Junior problems

J241. Determine all positive integers that can be represented as

$$\frac{ab+bc+ca}{a+b+c+\min(a,b,c)}$$

for some positive integers a, b, c.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J242. Let ABC be a triangle and let D, E, F be the feet of the altitudes from A, B, C to the sides BC, CA, AB, respectively. Let X, Y, Z be the midpoints of segments EF, FD, DE and let x, y, z be the perpendiculars from X, Y, Z to BC, CA, and AB, respectively. Prove that the lines x, y, z are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

J243. Let a, b, c be real numbers such that

$$\left(-\frac{a}{2}+\frac{b}{3}+\frac{c}{6}\right)^3+\left(\frac{a}{3}+\frac{b}{6}-\frac{c}{2}\right)^3+\left(\frac{a}{6}-\frac{b}{2}+\frac{c}{3}\right)^3=\frac{1}{8}.$$

Prove that

$$(a-3b+2c)(2a+b-3c)(-3a+2b+c)=9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J244. Let a and b be positive real numbers. Prove that

$$1\le\frac{\sqrt[n]{a^n+b^n}}{\sqrt[n+1]{a^{n+1}+b^{n+1}}}\le\sqrt[n(n+1)]{2}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J245. Find all triples (x,y,z) of positive real numbers satifying simultaneously the inequalities $x+y+z-2xyz\le 1$ and

$$xy+yz+zx+\frac{1}{xyz}\le 4.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J246. Let ABC be a triangle with circumcircle Γ and let P be a point on the side BC. Let Ω be the circle tangent to BC at P and to Γ internally. Let τ be the length of the tangents from A to Ω and let U and V be the intersections of Γ with the circle centered at A and radius τ. Prove that UV is tangent to Γ.

Proposed by Cosmin Pohoata, Princeton University, USA

Senior problems

S241. Let p and q be odd primes such that $\frac{p^3-q^3}{3} \geq 2pq+3$. Prove that

$$\frac{p^3-q^3}{4} \geq 3pq+16.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S242. Let ABC be a triangle and denote by K, L, M the midpoints of the arcs BC, CA, AB, respectively (the ones not containing the vertices of the triangle). Show that the perimeter of the hexagon $AMBKCL$ is greater than or equal to $4(R+r)$.

Proposed by Michal Rolinek, Charles University, Czech Republic

S243. A group of boys and girls went to a dance party. It is known that for every pair of boys, there are exactly two girls who danced with both of them; and for every pair of girls there are exactly two boys who danced with both of them. Prove that the numbers of girls and boys are equal.

Proposed by Iurie Boreico, Stanford University, USA

S244. Let ABC be an acute-angled triangle and let τ be the inradius of its orthic triangle. Prove that

$$r \geq \frac{R+\tau}{2},$$

where r and R are the inradius and circumradius of triangle ABC.

Proposed by Luis Gonzalez, Maracaibo, Venezuela

S245. Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Prove that

$$\frac{1}{(a-1)(b-1)(c-1)}+\frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S246. Let ABC be a triangle with circumcircle Ω and let X, Y, Z be points on the sides BC, CA, AB, respectively. Let α, β, γ be the circles to Γ that are also tangent to BC, CA, AB at points X, Y, Z, respectively. Let X', Y', Z' be the tangency points of Ω with α, β, and γ. Prove that the lines AX, BY, CZ are concurrent if and only if the lines AX', BY', CZ' are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

Undergraduate problems

U241. Let $a > b$ be positive real numbers. Prove that

$$c_n = \frac{\sqrt[n+1]{a^{n+1} - b^{n+1}}}{\sqrt[n]{a^n - b^n}}$$

is a decreasing sequence and find its limit.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U242. Find all continuous functions $f \colon \mathbb{R} \to \mathbb{R}$ that satisfy the following condition: whenever $x + y + z = 2k\pi$ is an integer multiple multiple of 2π,

$$f^2(x) + f^2(y) + f^2(z) - 2f(x)f(y)f(z) = 1$$

Proposed by Iurie Boreico, Stanford University, USA

U243. Let $f \ : \ (a, b) \to \mathbb{R}$ be a differentiable function such that $f'(a) = f'(b) = 0$ and with the property that there is a real valued function g for which $g(f'(x)) = f(x)$ for all x in $\mathbb{R}$. Prove that f is constant.

Proposed by Mihai Piticari and Sorin Radulescu, Bucharest, Romania

U244. Jimmy is analyzing a random variable X with infinite mean and variance. He needs to come up with a tentative mean and variance for his approximation model. Jimmy knows that X has symmetric distribution around 0. He decides that his tentative mean is going to be $\mu_X^* = 0$ and his tentative variance is going to be

$$(\sigma_X^*)^2 = E_{|X|\le 1}[X^2] + \exp\left(E_{|X|\ge 1}[\ln(X^2)]\right).$$

Prove that the tentative variance that Jimmy came up with for a standard Cauchy random variable $C(0,1)$ is

$$(\sigma_X^*)^2 = 2 - \frac{\pi}{2} + \exp\left(\frac{2\pi^2}{3}\right).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U245. Let K be a finite field of characteristic $p > 2$ and let $a \in K - \{0\}$. Let f be any polynomial over K. Prove that the following statements are equivalent: i) $f(X) = f(X + a)$; ii) there is $g \in K[X]$ such that $f(X) = g(X^p - a^{p-1}X)$.

Proposed by Mihai Piticari and Sorin Radulescu, Bucharest, Romania

U246. Find all pairs m, n of positive integers for which there is a non-abelian group G such that the maps $x \to x^m$ and $x \to x^n$ are endomorphisms of G.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Olympiad problems

O241. Let a and b be real numbers such that $3 \leq a^2 + ab + b^2 \leq 6$. Prove that $2 \leq a^4 + b^4 \leq 72$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O242. Let $n \geq 3$ be an odd integer. Consider a regular n-gon $\mathcal{A} = A_1A_2..., A_n$. Find the locus of points, P, inside $\mathcal{A}$ such that

$$\angle PA_1A_2 + \angle PA_2A_3 + \cdots + \angle PA_nA_1 = 90^\circ + 180^\circ \cdot k$$

for some integer k, where the angles are directed.

Proposed by Alex Anderson, University of California, Berkeley, USA

O243. Let m, n be positive integers with $n > m$. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+n-2k}{n-1} = \binom{n}{m+1}.$$

Proposed by Iurie Boreico, Stanford University, USA

O244. Let ABC be a triangle and let D, E, F be the tangency points of the incircle with BC, CA, AB, respectively. Let EF meet the circumcircle Γ of ABC at X and Y. Furthermore, let T be the second intersection of the circumcircle of DXY with the incircle. Prove that AT passes through the tangency point A' of the A-mixtilinear incircle with Γ.

Proposed by Sammy Luo, North Carolina School of Science and Mathematics and Cosmin Pohoata, Princeton University, USA

O245. Prove that in a $(1+\sqrt{2}) \times (1+\sqrt{2})$ square we cannot fit five 1×1 squares without overlapping, but we can fit them in a $(2+\frac{1}{\sqrt{2}}) \times (2+\frac{1}{\sqrt{2}})$ square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O246. Let P be a point inside or on the boundary of a convex polygon $A_1A_2 \cdots A_n$. Prove that the maximum value of $\sum_{i=1}^{n} PA_i$ is achieved when P is a vertex of $A_1A_2 \cdots A_n$.

Proposed by Cosmin Pohoata, Princeton University, USA