Mathematical Excalibur

Volume 13, Number 1 — February-April, 2008

Olympiad Corner

The 2008 APMO was held in March. Here are the problems.

Problem 1. Let ABC be a triangle with $\angle A < 60°$. Let X and Y be the points on the sides AB and AC, respectively, such that $CA+AX = CB+BX$ and $BA+AY = BC+CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC, respectively. Prove that $\angle BPC < 120°$.

Problem 2. Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

Problem 3. Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 20, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Point Set Combinatorics

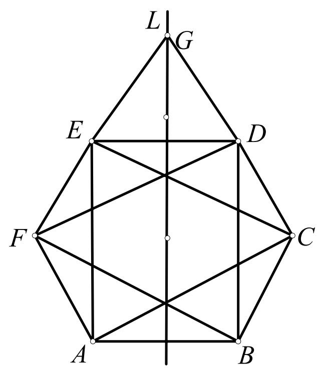
Kin Y. Li

Problems involving sets of points in the plane or in space often appear in math competitions. We will look at some typical examples. The solutions of these problems provide us the basic ideas to attack similar problems.

The following are some interesting examples.

Example 1. (*2001 USA Math Olympiad*) Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

Solution. Let A, B be arbitrary distinct points and consider a regular hexagon $ABCDEF$ in the plane. Let lines CD and EF intersect at G. Let L be the line through G perpendicular to line DE.



Observe that ΔCEG and ΔDFG are symmetric with respect to L and hence they have the same incenter. So $c+e+g = d+f+g$. Also, ΔACE and ΔBDF are symmetric with respect to L and have the same incenter. So $a+c+e=b+d+f$. Subtracting these two equations, we see $a=b$.

Comments: This outstanding elegant solution was due to Michael Hamburg, who was given a handsome cash prize as a Clay Math Institute award by the USAMO Committee.

Example 2. (*1987 IMO Shortlisted Problem*) In space, is there an infinite set M of points such that the intersection of M with every plane is nonempty and finite?

Solution. Yes, there is such a set M. For example, let

$$M = \{(t^5, t^3, t) : t \in \mathbb{R}\}.$$

Then, for every plane with equation $Ax + By + Cz + D = 0$, the intersection points are found by solving

$$At^5 + Bt^3 + Ct + D = 0,$$

which has at least one solution (since A or B or C is nonzero) and at most five solutions (since the degree is at most five).

Example 3. (*1963 Beijing Mathematics Competition*) There are $2n + 3$ ($n \ge 1$) given points on a plane such that no three of them are collinear and no four of them are concyclic.

Is it always possible to draw a circle through three of them so that half of the other $2n$ points are inside and half are outside the circle?

Solution. Yes, it is always possible.

Take the *convex hull* of these points, i.e. the smallest convex set containing them. The boundary is a polygon with vertices from the given points.

Let AB be a side of the polygon. Since no three are collinear, no other given points are on AB. By convexity, the other points $C_1, C_2, \cdots, C_{2n+1}$ are on the same side of line AB. Since no four are collinear, angles AC_iB are all distinct, say

$$\angle AC_1B < \angle AC_2B < \cdots < \angle AC_{2n+1}B.$$

Then $C_1, C_2, \cdots, C_n$ are inside the circle through A, B and C_{n+1} and $C_{n+2}, C_{n+3}, \cdots, C_{2n+1}$ are outside.

Example 4. (*1941 Moscow Math. Olympiad*) On a plane are given n points such that every three of them is inside some circle of radius 1. Prove that all these points are inside some circle of radius 1.

Solution. For every three of the n given points, consider the triangle they formed. If the triangle is an acute triangle, then draw their circumcircle, otherwise take the longest side and draw the circle having that side as the diameter. By the given condition, all these circles have radius less than 1.

Let S be one of these circles with minimum radius, say S arose from considering points A, B, C.

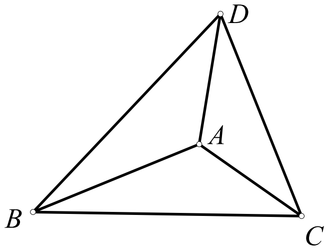
Assume one of the given points D is not inside S.

If ΔABC is acute, then D is on the same side as one of A,B,C with respect to the line through the other two points, say D and A are on the same side of line BC. Then the circle drawn for B,C,D would be their circumcircle and would have a radius greater than the radius of S, a contradiction.

If ΔABC is not acute and S is the circle with diameter AB, then the circle drawn for A, B, D would have AB as a chord and not as a diameter, which implies that circle has a radius greater than the radius of S, a contradiction.

Therefore, all n points are inside or on S. Since the radius of S was less than 1, we can take the circle of radius 1 at the same center as S to contain all n points.

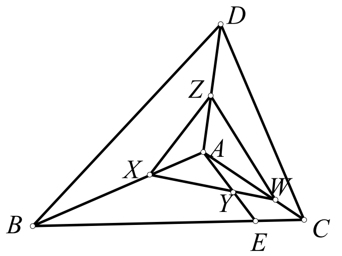
In the next example, we will consider a problem in space and the solution will involve a basic fact from solid geometry. Namely,



about vertex A of a tetrahedron $ABCD$, we have

$$\angle BAC \le \angle BAD + \angle DAC \le 360^\circ.$$

Nowadays, very little solid geometry is taught in school. So let's recall Euclid's proofs in Book XI, Problems 20 and 21 of his *Elements*.



For the left inequality, we may assume that $\angle BAC$ is the largest of the three angles about vertex A. Let E be on side BC so that $\angle BAD = \angle BAE$. Let X, Y, Z be on rays AB, AC, AD respectively, and $AX=AY=AZ$. Then $\Delta AXZ \cong \Delta AXY$ and we have $XZ=XY$. Let line XY intersect line AC at W. Since $XZ + ZW > XW$, cancelling $XZ = XY$ from both sides, we have $ZW > YW$. Comparing triangles WAZ and WAY, we have $WA=WA$, $AZ=AY$, so $ZW > YW$ implies $\angle ZAW > \angle YAW$. Then

$$\begin{aligned}\angle BAC &= \angle XAY + \angle YAW \\ &< \angle XAZ + \angle ZAW \\ &= \angle BAD + \angle DAC.\end{aligned}$$

For the right inequality, by the left inequality, we have

$$\begin{aligned}\angle DBC &\le \angle DBA + \angle ABC, \\ \angle BCD &\le \angle BCA + \angle ACD, \\ \angle CDB &\le \angle CDA + \angle ADB.\end{aligned}$$

Adding them, we get 180˚ is less than or equal to the sum of the six angles on the right. Now the sum of these six angles and the three angles about A is $3\times180^\circ$. So the sum of the three angles about A is less than or equal to 360˚.

Example 5. (*1969 All Soviet Math. Olympiad*) There are n given points in space with no three collinear. For every three of them, they form a triangle having an angle greater than 120°. Prove that there is a way to order the points as A_1, A_2, …, A_n such that whenever $1 \le i < j < k \le n$, we have

$$\angle A_iA_jA_k > 120^\circ.$$

Solution. Take two furthest points among these n points and call them A_1 and A_n.

For every two points X, Y among the other $n-2$ points, since A_1A_n is the longest side in both ΔA_1XA_n and ΔA_1YA_n, we have $\angle XA_1A_n < 60^\circ$ and $\angle YA_1A_n < 60^\circ$. About vertex A_1 of the tetrahedron A_1A_nXY, we have

$$\begin{aligned}\angle XA_1Y &\le \angle XA_1A_n + \angle YA_1A_n \\ &< 60^\circ + 60^\circ = 120^\circ.\end{aligned}$$

Similarly, $\angle XA_nY < 120^\circ$.

Also, $A_1X \ne A_1Y$ (since otherwise, the two equal angles in ΔXA_1Y cannot be greater than 90˚ and so only $\angle XA_1Y$ can be greater than 120°, which will contradict the inequality above). Now order the points by its distance to A_1 so that $A_1A_2 < A_1A_3 < \cdots < A_1A_n$.

For $1 < j < k \le n$, taking $X=A_j$ and $Y=A_k$ in the inequality above, we get $\angle A_jA_1A_k < 120°$. Since $A_1A_k > A_1A_j$, so in $\Delta A_1A_jA_k$, $\angle A_1A_jA_k > 120°$.

For $1 < i < j < k \le n$, we have $\angle A_1A_iA_j > 120°$ and $\angle A_1A_iA_k > 120°$ by the last paragraph. Then, about vertex A_i of the tetrahedron $A_iA_jA_kA_1$, we have $\angle A_jA_iA_k < 120°$. Next since $A_1A_k > A_1A_j > A_1A_i$, about vertex A_k of the tetrahedron $A_kA_jA_iA_1$, we have

$$\begin{aligned}\angle A_iA_kA_j &\le \angle A_iA_kA_1 + \angle A_jA_kA_1 \\ &< 60^\circ + 60^\circ = 120^\circ.\end{aligned}$$

Hence, in $\Delta A_iA_jA_k$, we have $\angle A_iA_jA_k > 120°$.

Example 6. (*1994 All Russian Math. Olympiad*) There are k points, $2 \le k \le 50$, inside a convex 100-sided polygon. Prove that we can choose at most $2k$ vertices from this 100-sided polygon so that the k points are inside the polygon with the chosen points as vertices.

Solution. Let $M = A_1A_2\cdots A_n$ be the boundary of the convex hull of the k points. Hence, $n \le k$. Let O be a point inside M. From i=1 to n, let ray OA_i intersect the 100-sided polygon at B_i. Let M' be the boundary of the convex hull of B_1, B_2, ···, B_n.

For every point P on or inside M, the line OP intersects M at two sides, say A_iA_{i+1} and A_jA_{j+1}. By the definition of the points B_i's, we see the line OP intersects B_iB_{i+1} and B_jB_{j+1}, say at points S and T respectively. Since B_i, B_{i+1}, B_j and B_{j+1} are in M', so S, T are in M'. Then O and P are in M'. Thus M' contains M.

Let $M' = C_1C_2\cdots C_m$. Then $m \le n \le k$. Observe that all C_i's are on the 100-sided polygon. Now each C_i is a vertex or between two consecutive vertices of the 100-sided polygon. Let G be the set of all these vertices. Then G has at most $2k$ points and the polygon with vertices from G contains the k points.

(Continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 20, 2008.***

Problem 296. Let $n > 1$ be an integer. From a $n \times n$ square, one 1×1 corner square is removed. Determine (with proof) the least positive integer k such that the remaining areas can be partitioned into k triangles with equal areas.
(*Source 1992 Shanghai Math Contest*)

Problem 297. Prove that for every pair of positive integers p and q, there exist an integer-coefficient polynomial $f(x)$ and an open interval with length $1/q$ on the real axis such that for every x in the interval, $|f(x) - p/q| < 1/q^2$.
(*Source:1983 Finnish Math Olympiad*)

Problem 298. The diagonals of a convex quadrilateral $ABCD$ intersect at O. Let M_1 and M_2 be the centroids of ΔAOB and ΔCOD respectively. Let H_1 and H_2 be the orthocenters of ΔBOC and ΔDOA respectively. Prove that $M_1M_2 \perp H_1H_2$.

Problem 299. Determine (with proof) the least positive integer n such that in every way of partitioning $S = \{1,2,\ldots,n\}$ into two subsets, one of the subsets will contain two distinct numbers a and b such that ab is divisible by $a+b$.

Problem 300. Prove that in base 10, every odd positive integer has a multiple all of whose digits are odd.

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Solutions

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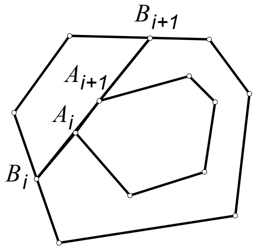
Problem 291. Prove that if a convex polygon lies in the interior of another convex polygon, then the perimeter of the inner polygon is less than the perimeter of the outer polygon.

Solution. **Jeff CHEN** (Virginia, USA), **HO Kin Fai** (HKUST, Math Year 3) and **Fai YUNG**.



We will define a sequence of convex polygons $P_0, P_1, \ldots, P_{n-1}$. Let the outer convex polygon be P_0 and the inner convex polygon be $A_1A_2\ldots A_n$. For $i = 1$ to $n-1$, let the line A_iA_{i+1} intersect P_{i-1} at B_i, B_{i+1}. The line A_iA_{i+1} divides P_{i-1} into two parts with one part enclosing $A_1A_2\ldots A_n$. Let P_i be the polygon formed by putting the segment B_iB_{i+1} together with the part of P_{i-1} enclosing $A_1A_2\ldots A_n$. Note P_{n-1} is $A_1A_2\ldots A_n$. Finally, the perimeter of P_i is less than the perimeter of P_{i-1} because the length of B_iB_{i+1}, being the shortest distance between B_i and B_{i+1}, is less than the length of the part of P_{i-1} removed to form P_i.

Commended solvers: **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), **Raúl A. SIMON** (Santiago, Chile) and **Simon YAU Chi-Keung** (City University of Hong Kong).

Problem 292. Let $k_1 < k_2 < k_3 < \cdots$ be positive integers with no two of them are consecutive. For every $m = 1, 2, 3, \ldots$, let $S_m = k_1+k_2+\cdots+k_m$. Prove that for every positive integer n, the interval $[S_n, S_{n+1})$ contains at least one perfect square number.
(*Source: 1996 Shanghai Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **HO Kin Fai** (HKUST, Math Year 3), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **Raúl A. SIMON** (Santiago, Chile).

There is a nonnegative integer a such that $a^2 < S_n \le (a+1)^2$. We have

$$\begin{aligned} S_n &= k_n + k_{n-1} + \cdots + k_1 \\ &< k_n + (k_n - 2) + \cdots + (k_n - 2n + 2) \\ &= n(k_n - n + 1). \end{aligned}$$

By the *AM-GM* inequality,

$$a < \sqrt{S_n} < \frac{n + (k_n - n + 1)}{2} = \frac{k_n + 1}{2}.$$

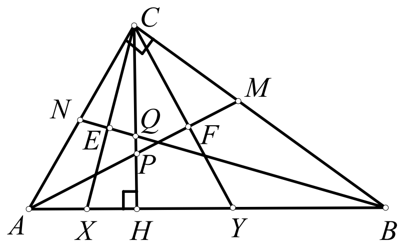
Then

$$\begin{aligned} (a+1)^2 &= a^2 + 2a + 1 < S_n + (k_n+1) + 1 \\ &\le S_n + k_{n+1} = S_{n+1}. \end{aligned}$$

Commended solvers: **Simon YAU Chi-Keung** (City University of Hong Kong).

Problem 293. Let CH be the altitude of triangle ABC with $\angle ACB = 90°$. The bisector of $\angle BAC$ intersects CH, CB at P, M respectively. The bisector of $\angle ABC$ intersects CH, CA at Q, N respectively. Prove that the line passing through the midpoints of PM and QN is parallel to line AB.
(*Source: 52nd Belorussian Math. Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA) and **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).



Let E, F be the midpoints of QN, PM respectively. Let X, Y be the intersection of CE, CF with AB respectively. Now

$$\begin{aligned} \angle CMP &= 90° - \angle CAM \\ &= 90° - \angle BAM \\ &= \angle APH = \angle CPM. \end{aligned}$$

So $CM=CP$. Then $CF \perp AF$. Since AF bisects $\angle CAY$, by ASA, $\Delta CAF \cong \Delta YAF$. So $CF=FY$. Similarly, $CE=EX$. By the midpoint theorem, we have EF parallel to line XY, which is the same as line AB.

Commended solvers: **Konstantine ZELATOR** (University of Toledo, Toledo, Ohio, USA).

Problem 294. For three nonnegative real numbers x, y, z satisfying the condition $xy+yz+zx = 3$, prove that

$$x^2 + y^2 + z^2 + 3xyz \ge 6.$$

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Ovidiu FURDUI** (Cimpia - Turzii, Cluj, Romania), **MA Ka Hei** (Wah Yan College, Kowloon) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Let $p = x+y+z$, $q = xy+yz+zx$ and $r = xyz$. Now

$$p^2 - 9 = x^2 + y^2 + z^2 - xy - yz - zx$$
$$= \frac{(x-y)^2 + (y-z)^2 + (z-x)^2}{2} \ge 0,$$

So $p \ge 3$. By Schur's inequality (see Math Excalibur, vol. 10, no. 5, p. 2, column 2), $12p = 4pq \le p^3 + 9r$. Since

$$p^2 = x^2+y^2+z^2+2(xy+yz+zx)$$
$$= x^2+y^2+z^2+6,$$

we get

$$3xyz = 3r \ge 9r/p$$
$$\ge 12 - p^2$$
$$= 6 - (x^2 + y^2 + z^2).$$

Problem 295. There are $2n$ distinct points in space, where $n \ge 2$. No four of them are on the same plane. If $n^2 + 1$ pairs of them are connected by line segments, then prove that there are at least n distinct triangles formed.
(*Source: 1989 Chinese IMO team training problem*)

Solution. **Jeff CHEN** (Virginia, USA) and **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England).

We prove by induction on n. For n=2, say the points are A,B,C,D. For five segments connecting them, only one pair of them is not connected, say they are A and B. Then triangles ACD and BCD are formed.

Suppose the case n=k is true. Consider the case n=k+1. We first claim there is at least one triangle. Suppose AB is one such connected segment. Let α, β be the number of segments connecting A, B to the other $2n$−2=$2k$ points respectively.

If $\alpha+\beta > 2k+1$, then A, B are both connected to one of the other $2k$ points, hence a triangle is formed.

If $\alpha+\beta \le 2k$, then the other $2k$ points have at least $(k+1)^2 + 1 - (2k+1) = k^2 + 1$ segments connecting them. By the case n=k, there is a triangle in these $2k$ points.

So the claim is established. Now take one such triangle, say ABC. Let α, β, γ be the number of segments connecting A, B, C to the other $2k$−1 points respectively.

If $\alpha+\beta+\gamma \ge 3k-1$, then let D_1, D_2, …, D_m ($m \le 2k-1$) be all the points among the other $2k$−1 points connecting to at least one of A or B or C. The number of segments to D_i from A or B or C is n_i = 1 or 2 or 3. Checking each of these three cases, we see there are at least n_i−1 triangles having D_i as a vertex and the two other vertices from A, B, C. So there are

$$\sum_{i=1}^{m}(n_i - 1) \ge 3k - 1 - m \ge k$$

triangles, each having one D_i vertex, plus triangle ABC, resulting in at least k+1 triangles.

If $\alpha+\beta+\gamma \le 3k-2$, then the sum of $\alpha+\beta$, $\gamma+\alpha$, $\beta+\gamma$ is at most $6k$−4. Hence the least of them cannot be $2k$−1 or more, say $\alpha+\beta \le 2k-2$. Then removing A and B and all segments connected to at least one of them, we have at least $(k+1)^2+1-(2k+1)=k^2+1$ segments left for the remaining $2k$ points. By the case n=k, we have k triangles. These plus triangle ABC result in at least k+1 triangles. The induction is complete.

Commended solvers: **Raúl A. SIMON** (Santiago, Chile) and **Simon YAU Chi-Keung** (City University of Hong Kong).

Olympiad Corner

(continued from page 1)

Problem 4. Consider the function $f: \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions:

(i) $f(0) = 0$, (ii) $f(2n) = 2f(n)$ and (iii) $f(2n+1) = n+2f(n)$ for all $n \ge 0$.

(a) Determine the three sets L:={ n | $f(n) < f(n+1)$}, E:={ n | $f(n) = f(n+1)$}, and G:={n | $f(n) > f(n+1)$}.

(b) For each $k \ge 0$, find a formula for a_k:= max{ $f(n)$ | $0 \le n \le 2^k$} in terms of k.

Problem 5. Let a, b, c be integers satisfying $0 < a < c-1$ and $1 < b < c$. For each k, $0 \le k \le a$, let r_k, $0 \le r_k < c$, be the remainder of kb when dived by c. Prove that the two sets $\{r_0, r_1, r_2, \ldots, r_a\}$ and $\{0, 1, 2, \ldots, a\}$ are different.

Point Set Combinatorics

(continued from page 2)

Example 7. (*1987 Chinese IMO Team Selection Test*) There are $2n$ distinct points in space, where $n \ge 2$. No four of them are on the same plane. If $n^2 + 1$ pairs of them are connected by line segments, then prove that there are two triangles sharing a common side.

Solution. We prove by induction on n. For n=2, say the points are A,B,C,D. For five segments connecting them, only one pair of them is not connected, say they are A and B. Then triangles ACD and BCD are formed and the side CD is common to them.

Suppose the case n=k is true. Consider the case n=k+1. Suppose AB is one such connected segment. Let α, β be the number of segments connecting A, B to the other $2n - 2 = 2k$ points respectively.

Case 1. If $\alpha+\beta \ge 2k+2$, then there are points C, D among the other $2k$ points such that AC, BC, AD, BD are connected. Then triangles ABC and ABD are formed and the side AB is common to them.

Case 2. If $\alpha+\beta \le 2k$, then removing A, B and all segments connecting to at least one of them, there would still be at least $(k+1)^2 + 1 - (2k+1) = k^2+1$ segments left for the remaining $2k$ points. By the case $n = k$, there would exist two triangles sharing a common side among them.

Case 3. Assume cases 1 and 2 do not occur for all the connected segments. Then take any connected segment AB and we have $\alpha+\beta = 2k+1$. There would then be a point C among the other $2k$ points such that triangle ABC is formed.

Let γ be the number of segments connecting C to the other $2k$−1 points respectively. Since cases 1 and 2 do not occur, we have

$$\beta + \gamma = 2k+1 \text{ and } \gamma + \alpha = 2k+1,$$

too. However, this would lead to

$$(\alpha + \beta) + (\beta + \gamma) + (\gamma + \alpha) = 6k + 3,$$

which is contradictory as the left side is even and the right side is odd.

One cannot help to notice the similarity between the last example and problem 295 in the problem corner. Naturally this raise the question: when n is large, again if $n^2 + 1$ pairs of the points are connected by line segments, would we be able to get more pairs of triangles sharing common sides? Any information or contribution for this question from the readers will be appreciated.

Mathematical Excalibur

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Olympiad Corner

The following are the four problems of the 2008 Balkan Mathematical Olympiad.

Problem 1. An acute-angled scalene triangle ABC is given, with $AC > BC$. Let O be its circumcenter, H its orthocenter and F the foot of the altitude from C. Let P be the point (other than A) on the line AB such that $AF=PF$ and M be the midpoint of AC. We denote the intersection of PH and BC by X, the intersection of OM and FX by Y and the intersection of OF and AC by Z. Prove that the points F, M, Y and Z are concyclic.

Problem 2. Does there exist a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ of positive real numbers satisfying both of the following conditions:

(i) $\sum_{i=1}^{n} a_i \le n^2$, for every positive integer n;

(ii) $\sum_{i=1}^{n} \frac{1}{a_i} \le 2008$, for every positive integer n ?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 20, 2008***.

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Geometric Transformations I

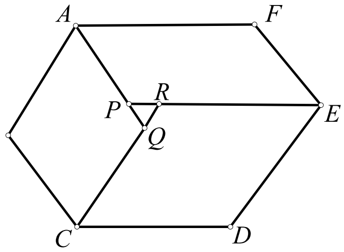
Kin Y. Li

Too often we <u>*stare*</u> at a figure in solving a geometry problem. In this article, we will <u>*move*</u> parts of the figure to better positions to facilitate the way to a solution.

Below we shall denote the vector from X to Y by the boldface italics $\boldsymbol{XY}$. On a plane, a <u>*translation*</u> by a vector v moves every point X to a point Y such that $\boldsymbol{XY} = v$. We denote this translation by $T(v)$.

<u>Example 1.</u> The opposite sides of a hexagon $ABCDEF$ are parallel. If $BC-EF = ED-AB = AF-CD > 0$, show that all angles of $ABCDEF$ are equal.

<u>Solution.</u> One idea is to move the side lengths closer to do the subtractions. Let $T(\boldsymbol{FA})$ move E to P, $T(\boldsymbol{BC})$ move A to Q and $T(\boldsymbol{DE})$ move C to R.



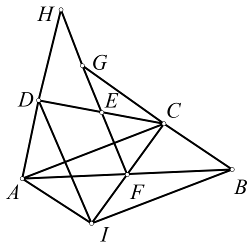
Hence, $EFAP$, $ABCQ$, $CDER$ are parallelograms. Since the opposite sides of the hexagon are parallel, P is on AQ, Q is on CR and R is on EP. Then, we get $BC - EF = AQ - AP = PQ$. Similarly, $ED - AB = QR$ and $AF - CD = RP$. Hence, ΔPQR is equilateral.

Now, $\angle ABC = \angle AQC = 120^\circ$. Also, $\angle BCD = \angle BCQ + \angle DCQ = 60^\circ + 60^\circ = 120^\circ$. Similarly, $\angle CDE = \angle DEF = \angle EFA = \angle FAB = 120^\circ$.

<u>Example 2.</u> $ABCD$ is a convex quadrilateral with $AD = BC$. Let E, F be midpoints of CD, AB respectively. Suppose rays AD, FE intersect at H and rays BC, FE intersect at G. Show that

$$\angle AHF = \angle BGF.$$

<u>Solution.</u> One idea is to move BC closer to AD. Let $T(\boldsymbol{CB})$ move A to I.



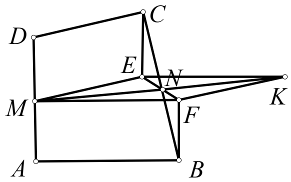
Then $BCAI$ is a parallelogram. Since F is the midpoint of AB, so F is also the midpoint of CI. Applying the midpoint theorem to ΔCDI, we get $EF||DI$. Using this and $CB||AI$, we get $\angle BGF = \angle AID$. From $AI = BC = AD$, we get $\angle AID = \angle ADI$. Since $EF \,||\, DI$, $\angle AHF = \angle ADI = \angle AID = \angle BGF$.

<u>Example 3.</u> Let M and N be the midpoints of sides AD and BC of quadrilateral $ABCD$ respectively. If

$$2MN = AB+CD,$$

then prove that $AB||CD$.

<u>Solution.</u> One idea is to move AB, CD closer to MN. Let $T(\boldsymbol{DC})$ move M to E and $T(\boldsymbol{AB})$ move M to F.



Then we can see $CDME$ and $BAMF$ are parallelograms. Since $\boldsymbol{EC} = ½\boldsymbol{AD} = \boldsymbol{BF}$, $BFCE$ is a parallelogram. Since N is the midpoint of BC, so N is also the midpoint of EF.

Next, let $T(\boldsymbol{ME})$ move F to K. Then $EMFK$ is a parallelogram and

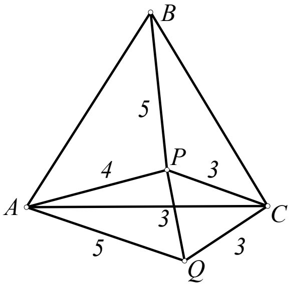
$$MK = 2MN = AB+CD = MF+EM = MF+FK.$$

So F, M, K, N are collinear and $AB||MN$. Similarly, $CD||MN$. Therefore, $AB||CD$.

On a plane, a <u>rotation</u> about a center O by angle α moves every point X to a point Y such that $OX = OY$ and $\angle XOY = \alpha$ (anticlockwise if $\alpha > 0$, clockwise if $\alpha < 0$). We denote this rotation by $R(O,\alpha)$.

Example 4. Inside an equilateral triangle ABC, there is a point P such that PC=3, PA=4 and PB=5. Find the perimeter of ΔABC.

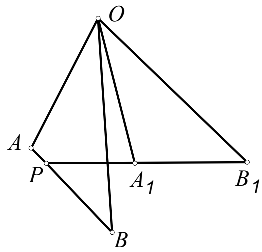
Solution. One idea is to move PC, PA, PB to form a triangle. Let $R(C,60°)$ move ΔCBP to ΔCAQ.



Now $CP=CQ$ and $\angle PCQ = 60°$ imply ΔPCQ is equilateral. As $AQ = BP = 5$, $AP = 4$ and $PQ = PC = 3$, so $\angle APQ = 90°$. Then $\angle APC = \angle APQ + \angle QPC = 90°+60° = 150°$. So the perimeter of ΔABC is

$$3AC = 3\sqrt{3^2+4^2-12\cos 150^\circ}$$
$$= 3\sqrt{25+12\sqrt{3}}.$$

For our next example, we will point out a property of rotation, namely



if $R(O,\alpha)$ moves a line AB to the line A_1B_1 and P is the intersection of the two lines, then these lines intersect at an angle α.

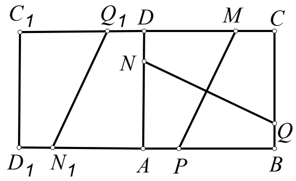
This is because $\angle OAB = \angle OA_1B_1$ implies O,A,P,A_1 are concyclic so that $\angle BPB_1 = \angle AOA_1 = \alpha$.

Example 5. $ABCD$ is a unit square. Points P,Q,M,N are on sides AB, BC, CD, DA respectively such that

$$AP + AN + CQ + CM = 2.$$

Prove that $PM \perp QN$.

Solution. One idea is to move AP, AN together and CQ, CM together. Let $R(A,90°)$ map $B \to D$, $C \to C_1$, $D \to D_1$, $Q \to Q_1$, $N \to N_1$ as shown below.



Then $AN=AN_1$ and $CQ=C_1Q_1$. So

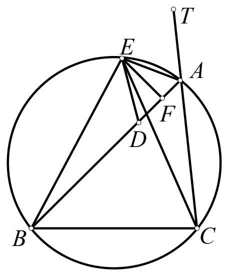
$$PN_1 = AP+AN_1 = AP+AN = 2-(CM+CQ)$$
$$= CC_1-(CM+C_1Q_1) = MQ_1.$$

Hence, PMQ_1N_1 is a parallelogram and $MP || Q_1N_1$. By the property before the example, lines QN and Q_1N_1 intersect at 90°. Therefore, $PM \perp QN$.

Example 6. (*1989 Chinese National Senoir High Math Competition*) In ΔABC, $AB > AC$. An external bisector of $\angle BAC$ intersects the circumcircle of ΔABC at E. Let F be the foot of perpendicular from E to line AB. Prove that

$$2AF = AB - AC.$$

Solution. One idea is to move AC to coincide with a part of AB. To do that, consider $R(E,\angle CEB)$.



Observe that $\angle EBC = \angle EAT = \angle EAB = \angle ECB$ implies $EC=EB$. So $R(E,\angle CEB)$ move C to B. Let $R(E,\angle CEB)$ move A to D. Since $\angle CAB = \angle CEB$, by the property above and $AB > AC$, D is on segment AB.

So $R(E,\angle CEB)$ moves ΔAEC to ΔDEB. Then $\angle DAE = \angle EAT = \angle EDA$ implies ΔAED is isosceles. Since $EF \perp AD$,

$$2AF = AD = AB - BD = AB - AC.$$

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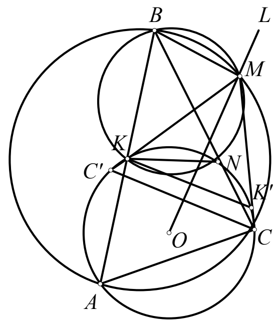
On a plane, a <u>reflection</u> across a line moves every point X to a point Y such that the line is the perpendicular bisector of segment XY. We say Y is the <u>mirror image</u> of X with respect to the line.

Example 7. (*1985 IMO*) A circle with center O passes through vertices A and C of ΔABC and cuts sides AB, BC at K, N respectively. The circumcircles of ΔABC and ΔKBN intersect at B and M. Prove that $\angle OMB = 90°$.

Solution. Let L be the line through O perpendicular to line BM. We are done if we can show M is on L.



Let the reflection across L maps $C \to C'$ and $K \to K'$. Then $CC' \perp L$ and $KK' \perp L$, which imply lines CC', KK', BM are parallel. We have

$$\angle KC'C = \angle KAC = \angle BNK = \angle BMK,$$

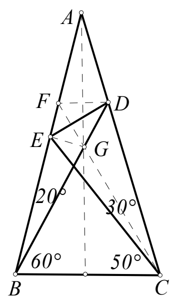
which implies C',K,M collinear. Now

$$\angle C'CK' = \angle CC'K = \angle CAK$$
$$= \angle CAB = 180° - \angle BMC$$
$$= \angle C'CM,$$

which implies C,K',M collinear. Then lines $C'K$ and CK' intersect at M. Since lines $C'K$ and CK' are symmetric with respect to L, so M is on L.

Example 8. Points D and E are on sides AB and AC of ΔABC respectively with $\angle ABD = 20°$, $\angle DBC = 60°$, $\angle ACE = 30°$ and $\angle ECB = 50°$. Find $\angle EDB$.

Solution. Note $\angle ABC = \angle ACB$. Consider the reflection across the perpendicular bisector of side BC. Let the mirror image of D be F. Let BD intersect CF at G. Since $BG = CG$, lines BD, CF intersect at 60° so that ΔBGC and ΔDGF are equilateral. Then $DF=DG$.



We claim $EF = EG$ (which implies $\Delta EFD \cong \Delta EGD$. So $\angle EDB = ½\angle FDG = 30°$). For the claim, we have $\angle EFG = \angle CDG = 40°$ and $\angle FGB = 120°$.

Next $\angle BEC = 50°$. So $BE=BC$. As ΔBGC is equilateral, so $BE = BC = BG$ .This gives $\angle EGB = 80°$. Then

$$\angle EGF = \angle FGB - \angle EGB$$
$$= 40° = \angle EFG,$$

which implies the claim.

(Continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 20, 2008.***

Problem 301. Prove that it is possible to decompose two congruent regular hexagons into a total of six pieces such that they can be rearranged to form an equilateral triangle with no pieces overlapping.

Problem 302. Let $\mathbb{Z}$ denotes the set of all integers. Determine (with proof) all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all x, y in $\mathbb{Z}$, we have $f(x+f(y)) = f(x) - y$.

Problem 303. In base 10, let N be a positive integer with all digits nonzero. Prove that there do not exist two permutations of the digits of N, forming numbers that are different (integral) powers of two.

Problem 304. Let M be a set of 100 distinct lattice points (i.e. coordinates are integers) chosen from the x-y coordinate plane. Prove that there are at most 2025 rectangles whose vertices are in M and whose sides are parallel to the x-axis or the y-axis.

Problem 305. A circle Γ_2 is internally tangent to the circumcircle Γ_1 of ΔPAB at P and side AB at C. Let E, F be the intersection of Γ_2 with sides PA, PB respectively. Let EF intersect PC at D. Lines PD, AD intersect Γ_1 again at G, H respectively. Prove that F, G, H are collinear.

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Solutions

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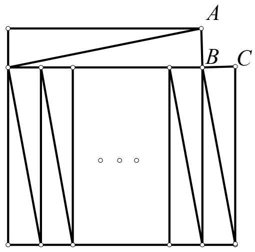
Problem 296. Let $n > 1$ be an integer. From a $n\times n$ square, one 1×1 corner square is removed. Determine (with proof) the least positive integer k such that the remaining areas can be partitioned into k triangles with equal areas.
(*Source 1992 Shanghai Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **O Kin Chit Alex** (GT Ellen Yeung College), **PUN Ying Anna** (HKU Math Year 2), **Simon YAU Chi-Keung** (City University of Hong Kong) and **Fai YUNG**.



The figure above shows the least k is at most $2n$+2. Conversely, suppose the required partition is possible for some k. Then one of the triangles must have a side lying in part of segment AB or in part of segment BC. Then the length of that side is at most 1. Next, the altitude perpendicular to that side is at most $n-1$. Hence, that triangle has an area at most $(n-1)/2$. That is $(n^2-1)/k \le (n-1)/2$. So $k \ge 2n+2$. Therefore, the least k is $2n+2$.

Problem 297. Prove that for every pair of positive integers p and q, there exist an integer-coefficient polynomial $f(x)$ and an open interval with length $1/q$ on the real axis such that for every x in the interval, $|f(x) - p/q| < 1/q^2$.
(*Source:1983 Finnish Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA) and **PUN Ying Anna** (HKU Math Year 2).

If $q = 1$, then take $f(x) = p$ works for any interval of length $1/q$. If $q > 1$, then define the interval $I = \left(\frac{1}{2q}, \frac{3}{2q}\right)$.

Choosing a positive integer m greater than $(\log q)/(\log 2q/3)$, we get $[3/(2q)]^m < 1/q$. Let $a = 1-[1/(2q)]^m$. Then for all x in I, we have $0 < 1 - qx^m < a < 1$.

Choosing a positive integer n greater than $-(\log pq)/(\log a)$, we get $a^n < 1/(pq)$. Let

$$f(x) = \frac{p}{q}[1-(1-qx^m)^n].$$

Now

$$f(x) = \frac{p}{q}[1-(1-qx^m)]\sum_{k=0}^{n-1}(1-qx^m)^k$$

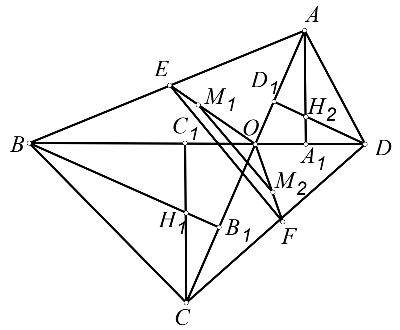
$$= px^m\sum_{k=0}^{n-1}(1-qx^m)^k$$

has integer coefficients. For x in I, we have

$$\left|f(x)-\frac{p}{q}\right| = \frac{p}{q}\left|(1-qx^m)^n\right| < \frac{p}{q}a^n < \frac{1}{q^2}.$$

Problem 298. The diagonals of a convex quadrilateral $ABCD$ intersect at O. Let M_1 and M_2 be the centroids of ΔAOB and ΔCOD respectively. Let H_1 and H_2 be the orthocenters of ΔBOC and ΔDOA respectively. Prove that $M_1M_2\perp H_1H_2$.

Solution. **Jeff CHEN** (Virginia, USA).



Let A_1, C_1 be the feet of the perpendiculars from A, C to line BD respectively. Let B_1, D_1 be the feet of the perpendiculars from B, D to line AC respectively. Let E, F be the midpoints of sides AB, CD respectively. Since

$$OM_1/OE = 2/3 = OM_2/OF,$$

we get $EF \parallel M_1M_2$. Thus, it suffices to show $H_1H_2\perp EF$.

Now the angles AA_1B and BB_1A are right angles. So A, A_1, B, B_1 lie on a circle Γ_1 with E as center. Similarly, C, C_1, D, D_1 lie on a circle Γ_2 with F as center.

Next, since the angles AA_1D and DD_1A are right angles, points A,D,A_1,D_1 are concyclic. By the intersecting chord theorem, $AH_2\cdot H_2A_1=DH_2\cdot H_2D_1$.

This implies H_2 has equal power with respect to Γ_1 and Γ_2. Similarly, H_1 has equal power with respect to Γ_1 and Γ_2. Hence, line H_1H_2 is the radical axis of Γ_1 and Γ_2. Since the radical axis is perpendicular to the line joining the centers of the circles, we get $H_1H_2\perp EF$.

Comments: For those who are not familiar with the concepts of power and radical axis of circles, please see <u>*Math. Excalibur*</u>, *vol.* 4, *no.* 3, *pp.* 2,4.

Commended solvers: **PUN Ying Anna** (HKU Math Year 2) and **Simon YAU Chi-Keung** (City University of Hong Kong).

Problem 299. Determine (with proof) the least positive integer n such that in every way of partitioning $S = \{1,2,\ldots,n\}$ into two subsets, one of the subsets will contain two distinct numbers a and b such that ab is divisible by $a+b$.

Solution. **Jeff CHEN** (Virginia, USA),

PUN Ying Anna (HKU Math Year 2).

Call a pair (a,b) of distinct positive integers a <u>*good*</u> pair if and only if ab is divisible by $a+b$. Here is a list of good pairs with $1 < a < b < 50$:

(3,6), (4,12), (5,20), (6,12), (6,30), (7,42), (8,24), (9,18), (10,15), (10,40), (12,24), (12,36), (14,35), (15,30), (16,48), (18,36), (20,30), (21,28), (21,42), (24,40), (24,48), (30,45), (36,45).

Now we try to put the positive integers from 1 to 39 into one of two sets S_1, S_2 so that no good pair is in the same set. If a positive integer is not in any good pair, then it does not matter which set it is in, say we put it in S_1. Then we get

S_1={1, 2, 3, 5, 8, 10, 12, 13, 14, 18, 19, 21, 22, 23, 30, 31, 32, 33, 34, 36} and S_2={4, 6, 7, 9, 11,15, 17, 20, 24, 25, 26, 27, 28, 29, 35, 37, 38, 39}.

So 1 to 39 do not have the property.

Next, for $n = 40$, we observe that any two consecutive terms of the sequence 6, 30, 15, 10, 40, 24, 12, 6 forms a good pair. So no matter how we divide the numbers 6, 30, 15, 10, 40, 24, 12 into two sets, there will be a good pair in one of them. So, $n = 40$ is the least case.

Problem 300. Prove that in base 10, every odd positive integer has a multiple all of whose digits are odd.

Solution. **Jeff CHEN** (Virginia, USA) and **G.R.A. 20 Problem Solving Group** (Roma, Italy), **PUN Ying Anna** (HKU Math Year 2).

We first show by induction that <u>for every positive integer k, there is a k-digit number n_k whose digits are all odd and n_k is a multiple of 5^k</u>. We can take n_1=5. Suppose this is true for k. We will consider the case $k+1$. If n_k is a multiple of 5^{k+1}, then take n_{k+1} to be $n_k + 5\times10^k$. Otherwise, n_k is of the form $5^k(5i+j)$, where i is a nonnegative integer and j = 1, 2, 3 or 4. Since $\gcd(5,2^k) = 1$, one of the numbers 10^k+n_k, $3\times10^k+n_k$, $7\times10^k+n_k$, $9\times10^k+n_k$ is a multiple of 5^{k+1}. Hence we may take it to be n_{k+1}, which completes the induction.

Now for the problem, let m be an odd number. Let $N(a,b)$ denote the number whose digits are those of a written b times in a row. For example, $N(27,3)$ = 272727.

Observe that m is of the form 5^kM, where k is a nonnegative integer and $\gcd(M,5) = 1$. Let $n_0 = 1$ and for $k > 0$, let n_k be as in the underlined statement above. Consider the numbers $N(n_k,1)$, $N(n_k,2)$, …, $N(n_k, M+1)$. By the pigeonhole principle, two of these numbers, say $N(n_k, i)$ and $N(n_k, j)$ with $1 \le i < j \le M+1$, have the same remainder when dividing by M. Then $N(n_k, j) - N(n_k, i) = N(n_k, j-i) \times 10^{ik}$ is a multiple of M and 5^k.

Finally, since $\gcd(M, 10) = 1$, $N(n_k, j-i)$ is also a multiple of M and 5^k. Therefore, it is a multiple of m and it has only odd digits.

Olympiad Corner

(continued from page 1)

Problem 3. Let n be a positive integer. The rectangle $ABCD$ with side lengths AB=$90n$+1 and BC=$90n$+5 is partitioned into unit squares with sides parallel to the sides of $ABCD$. Let S be the set of all points which are vertices of these unit squares. Prove that the number of lines which pass through at least two points from S is divisible by 4.

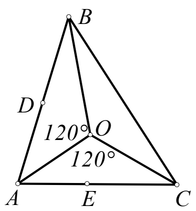
Problem 4. Let c be a positive integer. The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by $a_1=c$ and $a_{n+1}=a_n^2+a_n+c$ for every positive integer n. Find all values of c for which there exist some integers $k \ge 1$ and $m \ge 2$ such that $a_k^2+c^3$ is the m^{th} power of some positive integer.

Geometric Transformations I

(continued from page 2)

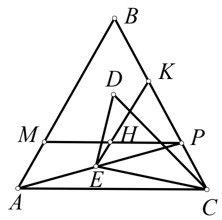
On a plane, a <u>*spiral similarity*</u> with center O, angle α and ratio k moves every point X to a point Y such that $\angle XOY = \alpha$ and $OY/OX = k$, i.e. it is a rotation with a homothety. We denote it by $S(O,\alpha, k)$.

<u>***Example 9.***</u> (*1996 St. Petersburg Math Olympiad*) In ΔABC, $\angle BAC$=60°. A point O is inside the triangle such that $\angle AOB = \angle BOC = \angle COA$. Points D and E are the midpoints of sides AB and AC, respectively. Prove that A, D, O, E are concyclic.



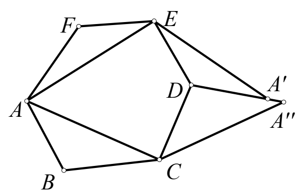
<u>***Solution.***</u> Since $\angle AOB=\angle COA$=120° and $\angle OBA$=60°−$\angle OAB=\angle OAC$, we see $\Delta AOB\sim\Delta COA$. Then the spiral similarity $S(O,120°,OC/OA)$ maps $\Delta AOB\to\Delta COA$ and also $D\to E$. Then $\angle DOE$ = 120° = 180°−$\angle BAC$, which implies A, D, O, E concyclic.

<u>***Example 10.***</u> (*1980 All Soviet Math Olympiad*) ΔABC is equilateral. M is on side AB and P is on side CB such that $MP||AC$. D is the centroid of ΔMBP and E is the midpoint of PA. Find the angles of ΔDEC.



<u>***Solution.***</u> Let H and K be the midpoints of PM and PB respectively. Observe that $S(D,-60°,1/2)$ maps $P\to H$, $B\to K$ and so $PB\to HK$. Now H, K, E are collinear as they are midpoints of PM, PB, PA. Note $BC/BP = BA/BM = KE/KH$, which implies $S(D,-60°,1/2)$ maps $C\to E$. Then $\angle EDC$ = 60° and DE=½DC. So we have $\angle DEC$ = 90° and $\angle DCE$ = 30°.

<u>***Example 11.***</u> (*1998 IMO Proposal by Poland*) Let $ABCDEF$ be a convex hexagon such that $\angle B+\angle D+\angle F$ = 360° and $(AB/BC)(CD/DE)(EF/FA)$=1. Prove $(BC/CA)(AE/EF)(FD/DB)$=1.



<u>***Solution.***</u> Since $\angle B+\angle D+\angle F$ =360°, $S(E, \angle FED, ED/EF)$ maps $\Delta FEA\to\Delta DEA'$ and $S(C,\angle BCD,CD/CB)$ maps $\Delta BCA\to\Delta DCA''$. So $\Delta FEA\sim\Delta DEA'$ and $\Delta BCA\sim\Delta DCA''$. These yield $BC/CA=DC/CA''$, $DE/EF=DA'/FA$ and using the given equation, we get

$$\frac{A''D}{DC}=\frac{AB}{BC}=\frac{DE}{CD}\frac{FA}{EF}=\frac{DA'}{CD},$$

which implies $A'=A''$. Next $\angle AEF$ = $\angle A'ED$ implies $\angle DEF$ = $\angle A'EA$. As $DE/FE=A'E/AE$, so $\Delta DEF\sim\Delta A'EA$ and $AE/FE=AA'/FD$. Similarly, we get $\Delta DCB\sim\Delta A'CA$ and $DC/A'C=DB/A'A$. Therefore,

$$\frac{BC}{CA}\frac{AE}{EF}\frac{FD}{DB}=\frac{DC}{CA''}\frac{AA'}{DB}=1.$$

Mathematical Excalibur

Volume 13, Number 3 *July-October, 2008*

Olympiad Corner

The following are the problems of the 2008 IMO held at Madrid in July.

Problem 1. An acute-angled triangle ABC has orthocenter H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with the centre the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

Problem 2. (a) Prove that

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\ge 1$$

for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

Problem 3. Prove that there exist infinitely many positive integers n such that n^2+1 has a prime divisor which is greater than $2n+\sqrt{2n}$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 31, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Geometric Transformations II

Kin Y. Li

Below the vector from X to Y will be denoted as $\boldsymbol{XY}$. The notation $\measuredangle ABC = \alpha$ means the ray BA after rotated an angle $|\alpha|$ (anticlockwise if $\alpha > 0$, clockwise if $\alpha < 0$) will coincide with the ray BC.

On a plane, a ___translation___ by a vector v (denoted as T(v)) moves every point X to a point Y such that $\boldsymbol{XY}$ = v. On the complex plane $\mathbb{C}$, if the vector v corresponds to the vector from 0 to v, then T(v) has the same effect as the function $f:\mathbb{C}\to\mathbb{C}$ given by $f(w)= w+v$.

A ___homothety___ about a center C and ratio r (denoted as $H(C,r)$) moves every point X to a point Y such that $\boldsymbol{CY} = r\,\boldsymbol{CX}$. If C corresponds to the complex number c in $\mathbb{C}$, then $H(C,r)$ has the same effect as $f(w) = r(w-c)+ c = rw + (1-r)c$.

A ___rotation___ about a center C by angle α (denoted as $R(C,\alpha)$) moves every point X to a point Y such that $CX = CY$ and $\measuredangle XCY = \alpha$. In $\mathbb{C}$, if C corresponds to the complex number c, then $R(C,\alpha)$ has the same effect as $f(w) = e^{i\alpha}(w-c) + c = e^{i\alpha}w + (1- e^{i\alpha})c$.

A ___reflection___ across a line ℓ (denoted as $S(\ell)$) moves every point X to a point Y such that the line ℓ is the perpendicular bisector of segment XY. In $\mathbb{C}$, let $S(\ell)$ send 0 to b. If $b = 0$ and ℓ is the line through 0 and $e^{i\theta/2}$, then $S(\ell)$ has the same effect as $f(w)=e^{i\theta}\overline{w}$. If $b \ne 0$, then let $b = |b|\, e^{i\beta}$, $e^{i\theta} = -e^{2i\beta}$ and L be the vertical line through $|b|/2$. In $\mathbb{C}$, $S(L)$ sends w to $|b|-\overline{w}$. Using that, $S(\ell)$ is

$$f(w)=e^{i\beta}(|b|-\overline{we^{-i\beta}})=e^{i\theta}\overline{w}+b.$$

We have the following useful facts:

Fact 1. If $\ell_1 \parallel \ell_2$, then

$$S(\ell_2)\circ S(\ell_1) = T(2\boldsymbol{A_1A_2}),$$

where A_1 is on ℓ_1 and A_2 is on ℓ_2 such that the length of $\boldsymbol{A_1A_2}$ is the distance d from ℓ_1 to ℓ_2.

(*Reason*: Say ℓ_1, ℓ_2 are vertical lines through $A_1 = 0$, $A_2 = d$. Then $S(\ell_1)$, $S(\ell_2)$ are $f_1(w)=-\overline{w}$ and $f_2(w)=-\overline{w}+2d$. So $S(\ell_2)\circ S(\ell_1)$ is

$$f_2(f_1(w))=-\overline{\left(-\overline{w}\right)}+2d=w+2d,$$

which is $T(2\boldsymbol{A_1A_2})$.)

Fact 2. If $\ell_1 \nparallel \ell_2$, then

$$S(\ell_2)\circ S(\ell_1) = R(O,\alpha),$$

where ℓ_1 intersects ℓ_2 at O and α is twice the angle from ℓ_1 to ℓ_2 in the anticlockwise direction.

(*Reason:* Say O is the origin, ℓ_1 is the x-axis. Then $S(\ell_1)$ and $S(\ell_2)$ are

$$f_1(w)=\overline{w}\ \text{ and }\ f_2(w)=e^{i\alpha}\overline{w},$$

so $S(\ell_2)\circ S(\ell_1)$ is $f_2(f_1(w)) = e^{i\alpha}w$, which is $R(O,\alpha)$.)

Fact 3. If $\alpha+\beta$ is not a multiple of 360°, then

$$R(O_2,\beta)\circ R(O_1,\alpha) = R(O,\alpha+\beta),$$

where $\measuredangle OO_1O_2 = \alpha/2$, $\measuredangle O_1O_2O = \beta/2$. If $\alpha+\beta$ is a multiple of 360°, then

$$R(O_2,\beta)\circ R(O_1,\alpha) = T(\boldsymbol{O_1O_3}),$$

where $R(O_2,\beta)$ sends O_1 to O_3.

(*Reason*: Say O_1 is 0, O_2 is -1. Then $R(O_1,\alpha)$, $R(O_2,\beta)$ are $f_1(w) = e^{i\alpha}w$, $f_2(w) = e^{i\beta}w+(e^{i\beta}-1)$, so $f_2(f_1(w)) = e^{i(\alpha+\beta)}w + (e^{i\beta}-1)$. If $e^{i(\alpha+\beta)}\ne 1$, this is a rotation about $c' = (e^{i\beta}-1)/(1-e^{i(\alpha+\beta)})$ by angle $\alpha+\beta$. We have

$$c'=\frac{\sin(\beta/2)}{\sin((\alpha+\beta)/2)}e^{i(\pi-\alpha/2)},$$

$$c'-1=\frac{\sin(\alpha/2)}{\sin((\alpha+\beta)/2)}e^{i\beta/2}.$$

If $e^{i(\alpha+\beta)} = 1$, this is a translation by $e^{i\beta}-1 = f_2(0)$.)

Fact 4. If O_1, O_2, O_3 are noncollinear, $\alpha_1, \alpha_2, \alpha_3 > 0$, $\alpha_1+\alpha_2+\alpha_3=360°$ and

$$R(O_3,\alpha_3)\circ R(O_2,\alpha_2)\circ R(O_1,\alpha_1) = I,$$

where I is the identity transformation, then $\measuredangle O_3O_1O_2= \alpha_1/2$, $\measuredangle O_1O_2O_3= \alpha_2/2$ and $\measuredangle O_2O_3O_1= \alpha_3/2$.

(This is just the case $\alpha_3=360°-(\alpha_1+\alpha_2)$ of fact 3.)

Fact 5. Let $O_1 \neq O_2$. For $r_1r_2 \neq 1$,

$$H(O_2,r_2) \circ H(O_1,r_1) = H(O,r_1r_2)$$

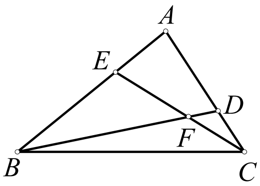
for some O on line O_1O_2. For $r_1r_2 = 1$,

$$H(O_2,r_2) \circ H(O_1,r_1) = T((1-r_2)\boldsymbol{O_1O_2}).$$

(*Reason*: Say O_1 is 0, O_2 is c. Then $H(O_1,r_1)$, $H(O_2,r_2)$ are $f_1(w) = r_1w$, $f_2(w) = r_2w+(1-r_2)c$, so $f_2(f_1(w)) = r_1r_2w+(1-r_2)c$. For $r_1r_2 \neq 1$, this is a homothety about $c' = (1-r_2)c/(1-r_1r_2)$ and ratio r_1r_2. For $r_1r_2 = 1$, this is a translation by $(1-r_2)c$.)

Next we will present some examples.

Example 1. In ΔABC, let E be onside AB such that AE:EB=1:2 and D be on side AC such that AD:DC = 2:1. Let F be the intersection of BD and CE. Determine FD:FB and FE:FC.



Solution. We have $H(E, -1/2)$ sends B to A and $H(C, 1/3)$ sends A to D. Since $(1/3)\times(-1/2) \neq 1$, by fact 5,

$$H(C, 1/3) \circ H(E, -1/2) = H(O, -1/6),$$

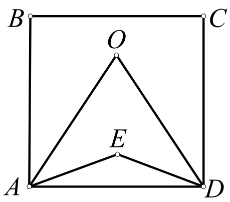
where the center O is on line CE. However, the composition on the left side sends B to D. So O is also on line BD. Hence, O must be F. Then we have FD: FB = OD: OB =1:6.

Similarly, we have

$$H(B, 2/3) \circ H(D, -2) = H(F, -4/3)$$

sends C to E, so FE:FC = 4:3.

Example 2. Let E be inside square $ABCD$ such that $\angle EAD = \angle EDA = 15°$. Show that ΔEBC is equilateral.

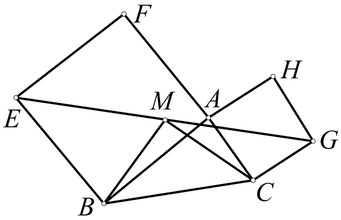


Solution. Let O be inside the square such that ΔADO is equilateral. Then $R(D, 30°)$ sends C to O and $R(A, 30°)$ sends O to B. Since $\measuredangle EDA = 15° = \measuredangle DAE$, by fact 3,

$$R(A, 30°) \circ R(D, 30°) = R(E, 60°),$$

So $R(E, 60°)$ sends C to B. Therefore, ΔEBC is equilateral.

Example 3. Let $ABEF$ and $ACGH$ be squares outside ΔABC. Let M be the midpoint of EG. Show that $MB = MC$ and $MB \perp MC$.

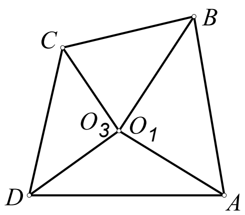


Solution. Since $GC = AC$ and $\measuredangle GCA = 90°$, so $R(C,90°)$ sends G to A. Also, $R(B, 90°)$ sends A to E. Then $R(B, 90°) \circ R(C,90°)$ sends G to E. By fact 3,

$$R(B, 90°) \circ R(C, 90°) = R(O, 180°),$$

where O satisfies $\measuredangle OCB = 45°$ and $\measuredangle CBO = 45°$. Since the composition on the left side sends G to E, O must be M. Now $\measuredangle BOC = 90°$. So $MB \perp MC$.

Example 4. On the edges of a convex quadrilateral $ABCD$, construct the isosceles right triangles ABO_1, BCO_2, CDO_3, DAO_4 with right angles at O_1, O_2, O_3, O_4 overlapping with the interior of the quadrilateral. Prove that if $O_1 = O_3$, then $O_2 = O_4$.



Solution. Now $R(O_1, 90°)$ sends A to B, $R(O_2, 90°)$ sends B to C, $R(O_3, 90°)$ sends C to D and $R(O_4, 90°)$ sends D to A. By fact 3,

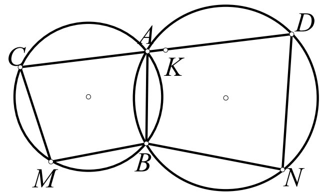
$$R(O_2, 90°) \circ R(O_1, 90°) = R(O, 180°),$$

where O satisfies $\angle OO_1O_2 = 45°$ and $\angle O_1O_2O = 45°$ (so $\angle O_2OO_1 = 90°$). Now the composition on the left side sends A to C, which implies O must be the midpoint of AC. Similarly, we have

$$R(O_4, 90°) \circ R(O_3, 90°) = R(O, 180°).$$

By fact 3, $\measuredangle O_4OO_3 = 90°$ and $\measuredangle OO_3O_4 = 45° = \measuredangle O_3O_4O$. Hence, $R(O, 90°)$ sends O_4O_2 to O_3O_1. Therefore, if $O_1 = O_3$, then $O_2 = O_4$.

Example 4. (*1999-2000 Iranian Math Olympiad, Round 2*) Two circles intersect in points A and B. A line ℓ that contains the point A intersects again the circles in the points C and D, respectively. Let M, N be the midpoints of the arcs BC and BD, which do not contain the point A, and let K be the midpoint of the segment CD. Show that $\angle MKN = 90°$.



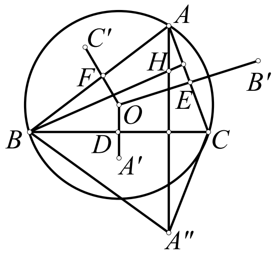
Solution. Since $\angle CAB + \angle BAD = 180°$, it follows that $\angle BMC + \angle DNB = 180°$.

Now $R(M, \measuredangle BMC)$ sends B to C, $R(K, 180°)$ sends C to D and $R(N, \measuredangle DNB)$ sends D to B. However, by fact 3,

$$R(N, \measuredangle DNB) \circ R(K, 180°) \circ R(M, \measuredangle BMC)$$

is a translation and since it sends B to B, it must be the identity transformation I. By fact 4, $\measuredangle MKN = 90°$.

Example 6. Let H be the orthocenter of ΔABC and lie inside it. Let A', B', C' be the circumcenters of ΔBHC, ΔCHA, ΔAHB respectively. Show that AA', BB', CC' are concurrent and identify the point of concurrency.



Solution. For ΔABC, let O be its circumcenter and G be its centroid. Let the reflection across line BC sends A to A''. Then $\angle BAC = \angle BA''C$. Now

$$\begin{aligned}&\angle BHC\\ =\ &\angle ABH + \angle BAC + \angle ACH\\ =\ &(90° - \angle BAC) + \angle BAC + (90° - \angle BAC)\\ =\ &180° - \angle BA''C.\end{aligned}$$

So A'' is on the circumcircle of ΔHBC.

Now the reflection across line BC sends O to A', the reflection across line CA sends O to B' and the reflection across line AB sends O to C'. Let D, E, F be the midpoints of sides BC, CA, AB respectively. Then $H(G, -1/2)$ sends ΔABC to ΔDEF and $H(O, 2)$ sends ΔDEF to $\Delta A'B'C'$. Since $(-1/2)\times 2 \neq 1$, by fact 5,

$$H(O, 2) \circ H(G, -1/2) = H(X, -1)$$

for some point X. Since the composition on the left side sends ΔABC to $\Delta A'B'C'$, segments AA', BB', CC' concur at X and in fact X is their common midpoint.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 31, 2008.***

Problem 306. Prove that for every integer $n \geq 48$, every cube can be decomposed into n smaller cubes, where every pair of these small cubes does not have any common interior point and has possibly different sidelengths.

Problem 307. Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

be a polynomial with real coefficients such that $a_0 \neq 0$ and for all real x,

$$f(x)f(2x^2) = f(2x^3+x).$$

Prove that $f(x)$ has no real root.

Problem 308. Determine (with proof) the greatest positive integer $n > 1$ such that the system of equations

$$(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = \cdots = (x+n)^2 + y_n^2.$$

has an integral solution $(x, y_1, y_2, \cdots, y_n)$.

Problem 309. In acute triangle ABC, $AB > AC$. Let H be the foot of the perpendicular from A to BC and M be the midpoint of AH. Let D be the point where the incircle of ΔABC is tangent to side BC. Let line DM intersect the incircle again at N. Prove that $\angle BND = \angle CND$.

Problem 310. (*Due to Pham Van Thuan*) Prove that if p, q are positive real numbers such that $p + q = 2$, then

$$3p^qq^p + p^pq^q \leq 4.$$

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Solutions

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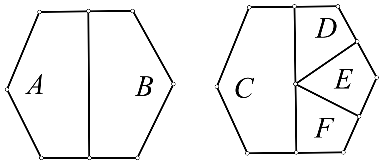
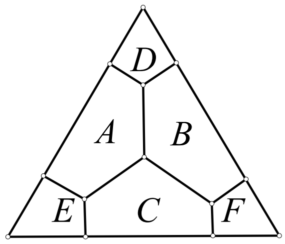
Problem 301. Prove that it is possible to decompose two congruent regular hexagons into a total of six pieces such that they can be rearranged to form an equilateral triangle with no pieces overlapping.

Solution. **G.R.A. 20 Problem Solving Group** (Roma, Italy).





Commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil), **Glenier L. BELLO- BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Victor FONG** (CUHK Math Year 2), **KONG Catherine Wing Yan** (G.T. Ellen Yeung College, Grade 9), **O Kin Chit Alex** (G.T. Ellen Yeung College) and **PUN Ying Anna** (HKU Math Year 3).

Problem 302. Let $\mathbb{Z}$ denotes the set of all integers. Determine (with proof) all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all x, y in $\mathbb{Z}$, we have $f(x+f(y)) = f(x) - y$.
(*Source: 2004 Spanish Math Olympiad*)

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Victor FONG** (CUHK Math Year 2), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **Ozgur KIRCAK** (Jahja Kemal College, Teacher, Skopje, Macedonia), **NGUYEN Tho Tung** (High School for Gifted Education, Ha Noi University of Education), **PUN Ying Anna** (HKU Math Year 3), **Salem MALIKIĆ** (Sarajevo College, Sarajevo, Bosnia and Herzegovina) and **Fai YUNG**.

Assume there is a function f satisfying

$$f(x+f(y)) = f(x) - y. \quad (*)$$

If $f(a) = f(b)$, then

$$f(x)-a = f(x+f(a)) = f(x+f(b)) = f(x)-b,$$

which implies $a = b$, i.e. f is injective. Taking $y = 0$ in (*), $f(x+f(0)) = f(x)$. By injectivity, we see $f(0) = 0$. Taking x=0 in (*), we get

$$f(f(y)) = -y. \quad (**)$$

Applying f to both sides of (*) and using (**), we have

$$f(f(x) - y) = f(f(x+f(y)) = -x - f(y).$$

Taking $x = 0$ in this equation, we get

$$f(-y) = -f(y). \quad (***)$$

Using (**), (*) and (***), we get

$$f(x+y) = f(x+f(f(-y)) = f(x) - f(-y) = f(x) + f(y).$$

Thus, f satisfies the Cauchy equation. By mathematical induction and (***), $f(n) = nf(1)$ for every integer n. Taking $n = f(1)$ in the last equation and $y = 1$ into (**), we get $f(1)^2 = -1$. This yields a contradiction.

Problem 303. In base 10, let N be a positive integer with all digits nonzero. Prove that there do not exist two permutations of the digits of N, forming numbers that are different (integral) powers of two.
(*Source: 2004 Spanish Math Olympiad*)

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Victor FONG** (CUHK Math Year 2), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **NGUYEN Tho Tung** (High School for Gifted Education, Ha Noi University of Education) and **PUN Ying Anna** (HKU Math Year 3).

Assume there exist two permutations of the digits of N, forming the numbers 2^k and 2^m for some positive integers k and m with $k > m$. Then $2^k < 10\times 2^m$. So $k \leq m+3$.

Since every number is congruent to its sum of digits (mod 9), we get $2^k \equiv 2^m$ (mod 9). Since 2^m and 9 are relatively prime, we get $2^{k-m} \equiv 1$ (mod 9). However, $k - m = 1$, 2 or 3, which contradicts $2^{k-m} \equiv 1$ (mod 9).

Problem 304. Let M be a set of 100 distinct lattice points (i.e. coordinates are integers) chosen from the x-y coordinate plane. Prove that there are at most 2025 rectangles whose vertices are in M and whose sides are parallel to the x-axis or the y-axis.
(*Source: 2003 Chinese IMO Team Training Test*)

Solution 1. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain) and **PUN Ying Anna** (HKU Math Year 3).

Let O be a point in M. We say a rectangle is good if all its sides are parallel to the x or y-axis and all its vertices are in M, one of which is O. We claim there are at most 81 good rectangles. (Once the claim is proved, we see there can only be at most $(81\times 100)/4=2025$ desired rectangles.

The division by 4 is due to such rectangle has 4 vertices, hence counted 4 times).

For the proof of the claim, we may assume O is the origin of the plane. Suppose the x-axis contains m points in M other than O and the y-axis contains n points in M other than O. For a point P in M not on either axis, it can only be a vertex of at most one good rectangle. There are at most $99-m-n$ such point P and every good rectangle has such a vertex.

If $m+n \geq 18$, then there are at most $99 - m - n \leq 81$ good rectangles. Otherwise, $m+n \leq 17$. Now every good rectangle has a vertex on the x-axis and a vertex on the y-axis other than O. So there are at most $mn \leq (m+n)^2/4 < 81$ rectangles by the *AM-GM* inequality. The claim follows.

Solution 2. **G.R.A. 20 Problem Solving Group** (Roma, Italy).

Let $f(x) = x(x-1)/2$. We will prove that if there are N lattice points, there are at most $[f(N^{1/2})]^2$ such rectangles. For $N=100$, we have $[f(10)]^2 = 45^2 = 2025$ (this bound is attained when the 100 points form a 10×10 square).

Suppose the N points are distributed on m lines parallel to an axis. Say the number of points in the m lines are $r_1, r_2, \ldots, r_m$, arranged in increasing order. Now the two lines with r_i and r_j points can form no more than $f(\min\{r_i,r_j\})$ rectangles. Hence, the number of rectangles is at most

$$\sum_{1\le i<j\le m} f(\min\{r_i,r_j\}) = \sum_{i=1}^{m-1}(m-i)f(r_i)$$
$$\le \sum_{i=1}^{m-1}(m-i)f\left(\frac{N}{m}\right) = f(m)f\left(\frac{N}{m}\right)$$
$$\le \left(f(\sqrt{N})\right)^2.$$

The second inequality follows by expansion and usage of the *AM-GM* inequality. The first one can be proved by expanding and simplifying it to

$$2m\sum_{i=1}^{m-1}(m-i)r_i(r_i-1) \le (m-1)\sum_{i=1}^{m} r_i \sum_{i=1}^{m}(r_i-1). \quad (*)$$

We will prove this by induction on m. For $m=2$, $4r_1(r_1-1) \le (r_1+r_2)(r_1-1+r_2-1)$ follows from $1 \le r_1 \le r_2$. For the inductive step, we suppose (*) is true. To do the $(m+1)$-st case of (*), observe that $r_i \le r_{m+1}$ implies

$$m\sum_{i=1}^{m} r_i(r_i-1) \le m(r_{m+1}-1)\sum_{i=1}^{m} r_i,$$

$$m\sum_{i=1}^{m} r_i(r_i-1) \le m r_{m+1}\sum_{i=1}^{m}(r_i-1),$$

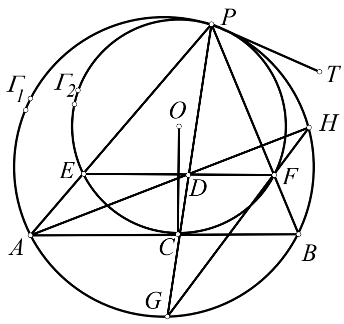
$$2\sum_{i=1}^{m}(m+1-i)r_i(r_i-1)$$
$$\le m r_{m+1}(r_{m+1}-1) + \sum_{i=1}^{m} r_i \sum_{i=1}^{m}(r_i-1).$$

Let $L(m)$ and $R(m)$ denote the left and right sides of (*) respectively. Adding the last three inequalities, it turns out we get $L(m+1) - L(m) \le R(m+1) - R(m)$. Now (*) holds, so $L(m) \le R(m)$. Adding these, we get $L(m+1) \le R(m+1)$.

Commended solvers: **Victor FONG** (CUHK Math Year 2), **O Kin Chit Alex** (G.T. Ellen Yeung College) and **Raúl A. SIMON** (Santiago, Chile).

Problem 305. A circle Γ_2 is internally tangent to the circumcircle Γ_1 of ΔPAB at P and side AB at C. Let E, F be the intersection of Γ_2 with sides PA, PB respectively. Let EF intersect PC at D. Lines PD, AD intersect Γ_1 again at G, H respectively. Prove that F, G, H are collinear.

Solution. **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **NGUYEN Tho Tung** (High School for Gifted Education, Ha Noi University of Education) and **PUN Ying Anna** (HKU Math Year 3).



Let PT be the external tangent to both circles at P. We have

$$\angle PAB = \angle BPT = \angle PEF,$$

which implies $EF||AB$. Let O be the center of Γ_2. Since $OC \perp AB$ (because AB is tangent to Γ_2 at C), we deduce that $OC\perp EF$ and therefore OC is the perpendicular bisector of EF. Hence C is the midpoint of arc ECF. Then PC bisects $\angle EPF$. On the other hand,

$$\angle HDF = \angle HAB = \angle HPB = \angle HPF,$$

which implies H, P, D, F are concyclic. Therefore,

$$\angle DHF = \angle DPF = \angle EPD$$
$$= \angle APG = \angle AHG = \angle DHG,$$

which implies F, G, H are collinear.

Remarks. A few solvers got $EF||AB$ by observing there is a homothety with center P sending Γ_2 to Γ_1 so that E goes to A and F goes to B.

Commended solvers: **Victor FONG** (CUHK Math Year 2) and **Salem MALIKIĆ** (Sarajevo College, Sarajevo, Bosnia and Herzegovina).

Olympiad Corner

(continued from page 1)

Problem 4. Find all functions f: $(0,\infty) \to (0,\infty)$ (so, f is a function from the positive real numbers to the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2)+f(z^2)} = \frac{w^2+x^2}{y^2+z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 5. Let n and k be positive integers with $k \ge n$ and $k-n$ an even number. Let $2n$ lamps labeled 1, 2, …, $2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamp 1 through n are all on, and lamps n+1 through $2n$ are all off.

Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps n+1 through $2n$ are all off, but where none of the lamps n+1 and $2n$ is ever switched on.

Determine the ratio N/M.

Problem 6. Let $ABCD$ be a convex quadrilateral with $|BA| \ne |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

Mathematical Excalibur

Volume 13, Number 4 *November-December, 2008*

Olympiad Corner

The following were the problems of the Hong Kong Team Selection Test 2, which was held on November 8, 2008 for the 2009 IMO.

Problem 1. Let $f{:}Z\to Z$ (Z is the set of all integers) be such that $f(1) = 1$, $f(2) = 20$, $f(-4) = -4$ and

$$f(x+y) = f(x) + f(y) + axy(x+y) + bxy + c(x+y) + 4$$

for all $x,y \in Z$, where a, b and c are certain constants.

(a) Find a formula for $f(x)$, where x is any integer.

(b) If $f(x) \geq mx^2 + (5m+1)x + 4m$ for all non-negative integers x, find the greatest possible value of m.

Problem 2. Define a *k-clique* to be a set of k people such that every pair of them know each other (knowing is mutual). At a certain party, there are two or more 3-cliques, but no 5-clique. Every pair of 3-cliques has at least one person in common. Prove that there exist at least one, and not more than two persons at the party, whose departure (or simultaneous departure) leaves no 3-clique remaining.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 10, 2009***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Double Counting

Law Ka Ho, Leung Tat Wing and Li Kin Yin

There are often different ways to count a quantity. By counting it in two ways (i.e. double counting), we thus obtain the same quantity in different forms. This often yields interesting equalities and inequalities. We begin with some simple examples.

Below we will use the notation $C_r^n = n!/(r!(n-r)!)$.

Example 1. (*IMO HK Prelim 2003*) Fifteen students join a summer course. Every day, three students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?

Solution. Let the answer be k. We count the total number of pairs of students were on duty together in the k days. Since every pair of students was on duty together exactly once, this is equal to $C_2^{15} \times 1 = 105$. On the other hand, since 3 students were on duty per day, this is also equal to $C_2^3 \times k = 3k$. Hence $3k = 105$ and so $k = 35$.

Example 2. (*IMO 1987*) Let $p_n(k)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

(*Remark*: A *permutation* f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a *fixed point* of the permutation f if $f(i) = i$.)

Solution. Note that the left hand side of the equality is the total number of fixed points in all permutations of $\{1,2,\ldots, n\}$. To show that this number is equal to $n!$, note that there are $(n-1)!$ permutations of $\{1, 2, \ldots, n\}$ fixing 1, $(n-1)!$ permutations fixing 2, and so on, and $(n-1)!$ permutations fixing n. It follows that the total number of fixed points in all permutations is equal to $n \cdot (n-1)! = n!$.

The simplest combinatorial identity is perhaps $C_r^n = C_{n-r}^n$. While this can be verified algebraically, we can give a proof in a more combinatorial flavour: to choose r objects out of n, it is equivalent to choosing $n-r$ objects out of n to be discarded. There are C_r^n ways to do the former and C_{n-r}^n ways to do the latter. So the two quantities must be equal.

Example 3. Interpret the following equalities from a combinatorial point of view:

(a) $C_k^n = C_{k-1}^{n-1} + C_k^{n-1}$

(b) $C_1^n + 2C_2^n + \cdots + nC_n^n = n \cdot 2^{n-1}$

Solution. (a) On one hand, the number of ways to choose k objects out of n objects is C_k^n. On the other hand, we may count by including the first object or not. If we include the first object, we need to choose $k-1$ objects from the remaining $n-1$ objects and there are C_{k-1}^{n-1} ways to do so.

If we do not include the first object, we need to choose k objects from the remaining $n-1$ objects and there are C_k^{n-1} ways to do so. Hence

$$C_k^n = C_{k-1}^{n-1} + C_k^{n-1}.$$

(b) Suppose that from a set of n people, we want to form a committee with a chairman of the committee. On one hand, there are n ways to choose a chairman, and for each of the remaining $n-1$ persons we may or may not include him in the committee. Hence there are $n \cdot 2^{n-1}$ ways to finish the task.

On the other hand, we may choose k people to form a committee ($1 \leq k \leq n$), which can be done in C_k^n ways, and for each of these ways there are k ways to select the chairman. Hence the number of ways to finish the task is also equal to

$$C_1^n + 2C_2^n + \cdots + nC_n^n.$$

Example 4. (*IMO 1989*) Let n and k be positive integers and let S be a set of n points in the plane such that:

(i) no three points of S are collinear, and

(ii) for every point P of S, there are at least k points of S equidistant from P.

Prove that $k < \frac{1}{2} + \sqrt{2n}.$

Solution. Solving for n, the desired inequality is equivalent to $n > k(k-1)/2 + 1/8$. Since n and k are positive integers, this is equivalent to $n - 1 \ge C_2^k$. Now we join any two vertices of S by an *edge* and count the number of edges in two ways.

On one hand, we have C_2^n edges. On the other hand, from any point of S there are at least k points equidistant from it. Hence if we draw a circle with the point as centre and with the distance as radius then there are at least C_2^k chords as edges. The total number of such chords, counted with multiplicities, is at least nC_2^k. Any two circles can have at most one common chord and hence there could be a maximum C_2^n chords (for every possible pairs of circles) counted twice. Therefore,

$$nC_2^k - C_2^n \le C_2^n,$$

which simplifies to $n - 1 \ge C_2^k$. (Note that collinearity was not needed.)

Example 5. (*IMO 1998*) In a competition, there are m contestants and n judges, where $n \ge 3$ is an odd integer. Each judge rates each contestant as either “pass” or “fail”. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{m} \ge \frac{n-1}{2n}.$$

Solution. We begin by considering pairs of judges who agree on certain contestants. We study this from two perspectives.

For contestant i, $1 \le i \le m$, suppose there are x_i judges who pass him, and y_i judges who fail him. On one hand, the number of pairs of judges who agree on him is

$$C_2^{x_i} + C_2^{y_i} = \frac{x_i^2 - x_i + y_i^2 - y_i}{2}$$

$$\ge \frac{(x_i + y_i)^2/2}{2} - \frac{x_i + y_i}{2}$$

$$= \frac{1}{4}n^2 - \frac{n}{2} = \frac{1}{4}\left[(n-1)^2 - 1\right].$$

Since n is odd and $C_2^{x_i} + C_2^{y_i}$ is an integer, it is at least $(n-1)^2/4$.

On the other hand, there are n judges and each pair of judges agree on at most k contestants. Hence the number of pairs of judges who agree on a certain contestant is at most kC_2^n. Thus,

$$kC_2^n \ge \sum_{i=1}^{m}(C_2^{x_i} + C_2^{y_i}) \ge \frac{m(n-1)^2}{4},$$

which can be simplified to obtain the desired result.

Some combinatorial problems in mathematical competitions can be solved by double counting certain ordered triples. The following are two such examples.

Example 6. (*CHKMO 2007*) In a school there are 2007 male and 2007 female students. Each student joins not more than 100 clubs in the school. It is known that any two students of opposite genders have joined at least one common club. Show that there is a club with at least 11 male and 11 female members.

Solution. Assume on the contrary that every club either has at most 10 male members or at most 10 female members. We shall get a contradiction via double counting certain ordered triples.

Let S be the number of ordered triples of the form (m, f, c), where m denotes a male student, f denotes a female student and c denotes a club. On one hand, since any two students of opposite genders have joined at least one common club, we have

$$S \ge 2007^2 = 4028049.$$

On the other hand, we can consider two types of clubs: let X be the set of clubs with at most 10 male members, and Y be the set of clubs with at least 11 male members (and hence at most 10 female members). Note that there are at most 10×2007×100=2007000 triples (m, f, c) with $c \in X$, because there are 2007 choices for f, then at most 100 choices for c (each student joins at most 100 clubs), and then at most 10 choices for m (each club $c \in X$ has at most 10 male members). In exactly the same way, we can show that there are at most 2007000 triples (m, f, c) with $c \in Y$. This gives

$$S \le 2007000 + 2007000 = 4014000,$$

a contradiction.

Example 7. (*2004 IMO Shortlisted Problem*) There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose the following conditions hold:

(i) Each pair of students is in exactly one club.

(ii) For each student and each society, the student is in exactly one club of the society.

(iii) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

Solution. An ordered triple (a, C, S) will be called <u>*acceptable*</u> if a is a student, C is a club and S is a society such that $a \in C$ and $C \in S$. We will count the number of acceptable ordered triples in two ways.

On one hand, for every student a and society S, by (ii), there is a unique club C such that (a, C, S) is acceptable. Hence, there are $10001k$ acceptable ordered triples.

On the other hand, for every club C, let the number of members in C be denoted by $|C|$. By (iii), C is in exactly $(|C|-1)/2$ societies. So there are $|C|(|C|-1)/2$ acceptable ordered triples with C as the second coordinates. Let Γ be the set of all clubs. Hence, there are

$$\sum_{C\in\Gamma} \frac{|C|(|C|-2)}{2}$$

acceptable ordered triples. By (i), this is equal to the number of pairs of students, which is 10001×5000. Therefore,

$$10001k = \sum_{C\in\Gamma} \frac{|C|(|C|-2)}{2}$$

$$= 10001 \times 5000,$$

which implies $k = 5000$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***January 10, 2009.***

Problem 311. Let $S = \{1,2,\ldots,2008\}$. Prove that there exists a function $f : S \to \{\text{red, white, blue, green}\}$ such that there does not exist a 10-term arithmetic progression $a_1,a_2,\ldots,a_{10}$ in S satisfying $f(a_1) = f(a_2) = \cdots = f(a_{10})$.

Problem 312. Let $x, y, z > 1$. Prove that

$$\frac{x^4}{(y-1)^2}+\frac{y^4}{(z-1)^2}+\frac{z^4}{(x-1)^2}\ge 48.$$

Problem 313. In ΔABC, $AB < AC$ and O is its circumcenter. Let the tangent at A to the circumcircle cut line BC at D. Let the perpendicular lines to line BC at B and C cut the perpendicular bisectors of sides AB and AC at E and F respectively. Prove that D, E, F are collinear.

Problem 314. Determine all positive integers x, y, z satisfying $x^3 - y^3 = z^2$, where y is a prime, z is not divisible by 3 and z is not divisible by y.

Problem 315. Each face of 8 unit cubes is painted white or black. Let n be the total number of black faces. Determine the values of n such that in every way of coloring n faces of the 8 unit cubes black, there always exists a way of stacking the 8 unit cubes into a 2×2×2 cube C so the numbers of black squares and white squares on the surface of C are the same.

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Solutions

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Problem 306. Prove that for every integer $n \ge 48$, every cube can be decomposed into n smaller cubes, where every pair of these small cubes does not have any common interior point and has possibly different sidelengths.

Solution. **G.R.A. 20 Problem Solving Group** (Roma, Italy) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

For such an integer n, we will say cubes are <u>*n-decomposable*</u>. Let <u>*r-cube*</u> mean a cube with sidelength r. If a r-cube C is n-decomposable, then we can first decompose C into 8 $r/2$-cubes and then decompose one of these $r/2$-cubes into n cubes to get a total of $n+7$ cubes so that C is $(n+7)$-decomposable.

Let C be a 1-cube. All we need to show is C is n-decomposable for $48 \le n \le 54$.

For n=48, decompose C to 27 1/3-cubes and then decompose 3 of these, each into 8 1/6-cubes.

For n=49, cut C by two planes parallel to the bottom at height 1/2 and 1/6 from the bottom, which can produce 4 1/2-cubes at the top layer, 9 1/3-cubes in the middle layer and 36 1/6-cubes at the bottom layer.

For n=50, decompose C to 8 1/2-cubes and then decompose 6 of these, each into 8 1/4-cubes.

For n=51, decompose C into 8 1/2-cubes, then take 3 of these 1/2-cubes on the top half to form a L-shaped prism and cut out 5 1/3-cubes and 41 1/6-cubes.

For n=52, decompose C into 1 3/4-cube and 37 1/4-cubes, then decompose 2 1/4-cubes, each into 8 1/8-cubes.

For n=53, decompose C to 27 1/3-cubes and then decompose 1 of these into 27 1/9-cubes.

For n=54, decompose C into 8 1/2-cubes, then take 2 of the adjacent 1/2-cubes, which form a 1×1/2×1/2 box, from which we can cut 2 3/8-cubes, 4 1/4-cubes and 42 1/8-cubes.

Comments: Interested readers may find more information on this problem by visiting mathworld.wolfram.com and by searching for <u>*Cube Dissection*</u> .

Problem 307. Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

be a polynomial with real coefficients such that $a_0 \ne 0$ and for all real x,

$$f(x)f(2x^2) = f(2x^3+x).$$

Prove that $f(x)$ has no real root.

Solution. **José Luis DÍAZ-BARRERO** (Universitat Politècnica de Catalunya, Barcelona, Spain), **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **Ozgur KIRCAK and Bojan JOVESKI** (Jahja Kemal College, Skopje, Macedonia)**, LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery)**, NG Ngai Fung** (STFA Leung Kau Kui College, Form 6), **O Kin Chit Alex** (G.T. Ellen Yeung College) and **Fai YUNG**.

For such polynomial $f(x)$, let k be largest such that $a_k\ne 0$. Then

$$f(x)f(2x^2) = a_0^2 2^n x^{3n} + \cdots + a_k^2 2^{n-k} x^{3(n-k)},$$

$$f(2x^3+x) = a_0 2^n x^{3n} + \cdots + a_k x^{n-k},$$

where the terms are ordered by decreasing degrees. This can happen only if $n - k = 0$. So $f(0) = a_n \ne 0$. Assume $f(x)$ has a real root $x_0 \ne 0$. The equation $f(x)f(2x^2) = f(2x^3+x)$ implies that if x_n is a real root, then $x_{n+1} = 2x_n^3 + x_n$ is also a real root. Since this sequence is strictly monotone, this implies $f(x)$ has infinitely many real roots, which is a contradiction.

Commended solvers: **Simon YAU Chi Keung** (City U).

Problem 308. Determine (with proof) the greatest positive integer $n > 1$ such that the system of equations

$$(x+1)^2+y_1^2=(x+2)^2+y_2^2=\cdots=(x+n)^2+y_n^2.$$

has an integral solution $(x, y_1, y_2, \cdots, y_n)$.

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **Ozgur KIRCAK and Bojan JOVESKI** (Jahja Kemal College, Skopje, Macedonia) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

We will show the greatest such n is 3. For $n = 3$, $(x, y_1, y_2, y_3) = (-2, 0, 1, 0)$ is a solution. For $n \ge 4$, assume the system has an integral solution. Since $x+1$, $x+2$, …, $x+n$ are of alternate parity, so $y_1, y_2, \ldots, y_n$ are also of alternate parity. Since $n \ge 4$, y_k is even for $k = 2$ or 3. Consider

$$(x+k-1)^2+y_{k-1}^2=(x+k)^2+y_k^2=(x+k+1)^2+y_{k+1}^2.$$

The double of the middle expression equals the sum of the left and right expressions. Eliminating common terms in that equation, we get

$$2y_k^2 = y_{k-1}^2 + y_{k+1}^2 + 2. \qquad (*)$$

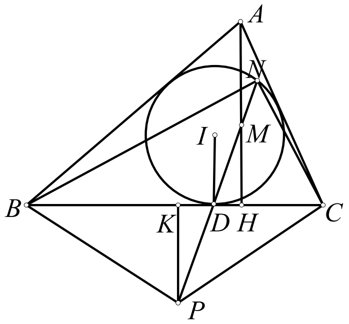
Now y_{k-1} and y_{k+1} are odd. Then the left side of (*) is 0 (mod 8), but the right side is 4 (mod 8), a contradiction.

Commended solvers: **O Kin Chit Alex** (G.T. (Ellen Yeung) College), **Raúl A. SIMON** (Santiago, Chile) and **Simon**

YAU Chi Keung (City U).

Problem 309. In acute triangle ABC, $AB > AC$. Let H be the foot of the perpendicular from A to BC and M be the midpoint of AH. Let D be the point where the incircle of ΔABC is tangent to side BC. Let line DM intersect the incircle again at N. Prove that $\angle BND = \angle CND$.

Solution.



Let I be the center of the incircle. Let the perpendicular bisector of segment BC cut BC at K and cut line DM at P. To get the conclusion, it is enough to show $DN{\cdot}DP=DB{\cdot}DC$ (which implies B,P,C,N are concyclic and since $PB = PC$, that will imply $\angle BND = \angle CND$).

Let sides $BC=a$, $CA=b$ and $AB=c$. Let $s = (a+b+c)/2$, then $DB = s-b$ and $DC = s-c$. Let r be the radius of the incircle and $[ABC]$ be the area of triangle ABC. Let $\alpha = \angle CDN$ and $AH = h_a$. Then $[ABC]$ equals

$$ah_a/2 = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

Now

$$DK = DB - KB = \frac{a+c-b}{2} - \frac{a}{2} = \frac{c-b}{2},$$

$$DH = DC - HC = \frac{a+b-c}{2} - b\cos\angle ACB$$

$$= \frac{a+b-c}{2} - \frac{a^2+b^2-c^2}{2a}$$

$$= \frac{(c-b)(b+c-a)}{2a} = \frac{(c-b)(s-a)}{a}.$$

Moreover, $DN = 2r\sin\alpha$, $DP = DK/(\cos\alpha) = (c-b)/(2\cos\alpha)$. So

$$DN\cdot DP = r(c-b)\tan\alpha = r(c-b)\frac{MH}{DH}$$

$$= r(c-b)\frac{h_a/2}{(c-b)(s-a)/a}$$

$$= r\frac{ah_a/2}{s-a} = \frac{rsrs}{s(s-a)} = \frac{[ABC]^2}{s(s-a)}$$

$$= (s-b)(s-c) = DB\cdot DC.$$

Problem 310. (*Due to Pham Van Thuan*) Prove that if p, q are positive real numbers such that $p + q = 2$, then

$$3p^q q^p + p^p q^q \le 4.$$

***Solution* 1. Proposer's Solution.**

As $p, q > 0$ and $p + q = 2$, we may assume $2 > p \ge 1 \ge q > 0$. Applying Bernoulli's inequality, which asserts that if $x > -1$ and $r \in [0,1]$, then $1+rx \ge (1+x)^r$, we have

$$p^p = pp^{p-1} \ge p(1+(p-1)^2) = p(p^2-2p+2),$$
$$q^q \le 1+q(q-1) = 1+(2-p)(1-p) = p^2-3p+3,$$
$$p^q \le 1+q(p-1) = 1+(2-p)(p-1) = -p^2+3p-1,$$
$$q^p = qq^{p-1} \le q(1+(p-1)(q-1)) = p(2-p)^2.$$

Then

$$\begin{aligned}&3p^q q^p + p^p q^q - 4\\ &\le 3(-p^2+3p-1)p(2-p)^2\\ &\quad + p(p^2-2p+2)(p^2-3p+3) - 4\\ &= -2p^5+16p^4-40p^3+36p^2-6p-4\\ &= -2(p-1)^2(p-2)((p-2)^2-5) \le 0.\end{aligned}$$

(To factor with $p-1$ and $p-2$ was suggested by the observation that $(p,q) = (1,1)$ and $(p,q) \to (2,0)$ lead to equality cases.)

Comments: The case $r = m/n \in \mathbb{Q}\cap[0,1]$ of Bernoulli's inequality follows by applying the *AM-GM* inequality to $a_1,\ldots,a_n$, where $a_1 = \cdots = a_m = 1+x$ and $a_{m+1} = \cdots = a_n = 1$. The case $r \in [0,1]\backslash\mathbb{Q}$ follows by taking rational m/n converging to r.

***Solution* 2. LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Suppose $2 > p \ge 1 \ge q > 0$. Applying Bernoulli's inequality with $1+x = p/q$ and $r = p/2$, we have

$$\left(\frac{p}{q}\right)^{p/2} \le 1 + \frac{p}{2}\left(\frac{p}{q}-1\right) = \frac{p^2+q^2}{2q}.$$

Multiplying both sides by q and squaring both sides, we have

$$p^p q^q \le (p^2+q^2)^2/4.$$

Similarly, applying Bernoulli's inequality with $1+x = q/p$ and $r = p/2$, we can get $p^q q^p \le p^2q^2$. So

$$\begin{aligned}3p^q q^p + p^p q^q &\le (p^4+14p^2q^2+q^4)/4\\ &= (p^4+6p^2q^2+q^4+4pq(2pq))/4\\ &\le (p^4+6p^2q^2+q^4+4pq(p^2+q^2))/4\\ &= (p+q)^4/4 = 4.\end{aligned}$$

Commended solvers: **Paolo Perfetti** (Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Olympiad Corner

(continued from page 1)

Problem 3. Prove that there are infinitely many primes p such that $N_p = p^2$, where N_p is the total number of solutions to the equation

$$3x^3+4y^3+5z^3-y^4z \equiv 0 \pmod{p}.$$

Problem 4. Two circles C_1, C_2 with different radii are given in the plane, they touch each other externally at T. Consider any points $A \in C_1$ and $B \in C_2$, both different from T, such that $\angle ATB = 90°$.

(a) Show that all such lines AB are concurrent.

(b) Find the locus of midpoints of all such segments AB.

Double Counting

(continued from page 2)

Example 8. (*2003 IMO Shortlisted Problem*) Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1\le i,j\le n}$ be the matrix with entries

$$a_{ij} = \begin{cases}1, & \text{if } x_i + y_j \ge 0;\\ 0, & \text{if } x_i + y_j < 0.\end{cases}$$

Suppose that B is an $n\times n$ matrix with entries 0 or 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A=B$.

Solution. Let $A = (a_{ij})_{1\le i,j\le n}$. Define

$$S = \sum_{i=1}^n \sum_{j=1}^n (x_i + y_j)(a_{ij} - b_{ij}).$$

On one hand, we have

$$S = \sum_{i=1}^n x_i\left(\sum_{j=1}^n a_{ij} - \sum_{j=1}^n b_{ij}\right) + \sum_{j=1}^n y_j\left(\sum_{i=1}^n a_{ij} - \sum_{i=1}^n b_{ij}\right) = 0.$$

On the other hand, if $x_i+y_j \ge 0$, then $a_{ij} = 1$, which implies $a_{ij}-b_{ij} \ge 0$; if $x_i+y_j < 0$, then $a_{ij} = 0$, which implies $a_{ij}-b_{ij} \le 0$. Hence, $(x_i+y_j)(a_{ij}-b_{ij}) \ge 0$ for all i,j. Since $S = 0$, all $(x_i+y_j)(a_{ij}-b_{ij}) = 0$.

In particular, if $a_{ij}=0$, then $x_i+y_j < 0$ and so $b_{ij} = 0$. Since a_{ij}, b_{ij} are 0 or 1, so $a_{ij} \ge b_{ij}$ for all i,j. Finally, since the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A, so $a_{ij} = b_{ij}$ for all i,j.

Mathematical Excalibur

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Olympiad Corner

The following were the problems of the Final Round (Part 2) of the Austrian Mathematical Olympiad 2008.

First Day: June 6th, 2008

Problem 1. Prove the inequality

$$\sqrt{a^{1-a}b^{1-b}c^{1-c}} \le \frac{1}{3}$$

holds for all positive real numbers a, b and c with $a+b+c = 1$.

Problem 2. (a) Does there exist a polynomial $P(x)$ with coefficients in integers, such that $P(d) = 2008/d$ holds for all positive divisors of 2008?

(b) For which positive integers n does a polynomial $P(x)$ with coefficients in integers exists, such that $P(d) = n/d$ holds for all positive divisors of n?

Problem 3. We are given a line g with four successive points P, Q, R, S, reading from left to right. Describe a straight-edge and compass construction yielding a square $ABCD$ such that P lies on the line AD, Q on the line BC, R on the line AB and S on the line CD.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 7, 2009***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Generating Functions

Kin Yin Li

In some combinatorial problems, we may be asked to determine a certain sequence of numbers $a_0, a_1, a_2, a_3, \ldots$. We can associate such a sequence with the following series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

This is called the *generating function* of the sequence. Often the geometric series $1/(1-t) = 1+t+t^2+t^3+\cdots$ for $|t| < 1$ and its square

$$1/(1-t)^2 = (1+t+t^2+t^3+\cdots)^2$$
$$= 1+2t+3t^2+4t^3+5t^4+\cdots$$

will be involved in our discussions.

Below we will provide examples to illustrate how generating functions can solve some combinatorial problems.

Example 1. Let a_0=1, a_1=1 and

$$a_n = 4a_{n-1} - 4a_n \text{ for } n \ge 2.$$

Find a formula for a_n in terms of n.

Solution. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then we have

$$\begin{aligned} f(x) - 1 - x &= a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\ &= (4a_1-4a_0)x^2 + (4a_2-4a_1)x^3 + \cdots \\ &= (4a_1x^2+4a_2x^3+\cdots)-(4a_0x^2+4a_1x^3+\cdots) \\ &= 4x(f(x)-1) - 4x^2f(x). \end{aligned}$$

Solving for $f(x)$ and taking $|x| < ½$,

$$\begin{aligned} f(x) &= (1-3x)/(1-2x)^2 \\ &= 1/(1-2x) - x/(1-2x)^2 \\ &= \sum_{n=0}^{\infty}(2x)^n - x\sum_{n=1}^{\infty}n(2x)^{n-1} \\ &= \sum_{n=0}^{\infty}(2^n - n2^{n-1})x^n. \end{aligned}$$

Therefore, $a_n = 2^n - n\,2^{n-1}$.

Example 2. Find the number a_n of ways n dollars can be changed into 1 or 2 dollar coins (regardless of order). For example, when $n = 3$, there are 2 ways, namely three 1 dollar coins or one 1 dollar coin and one 2 dollar coin.

Solution. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. To study this infinite series, let $|x| < 1$. For each way of changing n dollars into r 1 dollar and s 2 dollar coins, we can record it as $x^r x^{2s} = x^n$. Now r and s may be any nonnegative integers. Adding all the recorded terms for all nonnegative integers n, then factoring, we get

$$\sum_{r=0}^{\infty}x^r\sum_{s=0}^{\infty}x^{2s} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}x^{r+2s} = \sum_{n=0}^{\infty}a_nx^n = f(x).$$

On the other hand,

$$\begin{aligned} \sum_{r=0}^{\infty}x^r\sum_{s=0}^{\infty}x^{2s} &= \frac{1}{1-x}\cdot\frac{1}{1-x^2} = \frac{1}{(1-x)^2(1-x)} \\ &= \frac{1}{2}\left(\frac{1}{(1-x)^2}+\frac{1}{1-x^2}\right) \\ &= \tfrac{1}{2}\Big((1+2x+3x^2+\cdots)+(1+x^2+x^4+\cdots)\Big) \\ &= 1+x+2x^2+2x^3+3x^4+3x^5+\cdots. \\ &= \sum_{n=0}^{\infty}([n/2]+1)x^n. \end{aligned}$$

Therefore, $a_n = [n/2] + 1$.

Example 3. Let n be a positive integer. Find the number a_n of polynomials $P(x)$ with coefficients in {0,1,2,3} such that $P(2) = n$.

Solution. Let $f(t)$ be the generating function of the sequence $a_0, a_1, a_2, a_3, \ldots$. Let $P(x) = c_0 + c_1x + \cdots + c_kx^k$ with $c_i \in \{0,1,2,3\}$. Now $P(2) = n$ if and only if $c_0 + 2c_1 + \cdots + 2^kc_k = n$. Taking $t \in (-1,1)$, we can record this as

$$t^n = t^{c_0}t^{2c_1}\cdots t^{2^kc_k}.$$

Note 2^ic_i is one of the four numbers 0, 2^i, 2^{i+1}, $3\cdot 2^i$. Adding all the recorded terms for all nonnegative integers n and all possible $c_0, c_1, \ldots, c_k \in \{0,1,2,3\}$, then factoring on the right, we have

$$f(t) = \sum_{n=0}^{\infty}a_nt^n = \prod_{i=0}^{\infty}(1+t^{2^i}+t^{2^{i+1}}+t^{3\cdot 2^i}).$$

Using $1+s+s^2+s^3=(1-s^4)/(1-s)$, we see

$$\begin{aligned} f(t) &= \frac{1-t^4}{1-t}\cdot\frac{1-t^8}{1-t^2}\cdot\frac{1-t^{16}}{1-t^4}\cdot\frac{1-t^{32}}{1-t^8}\cdots \\ &= \frac{1}{1-t}\cdot\frac{1}{1-t^2}. \end{aligned}$$

As in example 2, we get $a_n = [n/2] + 1$.

For certain problems, instead of using the generating function of a_0, a_1, a_2, a_3, …, we may consider the series

$$x^{a_0}+x^{a_1}+x^{a_2}+x^{a_3}+\cdots.$$

Example 4. (*1998 IMO Shortlisted Problem*) Let a_0, a_1, a_2, … be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i+2a_j+4a_k$, where i, j and k are not necessarily distinct. Determine a_{1998}.

Solution. For $|x|<1$, let $f(x)=\sum_{i=0}^{\infty} x^{a_i}$.

The given condition implies

$$f(x)f(x^2)f(x^4)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}.$$

Replacing x by x^2, we get

$$f(x^2)f(x^4)f(x^8)=\frac{1}{1-x^2}.$$

From these two equations, we get $f(x)=(1+x)\,f(x^8)$. Repeating this recursively, we get

$$f(x)=(1+x)(1+x^8)(1+x^{8^2})(1+x^{8^3})\cdots.$$

In expanding the right side, we see the exponents a_0, a_1, a_2, … are precisely the nonnegative integers whose base 8 representations have only digit 0 or 1. Since $1998=2+2^2+2^3+2^6+2^7+2^8+2^9+2^{10}$, so $a_{1998}=8+8^2+8^3+8^6+8^7+8^8+8^9+8^{10}$.

For our next examples, we need some identities involving p-th roots of unity, where p is a positive integer. These are complex numbers λ, which are all the solutions of the equation $z^p=1$. For a real θ, we will use the common notation $e^{i\theta}=\cos\theta+i\sin\theta$. Since the equation is of degree p, there are exactly p p-th roots of unity. We can easily check that they are $e^{i\theta}$ with $\theta=0$, $2\pi/p$, $4\pi/p$, …, $2(p-1)\pi/p$.

Below let λ be any p-th root of unity, other than 1. When we have a series

$$B(z)=b_0+b_1z+b_2z^2+b_3z^3+\cdots,$$

sometimes we need to find the value of $b_p+b_{2p}+b_{3p}+\cdots$. We can use the fact

$$1+\lambda^j+\lambda^{2j}+\cdots+\lambda^{(p-1)j}=\frac{1-\lambda^{pj}}{1-\lambda^j}=0$$

(for any j not divisible by p) to get

$$\frac{1}{p}\sum_{j=0}^{p-1}B(\lambda^j)=b_p+b_{2p}+b_{3p}+\cdots. \quad (*)$$

For p odd, we have the factorization

$$1+t^p=(1+t)(1+\lambda t)\cdots(1+\lambda^{p-1}t) \quad (**)$$

since both sides have $-1/\lambda^i$ ($i=0,1,\ldots,p-1$) as roots and are monic of degree p.

Example 5. Can the set $\mathbb{N}$ of all positive integers be partitioned into more than one, but still a finite number of arithmetic progressions with no two having the same common differences?

Solution. (*Due to Donald J. Newman*) Assume the set $\mathbb{N}$ can be partitioned into sets $S_1, S_2, \ldots, S_k$, where $S_i=\{a_i+nd_i: n\in\mathbb{N}\}$ with $d_1>d_2>\cdots>d_k$. Then for $|z|<1$,

$$\sum_{n=1}^{\infty}z^n=\sum_{n=1}^{\infty}z^{a_1+nd_1}+\sum_{n=1}^{\infty}z^{a_2+nd_2}+\cdots+\sum_{n=1}^{\infty}z^{a_k+nd_k}.$$

Summing the geometric series, this gives

$$\frac{z}{1-z}=\frac{z^{a_1}}{1-z^{d_1}}+\frac{z^{a_2}}{1-z^{d_2}}+\cdots+\frac{z^{a_k}}{1-z^{d_k}}.$$

Letting z tend to $e^{2\pi i/d_1}$, we see the left side has a finite limit, but the right side goes to infinity. That gives a contradiction.

Example 6. (*1995 IMO*) Let p be an odd prime number. Find the number of subsets A of the set $\{1,2,\ldots,2p\}$ such that

(i) A has exactly p elements, and

(ii) the sum of all the elements in A is divisible by p.

Solution. Consider the polynomial

$$F_a(x)=(1+ax)(1+a^2x)(1+a^3x)\cdots(1+a^{2p}x)$$

When the right side is expanded, let $c_{n,k}$ count the number of terms of the form $(a^{i_1}x)(a^{i_2}x)\cdots(a^{i_k}x)$, where $i_1, i_2, \ldots, i_k$ are integers such that $1\le i_1<i_2<\cdots<i_k\le 2p$ and $i_1+i_2+\cdots+i_k=n$. Then

$$F_a(x)=1+\sum_{k=1}^{2p}\left(\sum_{n=1}^{\infty}c_{n,k}a^n\right)x^k.$$

Now, in terms of $c_{n,k}$, the answer to the problem is $C=c_{p,p}+c_{2p,p}+c_{3p,p}+\cdots$.

To get C, note the coefficient of x^p in $F_a(x)$ is $\sum_{n=1}^{\infty}c_{n,p}a^n$. By (*) above, we see

$$C=\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty}c_{n,p}\omega^{nj}.$$

Now the right side is the coefficient of x^p in $\frac{1}{p}\sum_{j=0}^{p-1}F_{\omega^j}(x)$, which equals

$$\frac{1}{p}\sum_{j=0}^{p-1}(1+\omega^jx)(1+\omega^{2j}x)\cdots(1+\omega^{2pj}x).$$

For $j=0$, the term is $(1+x)^{2p}$. For $1\le j\le p-1$, using (**) with $\lambda=\omega^j$ and $t=\lambda x$, we see the j-th term is $(1+x^p)^2$. Using these, we have

$$\frac{1}{p}\sum_{j=0}^{p-1}F_{\omega^j}(x)=\frac{1}{p}[(1+x)^{2p}+(p-1)(1+x^p)^2].$$

Therefore, the coefficient of x^p is

$$C=\frac{1}{p}\left[\binom{2p}{p}+2(p-1)\right].$$

So far all generating functions were in one variable. For the curious mind, next we will look at an example involving a two variable generating function

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}a_{i,j}x^iy^j$$

of the simplest kind.

Example 7. An $a\times b$ rectangle can be tiled by a number of $p\times 1$ and $1\times q$ types of rectangles, where a, b, p, q are fixed positive integers. Prove that a is divisible by p or b is divisible by q. (Here a $k\times 1$ and a $1\times k$ rectangles are considered to be different types.)

Solution. Inside the (i,j) cell of the $a\times b$ rectangle, let us put the term x^iy^j for $i=1,2,\ldots,a$ and $j=1,2,\ldots,b$. The sum of the terms inside a $p\times 1$ rectangle is

$$x^iy^j+\cdots+x^{i+p-1}y^j=(1+x+\cdots+x^{p-1})\,x^iy^j,$$

if the top cell is at (i,j), while the sum of the terms inside a $1\times q$ rectangle is

$$x^iy^j+\cdots+x^iy^{j+q-1}=x^iy^j\,(1+y+\cdots+y^{q-1}),$$

if the leftmost cell is at (i,j). Now take

$$x=e^{2\pi i/p} \text{ and } y=e^{2\pi i/q}.$$

Then both sums become 0. If the desired tiling is possible, then the total sum of all terms in the $a\times b$ rectangle would be

$$0=\sum_{i=1}^{a}\sum_{j=1}^{b}x^iy^j=xy\frac{(1-x^a)(1-y^b)}{(1-x)(1-y)}.$$

This implies that a is divisible by p or b is divisible by q.

For the readers who like to know more about generating functions, we recommend two excellent references:

T. Andreescu and Z. Feng, *A Path to Combinatorics for Undergraduates*, Birkhäuser, Boston, 2004.

M. Novaković, *Generating Functions*, The IMO Compendium Group, 2007 (www.imomath.com)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***March 7, 2009.***

Problem 316. For every positive integer $n > 6$, prove that in every n-sided convex polygon $A_1A_2\ldots A_n$, there exist $i \neq j$ such that

$$|\cos\angle A_i - \cos\angle A_j| < \frac{1}{2(n-6)}.$$

Problem 317. Find all polynomial $P(x)$ with integer coefficients such that for every positive integer n, 2^n-1 is divisible by $P(n)$.

Problem 318. In ΔABC, side BC has length equal to the average of the two other sides. Draw a circle passing through A and the midpoints of AB, AC. Draw the tangent lines from the centroid of the triangle to the circle. Prove that one of the points of tangency is the incenter of ΔABC.
(*Source: 2000 Chinese Team Training Test*)

Problem 319. For a positive integer n, let S be the set of all integers m such that $|m| < 2n$. Prove that whenever $2n+1$ elements are chosen from S, there exist three of them whose sum is 0.
(*Source: 1990 Chinese Team Training Test*)

Problem 320. For every positive integer $k > 1$, prove that there exists a positive integer m such that among the rightmost k digits of 2^m in base 10, at least half of them are 9's.
(*Source: 2005 Chinese Team Training Test*)

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Solutions

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Problem 311. Let $S = \{1,2,\ldots,2008\}$. Prove that there exists a function $f : S \to$ {red, white, blue, green} such that there does not exist a 10-term arithmetic progression $a_1,a_2,\ldots,a_{10}$ in S satisfying $f(a_1) = f(a_2) = \cdots = f(a_{10})$.

Solution 1. **Kipp JOHNSON** (Valley Catholic School, teacher, Beaverton, Oregon, USA) and **PUN Ying Anna** (HKU Math, Year 3).

The number of 10-term arithmetic progressions in S is the same as the number of ordered pairs (a,d) such that a, d are in S and $a+9d \le 2008$. Since $d \le 2007/9=223$ and for each such d, a goes from 1 to $2008-9d$, so there are at most

$$4^{(2008-10)} \times 4 \times \sum_{d=1}^{223}(2008-9d)$$

$$= 4^{1999} \times 223000$$

functions $f : S \to$ {red, white, blue, green} such that there exists a 10-term arithmetic progression $a_1,a_2,\ldots,a_{10}$ in S satisfying $f(a_1) = f(a_2) = \cdots = f(a_{10})$, while there are more (namely 4^{2008}) functions from S to {red, white, blue, green}. So the desired function exists.

Solution 2. **G.R.A. 20 Problem Solving Group** (Roma, Italy).

Replace red, white, blue, green by 0, 1, 2, 3 respectively. It can be seen by a direct checking that $f:\{1,2,\ldots,2048\}\to\{0,1,2,3\}$ given by

$$f(n) = \left[\frac{n-1}{8}\right]_{\text{mod }2} + 2\left[\frac{n-1}{128}\right]_{\text{mod }2}$$

avoids any 9-term arithmetic progression having the same value (where $k_{\text{mod }2}$ is 0 if k is even and 1 if k is odd). The range of f is $((0^8 1^8)^8(2^8 3^8)^8)^8$, where for any string x, x^8 denotes the string obtained by putting eight copies of the string x one after another in a row and $f(n)$ is the n-th digit in the specified string.

Commended solvers: **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Problem 312. Let $x, y, z > 1$. Prove that

$$\frac{x^4}{(y-1)^2} + \frac{y^4}{(z-1)^2} + \frac{z^4}{(x-1)^2} \ge 48.$$

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **Kipp JOHNSON** (Valley Catholic School, teacher, Beaverton, Oregon, USA), **Kelvin LEE** (Trinity College, University of Cambridge, Year 2), **LEUNG Kai Chung** (HKUST Math, Year 2), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery), **MA Ka Hei** (Wah Yan College, Kowloon), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **PUN Ying Anna** (HKU Math, Year 3).

Let $x = a + 1$, $y = b + 1$ and $z = c + 1$. Applying the *AM-GM* inequality twice, we have

$$\frac{x^4}{(y-1)^2} + \frac{y^4}{(z-1)^2} + \frac{z^4}{(x-1)^2}$$

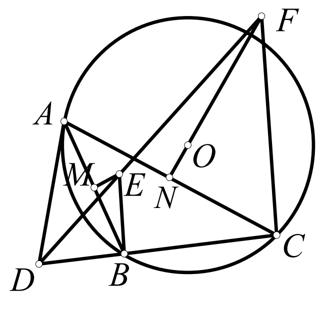
$$= \frac{(a+1)^4}{b^2} + \frac{(b+1)^4}{c^2} + \frac{(c+1)^4}{a^2}$$

$$\ge 3\left(\frac{(a+1)^4(b+1)^4(c+1)^4}{a^2b^2c^2}\right)^{1/3}$$

$$\ge 3\left(\frac{(2\sqrt{a})^4(2\sqrt{b})^4(2\sqrt{c})^4}{a^2b^2c^2}\right)^{1/3} = 48.$$

Commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 5), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 6), **Paolo PERFETTI** (Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Dimitar TRENEVSKI** (Yahya Kemal College, Skopje, Macedonia) and **TSOI Kwok Wing** (PLK Centenary Li Shiu Chung Memorial College, Form 6).

Problem 313. In ΔABC, $AB < AC$ and O is its circumcenter. Let the tangent at A to the circumcircle cut line BC at D. Let the perpendicular lines to line BC at B and C cut the perpendicular bisectors of sides AB and AC at E and F respectively. Prove that D, E, F are collinear.



Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **CHUNG Ping Ngai** (La Salle College, Form 5), **Kelvin LEE** (Trinity College, University of Cambridge, Year 2), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 6) and **PUN Ying Anna** (HKU Math, Year 3).

Let M be the midpoint of AB and N be the midpoint of AC. Using $\angle ABE = \angle ABC - 90°$, $\angle FCA = 90° - \angle ABC$ and the sine law, we have

$$\frac{BE}{CF}=\frac{BM/\cos\angle ABE}{CN/\cos\angle FCA}$$

$$=\frac{\frac{1}{2}AB/\sin\angle ABC}{\frac{1}{2}AC/\sin\angle BCA}=\frac{AB^2}{AC^2}.$$

From $\Delta DCA \sim \Delta DAB$, we see

$$\frac{DA}{DC}=\frac{DB}{DA}=\frac{\sin\angle DAB}{\sin\angle DBA}=\frac{\sin\angle ACB}{\sin\angle ABC}=\frac{AB}{AC}.$$

So

$$\frac{BE}{CF}=\frac{AB^2}{AC^2}=\frac{DA}{DC}\cdot\frac{DB}{DA}=\frac{DB}{DC}.$$

Then $\angle BDE = \angle CDF$. Therefore D,E,F are collinear.

Commended solvers: **Stefan LOZANOVSKI and Bojan JOVESKI** (Private Yahya Kemal College, Skopje, Macedonia)**.**

Problem 314. Determine all positive integers x, y, z satisfying $x^3 - y^3 = z^2$, where y is a prime, z is not divisible by 3 and z is not divisible by y.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5) and **PUN Ying Anna** (HKU Math, Year 3).

Suppose there is such a solution. Then

$$z^2 = x^3 - y^3 = (x-y)(x^2+xy+y^2)$$

$$= (x-y)((x-y)^2+3xy). \qquad (*)$$

Since y is a prime, z is not divisible by 3 and z is not divisible by y, (*) implies $(x,y)=1$ and $(x-y,3)=1$. Then

$$(x^2+xy+y^2, x-y)=(3xy, x-y)=1. \qquad (**)$$

Now (*) and (**) imply

$$x-y=m^2,\ x^2+xy+y^2=n^2 \text{ and } z=mn$$

for some positive integers m and n. Consequently,

$$4n^2= 4x^2+4xy+4y^2=(2x+y)^2+3y^2.$$

Then $3y^2=(2n+2x+y)(2n-2x-y)$. Since y is prime, there are 3 possibilities:

(1) $2n+2x+y = 3y^2$, $2n-2x-y = 1$
(2) $2n+2x+y = 3y$, $2n-2x-y = y$
(3) $2n+2x+y = y^2$, $2n-2x-y = 3$.

In (1), subtracting the equations leads to $3y^2-1 = 2(2x+y) = 2(2m^2+3y)$. Then

$$m^2 + 1 = 3y^2 - 6y - 3m^2 \equiv 0 \pmod 3.$$

However, $m^2 + 1 \equiv 1$ or $2 \pmod 3$. We get a contradiction.

In (2), subtracting the equations leads to $x = 0$, contradiction.

In (3), subtracting the equations leads to $y^2-3 = 2(2x+y) = 2(2m^2+3y)$, which can be rearranged as $(y-3)^2-4m^2=12$. This leads to $y = 7$ and $m = 1$. Then $x = 8$ and $z = 13$. Since $8^3-7^3=13^2$, this gives the only solution.

Commended solvers: **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Problem 315. Each face of 8 unit cubes is painted white or black. Let n be the total number of black faces. Determine the values of n such that in every way of coloring n faces of the 8 unit cubes black, there always exists a way of stacking the 8 unit cubes into a 2×2×2 cube C so the numbers of black squares and white squares on the surface of C are the same.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5) and **PUN Ying Anna** (HKU Math, Year 3).

The answer is n = 23 or 24 or 25. First notice that if n is a possible value, then so is $48-n$. This is because we can interchange all the black and white coloring and the condition can still be met by symmetry. Hence, without loss of generality, we may assume $n \le 24$.

For the 8 unit cubes, there are totally 24 pairs of opposite faces. In each pair, no matter how the cubes are stacked, there is one face on the surface of C and one face hidden.

If $n \le 22$, there is a coloring that has $[n/2]$ pairs with both opposite faces black. Then at least $[n/2]$ black faces will be hidden so that there can be at most $n-[n/2] \le 11$ black faces on the surface of C. This contradicts the existence of a stacking with 12 black and 12 white squares on the surface of C. So only n = 23 or 24 is possible.

Next, start with an arbitrary stacking. Let b be the number of black squares on the surface of C. For each of the 8 unit cubes, take an axis formed by the centers of a pair of opposite faces and rotate the cube about that axis by 90˚. Then take an axis formed by the centers of another pair of opposite faces of the same cube and rotate the cube about the axis by 90˚ twice. These three 90˚ rotations switch the three exposed faces with the three hidden faces. So after doing the twenty-four 90˚ rotations for the 8 unit cubes, there will be $n-b$ black squares on the surface of C.

For n = 23 or 24 and $b \le n$, the average of b and $n-b$ is 11.5 or 12, hence 12 is between b and $n-b$ inclusive.

Finally, observe that after each of the twenty-four 90˚ rotations, one exposed square will be hidden and one hidden square will be exposed. So the number of black squares on the surface of C can only increase by one, stay the same or decrease by one.

Therefore, at a certain moment, there will be exactly 12 black squares (and 12 white squares) on the surface of C.

Commended solvers: **G.R.A. 20 Problem Solving Group** (Roma, Italy) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery).

Olympiad Corner

(continued from page 1)

<u>Second Day: June 7th, 2008</u>

Problem 4. Determine all functions f mapping the set of positive integers to the set of non-negative integers satisfying the following conditions:

(1) $f(mn) = f(m)+f(n)$,
(2) $f(2008) = 0$, and
(3) $f(n) = 0$ for all $n \equiv 39 \pmod{2008}$.

Problem 5. Which positive integers are missing in the sequence $\{a_n\}$, with

$$a_n = n + [\sqrt{n}] + [\sqrt[3]{n}]$$

for all $n \ge 1$? ($[x]$ denotes the largest integer less than or equal to x, i.e. g with $g \le x < g+1$.)

Problem 6. We are given a square $ABCD$. Let P be a point not equal to a corner of the square or to its center M. For any such P, we let E denote the common point of the lines PD and AC, if such a point exists. Furthermore, we let F denote the common point of the lines PC and BD, if such a point exists.

All such points P, for which E and F exist are called acceptable points. Determine the set of all acceptable points, for which the line EF is parallel to AD.