Mathematical Excalibur

Volume 21, Number 1 — *October 2016 – March 2017*

Olympiad Corner

Below are the problems of the Final Round of the 65th Czech and Slovak Math Olympiad (April 4-5, 2016).

Problem 1. Let p>3 be a prime. Find the number of ordered sextuples (a,b,c,d,e,f) of positive integers, whose sum is $3p$, and all the fractions

$$\frac{a+b}{c+d}, \frac{b+c}{d+e}, \frac{c+d}{e+f}, \frac{d+e}{f+a}, \frac{e+f}{a+b}$$

are integers.

Problem 2. Let r and r_a be the radii of the inscribed circle and excircle opposite A of the triangle ABC. Show that if $r+r_a=|BC|$, then the triangle is right-angled.

Problem 3. Mathematics clubs are very popular in certain city. Any two of them have at least one common member. Prove that one can distribute rulers and compasses to the citizens in such a way that only one citizen get both (compass and ruler) and any club has to his disposal both, compass and ruler, from its members.

(continued on page 4)

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 15, 2017***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Miscellaneous Problems

Kin Y. Li

There are many Math Olympiad problems. Some are standard problems in algebra or in geometry or in number theory or in combinatorics, where there are some techniques for solving them. Then, there are problems that are not so standard, which cross two or more categories. In math problem books, they go under the category of miscellaneous problems. Some of these may arise due to curiosity. Then one may need to combine different facts to explain them. Below are some such problems we hope the readers will enjoy.

<u>*Example 1 (1995 USA Math Olympiad).*</u> A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational numbers q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculation with infinite precision. All functions are in terms of radians.

<u>*Solution.*</u> We will show that all numbers of the form $\sqrt{m/n}$, where m, n are positive integers, can be displayed by doing induction on $k=m+n$. (Since $r/s = \sqrt{r^2/s^2}$, these include all positive rational numbers.)

For k=2, pressing cos will display 1. Suppose the statement is true for integer less than k. Observe that if x is displayed, then letting $\theta=\tan^{-1}x$, we see

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - \theta \quad and \quad \tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{x}.$$

So we can display $1/x=\tan(\cos^{-1}(\sin x))$. Therefore, to display $\sqrt{m/n}$ with $k=m+n$, we may assume $m<n$. By the induction step, since $(n-m)+m = n < k$, $\sqrt{(n-m)/m}$ can be displayed. Then using

$$\phi = \tan^{-1}\sqrt{(n-m)/m} \quad and \quad \cos\phi = \sqrt{m/n},$$

we can display $\sqrt{m/n}$. This completes the induction.

<u>*Example 2 (1986 Brazilian Math Olympiad).*</u> A ball moves endlessly on a circular billiard table. When it hits the edge it is reflected. Show that if it passes through a point on the table three times, then it passes through it infinitely many times.

<u>*Solution.*</u> Suppose AB and BC are two successive chords of the ball's path. By the reflection law, $\angle ABO = \angle OBC$. Now $\triangle OAB$ and $\triangle OBC$ are isosceles. So $\angle AOB = \angle BOC$. Hence, $AB = BC$. Then every chord of the path has the same length d.

We now claim that through any given point P inside the circle there are at most two chords with length d. Let AB and CD be a chord containing P, with $AP=a$ and $CP=b$. The power of P with respect to the circle is $PA\cdot PB=PC\cdot PD$, which is $a(d-a)=b(d-b)$. Hence, $a=b$ or $a+b=d$. This means that P always divides the chord containing it in two segments of fixed lengths a and $d-a$. Now if three chords passes through P, the circle with center P and radius a would cut the circle of the billiard table three times, a contradiction.

Thus if the path passes through P more than twice, then on two occasions it must be moving along the same chord AB. That implies $\angle AOB$ is a rational multiple of 2π and hence the path will traverse AB repeatedly.

<u>*Example 3.*</u> Is there a way to pack 250 $1\times1\times4$ bricks into a $10\times10\times10$ box?

<u>*Solution.*</u> Assign coordinate (x,y,z) to each of the cells, where x,y,z= 0,1,…,9. Let the cell (x,y,z) be given color $x+y+z$ (mod 4). Note each $1\times1\times4$ brick contain all 4 colors exactly once. If the packing is possible, then there are exactly 250 cells of each color. However, a direct counting shows there are 251 cells of color 1, a contradiction. So such packing is impossible.

(continued on page 2)

Example 4 (2013 Singapore Math Olympiad). Six musicians gathered at a chamber music festival. At each scheduled concert some of the musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled so that every two musicians each must play for the other in some concert?

Solution. Let the musicians be A,B,C, D,E,F. We first show that four concerts are sufficient. The four concerts with the performing musicians: $\{A,B,C\}$, $\{A,D,E\}$, $\{B,D,F\}$ and $\{C,E,F\}$ satisfy the requirement. We shall now prove that three concerts are not sufficient. Suppose there are only three concerts. Since everyone must perform at least once, there is a concert where two of the musicians, say A, B, played. But they must also played for each other. Thus we have A played and B listened in the second concert and vice versa in the third. Now C,D,E,F must all perform in the second and third concerts since these are the only times when A and B are in the audience. It is not possible for them to perform for each other in the first concert. Thus the minimum is 4.

Example 5 (1999 Brazilian Math Olympiad). Prove that there is at least one nonzero digit between the 1,000,000th and the 3,000,000th decimal digits of $\sqrt{2}$.

Solution. Let us suppose that all digits between the 1,000,000th and the 3,000,000th decimal digits of $\sqrt{2}$ are zeros. Then

$$\sqrt{2}=\frac{n}{10^{1,000,000}}+\varepsilon, \qquad (*)$$

where n is a positive integer and $\varepsilon > 0$ satisfy

$$n<2\cdot 10^{1,000,000} \text{ and } \quad \varepsilon<(10^{-3})^{10^{1,000,000}}.$$

By squaring (*), we can get

$$2\cdot 10^{2,000,000}-n^2 = 2n\varepsilon 10^{1,000,000}+\varepsilon^2 10^{2.000.000}.$$

However, the left side is a positive integer and the right side is less than 1, which is a contradiction.

Example 6 (1995 Russian Math Olympiad). Is it possible to fill in the cells of a 9×9 table with positive integers ranging from 1 to 81 in such a way that the sum of the elements of every 3×3 square is the same?

Solution. Place 0,1,2,3,4,5,6,7,8 on the first, fourth and seventh rows. Place 3,4,5,6,7,8,0,1,2 on the second, fifth and eigth rows. Place 6,7,8,0,1,2,3,4,5 on the third, sixth and ninth rows. Then every 3×3 square contains 0 to 8. Consider this table and its 90° rotation. For each cell, fill it with the number $9a+b+1$, where a is the number in the cell originally and b is the number in the cell after the table is rotated by 90°. By inspection, 1 to 81 appears exactly once and every 3×3 square has sum 9×36+36+9=369.

Example 7. Can the positive integers be partitioned into infinitely many subsets such that each subset is obtained from any other subset by adding the same integer to each element of the other subset?

Solution. Yes. Let A be the set of all positive integers whose odd digit positions (from the right) are zeros. Let B be the set of all positive integers whose even digit positions (from the right) are zeros. Then A and B are infinite set and the set of all positive integers is the union of $a+B=\{a+b\colon b\in B\}$ as a range over the element of A. (For example, 12345 = 2040+10305 ∈ 2040+B.)

Example 8 (2015 IMO Shortlisted Problem proposed by Estonia). In Lineland there are $n\geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a right and left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so if a bulldozers reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B being to the right of A. We say that town A can *sweep* town B *away* if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly, B can sweep A away if the left bulldozer of B can move to A pushing off all bulldozers of the towns on its way.

Prove that there is exactly one town which cannot be swept away by any other one.

Solution. Let T_1, $T_2,\ldots,T_n$ be the towns enumerated from left to right. Observe first that, if town T_a can sweep away town T_b, then T_a also can sweep away every town located between T_a and T_b.

We prove by induction on n. The case n=1 is trivial. For the induction step, we first observe that the left bulldozer in T_1 and the right bulldozer in T_n are completely useless, so we may forget them forever. Among the other $2n-2$ bulldozers, we choose the largest one. Without loss of generality, it is the right bulldozer of some town T_k with $k<n$.

Surely, with this right bulldozer T_k can sweep away all towns to the right of it. Moreover, none of these towns can sweep T_k away; so they also cannot sweep away any town to the left of T_k. Thus, if we remove the towns T_{k+1}, $T_{k+2},\ldots,T_n$, none of the remaining towns would change its status of being (un)sweepable away by the others.

Applying the induction hypothesis to the remaining towns, we find a unique town among $T_1,T_2,\ldots,T_k$ which cannot be swept away. By the above reasons, it is also the unique such town in the initial situation. Thus the inductive step is established.

Example 9 (1991 Brazilian Math Olympiad). At a party every woman dances with at least one man, and no man dances with every woman. Show that there are men M and M' and women W and W' such that M dances with W, M' dances with W', but M does not dance with W', and M' does not dance with W.

Solution. Let M be one of the men who dance with the maximal number of women, W' one of the women he doesn't dance with, and M' one of the men W' dances with. If M' were to dance with every woman that M dances with, then the maximality of the number of women that M dances with would be contradicted, so there is a woman W that dances with M but not with M'.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 15, 2017.***

Problem 496. Let a,b,c,d be real numbers such that $a+\sin b > c+\sin d$, $b+\sin a > d+\sin c$. Prove that $a+b>c+d$.

Problem 497. Let there be three line segments with lengths 1, 2, 3. Let the segment of length 3 be cut into $n\geq 2$ line segments. Prove that among these $n+2$ segments, there exist three of them that can be put to form a triangle where each side is one of the three segments.

Problem 498. Determine all integers $n>2$ with the property that there exists one of the numbers $1,2,\ldots,n+1$ such that after its removal, the n numbers left can be arranged as $a_1,a_2,\ldots,a_n$ with no two of $|a_1-a_2|$, $|a_2-a_3|$, …, $|a_{n-1}-a_n|$, $|a_n-a_1|$ being equal.

Problem 499. Let ABC be a triangle with circumcenter O and incenter I. Let Γ be the escribed circle of $\triangle ABC$ meeting side BC at L. Let line AB meet Γ at M and line AC meet Γ at N. If the midpoint of line segment MN lies on the circumcircle of $\triangle ABC$, then prove that points O, I, L are collinear.

Problem 500. Determine all positive integers n such that there exist $k\geq 2$ positive rational numbers such that the sum and the product of these k numbers are both equal to n.

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Solutions

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Problem 491. Is there a prime number p such that both p^3+2008 and p^3+2010 are prime numbers? Provide a proof.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Prithwijit DE** (HBCSE, Mumbai, India), **EVGENIDIS Nikolaos** (M. N. Raptou High School, Palaiokastrou 10, Agia, Greece), **Karaganda** (Nazarbaev iIntellectual School, Nurligenov Temirlan - 9 grade student), **Koopa KOO**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S6), **Mark LAU**, **Toshihiro SHIMIZU** (Kawasaki, Japan), **Anderson TORRES**, **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

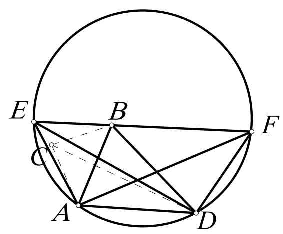
Let p be a prime. If $p\neq 7$, then $p^3\equiv -1$ or 1 (mod 7). Since $2008\equiv -1$ (mod 7) and $2010\equiv 1$ (mod 7), so either p^3+2008 or p^3+2010 is divisible by 7, hence composite. If $p = 7$, then $p^3+2010 = 2353 = 13\times 181$ is composite. Therefore, there is no such prime.

Problem 492. In convex quadrilateral $ADBE$, there is a point C within $\triangle ABE$ such that

$$\angle EAD+\angle CAB=180^\circ=\angle EBD+\angle CBA.$$

Prove that $\angle ADE=\angle BDC$.

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S6).



Let F be the second intersection of the circumcircle of $\triangle EAD$ and line EB. Then $\angle DBF=180^\circ-\angle EBD=\angle CBA$. Moreover,

$$\begin{aligned}\angle BDF &= 180^\circ-\angle AEB-\angle ADB\\ &= 180^\circ-(360^\circ-\angle EAD-\angle EBD)\\ &= 180^\circ-(\angle CAB+\angle CBA)=\angle BCA.\end{aligned}$$

These two relations give $\triangle BDF\sim\triangle BCA$. So $BD/BF=BC/BA$. Together with $\angle DBF=\angle CBA$, we have $\triangle BDC\sim\triangle BFA$. Then $\angle ADE=\angle AFE=\angle BFA=\angle BDC$.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 493. For $n\geq 4$, prove that $x^n-x^{n-1}-x^{n-2}-\cdots-x-1$ cannot be factored into a product of two polynomials with rational coefficients, both with degree greater than 1.

Solution. **Prithwijit DE** (HBCSE, Mumbai, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let $P_n(x) = x^n-x^{n-1}-x^{n-2}-\cdots-x-1$ and $Q_n(x)=(x-1)P_n(x)=x^{n+1}-2x^n+1$. The cases $n = 2$ or 3 follow directly from the rational root theorem. For $n\geq 4$, the Descartes' rule of signs shows there is a positive root r. It is easy to check $P_n(\sqrt{3})<0$. So $r>\sqrt{3}$.

If $P_n(s)=0$ with $|s|>1$, then $Q_n(s)=0$, which implies $|s|^n\,|s-2|=1$. We get $2\leq|s-2|+|s|=|s|^{-n}+|s|$. So $Q_n(|s|)\geq 0$. Since $Q_n(x)<0$ for $1<x<r$, we must have $|s|\geq r$. On the other hand, if $P_n(t)=0$ and $|t|<1$, then $1=|t-2||t|^n\leq 3|t|^n$. It follows that the absolute value of the product of all roots t of $P_n(x)$ with $|t|<1$ is at least 1/3. So r is the only root of $P_n(x)$ with absolute value greater than 1.

Assume $P_n(x)=f(x)g(x)$, where $f(x)$, $g(x)$ are monic polynomials with integer coefficients and $f(r)=0$. Then if $g(x)$ has positive degree, its roots would have absolute value less than 1 and so $|g(0)|<1$. This contradicts the constant term of $g(x)$, being $g(0)$, must be ± 1.

Other commended solvers: **Anderson TORRES**.

Problem 494. In a regular n-sided polygon, either 0 or 1 is written at each vertex. By using non-intersecting diagonals, Bob divides this polygon into triangles. Then he writes the sum of the numbers at the vertices of each of these triangles inside the triangle. Prove that Bob can choose the diagonals in such a way that the maximal and minimal numbers written in the triangles differ by at most 1.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

If all numbers written at the vertices of the polygon are equal, then the claim holds trivially. Hence assume that there are both zeros and ones among the numbers at the vertices. We prove by induction that, for every convex polygon, the partition into triangles can be chosen in such a way that Bob writes either 1 or 2 to each triangle.

If n=3, then this claim holds since the sum of the numbers at the vertices of a triangle can be neither 0 nor 3. If n=4, then draw the diagonal that connects the vertices where 0 and 1 are written, respectively, or, if such a diagonal does not exist, then an arbitrary diagonal. In both cases, only sums 1 and 2 can arise. If $n\geq 5$, then choose two consecutive vertices with different labels and a third vertex P that is not neighbor to either of them. Irrespective of whether the label of P is 0 or 1, we can draw the diagonal from it to one of the two consecutive vertices chosen before so that the labels of its endpoints are different. Now the polygon is divided into two convex polygons with smaller number of vertices so that both 0 and 1 occur among their vertex labels. By the induction hypothesis, both polygons can be partitioned into triangles with sum of labels of vertices either 1 or 2.

Other commended solvers: **William FUNG**.

Problem 495. The lengths of each side and diagonal of a convex polygon are rational. After all the diagonals are drawn, the interior of the polygon is partitioned into many smaller convex polygonal regions. Prove that the sides of each of these smaller convex polygons are rational numbers.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Anderson TORRES**.

We only need to show the quadrilateral case, since if this is showed, then the length of any segment of a diagonal connecting a vertex to an intersection point with other diagonal would be rational. Let $ABCD$ be a quadrilateral with all sides and diagonals have rational lengths. Let $\alpha=\angle ABD$ and $\beta=\angle DBC$. Let P be the intersection of AC and BD. Since

$$\cos\alpha=\frac{AB^2+BD^2-AD^2}{2AB\cdot BD},$$

$\cos\alpha$ is rational. Similarly, $\cos\beta$ and $\cos(\alpha+\beta)=\cos\angle ABC$ are rational. Then, since $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, so $\sin\alpha\sin\beta$ is also rational. Also, $\sin^2\beta=1-\cos^2\beta$ is rational. Thus, $\sin\alpha/\sin\beta=\sin\alpha\sin\beta/\sin^2\beta$ is rational. Then, AP/PC = area(ABD)/area(DBC) = $(AB\cdot BD\sin\alpha)/(BD\cdot BC\sin\beta)$ is rational. Therefore, AP and PC are rational. Similarly, PB and PD are rational.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania).

Olympiad Corner

(Continued from page 1)

Problem 4. For positive a, b, c, it holds $(a+c)(b^2+ac)=4a$. Find the maximal possible value of $b+c$ and find all triples (a,b,c), for which the value is attained.

Problem 5. There is $|BC|=1$ in a triangle ABC and there is a unique point D on BC such that $|DA|^2=|DB|\cdot|DC|$. Find all possible values of the perimeter of ABC.

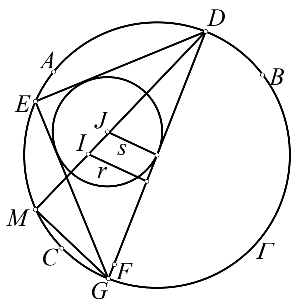
Problem 6. There is a figure of a prince on a field of a 6×6 square chessboard. The prince can in one move jump either horizontally or vertically. The lengths of the jumps are alternately either one or two fields, and the jump on the next field is the first one. Decide whether one can choose the initial field for the prince, so that the prince visits in an appropriate sequence of 35 jumps every field of the chessboard.

Miscellaneous Problems

(Continued from page 2)

Example 10. Two triangles have the same incircle. If a circle passes through five of the six vertices of the two triangles, then must it also pass the sixth vertex?

Solution. Let ABC and DEF be the triangles. Let A, B, C, D, E be on the same circle Γ, with radius R and center O. Suppose that F does not belong to Γ. Let $G\neq D$ be the intersection of DF with Γ. Let $\theta=\angle EDF=\angle EDG$. Let I and r be the common incenter and the inradius of $\triangle ABC$ and $\triangle DEF$. Let J and s be the incenter and the inradius of $\triangle DEG$.



We will prove that the incircle of $\triangle ABC$ and $\triangle DEG$ coincide. First, we prove that $I=J$ by showing $IM=JM$. It is well known that $IM=2R\sin(\theta/2)=EM$. From Euler's formula, $OI^2=R^2-2Rr$, which implies that the power of I with respect to Γ is $IM\cdot ID=2Rr$. Since $ID=r/\sin(\theta/2)$, we have $IM=2R\sin(\theta/2)=JM$. So $I=J$. This also proves $r=s$. Hence, the incircle of $\triangle ABC$ and $\triangle DEG$ are the same. Then $F=G$ follows.

Example 11 (1988 Brazilian Math Olympiad). A figure on a computer screen shows n points on a sphere, no four coplanar. Some pairs of points are joined by segments. Each segment is colored red or blue. For each point there is a key that switches the colors of all segments with that point as endpoint. For every three points there is a sequence of key presses that make all three segments between them red. Show that it is possible to make all the segments on the screen red. Find the smallest number of key presses that can turn all the segments red, starting from the worst case.

Solution. Consider three of the points. The parity of the number of blue segments of the triangle with these points as vertices doesn't change while switching the keys. Since it is possible to make all three segments red, the number of blue segments in each triangle is even.

Let P be one of the n points. Let A be the set of points connected to P by red points and B be the set of points connected to P by blue segments. Let $A_1, A_2\in A$. So PA_1 and PA_2 are both red and thus A_1A_2 is red. Now consider $B_1B_2\in B$. Then PB_1 and PB_2 are both blue and B_1B_2 is red. Finally consider $A\in A$ and $B\in B$. PA is red and PB is blue, so AB is blue. Put P in A. All this reasoning shows that segments in the same set are red and segments connecting points in different sets are blue.

Switching all points in set A will make all segments red. Indeed, all segments in A will change twice, one time from each of its edges, all segments connecting points from A and B will change once, turning from blue to red and segments in B won't change. This proves the first part.

For the second part, notice first that one needs to switch each point at most once. Let $|A|=k$ and $|B|=n-k$. If we switch a point from A and b points from B, we change at most $a(n-k)+bk$ blue segments. Suppose without loss of generality that $k\leq n-k$, hence $k\leq[n/2]$. Then $k(n-k)\leq a(n-k)+bk\leq a(n-k)+b(n-k)$, hence $k\leq a+b$. So the number of key presses is at most k and in the worst case, $[n/2]$. This number is needed to make all segments red if $|A|=[n/2]$.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2017 International Mathematical Olympiad (July 18-19, 2017) held in Brazil.

Problem 1. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & if \ \sqrt{a_n} \ is \ an \ \text{integer}, \\ a_n + 3 & otherwise, \end{cases}$$

for each $n \geq 0$. Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

Problem 2. Let $\mathbb{R}$ be the set of real numbers. Determine all functions f : $\mathbb{R} \rightarrow \mathbb{R}$ such that, for all real numbers x and y, $f(f(x)f(y)) + f(x+y) = f(xy)$.

Problem 3. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 21, 2017***.

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Notes on IMO2017

Tat Wing Leung

International Mathematical Olympiad (IMO) 2017 was held in Rio De Janeiro, Brazil from 12 to 24 July, 2017. Members of Hong Kong Team are as follows.

Tat Wing Leung (Leader)

Tak Wing Ching (Deputy Leader)

Man Yi Mandy Kwok, Shun Ming Samuel Lee, Yui Hin Arvin Leung, Cheuk Hin Alvin Tse, Jeff York Ye, Hoi Wai Yu (Contestants)

All contestants except Alvin Tse are entering universities during the academic year 2017-18. Thus we will have an essentially new team next year.

I went first to Brazil in July 13. Professor Shum Kar Ping, chairman of our Committee also went with me. He was to present the report of IMO2016. It was over quickly. Apparently members of the Advisory Board had nothing more to ask. Luckily it was done.

Upon arrival, I just had to follow the program closely and to attend Jury meetings. As claimed, I did experience *the famous Brazilian Hospitality* (this clause was copied from the program book) and I was quite happy in general.

As in these few years, in choosing the problems, first 4 easy problems, 1 from each of the four categories (Algebra, Combinatorics, Geometry and Number Theory) were selected. Then 4 medium problems, again 1 from each category was selected. Then members of the Jury (leaders) selected two easy problems of two categories, and the 2 medium problems from the two complementary categories were selected. It was claimed this scheme will help to produce a more balanced paper. But after a few years, I do think it is not necessarily true. First almost certain an easy geometry problem will be selected, thus all medium but nice geometry problems will be discarded. It is also almost certain two combinatorics problems will be selected. The papers will then become more predictable. Anyway members still chose this scheme.

Our contestants arrived on July 16. During the opening ceremony, July 17, I had a chance to look at them (from far away). In the opening ceremony, the speech of Marcelo Viana, director of IMPA (Instituto de Mathematica Pura e Applicada) was particularly genuine and moving. He talked about the IMO training and selection in Brazil in these 38 years. (Certainly it was not an easy task to select a team of 6 from 18 million youngsters). Then he also talked about Maryam Mirzakhani, the Iranian Mathematician, who was a 1994 and 1995 IMO gold medalist, 2014 Fields' medalist and passed away prematurely at age 40. Finally, he also talked about the upcoming International Congress of Mathematicians (ICM) 2018, to be held in Brazil.

The next two days (July 18 and 19) are contest days. The contestants had to sit for two 4.5 hours exam during the mornings. In the first half hours of the exams, there were Q&A times. In this year again they adapted a new scheme, namely they had 4 tables, 3 tables for problems 1, 2 and 3 (problems 4, 5 and 6 the next day), and so they were 4 queues. Clearly this is a more efficient scheme than before.

Again the next two days (July 20 and 21) were days of coordination, namely leaders and coordinators would decide the score of a particular problem. We followed the schedule to go to a particular table. We had a very capable deputy leader this year and so he knew well what our team members had done. So the process became relatively easy.

(continued on page 2)

Since the problems of IMO2017 is listed in this issue of Excalibur, I shall not reproduce them here nor to copy the proofs. I will only give a few comments of this year's problems. First a few key words came to my mind. My first word is algorithm (or construction). Indeed the proposer has been trying hard to think of a new scenario that when you try to solve the problem, you need to invent a new algorithm to solve the problem. For example, problems 5 and 6 do not need to know a lot of higher math, but you do need to have some sense of ingenuity to think of a new scheme or method to solve a particular problem. In problem 3, you had to show an algorithm does not exist. The second word is induction, namely in these problems, small cases (cases with smaller parameters) were easy. So one might try to consider if the method of induction does work. It was not obvious. The third word is geometry. In this year, only 3 of us could solve the relatively easy geometry problem. Indeed this year's geometry problem (P4), no new constructions are required, no new transformations (inversion, homothety, etc.) are needed. It is simply correct drawing and angle chasing. So I must admit that we have reverted back to our usual tradition.

Now I will say a few more words on the individual problems and the performance of our team. Problem 1 is a number theory problem. Once a contestant tries a few cases and guess the correct answer ($a_0 \equiv 0 \pmod 3$), then it is not too hard to prove $a_0 \equiv 1,2 \pmod 3$ do not work but $a_0 \equiv 0 \pmod 3$ works. Our team this year is relatively mature and relatively well trained. So all of them solved the problem and we have a perfect score.

Problem 2 is a functional equation, showing $f(f(x)f(y))+f(x+y)=f(xy)$ for all real x and y will imply $f(x)=0$ or $f(x) = \pm(x-1)$. The most troublesome thing is the marking scheme. It is easy to get the first 3 points, but it is real hard to get an extra point, i.e., proving injectivity and onward. A leader secretly showed me the scores of problem 2 of his team, apparently he was dismayed by the performance. I was not sure. At the end I found their team scored 1 more point than us.

For problem 3, I had (and still have) a very serious concern about it. Observe only two contestants scored 7 points (a Russian and an Australian contestant), and also none of the USA team and the Chinese team (plus other teams) together scored any point at all. I suspect many contestants are like me and simply don't know what exactly is going on. Indeed it is not quite sure what it means by "no matter how" and what exactly it means by a tracking device, I was told it is not like the "best strategy". Indeed when you look at the solution, you get the idea such a strategy (or algorithm) does not exist. The solution is roughly as follows. Assume the rabbit moves in a straight line, and with luck (this term appears quite a few times in the solution) the tracking device also moves in a straight line. Because of this happening, the hunter can only move along a straight line (also with no justification but intuition) and follow the rabbit, and after finitely many steps, the distance between the rabbit and the hunter will only increase (easy to show by simple geometry). Thus there is no best strategy. I am still awaiting members to educate me on this problem.

Problem 4 was a relatively easy geometry exercise.

We did best in problem 5 among all teams, (our deputy leader reminded me about this point). Indeed altogether we scored 26 points. So essentially 4 of us solved the problem, while other teams scored at most 23 points. This shows our team does know something about problem solving. Indeed the problem is equivalent to say there are $N(N+1)$ distinct integers randomly placed in a row, say, you can throw away $N(N-1)$ of them, and among the remaining integers, the largest integer and the second largest integer will stick together, so are the third largest and the fourth largest integer will stick together, and so on. Not too hard?

For Problem 6, an ordered pair (x,y) of integers is a primitive point if $\gcd(x,y)=1$. Now given a set of finitely many primitive points (x_i, y_i), $1\le i\le n$, we need to find a homogeneous polynomial $g(x,y)$ such that $g(x_i,y_i)=1$. If there is only one primitive point, then it is trivial, by Euclidean algorithm. The hard part is how to move on by induction. But it is not at all easy.

At the end Shun Ming was awarded a gold medal (25 points), Mandy a silver (23 points), Jeff (18 points), Hoi Wai (17 points) and Cheuk Hin (17 points) all received Bronze medals. Yui Hin (11 points) managed to get a honorable mention. Our rank is 26 among 111 countries/regions. Surely the result was not as good as last year nor as we had hoped for. Nevertheless there were certain things we can say. Indeed this was the 30th consecutive year that we sent teams to IMOs. No matter what, it is not an easy matter and it should be a date to remember. (Better still, we hosted the event in 1994 and 2016). Also Mandy Kwok was the second girls among all girl contestants. IMPA this year gave out 5 prizes to female contestants. Initially I thought Mandy should have a chance to get a prize. Later I found out the prizes were for the top female students who contribute the most to their respective team's score. So I understand why she was not eligible for the prize. Nevertheless I must say we are very glad to see her improving very well in these few years. Finally we managed to get the highest score in Problem 5. I think this is an indication that our team is comparable with any other team. They really don't have much special recipe we don't envisage.

I hasten to say the cut scores of IMO this year cannot be said to be ideal. Indeed the cut scores for gold is 25, for silver 19, and bronze 16. One may say the easy problems (problems 1 and 4) were too easy and the four other problems too hard. The easy problem were too easy. Hence 14 points was not enough for a bronze and the hard problems too hard. Thus 25 points was good enough to get a gold. Really we expect a contestant to solve at least 2 problems ($\geq$14 points) to get a bronze, at least 3 problems ($\geq$21 points) a silver, and at least 4 problems ($\geq$28 points) to get a gold. Some people expect a contestant should solve nearly at least 5 problems to get a gold. Really what is the point to set a problem so that only 2 out of 615 contestants can solve it?

Since we are trailing behind some other Asian countries this year, it was suggested that more money should be put into this activity. In my opinion the stakeholders (members of the Committee, the Academy and the Gifted Section of Education, but most important of all, past and present trainees) should sit together and sort out what exactly do we want, how much money/resource should be put into it and who will contribute what, etc. I suppose it is time to start thinking.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to Dr. *Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***October 21, 2017.***

Problem 501. Let x, y, s, m, n be positive integers such that $x+y=s^m$ and $x^2+y^2=s^n$. Determine the number of digits s^{300} has in base 10.

Problem 502. Let O be the center of the circumcircle of acute $\triangle ABC$. Let P be a point on arc BC so that A, P are on opposite sides of side BC. Point K is on chord AP such that BK bisects $\angle ABC$ and $\angle AKB > 90°$. The circle Ω passing through C, K, P intersect side AC at D. Line BD meets Ω at E and line PE meets side AB at F. Prove that $\angle ABC = 2\angle FCB$.

Problem 503. Let S be a subset of $\{1,2,\ldots,2015\}$ with 68 elements. Prove that S has three pairwise disjoint subsets A, B, C such that they have the same number of elements and the sums of the elements in A, B, C are the same.

Problem 504. Let $p>3$ be a prime number. Prove that there are infinitely many positive integers n such that the sum of k^n for $k=1,2,\ldots,p-1$ is divisible by p^3.

Problem 505. Determine (with proof) the least positive real number r such that if z_1, z_2, z_3 are complex numbers having absolute values less than 1 and sum 0, then

$$|z_1z_2+z_2z_3+z_3z_1|^2 + |z_1z_2z_3|^2 < r.$$

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Solutions

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Problem 496. Let a,b,c,d be real numbers such that $a+\sin b > c+\sin d$, $b+\sin a > d+\sin c$. Prove that $a+b>c+d$.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

For $x\ge 0$, $|\sin x|\le x$. Let $s = a-c$ and $t = d-b$. We have

$$\begin{aligned} s = a-c &> \sin d - \sin b \\ &= 2\cos[(d+b)/2]\sin[(d-b)/2] \\ &\ge -2|\sin(t/2)| \end{aligned}$$

and

$$\begin{aligned} t = d-b &< \sin a - \sin c \\ &= 2\cos[(a+c)/2]\sin[(a-c)/2] \\ &\le 2\,|\sin(s/2)|. \end{aligned}$$

If $s\ge 0$, then $t< 2|\sin(s/2)|\le s$. Similarly, if $t\le 0$, then $s > -2|\sin(-t/2)| \ge -2(-t/2) = t$.

Finally, if $s < 0 < t$, then $-s < 2|\sin(t/2)| \le t$ and $t < 2|\sin(s/2)| = |\sin(-s/2)| \le -s$, which leads to a contradiction.

Comment: The above solution avoided calculus as it used $\sin x \le x$ for $0\le x\le 1$, which followed by taking points A, B on a unit circle with center O such that $\angle AOB = 2x$, then the length $2x$ of arc AB is greater than the length $2\sin x$ of chord AB.

Other commended solvers: **Jason FONG** and **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School).

Problem 497. Let there be three line segments with lengths 1, 2, 3. Let the segment of length 3 be cut into $n\ge 2$ line segments. Prove that among these $n+2$ segments, there exist three of them that can be put to form a triangle where each side is one of the three segments.

Solution. **William FUNG**, **Mark LAU** (Pui Ching Middle School), **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Note line segments with lengths $a\le b\le c$ form a triangle if and only if $a+b>c$. Let $a_1\le a_2 \le\cdots\le a_n$ be the lengths of such n segments with sum equals to 3. Assume there exists i such that $a_i>1$. If $1<a_i<2$, then the segments with length $1,a_i,2$ forms a triangle since $1+a_i>2$. If $2\le a_i$, then the segments with length $1,2,a_i$ forms a triangle since $1+2>a_i$. It remains to consider the case all $a_i\le 1$. Then $i\ge 3$.

Assume no 3 of these segments form a triangle. Then $a_1+a_2\le a_3$, $a_2+a_3\le a_4$, …, $a_{n-2}+a_{n-1}\le a_n$, $a_n+1\le 2$. Adding these and cancelling $a_3,\ldots,a_n,1$ on both sides, we have

$$3+a_2 = (a_1+a_2+\cdots+a_n)+a_2 \le 2,$$

which yields $a_2\le -1$, a contradiction.

Problem 498. Determine all integers $n>2$ with the property that there exists one of the numbers $1,2,\ldots,n+1$ such that after its removal, the n numbers left can be arranged as $a_1,a_2,\ldots,a_n$ with no two of $|a_1-a_2|, |a_2-a_3|, \ldots, |a_{n-1}-a_n|, |a_n-a_1|$ being equal.

Solution. **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School), **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

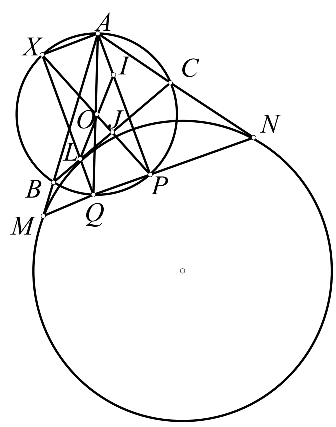
Since no two of $|a_1-a_2|, |a_2-a_3|, \ldots, |a_{n-1}-a_n|, |a_n-a_1|$ being equal and each is at most n, they must be $1,2,\ldots,n$ in some order. So $|a_1-a_2| + |a_2-a_3| + \cdots + |a_{n-1}-a_n|+|a_n-a_1|=1+2+\cdots+n=n(n+1)/2$. From $a \equiv |a| \pmod 2$ and $(a_1-a_2)+(a_2-a_3)+\cdots+(a_{n-1}-a_n)+(a_n-a_1)=0$, we see $|a_1-a_2|+ |a_2-a_3| + \cdots + |a_{n-1}-a_n| + |a_n-a_1|$ is even. For $n(n+1)/2$ to be even, this implies $n \equiv 0$ or $-1 \pmod 4$.

In the case $n=4k$, remove $k+1$ and let $a_1=4k+1$, $a_2=1$, $a_3=4k$, $a_4=2$, …, $a_{2k-1}=3k+2$, $a_{2k}=k$, $a_{2k+1}=3k+1$, $a_{2k+2}=k+2$, $a_{2k+3}=3k$, $a_{2k+4}= k+3$, …, $a_{4k-1}=2k+2$ and $a_{4k}=2k+1$.

In the case $n=4k-1$, remove $3k$ and let $a_1=4k$, $a_2=1$, $a_3=4k-1$, $a_4=2$, …, $a_{2k-1}=3k+1$, $a_{2k}=k$, $a_{2k+1}=3k-1$, $a_{2k+2}=k+1$, $a_{2k+3}=3k-2$, …, $a_{4k-2}= 2k-1$, $a_{4k-1}=2k$.

Problem 499. Let ABC be a triangle with circumcenter O and incenter I. Let Γ be the escribed circle of $\triangle ABC$ meeting side BC at L. Let line AB meet Γ at M and line AC meet Γ at N. If the midpoint of line segment MN lies on the circumcircle of $\triangle ABC$, then prove that points O, I, L are collinear.

Solution. **George SHEN.**



Let P be the midpoint of MN. From $AM=AN$, we see $AP\perp MN$. So A,I,P are collinear. Let Q be on MN such that $LQ\perp MN$. Now $\angle BMQ=\angle CNQ$ and

$$\frac{MQ}{NQ} = \frac{ML\cos\angle LMQ}{NL\cos\angle LNQ} = \frac{2MB\cos\angle LMB\cos\angle LNC}{2NC\cos\angle LNC\cos\angle LMB} = \frac{MB}{NC}.$$

This implies $\triangle BMQ$, $\triangle CNQ$ are similar.

Let $a=BC$, $b=CA$, $c=AB$, $s=(a+b+c)/2 = AM = AN$ and $\alpha = \angle BAC$.

We have

$$AP = AM\cos(\alpha/2) = s\cos(\alpha/2).$$

By extended sine law, $BC = a = 2R\sin\alpha$. From $IP=BP=CP$ [see <u>Math Excalibur</u>, *vol.* 11, *no.* 2, *page* 1, *Theorem in middle column–Ed.*], we have

$$a = BC = 2BP\sin\frac{180^\circ-\alpha}{2} = 2BP\cos\frac{\alpha}{2},$$

$$\cos\frac{\alpha}{2} = \frac{a}{2IP} = \frac{a}{2(AP-AI)} = \frac{a}{2(s\cos\frac{\alpha}{2}-AI)}.$$

Applying $AI\cos(\alpha/2)=s-a$ and the last equation, we can get

$$2s\cos^2\frac{\alpha}{2} = 2s-a = b+c,$$

$$2s\sin^2\frac{\alpha}{2} = a.$$

Next $MN = 2AM\sin(\alpha/2) = 2s\sin(\alpha/2)$ and $(MQ+NQ)\sin(\alpha/2) = MB+NC$. Using $MQ/NQ=MB/NC$, we get

$$MQ\sin(\alpha/2)=MB$$

and

$$NQ\sin(\alpha/2)=NC,$$

which says $\angle QBA = 90° = \angle QCA$. Then Q is on Γ and AQ is a diameter of Γ.

Let line LQ meet the circumcircle Γ of $\triangle ABC$ at X as labeled in the figure. Observe that $APQX$ is a rectangle and AQ, XP are diameters of Γ intersecting at O. We claim $LQ=AI$ (then $LI \cap AQ$ at O and so O,I,L are collinear).

Now $BO=CO$, $BJ=CJ$ and $\angle BAP = \angle CAP$ implies $BP=CP$. Hence, O, J, P are collinear. Next $OJ \perp BC$ implies $\angle LJP = 90° = \angle LQP$. Then, J,P,Q,L are concyclic. Hence,

$$XL\cdot XQ = XJ\cdot XP$$

Let R be the circumradius of $\triangle ABC$. From

$$XJ = \frac{a}{2}\cot\frac{\alpha}{2},\ XP = 2R,$$

$$IP = 2R\sin\frac{\alpha}{2},\ AP = s\cos\frac{\alpha}{2},$$

We get $XJ\cdot XP = IP\cdot AP$. Then $XL\cdot XQ = IP\cdot AP$. Since $XQ=AP$, so $XL=IP$. Then $QL=XQ-XL=AP-IP=AI$. The conclusion follows.

Other commended solvers: **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 500. Determine all positive integers n such that there exist $k\geq 2$ positive rational numbers such that the sum and the product of these k numbers are both equal to n.

Solution. **Mark LAU** (Pui Ching Middle School), **LW Solving Team** (S.K.H. Lam Woo Memorial Secondary School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Observe that for a composite number n, there exist integer $s,t\geq 2$ such that $n=st$, the sequence $s,t,1,1,\ldots,1$ (with $st-s-t$ 1's) has sum and product equals $st=n$.

For prime numbers $n\geq 11$, the sequence $n/2,1/2,2,2,1,1,\ldots,1$ (with $n-4-(n+1)/2$ 1's) satisfies the condition by a simple checking.

For $n=7$, the sequence 9/2, 4/3, 7/6, satisfies the condition by a simple checking.

Next we claim the cases n=1,2,3,5 have no solution. Assume $a_1, a_2,\ldots,a_k$ are positive rational numbers with sum and product equals to n. By the AM-GM inequality, we have

$$\frac{n}{k} = \frac{a_1+\cdots+a_k}{k} \geq \sqrt[k]{a_1\cdots a_k} = \sqrt[k]{n}.$$

Then $n\geq k^{k/(k-1)}>k$. Since $n>k\geq 2$, cases n=1 or 2 are impossible.

Finally, for n=3 or 5, since $3^{3/(3-1)}=5.1\ldots$ implies k=2, so only cases (n,k) = (3,2) and (5,2) remain. Now

$$\begin{aligned}(a_1-a_2)^2 &= (a_1+a_2)^2-4a_1a_2\\ &= n^2-4n = -3 \text{ or } 5,\end{aligned}$$

which have no rational solutions a_1, a_2. Therefore, the answers are all positive integers except 1,2,3,5.

Olympiad Corner

(Continued from page 1)

Problem 3 (Cont'd).

(i) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

(ii) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

(iii) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds she can ensure that the distance between her and the rabbit is at most 100?

Problem 4. Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Problem 5. An integer $N\geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

⋮

(N) no one stands between the two shortest players.

Show that this is always possible.

Problem 6. An ordered pair (x,y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x,y) in S, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Mathematical Excalibur

Volume 21, Number 3 — *October 2017 – January 2018*

Olympiad Corner

Below are the problems of the 20th Hong Kong (China) Mathematical Olympiad held on December 2, 2017. Time allowed is 3 hours.

Problem 1. The sequence $\{x_n\}$ is defined by $x_1=5$ and $x_{k+1}=x_k^2-3x_k+3$ for $k=1,2,3,\ldots$. Prove that $x_k>3^{2^{k-1}}$ for all positive integer k.

Problem 2. Suppose $ABCD$ is a cyclic quadrilateral. Produce DA and DC to P and Q respectively such that $AP=BC$ and $CQ=AB$. Let M be the midpoint of PQ. Show that $MA\perp MC$.

Problem 3. Let k be a positive integer. Prove that there exists a positive integer ℓ with the following property: if m and n are positive integers relatively prime to ℓ such that $m^m\equiv n^n \pmod{\ell}$, then $m\equiv n \pmod{k}$.

Problem 4. Suppose 2017 points in a plane are given such that no three points are collinear. Among the triangles formed by any three of these 2017 points, those triangles having the largest area are said to be *good*. Prove that there cannot be more than 2017 good triangles.

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

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Functional Inequalities

Kin Y. Li

In the volume 8, number 1 issue of Math Excalibur, we provided a number of examples of functional equation problems. In the volume 10, number 5 issue of Math Excalibur, problem 243 in the problem corner section was the first functional inequality problem we posed. That one was from the 1998 Bulgarian Math Olympiad. In this article, we would like to look at some functional inequality problems that appeared in various math Olympiads.

<u>*Example 1 (2016 Chinese Taipei Math Olympiad Training Camp.*</u> Let function $f:[0,+\infty)\to[0,+\infty)$ satisfy

(1) for arbitrary $x,y\ge 0$, we have

$$f(x)f(y)\le y^2 f\left(\frac{x}{2}\right)+x^2 f\left(\frac{y}{2}\right);$$

(2) for arbitrary $0\le x\le 1$, we have $f(x)\le 2016$.

Prove that for arbitrary $x\ge 0$ we have $f(x)\le x^2$.

<u>*Solution.*</u> In (1), let $x=y=0$, then $f(0)=0$. Assume there is $x_0>0$ such that $f(x_0)>x_0^2$. By (1), we see $f(x_0/2)>x_0^2/2$. By math induction, for all positive integer k, we have

$$f\left(x_0/2^k\right)>2^{2^k-2k-1}x_0^2.$$

As k gets large, eventually we have $x_0/2^k$ is in $[0,1]$, but $f(x_0/2^k)>2016$. This contradicts (2). So for all $x\ge 0$, $f(x)\le x^2$.

<u>*Example 2 (2005 Russian Math Olympiad).*</u> Does there exist a *bounded* function $f:\mathbb{R}\to\mathbb{R}$ such that $f(1)>0$ and for all $x,y\in\mathbb{R}$, it satisfies the inequality

$$f^2(x+y)\ge f^2(x)+2f(xy)+f^2(y)\ ?$$

<u>*Solution.*</u> Assume such f exists. Let $a=2f(1)>0$. For $x_1\ne 0$, let $y_1=1/x_1$, then

$$f^2(x_1+y_1)\ge f^2(x_1)+2f(1)+f^2(y_1)\ge f^2(x_1)+a.$$

For $n>1$, let $x_n=x_{n-1}+y_{n-1}$, $y_n=1/x_n$. Then

$$f^2(x_n+y_n)\ge f^2(x_n)+a=f^2(x_{n-1}+y_{n-1})+a \ge f^2(x_{n-1})+2a\ge\cdots\ge f^2(x_1)+na.$$

As $n\to\infty$, f becomes unbounded, which is a contradiction.

<u>*Example 3 (2016 Ukranian Math Olympiad).*</u> Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that for arbitrary real numbers x, y, we have

$$f(x-f(y))\le x-yf(x)\ ?$$

<u>*Solution.*</u> Assume such function exists. Let $y=0$. Then $f(x-f(0))\le x$. Replacing x by $x+f(0)$, we get $f(x)\le x+f(0)$. Then setting $x=f(y)$, we get

$$f(0)\le f(y)-yf(f(y))\le y+f(0)-yf(f(y)),$$

which implies $yf(f(y))\le y$. If $y<0$, then

$$1\le f(f(y))\le f(y)+f(0)\le y+2f(0).$$

The last inequality is satisfied for all $y<0$, which is a contradiction.

<u>*Example 4 (The Sixth IMAR Math Competition, 2008).*</u> Show that for any function $f:(0,+\infty)\to(0,+\infty)$ there exists real numbers $x>0$ and $y>0$ such that $f(x+y)<yf(f(x))$.

<u>*Solution.*</u> Assume $f(x+y)\ge yf(f(x))$ for all $x,y>0$. Let $a>1$, then $t=f(f(a))>0$. Now for $b\ge a(1+t^{-1}+t^{-2})>a$, we have

$$f(b)=f(a+(b-a))\ge(b-a)f(f(a))=(b-a)t\ge a(1+t^{-1})>a.$$

Then

$$f(f(b))=f(a+(f(b)-a))\ge(f(b)-a)t\ge a.$$

If we take $x\ge(ab+2)/(a-1)>b$, then

$$f(x)=f(b+(x-b))\ge(x-b)f(f(b))\ge(x-b)a\ge x+2.$$

Hence, $f(x)>x+1$ (*). However,

$$f(f(x))=f(x+(f(x)-x))\ge(f(x)-x)f(f(x)).$$

Cancelling $f(f(x))$ on both sides, we get $f(x)\le x+1$, which contradicts (*).

(continued on page 2)

Example 5 (2016 Romanian Math Olympiad). Determine all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying for arbitrary $a, b\in\mathbb{R}$, we have

$$f(a^2)-f(b^2)\le(f(a)+b)(a-f(b)).\quad (1)$$

Solution. Let $a=b=0$, then $f^2(0)\le 0$, so $f(0)=0$. Let $b=0$, then $f(a^2)\le af(a)$. Let $a=0$, then $f(b^2)\ge bf(b)$. So for all x, we have (2) $f(x^2)=xf(x)$. Using this on the left side of (1), we get (3) $f(a)f(b)\le ab$. Next, by (2), we have

$$-xf(-x)=f((-x)^2)=f(x^2)=xf(x).$$

So f is an odd function. This implies

$$f(a)f(b)=-f(a)f(-b)\ge -(-ab)=ab.$$

Using (3), we have $f(a)f(b)=ab$. Then $f^2(1)=1$. So $f(1)=\pm1$. Hence, for all x, $f(x)f(1)=x$, i.e. either $f(x)=x$ for all x or $f(x)=-x$ for all x. Simple checking shows both of these satisfy (1).

Example 6 (1994 APMO). Let f: $\mathbb{R}\to\mathbb{R}$ be a function such that

(i) for all $x,y\in\mathbb{R}$

$$f(x)+f(y)+1\ge f(x+y)\ge f(x)+f(y),$$

(ii) for all $x\in[0,1)$, $f(0)\ge f(x)$,

(iii) $-f(-1)=f(1)=1$.

Find all such functions.

Solution. By (iii), $f(-1)=-1$, $f(1)=1$. So $f(0)=f(-1+1)\ge f(1)+f(-1)=0$. By (i), $f(1)=f(1+0)\ge f(1)+f(0)$. So $f(0)\le 0$. Then $f(0)=0$.

Next we claim $f(x)=0$ for all x in $(0,1)$. Since $f(0)=0$, by (ii), $f(x)\le 0$ for all x in $(0,1)$. By (i) and (ii), $f(x)+f(1-x)+1 f(1)=1$. So $f(x)\ge -f(1-x)$. If $x\in(0,1)$, then $1-x\in(0,1)$. So $f(1-x)\le 0$ and $f(x)\ge -f(1-x)\ge 0$. Then $f(x)=0$.

Next by (i) and (iii), we have $f(x+1)\ge f(x)+f(1)=f(x)+1$ and $f(x)\ge f(x+1)+f(-1)=f(x+1)-1$. These give $f(x+1)=f(x)+1$.

So $f(x)=0$ for $x\in[0,1)$ and $f(x+1)=f(x)+1$. Hence, $f(x)=[x]$. We can check directly $[x]$ satisfies (i), (ii) and (iii).

Example 7 (2007 Chinese IMO Team Training Test). Does there exist any function $f:\mathbb{R}\to\mathbb{R}$ satisfy $f(0)>0$ and

$$f(x+y)\ge f(x)+y\,f(f(x)) \text{ for all } x,y\in\mathbb{R}?$$

Solution. Assume such function exists. In that case, we claim there would exist real z such that $f(f(z))>0$. (Otherwise, for all x, $f(f(x))\le 0$. So for all $y\le 0$, we have $f(x+y)\ge f(x)+yf(f(x))\ge f(x)$. Then f is a decreasing function. So for all $x\in\mathbb{R}$, $f(0)>0\ge f(f(x))$, which implies $f(x)>0$. This contradicts $f(f(x))\le 0$.)

From the claim, we see as $x\to+\infty$, $f(z+x)\ge f(z)+xf(f(z))\to+\infty$. So we get

$$f(x)\to+\infty \text{ as well as } f(f(x))\to+\infty.$$

Then there are $x, y>0$ such that $f(x)\ge 0$, $f(f(x))>1$, $f(x+y)>0$, $f(f(x+y+1))>0$ and (*) $y\ge(x+1)/(f(f(x))-1)$. Define $A=x+y+1$, $B=f(x+y)-(x+y+1)$. Then $f(f(A))>0$ and

$$f(x+y)\ge f(x)+yf(f(x))\ge x+y+1 \text{ by (*)}.$$

So $B\ge 0$. Next,

$$\begin{aligned} f(f(x+y)) &= f(A+B)\ge f(A)+Bf(f(A))\\ &\ge f(A)=f((x+y)+1)\\ &= f(x+y)+f(f(x+y))\\ &> f(f(x+y)),\end{aligned}$$

which is a contradiction.

Example 8 (2015 Greek IMO Team Selection Test). Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for arbitrary $x, y\in\mathbb{R}$, we have

$$f(xy)\le yf(x)+f(y).\qquad (1)$$

Solution. In (1), using $-y$ to replace y, we get

$$f(-xy)\le -yf(x)+f(-y).\qquad (2)$$

Adding (1) and (2), we get

$$f(xy)+f(-xy)\le f(y)+f(-y).\qquad (3)$$

Setting $y=1$, we get

$$f(x)+f(-x)\le f(1)+f(-1).\qquad (4)$$

In (3), using $1/y$ with $y\ne 0$ to replace x, we get

$$f(1)+f(-1)\le f(y)+f(-y).\qquad (5)$$

By (4) and (5), for $y\ne 0$, we have

$$f(y)+f(-y)=f(1)+f(-1).$$

Let $c=f(1)+f(-1)$. Then (2) becomes

$$c-f(xy)\le -yf(x)+c-f(y).$$

Then

$$f(xy)\ge yf(x)+f(y).\qquad (6)$$

By (1) and (6), for all $x,y\ne 0$,

$$f(xy)=yf(x)+f(y).\qquad (7)$$

Setting $x=y=1$, we get $f(1)=0$. In (7), interchanging x and y, we get

$$f(yx)=xf(y)+f(x).\qquad (8)$$

Subtracting (7) and (8), we get

$$(y-1)f(x)=(x-1)f(y).$$

Then for $x, y\ne 0,1$, we get $\dfrac{f(x)}{x-1}=\dfrac{f(y)}{y-1}$.

Since $f(1)=0$, we see there exists a such that $f(x)=a(x-1)$ for all $x\ne 0$. Setting $x=0$ in (1), we get $f(y)\ge(1-y)f(0)$. Then for $y\ne 0$, we get $a(y-1)\ge(1-y)f(0)$, which is $(y-1)(a+f(0))\ge 0$. Then $a=-f(0)$ and we get for all real x, $f(x)=f(0)(1-x)$. Setting $f(0)$ to be any real constant, we can check all such functions satisfy (1).

Example 9 (2013 Croatian IMO Team Selection Test). Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real numbers x, y, we have $f(1)\ge 0$ and

$$f(x)-f(y)\ge(x-y)f(x-y).\quad (*)$$

Solution. Setting $y=x-1$, we get $f(x)-f(x-1)\ge f(1)\ge 0$. So

$$f(x)\ge f(x-1).\qquad (1)$$

Setting $y=0$, we get

$$f(x)-f(0)\ge xf(x).\qquad (2)$$

Replacing y by x and x by 0, we get

$$f(0)-f(x)\ge -xf(-x).\qquad (3)$$

Adding (2) and (3), we get

$$0\ge xf(x)-xf(-x).$$

Then for every $x>0$, we get

$$f(-x)\ge f(x).\qquad (4)$$

Setting $x=1$, $y=0$ in (0), we get

$$f(0)\le 0.\qquad (5)$$

By (5), (1), (4), we get $0\ge f(0)\ge f(-1)\ge f(1)\ge 0$. So $f(0)=f(-1)=f(1)=0$. Using (1) repeatedly, we get

$$f(x)\ge f(x-1)\ge f(x-2)\ge\cdots,\quad (6)$$

i.e. $f(x)\ge f(x-k)$ for all real x, positive integer k. Using (6), (1) and replacing x by $x-1$ and y by -1 in (*), we get

$$f(x)\ge f(x-1)=f(x-1)-f(-1)\ge xf(x).$$

Then $f(x)(x-1)\le 0$. So if $x>1$, then $f(x)\le 0$. If $x<1$, then $f(x)\ge 0$.

For $x>1$, there is $y<1$ such that $k=x-y$ is a positive integer. Then

$$0\ge f(x)\ge f(x-k)=f(y)\ge 0.$$

So for $x>1$, $f(x)=0$. Similarly, for $x<1$, there is $y>1$ such that $k=y-x$ is a positive integer. Then as above, all $f(x)=0$. We can check directly $f(x)=0$ satisfies (*).

Example 10 (2011 IMO Problem 3 proposed by Belarus). Let $f:\mathbb{R}\to\mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y)\le yf(x)+f(f(x))\qquad (1)$$

for all real numbers x and y. Prove that $f(x)=0$ for all $x\le 0$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***February 10, 2018.***

Problem 506. Points A and B are on a circle Γ_1. Line AB is tangent to another circle Γ_2 at B and the center O of Γ_2 is on Γ_1. A line through A intersects Γ_2 at points D and E (with D between A and E). Line BD intersects Γ_1 at a point F, different from B. Prove that D is the midpoint of BF if and only if BE is tangent to Γ_1.

Problem 507. Determine all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that for all $x, y \in\mathbb{R}$,

$$(x-2)f(y) + f(y+2f(x)) = f(x+yf(x)).$$

Problem 508. Determine the largest integer k such that for all integers x,y, if $xy+1$ is divisible by k, then $x+y$ is also divisibly by k.

Problem 509. In $\triangle ABC$, the angle bisector of $\angle CAB$ intersects BC at a point L. On sides AC, AB, there are points M, N respectively such that lines AL, BM, CN are concurrent and $\angle AMN=\angle ALB$. Prove that $\angle NML= 90°$.

Problem 510. Numbers 1 to 20 are written on a board. A person randomly chooses two of these numbers with a difference of at least 2. He adds 1 to the smaller one and subtracts 1 from the larger one. Then he performs an operation by replacing the original two chosen numbers on the board with the two new numbers. Determine the maximum number of times he can do this operation.

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Solutions

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Problem 501. Let x, y, s, m, n be positive integers such that $x+y=s^m$ and $x^2+y^2=s^n$. Determine the number of digits s^{300} has in base 10.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **Soham GHOSH** (RKMRC Narendrapur, Kalkata, India), **Mark LAU, LEE Jae Woo** (Hamyang High School, South Korea), **Toshihiro SHIMIZU** (Kawasaki, Japan).

Since $s^{2m} = (x+y)^2 > x^2+y^2 = s^n$, so $2m > n$. Then

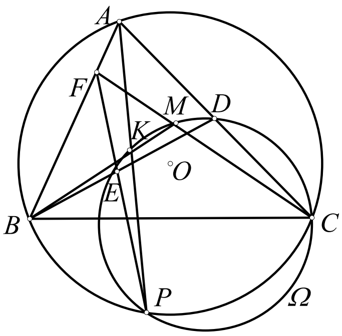
$$0 \le (x-y)^2 = 2(x^2+y^2)-(x-y)^2 = 2s^n-s^{2m} = s^n(2-s^{2m-n}).$$

If $s\ge3$, then we have $2-s^{2m-n}\le2-s<0$, a contradiction. If $s=1$, then we have $1+1\le x+y=s^m=1$, a contradiction. So s must be 2. Since $\log_{10}2^{300} = 300\log_{10}2 = 0.3010\ldots\times300= 90.3\ldots$, 2^{300} has 91 digits.

Other commended solvers: **DBS Maths Solving Team** (Diocesan Boys' School), **Akash Singha ROY** (Hariyana Vidya Mandir High School, India) and **George SHEN**.

Problem 502. Let O be the center of the circumcircle of acute $\triangle ABC$. Let P be a point on arc BC so that A, P are on opposite sides of side BC. Point K is on chord AP such that BK bisects $\angle ABC$ and $\angle AKB > 90°$. The circle Ω passing through C, K, P intersect side AC at D. Line BD meets Ω at E and line PE meets side AB at F. Prove that $\angle ABC = 2\angle FCB$.

Solution. **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Take point M on line KB such that $MB=MC$. Then we have $\triangle BMC$ is isosceles and

$$\begin{aligned}\angle KPC &= \angle APC=\angle ABC\\ &= \angle MBC+\angle MCB\\ &= 180°-\angle BMC\\ &= 180°-\angle KMC.\end{aligned}$$

This implies M is on the circle Ω. Applying Pascal's theorem to the points P, E, D, C, M, K on Ω, we have $PE\cap CM$, $ED\cap MK=B$ and $DC\cap KP=A$ are collinear. Since this line coincides with line AB, so $PE\cap CM=F$. Then

$$2\angle FCB=2\angle MCB=2\angle MBC=\angle ABC.$$

Other commended solvers: **LEE Jae Woo** (Hamyang High School, South Korea), **Vijaya Prasad NALLURI** (Retd Principal APES, Rajahmundry, India) and **Akash Singha ROY** (Hariyana Vidya Mandir High School, India).

Problem 503. Let S be a subset of $\{1,2,\ldots,2015\}$ with 68 elements. Prove that S has three pairwise disjoint subsets A, B, C such that they have the same number of elements and the sums of the elements in A, B, C are the same.

Solution. **Mark LAU** and **George SHEN**.

There are totally $(68\times67\times66)/6=50116$ 3-element subsets of S. The possible sums of the three elements in these subsets of S are from $1+2+3=6$ to $2013+2014+2015=6042$. Now $50116 > 8\times(6042-6+1)$. So by the pigeonhole principle, there are 9 distinct 3-element subsets $A_1, A_2,\ldots, A_9$ of S with the same sum of elements.

Assume $x\in S$ appears in $A_1, A_2,\ldots, A_9$ at least 3 times, say in A_1, A_2, A_3. Then no two of the sets $U=A_1\setminus\{x\}$, $V=A_2\setminus\{x\}$, $W=A_3\setminus\{x\}$ are the same. Otherwise say $U=V$, then $A_1=A_2$, contradiction.

So every $x\in S$ appear at most twice among $A_1, A_2,\ldots, A_9$. Then there can only be at most 3 of $A_2,\ldots, A_9$ (say A_2, A_3, A_4) having an element in common with A_1 (as every element of A_1 can only appear in at most one of $A_2,\ldots, A_9$). Without loss of generality, say each of $A_5,\ldots, A_9$ is disjoint with A_1. Similarly, among $A_6,\ldots, A_9$, there are at most three of them (say A_6, A_7, A_8) have a common element with A_5. Then A_9 and A_5 are disjoint. So the pairwise disjoint sets $A=A_1$, $B=A_5$, $C=A_9$ have the same sum of elements.

Other commended solvers: **LEE Jae Woo** (Hamyang High School, South Korea), **Akash Singha ROY** (Hariyana Vidya Mandir High School, India), and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 504. Let $p>3$ be a prime number. Prove that there are infinitely many positive integers n such that the sum of k^n for $k=1,2,\ldots,p-1$ is divisible by p^3.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **DBS Maths Solving Team**

(Diocesan Boys' School), **Mark LAU**, **LEE Jae Woo** (Hamyang High School, South Korea), **LEUNG Hei Chun** (SKH Tang Shiu Kin Secondary School), **Akash Singha ROY** (Hariyana Vidya Mandir High School, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

As $\varphi(p^3)=p^2(p-1)$, by Euler's theorem, for all positive integers r,s, we have

$$\sum_{k=1}^{p-1} k^{r+p^2(p-1)s} \equiv \sum_{k=1}^{p-1} k^r \pmod{p^3}.$$

In the case $r=p^2$, we have

$$\sum_{k=1}^{p-1} k^{p^2} = \sum_{k=1}^{(p-1)/2} \left(k^{p^2}+(p-k)^{p^2}\right)$$

$$= \sum_{k=1}^{(p-1)/2} \left(k^{p^2} + \sum_{t=0}^{p^2} \binom{p^2}{t} p^t (-k)^{p^2-t} \right)$$

$$\equiv \sum_{k=1}^{(p-1)/2} \left(p^3 k^{p^2-1} - \frac{p^4(p^2-1)}{2} k^{p^2-2} \right)$$

$$\equiv 0 \pmod{p^3}.$$

So all cases $n=p^2+p^2(p-1)s$ works.

Other commended solvers: **Soham GHOSH** (RKMRC Narendrapur, Kalkata, India) and **George SHEN.**

Problem 505. Determine (with proof) the least positive real number r such that if z_1, z_2, z_3 are complex numbers having absolute values less than 1 and sum 0, then

$$|z_1z_2+z_2z_3+z_3z_1|^2 + |z_1z_2z_3|^2 < r.$$

Solution. **Akash Singha ROY** (Hariyana Vidya Mandir High School, India) and **George SHEN**.

For i=1,2,3, let $a_i=|z_i|^2$, then $0\le a_i<1$. Since $z_1+z_2+z_3=0$, we have

$$\begin{aligned} & z_2\overline{z_3}+\overline{z_2}z_3 \\ &= (z_2+z_3)(\overline{z_2}+\overline{z_3}) - |z_2|^2 - |z_3|^2 \\ &= (-z_1)(-\overline{z_1}) - a_2 - a_3 \\ &= a_1 - a_2 - a_3. \end{aligned}$$

Let $b= z_1z_2+z_2z_3+z_3z_1$ and $c= z_1z_2z_3$. Let the notation $\Sigma f(u,v,w)$ denote the sum of $f(u,v,w)$, $f(v,w,u)$ and $f(w,u,v)$. We have

$$\begin{aligned} & |z_1z_2+z_2z_3+z_3z_1|^2 + |z_1z_2z_3|^2 \\ &= b\bar{b}+c\bar{c} \\ &= \sum |z_1z_2|^2 + \sum |z_1|^2 (z_2\overline{z_3}+\overline{z_2}z_3) + |z_1z_2z_3|^2 \\ &= \sum a_1a_2 + \sum a_1(z_2\overline{z_3}+\overline{z_2}z_3) + a_1a_2a_3 \\ &= \sum a_1a_2 + \sum a_1(a_1-a_2-a_3) + a_1a_2a_3 \\ &= a_1^2+a_2^2+a_3^2 - a_1a_2 - a_2a_3 - a_3a_1 + a_1a_2a_3 \\ &\le a_1+a_2+a_3 - a_1a_2 - a_2a_3 - a_3a_1 + a_1a_2a_3 \\ &= 1-(1-a_1)(1-a_2)(1-a_3) < 1. \end{aligned}$$

Next, for $0<x<1$, consider $z_1=x$, $z_2=-x$ and $z_3=0$. Then $|z_1z_2+z_2z_3+z_3z_1|^2 + |z_1z_2z_3|^2 = x^4 < r$. Letting x tend to 1, we get $1\le r$. Therefore, the least positive r is 1.

Other commended solvers: **DBS Maths Solving Team** (Diocesan Boys' School), **LEE Jae Woo** (Hamyang High School, South Korea) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Functional Inequalities

(Continued from page 2)

<u>Solution.</u> In (1), let $y=t-x$, then

$$f(t)\le tf(x)-xf(x)+f(f(x)). \qquad (2)$$

Consider $a,b\in\mathbb{R}$. Using (2) to $t=f(a)$, $x=b$ and $t=f(b)$, $x=a$, we get

$$f(f(a))-f(f(b)) \le f(a)f(b) - bf(b),$$
$$f(f(b))-f(f(a)) \le f(b)f(a) - af(a).$$

Adding these, we get

$$2f(a)f(b)\ge af(a)+bf(b).$$

Setting $b=2f(a)$, we get

$$2f(a)f(b)\ge af(a)+2f(a)f(b) \text{ or } af(a)\le 0.$$

Then for $a < 0, f(a) \ge 0.$ (3)

Now suppose $f(x)>0$ for some x. By (2), we see for every $t < (xf(x)-f(f(x)))/f(x)$, we have $f(t)<0$. This contradicts (3). So

$$f(x) \le 0 \quad \text{for all real } x. \qquad (4)$$

By (3) again, we get $f(x)=0$ for all $x < 0$. Finally setting $t=x<0$ in (2), we get $f(x) \le f(f(x))$. As $f(x)=0$, this implies $0\le f(0)$. This together with (4) give $f(0)=0$.

<u>Example 11 (2009 IMO Shortlisted Problem proposed by Belarus).</u> Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f(x-f(y)) > yf(x)+x. \qquad (1)$$

<u>Solution.</u> Assume the contrary, i.e. $f(x-f(y)) \le yf(x)+x$ for all real x and y. Let $a=f(0)$. Setting y=0 in (1) gives $f(x-a)\le x$ for all real x. This is equivalent to

$$f(y)\le y+a \quad \text{for all real } y. \qquad (2)$$

Setting $x=f(y)$ in (1) and using (2), we get

$$a=f(0)\le yf(f(y))+f(y)\le yf(f(y))+y+a.$$

This implies $0\le y(f(f(y))+1)$ and so

$$f(f(y))\ge -1 \text{ for all } y>0. \qquad (3)$$

By (2) and (3), we get $-1\le f(f(y))\le f(y)+a$ for all $y>0$. So

$$f(y)\ge -a-1 \text{ for all } y > 0. \qquad (4)$$

Next, we <u>*claim*</u> $f(x)\le 0$ for all real x. (5) Assume the contrary, i.e. there is some $f(x)>0$. Now take y such that $y<x-a$ and

$$y < (-a-x-1)/f(x). \qquad (6)$$

By (2), we get $x-f(y) \ge x-(y+a) > 0$. By (1) and (4), we get

$$yf(x)+x \ge f(x-f(y)) \ge -a-1.$$

Then $y \ge (-a-x-1)/f(x)$, contradicting (6). So (5) is true.

Now setting y=0 in (5) leads to $a=f(0)\le 0$ and using (2), we get

$$f(x)\le x \quad \text{for all real } x. \qquad (7)$$

Now choose $y > 0$, $y > -f(-1)-1$ and set $x=f(y)-1$. By (1),(5) and (7), we get

$$\begin{aligned} f(-1) &= f(x-f(y)) \\ &\le yf(x)+x = yf(f(y)-1)+f(y)-1 \\ &\le y(f(y)-1)-1 \le -y-1. \end{aligned}$$

Then $y\le -f(-1)-1$, which contradicts the choice of y.

<u>Example 12 (64th Bulgarian Math Olympiad in 2015).</u> Determine all functions $f:(0,+\infty)\to(0,+\infty)$ such that for arbitrary positive real numbers x, y, we have

(1) $f(x+y)\ge f(x)+y$;
(2) $f(f(x))\le x$.

<u>Solution.</u> As $y>0$, (1) implies f is strictly increasing on $(0,+\infty)$. By (2) and (1), we have

$$x+y\ge f(f(x+y))\ge f(f(x)+y). \qquad (*)$$

Using (*) and in (1), replacing x by y and y by $f(x)$, we get

$$x+y \ge f(f(x)+y) \ge f(x)+f(y). \qquad (**)$$

Since f is strictly increasing and $f(x)>0$, so the limit of $f(x)$ as $x\to 0^+$ is a nonnegative number c. By (2), the limit of $f(f(x))$ as $x\to 0^+$ is 0.

If $c>0$, then since f is strictly increasing, $f(f(x)) \ge f(c) > 0$. Taking the limit of $f(f(x))$ as $x\to 0^+$ leads to $0 \ge f(c) > 0$, contradiction. So c=0.

Now taking limit as $y\to 0^+$ in (**), we get $x\ge f(x)$ for all $x>0$. This and (1) lead to

$$x+y\ge f(x+y)\ge f(x)+y. \qquad (***)$$

Subtracting $f(x)+y$ in (***), we get $x-f(x) \ge f(x+y)-f(x)-y \ge 0$. Letting $w=x+y$ in (***) and taking limit of $w\ge f(w)\ge f(x)+w-x$ as $x\to 0^+$, we get $w=f(w)$. So $f(x+y)=f(w)=w=x+y$. Then f is the identity function on $(0,+\infty)$, which certainly satisfy (1) and (2).

Mathematical Excalibur

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Olympiad Corner

Below were the problems of the 2017 Serbian Mathematical Olympiad for high school students. The event was held in Belgrade on March 31 and April 1, 2017.

Time allowed was 270 minutes.

First Day

Problem 1. (*Nikola Petrović*) Let a, b and c be positive real numbers with $a+b+c=1$. Prove the inequality

$$a\sqrt{2b+1}+b\sqrt{2c+1}+c\sqrt{2a+1} \le \sqrt{2-(a^2+b^2+c^2)}.$$

Problem 2. (*Dušan Djukić*) A convex quadrilateral $ABCD$ is inscribed in a circle. The lines AD and BC meet at point E. Points M and N are taken on the sides AD, BC respectively, so that $AM:MD=BN:NC$. Let the circum-circles of triangle EMN and quadrilateral $ABCD$ intersect at points X and Y. Prove that either the lines AB, CD and XY have a common point or they are all parallel.

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 21, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Perfect Squares

Kin Y. Li

In this article, we will be looking at one particular type of number theory problems, namely problems on integers that have to do with the set of perfect squares 1, 4, 9, 16, 25, 36, …. This kind of problems have appeared in many Mathematical Olympiads from different countries for over 50 years. Here are some examples.

Example 1 (1953 Kürschák Math Competition Problems). Let n be a positive integer and let d be a positive divisor of $2n^2$. Prove that n^2+d is not a perfect square.

Solution. We have $2n^2=kd$ for some positive integer k. Suppose $n^2+d=m^2$ for some positive integer m. Then $m^2 = n^2+2n^2/k$ so that $(mk)^2=n^2(k^2+2k)$. Then k^2+2k must also be the square of a positive integer, but $k^2<k^2+2k<(k+1)^2$ leads to a contradiction.

Example 2 (1980 Leningrad Math Olympiad). Find all prime numbers p such that $2p^4-p^2+16$ is a perfect square.

Solution. For $p=2$, $2p^4-p^2+16=44$ is not a perfect square. For $p=3$, $2p^4-p^2+16=169=13^2$. For prime $p>3$, $p\equiv 1$ or 2 (mod 3) and $2p^4-p^2+16\equiv 2$ (mod 3). Assume $2p^4-p^2+16=k^2$. Then $k^2\equiv 0^2,1^2$ or $2^2\equiv 0$ or 1 (mod 3). So $2p^4-p^2+16\neq k^2$. Then $p=3$ is the only solution.

Example 3 (2008 Singapore Math Olympiad). Find all prime numbers p satisfying 5^p+4p^4 is a perfect square.

Solution. Suppose $5^p+4p^4=q^2$ for some integer q. Then

$$5^p = q^2-4p^4 = (q-2p^2)\,(q+2p^2).$$

Since 5 is a prime number, we have

$$q-2p^2 = 5^s \text{ and } q+2p^2 = 5^t$$

for some integers s, t with $t > s \ge 0$ and $s+t = p$. Eliminating q, we have

$$4p^2 = 5^s(5^{t-s}-1).$$

If $s>0$, then from 5 divides $4p^2$, we get $p=5$. So $5^p+4p^4=5625=75^2$ and $q=75$ is a solution. If $s=0$, then $t=p$. So $5^p=4p^2+1$. Now, for integer $k\ge 2$, we *claim* $5^k>4k^2+1$. The case $k=2$ is clear. Suppose the case k is true. Then

$$\frac{4(k+1)^2+1}{4k^2+1} = 1+\frac{8k}{4k^2+1}+\frac{4}{4k^2+1} < 1+1+1<5.$$

So $5^{k+1}=5\times 5^k > 5(4k^2+1) > 4(k+1)^2+1$. By mathematical induction, the claim is true. Therefore, $5^p = 4p^2+1$ has no prime solution p.

Example 4 (2009 Croatian Math Olympiad). Find all positive integers m, n such that 6^m+2^n+2 is a perfect square.

Solution. If

$$6^m+2^n+2=2(3^m\times 2^{m-1}+2^{n-1}+1)$$

is a perfect square, then $3^m\times 2^{m-1}+2^{n-1}+1$ is even. So one of the integers $3^m\times 2^{m-1}$ and 2^{n-1} is odd and the other is even.

Suppose $3^m\times 2^{m-1}$ is odd, then $m = 1$ and $6^m+2^n+2 = 8+2^n = 4(2^{n-2}+2)$. So $2^{n-2}+2$ is a perfect square. Since every perfect square dived by 4 has remainder 0 or 1, so $2^{n-2}+2$ cannot be of the form 4k+2. Hence, $n-2=1$, i.e. $n=3$. So $(m,n)=(1,3)$ is a solution.

If 2^{n-1} is odd, then $n=1$ and

$$6^m+2^n+2 = 6^m+4 \equiv (-1)^m+4 \text{ (mod 7)}.$$

This means 6^m+2^n+2 divided by 7 has remainder 3 or 5. However,

$(7k)^2\equiv 0$ (mod 7), $(7k\pm1)^2\equiv 1$ (mod 7), $(7k\pm2)^2\equiv 4$ (mod 7), $(7k\pm3)^2\equiv 2$ (mod 7).

So every perfect square divided by 7 cannot have remainder 3, 5 or 6. Therefore, $(m,n) = (1,3)$ is the only solution.

(continued on page 2)

Example 5 (2008 German Math Olympiad). Determine all real numbers x such that $4x^5-7$ and $4x^{13}-7$ are perfect squares.

Solution. Suppose there are positive integers a and b such that

$$4x^5-7=a^2 \text{ and } 4x^{13}-7=b^2.$$

Then $x^5 = (a^2+7)/4 > 1$ is rational and $x^{13} = (b^2+7)/4 > 1$ is rational. So $x = (x^5)^8/(x^{13})^3$ is rational. Suppose $x = p/q$ with p and q positive relatively prime integers. Then from $(p/q)^5 = (a^2+7)/4$, it follows q^5 divides $4p^5$ and so q=1. So x must be a positive integer and $x\geq 2$.

In the case x is an odd integer, we have $a^2\equiv 0, 1, 4 \pmod 8$, but $a^2 = 4x^5-7 \equiv 5 \pmod 8$, contradiction. So x is even. In the case x=2, we have $4x^5-7=11^2$ and $4x^{13}-7=181^2$. For an even $x\geq 4$, $(ab)^2=(4x^5-7)(4x^{13}-7)=16x^{18}-28x^{13}-28x^7+49$. However, expanding $(4x^9-7x^4/2-1)^2$ and $(4x^9-7x^4/2)^2$ and using $x^9 \geq 4x^8 \geq 4^2x^7 \geq 4^5x^4$, we see $(ab)^2$ is strictly between them. Then x=2 is the only solution.

Example 6 (2011 Iranian Math Olympiad). Integers a, b satisfy $a>b$. Also $ab-1$, $a+b$ are relatively prime and $ab+1$, $a-b$ are relatively prime. Prove that $(a+b)^2+(ab-1)^2$ is not a perfect square.

Solution. Assume $(a+b)^2+(ab-1)^2=c^2$ for some integer c. Then

$$c^2=a^2+b^2+a^2b^2+1=(a^2+1)(b^2+1).$$

Assume (*) there is a prime p such that $p \mid a^2+1$ and $p \mid b^2+1$, then $p \mid a^2+1-b^2+1 = a^2-b^2$. So (**) $p \mid a-b$ or $p \mid a+b$.

Assume $p \mid a-b$. Then $p \mid ab-b^2$. Since $p \mid b^2+1$, so $p \mid ab-b^2+b^2+1 = ab+1$, which contradicts $ab+1$, $a-b$ are relatively prime. Similarly, assume $p \mid a+b$. Then $p \mid ab+b^2$. Since $p \mid a^2+1$, so $p \mid ab+b^2+b^2-1 = ab-1$, which contradicts $ab-1$, $a+b$ are relatively prime. So (**) as well as (*) are wrong.

Then a^2+1, b^2+1 are relatively prime. Since $a>b$, not both of them are 0. So $(a+b)^2+(ab-1)^2$ equals a^2+1 (if b=0) or b^2+1 (if a=0) or $(a^2+1)(b^2+1)$. Then $(a+b)^2+(ab-1)^2$ is not a perfect square.

Example 7 (2000 Polish Math Olympiad). Let m, n be positive integers such that m^2+n^2+m is divisible by mn. Prove that m is a perfect square.

Solution. Since m^2+n^2+m is divisible by mn, so for some positive integer k, $m^2+n^2+m=kmn$. Then $n^2-kmn+(m^2+m) = 0$, which can be viewed as a quadratic equation in n. Then the discriminant $\Delta=k^2m^2-4m^2-4m$ is a perfect square. Suppose d is $\gcd(m, k^2m-4m-4)=1$. If d=1, then m (and $k^2m-4m-4$) are both perfect squares. If $d > 1$, then

$$d = \gcd(m, k^2m-4m-4) = \gcd(m,4).$$

Since $d >1$ divides 4, so d is even. Then m is even. Also, $n^2\equiv m^2+n^2+m \pmod 2$. So n is even. Then mn, m^2+n^2 are divisible by 4.

As m^2+n^2+m is given to be divisible by mn, so m^2+n^2+m is divisible by 4. Then $m = m^2+n^2+m-(m^2+n^2)$ is divisible by 4. So we get $d = 4$. Then

$$1 = \gcd(m/4, k^2(m/4)-m-1).$$

Now $\Delta/16 = (m/4)(k^2(m/4)-m-1)$ is a perfect square. So $m/4$ and $k^2(m/4)-m-1$ are perfect squares. Therefore, m is a perfect square.

Example 8 (2006 British Math Olympiad). Let n be an integer If $2+2\sqrt{1+12n^2}$ is an integer, then it is a perfect square.

Solution. If $2+2\sqrt{1+12n^2}$ is an integer, then $1+12n^2$ is a perfect square. Suppose $1+12n^2=m^2$ for some odd positive integer m. Then $12n^2 = (m+1)(m-1)$. Let t be the integer $(m+1)/2$ and we have (*) $t(t-1)=3n^2$.

Now we claim $2+2\sqrt{1+12n^2} = 2 + 2m = 4t$ is a perfect square. By (*), we see $t-1$ or t is divisible by 3. Now $\gcd(t-1, t)= 1$. Assume t is divisible by 3, then $(t/3)(t-1)= n^2$ and both $t/3$ and $t-1$ are perfect squares. Let $t/3=k^2$ for some integer k, Then $t-1=3k^2-1\equiv 2 \not\equiv 0^2, 1^2$ or $2^2 \pmod 3$, contradiction. So $t-1$ is divisible by 3. Then we have $\gcd(t, (t-1)/3)=1$. From $t\times(t-1)/3=n^2$, we see t is a perfect squares. So the claim is true.

Example 9 (2002 Australian Math Olympiad). Find all prime numbers p, q, r such that p^q+p^r is a perfect square.

Solution. If $q=r$, then $p^q+p^r=2p^q$. So p=2 and q is an odd prime at least 3. All prime triples $(p,q,r)=(2,q,q)$ are solutions.

If $q\neq r$, then without loss of generality, let $q<r$ and so $p^q+p^r = p^q(1+p^s)$, where $s=r-q$ is at least 1. Since p^q and $1+p^s$ are relatively prime, so they are both perfect squares. Then, the prime q is 2. Also, since $1+ p^s$ is a perfect square, $1+ p^s=u^2$ for some positive integer u. Then

$$p^s=u^2-1=(u+1)(u-1).$$

Since $\gcd(u+1,u-1)=1$ or 2, so if it is 2, then u is odd and p is even. Hence, p=2 and both $u+1$ and $u-1$ are powers of 2. Then u can only be 3 and $1+ p^s=3^2$ so that p=2, s=3, $r=q+s$=2+3=5. These lead to the solutions (p,q,r)=(2,2,5) or (2,5,2).

If $\gcd(u+1,u-1)=1$, then u is even and $u-1$ must be 1 (otherwise $u+1$ and $u-1$ have different odd prime factors and cannot be powers of the same prime). Then u=2, $p^s=(u-1)(u+1)=3$, p=3, s=1, $r=q+s$=3. The only such prime triples are (p,q,r) = (3,2,3) or (3,3,2).

Then all the solutions are (p,q,r) = (2,2,5), (2,5,2), (3,2,3), (3,3,2) and $(2,q,q)$ with q being a prime at least 3.

Example 10 (2008 USA Team Selection Test). Let n be a positive integer. Prove that n^7+7 is not a perfect square.

Solution. Assume $n^7+7=x^2$ for some positive integer x. Then

(1) n is odd (for otherwise $x^2\equiv 3 \pmod 4$, which is false).
(2) $n\equiv 1 \pmod 4$ (due to n odd and $x^2\not\equiv 2 \pmod 4$).

(3) $x^2+11^2 = n^7+128 =(n+2)N$, where N is $n^6-2n^5+4n^4-8n^3+16n^2-32n+64$.

(4) If $11 \nmid x$, then every prime factor p of x^2+11^2 must be odd and $p\equiv 1 \pmod 4$ (for if $p = 4k+3$, then $x^2 \equiv -11^2 \pmod p$ and by Fermat's little theorem, $x^{p-1} \equiv -11^{p-1} \equiv -1 \pmod p$, contradiction).

From (3), we get $n+2 \mid x^2+11^2$, $n+2\equiv 3 \pmod 4$ implies x^2+11^2 has a prime factor congruent 3 (mod 4), which contradicts (4).

If $x=11y$ for some integer y, then (3) becomes $121(y^2+1) = (n+2)N$, but checking $n \equiv -5$ to 5 (mod 11), we see N is not a multiple of 11. So n+2 is a multiple of 121, say $M = (n+2)/121$. Then $y^2+1 = MN$. Similarly, it can be checked that every prime factor of y^2+1 is congruent to 1 (mod 4). Hence, every odd factor of y^2+1 is congruent to 1 (mod 4). However, $M \equiv 3 \pmod 4$, so $y^2+1 = MN$ cannot be true. Therefore, n^7+7 is not a perfect square.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 21, 2018.***

Problem 511. Let $x_1,x_2,\ldots,x_{40}$ be positive integers with sum equal to 58. Find the maximum and minimum possible value of $x_1^2+x_2^2+\cdots+x_{40}^2$.

Problem 512. Let AD, BE, CF be the altitudes of acute ΔABC. Points P and Q are on segments DF and EF respectively. If $\angle PAQ=\angle DAC$, then prove that AP bisects $\angle FPQ$.

Problem 513. Let $a_0,a_1,a_2,\ldots$ be a sequence of nonnegative integers satisfying the conditions:

(1) $a_{n+1}=3a_n-3a_{n-1}+a_{n-2}$ for $n>1$,
(2) $2a_1=a_0+a_2-2$,
(3) for every positive integer m, in the sequence $a_0,a_1,a_2,\ldots$, there exist m terms $a_k,a_{k+1},\ldots,a_{k+m-1}$, which are perfect squares.

Prove that every term in $a_0,a_1,a_2,\ldots$ is a perfect square.

Problem 514. Let n be a positive integer and let $p(x)$ be a polynomial with real coefficients on the interval $[0,n]$ such that $p(0)=p(n)$. Prove that there are n distinct ordered pairs (a_i, b_i) with $i=1,2,\ldots,n$ such that $0\le a_i<b_i\le n$, b_i-a_i is an integer and $p(a_i)=p(b_i)$.

Problem 515. There are ten distinct nonzero real numbers. It is known that for every two of the numbers, either the sum or the product of them is rational. Prove that the square of each of the ten numbers is rational.

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Solutions

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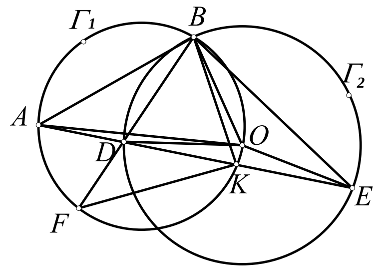
Problem 506. Points A and B are on a circle Γ_1. Line AB is tangent to another circle Γ_2 at B and the center O of Γ_2 is on Γ_1. A line through A intersects Γ_2 at points D and E (with D between A and E). Line BD intersects Γ_1 at a point F, different from B. Prove that D is the midpoint of BF if and only if BE is tangent to Γ_1.

Solution. **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School) and **George SHEN**.



Let point K be the intersection of Γ_1 with line DE. Then $\Delta KFD\sim\Delta ABD$. Since $\angle ABD=\angle AEB$, so $\Delta ABD\sim\Delta AEB$. Then $\Delta KFD\sim\Delta AEB$. Hence, $FD/DK=AB/BE$.

Let O be the center of Γ_2. Since $OB\perp AB$, AO is a diameter of Γ_1. So $AK\perp OK$. Then $\angle DKO=\angle AKO=90^\circ$. So $DK=EK$. Now BE is tangent to $\Gamma_1\Leftrightarrow\angle EBK=\angle BAD\Leftrightarrow\Delta EBK\sim\Delta BAD\Leftrightarrow AB/BE=DB/KE\Leftrightarrow FD/DK=DB/KE\Leftrightarrow FD=DB$ (i.e. D is the midpoint of BF).

Other commended solvers: **DBS Maths Solving Team** (Diocesan Boys' School), **Jae Woo LEE** (Hamyang High School, South Korea), **LIN Meng Fei**, **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 507. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x, y\in\mathbb{R}$,

$$(x-2)f(y)+f(y+2f(x))=f(x+yf(x)).\quad (*)$$

Solution. **DBS Maths Solving Team** (Diocesan Boys' School), **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School), **Jae Woo LEE** (Hamyang High School, South Korea) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

If $f(0)=0$, then setting $x=0$ in (*) yields $f(y)=0$ for all $y\in\mathbb{R}$, i.e. f is the zero function, which is a solution of (*).

If $f(0)\neq0$, then setting $y=0$ in (*) yields $(x-2)f(0)+f(2f(x))=f(x)$ for all $x\in\mathbb{R}$. Now $f(x)=f(y)$ implies $(x-2)f(0)+f(2f(x))=f(x)=f(y)=(y-2)f(0)+f(2f(y))=(y-2)f(0)+f(2f(x))$ yielding $x=y$. So f is injective.

Setting $x=2$ in (*) yields $f(y+2f(2))=f(2+yf(2))$ for all $y\in\mathbb{R}$. Since f is injective, $y+2f(2)=2+yf(2)$ for all $y\in\mathbb{R}$. Setting $y=0$, we get $f(2)=1$. Since f is injective, $f(3)\neq1$. Setting $x=3$, $y=3/(1-f(3))$ in (*), we get $f(3/(1-f(3))+2f(3))=0$. Thus, f has a root at $a=3/(1-f(3))+2f(3)$. Setting $y=a$ in (*), we get $f(a+2f(x))=f(x+af(x))$ for all $x\in\mathbb{R}$. Since f is injective, we get $a+2f(x)=x+af(x)$. Now $a\neq2$. So $f(x)=(x-a)/(2-a)$. Putting this in (*), we get $a=1$. Then the function can only be (1) $f(x)=0$ for all $x\in\mathbb{R}$ or (2) $f(x)=x-1$ for all $x\in\mathbb{R}$. Putting these in (*) show they are in fact solutions of (*).

Other commended solvers: **Yagub N. ALIYEV** (Problem Solving Group of ADA University, Baku, Azerbaijan) and **Akash Singha ROY** (West Bengal, India).

Problem 508. Determine the largest integer k such that for all integers x,y, if $xy+1$ is divisible by k, then $x+y$ is also divisibly by k.

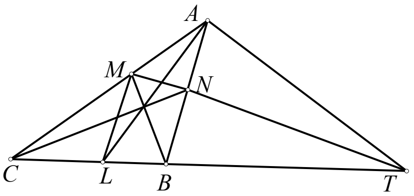
Solution. **George SHEN.**

Let k be such an integer. Let S be the set of all integers x such that $\gcd(x,k)=1$. For x in S, choose integer m in $[1,k-1]$ such that $mx^2\equiv-1 \pmod k$. Let $y=mx$, then $k\mid xy+1$. So $k\mid x+y$ and $k\mid(x+y)x-(xy+1)=x^2-1$. Then for every x in S, every prime factor p of k satisfies $x^2\equiv1 \pmod p$. If all prime factors p of k are at least 5, then $x=2, 3$ are in S, but $x^2\equiv1 \pmod p$ fails due to $p\nmid 2^2-1, 3^2-1$. So the prime factors of k can only be 2 or 3. So k is of the form 2^r3^s and $S=\{x: \gcd(x,2)=1=\gcd(x,3)\}$ Then for $x=5$ in S, $x^2\equiv1 \pmod{2^r}$ implies $2^r\mid 24$ and so $r\le3$. Also, for $x=5$ in S, $x^2\equiv1 \pmod{3^s}$ implies $3^s\mid 24$ and so $s\le1$. Then $k\le 2^33=24$.

Finally, for $k=24$, $xy\equiv-1 \pmod{24}$ implies $\gcd(x,24)=1=\gcd(y,24)$. Then $x,y\equiv 1, 5, 7, 11, 13, 17, 19$ or $23 \pmod{24}$. The only possible cases for $xy\equiv-1 \pmod{24}$ are $\{x,y\}=\{1,23\}$, $\{5,19\}$, $\{7,17\}$, $\{11,23\}$. Then $24\mid x+y$. So $k=24$ is the required largest integer.

Other commended solvers: **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4) and **DBS Maths Solving Team** (Diocesan Boys' School), **Jae Woo LEE** (Hamyang High School, South Korea), **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 509. In ΔABC, the angle bisector of $\angle CAB$ intersects BC at a point L. On sides AC, AB, there are points M, N respectively such that lines AL, BM, CN are concurrent and $\angle AMN=\angle ALB$. Prove that $\angle NML=90^\circ$.



Solution 1. **Apostolis MANOLOUDIS** and **George SHEN**.

Let $T=MN\cap BC$. From $\angle AMT=\angle AMN=\angle ALB=\angle ALT$, we get A, M, L, T are concyclic. So $\angle NML=\angle TML=\angle TAL$. To get $\angle TAL=90°$, it suffices to show AT is the exterior bisector of $\angle CAB$.

By Menelaos' theorem, as M,N,T are collinear, $(AM/MC)(CT/TB)(BN/NA)=1$. By Ceva's theorem, as AL, BM, CN concur, $(AM/MC)(CL/LB)(BN/NA)=1$. Then $CL/LB=CT/TB$. By the angle bisector theorem, $CA/AB=CL/LB=CT/TB$. So AT is the external bisector of $\angle CAB$.

Solution 2. **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School), **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

AL, BM, CN concurrent implies T, B, L, C is a harmonic range of points. Then $\angle AMT=\angle AMN=\angle ALB=\angle ALT$ led to T, A, M, L concyclic. By Apollonius' Theorem, $90°=\angle TAL=\angle NML$.

Other commended solvers: **Jae Woo LEE** (Hamyang High School, South Korea), **LEUNG Hei Chun** (SKH Tang Shiu Kin Secondary School), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" School, Buzău, Romania).

Problem 510. Numbers 1 to 20 are written on a board. A person randomly chooses two of these numbers with a difference of at least 2. He adds 1 to the smaller one and subtracts 1 from the larger one. Then he performs an operation by replacing the original two chosen numbers on the board with the two new numbers. Determine the maximum number of times he can do this operation.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School), **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Note after each operation, the sum of the numbers is always 210. Suppose the person chooses m,n with $m-n\geq 2$, then $(m-1)^2+(n+1)^2=n^2+m^2+2-2(m-n)\leq n^2+m^2-2$ with equality only for $m-n=2$. If the absolute value of the difference of the two numbers is 1, then the operation does not change anything. At the end, the board has ten 10's and ten 11's.

In the beginning, the sum of the squares is $1^2+2^2+\cdots+20^2=2870$ and at the end, it is $10\times(10^2+11^2)=2210$. After each operation, the sum of squares reduces by at least 2, so the number of operation that can be done is at most $(2870-2210)/2=330$. Below we will show the person can do 330 operations with the absolute values of the difference of the two numbers is 2.

The plan is to eliminate the minimum and the maximum of the remaining numbers until we get only 10's and 11's. In round 1, we eliminate 1's and 20's by operating on pairs (1,3), (2,4), …, (18,20) one time for every pair. In round 2, we eliminate 2's and 19's by operating on pairs (2,4), (3,5), …, (17,19) two times for every pair. Keep on eliminating in this way until we have only 9's, 10's, 11's and 12's. In round 9, we eliminate 9's and 12's by operating on pairs (9,11) and (10,12) nine times. The total number of operations is $18\times1+16\times2+\cdots+2\times9=330$.

Olympiad Corner

(Continued from page 1)

Problem 3. (*Dušan Djukić*) There are $2n-1$ bulbs in a line. Initially, the central (n-th) bulb is on, whereas all others are off. A step consists of choosing a string of at least three (consecutive) bulbs, the leftmost and rightmost ones being off and all between them being on, and changing the states of all bulbs in the string (for instance, the configuration ●○○○● will turn into ○●●●○). At most how many steps can be performed?

<u>Second Day</u>

Problem 4. (*Dušan Djukić*) Suppose that a positive integer a is such that, for any positive integer n, the number n^2a-1 has a divisor greater than 1 and congruent to 1 modulo n. Prove that a is a perfect square.

Problem 5. (*Bojan Bašić* and *PSC*) Determine the maximum number of queens that can be placed on a 2017×2017 chessboard so that each queen attacks at most one of the others.

Problem 6. (*Dušan Djukić*) Let k be the circumcircle of triangle ABC, and let k_a be its excircle opposite to A. The two common tangents of k and k_a meet the line BC at points P and Q. Prove that $\angle PAB=\angle QAC$.

Perfect Squares

(Continued from page 2)

<u>*Example 11 (2006 Thai Math Olympiad).*</u> Determine all prime numbers p such that $(2^{p-1}-1)/p$ are perfect squares.

<u>*Solution.*</u> For every prime number p, let $f(p)=(2^{p-1}-1)/p$. We will show for $p>7$, $f(p)$ is not a perfect square.

Assume there is a prime $p>7$ such that $2^{p-1}-1=pm^2$ for some positive integer m. Then m must be odd. Now there are two cases, (1) p is of the form $4k+1$ with $k>1$ or (2) p is of the form $4k+3$ with $k>1$.

In case (1), we have $2^{p-1}-1=pm^2=(4k+1)m^2\equiv 1 \pmod 4$, but also $2^{p-1}-1=2^{4k}-1\equiv 3 \pmod 4$, which is a contradiction.

In case (2), we have $2^{p-1}-1=2^{4k+2}-1=(2^{2k+1}-1)(2^{2k+1}+1)=pm^2$.

Since $\gcd(2^{2k+1}-1, 2^{2k+1}+1)=1$, again we have two subcases:

(a) $2^{2k+1}-1=u^2$, $2^{2k+1}+1=pv^2$ for some positive integers u, v;

(b) $2^{2k+1}-1=pu^2$, $2^{2k+1}+1=v^2$ for some positive integers u, v.

In subcase (a), since $k>1$, $2^{2k+1}+1\equiv 1 \pmod 4$, but $pv^2\equiv 3\times1=4 \pmod 4$, which is a contradiction.

In subcase (b), we have $2^{2k+1}=v^2-1=(v-1)(v+1)$. Then $v-1=2^s$, $v+1=2^t$ for some positive integers $s<t$. Observe that $2^{t-s}=(v+1)/(v-1)=2/(v-1)+1$. Then $v=2$ or 3. If $v=2$, then $2^{2k+1}+1=v^2=4$, which is a contradiction. If $v=3$, then $2^{2k+1}=v^2-1=8$ leads to $k=1$, which is a contradiction as $k>1$.

Finally, checking the cases $p=2,3,5,7$, we see only cases $p=3$ and 7 have solutions $(2^{3-1}-1)/3=1^2$ and $(2^{7-1}-1)/7=3^2$.

Mathematical Excalibur

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Olympiad Corner

Below were the problems of the 2017 Serbian IMO Team Selection Competition for high school students. The event was held in Belgrade on May 21 and 22, 2017.

Time allowed was 270 minutes per day.

First Day

Problem 1. (*Dušan Djukić*) Let D be the midpoint of side BC of a triangle ABC. Points E and F are taken on the respective sides AC and AB such that $DE=DF$ and $\angle EDF=\angle BAC$. Prove that

$$DE \geq \frac{AB+AC}{4}.$$

Problem 2. (*Bojan Bašić*) Given an ordered pair of positive integers (x,y) with exactly one even coordinate, a *step* maps this pair to $(x/2, y+x/2)$ if $2|x$, and to $(x+y/2, y/2)$ if $2|y$. Prove that for every odd positive integer $n>1$ there exists an even positive integer b, $b<n$, such that after finitely many steps the pair (n,b) maps to the pair (b,n).

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 31, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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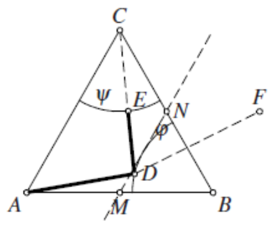
Strategies and Plans

Kin Y. Li

In this article, we will be looking at some Math Olympiad problems from different countries and regions. Some require strategies or plans to perform certain tasks. We hope these arouse your interest. Here are the examples.

<u>Example 1 (1973 IMO).</u> A soldier has to investigate whether there are mines in an area that has the form of an equilateral triangle. The radius of his detector is equal to one-half of an altitude of the triangle. The soldier starts from one vertex of the triangle. Determine the shortest path that the soldier has to traverse in order to check the whole region.

<u>Solution.</u> Suppose that the soldier starts at the vertex A of the equilateral triangle ABC of side length a. Let φ and ψ be the arcs of circles with centers B and C and radii $a\sqrt{3}/4$ respectively, that lie inside the triangle. In order to check the vertices B and C he must visit some point D in φ and E in ψ.



Thus his path cannot be shorter than the path ADE (or AED) itself. The length of the path ADE is $AD+DE\geq AD+DC-a\sqrt{3}/4$. Let F be the reflection of C across the line MN, where M and N are the midpoints of AB and BC respectively. Then $DC\geq DF$ and hence $AD+DC\geq AD+DF\geq AF$. So

$$AD+DE \geq AF-\frac{a\sqrt{3}}{4}=a\left(\frac{\sqrt{7}}{2}-\frac{\sqrt{3}}{4}\right)$$

with equality if and only if D is the midpoint of arc φ and E is the intersection point of CD and arc ψ. In following the path ADE, the soldier will check the whole region. Therefore, this path (as well as the one symmetric to it) is the shortest path the soldier can check the whole field.

<u>Example 2 (2011 Saudi Arabia Math Competition).</u> A Geostationary Earth Orbit is situated directly above the equator and has a period equal to the Earth's rotational period. It is at the precise distance of 22,236 miles above the Earth that a satellite can maintain an orbit with a period of rotation around the Earth exactly equal to 24 hours. Because the satellites revolve at the same rotational speed of the Earth, they appear stationary from the Earth surface. That is why most stationary antennas (satellite dishes) do not need to move once they have been properly aimed at a target satellite in the sky. In an international project, a total of ten stations were equally spaced on this orbit (at the precise distance of 22,236 miles above the equator). Given that the radius of the Earth is 3960 miles, find the exact straight distance between two neighboring stations. Write your answer in the form $a+b\sqrt{c}$, where a, b, c are integers and $c>0$ is square-free.

<u>Solution.</u> Let A and B be neighboring stations and O be the center of the Earth. Now $\angle AOB=36°$. Let $\theta=18°$. Then $AB=2R\sin\theta$, where $R=22236+3960=26196$. Since we have $\sin 36°=\cos 54°$, so $\sin 2\theta=\cos 3\theta$. That is, $2\cos\theta\sin\theta=4\cos^3\theta-3\cos\theta$. Dividing by $\cos\theta$ and expressing in terms of $\sin\theta$, we get $4\sin^2\theta+2\sin\theta-1=0$. Using the quadratic formula, we have $\sin\theta=(\sqrt{5}-1)/4$. Then $AB=2R\sin\theta=13098(\sqrt{5}-1)$. So $a=-13098$, $b=13098$ and $c=5$.

<u>Example 3 (2008 German National Math Competition).</u> On a bookshelf, there are n books ($n\geq 3$) from different authors standing side by side. A librarian inspects the two leftmost books and changes their places if and

(continued on page 2)

only if they are not in alphabetical order. Afterward, he does the same to the second and the third book from the left and so on. Acting this way, he passes the whole row of books three times in total. Determine the number of different starting arrangements for which the books will finally be ordered alphabetically.

<u>*Solution.*</u> There are exactly $6 \cdot 4^{n-3}$ arrangements for which the books are in order after 3 runs. For a proof, we number the positions and the books in alphabetical order from 1 to n. Obviously, for the position of $p(k)$ of book number k at the beginning it is necessary that $p(k)-k \le 3$. Now this condition is also sufficient: At every ordering run, all of the books standing right to their correct place are shifted one place to the left. On the other hand, no book can be shifted to the right beyond its correct place because if there is a book at position $p(k)$ with $p(k)>k$, there must be at least one book on the left side of $p(k)$ with its number larger than $p(k)$. Such a book takes over any book with a number smaller than $p(k)$.

The number given in the answer is then calculated by regarding that each of the books with numbers $1,2,\ldots, n-4$ that is not occupied by a book with a smaller number. For the last three books there are only 3, 2 and 1 places left. Hence the result follows.

<u>*Example 4 (2000 Russian Math Olympiad).*</u> Two pirates divide their loot, consisting of two sacks of coins and one diamond. They decide to use the following rules. On each turn, one pirate chooses a sack and takes $2m$ coin from it, keeping m for himself and putting the rest into the other sack. The pirates alternatively taking turns until no more moves are possible; the first pirate unable to make a move loses the diamond, and the other pirate takes it. For which initial numbers of coins can the first pirate guarantee that he will obtain the diamond?

<u>*Solution.*</u> We claim that if there are x and y coins left in the two sacks, respectively, then the next player P_1 to move has a winning strategy if and only if $|x-y|>1$. Otherwise, the other player P_2 has a winning strategy.

We prove the claim by induction on the total numbers of coins, $x+y$. If $x+y=0$, then no moves are possible and the next player does not have a winning strategy. Now assuming that the claim is true when $x+y \le n$ for some nonnegative n, we prove that it is true when $x+y=n+1$.

First consider the case $|x-y| \le 1$. Assume that a move is possible. Otherwise, the next player P_1 automatically loses, in accordance with our claim. The next player must take $2m$ coins from one sack, say the one containing x coins, and put m coins into the sack containing y coins. Hence the new difference between the number of coins in the sacks is

$$|(x-2m)-(y+m)| \ge |-3m| - |y-x| \ge 3-1=2.$$

At this point, there are now a total of $x+y-m$ coins in the sacks, and the difference between the numbers of coins in the two sacks is at least 2. Thus, by induction hypothesis, P_2 has a winning strategy. This proves the claim when $|x-y| \le 1$.

Now consider the case $|x-y| \ge 2$. Without loss of generality, let $x>y$. P_1 would like to find a m such that $2m \le x$, $m \ge 1$ and

$$|(x-2m)-(y+m)| \le 1.$$

The number $m=[(x-y-1)/3]$ satisfies the last two inequalities above and we claim $2m \le x$ as well. Indeed, $x-2m$ is nonnegative because it differs by at most 1 from the positive number $y+m$. After taking $2m$ coins from the sack with x coins, P_1 leaves a total of $x+y-m$ coins, where the difference between the numbers of coins in the sacks is at most 1. Hence, by the induction hypothesis, the other player P_2 has no winning strategy. It follows that P_1 has a winning strategy, as desired.

This completes the proof of the induction and of the claim. It follows that the first pirate can guarantee that he will obtain the diamond if and only if the number of coins initially in the sacks differs by at least 2.

<u>*Example 5 (2015 Croatian National Math Competition).*</u> In a country between every two cities there is a direct bus or a direct train line (all lines are two-way and they don't pass through any other city). Prove that all cities in that country can be arranged in two disjoint sets so that all cities in one set can be visited using only train so that no city is visited twice, and all cities in the other set can be visited using only bus so that no city is visited twice.

<u>*Solution.*</u> Let G be the set of all cities in the country. For disjoint subsets A, Z of G, we call a pair (A,Z) *good* if all cities in the set A can be visited using only bus such that no city is visited twice and all cities in the set Z can be visited using only train such that no city is visited twice.

Let (A,Z) be a good pair such that $A \cup Z$ has the maximum number of elements. If we prove $A \cup Z=G$, then the statement of the problem will follow.

Let us assume the opposite, i.e. there is a city g which is not from A nor Z. Without loss of generality we can assume that A and Z are non-empty because otherwise we can transfer any city from a non-empty set to an empty one.

Let n be the number of cities in the set A and m be the number of cities in the set Z. Let us arrange the cities from A in the series $a_1,\ldots,a_n$ such that every two consecutive cities in that series are connected by a direct bus line. Also, let us arrange the cities from Z in the series $z_1,\ldots,z_m$ such that every two consecutive cities in that series are connected by a direct train line.

Since we assumed that the pair (A,Z) is maximum, the cities g and a_1 have to be connected by train (otherwise the pair $(A \cup \{g\},Z)$ would be a good pair whose union would have more elements than $A \cup Z$, and g and z_1 have to be connected by bus (otherwise the pair $(A,Z \cup \{g\})$ would be a good pair whose union would have more element than $A \cup Z$).

The cities a_1 and z_1 have to be connected by bus or by train. If a_1 and z_1 are connected by bus, let us put $A'=\{z_1,g,a_1,\ldots,a_n\}$ and $Z'=\{z_2,\ldots,z_m\}$. Then (A',Z') is a good pair and the number of elements of $A' \cup Z'$ is greater than the number of elements of $A \cup Z$, which contradicts the assumption.

If a_1 and z_1 are connected by train, let us put $A''=\{a_2,\ldots,a_n\}$ and $Z''=\{a_1,g, z_1,\ldots,z_m\}$. Then (A'',Z'') is a good pair and the number of elements of $A'' \cup Z''$ is greater than the number of elements of $A \cup Z$, which contradicts the assumption.

Since all cases lead to contradiction, we conclude that the assumption was wrong and that every city is either in the set A or in the set Z.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***August 31, 2018.***

Problem 516. Determine all triples (p,m,n) of positive integers such that p is prime and $2^m p^2+1=n^5$ holds.

Problem 517. For all positive x and y, prove that

$$x^2y^2(x^2+y^2-2) \geq (xy-1)(x+y).$$

Problem 518. Let I be the incenter and AD be a diameter of the circumcircle of ΔABC. Let point E be on the ray BA and point F be on the ray CA. If the lengths of BE and CF are both equal to the semiperimeter of ΔABC, then prove that lines EF and DI are perpendicular.

Problem 519. Let A and B be subsets of the positive integers with 10 and 9 elements respectively. Suppose for every $x,y,u,v\in A$ satisfying $x+y=u+v$, we have $\{x,y\}=\{u,v\}$. Prove that the set $A+B=\{a+b: a\in A, b\in B\}$ has at least 50 elements.

Problem 520. Let P be the set of all polynomials $f(x)=ax^2+bx$,where a,b are nonnegative integers less than 2010^{18}. Find the number of polynomials f in P for which there is a polynomial g in P such that $g(f(k))\equiv k \pmod{2010^{18}}$ for all integers k.

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Solutions

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Problem 511. Let $x_1,x_2,\ldots,x_{40}$ be positive integers with sum equal to 58. Find the maximum and minimum possible value of $x_1^2+x_2^2+\cdots+x_{40}^2$.

Solution. **Arpon BASU** (AECS-4, Mumbai, India), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **William KAHN** (Sidney, Australia), **LAI Wai Lok** (La Salle Primary School), **LEUNG Hei Chun**, **LUI On Ki**, **George SHEN**, **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

If there exist $x_m, x_n\geq 2$, then we can replace them by x_m+x_n-1, 1 due to

$$(x_m+x_n-1)^2+1^2-(x_m^2+x_n^2)$$
$$= 2(x_m-1)(x_n-1) \geq 0.$$

So the maximum case can be attained by one 19 and thirty-nine 1's. This gives the maximum value $39\times 1^2+1\times 19^2=400$.

For the minimum case, there exists at least one 1, otherwise $58=x_1+x_2+\cdots+x_{40}\geq 2\times 40 =80$, contradiction. Let x_k be a largest term. If $x_k\geq 3$, then we can replace x_k and 1 by x_k-1 and 2 to lower the square sums since

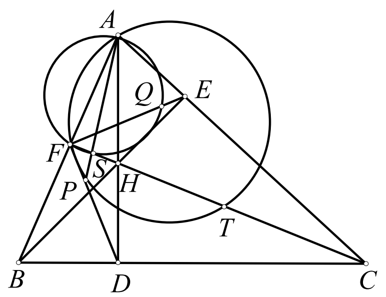
$$(x_k^2+1^2)-[(x_k-1)^2+2^2] = 2(x_k-2) > 0.$$

So in the minimum case, there are twenty-two 1's and eighteen 2's yielding $22\times 1^2+18\times 2^2=94$.

Other commended solvers: **George SHEN** and **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Problem 512. Let AD, BE, CF be the altitudes of acute ΔABC. Points P and Q are on segments DF and EF respectively. If $\angle PAQ=\angle DAC$, then prove that AP bisects $\angle FPQ$.

Solution. **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Let H be the orthocenter of ΔABC. Let S be the intersection of AP and CF. Let T be the intersection of AQ and CF. Now $\angle AFC=90°=\angle ADC$. As $AFDC$ is cyclic,

$$\angle PAT=\angle PAQ=\angle DAC=\angle DFC=\angle PFT,$$

points A, T, F, P are concyclic. Also, since

$$\angle SFQ=\angle HFE=\angle HAE$$
$$=\angle DAC=\angle PAQ=\angle SAQ,$$

points A, F, S, Q are concyclic. Then since

$$\angle SQT=\angle SFA=90^\circ$$
$$=\angle AFT=\angle APT=\angle SPT=\angle SAQ,$$

points S,P,T,Q are concyclic. Therefore, we have

$$\angle FPA=\angle FTA=\angle STQ=\angle SPQ,$$

which implies AP bisects $\angle FPQ$.

Other commended solvers: **Andrea FANCHINI** (Cantù, Italy), **William KAHN** (Sidney, Australia), **LEUNG Hei Chun**, **George SHEN** and **ZHANG Yupei** (HKUST).

Problem 513. Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative integers satisfying the conditions:

(1) $a_{n+1}=3a_n-3a_{n-1}+a_{n-2}$ for $n>1$,

(2) $2a_1=a_0+a_2-2$,

(3) for every positive integer m, in the sequence $a_0, a_1, a_2, \ldots$, there exist m terms $a_k, a_{k+1}, \ldots, a_{k+m-1}$, which are perfect squares.

Prove that every term in $a_0,a_1,a_2,\ldots$ is a perfect square.

Solution. **William KAHN** (Sidney, Australia), **LEUNG Hei Chun**, **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

We show we can select integers α, β, γ such that $a_n=n(n-1)\alpha/2+n\beta+\gamma$. For $n=0$, we must have $\gamma=a_0$. For $n=1$, we must have $a_1=\beta+\gamma$ and we can set integer β as $a_1-\gamma = a_1-a_0$. Finally for $n=2$, we must have $a_2=\alpha+2\beta+\gamma$ and we can set integer $\alpha = a_2-2\beta-\gamma = a_2-2(a_1-a_0) - a_0$. Then since all three sequences $b_n=n^2$, $b_n=n$ and $b_n=1$ satisfy the relation $b_{n+1}=3b_n-3b_{n-1}+b_{n-2}$, we also have $a_n= n(n-1)\alpha/2 + n\beta + \gamma = n^2\alpha/2+n(\beta-\alpha/2)+\gamma$ satisfies the relation.

From (2), we get $2(\beta+\gamma) = \gamma + \alpha + 2\beta + \gamma - 2$ or $\alpha = 2$. Therefore, we have $a_n= n(n-1)+n\beta+\gamma$, which can be put in the form $[(2n+t)^2+s]/4$ for some integers s and t.

Assume $s\neq 0$. If

$$(2n+t-1)^2<(2n+t)^2+s<(2n+t+1)^2 \quad (*),$$

then a_n cannot be a perfect square. However, (*) is equivalent to

$$-(4n+2t-1) < s < 4n+2t+1$$

or $-2t+1-s<4n$ and $s-2t-1<4n$, which is valid for sufficiently large n. Therefore, (3) would lead to $s=0$.

Since $a_0 = t^2/4$ must be an integer, so t must be even. Let $t=2t'$, then

$$a_n = \frac{(2n+2t')^2}{4} = (n+t')^2,$$

which implies that every term in a_n is a perfect square.

Other commended solvers: **Arpon BASU** (AECS-4, Mumbai, India), **George SHEN** and **ZHANG Yupei** (HKUST).

Problem 514. Let n be a positive integer and let $p(x)$ be a polynomial with real coefficients on the interval $[0,n]$ such that $p(0)=p(n)$. Prove that there are n distinct ordered pairs (a_i, b_i) with $i=1,2,\ldots,n$ such that $0\le a_i<b_i\le n$, b_i-a_i is an integer and $p(a_i)=p(b_i)$.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

We can solve the problem with continuous functions in place of polynomials. We will prove this by using mathematical induction. The case $n=1$ is trivial. Suppose the case $n-1$ is true. Define $f(x)=p(x+1)-p(x)$. Then

$f(0)+f(1)+\cdots+f(n-1)=p(n)-p(0)=0.$ (*)

First we show there exists $w\in[0,n-1]$ such that $p(w)=p(w+1)$. In fact, if there exists $k\in\{0,1,2,\ldots,n-1\}$ such that $f(k)=0$, then taking $w=k$, we are done. Otherwise, from (*), we know there exists $j\in\{0,1,2,\ldots,n-1\}$ such that $f(j)f(j+1)<0$. Then there is $w\in(j, j+1)$ such that $f(w)=0$. So $p(w) = p(w+1)$.

Next, define $g(x)=p(x)$ for $x\in[0,w]$ and $g(x)=p(x+1)$ for $x\in[w,n-1]$. Then $g(x)$ is continuous on $[0,n-1]$ and $g(0) = g(n-1)$. From induction hypothesis, there exist x_i and y_i with $y_i-x_i\in\mathbb{N}$ satisfying $g(x_i)=g(y_i)$ for $i=1,2,\ldots,n-1$. Then there are three cases:

(1) for $y_i<w$, $0 = g(y_i) - g(x_i) = p(y_i) - p(x_i)$,

(2) for $x_i \le w \le y_i$, $0 = g(y_i) - g(x_i) = p(y_i+1) - p(x_i)$ and

(3) for $w < x_i$, $0 = g(y_i) - g(x_i) = p(y_i+1) - p(x_i+1)$.

Together with $p(0) = p(n)$, we get the case n completing the induction step.

Other commended solvers: **William KAHN** (Sidney, Australia) and **George SHEN**.

Problem 515. There are ten distinct nonzero real numbers. It is known that for every two of the numbers, either the sum or the product of them is rational. Prove that the square of each of the ten numbers is rational.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Pick six of the nonzero distinct real numbers, say $A_1, A_2, \cdots, A_6$ (with the property that for $i\ne j$, either $A_iA_j\in\mathbb{Q}$ or $A_i+A_j\in\mathbb{Q}$). Consider a graph with $A_1, A_2, \cdots, A_6$ as vertices and color the edge with vertices A_i, A_j blue if $A_i+A_j\in\mathbb{Q}$, otherwise red for $A_iA_j\in\mathbb{Q}$. By Ramsay's Theorem, there is a red or a blue triangle in the complete graph with $A_1, A_2, \cdots, A_6$ as vertices.

There are two cases. In case 1, there is a blue triangle with vertices, say A_1, A_2 and A_3. Then $A_1+A_2,\ A_2+A_3,\ A_3+A_1\in\mathbb{Q}$. So $2A_1=(A_1+A_2)+(A_3+A_1)-(A_2+A_3)\in\mathbb{Q}$. Then $A_1\in\mathbb{Q}$ and similarly $A_2,A_3\in\mathbb{Q}$.

Next, for any $B\in\{A_4, A_5, \ldots, A_{10}\}$, we see $A_1+B\in\mathbb{Q}$ or $A_1B\in\mathbb{Q}$. So $B=(A_1+B)-A_1\in\mathbb{Q}$ or $B=(A_1B)/A_1\in\mathbb{Q}$. Then all ten $A_i\in\mathbb{Q}$.

In case 2, there is a red triangle with vertices, say A_1, A_2 and A_3. Then A_1A_2, A_2A_3, $A_3A_1\in\mathbb{Q}$. Now

$$A_1^2=(A_1A_2)(A_3A_1)/(A_2A_3)\in\mathbb{Q}$$

and similarly $A_2^2, A_3^2\in\mathbb{Q}$. If at least one of $A_1, A_2, A_3\in\mathbb{Q}$, say $A_1\in\mathbb{Q}$, then pick any $C\in\{A_2, A_3, \ldots, A_{10}\}$. Observe that $A_1+C\in\mathbb{Q}$ or $A_1C\in\mathbb{Q}$. It follows that we get $C=(A_1+C)-A_1\in\mathbb{Q}$ or $C=(A_1C)/A_1\in\mathbb{Q}$. Then all ten $A_i\in\mathbb{Q}$.

Otherwise, if $A_1^2\in\mathbb{Q}$, but $A_1\notin\mathbb{Q}$, then $A_1=m\sqrt{x}$, where $m=1$ or $m=-1$ and $x\in\mathbb{Q}$. Since $A_1A_2\in\mathbb{Q}$, we get $A_1A_2=(m\sqrt{x})A_2=b$ for some $b\in\mathbb{Q}$. Then we get $A_2=b/(m\sqrt{x})=r\sqrt{x}$, where $r=b/(mx)\in\mathbb{Q}$ and $m\ne r$ due to $A_1\ne A_2$. For $A_i\ne A_1, A_2$, if $A_1+A_i\in\mathbb{Q}$ and $A_2+A_i\in\mathbb{Q}$, then $(A_1+A_i)-(A_2+A_i)\in\mathbb{Q}$, but $(A_1+A_i)-(A_2+A_i)= A_1- A_2 = (m-r)\sqrt{x}\notin\mathbb{Q}$. Finally, if $A_1A_i\in\mathbb{Q}$ or $A_2A_i\in\mathbb{Q}$, then as above we get $A_i= s_i\sqrt{x}$ for some $s_i\in\mathbb{Q}$ with $s_i\ne m,r$. Then we have $A_i^2=s_i^2x\in\mathbb{Q}$.

Other commended solvers: **Arpon BASU** (AECS-4, Mumbai, India), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **William KAHN** (Sidney, Australia), **LUO On Ki** and **George SHEN**.

Olympiad Corner

(Continued from page 1)

Problem 3. (*Marko Radovanović*) Call a function $f:\mathbb{N}\to\mathbb{N}$ *lively* if

$f(a+b-1)=f(f(\cdots f(b)\cdots))$ for all $a,b\in\mathbb{N}$, where f appears a times on the right side.

Suppose that g is a lively function such that $g(A+2018)=g(A)+1$ holds for some $A\ge2$.

(a) Prove that $g(n+2017^{2017})=g(n)$ for all $n\ge A+2$.

(b) If $g(A+2017^{2017})\ne g(A)$, determine $g(n)$ for $n\le A-1$.

<u>Second Day</u>

Problem 4. (*Dušan Djukić*) An $n\times n$ square is divided into unit squares. One needs to place a number of isosceles right triangles with hypotenuse 2, with vertices at grid points, in such a way that every side of every unit square belongs to exactly one triangle (i.e. lies inside it or on its boundary). Determine all numbers n for which this is possible.

Problem 5. (*Dušan Djukić*) For a positive integer $n\ge2$, let $C(n)$ be the smallest positive real constant such that there is a sequence of n real numbers $x_1,x_2,\ldots,x_n$, not all zero, satisfying the following conditions:

(i) $x_1+x_2+\cdots+x_n=0$;

(ii) for each $i=1,2,\ldots,n$, it holds that $x_i\le x_{i+1}$ or $x_i\le x_{i+1}+C(n)x_{i+2}$ (the indices are taken modulo n).

Prove that:

(a) $C(n)\ge2$ for all n;

(b) $C(n)=2$ if and only if n is even.

Problem 6. (*Bojan Bašić*) Let k be a positive integer and let n be the smallest positive integer having exactly k divisors. If n is a perfect cube, can the number k have a prime divisor of the form $3j+2$?