Forum Geometricorum
Volume 11 (2011) 1–7.

FORUM GEOM
ISSN 1534-1178

A Note on the Hervey Point of a Complete Quadrilateral

Alain Levelut

Abstract. Using the extension of the Sylvester relation to four concyclic points we define the Hervey point of a complete quadrilateral and show that it is the center of a circle congruent to the Miquel circle and that it is the point of concurrence of eight remarkable lines of the complete quadrilateral. We also show that the Hervey point and the Morley point coincide for a particular type of complete quadrilaterals.

1. Introduction

In the present paper we report on several properties of a remarkable point of a complete quadrilateral, the so-called Hervey point. [1] This originated from a problem proposed and solved by F. R. J. Hervey [2] on the concurrence of the four lines drawn through the centers of the nine point circles perpendicular to the Euler lines of the four associated triangles. Here, we use a different approach which seems more efficient than the usual one. We first define the Hervey point by the means of a Sylvester type relation, and, only after that, we look for its properties. Following this way, several interesting results are easily obtained: the Hervey point is the center of a circle congruent to the Miquel circle, and it is the point of concurrence of eight remarkable lines of the complete quadrilateral, obviously including the four lines quoted above.

2. Some properties of the complete quadrilateral and notations

Before we present our approach to the Hervey point, we recall some properties of the complete quadrilateral. Most of them were given by J. Steiner in 1827 [6]. An analysis of the Steiner's note was recently published by J.-P. Ehrmann [1]. In what follows we quote the theorems as they were labelled by Steiner himself and reported in the Ehrmann's review. They are referred to as theorems (S-n). The three theorems used in the course of the present paper are:

Theorem S-1: *The four lines of a complete quadrilateral form four associated triangles whose circumcircles pass through the same point* (Miquel point M).

Theorem S-2: *The centers of the four circumcircles and the Miquel point M lie on the same circle* (Miquel circle).

Publication Date: February 4, 2011. Communicating Editor: Paul Yiu.

[1]Erroneously rendered “the Harvey point” in [5].

Theorem S-4: *The orthocenters of the four associated triangles lie on the same line* (orthocenter or Miquel-Steiner line).

In what follows the indices run from 1 to 4.

The complete quadrilateral is formed with the lines d_n. The associated triangle T_n is formed with three lines other than d_n, and has circumcenter O_n and orthocenter is H_n. The four circumcenters are all distinct, otherwise the complete quadrilateral would be degenerate. According to Theorem S-2, the four centers O_n are concyclic on the Miquel circle with center O.

The triangle Θ_n is formed with the three circumcenters other than O_n. It has circumcenter O and orthocenter is h_n.

The Euler line O_nH_n cuts the Miquel circle at O_n and another point N_n. The triangle MO_nN_n is inscribed in the Miquel circle.

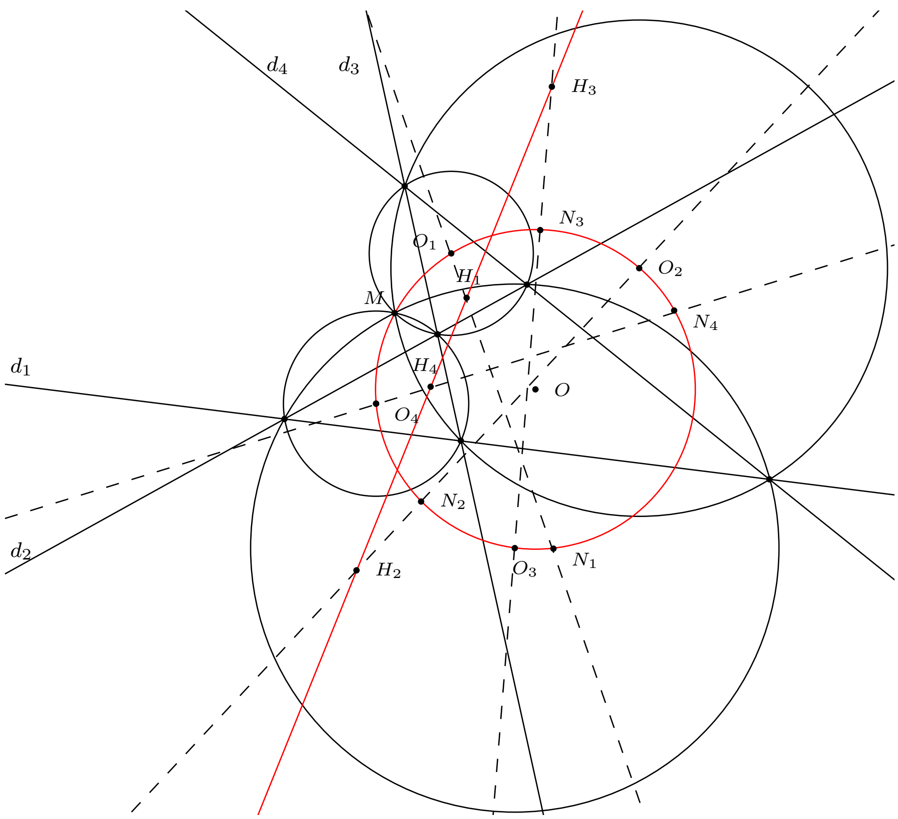

Figure 1. The Miquel circle and the Miquel-Steiner line

It is worthwhile to mention two more properties:

(1) The triangles Θ_n and T_n are directly similar [1];

(2) Each orthocenter h_n is on the corresponding line d_n.

3. The Hervey point of a complete quadrilateral

In order to introduce the Hervey point, we proceed in three steps, successively defining the orthocenter of a triangle, the pseudo-orthocenter of an inscriptable quadrangle and the Hervey point of a complete quadrilateral.

We define the orthocenter H of triangle ABC by the means of the standard Sylvester relation [3, §416]: the vector joining the circumcenter O to the orthocenter H is equal to the sum of the three vectors joining O to the three vertices, namely,

$$\mathbf{OH} = \mathbf{OA} + \mathbf{OB} + \mathbf{OC}.$$

It is easy to see that the resultant $\mathbf{AH}$ of the two vectors $\mathbf{OB}$ and $\mathbf{OC}$ is perpendicular to the side BC. Therefore the line AH is the altitude drawn through the vertex A and, more generally, the point H is the point of concurrence of the three altitudes. The usual property of the orthocenter is recovered.

We define the pseudo-orthocenter H of the inscriptable quadrangle with the following extension of the Sylvester relation

$$\mathbf{OH} = \mathbf{OP_1} + \mathbf{OP_2} + \mathbf{OP_3} + \mathbf{OP_4},$$

where P_1, P_2, P_3, P_4 are four distinct points lying on a circle with center O. We do not go further in the study of this point since many of its properties are similar to those of the Hervey point.

We define the Hervey point h of a complete quadrilateral as the pseudo-orthocenter of the inscriptable quadrangle made up with the four concyclic circumcenters O_1, O_2, O_3, O_4 of the triangles T_1, T_2, T_3, T_4, which lie on the Miquel circle with center O. The extended Sylvester relation reads

$$\mathbf{Oh} = \mathbf{OO_1} + \mathbf{OO_2} + \mathbf{OO_3} + \mathbf{OO_4}.$$

The standard Sylvester relation used for the triangle Θ_n reads

$$\mathbf{Oh_n} = \sum_{i \neq n} \mathbf{OO_i}.$$

This leads to

$$\mathbf{Oh} = \mathbf{Oh_n} + \mathbf{OO_n}$$

for each $n = 1, 2, 3, 4$. From these, we deduce

$$\mathbf{O_n h} = \mathbf{Oh_n}, \quad \mathbf{h_n h} = \mathbf{OO_n}$$

and

$$\mathbf{h_n h_m} = \mathbf{O_m O_n} \quad \text{for } m \neq n. \tag{1}$$

The last relation implies that the four orthocenters h_n are distinct since the centers O_n are distinct. Figure 2 shows the construction of the Hervey point starting with the orthocenter h_4.

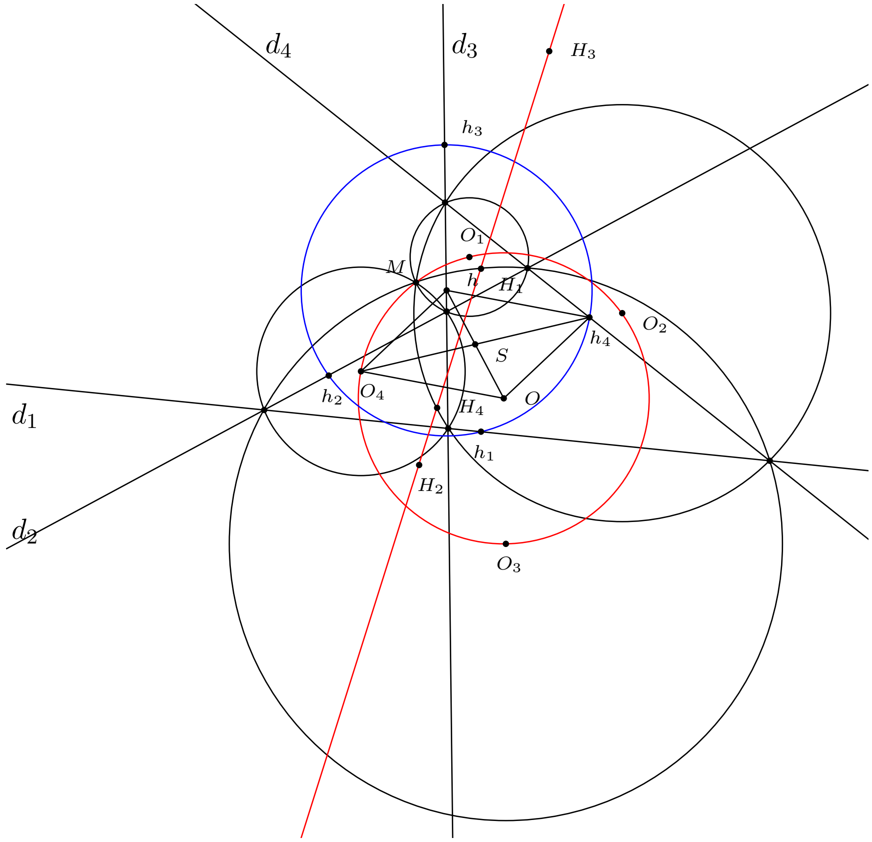

Figure 2.

4. Three theorems on the Hervey point

Theorem 1. *The Hervey point h is the center of the circle which bears the four orthocenters h_n of the triangles Θ_n.*

Proof. From the relation (1) we deduce that the quadrilateral $O_mO_nh_mh_n$ is a parallelogram, and its two diagonals O_mh_m and O_nh_n intersect at their common midpoints. The four triangles Θ_n are endowed with the same role. Therefore the four midpoints of O_ih_i coincide in a unique point S. By reflection in S the four points O_n are transformed into the four orthocenters h_n. Then the orthocenter circle $(h_1h_2h_3h_4)$ is congruent to the Miquel circle $(O_1O_2O_3O_4)$ and its center Ω corresponds to the center O.

The Hervey point h and the center Ω coincide since, on the one hand S is the midpoint of $O\Omega$, and on the other hand

$$\mathbf{Oh} = \mathbf{Oh_n} + \mathbf{OO_n} = \mathbf{OS} + \mathbf{Sh_n} + \mathbf{OS} + \mathbf{SO_n} = 2\mathbf{OS}.$$

This is shown in Figure 2. $\square$

A part of the theorem (the congruence of the two circles) was already contained in a theorem proved by Lemoine [3, §§265, 417].

Theorem 2. *The Hervey point h is the point of concurrence of the four altitudes of the triangles MO_nN_n drawn through the vertices O_n.*

Proof. Since the triangles Θ_n and T_n are directly similar, the circumcenter O and the orthocenter h_n respectively correspond to the circumcenter O_n and orthocenter H_n. We have the following equality of directed angles

$$(MO, MO_n) = (Oh_n, O_nH_n) \pmod{\pi}.$$

On the one hand, the triangle MOO_n is isoceles because the sides OM and OO_n are radii of the Miquel circle. Therefore we have

$$(MO, MO_n) = (O_nM, O_nO) \pmod{\pi}.$$

On the other hand, since the point N_n is on the Euler line O_nH_n, we have

$$(Oh_n, O_nH_n) = (Oh_n, O_nN_n) \pmod{\pi}.$$

Moreover, it is shown above that $\mathbf{Oh_n} = \mathbf{O_nH}$. Finally we obtain

$$(O_nM, O_nO) = (O_nh, O_nN_n) \pmod{\pi}.$$

This equality means that the lines O_nO and O_nh are isogonal with respect to the sides O_nM and O_nN_n of triangle MO_nN_n. Since the line O_nO passes through the circumcenter, the line O_nh passes through the orthocenter [3, §253]. Therefore, the line O_nh is the altitude of the triangle MO_nN_n drawn through the vertex O_n. In other words, the line O_nh is perpendicular to the chord MN_n This is shown in Figure 3 for the triangle MO_4N_4. This result is valid for the four lines O_nh. Consequently, they concur at the Hervey point h. □

Theorem 3. *The Hervey point h is the point of concurrence of the four perpendicular bisectors of the segments O_nH_n of the Euler lines of the triangles T_n.*

Proof. Since the triangles Θ_n and T_n are directly similar, the circumcenter O and the orthocenter h_n respectively correspond to the circumcenter O_n and orthocenter H_n and we have the following equality between oriented angles of oriented lines

$$(\mathbf{MO}, \mathbf{MO_n}) = (\mathbf{Oh_n}, \mathbf{O_nH_n}) \pmod{2\pi}$$

and the the following equality between the side ratios

$$\frac{MO_n}{MO} = \frac{O_nH_n}{Oh_n}.$$

Since $\mathbf{Oh_n} = \mathbf{O_nh}$, we convert these equalities into

$$(\mathbf{MO}, \mathbf{MO_n}) = (\mathbf{O_nh}, \mathbf{O_nH_n}) \pmod{2\pi}$$

and

$$\frac{MO_n}{MO} = \frac{O_nH_n}{O_nh}.$$

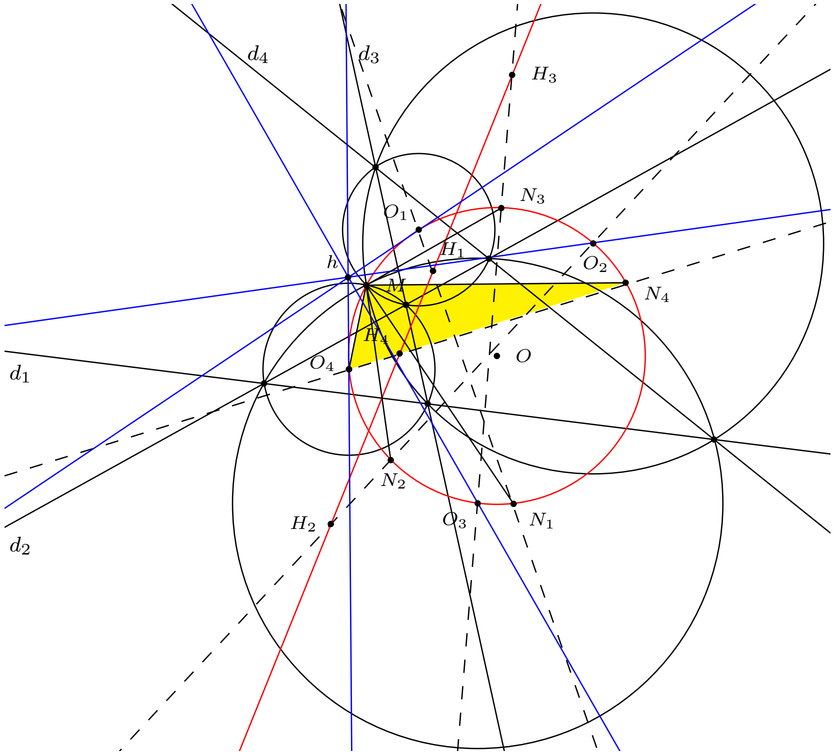

Figure 3.

These show that triangles MOO_n and O_nhH_n are directly similar. Since the triangle MOO_n is isoceles, the triangle O_nhH_n is isoceles too. Therefore, the perpendicular bisector of the side O_nH_n passes through the third vertex h. This is shown in Figure 4 for the triangles MOO_4 and O_4hH_4. This result is valid for the perpendicular bisectors of the four segments O_nH_n. Consequently they concur at the Hervey point h. □

5. The Hervey point of a particular type of complete quadrilaterals

Morley [4] has shown that the four lines drawn through the centers ω_n of the nine-point circles of the associated triangles T_n perpendicular to the corresponding lines d_n, concur at a point m (the Morley point of the complete quadrilateral) on the Miquel-Steiner line.

This result is valid for any complete quadrilateral. This is in contrast with the following theorem which is valid only for a particular type of complete quadrilaterals. Zeeman [7] has shown that if the Euler line of one associated triangle T_n is parallel to the corresponding line d_n, then this property is shared by all four associated triangles.

Our Theorem 3 also reads: the four lines drawn through the center ω_n of the nine point circle perpendicular to the Euler line of the associated triangle concur in the point h.

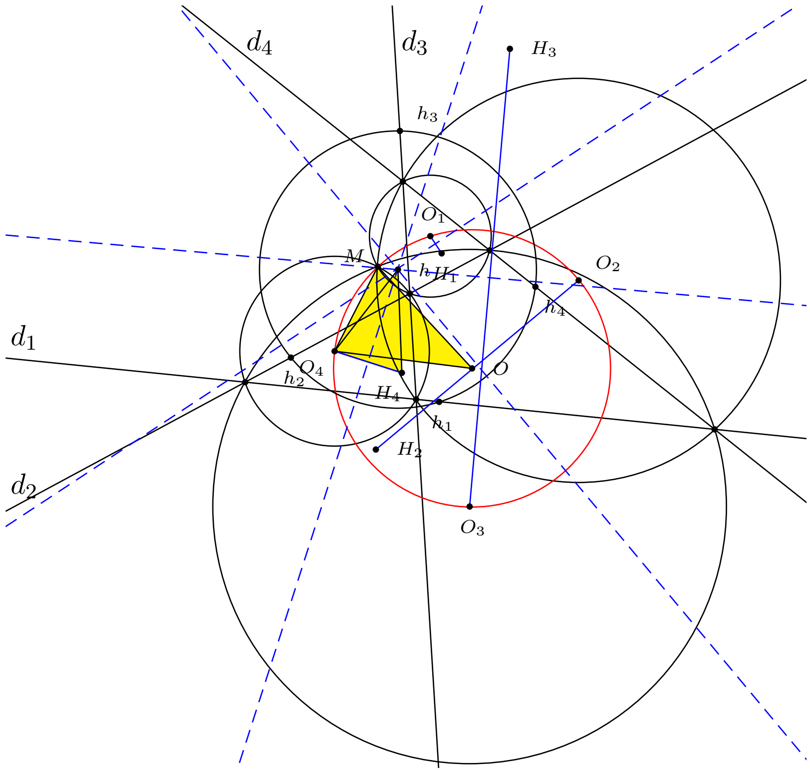

Figure 4.

Putting together these three theorems, we conclude that if a complete quadrilateral fulfills the condition of the Zeeman theorem, then the four pairs of lines $\omega_n h$ and $\omega_n m$ coincide. Consequently, their respective points of concurrence h and m coincide too. In other words, the center of the orthocenter circle $(h_1 h_2 h_3 h_4)$ lies on the orthocenter line $H_1 H_2 H_3 H_4$.

Theorem 4. *If the Euler line of one associated triangle T_n is parallel to the corresponding line d_n, then the Hervey point and the Morley point coincide.*

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Forum Geometricorum
Volume 11 (2011) 9–12.

FORUM GEOM
ISSN 1534-1178

The Incenter and an Excenter as Solutions to an Extremal Problem

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Abstract. Given a triangle ABC with a point X on the bisector of angle A, we show that the extremal values of $\frac{BX}{CX}$ occur at the incenter and the excenter on the opposite side of A.

Many centers of the triangle are solutions to a variety of extremal problems. For example, the Fermat point minimizes the sum of the distances of a point to the vertices of a triangle (provided the angles are all less than $120°$), and the centroid minimizes the sum of the squares of the distances to the vertices (see [1]). For recent results along these lines, see [3], and [2]. The following problem was brought to Dora's attention about 35 years ago.

Problem. Let ABC be a triangle and AT the angle bisector of angle A. Determine the points X on the line AT for which the ratio $\frac{BX}{CX}$ is extremal.

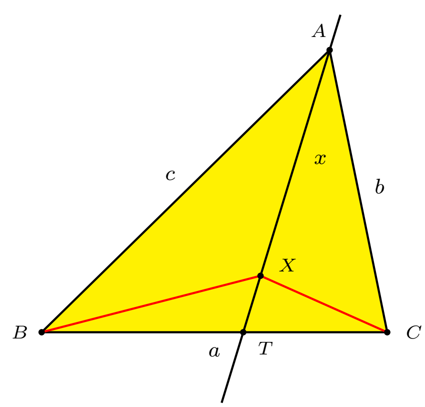

Figure 1.

Denote the lengths of the sides BC, CA, AB by a, b, c respectively. Let X be a point on AT and denote by x the directed length of AX. It is sufficient to consider

Publication Date: February 10, 2011. Communicating Editor: Paul Yiu.

This paper was conceived by Dora Bialostocki who passed away on July 25, 2009. After a failed attempt to find the results in the literature, her notes were supplemented and edited by her surviving husband Arie Bialostocki.

Thanks are due to an anonymous referee for generous suggestions leading to improvements on the methods and results of the paper.

the function

$$g(x) := \frac{BX^2}{CX^2} = \frac{x^2 - 2cx\cos\frac{A}{2} + c^2}{x^2 - 2bx\cos\frac{A}{2} + b^2}.$$

The derivative of $g(x)$ is given by

$$g'(x) = \frac{-2(b-c)\left(x^2\cos\frac{A}{2} - (b+c)x + bc\cos\frac{A}{2}\right)}{\left(x^2 - 2bx\cos\frac{A}{2} + b^2\right)^2}. \tag{1}$$

This is zero if and only if

$$x = \frac{b + c \pm \sqrt{(b+c)^2 - 4bc\cos^2\frac{A}{2}}}{2\cos\frac{A}{2}}.$$

Now,

$$\begin{aligned}(b+c)^2 - 4bc\cos^2\frac{A}{2} &= b^2 + c^2 - 2bc\left(2\cos^2\frac{A}{2} - 1\right)\\ &= b^2 + c^2 - 2bc\cos A\\ &= a^2. \end{aligned}\tag{2}$$

This means that g is extremal when

$$x = \frac{b+c-a}{2\cos\frac{A}{2}} = \frac{s-a}{\cos\frac{A}{2}} \quad \text{or} \quad x = \frac{b+c+a}{2\cos\frac{A}{2}} = \frac{s}{\cos\frac{A}{2}}, \tag{3}$$

where $s = \frac{a+b+c}{2}$ is the semiperimeter of the triangle ABC.

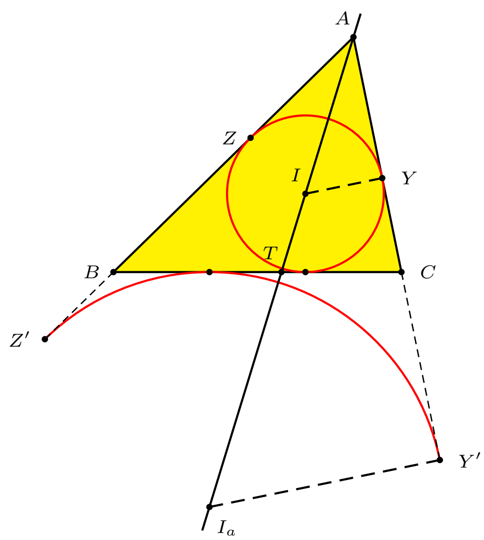

Figure 2. The incenter I and A-excenter I_a

Consider the incircle of the triangle tangent to the sides AC and AB at Y and Z respectively (see Figure 2). It is well known that

$$AY = AZ = \frac{1}{2}(b + c - a) = s - a.$$

Likewise, if the excircle on BC touches AC and AB at Y' and Z' respectively, then

$$AY' = AZ' = \frac{1}{2}(a+b+c) = s.$$

Therefore, according to (3), the extremal values of $g(x)$ occur at the incenter and A-excenter, whose orthogonal projections on the line AC are Y and Y', respectively. Note that

$$\frac{BI}{CI} = \frac{\sin\frac{C}{2}}{\sin\frac{B}{2}} \quad \text{and} \quad \frac{BI_a}{CI_a} = \frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}.$$

These are in fact the global maximum and minimum.

Figure 3 shows the graph of $g(x)$ for $b = 5$, $c = 6$ and $A = \frac{\pi}{3}$.

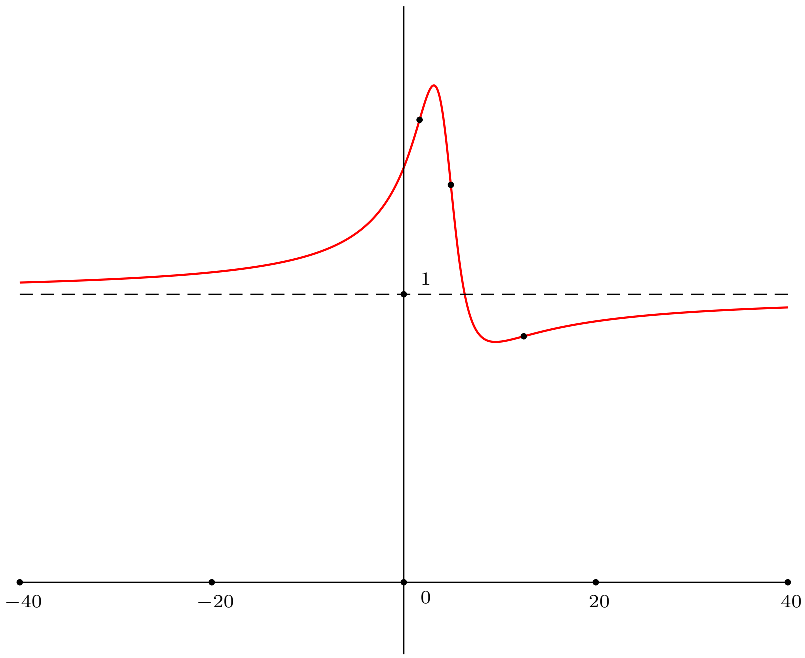

Figure 3. Graph of $g(x) = \frac{BX^2}{CX^2}$ with inflection points

We determine the inflection points of the graph of $g(x)$ (in the general case), and show that they are related to the problem of trisection of angles. Differentiating (1), we have

$$g''(x) = \frac{2(b-c)\left(2x^3\cos\frac{A}{2} - 3(b+c)x^2 + 6bcx\cos\frac{A}{2} + b^2\left(b+c-4c\cos^2\frac{A}{2}\right)\right)}{\left(x^2+b^2-2bx\cos\frac{A}{2}\right)^3}.$$

The inflection points are the roots of

$$\left(2\cos\frac{A}{2}\right)x^3 - 3(b+c)x^2 + \left(6bc\cos\frac{A}{2}\right)x + b^2\left(b+c-4c\cos^2\frac{A}{2}\right) = 0. \quad (4)$$

With the substitution

$$x = u + \frac{b+c}{2\cos\frac{A}{2}}, \tag{5}$$

equation (4) becomes

$$4u^3\cos^3\frac{A}{2} - \left((b+c)^2 - 4bc\cos^2\frac{A}{2}\right)\left(3u\cos\frac{A}{2} + b + c - 2b\cos^2\frac{A}{2}\right) = 0. \tag{6}$$

Making use of (2) and

$$b + c - 2b\cos^2\frac{A}{2} = c - b\cos A = a\cos B,$$

we rewrite (6) as

$$4u^3\cos^3\frac{A}{2} - a^2\left(3u\cos\frac{A}{2} + a\cos B\right) = 0. \tag{7}$$

If we further substitute

$$u\cos\frac{A}{2} = a\cos\theta, \tag{8}$$

the identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

transforms (7) into

$$\cos 3\theta = \cos B. \tag{9}$$

Combining (5) and (8), we conclude that the inflection points of $g(x)$ are at

$$\begin{aligned} x &= \frac{b+c+2a\cos\theta}{2\cos\frac{A}{2}} \\ &= \frac{(s-a) + a\left(\frac{1}{2} + \cos\theta\right)}{\cos\frac{A}{2}}, \end{aligned}$$

where θ satisfies (9).

Let F_1, F_2, F_3 be the inflection points on AT and denote their projections on AC by P_1, P_2, P_3. The last expression implies that the distances of P_1, P_2, P_3 from Y are given by $\left(\frac{1}{2} + \cos\theta\right)a$, where the angle θ satisfies (9).

Remark. It would be interesting to find in physics an interpretation of the two extrema.

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Forum Geometricorum
Volume 11 (2011) 13–26.

FORUM GEOM
ISSN 1534-1178

Translation of Fuhrmann's "Sur un nouveau cercle associé à un triangle"

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Abstract. We provide a translation (from the French) of Wilhelm Fuhrmann's influential 1890 paper in which he introduced what is today called the Fuhrmann triangle. We have added footnotes that indicate current terminology and references, ten diagrams to assist the reader, and an appendix.

Introductory remarks

Current widespread interest concerning the Fuhrmann triangle motivated us to learn what, exactly, Fuhrmann did in his influential 1890 article, "Sur un nouveau cercle associé à un triangle" [5]. We were quite surprised to see how much he accomplished in that paper. It was clear to us that anybody who has an interest in triangle geometry might appreciate studying the work for himself. We provide here a faithful translation of Fuhrmann's article (originally in French), supplementing it with footnotes that indicate current terminology and references. Two of the footnotes, however, (numbers 2 and 12) were provided in the original article by Joseph Neuberg (1840-1926), the cofounder (in 1881) of *Mathesis* and its first editor. Instead of reproducing the single diagram from the original paper, we have included ten figures showing that portion of the configuration relevant to the task at hand. At the end of our translation we have added an appendix containing further comments. Most of the background results that Fuhrmann assumed to be known can be found in classic geometry texts such as [7].

Biographical sketch.[1] Wilhelm Ferdinand Fuhrmann was born on February 28, 1833 in Burg bei Magdeburg. Although he was first attracted to a nautical career, he soon yielded to his passion for science, graduating from the Altstädtischen Gymnasium in Königsberg in 1853, then studying mathematics and physics at the University of Königsberg, from which he graduated in 1860. From then until his

Publication Date: February 18, 2010. Communicating Editor: Paul Yiu.

[1]Translated from Franz Kössler, *Personenlexikon von Lehrern des 19 Jahrhunderts*: *Berufsbiographien aus Schul-Jahresberichten und Schulprogrammen 1825-1918 mit Veröffentlichungsverzeichnissen*. We thank Professor Rudolf Fritsch of the Ludwig-Maximilians-University of Munich and his contacts on the genealogy chat line, Hans Christoph Surkau in particular, for providing us with this reference.

death 44 years later he taught at the Oberrealschule auf der Burg in Königsberg, receiving the title distinguished teacher (Oberlehrer) in 1875 and professor in 1887. In 1894 he obtained the Order of the Red Eagle, IV Class. He died in Königsberg on June 11, 1904.

On a New Circle Associated with a Triangle

Wilhelm Fuhrmann [2]

Preliminaries. Let ABC be a triangle; O its circumcenter; I, r its incenter and inradius; I_a, I_b, I_c its excenters; H_a, H_b, H_c, H the feet of its altitudes and the orthocenter; A', B', C' the midpoints of its sides; $A''B''C''$ the triangle whose sides pass through the vertices of ABC and are parallel to the opposite sides.

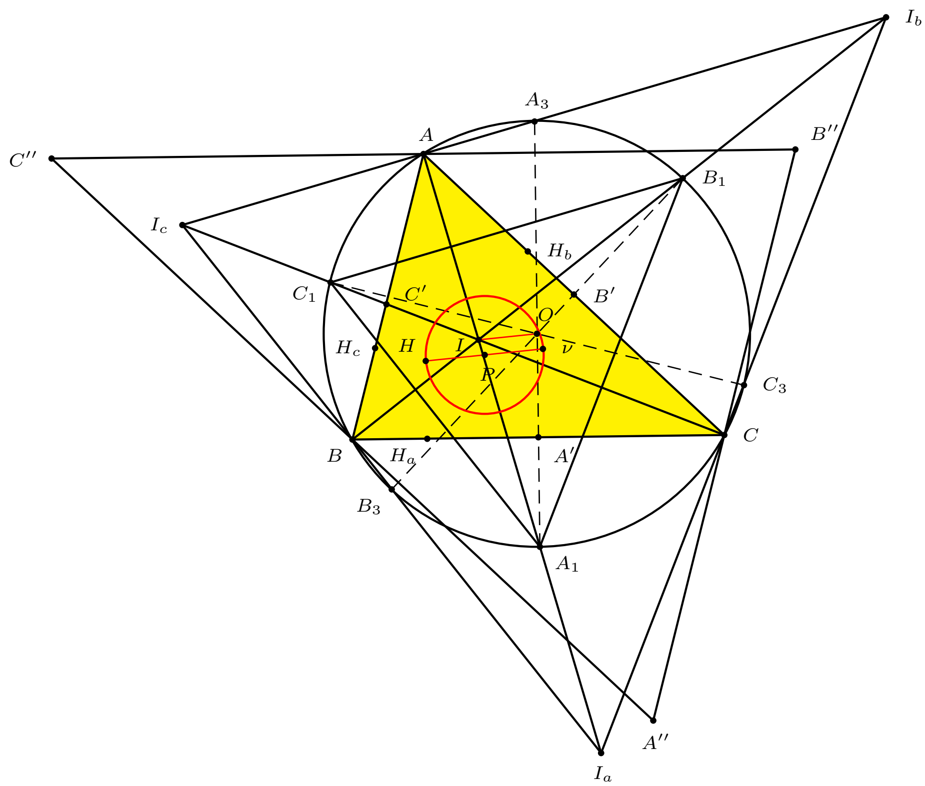

Figure 1. The initial configuration

The internal angle bisectors AI, BI, CI meet the circle O at the midpoints A_1, B_1, C_1 of the arcs that subtend the angles of triangle ABC. One knows that the triangles $A_1B_1C_1$ and $I_aI_bI_c$ are homothetic with respect to their common orthocenter I.

[2] [Neuberg's footnote.] Mr. Fuhrmann has just brought out, under the title *Synthetische Beweise planimetrischer Sätze* (Berlin, Simion, 1890, XXIV-190p. in-8°, 14 plates), an excellent collection of results related, in large part, to the recent geometry of the triangle. We will publish subsequently the results of Mr. Mandart, which complement the article of Mr. Fuhrmann. (J.N.)

The circle O passes through the midpoints A_3, B_3, C_3 of the sides of triangle $I_aI_bI_c$; the lines A_1A_3, B_1B_3, C_1C_3 are the perpendicular bisectors of the sides BC, CA, AB of triangle ABC.

We denote by ν the incenter of the anticomplementary triangle $A''B''C''$; this point is called the *Nagel point* of triangle ABC (see *Mathesis*, v. VII, p. 57).[3]

We denote the circle whose diameter is $H\nu$ by the letter P, which represents its center;[4] $H\nu$ and OI, homologous segments of the triangles $A''B''C''$, ABC are parallel and in the ratio $2 : 1$.

1. *Let A_2, B_2, C_2 be the reflections of the points A_1, B_1, C_1 in the lines BC, CA, AB: triangle $A_2B_2C_2$ is inscribed in the circle P and is oppositely similar to triangles $A_1B_1C_1$, $I_aI_bI_c$.*[5]

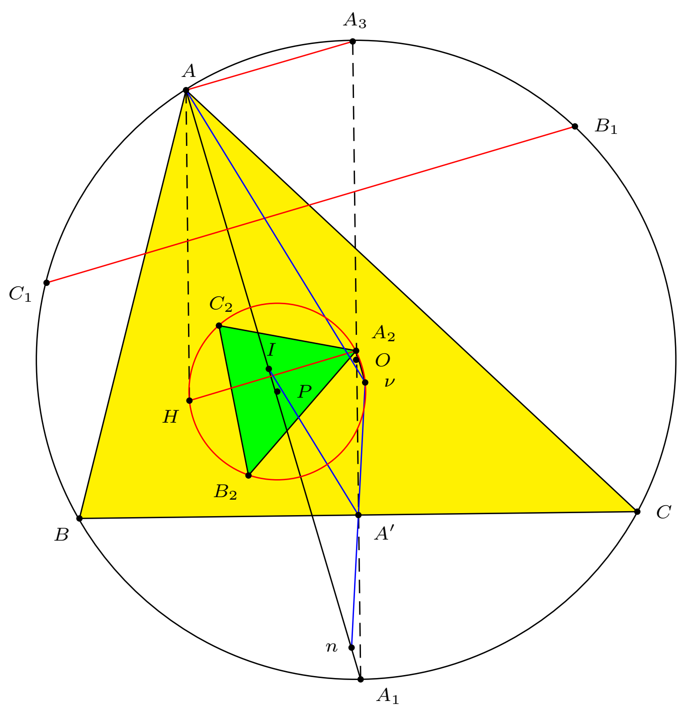

Figure 2. Proof of Theorem 1

One has

$$A_2A_3 = A'A_3 - A'A_1 = 2A'O = AH;$$

[3]Em. Vigarié, sur les points complémentaires. *Mathesis* VII (1887) 6-12, 57-62, 84-89, 105-110; Vigarié produced in French, Spanish, and German, more than a dozen such summaries of the latest results in triangle geometry during the final decades of the nineteenth century.

[4]Nowadays the circle P is called the *Fuhrmann circle* of triangle ABC. Its center P appears as X_{355} in Kimberling's *Encyclopedia of Triangle Centers* [11].

[5]Triangle $A_2B_2C_2$ is now called the *Fuhrmann triangle* of triangle ABC.

consequently, HA_2 is parallel to AA_3 and B_1C_1. $A\nu$ and $A'I$, homologous segments of the homothetic triangles $A''B''C''$, ABC, are parallel and $A\nu = 2A'I$; thus $\nu A'$ meets AI in a point n such that $\nu A' = A'n$, and as one also has $A_2A' = A'A_1$, the figure $A_1\nu A_2 n$ is a parallelogram.

Thus, the lines $A_2H, A_2\nu$ are parallel to the perpendicular lines AA_3, AA_1; consequently, $\angle HA_2\nu$ is right, and the circle P passes through A_2.

If, in general, through a point on a circle one takes three chords that are parallel to the sides of a given triangle, the noncommon ends of these chords are the vertices of a triangle that is oppositely similar to the given triangle. From this observation, because the lines HA_2, HB_2, HC_2 are parallel to the sides of triangle $A_1B_1C_1$, triangle $A_2B_2C_2$ is oppositely similar to $A_1B_1C_1$.

Remark. The points A_2, B_2, C_2 are the projections of H on the angle bisectors $A''\nu, B''\nu, C''\nu$ of triangle $A''B''C''$.

2. *From the altitudes AH, BH, CH of triangle ABC, cut segments AA_4, BB_4, CC_4 that are equal to the diameter $2r$ of the incircle: triangle $A_4B_4C_4$ is inscribed in circle P, and is oppositely similar to ABC.*

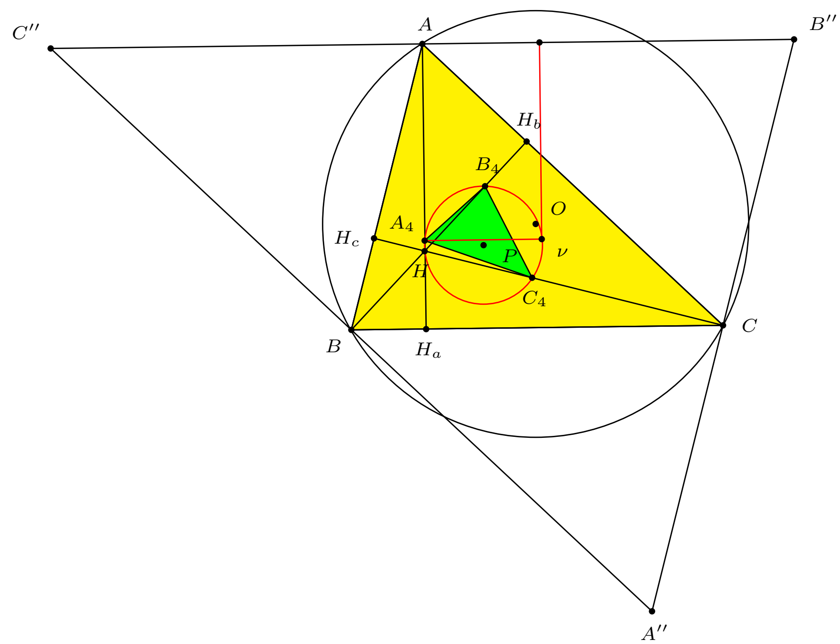

Figure 3. Proof of Theorem 2

Because the distances from ν to the sides of triangle $A''B''C''$ equal $2r$, the lines $\nu A_4, \nu B_4, \nu C_4$ are parallel to the sides of triangle ABC, and the angles νA_4H, etc. are right. Circle P passes, therefore, through the points A_4, B_4, C_4 and triangle $A_4B_4C_4$ is oppositely similar to ABC.

Remark. The points A_4, B_4, C_4 are the projections of the incenter of triangle $A''B''C''$ on the perpendicular bisectors of this triangle.

3. *The triangles $A_2B_2C_2$ and $A_4B_4C_4$ are perspective from I.*

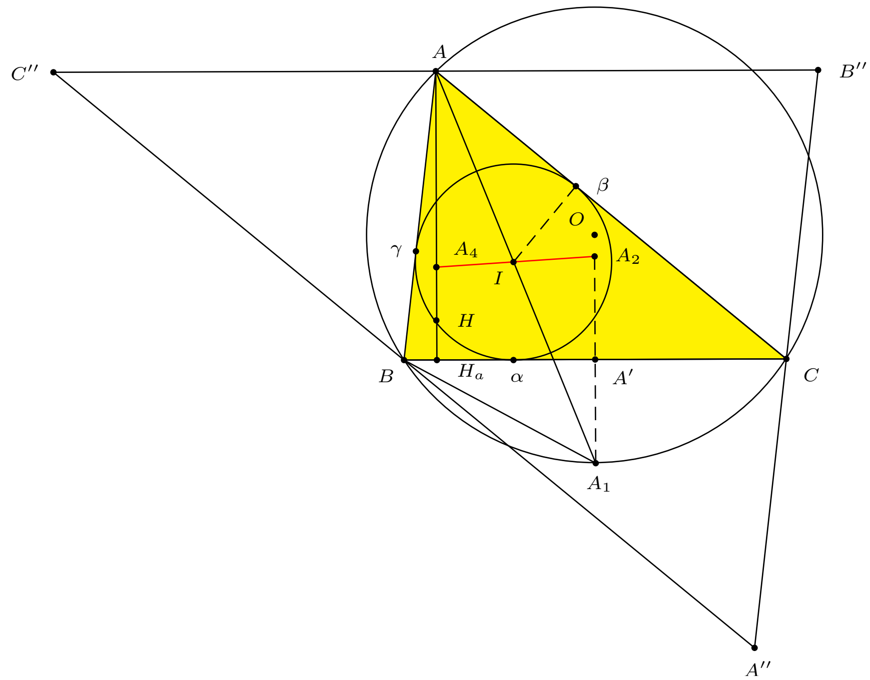

Figure 4. The proof of Theorems 3 and 4

Let α, β, γ be the points where the circle I touches BC, CA, AB. Recalling that triangles $AI\beta$, BA_1A' are similar, one finds that

$$\frac{AA_4}{A_2A_1} = \frac{2r}{2A_1A'} = \frac{I\beta}{A_1A'} = \frac{AI}{BA_1} = \frac{AI}{IA_1}.$$

The triangles AA_4I, A_1A_2I are therefore similar, and the points A_4, I, A_2 are collinear.

Remark. The axis of the perspective triangles ABC, $A_1B_1C_1$ is evidently the polar of the point I with respect to the circle O; that of triangles $A_2B_2C_2$, $A_4B_4C_4$ is the polar of I with respect to the circle P.

4. *I is the double point*[6] *of the oppositely similar triangles ABC, $A_4B_4C_4$, and also of the triangles $A_1B_1C_1$, $A_2B_2C_2$.*

Triangles ABC, $A_4B_4C_4$ are oppositely similar, and the lines that join their vertices to the point I meet their circumcircles in the vertices of the two oppositely similar triangles $A_1B_1C_1$, $A_2B_2C_2$. It follows that I is its own homologue in ABC and $A_4B_4C_4$, $A_1B_1C_1$ and $A_2B_2C_2$.

[6]That is, I is the center of the opposite similarity that takes the first triangle to the second. See the appendix for another proof of Theorem 4.

Remarks. (1) The two systems of points $ABCA_1B_1C_1$, $A_4B_4C_4A_2B_2C_2$ are oppositely similar; their double point is I, their double lines are the bisectors (interior and exterior) of angle AIA_4.

(2) I is the incenter of triangle $A_4B_4C_4$, and the orthocenter of triangle $A_2B_2C_2$.[7]

5. *Let I'_a, I'_b, I'_c be the reflections of the points I_a, I_b, I_c in the lines BC, CA, AB, respectively. The lines AI'_a, BI'_b, CI'_c, and the circumcircles of the triangles BCI'_a, CAI'_b, ABI'_c all pass through a single point R. The sides of triangle $A_2B_2C_2$ are the perpendicular bisectors of AR, BR, CR.*

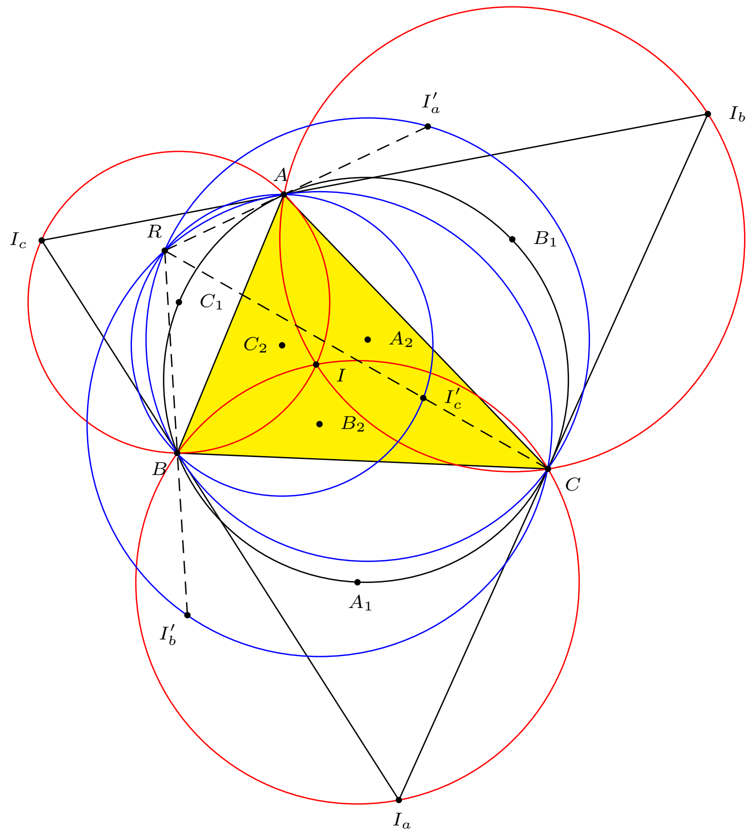

Figure 5. The proof of Theorem 5

Triangles I_aBC, AI_bC, ABI_c are directly similar; their angles equal

$$90^\circ - \frac{A}{2},\ 90^\circ - \frac{B}{2},\ 90^\circ - \frac{C}{2}.$$

The circumcircles of these triangles have centers A_1, B_1, C_1, and they pass through the same point I.

Triangles I'_aBC, AI'_bC, ABI'_c are oppositely similar to I_aBC, AI_bC, ABI_c; their circumcircles have, evidently, for centers the points A_2, B_2, C_2, and they pass

[7] Milorad Stevanovic [13] recently rediscovered that I is the orthocenter of the Fuhrmann triangle. Yet another recent proof can be found in [1].

through a single point R, say, which is the center of a pencil $R(ABC)$ that is congruent and oppositely oriented to the pencil $I(ABC)$. One shows easily, by considering the cyclic quadrilaterals $ABRI'_c$, $BCRI'_a$, that AR and RI'_a lie along the same line.

The line of centers C_2B_2 of the circles ABI'_c, ACI'_b is the perpendicular bisector of the common chord AR.

Remark. Because the pencils $R(ABC), I(ABC)$ are symmetric, their centers R, I are, in the terminology proposed by Mr. Artzt, *twin points* with respect to triangle ABC.[8]

6. *The axis of the perspective triangles $A_1B_1C_1$, $A_2B_2C_2$ is perpendicular to IR at its midpoint, and it touches the incircle of triangle ABC at this point.*[9]

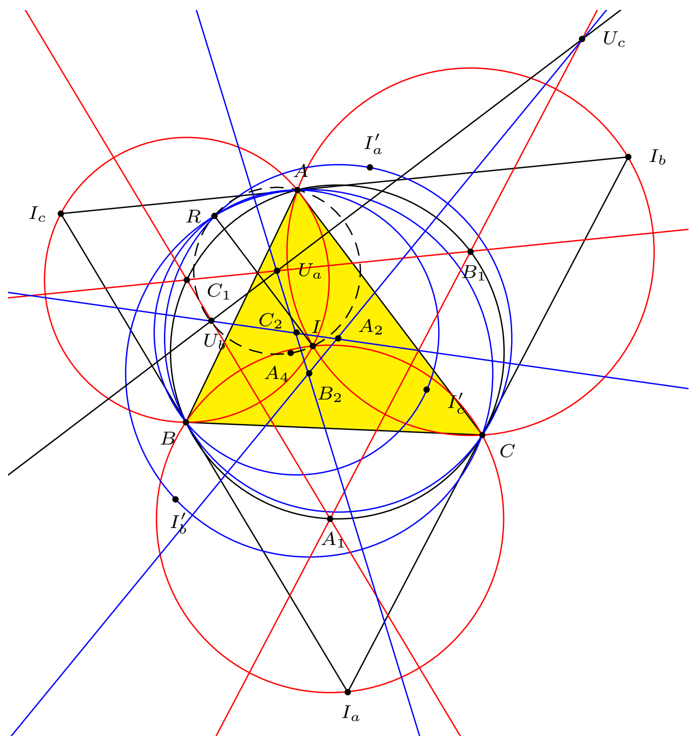

Figure 6. The proof of Theorems 6 and 7

Let U_a, U_b, U_c be the intersection points of the homologous sides of the two triangles; Since I is the orthocenter of both $A_1B_1C_1$ and $A_2B_2C_2$, B_1C_1 is the

[8]August Artzt (1835-1899) is known today for the Artzt parabolas of a triangle. (See, for example, [4, p.201].) Instead of *twin points*, Darij Grinberg uses the terminology *antigonal conjugates* [9]. Hatzipolakis and Yiu call these reflection conjugates; see [6, §3].

[9]The following Theorem 7 states that the midpoint of IR is what is now called the Feuerbach point of triangle ABC. The point R appears in [11] as X_{80}, which is called the *anti-Gray point* of triangle ABC in [8]. See the appendix for further information.

perpendicular bisector of IA and B_2C_2 the perpendicular bisector of IA_4; B_2C_2 is also the perpendicular bisector of RA. Thus, U_a is the center of a circle through the four points A, I, A_4, R; consequently, it lies on the perpendicular bisector of IR. Similarly, the points U_b, U_c lie on this line.

The lines AA_4, IR are symmetric with respect to B_2C_2; whence, $IR = 2r$. The midpoint Q of IR therefore lies on the incircle of ABC, and the tangent at this point is the line $U_aU_bU_c$.

7. *The incircle of ABC and the nine-point circle are tangent at the midpoint of IR.*

In the oppositely similar triangles $A_4B_4C_4$ and ABC, I is its own homologue and P corresponds to O: thus $IAO,\ IA_4P$ are homologous angles and, therefore, equal and oppositely oriented. As AI bisects angle A_4AO, the angles $IA_4P,\ IAA_4$ are directly equal, whence the radius PA_4 of circle P touches at A_4 circle AIA_4R, and the two circles are orthogonal. Circles $BIB_4R,\ CIC_4R$ are likewise orthogonal to circle P; moreover, the common points I, R harmonically divide a diameter of circle P. Consequently,

$$\overline{PH}^2 = PI \cdot PR = PR(PR - 2r);$$

from a known theorem,

$$\overline{PH}^2 = \overline{OI}^2 = OA(OA - 2r).$$

Comparison of these equations gives the relation $PR = OA$. Let O_9 be the center of the nine-point circle of triangle ABC; this point is the common midpoint of HO, IP (HP and IO are equal and parallel). Because Q is the midpoint of IR, the distance $O_9Q = \frac{1}{2}PR = \frac{1}{2}OA$. Thus, Q belongs to the circle O_9, and as it is on the line of centers of the circles I and O_9, these circles are tangent at Q.

Remark. The point R lies on the axis of the perspective triangles $A_2B_2C_2,\ A_4B_4C_4$ (no. **3**, *Remark*).

8. *Lemma*. If X is the harmonic conjugate of X_1 with respect to Y and Y_1, while M is the midpoint of XX_1 and N is the harmonic conjugate of Y with respect to X and M, then one can easily show that[10]

$$NY_1 \cdot NM = \overline{NX}^2.$$

9. *The lines A_2A_4, B_2B_4, C_2C_4 meet respectively BC, CA, AB in three points V_a, V_b, V_c situated on the common tangent to the circles I and O_9.*

Let D be the intersection of AI with BC, and α, α' the points where the circles I, I_a are tangent to BC. The projection of the harmonic set $(ADII_a)$[11] on BC is the harmonic set $(H_aD\alpha\alpha')$. Since A' is the midpoint of A_1A_2, one has a harmonic pencil $I(A_1A_2A'\alpha)$, whose section by BC is the harmonic set $(DV_aA'\alpha)$. The two

---

[10] One way to show this is with coordinates: With $-1, 1, y$ attached to the points X, X_1, Y we would have $Y_1 = \frac{1}{y}$, $M = 0$, and $N = \frac{-y}{2y+1}$.

[11] meaning A is the harmonic conjugate of D with respect to I and I_a.

sets $(H_aD\alpha\alpha')$, $(DV_aA'\alpha)$ satisfy the hypothesis of Lemma 8 because A' is the midpoint of $\alpha\alpha'$. Consequently,

$$\overline{V_a\alpha}^2 = V_aH_a \cdot V_aA';$$

whence, V_a is of equal power with respect to the circles I, O_9. This point belongs, therefore, to the common tangent through Q.

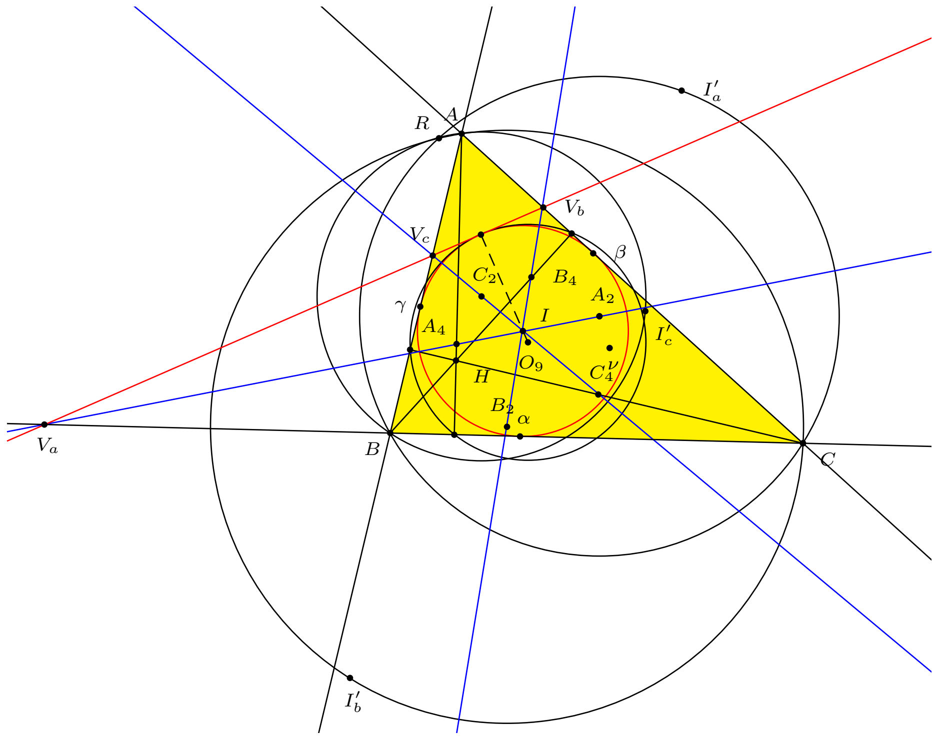

Figure 7. Theorem 9

Remark. From the preceding equality one deduces that

$$\overline{V_aI}^2 = V_aA_4 \cdot V_aA_2;$$

this says that the circle with center V_a, radius V_aI, is orthogonal to the circle P, a property that follows also from V_a being equidistant from the points I, R.

10. *The perpendiculars from A_1, B_1, C_1 to the opposite sides of triangle $A_2B_2C_2$ concur in a point S on the circle O. This point lies on the line $O\nu$ and is, with respect to triangle ABC, the homologue of the point ν with respect to triangle $A_4B_4C_4$.* [12]

[12] [Neuberg's footnote.] The normal coordinates of ν with respect to triangle $A_4B_4C_4$ are inversely proportional to the distances $\nu A_4, \nu B_4, \nu C_4$ or to the projections of OI on BC, CA, AB; namely, $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$. These are also the coordinates of S with respect to triangle ABC. (J.N.)

The first part of the theorem follows from the triangles $A_1B_1C_1$, $A_2B_2C_2$ being oppositely similar. One knows (§1) that the perpendiculars from the vertices of $A_2B_2C_2$ to the sides of $A_1B_1C_1$ concur at ν; thus ν, S are homologous points of $A_2B_2C_2$, $A_1B_1C_1$. P and O are likewise homologous points, and I corresponds to itself; consequently, $IP\nu$ and IOS are homologous angles, and as $PIO\nu$ is a parallelogram, the points O, ν, S are collinear.

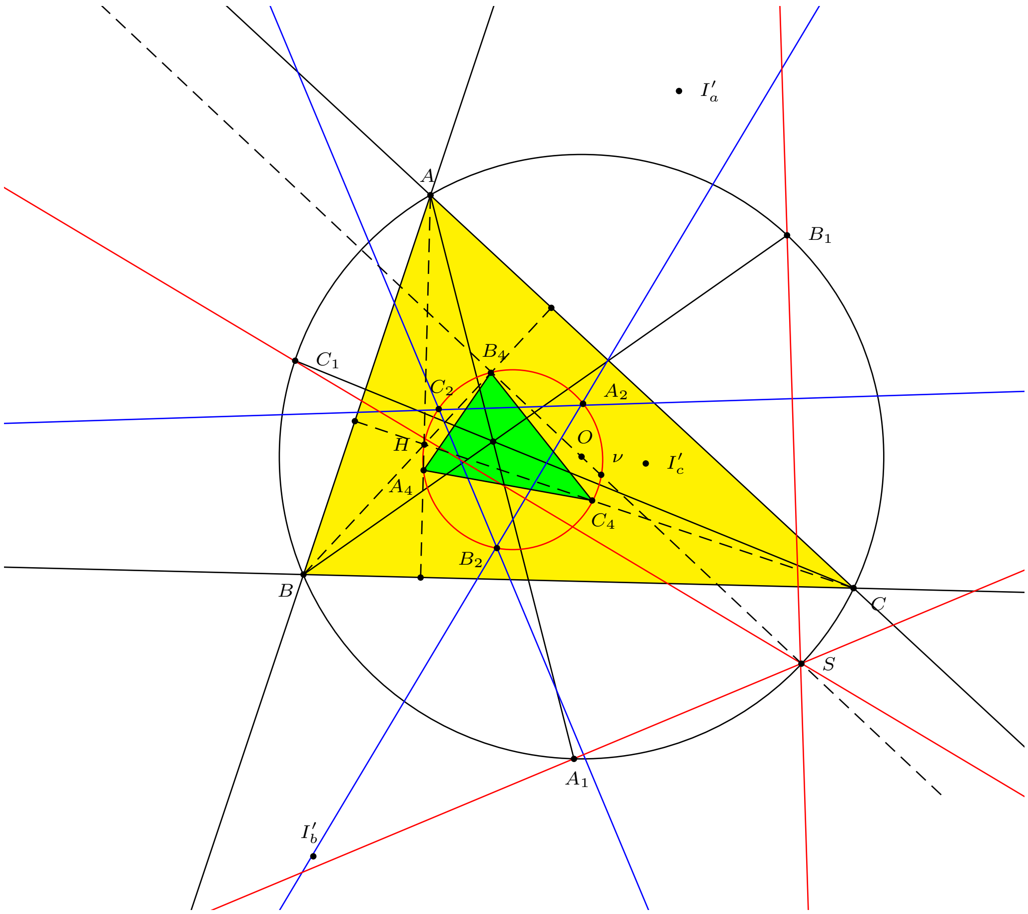

Figure 8. Theorem 10

11. *The perpendiculars from A, B, C to the opposite sides of triangle $A_4B_4C_4$ concur in a point T on the circle O; T and H are homologous points of the triangles ABC, $A_4B_4C_4$.*

The proof of this theorem is similar to the preceding proof.

Remarks. (1) The inverse [13] of the point S with respect to triangle ABC is the point at infinity in the direction perpendicular to the line $H\nu$. Indeed, let SS' be

[13]In today's terminology, the *inverse* of a point with respect to a triangle is called the *isogonal conjugate* of the point.

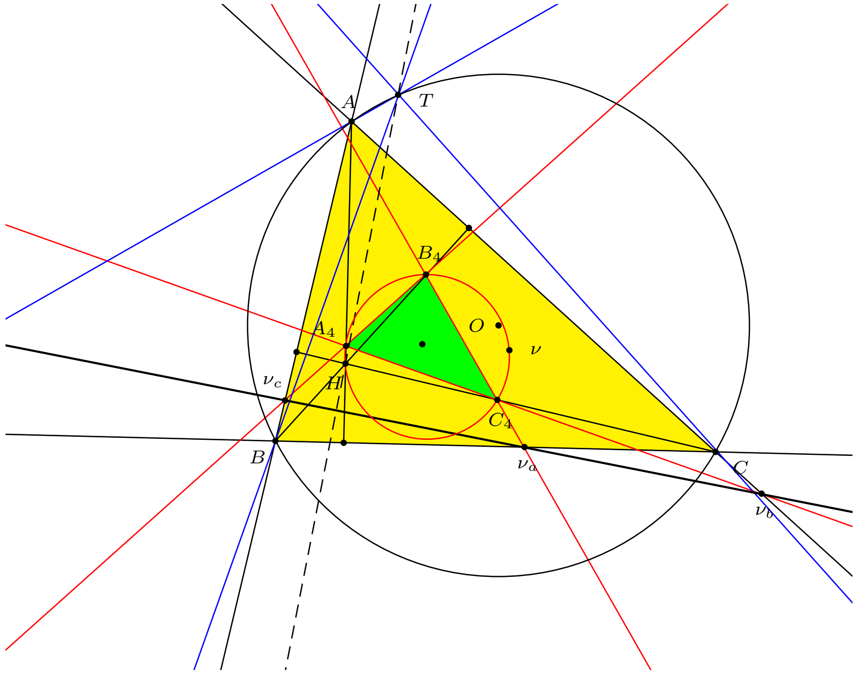

Figure 9. Theorems 11 and 12

the chord of circle O through S parallel to AB. In the oppositely similar figures $ABC, A_4B_4C_4$ the directions ST and $H\nu$, CT and C_4H correspond; thus angle $CH\nu = STC = SS'C$, and as SS' is perpendicular to CH, it follows that CS' is perpendicular to $H\nu$. But, CS' is the isogonal of CS, and the claim follows.

(2) Because $H\nu$ is a diameter of circle $A_4B_4C_4$, S and T are the ends of the diameter of circle ABC determined by the line $O\nu$. It follows that the lines AS, BS, CS are parallel to B_4C_4, C_4A_4, A_4B_4.

12. *The axis of the perspective triangles $ABC, A_4B_4C_4$ is perpendicular to the line HT.*

Let ν_a, ν_b, ν_c be the intersections of the corresponding sides of these triangles. The triangles $THB, CC_4\nu_a$ have the property that the perpendiculars from the vertices of the former to the sides of the latter concur at the point A; thus, the perpendiculars from C, C_4, ν_a, respectively on HB, TB, TH also concur; the first two of these lines meet at ν_b; thus, $\nu_a\nu_b$ is perpendicular to TH. [14]

More simply, the two complete quadrangles $ABHT, \nu_b\nu_aC_4C$ have five pairs of perpendicular sides; thus the remaining pair $HT, \nu_a\nu_b$ is perpendicular as well.

13. *The double lines of the oppositely similar triangles $ABC, A_4B_4C_4$ meet the altitudes of ABC in points (X_a, X_b, X_c), (X'_a, X'_b, X'_c) such that*

$$AX_a = BX_b = CX_c = R - d, \quad AX'_a = BX'_b = CX'_c = R + d,$$

[14] THB and $CC_4\nu_a$ are called *orthologic triangles*; see, for example, Dan Pedoe, *Geometry: A Comprehensive Course*, Dover (1970), Section 8.3.

where R is the circumradius and $d = OI$.

Indeed, the similar triangles ABC and $A_4B_4C_4$ are in the ratio of their circumradii, namely $R : d$; since IX_a bisects angle AIA_4, one has successively,

$$\frac{AX_a}{X_aA_4} = \frac{R}{d}, \qquad \frac{AX_a}{AX_a + X_aA_4} = \frac{R}{R+d}, \qquad AX_a = \frac{2Rr}{R+d} = R - d.$$

Analogously, $AX'_a = R + d$.

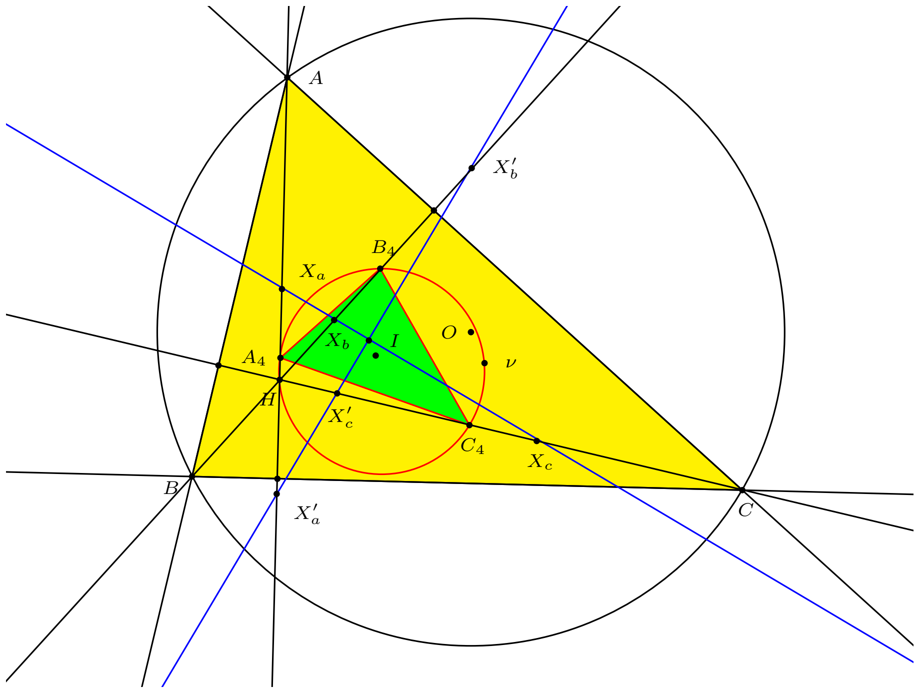

Figure 10. Theorem 13

Remarks. (1) Proposition 13 resolves the following problem: Take on the altitudes AH, BH, CH three equal segments $AX_a = BX_b = CX_c$ such that the points X_a, X_b, X_c are collinear. One sees that there exist two lines $X_aX_bX_c$ that satisfy these conditions; they are perpendicular and pass through the incenter of ABC.

(2) Let us take the lengths $AY_a = BY_b = CY_c = OA$ on AH, BH, CH. The triangles $Y_aY_bY_c,\ A_2B_2C_2$ are symmetric with respect to O_9.

Appendix

Comments on Theorem 4. Here is an alternative proof that I *is the center of the opposite similarity that takes triangle* $A_1B_1C_1$ *to triangle* $A_2B_2C_2$ (given that they are oppositely similar by Theorem 1). The isosceles triangles OA_1C and CA_2A_1 are similar because they share the angle at A_1; thus $\frac{OA_1}{A_1C} = \frac{A_1C}{A_1A_2}$. But $A_1C = A_1I$, so $\frac{OA_1}{A_1I} = \frac{A_1I}{A_1A_2}$. Because the angle at A_1 is shared, triangles OIA_1

and IA_2A_1 are similar (by SAS). Consequently,

$$\frac{IA_1}{IA_2} = \frac{OA_1}{OI} = \frac{R}{OI},$$

where R is the circumradius of triangle ABC. Similarly $\frac{R}{OI} = \frac{IB_1}{IB_2} = \frac{IC_1}{IC_2}$, whence I is the fixed point of the similarity, as claimed. Note that this argument proves, again, that the radius of the Fuhrmann circle $A_2B_2C_2$ equals OI.

The figure used in our argument suggests a way to construct the geometric mean of two given segments as a one-step-shorter alternative to the method handed down to us by Euclid. Copy the given lengths AC and BC along a line with B between A and C: Construct the perpendicular bisector of BC and call D either point where it meets the circle with center A and radius AC. Then CD is the geometric mean of AC and BC (because, $\frac{AC}{CD} = \frac{CD}{BC}$).

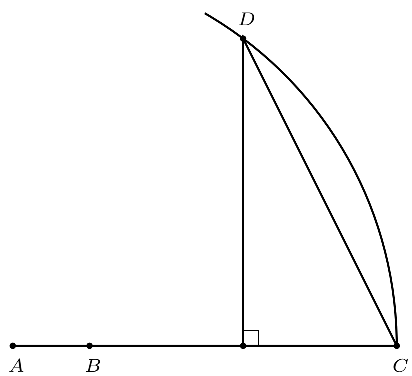

Figure 11. CD is the geometric mean of AC and BC

Further comments on the point R from Theorems 5 through 7.

From the definition of twin points in Section 5 we can immediately deduce that the twin of a point D with respect to triangle ABC is the point common to the three circles ABD_c, BCD_a, and CAD_b, where D_c, D_a, and D_b are the reflections of D in the sides of the triangle. From this construction we see that except for the orthocenter (whose reflection in a side of the triangle lies on the circumcircle), each point in the plane is paired with a unique twin. The point R, which by Theorem 5 can be defined to be the incenter's twin point (or the incenter's antigonal conjugate if you prefer Grinberg's terminology), is listed as X_{80} in [11], where it is defined to be the reflection of the incenter in the Feuerbach point (which agrees with Fuhrmann's Theorem 7).

Numerous properties of twin points are listed by Grinberg in his note [9]. There he quotes that the twin of a point D with respect to triangle ABC is the isogonal conjugate of the inverse (with respect to the circumcircle) of the isogonal conjugate of D. One can use that property to help show that O (the circumcenter of triangle ABC) and R are isogonal conjugates with respect to the Fuhrmann triangle $A_2B_2C_2$ as follows: Indeed, we see that P (the center of the Fuhrmann circle) and O are twin points with respect to the Fuhrmann triangle because $\angle B_2OC_2 =$

$\angle B_1OC_1 = -\angle B_2PC_2$, where the last equality follows from the opposite similarity of triangles $A_1B_1C_1$ and $A_2B_2C_2$; similarly $\angle A_2OC_2 = -\angle A_2PC_2$, and $\angle A_2OB_2 = -\angle A_2PB_2$. Because P is the circumcenter and I the orthocenter of triangle $A_2B_2C_2$, they are isogonal conjugates. According to the proof of Theorem 7, R is the inverse of I with respect to the Fuhrmann circle. It follows that R and O are isogonal conjugates with respect to the Fuhrmann triangle $A_2B_2C_2$.

The triangle $\alpha\beta\gamma$, whose vertices (which appeared in Sections 3 and 9) are the points where the incircle of triangle ABC touches the sides, is variously called the *Gergonne*, or *intouch*, or *contact triangle of triangle* ABC. A theorem attributed to the Japanese mathematician Kariya Yosojirou says that *for any real number* k*, if* XYZ *is the image of the contact triangle under the homothecy* $\mathsf{h}(I, k)$*, then triangles* ABC *and* XYZ *are perspective*. [15] Grinberg [8] calls their perspective center the k*-Kariya point of triangle* ABC. The 0-Kariya point is I and the 1-Kariya point is the Gergonne point X_7, while the -1-Kariya point is the Nagel point X_8, called ν by Fuhrmann. We will prove that R is the -2-Kariya point: In this case $|IX| = 2r = |AA_4|$ and $IX \parallel AA_4$, so it follows that $IA_4 \parallel XA$, which implies (by the final claim of Theorem 5) that $X \in AR$.

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[15] This result was independently discovered on at least three occasions: First by Auguste Boutin [2, 3], then by V. Retali [12], and then by J. Kariya [10]. Kariya's paper inspired numerous results appearing in *L'Enseignement mathématique* over the following two years. We thank Hidetoshi Fukagawa for his help in tracking down these references.

Forum Geometricorum
Volume 11 (2011) 27–51.

FORUM GEOM
ISSN 1534-1178

Triangles with Given Incircle and Centroid

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Abstract. In this article we study the set of triangles sharing a common incircle and a common centroid. It is shown that these triangles have their vertices on a conic easily constructible from the given data. It is also shown that the circumcircles of all these triangles are simultaneously tangent to two fixed circles, hence their centers lie also on a conic.

1. Introduction

The object of study here is on triangles sharing a common incircle and centroid. It belongs to the wider subject of (one-parameter) families of triangles, initiated by the notion of *poristic* triangles, which are triangles sharing a common incircle and also a common circumcircle [7, p.22]. There is a considerable literature on poristic triangles and their variations, which include triangles sharing the same circumcircle and Euler circle [16], triangles sharing the same incircle and Euler circle [6], or incircle and orthocenter [11, vol.I, p.15], or circumcircle and centroid [8, p.6]. The starting point of the discussion here, and the content of §2, is the exploration of a key configuration consisting of a circle c and a homothety f. This configuration generates in a natural way a homography H and the conic $c^* = H(c)$. §3 looks a bit closer at homography H and explores its properties and the properties of the conic c^*. §4 uses the results obtained previously to the case of the incircle and the homothety with ratio $k = -2$ centered at the centroid G of the triangle of reference to explore the kind of the conic $c^{**} = f(c^*)$ on which lie the vertices of all triangles of the poristic system. §5 discusses the locus of circumcenters of the triangles of the poristic system locating, besides the assumed fixed incircle c, also another fixed circle c_0, called *secondary*, to which are tangent all Euler circles of the system. Finally, §6 contains miscellaneous facts and remarks concerning the problem.

2. The basic configuration

The basic configuration of this study is a circle c and a homothety f centered at a point G and having a ratio $k \neq 1$. In the following the symbol p_X throughout denotes the polar line of point X with respect to c, this line becoming tangent to c when $X \in c$. Lemma 1 handles a simple property of this configuration.

Lemma 1. *Let c be a circle and G a point. Let further D be a point on the circle and L a line parallel to the tangent p_D at D and not intersecting the circle. Then there is exactly one point A on L such that the tangents $\{t, t'\}$ to c from A and line AG intersect on p_D equal segments $BM = MC$.*

Publication Date: February 25, 2011. Communicating Editor: Paul Yiu.

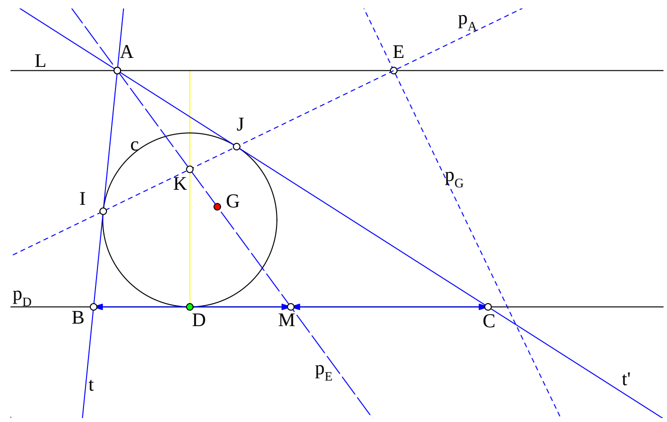

Figure 1. A basic lemma

Figure 1 suggests a proof. Let $\{E, A\}$ be the intersections of lines $E = L \cap p_G$ and $A = L \cap p_E$. Draw from A the tangents $\{t = AI, t' = AJ\}$ to c intersecting line p_D respectively at B and C. Let also M be the intersection $M = p_D \cap AG$. Then $BM = MC$. This follows immediately from the construction, since by the definitions made the pencil of lines $A(I, J, K, E)$ at A is harmonic. Thus, this pencil defines on p_D a harmonic division and, since the intersection of lines $\{p_D, L\}$ is at infinity, M is the middle of BC. Conversely, the equality $BM = MC$ implies that the pencil of four lines AB, AC, AM, L is harmonic. If E is the intersection $E = L \cap p_A$ then p_E coincides with AM and $G \in p_E \Rightarrow E \in p_G$. Thus A is uniquely determined as $A = L \cap p_E$, where $E = L \cap p_G$.

Remarks. (1) Point K, defined as the intersection $K = AG \cap p_A$, is on the diameter of c through D. This follows from the fact that K is on the polar of A and also on the polar of E. Hence its polar is $p_K = AE$.

(2) In other words the lemma says that there is exactly one triangle with vertex $A \in L$ such that the circumscribed to c triangle ABC has AG as median for side BC. Consequently if this happens also for another vertex and BG is the median also of CA then G coincides with the centroid of ABC and the third line CG is also a median of this triangle.

The next proposition explores the locus of point $M = M(D)$ uniquely determined from D in the previous lemma by letting D vary on the circle and taking the parallel L to p_D to be identical with the homothetic to p_D line $L_D = f(p_D)$ (see Figure 2).

Proposition 2. *Consider a circle c and a homothety f with ratio $k \neq 1$ and center at a point G. For each point $D \in c$ let $L_D = f(p_D)$ be the homothetic image of p_D and $E = L_D \cap p_G,\ M = p_E \cap p_D$. Then M describes a conic c^* as D moves on the circle.*

The proof of the proposition is given using homogeneous cartesian coordinates with the origin set at G and the x-axis identified with line IG, where I is the center

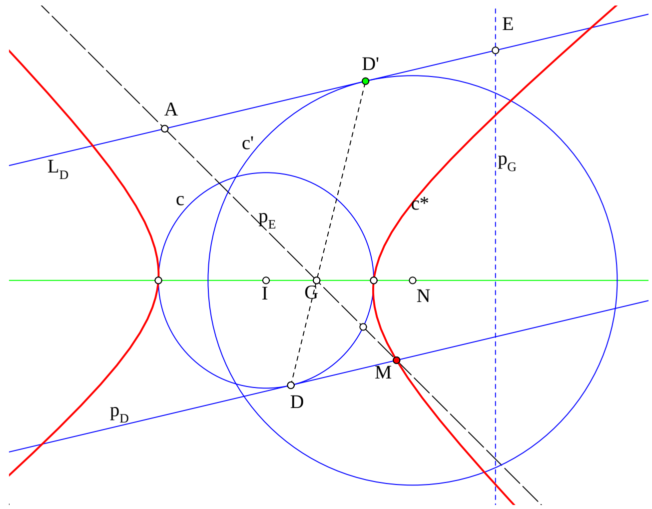

Figure 2. Locus of M=$p_D \cap p_E$

of the given circle c. In such a system circle c is represented by the equation

$$x^2+y^2-2uxz+(u^2-r^2)z^2=0 \Longleftrightarrow \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 & 0 & -u \\ 0 & 1 & 0 \\ -u & 0 & u^2-r^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0.$$

Here $(u,0)$ are the corresponding cartesian coordinates of the center and r is the radius of the circle. The polar p_G of point $G(0,0,1)$ is computed from the matrix and is represented by equation

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -u \\ 0 & 1 & 0 \\ -u & 0 & u^2-r^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \Longleftrightarrow -ux+(u^2-r^2)z=0.$$

The homothety f is represented by the matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Hence the tangent p_D at $D(x,y,z)$ and its image L_D under f have correspondingly coefficients

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 & 0 & -u \\ 0 & 1 & 0 \\ -u & 0 & u^2-r^2 \end{pmatrix}=\begin{pmatrix} x-uz & y & -ux+(u^2-r^2)z \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{1}{k}(x-uz) & \frac{1}{k}y & -ux+(u^2-r^2)z \end{pmatrix}.$$

The homogeneous coordinates of $E=p_G \cap L_D$ are then computed through the vector product of the coefficients of these lines which is:

$$\begin{pmatrix} -\frac{1}{k}y(u^2-r^2), & u(-ux+(u^2-r^2)z)+\frac{1}{k}(x-uz)(u^2-r^2), & -\frac{u}{k}y \end{pmatrix}.$$

The coefficients of the polar p_E are then seen to be

$$\begin{aligned}
&\begin{pmatrix} -\frac{1}{k}y(u^2-r^2) & u(-ux+(u^2-r^2)z)+\frac{1}{k}(x-uz)(u^2-r^2) & -\frac{u}{k}y \end{pmatrix} \begin{pmatrix} 1 & 0 & -u \\ 0 & 1 & 0 \\ -u & 0 & u^2-r^2 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{k}y(u^2-r^2)+\frac{u^2}{k}y & u(-ux+(u^2-r^2)z)+\frac{1}{k}(x-uz)(u^2-r^2) & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{k}yr^2 & u(-ux+(u^2-r^2)z)+\frac{1}{k}(x-uz)(u^2-r^2) & 0 \end{pmatrix} \\
&\cong \begin{pmatrix} yr^2 & (u^2(1-k)-r^2)x+u(u^2-r^2)(k-1)z & 0 \end{pmatrix}.
\end{aligned}$$

The homogeneous coordinates of $M = p_E \cap p_D$ are again computed through the vector product of the coefficients of the corresponding lines:

$$\begin{aligned}
&\begin{pmatrix} x-uz \\ y \\ -ux+(u^2-r^2)z \end{pmatrix} \times \begin{pmatrix} yr^2 \\ (u^2(1-k)-r^2)x+u(u^2-r^2)(k-1)z \\ 0 \end{pmatrix} \\
=\ &\begin{pmatrix} ((u^2(1-k)-r^2)x+u(u^2-r^2)(k-1)z)(ux-(u^2-r^2)z) \\ (-yr^2)(ux-(u^2-r^2)z) \\ (-ux+(u^2-r^2)z)(r^2z+u(1-k)(uz-x)) \end{pmatrix} \\
\cong\ &\begin{pmatrix} (u^2(1-k)-r^2)x+u(u^2-r^2)(k-1)z \\ -yr^2 \\ u(1-k)x-(r^2+u^2(1-k))z \end{pmatrix} \\
=\ &\begin{pmatrix} u^2(1-k)-r^2 & 0 & u(u^2-r^2)(k-1) \\ 0 & -r^2 & 0 \\ u(1-k) & 0 & -(r^2+u^2(1-k)) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}$$

The matrix product in the last equation defines a homography H and shows that the locus of points M is the image $c^* = H(c)$ of the circle c under this homography. This completes the proof of Proposition 2

Remarks. (3) The properties of the conic c^* are of course tightly connected to the properties of the homography H appearing at the end of the proposition and denoted by the same letter

$$H = \begin{pmatrix} u^2(1-k)-r^2 & 0 & u(u^2-r^2)(k-1) \\ 0 & -r^2 & 0 \\ u(1-k) & 0 & -(r^2+u^2(1-k)) \end{pmatrix}.$$

These properties will be the subject of study in the next section.

(4) Composing with the homothety f we obtain the locus of $A = f(M)$ which is the homothetic conic $c^{**} = f(c^*) = (f \circ H)(c)$. It is this conic rather, than c^*, that relates to our original problem. Since though the two conics are homothetic, work on either leads to properties for both of them.

(5) Figure 2 underlines the symmetry between these two conics. In it c' denotes the homothetic image $c' = f(c)$ of c. Using this circle instead of c and the inverse homothety $g = f^{-1}$ we obtain a basic configuration in which the roles of c^* and c^{**} are interchanged.

(6) If $D \in c^*$ and point $A = f(M)$ is outside the circle c then triangle ABC constructed by intersecting p_D with the tangents from A has, according to the lemma, line AG as median. Inversely, if a triangle ABC is circumscribed in c and has its median AM passing through G and divided by it in ratio $\frac{GA}{GM} = k$, then, again according to the lemma, it has M on the conic c^*. In particular if a triangle ABC is circumscribed in c and has two medians passing through G then it has all three of them passing through G, the ratio $k = -2$ and the middles of its sides are points of the conic c^*.

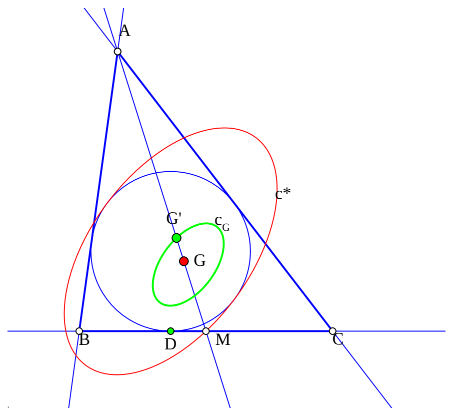

Figure 3. Locus of centroids G'

(7) The previous remark implies that if ratio $k \neq -2$ then every triangle ABC circumscribed in c and having its median AM passing through G has its other medians intersecting AG at the same point $G' \neq G$. Figure 3 illustrates this remark by displaying the locus of the centroid G' of triangles ABC which are circumscribed in c, their median AM passes through G and is divided by it in ratio k but later does not coincide with the centroid (*i.e.*, $k \neq -2$). It is easily seen that the locus of the centroid G' in such a case is part of a conic c_G which is homothetic to c^* with respect to G and in ratio $k' = \frac{2+k}{3}$. Obviously for $k = -2$ this conic collapses to the point G and triangles like ABC circumscribed in c have all their vertices on c^{**}.

(8) Figure 4 shows a case in which the points $A \in c^{**}$ which are outside c fall into four connected arcs of a hyperbola. In this example $k = -2$ and, by the symmetry of the condition, it is easily seen that if a point A is on one of these arcs then the other vertices of ABC are also on respective arcs of the same hyperbola. The fact to notice here is that conics c^* and c^{**} are defined directly from the basic configuration consisting of the circle c and the homothety f. It is though not possible for every point of c^{**} to be vertex of a triangle circumscribed in c and with centroid at G. I summarize the results obtained so far in the following proposition.

Proposition 3. *All triangles ABC sharing the same incircle c and centroid G have their side-middles on a conic c^* and their vertices on a conic $c^{**} = f(c^*)$ homothetic to c^* by the homothety f centered at the centroid with ratio $k = -2$.*

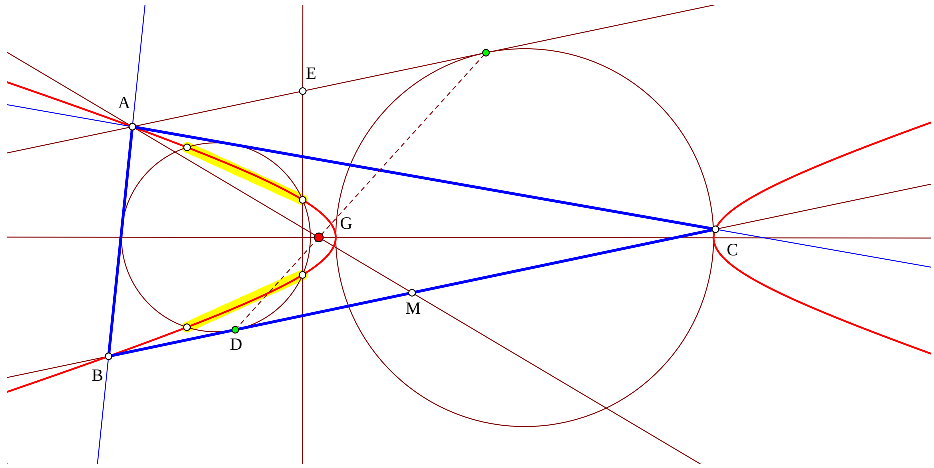

Figure 4. Arcs: $A = f(M) \in c^{**}$ and A outside c

The proof follows from the previous remarks and the fact that in this setting the (anti-)homothety f has center at G and ratio $k = -2$. By Lemma 5 below, the median AM of side BC coincides with the polar p_E of point $E = L \cap p_G$, where L is the parallel to BC from A. The conic c^* generated by point M, as in Lemma 5 , passes now through the middle of BC but also through the middles of the other sides, since the configuration, as already remarked, is independent of the preferred vertex A and depends only on the circle c and the homothety f (see Figure 5).

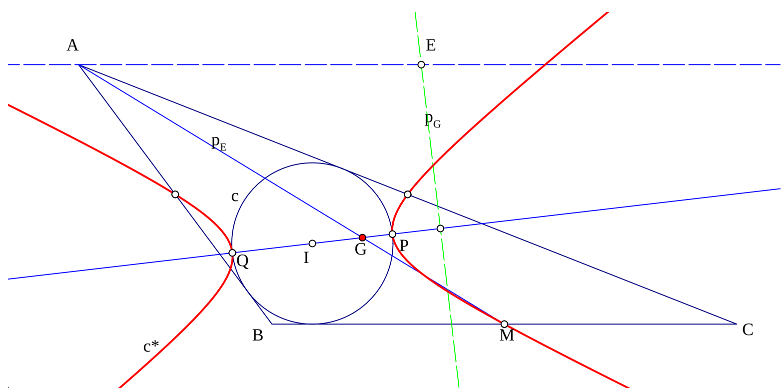

Figure 5. Conic c^*

The conic $c^{**} = f(c^*)$ which is homothetic to c^* will pass through all three vertices of triangle ABC. Since the same argument can be applied to any triangle sharing with ABC the same incircle c and centroid G it follows that all these triangles have their vertices on this conic. Figure 6 displays two triangles sharing the same incircle c and centroid G and having their vertices on conic c^{**} in a case in which this happens to be elliptic. Note that in this case the entire conic consists of vertices of triangles circumscribing c and having their centroid at G. This follows from the fact that in this case c^{**} does not intersect c (see §4) and by applying then the Poncelet's porism [3, p.68].

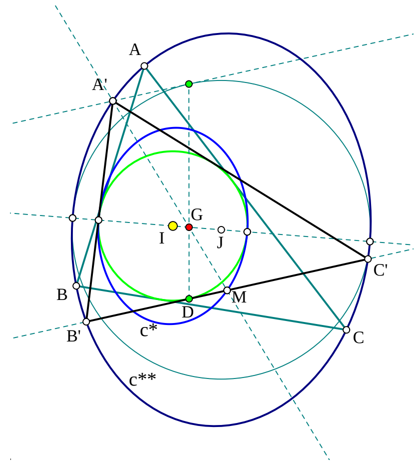

Figure 6. The two conics c^* and c^{**}

Before going further into the study of these conics I devote the next section to a couple of remarks concerning the homography H.

3. The homography H

Proposition 4. *The homography H defined in the previous section is uniquely characterized by its properties:*
(i) *H fixes points $\{P, Q\}$ which are the diameter points of c on the x-axis,*
(ii) *H maps points $\{S, T\}$ to $\{S', T'\}$. Here $\{S, T\}$ are the diameter points of c on a parallel to the y-axis and $\{S', T'\}$ are their orthogonal projections on the diameter of $c' = f(c)$ which is parallel to the y-axis.*

The proof of the proposition (see Figure 7) follows by applying the matrix to the coordinate vectors of these points which are $P(u+r, 0, 1)$, $Q(u-r, 0, 1)$, $S(u, r, 1)$, $T(u, -r, 1)$,
$S'(ku, r, 1)$ and $T'(ku, -r, 1)$, and using the well-known fact that a homography is uniquely determined by prescribing its values at four points in general position [2, Vol.I, p.97].

Remarks. (1) By its proper definition, line XX' for $X \in c$ and $X' = H(X) \in c^* = H(c)$ is tangent to circle c at X.

(2) The form of the matrix H implies that the conic c^* is symmetric with respect to the x-axis and passes through points P and Q having there tangents coinciding respectively with the tangents of circle c. Thus P and Q are vertices of c^* and c coincides with the *auxiliary* circle of c^* if this is a hyperbola. In the case c^* is an ellipse, by the previous remark, follows that it lies entirely outside c hence later is the maximal circle inscribed in the ellipse.

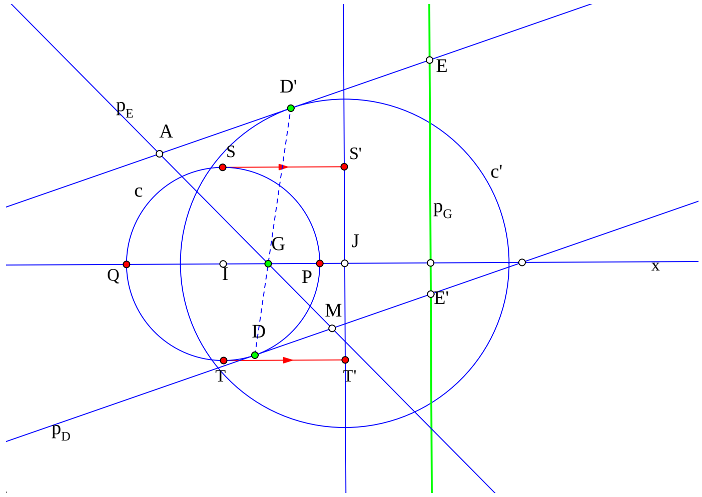

Figure 7. Homography $H : \{P, Q, S, T\} \mapsto \{P, Q, S', T'\}$

(3) The kind of c^* depends on the location of the line L_0 mapped to the line at infinity by H.

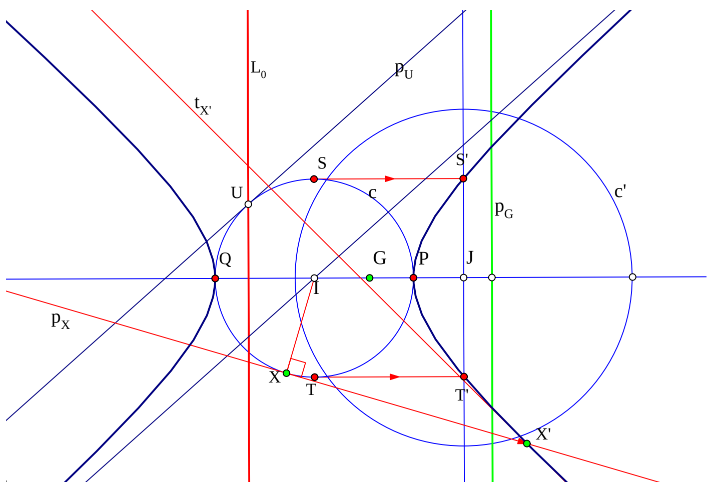

Figure 8. XX' is tangent to c

This line is easily determined by applying H to a point (x, y, z) and requiring that the resulting point (x', y', z') has $z' = 0$, thus leading to the equation of a line parallel to y-axis:

$$u(1 - k)x - (r^2 + u^2(1 - k))z = 0.$$

The conic c^*, depending on the number n of intersection points of L_0 with circle c, is a hyperbola $(n = 2)$, ellipse $(n = 0)$ or becomes a degenerate parabola consisting of the pair of parallel tangents to c at $\{P, Q\}$.

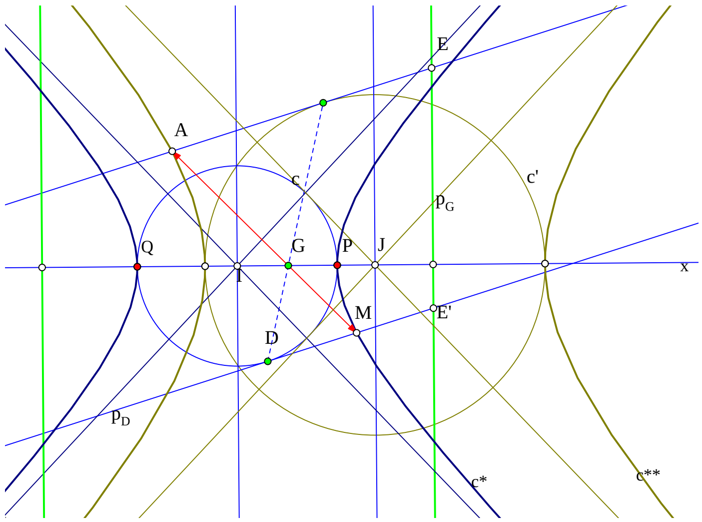

Figure 9. Homothetic conics c^* and c^{**}

(4) The tangent p_X of the circle c at X maps via H to the tangent $t_{X'}$ at the image point $X' = H(X)$ of c^*. In the case c^* is a hyperbola this implies that each one of its asymptotes is parallel to the tangent p_U of c at an intersection point $U \in c \cap L_0$ (see Figure 8).

(5) Figure 9 displays both conics c^* and $c^{**} = f(c^*)$ in a case in which these are hyperbolas and suggests that circle c is tangent to the asymptotes of c^{**}. This is indeed so and is easily seen by first observing that H maps the center I of c to the center J of $c' = f(c)$. In fact, line ST maps by H to $S'T'$ (see Figure 10) and line PQ is invariant by H. Thus $H(I) = J$. Thus all lines through I map under H to lines through J. In particular the antipode U' of U on line IU maps to a point $H(U')$ on the tangent to U' and U maps to the point at infinity on p_U. Thus line UU' maps to $U'J$ which coincides with $p_{U'}$ and is parallel to the asymptote of c^*. Since c^* and c^{**} are homothetic by f line $U'J$ is an asymptote of c^{**} thereby proving the claim.

(6) The arguments of the last remark show that line L_0 is the symmetric with respect to the center I of c (see Figure 10) of line $U'V'$ which is the polar of point J with respect to circle c. They show also that the intersection point K of $U'V'$ with the x-axis is the image via H of the axis point at infinity. An easy calculation using the matrices of the previous section shows that these remarks about the symmetry of $\{UV, p_J\}$ and the location of K is true also in the cases in which J is inside c and there are no real tangents from it to c.

(7) The homography H demonstrates a remarkable behavior on lines parallel to the coordinate axes (see Figure 11). As is seen from its matrix it preserves the point at infinity of the y-axis hence permutes the lines parallel to this axis. In particular, the arguments in (5) show that the parallel to the y-axis from I is simply orthogonally projected onto the parallel through point J. More generally points X moving on a parallel L to the y-axis map via H to points $X' \in L' = H(L)$,

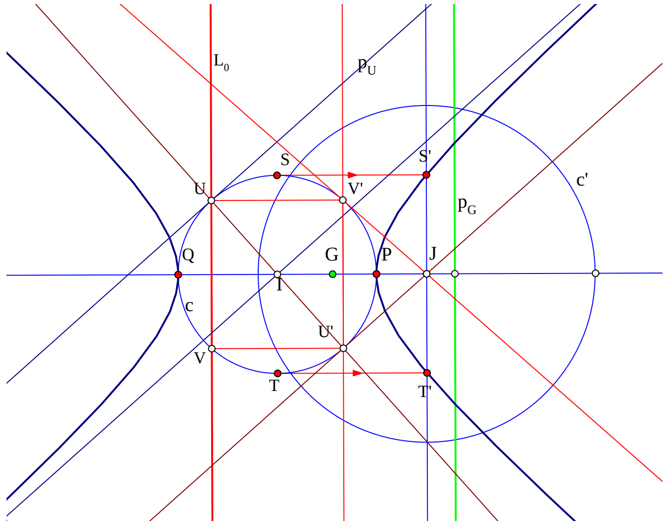

Figure 10. Location of asymptotes

such that the line XX' passes through a point X_L depending only on L and being harmonic conjugate with respect to $\{P, Q\}$ to the intersection X_0 of L with the x-axis.

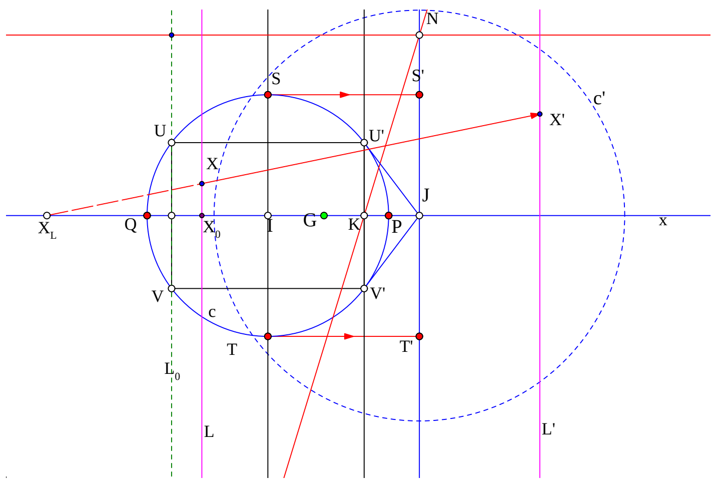

Figure 11. Mapping parallels to the axes

(8) Regarding the parallels to the x-axis and different from it their images via H are lines passing through K and also through their intersection N with the parallel to y-axis through J (see Figure 11). The x-axis itself is invariant under H and the action of this map on it is completely determined by the triple of points $\{P, Q, I\}$ and their images $\{P, Q, J\}$. Next lemma and its corollary give an insight into the difference of a general ratio $k \neq -2$ from the centroid case, in which $k =$

-2, by focusing on the behavior of the tangents to c from $A = f(M)$ and their intersections with the variable tangent p_D of circle c.

Lemma 5. *Consider a hyperbola c^* and its auxiliary circle c. Draw two tangents $\{t, t'\}$ parallel to the asymptotes intersecting at a point J. Then every tangent p to the auxiliary circle intersects lines $\{t, t'\}$ correspondingly at points $\{Q, R\}$ and the conic at two points $\{S, T\}$ of which one, S say, is the* middle *of QR.*

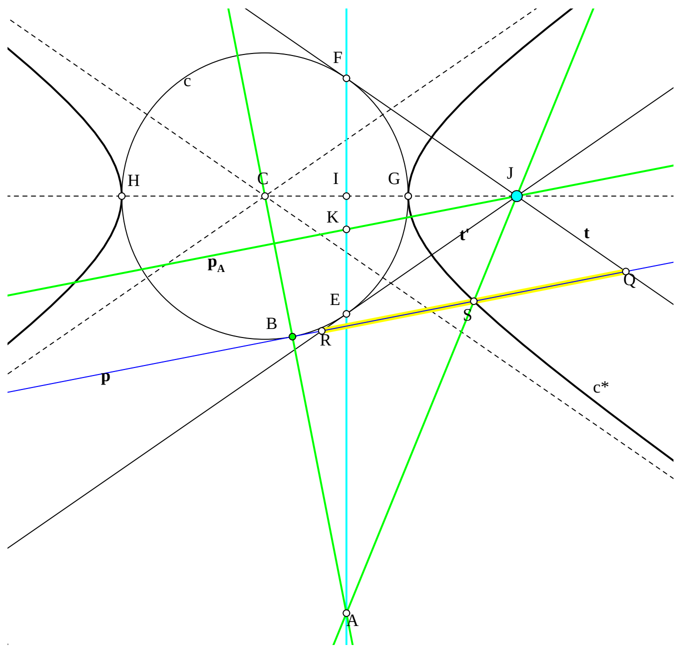

Figure 12. Hyperbola property

The proof starts by defining the polar FE of J with respect to the conic c^*. This is simultaneously the polar of J with respect to circle c since $(G, H, I, J) = -1$ are harmonic with respect to either of the curves (see Figure 12). Take then a tangent of c at B as required and consider the intersection point A of CB with the polar EF. The polar p_A of A with respect to the circle passes through J (by the reciprocity of relation pole-polar) and is parallel to tangent p. Besides $(E, F, K, A) = -1$ build a harmonic division, thus the pencil of lines at $J : J(E, F, K, A)$ defines a harmonic division on every line it meets. Apply this to the tangent p. Since JK is parallel to this tangent, S is the harmonic conjugate with respect to $\{R, Q\}$ of the point at infinity of line p. Hence it is the middle of RQ.

Corollary 6. *Under the assumptions of Lemma 5 , in the case conic c^* is a hyperbola, point $M = c^* \cap p_D$ is the common middle of segments $\{NO, KL, BC\}$ on the tangent p_D of circle c. Of these segments the first NO is intercepted by the asymptotes of c^{**}, the second KL is intercepted by the conic c^{**} and the third BC is intercepted by the tangents to c from A.*

The proof for the first segment NO follows directly from Lemma 5. The proof for the second segment KL results from the well known fact [4, p.267], according to which KL and NO have common middle for every secant of the hyperbola. The proof for the third segment BC follows from the definition of A and its properties as these are described by Lemma 1.

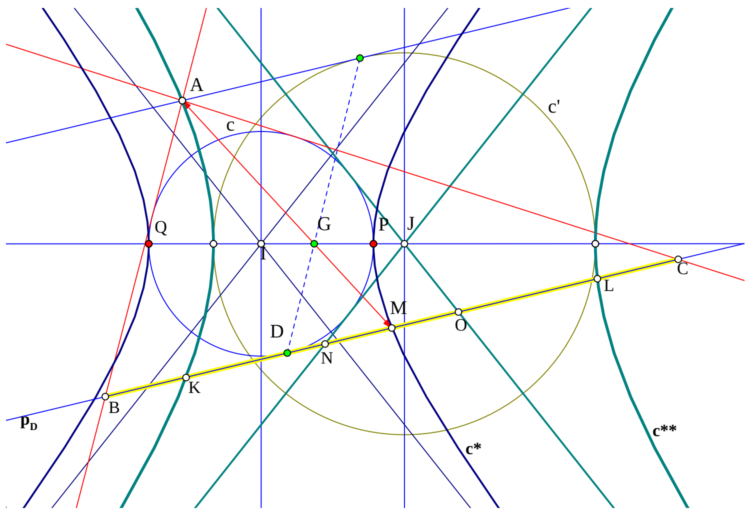

Figure 13. Common middle M

Remark. (9) When the homothety ratio $k = -2$ point B coincides with K (see Figure 13) and point L coincides with C. Inversely if $B \equiv K$ and $C \equiv L$ are points of c^{**} then the corresponding points $f^{-1}(B) \in AC$ and $f^{-1}(C) \in AB$ are middles of the sides, $k = -2$ and G is the centroid of ABC. It even suffices one triangle satisfying this identification of points to make this conclusion.

4. The kind of the conic c^*

In this section I examine the kind of the conic c^* by specializing the remarks made in the previous sections for the case of ratio $k = -2$, shown equivalent to the fact that there is a triangle circumscribed in c having its centroid at G, its side-middles on conic c^* and its vertices on $c^{**} = f(c^*)$. First notice that circle $c' = f(c)$ is the Nagel circle ([13]), its center $J = f(I)$ is the *Nagel* point ([10, p. 8]), its radius is twice the radius r of the incircle and it is tangent to the circumcircle. Line IG is the *Nagel* line of the triangle ([15]). The homography H defined in the second section obtains in this case $(k = -2, u = GI)$ the form.

$$H = \begin{pmatrix} 3u^2 - r^2 & 0 & -3u(u^2 - r^2) \\ 0 & -r^2 & 0 \\ 3u & 0 & -(r^2 + 3u^2) \end{pmatrix}.$$

As noticed in §2 the line L_0 sent to the line at infinity by H is orthogonal to line IG and its x-coordinate is determined by

$$x_0 = \frac{r^2 + 3u^2}{3u},$$

where $|u|$ is the distance of G from the incenter I. The kind of conic c^{**} is determined by the location of x_0 relative to the incircle. In the case $|x_0 - u| < r \iff |u| > \frac{r}{3}$ we obtain hyperbolas. In the case $|u| < \frac{r}{3}$ we obtain ellipses and in the case $|u| = \frac{r}{3}$ we obtain two parallel lines orthogonal to line GI. Since the hyperbolic case was discussed in some extend in the previous sections here I examine the two other cases.

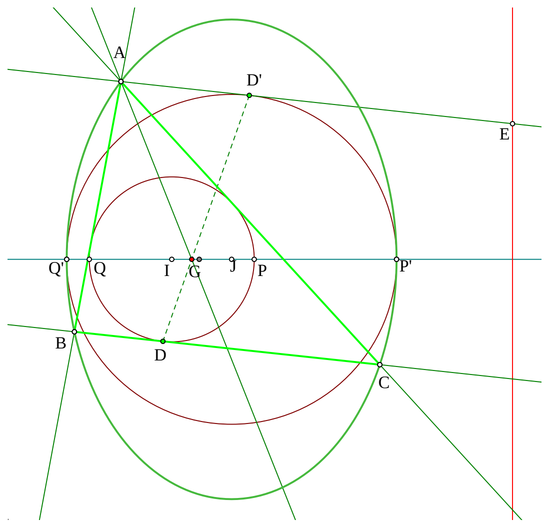

Figure 14. The elliptic case

First, the elliptic case characterized by the condition $u < \frac{r}{3}$ and illustrated by Figure 14. In this case it is easily seen that circle $c' = f(c)$ encloses entirely circle c and since c' is the maximal inscribed in c^{**} circle the points of this conic are all on the outside of circle c. Thus, from all points A of this conic there exist tangents to the circle c defining triangles ABC with incircle c and centroid G. Besides these common elements triangles ABC share also the same Nagel point which is the center J of the ellipse. Note that this ellipse can be easily constructed as a conic passing through five points $\{A, B, C, P', Q'\}$, where $\{P', Q'\}$ are the diametral points of its Nagel circle c' on IG.

Figure 15 illustrates the case of the singular conic, which can be considered as a degenerate parabola.

From the condition $|u| = \frac{r}{3}$ follows that c and $c' = f(c)$ are tangent at one of the diametral points $\{P, Q\}$ of c and the tangent there carries two of the vertices of the triangle.

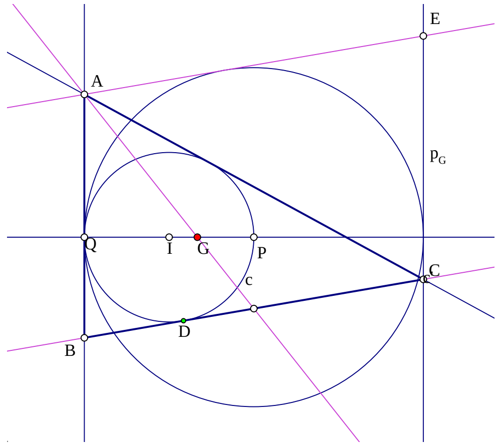

Figure 15. The singular case

Figure 16 displays two particular triangles of such a degenerate case. The isosceles triangle ABC characterized by the ratio of its sides $\frac{CA}{AB} = \frac{3}{2}$ and the right-angled $A'B'C'$ characterized by the ratio of its orthogonal sides $\frac{C'A'}{A'B'} = \frac{4}{3}$, which is similar to the right-angled triangle with sides $\{3, 4, 5\}$.

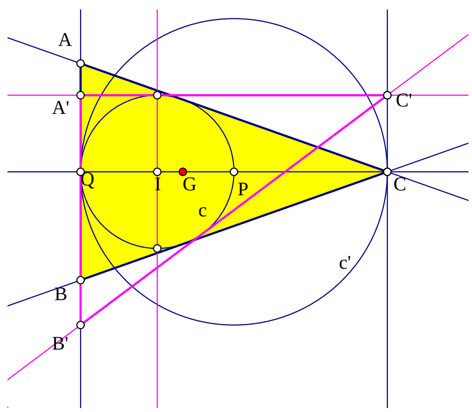

Figure 16. Two special triangles

It can be shown [3, p. 82] that triangles belonging to this degenerate case have the property to posses sides such that the sum of two of them equals three times the length of the third. This class of triangles answers also the problem [9] of finding all triangles such that their Nagel point is a point of the incircle.

5. The locus of the circumcenter

In this section I study the locus c_2 described by the circumcenter of all triangles sharing the same incircle c and centroid G. Since the homothety f, centered at G with ratio $k = -2$, maps the circumcenter O to the center E_u of the *Euler* circle

c_E, the problem reduces to that of finding the locus c_1 of points E_u. The clue here is the tangency of the Euler circle c_E to the incircle c at the *Feuerbach* point F_e of the triangle ([12]). Next proposition indicates that the Euler circle is also tangent to another fixed circle $c_0(S, r_0)$, whose center S (see Figure 17 and Figure 19) is on the Nagel line IG. I call this circle the *secondary* circle of the configuration (or of the triangle). As will be seen below this circle is homothetic to the incircle c with respect to the Nagel point N_a and at a certain ratio κ determined from the given data.

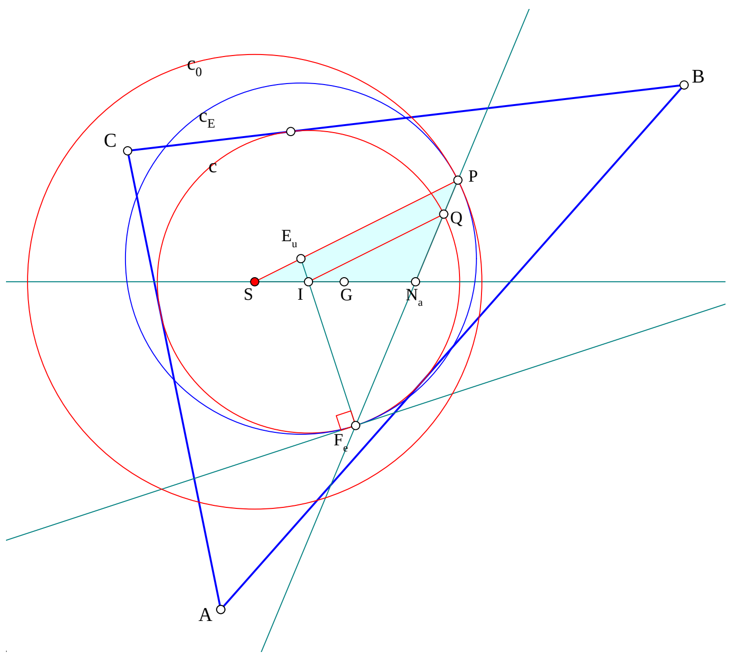

Figure 17. Invariant distance SP

Proposition 7. *Let $\{E_u, F_e, N_a\}$ be correspondingly the center of the Euler circle, the Feuerbach and the Nagel point of the triangle ABC. Let Q be the second intersection point of line F_eN_a and the incircle c. Then the parallel E_uP from E_u to line IQ intersects the Nagel line IG at a point S such that segments SN_a, SP have constant length for all triangles ABC sharing the same incircle c and centroid G. As a result all these triangles have their Euler circles c_E simultaneously tangent to the incircle c and a second fixed circle c_0 centered at S.*

The proof proceeds by showing that the ratio of oriented segments

$$\kappa = \frac{N_aS}{N_aI} = \frac{N_aP}{N_aQ} = \frac{N_aP \cdot N_aF_a}{N_aQ \cdot N_aF_a}$$

is constant. Since the incircle and the Euler circle are homothetic with respect to F_e, point P, being the intersection of line F_eN_a with the parallel to IQ from E_u, is on the Euler circle. Hence the last quotient is the ratio of powers of N_a with respect to the two circles: the variable Euler circle and the fixed incircle c. Denoting by r and R respectively the inradius and the circumradius of triangle ABC ratio κ can be expressed as

$$\kappa = \frac{|N_aE_u|^2 - (R/2)^2}{|N_aI|^2 - r^2}.$$

The computation of this ratio can be carried out using standard methods ([1, p.103], [17, p.87]). A slight simplification results from the fact ([17, p.30]) that homothety f maps the incenter I to the Nagel point N_a, producing the constellation displayed in Figure 18, in which H is the orthocenter and N' is the point on line GO such that $|GN'| : |N'O| = 1 : 3$ and $|IN'| = \frac{|E_uN_a|}{2}$.

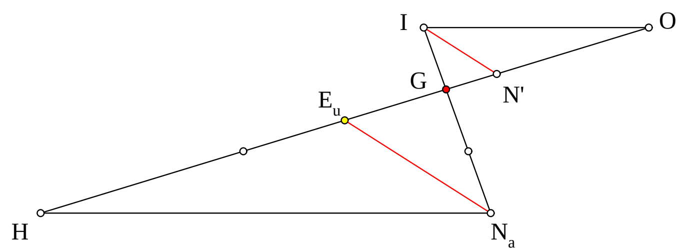

Figure 18. Configuration's symmetry

Thus, the two lengths needed for the determination of κ are $|N_aE_u| = 2|IN'|$ and $|N_aI| = 3|IG|$. In the next four equations $|OI|$ is given by Euler's relation ([17, p.10]), $|IG|, |GO|$ result by a standard calculation ([1, p.111], [14, p.185]), and $|IN'|$ results by applying Stewart's theorem to triangle GIO [5, p.6].

$$\begin{aligned}
|OI|^2 &= R(R-2r),\\
|OG|^2 &= \frac{1}{9}(9R^2 + 2r^2 + 8Rr - 2s^2),\\
|IG|^2 &= \frac{1}{9}(5r^2 + (s^2 - 16Rr)),\\
|IN'|^2 &= \frac{1}{16}(6r^2 - 32Rr + 2s^2 + R^2),
\end{aligned}$$

where s is the semiperimeter of the triangle. Using these relations ratio κ is found to be

$$\kappa = \frac{3r^2 + (s^2 - 16Rr)}{2(4r^2 + (s^2 - 16Rr))},$$

which using the above expression for $u^2 = |GI|^2$ becomes

$$\kappa = \frac{(3u)^2 - 2r^2}{2((3u)^2 - r^2)}.$$

By our assumptions this is a constant quantity, thereby proving the proposition.

The denominator of κ becomes zero precisely when $3|u| = r$. This is the case when point S (see Figure 17) goes to infinity and also the case in which, according to the previous section, the locus of the vertices of ABC is a degenerate parabola of two parallel lines. Excluding this exceptional case of infinite κ, to be handled below, last proposition implies that segments SP, SN_a (see Figure 17) have (constant) corresponding lengths $|SN_a| = |\kappa| \cdot |IN_a|$, $|SP| = |\kappa| \cdot r$. Hence the Euler circle of ABC is tangent simultaneously to the fixed incircle c as well as to the *secondary* circle c_0 with center at S and radius equal to $r_0 = |\kappa| \cdot r$. In the case $\kappa = 0$ the secondary circle collapses to a point coinciding with the Nagel point of the triangle and the Euler circle of ABC passes through that point.

Proposition 8. *The centers E_u of the Euler circles of triangles ABC which share the same incircle $c(I, r)$ and centroid G with $|IG| \neq \frac{r}{3}$ are on a central conic c_1 with one focus at the incenter I of the triangle and the other focus at the center S of the secondary circle. The kind of this conic depends on the value of κ as follows.*
(1) *For $\kappa > 1$ which corresponds to $3|u| < r$ conic c_1 is an ellipse similar to c^* and has great axis equal to $\frac{r_0 - r}{2}$, where r_0 the radius of the secondary circle.*
(2) *The other cases correspond to values of $\kappa < 0$, $0 < \kappa < \frac{1}{2}$ and $\kappa = 0$. In the first two cases the conic c_1 is a hyperbola similar to the conjugate one of the hyperbola c^*.*
(3) *In the case $\kappa = 0$ the Euler circles pass through the Nagel point and the conic c_1 is a rectangular hyperbola similar to c^*.*

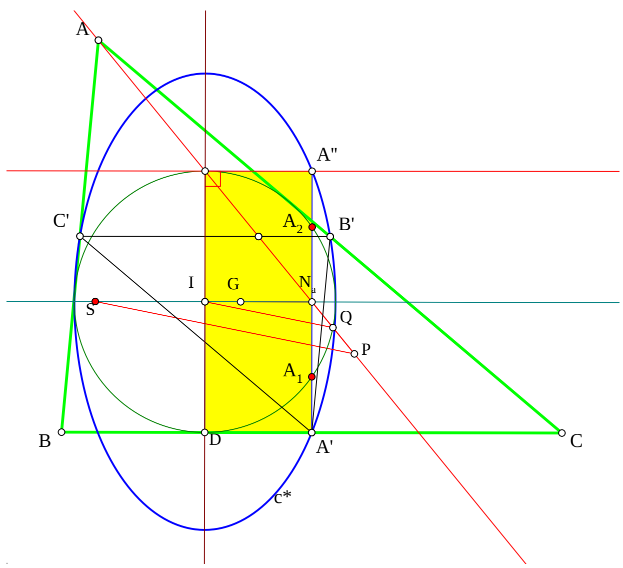

Figure 19. Axes ratio of c^*

The fact that the locus c_1 of points E_u is part of the central conic with foci at $\{I, S\}$ and great axis equal to $\frac{|r_0 \mp r|}{2}$ is a consequence of the simultaneous tangency of the Euler circles with the two fixed circles c and c_0 ([11, vol.I, p.42]). In the case

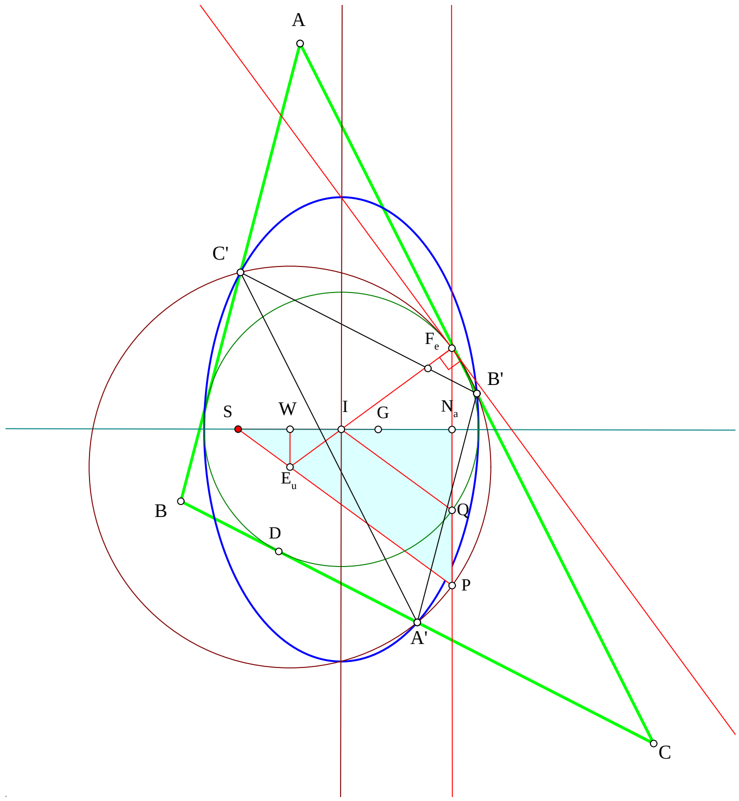

Figure 20. Axes ratio of c_1 (elliptic case)

$\kappa > 1$ it is readily verified that the Euler circles contain circle c and are contained in c_0 and the sum of the distances of E_u from $\{I, S\}$ is $r_0 - r$. In the case $\kappa < 0$ the Nagel point is between S and I and the Euler circle contains c and is outside c_0. Thus the difference of distances of E_u from $\{I, S\}$ is equal to $r_0 + r$. In the case $0 < \kappa < \frac{1}{2}$ the Euler circle contains both circles c and c_0 and the difference of distances of E_u from $\{I, S\}$ is $r - r_0$. Finally in the case $\kappa = 0$ the difference of the distances from $\{I, S\}$ is equal to r. This and the condition $0 = \kappa = (3u)^2 - 2r^2$ imply easily that c_1 is a rectangular hyperbola. To prove the similarity to c^* in the case $\kappa > 1$ it suffices to compare the ratios of the axes of the two conics. By the remarks of §3, in this case ($|IG| = |u| < \frac{r}{3}$), the incircle c is the maximal circle contained in c^* and the ratio of its axes is (see Figure 19) $q = \frac{|A_1A_2|}{|A'A''|}$. Here $A'A''$ is the chord parallel to the great axis and equal to the diameter of the incircle and A_1A_2 is the intersection of this chord with the incircle. It is readily seen that this ratio is equal to $\sqrt{1 - \frac{IN_a^2}{ID^2}} = \sqrt{1 - \frac{(3u)^2}{r^2}}$. The corresponding ratio for c_1 can be realized by considering the special position of ABC for which the corresponding point E_u becomes a vertex of c_1 *i.e.*, the position for which the projection W of E_u on line IG is the middle of the segment IS (see Figure 20). By this position of ABC the ratio of axes of conic c_1 is realized as $\frac{E_uW}{E_uI}$. By the resulting similarity at I this is equal to $\frac{F_eN_a}{F_eI} = \frac{\sqrt{r^2-IN_a^2}}{r} = \sqrt{1-\frac{(3u)^2}{r^2}}$, thereby proving the claim in the case $3|u| < r$. In the case $3|u| > r$, *i.e.*, when c^* is a hyperbola, it was seen in Remark (5) of §3 that the asymptotes of the hyperbola are the tangents to c from the Nagel point N_a (see Figure 21). The proof results in this case by taking the position of the variable triangle ABC in such a way that the center of the Euler circle goes to infinity. In this case points $\{P, Q\}$ defining the parallels $\{IP, E_uQ\}$ tend respectively to points $\{P_0, Q_0\}$, which are the projections of $\{I, S\}$ on the asymptote and the parallels tend respectively to $\{IP_0, SQ_0\}$, which are parallel to an asymptote of the conic c_1. Analogously is verified also that the other asymptote of c_1 is orthogonal to the corresponding other asymptote of c^*. This proves the claim on the conjugacy of c_1 to c.

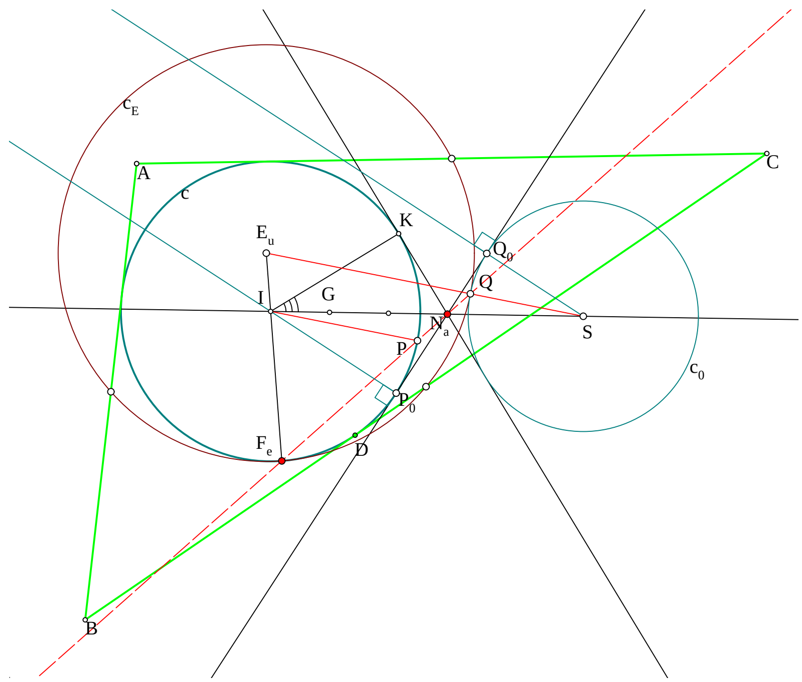

Figure 21. Axes ratio of c_1 (hyperbolic case)

Remarks. (1) The fact that the interval $(\frac{1}{2}, 1)$ represents a gap for the values of κ is due to the formula representing κ as a Moebius transformation of $(3u)^2$, whose the asymptote parallel to the x-axis is at $y = \frac{1}{2}$.

(2) An easy exploration of the same formula shows that $\{c, c_0\}$ are always disjoint, except in the case $u = -\frac{2r}{3}$, in which $\kappa = -\frac{1}{3}$ and the two circles become externally tangent at the Spieker point S_p of the triangle ABC, which is the middle of IN_a.

I turn now to the singular case corresponding to $|GI| = |u| = \frac{r}{3}$ for which κ becommes infinite. Following lemma should be known, I include though its proof for the sake of completeness of the exposition.

Lemma 9. *Let circle c, be tangent to line L_0 at its point C and line L_1 be parallel to L_0 and not intersecting c. From a point H on L_1 draw the tangents to c which intersect L_0 at points $\{K, L\}$. Then $CL \cdot CK$ is constant.*

Figure 22, illustrating the lemma, represents an interesting configuration and the problem at hand gives the opportunity to list some of its properties.
(1) Circle c', passing through the center A of the given circle c and points $\{K, L\}$, has its center P on the line AH.
(2) The other intersection points $\{Q, R\}$ of the tangents $\{HL, HK\}$ with circle c' define line QR, which is symmetric to L_0 with respect to line AH.
(3) Quadrangle $LRKQ$, which is inscribed in c', is an isosceles trapezium.
(4) Points $(A, M, O, H) = -1$ define a harmonic division. Here M is the diametral of A and $O = L_0 \cap QR$.
(5) Point M, defined above, is an excenter of triangle HLK.
(6) Points $(A, N, C, F) = -1$ define also a harmonic division, where N is the other intersection point with c' of AC and $F = AC \cap L_1$.

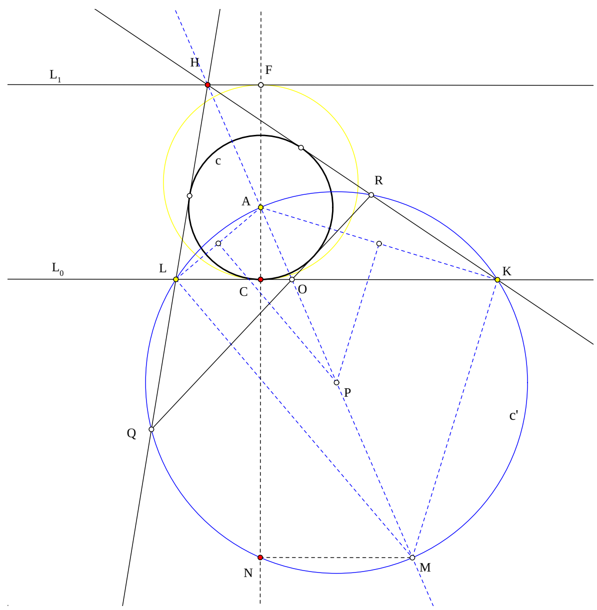

Figure 22. $CL \cdot CK$ is constant

(1) is seen by drawing first the medial lines of segments $\{AL, AK\}$ which meet at P and define there the circumcenter of ALK. Their parallels from $\{L, K\}$ respectively meet at the diametral M of A on the circumcircle c' of ALK. Since they are orthogonal to the bisectors of triangle HLK they are external bisectors of its angles and define an excenter of triangle HLK. (2), (3), (5) are immediate consequences. (4) follows from the standard way ([5, p.145]) to construct the polar of a point H with respect to a conic c'. (6) follows from (4) and the parallelism of lines $\{L_0, L_1, CO, NM\}$. The initial claim is a consequence of (6). This claim is also equivalent to the orthogonality of circle c' to the circle with diameter FC.

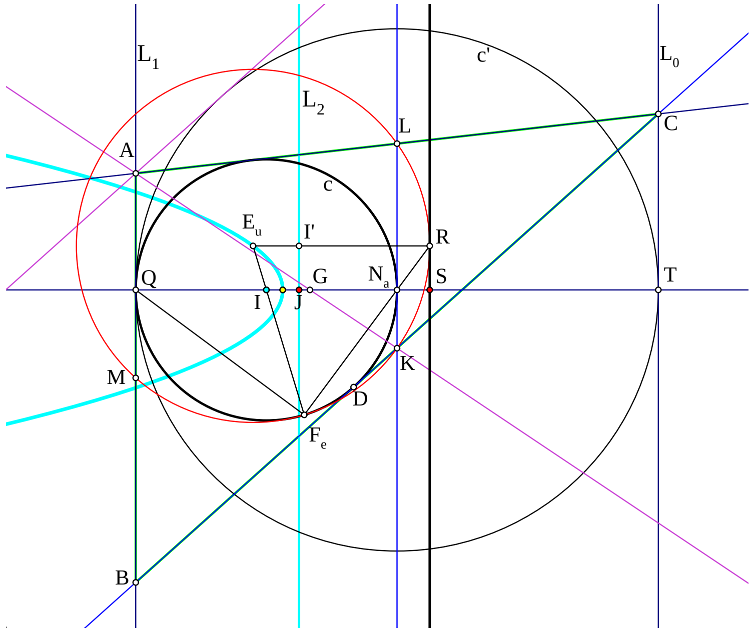

Figure 23. Locus of E_u for $|IG| = \frac{r}{3}$

Proposition 10. *Let $c(I, r)$ be a circle with center at point I and radius r. Let also G be situated at distance $|IG| = \frac{r}{3}$ from I. Then the following statements are valid:*
(1) *The homothety f centered at G and with ratio $k = -2$ maps circle c to circle $c'(N_a, 2r)$ of radius $2r$ and tangent to c at its intersection point Q with line IG such that $GQ : IQ = 4 : 3$.*
(2) *Let T be the diametral point of Q on c' and C be an arbitrary point on the line L_0, which is orthogonal to QT at T. Let ABC be the triangle formed by drawing the tangents to c from C and intersecting them with the parallel L_1 to L_0 from Q. Then G is the centroid of ABC.*
(3) *The center E_u of the Euler circle of ABC, as C varies on line L_0, describes a parabola with focus at I and directrix L_2 orthogonal to GI at a point J such that $IJ : IG = 3 : 4$.*

Statement (1) is obvious. Statement (2) follows easily from the definitions, since the middles $\{L, K\}$ of $\{CA, CB\}$ respectively define line KL which is tangent to c, orthogonal to GI and passes through the center N_a of c' (see Figure 23). Denoting by G' the intersection of AK with IG and comparing the similar triangles AQG' and KN_aG' one identifies easily G' with G. To prove (3) use first Feuerbach's theorem ([12]), according to which the Euler circle is tangent to the incircle c at a point F_e. Let then E_uR be the radius of the Euler circle parallel to GI. Line RF_e passes through N_a, which is the diametral of Q on circle c and which coincides with the Nagel point of the triangle ABC. This follows from Thales theorem for the similar isosceles triangles IF_eN_a and

E_uF_eR. Point R projects on a fixed point S on GI. This follows by first observing that RSF_eQ is a cyclic quadrangle, points $\{F_e, S\}$ viewing RQ under a right angle. It follows that $N_aS \cdot N_aQ = N_aR \cdot N_aF_u$ which is equal to $N_aK \cdot N_aL = \frac{1}{4}QA \cdot QB$ later, according to previous lemma, being constant and independent of the position of C on line L_0. Using the previous facts we see that $E_uI = E_uF_e - IF_e = E_uF_e - r = E_uR - r$. Let point J on GI be such that $SJ = r$ and line L_2 be orthogonal to GI at J. Then the projection I' of E_u on L_2 satisfies $E_uI = E_uI'$, implying that E_u is on the parabola with focus at I and directrix L_2. The claim on the ratio $IJ : IG = 3 : 4$ follows trivially.

6. Miscellanea

In this section I discuss several aspects of the structures involved in the problem at hand. I start with the determination of the perspector of the conic c^{**} in barycentric coordinates. The clue here is the incidence of the conic at the diametral points $\{P, Q\}$ of the Nagel circle c' on the Nagel line GI (see Figure 24). These points can be expressed as linear combinations of $\{G, I\}$:

$$P = p' \cdot G + p'' \cdot I, \qquad Q = q' \cdot G + q'' \cdot I.$$

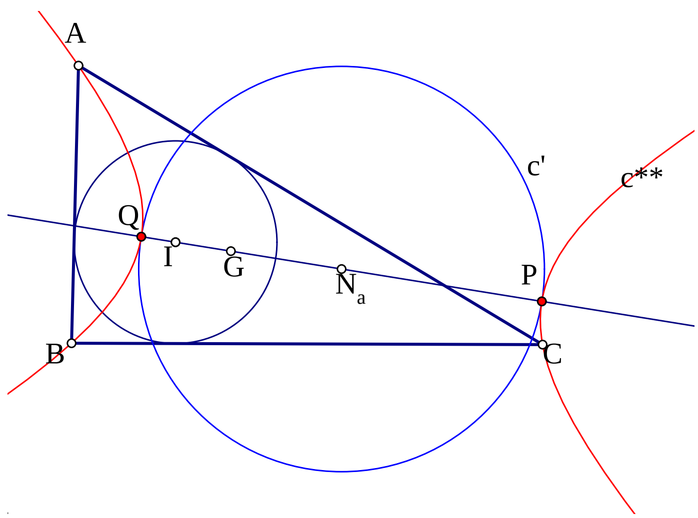

Figure 24. Triangle conic c^{**}

Here the equality is meant as a relation between the barycentric coordinates of the points and on the right are meant the normalized barycentric coordinates $G = \frac{1}{3}(1,1,1)$, $I = \frac{1}{2s}(a,b,c)$, where $\{a,b,c\}$ denote the side-lengths of triangle ABC and s denotes its half-perimeter. From the assumptions follows that

$$\frac{p''}{p'} = \frac{GP}{PI} = \frac{2(r-u)}{-2r+3u}, \qquad \frac{q''}{q'} = \frac{GQ}{QI} = -\frac{2(r+u)}{2r+3u}.$$

The equation of the conic is determined by assuming its general form

$$\alpha yz + \beta zx + \gamma xy = 0,$$

and computing $\{\alpha, \beta, \gamma\}$ using the incidence condition at P and Q. This leads to the system of equations

$$\begin{aligned} \alpha p_y p_z + \beta p_z p_x + \gamma p_x p_y &= 0, \\ \alpha q_y q_z + \beta q_z q_x + \gamma q_x q_y &= 0, \end{aligned}$$

implying

$$(\alpha, \beta, \gamma) = \lambda((q_x q_y p_z p_x - q_z q_x p_x p_y), (q_y q_z p_x p_y - q_x q_y p_y p_z), (q_z q_x p_y p_z - q_y q_z p_z p_x)),$$

where the barycentric coordinates of P and Q are given by

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = p'G + p''I = \frac{-2r+3u}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{r-u}{s}\begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

$$\begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = q'G + q''I = \frac{2r+3u}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{r+u}{s}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Making the necessary calculations and elliminating common factors we obtain

$$(\alpha, \beta, \gamma) = ((c-b)((3u(s-a))^2 - r^2(2s-3a)^2),\ \cdots,\ \cdots),$$

the dots meaning the corresponding formulas for β, γ resulting by cyclic permutation of the letters $\{a, b, c\}$.

The next proposition depicts another aspect of the configuration, related to a certain pencil of circles passing through the (fixed) Spieker point of the triangle.

Proposition 11. *Let $c(I, r)$ be a circle at I with radius r and G a point. Let also ABC be a triangle having c as incircle and G as centroid. Let further $\{c_E(E_u, r_E), c_0(S, r_0)\}$ be respectively the Euler circle and the secondary circle of ABC and $\{E_u, F_e, N_a, S_p\}$ be respectively the center of the Euler circle, the Feuerbach point, the Nagel point and the Spieker point of ABC. Then the following statements are valid.*

(1) *The pencil $\mathcal{I}$ of circles generated by c and c_0 has limit points $\{S_p, T\}$, where T is the inverse of S_p with respect to c.*

(2) *If P is the other intersection point of the Euler circle c_E with the line F_eN_a, then circle c_t tangent to the radii $\{IF_e, E_uP\}$ respectively at points $\{F_e, P\}$ belongs to the pencil of circles $\mathcal{J}$ which is orthogonal to $\mathcal{I}$ and passes through $\{S_p, T\}$.*

(3) *The polar p_T of T with respect to circles $\{c, c_0\}$ as well as the conic c^* is the same line UV which passes through S_p.*

(4) *In the case $\kappa > 1$ the conic c^* is an ellipse tangent to c_0 at its intersection points with line $p_T = UV$.*

The orthogonality of circle c_t to the three circles $\{c, c_0, c_E\}$ follows from its definition. This implies also the main part of the second claim. To complete the proof of the two first claims it suffices to identify the limit points of the pencil of circles generated by c and c_0. For this use can be made of the fact that these two points are simultaneously harmonic conjugate with respect to the diameter points of the circles c and c_0 on line IG. Denoting provisorily the intersection points of c_t with

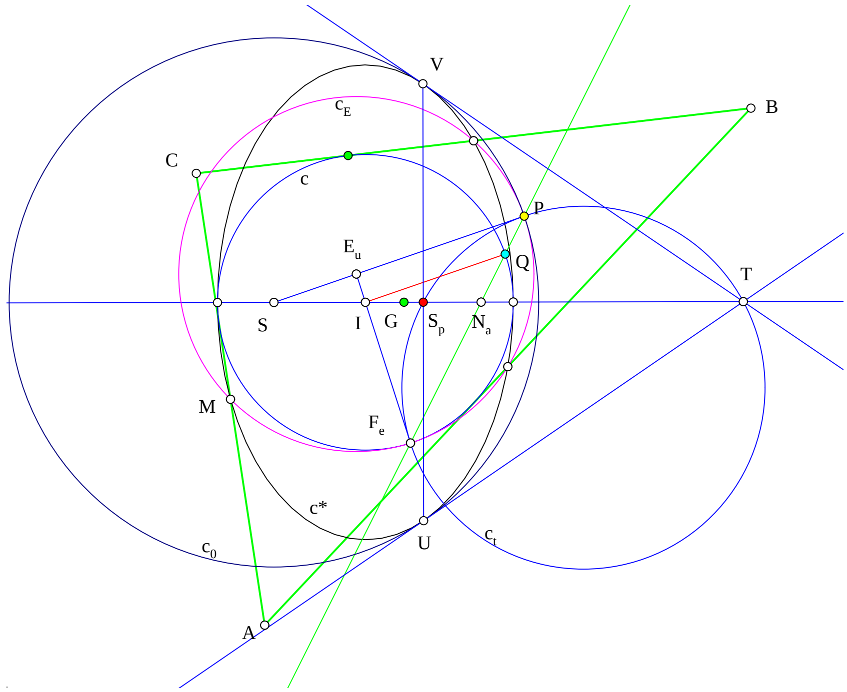

Figure 25. Circle pencil associated to $c(I, r)$ and G

line IG by $\{S', T'\}$ and their distance by d last property translates to the system of equations

$$r^2 = |IS'|(|IS'| + d),$$
$$r_0^2 = |SS'|(|SS'| + d).$$

Eliminating d from these equations, setting $x = IS'$, $SI = SN_a - IN_a = (\kappa - 1)IN_a = (1-\kappa)(3u)$, and $r_0 = \kappa \cdot r$, we obtain after some calculation, equation

$$6ux^2 + (9u^2 + 4r^2)x + 6ur^2 = 0.$$

One of the roots of this equation is $x = -\frac{3u}{2}$, identifying S' with the Spieker point S_p of the triangle. This completes the proof of the first two claims of the proposition. The third claim is an immediate consequence of the first two and the fact that c^* is bitangent to c_0 at its diametral points with line IG. The fourth claim results from a trivial verification of $\{U, V\} \subset c^*$ using the coordinates of the points.

Remark. Since points $\{G, I\}$ remain fixed for all triangles considered, the same happens for every other point X of the Nagel line IG which can be written as a linear combination $X = \lambda G + \mu I$ of the normalized barycentric coordinates of $\{G, I\}$ and with constants $\{\lambda, \mu\}$ which are independent of the particular triangle. Also fixing such a point X and expressing G as a linear combination of $\{I, X\}$ would imply the constancy of G. This, in particular, considering triangles with the same incircle and Nagel point or the same incircle and Spieker point would convey the discussion back to the present one and force all these triangles to have their vertices on the conic c^{**}.

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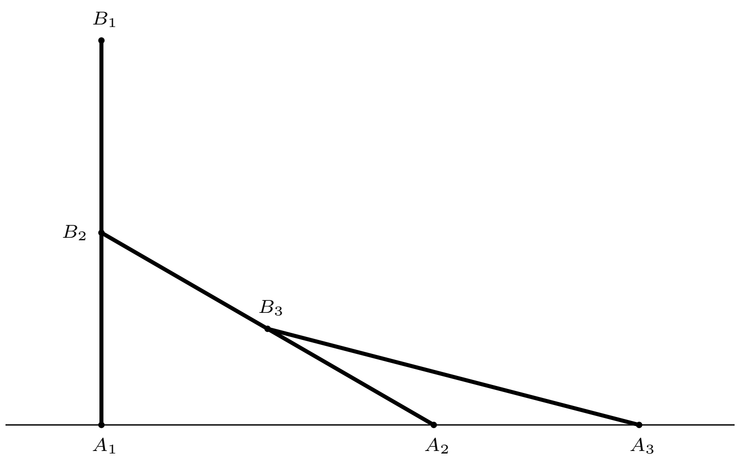
Forum Geometricorum
Volume 11 (2011) 53.

FORUM GEOM
ISSN 1534-1178

A Simple Construction of the Golden Section

Jo Niemeyer

Three equal segments A_1B_1, A_2B_2, A_3B_3 are positioned in such a way that the endpoints B_2, B_3 are the midpoints of A_1B_1, A_2B_2 respectively, while the endpoints A_1, A_2, A_3 are on a line perpendicular to A_1B_1.



In this arrangement, A_2 divides A_1A_3 in the golden ratio, namely,

$$\frac{A_1A_3}{A_1A_2} = \frac{\sqrt{5}+1}{2}.$$

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Publication Date: February 25, 2011. Communicating Editor: Paul Yiu.

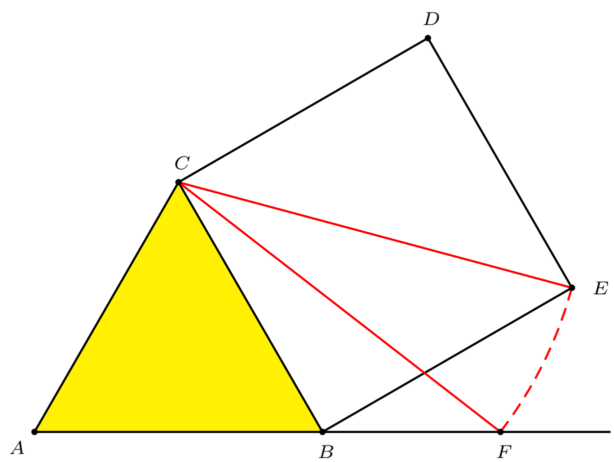
Forum Geometricorum
Volume 11 (2011) 55.

FORUM GEOM
ISSN 1534-1178

Another Simple Construction of the Golden Section

Michel Bataille

Given an equilateral triangle ABC, erect a square $BCDE$ externally on the side BC. Construct the circle, center C, passing through E, to intersect the line AB at F. Then, B divides AF in the golden ratio.



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Publication Date: March 4, 2011. Communicating Editor: Paul Yiu.

Forum Geometricorum
Volume 11 (2011) 57–63.

FORUM GEOM
ISSN 1534-1178

On the Foci of Circumparabolas

Michel Bataille

Abstract. We establish some results about the foci of the infinitely many parabolas passing through three given points A, B, C. A very simple construction directly leads to their barycentric coordinates which provide, besides their locus, a nice and unexpected link with the foci of the parabolas tangent to the sidelines of triangle ABC.

1. Introduction

The parabolas tangent to the sidelines of a triangle are well-known: the pair focus-directrix of such an *in*parabola is formed by a point of the circumcircle other than the vertices and its Steiner line (see [2] for example). In view of such a simple result, one is encouraged to consider the parabolas passing through the vertices or *circum*parabolas. The purpose of this note is to show how an elementary construction of their foci leads to some interesting results.

2. A construction

Given a triangle ABC, a ruler and compass construction of the focus of a circumparabola follows from two well-known results that we recall as lemmas.

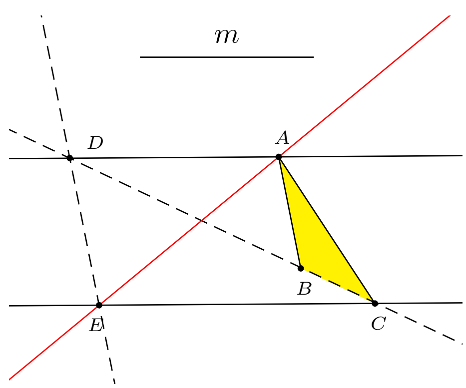

Figure 1.

Lemma 1. *Let P be a point on a parabola. The symmetric of the diameter through P in the tangent at P passes through the focus.*

Lemma 2. *Given three points A, B, C of a parabola and the direction m of its axis, the tangents to the parabola at A, B, C are constructible with ruler and compass.*

Publication Date: March 4, 2011. Communicating Editor: Paul Yiu.

Lemma 1 is an elementary property of the tangents to a parabola. Lemma 2 follows by applying Pascal's theorem to the six points m, B, A, A, m, C on the parabola. The intersections $D = BC \cap Am$, $E = mC \cap AA$ and $F = mm \cap BA$ are collinear. Note that F is the infinite point of AB. Therefore, by constructing
(i) the parallel to m through A to meet BC at D,
(ii) the parallels to AB through D to meet the parallel to m through C at E,
we obtain the line AE as the tangent to the parabola at A (see Figure 1).

Now, let ABC be a triangle and m be a direction other than the directions of its sides. It is an elementary fact that a unique parabola $\mathcal{P}_m$ with axis of direction m passes through A, B, C. The tangents to $\mathcal{P}_m$ at A, B, C can be drawn in accordance with Figure 1. Then, Lemma 1 indicates how to complete the construction of the focus F of this circumparabola $\mathcal{P}_m$ (see Figure 2). Note that the directrix is the line through the symmetrics $F_a,\ F_b,\ F_c$ of F in the tangents at A, B, C.

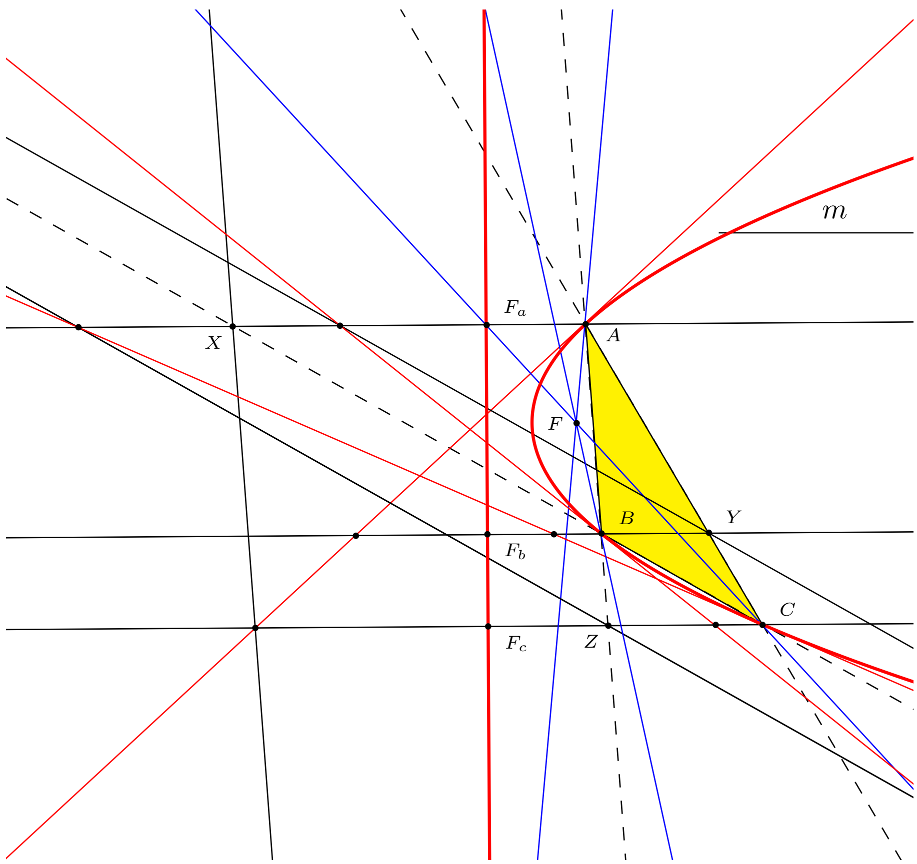

Figure 2.

3. The barycentric coordinates of F

In what follows, all barycentric coordinates are relatively to (A, B, C), and as usual, $a = BC, b = CA, c = AB$. First, we give a short proof of a result that will be needed later:

Lemma 3. *If $(f : g : h)$ is the infinite point of the line ℓ, then the infinite point $(f' : g' : h')$ of the perpendiculars to ℓ is given by*

$$f' = gS_B - hS_C, \quad g' = hS_C - fS_A, \quad h' = fS_A - gS_B$$

where $S_A = \frac{b^2+c^2-a^2}{2}$, $S_B = \frac{c^2+a^2-b^2}{2}$, $S_C = \frac{a^2+b^2-c^2}{2}$.

Proof. Note that $f + g + h = 0 = f' + g' + h'$ and that S_A, S_B, S_C are just the dot products $\overrightarrow{AB} \cdot \overrightarrow{AC}, \overrightarrow{BC} \cdot \overrightarrow{BA}, \overrightarrow{CA} \cdot \overrightarrow{CB}$, respectively. Expressing that the vectors $g\overrightarrow{AB} + h\overrightarrow{AC}$ and $g'\overrightarrow{AB} + h'\overrightarrow{AC}$ are orthogonal yields

$$0 = (g\overrightarrow{AB} + h\overrightarrow{AC}) \cdot (g'\overrightarrow{AB} + h'\overrightarrow{AC}) = g'(gc^2 + hS_A) + h'(gS_A + hb^2)$$

so that

$$\frac{g'}{gS_A + hb^2} = \frac{-h'}{hS_A + gc^2} = \frac{f'}{-gS_A - hb^2 + hS_A + gc^2}$$

or, as it is easily checked,

$$\frac{f'}{gS_B - hS_C} = \frac{g'}{hS_C - fS_A} = \frac{h'}{fS_A - gS_B}.$$

□

For an alternative proof of this lemma, see [3].

From now on, we identify direction and infinite point and denote by $\mathcal{P}_m$ or $\mathcal{P}_{u,v,w}$ the circumparabola whose axis has direction $m = (u : v : w)$ (distinct from the directions of the sides of triangle ABC). Translating the construction of the previous paragraph analytically, we will obtain the coordinates of F.

Theorem 4. *Let u, v, w be real numbers with $u, v, w \neq 0$ and $u + v + w = 0$. Barycentric coordinates of the focus F of the circumparabola $\mathcal{P}_{u,v,w}$ are*

$$\left(\frac{u^2}{vw} + \frac{a^2vw}{\rho} \; : \; \frac{v^2}{wu} + \frac{b^2wu}{\rho} \; : \; \frac{w^2}{uv} + \frac{c^2uv}{\rho}\right)$$

where $\rho = a^2vw + b^2wu + c^2uv$.

Proof. First, consider the conic with equation

$$u^2yz + v^2zx + w^2xy = 0. \tag{1}$$

Clearly, this conic passes through A, B, C and also through the infinite point $m = (u : v : w)$. Moreover, the tangent at this point is $x(wv^2 + vw^2) + y(uw^2 + wu^2) + z(vu^2 + uv^2) = 0$ that is, the line at infinity (since $wv^2 + vw^2 = -uvw = uw^2 + wu^2 = vu^2 + uv^2$). It follows that (1) is the equation of the circumparabola $\mathcal{P}_m$.

From (1), standard calculations give the tangent t_A to $\mathcal{P}_m$ at A and the diameter m_A through A:

$$t_A : \quad w^2y + v^2z = 0, \qquad m_A : \quad wy - vz = 0.$$

Since the infinite point of t_A is $(w^2 - v^2 : v^2 : -w^2)$, the lemma readily provides the normal n_A to $\mathcal{P}_m$ at A:

$$y(w^2S_A - v^2c^2) - z(v^2S_A - w^2b^2) = 0.$$

Now, the symmetric m'_A of m_A in t_A is the polar of the point $(0 : v : w)$ of m_A with respect to the pair of lines (t_A, n_A). The equation of this pair being $(w^2y + v^2z)(y(w^2S_A - v^2c^2) - z(v^2S_A - b^2w^2)) = 0$ that is,

$$y^2w^2(w^2S_A - v^2c^2) + yz(w^4b^2 - v^4c^2) + z^2v^2(b^2w^2 - v^2S_A) = 0,$$

the equation of m'_A is easily found to be

$$y[w^4(ub^2 + va^2) + c^2uvw(uv + w^2)] = z[v^4(uc^2 + wa^2) + b^2uvw(uw + v^2)]$$

From similar equations for the corresponding lines m'_B and m'_C, we immediately see that the common point F of m'_A, m'_B, m'_C has barycentric coordinates

$$x_1 = u^4(vc^2+wb^2)+a^2uvw(vw+u^2), \qquad y_1 = v^4(uc^2+wa^2)+b^2uvw(uw+v^2),$$

$$z_1 = w^4(ub^2 + va^2) + c^2uvw(uv + w^2)$$

or, observing that $u^4(vc^2 + wb^2) = \rho u^3 - a^2u^3vw$,

$$x_1 = \rho u^3 + a^2uv^2w^2, \quad y_1 = \rho v^3 + b^2u^2vw^2, \quad z_1 = \rho w^3 + c^2u^2v^2w. \tag{2}$$

Using $u^3 + v^3 + w^3 = 3uvw$ (since $u + v + w = 0$), an easy computation yields $x_1 + y_1 + z_1 = 4uvw\rho$ and dividing out by $uvw\rho$ in (2) leads to

$$4F = \left(\frac{u^2}{vw} + \frac{a^2vw}{\rho}\right)A + \left(\frac{v^2}{wu} + \frac{b^2wu}{\rho}\right)B + \left(\frac{w^2}{uv} + \frac{c^2uv}{\rho}\right)C. \tag{3}$$

$\square$

4. The locus of F

Theorem 4 gives a parametric representation of the locus of F, a not well-known curve, to say the least! (see Figure 3 below). Clearly, this curve must have asymptotic directions parallel to the sides of triangle ABC. Actually, it is a quintic that can be found in [1] under the reference Q077 with more information, in particular about the asymptotes. A barycentric equation of the curve is also given, but this equation is almost two-page long!

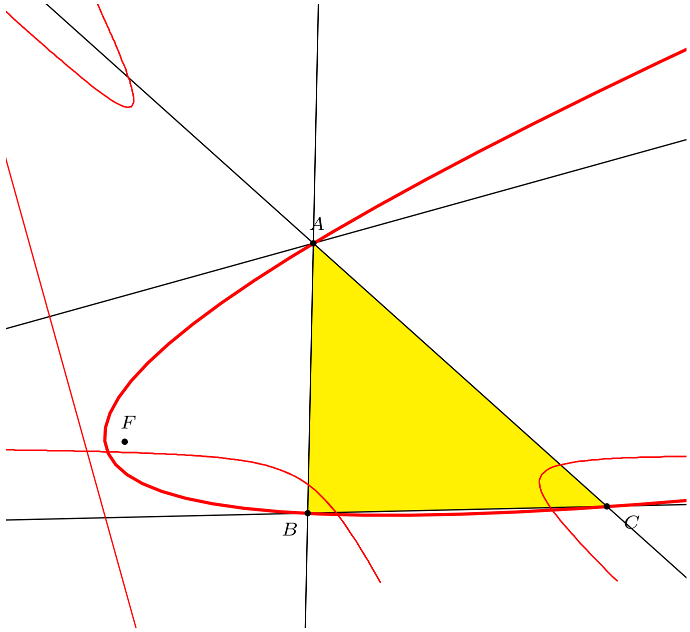

Figure 3.

5. A connection with the inparabolas

Let us write (3) as $4F = 3P + F'$ with

$$3P = \frac{u^2}{vw}A + \frac{v^2}{wu}B + \frac{w^2}{uv}C \quad \text{and} \quad F' = \frac{uvw}{\rho}\left(\frac{a^2}{u}A + \frac{b^2}{v}B + \frac{c^2}{w}C\right).$$

We observe that P is the centroid of the cevian triangle of the infinite point m (triangle XYZ in Figure 2), [1] and F' is the isogonal conjugate of this infinite point. Thus, F' is the focus of the inparabola $\mathcal{P}'_m$ whose axis has direction $m = (u : v : w)$. This point F' is easily constructed as the point of the circumcircle of triangle ABC whose Simson line is perpendicular to m. Since $4\overrightarrow{PF} = \overrightarrow{PF'}$, we see that F' is the image of F under the homothety $\mathsf{h} = \mathsf{h}(P, 4)$ with center P and ratio 4. More can be said.

Theorem 5. *With the notations above,*
(a) *$\mathcal{P}'_m$ is the image of $\mathcal{P}_m$ under $\mathsf{h}(P, 4)$,*
(b) *the locus of P is the cubic $\mathcal{C}$ with barycentric equation $(x + y + z)^3 = 27xyz$, the centroid G of triangle ABC being excluded.*

Proof. (a) As in §3, it is readily checked that a barycentric equation of $\mathcal{P}'_m$ is

$$u^2x^2 + v^2y^2 + w^2z^2 - 2vwyz - 2wuzx - 2uvxy = 0.$$

[1] I thank P. Yiu for this nice observation.

Let $A' = \mathrm{h}(A)$ so that $uvwA' = (4uvw - u^3)A - v^3B - w^3C$. Using $u = -v - w$ repeatedly, a straightforward computation yields

$$u^2(4uvw - u^3)^2 + v^8 + w^8 - 2v^4w^4 + 2uw^4(4uvw - u^3) + 2uv^4(4uvw - u^3) = 0$$

hence A' is on $\mathcal{P}'_m$. By symmetry, the homothetic B' and C' of B and C are on $\mathcal{P}'_m$ as well and therefore the parabolas $\mathcal{P}'_m$ and $\mathrm{h}(\mathcal{P}_m)$ both pass through A', B', C' and $(u : v : w)$. As a result, $\mathrm{h}(\mathcal{P}_m) = \mathcal{P}'_m$.

(b) Since $P(u^3 : v^3 : w^3)$ and $(u^3 + v^3 + w^3)^3 = (u^3 + v^3 + (-u - v)^3)^3 = (-3u^2v - 3uv^2)^3 = 27u^3v^3w^3$, P is on the cubic $\mathcal{C}$. Clearly, $P \neq G$.

Conversely, supposing that $P(x, y, z)$ is on $\mathcal{C}$ and $P \neq G$, we can set $x = u^3, y = v^3, z = w^3$. Then w is given by $w^3 - 3uvw + u^3 + v^3 = 0$ or $(w + u + v)(w^2 - (u + v)w + (u^2 - uv + v^2)) = 0$. If $u \neq v$, the second factor does not vanish, hence $w = -u - v$ and $u + v + w = 0$. If $u = v$, then $w \neq u$ (since $P \neq G$) and we must have $w = -2u = -u - v$ again. $\square$

Note that setting $\frac{u}{w} = t$, $\frac{v}{w} = -1 - t$, the locus of P can be also be constructed as the set of points P defined by

$$\overrightarrow{CP} = -\frac{t^2}{3(1+t)}\overrightarrow{CA} + \frac{(1+t)^2}{3t}\overrightarrow{CB}.$$

Figure 4 shows $\mathcal{C}$ and the parabolas $\mathcal{P}_m$ and $\mathcal{P}'_m$.

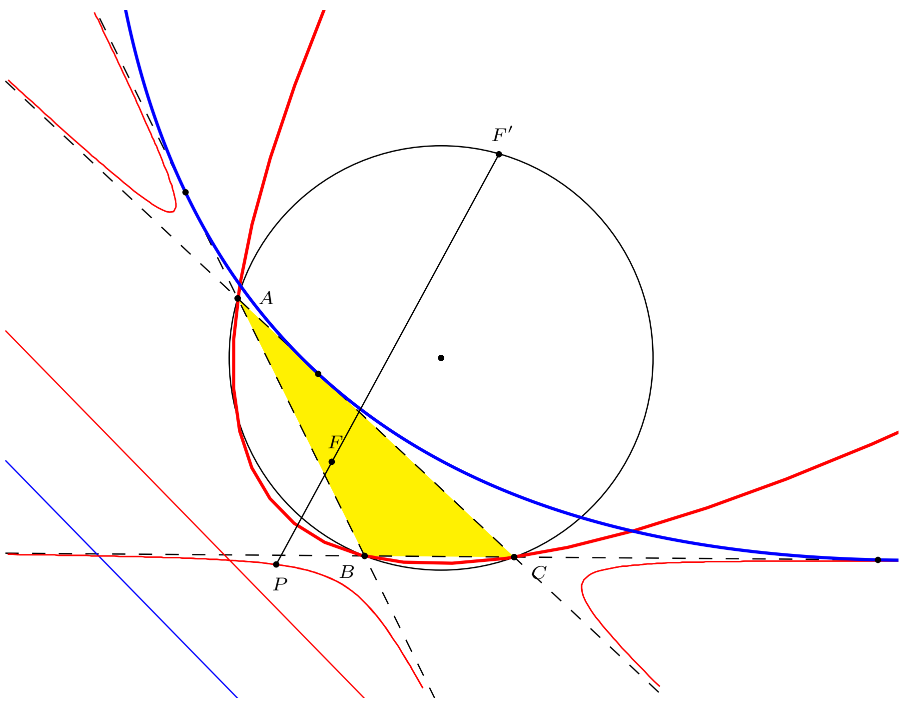

Figure 4.

In some way, the quintic of §4 can be considered as a combination of a quadratic (the circumcircle) and a cubic ($\mathcal{C}$).

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Forum Geometricorum
Volume 11 (2011) 65–82.

FORUM GEOM
ISSN 1534-1178

More Characterizations of Tangential Quadrilaterals

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Abstract. In this paper we will prove several not so well known conditions for a quadrilateral to have an incircle. Four of these are different excircle versions of the characterizations due to Wu and Vaynshtejn.

1. Introduction

In the wonderful paper [13] Nicuşor Minculete gave a survey of some known characterizations of tangential quadrilaterals and also proved a few new ones. This paper can to some extent be considered a continuation to [13].

A tangential quadrilateral is a convex quadrilaterals with an incircle, that is a circle tangent to all four sides. Other names for these quadrilaterals are[1] tangent quadrilateral, circumscribed quadrilateral, circumscribable quadrilateral, circumscribing quadrilateral, inscriptable quadrilateral and circumscriptible quadrilateral. The names inscriptible quadrilateral and inscribable quadrilateral have also been used, but sometimes they refer to a quadrilateral with a circumcircle (a cyclic quadrilateral) and are not good choices because of this ambiguity. To avoid confusion with so many names we suggest that only the names *tangential quadrilateral* (or tangent quadrilateral) and circumscribed quadrilateral be used. This is supported from the number of hits on Google[2] and the fact that both MathWorld and Wikipedia uses the name tangential quadrilateral in their encyclopedias.

Not all quadrilaterals are tangential,[3] hence they must satisfy some condition. The most important and perhaps oldest such condition is the *Pitot theorem*, that a quadrilateral $ABCD$ with consecutive sides a, b, c and d is tangential if and only if the sums of opposite sides are equal: $AB + CD = BC + DA$, that is

$$a + c = b + d. \tag{1}$$

It is named after the French engineer Henri Pitot (1695-1771) who proved that this is a necessary condition in 1725; that it is also a sufficient condition was proved by the Swiss mathematician Jakob Steiner (1796-1863) in 1846 according to F. G.-M. [7, p.319].

Publication Date: March 18, 2011. Communicating Editor: Paul Yiu.

[1]In decreasing order of the number of hits on Google.

[2]Tangential, tangent and circumscribed quadrilateral represent about 80 % of the number of hits on Google, so the other six names are rarely used.

[3]For example, a rectangle has no incircle.

The proof of the direct theorem is an easy application of the *two tangent theorem*, that two tangents to a circle from an external point are of equal length. We know of four different proofs of the converse to this important theorem, all beautiful in their own way. The first is a classic that uses a property of the perpendicular bisectors to the sides of a triangle [2, pp.135-136], the second is a proof by contradiction [10, pp.62-64], the third uses an excircle to a triangle [12, p.69] and the fourth is an exquisite application of the Pythagorean theorem [1, pp.56-57]. The first two of these can also be found in [3, pp.65-67].

Two similar characterizations are the following ones. If $ABCD$ is a convex quadrilateral where opposite sides AB and CD intersect at E, and the sides AD and BC intersect at F (see Figure 1), then $ABCD$ is a tangential quadrilateral if and only if either of

$$BE + BF = DE + DF,$$
$$AE - AF = CE - CF.$$

These are given as problems in [3] and [14], where the first condition is proved using contradiction in [14, p.147]; the second is proved in the same way[4].

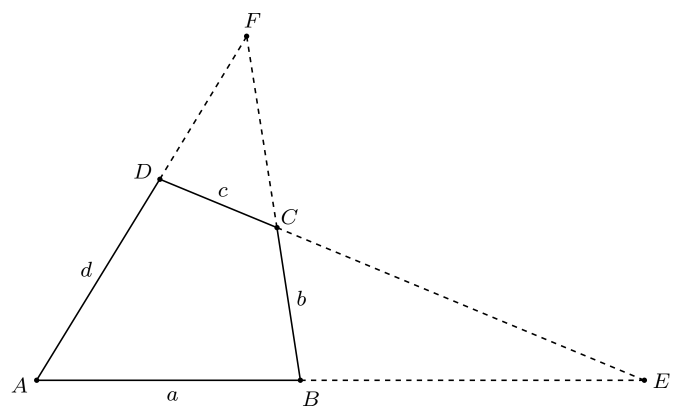

Figure 1. The extensions of the sides

2. Incircles in a quadrilateral and its subtriangles

One way of proving a new characterization is to show that it is equivalent to a previously proved one. This method will be used several times henceforth. In this section we prove three characterizations of tangential quadrilaterals by showing that they are equivalent to (1). The first was proved in another way in [19].

Theorem 1. *A convex quadrilateral is tangential if and only if the incircles in the two triangles formed by a diagonal are tangent to each other.*

[4]In [3, pp.186-187] only the direct theorems (not the converses) are proved.

Proof. In a convex quadrilateral $ABCD$, let the incircles in triangles ABC, CDA, BCD and DAB be tangent to the diagonals AC and BD at the points X, Y, Z and W respectively (see Figure 2). First we prove that

$$ZW = \tfrac{1}{2}\big|a - b + c - d\big| = XY.$$

Using the two tangent theorem, we have $BW = a - w$ and $BZ = b - z$, so

$$ZW = BW - BZ = a - w - b + z.$$

In the same way $DW = d - w$ and $DZ = c - z$, so

$$ZW = DZ - DW = c - z - d + w.$$

Adding these yields

$$2ZW = a - w - b + z + c - z - d + w = a - b + c - d.$$

Hence

$$ZW = \tfrac{1}{2}\big|a - b + c - d\big|$$

where we put an absolute value since Z and W can "change places" in some quadrilaterals; that is, it is possible for W to be closer to B than Z is. Then we would have $ZW = \frac{1}{2}(-a + b - c + d)$.

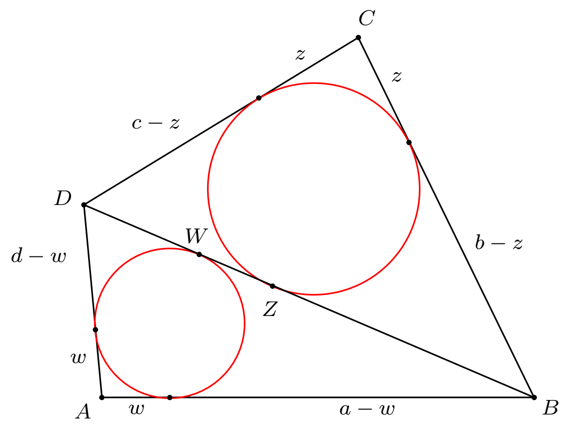

Figure 2. Incircles on both sides of one diagonal

The formula for XY is derived in the same way.

Now two incircles on different sides of a diagonal are tangent to each other if and only if $XY = 0$ or $ZW = 0$. These are equivalent to $a + c = b + d$, which proves the theorem according to the Pitot theorem. $\square$

Another way of formulating this result is that the incircles in the two triangles formed by one diagonal in a convex quadrilateral are tangent to each other if and only if the incircles in the two triangles formed by the other diagonal are tangent to each other. These two tangency points are in general not the same point, see Figure 3, where the notations are different from those in Figure 2.

Theorem 2. *The incircles in the four overlapping triangles formed by the diagonals of a convex quadrilateral are tangent to the sides in eight points, two per side, making one distance between tangency points per side. It is a tangential quadrilateral if and only if the sums of those distances at opposite sides are equal.*

Proof. According to the two tangent theorem, $AZ = AY$, $BS = BT$, $CU = CV$ and $DW = DX$, see Figure 3. Using the Pitot theorem, we get

$$\begin{aligned} & AB + CD = BC + DA \\ \Leftrightarrow\ & AZ + ZS + BS + CV + VW + DW = BT + TU + CU + DX + XY + AY \\ \Leftrightarrow\ & ZS + VW = TU + XY \end{aligned}$$

after cancelling eight terms. This is what we wanted to prove. $\square$

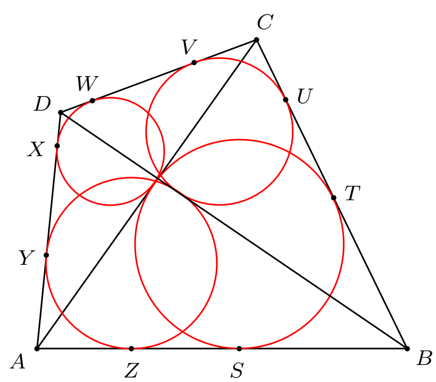

Figure 3. Incircles on both sides of both diagonals

The configuration with the four incircles in the last two theorems has other interesting properties. If the quadrilateral $ABCD$ is cyclic, then the four incenters are the vertices of a rectangle, see [2, p.133] or [3, pp.44-46].

A third example where the Pitot theorem is used to prove another characterization of tangential quadrilaterals is the following one, which is more or less the same as one given as a part of a Russian solution (see [18]) to a problem we will discuss in more detail in Section 4.

Theorem 3. *A convex quadrilateral is subdivided into four nonoverlapping triangles by its diagonals. Consider the four tangency points of the incircles in these triangles on one of the diagonals. It is a tangential quadrilateral if and only if the distance between two tangency points on one side of the second diagonal is equal to the distance between the two tangency points on the other side of that diagonal.*

Proof. Here we cite the Russian proof given in [18]. We use notations as in Figure 4 and shall prove that the quadrilateral has an incircle if and only if $T_1'T_2' = T_3'T_4'$.

By the two tangent theorem we have

$$AT_1 = AT_1'' = AP - PT_1'',$$
$$BT_1 = BT_1' = BP - PT_1',$$

so that

$$AB = AT_1 + BT_1 = AP + BP - PT_1'' - PT_1'.$$

Since $PT_1'' = PT_1'$,

$$AB = AP + BP - 2PT_1'.$$

In the same way

$$CD = CP + DP - 2PT_3'.$$

Adding the last two equalities yields

$$AB + CD = AC + BD - 2T_1'T_3'.$$

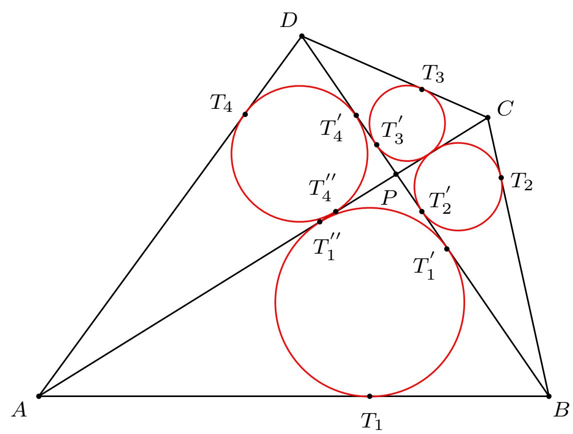

Figure 4. Tangency points of the four incircles

In the same way we get

$$BC + DA = AC + BD - 2T_2'T_4'.$$

Thus

$$AB + CD - BC - DA = -2\left(T_1'T_3' - T_2'T_4'\right).$$

The quadrilateral has an incircle if and only if $AB + CD = BC + DA$. Hence it is a tangential quadrilateral if and only if

$$T_1'T_3' = T_2'T_4' \quad \Leftrightarrow \quad T_1'T_2' + T_2'T_3' = T_2'T_3' + T_3'T_4' \quad \Leftrightarrow \quad T_1'T_2' = T_3'T_4'.$$

Note that both $T_1'T_3' = T_2'T_4'$ and $T_1'T_2' = T_3'T_4'$ are characterizations of tangential quadrilaterals. It was the first of these two that was proved in [18]. □

3. Characterizations with inradii, altitudes and exradii

According to Wu Wei Chao (see [20]), a convex quadrilateral $ABCD$ is tangential if and only if

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4},$$

where r_1, r_2, r_3 and r_4 are the inradii in triangles ABP, BCP, CDP and DAP respectively, and P is the intersection of the diagonals, see Figure 5.

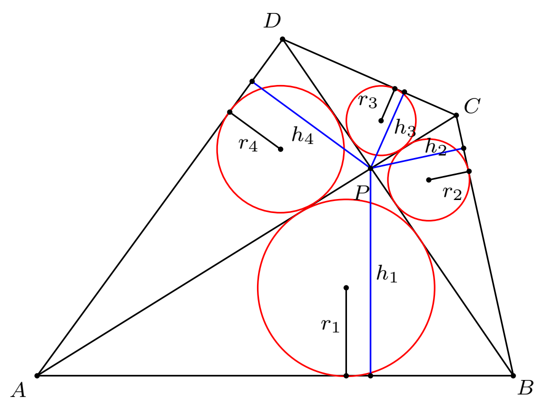

Figure 5. The inradii and altitudes

In [13] Nicuşor Minculete proved in two different ways that another characterization of tangential quadrilaterals is[5]

$$\frac{1}{h_1}+\frac{1}{h_3}=\frac{1}{h_2}+\frac{1}{h_4}, \tag{2}$$

where h_1, h_2, h_3 and h_4 are the altitudes in triangles ABP, BCP, CDP and DAP from P to the sides AB, BC, CD and DA respectively, see Figure 5. These two characterizations are closely related to the following one.

Theorem 4. *A convex quadrilateral $ABCD$ is tangential if and only if*

$$\frac{1}{R_1}+\frac{1}{R_3}=\frac{1}{R_2}+\frac{1}{R_4}$$

where R_1, R_2, R_3 and R_4 are the exradii to triangles ABP, BCP, CDP and DAP opposite the vertex P, the intersection of the diagonals AC and BD.

Proof. In a triangle, an exradius R_a is related to the altitudes by the well known relation

$$\frac{1}{R_a}=-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}. \tag{3}$$

[5] Although he used different notations.

If we denote the altitudes from A and C to the diagonal BD by h_A and h_C respectively and similar for the altitudes to AC, see Figure 6, then we have

$$\begin{aligned}\frac{1}{R_1} &= -\frac{1}{h_1}+\frac{1}{h_A}+\frac{1}{h_B},\\ \frac{1}{R_2} &= -\frac{1}{h_2}+\frac{1}{h_B}+\frac{1}{h_C},\\ \frac{1}{R_3} &= -\frac{1}{h_3}+\frac{1}{h_C}+\frac{1}{h_D},\\ \frac{1}{R_4} &= -\frac{1}{h_4}+\frac{1}{h_D}+\frac{1}{h_A}.\end{aligned}$$

Using these, we get

$$\frac{1}{R_1}+\frac{1}{R_3}-\frac{1}{R_2}-\frac{1}{R_4} = -\left(\frac{1}{h_1}+\frac{1}{h_3}-\frac{1}{h_2}-\frac{1}{h_4}\right).$$

Hence

$$\frac{1}{R_1}+\frac{1}{R_3}=\frac{1}{R_2}+\frac{1}{R_4} \quad\Leftrightarrow\quad \frac{1}{h_1}+\frac{1}{h_3}=\frac{1}{h_2}+\frac{1}{h_4}.$$

Since the equality to the right is a characterization of tangential quadrilaterals according to (2), so is the equality to the left. $\square$

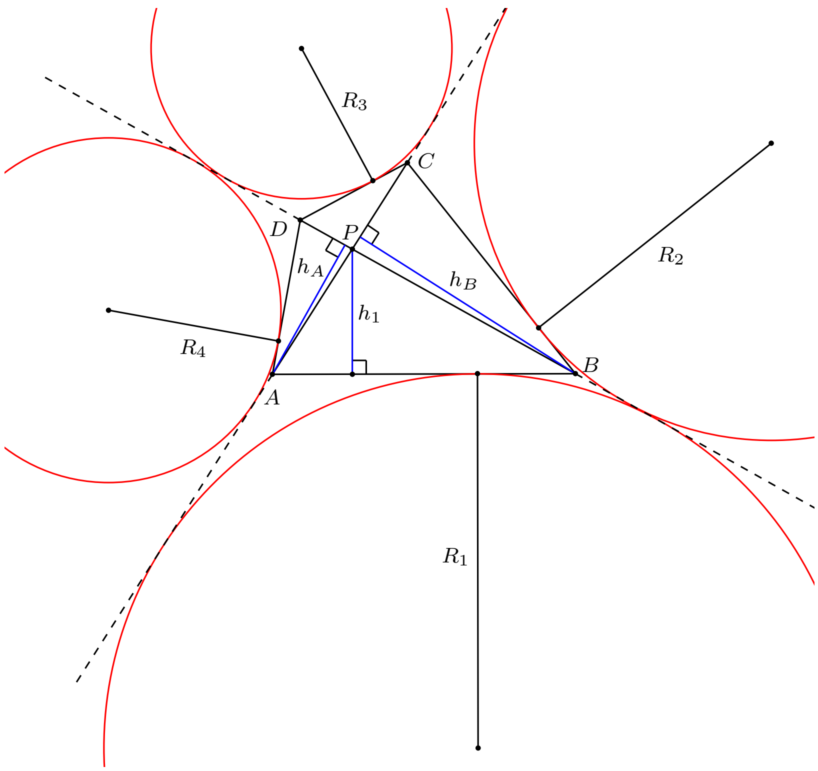

Figure 6. Excircles to four subtriangles

4. Christopher Bradley's conjecture and its generalizations

Consider the following problem:

In a tangential quadrilateral $ABCD$, let P be the intersection of the diagonals AC and BD. Prove that the incenters of triangles ABP, BCP, CDP and DAP form a cyclic quadrilateral. See Figure 7.

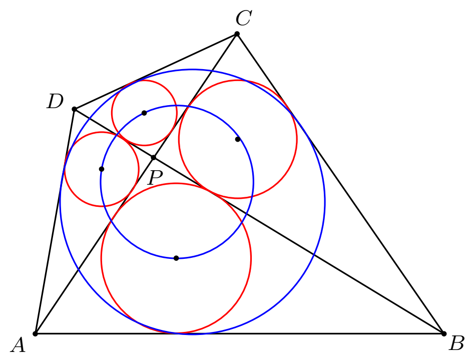

Figure 7. Christopher Bradley's conjecture

This problem appeared at the CTK Exchange[6] on September 17, 2003 [17], where it was debated for a month. On Januari 2, 2004, it migrated to the *Hyacinthos* problem solving group at Yahoo [15], and after a week a full synthetic solution with many extra properties of the configuration was given by Darij Grinberg [8] with the help of many others.

So why was this problem called *Christopher Bradley's conjecture*? In November 2004 a paper about cyclic quadrilaterals by the British mathematician Christopher Bradley was published, where the above problem was stated as a conjecture (see [4, p.430]). Our guess is that the conjecture was also published elsewhere more than a year earlier, which explains how it appeared at the CTK Exchange.

A similar problem, that is almost the converse, was given in 1998 by Toshio Seimiya in the Canadian problem solving journal Crux Mathematicorum [16]:

Suppose $ABCD$ is a convex cyclic quadrilateral and P is the intersection of the diagonals AC and BD. Let I_1, I_2, I_3 and I_4 be the incenters of triangles PAB, PBC, PCD and PDA respectively. Suppose that I_1, I_2, I_3 and I_4 are concyclic. Prove that $ABCD$ has an incircle.

The next year a beautiful solution by Peter Y. Woo was published in [16]. He generalized the problem to the following very nice characterization of tangential quadrilaterals:

When a convex quadrilateral is subdivided into four nonoverlapping triangles by its two diagonals, then the incenters of the four triangles are concyclic if and only if the quadrilateral has an incircle.

[6]It was formulated slightly different, where the use of the word inscriptable led to a misunderstanding.

There was however an even earlier publication of Woo's generalization. According to [8], the Russian magazine *Kvant* published in 1996 (see [18]) a solution by I. Vaynshtejn to the problem we have called Christopher Bradley's conjecture and its converse (see the formulation by Woo). [18] is written in Russian, so neither we nor many of the readers of Forum Geometricorum will be able to read that proof. But anyone interested in geometry can with the help of the figures understand the equations there, since they are written in the Latin alphabet.

Earlier we saw that Minculete's characterization with incircles was also true for excircles (Theorem 4). Then we might wonder if Vaynshtejn's characterization is also true for excircles? The answer is yes and it was proved by Nikolaos Dergiades at [6], even though he did not state it as a characterization of tangential quadrilaterals. The proof given here is a small expansion of his.

Theorem 5 (Dergiades)**.** *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if the four excenters to triangles ABP, BCP, CDP and DAP opposite the vertex P are concyclic.*

Proof. In a triangle ABC with sides a, b, c and semiperimeter s, where I and J_1 are the incenter and excenter opposite A respectively, and where r and R_a are the radii in the incircle and excircle respectively, we have $AI = \frac{r}{\sin\frac{A}{2}}$ and $AJ_1 = \frac{R_a}{\sin\frac{A}{2}}$. Using Heron's formula $T^2 = s(s-a)(s-b)(s-c)$ and other well known formulas[7], we have

$$AI \cdot AJ_1 = r \cdot R_a \cdot \frac{1}{\sin^2\frac{A}{2}} = \frac{T}{s} \cdot \frac{T}{s-a} \cdot \frac{bc}{(s-b)(s-c)} = bc. \tag{4}$$

Similar formulas hold for the other excenters.

Returning to the quadrilateral, let I_1, I_2, I_3 and I_4 be the incentes and J_1, J_2, J_3 and J_4 the excenters opposite P in triangles ABP, BCP, CDP and DAP respectively. Using (4) we get (see Figure 8)

$$\begin{aligned} PI_1 \cdot PJ_1 &= PA \cdot PB, \\ PI_2 \cdot PJ_2 &= PB \cdot PC, \\ PI_3 \cdot PJ_3 &= PC \cdot PD, \\ PI_4 \cdot PJ_4 &= PD \cdot PA. \end{aligned}$$

From these we get

$$PI_1 \cdot PI_3 \cdot PJ_1 \cdot PJ_3 = PA \cdot PB \cdot PC \cdot PD = PI_2 \cdot PI_4 \cdot PJ_2 \cdot PJ_4.$$

Thus

$$PI_1 \cdot PI_3 = PI_2 \cdot PI_4 \qquad \Leftrightarrow \qquad PJ_1 \cdot PJ_3 = PJ_2 \cdot PJ_4.$$

In his proof [16], Woo showed that the quadrilateral has an incircle if and only if the equality to the left is true. Hence the quadrilateral has an incircle if and only if the equality to the right is true. Both of these equalities are conditions for the four points I_1, I_2, I_3, I_4 and J_1, J_2, J_3, J_4 to be concyclic according to the converse of the intersecting chords theorem. $\square$

[7] Here and a few times later on we use the half angle theorems. For a derivation, see [9, p.158].

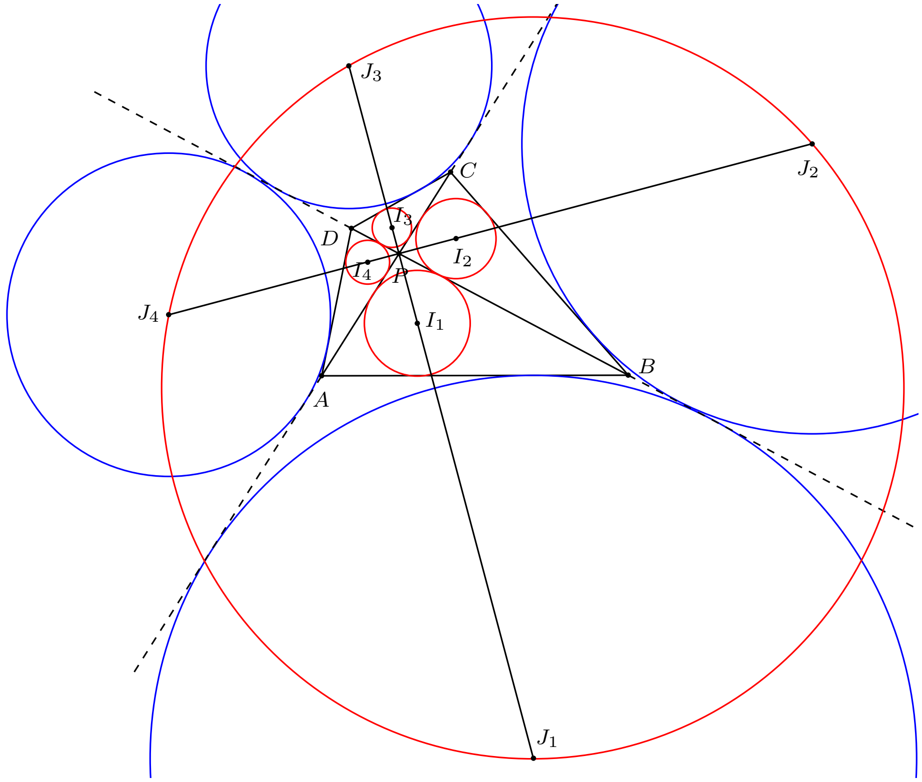

Figure 8. An excircle version of Vaynshtejn's characterization

Figure 8 suggests that $J_1J_3 \perp J_2J_4$ and $I_1I_3 \perp I_2I_4$. These are true in all convex quadrilaterals, and the proof is very simple. The incenters and excenters lies on the angle bisectors to the angles between the diagonals. Hence we have $\angle J_4PJ_1 = \angle I_4PI_1 = \frac{1}{2}\angle DPB = \frac{\pi}{2}$.

Another characterization related to the configuration of Christopher Bradley's conjecture is the following one. This is perhaps not one of the nicest characterizations, but the connection between opposite sides is present here as well as in many others. That the equality in the theorem is true in a tangential quadrilateral was established at [5].

Theorem 6. *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if*

$$\frac{(AP+BP-AB)(CP+DP-CD)}{(AP+BP+AB)(CP+DP+CD)} = \frac{(BP+CP-BC)(DP+AP-DA)}{(BP+CP+BC)(DP+AP+DA)}.$$

Proof. In a triangle ABC with sides a, b and c, the distance from vertex A to the incenter I is given by

$$AI = \frac{r}{\sin\frac{A}{2}} = \frac{\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}}{\sqrt{\frac{(s-b)(s-c)}{bc}}} = \sqrt{\frac{bc(s-a)}{s}} = \sqrt{\frac{bc(-a+b+c)}{a+b+c}}. \quad (5)$$

In a quadrilateral $ABCD$, let the incenters in triangles ABP, BCP, CDP and DAP be I_1, I_2, I_3 and I_4 respectively. Using (5), we get

$$PI_1 = \sqrt{\frac{PA \cdot PB(AP + BP - AB)}{AP + BP + AB}},$$
$$PI_3 = \sqrt{\frac{PC \cdot PD(CP + DP - CD)}{CP + DP + CD}}.$$

Thus in all convex quadrilaterals

$$(PI_1 \cdot PI_3)^2 = \frac{AP \cdot BP \cdot CP \cdot DP(AP + BP - AB)(CP + DP - CD)}{(AP + BP + AB)(CP + DP + CD)}$$

and in the same way we have

$$(PI_2 \cdot PI_4)^2 = \frac{AP \cdot BP \cdot CP \cdot DP(BP + CP - BC)(DP + AP - DA)}{(BP + CP + BC)(DP + AP + DA)}.$$

In [16], Woo proved that $PI_1 \cdot PI_3 = PI_2 \cdot PI_4$ if and only if $ABCD$ has an incircle. Hence it is a tangential quadrilateral if and only if

$$\frac{AP \cdot BP \cdot CP \cdot DP(AP + BP - AB)(CP + DP - CD)}{(AP + BP + AB)(CP + DP + CD)}$$
$$= \frac{AP \cdot BP \cdot CP \cdot DP(BP + CP - BC)(DP + AP - DA)}{(BP + CP + BC)(DP + AP + DA)}$$

from which the theorem follows. □

5. Iosifescu's characterization

In [13] Nicuşor Minculete cites a trigonometric characterization of tangential quadrilaterals due to Marius Iosifescu from the old Romanian journal [11]. We had never seen this nice characterization before and suspect no proof has been given in English, so here we give one. Since we have had no access to the Romanian journal we don't know if this is the same proof as the original one.

Theorem 7 (Iosifescu)**.** *A convex quadrilateral $ABCD$ is tangential if and only if*

$$\tan\frac{x}{2} \cdot \tan\frac{z}{2} = \tan\frac{y}{2} \cdot \tan\frac{w}{2}$$

where $x = \angle ABD$, $y = \angle ADB$, $z = \angle BDC$ and $w = \angle DBC$.

Proof. Using the trigonometric formula

$$\tan^2\frac{u}{2} = \frac{1 - \cos u}{1 + \cos u},$$

we get that the equality in the theorem is equivalent to

$$\frac{1 - \cos x}{1 + \cos x} \cdot \frac{1 - \cos z}{1 + \cos z} = \frac{1 - \cos y}{1 + \cos y} \cdot \frac{1 - \cos w}{1 + \cos w}.$$

This in turn is equivalent to

$$(1 - \cos x)(1 - \cos z)(1 + \cos y)(1 + \cos w)$$
$$= (1 - \cos y)(1 - \cos w)(1 + \cos x)(1 + \cos z). \qquad (6)$$

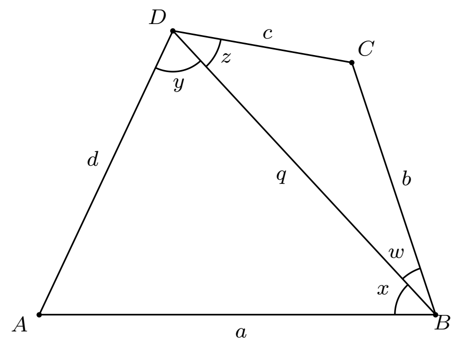

Figure 9. Angles in Iosifescu's characterization

Let $a = AB$, $b = BC$, $c = CD$, $d = DA$ and $q = BD$. From the law of cosines we have (see Figure 9)

$$\cos x = \frac{a^2 + q^2 - d^2}{2aq},$$

so that

$$1 - \cos x = \frac{d^2 - (a-q)^2}{2aq} = \frac{(d+a-q)(d-a+q)}{2aq}$$

and

$$1 + \cos x = \frac{(a+q)^2 - d^2}{2aq} = \frac{(a+q+d)(a+q-d)}{2aq}.$$

In the same way

$$1 - \cos y = \frac{(a+d-q)(a-d+q)}{2dq}, \qquad 1 + \cos y = \frac{(d+q+a)(d+q-a)}{2dq},$$

$$1 - \cos z = \frac{(b+c-q)(b-c+q)}{2cq}, \qquad 1 + \cos z = \frac{(c+q+b)(c+q-b)}{2cq},$$

$$1 - \cos w = \frac{(c+b-q)(c-b+q)}{2bq}, \qquad 1 + \cos w = \frac{(b+q+c)(b+q-c)}{2bq}.$$

Thus (6) is equivalent to

$$\frac{(d+a-q)(d-a+q)^2}{2aq} \cdot \frac{(b+c-q)(b-c+q)^2}{2cq} \cdot \frac{(d+q+a)}{2dq} \cdot \frac{(b+q+c)}{2bq}$$
$$= \frac{(a+d-q)(a-d+q)^2}{2dq} \cdot \frac{(c+b-q)(c-b+q)^2}{2bq} \cdot \frac{(a+q+d)}{2aq} \cdot \frac{(c+q+b)}{2cq}.$$

This is equivalent to

$$P\left((d-a+q)^2(b-c+q)^2 - (a-d+q)^2(c-b+q)^2\right) = 0 \qquad (7)$$

where

$$P = \frac{(d+a-q)(b+c-q)(d+q+a)(b+q+c)}{16abcdq^4}$$

is a positive expression according to the triangle inequality applied in triangles ABD and BCD. Factoring (7), we get

$$P\big((d-a+q)(b-c+q)+(a-d+q)(c-b+q)\big)$$
$$\cdot\big((d-a+q)(b-c+q)-(a-d+q)(c-b+q)\big)=0.$$

Expanding the inner parentheses and cancelling some terms, this is equivalent to

$$4qP(b+d-a-c)\left((d-a)(b-c)+q^2\right)=0. \tag{8}$$

The expression in the second parenthesis can never be equal to zero. Using the triangle inequality, we have $q>a-d$ and $q>b-c$. Thus $q^2 \gtrless (a-d)(b-c)$.

Hence, looking back at the derivation leading to (8), we have proved that

$$\tan\frac{x}{2}\cdot\tan\frac{z}{2}=\tan\frac{y}{2}\cdot\tan\frac{w}{2} \qquad \Leftrightarrow \qquad b+d=a+c$$

and Iosifescu's characterization is proved according to the Pitot theorem. □

6. Characterizations with other excircles

We have already seen two characterizations concerning the four excircles opposite the intersection of the diagonals. In this section we will study some other excircles. We begin by deriving a characterization similar to the one in Theorem 6, not for its own purpose, but because we will need it to prove the next theorem.

Theorem 8. *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if*

$$\frac{(AB+AP-BP)(CD+CP-DP)}{(AB-AP+BP)(CD-CP+DP)}=\frac{(BC-BP+CP)(DA-DP+AP)}{(BC+BP-CP)(DA+DP-AP)}.$$

Proof. It is well known that in a triangle ABC with sides a, b and c,

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{(a-b+c)(a+b-c)}{(a+b+c)(-a+b+c)}}$$

where s is the semiperimeter [9, p.158]. Now, if P is the intersection of the diagonals in a quadrilateral $ABCD$ and x, y, z, w are the angles defined in Theorem 7, we have

$$\tan\frac{x}{2}=\sqrt{\frac{(AB+AP-BP)(BP+AP-AB)}{(AB+AP+BP)(BP-AP+AB)}},$$
$$\tan\frac{z}{2}=\sqrt{\frac{(CD+CP-DP)(DP+CP-CD)}{(CD+CP+DP)(DP-CP+CD)}},$$
$$\tan\frac{y}{2}=\sqrt{\frac{(DA+AP-DP)(DP+AP-DA)}{(DA+AP+DP)(DP-AP+DA)}},$$
$$\tan\frac{w}{2}=\sqrt{\frac{(BC+CP-BP)(BP+CP-BC)}{(BC+CP+BP)(BP-CP+BC)}}.$$

From Theorem 7 we have the equality[8]

$$\tan^2\frac{x}{2}\cdot\tan^2\frac{z}{2}=\tan^2\frac{y}{2}\cdot\tan^2\frac{w}{2}$$

and putting in the expressions above we get

$$\frac{(AB+AP-BP)(BP+AP-AB)(CD+CP-DP)(DP+CP-CD)}{(AB+AP+BP)(BP-AP+AB)(CD+CP+DP)(DP-CP+CD)}$$
$$=\frac{(DA+AP-DP)(DP+AP-DA)(BC+CP-BP)(BP+CP-BC)}{(DA+AP+DP)(DP-AP+DA)(BC+CP+BP)(BP-CP+BC)}.$$

Now using Theorem 6, the conclusion follows.[9] □

Lemma 9. *If J_1 is the excenter opposite A in a triangle ABC with sides a, b and c, then*

$$\frac{(BJ_1)^2}{ac}=\frac{s-c}{s-a}$$

where s is the semiperimeter.

Proof. If R_a is the radius in the excircle opposite A, we have (see Figure 10)

$$\sin\frac{\pi-B}{2}=\frac{R_a}{BJ_1},$$
$$BJ_1\cos\frac{B}{2}=\frac{T}{s-a},$$
$$(BJ_1)^2\cdot\frac{s(s-b)}{ac}=\frac{s(s-a)(s-b)(s-c)}{(s-a)^2},$$

and the equation follows. Here T is the area of triangle ABC and we used Heron's formula. □

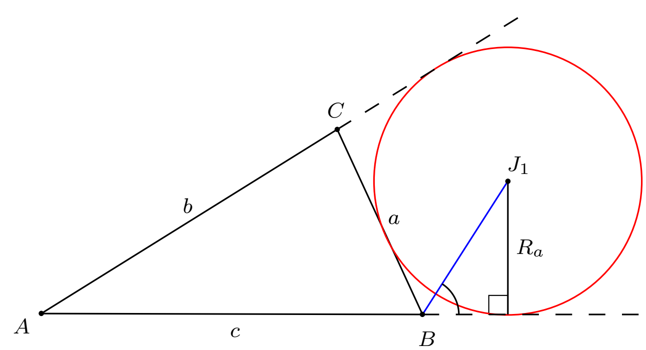

Figure 10. Distance from an excenter to an adjacent vertex

[8]This is a characterization of tangential quadrilaterals, but that's not important for the proof of this theorem.

[9]Here it is important that Theorem 6 is a characterization of tangential quadrilaterals.

Theorem 10. *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if the four excenters to triangles ABP, BCP, CDP and DAP opposite the vertices B and D are concyclic.*

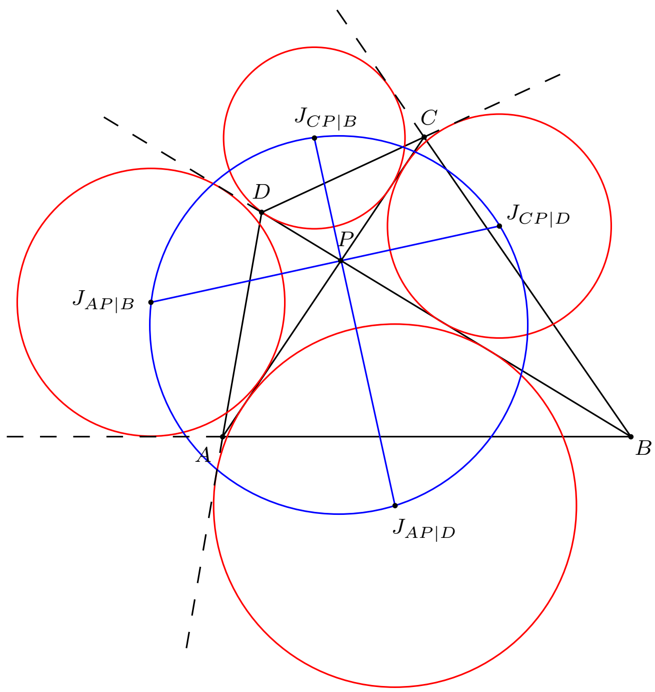

Figure 11. Excircles to subtriangles opposite the vertices B and D

Proof. We use the notation $J_{AP|B}$ for the excenter in the excircle tangent to side AP opposite B in triangle ABP. Using the Lemma in triangles ABP, BCP, CDP and DAP yields (see Figure 11)

$$\frac{(PJ_{AP|B})^2}{AP\cdot BP}=\frac{AB+AP-BP}{AB-AP+BP},$$
$$\frac{(PJ_{CP|D})^2}{CP\cdot DP}=\frac{CD+CP-DP}{CD-CP+DP},$$
$$\frac{(PJ_{CP|B})^2}{CP\cdot BP}=\frac{BC+CP-BP}{BC-CP+BP},$$
$$\frac{(PJ_{AP|D})^2}{AP\cdot DP}=\frac{DA+AP-DP}{DA-AP+DP}.$$

From Theorem 8 we get that $ABCD$ is a tangential quadrilateral if and only if

$$\frac{(PJ_{AP|B})^2}{AP\cdot BP}\cdot\frac{(PJ_{CP|D})^2}{CP\cdot DP}=\frac{(PJ_{CP|B})^2}{CP\cdot BP}\cdot\frac{(PJ_{AP|D})^2}{AP\cdot DP},$$

which is equivalent to

$$PJ_{AP|B} \cdot PJ_{CP|D} = PJ_{CP|B} \cdot PJ_{AP|D}. \tag{9}$$

Now $J_{AP|B}J_{CP|D}$ and $J_{CP|B}J_{AP|D}$ are straight lines through P since they are angle bisectors to the angles between the diagonals in $ABCD$. According to the intersecting chords theorem and its converse, (9) is a condition for the excenters to be concyclic. □

There is of course a similar characterization where the excircles are opposite the vertices A and C.

We conclude with a theorem that resembles Theorem 4, but with the excircles in Theorem 10.

Theorem 11. *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if*

$$\frac{1}{R_a} + \frac{1}{R_c} = \frac{1}{R_b} + \frac{1}{R_d},$$

where R_a, R_b, R_c and R_d are the radii in the excircles to triangles ABP, BCP, CDP and DAP respectively opposite the vertices B and D.

Proof. We use notations on the altitudes as in Figure 12, which are the same as in the proof of Theorem 4. From (3) we have

$$\begin{aligned}
\frac{1}{R_a} &= -\frac{1}{h_B} + \frac{1}{h_A} + \frac{1}{h_1}, \\
\frac{1}{R_b} &= -\frac{1}{h_B} + \frac{1}{h_C} + \frac{1}{h_2}, \\
\frac{1}{R_c} &= -\frac{1}{h_D} + \frac{1}{h_C} + \frac{1}{h_3}, \\
\frac{1}{R_d} &= -\frac{1}{h_D} + \frac{1}{h_A} + \frac{1}{h_4}.
\end{aligned}$$

These yield

$$\frac{1}{R_a} + \frac{1}{R_c} - \frac{1}{R_b} - \frac{1}{R_d} = \frac{1}{h_1} + \frac{1}{h_3} - \frac{1}{h_2} - \frac{1}{h_4}.$$

Hence

$$\frac{1}{R_a} + \frac{1}{R_c} = \frac{1}{R_b} + \frac{1}{R_d} \quad \Leftrightarrow \quad \frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}.$$

Since the equality to the right is a characterization of tangential quadrilaterals according to (2), so is the equality to the left. □

Even here there is a similar characterization where the excircles are opposite the vertices A and C.

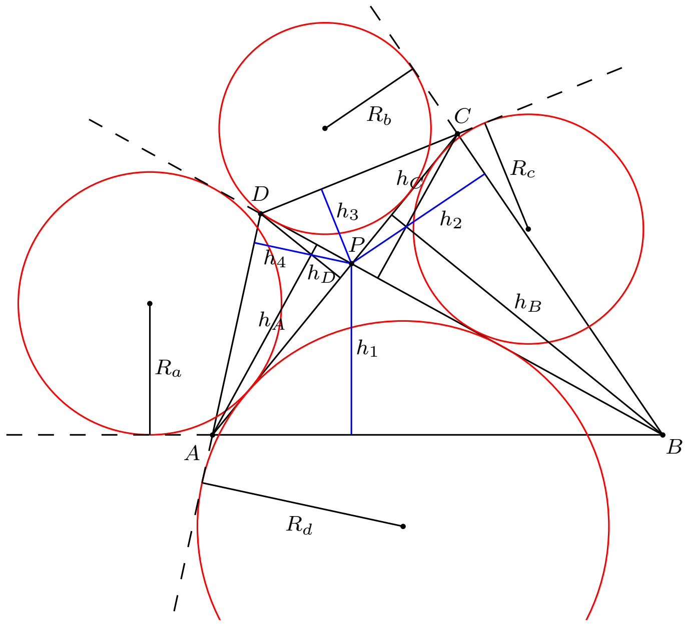

Figure 12. The exradii and altitudes

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Forum Geometricorum
Volume 11 (2011) 83–93.

FORUM GEOM
ISSN 1534-1178

Perspective Isoconjugate Triangle Pairs, Hofstadter Pairs, and Crosssums on the Nine-Point Circle

Peter J. C. Moses and Clark Kimberling

Abstract. The r-Hofstadter triangle and the $(1-r)$-Hofstadter triangle are proved perspective, and homogeneous trilinear coordinates are found for the perspector. More generally, given a triangle DEF inscribed in a reference triangle ABC, triangles $A'B'C'$ and $A''B''C''$ derived in a certain manner from DEF are perspective to each other and to ABC. Trilinears for the three perspectors, denoted by P^*, P_1, P_2 are found (Theorem 1) and used to prove that these three points are collinear. Special cases include (Theorems 4 and 5) this: if X and X' are an antipodal pair on the circumcircle, then the perspector $P^* = X \oplus X'$, where $\oplus$ denotes crosssum, is on the nine-point circle. Taking X to be successively the vertices of a triangle DEF inscribed in the circumcircle thus yields a triangle $D'E'F'$ inscribed in the nine-point circle. For example, if DEF is the circumtangential triangle, then $D'E'F'$ is an equilateral triangle.

1. Introduction and main theorem

We begin with a very general theorem about three triangles, one being the reference triangle ABC with sidelengths a, b, c, and the other two, denoted by $A'B'C'$ and $A''B''C''$, which we shall now proceed to define. Suppose DEF is a triangle inscribed in ABC; that is, the vertices are given by homogeneous trilinear coordinates (henceforth simply *trilinears*) as follows:

$$D = 0 : y_1 : z_1, \quad E = x_2 : 0 : z_2, \quad F = x_3 : y_3 : 0, \tag{1}$$

where $y_1z_1x_2z_2x_3y_3 \neq 0$ (this being a quick way to say that none of the points is A, B, C). For any point $P = p : q : r$ for which $pqr \neq 0$, define

$$D' = 0 : \frac{1}{qy_1} : \frac{1}{rz_1}, \quad E' = \frac{1}{px_2} : 0 : \frac{1}{rz_2}, \quad F' = \frac{1}{px_3} : \frac{1}{qy_3} : 0.$$

Define $A'B'C'$ and introduce symbols for trilinears of the vertices A', B', C' :

$$\begin{aligned} A' =&\ CF \cap BE' = u_1 : v_1 : w_1, \\ B' =&\ AD \cap CF' = u_2 : v_2 : w_2, \\ C' =&\ BE \cap AD' = u_3 : v_3 : w_3, \end{aligned}$$

Publication Date: April 1, 2011. Communicating Editor: Paul Yiu.

and similarly,

$$\begin{aligned} A'' &= CF' \cap BE = \frac{1}{pu_1} : \frac{1}{qv_1} : \frac{1}{rw_1}, \\ B'' &= AD' \cap CF = \frac{1}{pu_2} : \frac{1}{qv_2} : \frac{1}{rw_2}, \\ C'' &= BE' \cap AD = \frac{1}{pu_3} : \frac{1}{qv_3} : \frac{1}{rw_3}. \end{aligned}$$

Thus, triangles $A'B'C'$ and $A''B''C''$ are a P-isoconjugate pair, in the sense that every point on each is the P-isoconjugate of a point on the other (except for points on a sideline of ABC). (The P-isoconjugate of a point $X = x : y : z$ is the point $1/(px) : 1/(qy) : 1/(rz)$; this is the isogonal conjugate of X if P is the incenter, and the isotomic conjugate of X if $P = X_{31} = a^2 : b^2 : c^2$. Here and in the sequel, the indexing of triangles centers in the form X_i follows that of [4].)

Theorem 1. *The triangles ABC, $A'B'C'$, $A''B''C''$ are pairwise perspective, and the three perspectors are collinear.*

Proof. The lines CF given by $-y_3\alpha + x_3\beta = 0$ and BE' given by $px_2\alpha - rz_2\gamma = 0$, and cyclic permutations, give

$$\begin{aligned} A' &= CF \cap BE' = u_1 : v_1 : w_1 = rx_3z_2 : ry_3z_2 : px_2x_3, \\ B' &= AD \cap CF' = u_2 : v_2 : w_2 = qy_1y_3 : px_3y_1 : px_3z_1, \\ C' &= BE \cap AD' = u_3 : v_3 : w_3 = qy_1x_2 : rz_2z_1 : qy_1z_2; \\ A'' &= CF' \cap BE = qx_2y_3 : px_2x_3 : qy_3z_2, \\ B'' &= AD' \cap CF = rz_1x_3 : ry_3z_1 : qy_3y_1, \\ C'' &= BE' \cap AD = rz_1z_2 : px_2y_1 : pz_1x_2. \end{aligned}$$

Then the line $B'B''$ is given by $d_1\alpha + d_2\beta + d_3\gamma = 0$, where

$$d_1 = px_3y_3(qy_1^2 - rz_1^2), \quad d_2 = prx_3^2y_1^2 - q^2y_1^2y_3^2, \quad d_3 = ry_1z_1(qy_3^2 - px_3^2),$$

and the line $C'C''$ by $d_4\alpha + d_5\beta + d_6\gamma = 0$, where

$$d_4 = px_2z_2(rz_1^2 - qy_1^2), \quad d_5 = qy_1z_1(rz_2^2 - px_2^2), \quad d_6 = pqx_2^2y_1^2 - r^2z_1^2z_2^2.$$

The perspector of $A'B'C'$ and $A''B''C''$ is $B'B'' \cap C'C''$, with trilinears

$$d_2d_6 - d_3d_5 : d_3d_4 - d_1d_6 : d_1d_5 - d_2d_4.$$

These coordinates share three common factors, which cancel, leaving the perspector

$$P^* = qx_2y_1y_3 + rx_3z_1z_2 : ry_3z_1z_2 + px_2x_3y_1 : px_2x_3z_1 + qy_1y_3z_2. \quad (2)$$

Next, we show that the lines AA', BB', CC' concur. These lines are given, respectively, by

$$-px_2x_3\beta + ry_3z_2\gamma = 0, \quad pz_1x_3\alpha - qy_3y_1\gamma = 0, \quad -rz_1z_2\alpha + qx_2y_1\beta = 0,$$

from which it follows that the perspector of ABC and $A'B'C'$ is the point

$$P_1 = AA' \cap BB' \cap CC' = qx_2y_1y_3 : ry_3z_1z_2 : px_2x_3z_1. \quad (3)$$

The same method shows that the lines AA'', BB'', CC'' concur in the P-isoconjugate of P_1 :

$$P_2 = AA'' \cap BB'' \cap CC'' = rx_3z_1z_2 : py_1x_2x_3 : qy_1y_3z_2. \tag{4}$$

Obviously,

$$\begin{vmatrix} qx_2y_1y_3 + rx_3z_1z_2 & ry_3z_1z_2 + px_2x_3y_1 & px_2x_3z_1 + qy_1y_3z_2 \\ qx_2y_1y_3 & ry_3z_1z_2 & px_2x_3z_1 \\ rx_3z_1z_2 & py_1x_2x_3 & qy_1y_3z_2 \end{vmatrix} = 0,$$

so that the three perspectors are collinear. □

Example 1. Let $P = 1 : 1 : 1$ (the incenter), and let DEF be the cevian triangle of the centroid, so that $D = 0 : ca : ab$, etc. Then

$$P_1 = \frac{c}{b} : \frac{a}{c} : \frac{b}{a} \quad \text{and} \quad P_2 = \frac{b}{c} : \frac{c}{a} : \frac{a}{b},$$

these being the 1st and 2nd Brocard points, and

$$P^* = a(b^2 + c^2) : b\left(c^2 + a^2\right) : c(a^2 + b^2),$$

the midpoint X_{39} of segment P_1P_2.

2. Hofstadter triangles

Suppose r is a nonzero real number. Following ([3], p 176, 241), regard vertex B as a pivot, and rotate segment BC toward vertex A through angle rB. Let L_{BC} denote the line containing the rotated segment. Similarly, obtain line L_{CB} by rotating segment BC about C through angle rC. Let $A' = L_{BC} \cap L_{CB}$, and obtain similarly points B' and C'. The r-Hofstadter triangle is $A'B'C'$, and the $(1-r)$-Hofstadter triangle $A''B''C''$ is formed in the same way using angles $(1-r)A$, $(1-r)B$, $(1-r)C$.

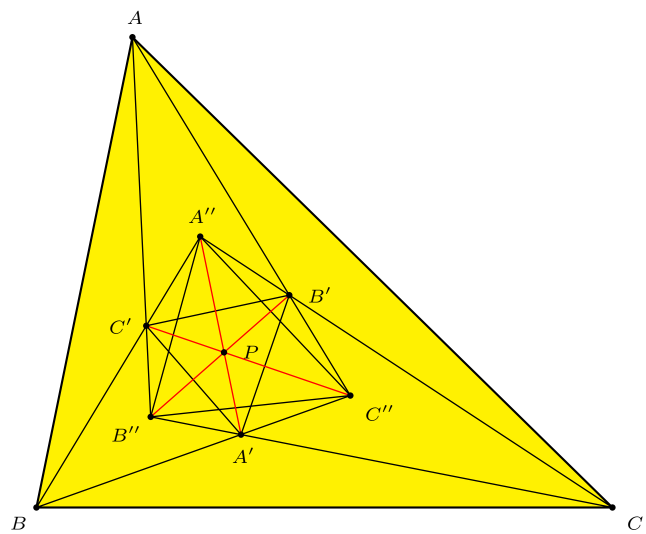

Figure 1. Hofstadter triangles $A'B'C'$ and $A''B''C''$

Trilinears for A' and A'' are easily found, and appear here in rows 2 and 3 of an equation for line $A'A''$:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \sin rB \sin rC & \sin rB \sin(C - rC) & \sin rC \sin(B - rB) \\ \sin(B - rB)\sin(C - rC) & \sin rC \sin(B - rB) & \sin rB \sin(C - rC) \end{vmatrix} = 0.$$

Lines $B'B''$ and $C'C''$ are similarly obtained, or obtained from $A'A''$ by cyclic permutations of symbols. It is then found by computer that the perspectivity determinant is 0 and that the perspector of the r- and $(1-r)$-Hofstadter triangles is the point

$$\begin{aligned} P(r) = \; & \sin(A - rA)\sin rB \sin rC + \sin rA \sin(B - rB)\sin(C - rC) \\ & : \sin(B - rB)\sin rC \sin rA + \sin rB \sin(C - rC)\sin(A - rA) \\ & : \sin(C - rC)\sin rA \sin rB + \sin rC \sin(A - rA)\sin(B - rB). \end{aligned}$$

The domain of P excludes 0 and 1. When r is any other integer, it can be checked that $P(r)$, written as $u : v : w$, satisfies

$$u \sin A + v \sin B + w \sin C = 0,$$

which is to say that $P(r)$ lies on the line L^∞ at infinity. For example, $P(2)$, alias $P(-1)$, is the point X_{30} in which the Euler line meets L^∞. Also, $P\left(\frac{1}{2}\right) = X_1$, the incenter, and $P\left(-\frac{1}{2}\right) = P\left(\frac{3}{2}\right) = X_{1770}$. Regarding $r = 1$ and $r = 0$, we obtain, *as limits*, the Hofstadter one-point and Hofstadter zero-point:

$$\begin{aligned} P(1) = \; & X_{359} = \frac{a}{A} : \frac{b}{B} : \frac{c}{C}, \\ P(0) = \; & X_{360} = \frac{A}{a} : \frac{B}{b} : \frac{C}{c}, \end{aligned}$$

remarkable because of the "exposed" vertex angles A, B, C. Another example is $P\left(\frac{1}{3}\right) = X_{356}$, the centroid of the Morley triangle.

This scattering of results can be supplemented by a more systematic view of selected points $P(r)$. In Figure 2, the specific triangle $(a, b, c) = (6, 9, 13)$ is used to show the points $P\left(r + \frac{1}{2}\right)$ for $r = 0, 1, 2, 3, \ldots, 5000$.

If the swing angles $rA, rB, rC, (1-r)B, (1-r)B, (1-r)C$ are generalized to $rA + \theta$, $rB + \theta$, $rC + \theta$, $(1-r)B + \theta$, $(1-r)B + \theta$, $(1-r)C + \theta$, then the perspector is given by

$$\begin{aligned} P(r, \theta) = \; & \sin(A - rA + \theta)\sin(rB - \theta)\sin(rC - \theta) \\ & + \sin(rA - \theta)\sin(B - rB + \theta)\sin(C - rC + \theta) \\ & : \sin(B - rB + \theta)\sin(rC - \theta)\sin(rA - \theta) \\ & + \sin(rB - \theta)\sin(C - rC + \theta)\sin(A - rA + \theta) \\ & : \sin(C - rC + \theta)\sin(rA - \theta)\sin(rB - \theta) \\ & = + \sin(rC - \theta)\sin(A - rA + \theta)\sin(B - rB + \theta). \end{aligned}$$

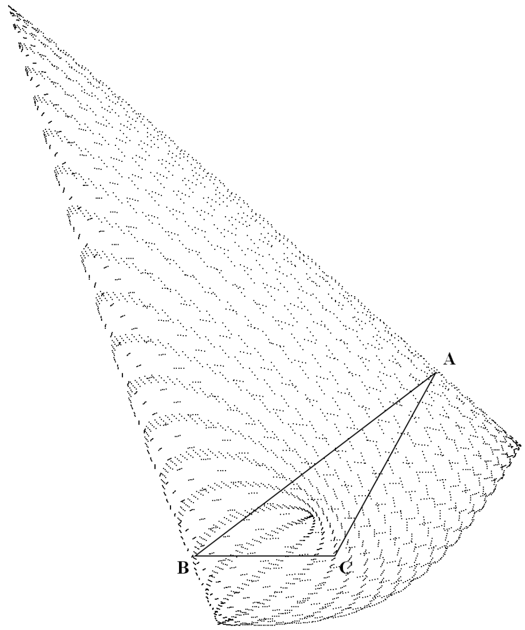

Figure 2

Trilinears for the other two perspectors are given by

$$\begin{aligned} P_1(r,\theta) =&\ \sin(rA-\theta)\csc(A-rA+\theta)\\ &: \sin(rB-\theta)\csc(B-rB+\theta) : \sin(rC-\theta)\csc(A-rC+\theta);\\ P_2(r,\theta) =&\ \sin(A-rA+\theta)\csc(rA-\theta)\\ &: \sin(B-rB+\theta)\csc(rB-\theta) : \sin(A-rC+\theta)\csc(rC-\theta). \end{aligned}$$

If $0 < \theta < 2\pi$, then $P(0,\theta)$ is defined, and taking the limit as $\theta \to 0$ enables a definition of $P(0,0)$. Then, remarkably, the locus of $P(0,\theta)$ for $0 \leq \theta < 2\pi$ is the Euler line. Six of its points are indicated here:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$(1/2)\arccos(5/2)$
$P(0,\theta)$	X_2	X_5	X_4	X_{30}	X_{20}	X_{549}

In general,

$$P\left(0,\frac{1}{2}\arccos t\right) = t\cos A+\cos B\cos C : t\cos B+\cos C\cos A : t\cos C+\cos A\cos B.$$

Among intriguing examples are several for which the angle θ, as a function of a,b,c (or A,B,C), is not constant:

if $\theta =$	then $P(0,\theta) =$
$\omega = \arctan \frac{4\sigma}{a^2+b^2+c^2}$(the Brocard angle)	X_{384}
$\frac{1}{2}\arccos\left(3 - \frac{\lvert OI\rvert}{2R^2}\right)$	X_{21}
$\frac{1}{2}\arccos\frac{\lvert OI\rvert}{2R^2}$	X_{441}
$\frac{1}{2}\arccos(-1 - 2\cos^2 A\cos^2 B\cos^2 C)$	X_{22}
$\frac{1}{2}\arccos\left(-\frac{1}{3} - \frac{4}{3}\cos^2 A\cos^2 B\cos^2 C\right)$	X_{23}

where

$$\begin{aligned} \sigma = &\ \text{area of } ABC, \\ |OI| = &\ \text{distance between the circumcenter and the incenter}, \\ R = &\ \text{circumradius of } ABC. \end{aligned}$$

3. Cevian triangles

The cevian triangle of a point $X = x : y : z$ is defined by (1) on putting $(x_i, y_i, z_i) = (x, y, z)$ for $i = 1, 2, 3$. Suppose that X is a triangle center, so that

$$X = g(a,b,c) : g(b,c,a) : g(c,a,b)$$

for a suitable function $g\,(a,b,c)$. Abbreviating this as $X = g_a : g_b : g_c$, the cevian triangle of X is then given by

$$D = 0 : g_b : g_c, \quad E = g_a : 0 : g_c, \quad F = g_a : g_b : 0,$$

and the perspector of the derived triangles $A'B'C'$ and $A''B''C''$ in Theorem 1 is given by

$$P^* = g_a\left(qg_b^2 + rg_c^2\right) : g_b\left(pg_a^2 + rg_c^2\right) : g_c\left(pg_a^2 + qg_b^2\right),$$

which is a triangle center, namely the crosspoint (defined in the Glossary of [4]) of U and the P-isoconjugate of U. In this case, the other two perspectors are

$$P_1 = qg_ag_b^2 : rg_bg_c^2 : pg_cg_a^2 \text{ and } P_2 = rg_ag_c^2 : pg_bg_a^2 : qg_cg_b^2.$$

4. Pedal triangles

Suppose $X = x : y : z$ is a point for which $xyz \neq 0$. The pedal triangle DEF of X is given by

$$D = 0 : y+xc_1 : z+xb_1, \quad E = x+yc_1 : 0 : z+ya_1, \quad F = x+zb_1 : y+xa_1 : 0,$$

where

$$\begin{aligned} (a_1,b_1,c_1) = &\ (\cos A, \cos B, \cos C) \\ = &\ \left(\frac{b^2+c^2-a^2}{2bc}, \frac{c^2+a^2-b^2}{2ca}, \frac{a^2+b^2-c^2}{2ab}\right). \end{aligned}$$

The three perspectors as in Theorem 1 are given, as in (2)-(4) by

$$P^* = u + u' : v + v' : w + w', \tag{5}$$
$$P_1 = u : v : w, \tag{6}$$
$$P_2 = u' : v' : w', \tag{7}$$

where

$$\begin{aligned} u &= q(x + yc_1)(y + xc_1)(y + za_1), \\ v &= r(y + za_1)(z + ya_1)(z + xb_1), \\ w &= p(z + xb_1)(x + zb_1)(x + yc_1); \\ u' &= r(x + zb_1)(z + xb_1)(z + ya_1), \\ v' &= p(y + xc_1)(x + yc_1)(x + zb_1), \\ w' &= q(z + ya_1)(y + za_1)(y + xc_1). \end{aligned}$$

The perspector P^* is notable in two cases which we shall now consider: when X is on the line at infinity, L^∞, and when X is on the circumcircle, Γ.

Theorem 2. *Suppose DEF is the pedal triangle of a point X on L^∞. Then the perspectors P^*, P_1, P_2 are invariant of X, and P^* lies on L^∞.*

Proof. The three perspectors as in Theorem 1 are given as in (5)-(7) by

$$P^* = a\left(b^2r - c^2q\right) : b\left(c^2p - a^2r\right) : c\left(a^2q - b^2p\right), \tag{8}$$
$$P_1 = qac^2 : rba^2 : pcb^2,$$
$$P_2 = rab^2 : pbc^2 : qca^2.$$

Clearly, the trilinears in (8) satisfy the equation $a\alpha + b\beta + c\gamma = 0$ for L^∞. □

Example 2. For $P = 1 : 1 : 1 = X_1$, we have $P^* = a(b^2 - c^2) : b(c^2 - a^2) : c(a^2 - b^2)$, indexed in ETC as X_{512}. This and other examples are included in the following table.

P	661	1	6	32	663	649	667	19	25	184	48	2
P^*	511	512	513	514	517	518	519	520	521	522	523	788

We turn now to the case that X is on Γ, so that pedal triangle of X is degenerate, in the sense that the three vertices D, E, F are collinear ([1], [3]**)**. The line DEF is known as the pedal line of X. We restrict the choice of P to the point X_{31} :

$$P = a^2 : b^2 : c^2,$$

so that the P-isoconjugate of a point is the isotomic conjugate of the point.

Theorem 3. *Suppose X is a point on the circumcircle of ABC, and $P = a^2 : b^2 : c^2$. Then the perspector P^*, given by*

$$\begin{aligned} P^* = \ & bcx\left(y^2 - z^2\right)\left(ax(bz - cy) + yz(b^2 - c^2)\right) \\ & : cay\left(z^2 - x^2\right)\left(by(cx - az) + zx(c^2 - a^2)\right) \\ & : abz\left(x^2 - y^2\right)\left(cz(ay - bx) + xy(a^2 - b^2)\right), \end{aligned} \tag{9}$$

lies on the nine-point circle.

Proof. Since X satisfies $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$, we can and do substitute $z=-\frac{cxy}{ay+bx}$ in (8), obtaining

$$P^* = \alpha : \beta : \gamma, \tag{10}$$

where

$$\begin{aligned}
\alpha =&\ ycb\,(ay+bx-cx)\,(ay+bx+cx)\left(2abx+a^2y+b^2y-c^2y\right),\\
\beta =&\ xca\left(2aby+a^2x+b^2x-c^2x\right)(ay+bx-cy)\,(ay+bx+cy)\,,\\
\gamma =&\ ab\,(ay+bx)\left(b^3x-a^3y-a^2bx+ab^2y+ac^2y-bc^2x\right)(x+y)\,(y-x)\,.
\end{aligned}$$

An equation for the nine-point circle [5] is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta - (1/2)(a\alpha+b\beta+c\gamma)(a_1\alpha+b_1\beta+c_1\gamma) = 0, \tag{11}$$

and using a computer, we find that P^* indeed satisfies (11). Using $ay+by=-\frac{cxy}{z}$, one can verify that the trilinears in (10) yield those in (9). □

A description of the perspector P^* in Theorem 3 is given in Theorem 4, which refers to the antipode X' of X, defined as the reflection of X in the circumcenter, O; i.e., X' is the point on Γ that is on the opposite side of the diameter that contains X. Theorem 4 also refers to the crosssum of two points, defined (Glossary of [4]) for points $U = u : v : w$ and $U' = u' : v' : w'$ by

$$U \oplus U' = vw' + wv' : wu' + uw' : uv' + vu'.$$

Theorem 4. *Suppose X is a point on the circumcircle of ABC, and let X' denote the antipode of X. Then $P^* = X \oplus X'$.*

Proof. [1] Since X is an arbitrary point on Γ, there exists θ such that

$$X = \csc\theta : \csc(C-\theta) : -\csc(B+\theta),$$

where θ, understood here a function of a, b, c, is defined ([6], [3, p. 39]) by

$$0 \le 2\theta = \measuredangle AOX < \pi,$$

so that the antipode of X is

$$X' = \sec\theta : -\sec(C-\theta) : -\sec(B+\theta).$$

The crosssum of the two antipodes is the point $X \oplus X' = \alpha : \beta : \gamma$ given by

$$\begin{aligned}
\alpha =&\ -\csc(C-\theta)\sec(B+\theta) + \csc(B+\theta)\sec(C-\theta)\\
\beta =&\ -\csc(B+\theta)\sec\theta - \csc\theta\sec(B+\theta)\\
\gamma =&\ -\csc\theta\sec(C-\theta) + \csc(C-\theta)\sec\theta.
\end{aligned}$$

It is easy to check by computer that $\alpha : \beta : \gamma$ satisfies (11). □

[1]This proof includes a second proof that P^* lies on the nine-point circle.

Example 3. In Theorems 3 and 4, let $X = X_{1113}$, this being a point of intersection of the Euler line and the circumcircle. The antipode of X is X_{1114}, and we have

$$X_{1113} \oplus X_{1114} = X_{125},$$

the center of the Jerabek hyperbola, on the nine-point circle.

Example 4. The antipode of the Tarry point, X_{98}, is the Steiner point, X_{99}, and

$$X_{98} \oplus X_{99} = X_{2679}.$$

Example 5. The antipode of the point, $X_{101} = \frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}$ is X_{103}, and

$$X_{101} \oplus X_{103} = X_{1566}.$$

Example 6. The Euler line meets the line at infinity in the point X_{30}, of which the isogonal conjugate on the circumcircle is the point

$$X_{74} = \frac{1}{\cos A - 2\cos B \cos C} : \frac{1}{\cos B - 2\cos C \cos A} : \frac{1}{\cos C - 2\cos A \cos B}.$$

The antipode of X_{74} is the center of the Kiepert hyperbola, given by

$$X_{110} = \csc(B - C) : \csc(C - A) : \csc(A - B),$$

and we have

$$X_{74} \oplus X_{110} = X_{3258}.$$

Our final theorem gives a second description of the perspector $X \oplus X'$ in (9). The description depends on the point $X(\text{medial})$, this being functional notation, read "X of medial (triangle)", in the same way that $f(x)$ is read "f of x"; the variable triangle to which the function X is applied is the cevian triangle of the centroid, whose vertices are the midpoint of the sides of the reference triangle ABC. Clearly, if X lies on the circumcircle of ABC, then $X(\text{medial})$ lies on the nine-point circle of ABC.

Theorem 5. *Let X' be the antipode of X. Then $X \oplus X'$ is a point of intersection of the nine-point circle and the line of the following two points: the isogonal conjugate of X and $X(\textit{medial})$.*

Proof. Trilinears for $X(\text{medial})$ are given ([3, p. 86]) by

$$(by + cz)/a : (cz + ax)/b : (ax + by)/c.$$

Writing $u : v : w$ for trilinears for $X \oplus X'$ and $yz : zx : xy$ for the isogonal conjugate of X, and putting $z = -cxy/(ay + bx)$ because $X \in \Gamma$, we find

$$\begin{vmatrix} u & v & w \\ yz & zx & xy \\ (by+cz)/a & (cz+ax)/b & (ax+by)/c \end{vmatrix} = 0,$$

so that the three points are collinear. □

As a source of further examples for Theorems 4 and 5, suppose D, E, F are points on the circumcircle. Let D', E', F' be the respective antipodes of D, E, F, so that the triangle $D'E'F'$ is the reflection in the circumcenter of triangle DEF. Let

$$D'' = D \oplus D', \quad E'' = E \oplus E', \quad F'' = F \oplus F',$$

so that $D''E''F''$ is inscribed in the nine-point circle.

Example 7. If DEF is the circumcevian triangle of the incenter, then $D''E''F''$ is the medial triangle.

Example 8. If DEF is the circumcevian triangle of the circumcenter, then $D''E''F''$ is the orthic triangle.

Example 9. If DEF is the circumtangential triangle, then $D''E''F''$ is homothetic to each of the three Morley equilateral triangles, as well as the circumtangential triangle (perspector X_2, homothetic ratio $-1/2$) the circumnormal triangle (perspector X_4, homothetic ratio $1/2$), and the Stammler triangle (perspector X_{381}). If DEF is the circumnormal triangle, then $D''E''F''$ is the same as for the circumtangential. (For descriptions of the various triangles, see [5].) The triangle $D''E''F''$ is the second of two equilateral triangles described in the article on the Steiner deltoid at [5]; its vertices are given as follows:

$$\begin{aligned}
D'' = &\cos(B-C) - \cos\left(\frac{B}{3} - \frac{C}{3}\right) \\
&: \cos(C-A) - \cos\left(B - \frac{2C}{3}\right) : \cos(A-B) - \cos\left(B - \frac{2C}{3}\right), \\
E'' = &\cos(B-C) - \cos\left(C - \frac{2A}{3}\right) \\
&: \cos(C-A) - \cos\left(\frac{C}{3} - \frac{A}{3}\right) : \cos(A-B) - \cos\left(C - \frac{2A}{3}\right), \\
F'' = &\cos(B-C) - \cos\left(A - \frac{2B}{3}\right) \\
&: \cos(C-A) - \cos\left(A - \frac{2B}{3}\right) : \cos(A-B) - \cos\left(\frac{A}{3} - \frac{B}{3}\right).
\end{aligned}$$

5. Summary and concluding remarks

If the point X in Section 4 is a triangle center, as defined at [5], then the perspector P^* is a triangle center. If instead of the cevian triangle of X, we use in Section 4 a central triangle of type 1 (as defined in [3], pp. 53-54), then P^* is clearly the same point as obtained from the cevian triangle of X.

Regarding pedal triangles, in Section 4, there, too, if X is a triangle center, then so is P^*, in (8). The same perspector is obtained by various central triangles of type 2. In all of these cases, the other two perspectors, P_1 and P_2, as in (6) and (7) are a bicentric pair [5].

In Examples 3-6, the antipodal pairs are triangle centers. The 90° rotation of such a pair is a bicentric pair, as in the following example.

Example 10. The Euler line meets the circumcircle in the points X_{1113} and X_{1114}. Let X^*_{1113} and X^*_{1114} be their 90° rotations about the circumcenter. Then $X^*_{1113} \oplus X^*_{1114}$ (the perspector of two triangles $A'B'C'$ and $A''B''C''$ as in Theorem 1) lies on the nine-point circle, in accord with Theorems 4 and 5. Indeed $X^*_{1113} \oplus X^*_{1114} = X_{113}$, which is the nine-point-circle-antipode of $X_{125} = X_{1113} \oplus X_{1114}$. Likewise, $X^*_{1379} \oplus X^*_{1380} = X_{114}$ and $X^*_{1381} \oplus X^*_{1382} = X_{119}$.

Example 10 illustrates the following theorem, which the interested reader may wish to prove: *Suppose X and Y are circumcircle-antipodes, with 90° rotations $X^*$and Y^*. Then $X^* \oplus Y^*$is the nine-point-circle-antipode of the center of the rectangular circumhyperbola formed by the isogonal conjugates of the points on the lineXY.*

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Forum Geometricorum
Volume 11 (2011) 95–107.

FORUM GEOM
ISSN 1534-1178

On a Theorem of Intersecting Conics

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Abstract. Given two conics over an infinite field that intersect at the origin, a line through the origin will, in general intersect both conic sections once more each, at points C and D. As the line varies we find that the midpoint of C and D traces out a curve, which is typically a quartic. Intuitively, this locus is the "average" of the two conics from the perspective of an observer at the origin. We give necessary and sufficient conditions for this locus to be a point, line, line minus a point, or a conic itself.

1. Introduction

Consider, in Figure 1, an observer standing on the point A . He wants to find the "average" of the two circles P_1, P_2. He could accomplish his task by facing towards an arbitrary direction, and then measuring his distance to P_1 and P_2 through that direction. The distances may be negative if either circle is behind him. He can then take the average of the two distances and mark it along his chosen direction. As our observer repeats this process, he will eventually trace out the circle ABE_1. Thus, in some sense ABE_1 is the "average" of our two original circles. In this paper we will consider analogous (weighted) averages of two nondegenerate plane conics meeting at a point A. This curve will be termed the "medilocus".

Definition 1 (Nondegenerate conic)**.** A nondegenerate conic is the zero set of a quadratic equation in two variables over an infinite field $\mathbb{F}$ which is not a point and does not contain a line.

In this paper, we shall assume all conics nondegenerate. We thus exclude lines and pairs of lines, for example $xy = 0$. We also remark here that if a conic consists of more than one point, it must be infinite: we will prove this in Proposition 6.

We will also define a tangent line to a point on a conic:

Definition 2 (Tangent line)**.** The tangent line to a conic represented by the equation $P(x, y) = 0$ at the point (x_0, y_0) is the line that contains the point (x_0, y_0) with (linear) equation $L(x, y) = 0$, whose substitution into $P(x, y) = 0$ gives a quadratic equation in one variable with a double root.

We remark here that the tangent line of a conic at the origin is the homogeneous linear part of the equation for the conic. We will prove that the homogeneous linear part is always nonzero in Lemma 4.

Publication Date: April 15, 2011. Communicating Editor: Paul Yiu.

This work is part of a senior thesis to satisfy the requirements for Departmental Honors in Mathematics at my undergraduate institution, Texas Christian University. I would like to thank my faculty mentor, Scott Nollet for his help and guidance. I wish also to thank the members of my Honors Thesis Committee, George Gilbert and Yuri Strzhemechny for their suggestions.

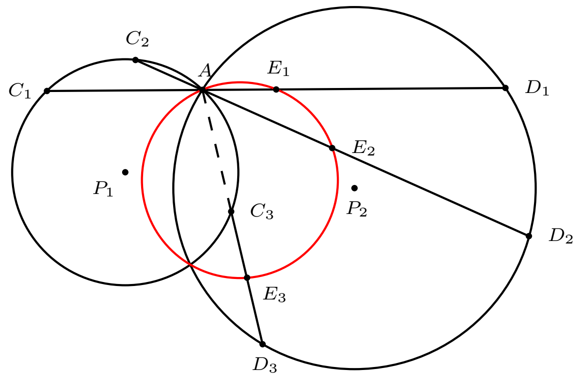

Figure 1. The medilocus of weight 1/2 of the two circles P_1, P_2 is the circle ABE_1. As we rotate a line CD around A into C_1D_1, C_2D_2, C_3D_3, the locus of the midpoint $E_1E_2E_3$ is the medilocus.

Definition 3 (Medilocus)**.** Let P_1, P_2 be two conics meeting at a distinguished intersection point A. The medilocus of weight k is the set of all points of the form $E = kC + (1-k)D$, where C, A, D are points on some line L through A, with $C \in P_1$ and $D \in P_2$. $C = A$ or $D = A$ is possible if and only if L is tangent to P_1 or P_2 respectively at A. The medilocus is denoted $M(P_1, P_2, A, k)$.

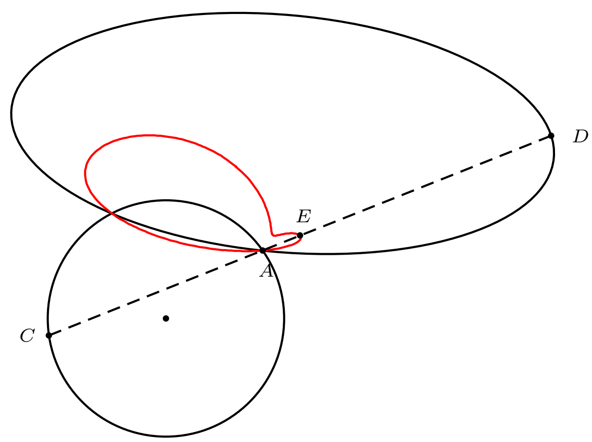

Figure 2. The red curve is the medilocus weight $1/2$ of the circle and the ellipse, with distinguished intersection point at A.

We comment here that given a line L, C and D are uniquely defined, if they exist. Clearly any line intersects a conic at most twice, and the following proposition will demonstrate that C or D cannot be multiply defined even when L is tangent to P_1 or P_2.

Proposition 1. *A line tangent to a conic at the origin cannot intersect the conic section again at another point.*

Proof. Choose coordinates so that the tangent line at the origin is $x = 0$. Restricting the conic to $x = 0$ gives us a quadratic polynomial in y, which has at most two zeroes with multiplicity. But since $x = 0$ is tangent to the conic at the origin, there is a double zero at $y = 0$, and so there cannot be any more. $\square$

If P_1 and P_2 are the same conic, we can see that their medilocus is the conic P_1. The medilocus of weight 0 is always the conic P_1, and the medilocus of weight 1 is always the conic P_2. Figure 1 shows the medilocus of weight $1/2$ of two circles. While the mediloci we have seen so far have been circles, if we make different choices of intersecting conics we usually obtain a more interesting medilocus. Figure 2 is an example. Note in this case that our medilocus is not even a conic. Our research began as an attempt to answer the question, when is the medilocus of two conic sections itself either a conic, or another "well-behaved" curve? We addressed this question by introducing an equivalence relation on conics.

Definition 4 (Medisimilarity)**.** Two conics are medisimilar if the homogeneous quadratic part of the equation of the first conic is a nonzero scalar multiple of the homogeneous quadratic part of the equation of the second conic. Equivalently, we can say that two conics are medisimilar if their equations can be written in such a way that their homogeneous quadratic parts are equal.

The proposition that follows describes medisimilarity in the real plane, so that the reader may gain some geometric intuition of what medisimilarity means. For the sake of brevity, this proposition is stated without proof. All that a proof requires is the understanding of the role that the homogeneous quadratic part of the equation of a conic plays in its geometry. Both [3] and [2] serve as useful references in this regard.

Proposition 2. *Two intersecting conics in $\mathbb{R}^2$ can be medisimilar only if they are both ellipses, both hyperbolas, or both parabolas.*
(1) *Two intersecting ellipses are medisimilar only if they have the same eccentricity, and their respective major axes are parallel.*
(2) *Two hyperbolas are medisimilar if and only if the two asymptotes of the first hyperbola have the same slopes as the two asymptotes of the second hyperbola.*
(3) *Two parabolas are medisimilar if and only if their directrices are parallel.*

Our main theorem can be stated thus:

Theorem 3. *The medilocus weight $k \neq 0, 1$ of two conics P_1, P_2 over an infinite field is a conic, a line, a line with a missing point, or a point if and only if P_1, P_2 are medisimilar.*

2. Preliminaries

Here we establish some conventions and prove certain lemmas that will streamline the proofs in the following section. Firstly, whenever two conics intersect and we wish to describe the medilocus, we shall choose coordinates so that the distinguished intersection point is at the origin. We will label the two conics P_1 and P_2

and express them as follows:

$$P_1 : \ Q_1(x,y) + L_1(x,y) = 0, \tag{1}$$

$$P_2 : \ Q_2(x,y) + L_2(x,y) = 0, \tag{2}$$

where Q_1, Q_2 are quadratic forms, and L_1, L_2 are linear forms.

We are now prepared to demonstrate some brief lemmas about the conics P_1, P_2.

Lemma 4. *Neither L_1 nor L_2 can be identically zero.*

Proof. If L_1 is identically zero, then the equation of P_1 is a homogeneous quadratic form. Thus if $P_1(a,b) = 0$, then either $a = b = 0$ or the line $\{(at, bt), t \in \mathbb{F}\}$ is contained in P_1. Thus P_1 is either a point or contains a line, neither of which is acceptable by our definition of conic. An analogous contradiction occurs when L_2 is identically zero. □

Lemma 5. *L_1 cannot divide Q_1, and L_2 cannot divide Q_2.*

Proof. If L_1 divides Q_1, then L_1 divides P_1 as well. But according to our definition, a conic cannot contain a zero set of a linear equation. A completely analogous proof works for P_2. □

Proposition 6. *If a quadratic equation in two variables over an infinite field has two distinct solutions, it has infinitely many.*

Proof. Let $P(x,y)$ be a quadratic polynomial with two distinct solutions. We may choose coordinates so that one of the solutions is the origin, and the other is (a,b), where a is nonzero. Substitute $y = mx$ into $P(x,y)$. We then get

$$P(x, mx) = xL(m) + x^2 Q(m),$$

where L, Q are linear and quadratic functions respectively. Note that P has no constant term since $(0,0)$ is a solution. We know when $m = b/a$, $P(x, mx)$ has two distinct solutions: this implies that $Q(b/a)$ is not zero. We also deduce that $-L(b/a)/Q(b/a) = a$, so $L(b/a)$ is not zero. In particular, we now know that neither Q nor L is identically zero. Q has at most two solutions, m_1, m_2. Thus for every choice of $m \neq m_1, m_2$, $P(x, mx)$ has solutions $x = 0$ and $x = -L(m)/Q(m)$. We have infinitely many choices for m, and so P must have infinitely many solutions. □

Lemma 7. *If we define conics P_1, P_2 intersecting at the origin as in* (1) *and* (2)*, then the medilocus is either the zero set of the equation*

$$1 = k\left(-\frac{L_1(x,y)}{Q_1(x,y)}\right) + (1-k)\left(-\frac{L_2(x,y)}{Q_2(x,y)}\right), \tag{3}$$

or the union of the zero set of this equation with a point at the origin. The origin is in the medilocus if and only if there exist $a, b \in \mathbb{F}$ not both zero for which

$$0 = k\left(-\frac{L_1(a,b)}{Q_1(a,b)}\right) + (1-k)\left(-\frac{L_2(a,b)}{Q_2(a,b)}\right), \tag{4}$$

in which case that medilocus point on the origin is given by the weighted average of the intersections of the line $\{(at, bt) | t \in \mathbb{F}\}$ with P_1 and P_2.

Proof. Firstly, we write a parametric equation for a line through the origin and a non-origin point (a, b) as

$$l = \{(at, bt), t \in \mathbb{F}\}. \tag{5}$$

We define points C, D as in Definition 3, such that the line l intersects P_1 at the origin and at C, and intersects P_2 at the origin and at D. We wish to find the coordinates of C, D in terms of t. Substituting (5) into (1), we obtain the quadratic

$$Q_1(at, bt) + L_1(at, bt) = 0.$$

Provided $Q_1(at, bt) \neq 0$, this means either $t = 0$ or

$$t = -\frac{L_1(a, b)}{Q_1(a, b)}.$$

If b/a is the slope of L_1 (if $a = 0$, we consider b/a to be the "slope" of $x = 0$) then l is tangent to P_1 at the origin, and so we set C at the origin. This is consistent with the equation since $L_1(a, b) = 0$. We cannot have $Q_1(a, b) = 0$ for that same a and b, otherwise $Q_1(x, y)$ is a multiple of $L_1(x, y)$, contradicting Lemma 5.

Thus l intersects P_1 at $t = 0$ and $t = -L_1(a, b)/Q_1(a, b)$, and so C is at $t = -L_1(a, b)/Q_1(a, b)$. Similarly, the t-coordinate of D is $t = -L_2(a, b)/Q_2(a, b)$.

Note that for certain choices of a, b the denominators can be zero. If this happens we know that the line through the origin with that slope b/a does not intersect P_1 or P_2, and so C or D is undefined and the medilocus does not intersect l except, perhaps, at the origin. For example, when we are working on the field $\mathbb{R}$ the denominator Q_1 will be zero when P_1 is a hyperbola and b/a is the slope its asymptote or when P_1 is a parabola and b/a is the slope of its axis of symmetry.

Let T be a function of a, b such that T is the t-coordinate of the non-origin point of intersection between the medilocus and the line $\{at, bt\}$. By the definition of the medilocus we know

$$T = k\left(-\frac{L_1(a, b)}{Q_1(a, b)}\right) + (1 - k)\left(-\frac{L_2(a, b)}{Q_2(a, b)}\right). \tag{6}$$

Substitute $T = t, x = at, y = bt$. If $T \neq 0$ we can divide by t, and so the points of the medilocus that are not on the origin satisfy (3). If $T = 0$, then the medilocus contains the origin and (4) holds for a, b.

Conversely, let (a, b) be any solution of (3). We consider the line expressed parametrically as $\{(at, bt) | t \in \mathbb{F}\}$. It intersects P_1 at the origin and when $t = -L_1(a, b)/Q_1(a, b)$, and it intersects P_2 at the origin and when $t = -L_2(a, b)/Q_2(a, b)$. Thus there is a point of the medilocus at

$$t = k\left(-\frac{L_1(a, b)}{Q_1(a, b)}\right) + (1 - k)\left(-\frac{L_2(a, b)}{Q_2(a, b)}\right) = 1,$$

but then $t = 1$ is the point (a, b), and so every solution to (3) is indeed in the medilocus.

If (a,b) is a non-origin solution for (4), then we note that the line $\{(at,bt)|t \in \mathbb{F}\}$ intersects P_1, P_2 at $t = -L_1(a,b)/Q_1(a,b)$, $t = -L_2(a,b)/Q_2(a,b)$ respectively. But by (4), the weighted average of the two values of t is 0, and so the medilocus does indeed contain the origin. $\square$

Some mediloci intersect the distinguished intersection point, and some do not. The medilocus in Figure 1 contains the distinguished intersection point. If we consider two parabolas in $\mathbb{R}^2$, $P_1 = x^2 - 2x - y, P_2 = x^2 - 2x + y$ with distinguished intersection point at the origin, we find that the medilocus weight $1/2$ is the line $x = 2$ which does not contain the origin.

3. A criterion for medisimilarity

Proposition 8. *The medilocus of two intersecting medisimilar conics is a conic section, a line, a line with a missing point, or a point.*

Proof. If $k = 0$ or $k = 1$ we are done, so let us assume k is not equal to 0 or 1. With a proper choice of coordinates, we may translate the distinguished intersection point to the origin. We may thus represent the two conics as in (1), (2). By Lemma 7, we know that the medilocus is the zero set of (3), possibly union the origin. And since P_1 is medisimilar to P_2, we may scale their equations appropriately so that $Q_1 = Q_2$. If we set $Q = Q_1 = Q_2$, $L = -kL_1 - (1-k)L_2$, (3) and (4) in Lemma 7 respectively reduce to

$$1 = \frac{L(x,y)}{Q(x,y)}, \tag{7}$$

$$0 = \frac{L(x,y)}{Q(x,y)}. \tag{8}$$

Five natural cases arise:

(1) If L is identically zero, then there are no solutions to (7). Thus the medilocus is either empty or the point at the origin. But since Q cannot be identically zero, there must exist (a,b) such that $Q(a,b) \neq 0$, in which case (a,b) would be a solution to (8), and hence the medilocus is precisely the point at the origin.

(2) If L is nonzero and Q is irreducible, we consider the line $L(x,y) = 0$ through the origin and write it in the form $\{(at,bt)\}$. But then $L(a,b) = 0$, and so (a,b) satisfies (8). Thus by Lemma 7 the medilocus contains a point at the origin. But then (7) simply reduces to $Q - L = 0$, and so again by Lemma 7 the medilocus is the zero set of $Q - L = 0$, and thus a conic.

(3) If L is nonzero and Q factors into linear terms M, N, neither of which is a multiple of L, we have by Lemma 7 that the medilocus is the zero set of $1 = L/MN$, possibly union a point in the origin. Reasoning identical to that of the previous case tells us that the origin is indeed in the medilocus. Since M, N aren't multiples of L, we add no erroneous solutions (except for the origin, which we know is in the medilocus) by clearing denominators. This shows that the medilocus is the zero set of $MN - L = 0$ and thus is a conic.

(4) If L is nonzero and Q factors into ML with M not a multiple of L, then Lemma 7 gives us that the medilocus is the zero set of $1 = L/ML$, possibly union

the origin. We show that the medilocus does not contain the origin. Assume instead for some a, b that (8) is satisfied. This implies $Q(a,b)$ nonzero $L(a,b) = 0$, which means that $L(x,y)$ has a root which is not a root of $Q(x,y)$. This is impossible since $L|Q$. Thus the medilocus cannot contain the origin. But then the zero set of $1 = L/ML$ is the zero set of $M = 1$, a line not through the origin, minus the zero set of $L = 0$. This is a line missing a point.

(5) If L nonzero, and $Q = L^2$, then by Lemma 7 the medilocus is the zero set of $1 = L/L^2$, possibly union a point at the origin. By reasoning similar to the previous part, we can conclude that the medilocus does not contain a point at the origin. Thus the medilocus is the zero set of $L = 1$, minus the zero set of $L = 0$. Since the lines $L = 1, L = 0$ are disjoint, we can conclude that the medilocus is just the zero set of the line $L = 1$. □

We will now demonstrate examples for these various cases in $\mathbb{R}^2$.

Example 1. Figure 1 gives us an example of two medisimilar conics having a medilocus that is a conic.

Example 2. If we consider two parabolas in $\mathbb{R}^2$, $P_1 = x^2 - 2x - y, P_2 = x^2 - 2x + y$ with distinguished intersection point at the origin, we find that the medilocus weight $1/2$ is the line $x = 2$.

Example 3. If we consider two hyperbolas in $\mathbb{R}^2$, $P_1 = yx - x - y, P_2 = yx + x - y$ with distinguished intersection point at the origin, we find that the medilocus weight $1/2$ is the line $x = 1$ missing the point $(1,0)$.

Example 4. The parabolas $P_1 = y - x^2, P_2 = y + x^2$ in $\mathbb{R}^2$ have the single point at the origin as their medilocus weight $1/2$.

Before we proceed to prove the other direction of Theorem 3, let us first demonstrate a case where that direction comes literally a point away from failing. This proposition will also be used in the proof for Proposition 11.

Proposition 9. *Consider two conics intersecting the origin,*

$$P_1 : h(x,y)c_1(x,y) + d_1(x,y) = 0,$$
$$P_2 : h(x,y)c_2(x,y) + d_2(x,y) = 0,$$

where h, c_1, c_2, d_1, d_2 are all linear forms of x, y through the origin. Let $h(x,y)$, $c_1(x,y)$ and $c_2(x,y)$ not be scalar multiples of each other, so that in particular P_1, P_2 are not medisimilar. Also assume $kc_1(x,y)d_2(x,y) + (1-k)c_2(x,y)d_1(x,y)$ is a multiple of $h(x,y)$. Then the medilocus weight k of P_1, P_2 is a conic missing exactly one point.

Proof. We invoke Lemma 7, so we know that points of the medilocus away from the origin can be expressed in the form

$$1 = \frac{-kc_1(x,y)d_2(x,y) - (1-k)c_2(x,y)d_1(x,y)}{h(x,y)c_1(x,y)c_2(x,y)}.$$

If we denote $kc_1(x,y)d_2(x,y) + (1-k)c_2(x,y)d_1(x,y) = h(x,y)g(x,y)$ for another linear equation $g(x,y)$ through the origin, we can write the medilocus except for the origin as the zero set of the quadratic equation

$$c_1(x,y)c_2(x,y) + g(x,y) = 0, \tag{9}$$

subtracting the points on the line $h(x,y)=0$. Since c_1, c_2 are not scalar multiples, $c_1(x,y), c_2(x,y)$ cannot divide $g(x,y)$ without contradicting Lemma 5, and so $c_1(x,y)=0, c_2(x,y)=0$ do not intersect the curve defined by (9) except at the origin.

First, assume that $g(x,y), h(x,y)$ are not scalar multiples of each other. Recall that $g(x,y)$ is not a multiple of $c_1(x,y)$ or $c_2(x,y)$. (4) is written as

$$0 = \frac{h(a,b)g(a,b)}{h(a,b)c_1(a,b)c_2(a,b)}. \tag{10}$$

If we express $g(x,y)=0$ as $\{(a't, b't)\}$, then $g(a',b')=0$ and since h is not a scalar multiple of g, $h(a',b') \neq 0$ and thus $a=a', b=b'$ will solve (10). Thus by Lemma 7 the medilocus contains the origin.

Additionally, assume that $h(x,y)$ is not simply a multiple of y. In that case, we can write $c_1(x,y), c_2(x,y), g(x,y)$ respectively as $m_1h(x,y)+\alpha y, m_2h(x,y)+\beta y, m_3h(x,y)+\gamma y$ where $m_1, m_2, m_3, \alpha, \beta, \gamma$ are constants, and α, β, γ are nonzero. But substituting these equations into (9), this implies that at a point on $y = -\gamma/\alpha\beta$, (a nonzero value for y) $h(x,y)=0$ intersects the curve (9). Thus the medilocus must be a conic subtracting one point.

If $h(x,y)$ is a multiple of y, we instead write $c_1(x,y), c_2(x,y), g(x,y)$ respectively as $m_1h(x,y)+\alpha x, m_2h(x,y)+\beta x, m_3h(x,y)+\gamma x$ and proceed analogously.

Now consider the case where $g = sh$, for some nonzero constant s. In this case certainly $h=0$ does not intersect (9) other than the origin, and so the solutions to (9) must be precisely the points of the medilocus away from the origin. We claim that the medilocus cannot contain the origin. By Lemma 7 the medilocus contains the origin if and only if for some a, b not both zero

$$0 = -\frac{kd_1(a,b)c_2(a,b) + (1-k)d_2(a,b)c_1(a,b)}{h(a,b)c_1(a,b)c_2(a,b)} = -\frac{sh(a,b)^2}{h(a,b)c_1(a,b)c_2(a,b)}.$$

But then clearly the denominator of the right hand side is zero whenever the numerator is zero, so this equation can never be satisfied. We conclude that the medilocus is the conic represented by (9) missing a point at the origin. $\square$

Example 5. The medilocus weight $1/2$ of $x^2 - yx + y = 0$, $x^2 + yx + y = 0$ is $y = y^2 - x^2$, missing the point at $(0,1)$.

We will now need to invoke the definition of variety in our next lemma.

Definition 5 (Variety)**.** For an algebraic equation $f=0$ over a field $\mathbb{F}$ we use the notation $V(f)$, the *variety* or *zero set* of f to represent the subset of $\mathbb{F}[x,y]$ for which f is zero.

This definition is given in [1, p.8].

Lemma 10. *Let $l, g, f \in \mathbb{F}[x, y]$ be polynomials with degree $l = 1$, degree $g = 2$ and $V(g)$ infinite.*
(i) *If $V(l) \subset V(f)$, then $l|f$.*
(ii) *If $V(g) \subset V(f)$ and $V(g)$ is not a line, then $g|f$.*

Proof. (i) First, we claim that if l is linear and $V(l) \subset V(f)$, then $l|f$. Corollary 1 of Proposition 2 in Chapter 1 of [1] says that if l is an irreducible polynomial in a closed field $\overline{\mathbb{F}}$, $V(l) \subset V(f)$ and $V(l)$ infinite, then we may conclude that $l|f$ in $\overline{\mathbb{F}}[x, y]$. Thus $l|f$ in $\mathbb{F}[x, y]$ as well.

(ii) We consider then the case where $g = l_1 l_2$ factors as a product of distinct linear factors. We then have $V(g) = V(l_1) \cup V(l_2) \subset V(f)$, so this implies $l_1|f$ and $l_2|f$, therefore $l_1 l_2|f$ and so $g|f$.

If g is irreducible over $\mathbb{F}$, suppose g is reducible over $\overline{\mathbb{F}}$. Then $g = st$ for some linear terms s, t, and we have $V(g) = V(s) \cup V(t)$ over $\overline{\mathbb{F}}^2$. Thus $V(g)$ over $\mathbb{F}^2$ is empty, one point, two points, a point and line, a line, or two lines. The first three cases are not possible because $V(g)$ over $\mathbb{F}^2$ is infinite. The last three cases are not possible by our observation in the first paragraph of this proof, because then a linear equation divides g, contradicting irreducibility in $\mathbb{F}$. We conclude that g is irreducible over $\overline{\mathbb{F}}$. $V(g) \cap V(f)$ infinite implies $g|f$ again by Corollary 1 of Proposition 2 in Chapter 1 of [1] $\square$

Proposition 11. *The medilocus weight k of two conics P_1, P_2 is a conic itself only if $k = 0$, $k = 1$, or P_1, P_2 are medisimilar.*

Proof. If $k = 0, 1$ we are done; we may henceforth assume k is neither 0 nor 1.

We may set the conics P_1, P_2 as in (1),(2). By Lemma 7, we know that we can represent the medilocus by (3) for every point except for the origin. Clearing the denominators, we get the quartic $f(x, y) = 0$ where

$$f(x, y) = Q_1(x, y)Q_2(x, y) + kL_1(x, y)Q_2(x, y) + (1 - k)L_2(x, y)Q_1(x, y). \tag{11}$$

We claim that the zero set of this equation contains the medilocus.

Note that we might have added some erroneous points by clearing denominators this way: for example, if $Q_1(x, y), Q_2(x, y)$ have a linear common factor, then $f(x, y) = 0$ will contain the solutions of that linear factor, even if the medilocus itself does not. It is clear, however that every point of the medilocus is in the zero set of $f(x, y) = 0$. In particular, the origin is in that zero set whether or not it is in the medilocus.

Whether $f(x, y) = 0$ is the medilocus or merely contains the medilocus, by Lemma 10 and Proposition 6, $f(x, y)$ contains a conic only if it has a quadratic factor. So assume that the quartic f factors into quadratics f_1, f_2. Let us define

$$f_1 = q_1 + w_1 + c_1,$$
$$f_2 = q_2 + w_2 + c_2.$$

The c_i are constants, w_i are homogeneous linear terms in x, y, and the q_i are homogeneous quadratic terms in x, y. First, we note that since $f = f_1 f_2$ has no quadratic, linear or constant terms, either $c_1 = 0$ or $c_2 = 0$. We claim that this implies c_1, c_2, w_1w_2 are all zero. Without loss of generality, let us start with the assumption that $c_1 = 0$. But then c_2w_1 is the homogeneous linear part of f, and so either $c_2 = 0$ or $w_1 = 0$. If $c_2 = 0$ then w_1w_2 is the homogeneous quadratic part of f, and so $w_1w_2 = 0$. If $w_1 = 0$, then $c_2q_1 = 0$, and since by this point $f_1 = q_1$, q_1 must be nonzero and we must have $c_2 = 0$.

And so we must have $c_1 = 0$, $c_2 = 0$ and $w_1w_2 = 0$. Without loss of generality we let $w_1 = 0$. We now have

$$\begin{aligned} f_1 &= q_1, \\ f_2 &= q_2 + w_2. \end{aligned}$$

Given that $f = f_1f_2$, the homogeneous quartic and homogeneous cubic parts must match (with reference to (11)). This gives us

$$q_1(x,y)q_2(x,y) = Q_1(x,y)Q_2(x,y), \tag{12}$$

$$q_1(x,y)w_2(x,y) = kL_1(x,y)Q_2(x,y) + (1-k)L_2(x,y)Q_1(x,y). \tag{13}$$

From (12), we have three possibilities: either $q_1(x,y)$ divides $Q_1(x,y)$, $q_1(x,y)$ divides $Q_2(x,y)$ or $q_1(x,y)$ factors into two homogeneous linear terms $u_1(x,y)$,$v_1(x,y)$ and we have both $u_1(x,y)|Q_1(x,y)$ and $v_1(x,y)|Q_2(x,y)$. We handle these three cases one by one.

(1) If $q_1(x,y)|Q_1(x,y)$, we consider (13) and conclude that it must be the case that

$$q_1(x,y)|kL_1(x,y)Q_2(x,y).$$

This is possible only if a linear factor of $q_1(x,y)$ is a scalar multiple of $L_1(x,y)$, or $q_1(x,y)|Q_2(x,y)$. If a linear factor of $q_1(x,y)$ is a scalar multiple of $L_1(x,y)$, then $L_1(x,y)$ divides $P_1(x,y)$, violating (5). If $q_1(x,y)|Q_2(x,y)$, then $Q_2(x,y)$ is a scalar multiple of $q_1(x,y)$ and hence of $Q_1(x,y)$, implying that P_1, P_2 are medisimilar.

(2) If $q_1(x,y)|Q_2(x,y)$, we consider (13) and conclude that it must be the case that

$$q_1(x,y) = (1-k)L_2(x,y)Q_1(x,y).$$

By a proof completely analogous to that of part (a), this implies that P_1, P_2 are medisimilar.

(3) If we have $u_1(x,y)|Q_1(x,y)$ and $v_1(x,y)|Q_2(x,y)$, then by (13) it must be the case that

$$u_1(x,y)|kL_1(x,y)Q_2(x,y).$$

This is possible only if $u_1(x,y)$ is a scalar multiple of $L_1(x,y)$, or $u_1(x,y)$ divides $Q_2(x,y)$. If $u_1(x,y)$ is a scalar multiple of $L_1(x,y)$, then $L_1(x,y)$ divides $P_1(x,y)$, violating Lemma 5.

As for $u_1(x,y)|Q_2(x,y)$, if $u_1(x,y)$ isn't a scalar multiple of $v_1(x,y)$ then we have case (b). If $u_1(x,y)$ is a scalar multiple of $v_1(x,y)$, this means that $u_1(x,y)^2$ divides the right hand side of (13). Note that $u_1(x,y)$ divides $Q_1(x,y)$ and $Q_2(x,y)$. If $u_1(x,y)^2$ does not divide each individual term of the right hand side of (13) we either have P_1, P_2 medisimilar, or we have the case described in

Proposition 9 (where u_1 is h in the notation of that proposition). If $u_1(x,y)^2$ divides each term of the right hand side, we have

$$u_1(x,y)^2 | kL_1(x,y)Q_2(x,y)$$

and

$$u_1(x,y)^2 | (1-k)L_2(x,y)Q_1(x,y).$$

The first equation implies that $u_1(x,y)^2$ is a scalar multiple of $Q_2(x,y)$. If $u_1(x,y)$ is a scalar multiple of $L_2(x,y)$, $u_1(x,y)$ divides $P_2(x,y)$ violating Lemma 5. If $u_1(x,y)$ is not a scalar multiple of $L_2(x,y)$, $Q_1(x,y)$ is a scalar multiple of $u_1(x,y)^2$ and hence $Q_2(x,y)$. Thus P_1, P_2 are medisimilar. □

Proposition 12. *The medilocus of two intersecting conics is a line or a line missing a point only if they are medisimilar.*

Proof. Clearly, $k \neq 0, 1$.

We will need to consider the coefficients of P_1, P_2, so let us again assume that the distinguished intersection points is the origin and that we can express our two conics as

$$P_1 : R_1x^2 + S_1xy + T_1y^2 + V_1x + W_1y = 0, \tag{14}$$

$$P_2 : R_2x^2 + S_2xy + T_2y^2 + V_2x + W_2y = 0, \tag{15}$$

with constants $R_1, S_1, T_1, V_1, W_1, R_2, S_2, T_2, V_2, W_2$.

Through a proper choice of coordinates, we may express the medilocus as a vertical line $x = c$ for some constant c, which may or may not be missing a point. First, we note that c cannot be zero. The line $x = 0$ intersects P_1, P_2 at most twice each, and so the medilocus cannot intersect $x = 0$ infinitely many times.

But if c is nonzero, the medilocus does not intersect $x = 0$ at all. But this means $x = 0$ cannot intersect twice both P_1 and P_2. In algebraic terms, we know that $x = 0$ intersects P_1, P_2 at the origin and at $-W_1/T_1$, $-W_2/T_2$ respectively. Thus at least one of W_1, T_1, W_2, T_2 must be zero. If W_1 is zero, P_1 is tangent to $x = 0$ at the origin, and so $x = 0$ now cannot intersect P_2 twice: in other words, $W_1 = 0$ implies $W_2 = 0$ or $T_2 = 0$. If W_2 is zero, both P_1, P_2 would be tangent to $x = 0$, and so the origin will be in the medilocus. Thus we must have $T_2 = 0$. In other words, $W_1 = 0$ implies $T_2 = 0$, and we similarly have $W_2 = 0$ implying $T_1 = 0$. We thus must have either $T_1 = 0$ or $T_2 = 0$. Without loss of generality we let $T_1 = 0$.

All the points in the medilocus must have x-value $c \neq 0$. By appropriate scaling, we may without loss of generality set $c = 1$. Thus with reference to Lemma 7 and in particular (3), we then find that the coefficients of P_1, P_2 must satisfy

$$1 = k\left(-\frac{V_1 + W_1m}{R_1 + S_1m}\right) + (1-k)\left(-\frac{V_2 + W_2m}{R_2 + S_2m + T_2m^2}\right), \tag{16}$$

for all but at most one m. We may simplify (16) into

$$\begin{aligned} &(R_1 + S_1m)(R_2 + S_2m + T_2m^2) \\ = &-k(R_2 + S_2m + T_2m^2)(V_1 + W_1m) - (1-k)(R_1 + S_1m)(V_2 + W_2m), \end{aligned}$$

which in turn reduces to

$$\begin{aligned}&((R_1+S_1m)+k(V_1+W_1m))(R_2+S_2m+T_2m^2)\\ =&-(1-k)(R_1+S_1m)(V_2+W_2m).\end{aligned} \tag{17}$$

Since this equality holds for infinitely many values of m, all the coefficients must be zero. Looking at the m^3 coefficient, we deduce that either $S_1 + kW_1 = 0$, or $T_2 = 0$.

If $T_2 \neq 0$, we must have $S_1 + kW_1 = 0$. Since (17) now implies $(R_1 + S_1m)(V_2+W_2m)$ is a scalar multiple of $R_2+S_2m+T_2m^2$, this means that S_1, W_2 both nonzero. But since we have $(R_1+S_1m)(V_2+W_2m) = C(R_2+S_2m+T_2m^2)$ for some constant C, we may write $m = y/x$ and then multiply both sides by x^2, to determine that $V_2x + W_2y$ divides $R_2x^2 + S_2xy + T_2y^2$, contradicting Lemma 5.

Thus it is necessary that $T_2 = 0$. However, as we reevaluate (16), we note that there appear to be no solutions of the medilocus at $m = -R_1/S_1$ and $m = -R_2/S_2$. Since the medilocus is the line $x = c$ missing at most one point, it is necessary that $S_1 = 0$, $S_2 = 0$, or $-R_1/S_1 = -R_2/S_2$. The last case immediately implies that P_1, P_2 are medisimilar, so we consider the first two cases.

If we start by assuming $S_1 = 0$, note now that R_1 must be nonzero, otherwise P_1 has no quadratic terms. Recall that $T_1 = T_2 = 0$. Then by comparing coefficients in (17) we have:

$$S_2W_1 = 0, \tag{18}$$

$$R_1S_2 = -k(R_2W_1 + S_2V_1) - (1-k)R_1W_2, \tag{19}$$

$$R_1R_2 = -kR_2V_1 - (1-k)R_1V_2. \tag{20}$$

Based on (18), we are dealing with two subcases: $W_1 = 0$ or $S_2 = 0$.

If $S_2 \neq 0$, $W_1 = 0$. If $R_2 = 0$ as well, then (20) implies that $V_2 = 0$. But then P_2 reduces to $W_2y + S_2xy$, contradicting either Lemma 4 or Lemma 5.

Thus we must have R_2 nonzero, and since we assumed $W_1 = 0$ (19) and (20) imply

$$\frac{R_1W_2}{S_2} = -\frac{R_1+kV_1}{1-k} = \frac{R_1V_2}{R_2}.$$

But this implies $W_2/S_2 = V_2/R_2$, again contradicting Lemma 5.

Thus we deduce $S_2 = 0$, and we now have $T_1 = T_2 = S_1 = S_2 = 0$, so P_1, P_2 must now be medisimilar. If we start by assuming $S_2 = 0$, we analogously deduce that $S_1 = 0$ as well, and that P_1, P_2 are medisimilar. □

Proposition 13. *The medilocus of two intersecting conics is a point only if they are medisimilar.*

Proof. We can discount the possibility that the weight k equals 0 or 1. We shall set the distinguished intersection point at the center, and so we may once again define our conics P_1, P_2 as in (1),(2). We claim that if the medilocus is a single point, that point must be in the center. Consider any line of the form $y = mx$ through the center. It intersects the first conic section, P_1 at the origin and at $x = -L_1(1,m)/Q_1(1,m)$, and it intersects the second conic, P_2 at the origin and at

$x = -L_2(1,m)/Q_2(1,m)$. Thus if $Q_1(1,m), Q_2(1,m)$ are both nonzero, then the medilocus has a point on the line $y = mx$ (note that $L_1(1,m) = 0, Q_1(1,m) \neq 0$ or $L_2(1,m) = 0, Q_2(1,m) \neq 0$ respectively imply that $y = mx$ is tangent to P_1 or P_2 at the origin).

But clearly $Q_1(1,m), Q_2(1,m)$ have at most two roots each, and so except for at most four values of m, the medilocus intersects $y = mx$. Thus if the medilocus consists of exactly one point, that point must be the origin.

By Lemma 7, (3) must have no solutions. This means that

$$\frac{kL_1(x,y)Q_2(x,y) + (1-k)L_2(x,y)Q_1(x,y)}{Q_1(x,y)Q_2(x,y)}, \tag{21}$$

is either zero or undefined for all choices of x, y. If $Q_1(x,y)$ is zero at (a,b), it must be zero on the line $M_1(x,y)$ through $(0,0)$ and (a,b). Thus $Q_1(x,y) = M_1(x,y)N_1(x,y)$ for some linear form N_1. Similarly, if $Q_2(x,y)$ has a non-origin solution, it must factor into linear forms $Q_2(x,y) = M_2(x,y)N_2(x,y)$ as well. Since $Q_1(x,y)Q_2(x,y)$ cannot be identically zero the expression (21) can be undefined on at most four lines through the origin. But then the expression $kL_1(x,y)Q_2(x,y) + (1-k)L_2(x,y)Q_1(x,y)$ is a homogeneous cubic in x and y . Note that if $(a,b) \neq (0,0)$ is a solution to a homogeneous cubic the entire line through $(0,0)$ and (a,b) is as well, and by Lemma 10 the equation for that line must divide the homogeneous cubic. Thus the numerator of (21) is either identically zero, or zero on at most three lines through the origin. We know $\mathbb{F}$ is an infinite field, and so there are infinitely many lines through the origin. Since the expression in (21) must be zero or undefined everywhere, we conclude that

$$kL_1(x,y)Q_2(x,y) + (1-k)L_2(x,y)Q_1(x,y) = 0. \tag{22}$$

By Lemma 5, we know that $L_1(x,y)$ cannot divide $Q_1(x,y)$, and that $L_2(x,y)$ cannot divide $Q_2(x,y)$. Thus it must be true that $L_1(x,y)$ and $L_2(x,y)$ are scalar multiples of each other. We note here that neither $L_1(x,y)$ nor $L_2(x,y)$ can be identically zero by Lemma 4. But then this implies that $Q_1(x,y)$ and $Q_2(x,y)$ are scalar multiples of each other. Thus P_1, P_2 are medisimilar. $\square$

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Forum Geometricorum
Volume 11 (2011) 109–119.

FORUM GEOM
ISSN 1534-1178

The Droz-Farny Circles of a Convex Quadrilateral

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Abstract. The Droz-Farny circles of a triangle are a pair of circles of equal radii obtained by particular geometric constructions. In this paper we deal with the problem to see whether and how analogous properties of concyclicity hold for convex quadrilaterals.

1. Introduction

The Droz-Farny circles of a triangle are a pair of circles of equal radii obtained by particular geometric constructions [4]. Let **T** be a triangle of vertices A_1, A_2, A_3, with circumcenter O and orthocenter H. Let H_i be the foot of the altitude of **T** at A_i, and M_i the middle point of the side A_iA_{i+1} (with indices taken modulo 3); see Figure 1.

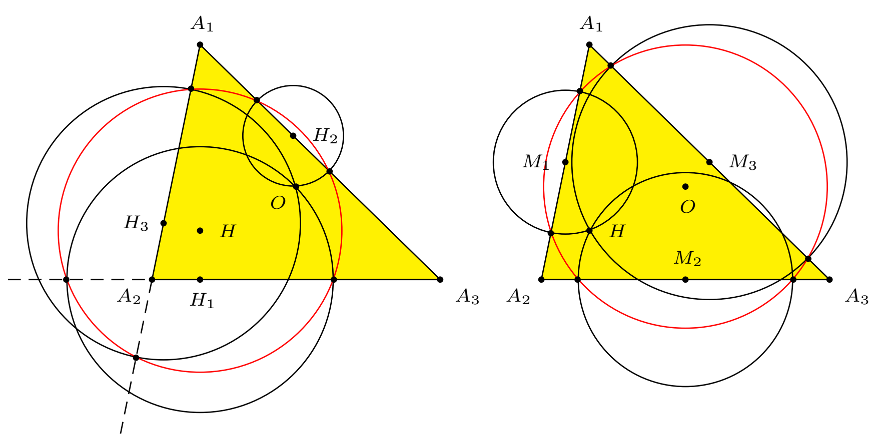

Figure 1.

(a) If we consider the intersections of the circle $H_i(O)$ (center H_i and radius H_iO) with the line $A_{i+1}A_{i+2}$, then we obtain six points which all lie on a circle with center H (first Droz-Farny circle).

(b) If we consider the intersections of the circle $M_i(H)$ (center M_i and radius M_iH) with the line A_iA_{i+1}, then we obtain six points which all lie on a circle with center O (second Droz-Farny circle).

The property of the first Droz-Farny circle is a particular case of a more general property (first given by Steiner and then proved by Droz-Farny in 1901 [2]). Fix a segment of length r, if for $i = 1, 2, 3$, the circle with center A_i and radius r

Publication Date: May 13, 2011. Communicating Editor: Paul Yiu.

intersects the line M_iM_{i+2} in two points, then we obtain six points all lying on a circle Γ with center H. When r is equal to the circumradius of $\mathbf{T}$ we obtain the first Droz-Farny circle (see Figure 2).

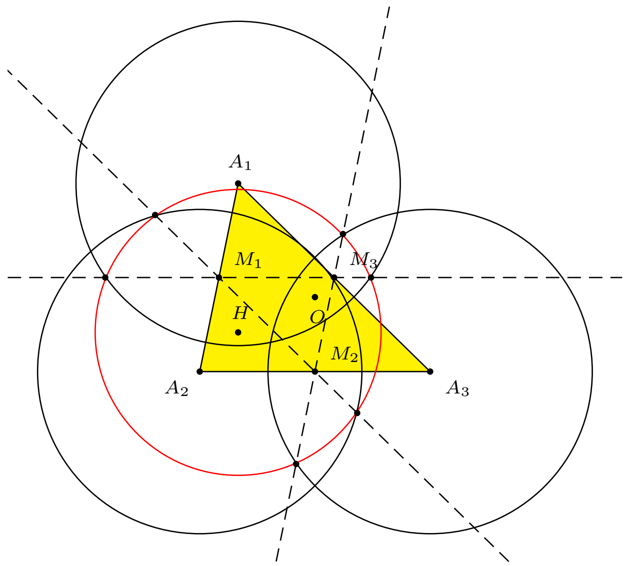

Figure 2.

In this paper we deal with the problem to see whether and how analogous properties of concyclicity hold for convex quadrilaterals.

2. An eight-point circle

Let $A_1A_2A_3A_4$ be a convex quadrilateral, which we denote by $\mathbf{Q}$, and let G be its centroid. Let $\mathbf{V}$ be the Varignon parallelogram of $\mathbf{Q}$, *i.e.*, the parallelogram $M_1M_2M_3M_4$, where M_i is the middle point of the side A_iA_{i+1}. Let H_i be the foot of the perpendicular drawn from M_i to the line $A_{i+2}A_{i+3}$. The quadrilateral $H_1H_2H_3H_4$, which we denote by $\mathbf{H}$, is called the principal orthic quadrilateral of $\mathbf{Q}$ [5], and the lines M_iH_i are the maltitudes of $\mathbf{Q}$. We recall that the maltitudes of $\mathbf{Q}$ are concurrent if and only if $\mathbf{Q}$ is cyclic [6]. If $\mathbf{Q}$ is cyclic, the point of concurrency of the maltitudes is called anticenter of $\mathbf{Q}$ [7]. Moreover, if $\mathbf{Q}$ is cyclic and orthodiagonal, the anticenter is the common point of the diagonals of $\mathbf{Q}$ (Brahmagupta theorem) [4]. In general, if $\mathbf{Q}$ is cyclic, O is its circumcenter and G its centroid, the anticenter H is the symmetric of O with respect to G, and the line containing the three points H, O and G is called the Euler line of $\mathbf{Q}$.

Theorem 1. *The vertices of the Varignon parallelogram and those of the principal orthic quadrilateral of* $\mathbf{Q}$*, that lie on the lines containing two opposite sides of* $\mathbf{Q}$*, belong to a circle with center* G*.*

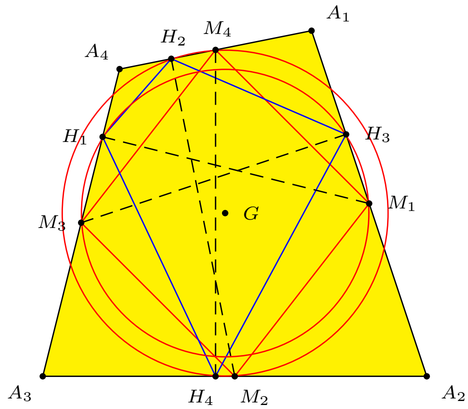

Figure 3.

Proof. The circle with diameter M_1M_3 passes through H_1 and H_3, because $\angle M_1H_1M_3$ and $\angle M_1H_3M_3$ are right angles (see Figure 3). Analogously, the circle with diameter M_2M_4 passes through H_2 and H_4. □

Theorem 1 states that the vertices of $\mathbf{V}$ and those of $\mathbf{H}$ lie on two circles with center G.

Corollary 2. *The vertices of the Varignon parallelogram and those of the principal orthic quadrilateral of* $\mathbf{Q}$ *all lie on a circle* (*with center* G) *if and only if* $\mathbf{Q}$ *is orthodiagonal.*

Proof. The two circles containing the vertices of V and H coincide if and only if $M_1M_3 = M_2M_4$, *i.e.*, if and only if V is a rectangle. This is the case if and only if $\mathbf{Q}$ is orthodiagonal. □

If $\mathbf{Q}$ is orthodiagonal (see Figure 4), the circle containing all the vertices of $\mathbf{V}$ and $\mathbf{H}$ is the eight-point circle of $\mathbf{Q}$ (see [1, 3]).

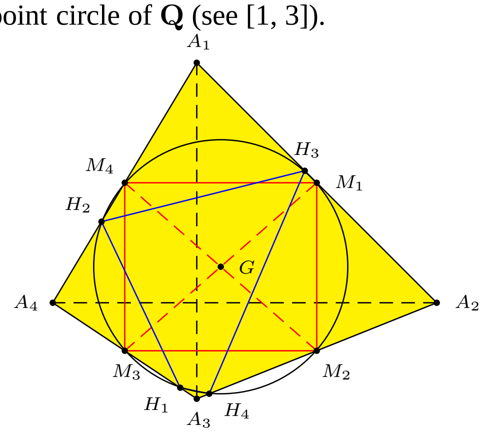

Figure 4.

3. The first Droz-Farny circle

Let **Q** be cyclic and let O and H be the circumcenter and the anticenter of **Q**, respectively. Consider the principal orthic quadrilateral **H** with vertices H_1, H_2, H_3, H_4. Let X_i and X_i' be the intersections of the circle $H_i(O)$ with the line $A_{i+2}A_{i+3}$ (indices taken modulo 4). Altogether there are eight points.

Theorem 3. *If* **Q** *is cyclic, the points X_i, X_i' that belong to the lines containing two opposite sides of* **Q** *lie on a circle with center H.*

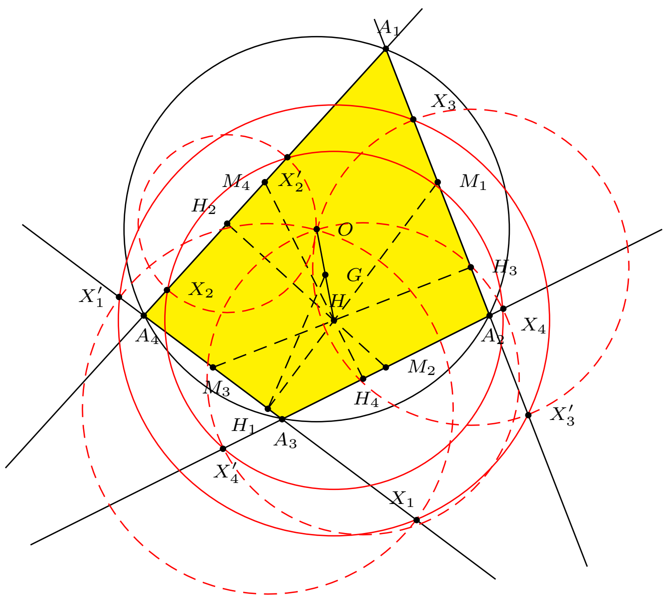

Figure 5.

Proof. Let us prove that the points X_1, X_1', X_3, X_3' are on a circle with center H (see Figure 5). Since H is on the perpendicular bisector of the segment X_1X_1', we have $HX_1 = HX_1'$. Moreover, since X_1 lies on the circle with center H_1 and radius OH_1, $H_1X_1 = OH_1$. By applying Pythagoras' theorem to triangle HH_1X_1, and Apollonius' theorem to the median H_1G of triangle OHH_1, we have

$$HX_1^2 = HH_1^2 + H_1X_1^2 = HH_1^2 + OH_1^2 = 2H_1G^2 + \frac{1}{2}OH^2.$$

Analogously,

$$HX_3^2 = 2H_3G^2 + \frac{1}{2}OH^2.$$

But from Theorem 1, H_1 and H_3 are on a circle with center G, then $H_1G = H_3G$. Consequently, $HX_1 = HX_3$, and it follows that the points X_1, X_1', X_3, X_3' are on a circle with center H.

The same reasoning shows that the points X_2, X_2', X_4, X_4' also lie on a circle with center H. □

Theorem 3 states that the points X_i, X_i', $i = 1, 2, 3, 4$, lie on two circles with center H.

Corollary 4. *For a cyclic quadrilateral* $\mathbf{Q}$, *the eight points* X_i, X_i', $i = 1, 2, 3, 4$, *all lie on a circle* (*with center* H) *if and only if* $\mathbf{Q}$ *is orthodiagonal.*

Proof. The two circles that contains the points X_i, X_i' and coincide if and only if $H_1G = H_2G = H_3G = H_4G$, *i.e.*, if and only if the principal orthic quadrilateral is inscribed in a circle with center G. From Corollary 2, this is the case if and only if $\mathbf{Q}$ is orthodiagonal. □

As in the triangle case, if $\mathbf{Q}$ is cyclic and orthodiagonal, we call the circle containing the eight points X_i, X_i', $i = 1, 2, 3, 4$, the first Droz-Farny circle of $\mathbf{Q}$.

Theorem 5. *If* $\mathbf{Q}$ *is cyclic and orthodiagonal, the radius of the first Droz-Farny circle of* $\mathbf{Q}$ *is the circumradius of* $\mathbf{Q}$.

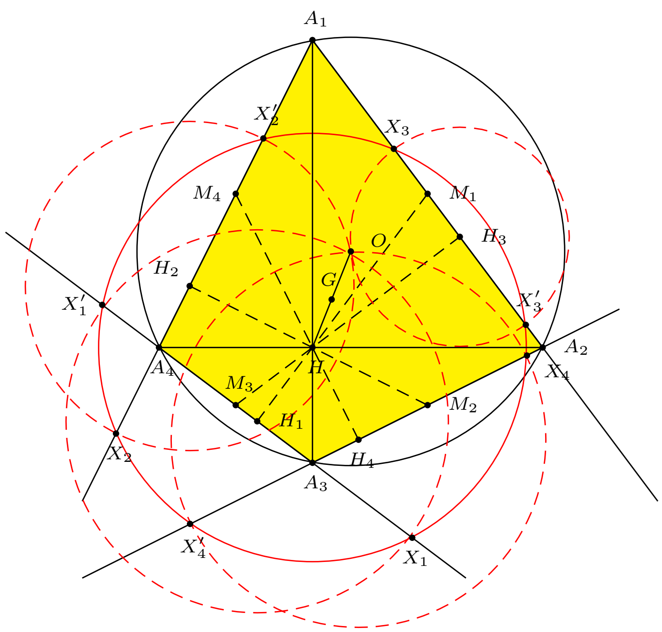

Figure 6.

Proof. From the proof of Theorem 3 we have

$$HX_1^2 = 2H_1G^2 + \frac{1}{2}OH^2. \tag{1}$$

Moreover, $M_3A_3 = M_3H$ since **Q** is orthodiagonal and $\angle A_3HA_4$ is a right angle (see Figure 6). By applying Pythagoras' theorem to the triangle OM_3A_3, and Apollonius' theorem to the median M_3G of triangle OM_3H, we have

$$OA_3^2 = OM_3^2 + M_3A_3^2 = OM_3^2 + M_3H^2 = 2M_3G^2 + \frac{1}{2}OH^2.$$

Since M_3G and H_1G are radii of the eight-points circle of **Q**,

$$OA_3^2 = 2H_1G^2 + \frac{1}{2}OH^2. \tag{2}$$

From (1) and (2) it follows that $HX_1 = OA_3$. □

4. The second Droz-Farny circle

Let **Q** be cyclic, with circumcenter O and anticenter H. For $i = 1, 2, 3, 4$, let Y_i and Y_i' be the intersection points of the line A_iA_{i+1} with the circle $M_i(H)$. Altogether there are eight points.

Theorem 6. *If* **Q** *is cyclic, the points Y_i, Y_i' that belong to the lines containing two opposite sides of* **Q** *lie on a circle with center O.*

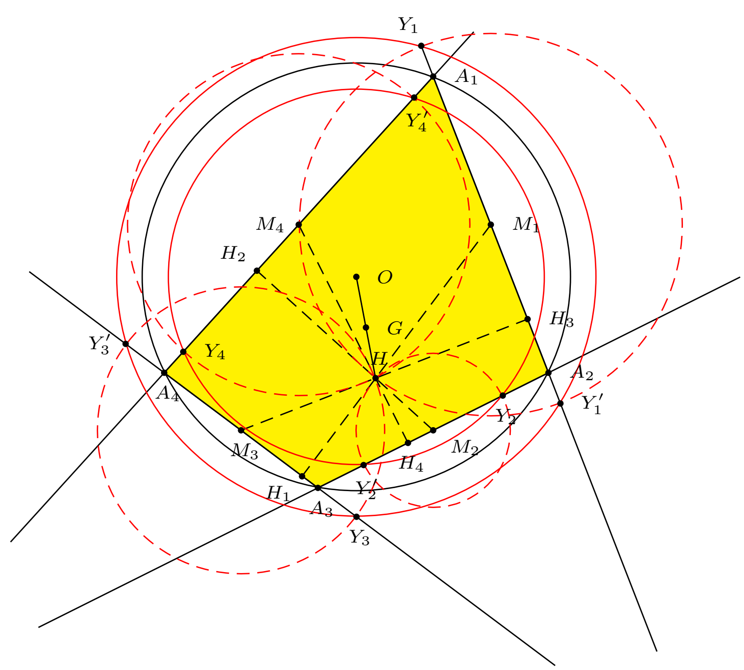

Figure 7.

Proof. Let us prove that the points Y_1, Y_1', Y_3, Y_3' are on a circle with center O (see Figure 7).

Since O is on the perpendicular bisector of the segment Y_1Y_1', $OY_1 = OY_1'$. Moreover, since Y_1 lies on the circle with center M_1 and radius HM_1, $M_1Y_1 = HM_1$. By applying Pythagoras' theorem to triangle OM_1Y_1, and Apollonius' theorem to the median M_1G of triangle OM_1H, we have

$$OY_1^2 = OM_1^2 + M_1Y_1^2 = OM_1^2 + HM_1^2 = 2M_1G^2 + \frac{1}{2}OH^2.$$

Analogously,

$$OY_3^2 = 2M_3G^2 + \frac{1}{2}OH^2.$$

Since G is the midpoint of the segment M_1M_3, $OY_1 = OY_3$. It follows that the points Y_1, Y_1', Y_3, Y_3' lie on a circle with center O.

The same reasoning shows that the points Y_2, Y_2', Y_4, Y_4' also lie on a circle with center O. □

Theorem 6 states that the points Y_i, Y_i', $i = 1, 2, 3, 4$, lie on two circles with center O.

Corollary 7. *For a cyclic quadrilateral* $\mathbf{Q}$*, the eight points* Y_i, Y_i', $i = 1, 2, 3, 4$, *all lie on a circle* (*with center* O) *if and only if* $\mathbf{Q}$ *is orthodiagonal.*

Proof. The two circles that contain the points Y_i', Y_i', $i = 1, 2, 3, 4$, coincide if and only if $M_1G = M_2G$, *i.e.*, if and only if $M_1M_3 = M_2M_4$. This is the case if and only if $\mathbf{Q}$ is orthodiagonal. □

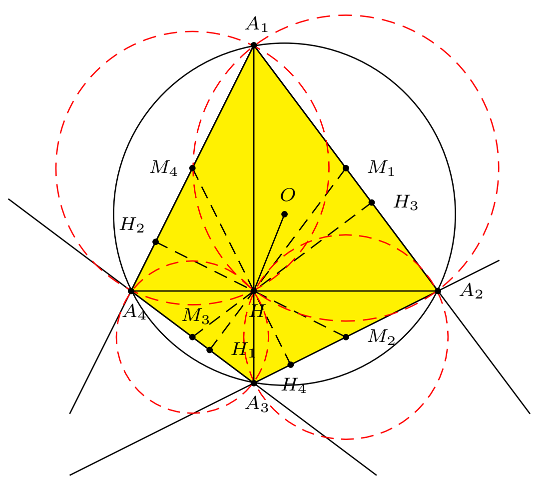

Figure 8.

If $\mathbf{Q}$ is cyclic and orthodiagonal, we call the circle containing the eight points Y_i', Y_i', $i = 1, 2, 3, 4$, the second Droz-Farny circle of $\mathbf{Q}$. But observe that the circle with diameter a side of $\mathbf{Q}$ passes through H, because the diagonals of $\mathbf{Q}$ are perpendicular. The points Y_i, Y_i' are simply the vertices A_i of $\mathbf{Q}$, each counted twice. The second Droz-Farny circle coincides with the circumcircle of $\mathbf{Q}$ (see Figure 8).

5. An ellipse through eight points

Suppose that $\mathbf{Q}$ is any convex quadrilateral and let K be the common point of the diagonals of $\mathbf{Q}$. Consider the Varignon parallelogram $M_1M_2M_3M_4$ of $\mathbf{Q}$. Let us fix a segment of length r, greater than the distance of A_i from the line $M_{i-1}M_i$, $i = 1, 2, 3, 4$. Let Z_i and Z_i' be the intersections of the circle with center A_i and radius r with the line $M_{i-1}M_i$. We obtain altogether eight points.

Let p_i be the perpendicular drawn from A_i to the line $M_{i-1}M_i$, and let C_i be the common point of p_i and p_{i+1} (see Figure 9). Since p_i and p_{i+1} are the perpendicular bisectors of the segments Z_iZ_i' and $Z_{i+1}Z_{i+1}'$ respectively, we have

Theorem 8. *The points Z_i, Z_i', Z_{i+1}, Z_{i+1}' lie on a circle with center C_i.*

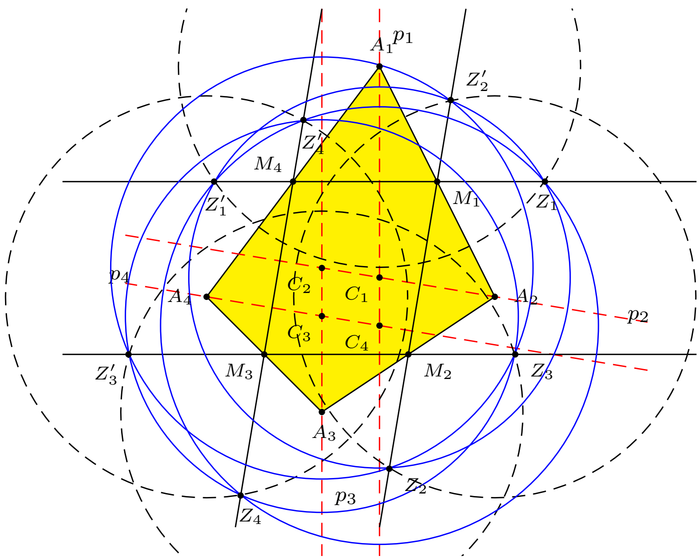

Figure 9.

Theorem 8 states that there are four circles, each passing through the points Z_i, Z_i', that belong to the lines containing two consecutive sides of the Varignon parallelogram of $\mathbf{Q}$ (see Figure 10).

Theorem 9. *The eight points Z_i, Z_i', $i = 1, 2, 3, 4$, all lie on a circle if and only if $\mathbf{Q}$ is orthodiagonal.*

Proof. Suppose first that the eight points Z_i, Z_i', $i = 1, 2, 3, 4$, all lie on a circle. If C is the center of the circle, then each C_i coincides with C. Since the lines A_1C_1 and A_3C_3 both are perpendicular to A_2A_4, the point C must lie on A_1A_3 and then $\mathbf{Q}$ is orthodiagonal.

Conversely, let $\mathbf{Q}$ be orthodiagonal. Since A_1A_3 is perpendicular to M_1M_4, the point C_1 lies on A_1A_3. Since A_2A_4 is perpendicular to M_1M_2, C_1 also lies on A_2A_4. It follows that C_1 coincides with K. Analogously, each of C_2, C_3, C_4 also coincides with K, and the four circles coincide each other in one circle with center K. □

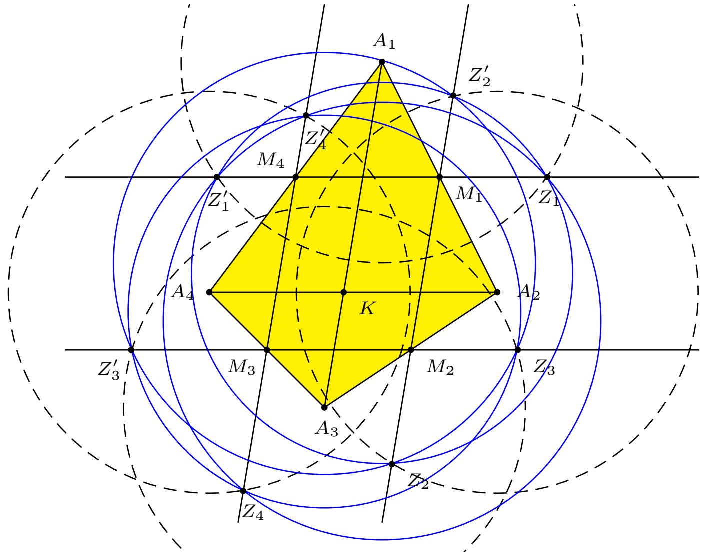

Figure 10.

Because of Theorem 9 we can state that if $\mathbf{Q}$ is orthodiagonal, the eight points Z_i, Z_i', $i = 1, 2, 3, 4$, all lie on a circle with center K (see Figure 11).

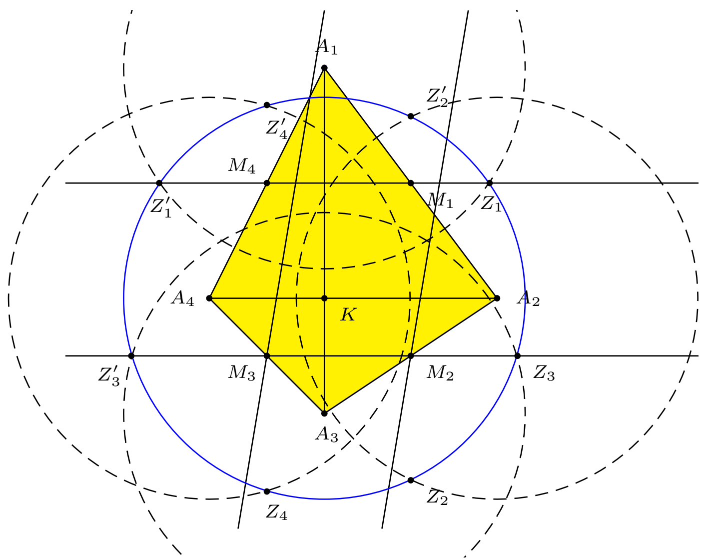

Figure 11.

Corollary 10 below follows from Theorem 5 and from the fact that in a cyclic and orthodiagonal quadrilateral $\mathbf{Q}$ the common point of the diagonals is the anticenter of $\mathbf{Q}$.

Corollary 10. *If* **Q** *is cyclic and orthodiagonal, the circle containing the eight points* Z_i, Z'_i, $i = 1, 2, 3, 4$, *obtained by getting the circumradius of* **Q** *as r, coincides with the first Droz-Farny circle of* **Q**.

We conclude the paper with the following general result.

Theorem 11. *If* **Q** *is a convex quadrilateral, the eight points* Z_i, Z'_i, $i = 1, 2, 3, 4$, *all lie on an ellipse whose axes are the bisectors of the angles between the diagonals of* **Q**. *Moreover, the area of the ellipse is equal to the area of a circle with radius* r.

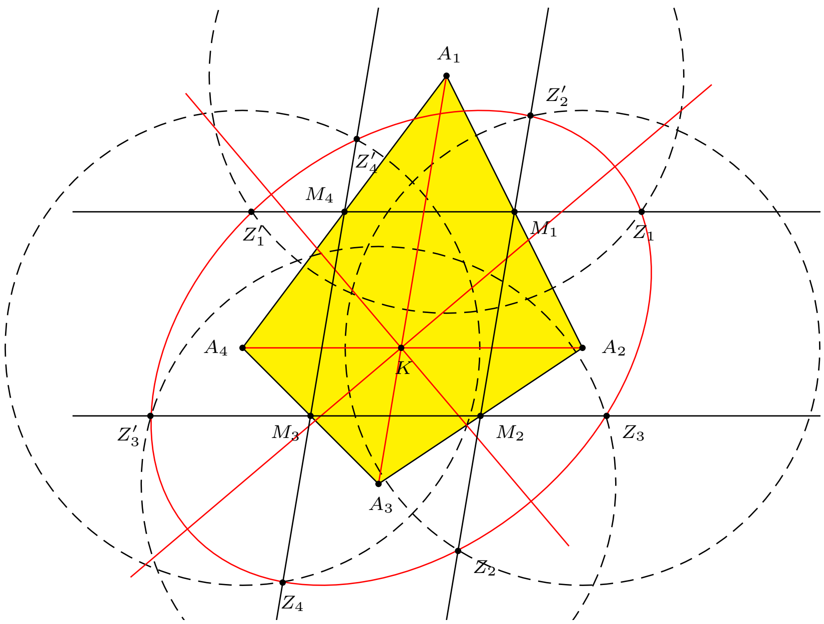

Figure 12.

Proof. We set up a Cartesian coordinate system with axes the bisectors of the angles between the diagonals of **Q**. The equations of the diagonals are of the form $y = mx$ and $y = -mx$, with $m > 0$. The vertices of **Q** have coordinates $A_1(a_1,\ ma_1)$, $A_2(a_2,\ -ma_2)$, $A_3(a_3,\ ma_3)$, $A_4(a_4,\ -ma_4)$, with $a_1, a_2 > 0$ and $a_3, a_4 < 0$. By calculations, the coordinates of the points Z_i and Z'_i are

$$\left(\frac{a_i \pm \sqrt{(m^2+1)r^2 - m^2 a_i^2}}{m^2+1},\ \frac{m^3 a_i \mp \sqrt{(m^2+1)r^2 - m^2 a_i^2}}{m^2+1} \right).$$

These eight points Z_i, Z'_i, $i = 1, 2, 3, 4$, lie on the ellipse

$$m^4 x^2 + y^2 = m^2 r^2. \tag{3}$$

Moreover, since the lengths of the semi axes of the ellipse are $\frac{r}{m}$ and mr, the area enclosed by the ellipse is equal to πr^2. □

Note that (3) is the equation of a circle if and only if $m = 1$. In other words, the ellipse is a circle if and only if $\mathbf{Q}$ is orthodiagonal.

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Forum Geometricorum
Volume 11 (2011) 121–129.

FORUM GEOM
ISSN 1534-1178

Solving Euler's Triangle Problems with Poncelet's Pencil

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Abstract. We determine the unique triangle given its orthocenter, circumcenter and another particular triangle point. The main technique is to realize the triangle as special intersection points of the circumcircle and a rectangular hyperbola in Poncelet's pencil.

1. Introduction

Euler's triangle problem asks one to determine a triangle when given its orthocenter H, circumcenter O and incenter I. This problem has received some recent attention. Some notable papers are those of Scimemi [10], Smith [11], and Yiu [12]. It is known that the solution to the problem is not (ruler-compass) constructible in general so other methods are necessary. For example, Yiu solves the Euler triangle problem with the auxiliary construction of a cubic curve and then realizes the solution as the intersection points of a rectangular hyperbola and the circumcircle.

From the work of Guinand [5] we know that a necessary and sufficient condition for a solution is that I lies inside the circle with diameter GH (G is the centroid) but different from N, the center of Euler's nine-point circle.

We approach these triangle determination problems by realizing the triangle vertices as the intersections of the circumcircle and a rectangular hyperbola in the Poncelet pencil. We can solve Euler's triangle problem using either Feuerbach's hyperbola or Jerabek's hyperbola. We establish some further properties of the Poncelet pencil in order to prove that the triangle is uniquely determined. For the solution using Jerabek's hyperbola we use some methods suggested by the work of Scimemi. As an aid we develop some of the relations between Wallace-Simson lines and the Poncelet pencil.

Also we use Kiepert's hyperbola to solve (uniquely) the triangle determination problem when given O, H and any one of following: the symmedian point K, the first Fermat point F_+, the Steiner point S_{t} or the Tarry point T_{a}.

2. Data

Suppose that the three points O, H, I are given. As Euler and Feuerbach showed, $OI^2 = R(R-2r)$ and $2NI = R-2r$, where R is the circumradius and r is the inradius, and we have $R = \frac{OI^2}{2NI}$. The nine-point circle has center N, the midpoint of OH and radius $\frac{R}{2}$. Thus the circumcircle $\mathcal{C}$ and the Euler circle can be constructed. The Feuerbach point F_{e} can now also be constructed as the intersection point of the nine-point circle and the extension of the ray NI beginning at the center N of the nine-point circle and passing through the incenter I ([12]).

Publication Date: May 26, 2011. Communicating Editor: Jean-Pierre Ehrmann.

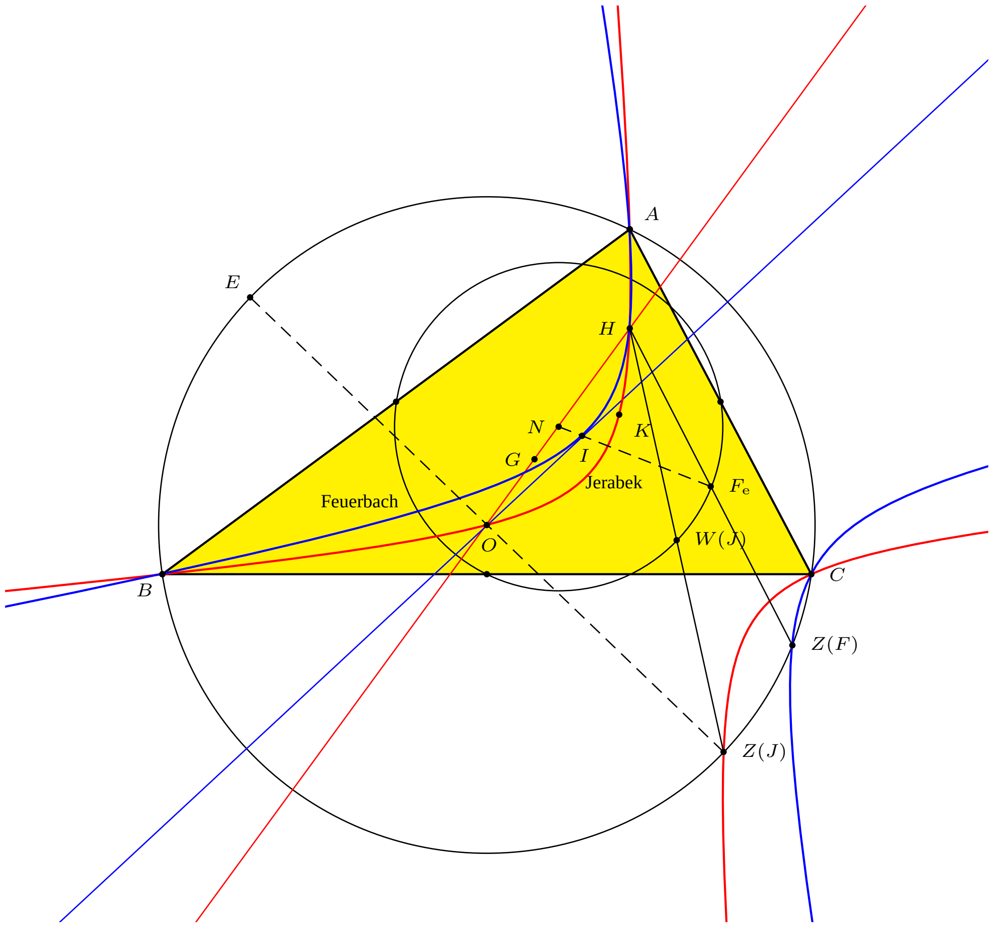

Figure 1. Feuerbach and Jerabek Hyperbolas

The Poncelet pencil is a pencil of rectangular hyperbolas determined by a triangle, namely, the conics are the isogonal transforms of the lines through O ([1]). It can be characterized as the pencil of conics through the vertices of the given triangle where each conic is a rectangular hyperbola. The Feuerbach hyperbola $\mathcal{F}$ is the isogonal transform of the line OI; this has the Feuerbach point $F_{\rm e}$ as its center. It is tangent to the line OI at the point I. Thus the five linear conditions: rectangular, duality of $F_{\rm e}$ and the line at infinity, duality of I and the line OI determine the equation for the rectangular hyperbola $\mathcal{F}$ explicitly, without knowing the vertices of the triangle.

Generally, two conics intersect in four points. The four intersections of the Feuerbach hyperbola and the circumcircle consists of the three triangle vertices together with a fourth point, called the circumcircle point of the hyperbola ([1]). The point Z on the line through $H, F_{\rm e}$ with $HF_{\rm e} = F_{\rm e}Z$ is a point on the circumcircle since the central similarity at H with scale 2 takes the nine-point circle to the circumcircle. We show (Proposition 3) that this point Z is the circumcircle point of Feuerbach's hyperbola, that is, Z is also on Feuerbach's hyperbola.

The solution to the Euler triangle problem for O, I, H is now given by the following.

Theorem 1. *Suppose that $I \neq N$ is interior to the open disk with diameter GH. Let $\mathcal{F}$ be the rectangular hyperbola with center $F_{\rm e}$ and duality of I with line OI. The intersection of $\mathcal{F}$ with the circumcircle $\mathcal{C}$ consists of the circumcircle point Z and the vertices of the unique triangle ABC with incenter I, orthocenter H and circumcenter O.*

The solution to Euler's problem is unique, when it exists, since there is no ambiguity in determining which three of the four points of intersection of the circumcircle and hyperbola are the triangle vertices.

However, if the circumcircle point Z is a triangle vertex, then Feuerbach's hyperbola is tangent to the circumcircle at that vertex. In this case we can construct a vertex and so the triangle is actually constructible by ruler-compass methods. This situation arises if and only if the line $HF_{\rm e}$ is perpendicular to OI as we show in Proposition 4.

Corollary 2. *We can solve the triangle problem when given O, H and either the Nagel point $N_{\rm a}$ or the Spieker center $S_{\rm p}$.*

Proof. We use the fact that the four points $I, G, S_{\rm p}, N_{\rm a}$ lie on a line with ratio $IG : GS_{\rm p} : S_{\rm p}N_{\rm a} = 2 : 1 : 3$. Given O, H we can construct G, and then given either $S_{\rm p}$ or $N_{\rm a}$, we can determine I. Thus we can solve the triangle problem with the hyperbola $\mathcal{F}$ as constructed in Theorem 1. □

3. Poncelet Pencil

In this section we develop the results about the Poncelet pencil used in the proof of Theorem 1.

Suppose Δ is a triangle with circumcircle $\mathcal{C}$. For Δ the isogonal transform of the lines through the circumcenter O gives the Poncelet pencil of rectangular hyperbolas discussed in [1]. The centers of these hyperbolas lie on the nine-point circle. The orthocenter H lies on every hyperbola of this pencil.

Let $\mathcal{P}$ be a hyperbola of the Poncelet pencil. The conic $\mathcal{P}$ and $\mathcal{C}$ meet at the vertices of Δ and a fourth point.

Thus we have the following result.

Proposition 3. *Let $\mathcal{P}$ be the hyperbola of the Poncelet pencil whose center is W. The point Z so that $HW = WZ$ on the line HW is on the circumcircle of Δ and the hyperbola $\mathcal{P}$.*

Proof. A rectangular hyperbola is symmetric about its center. Hence, any line through the center meets the hyperbola in two points of equal distance from the center. Thus the line HW meets $\mathcal{P}$ at another point Z so that $WZ = HW$.

Since the central similarity at H with scale factor 2 takes the Euler circle to the circumcircle then the point Z is also on the circumcircle. $\square$

The Z is called the circumcircle point of the hyperbola $\mathcal{P}$. It may happen that this point Z is one of the vertices of Δ. The center of the hyperbola is denoted W.

We analyze that situation. A point Y on $\mathcal{K}$ is a vertex if and only if HY is an altitude. So if Z is a vertex then HZ is an altitude and hence W also lies on an altitude since Z lies on HW. Conversely, if W lies on the altitude then Z also lies on that altitude and hence is a vertex.

Proposition 4. *A hyperbola $\mathcal{P}$ of the Poncelet pencil with center W is tangent to the circumcircle if and only if HW is perpendicular to $\mathcal{L} = \mathcal{P}^*$ if and only if the circumcircle point Z is a vertex of the triangle.*

Proof. If Z is a vertex then its isogonal transform is at infinity on BC and on the transform $\mathcal{L} = \mathcal{P}^*$. Hence $\mathcal{L}$ is parallel to BC. Thus from the remarks above, HW is an altitude if and only if the circumcircle point is a vertex. Then tangency of the circumcircle and hyperbola occurs if and only if HW is perpendicular to $\mathcal{L}$. $\square$

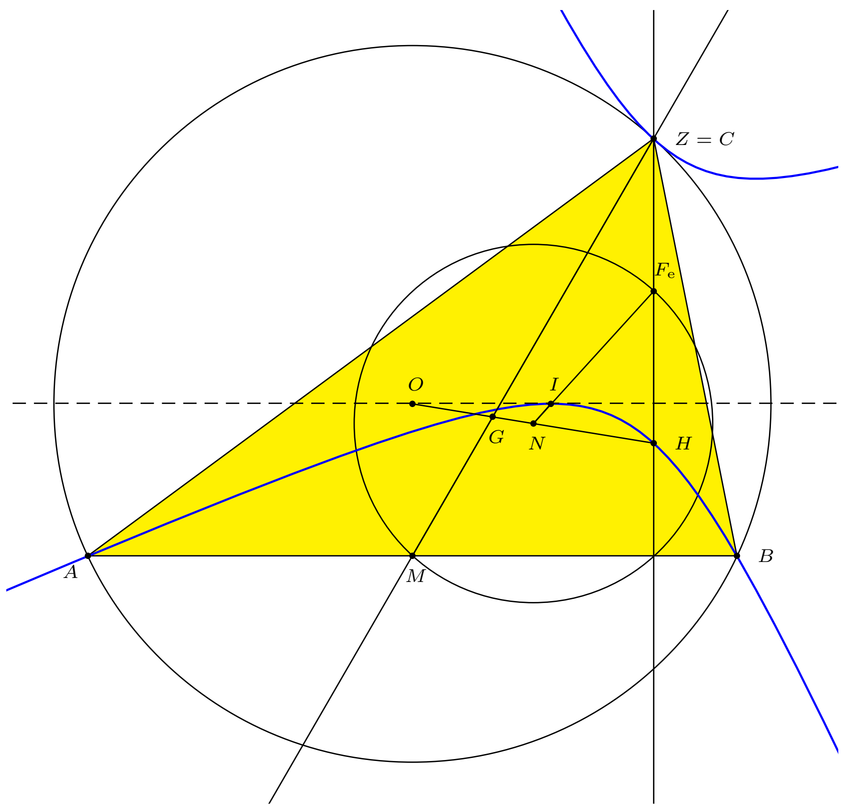

Figure 2. Triangle is constructible when OI is perpendicular to HF_e. Relation to Feuerbach hyperbola.

In this tangent case we can solve the Euler triangle problem easily with ruler and compass if we have the hyperbola's center W since then we can construct the vertex $C = Z$. The other vertices are then also easy to obtain: on the line CG, construct the point M with ratio $CG : GM = 2 : 1$. The line through M, perpendicular to HC meets the circumcircle at two other vertices of the triangle.

For the case of Feuerbach hyperbola, $W = F_{\mathrm{e}}$; the point Z is a vertex if and only if OI is perpendicular to HF_{e}.

4. Solution with Jerabek's Hyperbola

We now develop some of the useful relations between the Poncelet pencil, circumcircle points and Simson lines. These allow us to prove the relation of Scimemi's Euler point to the circumcircle point of Jerabek's hyperbola.

4.1. *Simson Lines and Orthopole.* See [6, §327-338, 408]. Denote the isogonal conjugate of X by X^*.

Theorem 5. *For S on the circumcircle of triangle ABC, SS^* is perpendicular to the Wallace-Simson line of S, i.e., S^* lies on the Wallace-Simson line of the antipode S' of S.*

Proof. From [4] the Wallace-Simson line of S passes through the isogonal conjugate T^* of its antipodal point $T = S'$. Thus it is perpendicular to SS^* since the angle between T^* and S^* is 90 degrees, the angle being halved by the isogonal transformation. □

Theorem 6. *Consider the line $\mathcal{L}$ though the circumcenter of triangle ABC, meeting the circumcircle at U, U'. Let $\mathcal{K} = \mathcal{L}^*$.*
(i) *The Wallace-Simson lines of U, U' are asymptotes of $\mathcal{K}$ and meet at the center $W(\mathcal{K})$ of $\mathcal{K}$ on the nine-point circle of triangle ABC.*
(ii) *The center $W(\mathcal{K})$ is the orthopole of $\mathcal{L}$.*
(iii) *The Wallace-Simson line of $C(\mathcal{K})$ is perpendicular to $\mathcal{L}$. This line bisects the segment from H to $C(\mathcal{K})$ at $W(\mathcal{K})$.*

Proof. The asymptotes of $\mathcal{K}$ are the Wallace-Simson lines of the isogonal conjugates of the points at infinity of $\mathcal{K}$, [4, p. 196]. Hence the center of $\mathcal{K}$ is the orthopole of $\mathcal{K}^* = \mathcal{L}$ (see [6, §406]).

A dilation at H by $\frac{1}{2}$ takes the the circumcircle to the nine-point circle. The Wallace-Simson line of any point S on the circumcircle bisects the segment HS and passes through a point of the nine-point circle [6, §327]. Thus midpoints of U, U' with H are antipodal points on the nine-point circle and lie on Wallace-Simson lines (asymptotes of $\mathcal{K}$).

The isogonal transform of the circumcircle point $C(K)$ lies on the line $\mathcal{L} = \mathcal{K}^*$ and the line at infinity. Thus the Wallace Simson line of $C(\mathcal{K})$ is perpendicular to $\mathcal{L}$ by Theorem 5. In [6, §406] a point W is constructed from UU^* so that its Wallace-Simson line is perpendicular to UU^*. Thus by uniqueness of the directions of Wallace-Simson lines this point W is $C(\mathcal{K})$. As shown there the Wallace-Simson line of W is also coincident with the Wallace-Simson lines of both U and U'. Thus

$W = C(\mathcal{K})$ is the dilation by 2 of the center of the right hyperbola and HW is bisected by the Wallace-Simson line of W at the center of $\mathcal{K}$. $\square$

4.2. Scimemi has introduced the Euler point E in [10]. He shows that it is constructible from given O, H, I. Following Scimemi's Theorem 2 ([10]) and the previous theorems we obtain the following.

Corollary 7. *Let $\mathcal{L}$ be a line through the circumcenter of triangle ABC, $\mathcal{K} = \mathcal{L}^*$, $W = C(\mathcal{K})$ the circumcircle point of $\mathcal{K}$. Then the reflection of W', the antipodal of W, in the sides of ABC lie on a line passing through H, which is parallel to $\mathcal{L}$ and perpendicular to the Wallace-Simson line of W.*

Scimemi's Euler point E is the point of coincidence of the reflections of the Euler line in the sides of the triangle. Thus it is antipodal to the circumcircle point of Jerabek's hyperbola defined by $\mathcal{K} = \mathcal{L}^*$, where $\mathcal{L}$ is the Euler line.

Hence we can construct the center of Jerabek's hyperbola since it is the midpoint of HE. Thus Jerabek's hyperbola is determined linearly from the data: it is a rectangular hyperbola; it passes through H and O; there is a duality of its center with the line at infinity.

4.3. *Construction with Feuerbach and Jerabek Hyperbolas.* We can use both Feuerbach's and Jerabek's hyperbolas to determine the triangle. The common points are the triangle vertices and the orthocenter H.

5. Construction with Kiepert's hyperbola

In [2] it is shown that the symmedian point K ranges over the open disk with diameter GH punctured at its center.

Theorem 8. *We can uniquely determine the triangle when given O, H and K.*

Proof. It is known that the center W of Kiepert's hyperbola $\mathcal{K}$ is the inverse of K in the orthocentroidal circle with diameter GH and center J [7]. Thus the center W is constructible given O, H, K. Since this point is the intersection of the ray JK with the Euler circle, we also can construct the radius of the Euler circle and hence also the radius of the circumcircle. Hence we may construct the circumcircle since the center O is given.

We can provide linear conditions to determine Kiepert's hyperbola $\mathcal{K}$: rectangular hyperbola, duality of W and the line at infinity, passing through G and H.

The circumcircle point Z is the intersection of HW with the circumcircle. The four intersections of Kiepert's hyperbola and the circumcircle are the points of the triangle and the point Z. $\square$

Thus also the triangle is uniquely determined and ruler-compass constructible if Kiepert's hyperbola is tangent to the circumcircle. This last condition is equivalent to HW is perpendicular to the line OK, where W is the center of Kiepert's hyperbola.

5.1. *Kiepert's Hyperbola and Fermat's Points.* Given O, H and the first Fermat point F_+, we can construct the second Fermat point F_- since it the inverse of F_+ in the orthocentroidal circle [2, p.63]. The center Z of Kiepert's hyperbola is the midpoint of these two Fermat points [4, p.195]. These conditions determine Kiepert's hyperbola: rectangular, center W, passing through G, H.

Now also we can construct the circumcircle point W of Kiepert's hyperbola since it is the symmetry about Z of the orthocenter point H. Now we have the center O and W a point of the circumcircle. Hence we can now determine the triangle uniquely.

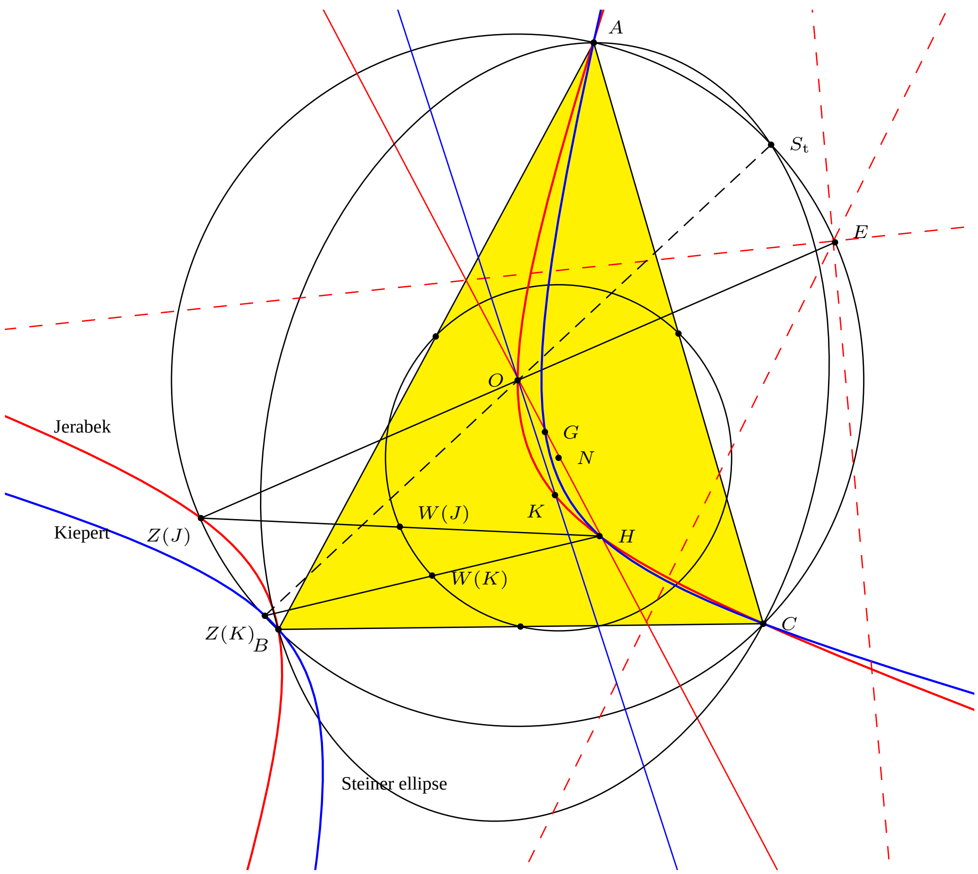

Figure 3. Kiepert Hyperbola, Jerabek Hyperbola, Steiner's Ellipse and Euler Point E

5.2. *Kiepert's Hyperbola and Steiner or Tarry points.* With O and either the Steiner or Tarry point we can construct the circumcircle of the desired triangle since each of these points lies on the circumcircle. Moreover, since the Tarry point is the circumcircle point of Kiepert's hyperbola [7] we may construct the center of the Kiepert hyperbola. Also Steiner's point is antipodal to the Tarry point on the circumcircle so we may use the Steiner point to determine the center. Thus using the duality of the line at infinity and the center of the Kiepert hyperbola, we may determine the rectangular hyperbola also passing through G and H. This is Kiepert's hyperbola, so the intersections of this with the circumcircle give the triangle and the Tarry point. Thus the triangle is uniquely determined.

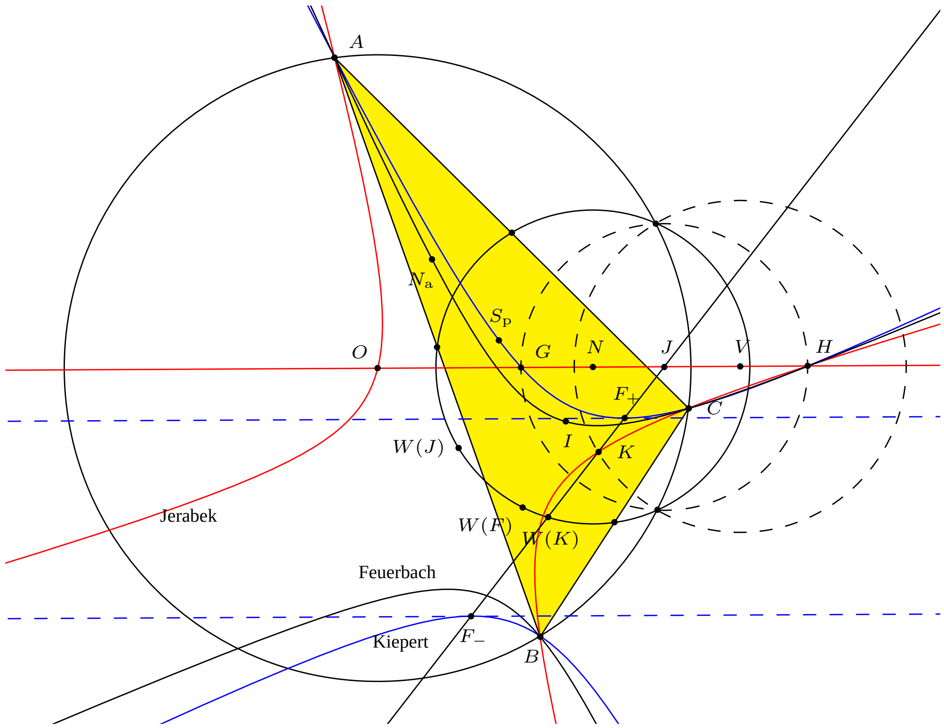

Figure 4. Relations to Orthocentroidal Disk

5.3. *Location of the Symmedian point.* Since the center of Kiepert's hyperbola lies on the nine-point circle, we can obtain the location of the symmedian point K by inversion in the orthocentroidal circle.

The familiar formula for inverting a circle $\mathcal{C}$ of radius c in a circle $\mathcal{K}$ of radius k gives a circle $\mathcal{C}'$ of radius c' with $c'^2(d^2 - c^2)^2 = k^4c^2$, where d is the distance of the center of $\mathcal{C}$ from the center of $\mathcal{K}$.

In the case of inverting the nine-point circle in the orthocentroidal circle, we have $k = 2d = \frac{OH}{3}$, $c = \frac{R}{2}$ and the circle $\mathcal{C}'$ has center V on the Euler line. Thus we have that $X = K$ satisfies the equation $VX^2 = c'^2 = (\frac{k^2c}{d^2-c^2})^2 = (\frac{2R\cdot OH^2}{OH^2-R^2})^2$. As a comparison, one knows that the incenter $X = I$ satisfies the quartic equation $OX^4 = 4R^2 \cdot JX^2$ [5]. In addition, it is known that the symmedian point ([2]) and incenter I ([5]) lie inside the orthocentroidal circle with center J.

5.4. *Tangent Lines at Fermat Points.* See Figure 4. The Fermat points lie on the line through the center $W(K)$ of Kiepert's hyperbola; also the symmedian K,

$W(K)$, J lie on the same line since F_+ and F_- are inverses in the orthocentroidal circle with center J [7].

The Kiepert hyperbola is the isotomic transform of the line through H^s and G, where H^s is the isotomic transform of H [4]. Since G is a fixed point of the isotomic transformation this line is tangent to the hyperbola at G. As shown in [3], H^s is the symmedian of the anticomplementary triangle, so in fact K is also on that line and $H^sG : GL = 2 : 1$.

Let Y be the dual of line GH in Kiepert's hyperbola; then Y lies on the tangent line H^sG to G; hence the dual of J passes through Y. Since J is the midpoint of GH then the line JY passes through the center $W(K)$. But we have already shown that K is on the line $JW(K)$ and H^sG; thus $Y = K$. Consequently the dual of any point on GH passes through K; in particular the dual of the point at infinity on GH passes through K and the center $W(K)$. Consequently the two intersections of this line with the conic, F_+ and F_- have their tangents parallel to the Euler line GH.

Remark. The editors have pointed out the very recent reference [9].

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Forum Geometricorum
Volume 11 (2011) 131–138.

FORUM GEOM
ISSN 1534-1178

More on the Extension of Fermat's Problem

Nguyen Minh Ha and Bui Viet Loc

Abstract. We give an elementary and complete solution to the Fermat problem with arbitrary nonzero weights.

1. Introduction

At the end of his famous 1643 essay on maxima and minima, Pierre de Fermat (1601–1665) threw out a challenge: "Let he who does not approve of my method attempt the solution of the following problem: given three points in a plane, find a fourth point such that the sum of its distances to the three given points is a minimum!" Our Problem 1 is the most interesting case of his problem.

Problem 1 (Fermat's problem)**.** *Let ABC be a triangle. Find a point P such that $PA + PB + PC$ is minimum.*

The first published solution came from Evangelist Torricelli, published posthumously in 1659. Numerous subsequent solutions are readily to be found in books and journals. Problem 1 has been generalized in different ways. The generalization below is presumably the most natural.

Problem 2. *Let ABC be a triangle and $x,\ y,\ z$ be positive real numbers. Find a point P such that $x \cdot PA + y \cdot PB + z \cdot PC$ is minimum.*

Various approaches to the solution of Problem 2 can be found in [5, 6, 7, 8, 9, 11].

In 1941, R. Courant and H. Robbins [1] posed another problem inspired by Problem 1, replacing the weights $1, 1, 1$ with $-1, 1, 1$. Unfortunately, their claimed solution is flawed. Afterwards, other problems were suggested in the same spirit. The following problem is among the most natural [2, 3, 4, 10].

Problem 3. *Let ABC be a triangle and $x,\ y,\ z$ be non-zero real numbers. Find a point P such that $x \cdot PA + y \cdot PB + z \cdot PC$ is minimum.*

The solution of problem 3 requires the solution of Problem 2 as well as solutions to Problems 4 and 5 below.

Problem 4. *Let ABC be a triangle and $x,\ y,\ z$ be positive real numbers. Find a point P such that $-x \cdot PA + y \cdot PB + z \cdot PC$ is minimum.*

Publication Date: June 14, 2011. Communicating Editor: J. Chris Fisher.
The authors thank Professor Chris Fisher for his valuable comments and suggestions.

Problem 5. *Let ABC be a triangle and $x,\ y,\ z$ be positive real numbers. Find a point P such that $-x \cdot PA - y \cdot PB + z \cdot PC$ is minimum.*

Here is an easy solution of Problem 5. If $z < x+y$, then $-x{\cdot}PA-y{\cdot}PB+z{\cdot}PC$ decreases without bound as PC goes to ∞, whence there exists no point P for which the minimum is attained; otherwise, when $z \geq x+y$ the minimum is attained when P coincides with C. The verification of our claim is straightforward.

Problem 4, on the other hand, is very tricky. A correct solution was first introduced in 1980 by L. N Tellier and B.Polanski employing trigonometry [10]. In 1998 J. Krarup gave a solution to problem 4 which was more specific in its conclusion [4]. In 2003 G. Jalal and J. Krarup [3] devised a different solution to problem 4 based on a geometric approach. Four years later, another solution was introduced by G. Ganchev and N. Nikolov using the concept of isogonal conjugacy [2]. However, these solutions are somewhat complicated and not elementary. In this article, we aim to deliver a synthetic and elementary solution to Problem 4.

2. Solution to Problem 4

Let a, b, c be the lengths of BC, CA, and AB of triangle ABC, respectively. Without loss of generality we assume $x = a$. Let $f(P) = -a{\cdot}PA+y{\cdot}PB+z{\cdot}PC$. The following is a summary of the main results.

(1) When a, y, z are the side-lengths of a triangle, construct triangle UBC such that $UC = y$, $UB = z$ and U, A are on the same side of line BC. There are three possibilities:

- **(1.1):** U is inside triangle ABC. $f(P)$ attains its minimum when P is the intersection point, other than U, of line AU and the circumcircle of triangle UBC.
- **(1.2):** $U = A$. $f(P)$ attains its minimum when P lies on the arc BC not containing A of the circumcircle of triangle ABC.
- **(1.3):** U is not inside triangle ABC and U is distinct from A. The minimum of $f(P)$ occurs when $P = B$ or C or both, according as $f(B) < f(C)$ or $f(B) > f(C)$ or $f(B) = f(C)$.

(2) When a, y, z are not the side-lengths of a triangle, we consider two possibilities.

- **(2.1):** $a \geq y + z$. There is no P such that $f(P)$ attains its minimum.
- **(2.2):** $a < y + z$. The minimum of $f(P)$ occurs when $P = B$ or $P = C$ according as $f(B) < f(C)$ or $f(B) > f(C)$.

We shall make use of the following four easy lemmas in the solution to the Problem 4.

Lemma 1 (Ptolemy's inequality)**.** *Let P be a point in the plane of triangle ABC.*
(a) *Unless P lies on the circumcircle of triangle ABC, the three numbers $BC{\cdot}PA$, $CA \cdot PB$, $AB \cdot PC$ are side lengths of some triangle.*
(b) *If P lies on the circumcircle of triangle ABC, then one of three numbers $BC \cdot PA$, $CA \cdot PB$, $AB \cdot PC$ is the sum of the other two numbers. Specifically,*
(i) *If P lies on the arc BC not containing A then $BC \cdot PA = CA \cdot PB + AB \cdot PC$.*

(ii) *If P lies on the arc CA not containing B, then $CA \cdot PB = AB \cdot PC + BC \cdot PA$.*
(iii) *If P lies on the arc AB not containing C, then $AB \cdot PC = BC \cdot PA + CA \cdot PB$.*

Lemma 2 (Euclid I.21)**.** *For any point P in the interior of triangle ABC, $PB + PC < AB + AC$.*

Lemma 3. *If $ABCD$ is a convex quadrilateral, then $AB + CD < AC + BD$.*

Lemma 4. . *Give an isosceles triangle ABC with $AB = AC$, Ax is the opposite ray of the ray AB. For every P lying inside xAC, we have $PB \geq PC$, with equality when P coincides with A.*

Lemma 3 is just the triangle inequality applied to the pair of triangles formed by the intersection point of the two diagonals together with the endpoints of the edges AB and CD. Lemma 4 holds because every point to the side of the perpendicular bisector of BC is closer to C than to B.

We now turn to the solution of Problem 4. There are two cases to consider.

Case 1. a, y, z are sides of some triangles. Construct triangle UBC such that $UC = y$, $UB = z$ and points U, A lie on the same side of line BC. There are three possibilities.
(1.1) U is inside triangle ABC (see Figure 1).

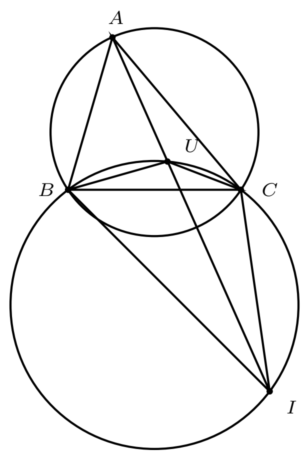

Figure 1.

Denote by I the intersection point, other than U, of line AU and the circumcircle of triangle UBC, which exists because U lies inside triangle ABC so that AU meets the interior of segment BC, which lies inside the circumcircle. Applying Lemma 1 to triangle UBC and an arbitrary point P, we have

$$\begin{aligned} f(P) &= -a \cdot PA + CU \cdot PB + UB \cdot PC \\ &\geq -a \cdot PA + BC \cdot PU = -a(PA - PU) \\ &\geq -a \cdot AU \\ &= f(I). \end{aligned}$$

The equality occurs when P lies on the arc BC not containing U of the circumcircle of triangle UBC and P lies on the opposite ray of the ray UA, that is, $P = I$. In conclusion, $f(P)$ attains its minimum value when $P = I$.

(1.2) $U = A$ (see Figure 2).

For every point P, applying Lemma 1 to triangle UBC and point P, we have

$$f(P) = -a \cdot PA + CU \cdot PB + UB \cdot PC \geq -a \cdot PA + BC \cdot PU = 0.$$

The equality occurs when P lies on the arc BC not containing A of the circumcircle of triangle ABC. From this, we conclude that $f(P)$ attains its minimum value when P lies on the arc BC not containing A of the circumcircle of triangle ABC.

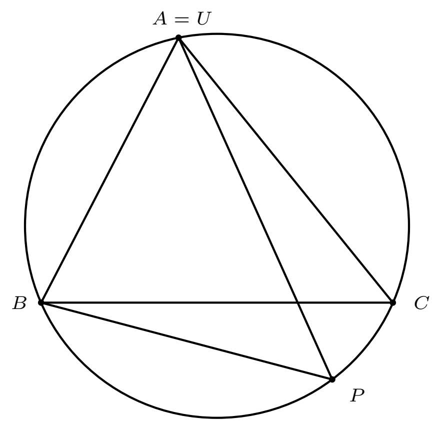

Figure 2.

(1.3) U is not inside triangle ABC and U is distinct from A. There are three situations to consider.

(a) $z - y = c - b$. Denote by Am and An, respectively the opposite rays of the rays AB and AC.

(i) If U lies inside angle mAC (see Figure 3a), then quadrilateral $AUCB$ is convex. By Lemma 3, we have $UB + AC > UC + AB$, which implies that $z - y = UB - UC > AB - AC = c - b$, a contradiction.

(ii) If U lies inside angle nAB (see Figure 3b), we apply Lemma 3 to convex quadrilateral $AUBC$: $UB + AC < UC + AB$. Hence, $z - y = UB - UC < AB - AC = c - b$, a contradiction.

(iii) If U lies inside angle mAn (see Figure 3c), then A lies inside triangle UBC. By Lemma 2, we have $UB + UC > AB + AC$, which implies that $(UB - AB) + (UC - AC) > 0$. Thus, by virtue of $UB - AB = z - c = y - b = UC - AC$, we have $UB - AB = UC - AC > 0$.

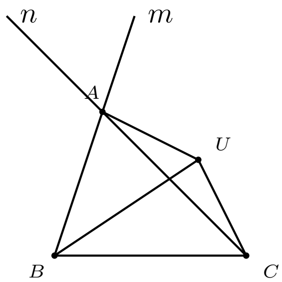

Figure 3a

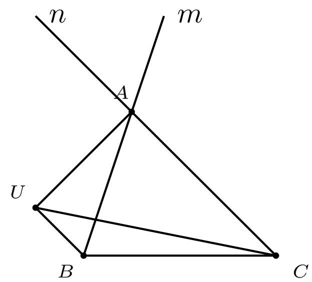

Figure 3b

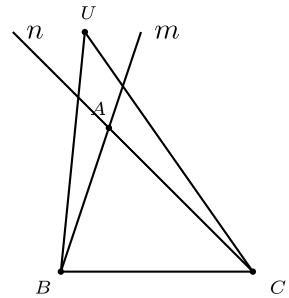

Figure 3c

Now, for any point P, applying Lemma 1 to triangle ABC and point P, taking into account that $PB + PC \geq BC$ we have

$$\begin{aligned} f(P) = &\ -a \cdot PA + UC \cdot PB + UB \cdot PC \\ = &\ -BC \cdot PA + AC \cdot PB + AB \cdot PC + (UC - AC)PB + (UB - AB)PC \\ \geq &\ (UB - AB)(PB + PC) \\ \geq &\ (UB - AB)BC \\ = &\ a(UB - AB) \\ = &\ f(B) = f(C). \end{aligned}$$

The equality occurs if and only if P lies on the arc BC not containing A of circumcircle of triangle ABC and P belongs to the segment BC, *i.e.*, $P = B$ or $P = C$. In conclusion, $f(P)$ attains its minimum value when $P = B$ or $P = C$.

(b) $z - y < c - b$.

(i) $UB - AB > 0$. Similar to (a), note that $UC - AC = y - b > z - c = UB - AB > 0$. Therefore, for any point P,

$$\begin{aligned} f(P) \geq &\ (UC - AC)PB + (UB - AB)PC \\ \geq &\ (UB - AB)(PB + PC) \\ \geq &\ (UB - AB)BC \\ = &\ a(UB - AB) \\ = &\ f(B). \end{aligned}$$

Equality holds if and only if P lies on the arc BC not containing A of the circumcircle of triangle ABC, P coincides with B, and P belongs to segment BC. This means that $P = B$.

(ii) $UB - AB \leq 0$. As before, denote by Am and An respectively the opposite rays of the rays AB and AC. If U lies inside angle mAC (see Figure 4a), then quadrilateral $AUCB$ is convex. By Lemma 3, we have $UB + AC > UC + AB$, implying that $z - y = UB - UC > AB - AC = c - b$, a contradiction.

Thus, U and C are on different sides of line AB (Figure 4b). Since $UB \leq AB$, there is a point T on the segment AB such that $TB = UB$. Hence, applying Lemma 4 to isosceles triangle BUT and point C, we have $UC \geq TC$.

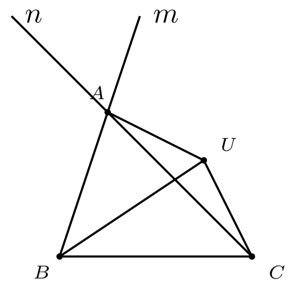

Figure 4a

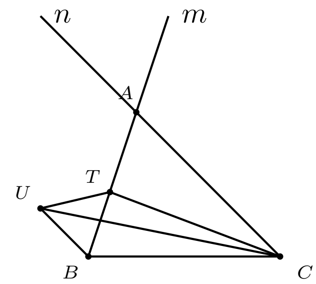

Figure 4b

From this, applying Lemma 1 to triangle TBC and an arbitrary point P, we have

$$\begin{aligned} f(P) &\geq -a\cdot PA + TC\cdot PB + TB\cdot PC & \text{(i)}\\ &\geq -a\cdot PA + BC\cdot TP & \text{(ii)}\\ &= a(-PA+TP) & \\ &\geq -a\cdot AT & \text{(iii)}\\ &= a(UB-AB) & \\ &= f(B). & \end{aligned}$$

Equality occurs if and only if (i) $P=B$, (ii) P lies on arc BC not containing T of the circumcircle of triangle TBC, and (iii) P lies on the opposite ray of the ray TA. Together these mean that $P=B$.

Now we can conclude that $f(P)$ attains its minimum value when $P=B$.

(c) $z-y>c-b$. Similar to (b), interchanging y with z and b with c we conclude that $f(P)$ attains its minimum value when $P=C$.

Case 2. $a,\ y,\ z$ are not side lengths of any triangle. There are two possibilities

(2.1) $a\geq y+z$.

(a) $a=y+z$. Point U is taken on the segment BC such that $y=UC$ and $z=UB$ (see Figure 5).

For every point P we have

$$\overrightarrow{PU}=\frac{UC}{BC}\cdot\overrightarrow{PB}+\frac{UB}{BC}\cdot\overrightarrow{PC}=\frac{y}{a}\cdot\overrightarrow{PB}+\frac{z}{a}\cdot\overrightarrow{PC}.$$

It follows that

$$a\cdot PU=|a\cdot\overrightarrow{PU}|=|y\cdot\overrightarrow{PB}+z\cdot\overrightarrow{PC}|\leq y\cdot PB+z\cdot PC.$$

Therefore,

$$f(P)=-a\cdot PA+y\cdot PB+z\cdot PC\geq -a\cdot PA+a\cdot PU=-a(PA-PU)\geq -a\cdot AU.$$

The equality occurs if and only if the vectors $\overrightarrow{PB},\ \overrightarrow{PC}$ have the same direction and point P belongs to the opposite ray of the ray UA. However, these conditions

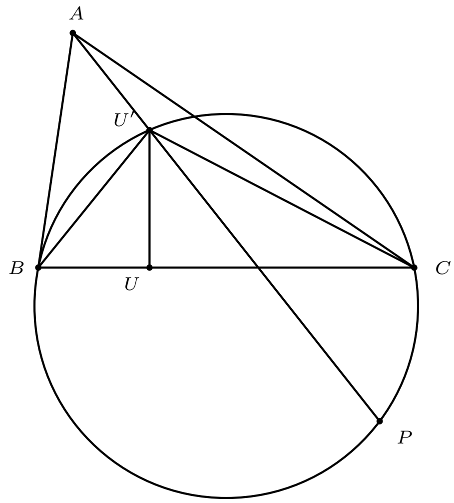

Figure 5.

are incompatible, whence the equality can not be attained. Thus

$$f(P) > -a \cdot AU. \tag{1}$$

For a sufficiently small positive value of ε, take U' inside triangle ABC such that $UU' \perp BC$ and $UU' = \frac{\varepsilon}{a}$. Let P be the intersection, distinct from U', of the line $U'A$ and the circumcircle of $U'BC$ (which exists because U' is inside triangle ABC as shown in Figure 5).

From (1), applying Lemma 1 to triangle $U'BC$ and P, we have

$$\begin{aligned} -a \cdot AU < \; & f(P) \\ = \; & -a \cdot PA + CU \cdot PB + UB \cdot PC \\ < \; & -a \cdot PA + CU' \cdot PB + U'B \cdot PC \\ = \; & -a \cdot PA + BC \cdot PU' \\ = \; & -a \cdot AU'. \end{aligned}$$

It follows that

$$|f(P) - (-a \cdot AU)| < |-a \cdot AU' - (-a \cdot AU)| = a|AU - AU'| < a \cdot UU' = a \cdot \frac{\varepsilon}{a} = \varepsilon. \tag{2}$$

From (1), (2), we can affirm that there does not exist a point P such that $f(P)$ attains its minimum value.

(b) $a > y + z$. For every point P, we have

$$\begin{aligned} f(P) = \; & -a \cdot PA + y \cdot PB + z \cdot PC \\ = \; & (-a + y + z)PA + y(PB - PA) + z(PC - PA) \\ \leq \; & (-a + y + z)PA + y \cdot AB + z \cdot AC. \end{aligned}$$

However, as PA tends to $+\infty$, $f(P)$ tends to $-\infty$. This implies that there does not exist a point P such that $f(P)$ attains its minimum value.

(2.2) $a < y + z$.
(a) $y \geq z + a$. For every point P we have

$$\begin{aligned} f(P) &= -a \cdot PA + y \cdot PB + z \cdot PC \\ &= a(PB - PA) + (y - z - a)PB + z(PB + PC) \\ &\geq -a \cdot BA + z \cdot BC \\ &= f(B). \end{aligned}$$

The equality holds if and only if $P = B$. Hence $f(P)$ attains its minimum value when $P = B$.
(b) $z \geq y + a$. Similarly, interchanging y with z and b with c, we conclude that $f(P)$ attains its minimum value when $P = C$.

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Forum Geometricorum
Volume 11 (2011) 139–144.

FORUM GEOM
ISSN 1534-1178

More on Twin Circles of the Skewed Arbelos

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Abstract. Two pairs of congruent circles related to the skewed arbelos are given.

1. Introduction

We showed several twin circles related to the skewed arbelos in [2]. In this article we give two more pairs, by generalizing two Archimedean circle pairs in [1]. We begin with a brief review of these circles. Let O be a point on a segment AB, and α, β, γ be the circles with diameters OA, OB, AB respectively. We denote one of the intersections of γ and the radical axis of α and β by I. The intersection of IA and α coincides with the tangency point of α and one of the external common tangents of α and β. The circle passing through this point and touching the line IO is Archimedean; so is the one tangent to IO and passing through the intersection of β with the external common tangent. These are the circles W_9 and W_{10} in [1] (see Figure 1). The circle touching α internally and touching the tangents of β from A is also Archimedean; so is the one tangent to β and the the tangents of α from B. These are the circles W_6 and W_7 in [1] (see Figure 2).

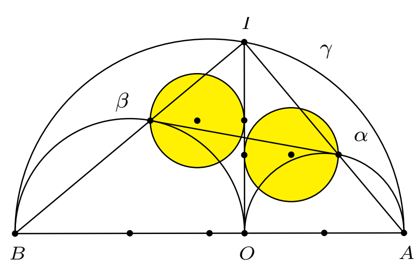

Figure 1

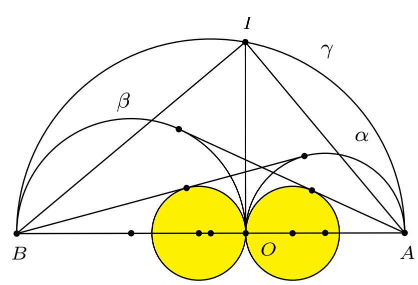

Figure 2

Suppose the circles α and β have radii a and b respectively. We set up rectangular coordinate system with origin O, so that A and B have coordinates $(2a, 0)$ and $(-2b,\ 0)$ respectively. Consider a variable circle touching α and β at points different from O. Such a circle is expressed by the equation

$$\left(x - \frac{b-a}{t^2-1}\right)^2 + \left(y - \frac{2t\sqrt{ab}}{t^2-1}\right)^2 = \left(\frac{a+b}{t^2-1}\right)^2 \qquad (1)$$

Publication Date: June 29, 2011. Communicating Editor: Paul Yiu.

for a real number $t \neq \pm 1$ [2]. We denote this circle by γ_t. It is tangent to α at

$$A_t = \left(\frac{2ab}{at^2+b},\ \frac{2at\sqrt{ab}}{at^2+b} \right),$$

and to β at

$$B_t = \left(-\frac{2ab}{a+bt^2},\ \frac{2bt\sqrt{ab}}{a+bt^2} \right).$$

The tangency (in both cases) is internal if $|t| < 1$ and external if $|t| > 1$. The configuration $(\alpha,\ \beta,\ \gamma_t)$ is called a skewed arbelos.

2. Archimedean twin circles in the skewed arbelos

The circle γ_t intersects the y-axis at the points

$$I_t^+ = \left(0,\ \frac{2\sqrt{ab}}{t+1} \right) \quad \text{and} \quad I_t^- = \left(0, \frac{2\sqrt{ab}}{t-1} \right)$$

respectively.

The lines $I_t^+ A_t$ and $I_t^- A_t$ have equations

$$(at-b)x - (t+1)\sqrt{ab}y + 2ab = 0,$$
$$(at+b)x + (t-1)\sqrt{ab}y - 2ab = 0.$$

These lines intersect the circle α again at the points

$$\left(\frac{2ab}{a+b}, \frac{2a\sqrt{ab}}{a+b} \right) \quad \text{and} \quad \left(\frac{2ab}{a+b}, -\frac{2a\sqrt{ab}}{a+b} \right).$$

Now, it is well known that the common radius of the twin Archimedean circles tangent to α, β, γ is $r_{\mathrm{A}} = \frac{ab}{a+b}$.

These intersections are the fixed points

$$\left(2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{a}{b}} \right) \quad \text{and} \quad \left(2r_{\mathrm{A}}, -2r_{\mathrm{A}}\sqrt{\frac{a}{b}} \right).$$

Similarly, with B_t, the lines $I_t^+ B_t$ and $I_t^- B_t$ intersect the circle β at the points

$$\left(-2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{b}{a}} \right) \quad \text{and} \quad \left(-2r_{\mathrm{A}}, -2r_{\mathrm{A}}\sqrt{\frac{b}{a}} \right).$$

From these we obtain the following result.

Theorem 1. *For $t \neq \pm 1$, if I_t is one of the intersections of the circle γ_t and the line IO, then the circle touching IO and passing through the remaining intersection of the line $I_t A_t$* (respectively $I_t B_t$) *and the circle α* (respectively β) *is an Archimedean circle of the arbelos* (formed by α, β and γ; see Figure 3).

Two of these circles are W_9 and W_{10}, independent of the value of t. If $t = \pm 1$, A_t and the remaining intersection coincide.

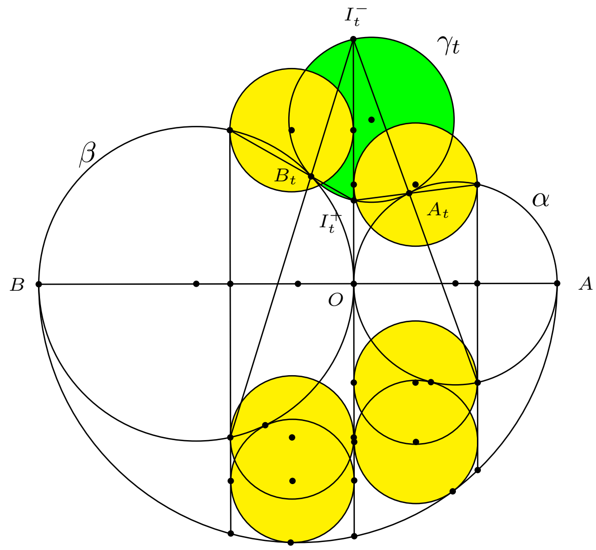

Figure 3.

3. Non-Archimedean twin circles

In this section we generalize the Archimedean circles W_6 and W_7.

We denote the lines $x = 2r_{\mathrm{A}}$ and $x = -2r_{\mathrm{A}}$ by $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ respectively. Let δ_t^α be the circle touching the line $\mathcal{L}_\alpha$ from the side opposite to A_t and the tangents of β from A_t. The circle δ_t^β is defined similarly. The circle δ_t^α lies in the region $x < 2r_{\mathrm{A}}$ if $|t| < 1$ (see Figure 4), and in the region $x > 2r_{\mathrm{A}}$ if $|t| > 1$ (see Figure 5).

If $t = \pm 1$, γ_t reduces to an external common tangent of α and β, and has equation

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0,$$

which are obtained from (1) by letting t approach to ± 1. In these cases, we regard δ_t^α as the tangency point of α with an external common tangent of α and β.

Lemma 2. *The circles δ_t^α and δ_t^β are congruent with common radii $|1-t^2|r_{\mathrm{A}}$.*

Proof. Let s be the radius of the circle δ_t^α. If γ_t touches α and β internally, then we get

$$\left(\frac{2ab}{at^2+b} - 2r_{\mathrm{A}}\right) : s = \frac{2ab}{at^2+b} : b.$$

Solving the equation we get $s = (1-t^2)r_{\mathrm{A}}$. Similarly we get $s = (t^2-1)r_{\mathrm{A}}$ in the case γ_t touching α and β externally. □

The point A_t divides the segment joining the centers of δ_t^α and β in the ratio $|1-t^2|r_{\mathrm{A}} : b$ externally (respectively internally) if γ_t touches α and β internally (respectively externally). Therefore the center of δ_t^α is the point $\left(r_{\mathrm{A}}(1+t^2), 2tr_{\mathrm{A}}\sqrt{\frac{a}{b}}\right)$.

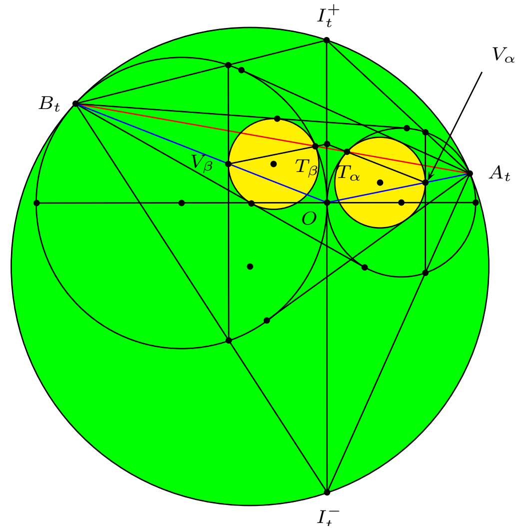

Figure 4.

The circles α and δ_t^α are tangent to each other, since they have only one point in common, namely,

$$T_\alpha = \left(\frac{2abt^2}{a+bt^2}, \frac{2at\sqrt{ab}}{a+bt^2} \right).$$

This point T_α lies on the line A_tB_t since

$$T_\alpha = \frac{at^2+b}{a+b} \cdot A_t + \frac{a(1-t^2)}{a+b} \cdot B_t.$$

Similarly, the circles δ_t^β is tangent to β at

$$T_\beta = \left(-\frac{2abt^2}{at^2+b}, \frac{2bt\sqrt{ab}}{at^2+b} \right),$$

which also lies on the line A_tB_t. The point T_α lies on the line A_tB_t, since

$$T_\beta = \frac{b(1-t^2)}{a+b} \cdot A_t + \frac{a+bt^2}{a+b} \cdot B_t.$$

We summarize the results (see Figures 4 and 5).

Theorem 3. *The circles δ_t^α and δ_t^β are congruent with common radius $|1-t^2|r_{\mathrm{A}}$. The points of tangency A_t, B_t, T_α, and T_β are collinear.*

The circles δ_t^α and δ_t^β are generalizations of the Archimedean circles W_6 and W_7, which correspond to $t = 0$.

The circle δ_t^α touches the line $\mathcal{L}_\alpha$ at the point

$$V_\alpha = \left(2r_{\mathrm{A}}, 2tr_{\mathrm{A}}\sqrt{\frac{a}{b}} \right).$$

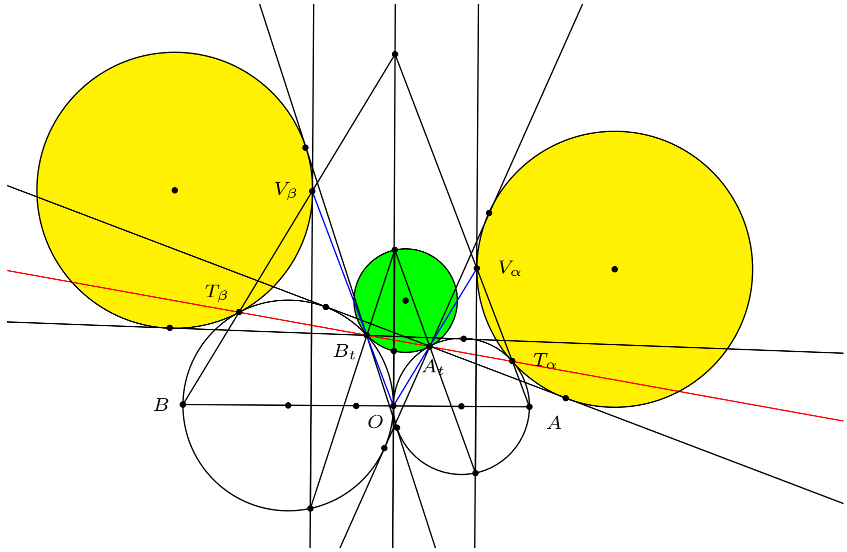

Figure 5.

From these coordinates it is clear that A_t, V_α and O are collinear (see Figures 4 and 5). Similarly, the circle δ_t^β touches the line $\mathcal{L}_\beta$ at

$$V_\beta = \left(-2r_{\mathrm{A}}, 2tr_{\mathrm{A}}\sqrt{\frac{b}{a}}\right)$$

collinear with B_t and O. Now, the line $T_\alpha V_\alpha$ is parallel to B_tO. Since the distances from V_α and V_β to the line IO (the radical axis of α and β) are equal, the lines $T_\alpha V_\alpha$ and $T_\beta V_\beta$ intersect on the radical axis. This intersection and O, V_α, V_β form the vertices of a parallelogram.

4. A special case

For real numbers t and w, the circles δ_t^α and δ_t^β and the circles δ_w^α and δ_w^β are congruent if and only if $|1-t^2| = |1-w^2|$, *i.e.*, $|t| = |w|$ or $t^2+w^2 = 2$. From the equation of the circle γ_t, it is clear that this is the same condition for the congruence of the circles γ_t and γ_w. Therefore we get the following corollary.

Corollary 4. *For real numbers t and w, the circles δ_t^α and δ_t^β and the circles δ_w^α and δ_w^β are congruent if and only if γ_t and γ_w are congruent.*

In the case $t^2+w^2 = 2$, one of the circles γ_t and γ_w touches α and β internally and the other externally. In particular, the circles δ_t^α and δ_t^β are Archimedean circles of the arbelos formed by α, β and γ if and only if γ_t is congruent to γ, *i.e.*, $t = \pm\sqrt{2}$ (see Figure 6).

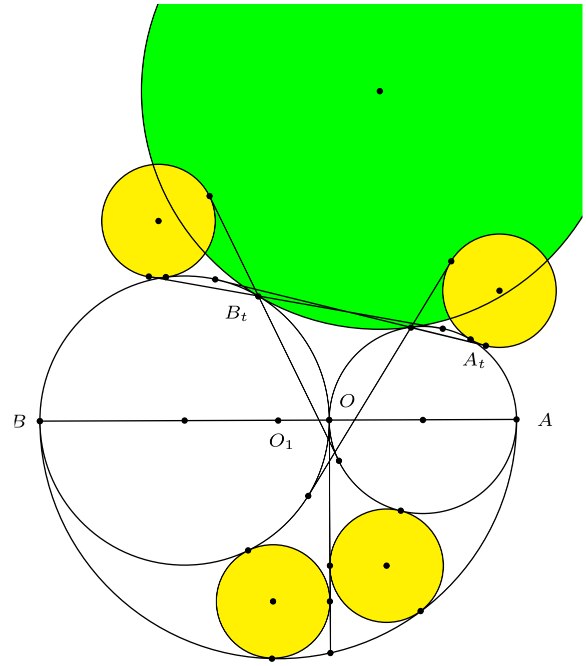

Figure 6.

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Forum Geometricorum
Volume 11 (2011) 145–154.

FORUM GEOM
ISSN 1534-1178

On a Triad of Circles Tangent to the Circumcircle and the Sides at Their Midpoints

Luis González

Abstract. With synthetic methdos, we study, for a given triangle ABC, a triad of circles tangent to the arcs BC, CA, AB of its circumcircle and the sides BC, CA, AB at their midpoints.

1. Introduction

Given a triangle ABC and an interior point T with cevian triangle $A_0B_0C_0$, consider the triad of circles each tangent to a sideline and the circumcircle internally at a point on the opposite side of the corresponding vertex. Lev Emelyanov [1] has shown that the inner Apollonius circle of the triad is also tangent to the incircle (see Figure 1).

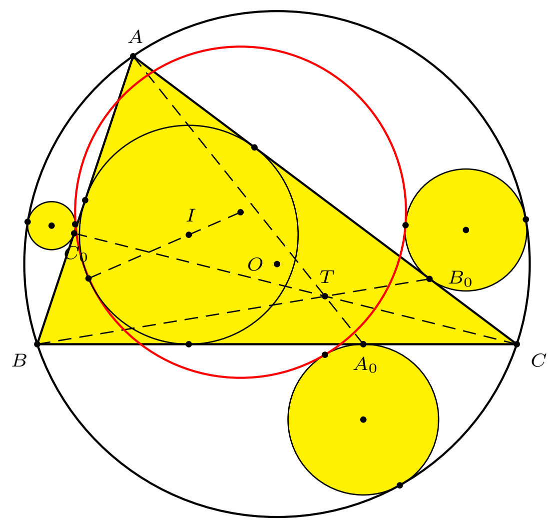

Figure 1.

Yiu [5] has studied this configuration in more details. The points of tangency with the circumcircle form the circumcevian triangle of the barycentric product $I \cdot T$, where I is the incenter. Let X, Y, Z be the intersections of the sidelines EF, FD, EF of the intouch triangle and the corresponding sidelines of the cevian triangle $A_0B_0C_0$. Then XYZ is perspective with DEF, and the perspector is the point F_T on the incircle tangent to the Emelyanov circle, *i.e.*, the inner Apollonius

Publication Date: July 13, 2011. Communicating Editor: Paul Yiu.

circle of the triad ([5, Proposition 12]). In particular, for $T = G$, the centroid of triangle ABC, this point of tangency is the Feuerbach point F_e, which is famously the point of tangency of the incircle with the nine-point circle. Also, in this case,
(i) the radical center of the triad of circles is the triangle center X_{1001} which divides GX_{55} in the ratio $GX_{1001} : X_{1001}X_{55} = R + r : 3R$, where X_{55} is the internal center of similitude of the circumcircle and incircle,
(ii) the center of the Emelyanov circle is the point which divides IN in the ratio $2 : 1$ (see Figure 2).

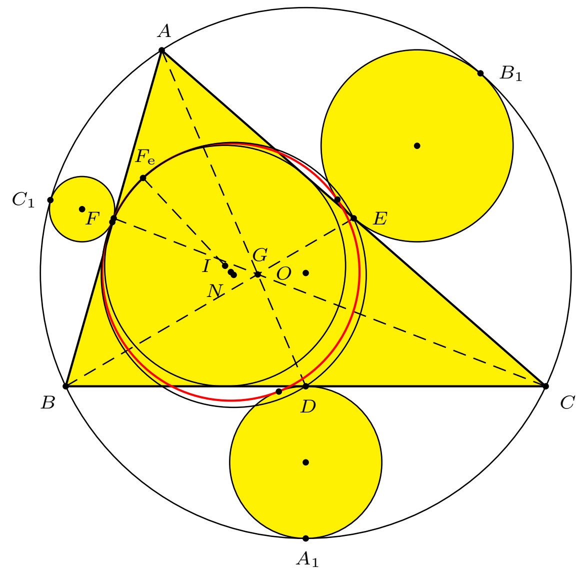

Figure 2.

Yiu obtained these conclusions by computation with barycentric coordinates. In this paper, we revisit the triad of circles $\Gamma(G)$ by synthetic methods.

2. Some preliminary results

Proposition 1. *Two circles $\Gamma_1(r_1)$ and $\Gamma_2(r_2)$ are tangent to a circle $\Gamma(R)$ through A, B, respectively. The length δ_{12} of the common external tangent of Γ_1, Γ_2 is given by*

$$\delta_{12} = \frac{AB}{R}\sqrt{(R \pm r_1)(R \pm r_2)},$$

where the sign is positive if the tangency is external, and negative if the tangency is internal.

Proof. Without loss of generality assume that $r_1 \geq r_2$. Let ε_1 (respectively ε_2 be $+1$ or -1 according as the tangency of (O) and (O_1) (respectively (O_2) is external or internal. Figure 3 shows the case when (O) is both tangent internally to (O_1) and (O_2). Let $\angle O_1OO_2 = \theta$. By the law of cosines,

$$O_1O_2^2 = (R + \varepsilon_1 r_1)^2 + (R + \varepsilon_2 r_2)^2 - 2(R + \varepsilon_1 r_1)(R + \varepsilon_2 r_2)\cos\theta. \quad (1)$$

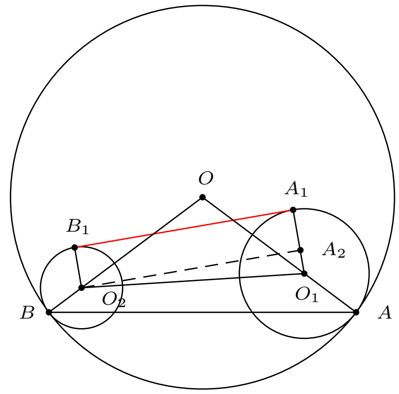

Figure 3.

From the isosceles triangle OAB, we have

$$AB^2 = 2R^2(1 - \cos\theta). \tag{2}$$

Let A_1B_1 be a common tangent of (O_1) and (O_2), external or internal according as $\varepsilon_1\varepsilon_2 = +1$ or -1. If A_2 is the orthogonal projection of O_2 on the line O_1A_1, applying the Pythagorean theorem to the right triangle $O_1O_2A_2$, we have

$$\delta_{12}{}^2 = A_1B_1^2 = O_1O_2^2 - (\varepsilon_1 r_1 - \varepsilon_2 r_2)^2.$$

Eliminating $\cos\theta$ and O_1O_2 from (1) and (2), we have

$$\begin{aligned}\delta_{12}^2 =&\ (R+\varepsilon_1 r_1)^2 + (R+\varepsilon_2 r_2)^2 - (\varepsilon_1 r_1 - \varepsilon_2 r_2)^2 - 2(R+\varepsilon_1 r_1)(R+\varepsilon_2 r_2)\left(1 - \frac{AB^2}{2R^2}\right)\\ =&\ \frac{AB^2}{R^2}\cdot(R+\varepsilon_1 r_1)(R+\varepsilon_2 r_2).\end{aligned}$$

From this the result follows. $\square$

We shall also make use of the following famous theorem.

Proposition 2 (Casey's theorem [2, §172])**.** *Given four circles* $\Gamma_i, i = 1,2,3,4$, *let* δ_{ij} *denote the length of a common tangent* (either internal or external) *between* Γ_i *and* Γ_j. *The four circles are tangent to a fifth circle* Γ *(or line) if and only if for appropriate choice of signs,*

$$\delta_{12}\cdot\delta_{34} \pm \delta_{13}\cdot\delta_{42} \pm \delta_{14}\cdot\delta_{23} = 0$$

3. A triad of circles

Consider a triangle ABC with D, E, F the midpoints of the sides BC, CA, AB respectively. The circle ω_1 tangent to BC at D, and to the arc of the circumcircle on the opposite side of A touches the circumcircle at A_1, the second intersection with the line AI. The circles ω_b and ω_c are similarly defined.

Lemma 3. *The lines through the incenter* I *parallel to* AB *and* AC *are tangent to the circle* $A_1(D)$.

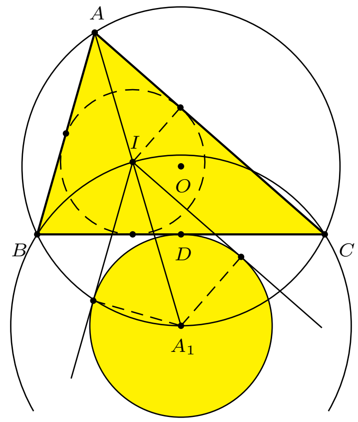

Figure 4.

Proof. Let O be the circumcenter, and R the circumradius. Since $OD = R\cos A$, it is enough to show that

$$R(1-\cos A) + r = AA_1 \sin\frac{A}{2}. \tag{3}$$

This follows from $r = IA\sin\frac{A}{2}$ and $1-\cos A = 2\sin^2\frac{A}{2}$. (3) is equivalent to $2R\sin\frac{A}{2} + IA = AA_1$, which follows from $A_1B = A_1I = 2R\sin\frac{A}{2}$. $\square$

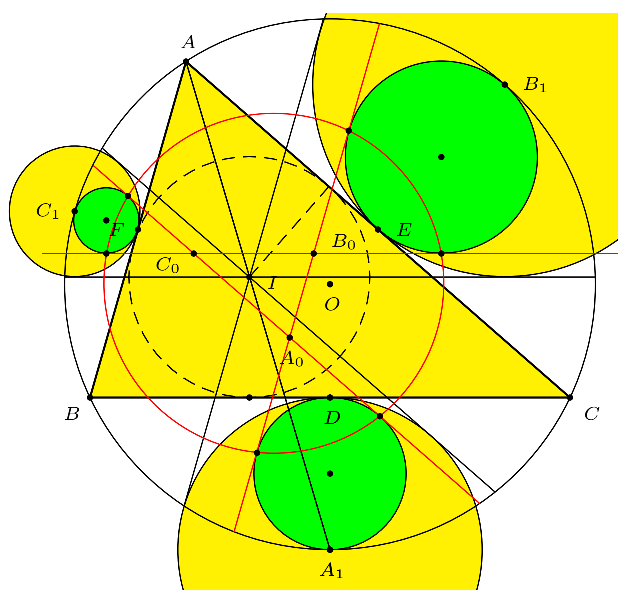

Figure 5.

Note that the length of the tangent from I to $A_1(D)$ is

$$A_1I\cos\frac{A}{2} = A_1B\cos\frac{A}{2} = 2R\sin\frac{A}{2}\cos\frac{A}{2} = R\sin A = \frac{a}{2}.$$

The homothety $\mathsf{h}\left(D, \frac{1}{2}\right)$ takes the two tangents through I to the circle $A'(D)$ to two tangents of ω_a through the midpoint A_2 of ID. These have lengths $\frac{a}{4}$.

Similarly, let B_2 and C_2 be the midpoints of IE and IF respectively. The two tangents from B_2 (respectively C_2 to ω_b (respectively ω_c) have lengths $\frac{b}{4}$ (respectively $\frac{c}{4}$). Now, since DE is parallel to BC, so is the line B_2C_2. These six tangents fall on three lines bounding a triangle $A_2B_2C_2$ homothetic to ABC with factor $-\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}$ and homothetic center J dividing IG in the ratio $IJ : JG = 3 : 2$. This leads to the configuration of three (pairwise) common tangents of the triad $(\omega_a,\ \omega_b,\ \omega_c)$ parallel to the sidelines of triangle ABC (see Figure 5). These tangents all have length $\frac{s}{2} = \frac{a+b+c}{4}$. It also follows that the 6 points of tangency lie on a circle, whose center is the incenter of triangle $A_2B_2C_2$, and radius $\frac{1}{4}\sqrt{r^2 + s^2}$. This latter fact follows from the proposition below, applied to triangle $A_2B_2C_2$.

Proposition 4. *The sides of a triangle ABC are extended to points X_b, X_c, Y_c, Y_a, Z_a, Z_b such that*

$$AY_a = AZ_a = a, \quad BZ_b = BX_b = b, \quad CX_c = CY_c = c.$$

The six points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on a circle concentric with the incircle and with radius $\sqrt{r^2 + s^2}$, where r is the inradius and s the semiperimeter of the triangle.

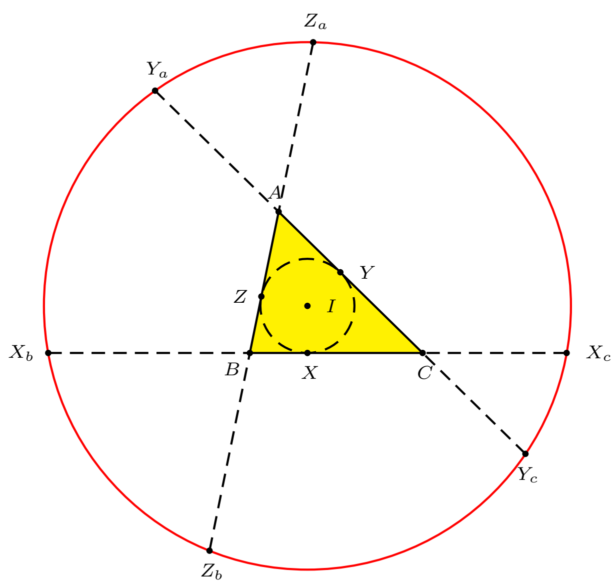

Figure 6.

Proof. If the incircle touches BC at X, then $BX = s - b$. It follows that $X_bX = b + (s - b) = s$, and $IX_b = \sqrt{r^2 + s^2}$. The same result holds for the other five points. □

4. On the radical center

Proposition 5. *The radical center of the circles $\omega_a, \omega_b, \omega_c$ is the midpoint between the incenter I and Mittenpunkt of $\triangle ABC$.*

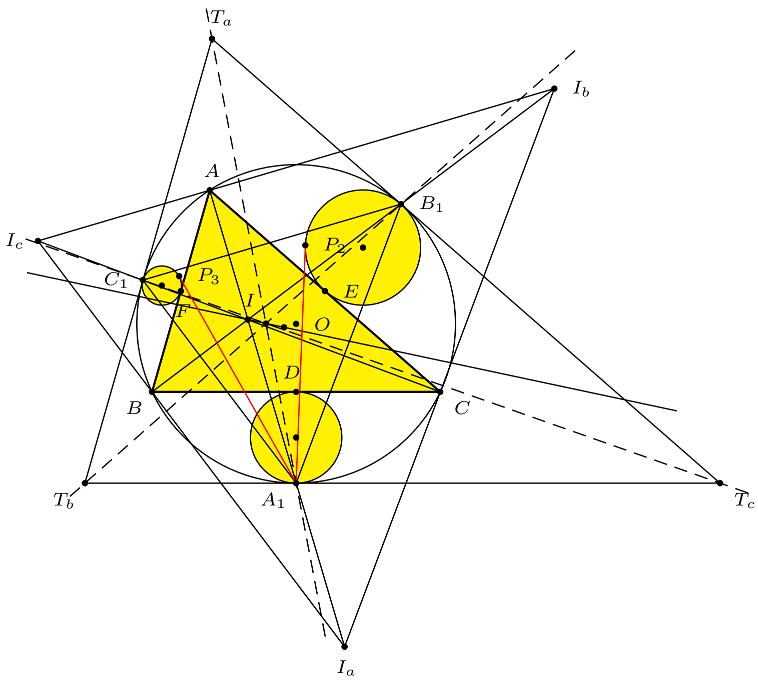

Figure 7.

Proof. Let A_1P_2 and A_1P_3 be the tangent segments from A_1 to ω_b and ω_c ($P_2 \in \omega_b$ and $P_3 \in \omega_c$) (See Figure 7). By Casey's theorem for $(A_1), (A), (C), \omega_b$ and $(A_1), (A), (B), \omega_c$, all tangent to the circumcircle, we have

$$A_1P_2 \cdot AC = A_1A \cdot CE + A_1C \cdot AE \Longrightarrow A_1P_2 = \frac{1}{2}(A_1A + A_1C) \qquad (4)$$

$$A_1P_3 \cdot AB = A_1A \cdot BF + A_1B \cdot AF \Longrightarrow A_1P_3 = \frac{1}{2}(A_1A + A_1B) \qquad (5)$$

Since $A_1B = A_1C$, from (4) and (5) we have $A_1P_2 = A_1P_3$, *i.e.*, A_1 has equal powers with respect to the circles ω_b and ω_c. If $T_aT_bT_c$ is the tangential triangle of $A_1B_1C_1$, then $T_aB_1 = T_aC_1$ implies that T_a has also equal powers with respect to ω_b and ω_c. Therefore, the line A_1T_a is the radical axis of ω_b and ω_c. Likewise, B_1T_b and C_1T_c are the radical axes of ω_c, ω_a and ω_a, ω_b respectively. Hence, the radical center L of $\omega_a, \omega_b, \omega_c$, being the intersection of A_1T_a, B_qT_b, C_1T_c, is the symmedian point of triangle $A_1B_1C_1$. Now, since the homothety $\mathsf{h}(I, 2)$ takes triangle $A_1B_1C_1$ into the excentral triangle $I_aI_bI_c$, and the latter has symmedian point X_9, the Mittenpunkt of triangle ABC, the symmedian point of $A_1B_1C_1$ is the midpoint of IX_9. According to [3], this is the triangle center X_{1001}. $\square$

5. On the Inner Apollonius circle

Emelyanov [1] has shown that the inner Apollonius circle of the triad $(\omega_a,\ \omega_b,\ \omega_c)$ is tangent to the incircle at the Feuerbach point; see also Yiu [5, §5]. The center of the inner Apollonius circle divides IN in the ratio $2:1$.

Proposition 6. *The Apollonius circle ω externally tangent to $\omega_a,\ \omega_b,\ \omega_c$ is also tangent to the incircle of triangle ABC through its Feuerbach point $F_{\rm e}$.*

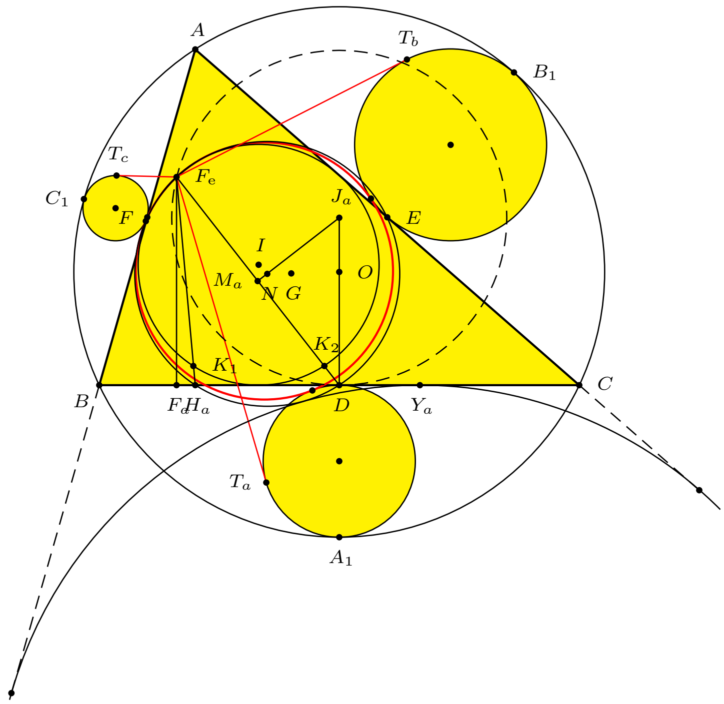

Figure 8.

Proof. Without loss of generality we assume that $b \geq a \geq c$. Let the incircle (I) touch BC, CA, AB at X_a, X_b, X_c respectively. By Casey's theorem there exists a circle ω tangent to $\omega_a, \omega_a, \omega_c$ externally and tangent to (I) internally if and only if

$$\delta_{bc} \cdot DX_a - \delta_{ca} \cdot EX_b - \delta_{ab} \cdot FX_c = 0. \tag{6}$$

Since $\delta_{bc} = \delta_{ca} = \delta_{ab} = \frac{s}{2}$ by Proposition 2, then (6) is an obvious identity because of $DX_a = \frac{1}{2}(b-c)$, $EX_b = \frac{1}{2}(a-c)$ and $FX_c = \frac{1}{2}(b-a)$. (In fact, this tangency is still true if we consider D, E, F as the feet of three concurring cevians. For a proof with similar arguments see [1].) Now, it remains to show that $\omega \cap (I)$ is the Feuerbach point $F_{\rm e}$ of triangle ABC.

Let $F_{\rm e}T_a$, $F_{\rm e}T_b$, $F_{\rm e}T_c$ be the tangent segments from $F_{\rm e}$ to $\omega_a, \omega_b, \omega_c$ respectively. If F_a is the orthogonal projection of $F_{\rm e}$ onto BC and $J_a(\rho_a)$ is the circle passing through $F_{\rm e}$ tangent to BC at D (see Figure 6). Let R_a denote the radius of ω_a, then using Proposititon 1 for ω_a and $(F_{\rm e})$ (with zero radius) externally tangent to $J_a(\rho_a)$, we get

$$F_{\rm e}D = F_{\rm e}T_a\sqrt{\frac{\rho_a}{\rho_a + R_a}}. \tag{7}$$

If M_a denotes the midpoint of $DF_{\rm e}$, then the right triangles DJ_aM_a and $F_{\rm e}DF_a$ are similar. This gives $2\rho_a \cdot F_{\rm e}F_a = F_{\rm e}D^2$. Thus, substituting ρ_a from this expression into (7) gives

$$F_{\rm e}D = \sqrt{\frac{F_{\rm e}D^2}{F_{\rm e}D^2 + 2F_{\rm e}F_a \cdot R_a}} \cdot F_{\rm e}T_a. \tag{8}$$

If H_a denotes the foot of the A-altitude of triangle ABC, then $F_{\rm e}F_a$ and the nine-point circle $N\left(\frac{R}{2}\right)$ become the $F_{\rm e}$-altitude and circumcircle of triangle $F_{\rm e}H_aD$. Consequently, $F_{\rm e}D \cdot F_{\rm e}H_a = R \cdot F_{\rm e}F_a$. Substituting $F_{\rm e}F_a$ from this expression into (8) and rearranging, we get

$$F_{\rm e}T_a = \sqrt{F_{\rm e}D\left(F_{\rm e}D + \frac{2F_{\rm e}H_a \cdot R_a}{R}\right)}. \tag{9}$$

Let $F_{\rm e}H_a$ and $F_{\rm e}D$ intersect the incircle (I) again at K_1 and K_2. Since $F_{\rm e}$ is the exsimilicenter of (I) and (N), K_1K_2 is parallel to DH_a. Therefore, the arcs X_aK_1 and X_aK_2 of (I) are equal, and $F_{\rm e}X_a$ bisects angle $H_aF_{\rm e}D$. Hence, by the angle bisector theorem, we have $\frac{F_{\rm e}H_a}{F_{\rm e}D} = \frac{X_aH_a}{X_aD}$. Substituting $F_{\rm e}H_a$ from this expression into (9) gives

$$F_{\rm e}T_a = \sqrt{1 + 2\cdot\frac{X_aH_a}{X_aD}\cdot\frac{R_a}{R}} \cdot F_{\rm e}D. \tag{10}$$

Since the incenter I and the A-excenter I_a divide harmonically A and the trace V_a of the A-angle bisector, the points of tangency X_a and Y_a of the line BC with the incircle (I) and the A-excircle divide harmonically H_a and V_a. Since D is also the midpoint of X_aY_a, $DX_a{}^2 = DY_a{}^2 = DH_a \cdot DV_a$. Equivalently,

$$\frac{H_aD}{X_aD} = \frac{X_aD}{V_aD} \Longrightarrow \frac{H_aD}{X_aD} - 1 = \frac{X_aH_a}{X_aD} = \frac{X_aD}{V_aD} - 1 = \frac{V_aX_a}{V_aD}.$$

Since V_a is the insimilicenter of (I) and the circle $A_1(D)$, it follows that

$$\frac{r}{2R_a} = \frac{V_aX_a}{V_aD} = \frac{X_aH_a}{X_aD}. \tag{11}$$

Substituting the ratio $\frac{X_aH_a}{X_aD}$ from (11) into (10), we have

$$F_{\rm e}T_a = \sqrt{\frac{R+r}{R}} \cdot F_{\rm e}D. \tag{12}$$

On the other hand, using Proposition 1 for the incircle (I) and the point circle (D), both internally tangent to the nine-point circle (N), we have

$$F_{\rm e}D = \sqrt{\frac{R}{R-2r}} \cdot DX_a = \sqrt{\frac{R}{R-2r}} \cdot \frac{b-c}{2}. \tag{13}$$

Combining equations (12) and (13), we have

$$F_eT_a = \sqrt{\frac{R+r}{R-2r}} \cdot \frac{b-c}{2}. \tag{14}$$

By similar reasoning, we have the expressions

$$F_eT_b = \sqrt{\frac{R+r}{R-2r}} \cdot \frac{a-c}{2}, \tag{15}$$

$$F_eT_c = \sqrt{\frac{R+r}{R-2r}} \cdot \frac{b-a}{2}. \tag{16}$$

Now, by Casey's theorem there exists a circle externally tangent to $\omega_a, \omega_b, \omega_c$ and (F_e) (with zero radius), if and only if

$$\delta_{bc} \cdot F_eT_a - \delta_{ca} \cdot F_eT_b - \delta_{ab} \cdot F_eT_c = 0.$$

Since $\delta_{bc} = \delta_{ca} = \delta_{ab} = \frac{s}{2}$ by Proposition 2, the latter condition becomes $F_eT_a - F_eT_b - F_eT_c = 0$, which is easily verified by (14), (15), (16). Hence, we conclude that ω is tangent to (I) through F_e, as desired. □

Proposition 7. *The center I_0 of the inner Apollonius circle ω of $\omega_a, \omega_b, \omega_c$ is the intersection of the lines X_3X_{1001}, X_1X_{11} and its radius ρ equals a third of the sum of the inradius and circumradius of triangle ABC.*

Proof. By Proposition 5, the radical center L of $\omega_a, \omega_b, \omega_c$ is the midpoint X_{1001} between the incenter I and the Mittenpunkt X_9 of triangle ABC. Hence, the inversion with center X_{1001} and power equal to the power of X_{1001} to $\omega_a, \omega_b, \omega_c$, carries these circles into themselves and swaps ω and the circumcircle of $\triangle ABC$ due to conformity. Since the center of the inversion is also a similitude center between the circle at its inverse, it follows that I_0 lies on the line connecting the circumcenter X_3 and X_{1001}. But, from Proposition 6, we deduce that I_0 lies on the line connecting I and the Feuerbach point F_e. Therefore $I_0 = OX_{1001} \cap IF_e$.

Let D_a be the tangency point of ω with ω_a. Applying Proposition 1 to the two triads of circles $(F_e), \omega_a, \omega$ and $(I), \omega_a, \omega$, respectively, we obtain

$$F_eT_a^2 = \frac{\rho + R_a}{\rho} \cdot F_eD_a{}^2,$$

$$X_aD^2 = \frac{(\rho + R_a)(\rho - r)}{\rho^2} \cdot F_eD_a{}^2.$$

Eliminating $(\rho + R_a)F_eD_a{}^2$ from these two latter expressions and using (14), we have

$$\left(\frac{F_eT_a}{X_aD}\right)^2 = \frac{\rho}{\rho - r} \Longrightarrow \frac{R+r}{R-2r} = \frac{\rho}{\rho - r} \Longrightarrow \rho = \frac{R+r}{3}.$$

□

Remark. Since F_e is the insimilicenter of (I) and ω,

$$\frac{F_eI_0}{F_eI} = \frac{\rho}{r} = \frac{R+r}{3r}.$$

Consequently, in absolute barycentric coordinates,

$$I_0 = \frac{R+r}{3r} \cdot I - \frac{R-2r}{3r} \cdot F_{\mathrm{e}} = \frac{R+r}{3r} \cdot I - \frac{R-2r}{3r} \cdot \frac{R \cdot I - 2r \cdot N}{R-2r} = \frac{I+2N}{3},$$

where N is the nine-point center. It does not appear in the current edition of [3], though its homogeneous barycentric coordinates are recorded in [5].

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Forum Geometricorum
Volume 11 (2011) 155–164.

FORUM GEOM
ISSN 1534-1178

The Area of a Bicentric Quadrilateral

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Abstract. We review and prove a total of ten different formulas for the area of a bicentric quadrilateral. Our main result is that this area is given by

$$K = \left|\frac{m^2 - n^2}{k^2 - l^2}\right| kl$$

where m, n are the bimedians and k, l the tangency chords.

1. The formula $K = \sqrt{abcd}$

A bicentric quadrilateral is a convex quadrilateral with both an incircle and a circumcircle, so it is both tangential and cyclic. It is well known that the square root of the product of the sides gives the area of a bicentric quadrilateral. In [12, pp.127–128] we reviewed four derivations of that formula and gave a fifth proof. Here we shall give a sixth proof, which is probably as simple as it can get if we use trigonometry and the two fundamental properties of a bicentric quadrilateral.

Theorem 1. *A bicentric quadrilateral with sides a, b, c, d has the area*

$$K = \sqrt{abcd}.$$

Proof. The diagonal AC divide a convex quadrilateral $ABCD$ into two triangles ABC and ADC. Using the law of cosines in these, we have

$$a^2 + b^2 - 2ab\cos B = c^2 + d^2 - 2cd\cos D. \tag{1}$$

The quadrilateral has an incircle. By the Pitot theorem $a + c = b + d$ [4, pp.65–67] we get $(a - b)^2 = (d - c)^2$, so

$$a^2 - 2ab + b^2 = d^2 - 2cd + c^2. \tag{2}$$

Subtracting (2) from (1) and dividing by 2 yields

$$ab(1 - \cos B) = cd(1 - \cos D). \tag{3}$$

In a cyclic quadrilateral opposite angles are supplementary, so that $\cos D = -\cos B$. We rewrite (3) as

$$(ab + cd)\cos B = ab - cd. \tag{4}$$

Publication Date: September 6, 2011. Communicating Editor: Paul Yiu.
The author would like to thank an anonymous referee for his careful reading and suggestions that led to an improvement of an earlier version of the paper.

The area K of a convex quadrilateral satisfies $2K = ab\sin B + cd\sin D$. Since $\sin D = \sin B$, this yields

$$2K = (ab + cd)\sin B. \tag{5}$$

Now using (4), (5) and the identity $\sin^2 B + \cos^2 B = 1$, we have for the area K of a bicentric quadrilateral

$$(2K)^2 = (ab + cd)^2(1 - \cos^2 B) = (ab + cd)^2 - (ab - cd)^2 = 4abcd.$$

Hence $K = \sqrt{abcd}$. □

Corollary 2. *A bicentric quadrilateral with sides a, b, c, d has the area*

$$K = ac\tan\frac{\theta}{2} = bd\cot\frac{\theta}{2}$$

where θ is the angle between the diagonals.

Proof. The angle θ between the diagonals in a bicentric quadrilateral is given by

$$\tan^2\frac{\theta}{2} = \frac{bd}{ac}$$

according to [8, p.30]. Hence we get

$$K^2 = (ac)(bd) = (ac)^2\tan^2\frac{\theta}{2}$$

and similar for the second formula. □

Corollary 3. *In a bicentric quadrilateral $ABCD$ with sides a, b, c, d we have*

$$\tan\frac{A}{2} = \sqrt{\frac{bc}{ad}} = \cot\frac{C}{2},$$
$$\tan\frac{B}{2} = \sqrt{\frac{cd}{ab}} = \cot\frac{D}{2}.$$

Proof. A well known trigonometric formula and (3) yields

$$\tan\frac{B}{2} = \sqrt{\frac{1-\cos B}{1+\cos B}} = \sqrt{\frac{cd}{ab}} \tag{6}$$

where we also used $\cos D = -\cos B$ in (3). The formula for D follows from $B = \pi - D$. By symmetry in a bicentric quadrilateral, we get the formula for A by the change $b \leftrightarrow d$ in (6). Then we use $A = \pi - C$ to complete the proof. □

Therefore, not only the area but also the angles have simple expressions in terms of the sides.

The area of a bicentric quadrilateral also gives a condition when a tangential quadrilateral is cyclic. Even though we did not express it in those terms, we have already proved the following characterization in the proof of Theorem 9 in [12]. Here we give another short proof.

Theorem 4. *A tangential quadrilateral with sides a, b, c, d is also cyclic if and only if it has the area $K = \sqrt{abcd}$.*

Proof. The area of a tangential quadrilateral is according to [8, p.28] given by

$$K = \sqrt{abcd}\sin\frac{B+D}{2}.$$

It's also cyclic if and only if $B + D = \pi$; hence a tangential quadrilateral is cyclic if and only if it's area is $K = \sqrt{abcd}$. □

This is not a new characterization of bicentric quadrilaterals. One quite long trigonometric proof of it was given by Joseph Shin in [15] and more or less the same proof of the converse can be found in the solutions to Problem B-6 in the 1970 William Lowell Putnam Mathematical Competition [1, p.69].

In this characterization the formulation of the theorem is important. The tangential and cyclic quadrilaterals cannot change roles in the formulation, nor can the formulation be that it's a bicentric quadrilateral if and only if the area is given by the formula in the theorem. This can be seen with an example. A rectangel is cyclic but not tangential. Its area satisfy the formula $K = \sqrt{abcd}$ since opposite sides are equal. Thus it's important that it must be a tangential quadrilateral that is also cyclic if and only if the area is $K = \sqrt{abcd}$, otherwise the conclution would be that a rectangle also has an incircle, which is obviously false.

2. Other formulas for the area of a bicentric quadrilateral

In this section we will prove three more formulas for the area of a bicentric quadrilateral, where the area is given in terms of other quantities than the sides. Let us first review a few other formulas and one double inequality for the area that can be found elsewhere.

In [12], Theorem 10, we proved that a bicentric quadrilateral has the area

$$K = \sqrt[4]{efgh}(e+f+g+h)$$

where e, f, g, h are the tangent lengths, that is, the distances from the vertices to the point where the incircle is tangent to the sides.

According to Juan Carlos Salazar [14], a bicentric quadrilateral has the area

$$K = 2\,MN\sqrt{EQ\cdot FQ}$$

where M, N are the midpoints of the diagonals; E, F are the intersection points of the extensions of opposite sides, and Q is the foot of the normal to EF through the incenter I (see Figure 1). This is a remarkable formula since the area is given in terms of only three distances. A short proof is given by "pestich" at [14]. He first proved that a bicentric quadrilateral has the area

$$K = 2\,MN\cdot IQ$$

which is even more extraordinary, since here the area is given in terms of only two distances!

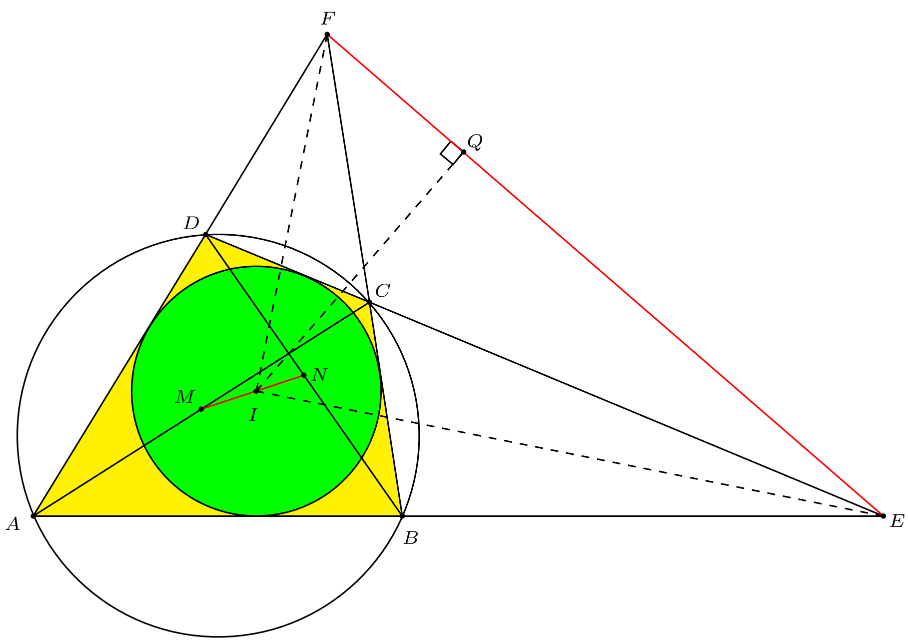

Figure 1. The configuration of Salazar's formula

The angle EIF (see Figure 1) is a right angle in a bicentric quadrilateral,[1] so we also get that the area of a bicentric quadrilateral is given by

$$K = \frac{2\,MN \cdot EI \cdot FI}{EF}$$

where we used the well known property that the product of the legs is equal to the product of the hypotenuse and the altitude in a right triangle. The last three formulas are not valid in a square since there we have $MN = 0$.

In [2, p.64] Alsina and Nelsen proved that the area of a bicentric quadrilateral satisfy the inequalities

$$4r^2 \leq K \leq 2R^2$$

where r, R are the radii in the incircle and circumcircle respectively. We have equality on either side if and only if it is a square.

Problem 1 on Quiz 2 at the China Team Selection Test 2003 [5] was to prove that in a tangential quadrilateral $ABCD$ with incenter I,

$$AI \cdot CI + BI \cdot DI = \sqrt{AB \cdot BC \cdot CD \cdot DA}.$$

The right hand side gives the area of a bicentric quadrilateral, so from this we get another formula for this area. It is easier to prove the following theorem than solving the problem from China,[2] since in a bicentric quadrilateral we can also use that opposite angles are supplementary angles.

[1] This is proved in Theorem 5 in [13], where the notations are different from here.

[2] One solution is given by Darij Grinberg in [9, pp.16–19].

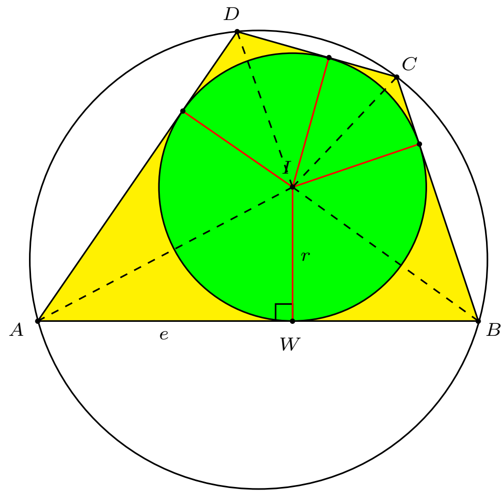

Figure 2. Partition of a bicentric quadrilateral into kites

Theorem 5. *A bicentric quadrilateral $ABCD$ with incenter I has the area*

$$K = AI \cdot CI + BI \cdot DI.$$

Proof. The quadrilateral has an incircle, so $\tan \frac{A}{2} = \frac{r}{e}$ where r is the inradius (see Figure 2). It also has a circumcircle, so $A+C = B+D = \pi$. Thus $\cot \frac{C}{2} = \tan \frac{A}{2}$ and $\sin \frac{C}{2} = \cos \frac{A}{2}$. A bicentric quadrilateral can be partitioned into four right kites by four inradii, see Figure 2.

Triangle AIW has the area $\frac{er}{2} = \frac{r^2}{2 \tan \frac{A}{2}}$. Thus the bicentric quadrilateral has the area

$$K = r^2 \left(\frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{B}{2}} + \frac{1}{\tan \frac{C}{2}} + \frac{1}{\tan \frac{D}{2}} \right).$$

Hence we get

$$\begin{aligned}
K &= r^2 \left(\frac{1}{\tan \frac{C}{2}} + \frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{D}{2}} + \frac{1}{\tan \frac{B}{2}} \right) \\
&= r^2 \left(\left(\tan \frac{A}{2} + \cot \frac{A}{2} \right) + \left(\tan \frac{B}{2} + \cot \frac{B}{2} \right) \right) \\
&= r^2 \left(\frac{1}{\sin \frac{A}{2} \cos \frac{A}{2}} + \frac{1}{\sin \frac{B}{2} \cos \frac{B}{2}} \right) \\
&= \frac{r^2}{\sin \frac{A}{2} \sin \frac{C}{2}} + \frac{r^2}{\sin \frac{B}{2} \sin \frac{D}{2}} \\
&= AI \cdot CI + BI \cdot DI
\end{aligned}$$

where we used that $\sin \frac{A}{2} = \frac{r}{AI}$ and similar for the other angles. $\square$

Corollary 6. *A bicentric quadrilateral $ABCD$ has the area*

$$K = 2r^2\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right)$$

where r is the inradius.

Proof. Using one of the equalities in the proof of Theorem 5, we get

$$K = r^2\left(\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}\cos\frac{B}{2}}\right) = r^2\left(\frac{1}{\frac{1}{2}\sin A} + \frac{1}{\frac{1}{2}\sin B}\right)$$

and the result follows. □

Here is an alternative, direct proof of Corollary 6:

In a tangential quadrilateral with sides a, b, c, d and semiperimeter s we have $K = rs = r(a+c) = r(b+d)$. Hence

$$\begin{aligned} K^2 &= r^2(a+c)(b+d) \\ &= r^2(ad+bc+ab+cd) \\ &= r^2\left(\frac{2K}{\sin A} + \frac{2K}{\sin B}\right) \end{aligned}$$

since in a cyclic quadrilateral $ABCD$, the area satisfies $2K = (ad+bc)\sin A = (ab+cd)\sin B$. Now factor the right hand side and then divide both sides by K. This completes the proof.

From Corollary 6 we get another proof of the inequality $4r^2 \leq K$, different form the one given in [2, p.64]. We have

$$K = 2r^2\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right) \geq 2r^2(1+1) = 4r^2$$

for $0 < A < \pi$ and $0 < B < \pi$.

In [12], Theorem 11, we proved that a bicentric quadrilateral with diagonals p, q and tangency chords [3] k, l has the area

$$K = \frac{klpq}{k^2+l^2}. \tag{7}$$

We shall use this to derive another beautiful formula for the area of a bicentric quadrilateral. In the proof we will also need the following formula for the area of a convex quadrilateral, which we have not found any reference to.

Theorem 7. *A convex quadrilateral with diagonals p, q and bimedians m, n has the area*

$$K = \tfrac{1}{2}\sqrt{p^2q^2 - (m^2-n^2)^2}.$$

[3] A tangency chord is a line segment connecting the points on two opposite sides where the incircle is tangent to those sides in a tangential quadrilateral.

Proof. A convex quadrilateral with sides a, b, c, d and diagonals p, q has the area

$$K = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2} \tag{8}$$

according to [6, p.243], [11] and [16]. The length of the bimedians [4] m, n in a convex quadrilateral are given by

$$m^2 = \tfrac{1}{4}(p^2 + q^2 - a^2 + b^2 - c^2 + d^2), \tag{9}$$

$$n^2 = \tfrac{1}{4}(p^2 + q^2 + a^2 - b^2 + c^2 - d^2). \tag{10}$$

according to [6, p.231] and post no 2 at [10] (both with other notations). From (9) and (10) we get

$$4(m^2 - n^2) = -2(a^2 - b^2 + c^2 - d^2)$$

so

$$(a^2 - b^2 + c^2 - d^2)^2 = 4(m^2 - n^2)^2.$$

Using this in (8), the formula follows. □

The next theorem is our main result and gives the area of a bicentric quadrilateral in terms of the bimedians and tangency chords (see Figure 3).

Theorem 8. *A bicentric quadrilateral with bimedians m, n and tangency chords k, l has the area*

$$K = \left|\frac{m^2 - n^2}{k^2 - l^2}\right| kl$$

if it is not a kite.

Proof. From Theorem 7 we get that in a convex quadrilateral

$$(m^2 - n^2)^2 = (pq)^2 - 4K^2. \tag{11}$$

Rewriting (7), we have in a bicentric quadrilateral

$$pq = \frac{k^2 + l^2}{kl}K.$$

Inserting this into (11) yields

$$\begin{aligned}(m^2 - n^2)^2 &= \frac{(k^2 + l^2)^2}{k^2l^2}K^2 - 4K^2 \\ &= K^2\left(\frac{(k^2 + l^2)^2 - 4k^2l^2}{k^2l^2}\right) \\ &= K^2\left(\frac{(k^2 - l^2)^2}{k^2l^2}\right).\end{aligned}$$

Hence

$$\left|m^2 - n^2\right| = K\frac{\left|k^2 - l^2\right|}{kl}$$

and the formula follows.

[4] A bimedian is a line segment connecting the midpoints of two opposite sides in a quadrilateral.

It is not valid in two cases, when $m = n$ or $k = l$. In the first case we have according to (9) and (10) that

$$-a^2 + b^2 - c^2 + d^2 = a^2 - b^2 + c^2 - d^2 \quad \Leftrightarrow \quad a^2 + c^2 = b^2 + d^2$$

which is a well known condition for when a convex quadrilateral has perpendicular diagonals. The second case is equivalent to that the quadrilateral is a kite according to Corollary 3 in [12]. Since the only tangential quadrilateral with perpendicular diagonals is the kite (see the proof of Corollary 3 in [12]), this is the only quadrilateral where the formula is not valid. [5] □

In view of the expressions in the quotient in the last theorem, we conclude with the following theorem concerning the signs of those expressions in a tangential quadrilateral. Let $m = EG$ and $n = FH$ be the bimedians, and $k = WY$ and $l = XZ$ be the tangency chords in a tangential quadrilateral, see Figure 3.

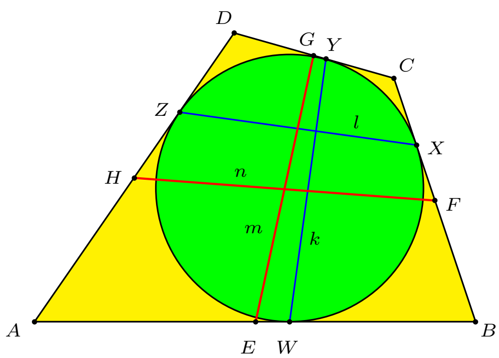

Figure 3. The bimedians m, n and the tangency chords k, l

Theorem 9. *Let a tangential quadrilateral have bimedians m, n and tangency chords k, l. Then*

$$m < n \quad \Leftrightarrow \quad k > l$$

where m and k connect the same pair of opposite sides.

Proof. Eulers extension of the parallelogram law to a convex quadrilateral with sides a, b, c, d states that

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2$$

where v is the distance between the midpoints of the diagonals p, q (this is proved in [7, p.107] and [3, p.126]). Using this in (9) and (10) we get that the length of the bimedians in a convex quadrilateral can also be expressed as

$$m = \tfrac{1}{2}\sqrt{2(b^2 + d^2) - 4v^2},$$
$$n = \tfrac{1}{2}\sqrt{2(a^2 + c^2) - 4v^2}.$$

[5]This also means it is not valid in a square since a square is a special case of a kite.

Thus in a tangential quadrilateral we have

$$\begin{aligned}
& m < n \\
\Leftrightarrow \quad & b^2 + d^2 < a^2 + c^2 \\
\Leftrightarrow \quad & (f+g)^2 + (h+e)^2 < (e+f)^2 + (g+h)^2 \\
\Leftrightarrow \quad & fg + he < ef + gh \\
\Leftrightarrow \quad & (e-g)(h-f) < 0
\end{aligned}$$

where $e = AW, f = BX, g = CY$ and $h = DZ$ are the tangent lengths.

In [12], Theorem 1, we proved that the lengths of the tangency chords in a tangential quadrilateral are

$$k = \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+f)(f+h)(h+g)(g+e)}},$$
$$l = \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+h)(h+f)(f+g)(g+e)}}.$$

Thus

$$\begin{aligned}
& k > l \\
\Leftrightarrow \quad & (e+f)(f+h)(h+g)(g+e) < (e+f)(f+h)(h+g)(g+e) \\
\Leftrightarrow \quad & eh + fg < ef + gh \\
\Leftrightarrow \quad & (e-g)(h-f) < 0.
\end{aligned}$$

Hence in a tangential quadrilateral

$$m < n \quad \Leftrightarrow \quad (e-g)(h-f) < 0 \quad \Leftrightarrow \quad k > l$$

which proves the theorem. □

We also note that the bimedians are congruent if and only if the tangency chords are congruent. Such equivalences will be investigated further in a future paper.

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Forum Geometricorum
Volume 11 (2011) 165–174.

FORUM GEOM
ISSN 1534-1178

When is a Tangential Quadrilateral a Kite?

Martin Josefsson

Abstract. We prove 13 necessary and sufficient conditions for a tangential quadrilateral to be a kite.

1. Introduction

A *tangential quadrilateral* is a quadrilateral that has an incircle. A convex quadrilateral with the sides a, b, c, d is tangential if and only if

$$a + c = b + d \tag{1}$$

according to the Pitot theorem [1, pp.65–67]. A *kite* is a quadrilateral that has two pairs of congruent adjacent sides. Thus all kites has an incircle since its sides satisfy (1). The question we will answer here concerns the converse, that is, what additional property a tangential quadrilateral must have to be a kite? We shall prove 13 such conditions. To prove two of them we will use a formula for the area of a tangential quadrilateral that is not so well known, so we prove it here first. It is given as a problem in [4, p.29].

Theorem 1. *A tangential quadrilateral with sides a, b, c, d and diagonals p, q has the area*

$$K = \tfrac{1}{2}\sqrt{(pq)^2 - (ac - bd)^2}.$$

Proof. A convex quadrilateral with sides a, b, c, d and diagonals p, q has the area

$$K = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2} \tag{2}$$

according to [6] and [14]. Squaring the Pitot theorem (1) yields

$$a^2 + c^2 + 2ac = b^2 + d^2 + 2bd. \tag{3}$$

Using this in (2), we get

$$K = \tfrac{1}{4}\sqrt{4(pq)^2 - (2bd - 2ac)^2}$$

and the formula follows. $\square$

Publication Date: September 9, 2011. Communicating Editor: Paul Yiu.

2. Conditions for when a tangential quadrilateral is a kite

In a tangential quadrilateral, a *tangency chord* is a line segment connecting the points on two opposite sides where the incircle is tangent to those sides, and the *tangent lengths* are the distances from the four vertices to the points of tangency (see [7] and Figure 1). A *bimedian* in a quadrilateral is a line segment connecting the midpoints of two opposite sides.

In the following theorem we will prove eight conditions for when a tangential quadrilateral is a kite.

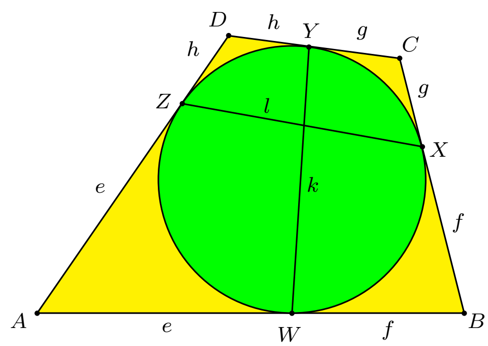

Figure 1. The tangency chords k, l and tangent lengths e, f, g, h

Theorem 2. *In a tangential quadrilateral the following statements are equivalent:*
(i) *The quadrilateral is a kite.*
(ii) *The area is half the product of the diagonals.*
(iii) *The diagonals are perpendicular.*
(iv) *The tangency chords are congruent.*
(v) *One pair of opposite tangent lengths are congruent.*
(vi) *The bimedians are congruent.*
(vii) *The products of the altitudes to opposite sides of the quadrilateral in the nonoverlapping triangles formed by the diagonals are equal.*
(viii) *The product of opposite sides are equal.*
(ix) *The incenter lies on the longest diagonal.*

Proof. Let the tangential quadrilateral $ABCD$ have sides a, b, c, d. We shall prove that each of the statements (i) through (vii) is equivalent to (viii); then all eight of them are equivalent. Finally, we prove that (i) and (ix) are equivalent.

(i) If in a kite $a = d$ and $b = c$, then $ac = bd$. Conversely, in [7, Corollary 3] we have already proved that a tangential quadrilateral with $ac = bd$ is a kite.

(ii) Using Theorem 1, we get

$$K = \tfrac{1}{2}\sqrt{(pq)^2 - (ac - bd)^2} = \tfrac{1}{2}pq \quad \Leftrightarrow \quad ac = bd.$$

(iii) We use the well known formula $K = \frac{1}{2}pq\sin\theta$ for the area of a convex quadrilateral,[1] where θ is the angle between the diagonals p, q. From

$$K = \tfrac{1}{2}\sqrt{(pq)^2 - (ac-bd)^2} = \tfrac{1}{2}pq\sin\theta$$

we get

$$\theta = \frac{\pi}{2} \quad \Leftrightarrow \quad ac = bd.$$

(iv) In a tangential quadrilateral, the tangency chords k, l satisfy

$$\left(\frac{k}{l}\right)^2 = \frac{bd}{ac}$$

according to Corollary 2 in [7]. Hence

$$k = l \quad \Leftrightarrow \quad ac = bd.$$

(v) Let the tangent lengths be e, f, g, h, where $a = e + f$, $b = f + g$, $c = g + h$ and $d = h + e$ (see Figure 1). Then we have

$$\begin{aligned} & ac = bd \\ \Leftrightarrow \quad & (e+f)(g+h) = (f+g)(h+e) \\ \Leftrightarrow \quad & ef - eh - fg + gh = 0 \\ \Leftrightarrow \quad & (e-g)(f-h) = 0 \end{aligned}$$

which is true when (at least) one pair of opposite tangent lengths are congruent.

(vi) In the proof of Theorem 7 in [9] we noted that the length of the bimedians m, n in a convex quadrilateral are

$$\begin{aligned} m &= \tfrac{1}{2}\sqrt{2(b^2+d^2) - 4v^2}, \\ n &= \tfrac{1}{2}\sqrt{2(a^2+c^2) - 4v^2} \end{aligned}$$

where v is the distance between the midpoints of the diagonals. Using these, we have

$$m = n \quad \Leftrightarrow \quad a^2 + c^2 = b^2 + d^2 \quad \Leftrightarrow \quad ac = bd$$

where the last equivalence is due to (3).

(vii) The diagonal intersection P divides the diagonals in parts w, x and y, z. Let the altitudes in triangles ABP, BCP, CDP, DAP to the sides a, b, c, d be h_1, h_2, h_3, h_4 respectively (see Figure 2). By expressing twice the area of these triangles in two different ways we get

$$\begin{aligned} ah_1 &= wy\sin\theta, \\ bh_2 &= xy\sin\theta, \\ ch_3 &= xz\sin\theta, \\ dh_4 &= wz\sin\theta, \end{aligned}$$

[1] For a proof, see [5] or [13, pp.212–213].

where θ is the angle between the diagonals and we used that $\sin(\pi - \theta) = \sin\theta$. These equations yields

$$ach_1h_3 = wxyz\sin^2\theta = bdh_2h_4.$$

Hence

$$h_1h_3 = h_2h_4 \quad \Leftrightarrow \quad ac = bd.$$

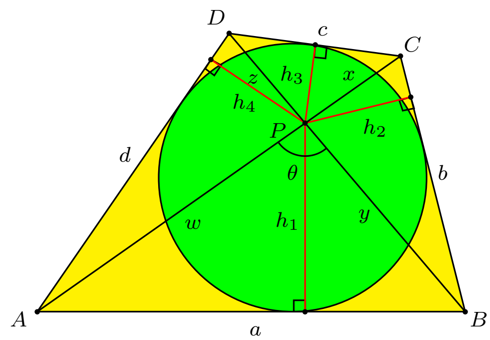

Figure 2. The subtriangle altitudes h_1, h_2, h_3, h_4

(ix) We prove that (i) $\Leftrightarrow$ (ix). A kite has an incircle and the incenter lies on the intersection of the angle bisectors. The longest diagonal is an angle bisector to two of the vertex angles since it divides the kite into two congruent triangles (SSS), hence the incenter lies on the longest diagonal.[2] Conversely, if the incenter lies on the longest diagonal in a tangential quadrilateral (see Figure 3) it directly follows that the quadrilateral is a kite since the longest diagonal divides the quadrilateral into two congruent triangles (ASA), so two pairs of adjacent sides are congruent. $\square$

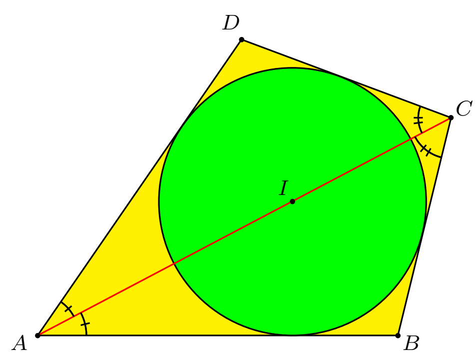

Figure 3. This tangential quadrilateral is a kite

[2]A more detailed proof not assuming that a kite has an incircle is given in [10, pp.92–93].

For those interested in further explorations we note that in a convex quadrilateral where $ac = bd$ there is an interesting angle relation concerning the angles formed by the sides and the diagonals, see [2] and [3]. Atzema calls these *balanced quadrilaterals*.

Theorem 2 (vii) has the following corollary.

Corollary 3. *The sums of the altitudes to opposite sides of a tangential quadrilateral in the nonoverlapping triangles formed by the diagonals are equal if and only if the quadrilateral is a kite.*

Proof. If h_1, h_2, h_3, h_4 are the altitudes from the diagonal intersection P to the sides AB, BC, CD, DA in triangles ABP, BCP, CDP, DAP respectively, then according to [12] and Theorem 1 in [11] (with other notations)

$$\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}. \tag{4}$$

From this we get

$$\frac{h_1 + h_3}{h_1 h_3} = \frac{h_2 + h_4}{h_2 h_4}.$$

Hence

$$h_1 h_3 = h_2 h_4 \quad \Leftrightarrow \quad h_1 + h_3 = h_2 + h_4$$

and the proof is compleate. □

Remark. In [8] we attributed (4) to Minculete since he proved this condition in [11]. After the publication of [8], Vladimir Dubrovsky pointed out that condition (4) in fact appeared earlier in the solution of Problem M1495 in the Russian magazine Kvant in 1995, see [12]. There it was given and proved by Vasilyev and Senderov together with their solution to Problem M1495. This problem, which was posed and solved by Vaynshtejn, was about proving a condition with inverse inradii, see (7) later in this paper. In [8, p.70] we incorrectly attributed this inradii condition to Wu due to his problem in [15].

3. Conditions with subtriangle inradii and exradii

In the proof of the next condition for when a tangential quadrilateral is a kite we will need the following formula for the inradius of a triangle.

Lemma 4. *The incircle in a triangle ABC with sides a, b, c has the radius*

$$r = \frac{a + b - c}{2} \tan \frac{C}{2}.$$

Proof. We use notations as in Figure 4, where there is one pair of equal tangent lengths x, y and z at each vertex due to the two tangent theorem. For the sides of the triangle we have $a = y + z$, $b = z + x$ and $c = x + y$; hence $a + b - c = 2z$. CI is an angle bisector, so

$$\tan \frac{C}{2} = \frac{r}{z}$$

and the formula follows. □

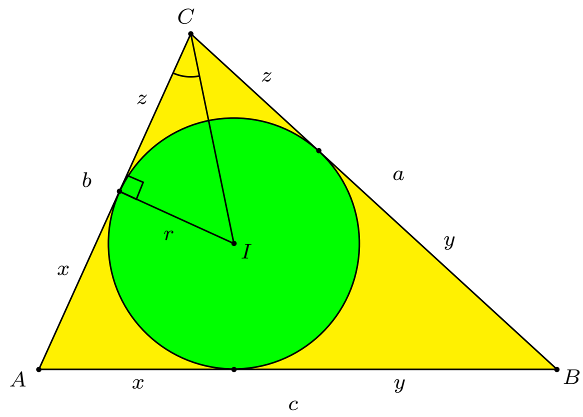

Figure 4. An incircle in a triangle

Theorem 5. *Let the diagonals in a tangential quadrilateral $ABCD$ intersect at P and let the inradii in triangles ABP, BCP, CDP, DAP be r_1, r_2, r_3, r_4 respectively. Then the quadrilateral is a kite if and only if*

$$r_1 + r_3 = r_2 + r_4.$$

Proof. We use the same notations as in Figure 2. The four incircles and their radii are marked in Figure 5. Since $\tan \frac{\pi-\theta}{2} = \cot \frac{\theta}{2}$, where θ is the angle between the diagonals, Lemma 4 yields

$$\begin{aligned} & r_1 + r_3 = r_2 + r_4 \\ \Leftrightarrow \quad & \frac{w+y-a}{2} \tan \frac{\theta}{2} + \frac{x+z-c}{2} \tan \frac{\theta}{2} = \frac{x+y-b}{2} \cot \frac{\theta}{2} + \frac{w+z-d}{2} \cot \frac{\theta}{2} \\ \Leftrightarrow \quad & (w+x+y+z-a-c) \tan \frac{\theta}{2} = (w+x+y+z-b-d) \cot \frac{\theta}{2}. \qquad (5) \end{aligned}$$

Using the Pitot theorem $a + c = b + d$, (5) is eqivalent to

$$(w+x+y+z-a-c)\left(\tan \frac{\theta}{2} - \cot \frac{\theta}{2}\right) = 0. \qquad (6)$$

According to the triangle inequality applied in triangles ABP and CDP, we have $w + y > a$ and $x + z > c$. Hence $w + x + y + z > a + c$ and (6) is equivalent to

$$\tan \frac{\theta}{2} - \cot \frac{\theta}{2} = 0 \quad \Leftrightarrow \quad \tan^2 \frac{\theta}{2} = 1 \quad \Leftrightarrow \quad \frac{\theta}{2} = \frac{\pi}{4} \quad \Leftrightarrow \quad \theta = \frac{\pi}{2}$$

where we used that $\theta > 0$, so the negative solution is invalid. According to Theorem 2 (iii), a tangential quadrilateral has perpendicular diagonals if and only if it is a kite. $\square$

Corollary 6. *If r_1, r_2, r_3, r_4 are the same inradii as in Theorem 5, then the tangential quadrilateral is a kite if and only if*

$$r_1 r_3 = r_2 r_4.$$

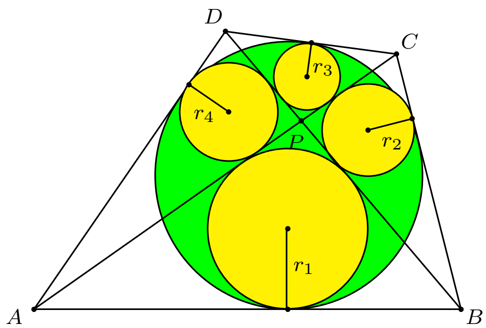

Figure 5. The incircles in the subtriangles

Proof. In a tangential quadrilateral we have according to [12] and [15]

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}. \tag{7}$$

We rewrite this as

$$\frac{r_1+r_3}{r_1r_3}=\frac{r_2+r_4}{r_2r_4}.$$

Hence

$$r_1+r_3=r_2+r_4 \qquad \Leftrightarrow \qquad r_1r_3=r_2r_4$$

which proves this corollary. □

Now we shall study similar conditions concerning the exradii to the same subtriangles.

Lemma 7. *The excircle to side $AB=c$ in a triangle ABC with sides a,b,c has the radius*

$$R_c=\frac{a+b+c}{2}\tan\frac{C}{2}.$$

Proof. We use notations as in Figure 6, where $u+v=c$. Also, according to the two tangent theorem, $b+u=a+v$. Hence $b+u=a+c-u$, so

$$u=\frac{a-b+c}{2}$$

and therefore

$$b+u=\frac{a+b+c}{2}.$$

For the exradius we have

$$\tan\frac{C}{2}=\frac{R_c}{b+u}$$

since CI is an angle bisector, and the formula follows. □

Theorem 8. *Let the diagonals in a tangential quadrilateral $ABCD$ intersect at P and let the exradii in triangles ABP, BCP, CDP, DAP opposite the vertex P be R_1, R_2, R_3, R_4 respectively. Then the quadrilateral is a kite if and only if*

$$R_1+R_3=R_2+R_4.$$

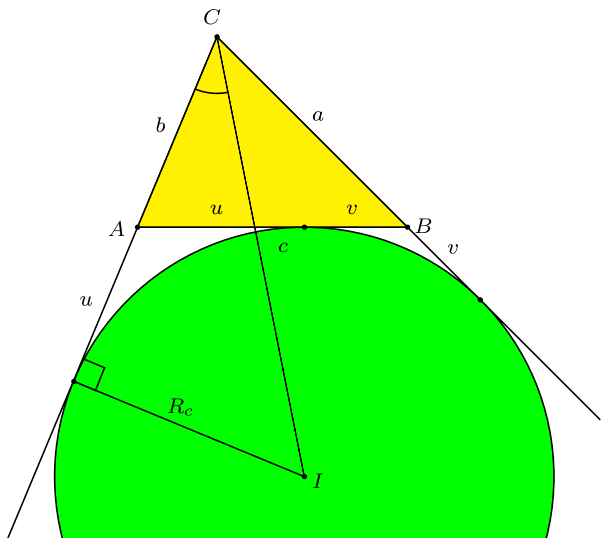

Figure 6. An excircle to a triangle

Proof. The four excircles and their radii are marked in Figure 7. Since $\tan\frac{\pi-\theta}{2} = \cot\frac{\theta}{2}$, where θ is the angle between the diagonals, Lemma 7 yields

$$\begin{aligned}
& R_1 + R_3 = R_2 + R_4 \\
\Leftrightarrow \quad & \frac{w+y+a}{2}\tan\frac{\theta}{2} + \frac{x+z+c}{2}\tan\frac{\theta}{2} = \frac{x+y+b}{2}\cot\frac{\theta}{2} + \frac{w+z+d}{2}\cot\frac{\theta}{2} \\
\Leftrightarrow \quad & (w+x+y+z+a+c)\tan\frac{\theta}{2} = (w+x+y+z+b+d)\cot\frac{\theta}{2}. \qquad (8)
\end{aligned}$$

Using the Pitot theorem $a + c = b + d$, (8) is eqivalent to

$$(w+x+y+z+a+c)\left(\tan\frac{\theta}{2} - \cot\frac{\theta}{2}\right) = 0. \qquad (9)$$

The first parenthesis is positive. Hence (9) is equivalent to that the second parenthesis is zero and the end of the proof is the same as in Theorem 5. □

Corollary 9. *If R_1, R_2, R_3, R_4 are the same exradii as in Theorem 8, then the tangential quadrilateral is a kite if and only if*

$$R_1R_3 = R_2R_4.$$

Proof. In a tangential quadrilateral we have according to Theorem 4 in [8]

$$\frac{1}{R_1} + \frac{1}{R_3} = \frac{1}{R_2} + \frac{1}{R_4}.$$

We rewrite this as

$$\frac{R_1 + R_3}{R_1R_3} = \frac{R_2 + R_4}{R_2R_4}.$$

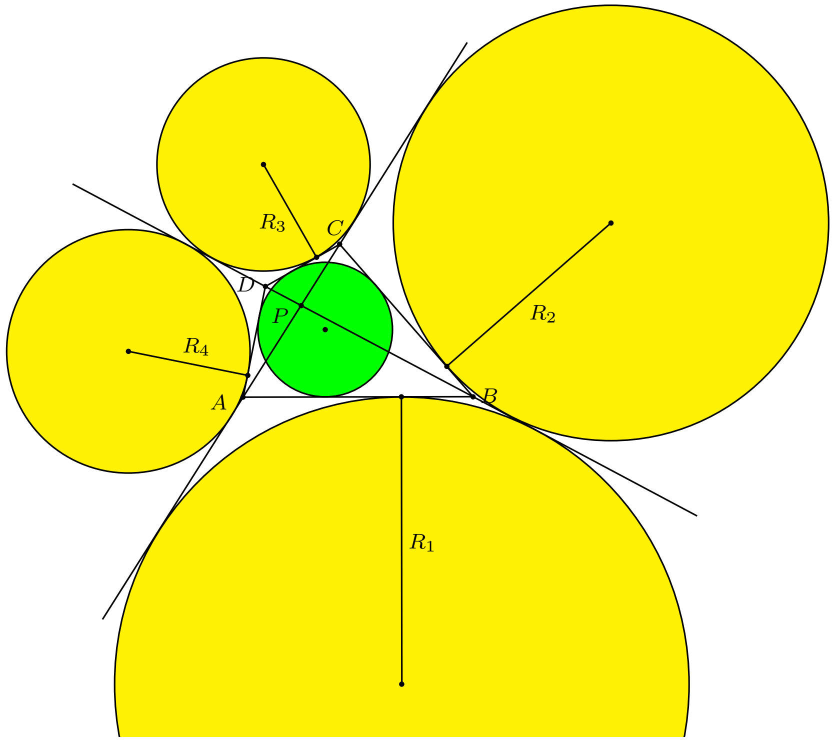

Figure 7. The excircles to the subtriangles

Hence

$$R_1 + R_3 = R_2 + R_4 \quad \Leftrightarrow \quad R_1R_3 = R_2R_4$$

completing the proof. □

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Forum Geometricorum
Volume 11 (2011) 175–181.

FORUM GEOM
ISSN 1534-1178

On a Certain Cubic Geometric Inequality

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Abstract. We provide a proof of a geometric inequality relating the cube of the distances of an arbitrary point from the vertices of a triangle to the cube of the inradius of the triangle. Comparable versions of the inequality involving second and fourth powers may be obtained by modifying our arguments.

1. Introduction

Given triangle ABC and a point P in its plane, let R_1, R_2, and R_3 denote the respective distances AP, BP, and CP and r denote the inradius of triangle ABC. In this note, we prove the following result.

Theorem 1. *If ABC is a triangle and P is a point, then*

$$R_1^3 + R_2^3 + R_3^3 + 6R_1R_2R_3 \geqslant 72r^3. \tag{1}$$

This answers a conjecture raised by Wu, Zhang, and Chu at the end of [5] in the affirmative. Furthermore, there is equality in (1) if and only if triangle ABC is equilateral with P its center. By homogeneity, one may take $r = 1$ in (1), which we will assume. Modifying our proof, one can generalize inequality (1) somewhat and establish versions of it for exponents 2 and 4. For other related inequalities of the Erdős-Mordell type involving powers of the R_i, see, for example, [1–5].

Note that in Theorem 1, one may assume that the point P lies on or within the triangle ABC since for any P lying outside triangle ABC, one can find a point Q lying on or within the triangle, all of whose distances to the vertices are strictly less than or equal the respective distances for P. To see this, consider the seven regions of the plane formed by extending the sides of triangle ABC indefinitely in both directions. Suppose first that P, say, lies in the region inside of $\angle BAC$ but outside of triangle ABC. Let K be the foot of P on line BC. If K lies between B and C, then take $Q = K$; otherwise, take Q to be the closer of B or C to the point K. On the other hand, suppose that the point P lies in a region bounded by one of the vertical angles opposite an interior angle of triangle ABC, say in the vertical angle to $\angle BAC$. Let J be the foot of P on the line ℓ passing through A and parallel to line BC; note that all of the distances from J to the vertices of triangle ABC are strictly less than the respective distances for P. Now proceed as in the prior case using the point J.

Publication Date: September 14, 2011. Communicating Editor: Paul Yiu.

To reduce the number of cases, we will assume that the point P lies on or within the triangle ABC.

2. Preliminary results

Lemma 2. *Suppose B and C are fixed points on the x-axis. Consider all possible triangles ABC having inradius 1. If $A = (x, y)$, where $y > 0$, then y achieves its minimum value only when triangle ABC is isosceles.*

Proof. Let P denote the point of tangency of the incircle of triangle ABC with side BC. Without loss of generality, we may let $B = (0, 0)$, $P = (a, 0)$ and $C = (a + b, 0)$, where a and b are positive numbers such that $ab > 1$. It is then a routine exercise to show that

$$A = \left(\frac{b(a^2-1)}{ab-1}, \frac{2ab}{ab-1}\right),$$

upon first noting that cot $IBP = a$ and cot $ICP = b$, where I denotes the incenter. Thus, the y-coordinate of A is $2 + \frac{2}{ab-1}$, and this has minimum value when ab is maximized, which occurs only when $a = b$ since $BC = a+b$ is of fixed length. $\square$

For a point P in the plane of triangle ABC, we define the *power of the point P* (with respect to triangle ABC) as

$$f^P_{\triangle ABC} := R_1^3 + R_2^3 + R_3^3 + 6R_1R_2R_3.$$

Lemma 3. *Fix base BC. Let triangle $A'BC$ have inradius 1, and P' be a point on or within this triangle. Then*

$$f^{P'}_{\triangle A'BC} \geq f^P_{\triangle ABC}$$

for an isosceles triangle ABC with inradius 1 and some point P on the altitude from A to BC.

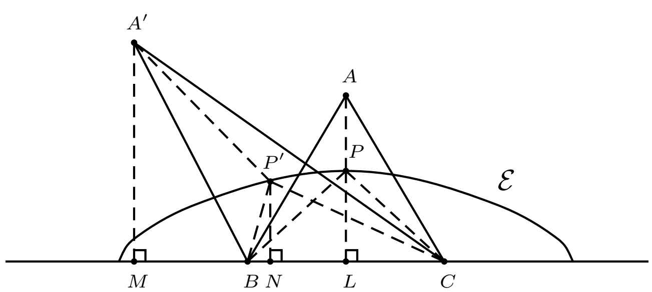

Figure 1. Triangles ABC and $A'BC$.

Proof. Let $R_1' = A'P'$, $R_2' = BP'$, and $R_3' = CP'$. Without loss of generality, we may assume $R_1' = \min\{R_1', R_2', R_3'\}$, for we may relabel the figure otherwise. Note that P' lies on the ellipse $\mathcal{E}$ with foci at points B and C having major axis length $d = R_2' + R_3'$. Let L denote the midpoint of BC and suppose the perpendicular to BC at L intersects ellipse $\mathcal{E}$ at the point P. Let A be the point (on the same side of BC as P) such that triangle ABC is isosceles and has inradius 1. See Figure 1.

Let $R_1 = AP$, $R_2 = BP$, and $R_3 = CP$; note that $2R_2 = R_2 + R_3 = d$ since P and P' lie on the same ellipse. Let M and N denote, respectively, the feet of points A' and P' on line BC; note that $P'N \leqslant PL$ since P is the highest point on the ellipse $\mathcal{E}$. We now consider two cases concerning the position of P:
(i) P lies between points A and L (as illustrated above), or
(ii) P lies past point A on line AL.

In case (i), we have, by Lemma 2,

$$R_1 = AL - PL \leqslant A'M - PL \leqslant A'M - P'N \leqslant R_1',$$

as the difference in the vertical heights of points A' and P' (with respect to BC) is no more than the distance between them.

Since $R_2 + R_3 = R_2' + R_3'$, with $R_1' = \min\{R_1', R_2', R_3'\}$ and $R_2 = R_3$, it is a routine exercise to verify

$$R_1'^3 + R_2^3 + R_3^3 + 6R_1'R_2R_3 \leqslant R_1'^3 + R_2'^3 + R_3'^3 + 6R_1'R_2'R_3'.$$

Since $R_1 \leqslant R_1'$, this implies

$$R_1^3 + R_2^3 + R_3^3 + 6R_1R_2R_3 \leqslant R_1'^3 + R_2'^3 + R_3'^3 + 6R_1'R_2'R_3',$$

which completes the proof in case (i).

In case (ii), we may replace P with vertex A since clearly $f^A_{\triangle ABC} \leqslant f^P_{\triangle ABC}$. Furthermore, note that $f^A_{\triangle ABC} \leqslant f^{P'}_{\triangle A'BC}$, by the argument in the prior case, since now $R_2 + R_3 = AB + AC \leqslant R_2' + R_3'$ with $R_1 = 0$. □

Lemma 4. *Suppose ABC is an isosceles triangle having inradius 1 and base BC. Let P be any point on the altitude from A to BC. Then for all possible P, we have*

$$f^P_{\triangle ABC} \geqslant 72.$$

Equality holds if and only if triangle ABC is equilateral with P the center.

Proof. Let $\angle ABC = 2\alpha$ and AD be the altitude from vertex A to side BC. Then $BD = \cot\alpha$ and thus

$$AD = BD \cdot \tan 2\alpha = \tan 2\alpha \cot\alpha = \frac{2}{1 - \tan^2\alpha}.$$

Let $x = \tan^2\alpha$ and $y = R_1 = AP$. Note that

$$\begin{aligned} R_2 = R_3 = \ & BP = \sqrt{DP^2 + BD^2} = \sqrt{(AD - AP)^2 + BD^2} \\ & = \sqrt{\left(\frac{2}{1-x} - y\right)^2 + \frac{1}{x}}. \end{aligned}$$

See Figure 2.

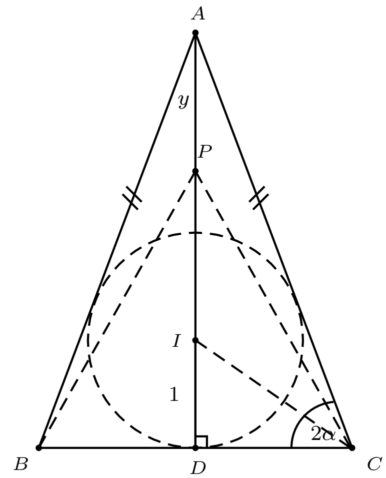

Figure 2. Isosceles triangle ABC.

To show $R_1^3+R_2^3+R_3^3+6R_1R_2R_3 \geqslant 72$, we need only establish the inequality

$$2\left[\left(\frac{2}{1-x}-y\right)^2+\frac{1}{x}\right]^{3/2}+6y\left[\left(\frac{2}{1-x}-y\right)^2+\frac{1}{x}\right]+y^3 \geqslant 72, \qquad (2)$$

for all $0<x<1$ and $y \geqslant 0$.

To prove (2), we consider the function

$$f(x,y)=2\left[\left(\frac{2}{1-x}-y\right)^2+\frac{1}{x}\right]^{3/2}+6y\left[\left(\frac{2}{1-x}-y\right)^2+\frac{1}{x}\right]+y^3$$

and study its extreme points. Direct calculations give

$$f_x(x,y)=3\cdot\frac{8x^2-(1-x)^3-4x^2(1-x)y}{(1-x)^3x^2}\left(\sqrt{\frac{1}{x}+\frac{(2-y+xy)^2}{(1-x)^2}}+2y\right),$$

$$f_y(x,y)=\frac{6(1+x)^2}{(1-x)^2x}-\frac{48}{1-x}y+21y^2-\frac{6(2-y+xy)}{1-x}\sqrt{\frac{1}{x}+\frac{(2-y+xy)^2}{(1-x)^2}}.$$

Since $0<x<1$ and $y \geqslant 0$, we see that the equation $f_x(x,y)=0$ implies $y=\frac{8x^2-(1-x)^3}{4x^2(1-x)}$. Substituting this expression for y into the equation $f_y(x,y)=0$, we obtain, after several straightforward algebraic operations, the equality

$$\begin{aligned}&(1+x)(7x^5+31x^4-70x^3+106x^2-49x+7)\\&=2(1-x)^4\sqrt{(1+x)(1-5x+11x^2+x^3)},\end{aligned}$$

which, since $0<x<1$, implies either $x=x_0=\frac{1}{3}$ or $g(x)=0$, where

$$\begin{aligned}g(x)=\ &15x^{10}+162x^9+279x^8-1256x^7+3150x^6-2644x^5\\&+694x^4+1752x^3-1533x^2+450x-45.\end{aligned}$$

By Maple or any mathematical programming (or, by hand, upon considering eight derivatives), it can be shown that $g'(x) > 0$ for all $0 < x < 1$; see graph above. Thus $g(x)$ is an increasing function with $g(0) < 0$ and $g(1) > 0$. Hence the equation $g(x) = 0$ has a unique root $x = x_1$ on the interval $(0, 1)$. By any numerical method, one may estimate this root to be $x_1 = 0.2558509455\cdots$.

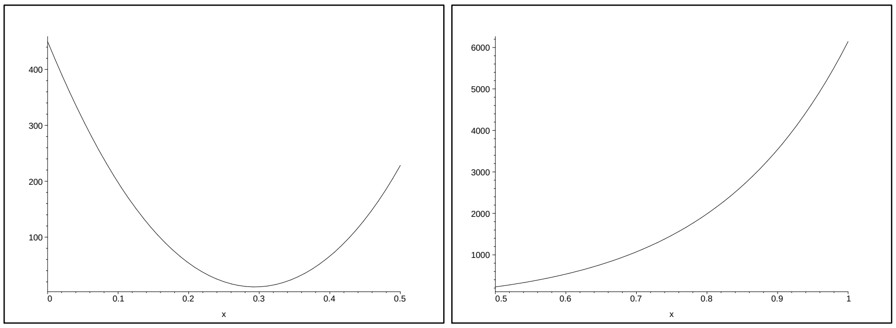

Figure 3. Graph of $g'(x)$ on the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

Note that $\lim_{x\to 0^+} f(x, y) = \lim_{x\to 1^-} f(x, y) = \lim_{y\to\infty} f(x, y) = \infty$. Observe further that the function

$$f(x, 0) = 2\left(\frac{4}{(1-x)^2} + \frac{1}{x}\right)^{3/2}$$

has a unique extreme point at $x = -2 + \sqrt{5}$ on the interval $0 < x < 1$, and in this case we have $f(-2 + \sqrt{5}, 0) = \frac{4+2\sqrt{5}}{(\sqrt{5}-2)^{\frac{3}{2}}} \approx 73.8647611$. On the other hand, the function $f(x, y)$ has two internal extreme points, namely $(x_0, y_0) = (\frac{1}{3}, 2)$ and $(x_1, y_1) = (0.2558509455, 0.5727531365)$, with $f(x_0, y_0) = 72$ and $f(x_1, y_1) \approx 77.5182420$. This shows $f(x, y) \geqslant 72$ for all $0 < x < 1$ and $y \geqslant 0$, as required. There is equality only when $(x, y) = (\frac{1}{3}, 2)$, which corresponds to the case when triangle ABC is equilateral with P its center. □

We invite the reader to seek a less technical proof of inequality (2). Meanwhile, Theorem 1 follows quickly from Lemmas 3 and 4.

3. Proof of Theorem 1

Suppose P is a point in the plane of the arbitrary triangle ABC having inradius 1, where BC is fixed. We may assume P lies on or within triangle ABC, by the observation in the introduction. From Lemma 3, we see that the power of P with respect to triangle ABC is at least as large as the power of some point Q with respect to the isosceles triangle having inradius 1 and the same base, where Q lies on the altitude to it, which, by Lemma 4, is at least 72. Since BC was arbitrary, the theorem is proven. We also have that there is equality only in the case when triangle ABC is equilateral and P is the center. □

4. Some corollaries

We now state a few simple consequences of Theorem 1. First, combining with the arithmetic-geometric mean inequality, we obtain the following result.

Corollary 5. *If ABC is a triangle and P is a point, then*

$$R_1^3 + R_2^3 + R_3^3 \geqslant 24r^3. \tag{3}$$

Taking P to be the circumcenter of triangle ABC, we see that Theorem 1 reduces to Euler's inequality $R \geq 2r$, where R denotes the circumradius. Taking P to be the incenter, we obtain the following trigonometric inequality.

Corollary 6. *If α, β, and γ denote the half-angles of a triangle ABC, then*

$$\csc^3 \alpha + \csc^3 \beta + \csc^3 \gamma + 6 \csc \alpha \csc \beta \csc \gamma \geqslant 72. \tag{4}$$

Inequality (4) may also be realized by observing

$$\csc^3 \alpha + \csc^3 \beta + \csc^3 \gamma + 6 \csc \alpha \csc \beta \csc \gamma \geqslant 9 \csc \alpha \csc \beta \csc \gamma = 9 \left(\frac{4R}{r}\right) \geqslant 72,$$

using Euler's inequality and the fact $\frac{r}{4R} = \sin \alpha \sin \beta \sin \gamma$.

Finally, taking P to be the centroid of triangle ABC, we obtain the following inequality.

Corollary 7. *If triangle ABC has median lengths m_a, m_b, and m_c, then*

$$m_a^3 + m_b^3 + m_c^3 + 6m_a m_b m_c \geqslant 243r^3. \tag{5}$$

Using the proof above of Theorem 1, it is possible to find lower bounds for the general sum $R_1^\alpha + R_2^\alpha + R_3^\alpha + k(R_1R_2R_3)^{\frac{\alpha}{3}}$, where α and k are positive constants, in several particular instances. For example, one may generalize Theorem 1 as follows, *mutatis mutandis*.

Theorem 8. *If ABC is a triangle and P is a point, then*

$$R_1^3 + R_2^3 + R_3^3 + kR_1R_2R_3 \geqslant 8(k + 3)r^3 \tag{6}$$

for $k = 0, 1, \ldots, 6$.

Furthermore, it appears that inequality (6) would hold for all *real numbers* k in the interval $[0, 6]$, though we do not have a complete proof. However, for any *given* number in this interval, one could apply the steps in the proof of Lemma 4 to test whether or not (6) holds. Even though it does not appear that Lemma 3 can be generalized to handle the case when $\alpha = 3$ and $k > 6$, numerical evidence suggests that inequality (6) still might hold over a larger range, such as $0 \leqslant k \leqslant 15$.

The proof above can be further modified to show the following results for exponents 2 and 4.

Theorem 9. *If ABC is a triangle and P is a point, then*

$$R_1^2 + R_2^2 + R_3^2 + (R_1R_2R_3)^{\frac{2}{3}} \geqslant 16r^2$$

and

$$R_1^2 + R_2^2 + R_3^2 + 2(R_1R_2R_3)^{\frac{2}{3}} \geqslant 20r^2.$$

Theorem 10. *If ABC is a triangle and P is a point, then*

$$R_1^4 + R_2^4 + R_3^4 + k(R_1R_2R_3)^{\frac{4}{3}} \geqslant 16(k+3)r^4$$

for $k = 0, 1, \ldots, 9$.

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Forum Geometricorum
Volume 11 (2011) 183–199.

FORUM GEOM
ISSN 1534-1178

The Lemniscatic Chessboard

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Abstract. Unit speed parameterization of the lemniscate of Bernoulli is given by a simple rational expression in the lemniscatic sine function. The beautiful structure of this parameterization becomes fully visible only when complex values of the arclength parameter are allowed and the lemniscate is viewed as a complex curve. To visualize such hidden structure, we will show squares turned to spheres, chessboards to lemniscatic chessboards.

1. Introduction

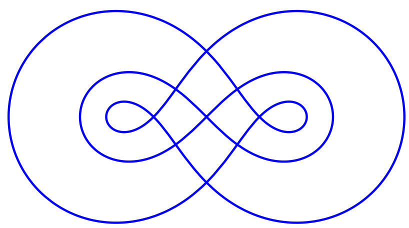

Figure 1. The lemniscate and two parallels.

The lemniscate of Bernoulli resembles the iconic notation for infinity ∞. The story of the remarkable discoveries relating elliptic functions, algebra, and number theory to this plane curve has been well told but sparsely illustrated. We aim to fill the visual void.

Figure 1 is a plot of the lemniscate, together with a pair of *parallel curves*. Let $\gamma(s) = x(s) + iy(s),\ 0 \leq s \leq L$, parameterize the lemniscate by arclength, as a curve in the complex plane. Then analytic continuation of $\gamma(s)$ yields parameterizations of the two shown parallel curves: $\gamma_\pm(s) = \gamma(s \pm iL/16),\ 0 \leq s \leq L$. Let $p_1, \dots, p_9$ denote the visible points of intersection. The figure and the following accompanying statements hint at the beautiful structure of $\gamma(s)$:

- The intersection at each p_j is orthogonal.
- The arclength from origin to closest point of intersection is $L/16$.
- Each point p_j is constructible by ruler and compass.

We briefly recall the definition of the lemniscate and the *lemniscatic integral*. Let $\mathcal{H}$ denote the rectangular hyperbola with foci $h_\pm = \pm\sqrt{2}$. Rectangular and

Publication Date: September 21, 2011. Communicating Editor: Paul Yiu.

polar equations of $\mathcal{H}$ are given by:

$$\begin{aligned} 1 &= x^2 - y^2 \\ 1 &= r^2\cos^2\theta - r^2\sin^2\theta = r^2\cos 2\theta \end{aligned}$$

Circle inversion, $(r,\theta) \mapsto (1/r,\theta)$, transforms $\mathcal{H}$ into the Bernoulli lemniscate $\mathcal{B}$:

$$\begin{aligned} r^4 &= r^2\cos 2\theta \\ (x^2+y^2)^2 &= x^2 - y^2 \end{aligned}$$

The foci of $\mathcal{H}$ are carried to the *foci* $b_\pm = \pm 1/\sqrt{2}$ of $\mathcal{B}$. While focal distances for the hyperbola $d_\pm(h) = \|h - h_\pm\|$ have constant difference $|d_+ - d_-| = 2$, focal distances for the lemniscate $d_\pm(b) = \|b - b_\pm\|$ have constant product $d_+d_- = 1/2$. (Other constant values of the product $d_+d_- = d^2$ define the non-singular Cassinian ovals in the confocal family to $\mathcal{B}$.)

Differentiation of the equation $r^2 = \cos 2\theta$ and elimination of $d\theta$ in the arclength element $ds^2 = dr^2 + r^2 d\theta^2$ leads to the elliptic integral for arclength along $\mathcal{B}$:

$$s(r) = \int \frac{dr}{\sqrt{1-r^4}} \tag{1}$$

The length of the full lemniscate $L = 4K$ is *four* times the complete elliptic integral $\int_0^1 dr/\sqrt{1-r^4} = K \approx 1.311$. One may compare K, formally, to the integral for arclength of a quarter-circle $\int_0^1 dx/\sqrt{1-x^2} = \frac{\pi}{2}$, though the integral in the latter case is based on the rectangular equation $y = \sqrt{1-x^2}$.

The lemniscatic integral $s(r)$ was considered by James Bernoulli (1694) in his study of elastic rods, and was later the focus of investigations by Count Fagnano (1718) and Euler (1751) which paved the way for the general theory of elliptic integrals and elliptic functions. In particular, Fagnano had set the stage with his discovery of methods for *doubling an arc* of $\mathcal{B}$ and subdivision of a quadrant of $\mathcal{B}$ into two, three, or five equal (length) sub-arcs by ruler and compass constructions; such results were understood via addition formulas for elliptic integrals and (ultimately) elliptic functions.

But a century passed before Abel (1827) presented a proof of the definitive result on subdivision of $\mathcal{B}$. Abel's result followed the construction of the 17-gon by Gauss (1796), who also showed that the circle can be divided into n equal parts when $n = 2^j p_1 p_2 \ldots p_k$, where the integers $p_i = 2^{2^{m_i}} + 1$ are distinct *Fermat primes*. Armed with his extensive new theory of elliptic functions, Abel showed: *The lemniscate can be n-subdivided for the very same integers $n = 2^j p_1 p_2 \ldots p_k$.* (See [3], [11], [10] and [13].)

2. Squaring the circle

It may seem curious at first to consider complex values of the "radius" r in the lemniscatic integral, but this is essential to understanding the integral's remarkable properties. Let $D = \{w = x + iy : x^2 + y^2 < 1\}$ be the open unit disk in the complex w-plane and, for each $w \in D$, consider the radial path $r = tw, 0 \le t \le 1$. The complex line integral

$$\zeta = s(w) = \int_0^w \frac{dr}{\sqrt{1-r^4}} = \int_0^1 \frac{wdt}{\sqrt{1-w^4t^4}} \tag{2}$$

defines a complex analytic function on D. (Here, $\sqrt{1-r^4}$ denotes the analytic branch with $\mathrm{Re}[\sqrt{1-r^4}] > 0$, for $r \in D$.) Then $s(w)$ maps D one-to-one and conformally onto the open square $\mathcal{S}$ with vertices $\pm K, \pm iK$—*the lemniscatic integral squares the circle!*

In fact, for $n = 3, 4, \dots$, the integral $s_n(w) = \int_0^w \frac{dr}{(1-r^n)^{2/n}}$ maps D conformally onto the regular n-gon with vertices $\sigma_n^k = e^{2\pi ki/n}$. This is a beautifully symmetrical (therefore tractable) example in the theory of the *Schwarz-Christoffel mapping*. Without recalling the general theory, it is not hard to sketch a proof of the above claim; for convenience, we restrict our discussion to the present case $n = 4$.

First, $s(w)$ extends analytically by the same formula to the unit circle, except at $\pm 1, \pm i$, where $s(w)$ extends continuously with values $s(e^{\pi ki/2}) = i^k K$, $k = 0, 1, 2, 3$. The four points $e^{\pi ki/2}$ divide the unit circle into four quarters, whose images $\gamma(t) = s(e^{it})$ we wish to determine. Differentiation of $\gamma'(t) = ie^{it}/\sqrt{1-e^{4it}}$ leads, after simplification, to $\gamma''(t)/\gamma'(t) = \cot 2t$. Since this is real, it follows that the velocity and acceleration vectors are parallel (or anti-parallel) along $\gamma(t)$, which is therefore locally straight. Apparently, s maps the unit circle to the square $\mathcal{S}$ with vertices $i^k K$, where the square root type singularities of $s(w)$ turn circular arcs into right angle bends. Standard principles of complex variable theory then show that D must be conformally mapped onto $\mathcal{S}$—see Figure 2.

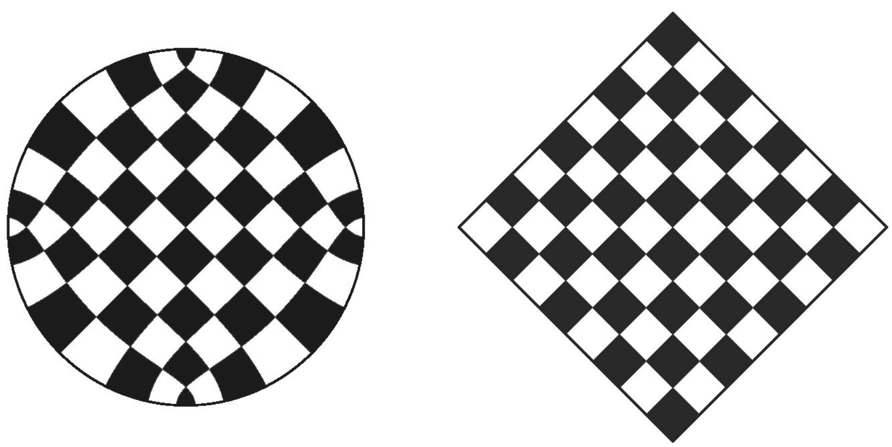

Figure 2. The lemniscatic integral $s(w) = \int_0^w \frac{dr}{\sqrt{1-r^4}}$ squares the circle.

The 64 subregions of the *circular chessboard* of Figure 2 (left) are the preimages of the squares in the standard chessboard (right). Four of the subregions in the disk appear to have only three sides apiece, but the bounding arcs on the unit circle each count as two sides, bisected by vertices $\pm 1, \pm i$.

It is visually obvious that the circular and standard chessboards have precisely the same symmetry—namely, that of a square. To be explicit, the mapping $\zeta = s(w)$ is easily seen to be equivariant with respect to the four rotations and four reflections of the dihedral group D_4:

$$s(i^k w) = i^k s(w), \quad s(i^k \bar{w}) = i^k \overline{s(w)}, \quad k = 0, 1, 2, 3. \tag{3}$$

Anticipating the next section, we note that the conformal map $\zeta = s(w)$ inverts to a function $w = s^{-1}(\zeta)$ which maps standard to circular chessboard and is also D_4-equivariant.

3. The lemniscatic functions

Let us return briefly to the analogy between arclength integrals for lemniscate and circle. The integral $t = \int dx/\sqrt{1-x^2}$ defines a monotone increasing function called $\arcsin x$ on the interval $[-1, 1]$. By a standard approach, $\sin t$ may be defined, first on the interval $[-\pi/2, \pi/2]$, by inversion of $\arcsin x$; repeated "reflection across endpoints" ultimately extends $\sin t$ to a smooth, 2π-periodic function on $\mathcal{R}$.

In the complex domain, one may define a branch of $\arcsin w$ on the slit complex plane $P = \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$ by the complex line integral: $\arcsin w = \int_0^w dr/\sqrt{1-r^2} = \int_0^1 \zeta dt/\sqrt{1-\zeta^2 t^2}$. It can be shown that $\zeta = \arcsin w$ maps P conformally onto the infinite vertical strip $V = \{w = u + iv : -\pi/2 < u < \pi/2\}$. Therefore, inversion defines $w = \sin\zeta$ as an analytic mapping of V onto P. One may then invoke the *Schwarz reflection principle*: Since $w = \sin\zeta$ approaches the real axis as ζ approaches the boundary of V, $\sin\zeta$ may be (repeatedly) extended *by reflection* to a 2π-periodic analytic function on $\mathbb{C}$.

A similar procedure yields the *lemniscatic sine function* $w = \mathrm{sl}\zeta$ (notation as in [13], §11.6), extending the function $s^{-1}(\zeta) : \mathcal{S} \to D$ introduced at the end of the last section. The important difference is that the boundary of the square $\mathcal{S}$ determines *two* "propagational directions," and the procedure leads to a *doubly periodic*, meromorphic function $\mathrm{sl} : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$. Shortly, we discuss some of the most essential consequences of this construction.

But first we note that the lemniscatic integral is an *elliptic integral of the first kind* $\int \frac{dr}{\sqrt{(1-r^2)(1-mr^2)}}$, for the special parameter value $m = k^2 = -1$. Inversion of the latter integral leads to the *Jacobi elliptic sine function with modulus* k, $\mathrm{sn}(\zeta, k)$. The *elliptic cosine* $\mathrm{cn}\zeta = \sqrt{1 - \mathrm{sn}^2\zeta}$ and $\mathrm{dn}\zeta = \sqrt{1 - m\,\mathrm{sn}^2\zeta}$ are two of the other basic Jacobi elliptic functions. All such functions are doubly periodic, but the lemniscatic functions

$$\mathrm{sl}\zeta = \mathrm{sn}(\zeta, i), \ \mathrm{cl}\zeta = \sqrt{1 - \mathrm{sl}^2\zeta}, \ \mathrm{dl}\zeta = \sqrt{1 + \mathrm{sl}^2\zeta} \tag{4}$$

posses additional symmetry which is essential to our story. In particular, Equations 3, 4 imply the exceptional identities:

$$\mathrm{sl}i\zeta = i\mathrm{sl}\zeta, \ \mathrm{cl}i\zeta = \mathrm{dl}\zeta, \ \mathrm{dl}i\zeta = \mathrm{cl}\zeta \tag{5}$$

It is well worth taking the time to derive further identities for slζ, "from scratch"—this we now proceed to do.

The Schwarz reflection principle allows us to extend $\mathrm{sl} : \mathcal{S} \to \mathbb{C}$ across the boundary $\partial\mathcal{S}$ according to the rule: *symmetric point-pairs map to symmetric point-pairs*. By definition, such pairs (in domain or range) are swapped by the relevant *reflection* R (antiholomorphic involution fixing boundary points). In the w-plane, we use inversion in the unit circle $R_C(w) = 1/\bar{w}$. In the $\zeta = s + it$-plane, a separate formula is required for each of the four edges making up $\partial\mathcal{S} = e_1 + e_2 + e_3 + e_4$. For e_1, we express reflection across $t = K - s$ by $R_1(\zeta) = -i\bar{\zeta} + (1 + i)K = -i\bar{\zeta} + e_+$, where we introduce the shorthand $e_\pm = (1 \pm i)K$. Then slζ may be defined on the square $\mathcal{S}_+ = \{\zeta + e_+ : \zeta \in \mathcal{S}\}$ by the formula:

$$\mathrm{sl}\zeta = R_C \mathrm{sl} R_1 \zeta = \frac{1}{\mathrm{sl}(i\zeta + e_-)}, \quad \zeta \in \mathcal{S}_+. \tag{6}$$

$\mathcal{S}_+$ is thus mapped conformally onto the exterior of the unit circle in the Riemann sphere (extended complex plane) $\mathbb{P} = \mathbb{C} \cup \{\infty\}$, with pole $sle_+ = \infty$.

Figure 3 illustrates the extended mapping $\mathrm{sl} : \mathcal{S} \cup \mathcal{S}_+ \to \mathbb{P}$. The circular chessboard is still recognizable within the right-hand figure, though "chessboard coloring" has not been used. Instead, one of the two orthogonal families of curves has been highlighted to display the image on the right as a *topological cylinder* obtained from the Riemann sphere by removing two slits—the quarter circles in second and fourth quadrants.

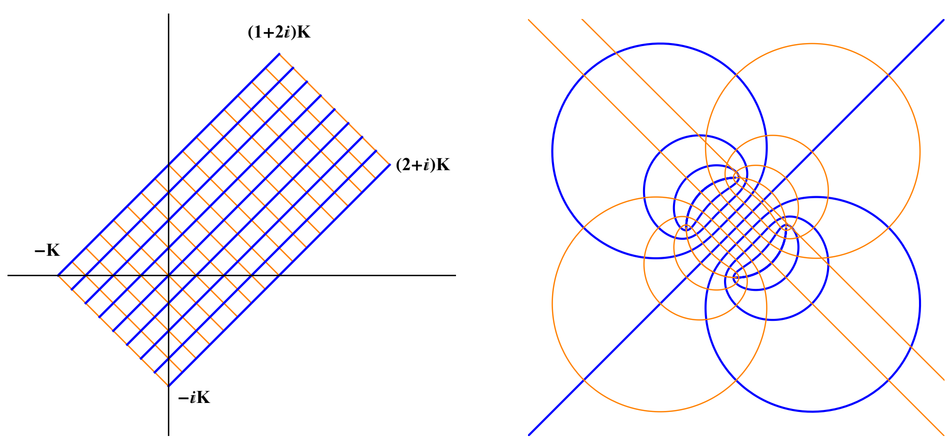

Figure 3. Mapping rectangle $\mathcal{S} \cup \mathcal{S}_+$ onto Riemann sphere by $w = \mathrm{sl}\zeta$.

We may similarly extend sl across e_2, e_3, e_4 and, by iteration, extend meromorphically to $\mathrm{sl} : \mathbb{C} \to \mathbb{P}$. The resulting extension is most conveniently described in terms of basic symmetries generated by pairs of reflections. Let $R_\pm\zeta = \pm i\bar{\zeta}$ be the reflection in the line $t = \pm s$; we use the same notation for reflections in the $w = u + iv$-plane, $R_\pm w = \pm i\bar{w}$. We may regard translations $\zeta \mapsto \zeta \pm e_+$ as composites of reflections ($R_1 R_-$ and $R_- R_1$) in parallel lines $t = K - s$ and $t = -s$, and thus deduce the effect on $w = \mathrm{sl}\zeta$. Here we may freely apply Equation 6 as well as D_4-equivariance—which meromorphic extension of sl must respect.

Thus, $\mathrm{sl}(\zeta + e_+) = \mathrm{sl}R_1R_-\zeta = R_C\mathrm{sl}R_1R_1R_-\zeta = R_CR_-\mathrm{sl}\zeta = \frac{1}{i\mathrm{sl}\zeta}$, and identities

$$\begin{aligned} \mathrm{sl}(\zeta \pm e_+) &= \frac{1}{i\mathrm{sl}\zeta}, \quad \mathrm{sl}(\zeta \pm e_-) = \frac{i}{\mathrm{sl}\zeta} && (7)\\ \mathrm{sl}(\zeta \pm 2e_\pm) &= \mathrm{sl}\zeta, && (8)\\ \mathrm{sl}(\zeta \pm 2K) &= \mathrm{sl}(\zeta \pm 2iK) = -\mathrm{sl}\zeta, && (9)\\ \mathrm{sl}(\zeta \pm 4K) &= \mathrm{sl}(\zeta \pm 4iK) = \mathrm{sl}\zeta && (10) \end{aligned}$$

follow easily in order. One should not attempt to read off corresponding identities for cl, dl using Equation 4—though we got away with this in Equation 5! (One may better appeal to angle addition formulas, to be given below.) Below, we will use the pair of equations $\mathrm{cl}(\zeta + 2K) = -\mathrm{cl}(\zeta)$ and $\mathrm{dl}(\zeta + 2K) = \mathrm{dl}(\zeta)$, which reveal the limitations of Equation 4. Actually, this points out one of the virtues of working exclusively with $\mathrm{sl}\zeta$, if possible, as it will eventually prove to be for us.

A *period* of a meromorphic function $w = f(\zeta)$ is a number $\omega \in \mathbb{C}$ such that $f(\zeta + \omega) = f(\zeta)$ for all $\zeta \in \mathbb{C}$. The *lattice of periods of* $f(\zeta)$ is the subgroup of the additive group $\mathbb{C}$ consisting of all periods of $f(\zeta)$. Let $\omega_0 = 2(1 + i)K = 2e_+ = 2ie_-$. Then the lattice of periods of $\mathrm{sl}\zeta$ is the *rescaled Gaussian integers*:

$$\Omega = \{(a + ib)\omega_0 : a, b \in \mathbb{N}\} \tag{11}$$

In fact, Equation 8 shows that any $(a + ib)\omega_0$ is a period; on the other hand, the assumption of an additional period leads quickly to a contradiction to the fact that $\mathrm{sl}\zeta$ is one-to-one on the (closed) chessboard $\mathcal{S}^c = \mathrm{closure}(\mathcal{S})$.

Using Ω, $\mathrm{sl}\zeta$ is most easily pictured by choosing a *fundamental region* for Ω and tiling the complex plane by its Ω-translates. To fit our earlier discussion, a convenient choice is the (closure of the) square consisting of four "chessboards" $\mathcal{R} = \mathcal{S} \cup \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_\pm$, where $\mathcal{S}_- = \{\zeta + e_- : \zeta \in \mathcal{S}\}$, $\mathcal{S}_\pm = \{\zeta + 2K : \zeta \in \mathcal{S}\}$. To say that $\mathcal{R}^c$ is a fundamental region means:

(1) Any $\zeta \in \mathbb{C}$ is equivalent to some $\zeta_0 \in \mathcal{R}^c$ in the sense that $\zeta - \zeta_0 \in \Omega$.
(2) No two elements in $\mathcal{R} = \mathrm{interior}(\mathcal{R}^c)$ are equivalent.

The double periodicity of $\mathrm{sl}\zeta$ is realized concretely in terms of the square tiling; $\mathrm{sl}\zeta$ maps $\mathcal{R}$ two-to-one onto $\mathbb{P}$, and repeats itself identically on every other tile.

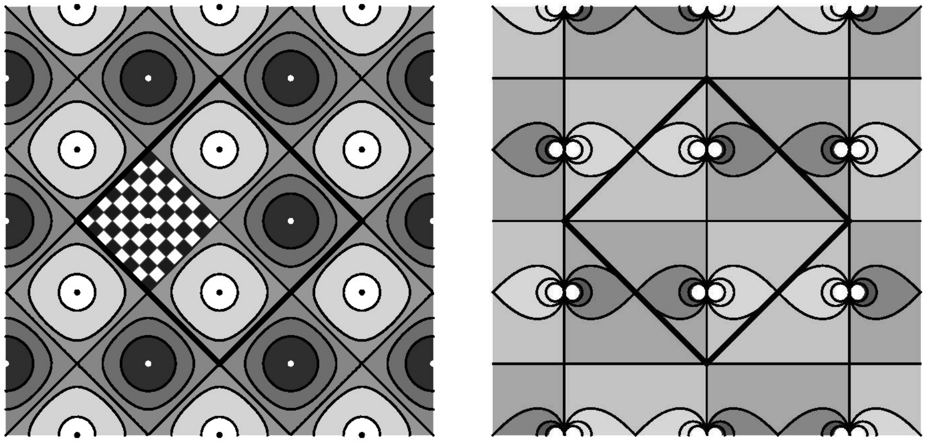

Figure 4. Contour plots of $|\mathrm{sl}\zeta|$ (left) and $\mathrm{Im}[\mathrm{sl}\zeta]$ (right).

The left side of Figure 4 illustrates the behavior just described using a contour plot of $|\mathrm{sl}z|$ (with tile $\mathcal{R}$ and chessboard subtile $\mathcal{S}$ superimposed). The critical level set $|\mathrm{sl}\zeta| = 1$ is recognizable as a square grid; half of the squares contain zeros (white dots), the other half contain poles (black dots). The right side of Figure 4 is a contour plot of $\mathrm{Im}[\mathrm{sl}\zeta]$. The level set $\mathrm{Im}[\mathrm{sl}\zeta] = 0$ is a larger square grid, this time formed by vertical and horizontal lines. We note that such a plot by itself (without superimposed tile $\mathcal{R}$) is visually misleading as to the underlying lattice.

A closely related idea is to regard $\mathrm{sl}\zeta$ as a function on the quotient $T^2 = \mathbb{C}/\Omega$—topologically, the torus obtained from the square $\mathcal{R}^c$ by identifying points on opposite edges (using $\zeta + i^k\omega_0 \equiv \zeta$). Then $\mathrm{sl} : T^2 \to \mathbb{P}$ is a *double branched cover* of the Riemann sphere by the torus. This point of view meshes perfectly with our earlier description of $\mathrm{sl} : \mathcal{S} \cup \mathcal{S}_+ \to \mathbb{P}$ as defining a cylinder. For the same description applies to the other *sheet* $\mathrm{sl} : \mathcal{S}_- \cup \mathcal{S}_\pm \to \mathbb{P}$, and the two identical sheets "glue" together along the *branch cuts* (quarter-circle slits) to form a torus T^2.

Yet another point of view uses the basic differential equation satisfied by $r = \mathrm{sl}s$. Applying the inverse function theorem to $\frac{ds}{dr} = 1/\sqrt{1-r^4}$ gives

$$(\frac{dr}{ds})^2 = 1 - r^4 \tag{12}$$

Here we have temporarily reverted to our original real notation, but it should be understood that Equation 12 is valid in the complex domain; r may be replaced by w and s by ζ. Also, we note that the equation implies the following derivative formulas for the lemniscatic functions:

$$\frac{d}{d\zeta}\mathrm{sl}\zeta = \mathrm{cl}\zeta\mathrm{dl}\zeta, \quad \frac{d}{d\zeta}\mathrm{cl}\zeta = -\mathrm{sl}\zeta\mathrm{dl}\zeta, \quad \frac{d}{d\zeta}\mathrm{dl}\zeta = \mathrm{sl}\zeta\mathrm{cl}\zeta \tag{13}$$

In particular, $\frac{d}{d\zeta}\mathrm{sl}\zeta$ vanishes at the chessboard corners $\pm K, \pm iK$, where the mapping of Figure 3 folds the long edges of the rectangle $\mathcal{S} \cup \mathcal{S}_+$ onto *doubled* quarter circles; these four critical points account for the branching behavior of $\mathrm{sl} : T^2 \to \mathbb{P}$.

In effect, $\mathrm{sl}\zeta$ defines a local coordinate on T^2, but is singular at the four points just mentioned. Actually, $\mathrm{sl}\zeta$ may be regarded as the first of two coordinates in a nonsingular parameterization of the torus. Here it is helpful to draw again on the analogy with $x = \sin t$ as a coordinate on the unit circle and the differential equation $(\frac{dx}{dt})^2 = 1 - x^2$. Just as the circle $x^2 + y^2 = 1$ is parameterized by $x = \sin t$, $y = \cos t$ ($t = \pi/2 - \theta$), the above formulas lead us to parameterize a quartic algebraic curve:

$$y^2 = 1 - x^4; \quad x = \mathrm{sl}\zeta,\ y = \mathrm{cl}\zeta\ \mathrm{dl}\zeta \tag{14}$$

This curve is in fact an *elliptic curve*—it is modeled on T^2—and we have effectively described its underlying analytic structure as a *Riemann surface of genus one*.

4. Parameterization of the lemniscate

In the previous two sections we developed the lemniscatic functions by way of two important applications; one a conformal mapping problem, the other a parameterization of an elliptic curve. The lemniscatic functions may also be used, fittingly enough, to neatly express arclength parameterization of the lemniscate $\mathcal{B}$ itself:

$$x(s) = \frac{1}{\sqrt{2}}\mathrm{dl}s\ \mathrm{sl}s,\ y(s) = \frac{1}{\sqrt{2}}\mathrm{cl}s\ \mathrm{sl}s,\ 0 \le s \le 4K \tag{15}$$

Equations 4 imply that $x = x(s), y = y(s)$ satisfy $(x^2 + y^2)^2 = x^2 - y^2$; further, Equations 13 lead to $x'(s)^2 + y'(s)^2 = 1$, as required. We note that s may be replaced by complex parameter ζ, as in Equation 14, for global parameterization of $\mathcal{B}$ as complex curve.

But first we focus on real points of $\mathcal{B}$ and subdivision of the plane curve into equal arcs. As a warmup, observe that Gauss's theorem on regular n-gons may be viewed as a result about constructible values of the sine function. For if n is an integer such that $y = \sin\frac{\pi}{2n}$ is a constructible number, so is $x = \cos\frac{\pi}{2n} = \sqrt{1 - y^2}$ constructible; then all points $(\sin\frac{\pi j}{2n}, \cos\frac{\pi j}{2n}),\ j = 1 \ldots 4n$ are constructible by virtue of the angle addition formulas for sine and cosine.

Likewise, Abel's result on uniform subdivision of the lemniscate is about constructible values $\mathrm{sl}(\frac{K}{n})$. For if it is known that a *radius* $r(K/n) = \mathrm{sl}(K/n)$ is a constructible number, it follows from Equation 15 that the corresponding point $(x(K/n), y(K/n))$ on $\mathcal{B}$ is constructible. Further, in terms of $r(K/n)$, $x(K/n)$, $y(K/n)$, only rational operations are required to divide the lemniscate into $4n$ arcs of length K/n (or n arcs of length $4K/n$). Specifically, one can compute the values $r(jK/n), x(jK/n), y(jK/n), j = 1, \ldots, n,$ using the *angle addition formulas for lemniscatic functions*:

$$\mathrm{sl}(\mu+\nu) = \frac{\mathrm{sl}\mu\,\mathrm{cl}\nu\,\mathrm{dl}\nu + \mathrm{sl}\nu\,\mathrm{cl}\mu\,\mathrm{dl}\mu}{1+\mathrm{sl}^2\mu\,\mathrm{sl}^2\nu} \tag{16}$$

$$\mathrm{cl}(\mu+\nu) = \frac{\mathrm{cl}\mu\,\mathrm{cl}\nu - \mathrm{sl}\mu\,\mathrm{sl}\nu\,\mathrm{dl}\mu\,\mathrm{dl}\nu}{1+\mathrm{sl}^2\mu\,\mathrm{sl}^2\nu} \tag{17}$$

$$\mathrm{dl}(\mu+\nu) = \frac{\mathrm{dl}\mu\,\mathrm{dl}\nu + \mathrm{sl}\mu\,\mathrm{sl}\nu\,\mathrm{cl}\mu\,\mathrm{cl}\nu}{1+\mathrm{sl}^2\mu\,\mathrm{sl}^2\nu} \tag{18}$$

(We note that the general angle addition formulas for the elliptic functions $\mathrm{sn}\zeta = \mathrm{sn}(\zeta,k), \mathrm{cn}\zeta = \mathrm{cn}(\zeta,k), \mathrm{dn}\zeta = \mathrm{dn}(\zeta,k)$ are very similar; the denominators are $1-k^2\mathrm{sn}^2\mu\,\mathrm{sn}^2\nu$ and the third numerator is $\mathrm{dn}\mu\,\mathrm{dn}\nu - k^2\mathrm{sn}\mu\,\mathrm{sn}\nu\,\mathrm{cn}\mu\,\mathrm{cn}\nu$.)

Everything so far seems to build on the analogy: *The lemniscatic sine function is to the lemniscate as the circular sine function is to the circle.*

But the moment we turn to constructibility of *complex points* on $\mathcal{B}$, an interesting difference arises: The integer j in the above argument can be replaced by a Gaussian integer $j+ik$. That is, one may use $\mu = jK/n, \nu = ikK/n$ in Equations 16-18, then apply Equations 5, then reduce integer multiple angles as before. Thus, a single constructible value $r(K/n) = \mathrm{sl}(K/n)$ ultimately yields n^2 constructible values $r((j+ik)K/n),\ 0 < j,k \le n$—hence n^2 points $(x((j+ik)K/n), y((j+ik)K/n)$ on $\mathcal{B}$ (actually four times as many points, but the remaining points are easily obtained by symmetry anyway).

Two remarks are in order here:

(1) No such phenomenon holds for the circle. Note that $(\sin\pi, \cos\pi) = (0,-1)$ is a constructible point on the real circle, while $(\sin i\pi, \cos i\pi)$ is a *non-constructible* point on the complex circle—otherwise, the transcendental number $e^\pi = \cos i\pi - i\sin i\pi$ would also be constructible!

(2) The constructibility of $r(K/n)$, $n = 2^j p_1 p_2 \dots p_k$, may be viewed as an *algebraic consequence* of its congruence class of n^2 values!

Though both remarks are interesting from the standpoint of algebra or number theory—in fact they belong to the deeper aspects of our subject—statements about complex points on a curve may be harder to appreciate geometrically. However, beautiful visualizations of such points may be obtained by *isotropic projection* onto the real plane $\mathbb{R}^2 \simeq \mathbb{C}$: $(x_0, y_0) \mapsto z_0 = x_0 + iy_0$. (The geometric meaning of this projection is not so different from stereographic projection: Point $(x_0, y_0) \in \mathbb{C}^2$ lies on a complex line $x+iy = z_0$ through the *circular point* c_+ with homogeneous coordinates $[1, i, 0]$, and the projection of (x_0, y_0) from c_+ is the point of intersection $(\mathrm{Re}[z_0], \mathrm{Im}[z_0])$ of line $x+iy = z_0$ and real plane $\mathrm{Im}[x] = \mathrm{Im}[y] = 0$.)

In particular, the (complexified) arclength parameterization $\Gamma(\zeta) = (x(\zeta), y(\zeta))$ of the lemniscate thus results in a meromorphic function on $\mathbb{C}$,

$$\gamma(\zeta) = x(\zeta) + iy(\zeta) = \frac{1}{\sqrt{2}}\mathrm{sl}\zeta\ (\mathrm{dl}\zeta\ + i\mathrm{cl}\zeta),\ \zeta \in \mathbb{C}, \tag{19}$$

by which we will plot, finally, the *lemniscate parallels* alluded to in the introduction: $\gamma(s+it_0),\ -2K \le s \le 2K,\ t_0 \in \mathbb{R}$. For instance, Figure 5 shows lemniscate parallels for the uniformly spaced "time"-values $t_0 = jK/4,\ -4 \le j < 4$.

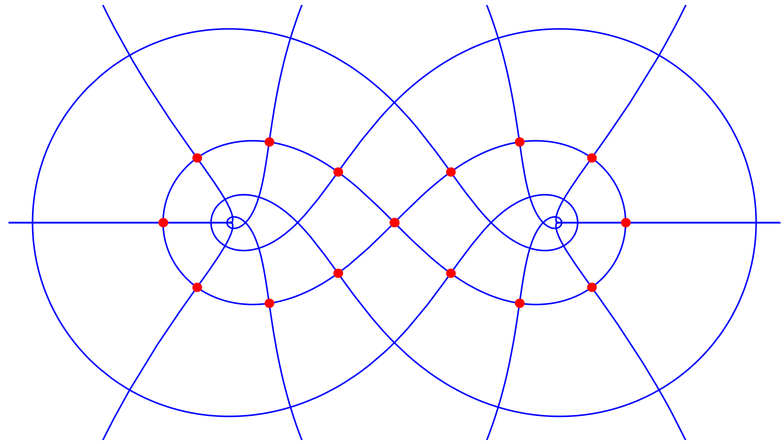

Figure 5. Lemniscate parallels for times $t_0 = \frac{jK}{4},\ -4 \le j < 4$.

Figure 5 is but a more complete version of Figure 1, but this time we are in a position to explain some of the previously mysterious features of the figure. The following statements apply not only to Figure 5 but also to similar plots obtained using Equation 19:

- The family of lemniscate parallels is *self-orthogonal*.
- The time increment Δt induces an equal curve increment Δs.
- For $\Delta t = K/n$, $n = 2^j p_1 p_2 \ldots p_k$, all intersections are constructible.

To explain the first two statements we combine Equations 5, 9 and above mentioned identities $\mathrm{cl}(\zeta + 2K) = -\mathrm{cl}(\zeta), \mathrm{dl}(\zeta + 2K) = \mathrm{dl}(\zeta)$ to obtain $\gamma(i\zeta) = \frac{1}{\sqrt{2}}\mathrm{sl}(i\zeta)(\mathrm{dl}(i\zeta) + i\mathrm{cl}(i\zeta)) = \frac{1}{\sqrt{2}}\mathrm{sl}(\zeta)(-\mathrm{dl}(\zeta) + i\mathrm{cl}(\zeta)) = \gamma(\zeta + 2K)$, hence:

$$\gamma(s + it) = \gamma(i(t - is)) = \gamma(t + 2K - is) \tag{20}$$

Thus, the γ-images of vertical lines $s = s_0$ and horizontal lines $t = t_0$ are one and the same family of curves! Since γ preserves orthogonality, the first claim is now clear. By the same token, since members of the family are separated by uniform time increment Δt, they are likewise separated by uniform s-increment. In particular, curves in the family divide the lemniscate into 16 arcs of equal length. (The last bulleted statement is a part of a longer story for another day!).

5. Squaring the sphere

The meromorphic lemniscate parameterization $\gamma(\zeta)$ arose, over the course of several sections, in a rather *ad hoc* manner. Here we describe a more methodical derivation which leads to a somewhat surprising, alternative expression for γ as a composite of familiar geometric mappings. Ignoring linear changes of variable, the three required mappings are: The lemniscatic sine $\mathrm{sl}\, z$, the *Joukowski map* $j(z)$, and the complex reciprocal $\iota(z) = 1/z$. The idea is to use j to parameterize the hyperbola $\mathcal{H}$, apply ι to turn $\mathcal{H}$ into $\mathcal{B}$, then compute the reparametrizing function required to achieve unit speed.

First recall the Joukowski map

$$j(z) = \frac{1}{2}(z + \frac{1}{z}), \tag{21}$$

a degree two map of the Riemann sphere with ramification points $\pm 1 = j(\pm 1)$; the unit circle is mapped two-to-one to the interval $[-1, 1]$, and the interior/exterior of the unit disk is mapped conformally onto the slit plane $\mathbb{C} \setminus [-1, 1]$. (Joukowski introduced j to study fluid motion around airfoils, building on the simpler flow around obstacles with circular cross sections.)

We note also that j maps circles $|z| = r$ to confocal ellipses (degenerating at $r = 1$) and rays $\mathrm{Arg}\, z = const$ to (branches of) the orthogonal family of confocal hyperbolas. In particular, $\mathrm{Arg}\, z = \pi/4$ is mapped by j to the rectangular hyperbola with foci ± 1: $x^2 - y^2 = 1/2$. We rescale to get $\mathcal{H}$, then take reciprocal: $\mathcal{B}$ is the image of the line $\mathrm{Re}[z] = \mathrm{Im}[z]$ under the map

$$\beta(z) = \iota(\sqrt{2} j(z)) = \frac{\sqrt{2} z}{1 + z^2} \tag{22}$$

(Inversion in the unit circle, discussed earlier, is given by conjugate reciprocal $z \mapsto 1/\bar{z}$; but by symmetry of $\mathcal{H}$, ι yields the same image.)

Let $\sigma = \sigma_8 = e^{\pi k i/4}$. Taking real and imaginary parts of $\beta(\sigma t) = \beta(\frac{1+i}{\sqrt{2}} t)$ results in the following parameterization of the real lemniscate:

$$x(t) = \frac{t + t^3}{1 + t^4}, \quad y(t) = \frac{t - t^3}{1 + t^4} \tag{23}$$

This rational parameterization possess beautiful properties of its own, but our current agenda requires only to relate its arclength integral to the lemniscatic integral by simple (complex!) substitution $t = \sigma\tau$:

$$s(t) = \int \sqrt{x'(t)^2 + y'(t)^2} dt = \int \frac{\sqrt{2}}{\sqrt{1 + t^4}} dt = \int \frac{\sqrt{2}\sigma}{\sqrt{1 - \tau^4}} d\tau \tag{24}$$

Thus we are able to invert $s(t)$ via the lemniscatic sine function, put $t = t(s)$ in $\beta(\sigma t)$, and simplify with the help of Equation 5: $\gamma(s) = \beta(\sigma t(s)) = \beta(i \mathrm{sl}\, \frac{s}{\sqrt{2}\sigma}) = \beta(\mathrm{sl}\, \frac{\sigma}{\sqrt{2}} s)$.

We are justified in using the same letter γ, as in Equation 19, to denote the resulting meromorphic arclength parameterization:

$$\gamma(\zeta) = \beta(\mathrm{sl}(\frac{\sigma}{\sqrt{2}}\zeta)) = \frac{\sqrt{2}\, \mathrm{sl}(\frac{1+i}{2}\zeta)}{1 + \mathrm{sl}^2(\frac{1+i}{2}\zeta)} \tag{25}$$

Comparing this expression with Equation 19, one could not be expected to recognize that these meromorphic functions are one and the same; but both satisfy $\gamma(0) = 0, \gamma'(0) = \sigma$—so it must be the case! (It requires a bit of work to verify this directly with the help of elliptic function identities.)

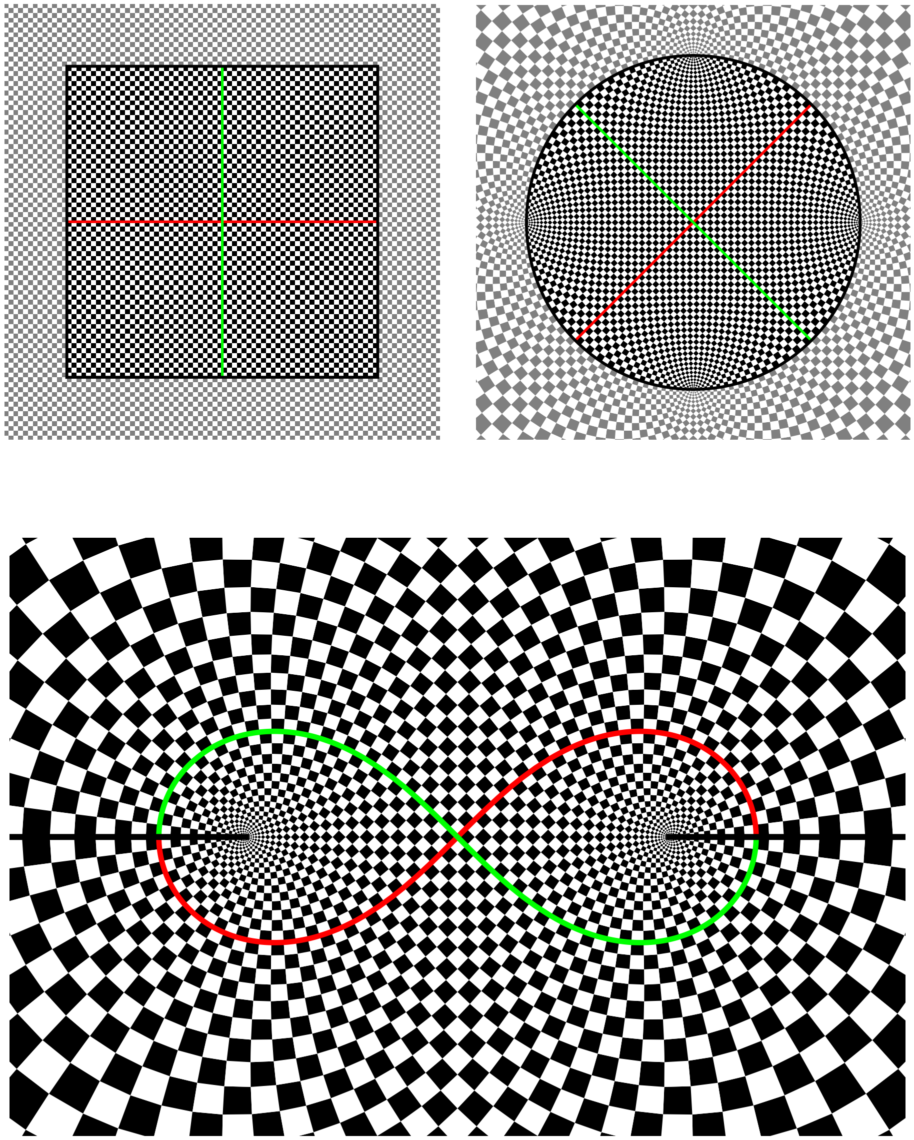

Figure 6. The factorization $\gamma(\zeta) = \beta(\mathrm{sl}(\frac{\sigma}{\sqrt{2}}\zeta))$: From square to round to lemniscatic chessboards.

Figure 6 illustrates the factorization $\gamma(\zeta) = \beta(\mathrm{sl}(\frac{\sigma}{\sqrt{2}}\zeta))$ with the aid of black and white $n \times n$ "chessboards" in domain, intermediate, and target spaces. (Though it may be more meaningful to speak of $n \times n$ "checkers" or "draughts," we have reason shortly to prefer the chess metaphor.) The present choice $n = 64$ is visually representative of integers $n >> 8$, but belongs again to the algebraically simplest class $n = 2^k$. The square board (SB) in the domain has corners $(\pm 1 \pm i)K$ (upper left). Rotation by $\pi/4$ and dilation by $\sqrt{2}$ takes SB to the square $\mathcal{S}$, which was earlier seen to be mapped conformally onto the unit disc by $\operatorname{sl} z$. The result is the round board (RB) in the intermediate space (upper right).

The *lemniscatic chess board* (bottom) is the conformal image $LB = \gamma(SB) = \beta(RB)$. As a domain, LB occupies the slit plane $\mathbb{C} \setminus \mathcal{I}$, $\mathcal{I} = (-\infty, -1/\sqrt{2}] \cup [1/\sqrt{2}, \infty)$. Note that the closure of LB is the sphere $LB^c = \gamma(SB^c) = \mathbb{P}$—so γ^{-1} *squares the sphere!* In the process, the horizontal (red) and vertical (green) centerlines on SB are carried by γ to the two halves of the real lemniscate in LB. (For n even, the lemniscate is already a "gridline" of LB.)

In addition to providing a more geometric interpretation for $\gamma(\zeta)$, Equation 25 represents an improvement over Equation 19 with respect to "number theory and numerics of LB." Here it's nice to consider discrete versions of RB and LB: Let each of the n^2 curvilinear squares be replaced by the quadrilateral with the same vertices. (This is in fact how the curvilinear squares in Figure 6 are rendered! Such a method reduces computation considerably and provides much sharper graphical results.) With this interpretation, we note that only rational operations are required to construct LB from RB.

6. Lemniscatic chess

In Figure 6 one detects yet unfinished business. What became of the gray and white checkered regions exterior to SB and RB? What to make of checks of the same color meeting along $\mathcal{I}$ in LB? For any n, symmetry forces this coloring *faux pas.* Something appears to be missing!

For amusement, let's return to case $n = 8$ and the spherical chessboard LB. Observe that in the game of lemniscatic chess, a bishop automatically changes from white to black or black to white each time he crosses $\mathcal{I}$. From the bishop's point of view, the board seems twice as big, since a neighborhood on LB feels different, depending on whether he's white or black.

In effect, the bishop's wanderings define a Riemann surface Σ, consisting of two oppositely colored, but otherwise identical, copies of LB, joined together along $\mathcal{I}$ to form a chessboard of $2n^2 = 128$ squares. In fact, it will be seen that Σ—itself a copy of the Riemann sphere $\Sigma \simeq \mathbb{P}$—may be regarded as the underlying Riemann surface, the intrinsic analytical model, of the algebraic curve $\mathcal{B}$.

To approach this from a slightly different angle, we note that $\mathcal{B}$ is a rational curve; its parameterization $b(t) = (x(t), y(t))$ given by Equation 23 shows that the totality of its (real, complex, and infinite) points may be identified with the space of parameter values $t \in \mathbb{P}$. In other words, $\mathcal{B}$ has *genus zero.* In declaring $\mathcal{B} \simeq \mathbb{P} = \Sigma$, the double point at the origin is treated as two separate points, $b(0), b(\infty)$, one for each *branch* of $\mathcal{B}$. Likewise, the lemniscate's four ideal points are double circular points $c_\pm$: $b(\sigma^3), b(\sigma^7)$, and $b(\sigma), b(\sigma^5)$. ($\mathcal{B}$ is said to be a *bicircular curve*.)

To understand the relationship between the two descriptions of Σ is to glimpse the beauty of isotropic coordinates, $z_1 = x + iy, z_2 = x - iy$. In these coordinates, $x = \frac{z_1+z_2}{2}, y = \frac{z_1-z_2}{2i}$ and the equation for $\mathcal{B}$ may be re-expressed $0 = f(x,y) = g(z_1, z_2) = z_1^2 z_2^2 - \frac{1}{2}(z_1^2 + z_2^2)$. A similar computation (which proves to be unnecessary) turns Equation 23 into a rational parameterization of $\mathcal{B}$ in isotropic coordinates, $(z_1(t), z_2(t))$.

To illustrate a general fact, we wish to express $(z_1(t), z_2(t))$ in terms of our original complex parameterization $z_1(t) = \beta(\sigma t)$. Let the *conjugate* of an analytic function $h(z)$ defined on U be the analytic function on $\bar{U}$ obtained by "complex conjugation of coefficients," $\bar{h}(z) = \overline{h(\bar{z})}$. For example, $\bar{\beta} = \beta$, consequently $h(z) = \beta(\sigma z)$ has conjugate $\bar{h}(z) = \beta(\zeta/\sigma)$. Now it is a fact that the corresponding parameterization of the complex curve $\mathcal{B}$ may be expressed simply:

$$z \mapsto (z_1(z), z_2(z)) = (h(z), \bar{h}(z)) = (\beta(\sigma z), \beta(z/\sigma)) \tag{26}$$

Given this parameterization, on the other hand, isotropic projection $(z_1, z_2) \mapsto z_1$ returns the original complex function $h(z) = \beta(\sigma z)$.

Now we've come to the source of the doubling: Isotropic projection determines a $2-1$ meromorphic function $\rho : \mathcal{B} \to \mathbb{P}$. Recall each isotropic line $z_1 = x_0 + iy_0$ meets the fourth degree curve $\mathcal{B}$ four times, counting multiplicity. Each such line passes through the double circular point $c_+ \in \mathcal{B}$, and the two finite intersections with $\mathcal{B}$ give rise to the two sheets of the isotropic image. Two exceptional isotropic lines are tangent to $\mathcal{B}$ at c_+, leaving only one finite intersection apiece; the resulting pair of projected points are none other than the foci of the lemniscate.

The same principles apply to the arclength parameterization. To cover $\mathcal{B}$ once, the parameter ζ may be allowed to vary over the *double chessboard* $2SR$ occupying the rectangle $-K \leq \mathrm{Re}[\zeta] < 3K$, $-K \leq \mathrm{Im}[\zeta] < K$. While $\gamma(\zeta)$ maps $2SR$ *twice* onto LB (according to the symmetry $\gamma(\zeta + 2K) = \gamma(i\zeta)$), the corresponding parameterization $(\gamma(\zeta), \bar{\gamma}(\zeta)) = (\beta(\mathrm{sl}(\frac{\sigma}{\sqrt{2}}\zeta)), \beta(\mathrm{sl}(\frac{1}{\sqrt{2}\sigma}\zeta))$ maps $2SR$ once onto the bishop's world—$\mathcal{B}$ divided into 128 squares. Note that this world is modeled by the full intermediate space (Figure 6, upper right), in which there are four "3-sided squares," but no longer any adjacent pairs of squares of the same color.

7. A tale of two tilings

By now it will surely have occurred to the *non-standard chess* enthusiast that the board may be doubled once again to the torus T^2 modeled by the fundamental square $4SR$: $-K \leq \mathrm{Re}[\zeta], \mathrm{Im}[\zeta] < 3K$. That is to say that LS (Figure 6, bottom) is *quadruply* covered by the elliptic curve Equation 14, which is itself infinitely covered by $\mathbb{C}$.

Thus, we have worked our way back to $\mathbb{C}$—the true *starting point*. Be it double, quadruple or infinite coverings, our images have displayed transference of structure *forwards*. A common thread and graphical theme, in any event, has been the hint or presence of square tiling of the plane and $*442$ *wallpaper symmetry* (see [2] to learn all about symmetry, tiling, and its notation). Likewise for subdivision of the lemniscate, the underlying symmetry and algebraic structure derive from $\mathbb{C}$ and the Gaussian integers; the intricate consequences are the business of Galois theory, number theory, elliptic functions—stories for another time ([11], [3], [9]).

It may sound strange, then: The symmetry group of the lemniscate itself, as a projective algebraic curve, is the octahedral group $O \simeq S_4$, the $*432$ tiling of the sphere (see Figure 7) is the skeleton of the lemniscate's intrinsic geometry. Projective O-symmetry characterizes $\mathcal{B}$ among genus zero curves of degree at most four ([7]). The natural Riemannian metric on $\mathcal{B} \subset \mathbb{C}P^2$ has precisely nine geodesics of reflection symmetry, which triangulate $\mathcal{B}$ as a disdyakis dodecahedron (which has 26 vertices, 72 edges and 48 faces); the 26 polyhedral vertices are the critical points of Gaussian curvature K ([8]). These are: 6 maxima $K = 2$, 8 minima $K = -7$ and 12 saddles $K = -1/4$, which are, respectively, the centers of 4-fold, 3-fold, and 2-fold rotational symmetry. It is also the case that our rational parameterization of $\mathcal{B}$ (given by Equation 23 or 26) realizes the full octahedral ($*432$) symmetry. (The 48-element *full octahedral group* O_h is distinct from the 48-element *binary octahedral group* $2O$, which is related to O like the bishop's world to LB—again another story.)

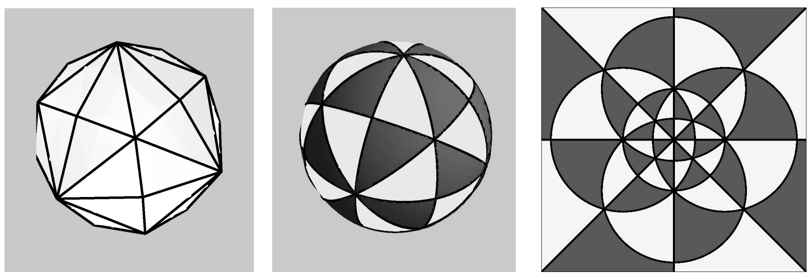

Figure 7. The disdyakis dodecahedron, $*432$ spherical tiling, and its stereographic planar image.

It is indeed a curious thing that these planar and spherical tilings and their symmetries somehow coexist in the world of the lemniscate. In an effort to catch both in the same place at the same time, we offer two final images, Figure 8. The idea here is to reverse course, transferring structure *backwards*—from lemniscate $\mathcal{B}$ to torus T^2—via the $2-1$ map $\gamma : T^2 \to \mathbb{P}$. On the left, the $*432$ spherical tiling is pulled back; on the right, a contour plot of K is pulled back. In both cases, we have chosen (for aesthetic reasons) to indicate a *wallpaper pattern* meant to fill the whole plane; however, for the discussion to follow, it should be kept in mind that T^2 is represented by the *central* 2×2 *square* in each (4×4) figure.

From a purely topological point of view, Figures 7 (middle) and 8 (left) together provide a perfect illustration of the proof of the *Riemann-Hurwitz Formula*: If $f : M \to N$ is a branched covering of Riemann surfaces, f has degree n and b branch points (counting multiplicities), and χ denotes Euler characteristic of a surface, then $\chi(M) = n\chi(N) - r$. Here we take $N = \mathbb{P}$, triangulated as in the former figure ($\chi(N) = V - E + F = 26 - 72 + 48 = 2$), and $M = T^2$ with triangulation pulled back from N by $f(\zeta) = \mathrm{sl}(\frac{1+i}{2}\zeta)$. Within T^2, f has four first order *ramification points*, where f fails to be locally $1-1$; these are the points where 16 curvilinear triangles meet. The mapping f doubles the angles $\pi/8$ of these triangles and wraps neighborhoods of such points twice around respective image points $\pm 1, \pm i \in \mathbb{P}$ (together with $0, \infty \in \mathbb{P}$, these may be identified as the "octahedral vertices" in

Figure 7, right). Since $f : T^2 \to \mathbb{P}$ has degree two, the resulting triangulation of T^2 has double the vertices, edges, and faces of PL, except for the vertices at ramification points which correspond one-to-one. Therefore, $\chi(T^2) = 2\chi(PL) - 4 = 0$, just as it should.

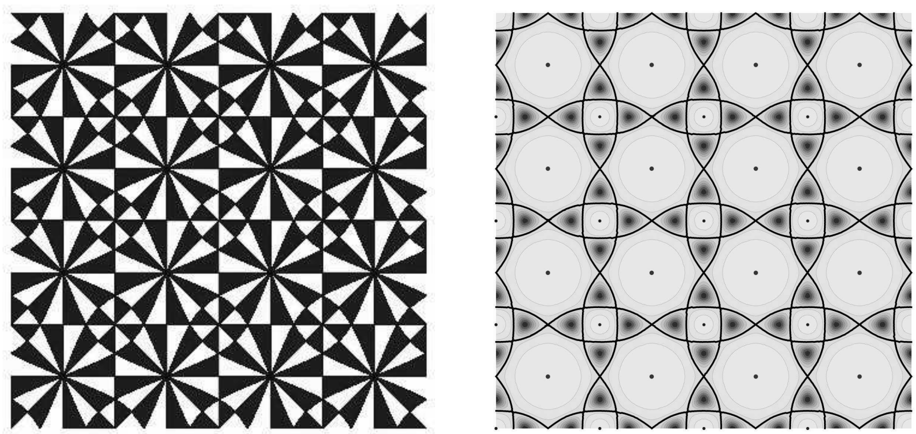

Figure 8. Wallpaper preimages of triangulated lemniscate (left) and lemniscate's curvature (right).

One may wish to contemplate the more subtle relationship between symmetries of the triangulated torus and those of the sphere. The group G of orientation-preserving (*translational* and *rotational*) symmetries of the former has order $|G| = 16$. To account for the smaller number $|G| < |O| = 24$, note that f breaks octahedral symmetry by treating the (unramified) points $0, \infty \in \mathbb{P}$ unlike $\pm 1, \pm i \in \mathbb{P}$. The former correspond to the four points in T^2 (count them!) where 8 triangles meet. But neither can all elements of G be "found" in O. Imagine a chain of "stones" $s_0, s_1, s_2, \ldots$ (vertices $s_j \in \mathbb{P}$), with $s_{j+1} = r_j s_j$ obtained by applying $r_j \in O$, and the corresponding chain of stones in $T^2 \ldots$ Anyone for a game of *disdyakis dodecahedral Go?*

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Forum Geometricorum
Volume 11 (2011) 201–203.

FORUM GEOM
ISSN 1534-1178

A Spatial View of the Second Lemoine Circle

Floor van Lamoen

Abstract. We consider circles in the plane as orthogonal projections of spheres in three dimensional space, and give a spatial characterization of the second Lemoine circle.

1. Introduction

In some cases in triangle geometry our knowledge of the plane is supported by a spatial view. A well known example is the way that David Eppstein found the Eppstein points, associated with the Soddy circles [2]. We will act in a similar way.

In the plane of a triangle ABC consider the tangential triangle $A'B'C'$. The three circles $A'(B)$, $B'(C)$, and $C'(A)$ form the A-, B-, and C-Soddy circles of the tangential triangle. We will, however, regard these as spheres T_A, T_B, and T_C in the three dimensional space. Let their radii be ρ_a, ρ_b, and ρ_c respectively. We consider the spheres that are tritangent to the triple of spheres T_A, T_B, and T_C externally. There are two such congruent spheres, symmetric with respect to the plane of ABC. We denote one of these by $T(\rho_t)$.

If we project $T(\rho_t)$ orthogonally onto the plane of ABC, then its center T is projected to the radical center of the three circles $A'(\rho_a+\rho_t)$, $B'(\rho_b+\rho_t)$ and $C'(\rho_c+\rho_t)$. In general, when a parameter t is added to the radii of three circles, their radical center depends linearly on t. Clearly the incenter of $A'B'C'$ (circumcenter of ABC) as well as the inner Soddy center of $A'B'C'$ are radical centers of $A'(\rho_a+t)$, $B'(\rho_b+t)$ and $C'(\rho_c+t)$ for particular values of t.

Proposition 1. *The orthogonal projection to the plane of ABC of the centers of spheres T tritangent externally to S_A, S_B, and S_C lie on the Soddy-line of $A'B'C'$, which is the Brocard axis of ABC.*

2. The second Lemoine circle as a sphere

Recall that the antiparallels through the symmedian point K meet the sides in six concyclic points (P, Q, R, S, U, and V in Figure 1), and that the circle through

Publication Date: September 28, 2011. Communicating Editor: Paul Yiu.

these points is called the second Lemoine circle, [1] with center K and radius

$$r_L = \frac{abc}{a^2 + b^2 + c^2}.$$

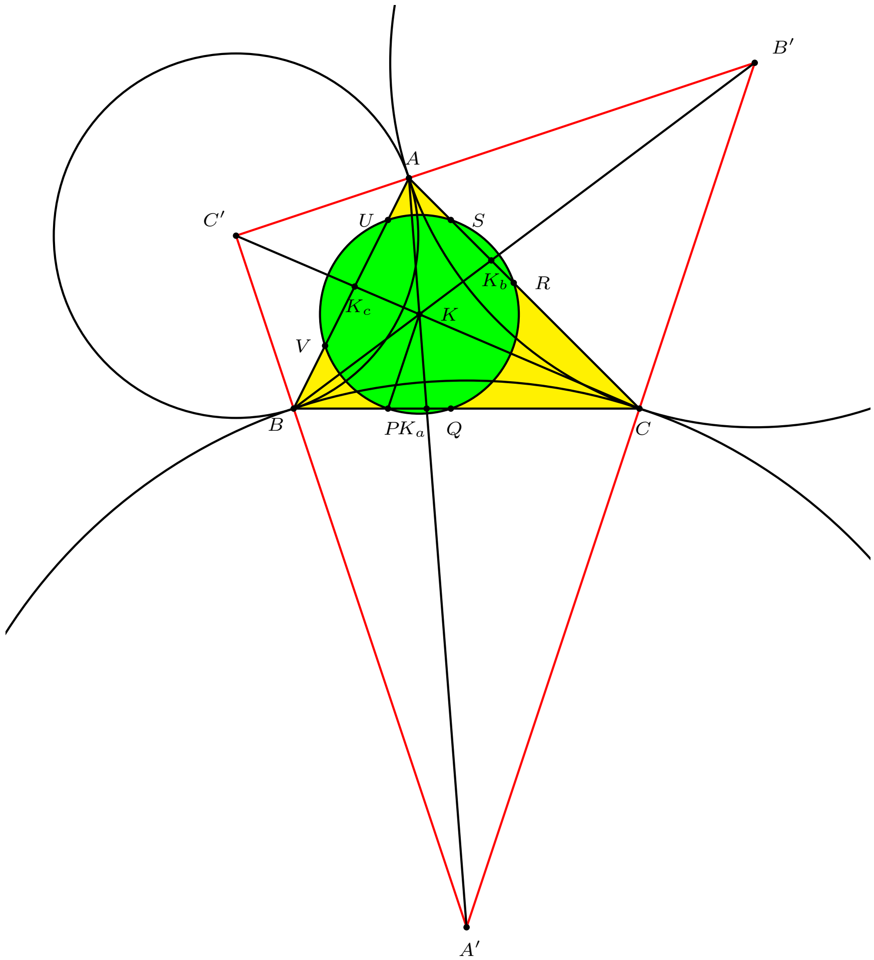

Figure 1. The second Lemoine circle and the circles $A'(B)$, $B'(C)$, $C'(A)$

Proposition 2. *The second Lemoine circle is the orthogonal projection on the plane of ABC of a sphere T tritangent externally to T_A, T_B, and T_C. Furthermore,*

(1) *the center T_K of T has a distance of $2r_L$ to the plane of ABC;*
(2) *the highest points of T, T_A, T_B, and T_C are coplanar.*

---

[1]An alternative name is the cosine circle, because the sides of ABC intercept chords of lengths are proportional to the cosines of the vertex angles. Since however there are infinitely many circles with this property, see [1], this name seems less appropriate.

Proof. Since A' is given in barycentrics by $(-a^2 : b^2 : c^2)$, we find with help of the distance formula (see for instance [3], where some helpful information on circles in barycentric coordinates is given as well):

$$\rho_a = d_{A',B} = \frac{abc}{b^2+c^2-a^2},$$
$$d^2_{A',K} = \frac{4a^4b^2c^2(2b^2+2c^2-a^2)}{(a^2+b^2+c^2)^2(b^2+c^2-a^2)^2}.$$

Now combining these, we see that the power K with respect to $A'(\rho_a + r_L)$ is equal to

$$\mathcal{P} = d^2_{A',K} - (\rho_a + r_L)^2 = -\frac{4a^2b^2c^2}{(a^2+b^2+c^2)^2} = -4r_L^2.$$

By symmetry, K is indeed the radical center of $A'(\rho_a + r_L)$, $B'(\rho_b + r_L)$, and $C'(\rho_c + r_L)$. Therefore, the second Lemoine circle is indeed the orthogonal projection of a sphere externally tritangent to T_A, T_B, and T_C. In addition $-\mathcal{P} = d^2_{K,T_K}$, which proves (1).

Now from

$$\rho_b \cdot A' - \rho_a \cdot B' \sim S_A \cdot A' - S_B \cdot B' = (-a^2 : b^2 : 0)$$

we see by symmetry that the plane through highest points of T_A, T_B, and T_C meets the plane of ABC in the Lemoine axis $\frac{x}{a^2} + \frac{y}{b^2} + \frac{x}{c^2} = 0$. The fact that $3r_L \cdot A' - \rho_a \cdot K \sim (-2a^2 : b^2 : c^2)$ lies on the Lemoine axis as well completes the proof of (2).

Note finally that from the similarity of triangles $A'CK_a$ and KPK_a in Figure 1 K_a divides $A'K$ in the ratio of the radii of T_A and T. This means that the vertices of the cevian triangle $K_aK_bK_c$ are the orthogonal projections of the points of contact of T with T_A, T_B, and T_C respectively. □

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Forum Geometricorum
Volume 11 (2011) 205–210.

FORUM GEOM
ISSN 1534-1178

A Simple Vector Proof of Feuerbach's Theorem

Michael J. G. Scheer

Abstract. The celebrated theorem of Feuerbach states that the nine-point circle of a nonequilateral triangle is tangent to both its incircle and its three excircles. In this note, we give a simple proof of Feuerbach's Theorem using straightforward vector computations. All required preliminaries are proven here for the sake of completeness.

1. Notation and background

Let ABC be a nonequilateral triangle. We denote its side-lengths by a, b, c, its semiperimeter by $s = \frac{1}{2}(a + b + c)$, and its area by Δ. Its *classical centers* are the circumcenter O, the incenter I, the centroid G, and the orthocenter H (Figure 1). The nine-point center N is the midpoint of OH and the center of the nine-point circle, which passes through the side-midpoints A', B', C' and the feet of the three altitudes. The Euler Line Theorem states that G lies on OH with $OG : GH = 1 : 2$.

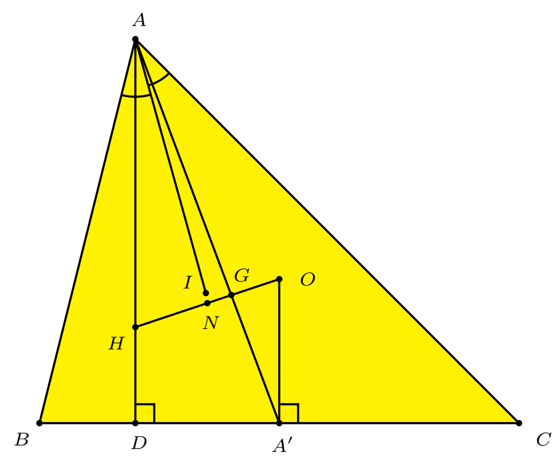

Figure 1. The classical centers and the Euler division $OG : GH = 1 : 2$.

We write I_a, I_b, I_c for the excenters opposite A, B, C, respectively; these are points where one internal angle bisector meets two external angle bisectors. Like I, the points I_a, I_b, I_c are equidistant from the lines AB, BC, and CA, and thus are centers of three circles each tangent to the three lines. These are the excircles. The *classical radii* are the circumradius R $(= OA = OB = OC)$, the inradius r, and

Publication Date: October 21, 2011. Communicating Editor: Paul Yiu.
The author wishes to thank his teacher, Mr. Joseph Stern, for offering numerous helpful suggestions and comments during the writing of this note.

the exradii r_a, r_b, r_c. The following area formulas are well known (see, e.g., [1] and [2]):

$$\Delta = \frac{abc}{4R} = rs = r_a(s-a) = \sqrt{s(s-a)(s-b)(s-c)}.$$

Feuerbach's Theorem states that *the nine-point circle is tangent internally to the incircle, and externally to each of the excircles* [3]. Two of the four points of tangency are shown in Figure 2.

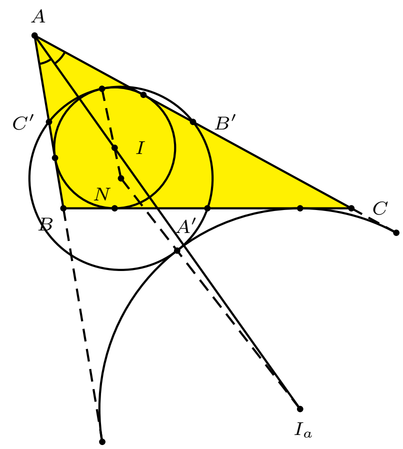

Figure 2. The excenter I_a and A-excircle; Feuerbach's theorem.

2. Vector formalism

We view the plane as $\mathbb{R}^2$ with its standard vector space structure. Given triangle ABC, the vectors $A-C$ and $B-C$ are linearly independent. Thus for any point X, we may write $X-C = \alpha(A-C)+\beta(B-C)$ for unique $\alpha, \beta \in \mathbb{R}$. Defining $\gamma = 1-\alpha-\beta$, we find that

$$X = \alpha A + \beta B + \gamma C, \qquad \alpha+\beta+\gamma = 1.$$

This expression for X is unique. One says that X has *barycentric coordinates* (α, β, γ) with respect to triangle ABC (see, e.g., [1]). The barycentric coordinates are particularly simple when X lies on a side of triangle ABC:

Lemma 1. *Let X lie on the sideline BC of triangle ABC. Then, with respect to triangle ABC, X has barycentric coordinates* $\left(0,\ \frac{XC}{a},\ \frac{BX}{a}\right)$.

Proof. Since X lies on line BC between B and C, there is a unique scalar t such that $X-B = t(C-B)$. In fact, the length of the directed segment $BX = t \cdot BC = ta$, *i.e.*, $t = \frac{BX}{a}$. Rearranging, $X = 0A + (1-t)B + tC$, in which the coefficients sum to 1. Finally, $1-t = \frac{a-BX}{a} = \frac{XC}{a}$. □

The next theorem reduces the computation of a distance XY to the simpler distances AY, BY, and CY, when X has known barycentric coordinates.

Theorem 2. *Let X have barycentric coordinates (α, β, γ) with respect to triangle ABC. Then for any point Y,*

$$XY^2 = \alpha AY^2 + \beta BY^2 + \gamma CY^2 - (\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2).$$

Proof. Using the well known identity $|V|^2 = V \cdot V$, we compute first that

$$\begin{aligned} XY^2 &= |Y - X|^2 \\ &= |Y - \alpha A - \beta B - \gamma C|^2 \\ &= |\alpha(Y - A) + \beta(Y - B) + \gamma(Y - C)|^2 \\ &= \alpha^2 AY^2 + \beta^2 BY^2 + \gamma^2 CY^2 + 2\alpha\beta(Y - A) \cdot (Y - B) \\ &\quad + 2\alpha\gamma(Y - A) \cdot (Y - C) + 2\beta\gamma(Y - B) \cdot (Y - C). \end{aligned}$$

On the other hand,

$$c^2 = |B - A|^2 = |(Y - A) - (Y - B)|^2 = AY^2 + BY^2 - 2(Y - A) \cdot (Y - B).$$

Thus,

$$2\alpha\beta(Y - A) \cdot (Y - B) = \alpha\beta(AY^2 + BY^2 - c^2).$$

Substituting this and its analogues into the preceding calculation, the total coefficient of AY^2 becomes $\alpha^2 + \alpha\beta + \alpha\gamma = \alpha(\alpha + \beta + \gamma) = \alpha$, for example. The result is the displayed formula. $\square$

3. Distances from N to the vertices

Lemma 3. *The centroid G has barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.*

Proof. Let G' be the point with barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and we will prove $G = G'$. By Lemma 1, $A' = \frac{1}{2}B + \frac{1}{2}C$. We calculate

$$\frac{1}{3}A + \frac{2}{3}A' = \frac{1}{3}A + \frac{2}{3}\left(\frac{1}{2}B + \frac{1}{2}C\right) = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C = G',$$

which implies that G' is on segment AA'. Similarly, G' is on the other two medians of triangle ABC. However the intersection of the medians is G, and so $G = G'$. $\square$

Lemma 4 (Euler Line Theorem)**.** $H - O = 3(G - O)$.

Proof. Let $H' = O + 3(G - O)$ and we will prove $H = H'$. By Lemma 3,

$$H' - O = 3(G - O) = A + B + C - 3O = (A - O) + (B - O) + (C - O).$$

And so,

$$\begin{aligned} (H' - A) \cdot (B - C) &= \{(H' - O) - (A - O)\} \cdot \{(B - O) - (C - O)\} \\ &= \{(B - O) + (C - O)\} \cdot \{(B - O) - (C - O)\} \\ &= (B - O) \cdot (B - O) - (C - O) \cdot (C - O) \\ &= |OB|^2 - |OC|^2 \\ &= 0, \end{aligned}$$

which implies H' is on the altitude from A to BC. Similarly, H' is on the other two altitudes of triangle ABC. Since H is defined to be the intersection of the altitudes, it follows that $H = H'$. $\square$

Lemma 5. $(A - O) \cdot (B - O) = R^2 - \frac{1}{2}c^2$.

Proof. One has

$$\begin{aligned} c^2 &= |A - B|^2 \\ &= |(A - O) - (B - O)|^2 \\ &= OA^2 + OB^2 - 2\,(A - O) \cdot (B - O) \\ &= 2R^2 - 2\,(A - O) \cdot (B - O). \end{aligned}$$ $\square$

We now find AN, BN, CN, which are needed in Theorem 2 for $Y = N$.

Theorem 6. $4AN^2 = R^2 - a^2 + b^2 + c^2$.

Proof. Since N is the midpoint of OH, we have $H - O = 2(N - O)$. Combining this observation with Theorem 2, and using Lemma 5, we obtain

$$\begin{aligned} 4AN^2 =\ & |2(A - O) - 2(N - O)|^2 \\ =\ & |(A - O) - (B - O) - (C - O)|^2 \\ =\ & AO^2 + BO^2 + CO^2 \\ & - 2(A - O) \cdot (B - O) - 2(A - O) \cdot (C - O) + 2(B - O) \cdot (C - O) \\ =\ & 3R^2 - 2\left(R^2 - \frac{1}{2}c^2\right) - 2\left(R^2 - \frac{1}{2}b^2\right) + 2\left(R^2 - \frac{1}{2}a^2\right) \\ =\ & R^2 - a^2 + b^2 + c^2. \end{aligned}$$ $\square$

4. Proof of Feuerbach's Theorem

Theorem 7. *The incenter I has barycentric coordinates $\left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right)$.*

Proof. Let I' be the point with barycentric coordinates $(a/2s, b/2s, c/2s)$, and we will prove $I = I'$. Let F be the foot of the bisector of angle A on side BC. Applying the law of sines to triangles ABF and ACF, and using $\sin(\pi - x) = \sin x$, we find that

$$\frac{BF}{c} = \frac{\sin BAF}{\sin BFA} = \frac{\sin CAF}{\sin CFA} = \frac{FC}{b}.$$

The equations $b \cdot BF = c \cdot FC$ and $BF + FC = a$ jointly imply that $BF = \frac{ac}{b+c}$. By Lemma 1, $F = (1 - t)B + tC$, where $t = \frac{BF}{a} = \frac{c}{b+c}$. Now,

$$\begin{aligned} \frac{b+c}{2s} \cdot F + \frac{a}{2s} \cdot A =\ & \frac{b+c}{2s}\left(\frac{b}{b+c} \cdot B + \frac{c}{b+c} \cdot C\right) + \frac{a}{2s} \cdot A \\ =\ & \frac{a}{2s} \cdot A + \frac{b}{2s} \cdot B + \frac{c}{2s} \cdot C \\ =\ & I', \end{aligned}$$

which implies that I' is on the angle bisector of angle A. Similarly, I' is on the other two angle bisectors of triangle ABC. Since I is the intersection of the angle bisectors, this implies $I = I'$. $\square$

Theorem 8 (Euler). $OI^2 = R^2 - 2Rr$.

Proof. We use $X = I$ and $Y = O$ in Theorem 2 to obtain

$$\begin{aligned}
OI^2 = &\ \frac{a}{2s}R^2 + \frac{b}{2s}R^2 + \frac{c}{2s}R^2 - \left(\frac{bc}{(2s)^2}a^2 + \frac{ca}{(2s)^2}b^2 + \frac{ab}{(2s)^2}c^2\right)\\
= &\ R^2 - \frac{a^2bc + b^2ac + c^2ab}{(2s)^2}\\
= &\ R^2 - \frac{abc}{2s}\\
= &\ R^2 - \left(\frac{abc}{2\Delta}\right)\left(\frac{\Delta}{s}\right)\\
= &\ R^2 - 2Rr.
\end{aligned}$$

The last step here uses the area formulas of §1—in particular $\Delta = rs = abc/4R$. $\square$

Theorem 9. $IN = \frac{1}{2}R - r$ *and* $I_aN = \frac{1}{2}R + r_a$

Proof. To find the distance IN, we set $X = I$ and $Y = N$ in Theorem 2, with Theorems 6 and 7 supplying the distances AN, BN, CN, and the barycentric coordinates of I. For brevity in our computation, we use *cyclic sums*, in which the displayed term is transformed under the permutations (a, b, c), (b, c, a), and (c, a, b), and the results are summed (thus, symmetric functions of a, b, c may be factored through the summation sign, and $\sum_{\circlearrowleft} a = a+b+c = 2s$). The following computation results:

$$\begin{aligned}
IN^2 = &\ \sum_{\circlearrowleft}\left(\frac{a}{2s}\right)\frac{R^2 - a^2 + b^2 + c^2}{4} - \sum_{\circlearrowleft}\left(\frac{b}{2s}\cdot\frac{c}{2s}\right)a^2\\
= &\ \frac{R^2}{8s}\left(\sum_{\circlearrowleft} a\right) + \frac{1}{8s}\left(\sum_{\circlearrowleft}(-a^3 + ab^2 + ac^2)\right) - \frac{abc}{(2s)^2}\left(\sum_{\circlearrowleft} a\right)\\
= &\ \frac{R^2}{4} + \frac{(-a+b+c)(a-b+c)(a+b-c) + 2abc}{8s} - \frac{abc}{2s}\\
= &\ \frac{R^2}{4} + \frac{(2s-2a)(2s-2b)(2s-2c)}{8s} - \frac{abc}{4s}\\
= &\ \frac{R^2}{4} + \frac{(\Delta^2/s)}{s} - \frac{4R\Delta}{4s}\\
= &\ \left(\frac{1}{2}R\right)^2 + r^2 - Rr\\
= &\ \left(\frac{1}{2}R - r\right)^2.
\end{aligned}$$

The two penultimate steps again use the area formulas of §1. Theorem 8 tells us that $OI^2 = 2R(\frac{1}{2}R - r)$, and so $\frac{1}{2}R - r$ is nonnegative. Thus we conclude $IN = \frac{1}{2}R - r$. A similar calculation applies to the A-excircle, with two modifications: (i) I_a has barycentric coordinates

$$\left(\frac{-a}{2(s-a)}, \frac{b}{2(s-a)}, \frac{c}{2(s-a)}\right),$$

and (ii) in lieu of $\Delta = rs$, one uses $\Delta = r_a(s-a)$. The result is $I_aN = \frac{1}{2}R + r_a$. □

We are now in a position to prove Feuerbach's Theorem.

Theorem 10 (Feuerbach, 1822)**.** *In a nonequilateral triangle, the nine-point circle is internally tangent to the incircle and externally tangent to the three excircles.*

Proof. Suppose first that the nine-point circle and the incircle are nonconcentric. Two nonconcentric circles are internally tangent if and only if the distance between their centers is equal to the positive difference of their radii. Since the nine-point circle is the circumcircle of the medial triangle $A'B'C'$, its radius is $\frac{1}{2}R$. Thus the positive difference between the radii of the nine-point circle and the incircle is $\frac{1}{2}R - r$ which is IN by Theorem 9. This implies that the nine-point circle and the incircle are internally tangent. Also, since the sum of the radii of the A-excircle and the nine-point circle is I_aN, by Theorem 9, the nine-point circle is externally tangent to the A-excircle. Suppose now that the nine-point circle and the incircle are concentric, that is $I = N$. Then $0 = IN = \frac{1}{2}R - r = OI^2/2R$ and so $I = O$. This clearly implies triangle ABC is equilateral. □

For historical details, see [3] and [4].

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Forum Geometricorum
Volume 11 (2011) 211–212.

FORUM GEOM
ISSN 1534-1178

A New Formula Concerning the Diagonals and Sides of a Quadrilateral

Larry Hoehn

Abstract. We derive a formula relating the sides and diagonal sections of a general convex quadrilateral along with the special case of a quadrilateral with an incircle.

Let $ABCD$ be a convex quadrilateral whose diagonals intersect at E. We use the notation $a = AB$, $b = BC$, $c = CD$, $d = DA$, $e = AE$, $f = BE$, $g = CE$, and $h = DE$ as seen in the figure below.

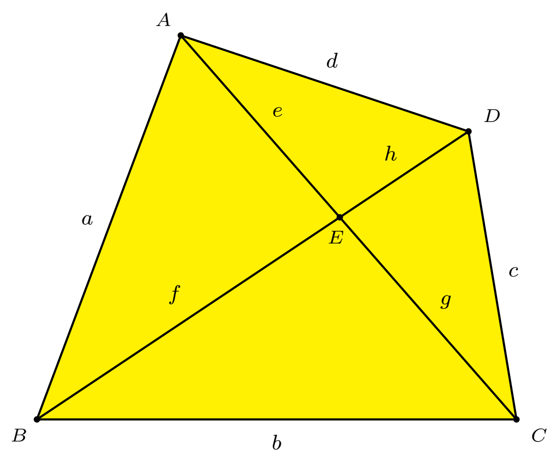

Figure 1. Convex quadrilateral

By the law of cosines in triangles ABE and CDE we obtain

$$\frac{e^2+f^2-a^2}{2ef} = \cos AEB = \cos CED = \frac{g^2+h^2-c^2}{2gh}.$$

This can be rewritten as

$$gh(e^2+f^2-a^2) = ef(g^2+h^2-c^2),$$

rewritten further as

$$fh(fg-eh) - eg(fg-eh) = a^2gh - c^2ef,$$

and still further as

$$fh - eg = \frac{a^2gh - c^2ef}{fg-eh}.$$

Publication Date: October 28, 2011. Communicating Editor: Paul Yiu.

In the same manner for triangles BCE and DAE we obtain

$$fh - eg = \frac{b^2eh - d^2fg}{ef - gh}.$$

By setting the right sides of these two equations equal to each other and some computation, we obtain

$$(ef - gh)(a^2gh - c^2ef) = (fg - eh)(b^2eh - d^2fg),$$

which can be rewritten as

$$efgh(a^2 + c^2 - b^2 - d^2) = a^2g^2h^2 + c^2e^2f^2 - b^2e^2h^2 - d^2f^2g^2.$$

By adding the same quantity to both sides we obtain

$$\begin{aligned} &efgh(a^2 + c^2 - b^2 - d^2) + efgh(2ac - 2bd) \\ &= a^2g^2h^2 + c^2e^2f^2 - b^2e^2h^2 - d^2f^2g^2 + 2aghcef - 2behdfg, \end{aligned}$$

which can be factored into

$$efgh((a + c)^2 - (b + d)^2) = (agh + cef)^2 - (beh + dfg)^2,$$

or

$$efgh(a+c+b+d)(a+c-b-d) = (agh+cef+beh+dfg)(agh+cef-beh-dfg).$$

This is a formula for an arbitrary quadrilateral. If the quadrilateral has an incircle, then the sums of the opposite sides are equal with $a + c = b + d$. By making this substitution in the last equation above we get the result that

$$agh + cef = beh + dfg.$$

This is a nice companion result to Ptolemy's theorem for a quadrilateral inscribed in a circle. In the notation of this paper Ptolemy's theorem states that

$$(e + g)(f + h) = ac + bd.$$

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Forum Geometricorum
Volume 11 (2011) 213–216.

FORUM GEOM
ISSN 1534-1178

The Area of the Diagonal Point Triangle

Martin Josefsson

Abstract. In this note we derive a formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral in terms of the sides of the quadrilateral, and prove a characterization of trapezoids.

1. Introduction

If the diagonals in a convex quadrilateral intersect at E and the extensions of the opposite sides intersect at F and G, then the triangle EFG is called the *diagonal point triangle* [3, p.79] or sometimes just the diagonal triangle [8], see Figure 1. Here it's assumed the quadrilateral has no pair of opposite parallel sides. If it's a cyclic quadrilateral, then the diagonal point triangle has the property that its orthocenter is also the circumcenter of the quadrilateral [3, p.197].

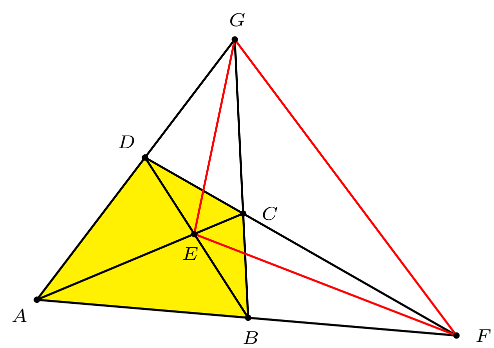

Figure 1. The diagonal point triangle EFG

How about the area of the diagonal point triangle? In a note in an old number of the MONTHLY [2, p.32] reviewing formulas for the area of quadrilaterals, we found a formula and a reference to an even older paper on quadrilaterals by Georges Dostor [4, p.272]. He derived the following formula for the area T of the diagonal point triangle belonging to a cyclic quadrilateral,

$$T = \frac{4a^2b^2c^2d^2K}{(a^2b^2 - c^2d^2)(a^2d^2 - b^2c^2)}$$

where a, b, c, d are the sides of the cyclic quadrilateral and K its area. To derive this formula was also a problem in [7, p.208].[1] However, the formula is wrong, and the mistake Dostor made in his derivation was assuming that two angles are equal when they in fact are not. The purpose of this note is to derive the correct formula.

Publication Date: November 4, 2011. Communicating Editor: Paul Yiu.

[1] But it had a misprint, the 4 was missing.

As far as we have been able to find out, this triangle area has received little interest over the years. But we have found one important result. In [1, pp.13–17] Hugh ApSimon made a delightful derivation of a formula for the area of the diagonal point triangle belonging to a general convex quadrilateral $ABCD$. His formula (which used other notations) is

$$T = \frac{2T_1T_2T_3T_4}{(T_1+T_2)(T_1-T_4)(T_2-T_4)}$$

where T_1, T_2, T_3, T_4 are the areas of the four overlapping triangles ABC, ACD, ABD, BCD respectively. In [6, p.163] Richard Guy rewrote this formula using $T_1 + T_2 = T_3 + T_4$ in the more symmetric form

$$T = \frac{2T_1T_2T_3T_4}{K(T_1T_2 - T_3T_4)} \tag{1}$$

using other notations, where $K = T_1 + T_2$ is the area of the convex quadrilateral.

2. The area of the diagonal point triangle belonging to a cyclic quadrilateral

We will use (1) to derive a formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral in terms of the sides of the quadrilateral.

Theorem 1. *If a, b, c, d are the consecutive sides of a cyclic quadrilateral, then its diagonal point triangle has the area*

$$T = \frac{2abcdK}{|a^2-c^2||b^2-d^2|}$$

if it's not an isosceles trapezoid, where

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

is the area of the quadrilateral and $s = \frac{a+b+c+d}{2}$ is its semiperimeter.

Proof. In a cyclic quadrilateral $ABCD$ opposite angles are supplementary, so $\sin C = \sin A$ and $\sin D = \sin B$. From these we get that $\sin A = \frac{2K}{ad+bc}$ and $\sin B = \frac{2K}{ab+cd}$ by dividing the quadrilateral into two triangles by a diagonal in two different ways. Using (1) and the formula for the area of a triangle, we have

$$\begin{aligned}
T &= \frac{2\cdot\frac{1}{2}ad\sin A\cdot\frac{1}{2}bc\sin C\cdot\frac{1}{2}ab\sin B\cdot\frac{1}{2}cd\sin D}{K\left(\frac{1}{2}ad\sin A\cdot\frac{1}{2}bc\sin C - \frac{1}{2}ab\sin B\cdot\frac{1}{2}cd\sin D\right)}\\
&= \frac{\frac{1}{2}abcd\sin^2 A\sin^2 B}{K\left(\sin^2 A - \sin^2 B\right)} = \frac{\frac{1}{2}abcd}{K\left(\frac{1}{\sin^2 B} - \frac{1}{\sin^2 A}\right)}\\
&= \frac{2abcdK}{(ab+cd)^2-(ad+bc)^2} = \frac{2abcdK}{(ab+cd+ad+bc)(ab+cd-ad-bc)}\\
&= \frac{2abcdK}{(a+c)(b+d)(a-c)(b-d)} = \frac{2abcdK}{(a^2-c^2)(b^2-d^2)}.
\end{aligned}$$

Since we do not know which of the opposite sides is longer, we put absolute values around the subtractions in the denominator to cover all cases. The area K of the cyclic quadrilateral is given by the well known Brahmagupta formula [5, p.24]. The formula for the area of the diagonal point triangle does not apply when two opposite sides are congruent. Then the other two sides are parallel, so the cyclic quadrilateral is an isosceles trapezoid.[2] □

3. A characterization of trapezoids

In (1) we see that if $T_1T_2 = T_3T_4$, then the area of the diagonal point triangle is infinite. This suggests that two opposite sides of the quadrilateral are parallel, giving a condition for when a quadrilateral is a trapezoid. We prove this in the next theorem.

Theorem 2. *A convex quadrilateral is a trapezoid if and only if the product of the areas of the triangles formed by one diagonal is equal to the product of the areas of the triangles formed by the other diagonal.*

Proof. We use the same notations as in the proof of Theorem 1. The following statements are equivalent.

$$
\begin{aligned}
&T_1T_2 = T_3T_4,\\
&\tfrac{1}{2}ad\sin A\cdot\tfrac{1}{2}bc\sin C = \tfrac{1}{2}ab\sin B\cdot\tfrac{1}{2}cd\sin D,\\
&\sin A\sin C = \sin B\sin D,\\
&\tfrac{1}{2}\left(\cos(A-C)-\cos(A+C)\right) = \tfrac{1}{2}\left(\cos(B-D)-\cos(B+D)\right),\\
&\cos(A-C) = \cos(B-D),\\
&A-C = B-D \quad\text{or}\quad A-C=-(B-D),\\
&A+D=B+C=\pi \quad\text{or}\quad A+B=C+D=\pi.
\end{aligned}
$$

Here we have used that $\cos(A+C) = \cos(B+D)$, which follows from the sum of angles in a quadrilateral. The equalities in the last line are respectively equivalent to $AB \parallel DC$ and $AD \parallel BC$ in a quadrilateral $ABCD$. □

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Forum Geometricorum
Volume 11 (2011) 217–221.

FORUM GEOM
ISSN 1534-1178

Collinearity of the First Trisection Points of Cevian Segments

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Abstract. We show that the first trisection points of the cevian segments of a finite point P are collinear if and only if P lies on the Steiner circum-ellipse. Some further results are obtained concerning the line containing these first trisection points.

This note answers a question of Paul Yiu [3]: *In the plane of a given triangle ABC, what is the locus of P for which the "first" trisection points of the three cevian segments (the ones nearer to the vertices) are collinear* ? We identify this locus and establish some further results.

Let $P = (u : v : w)$ be in homogeneous barycentric coordinates with respect to triangle ABC. Its cevian triangle $A'B'C'$ have vertices

$$A' = (0 : v : w), \qquad B = (u : 0 : w), \qquad C' = (u : v : 0).$$

The first trisection points of the cevian segments AA', BB', CC' are the points dividing these segments in the ratio 1 : 2. These are the points

$$X = (2(v+w) : v : w), \qquad Y = (u : 2(w+u) : w), \qquad Z = (u : v : 2(u+v)).$$

These three points are collinear if and only if

$$0 = \begin{vmatrix} 2(v+w) & v & w \\ u & 2(w+u) & w \\ u & v & 2(u+v) \end{vmatrix} = 6(u+v+w)(uv+vw+wu).$$

It follows that, for a finite point P, the first trisection points are collinear if and only if P lies on the Steiner circum-ellipse

$$uv + vw + wu = 0.$$

Proposition 1. *If $P = (u : v : w)$ lies on the Steiner circum-ellipse, the line $\mathcal{L}_P$ containing the three "first" trisection points of the three cevian segments AA', BB', CC' has equation*

$$\left(\frac{1}{v} - \frac{1}{w}\right) x + \left(\frac{1}{w} - \frac{1}{u}\right) y + \left(\frac{1}{u} - \frac{1}{v}\right) z = 0.$$

Publication Date: November 10, 2011. Communicating Editor: Paul Yiu.

Proof. The line $\mathcal{L}_P$ contains the point X since

$$\begin{aligned}&\left(\frac{1}{v}-\frac{1}{w}\right)\cdot 2(v+w)+\left(\frac{1}{w}-\frac{1}{u}\right)v+\left(\frac{1}{u}-\frac{1}{v}\right)w\\ =&-\frac{(v-w)(uv+vw+wu)}{uvw}\\ =&\,0;\end{aligned}$$

similarly for Y and Z. $\square$

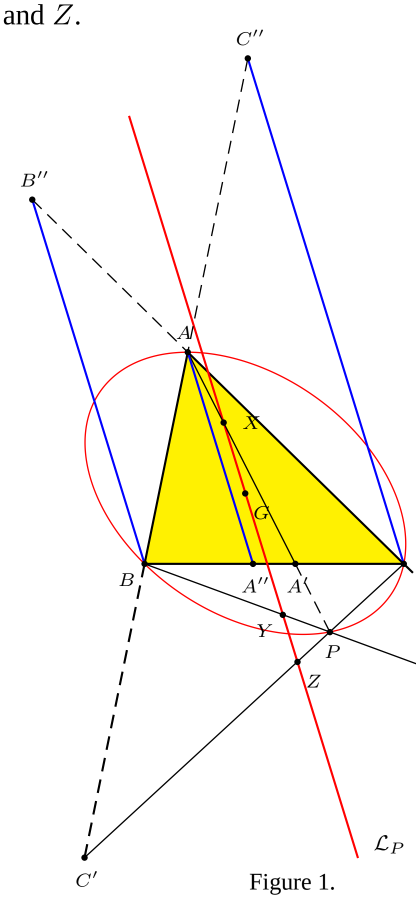

Figure 1.

Let P be a point on the Steiner circum-ellipse, with cevian triangle $A'B'C'$. Construct points A'', B'', C'' on the lines BC, CA, AB respectively such that

$$BA' = A''C, \qquad CB' = B''A, \qquad AC' = C''B.$$

The lines AA'', BB'', CC'' are parallel, and their common infinite point is the isotomic conjugate of P. It is clear from Proposition 1 that the line $\mathcal{L}_P$ contains the isotomic conjugate of P:

$$\left(\frac{1}{v}-\frac{1}{w}\right)\cdot\frac{1}{u}+\left(\frac{1}{w}-\frac{1}{u}\right)\cdot\frac{1}{v}+\left(\frac{1}{u}-\frac{1}{v}\right)\cdot\frac{1}{w}=0.$$

It follows that $\mathcal{L}_P$ is parallel to AA'', BB'', and CC''; see Figure 1.

Corollary 2. *Given a line ℓ containing the centroid G, there is a unique point P on the Steiner circum-ellipse such that $\mathcal{L}_P = \ell$, namely, P is the isotomic conjugate of the infinite point of the line ℓ.*

Lemma 3. *Let $\ell : px+qy+rz = 0$ be a line through the centroid G. The conjugate diameter in the Steiner circum-ellipse is the line $(q-r)x+(r-p)y+(p-q)z = 0$.*

Proof. The parallel of ℓ through A intersects the ellipse again at the point

$$N = \left(\frac{1}{q-r} : \frac{1}{p-q} : \frac{1}{r-p}\right).$$

The midpoint of AN is the point

$$(2(r-p)(p-q) - (q-r)^2 : (q-r)(r-p) : (q-r)(p-q)).$$

This point lies on the line

$$(q-r)x + (r-p)y + (p-q)z = 0,$$

which also contains the centroid. This line is therefore the conjugate diameter of ℓ. □

Corollary 4. *Let $\ell : px + qy + rz = 0$ be a line through the centroid G, so that $P_\ell := \left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$ is a point on the Steiner circum-ellipse. The conjugate diameter of ℓ in the ellipse is the line $\mathcal{L}_{P_\ell}$.*

Proposition 5. *Let P and Q be points on the Steiner circum-ellipse. The lines $\mathcal{L}_P$ and $\mathcal{L}_Q$ are conjugate diameters if and only if P and Q are antipodal.*

Proof. Suppose $\mathcal{L}_P$ has equation $px+qy+rz = 0$ with $p+q+r = 0$. The line $\mathcal{L}_Q$ is the conjugate diameter of $\mathcal{L}_P$ if and only if its has equation $(q-r)x+(r-p)y+(p-q)z = 0$. Thus, P and Q are the points $\left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$ and $\left(\frac{1}{q-r} : \frac{1}{r-p} : \frac{1}{p-q}\right)$. These are antipodal points since the line joining them, namely,

$$p(q-r)x + q(r-p)y + r(p-q)z = 0,$$

contains the centroid G. □

It is easy to determine the line through G whose intersections with the Steiner circum-ellipse are two points P and Q such that $\mathcal{L}_P : px + qy + rz = 0$ and $\mathcal{L}_Q : (q-r)x + (r-p)y + (p-q)z = 0$ are the axes of the ellipse. This is the case if the two conjugate axes are perpendicular. Now, the infinite points of the lines $\mathcal{L}_P$ and $\mathcal{L}_Q$ are respectively $q-r : r-p : p-q$ and $p : q : r$. They are perpendicular, according to [4, §4.5, Theorem] if and only if

$$(b^2+c^2-a^2)p(q-r) + (b^2+c^2-a^2)q(r-p) + (a^2+b^2-c^2)r(p-q) = 0.$$

Equivalently,

$$(b^2+c^2-a^2)\left(\frac{1}{r} - \frac{1}{q}\right) + (c^2+a^2-b^2)\left(\frac{1}{p} - \frac{1}{r}\right) + (a^2+b^2-c^2)\left(\frac{1}{q} - \frac{1}{p}\right) = 0,$$

or

$$\frac{b^2-c^2}{p} + \frac{c^2-a^2}{q} + \frac{a^2-b^2}{r} = 0.$$

Therefore, $P = \left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$ is an intersection of the Steiner circum-ellipse and the line

$$(b^2 - c^2)x + (c^2 - a^2)y + (a^2 - b^2)z = 0,$$

which contains the centroid G and the symmedian point $K = (a^2 : b^2 : c^2)$.

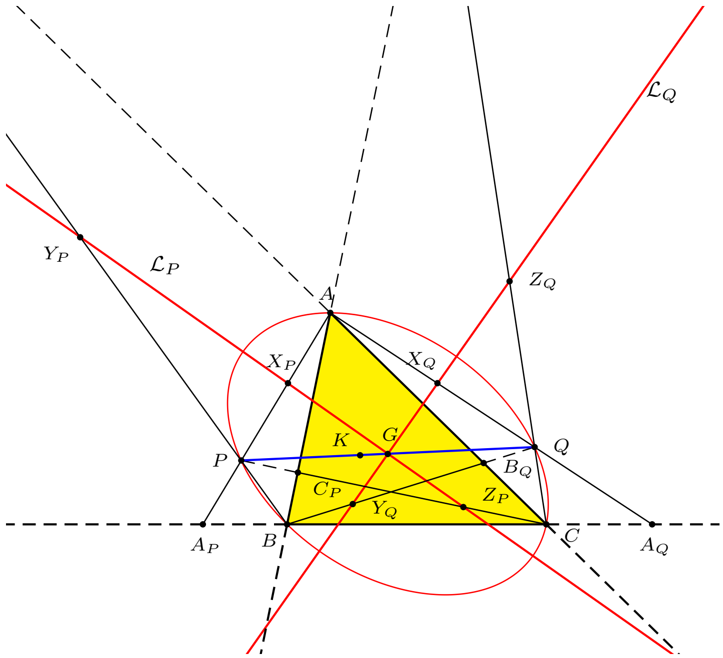

Figure 2.

A point on the line GK has homogeneous barycentric coordinates $(a^2 + t : b^2 + t : c^2 + t)$ for some t. If this point lies on the Steiner circum-ellipse, then

$$(b^2 + t)(c^2 + t) + (c^2 + t)(a^2 + t) + (a^2 + t)(b^2 + t) = 0.$$

From this,

$$t = -\frac{1}{3}(a^2 + b^2 + c^2 \pm \sqrt{D}),$$

where

$$D = a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2.$$

We obtain, with $\varepsilon = \pm 1$, the two points P and Q

$$(b^2 + c^2 - 2a^2 + \varepsilon\sqrt{D} : \ c^2 + a^2 - 2b^2 + \varepsilon\sqrt{D} : \ a^2 + b^2 - 2c^2 + \varepsilon\sqrt{D})$$

for which $\mathcal{L}_P$ and $\mathcal{L}_Q$ are the axes of the Steiner circum-ellipse. Their isotomic conjugates are the points X_{3413} and X_{3414} in [2], which are the infinite points of the axes of the ellipse.

We conclude the present note with two remarks.

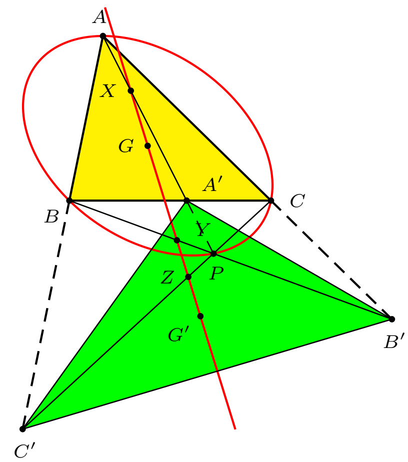

Figure 3.

Remarks. (1) The line $\mathcal{L}_P$ contains the centroids of triangle ABC and the cevian triangle $A'B'C'$ of P.

Proof. Clearly $\mathcal{L}_P$ contains the centroid $G = (1:1:1)$. Since X, Y, Z are on the line $\mathcal{L}_P$, so is the centroid $G' = \frac{1}{3}(X+Y+Z)$ of the degenerate triangle XYZ. Since

$$X = \frac{2A+A'}{3}, \qquad Y = \frac{2B+B'}{3}, \qquad Z = \frac{2C+C'}{3},$$

it follows that $\mathcal{L}_P$ also contains the point $\frac{1}{3}(A'+B'+C')$, the centroid of the cevian triangle of P.

(2) More generally, if we consider X, Y, Z dividing the cevian segments AA', BB', CC' in the ratio $m:n$, these points are collinear if and only if

$$n(n-m)(u+v+w)(uv+vw+wu)+(2m-n)(m+n)uvw = 0.$$

In particular, for the "second" trisection points, we take $m=2,\ n=1$ and obtain for the locus of P the cubic

$$u(v^2+w^2)+v(w^2+u^2)+w(u^2+v^2)-6uvw = 0,$$

which is the Tucker nodal cubic $\mathsf{K}015$ in Gibert's catalogue of cubic curves [1].

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Forum Geometricorum
Volume 11 (2011) 223–229.

FORUM GEOM
ISSN 1534-1178

Cyclic Quadrilaterals Associated With Squares

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Abstract. We discuss a family of problems asking to find the geometrical locus of points P in the plane of a square $ABCD$ having the property that the i-th triangle center in Kimberling's list with respect to triangles ABP, BCP, CDP, DAP are concyclic.

1. A family of problems

A fruitful research line in the Euclidean geometry of the plane refers to a given quadrilateral, to which another one is associated, and the question is whether the new quadrilateral has a specific property, like being a trapezium, or parallelogram, rectangle, rhombus, cyclic quadrilateral, and so on. Besides the target property, the results of this kind differ by the procedure used to generate the new points. A common selection procedure involves two choices: on the one hand, one produces four triangles out of the given quadrilateral, on the other hand, one identifies a point in each resulting triangle. We restrain our discussion to the handiest ways to fulfill each task.

With regards to the choices of the first kind, one option is to consider the four triangles defined by the vertices of the given quadrilateral taken three at a time. A known result in this category is illustrated in [4]. Another frequently used construction starts from a triangulation determined by a point in the given quadrilateral.

When it comes to select a point in the four already generated triangles, the most convenient approach is to pick one out of the triangle centers listed in [3]. The alternative is to invoke a more complicated construction.

Changing slightly the point of view, one has another promising research line, whose basic theme is to find the geometric locus of points P for which the quadrilateral obtained by choosing a point in each of the triangles determined by P and an edge of the given quadrilateral has a specific property. The chances to obtain interesting results in this manner are improved if one starts from a configuration richer in potentially useful properties.

To conclude the discussion, we end this section by stating the following research problem.

Publication Date: November 15, 2011. Communicating Editor: Paul Yiu.

The author thanks an anonymous referee for valuable suggestions incorporated in the present version of the paper.

Problem A. *For a point P in the plane of a square $ABCD$, denote by E, F, G, and H the triangle center $X(i)$ in Kimberling's Encyclopedia of triangle centers* [3] *of triangles ABP, BCP, CDP, DAP, respectively. Find the geometrical locus of points P for which $EFGH$ is a cyclic quadrilateral.*

This problem has as many instances as entries in Kimberling's list. It is plausible that they are of various degrees of difficulty to solve. Indeed, in the next section we present solutions for four of the possible specializations of Problem A, and we shall have convincing samples of different techniques needed in the proofs. The goal is to persuade the reader that Problem A is worth studying. Some results are valid for more general quadrangle $ABCD$. From their proofs we learn to what extent the requirement "$ABCD$ square" is a sensible one.

2. Four results

The instance of Problem A corresponding to $X(2)$ (centroid) is easily disposed of. The answer follows from the next result, probably already known. Having no suitable reference, we present its simple proof.

Theorem 1. *Let P be a point in the plane of a quadrilateral $ABCD$, and let E, F, G, and H be the centroid of triangles ABP, BCP, CDP, DAP, respectively. Then $EFGH$ is a parallelogram. In particular, $EFGH$ is cyclic if and only if AC and BD are perpendicular.*

Proof. The quadrilateral $EFGH$ is the image of the Varignon parallelogram under the homothety $\mathsf{h}(P, \frac{2}{3})$, hence is a parallelogram. A parallelogram is cyclic exactly when it is rectangle. □

Notice that the condition "perpendicular diagonals" is necessary and sufficient for $EFGH$ to be a cyclic quadrilateral. This remark shows that for arbitrary quadrilaterals the sought-for geometrical locus may be empty. Therefore, Problem A is interesting under a minimum of conditions.

Proving the next theorem is more complicated and requires a different approach. As the proof is based on the intersecting chords theorem, we recall its statement: In the plane, let points I, J, K, L with $I \neq J$ and $K \neq L$ be such that lines IJ and KL intersect at Ω. Then, I, J, K, L are concyclic if and only if $\overrightarrow{\Omega I} \cdot \overrightarrow{\Omega J} = \overrightarrow{\Omega K} \cdot \overrightarrow{\Omega L}$.

Theorem 2. *For a point P in the plane of a rectangle $ABCD$, denote by E, F, G, H the circumcenters of triangles ABP, BCP, CDP, DAP, respectively. The geometrical locus of points P for which $EFGH$ is a cyclic quadrilateral consists of the circumcircle of the rectangle and the real hyperbola passing through A, B, C, D and whose asymptotes are the bisectors of rectangle's symmetry axes. In particular, if $ABCD$ is a square, the geometrical locus consists of the diagonals and the circumcircle of the square.*

Proof. We first look for P lying on a side of the rectangle. For the sake of definiteness, suppose that P belongs to the line AB and $P \neq A$, B. Then E is at infinity, F and H sit on the perpendicular bisector of side BC, and G is on the perpendicular bisector of side CD. Suppose P belongs to the desired geometrical locus.

The condition E, F, G, H concyclic is tantamount to F, G, H collinear, which in turn amounts to G being the rectangle's center O. It follows that $PO = OA$, which is true only when P is one of the vertices A, B. If $P = A$, say, then E is not determined, it can be anywhere on the perpendicular bisector of the segment AB. Also, H is an arbitrary point on the perpendicular bisector of the segment DA, while both F and G coincide with the center of the rectangle. Therefore, $EFGH$ degenerates to a rectangular triangle. Thus, the only points common to the geometric locus and to sides are the vertices of the rectangle.

We now examine the points from the geometrical locus which do not lie on the sides of the rectangle. We choose a coordinate system with origin at the center and axes parallel to the sides of the given rectangle. Then the vertices have the coordinates $A(-a,-b)$, $B(a,-b)$, $C(a,b)$, $D(-a,b)$ for some a, $b > 0$. Let $P(u,v)$ be a point, not on the sides of the rectangle. Routine computations yield the circumcenters:

$$E\left(0, \frac{u^2+v^2-a^2-b^2}{2(v+b)}\right), \quad F\left(\frac{u^2+v^2-a^2-b^2}{2(u-a)}, 0\right),$$
$$G\left(0, \frac{u^2+v^2-a^2-b^2}{2(v-b)}\right), \quad H\left(\frac{u^2+v^2-a^2-b^2}{2(u+a)}, 0\right).$$

If P is on the circle $(ABCD)$, *i.e.*, $u^2+v^2-a^2-b^2=0$, then E, F, G, H coincide with $O(0,0)$. If P is not on the circle $(ABCD)$, then $E \neq G$, $F \neq H$, and EG, FH intersect at O. Thus, E, F, G, H are concyclic if and only if $\overrightarrow{OE} \cdot \overrightarrow{OG} = \overrightarrow{OF} \cdot \overrightarrow{OH}$. Since $u^2+v^2-a^2-b^2 \neq 0$, we readily see that this is equivalent to $u^2-v^2=a^2-b^2$. In conclusion, the desired locus is the union of the circle $(ABCD)$ and the hyperbola described in the statement.

In the case of a square, the hyperbola becomes $u^2=v^2$, which means that P belongs to one of the square's diagonals. $\square$

The above proof shows that one can ignore from the beginning points on the sidelines of $ABCD$: if P is on such a sideline, one of the triangles is degenerate. Such degeneracy would make difficult even to understand the statement of some instances of Problem A.

The differences between rectangles and squares are more obvious when examining the case of $X(4)$ (orthocenter) in Problem A.

Theorem 3. *In a plane endowed with a Cartesian coordinate system centered at O, consider a rectangle $ABCD$ with sides parallel to the axes and of length $|AB| = 2a$, $|BC| = 2b$, and diagonals intersecting at O. For a point P in the plane, not on the sidelines of $ABCD$, denote by E, F, G, H the orthocenters of triangles ABP, BCP, CDP, DAP, respectively. Let $\mathcal{C}$ denote the geometrical locus of points P for which $EFGH$ is a cyclic quadrilateral. Then:*

(a) *If $a \neq b$, then $\mathcal{C}$ is the union of the hyperbola $u^2-v^2=a^2-b^2$ and the sextic*

$$\begin{aligned}&(u^2+v^2+a^2-b^2)^2(v^2-b^2)-(u^2+v^2-a^2+b^2)^2(u^2-a^2)\\&=4a^2u^2(v^2-b^2)-4b^2v^2(u^2-a^2)\end{aligned} \tag{1}$$

from which points on the sidelines of $ABCD$ are excluded.
(b) *If $ABCD$ is a square, then $\mathcal{C}$ consists of the points on the diagonals or on the circumcircle of the square which are different from the vertices of the square.*

Proof. To take advantage of computations already performed, we use the fact that the coordinates of the orthocenter are the sums of the coordinates of the vertices with respect to a coordinate system centered in the circumcenter. Having in view the previous proof and the restriction on P, we readily find

$$E\left(u, \frac{a^2-u^2}{v+b}-b\right), \quad F\left(\frac{b^2-v^2}{u-a}+a, v\right),$$
$$G\left(u, \frac{a^2-u^2}{v-b}+b\right), \quad H\left(\frac{b^2-v^2}{u+a}-a, v\right).$$

First, using the coordinates of E, F, G, H, a short calculation shows that

$$E = G \Longleftrightarrow F = H \Longleftrightarrow u^2 - v^2 = a^2 - b^2.$$

It follows that the points on the hyperbola $u^2 - v^2 = a^2 - b^2$ which are different from A, B, C, D belong to the locus. Now, consider a point P not on this hyperbola. Then $E \neq G$, $F \neq H$, and lines EG, FH intersect at P. Thus, E, F, G, H are concyclic if and only if $\overrightarrow{PE} \cdot \overrightarrow{PG} = \overrightarrow{PF} \cdot \overrightarrow{PH}$. This yields the equation (1) of the sextic.

It is worth noting that the points of intersection (other than A, B, C, D) of any two of the circles with diameters AB, BC, CD, DA are in $\mathcal{C}$, either on the hyperbola or on the sextic.

If $ABCD$ is a square of sidelength 2, the equations of these curves simplify to $u^2 = v^2$ and

$$(u^2 - v^2)\big((u^2 + v^2)^2 - 4\big) = 0,$$

giving the diagonals and the circumcircle of the square. □

Figures 1 to 3 illustrate Theorem 3. These graphs convey the idea that the position of real points on the sextic depends on the rectangle's shape.

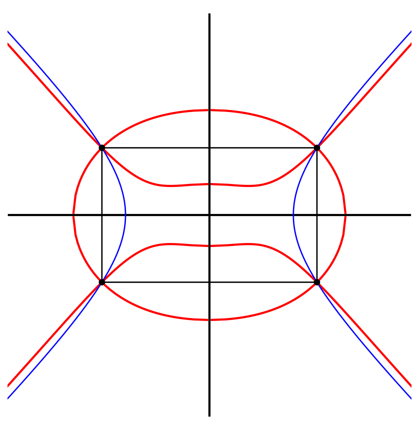

Figure 1. Locus of P in Theorem 3: Rectangle with lengths $\frac{16}{5}$ and 2

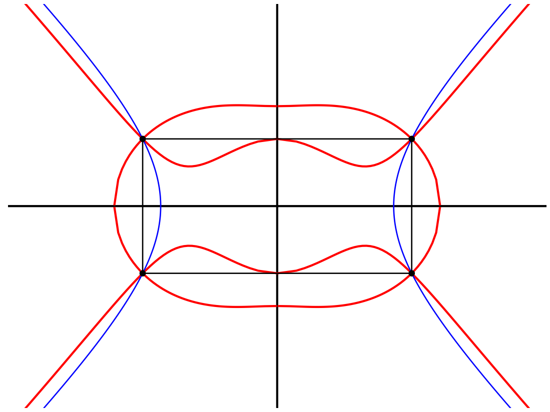

Figure 2. Locus of P in Theorem 3: Rectangle with sides 4 and 2

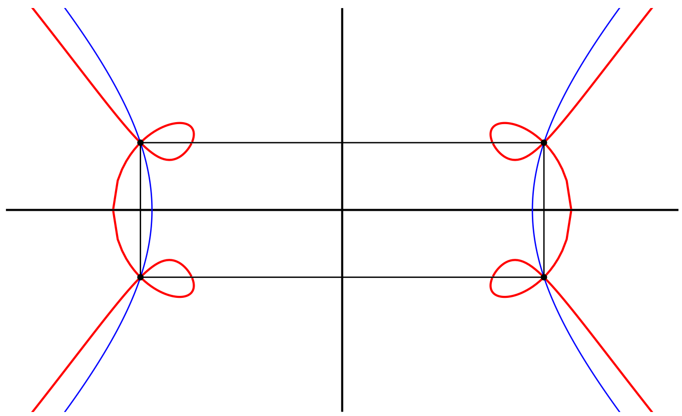

Figure 3. Locus of P in Theorem 3: Rectangle with sides 6 and 2

Solving Problem A for rectangle instead of square would rise great difficulties. As seen above, we might loose the support of geometric intuition, the answer might be phrased in terms of algebraic equation whose complexity would make a complete analysis very laborious. It would be very difficult to decide whether various algebraic curves appearing from computation have indeed points in the sought-for geometrical locus, or even if the locus is nonempty. Therefore we chose to state Problem A for square only. This hypothesis assures that in any instance of Problem A there are points P with the desired property. Namely, if P is on one of the diagonals of the square, that diagonal is a symmetry axis for the configuration. Thus $EFGH$ is an isosceles trapezium, which is certainly cyclic. The argument is valid for P different from the vertices of the square. Therefore, the diagonals (with the possible exception of square's vertices) are included in the geometrical locus.

The proof of Theorem 3 is more difficult than the previous ones. Yet it is much easier than the proof of the next result, which gives a partial answer to the problem obtained by specializing Problem A to incenters.

Theorem 4. *For a point P in the interior of a square $ABCD$, one denotes by E, F, G, and H the incenter of triangle ABP, BCP, CDP, DAP, respectively. The geometrical locus of points P for which $EFGH$ is a cyclic quadrilateral consists of the diagonals AC and BD.*

This settles a conjecture put forward more than 25 years ago by Daia [2]. The only proof we are aware of has been just published [1] and is similar to the proof of Theorem 3. In order to decide whether E, F, G, H are concyclic, Ptolemy's theorem rather than the intersecting chords theorem was used. The crucial difference is the complexity of expressions yielding the coordinates of the incenters in terms of the coordinates of the additional point P. This difference is huge, the required computations can not be performed without computer assistance. For instance, after squaring twice the equality stated by Ptolemy's theorem, one gets $16f^2 = 0$, where f is a polynomial in 14 variables having 576 terms. The algorithms employed to manipulate such large expressions belong to the field generally known as symbolic computation. Even powerful computer algebra systems like MAPLE and SINGULAR running on present-day machines needed several hours and a large amount of memory to complete the task. The output consists of several dozens of polynomial relations satisfied by the variables describing the geometric configuration. In order to obtain a geometric interpretation for the algebraic translation of the conclusion it was essential to use the hypothesis that P sits in the interior of the square. In algebraic terms, this means that the real roots of the polynomials are positive and less than one. In the absence of such an information, it is not at all clear that the real variety describing the asked geometrical locus is contained in the union of the two diagonals.

Full details of the proof for Theorem 4 are given in [1]. As the quest for elegance and simplicity is still highly regarded by many mathematicians, we would like to have a less computationally involved proof to Theorem 4. It is to be expected that a satisfactory solution to this problem will be not only more conceptual but also more enlightening than the approach sketched in the previous paragraph. Moreover, it is hoped that the new ideas needed for such a proof will serve to remove the hypothesis "P in the interior of the square" from the statement of Theorem 4.

3. Conclusions

Treating only four instances, this article barely scratches the surface of a vast research problem. Since as of June 2011 the Encyclopedia of Triangle Centers lists more than 3600 items, it is apparent that a huge work remains to be done.

The approach employed in the proofs of Theorems 2 and 3 seems promising. It consists of two phases. First, one identifies the set containing the points P for which the four centers fail to satisfy the hypothesis of the intersecting chords theorem. This set is contained in the desired locus unless the corresponding quadrilateral $EFGH$ is a non-isosceles trapezium. Next, for points P outside the set one uses the intersecting chords theorem.

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Forum Geometricorum
Volume 11 (2011) 231–236.

FORUM GEOM
ISSN 1534-1178

The Distance from the Incenter to the Euler Line

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Abstract. It is well known that the incenter of a triangle lies on the Euler line if and only if the triangle is isosceles. A natural question to ask is how far the incenter can be from the Euler line. We find least upper bounds, across all triangles, for that distance relative to several scales. Those bounds are found relative to the semi-perimeter of the triangle, the length of the Euler line and the circumradius, as well as the length of the longest side and the length of the longest median.

1. Introduction

A quiet thread of interest in the relationship of the incenter to the Euler line has persisted to this day. Given a triangle, the Euler line joins the circumcenter, O, to the orthocenter, H. The centroid, G, trisects this line (being closer to O) and the center of the nine-point circle, N, bisects it. It is known that the incenter, I, of a triangle lies on the Euler line if and only if the triangle is isosceles (although proofs of this fact are thin on the ground). But you can't just choose any point, on or off the Euler line, to be the incenter of a triangle. The points you can choose are known, as will be seen. An obvious question asks how far can the incenter be from the Euler line. For isosceles triangles the distance is 0. Clearly this question can only be answered relative to some scale, we will consider three scales: the length of the Euler line, $\mathcal{E}$, the circumradius, R, and the semiperimeter, s. Along the way we will see that the answer for the semiperimeter also gives us the answer relative to the longest side, μ, and the longest median, ν. To complete the list of lengths, let d be the distance of the incenter from the Euler line.

Time spent playing with triangles using any reasonable computer geometry package will convince you that the following are reasonable conjectures.

$$\frac{d}{\mathcal{E}} \le \frac{1}{3}, \qquad \frac{d}{R} \le \frac{1}{2} \qquad \text{and} \qquad \frac{d}{s} \le \frac{1}{3}$$

Maybe with strict inequalities, but then again the limits might be attained.

A large collection of relationships between the centers of a triangle is known, for example, if R is the radius of the circumcircle and r the radius of the incircle,

Publication Date: November 21, 2011. Communicating Editor: Paul Yiu.

then we have

$$\begin{aligned} OI^2 &= R(R-2r) \\ IN &= \frac{1}{2}(R-2r) \end{aligned}$$

Before moving on, it is worth noting that the second of the above gives an immediate upper bound for the distance relative to the circumradius. As the inradius of a non-degenerate triangle must be positive we have $d \leq IN = \frac{R}{2} - r < \frac{R}{2}$, and hence

$$\frac{d}{R} < \frac{1}{2}.$$

2. Relative to the Euler line

The relationships given above, and others, can be used to show that for any triangle the incenter, I, must lie within the *orthocentroidal circle* punctured at the center of the nine-point circle, N, namely, the disk with diameter GH except for the circumference and the point N.

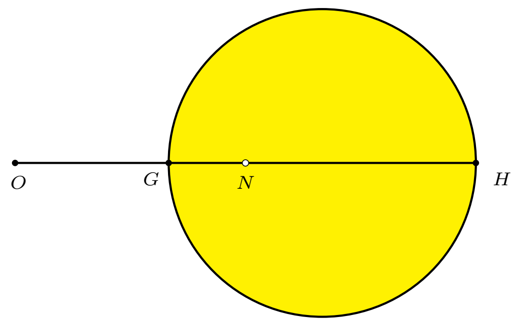

Figure 1

In 1984 Guinand [1] showed that every such point gives rise to a triangle which has the nominated points as its centers. Guinand shows that if $OI = \rho$, $IN = \sigma$ and $OH = \kappa$ then the cosines of the angles of the triangle we seek are the zeros of the following cubic.

$$p(c) = c^3 + \frac{3}{2}\left(\frac{4\sigma^2}{3\rho^2} - 1\right)c^2 + \frac{3}{4}\left(-\frac{2\kappa^2\sigma^2}{3\rho^4} + \frac{8\sigma^4}{3\rho^4} - \frac{4\sigma^2}{\rho^2} + 1\right)c + \frac{1}{8}\left(\frac{4\kappa^2\sigma^2}{\rho^4} - 1\right).$$

Stern [2] approached the problem using complex numbers and provides a simpler derivation of a cubic, and explicitly demonstrates that the triangle found is unique. His approach also provided the vertices directly, as complex numbers.

Consideration of the orthocentroidal circle provides the answer to our question relative to $\mathcal{E}$, the length of the Euler line. The incenter must lie strictly within the orthocentroidal circle which has radius one third the length of the Euler line. Guinand has proved that all such points, except N, lead to a suitable triangle. Thus the least upper bound, over all non-degenerate triangles, of the ratio $\frac{d}{\mathcal{E}}$ is $\frac{1}{3}$, with triangles approaching this upper-bound being defined by having incenters close to the points on circumference of the orthocentroidal circle on a radius perpendicular to the Euler line. For any given non-degenerate triangle we obtain the strict inequality $\frac{d}{\mathcal{E}} < \frac{1}{3}$.

Consideration of Figure 2 gives us more information. Taking $OH = 3$ as our scale. For triangles with I close to the limit point above, the angle IGH is close to $\frac{\pi}{4}$. Moreover, with I near that point, a calculation using the inferred values of $OI \approx \sqrt{5}$ and $IN \approx \frac{\sqrt{5}}{2}$ shows that the circumradius will be close to $\sqrt{5}$, and the inradius will be close to 0.

We observed above that $IN < \frac{R}{2}$. This distance only becomes relevant for us if IN is perpendicular to the Euler line. Consideration of the orthocentroidal circle again allows us to see that this may happen, with the angle IGH being close to $\frac{\pi}{3}$. In this case the circumradius will be close to $\sqrt{3}$.

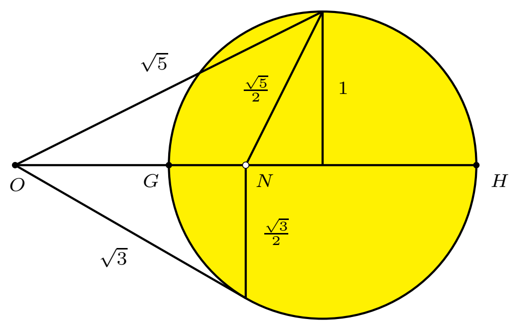

Figure 2

Remark. It is easy to see that the last case also gives the least upper bound of the angle IOH as $\frac{\pi}{6}$.

3. Relative to the triangle

We now wish to find the maximal distance relative to the dimensions of the triangle itself. The relevant dimensions will be the length of the longest median, ν, the length of the longest side, μ, and the semiperimeter, s. It is clear that $\nu < \mu < s$ (see Lemma 4 below).

The following are well-known, and show that the incenter and centroid lie within the medial triangle, the triangle formed by the three midpoints of the sides.

Lemma 1. *The incenter, I, lies in the medial triangle.*

Lemma 2. *The centroid of triangle ABC is the centroid of the medial triangle.*

Lemma 3. *The distance from the incenter to the centroid is less than one third the length of the longest median of the triangle.*

Proof. We have just shown that both the incenter and centroid lie inside the medial triangle. Therefore the distance from the incenter to the centroid is less than the largest distance from the centroid to a vertex of the medial triangle. (Consider the circle centered at O passing through the most distant vertex.)

Now the distance of the centroid from the vertices of the medial triangle is, by definition, the distance from the centroid to the mid-points of the side of triangle ABC. Those distances are equal to one third the lengths of the medians, and the result follows. □

Lemma 4. *The length of a median is less than μ. Hence, $\nu < \mu < s$.*

Proof. Consider the median from A. If we rotate the triangle through π about M_A, the mid-point of the side opposite A, we obtain the parallelogram $ABDC$. The diagonal AD has twice the length AM_A. As A, B and D form a non-degenerate triangle we have

$$2AM_A = AD < AB + BD = AB + AC \leq 2\mu,$$

where μ is the length of the longest side. Thus the median $AM_A < \mu$. This is also true for the other two medians. Thus, $\nu \leq \mu$. □

Proposition 5. *The distance, d, from the incenter to the Euler line satisfies*

$$\frac{d}{s} < \frac{d}{\mu} < \frac{d}{\nu} < \frac{1}{3},$$

where ν is the length of the longest median, μ is the length of the longest side and s is the semi-perimeter of the triangle.

Proof. As the centroid lies on the Euler line, the distance from the incenter to the Euler line is at most the distance from the incenter to the centroid. By Lemma 3, this distance is one third the length of the longest median. But, by Lemma 4, the length of each median is less than $\mu < s$, and the result follows. □

4. In the limit

As the expressions $\frac{d}{\mathcal{E}}$, $\frac{d}{R}$ and $\frac{d}{s}$ are dimensionless we may choose our scale as suits us best. Consider the triangle with vertices $(0,0)$, $(1,0)$ and (ε,δ), where ε and δ are greater than but approximately equal to 0. The following information may be easily checked.

The coordinates of the orthocenter are

$$H\left(\varepsilon, \frac{\varepsilon-\varepsilon^2}{\delta}\right).$$

The coordinates of the circumcenter are

$$O\left(\frac{1}{2}, \frac{\delta^2+\varepsilon^2-\varepsilon}{2\delta}\right).$$

The Euler line has equation

$$l_{OH}: \quad \left(-\delta^2+3(1-\varepsilon)\varepsilon\right)x + (1-2\varepsilon)\delta y + \varepsilon\left(\delta^2+\varepsilon^2-1\right) = 0.$$

If we let $p = 2s = \sqrt{\delta^2+\varepsilon^2} + \sqrt{\delta^2+(1-\varepsilon)^2} + 1$, then the coordinates of the incenter are

$$I\left(\frac{\sqrt{\delta^2+\epsilon^2}+\epsilon}{p}, \frac{\delta}{p}\right).$$

We may now write down the value of d, being the perpendicular distance from I to l_{OH}.

$$d = \frac{\left|\left(-\delta^2+3(1-\varepsilon)\varepsilon\right)\left(\sqrt{\delta^2+\varepsilon^2}+\varepsilon\right) + (1-2\varepsilon)\delta^2 + p\varepsilon\left(\delta^2+\varepsilon^2-1\right)\right|}{p\sqrt{\left(-\delta^2+3(1-\varepsilon)\varepsilon\right)^2+(1-2\varepsilon)^2\delta^2}}.$$

Suppose we let $\delta = \varepsilon^2$, then the expression for the ratio $\frac{d}{s}$ is

$$\frac{2\varepsilon\left|(-\varepsilon^3-3\varepsilon+3)(\sqrt{\varepsilon^4+\varepsilon^2}+\varepsilon)+\varepsilon(\varepsilon^2-2\varepsilon^3)+p(\varepsilon^4-\varepsilon^2-1)\right|}{p^2\varepsilon\sqrt{(-\varepsilon^3-3(\varepsilon-1))^2+(\varepsilon-2\varepsilon^2)^2}}$$

We cancel the common factor of ε and take the limit as $\varepsilon \to 0$. Noting that $p \to 2$ we see that the numerator approaches 4 while the denominator approaches 12, and we have proved the following.

Theorem 6. *If d is the distance from the incenter to the Euler line, s the semi-perimeter, μ the length of the longest side and ν the length of the longest median, then the least upper bound of $\frac{d}{s}$, and hence $\frac{d}{\mu}$ and $\frac{d}{\nu}$, over all non-degenerate triangles is $\frac{1}{3}$.*

Remark. In those cases where the distance ratio is close to the maximum, the line IG is nearly perpendicular to the Euler line. Thus the angle IGH will be close to $\frac{\pi}{2}$. In these cases the Euler line is extremely large compared to the triangle.

Similar calculations can be carried out for the ratios $\frac{d}{\mathcal{E}}$ and $\frac{d}{R}$. In those cases we take the point (ε, δ) to be a point on the circle through $(0,0)$ and $(1,0)$ with radius $\frac{\sqrt{10}}{6}$, or $\frac{\sqrt{3}}{3}$ respectively (remember that the values of $\sqrt{5}$ and $\sqrt{3}$ met earlier were relative to the length of the Euler line, not the length of a side).

5. Demonstrating the limits

We now have enough information to assist us in constructing diagrams that will demonstrate these limits using a suitable computer geometry package.

Taking the case of triangles with the ratio $\frac{d}{R}$ approaching $\frac{1}{2}$. Let AB be a line segment and define its length to be 1. Let G' be the point on AB one third of the way from A to B. Construct the line $G'T$ such that $\angle BG'T = \frac{\pi}{3}$ and let O be the point where this line meets the perpendicular bisector of AB. Draw the arc AB centered at O and let C be a point on that arc. Constructing the Euler line and incenter of triangle ABC will demonstrate that the ratio $\frac{d}{R}$ approaches $\frac{1}{2}$ as C approaches A. This construction is explained if you note that G' is the limiting position of the centroid, G, as C approaches A (see Figure 3).

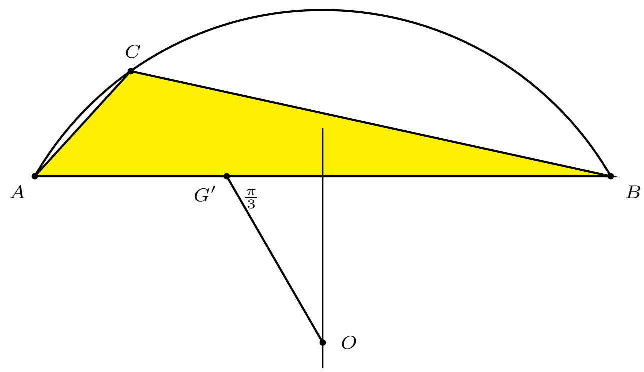

Figure 3.

A similar construction, except with $\angle BG'T = \frac{\pi}{4}$ will give a demonstration that $\frac{d}{\mathcal{E}}$ approaches $\frac{1}{3}$ as C approaches A.

Something different is required to demonstrate that $\frac{d}{s}$ approaches $\frac{1}{3}$. Given AB above, choose a point C' between A and B and let the length $AC' = \varepsilon$, with $0 < \varepsilon < 1$. Construct the perpendicular at C' and find the point C on the perpendicular with $CC' = \varepsilon^2$. Constructing the Euler line and incenter of this triangle will demonstrate that the ratio $\frac{d}{s}$ approaches $\frac{1}{3}$ as C approaches A.

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Forum Geometricorum
Volume 11 (2011) 237–249.

FORUM GEOM
ISSN 1534-1178

Rational Steiner Porism

Paul Yiu

Abstract. We establish formulas relating the radii of neighboring circles in a Steiner chain, a chain of mutually tangent circles each tangent to two given ones, one in the interior of the other. From such we parametrize, for $n = 3,\ 4,\ 6$, all configurations of Steiner n-cycles of rational radii.

1. Introduction

Given a circle $O(R)$ and a circle $I(r)$ in its interior, we write the distance d between their centers in the form

$$d^2 = (R-r)^2 - 4qRr. \tag{1}$$

It is well known [3, pp.98–100] that there is a closed chain of n mutually tangent circles each tangent internally to (O) and externally to (I) if and only if $q = \tan^2\frac{\pi}{n}$. In this case we have a Steiner n-cycle, and such an n-cycle can be constructed beginning with any circle tangent to both (O) and (I). In this note we study the possibilities that the two given circles have *rational* radii and distance between their centers. Such is called a rational Steiner pair. Since $\cos\frac{2\pi}{n} = \frac{1-q}{1+q}$, we must have $\cos\frac{2\pi}{n}$ rational. By a classic theorem in algebraic number theory, $\cos\frac{2}{n}\pi$ is rational only for $n = 3, 4, 6$ (see, for example, [2, p.41, Corollary 3.12]). It follows that rational Steiner pairs exist only for $q = 3,\ 1,\ \frac{1}{3}$, corresponding to $n = 3, 4, 6$. We shall give a parametrization of such pairs and proceed to show how to construct Steiner n-cycles consisting of circles of rational radii. Here are some examples of symmetric rational Steiner n-cycles for these values of n.

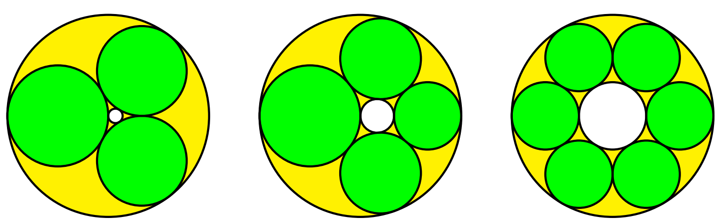


Figure 1. Symmetric Steiner n-cycles for $n = 3,\ 4,\ 6$

q	n	(R,r,d)	Radii $(\rho_1,\dots,\rho_n)$
3	3	$(14,1,1)$	$\left(7,\ \frac{56}{9},\ \frac{56}{9}\right)$
1	4	$(6,1,1)$	$\left(3,\ \frac{12}{5},\ 2,\ \frac{12}{5}\right)$
$\frac{1}{3}$	6	$(6,1,0)$	$(1,1,1,1,1,1)$

Publication Date: November 30, 2011. Communicating Editor: Li Zhou.
The author thanks Li Zhou for comments and suggestions leading to improvements over an earlier version of this paper.

2. Construction of Steiner chains

In this section we consider two circles $O(R)$ and $I(r)$ with centers at a distance d apart, without imposing any relation on R, r, d, nor rationality assumption. By a Steiner circle we mean one which is tangent to both (O) and (I). We shall assume $d \neq 0$ so that the circles (O) and (I) are not concentric. Clearly there are unique Steiner circles of radii $\rho_0 := \frac{R-r-d}{2}$ and $\rho_1 := \frac{R-r+d}{2}$. For each $\rho \in (\rho_0,\ \rho_1)$, there are exactly two Steiner circles of radius ρ symmetric in the center line OI. The center of each is at distances $R - \rho$ from O and $r + \rho$ from I.

Proposition 1. *If $A(\rho)$ is a Steiner circle tangent to (O) at P and (I) at Q, then the line PQ contains T_+, the internal center of similitude of (O) and (I).*

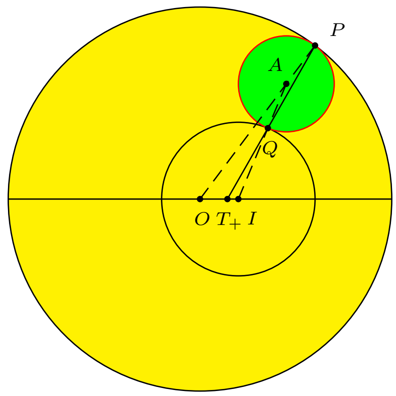


Figure 2.

Proof. Note that A divides OP internally in the ratio $OA : AP = R - \rho : \rho$, so that

$$A = \frac{\rho \cdot O + (R-\rho)P}{R}.$$

Similarly, the same point A divides IQ externally in the $IA : AQ = r + \rho : -\rho$, so that

$$A = \frac{-\rho \cdot I + (r+\rho)Q}{r}.$$

Eliminating A from these two equations, and rearranging, we obtain

$$\frac{-r(R-\rho)P + R(r+\rho)Q}{(R+r)\rho} = \frac{R \cdot I + r \cdot O}{R+r}.$$

This equation shows that a point on the line PQ is the same as a point on the line OI, which is the intersection of the lines PQ and OI. Note that the point on the line OI is independent of P. It is the internal center of similitude T_+ of the two circles, dividing O and I internally in the ratio $OT_+ : T_+I = R : r$. □

Remark. The point T_+ can be constructed as the intersection of the line OI with the line joining the endpoints of a pair of oppositely parallel radii of the circles.

Two Steiner circles are neighbors if they are tangent to each other externally.

Proposition 2. *If two neighboring Steiner circles are tangent to each other at T, then T lies on a circle with center T_+.*

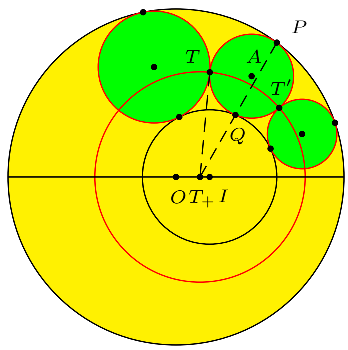


Figure 3

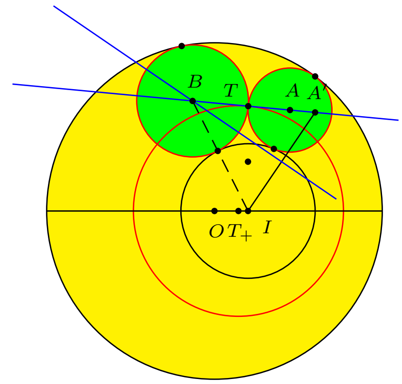


Figure 4

Proof. Applying the law of cosines to triangles POT_+ and AOI, we have

$$\frac{R^2 + \left(\frac{Rd}{R+r}\right)^2 - T_+P^2}{2R \cdot \frac{Rd}{R+r}} = \frac{(R-\rho)^2 + d^2 - (r+\rho)^2}{2(R-\rho)d}.$$

From this,

$$T_+P^2 = \frac{R^2((R+r)^2 - d^2)}{(R+r)^2} \cdot \frac{r+\rho}{R-\rho}.$$

Similarly,

$$T_+Q^2 = \frac{r^2((R+r)^2 - d^2)}{(R+r)^2} \cdot \frac{R-\rho}{r+\rho}.$$

It follows that

$$T_+P \cdot T_+Q = Rr \cdot \frac{(R+r)^2 - d^2}{(R+r)^2}.$$

If we put $t^2 = Rr \cdot \frac{(R+r)^2 - d^2}{(R+r)^2}$ (independent on ρ), then the circle $T_+(t)$ intersects the Steiner circle $A(\rho)$ at a point T such that TT_+ is tangent to the Steiner circle (see Figure 3). □

This leads to an easy construction of the neighbor of $A(\rho)$ tangent at T (see Figure 4):

(1) Extend TA to A' such that $TA' = r$.

(2) Construct the perpendicular bisector of IA' to intersect the line AT at B.

Then the circle $B(T)$ is the Steiner circle tangent to $A(\rho)$ at T.

3. Radii of neighboring Steiner circles

Henceforth we write

$$(R, r, d;\ \rho_1,\ \dots,\ \rho_n)$$

for a rational Steiner pair $(R, r, d)_q$ and an n-cycles of Steiner circles with rational radii $\rho_1, \dots, \rho_n$. To relate the radii of neighboring Steiner circles, we make use of the following results.

Lemma 3. (a) $\rho_0\rho_1 = qRr$,
(b) $(R - \rho_0)(R - \rho_1) = (q + 1)Rr$.

Proposition 4 (Bottema [1]). *Given a triangle with sidelengths a_1, a_2, a_3, the distances d_1, d_2, d_3 from the opposite vertices of these sides to a point in the plane of the triangle satisfy the relation*

$$\begin{vmatrix} 2d_1^2 & -a_3^2 + d_1^2 + d_2^2 & -a_2^2 + d_3^2 + d_1^2 \\ -a_3^2 + d_1^2 + d_2^2 & 2d_2^2 & -a_1^2 + d_2^2 + d_3^2 \\ -a_2^2 + d_3^2 + d_1^2 & -a_1^2 + d_2^2 + d_3^2 & 2d_3^2 \end{vmatrix} = 0.$$

Proposition 5. *Let $A(\rho)$ be a Steiner circle between (O) and (I). The radii of its two neighbors are the roots of the quadratic polynomial $a\sigma^2 + b\sigma + c$, where*

$$\begin{aligned} a =&\ ((q + 1)Rr - (R - r)\rho)^2 + 4Rr\rho^2, \\ b =&\ 2(q + 1)Rr\rho((q - 1)Rr - (R - r)\rho), \\ c =&\ (q + 1)^2R^2r^2\rho^2. \end{aligned} \tag{2}$$

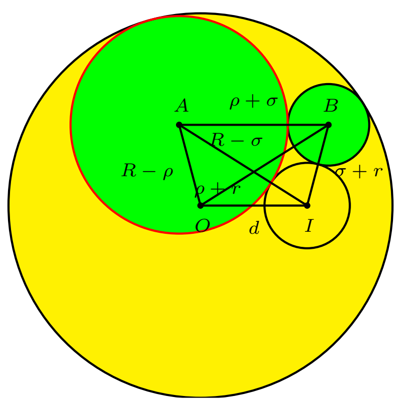


Figure 5

Proof. Let $B(\sigma)$ be a neighbor of $A(\rho)$. Apply Proposition 4 to triangle IAB with sides $\rho + \sigma$, $r + \sigma$, $r + \rho$, and the point O whose distances from I, A, B are respectively d, $R - \rho$, $R - \sigma$.

$$\begin{vmatrix} 2d^2 & R^2 - r^2 + d^2 - 2(R + r)\rho & R^2 - r^2 + d^2 - 2(R + r)\sigma \\ R^2 - r^2 + d^2 - 2(R + r)\rho & 2(R - \rho)^2 & 2(R(R - \rho) - (R + \rho)\sigma) \\ R^2 - r^2 + d^2 - 2(R + r)\sigma & 2(R(R - \rho) - (R + \rho)\sigma) & 2(R - \sigma)^2 \end{vmatrix} = 0.$$

It is clear that the determinant is a quadratic polynomial in σ. With the relation (1), we eliminate d and obtain

$$\begin{vmatrix} (R-r)^2-4qRr & R^2-Rr-2qRr-(R+r)\rho & R^2-Rr-2qRr-(R+r)\sigma \\ R^2-Rr-2qRr-(R+r)\rho & (R-\rho)^2 & R(R-\rho)-(R+\rho)\sigma \\ R^2-Rr-2qRr-(R+r)\sigma & R(R-\rho)-(R+\rho)\sigma & (R-\sigma)^2 \end{vmatrix} = 0.$$

This determinant, apart from a factor -4, is $a\sigma^2+b\sigma+c$, with coefficients given by (2) above. □

Lemma 6. $b^2-4ac = 16(q+1)^2R^3r^3\rho^2(\rho_1-\rho)(\rho-\rho_0)$.

Proof. $b^2-4ac = 4(q+1)^2R^2r^2\rho^2\cdot D$, where

$$\begin{aligned} D =& \; ((q-1)Rr-(R-r)\rho)^2-(((q+1)Rr-(R-r)\rho)^2+4Rr\rho^2) \\ =& \; 4Rr(-qRr+(R-r)\rho-\rho^2) \\ =& \; 4Rr(-\rho_0\rho_1+(\rho_0+\rho_1)\rho-\rho^2) \\ =& \; 4Rr(\rho_1-\rho)(\rho-\rho_0). \end{aligned}$$

□

Proposition 7. *The radius ρ of a Steiner circle and those of its two neighbors are rational if and only if*

$$\rho = \mathcal{R}(\tau) := \frac{\tau^2\rho_0+Rr\rho_1}{\tau^2+Rr} \tag{3}$$

for some rational number τ.

Proof. The roots of the quadratic polynomial $a\sigma^2+b\sigma+c$ are rational if and only if b^2-4ac is the square of a rational number. With a, b, c given in (2), this discriminant is given by Lemma 6. Writing $Rr(\rho_1-\rho)(\rho-\rho_0) = \tau^2(\rho-\rho_0)^2$ for a rational τ leads to the rational expression (3) above. □

Theorem 8. *For a Steiner circle with rational radius $\rho=\mathcal{R}(\tau)$, the two neighbors have radii $\rho_+=\mathcal{R}(\tau_+)$ and $\rho_-=\mathcal{R}(\tau_-)$ where*

$$\tau_+ = \frac{Rr(\tau-\rho_1)}{Rr+\tau\rho_0} \quad \text{and} \quad \tau_- = \frac{Rr(\tau+\rho_1)}{Rr-\tau\rho_0}.$$

Proof. With ρ given by (3), we have
(i) $\rho_1-\rho = \frac{\tau^2(\rho_1-\rho_0)}{\tau^2+Rr}$ and $\rho-\rho_0 = \frac{Rr(\rho_1-\rho_0)}{\tau^2+Rr}$,
(ii) from (2)

$$\begin{aligned} b =& \; 2(q+1)Rr\rho((q-1)Rr-(\rho_0+\rho_1)\rho), \\ =& \; 2(q+1)Rr\rho\left(-R^2+R(\rho_0+\rho_1)+\rho_0\rho_1-(\rho_0+\rho_1)\cdot\frac{\tau^2\rho_0+Rr\rho_1}{\tau^2+Rr}\right) \\ =& \; -2(q+1)Rr\rho\cdot\frac{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr}{\tau^2+Rr}, \end{aligned}$$

(iii) by Lemma 6,

$$b^2-4ac=\frac{16(q+1)^2R^4r^4\rho^2(\rho_1-\rho_0)^2\tau^2}{(\tau^2+Rr)^2}.$$

The roots of the quadratic polynomial $a\sigma^2+b\sigma+c$ are

$$\begin{aligned}\frac{2c}{-b-\varepsilon\sqrt{b^2-4ac}} &= \frac{2(q+1)^2R^2r^2\rho^2}{2(q+1)Rr\rho\cdot\frac{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr}{\tau^2+Rr}-\varepsilon\frac{4(q+1)R^2r^2\rho(\rho_1-\rho_0)\tau}{\tau^2+Rr}} \\ &= \frac{(q+1)Rr\rho}{\frac{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr}{\tau^2+Rr}-\varepsilon\frac{2Rr(\rho_1-\rho_0)\tau}{\tau^2+Rr}} \\ &= \frac{(q+1)Rr(\tau^2\rho_0+Rr\rho_1)}{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr-2\varepsilon Rr(\rho_1-\rho_0)\tau}, \qquad (4)\end{aligned}$$

where $\varepsilon=\pm1$. On the other hand,

$$\begin{aligned}\mathcal{R}\left(\frac{Rr(\tau-\varepsilon\rho_1)}{Rr+\varepsilon\tau\rho_0}\right) &= \frac{R^2r^2(\tau-\varepsilon\rho_1)^2\rho_0+Rr(Rr+\varepsilon\tau\rho_0)^2\rho_1}{R^2r^2(\tau-\varepsilon\rho_1)^2+Rr(Rr+\varepsilon\tau\rho_0)^2} \\ &= \frac{Rr(\tau-\varepsilon\rho_1)^2\rho_0+(Rr+\varepsilon\tau\rho_0)^2\rho_1}{Rr(\tau-\varepsilon\rho_1)^2+(Rr+\varepsilon\tau\rho_0)^2} \\ &= \frac{(Rr+\rho_0\rho_1)(\tau^2\rho_0+Rr\rho_1)}{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr-2\varepsilon Rr(\rho_1-\rho_0)\tau} \\ &= \frac{(q+1)Rr(\tau^2\rho_0+Rr\rho_1)}{(Rr+\rho_0^2)\tau^2+(Rr+\rho_1^2)Rr-2\varepsilon Rr(\rho_1-\rho_0)\tau}.\end{aligned}$$

These are, according to (4) above, the radii of the two neighbors. □

4. Parametrizations

A rational Steiner pair (R,r,d) is *standard* if $R=1$.

Proposition 9. *The standard rational Steiner pairs are parametrized by*

$$R=1,\quad r=\frac{t}{(q+t)(q+1+t)},\quad d=\frac{q(q+1)-t^2}{(q+t)(q+1+t)}. \qquad (5)$$

Proof. Since $(R,r,d)=(1,0,1)$ is a rational solution of

$$d^2=(1-r)^2-4qr,$$

every rational solution is of the form $d=1-(2q+1+2t)r$ for some rational number t. Direct substitution leads to

$$(q+t)(q+1+t)r-t=0.$$

From this, the expressions of r and d follow. □

Remark. $\rho_0=\frac{t}{q+1+t}$ and $\rho_1=\frac{q}{q+t}$.

Proposition 10. *In a standard rational Steiner pair $(R, r, d)_q$, a Steiner circle $A(\rho)$ and its neighbors have rational radii if and only if*

$$\rho = \mathcal{R}(\tau) = \frac{t(q + (q+t)^2\tau^2)}{(q+t)(t + (q+t)(q+1+t)\tau^2)}$$

for a rational number τ. The radii of the two neighbors are $\mathcal{R}(\tau_\pm)$, where

$$\tau_+ = \frac{-q + (q+t)\tau}{(q+t)(1 + (q+t)\tau)} \quad \text{and} \quad \tau_- = \frac{q + (q+t)\tau}{(q+t)(1 - (q+t)\tau)}.$$

Proof. The neighbors of $A(\rho)$ have rational radii if and only if $a\sigma^2 + b\sigma + c$ (with a, b, c given in (2)) has rational roots. Therefore, the two neighbors have rational radii if and only if $Rr(\rho_1 - \rho)(\rho - \rho_0)$ is the square of a rational number by Proposition 7, Theorem 8, and Proposition 9. □

Proposition 11. *The iterations of* τ_+ (respectively τ_-) *have periods* 3, 4, 6 *according as* $q = 3$, 1, *or* $\frac{1}{3}$.

Proof. The iterations of τ_+ are as follows.

$$\begin{tikzcd}
\frac{-3+(3+t)\tau}{(3+t)(1+(3+t)\tau)} \arrow[dd] & \\
& \tau \arrow[ul] \\
\frac{3+(3+t)\tau}{(3+t)(1-(3+t)\tau)} \arrow[ur] &
\end{tikzcd}$$

Figure 6. 3-cycle for $q = 3$

$$\begin{tikzcd}
& \frac{-1+(1+t)\tau}{(1+t)(1+(1+t)\tau)} \arrow[dl] & \\
\frac{-1}{(1+t)^2\tau} \arrow[dr] & & \tau \arrow[ul] \\
& \frac{1+(1+t)\tau}{(1+t)(1-(1+t)\tau)} \arrow[ur] &
\end{tikzcd}$$

Figure 7. 4-cycle for $q = 1$

$$\begin{tikzcd}
& \frac{-3+(1+3t)\tau}{(1+3t)(1+(1+3t)\tau)} \arrow[dl] & \frac{3(-1+(1+3t)\tau)}{(1+3t)(3+(1+3t)\tau)} \arrow[l] & \\
\frac{-3}{(1+3t)^2\tau} \arrow[dr] & & & \tau \arrow[ul] \\
& \frac{3+(1+3t)\tau}{(1+3t)(1-(1+3t)\tau)} \arrow[r] & \frac{3(1+(1+3t)\tau)}{(1+3t)(3-(1+3t)\tau)} \arrow[ur] &
\end{tikzcd}$$

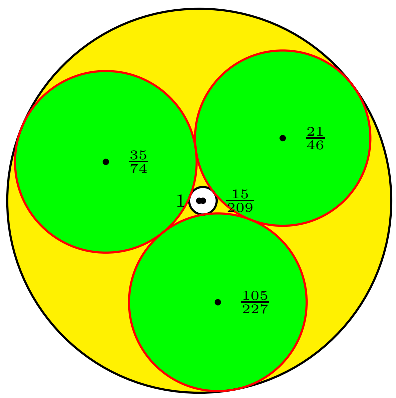
Figure 8. 6-cycle for $q = \frac{1}{3}$

The iterations of τ_- simply reverse the orientations of these cycles. □

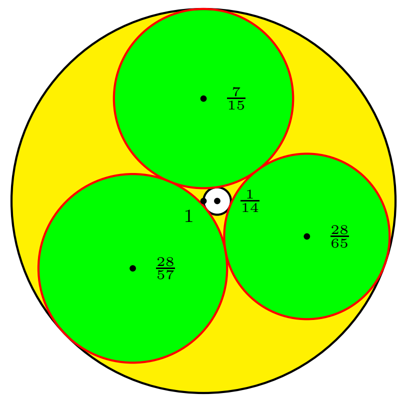
Example 1. Rational Steiner 3-cycles with simple rational radii:

t	(R, r, d)	τ	Steiner cycle
2	$\left(1, \frac{1}{15}, \frac{4}{15}\right)$	1	$\left(\frac{7}{12}, \frac{7}{17}, \frac{7}{20}\right)$
		$\frac{1}{3}$	$\left(\frac{13}{23}, \frac{13}{38}, \frac{13}{30}\right)$
		$\frac{4}{5}$	$\left(\frac{19}{32}, \frac{19}{48}, \frac{19}{53}\right)$
		$\frac{9}{5}$	$\left(\frac{21}{38}, \frac{21}{47}, \frac{21}{62}\right)$
		$\frac{11}{5}$	$\left(\frac{31}{57}, \frac{31}{68}, \frac{31}{92}\right)$
3	$\left(1, \frac{1}{14}, \frac{1}{14}\right)$	$\frac{3}{2}$	$\left(\frac{28}{57}, \frac{7}{15}, \frac{28}{65}\right)$
$\frac{3}{2}$	$\left(1, \frac{2}{33}, \frac{13}{33}\right)$	$\frac{4}{3}$	$\left(\frac{8}{13}, \frac{8}{21}, \frac{2}{7}\right)$
		$\frac{1}{9}$	$\left(\frac{24}{61}, \frac{24}{85}, \frac{3}{5}\right)$

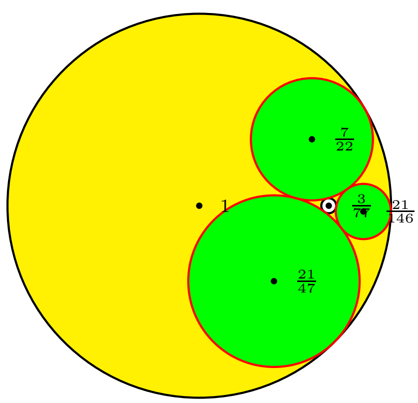
t	(R, r, d)	τ	Steiner cycle
$\frac{2}{3}$	$\left(1, \frac{3}{77}, \frac{52}{77}\right)$	$\frac{5}{11}$	$\left(\frac{3}{5}, \frac{3}{20}, \frac{1}{4}\right)$
		$\frac{1}{3}$	$\left(\frac{7}{22}, \frac{21}{47}, \frac{21}{146}\right)$
$\frac{3}{4}$	$\left(1, \frac{4}{95}, \frac{61}{95}\right)$	$\frac{4}{13}$	$\left(\frac{16}{45}, \frac{4}{9}, \frac{16}{101}\right)$
$\frac{3}{5}$	$\left(1, \frac{5}{138}, \frac{97}{138}\right)$	$\frac{11}{6}$	$\left(\frac{20}{33}, \frac{5}{22}, \frac{4}{29}\right)$
$\frac{9}{5}$	$\left(1, \frac{15}{232}, \frac{73}{232}\right)$	$\frac{1}{2}$	$\left(\frac{60}{97}, \frac{20}{59}, \frac{3}{8}\right)$
$\frac{12}{5}$	$\left(1, \frac{5}{72}, \frac{13}{72}\right)$	$\frac{1}{3}$	$\left(\frac{20}{37}, \frac{5}{13}, \frac{4}{9}\right)$
$\frac{9}{7}$	$\left(1, \frac{21}{370}, \frac{169}{370}\right)$	$\frac{7}{5}$	$\left(\frac{12}{19}, \frac{28}{79}, \frac{21}{82}\right)$
$\frac{10}{3}$	$\left(1, \frac{15}{209}, \frac{4}{209}\right)$	$\frac{1}{19}$	$\left(\frac{35}{74}, \frac{105}{227}, \frac{21}{46}\right)$



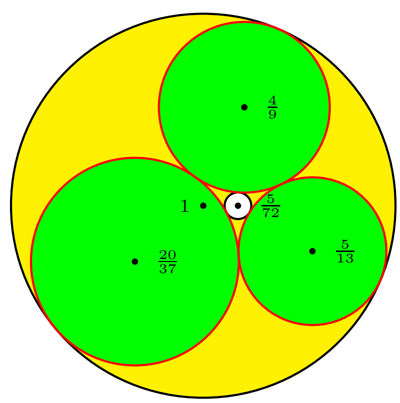
A: $\left(1, \frac{15}{209}, \frac{4}{209}; \frac{35}{74}, \frac{105}{227}, \frac{21}{46}\right)$



B: $\left(1, \frac{1}{14}, \frac{1}{14}; \frac{28}{57}, \frac{7}{15}, \frac{28}{65}\right)$



C: $\left(1, \frac{3}{77}, \frac{52}{77}; \frac{7}{22}, \frac{21}{47}, \frac{21}{146}\right)$

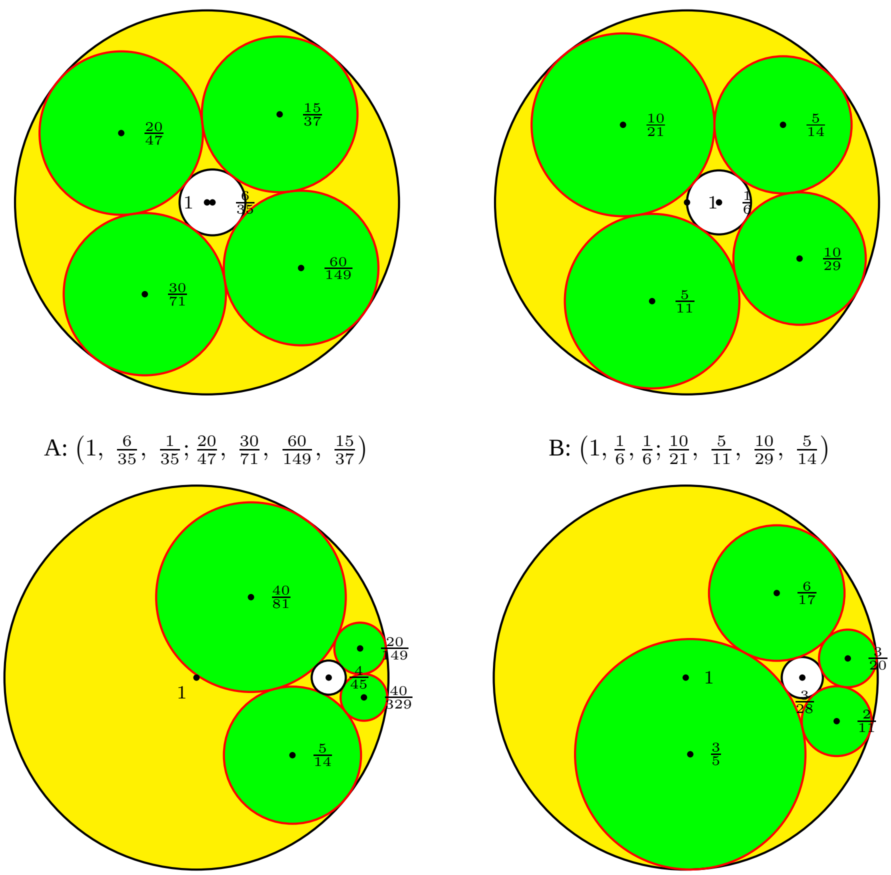


D: $\left(1, \frac{5}{72}, \frac{13}{72}; \frac{20}{37}, \frac{5}{13}, \frac{4}{9}\right)$

Figure 9. Rational Steiner 3-cycles

Example 2. Rational Steiner 4-cycles with simple rational radii:

t	(R,r,d)	τ	Steiner cycle
1	$\left(1,\frac{1}{6},\frac{1}{6}\right)$	1	$\left(\frac{10}{21},\ \frac{5}{11},\ \frac{10}{29},\ \frac{5}{14}\right)$
		2	$\left(\frac{34}{77},\ \frac{17}{35},\ \frac{34}{93},\ \frac{17}{50}\right)$
		$\frac{1}{3}$	$\left(\frac{26}{53},\ \frac{13}{35},\ \frac{26}{77},\ \frac{13}{30}\right)$
$\frac{1}{2}$	$\left(1,\frac{2}{15},\frac{7}{15}\right)$	2	$\left(\frac{5}{11},\ \frac{20}{37},\ \frac{10}{43},\ \frac{20}{93}\right)$
$\frac{1}{3}$	$\left(1,\ \frac{3}{28},\ \frac{17}{28}\right)$	3	$\left(\frac{6}{17},\ \frac{3}{5},\ \frac{2}{11},\ \frac{3}{20}\right)$
		$\frac{1}{4}$	$\left(\frac{15}{37},\ \frac{30}{193},\ \frac{15}{88},\ \frac{10}{19}\right)$
$\frac{1}{4}$	$\left(1,\ \frac{4}{35},\ \frac{31}{45}\right)$	$\frac{2}{5}$	$\left(\frac{5}{14},\ \frac{40}{81},\ \frac{20}{149},\ \frac{40}{329}\right)$
$\frac{3}{5}$	$\left(1,\ \frac{15}{104},\ \frac{41}{104}\right)$	$\frac{1}{2}$	$\left(\frac{30}{49},\ \frac{15}{49},\ \frac{10}{43},\ \frac{3}{8}\right)$
$\frac{4}{5}$	$\left(1,\frac{10}{63},\frac{17}{63}\right)$	$\frac{1}{3}$	$\left(\frac{10}{19},\ \frac{20}{61},\ \frac{5}{17},\ \frac{4}{9}\right)$
$\frac{4}{3}$	$\left(1,\frac{6}{35},\frac{1}{35}\right)$	$\frac{1}{7}$	$\left(\frac{20}{47},\ \frac{30}{71},\ \frac{60}{149},\ \frac{15}{37}\right)$



A: $\left(1,\ \frac{6}{35},\ \frac{1}{35};\ \frac{20}{47},\ \frac{30}{71},\ \frac{60}{149},\ \frac{15}{37}\right)$

B: $\left(1,\ \frac{1}{6},\ \frac{1}{6};\ \frac{10}{21},\ \frac{5}{11},\ \frac{10}{29},\ \frac{5}{14}\right)$

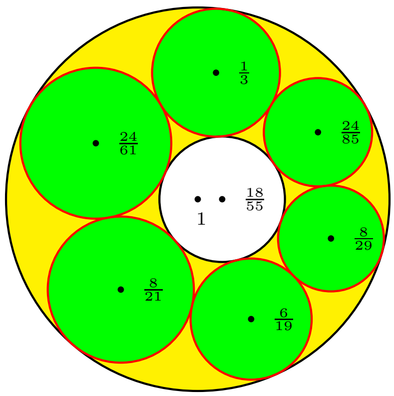
C: $\left(1,\ \frac{4}{45},\ \frac{31}{45};\ \frac{5}{14},\ \frac{40}{81},\ \frac{20}{149},\ \frac{40}{329}\right)$

D: $\left(1,\ \frac{3}{28},\ \frac{17}{28};\ \frac{6}{17},\ \frac{3}{5},\ \frac{2}{11},\ \frac{3}{20}\right)$

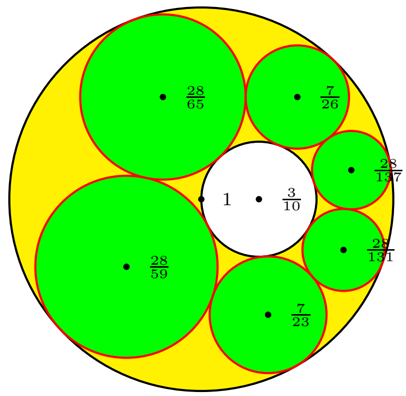
Figure 10. Rational Steiner 4-cycles

Example 3. Rational Steiner 6-cycles with simple rational radii:

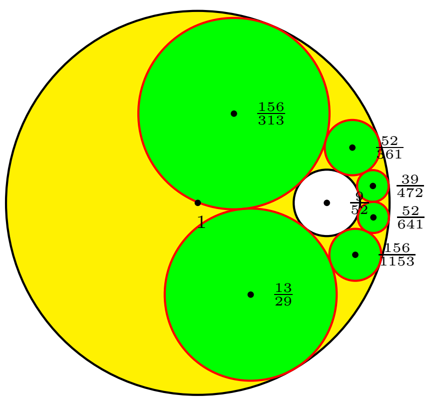
t	(R, r, d)	τ	Steiner cycle
$\frac{1}{2}$	$\left(1, \frac{18}{55}, \frac{7}{55}\right)$	2	$\left(\frac{1}{3}, \frac{24}{61}, \frac{8}{21}, \frac{6}{19}, \frac{8}{29}, \frac{24}{85}\right)$
$\frac{1}{3}$	$\left(1, \frac{3}{10}, \frac{3}{10}\right)$	1	$\left(\frac{28}{65}, \frac{28}{59}, \frac{7}{23}, \frac{28}{131}, \frac{28}{137}, \frac{7}{26}\right)$
$\frac{1}{5}$	$\left(1, \frac{45}{184}, \frac{91}{184}\right)$	$\frac{5}{3}$	$\left(\frac{20}{57}, \frac{60}{97}, \frac{5}{17}, \frac{12}{77}, \frac{20}{153}, \frac{15}{88}\right)$
$\frac{1}{9}$	$\left(1, \frac{9}{52}, \frac{35}{52}\right)$	$\frac{3}{2}$	$\left(\frac{12}{31}, \frac{4}{7}, \frac{3}{19}, \frac{4}{47}, \frac{12}{151}, \frac{1}{8}\right)$
		$\frac{5}{4}$	$\left(\frac{52}{361}, \frac{156}{313}, \frac{13}{29}, \frac{156}{1153}, \frac{52}{641}, \frac{39}{472}\right)$
$\frac{2}{9}$	$\left(1, \frac{9}{35}, \frac{16}{35}\right)$	3	$\left(\frac{21}{83}, \frac{7}{13}, \frac{21}{47}, \frac{7}{33}, \frac{21}{143}, \frac{7}{45}\right)$
$\frac{4}{15}$	$\left(1, \frac{5}{18}, \frac{7}{18}\right)$	5	$\left(\frac{5}{21}, \frac{4}{9}, \frac{20}{39}, \frac{5}{18}, \frac{20}{111}, \frac{20}{117}\right)$



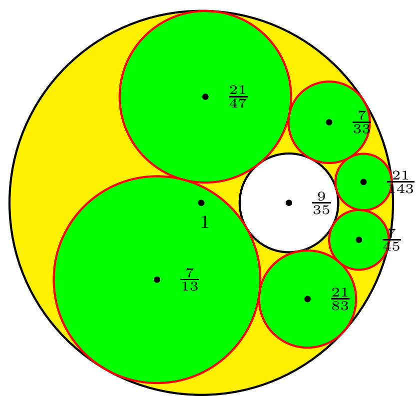
A: $\left(1, \frac{18}{55}, \frac{7}{55}; \frac{1}{3}, \frac{24}{61}, \frac{8}{21}, \frac{6}{19}, \frac{8}{29}, \frac{24}{85}\right)$



B: $\left(1, \frac{3}{10}, \frac{3}{10}; \frac{28}{65}, \frac{28}{59}, \frac{7}{23}, \frac{28}{131}, \frac{28}{137}, \frac{7}{26}\right)$



C: $\left(1, \frac{9}{52}, \frac{35}{52}; \frac{52}{361}, \frac{156}{313}, \frac{13}{29}, \frac{156}{1153}, \frac{52}{641}, \frac{39}{472}\right)$



D: $\left(1, \frac{9}{35}, \frac{16}{35}; \frac{21}{83}, \frac{7}{13}, \frac{21}{47}, \frac{7}{33}, \frac{21}{143}, \frac{7}{45}\right)$

Figure 11. Rational Steiner 6-cycles

5. Inversion

By inverting a Steiner n-cycle configuration $(R, r, d; \rho_1, \dots, \rho_n)$ in the circle (O), we obtain a new n-cycle $(R, r', d'; \rho'_1, \dots, \rho'_n)$ in which

$$r' = \frac{-R^2 r}{d^2 - r^2}, \qquad d' = \frac{R^2 d}{d^2 - r^2}, \qquad \rho'_i = \frac{R\rho_i}{2\rho_i - R},\ i = 1, \dots, n.$$

Regarding the circles in a Steiner configuration with (I) in the interior of (O) all positively oriented, we interpret circles with negative radii as those oppositely oriented to the circle (O).

The tables below show the rational Steiner n-cycles obtained by inverting those in Figures 9-11 in the circle (O). Figure 12 illustrates those obtained from the 4-cycles in Figure 10.

	Steiner 3-cycle	Inversive images in (O)
A	$\left(1,\ \frac{15}{209},\ \frac{4}{209};\ \frac{35}{74},\ \frac{105}{227},\ \frac{21}{46}\right)$	$\left(1, 15, -4;\ -\frac{35}{4},\ -\frac{105}{17},\ -\frac{21}{4}\right)$
B	$\left(1, \frac{1}{14}, \frac{1}{14};\ \frac{28}{57},\ \frac{7}{15},\ \frac{28}{65}\right)$	$\left(1, \infty, 7;\ -28, -7, -\frac{28}{9}\right)$
C	$\left(1, \frac{3}{77}, \frac{52}{77};\ \frac{7}{22},\ \frac{21}{47},\ \frac{21}{146}\right)$	$\left(1, -\frac{3}{35}, \frac{52}{35};\ -\frac{7}{8}, -\frac{21}{5}, -\frac{21}{104}\right)$
D	$\left(1,\ \frac{5}{72}, \frac{13}{72};\ \frac{20}{37}, \frac{5}{13},\ \frac{4}{9}\right)$	$\left(1, -\frac{5}{2}, \frac{13}{2};\ \frac{20}{3}, -\frac{5}{3}, -4\right)$

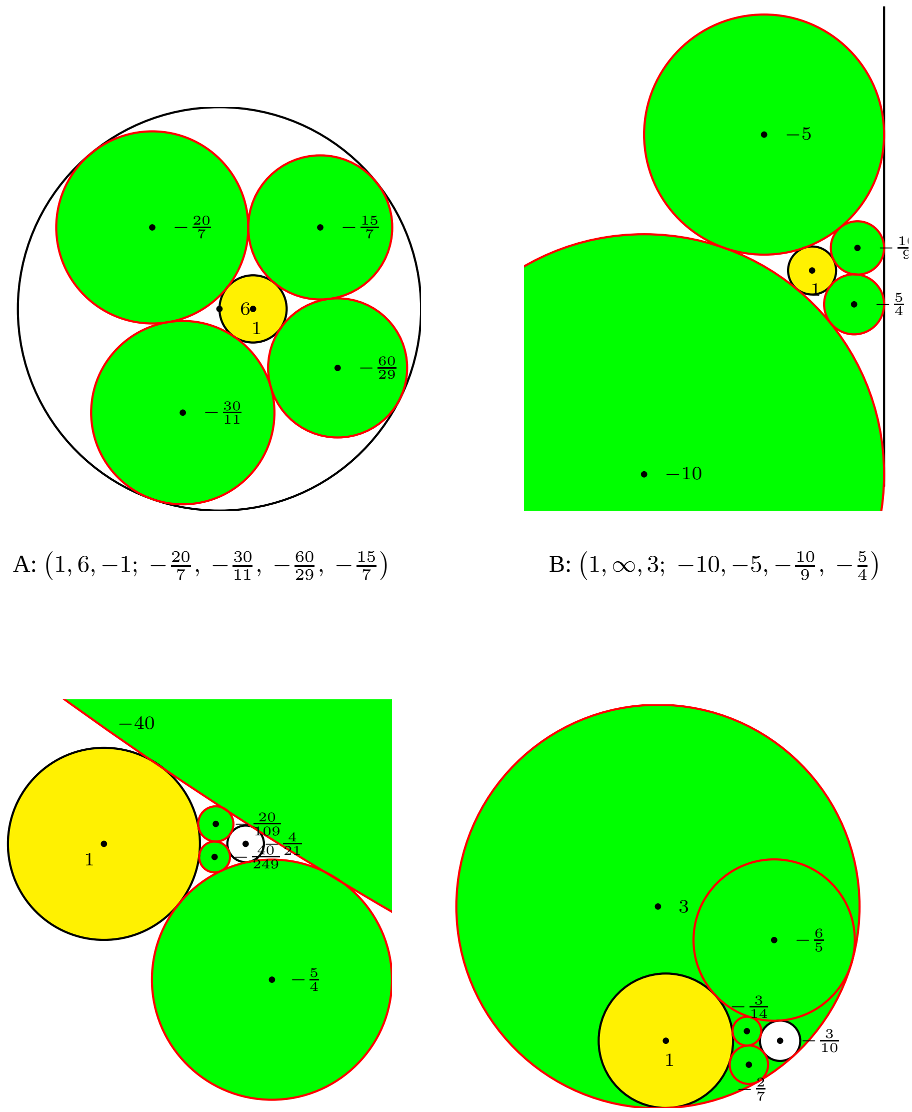
	Steiner 4-cycle	Inversive images in (O)
A	$\left(1, \frac{6}{35}, \frac{1}{35};\ \frac{20}{47},\ \frac{30}{71},\ \frac{60}{149},\ \frac{15}{37}\right)$	$\left(1, 6, -1;\ -\frac{20}{7},\ -\frac{30}{11},\ -\frac{60}{29},\ -\frac{15}{7}\right)$
B	$\left(1, \frac{1}{6}, \frac{1}{6};\ \frac{10}{21},\ \frac{5}{11},\ \frac{10}{29},\ \frac{5}{14}\right)$	$\left(1, \infty, 3;\ -10, -5, -\frac{10}{9},\ -\frac{5}{4}\right)$
C	$\left(1, \frac{4}{45}, \frac{31}{45};\ \frac{5}{14},\ \frac{40}{81},\ \frac{20}{149},\ \frac{40}{329}\right)$	$\left(1, -\frac{4}{21}, \frac{31}{21};\ -\frac{5}{4}, -40, -\frac{20}{109},\ -\frac{40}{249}\right)$
D	$\left(1,\ \frac{3}{28},\ \frac{17}{28};\ \frac{6}{17},\ \frac{3}{5},\ \frac{2}{11},\ \frac{3}{20}\right)$	$\left(1,\ -\frac{3}{10},\ \frac{17}{10};\ -\frac{6}{5},\ 3,\ -\frac{2}{7},\ -\frac{3}{14}\right)$

	Steiner 6-cycle
A	$\left(1,\ \frac{18}{55},\ \frac{7}{55};\ \frac{1}{3},\ \frac{24}{61},\ \frac{8}{21},\ \frac{6}{19},\ \frac{8}{29},\ \frac{24}{85}\right)$
Inversive images	$\left(1,\ \frac{18}{5},\ -\frac{7}{5};\ -1,\ -\frac{24}{13},\ -\frac{8}{5},\ -\frac{6}{7},\ -\frac{8}{13},\ -\frac{24}{37}\right)$
B	$\left(1,\ \frac{3}{10},\ \frac{3}{10};\ \frac{28}{65},\ \frac{28}{59},\ \frac{7}{23},\ \frac{28}{131},\ \frac{28}{137},\ \frac{7}{26}\right)$
Inversive images	$\left(1, \infty, \frac{5}{3};\ -\frac{28}{9},\ -\frac{28}{3},\ -\frac{7}{9},\ -\frac{28}{75},\ -\frac{28}{81},\ -\frac{7}{12}\right)$
C	$\left(1,\ \frac{9}{52},\ \frac{35}{52};\ \frac{52}{361},\ \frac{156}{313},\ \frac{13}{29},\ \frac{156}{1153},\ \frac{52}{641},\ \frac{39}{472}\right)$
Inversive images	$\left(1,\ -\frac{9}{22},\ \frac{35}{22};\ -\frac{52}{257},\ -156,\ -\frac{13}{3},\ -\frac{156}{841},\ -\frac{52}{537},\ -\frac{39}{394}\right)$
D	$\left(1,\ \frac{9}{35},\ \frac{16}{35};\ \frac{21}{83},\ \frac{7}{13},\ \frac{21}{47},\ \frac{7}{33},\ \frac{21}{143},\ \frac{7}{45}\right)$
Inversive images	$\left(1,\ -\frac{9}{5},\ \frac{16}{5};\ -\frac{21}{41},\ 7,\ -\frac{21}{5},\ -\frac{7}{19},\ -\frac{21}{101},\ -\frac{7}{31}\right)$

The rational Steiner pairs in Figures 9-11A all have $r > d$. In the inversive images, (I) contains (O) in its interior. The new configuration is equivalent to a standard one with $r < 1$. For example, the Steiner pair in Figure 12A is equivalent to $\left(1, \frac{1}{6}, \frac{1}{6}\right)$.

The Steiner pairs in Figures 9-11B all have $r = d$. The inversive image of (I) is a line at a distance $\frac{1}{2r}$ from O.

In Figures 9-11C and D, $r < d$. The images of (O) and (I) are disjoint circles. For the cycles in C, none of the Steiner circles contains O. Their images are all externally tangent to the images of (O) and (I). On the other hand, for the cycles in D, one of the Steiner circles contains O in its interior. Therefore, its inversive image contains those of (O) and (I) in its interior.



A: $\left(1, 6, -1;\ -\frac{20}{7}, -\frac{30}{11}, -\frac{60}{29}, -\frac{15}{7}\right)$

B: $\left(1, \infty, 3;\ -10, -5, -\frac{10}{9}, -\frac{5}{4}\right)$

C: $\left(1, -\frac{4}{21}, \frac{31}{21};\ -\frac{5}{4}, -40, -\frac{20}{109}, -\frac{40}{249}\right)$

D: $\left(1, -\frac{3}{10}, \frac{17}{10};\ -\frac{6}{5}, 3, -\frac{2}{7}, -\frac{3}{14}\right)$

Figure 12. Rational Steiner 4-cycles by inversion

6. Relations among standard rational Steiner pairs

We conclude this note with a brief explanation of the relations among Steiner pairs with different parameters. Denote by $\mathcal{S}_q(t)$ the standard rational Steiner pair given in Proposition 9. By allowing t to take on negative values, we also include the disjoint pairs and those with (I) containing (O).

Proposition 12. *Let (O) and (I) be the circles in $\mathcal{S}_q(t)$.*
(a) *(I) is in the interior of (O) if and only if $t > 0$.*
(b) *(I) contains (O) in its interior if and only if $-(q+1) < t < -q$.*
(c) *(O) and (I) are disjoint if and only if $-q < t < 0$ or $t < -(q+1)$.*

Proof. (a) The circle (I) is contained in the interior of (O) if and only if $d+r < R$ and $d - r > -R$. This means $t(q+1+t) > 0$ and $q+t > 0$. Therefore, $t > 0$.

(b) The circle (I) contains (O) in its interior if and only if $d+r > R$ and $d-r < -R$. This means $t(q+1+t) < 0$ and $q+t < 0$. Therefore, $-(q+1) < t < -q$.

(c) follows from (a) and (b). □

Remarks. (1) If $-(q+1) < t < -q$, $\mathcal{S}_q(t)$ is homothetic, by the homothety at I with ratio $\frac{R}{r}$, to $\mathcal{S}_q(t')$, where $t' = -\frac{(q+1)(q+t)}{q+1+t} > 0$.

(2) For standard pairs $\mathcal{S}_q(t)$ with $t > 0$, we may restrict to $0 < t < \sqrt{q(q+1)}$. If $t^2 > q(q+1)$, $\mathcal{S}_q(t)$ is the reflection of $\mathcal{S}_q\left(\frac{q(q+1)}{t}\right)$ in O.

Proposition 13. *The inversive image of $\mathcal{S}_q(t)$ in the circle (O) is $\mathcal{S}_q(-t)$.*

Proof. Let $I'(r')$ be the inversive image of (I) in (O), with $d' = OI'$.

$$r' = \frac{1}{2}\left(\frac{R^2}{d+r} - \frac{R^2}{d-r}\right) = \frac{-t}{(q-t)(q+1-t)},$$
$$d' = \frac{1}{2}\left(\frac{R^2}{d+r} + \frac{R^2}{d-r}\right) = \frac{q(q+1)-t^2}{(q-t)(q+1-t)}.$$

From this it is clear that the inversive image of the pair $\mathcal{S}_q(t)$ is $\mathcal{S}_q(-t)$. □

Remarks. (1) If $t = 0$, (I) reduces to a point on the circle (O).

(2) If $t^2 = q(q+1)$, then $d = 0$. The circles (O) and (I) are concentric.

(3) If $t = -q$ or $-(q+1)$, the circle (I) degenerates into a line. This means that with $t = q$ or $q+1$, the circle (I) passes through O. It has radius $\frac{1}{2(2q+1)}$. Therefore, the line in $\mathcal{S}_q(-q)$ is at a distance $2q+1$ from O.

(4) If the circle (I) contains the center O, then the inversive image of (I) contains (O). This means that $-(q+1) < -t < -q$, and $q < t < q+1$. It follows that if $0 < t < q$, the inversive images of the Steiner pair of circles are disjoint.

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Forum Geometricorum
Volume 11 (2011) 251–254.

FORUM GEOM
ISSN 1534-1178

Golden Sections in a Regular Hexagon

Quang Tuan Bui

Abstract. We relate a golden section associated with a regular hexagon to two recent simple constructions by Hofstetter and Bataille, and give a large number of golden sections of segments in a regular hexagon.

Consider a regular hexagon $ABCDEF$ with center O. Let M be a point on the side BC. An equilateral triangle constructed on AM has its third vertex N on the radius OE, such that $ON = BM$.

Proposition 1. *The area of the regular hexagon $ABCDEF$ is 3 times the area of triangle AMN if and only if M divides BC in the golden ratio.*

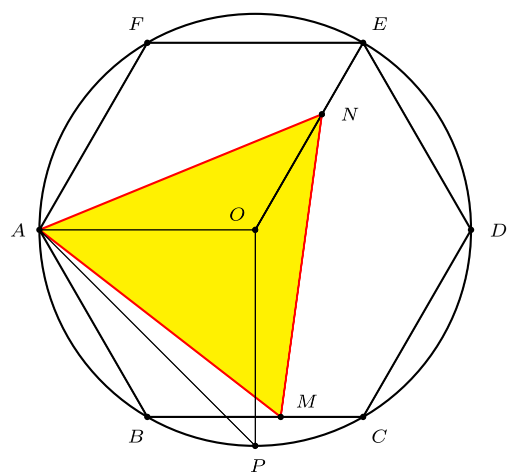

Figure 1.

Proof. Let P is midpoint of the minor BC. It is clear that AOP is an isosceles right triangle and $AP = \sqrt{2} \cdot AO$ (see Figure 1). The area of the regular hexagon $ABCDEF$ is three times that of triangle AMN if and only if

$$\frac{\Delta AMN}{\Delta ABO} = 2 \Leftrightarrow \frac{AM}{AO} = \sqrt{2} \Leftrightarrow AM = AP.$$

Equivalently, M is an intersection of BC with the circle $O(P)$. We fill in the circles $B(C)$, $C(B)$ and $A(P)$. These three circles execute exactly Hofstetter's division of the segment BC in the golden ratio at the point M [2] (see Figure 2).[1] □

Publication Date: December 5, 2011. Communicating Editor: Paul Yiu.

[1]Hofstetter has subsequently noted [3] that this construction was known to E. Lemoine and J. Reusch one century ago.

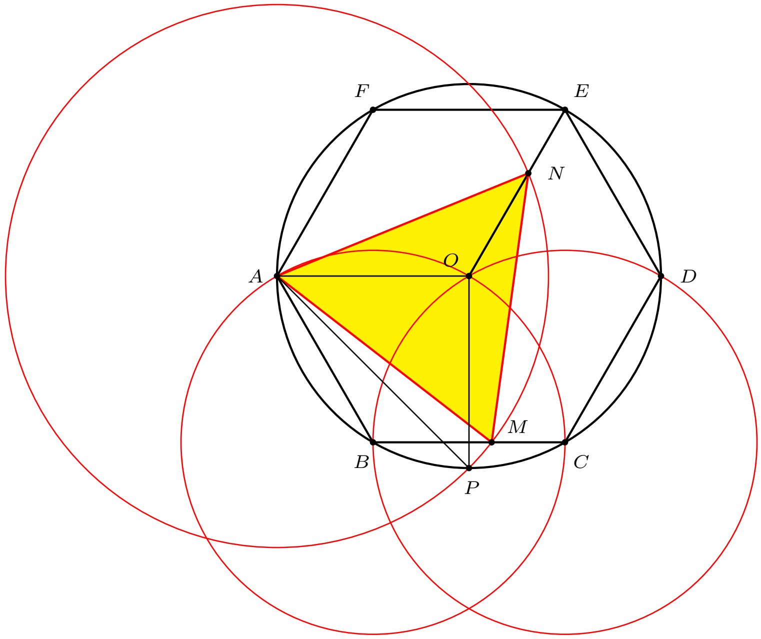

Figure 2.

Since triangles AON and ABM are congruent, N also divides OE in the golden ratio. This fact also follows independently from a construction given by M. Bataille. Let Q be the antipodal point of P, and complete the square $AOQR$. According to [1], O divides BN in the golden ratio. Since O is the midpoint of BE, it follows easily that N divides OE in the golden ratio as well.

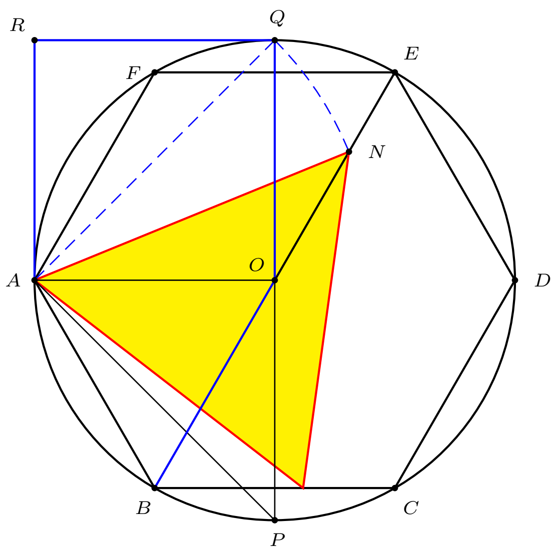

Figure 3.

Proposition 2. *The sides of the equilateral triangle AMN are divided in the golden ratio as follows.*

Directed segment	MN	NM	AM	AN	NA	MA
divided by	OD	OC	OB	OF	BF	perp. from O to AB
in golden ratio at	A_n	A_m	N_m	M_n	M_a	N_a

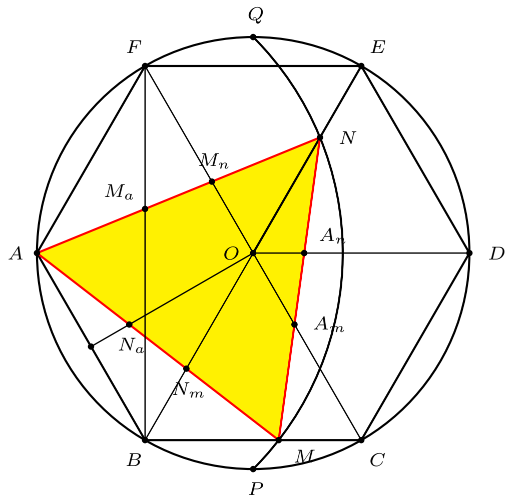

Figure 4

Proposition 3. *Each of the six points A_m, A_n, M_n, M_a, N_a, N_m divides a segment, apart from the sides of the equilateral triangle AMN, in the golden ratio.*

Point	A_m	A_n	M_n	M_a	N_a	N_m
Segment	CO	AD	FO	BF	SR	OB

Here, S is the midpoint of ON_a.

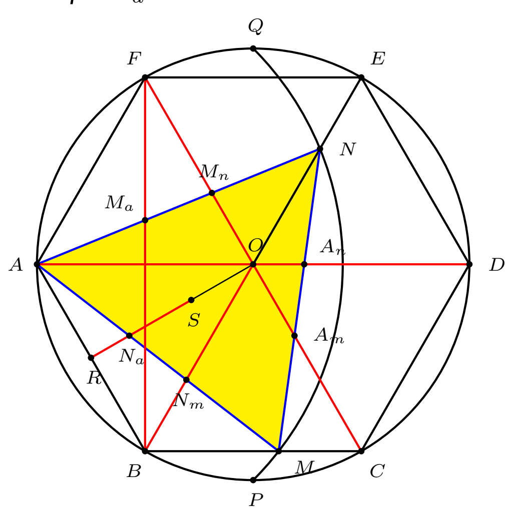

Figure 5.

Proposition 4. *The segments N_aM_a, N_mA_m and M_nA_n are divided in the golden ratio by the lines OA, OP, OE respectively.*

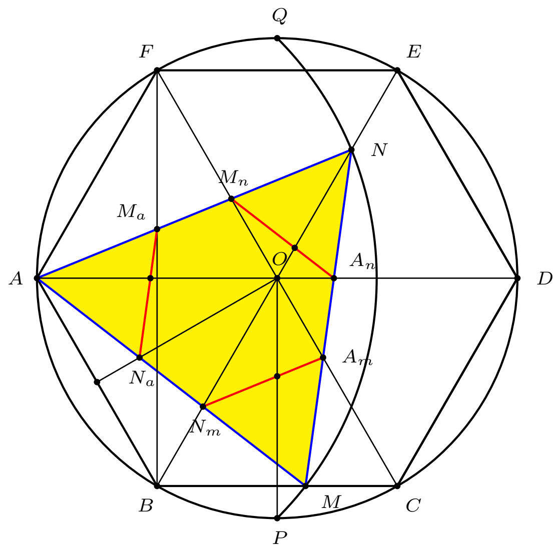

Figure 6.

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Forum Geometricorum
Volume 11 (2011) 255–259.

FORUM GEOM
ISSN 1534-1178

The Golden Section with a Collapsible Compass Only

Nikolaos Dergiades and Paul Yiu

Abstract. With the use of a collapsible compass, we divide a given segment in the golden ratio by drawing ten circles.

Kurt Hofstetter [4] has given an elegant euclidean construction in *five* steps for the division of a segment in the golden ratio. In Figure 1, AB is a given segment. The circles $c_1 := A(B)$ and $c_2 := B(A)$ intersect at C and C'. The circle $c_3 := C(A)$ intersects c_1 at D. Join C and C' to intersect c_3 at the midpoint M of the arc AB. Then the circle $c_4 := D(M)$ intersects the segment AB at a point G which divides it in the golden ratio. The validity of this construction depends on the fact that DM is a side of a square inscribed in the circle $C(D)$.

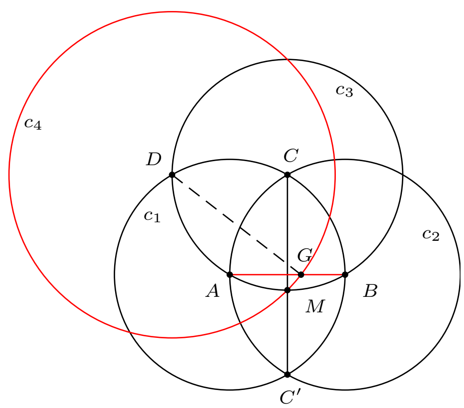

Figure 1. Hofstetter's division of a segment in the golden ratio

We shall modify this construction to one using only a collapsible compass, in *ten* steps (see Construction 5 below). Euclid, in his *Elements* I.2, shows how to construct, in *seven* steps, using a collapsible compass with the help of a straightedge, a circle with given center A and radius equal to a given segment BC (see [2, p.244]). His interest was not on the parsimoniousness of the construction, but rather on the justification of how his Postulate 3 (to describe a circle with any given center and distance) can be put into effect by the use of a straightedge (Postulates 1 and 2) and a collapsible compass (*Elements* I.1). Since we restrict to the use of a collapsible compass only, we show that this can be done without the use of a straightedge, more simply, in *five steps* (see Figure 2). There were two prior publications in this

Publication Date: December 8, 2011. Communicating Editor: Floor van Lamoen.

Forum on compass-only constructions. Note that Hofstetter [3] did not divide a given segment in the golden ratio. On the other hand, the one given by Bataille [1] requires a rusty compass.

The proof of Construction 1 below, though simple, makes use of *Elements* I.8.

Construction 1. *Given three points A, B, C, construct*
(1,2) $c_1 := A(B)$ *and* $c_2 := B(A)$ *to intersect at* P *and* Q,
(3,4) $c_3 := P(C)$ *and* $c_4 := Q(C)$ *to intersect at* D,
(5) $c_5 := A(D)$.

The circle c_5 has radius congruent to BC.

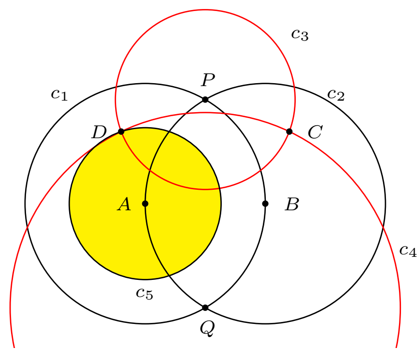

Figure 2. Construction of $A(BC)$ with a collapsible compass only

Lemma 2 helps simplify the proof of Construction 5.

Lemma 2. *Given a unit segment AB, let the circles $A(B)$ and $B(A)$ intersect at C. If the circle $C(A)$ intersects $A(B)$ at D and $B(A)$ at E, and the circles $D(B)$ and $E(A)$ intersect at H, then $CH = \sqrt{2}$, the side of a square inscribed in the circle $A(B)$.*

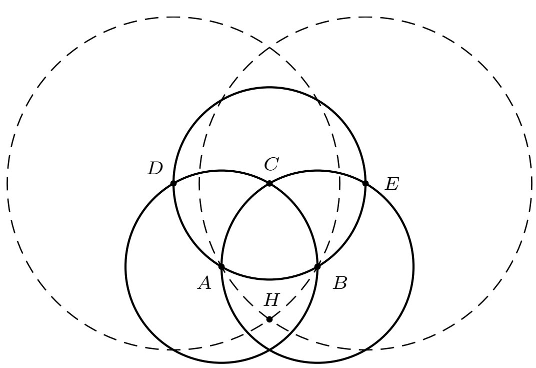

Figure 3.

We present two simple applications of Lemma 2.

Construction 3. *Given two points O and A, to construct a square $ABCD$ inscribed in the circle $O(A)$, construct*

(1,2) $c_1 := O(A)$ *and* $c_2 := A(O)$ *intersecting at* E *and* F,
(3) $c_3 := E(O)$ *intersecting* c_2 *at* F',
(4) $c_4 := F'(O)$,
(5) $c_5 := F(E)$ *intersecting at* c_1 *at* C *and* c_4 *at* H,
(6) $c_6 := A(H)$ *intersecting* c_1 *at* B *and* D.

$ABCD$ *is a square inscribed in the circle* $c_1 = O(A)$ (see Figure 4).

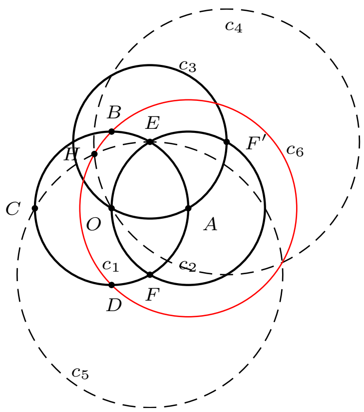

Figure 4

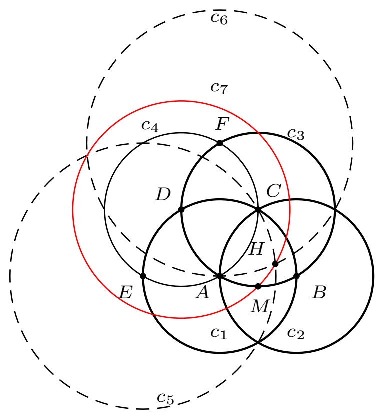

Figure 5

Construction 4. *To construct the midpoint M of the arc AB of the circle c_3 in Figure 1, we construct*

(1,2) $c_1 := A(B)$ *and* $c_2 := B(A)$ *to intersect at* C,
(3) $c_3 := C(A)$ *to intersect* c_1 *at* D,
(4) $c_4 := D(A)$ *to intersect* c_1 *at* E *and* c_3 *at* F,
(5,6) $c_5 := E(C)$ *and* $c_6 := F(A)$ *to intersect at* H,
(7) $c_7 := D(H)$ *to intersect* c_3 *at* M.

M *is the midpoint of the arc* AB (see Figure 5).

Finally, we present a division of a segment in the golden ratio in *ten* steps.

Construction 5. *Given two points A and B, construct*

(1,2) $c_1 := A(B)$ *and* $c_2 := B(A)$ *to intersect at* C.
(3) $c_3 := C(A)$ *to intersect* c_1 *at* D *and* c_2 *at* E.
(4) $c_4 := E(A)$ *intersects* c_3 *at* A'.
(5) $c_5 := D(B)$ *intersects* c_1 *at* D' *and* c_4 *at* H.
(6,7) $c_6 := A(H)$ *and* $c_7 := A'(H)$ *to intersect at* F.
(8) $c_8 := B(F)$ *intersects* c_6 (which is also $A(F)$) *at* F'.
(9,10) $c_9 := D(F)$ *and* $c_{10} := D'(F')$ *to intersect at* G.

The point G divides AB in the golden ratio (see Figure 6).

In Step (5), either of the intersections may be chosen as H. In Figure 6, H and C are on opposite sides of AB.

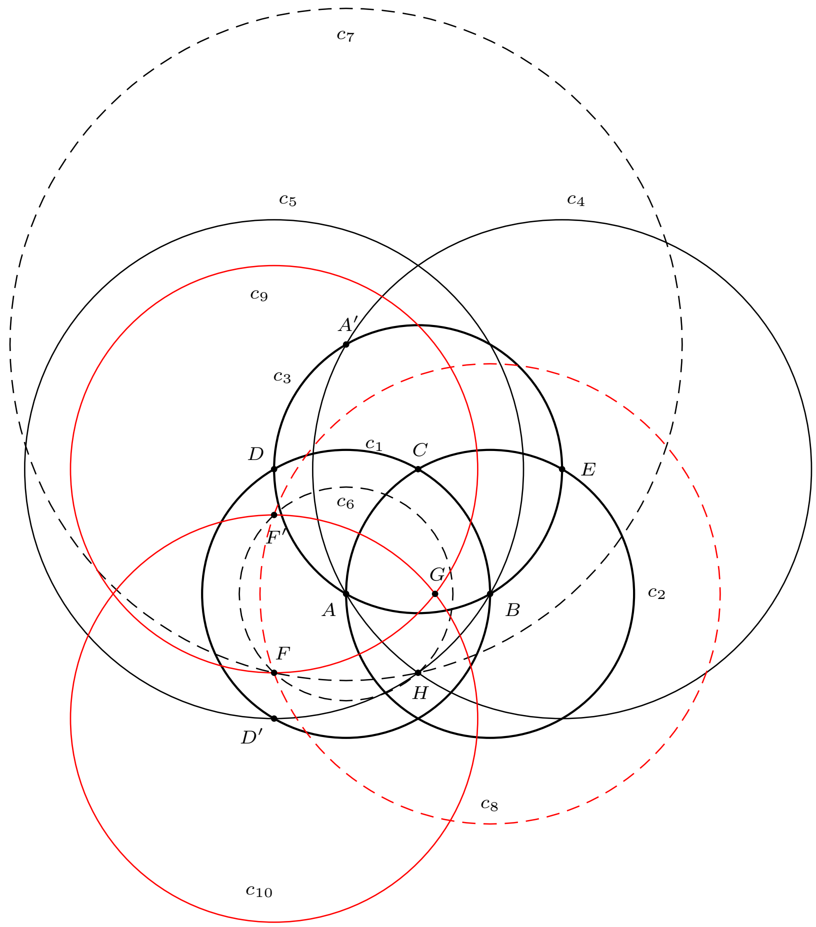

Figure 6. Golden section with collapsible compass only

Proof. Assume AB has unit length. We have
(i) $CH = \sqrt{2}$ by Lemma 2.
(ii) D and C are symmetric in the line AA'.
(iii) F and H are also symmetric in the line AA' by construction.
(iv) Therefore, $DF = \sqrt{2}$.
(v) D' and F' are the reflections of D and F in the line AB by construction. Therefore, $D(F)$ and $D'(F')$ intersect at a point G on the line AB. The fact that G divides AB in the golden ratio follows from Hofstetter's construction. $\square$

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Forum Geometricorum
Volume 11 (2011) 261–268.

FORUM GEOM
ISSN 1534-1178

Construction of Circles Through Intercepts of Parallels to Cevians

Jean-Pierre Ehrmann, Francisco Javier García Capitán, and Alexei Myakishev

Abstract. From the traces of the cevians of a point in the plane of a given triangle, construct parallels to the cevians to intersect the sidelines at six points. We determine the points for which these six intersections are concyclic.

Given a point P in the plane of triangle ABC, with cevian triangle XYZ, construct parallels through X, Y, Z to the cevians to intersect the sidelines at the following points.

Point	Intersection of	with the parallel to	through	Coordinates
B_a	CA	CZ	X	$(-u : 0 : u+v+w)$
C_a	AB	BY	X	$(-u : u+v+w : 0)$
C_b	AB	AX	Y	$(u+v+w : -v : 0)$
A_b	BC	CZ	Y	$(0 : -v : u+v+w)$
A_c	BC	BY	Z	$(0 : u+v+w : -w)$
B_c	CA	AX	Z	$(u+v+w : 0 : -w)$

A simple application of Carnot's theorem shows that these six points lie on a conic $\mathcal{C}(P)$ (see Figure 1). In this note we inquire the possibility for this conic to be a circle, and give a complete answer. We work with homogeneous barycentric coordinates with reference to triangle ABC. Suppose the given point P has coordinates $(u : v : w)$. The coordinates of the six points are given in the rightmost column of the table above. It is easy to verify that these points are all on the conic

$$\begin{aligned}&u(u+v)(u+w)yz + v(v+w)(v+u)zx + w(w+u)(w+v)xy\\&+ (u+v+w)(x+y+z)(vwx+wuy+uvz) = 0.\end{aligned} \tag{1}$$

Proposition 1. *The conic $\mathcal{C}(P)$ through the six points is a circle if and only if*

$$\frac{u}{v+w} : \frac{v}{w+u} : \frac{w}{u+v} = a^2 : b^2 : c^2. \tag{2}$$

Proof. Note that the lines B_aC_a, C_bA_b, A_cB_c are parallel to the sidelines of ABC. These three lines bound a triangle homothetic to ABC at the point

$$\left(\frac{u}{v+w} : \frac{v}{w+u} : \frac{w}{u+v}\right).$$

Publication Date: December 29, 2011. Communicating Editor: Paul Yiu.

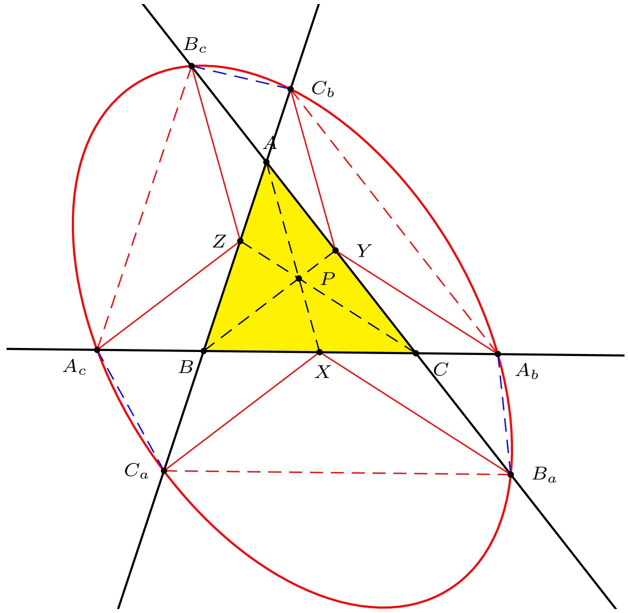

Figure 1.

It is known (see, for example, [2, §2]) that the hexagon $B_aC_aA_cB_cC_bA_b$ is a Tucker hexagon, *i.e.*, B_cC_b, C_aA_c, A_bB_a are antiparallels and the conic through the six points is a circle, if and only if this homothetic center is the symmedian point $K = (a^2 : b^2 : c^2)$. Hence the result follows. □

Corollary 2. *If $\mathcal{C}(P)$ is a circle, then it is a Tucker circle with center on the Brocard axis (joining the circumcenter and the symmedian point).*

Proposition 3. *If ABC is a scalene triangle, there are three distinct real points P for which the conic $\mathcal{C}(P)$ is a circle.*

Proof. Writing

$$\frac{u}{v+w} = \frac{a^2}{t}, \qquad \frac{v}{w+u} = \frac{b^2}{t}, \qquad \frac{w}{u+v} = \frac{c^2}{t}, \tag{3}$$

we have

$$\begin{array}{rcrcrcl} -tu & + & a^2v & + & a^2w & = & 0, \\ b^2u & - & tv & + & b^2w & = & 0, \\ c^2u & + & c^2v & - & tw & = & 0. \end{array}$$

Hence,

$$\begin{vmatrix} -t & a^2 & a^2 \\ b^2 & -t & b^2 \\ c^2 & c^2 & -t \end{vmatrix} = 0,$$

or

$$F(t) := -t^3 + (a^2b^2 + b^2c^2 + c^2a^2)t + 2a^2b^2c^2 = 0. \tag{4}$$

Note that $F(0) > 0$ and $F(+\infty) = -\infty$. Furthermore, assuming $a > b > c$, we easily note that

$$F(-a^2) > 0, \qquad F(-b^2) < 0, \qquad F(-c^2) > 0.$$

Therefore, F has one positive and two negative roots. □

Theorem 4. *For a scalene triangle ABC with $\rho = \frac{2}{\sqrt{3}}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$ and $\theta_0 := \frac{1}{3}\arccos\frac{8a^2b^2c^2}{\rho^3}$, the three points for which the corresponding conics $\mathcal{P}$ are circle are*

$$P_k = \left(\frac{a^2}{a^2 + \rho\cos\left(\theta_0 + \frac{2k\pi}{3}\right)} : \frac{b^2}{b^2 + \rho\cos\left(\theta_0 + \frac{2k\pi}{3}\right)} : \frac{c^2}{c^2 + \rho\cos\left(\theta_0 + \frac{2k\pi}{3}\right)}\right)$$

for $k = 0, \pm 1$.

Proof. From (3) the coordinates of P are

$$u : v : w = \frac{a^2}{a^2 + t} : \frac{b^2}{b^2 + t} : \frac{c^2}{c^2 + t},$$

with t a real root of the cubic equation (4). Writing $t = \rho\cos\theta$ we transform (4) into

$$\frac{1}{4}\rho^3\left(4\cos^3\theta - \frac{4(a^2b^2 + b^2c^2 + c^2a^2)}{\rho^2}\cos\theta\right) = 2a^2b^2c^2.$$

If $\rho = \frac{2}{\sqrt{3}}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$, this can be further reduced to

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = \frac{8a^2b^2c^2}{\rho^3}.$$

The three real roots of (4) $t_k = \rho\cos\left(\theta_0 + \frac{2k\pi}{3}\right)$ for $k = 0, \pm 1$.

□

Remarks. (1) If the triangle is equilateral, the roots of the cubic equation (4) are $t = -a^2,\ -a^2,\ \frac{a^2}{2}$.

(2) If the triangle is isosceles at A (but not equilateral), we have two solutions P_1, P_2 on the line AG. The third one degenerates into the infinite point of BC. The two finite points can be constructed as follows. Let the tangent at B to the circumcircle intersects AC at U, and T be the projection of U on AG, $AW = \frac{3}{2}\cdot AT$. The circle centered at W and orthogonal to the circle $G(A)$ intersects AG at P_1 and P_2.

Henceforth, we shall assume triangle ABC scalene.

Proposition 5. *The conic $\mathcal{C}(P)$ is a circle if and only if P is an intersection, apart from the centroid G, of*

(i) *the rectangular hyperbola through G and the incenter I and their anticevian triangles,*

(ii) *the circum-hyperbola through G and the symmedian point K.*

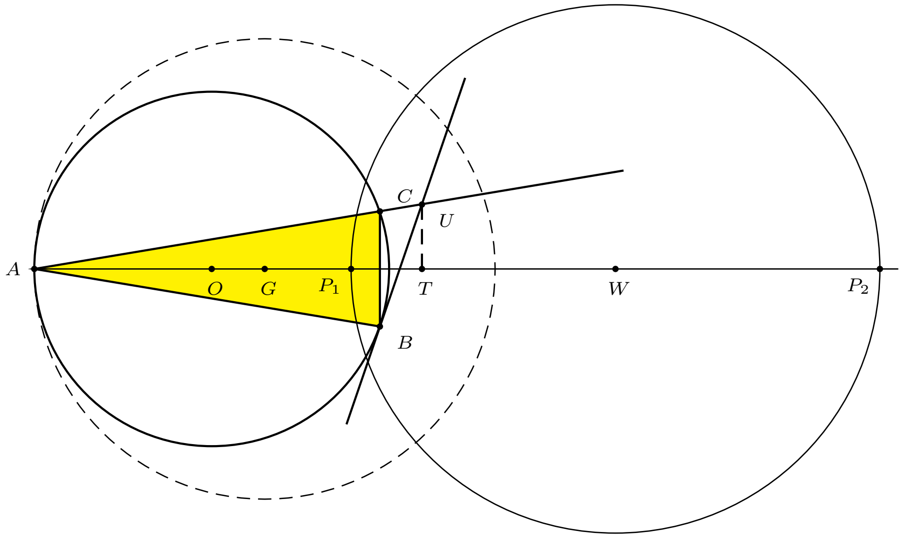

Figure 2

Proof. From (2), we have

$$f := c^2v(u+v) - b^2w(w+u) = 0, \tag{5}$$

$$g := a^2w(v+w) - c^2u(u+v) = 0, \tag{6}$$

$$h := b^2u(w+u) - a^2v(v+w) = 0. \tag{7}$$

From these,

$$0 = f + g + h = (b^2 - c^2)u^2 + (c^2 - a^2)v^2 + (a^2 - b^2)w^2.$$

This is the conic through the centroid $G = (1 : 1 : 1)$, the incenter $I = (a : b : c)$, and the vertices of their anticevian triangles.

Also, from (5)–(7),

$$0 = a^2f + b^2g + c^2h = a^2(b^2 - c^2)vw + b^2(c^2 - a^2)wu + c^2(a^2 - b^2)uv = 0.$$

This shows that the point P also lies on the circumconic through G and the symmedian point $K = (a^2 : b^2 : c^2)$.

If P is the centroid, the conic through the six points has equation

$$4(yz + zx + xy) + 3(x + y + z)^2 = 0.$$

This is homothetic to the Steiner circum-ellipse and is not a circle since the triangle is scalene. Therefore, if $\mathcal{C}(P)$ is a circle, P is an intersection of the two conics above, apart from the centroid G. □

Remark. The positive root corresponds to the intersection which lies on the arc GK of the circum-hyperbola through these two points.

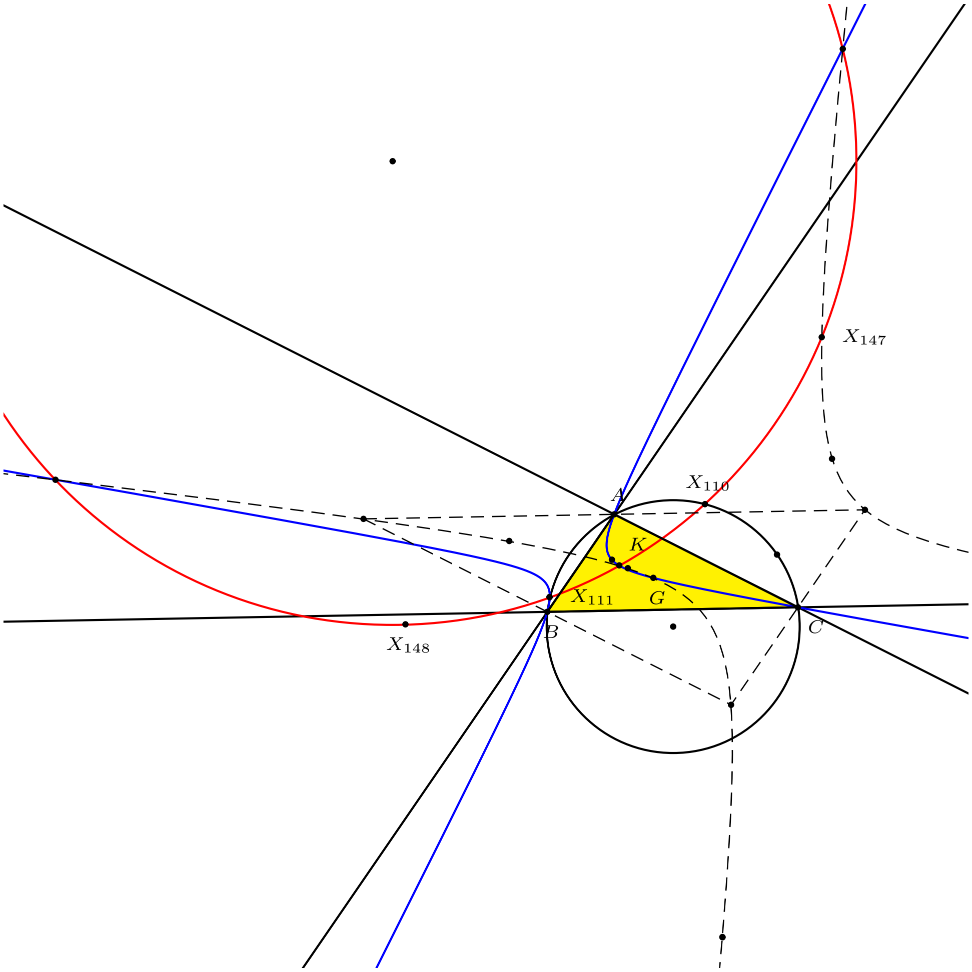

Figure 3.

Proposition 6. *The three real points P for which $\mathcal{C}(P)$ is a Tucker circle lie on a circle containing the following triangle centers:* (i) *the Euler reflection point*

$$X_{110} = \left(\frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2}\right),$$

(ii) *the Parry point*

$$X_{111} = \left(\frac{a^2}{b^2 + c^2 - 2a^2} : \frac{b^2}{c^2 + a^2 - 2b^2} : \frac{c^2}{a^2 + b^2 - 2c^2}\right),$$

(iii) *the Tarry point of the superior triangle X_{147},*
(iv) *the Steiner point of the superior triangle X_{148}.*

Proof. The combination

$$a^2(c^2 - a^2)(a^2 - b^2)f + b^2(a^2 - b^2)(b^2 - c^2)g + c^2(b^2 - c^2)(c^2 - a^2)h \quad (8)$$

of (5)–(7) (with x, y, z replacing u, v, w) yields the circle through the three points:

$$(a^2-b^2)(b^2-c^2)(c^2-a^2)(a^2yz+b^2zx+c^2xy) + (x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(b^2+c^2-2a^2)x\right) = 0. \tag{9}$$

Since the line

$$\sum_{\text{cyclic}} b^2c^2(b^2-c^2)(b^2+c^2-2a^2)x = 0$$

contains the Euler reflection point and the Parry point, as is easily verified, so does the circle (9).

If we replace in (5)–(7) u, v, w by $y+z-x$, $z+x-y$, $x+y-z$ respectively, the combination (8) yields the circle

$$2(a^2-b^2)(b^2-c^2)(c^2-a^2)(a^2yz+b^2zx+c^2xy) - (x+y+z)\left(\sum_{\text{cyclic}} a^2(b^2-c^2)(b^4+c^4-a^2(b^2+c^2))x\right) = 0, \tag{10}$$

which is the inferior of the circle (9). Since the line

$$\sum_{\text{cyclic}} a^2(b^2-c^2)(b^4+c^4-a^2(b^2+c^2))x = 0$$

clearly contains the Tarry point

$$\left(\frac{1}{b^4+c^4-a^2(b^2+c^2)} : \frac{1}{c^4+a^4-b^2(c^2+a^2)} : \frac{1}{a^4+b^4-c^2(a^2+b^2)}\right),$$

and the Steiner point

$$\left(\frac{1}{b^2-c^2} : \frac{1}{c^2-a^2} : \frac{1}{a^2-b^2}\right),$$

so does the circle (10). It follows that the circle (9) contains these two points of the superior triangle. □

Remark. (1) The triangle center X_{147} also lies on the hyperbola through the hyperbola in Proposition 5(i).

(2) The Parry point X_{111} also lies on the circum-hyperbola through G and K (in Proposition 5(ii)). It is the isogonal conjugate of the infinite point of the line GK.

We conclude this note by briefly considering a conic companion to $\mathcal{C}(P)$.

With the same parallel lines through the traces of P on the sidelines, consider the intersections

Point	Intersection of	with the parallel to	through	Coordinates
B'_a	CA	BY	X	$(uv : 0 : w(u+v+w))$
C'_a	AB	CZ	X	$(wu : v(u+v+w) : 0)$
C'_b	AB	CZ	Y	$(u(u+v+w) : vw : 0)$
A'_b	BC	AX	Y	$(0 : uv : w(u+v+w))$
A'_c	BC	AX	Z	$(0 : v(u+v+w) : uw)$
B'_c	CA	BY	Z	$(u(u+v+w) : 0 : vw)$

These six points also lie on a conic $\mathcal{C}'(P)$, which has equation

$$(u+v)(v+w)(w+u)\sum_{\text{cyclic}} u(v+w)yz-(u+v+w)(x+y+z)\sum_{\text{cyclic}} v^2w^2x=0.$$

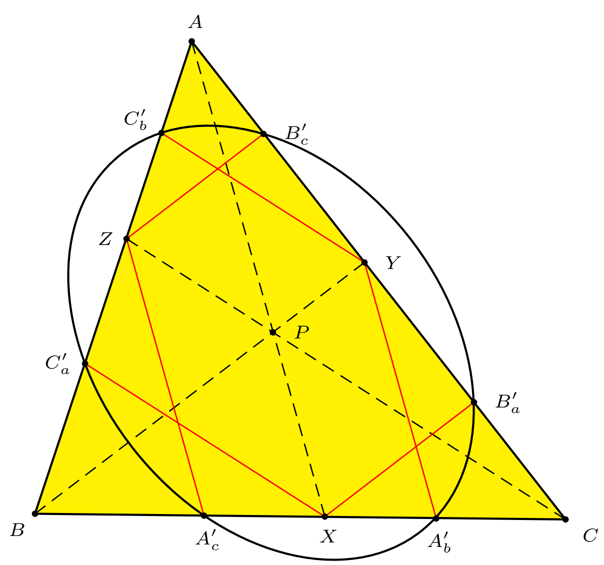

Figure 4.

In this case, the lines $B'_cC'_b$, $C'_aA'_c$, $A'_bB'_a$ are parallel to the sidelines, and bound a triangle homothetic to ABC at the point

$$(u(v+w):\ v(w+u):\ w(u+v)),$$

which is the inferior of the isotomic conjugate of P. The lines $B'_aC'_a$, $C'_bA'_b$, $A'_cB'_c$ are antiparallels if and only if the homothetic center is the symmedian point. Therefore, the conic $\mathcal{C}'(P)$ is a circle if and only if P is the isotomic conjugate of the superior of K, namely, the orthocenter H. The resulting circle is the Taylor circle.

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Forum Geometricorum
Volume 11 (2011) 269–275.

FORUM GEOM
ISSN 1534-1178

On Six Circumcenters and Their Concyclicity

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Abstract. Given triangle ABC, let P be a point with circumcevian triangle $A'B'C'$. We determine the positions of P such that the circumcenters of the six circles PBC', $PB'C$, PCA', $PC'A$, PAB', $PA'B$ are concyclic. There are two such real points P which lie on the Euler line of ABC provided the triangle is acute-angled. We provide two simple constructions of such points.

In the plane of a given triangle ABC with circumcenter O, consider a point P with its circumcevian triangle $A'B'C'$. In Theorem 1 below we show that the centers of the six circles PBC', $PB'C$, PCA', $PC'A$, PAB', $PA'B$ form three segments sharing a common midpoint M with OP. It follows that these six circumcenters lie on a conic $\mathcal{C}(P)$. We proceed to identify the point P for which this conic is a circle. It turns out (Theorem 1 below) that there are two such real points lying on the Euler line when the given triangle is acute-angled, and these points can be easily constructed with ruler and compass.

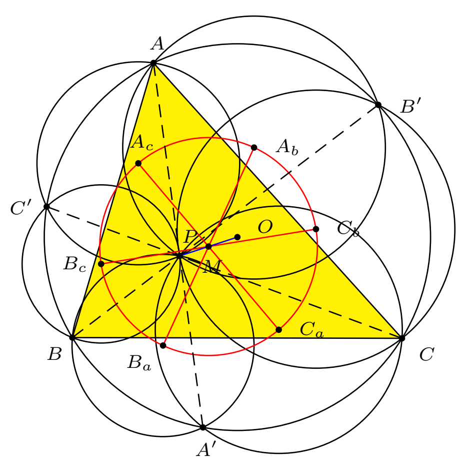

Figure 1. Six centers on a conic

Denote by B_c, C_b, C_a, A_c, A_b, B_a the centers of the circles PBC', PCB', PCA', PAC', PAB', PBA' respectively, and by r_{bc}, r_{cb}, r_{ca}, r_{ac}, r_{ab}, r_{ba} their radii. Let R be the circumradius of triangle ABC.

Publication Date: December 29, 2011. Communicating Editor: Paul Yiu.
The authors thank the editor for his help in finding the coordinates of P_+ and P_- in Theorem 5 and putting things together.

Theorem 1. *The segments B_cC_b, C_aA_c, A_bB_a, and OP share a common midpoint.*

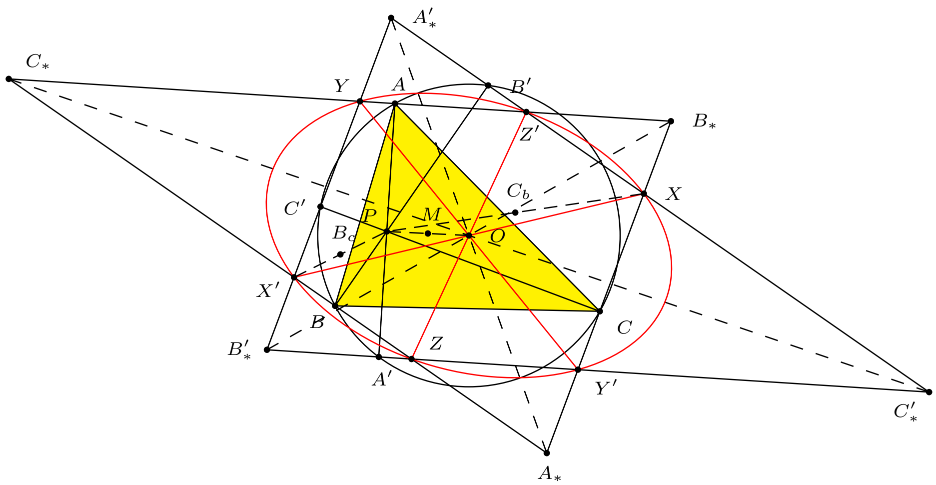

Figure 2. Antipedal triangle of P and its reflection in O

Proof. Consider the lines perpendicular to AP, BP, CP at A, B, C respectively. These lines bound the antipedal triangle $A_*B_*C_*$ of P. If we draw the corresponding lines at A', B', C' perpendicular to $A'P$, $B'P$, $C'P$, we obtain a triangle $A'_*B'_*C'_*$ oppositely homothetic to $A_*B_*C_*$. Since the parallel through the circumcenter O (of triangle ABC) to B_*C_* and $B'_*C'_*$ passes through the midpoint of AA', O is equidistant from the parallel lines B_*C_* and $B'_*C'_*$. The same is true for the other two pairs of lines C_*A_*, $C'_*A'_*$, and A_*B_*, $A'_*B'_*$. Therefore, the two triangles $A_*B_*C_*$ and $A'_*B'_*C'_*$ are oppositely congruent at O (see Figure 2). By symmetry, their sidelines intersect at six points which are pairwise symmetric in O. These are the points

$$\begin{aligned} X &:= A_*B_* \cap C'_*A'_*, & X' &:= A'_*B'_* \cap C_*A_*; \\ Y &:= B_*C_* \cap A'_*B'_*, & Y' &:= B'_*C'_* \cap A_*B_*; \\ Z &:= C_*A_* \cap B'_*C'_*, & Z' &:= C'_*A'_* \cap B_*C_*. \end{aligned}$$

The six circumcenters B_c, C_b, C_a, A_c, A_b, B_a are the images of X', X, Y', Y, Z', Z under the homothety $\mathsf{h}\left(P, \frac{1}{2}\right)$. It follows that the segments B_cC_b, C_aA_c, A_bB_a share a common midpoint, which is $\mathsf{h}\left(P, \frac{1}{2}\right)(O)$, the midpoint of OP. $\square$

We determine the location of P for which the conic through these six circumcenters is a circle. Clearly, this is case if and only if $B_cC_b = C_aA_c = A_bB_a$.

Lemma 2. *The radii of the circles PBC', PCB', PCA', PAC', PAB', PBA' are*

$$r_{bc} = \frac{R}{a}\cdot BP, \qquad r_{cb} = \frac{R}{a}\cdot CP;$$
$$r_{ca} = \frac{R}{b}\cdot CP, \qquad r_{ac} = \frac{R}{b}\cdot AP;$$
$$r_{ab} = \frac{R}{c}\cdot AP, \qquad r_{ba} = \frac{R}{c}\cdot BP.$$

Proof. It is enough to establish the expression for r_{bc}. The others follow similarly.

Note that $\angle BC'P = \angle BC'C = \angle BAC$. Applying the law of sines to triangles PBC' and BCC', we have

$$r_{bc} = \frac{BP}{2\sin BC'P} = \frac{BP}{2\sin BAC} = \frac{R}{BC}\cdot BP = \frac{R}{a}\cdot BP.$$

□

Theorem 3. *Let A_1, B_1, C_1 be the midpoints of BC, CA, AB respectively. The six circumcenters lie on a circle if and only if*

$$A_1P : B_1P : C_1P = B_1C_1 : C_1A_1 : A_1B_1. \tag{1}$$

Proof. Let M be the common midpoint of OP, B_cC_b, C_aA_c, A_bB_a. Clearly the conic through the six circumcenter is a circle if and only if $B_cC_b = C_aA_c = A_bB_a$. Applying Apollonius' theorem to the triangles PB_cC_b and PBC, making use of Lemma 2, we have

$$2PM^2 + \frac{B_cC_b^2}{2} = r_{bc}^2 + r_{cb}^2 = \frac{R^2}{a^2}(BP^2 + CP^2) = \frac{R^2}{2}\left(\frac{A_1P^2}{B_1C_1^2} + 1\right). \tag{2}$$

Similarly,

$$2PM^2 + \frac{C_aA_c^2}{2} = \frac{R^2}{2}\left(\frac{B_1P^2}{C_1A_1^2} + 1\right), \tag{3}$$

$$2PM^2 + \frac{A_bB_a^2}{2} = \frac{R^2}{2}\left(\frac{C_1P^2}{A_1B_1^2} + 1\right). \tag{4}$$

Comparison of (2), (3) and (4) yields (1) as a necessary and sufficient condition for $B_cC_b = C_aA_c = A_bB_a$; hence for the six circumcenters to lie on a circle. □

Now we identify the points P satisfying the condition (1).

Let A_2, B_2, C_2 be the midpoints of B_1C_1, C_1A_1, A_1B_1 respectively. Consider the reflections P_a, P_b, P_c of P in A_2, B_2, C_2 respectively. Since $P_aB_1 = PC_1$ and $P_aC_1 = PB_1$, we have $\frac{P_aC_1}{C_1A_1} = \frac{P_aB_1}{A_1B_1}$ or $\frac{P_aB_1}{P_aC_1} = \frac{A_1B_1}{A_1C_1}$. This means that P_a is on the A_1-Apollonian circle of triangle $A_1B_1C_1$. Equivalently, P is a point on the circle $\mathscr{C}_a$ which is the reflection of the A_1-Apollonian circle of $A_1B_1C_1$ in the perpendicular bisector of B_1C_1. For the same reason, P also lies on the two circles $\mathscr{C}_b$ and $\mathscr{C}_c$, which are the reflections of the B_1- and C_1-Apollonian circles in the perpendicular bisectors of C_1A_1 and A_1B_1 respectively.

The circle $\mathscr{C}_a$ passes through A, H_a the trace of H on BC, and the intersection of B_1C_1 with the internal bisector of angle A of ABC. Hence the diameter AO of ABC is tangent to this circle. This leads to the following simple barycentric equations of $\mathscr{C}_a$, and the other two circles.

$$\begin{aligned}
\mathscr{C}_a: &\quad (S_B - S_C)(a^2yz + b^2zx + c^2xy) - (x+y+z)(c^2S_By - b^2S_Cz) = 0,\\
\mathscr{C}_b: &\quad (S_C - S_A)(a^2yz + b^2zx + c^2xy) - (x+y+z)(a^2S_Cz - c^2S_Ax) = 0,\\
\mathscr{C}_c: &\quad (S_A - S_B)(a^2yz + b^2zx + c^2xy) - (x+y+z)(b^2S_Ax - a^2S_By) = 0.
\end{aligned}$$

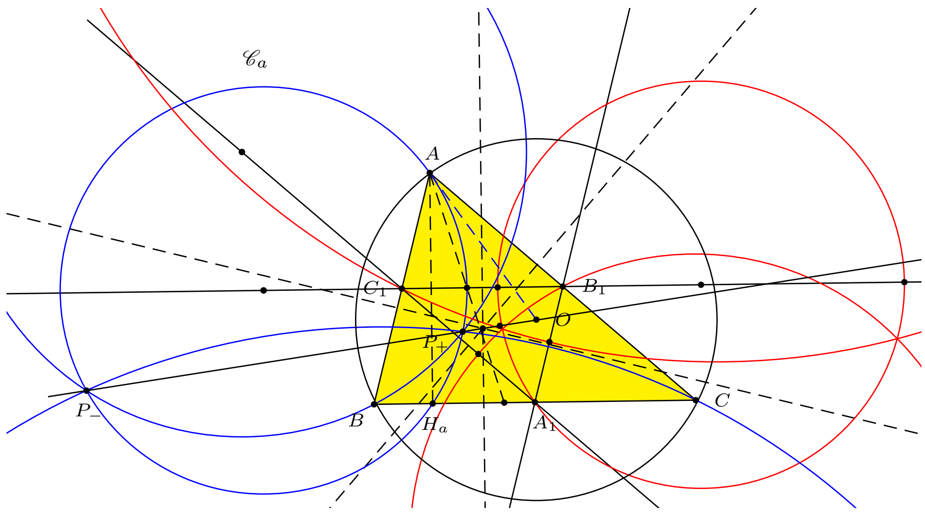

Figure 3. $P_\pm$ as intersections of reflections of Apollonian circles

Proposition 4. *The two points $P_\pm$ lie on the Euler line of triangle ABC.*

Proof. If $AP = \lambda$, $BP = \mu$, $CP = \nu$ are the tripolar coordinates of P with reference to ABC (see [1]), then from (1), we have

$$\frac{2(\mu^2+\nu^2)-a^2}{a^2} = \frac{2(\nu^2+\lambda^2)-b^2}{b^2} = \frac{2(\lambda^2+\mu^2)-c^2}{c^2},$$

or

$$\frac{\mu^2+\nu^2}{a^2} = \frac{\nu^2+\lambda^2}{b^2} = \frac{\lambda^2+\mu^2}{c^2} = k,$$

for some k. Hence,

$$\lambda^2 = \frac{k(b^2+c^2-a^2)}{2}, \qquad \mu^2 = \frac{k(c^2+a^2-b^2)}{2}, \qquad \nu^2 = \frac{k(a^2+b^2-c^2)}{2}, \tag{5}$$

and

$$(b^2-c^2)\lambda^2 + (c^2-a^2)\mu^2 + (a^2-b^2)\nu^2 = 0.$$

This is the equation of the Euler line in tripolar coordinates ([1, Proposition 3]). $\square$

Representing the circle $\mathscr{C}_a$ by the matrix

$$M_a = \begin{pmatrix} 0 & -S_C(S_A+S_B) & S_B(S_C+S_A) \\ -S_C(S_A+S_B) & -2S_B(S_A+S_B) & -S_A(S_B-S_C) \\ S_B(S_C+S_A) & -S_A(S_B-S_C) & 2S_C(S_C+S_A) \end{pmatrix},$$

we compute the equation of the polar of G in the circle. This gives

$$\begin{pmatrix} x & y & z \end{pmatrix} M_a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0,$$

or

$$S_A(S_B-S_C)x - S_B(3S_A+2S_B+S_C)y + S_C(3S_A+S_B+2S_C)z = 0.$$

Clearly, this polar contains the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$. This shows that G and H are conjugate in the circle $\mathscr{C}_a$; similarly also in the circles $\mathscr{C}_b$ and $\mathscr{C}_c$. Therefore, G and H divide P_+ and P_- harmonically.

Theorem 5. *The two points satisfying* (1) *are*

$$P_\varepsilon := (\sqrt{S_A+S_B+S_C}\cdot S_{BC} + \varepsilon S \cdot \sqrt{S_{ABC}} : \cdots : \cdots), \quad \varepsilon = \pm 1$$

in homogeneous barycentric coordinates.

Proof. Let $P = (S_{BC}+t : S_{CA}+t : S_{AB}+t)$. We have

$$\begin{aligned} 0 = & \begin{pmatrix} S_{BC}+t & S_{CA}+t & S_{AB}+t \end{pmatrix} M_a \begin{pmatrix} S_{BC}+t \\ S_{CA}+t \\ S_{AB}+t \end{pmatrix} \\ = & \begin{pmatrix} S_{BC} & S_{CA} & S_{AB} \end{pmatrix} M_a \begin{pmatrix} S_{BC} \\ S_{CA} \\ S_{AB} \end{pmatrix} + \begin{pmatrix} t & t & t \end{pmatrix} M_a \begin{pmatrix} t \\ t \\ t \end{pmatrix} \\ = & 2S_{ABC}(S_B-S_C)(S_{BC}+S_{CA}+S_{AB}) - 2t^2(S_A+S_B+S_C)(S_B-S_C). \end{aligned}$$

It follows that $t^2 = \frac{S^2 \cdot S_{ABC}}{S_A+S_B+S_C}$. From these we obtain the coordinates of the two points $P_\pm$ given above. □

Proposition 6. *The midpoint of the segment P_+P_- is the point*

$$Q = (S_{BC}(S_B+S_C-2S_A) : S_{CA}(S_C+S_A-2S_B) : S_{AB}(S_A+S_B-2S_C)).$$

Proof. The midpoint between the two points $(S_{BC}+t : S_{CA}+t : S_{AB}+t)$ and $(S_{BC}-t : S_{CA}-t : S_{AB}-t)$ has coordinates

$$\begin{aligned} &(S_{BC}+S_{CA}+S_{AB}-3t)(S_{BC}+t, S_{CA}+t, S_{AB}+t)) \\ +&(S_{BC}+S_{CA}+S_{AB}+3t)(S_{BC}-t, S_{CA}-t, S_{AB}-t)) \\ =&(S_{BC}S^2-3t^2, S_{CA}S^2-3t^2, S_{AB}S^2-3t^2). \end{aligned}$$

For $t^2 = \frac{S_{ABC}\cdot S^2}{S_A+S_B+S_C}$, this simplifies into the form given above. □

The point Q is the intersection of the Euler line and the orthic axis. [1]

This leads to the simple construction of the two points P_+ and P_- as the intersections of the circle with center Q, orthogonal to the orthocentroidal circle (see Figure 4).

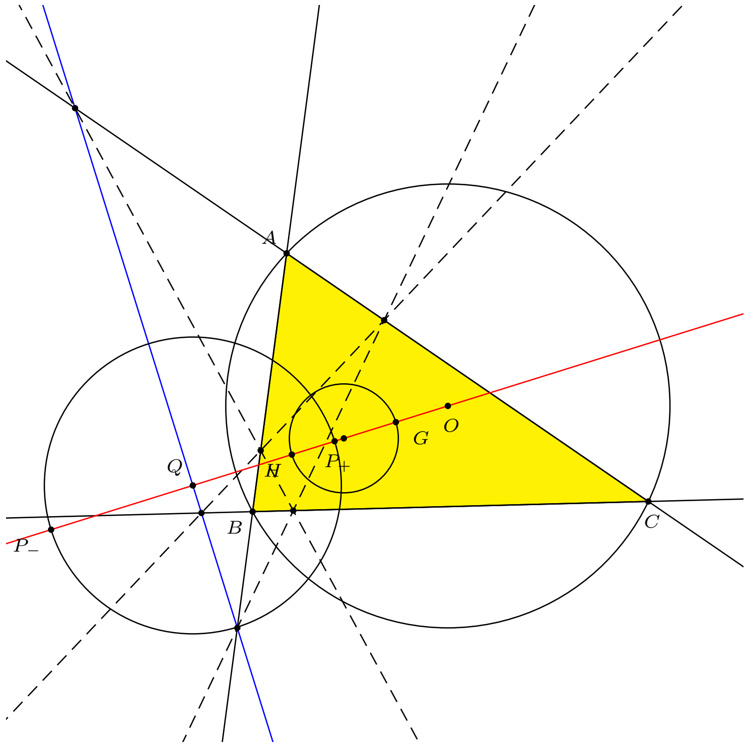

Figure 4. Construction of P_+ and P_-

We conclude this note with the remark that the problem of construction of points whose distances from the vertices of a given triangle are proportional to the lengths of the opposite sides was addressed in [4]. Also, according to [5], these two points can also be constructed as the common points of the triad of generalized Apollonian circles for the isotomic conjugate of the incenter, namely, the triangle center $X_{75} = \left(\frac{1}{a} : \frac{1}{b} : \frac{1}{c}\right)$.

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