# Mathematical Excalibur

Volume 5, Number 1 | January 2000 – February 2000


## Olympiad Corner

*8th Taiwan (ROC) Mathematical Olympiad, April 1999:*

*Time allowed: 4.5 Hours*
*Each problem is worth 7 points.*

**Problem 1.** Determine all solutions ($x$, $y$, $z$) of positive integers such that

$$(x+1)^{y+1}+1=(x+2)^{z+1}.$$

**Problem 2.** Let $a_1, a_2, .., a_{1999}$ be a sequence of nonnegative integers such that for any integers $i$, $j$, with $i + j \le 1999$,

$$a_i + a_j \le a_{i+j} \le a_i + a_j + 1.$$

Prove that there exists a real number $x$ such that $a_n = [nx]$ for each $n$ = 1, 2, …, 1999, where $[nx]$ denotes the largest integer less than or equal to $nx$.

**Problem 3.** There are 1999 people participating in an exhibition. Two of any 50 people do not know each other. Prove that there are at least 41 people, and each of them knows at most 1958 people.

*(continued on page 4)*




**Acknowledgment**: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

**On-line:** http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is March 4, 2000.

For individual subscription for the next five issues for the 00-01 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:



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# The Road to a Solution

*Kin Y. Li*

Due to family situation, I missed the trip to the 1999 IMO at Romania last summer. Fortunately, our Hong Kong team members were able to send me the problems by email. Of course, once I got the problems, I began to work on them. The first problem is the following.

*Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry of S.*

This was a nice problem. I spent sometime on it and got a solution. However, later when the team came back and I had a chance to look at the official solution, I found it a little beyond my expectation. Below I will present my solution and the official solution for comparison.

Here is the road I took to get a solution. To start the problem, I looked at the case of three points, say $P_1, P_2, P_3$, satisfying the condition. Clearly, the three points cannot be collinear (otherwise considering the perpendicular bisector of the segment joining two consecutive points on the line will yield a contradiction). Now by the condition, it follows that $P_2$ must be on the perpendicular bisector of segment $P_1\ P_3$. Hence, $P_1\ P_2 = P_2\ P_3$. By switching indices, $P_3$ should be on the perpendicular bisector of $P_2\ P_1$ and so $P_2\ P_3 = P_3\ P_1$. Thus, $P_1$, $P_2$, $P_3$ are the vertices of an equliateral triangle.

Next the case of four points required more observations. Again no three points are collinear. Also, from the condition, none of the point can be inside the triangle having the other three points as vertices. So the four points are the vertices of a convex quadrilateral. Then the sides have equal length as in the case of three points. Considering the perpendicular bisector of any side, by symmetry, the angles at the other two vertices must be the same. Hence all four angles are the same. Therefore, the four points are the vertices of a square.

After the cases of three and four points, it is quite natural to guess such sets are the vertices of regular polygons. The proof of the general case now follows from the reasonings of the two cases we looked at. First, no three points are collinear. Next, the smallest convex set enclosing the points must be a polygonal region with all sides having the same length and all angles the same. So the boundary of the region is a regular polygon. Finally, one last detail is required. In the case of four points, no point is inside the triangle formed by the other three points by inspection. However, for large number of points, inspection is not good enough. To see that none of the points is inside the polygonal region takes a little bit more work.

Again going back to the case of four points, it is natural to look at the situation when one of the point, say $P$, is inside the triangle formed by the other three points. Considering the perpendicular bisectors of three segments joining $P$ to the other three points, we see that we can always get a contradiction.

Putting all these observations together, here is the solution I got:

Clearly, no three points of such a set is collinear (otherwise considering the perpendicular bisector of the two furthest points of $S$ on that line, we will get a contradiction). Let $H$ be the convex hull of such a set, which is the smallest convex set containing $S$. Since $S$ is finite, the boundary of $H$ is a polygon with the vertices $P_1$, $P_2, ..., P_n$ belonging to $S$. Let $P_i = P_j$ if $i \equiv j \pmod n$. For $i$ = 1, 2, …, $n$, the condition on the set implies $P_i$ is on the perpendicular bisector of $P_{i-1}\ P_{i+1}$. So $P_{i-1}\ P_i = P_i\ \ P_{i+1}$. Considering the perpendicular bisector of $P_{i-1}\ P_{i+2}$, we see that $\angle P_{i-1}\ \ P_i\ P_{i+1} = \angle P_i P_{i+1} P_{i+2}$. So the boundary of $H$ is a regular polygon.

Next, there cannot be any point $P$ of $S$ inside the regular polygon. (To see this, assume such a $P$ exists. Place it at the origin and the furthest point $Q$ of $S$ from $P$ on the positive real axis. Since the origin $P$ is in the interior of the convex polygon, not all the vertices can lie on or to the right of the $y$-axis. So there exists a vertex $P_j$ to the left of the $y$-axis. Since the perpendicular bisector of $PQ$ is an axis of symmetry, the mirror image of $P_j$ will be a point in $S$ further than $Q$ from $P$, a contradiction.) So $S$ is the set of vertices of some regular polygon. Conversely, such a set clearly has the required property.

Next we look at the official solution, which is shorter and goes as follows: Suppose $S = \{X_1, \ldots, X_n\}$ is such a set. Consider the *barycenter* of $S$, which is the point $G$ such that

$$\overrightarrow{OG} = \frac{\overrightarrow{OX}_1 + \cdots + \overrightarrow{OX}_n}{n}.$$

Note the barycenter does not depend on the origin. To see this, suppose we get a point $G'$ using another origin $O'$, i.e. $\overrightarrow{O'G'}$ is the average of $\overrightarrow{O'X_i}$ for $i = 1, \ldots, n$. Subtracting the two averages, we get $\overrightarrow{OG} - \overrightarrow{O'G'} = \overrightarrow{OO'}$. Adding $\overrightarrow{O'G'}$ to both sides, $\overrightarrow{OG} = \overrightarrow{OG'}$, so $G = G'$.

By the condition on $S$, after reflection with respect to the perpendicular bisector of every segment $X_iX_j$, the points of S are permuted only. So $G$ is unchanged, which implies $G$ is on every such perpendicular bisector. Hence, $G$ is equidistant from all $X_i$'s. Therefore, the $X_i$'s are concyclic. For three consecutive points of $S$, say $X_i, X_j, X_k$, on the circle, considering the perpendicular bisector of segment $X_iX_k$, we have $X_iX_j = X_jX_k$. It follows that the points of $S$ are the vertices of a regular polygon and the converse is clear.

# Cavalieri's Principle

*Kin Y. Li*

Have you ever wondered why the volume of a sphere of radius $r$ is given by the formula $\frac{4}{3}\pi r^3$? The $r^3$ factor can be easily accepted because volume is a three dimensional measurement. The $\pi$ factor is probably because the sphere is round. Why then is there $\frac{4}{3}$ in the formula?

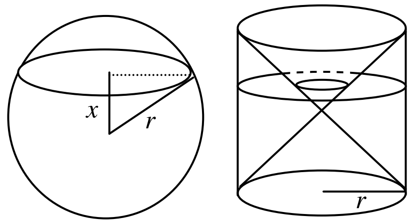
In school, most people told you it came from calculus. Then, how did people get the formula before calculus was invented? In particular, how did the early Egyptian or Greek geometers get it thousands of years ago?

Those who studied the history of mathematics will be able to tell us more of the discovery. Below we will look at one way of getting the formula, which may not be historically the first way, but it has another interesting application as we will see. First, let us introduce

**Cavalieri's Principle:** *Two objects having the same height and the same cross sectional area at each level must have the same volume.*

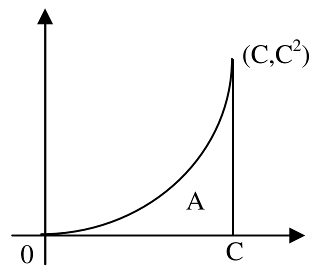
To understand this, imagine the two objects are very large, like pyramids that are built by piling bricks one level on top of another. By definition, the volume of the objects are the numbers of $1\times1\times1$ bricks used to build the objects. If at *each level* of the construction, the number of bricks used (which equals the cross sectional area numerically) is the same for the two objects, then the volume (which equals the total number of bricks used) would be the same for both objects.

To get the volume of a sphere, let us apply Cavalieri's principle to a solid sphere $S$ of radius $r$ and an object $T$ made out from a solid right circular cylinder with height $2r$ and base radius $r$ removing a pair of right circular cones with height $r$ and base radius $r$ having the center of the cylinder as the apex of each cone.




Both $S$ and $T$ have the same height $2r$. Now consider the cross sectional area of each at a level $x$ units from the equatorial plane of $S$ and $T$. The cross section for $S$ is a circular disk of radius $\sqrt{r^2 - x^2}$ by Pythagoras' theorem, which has area $\pi(r^2 - x^2)$. The cross section for $T$ is an annular ring of outer radius $r$ and inner radius $x$, which has the same area $\pi r^2 - \pi x^2$. By Cavalieri's principle, $S$ and $T$ have the same volume. Since the volume of $T$ is $\pi r^2(2r) - 2\times\frac{1}{3}\pi r^2 r = \frac{4}{3}\pi r^3$, so the volume of $S$ is the same.

Cavalieri's principle is not only useful in getting the volume of special solids, but it can also be used to get the area of special regions in a plane! Consider the region $A$ bounded by the graph of $y = x^2$, the $x$-axis and the line $x = c$ in the first quadrant.




The area of this region is less than the area of the triangle with vertices at (0, 0), ($c$, 0), ($c$, $c^2$), which is $\frac{1}{2}c^3$. If you ask a little kid to guess the answer, you may get $\frac{1}{3}c^3$ since he knows $\frac{1}{3} < \frac{1}{2}$. For those who know calculus, the answer is easily seen to be correct. How can one explain this without calculus?

*(continued on page 4)*

## Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *March 4, 2000.*

**Problem 96.** If every point in a plane is colored red or blue, show that there exists a rectangle all of its vertices are of the same color.

**Problem 97.** A group of boys and girls went to a restaurant where only big pizzas cut into 12 pieces were served. Every boy could eat 6 or 7 pieces and every girl 2 or 3 pieces. It turned out that 4 pizzas were not enough and that 5 pizzas were too many. How many boys and how many girls were there? (*Source: 1999 National Math Olympiad in Slovenia*)

**Problem 98.** Let $ABC$ be a triangle with $BC > CA > AB$. Select points $D$ on $BC$ and $E$ on the extension of $AB$ such that $BD = BE = AC$. The circumcircle of $BED$ intersects $AC$ at point $P$ and $BP$ meets the circumcircle of $ABC$ at point $Q$. Show that $AQ + CQ = BP$. (*Source: 1998-99 Iranian Math Olympiad*)

**Problem 99.** At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered. (*Source: 1996 Spanish Math Olympiad*)

**Problem 100.** The arithmetic mean of a number of pairwise distinct prime numbers equals 27. Determine the biggest prime that can occur among them. (*Source: 1999 Czech and Slovak Math Olympiad*)

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### *Solutions*

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**Problem 91.** Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1-\frac{1}{x+y}\right)=4\sqrt{2}\,.$$

(*This is the corrected version of problem 86.*)

**Solution. (CHENG Kei Tsi, LEE Kar Wai, TANG Yat Fai)** (La Salle College, Form 5), **CHEUNG Yui Ho Yves** (University of Toronto), **HON Chin Wing** (Pui Ching Middle School, Form 5) **KU Hong Tung** (Carmel Divine Grace Foundation Secondary School, Form 6), **LAU Chung Ming Vincent** (STFA Leung Kau Kui College, Form 5), **LAW Siu Lun Jack** (Ming Kei College, Form 5), **Kevin LEE** (La Salle College, Form 4), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **MAK Hoi Kwan Calvin** (Form 4), **NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 6), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4), **TANG King Fun** (Valtorta College, Form 5), **WONG Chi Man** (Valtorta College, Form 5) and **WONG Chun Ho Terry** (STFA Leung Kau Kui College, Form 5). (All solutions received were essentially the same.) Clearly, if $(x, y)$ is a solution, then $x, y > 0$ and

$$1+\frac{1}{x+y}=\frac{2}{\sqrt{3x}}$$

$$1-\frac{1}{x+y}=\frac{4\sqrt{2}}{\sqrt{7y}}\,.$$

Taking the difference of the squares of both equations, we get

$$\frac{4}{x+y}=\frac{4}{3x}-\frac{32}{7y}\,.$$

Simplifying this, we get $0 = 7y^2 - 38xy - 24x^2 = (7y+4x)(y-6x)$. Since $x, y > 0$, $y = 6x$. Substituting this into the first given equation, we get $\sqrt{3x}\left(1+\frac{1}{7x}\right)=2$, which simplifies to $7\sqrt{3x}-14\sqrt{x}+\sqrt{3}=0$. By the quadratic formula, $\sqrt{x}=(7\pm 2\sqrt{7})/(7\sqrt{3})$. Then $x=(11\pm 4\sqrt{7})/21$ and $y = 6x = (22 \pm 8\sqrt{7})/7$. Direct checking shows these are solutions.

*Comments*: An alternative way to get the answers is to substitute $u = \sqrt{x}$, $v = \sqrt{y}$, $z = u + iv$, then the given equations become the real and imaginary parts of the complex equation $z+\frac{1}{z}=c$, where $c = \frac{2}{\sqrt{3}}+i\frac{4\sqrt{2}}{\sqrt{7}}$. Multiplying by $z$, we can apply the quadratic formula to get $u + iv$, then squaring $u$, $v$, we can get $x$, $y$.

**Problem 92.** Let $a_1, a_2, \ldots, a_n\ (n > 3)$ be real numbers such that $a_1+a_2+\cdots+a_n \ge n$ and $a_1^2+a_2^2+\cdots+a_n^2 \ge n^2$. Prove that $\max(a_1, a_2, \ldots, a_n) \ge 2$. (*Source: 1999 USA Math Olympiad*).

**Solution. FAN Wai Tong Louis** (St. Mark's School, Form 7).

Suppose $\max(a_1, a_2, \ldots, a_n) < 2$. By relabeling the indices, we may assume $2 > a_1 \ge a_2 \ge \cdots \ge a_n$. Let $j$ be the largest index such that $a_j \ge 0$. For $i > j$, let $b_i = -a_i > 0$. Then

$$2j-n > (a_1+\cdots+a_j)-n \ge b_{j+1}+\cdots+b_n\,.$$

So $(2j-n)^2 > b_{j+1}^2+\cdots+b_n^2$. Then

$$4j+(2j-n)^2 > a_1^2+\cdots+a_n^2 \ge n^2,$$

which implies $j > n - 1$. Therefore, $j = n$ and all $a_i \ge 0$. This yields $4n > a_1^2+\cdots+a_n^2 \ge n^2$, which gives the contradiction that $3 \ge n$.

*Other recommended solvers:* **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7) and **WONG Wing Hong** (La Salle College, Form 2).

**Problem 93.** Two circles of radii $R$ and $r$ are tangent to line $L$ at points $A$ and $B$ respectively and intersect each other at $C$ and $D$. Prove that the radius of the circumcircle of triangle $ABC$ does not depend on the length of segment $AB$. (*Source: 1995 Russian Math Olympiad*).

**Solution. CHAO Khek Lun** (St. Paul's College, Form 5).

Let $O$, $O'$ be the centers of the circles of radius $R$, $r$, respectively. Let $\alpha = \angle CAB = \angle AOC/2$ and $\beta = \angle CBA = \angle BO'C/2$. Then $AC = 2R\sin\alpha$ and $BC = 2r\sin\beta$. The distance from $C$ to $AB$ is $AC\sin\alpha = BC\sin\beta$, which implies $\sin\alpha / \sin\beta = \sqrt{r/R}$. The circumradius of triangle

*ABC* is

$$\frac{AC}{2\sin\beta}=\frac{R\sin\alpha}{\sin\beta}=\sqrt{Rr}\ ,$$

which does not depend on the length of *AB*.

*Other recommended solvers:* **CHAN Chi Fung** (Carmel Divine Grace Foundation Secondary School, Form 6), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Chun Bartholomew** (Queen Elizabeth School), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4).

**Problem 94**. Determine all pairs (*m, n*) of positive integers for which $2^m+3^n$ is a square.

**Solution. NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7) and **YEUNG Kai Sing** (La Salle College, Form 3).

Let $2^m+3^n=a^2$. Then $a$ is odd and $a^2=2^m+3^n\equiv(-1)^m$ (mod 3). Since squares are 0 or 1 (mod 3), $m$ is even. Next $(-1)^n\equiv 2^m+3^n=a^2\equiv 1$ (mod 4) implies *n* is even, say $n = 2k$, $k \geq 1$. Then $2^m=(a+3^k)(a-3^k)$. So $a+3^k=2^r$, $a-3^k=2^s$ for integers $r>s\geq 0$ with $r + s = m$. Then $2\cdot 3^k=2^r-2^s$ implies $s = 1$, so $2^{r-1}-1=3^k$. Now $r+1=m$ implies *r* is odd. So

$$\left(2^{(r-1)/2}+1\right)\left(2^{(r-1)/2}-1\right)=3^k.$$

Since the difference of the factors is 2, not both are divisible by 3. Then the factor $2^{(r-1)/2}-1=1$. Therefore, $r=3,\ k=1,\ (m,n)=(4,2)$, which is easily checked to be a solution.

*Other recommended solvers:* **CHAO Khek Lun** (St. Paul's College, Form 5), **CHENG Kei Tsi** (La Salle College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **KU Hong Tung** (Carmel Divine Grace Foundation Secondary School, Form 6), **LAW Siu Lun Jack** (Ming Kei College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Chun Batholomew** (Queen Elizabeth School), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NG Ting Chi** (TWGH Chang Ming Thien College, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4).

**Problem 95.** Pieces are placed on an $n\times n$ board. Each piece "attacks" all squares that belong to its row, column, and the northwest-southeast diagonal which contains it. Determine the least number of pieces which are necessary to attack all the squares of the board. (*Source: 1995 Iberoamerican Olympiad*).

**Solution. LEUNG Wai Ying** (Queen Elizabeth School, Form 5).

Assign coordinates to the squares so (*x, y*) represents the square on the *x*-th column from the west and *y*-th row from the south. Suppose *k* pieces are enough to attack all squares. Then at least *n* – *k* columns, say columns $x_1,...,x_{n-k}$, and *n* – *k* rows, say $y_1,...,y_{n-k}$, do not contain any of the *k* pieces. Consider the 2(*n* - *k*) – 1 squares $(x_1,y_1)$, $(x_1,y_2)$, …, $(x_1, y_{n-k})$, $(x_2,y_1)$, $(x_3,y_1)$, …, $(x_{n-k}, y_1)$. As they are on different diagonals and must be attacked diagonally by the *k* pieces, we have $k\geq 2(n-k)-1$. Solving for *k*, we get $k \geq (2n-1)/3$. Now let *k* be the least integer such that $k\geq(2n-1)/3$. We will show *k* is the answer. The case *n* = 1 is clear. Next if *n* = 3*a* + 2 for a nonnegative integer *a*, then place *k* = 2*a* + 1 pieces at (1, *n*), (2, *n* – 2), (3, *n* – 4), …, (*a* + 1, *n* – 2*a*), (*a* + 2, *n* – 1), (*a* + 3, *n* – 3), (*a* + 4, *n* – 5), …, (2*a* + 1, *n* – 2*a* + 1). So squares with $x\leq 2a+1$ or $y\geq n-2a$ are under attacked horizontally or vertically. The other squares, with $2a + 2 \leq x \leq n$ and $1 \leq y \leq n - 2a - 1$, have $2a + 3 \leq x + y \leq 2n - 2a - 1$. Now the sums *x* + *y* of the *k* pieces range from *n* – *a* + 1 = 2*a* + 3 to *n* + *a* + 1 = 2*n* – 2*a* – 1. So the *k* pieces also attack the other squares diagonally.

Next, if *n* = 3*a* + 3, then *k* = 2*a* + 2 and we can use the 2*a* + 1 pieces above and add a piece at the southeast corner to attack all squares. Finally, if *n* = 3*a* + 4, then *k* = 2*a* + 3 and again use the 2*a* + 2 pieces in the last case and add another piece at the southeast corner.

*Other recommended solvers:* **(LEE Kar Wai Alvin, CHENG Kei Tsi Daniel, LI Chi Pang Bill, TANG Yat Fai Roger)** (La Salle College, Form 5), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7).

## Olympiad Corner

*(continued from page 1)*

**Problem 4.** Let $P^*$ denote all the odd primes less than 10000. Determine all possible primes $p\in P^*$ such that for each subset *S* of $P^*$, say *S* = { $p_1$, $p_2$, …, $p_k$ }, with $k \geq 2$, whenever $p\notin S$, there must be some *q* in $P^*$, but not in *S*, such that *q* + 1 is a divisor of $(p_1+1)\ (p_2+1)\ \ldots\ (p_k+1)$.

**Problem 5.** The altitudes through the vertices *A, B, C* of an acute-angled triangle *ABC* meet the opposite sides at *D, E, F*, respectively, and *AB* > *AC*. The line *EF* meets *BC* at *P*, and the line through *D* parallel to *EF* meets the lines *AC* and *AB* at *Q* and *R*, respectively. *N* is a point on the side *BC* such that $\angle NQP + \angle NRP < 180^\circ$. Prove that *BN* > *CN*.

**Problem 6.** There are 8 different symbols designed on *n* different T-shirts, where $n \geq 2$. It is known that each shirt contains at least one symbol, and for any two shirts, the symbols on them are not all the same. Suppose that for any *k* symbols, $1 \leq k \leq 7$, the number of shirts containing at least one of the *k* symbols is even. Find the value on *n*.

## Cavalieri

*(continued from page 2)*

To get the answer, we will apply Cavalieri's principle. Consider a solid right cylinder with height 1 and base region *A*. Numerically, the volume of this solid equals the area of the region *A*. Now rotate the solid so that the $1\times c^2$ rectangular face becomes the base. As we expect the answer to be $\frac{1}{3}c^3$, we compare this rotated solid with a solid right pyramid with height *c* and square base of side *c*.

Both solids have height *c*. At a level *x* units below the top, the cross section of the rotated solid is a $1\times x^2$ rectangle. The cross section of the right pyramid is a square of side *x*. So both solids have the same cross sectional areas at all levels. Therefore, the area of *A* equals numerically to the volume of the pyramid, which is $\frac{1}{3}c^3$.

# Mathematical Excalibur

Volume 5, Number 2 — *March 2000 – April 2000*

## Olympiad Corner

*28th United States of America Mathematical Olympiad, April 1999:*

*Time allowed: 6 Hours*

**Problem 1.** Some checkers placed on an $n\times n$ checkerboard satisfy the following conditions:

(a) every square that does not contain a checker shares a side with one that does;

(b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2-2)/3$ checkers have been placed on the board.

**Problem 2.** Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB-CD|+|AD-BC|\geq 2|AC-BD|.$$

**Problem 3.** Let $p>2$ be a prime and let $a, b, c, d$ be integers not divisible by $p$, such that

$$\{ra/p\}+\{rb/p\}+\{rc/p\}+\{rd/p\}=2$$

*(continued on page 4)*



**Acknowledgment**: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

**On-line:** http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is May 20, 2000.

For individual subscription for the next five issues for the 00-01 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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# 漫談質數

梁子傑
香港道教聯合會青松中學

我們從小學開始就已經認識甚麼是質數。一個大於 1 的整數，如果祇能被 1 或自己整除，則我們稱該數為「質數」。另外，我們叫 1 做「單位」，而其他的數字做「合成數」。例如：2、3、5、7 … 等等，就是質數，4、6、8、9 … 等等就是合成數。但是除了這個基本的定義之外，一般教科書中，就很少提到質數的其他性質了。而本文就為大家介紹一些與質數有關的人和事。

有人相信，人類在遠古時期，就經已發現質數。不過最先用文字紀錄質數性質的人，就應該是古希臘時代的偉大數學家歐幾里得 (Euclid) 了。

歐幾里得，約生於公元前 330 年，約死於公元前 275 年。他是古代亞歷山大里亞學派的奠基者。他的著作《幾何原本》，集合了平面幾何、比例論、數論、無理量論和立體幾何之大成，一致公認為數學史上的一本鉅著。

《幾何原本》全書共分十三卷，一共包含 465 個命題，當中的第七、八、九卷，主要討論整數的性質，後人又稱這學問為「數論」。第九卷的命題 20 和質數有關，它是這樣寫的：「預先任意給定幾個質數，則有比它們更多的質數。」

歐幾里得原文的證明並不易懂，但改用現代的數學符號，他的證明大致如下：

首先，假如 $a$、$b$、$c$ … $k$ 是一些質數。那麼 $abc\cdots k+1$ 或者是質數，或者不是。如果它是質數，那麼就加添了一個新的質數。如果它不是質數，那麼這個數就有一個質因子 $p$。如果 $p$ 是 $a$、$b$、$c$ … $k$ 其中的一個數，由於它整除 $abc\cdots k$，於是它就能整除 1。但這是不可能的，因為 1 不能被其他數整除。因此 $p$ 就是一個新的質數。總結以上兩個情況，我們總獲得一個新的質數。命題得證。

命題 20 提供了一個製造質數的方法，而且可以無窮無盡地製造下去。由此可知，命題 20 實際上是證明了質數有無窮多個。

到了十七世紀初，法國數學家默森 (Mersenne)（1588 – 1648）提出了一條計算質數的「公式」，相當有趣。

因為 $x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$，所以如果 $x^n-1$ 是質數，$x-1$ 必定要等於 1。由此得 $x=2$。另外，假如 $n=ab$ 並且 $a\leq b$，又令 $x=2^a$，則 $2^n-1=(2^a)^b=x^b-1=(x-1)(x^{b-1}+x^{b-2}+\cdots+x+1)$。所以，如果 $2^n-1$ 是質數，那麼 $x-1$ 必定又要等於 1。由此得 $2^a=2$，即 $a=1$，$n$ 必定是質數。

綜合上述結果，默森提出了一條計算質數的公式，它就是 $2^p-1$，其中 $p$ 為質數。例如：$2^2-1=3$，$2^3-1=7$，$2^5-1=31$ 等等。但默森的公式祇是計算質數時的「必要」條件，並不是一個「充分」條件；即是說，在某些情況下，由 $2^p-1$ 計算出來的結果，未必一定是質數。例如：$2^{11}-1=2047=23\times 89$，這就不是質數了。因此由默森公式計算出來的數，其實也需要進一步的驗算，才可以知道它是否真正是一個質數。

由於現代的電腦主要用二進數來進行運算，而這又正好和默森公式配合，所以在今天，當人類找尋更大的質數時，往往仍會用上默森的方法。跟據互聯網上的資料，（網址為：

www.utm.edu/research/primes/largest.html），現時發現的最大質數為 $2^{6972593}-1$，它是由三位數學家在 1999 年 6 月 1 日發現的。

默森的好朋友費馬 (Fermat) (1601–1665) 亦提出過一條類似的質數公式。

設 $n=ab$ 並且 $b$ 是一個奇數。令 $x=2^a$，則 $2^n+1=(2^a)^b+1=x^b+1=(x+1)(x^{b-1}-x^{b-2}+\cdots-x+1)$。注意：祇有當 $b$ 為奇數時，上式才成立。很明顯，$2^n+1$ 並非一個質數。故此，如果 $2^n+1$ 是質數，那麼 $n$ 必定不能包含奇因子，即 $n$ 必定是 2 的乘冪。換句話說，費馬的質數公式為 $2^{2^n}+1$。

不難驗證，$2^{2^0}+1=3$，$2^{2^1}+1=5$，$2^{2^2}+1=17$，$2^{2^3}+1=257$，$2^{2^4}+1=65537$，它們全都是質數。問題是：跟著以後的數字，又是否質數呢？由於以後的數值增長得非常快，就連費馬本人，也解答不到這個問題了。

最先回答上述問題的人，是十八世紀瑞士大數學家歐拉 (Euler)（1707 – 1783）。歐拉出生於一個宗教家庭，17 歲已獲得碩士學位，一生都從事數學研究，縱使晚年雙目失明，亦不斷工作，可算是世上最多產的數學家。歐拉指出，$2^{2^5}+1$ 並非質數。他的證明如下：

記 $a=2^7$ 和 $b=5$。那麼 $a-b^3=3$ 而 $1+ab-b^4=1+(a-b^3)b=1+3b=2^4$。所以

$$
\begin{aligned}
& 2^{32}+1=(2a)^4+1 \\
=\ & 2^4a^4+1=(1+ab-b^4)a^4+1 \\
=\ & (1+ab)a^4+(1-a^4b^4) \\
=\ & (1+ab)(a^4+(1-ab)(1+a^2b^2)),
\end{aligned}
$$

即 $1+ab=641$ 可整除 $2^{32}+1$，$2^{32}+1$ 並不是質數！

事實上，到了今天，祇要用一部電子計算機就可以知道：$2^{32}+1=4294967297=641\times6700417$。同時，跟據電腦的計算，當 $n$ 大於 4 之後，由費馬公式計算出來的數字，再沒有發現另一個是質數了！不過，我們同時亦沒有一個數學方法來證明，費馬質數就祇有上述的五個數字。

自從歐拉證實 $2^{2^5}+1$ 並非質數之後，人們對費馬公式的興趣也隨之大減。不過到了 1796 年，當年青的數學家高斯發表了他的研究結果後，費馬質數又一再令人關注了。

高斯 (Gauss)（1777 – 1855），德國人。一個數學天才。3 歲已能指出父親帳簿中的錯誤。22 歲以前，已經成功地證明了多個重要而困難的數學定理。由於他的天份，後世人都稱他為「數學王子」。

高斯在 19 歲的時候發現，一個正質數多邊形可以用尺規作圖的充分和必要條件是，該多邊形的邊數必定是一個費馬質數！換句話說，祇有正三邊形（即正三角形）、正五邊形、正十七邊形、正 257 邊形和正 63357 邊形可以用尺規構作出來，其他的正質數多邊形就不可以了。（除非我們再發現另一個費馬質數。）高斯同時更提出了一個繪畫正十七邊形方案，打破了自古希臘時代流傳下來，最多祇可構作正五邊形的紀錄。

提到和質數有關的故事，就不可不提「哥德巴赫猜想」了。

哥德巴赫 (Goldbach) 是歐拉的朋友。1742 年，哥德巴赫向歐拉表示他發現每一個不小於 6 的偶數，都可以表示為兩個質數之和，例如：$8=3+5$、$20=7+13$、$100=17+83$ … 等。哥德巴赫問歐拉這是否一個一般性的現象。

歐拉表示他相信這是一個事實，但他無法作出一個證明。自此，人們就稱這個現象為「哥德巴赫猜想」。

自從「哥德巴赫猜想」被提出後，經過了整個十九世紀，對這方面研究的進展都很緩慢。直到 1920 年，挪威數學家布朗 (Brun) 證實一個偶數可以寫成兩個數字之和，其中每一個數字都最多祇有 9 個質因數。這可以算是一個重大的突破。

1948 年，匈牙利的瑞尼 (Renyi) 證明了一個偶數必定可以寫成一個質數加上一個有上限個因子所組成的合成數。1962 年，中國的潘承洞證明了一個偶數必定可以寫成一個質數加上一個由 5 個因子所組成的合成數。後來，有人簡稱這結果為 $(1+5)$。

1963 年，中國的王元和潘承洞分別證明了 $(1+4)$。1965 年，蘇聯的維諾格拉道夫 (Vinogradov) 證實了 $(1+3)$。1966 年，中國的陳景潤就證明了 $(1+2)$。這亦是世上現時對「哥德巴赫猜想」證明的最佳結果。

陳景潤（1933 – 1996），福建省福州人。出生於貧窮的家庭，由於戰爭的關係，自幼就在非常惡劣的環境下學習。1957 年獲得華羅庚的提拔，進入北京科學院當研究員。在「文化大革命」的十年中，陳景潤受到了批判和不公正的待遇，使他的工作和健康都大受傷害。1980 年，他當選為中國科學院學部委員。1984 年證實患上了「帕金遜症」，直至 1996 年 3 月 19 日，終於不治去世。

其實除了對「哥德巴赫猜想」的證明有貢獻外，陳景潤的另一個成就，就是對「孿生質數猜想」證明的貢獻。在質數世界中，我們不難發現有時有兩個質數，它們的距離非常接近，它們的差祇有 2，例如：3 和 5、5 和 7、11 和 13 … 10016957 和 10016959 … 等等。所謂「孿生質數猜想」，就是認為這些質數會有無窮多對。而在 1973 年，陳景潤就證得：「存在無窮多個質數 $p$，使得 $p+2$ 為不超過兩個質數之積。」

其實在質數的世界之中，還有很多更精彩更有趣的現象，但由於篇幅和個人能力的關係，未能一一盡錄。以下有一些書籍，內容豐富，值得對本文內容有興趣的人士參考。

**參考書目**

《數學和數學家的故事》
作者：李學數　出版社：廣角鏡

《天才之旅》
譯者：林傑斌　出版社：牛頓出版公司

《哥德巴赫猜想》
作者：陳景潤　出版社：九章出版社

《素數》
作者：王元　　出版社：九章出版社

# Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *May 20, 2000.*

**Problem 101.** A triple of numbers $(a_1, a_2, a_3) = (3, 4, 12)$ is given. We now perform the following operation: choose two numbers $a_i$ and $a_j$, $(i \neq j)$, and exchange them by $0.6\,a_i - 0.8\,a_j$ and $0.8\,a_i + 0.6\,a_j$. Is it possible to obtain after several steps the (unordered) triple (2, 8, 10)? *(Source: 1999 National Math Competition in Croatia)*

**Problem 102.** Let $a$ be a positive real number and $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $x_1 = a$ and

$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} kx_k \text{ , for all } n \geq 1.$$

Show that there exists a positive integer $n$ such that $x_n > 1999!$ (*Source: 1999 Romanian Third Selection Examination*)

**Problem 103.** Two circles intersect in points $A$ and $B$. A line $l$ that contains the point $A$ intersects the circles again in the points $C, D$, respectively. Let $M, N$ be the midpoints of the arcs $BC$ and $BD$, which do not contain the point $A$, and let $K$ be the midpoint of the segment $CD$. Show that $\angle MKN = 90^\circ$. (*Source: 1999 Romanian Fourth Selection Examination*)

**Problem 104.** Find all positive integers $n$ such that $2^n$-1 is a multiple of 3 and ($2^n$-1)/3 is a divisor of $4m^2 + 1$ for some integer $m$. (*Source: 1999 Korean Mathematical Olympiad*)

**Problem 105.** A rectangular parallelopiped (box) is given, such that its intersection with a plane is a regular hexagon. Prove that the rectangular parallelopiped is a cube. (*Source: 1999 National Math Olympiad in Slovenia*)

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## *Solutions*

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**Problem 96.** If every point in a plane is colored red or blue, show that there exists a rectangle all of its vertices are of the same color.

***Solution.*** **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7).

Consider the points (*x, y*) on the co-ordinate plane, where $x$ = 1, 2, …, 7 and y = 1, 2, 3. In row 1, at least 4 of the 7 points are of the same color, say color $A$. In each of row 2 or 3, if 2 or more of the points directly above the $A$-colored points in row 1 are also $A$-colored, then there will be a rectangle with $A$-colored vertices. Otherwise, at least 3 of the points in each of row 2 and 3 are $B$-colored and they are directly above four $A$-colored points in row 1. Then there will be a rectangle with $B$-colored vertices.

*Other recommended solvers:* **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **CHEUNG Chi Leung** (Carmel Divine Grace Foundation Secondary School, Form 6), **FAN Wai Tong** (St. Mark's School, Form 7), **LAM Shek Ming Sherman** (La Salle College), **LEE Kar Wai Alvin**, **LI Chi Pang Bill**, **TANG Yat Fai Roger** (La Salle College, Form 5), **LEE Kevin** (La Salle College, Form 4), **LEUNG Wai Ying**, **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 5), **NG Wing Ip** (Carmel Divine Grace Foundation Secondary School, Form 6), **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7), **WONG Wing Hong** (La Salle College, Form 2) and **YEUNG Kai Sing Kelvin** (La Salle College, Form 3).

**Problem 97.** A group of boys and girls went to a restaurant where only big pizzas cut into 12 pieces were served. Every boy could eat up to 6 or 7 pieces and every girl 2 or 3 pieces. It turned out that 4 pizzas were not enough and that 5 pizzas were too many. How many boys and how many girls were there? (*Source: 1999 National Math Olympaid in Slovenia*).

***Solution.*** **TSE Ho Pak** (SKH Bishop Mok Sau Tseng Secondary School, Form 6).

Let the number of boys and girls be $x$ and $y$, respectively. Then $7x + 3y \leq 59$ and $6x + 2y \geq 49$. Subtracting these, we get $x + y \leq 10$. Then $6x + 2(10 - x) \geq 49$ implies $x \geq 8$. Also, $7x + 3y \leq 59$ implies $x \leq 8$. So $x = 8$. To satisfy the inequalities then $y$ must be 1.

*Other recommended solvers:* **AU Cheuk Yin Eddy** (Ming Kei College, Form 7), **CHAN Chin Fei** (STFA Leung Kau Kui College,), **CHAN Hiu Fai** (STFA Leung Kau Kui College, Form 6), **CHAN Man Wai** (St. Stephen's Girls' College, Form 5), **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **CHUNG Ngai Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **CHUNG Wun Tung Jasper** (Ming Kei College, Form 6), **FAN Wai Tong** (St. Mark's School, Form 7), **HONG Chin Wing** (Pui Ching Middle School, Form 5), **LAM Shek Ming Sherman** (La Salle College), **LEE Kar Wai Alvin, LI Chin Pang Bill, TANG Yat Fai Roger** (La Salle College, Form 5), **LEE Kevin** (La Salle College, Form 4), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **LEUNG Yiu Ka** (STFA Leung Kau Kui College, Form 5), **LYN Kwong To** (Wah Yan College, Form 6), **MOK Ming Fai** (Carmel Divine Grace Foundation Secondary School, Form 6), **NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 6), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 5), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **POON Wing Sze Jessica** (STFA Leung Kau Kui College), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4), **WONG Chi Man** (Valtorta College, Form 5), **WONG Chun Ho** (STFA Leung Kau Kui College), **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7), **WONG So Ting** (Carmel Divine Grace Foundation Secondary School, Form 6), **WONG Wing Hong** (La Salle College, Form 2) and **YEUNG Kai Sing Kelvin** (La Salle College, Form 3).

**Problem 98.** Let $ABC$ be a triangle with $BC > CA > AB$. Select points $D$ on $BC$ and $E$ on the extension of $AB$ such that $BD = BE = AC$. The circumcircle of $BED$ intersects $AC$ at point $P$ and $BP$ meets the circumcircle of $ABC$ at point $Q$. Show that $AQ + CQ = BP$. (*Source: 1998-99 Iranian Math Olympiad*)

***Solution.*** **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7) and **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7).

Since $\angle CAQ = \angle CBQ = \angle DEP$ and $\angle AQC = 180^\circ - \angle ABD = \angle EPD$, so $\Delta AQC \sim \Delta EPD$. By Ptolemy's theorem, $BP \times ED = BD \times EP + BE \times DP$. So

$$BP = BD \times \frac{EP}{ED} + BE \times \frac{DP}{ED} =$$

$$AC \times \frac{AQ}{AC} + AC \times \frac{CQ}{AC} = AQ + CQ.$$

*Other recommended solvers:* **AU Cheuk Yin Eddy** (Ming Kei College, Form 7), **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **LAM Shek Ming Sherman** (La Salle College), **LEE Kevin** (La Salle College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4) and **YEUNG Kai Sing Kelvin** (La Salle College, Form 3).

**Problem 99**. At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered. (*Source: 1996 Spanish Math Olympiad*)

***Solution.*** **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 5) and **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7).

If some agent watches less than 7 other agents, then he will miss at least 9 agents. The agent himself and these 9 agents will form a group violating the cycle condition. So every agent watches at least 7 other agents. Similarly, every agent is watched by at least 7 agents. (Then each agent can watch at most 15 –7 = 8 agents and is watched by at most 8 agents)

Define two agents to be "connected" if one watches the other. From above, we know that each agent is connected with at least 14 other agents. So each is "disconnected" to at most 1 agent. Since disconnectedness comes in pairs, among 11 agents, at least one, say *X*, will not disconnected to any other agents. Removing *X* among the 11 agents, the other 10 will form a cycle, say

$$X_1, X_2, ..., X_{10}, X_{11} = X_1.$$

Going around the cycle, there must be 2 agents $X_i, X_{i+1}$ in the cycle such that $X_i$ also watches *X* and $X_{i+1}$ is watched by *X*. Then *X* can be inserted to the cycle between these 2 agents.

*Other commended solvers:* **CHAN Hiu Fai Philip, NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 6) and **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7).

**Problem 100.** The arithmetic mean of a number of pairwise distinct prime numbers equals 27. Determine the biggest prime that can occur among them. (*Source: 1999 Czech and Slovak Math Olympiad*)

***Solution.*** **FAN Wai Tong** (St. Mark's School, Form 7) and **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7)

Let $p_1 < p_2 < \cdots < p_n$ be distinct primes such that $p_1 + p_2 + \cdots + p_n = 27n$. Now $p_1 \neq 2$ (for otherwise $p_1 + p_2 + \cdots + p_n$ - $27n$ will be odd no matter *n* is even or odd). Since the primes less than 27 are 2, 3, 5, 7, 11, 13, 17, 19, 23, so $p_n = 27n$ - $(p_1 + \cdots + p_{n-1}) = 27 + (27 - p_1) + \cdots + (27 - p_{n-1}) \leq 27+(27-2)+(27-3) + \cdots + (27 - 23) = 145$. Since $p_n$ is prime, $p_n \leq 139$. Since the arithmetic mean of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 139 is 27. The answer to the problem is 139.

*Other recommended solvers:* **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **CHEUNG Ka Chung**, **LAM Shek Ming Sherman**, **LEE Kar Wai Alvin**, **TANG Yat Fai Roger**, **WONG Wing Hong**, **YEUNG Kai Sing Kelvin** (La Salle College), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), and **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7).

## Olympiad Corner

*(continued from page 1)*

**Problem 3.** ***(cont'd)***
for any integer *r* not divisible by *p*. Prove that at least two of the numbers *a+b, a+c, a+d, b+c, b+d, c+d* are divisible by *p*. (Note: $\{x\} = x - [x]$ denotes the fractional part of *x*.)

**Problem 4.** Let $a_1, a_2, ..., a_n (n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, ..., a_n) \geq 2$.

**Problem 5.** The Y2K Game is played on a $1 \times 2000$ grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

**Problem 6.** Let *ABCD* be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle $\omega$ of triangle *BCD* meets *CD* at *E*. Let *F* be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle *ACF* meet line *CD* at *C* and *G*. Prove that the triangle *AFG* is isosceles.

## Interesting Theorems About Primes

Below we will list some interesting theorem concerning prime numbers.

**Theorem** (due to Fermat in about 1640) *A prime number is the sum of two perfect squares if and only if it is 2 or of the form 4n + 1. A positive integer is the sum of two perfect squares if and only if in the prime factorization of the integer, primes of the form 4n + 3 have even exponents.*

***Dirichlet's Theorem on Primes in Progressions*** (1837) *For every pair of relatively prime integers a and d, there are infinitely many prime numbers in the arithmetic progression* $a$, $a+d$, $a+2d$, $a+3d$, …, *(In particular, there are infinitely many prime numbers of the form 4n + 1, of the form 6n + 5, etc.)*

**Theorem** *There is a constant C such that if* $p_1$, $p_2$, …, $p_n$ *are all the prime numbers less than x, then*

$$\ln(\ln x) - 1 < \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} < \ln(\ln x) + C\ln(\ln(\ln x)).$$

*In particular, if* $p_1, p_2, p_3, \ldots$ *are all the prime numbers, then*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots = \infty.$$

(The second statement was obtained by Euler in about 1735. The first statement was proved by Chebysev in 1851.)

***Chebysev's Theorem*** (1852) *If* $x > 1$, *then there exists at least one prime number between x and 2x.* (This was known as *Bertrand's postulate* because J. Bertrand verified this for *x* less than six million in 1845.)

***Prime Number Theorem*** (due to J. Hadamard and Ch. de la Vallée Poussin independently in 1896) *Let* $\pi(x)$ *be the number of prime numbers not exceeding x, then*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1.$$

*If* $p_n$ *is the n-th prime number, then*

$$\lim_{x\to\infty} \frac{p_n}{n\ln n} = 1.$$

(This was conjectured by Gauss in 1793 when he was about 15 years old.)

***Brun's Theorem on Twin Primes*** (1919) *The series of reciprocals of the twin primes either is a finite sum or forms a convergent infinite series, i.e.*

$$\left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{11} + \frac{1}{13}\right) + \cdots < \infty.$$

As a general reference to these results, we recommend the book *Fundamentals of Number Theory* by William J. Le Veque, published by Dover.

# Mathematical Excalibur

Volume 5, Number 3 | May 2000 – Septl 2000

## Olympiad Corner

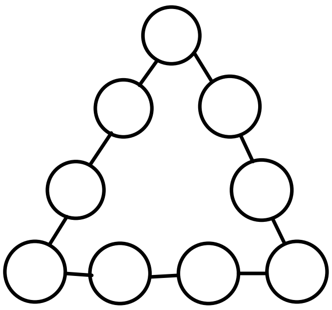
*XII Asia Pacific Math Olympiad, March 2000:*

*Time allowed: 4 Hours*

**Problem 1.** Compute the sum

$$S=\sum_{i=0}^{101}\frac{x_i^3}{1-3x_i+3x_i^2}\ \text{ for } x_i=\frac{i}{101}.$$

**Problem 2.** Given the following triangular arrangement of circles:



Each of the numbers 1,2,...,9 is to be written into one of these circles, so that each circle contains exactly one of these numbers and

(i) the sums of the four numbers on each side of the triangle are equal;

*(continued on page 4)*



**Acknowledgment**: Thanks to Janet Wong, MATH Dept, HKUST for general assistance.

**On-line:** http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is October 10, 2000.

For individual subscription for the next five issues for the 00-01 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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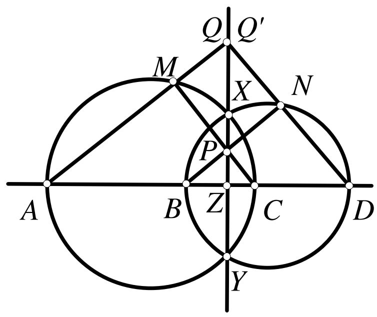
# Coordinate Geometry

Kin Y. Li

When we do a geometry problem, we should first look at the given facts and the conclusion. If all these involve intersection points, midpoints, feet of perpendiculars, parallel lines, then there is a good chance we can solve the problem by coordinate geometry. However, if they involve two or more circles, angle bisectors and areas of triangles, then sometimes it is still possible to solve the problem by choosing a good place to put the origin and the x-axis. Below we will give some examples. *It is important to stay away from messy computations !*

**Example 1**. (1995 IMO) Let $A, B, C$ and $D$ be four distinct points on a line, in that order. The circles with diameters $AC$ and $BD$ intersect at the points $X$ and $Y$. The line $XY$ meets $BC$ at the point $Z$. Let $P$ be a point on the line $XY$ different from $Z$. The line $CP$ intersects the circle with diameter $AC$ at the points $C$ and $M$, and the line $BP$ intersects the circle with diameter $BD$ at the points $B$ and $N$. Prove that the lines $AM$, $DN$, and $XY$ are concurrent.

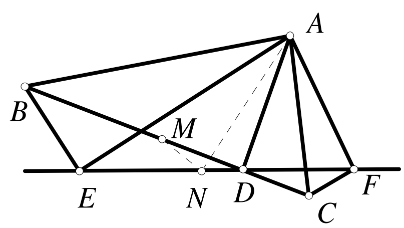



(*Remarks*. Quite obvious we should set the origin at $Z$. Although the figure is not symmetric with respect to line $XY$, there are pairs such as $M, N$ and $A, D$ and $B, C$ that are symmetric in *roles*! So we work on the left half of the figure, the computations will be similar for the right half.)

**Solution**. (*Due to Mok Tze Tao, 1995 Hong Kong Team Member*) Set the origin at $Z$ and the $x$-axis on line $AD$. Let the coordinates of the circumcenters of triangles $AMC$ and $BND$ be $(x_1, 0)$ and $(x_2, 0)$, and the circumradii be $r_1$ and $r_2$, respectively. Then the coordinates of A and $C$ are $(x_1-r_1, 0)$ and $(x_1+r_1, 0)$, respectively. Let the coordinates of $P$ be $(0, y_0)$. Since $AM\perp CP$ and the slope of $CP$ is $-y_0/(x_1+r_1)$, the equation of $AM$ works out to be $(x_1+r_1)x-y_0y=x_1^2-r_1^2$. Let $Q$ be the intersection of $AM$ with $XY$, then $Q$ has coordinates $(0,(r_1^2-x_1^2)/y_0)$.

Similarly, let $Q'$ be the intersection of $DN$ with $XY$, then $Q'$ has coordinates $(0,(r_2^2-x_2^2)/y_0)$. Since $r_1^2-x_1^2=ZX^2$ $=r_2^2-x_2^2$, so $Q=Q'$.

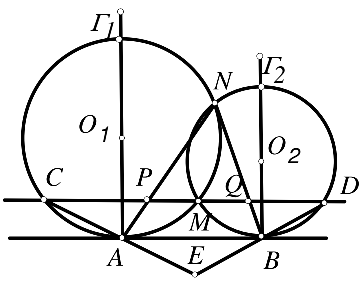
**Example 2**. (1998 APMO) Let $ABC$ be a triangle and $D$ the foot of the altitude from $A$. Let $E$ and $F$ be on a line passing through $D$ such that $AE$ is perpendicular to $BE$, $AF$ is perpendicular to $CF$, and $E$ and $F$ are different from $D$. Let $M$ and $N$ be the midpoints of the line segments $BC$ and $EF$, respectively. Prove that $AN$ is perpendicular to $NM$.




(*Remarks*. We can set the origin at $D$ and the $x$-axis on line $BC$. Then computing the coordinates of $E$ and $F$ will be a bit messy. A better choice is to set the line through $D,E.F$ horizontal.)

**Solution.** (*Due to Cheung Pok Man, 1998 Hong Kong Team Member*) Set the origin at $A$ and the $x$-axis parallel to line $EF$. Let the coordinates of $D, E, F$ be $(d, b)$, $(e, b)$, $(f, b)$, respectively. The case $b=0$ leads to $D=E$, which is not allowed. So we may assume $b\neq 0$. Since $BE \perp AE$ and the slope of $AE$ is $b/e$, so the equation of line $BE$ works out to be $ex+by=e^2+b^2$. Similarly, the equations of lines $CF$ and $BC$ are $fx+by=f^2+b^2$ and $dx+by=d^2+b^2$, respectively. Solving the equations for $BE$ and $BC$, we find $B$ has coordinates $(d+e, b-(de/b))$. Similarly, $C$ has coordinates $(d+f, b-(df/b))$. Then $M$ has coordinates $(d+(e+f)/2, b-(de+df)/(2b))$ and $N$ has coordinates $((e+f)/2, b)$. So the slope of $AN$ is $2b/(e+f)$ and the slope of $MN$ is $-(e+f)/(2b)$. Therefore, $AN \perp MN$.

**Example 3.** (2000 IMO) Two circles $\Gamma_1$ and $\Gamma_2$ intersect at M and N. Let $\ell$ be the common tangent to $\Gamma_1$ and $\Gamma_2$ so that $M$ is closer to $\ell$ than $N$ is. Let $\ell$ touch $\Gamma_1$ at $A$ and $\Gamma_2$ at $B$. Let the line through $M$ parallel to $\ell$ meet the circle $\Gamma_1$ again at $C$ and the circle $\Gamma_2$ again at $D$. Lines $CA$ and $DB$ meet at $E$; lines $AN$ and $CD$ meet at $P$; lines $BN$ and $CD$ meet at $Q$. Show that $EP=EQ$.




(*Remarks.* Here if we set the $x$-axis on the line through the centers of the circles, then the equation of the line $AB$ will be complicated. So it is better to have line $AB$ on the $x$-axis.)

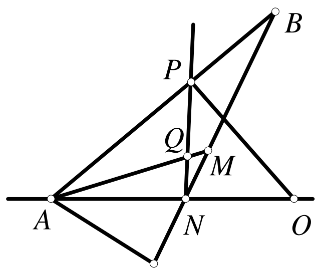
**Solution.** Set the origin at $A$ and the $x$-axis on line $AB$. Let $B$, $M$ have coordinates $(b,0)$, $(s,t)$, respectively. Let the centers $O_1$, $O_2$ of $\Gamma_1$, $\Gamma_2$ be at $(0, r_1)$, $(b, r_2)$, respectively. Then $C$, $D$ have coordinates $(-s, t)$, $(2b-s,t)$, respectively. Since $AB$, $CD$ are parallel, $CD=2b=2AB$ implies $A$, $B$ are midpoints of $CE$, $DE$, respectively. So $E$ is at $(s, -t)$. We see $EM \perp CD$.

To get $EP=EQ$, it is now left to show $M$ is the midpoint of segment $PQ$. Since $O_1O_2 \perp MN$ and the slope of $O_1O_2$ is $(r_2- r_1)/b$, the equation of line $MN$ is $bx+(r_2-r_1)y=bs+(r_2-r_1)t$. (This line should pass through the midpoint of segment $AB$.) Since $O_2M=r_2$ and $O_1M=r_1$, we get

$$(b-s)^2+(r_2-t)^2=r_2^2 \quad \text{and}$$
$$s^2+(r_1-t)^2=r_1^2.$$

Subtracting these equations, we get $b^2/2=bs+(r_2- r_1)t$, which implies $(b/2, 0)$ is on line $MN$. Since $PQ$, $AB$ are parallel and line $MN$ intersects $AB$ at its midpoint, then $M$ must be the midpoint of segment $PQ$. Together with $EM \perp PQ$, we get $EP=EQ$.

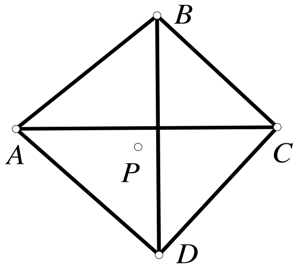
**Example 4.** (2000 APMO) Let $ABC$ be a triangle. Let $M$ and $N$ be the points in which the median and the angle bisector, respectively, at $A$ meet the side $BC$. Let $Q$ and $P$ be the points in which the perpendicular at $N$ to $NA$ meets $MA$ and $BA$, respectively, and $O$ the point in which the perpendicular at $P$ to $BA$ meets $AN$ produced. Prove that $QO$ is perpendicular to $BC$.




(*Remarks.* Here the equation of the angle bisector is a bit tricky to obtain unless it is the $x$-axis. In that case, the two sides of the angle is symmetric with respect to the $x$-axis.)

**Solution.** *(Due to Wong Chun Wai, 2000 Hong Kong Team Member)* Set the origin at $N$ and the $x$-axis on line $NO$. Let the equation of line $AB$ be $y=ax+b$, then the equation of lines $AC$ and $PO$ are $y=-ax-b$ and $y=(-1/a)x+b$, respectively. Let the equation of $BC$ be $y=cx$. Then $B$ has coordinates $(b/(c-a), bc/(c-a))$, $C$ has coordinates $(-b/(c+a), -bc(c+a))$, $M$ has coordinates $(ab/(c^2-a^2), abc/(c^2-a^2))$, $A$ has coordinates $(-b/a, 0)$, $O$ has coordinates $(ab, 0)$ and $Q$ has coordinates $(0, ab/c)$. Then $BC$ has slope $c$ and $QO$ has slope $-1/c$. Therefore, $QO \perp BC$.

**Example 5.** (1998 IMO) In the convex quadrilateral $ABCD$, the diagonals $AC$ and $BD$ are perpendicular and the opposite sides $AB$ and $DC$ are not parallel. Suppose that the point $P$, where the perpendicular bisectors of $AB$ and $DC$ meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles $ABP$ and $CDP$ have equal areas.




(*Remarks.* The area of a triangle can be computed by taking the half length of the cross product. A natural candidate for the origin is $P$ and having the diagonals parallel to the axes will be helpful.)

**Solution.** (*Due to Leung Wing Chung, 1998 Hong Kong Team Member*) Set the origin at $P$ and the $x$-axis parallel to line $AC$. Then the equations of lines $AC$ and $BD$ are $y=p$ and $x=q$, respectively. Let $AP=BP=r$ and $CP=DP=s$. Then the coordinates of $A$, $B$, $C$, $D$ are $(-\sqrt{r^2-p^2}, p)$, $(q, \sqrt{r^2-q^2})$, $(\sqrt{s^2-p^2}, p)$, $(q, -\sqrt{s^2-q^2})$, respectively. Using the determinant formula for finding the area of a triangle, we see that the areas of triangles $ABP$ and $CDP$ are equal if and only if

$$-\sqrt{r^2-p^2}\sqrt{r^2-q^2}-pq=-\sqrt{s^2-p^2}\sqrt{s^2-q^2}-pq.$$

Since $f(x) = -\sqrt{x^2-p^2}\ \sqrt{x^2-q^2}-pq$ is strictly decreasing when $x \ge |p|$ *and* $|q|$, equality of areas hold if and only if $r=s$, which is equivalent to $A$, $B$, $C$, $D$ concyclic (since $P$ being on the perpendicular bisectors of $AB$, $CD$ is the only possible place for the center).

After seeing these examples, we would like to remind the readers that there are pure geometric proofs to each of the problems. For examples (1) and (3), there are proofs that only take a few lines. We encourage the readers to discover these simple proofs.

Although in the opinions of many people, a pure geometric proof is better and more beautiful than a coordinate geometric proof, we should point out that sometimes the coordinate geometric proofs may be preferred when there are many cases. For example (2), the different possible orderings of the points $D$, $E$, $F$ on the line can all happen as some pictures will show. The coordinate geometric proofs above cover all cases.

# Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *October 10, 2000.*

**Problem 106.** Find all positive integer ordered pairs *(a,b)* such that

$$gcd(a,b)+lcm(a,b)=a+b+6,$$

where gcd stands for greastest common divisor (or highest common factor) and lcm stands for least common multiple.

**Problem 107.** For $a, b, c > 0$, if $abc=1$, then show that

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}} \ge \sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

**Problem 108.** Circles $C_1$ and $C_2$ with centers $O_1$ and $O_2$ (respectively) meet at points *A, B*. The radii $O_1B$ and $O_2B$ intersect $C_1$ and $C_2$ at *F* and *E*. The line parallel to *EF* through *B* meets $C_1$ and $C_2$ at *M* and *N*, respectively. Prove that *MN=AE+AF*. (Source: 17th Iranian Mathematical Olympiad)

**Problem 109.** Show that there exists an increasing sequence $a_1, a_2, a_3, \ldots$ of positive integers such that for every nonnegative integer *k*,the sequence $k+a_1, k+a_2, k+a_3 \ldots$ contains only finitely many prime numbers. (Source: 1997 Math Olympiad of Czech and Slovak Republics)

**Problem 110.** In a park, 10000 trees have been placed in a square lattice. Determine the maximum number of trees that can be cut down so that from any stump, you cannot see any other stump. (Assume the trees have negligible radius compared to the distance between adjacent trees.) (Source: 1997 German Mathematical Olympiad)

*Comments*. You may think of the trees being placed at $(x,y)$, where $x, y = 0, 1, 2, \ldots, 99$.

*****************

## *Solutions*

*****************

**Problem 101.** A triple of numbers $(a_1, a_2, a_3)$=(3, 4, 12) is given. We now perform the following operation: choose two numbers $a_i$ and $a_j$, $(i \neq j)$, and exchange them by $0.6a_i–0.8a_j$ and $0.8a_i+0.6a_j$. Is it possible to obtain after several steps the (unordered) triple (2, 8, 10) ? (*Source:* 1999 National Math Competition in Croatia)

***Solution.*** **FAN Wai Tong** (St. Mark's School, Form 7), **KO Man Ho** (Wah Yan College, Kowloon, Form 6) and **LAW Hiu Fai** (Wah Yan College, Kowloon, Form 6).

Since $(0.6a_i–0.8a_j)^2$ + $(0.8a_i+0.6a_j)^2$ = $a_i^2+a_j^2$, the sum of the squares of the triple of numbers before and after an operation stays the same. Since $3^2 + 4^2 + 12^2 \neq 2^2 + 8^2 + 10^2$, so (2,8,10) cannot be obtained.

**Problem 102.** Let $a$ be a positive real number and $(x_n)_{n\ge 1}$ be a sequence of real numbers such that $x_1=a$ and

$$x_{n+1} \ge (n+2)x_n - \sum_{k=1}^{n-1} kx_k, \text{ for all } n \ge 1.$$

Show that there exists a positive integer $n$ such that $x_n > 1999!$ (*Source: 1999 Romanian Third Selection Examination*)

***Solution.*** **FAN Wai Tong** (St. Mark's School, Form 7).

We will prove by induction that $x_{j+1} \ge 3x_j$ for every positive integer *j*. The case $j=1$ is true by the given inequality. Assume the cases $j$ =1, ..., $n-1$ are true. Then $x_n \ge 3x_{n-1} \ge 9x_{n-2} \ge \ldots$ and

$$\frac{x_{n+1}}{x_n} \ge (n+2)-\sum_{k=1}^{n-1}\frac{kx_k}{x_n}$$

$$\ge (n+2)-\sum_{k=1}^{n-1}\frac{n-1}{3^{n-k}}$$

$$\ge (n+2)-(n-1)(\frac{1}{3}+\frac{1}{9}+\ldots)$$

$$=\frac{n+5}{2}$$

$$\ge 3.$$

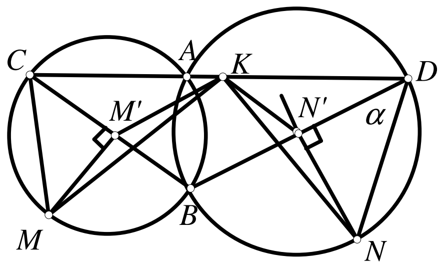
So the case $j = n$ is also true.

Since $a > 0$, we can take

$$n > 1+ \log_3 (1999!/a).$$

Then $x_n \ge 3^{n-1}x_1 = 3^{n-1}a > 1999!$.

**Problem 103.** Two circles intersect in points *A* and *B*. A line *l* that contains the point *A* intersects again the circles in the points *C, D*, respectively. Let *M, N* be the midpoints of the arcs *BC* and *BD*, which do not contain the point *A*, and let *K* be the midpoint of the segment *CD*. Show that $\angle MKN=90°$. (*Source:* 1999 Romanian Fourth Selection Examination)




***Solution.*** **FAN Wai Tong** (St. Mark's School, Form 7)

Let *M'* and *N'* be the midpoints of chords *BC* and *BD* respectively. From the midpoint theorem, we see that *BM'KN'* is a parallelogram. Now

$$\angle KN'N = \angle KN'B + 90^\circ$$
$$= \angle KM'B + 90^\circ$$
$$= \angle KM'M.$$

Let $\alpha = \angle NDB = \angle NAB$. Then

$$\frac{KN'}{N'N}=\frac{M'B}{N'D\tan\alpha}=\frac{\frac{1}{2}BC}{\frac{1}{2}BD\tan\alpha}.$$

Now

$$\angle MCB = \angle MCB = \frac{1}{2}\angle CAB$$
$$=\frac{1}{2}(180^\circ - \angle DAB)$$
$$= 90^\circ - \angle NAB$$
$$= 90^\circ - \alpha.$$

So

$$\frac{MM'}{M'K} = \frac{CM'\cot\alpha}{BN'} = \frac{\frac{1}{2}BC\cot\alpha}{\frac{1}{2}BD}.$$

Then $KN'/N'N = MM'/M'K$. So triangles $MM'K, KN'N$ are similar. Then $\angle M'KM = \angle N'NK$ and

$$\begin{aligned}\angle MKN &= \angle M'KN' - \angle M'KM - \angle N'KN \\ &= \angle KN'D - (\angle N'NK + \angle N'KN) \\ &= 90^{\circ}\end{aligned}$$

*Other commended solvers:* **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7).

**Problem 104**. Find all positive integers $n$ such that $2^n-1$ is a multiple of 3 and $(2^n-1)/3$ is a divisor of $4m^2+1$ for some integer $m$. (*Source:* 1999 Korean Mathematical Olympiad)

***Solution.*** (Official Solution)

(Some checkings should suggest $n$ is a power of 2.) Now $2^n-1$ is a multiple of 3 if and only if $(-1)^n \equiv 2^n \equiv (\text{mod } 3)$, that is $n$ is even. Suppose for some even $n$, $(2^n-1)/3$ is a divisor of $4m^2+1$ for some $m$. Assume $n$ has an odd prime divisor $d$. Now $2^d-1\equiv 3 \pmod 4$ implies one of its prime divisor $p$ is of the form $4k+3$. Then $p$ divides $2^d-1$, which divides $2^n-1$, which divides $4m^2+1$. Then $p$ and $2m$ are relatively prime and so

$$1\equiv(2m)^{p-1}=(4m^2)^{2k+1}\equiv-1 \pmod p,$$

a contradiction. So $n$ cannot have any odd prime divisor. Hence $n=2^j$ for some positive integer $j$.

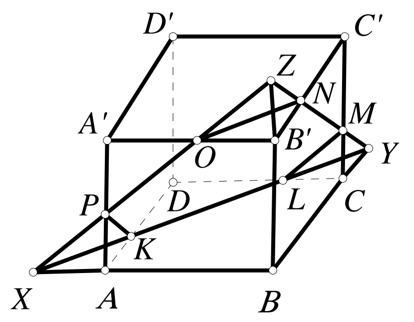
Conversely, suppose $n=2^j$. Let $F_i=2^{2^i}+1$. Using the factorization $2^{2b}-1=(2^b-1)\times(2^b+1)$ repeatedly on the numerator, we get

$$\frac{2^n-1}{3} = F_1F_2\cdots F_{j-1}.$$

Since $F_i$ divides $F_j-2$ for $i<j$, the $F_i$'s are pairwise relatively prime. By the Chinese remainder theorem, there is a positive integer $x$ satisfying the simultaneous equations $x\equiv 0 \pmod 2$ and $x \equiv 2^{2^{i-1}} \pmod{F_i}$ for $i=1, 2, \ldots, j-1$. Then $x=2m$ for some positive integer $m$ and $4m^2+1= x^2+1\equiv 0 \pmod{F_i}$ for $i=1, 2,\ldots j-1$. So $4m^2+1$ is divisible by $F_1F_2\ldots F_{j-1}=(2^n-1)/3$.

**Problem 105.** A rectangular parallelepiped (box) is given, such that its intersection with a plane is a regular hexagon. Prove that the rectangular parallelepiped is a cube. (*Source:* 1999 National Math Olympiad in Slovenia)

***Solution.*** (Official Solution)




As in the figure, an equilateral triangle $XYZ$ is formed by extending three alternate sides of the regular hexagon.

The right triangles $XBZ$ and $YBZ$ are congruent as they have a common side $BZ$ and the hypotenuses have equal length. So $BX=BY$ and similarly $BX=BZ$. As the pyramids $XBYZ$ and $OB'NZ$ are similar and $ON=\frac{1}{3}XY$, it follows $B'Z=\frac{1}{3}BZ$. Thus we have $BB'=\frac{2}{3}BZ$ and similarly $AB=\frac{2}{3}BX$ and $CB=\frac{2}{3}BY$. Since $BX=BY=BZ$, we get $AB=BC=BB'$.

*Other commended solvers:* **FAN Wai Tong** (St. Mark's School, Form 7).

**Olympiad Corner**

*(continued from page 1)*

(ii) the sums of the squares of the four numbers on each side of the triangle are equal.
Find all ways in which this can be done.

**Problem 3.** Let $ABC$ be a triangle. Let $M$ and $N$ be the points in which the median and the angle bisector, respectively, at $A$ meet the side $BC$. Let $Q$ and $P$ be the points in which the perpendicular at $N$ to $NA$ meets $MA$ and $BA$, respectively, and $O$ the point in which the perpendicular at $P$ to $BA$ meets $AN$ produced. Proved that $QO$ is perpendicular to $BC$.

**Problem 4.** Let $n$, $k$ be given positive integers with n > k. Prove that

$$\frac{1}{n+1}\cdot\frac{n^n}{k^k(n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k(n-k)^{n-k}}.$$

**Problem 5.** Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence $0,1,\ldots, n$. A transposition of $a_i$ with $a_j$ is called *legal* if $i > 0$, $a_i = 0$ and $a_{i-1} + 1 = a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called *regular* if after a number of legal transpositions it becomes $(1,2, \ldots,n,0)$. For which numbers $n$ is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular ?

**2000 APMO and IMO**

In April this year, Hong Kong IMO trainees participated in the XII Asia Pacific Mathematical Olympiad. The winners were

Gold Award

Fan Wai Tong (Form 7, St Mark's School)

Silver Award

Wong Chun Wai (Form 7, Choi Hung Estate Catholic Secondary School)
Chao Khek Lun (Form 5, St. Paul's College)

Bronze Award

Law Ka Ho (Form 7, Queen Elizabeth School)
Ng Ka Chun (Form 5, Queen Elizabeth School)
Yu Hok Pun (Form 4, SKH Bishop Baker Secondary School)
Chan Kin Hang (Form 6, Bishop Hall Jubilee School)

Honorable Mention

Ng Ka Wing (Form 7, STFA Leung Kau Kui College)
Chau Suk Ling (Form 5, Queen Elizabeth School)
Choy Ting Pong (Form 7, Ming Kei College)

Based on the APMO and previous test results, the following trainees were selected to be the Hong Kong team members to the 2000 International Mathematical Olympiad, which was held in July in South Korea.

Wong Chun Wai (Form 7, Choi Hung Estate Catholic Secondary School)
Ng Ka Wing (Form 7, STFA Leung Kau Kui College)
Law Ka Ho (Form 7, Queen Elizabeth School)
Chan Kin Hang (Form 6, Bishop Hall Jubilee School)
Yu Hok Pun (Form 4, SKH Bishop Baker Secondary School)
Fan Wai Tong (Form 7, St. Mark's School)

# Mathematical Excalibur

Volume 5, Number 4 September 2000 – November 2000

## Olympiad Corner

*The 41st International Mathematical Olympiad, July 2000:*

*Time allowed: 4 hours 30 minutes*
*Each problem is worth 7 points.*

**Problem 1.** Two circles $\Gamma_1$ and $\Gamma_2$ intersect at $M$ and $N$. Let $\ell$ be the common tangent to $\Gamma_1$ and $\Gamma_2$ so that $M$ is closer to $\ell$ than $N$ is. Let $\ell$ touches $\Gamma_1$ at $A$ and $\Gamma_2$ at $B$. Let the line through $M$ parallel to $\ell$ meets the circle $\Gamma_1$ again at $C$ and the circle $\Gamma_2$ at $D$. Lines $CA$ and $DB$ meet at $E$; lines $AN$ and $CD$ meet at $P$; lines $BN$ and $CD$ meet at $Q$. Show that $EP = EQ$.

**Problem 2.** Let $a, b, c$ be positive real numbers such that $abc = 1$. Prove that
$(a-1+1/b)(b-1+1/c)(c-1+1/a) \le 1$

**Problem 3.** Let $n \ge 2$ be a positive integer. Initially, there are $n$ fleas on a horizontal line, not all at the same point. For a positive real number $\lambda$, define a move as follows:
Choose any two fleas, at points $A$ and $B$, with $A$ to the left of $B$; let the flea at $A$ jump to the point $C$ on the line to the line to the right of $B$ with $BC/AB = \lambda$.

(continued on page 4)

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**Acknowledgment**: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

**On-line:** http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is December 10, 2000.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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# Jensen's Inequality

*Kin Y. Li*

In comparing two similar expressions, often they involve a common function. To see which expression is greater, the shape of the graph of the function on an interval is every important. A function $f$ is said to be *convex* on an interval $I$ if for any two points ($x_1$, $f(x_1)$) and ($x_2$, $f(x_2)$) on the graph, the segment joining these two points lie on or above the graph of the function over [$x_1$, $x_2$]. That is,

$$f((1-t)x_1 + tx_2) \le (1-t)\, f(x_1) + tf(x_2)$$

for every $t$ in [0, 1]. If $f$ is continuous on $I$, then it is equivalent to have

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$$

for every $x_1$, $x_2$ in $I$. If furthermore $f$ is differentiable, then it is equivalent to have a nondecreasing derivative. Also, $f$ is *strictly convex on I* if $f$ is convex on $I$ and equality holds in the inequalities above only when $x_1 = x_2$. We say a function $g$ is *concave on an interval I* if the function $-g$ is convex on $I$. Similarly, $g$ is strictly concave on $I$ if $-g$ is strictly convex on $I$.

The following are examples of strictly convex functions on intervals:

$$x^p \text{ on } [0, \infty) \text{ for } p > 1,$$

$$x^p \text{ on } (0, \infty) \text{ for } p < 0,$$

$$a^x \text{ on } (-\infty, \infty) \text{ for } a > 1,$$

$$\tan x \text{ on } [0, \tfrac{\pi}{2}).$$

The following are examples of strictly concave functions on intervals:

$$x^p \text{ on } [0, \infty) \text{ for } 0 < p < 1,$$

$$\log_a x \text{ on } (0, \infty) \text{ for } a > 1,$$

$$\cos x \text{ on } [-\pi/2, \pi/2],$$

$$\sin x \text{ on } [0, \pi].$$

The most important inequalities con-cerning these functions are the following.

**Jensen's Inequality.** *If $f$ is convex on an interval I and $x_1, x_2, \ldots, x_n$ are in I, then*

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \le \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}.$$

*For strictly convex functions, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.*

**Generalized Jensen's Inequality.** *Let f be continuous and convex on an interval I. If $x_1, \ldots, x_n$ are in I and $0 < t_1, t_2, \ldots, t_n < 1$ with $t_1 + t_2 + \cdots + t_n = 1$, then*

$$f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \le t_1 f(x_1) + t_2 f(x_2) + \cdots + t_n f(x_n)$$

*(with the same equality condition for strictly convex functions).*

Jensen's inequality is proved by doing a forward induction to get the cases $n = 2^k$, then a backward induction to get case $n - 1$ from case $n$ by taking $x_n$ to be the arithmetic mean of $x_1$, $x_2$, …, $x_{n-1}$. For the generalized Jensen's inequality, the case all $t_i$'s are rational is proved by taking common denominator and the other cases are obtained by using continuity of the function and the density of rational numbers.

There are similar inequalities for concave and strictly concave functions by reversing the inequality signs.

***Example 1.*** For a triangle $ABC$, show that $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$ and determine when equality holds.

**Solution.** Since $f(x) = \sin x$ is strictly concave on [0, $\pi$], so

$$\begin{aligned} &\sin A + \sin B + \sin C \\ &= f(A) + f(B) + f(C) \\ &\le 3f\left(\frac{A+B+C}{3}\right) \\ &= 3\sin\left(\frac{A+B+C}{3}\right) \\ &= \frac{3\sqrt{3}}{2}. \end{aligned}$$

Equality holds if and only if $A = B = C = \pi/3$, i.e. $\Delta ABC$ is equilateral.

***Example 2.*** If $a, b, c > 0$ and

$$a + b + c = 1,$$

then find the minimum of

$$\left(a+\frac{1}{a}\right)^{10}+\left(b+\frac{1}{b}\right)^{10}+\left(c+\frac{1}{c}\right)^{10}.$$

**Solution.** Note $0 < a, b, c < 1$. Let $f(x) = \left(x+\frac{1}{x}\right)^{10}$ on $I = (0, 1)$, then $f$ is strictly convex on $I$ because its second derivative

$$90\left(x+\frac{1}{x}\right)^8\left(1-\frac{1}{x^2}\right)^2+10\left(x+\frac{1}{x}\right)^9\left(\frac{2}{x^3}\right)$$

is positive on $I$. By Jensen's inequality,

$$\frac{10^{10}}{3^9}=3f\left(\frac{a+b+c}{3}\right)$$
$$\le f(a)+f(b)+f(c)$$
$$=\left(a+\frac{1}{a}\right)^{10}+\left(b+\frac{1}{b}\right)^{10}+\left(c+\frac{1}{c}\right)^{10}.$$

So the minimum is $10^{10}/3^9$, attained when $a = b = c = 1/3$.

***Example 3.*** Prove that AM-GM in-equality, which states that if $a_1$, $a_2, \ldots, a_n \ge 0$, then

$$\frac{a_1+a_2+\cdots+a_n}{n} \ge \sqrt[n]{a_1a_2\cdots a_n}.$$

**Solution.** If one of the $a_i$'s is 0, then the right side is 0 and the inequality is clear. If $a_1, a_2, \ldots, a_n > 0$, then since $f(x) = \log x$ is strictly concave on $(0, \infty)$, by Jensen's inequality,

$$\log\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$
$$\ge \frac{\log a_1+\log a_2+\cdots+\log a_n}{n}$$
$$=\log\left(\sqrt[n]{a_1a_2\cdots a_n}\right).$$

Exponentiating both sides, we get the AM-GM inequality.

*Remarks.* If we use the generalized Jensen's inequality instead, we can get the weighted AM-GM inequality. It states that if $a_1, \ldots, a_n > 0$ and $0 < t_1, \ldots, t_n < 1$ satisfying $t_1 + \cdots + t_n = 1$, then $t_1 a_1 + \cdots + t_n a_n \ge a_1^{t_1}\cdots a_n^{t_n}$ with equality if and only if all $a_i$'s are equal.

***Example 4.*** Prove the power mean inequality, which states that for $a_1$, $a_2, \ldots, a_n > 0$ and $s < t$, if

$$S_r=\left(\frac{a_1^r+a_2^r+\cdots+a_n^r}{n}\right)^{1/r},$$

then $S_s \le S_t$. Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

*Remarks.* $S_1$ is the arithmetic mean (AM) and $S_{-1}$ is the harmonic mean (HM) and $S_2$ is the root-mean-square (RMS) of $a_1$, $a_2, \cdots, a_n$. Taking limits, it can be shown that $S_{+\infty}$ is the maximum (MAX), $S_0$ is the geometric mean (GM) and $S_{-\infty}$ is the minimum (MIN) of $a_1, a_2, \cdots, a_n$.

**Solution.** In the cases $0 < s < t$ or $s < 0 < t$, we can apply Jensen's inequality to $f(x) = x^{t/s}$. In the case $s < t < 0$, we let $b_i = 1/a_i$ and apply the case $0 < -t < -s$. The other cases can be obtained by taking limit of the cases proved.

***Example 5.*** Show that for $x, y, z > 0$,

$$x^5+y^5+z^5$$
$$\le x^5\sqrt{\frac{x^2}{yz}}+y^5\sqrt{\frac{y^2}{zx}}+z^5\sqrt{\frac{z^2}{xy}}.$$

**Solution.** Let $a = \sqrt{x}$, $b = \sqrt{y}$, $c = \sqrt{z}$, then the inequality becomes

$$a^{10}+b^{10}+c^{10}\le\frac{a^{13}+b^{13}+c^{13}}{abc}.$$

By the power mean inequality,

$$a^{13}+b^{13}+c^{13}=3S_{13}^{13}$$
$$=3S_{13}^{10}\,S_{13}^{3}\ge 3S_{10}^{10}\,S_0^3$$
$$=\left(a^{10}+b^{10}+c^{10}\right)abc.$$

***Example 6.*** Prove Hölder's inequality, which states that if $p, q > 1$ satisfy $\frac{1}{p}+\frac{1}{q} = 1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real (or complex) numbers, then

$$\sum_{i=1}^n|a_ib_i|\le\left(\sum_{i=1}^n|a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n|b_i|^q\right)^{\frac{1}{q}}.$$

(The case $p = q = 2$ is the Cauchy-Schwarz inequality.)

**Solution.** Let

$$A=|a_1|^p+\cdots+|a_n|^p.$$
$$B=|b_1|^p+\cdots+|b_n|^q.$$

If $A$ or $B$ is 0, then either all $a_i$'s or all $b_i$'s are 0, which will make both sides of the inequality 0.

So we need only consider the case $A \ne 0$ and $B \ne 0$. Let $t_1 = 1/p$ and $t_2 = 1/q$, then $0 < t_1, t_2 < 1$ and $t_1 + t_2 = 1$. Let $x_i = |a_i|^p/A$ and $y_i = |b_i|^q/B$, then

$$x_1+\cdots+x_n=1,\qquad y_1+\cdots+y_n=1.$$

Since $f(x) = e^x$ is strictly convex on $(-\infty, \infty)$, by the generalized Jensen's inequality,

$$x_i^{1/p}y_i^{1/q}=f(t_1\ln x_i+t_2\ln y_i)$$
$$\le t_1f(\ln x_i)+t_2f(\ln y_i)=\frac{x_i}{p}+\frac{y_i}{q}.$$

Adding these for $i = 1, \ldots, n$, we get

$$\sum_{i=1}^n\frac{|a_i||b_i|}{A^{1/p}B^{1/q}}=\sum_{i=1}^n x_i^{1/p}y_i^{1/q}$$
$$\le\frac{1}{p}\sum_{i=1}^n x_i+\frac{1}{q}\sum_{i=1}^n y_i=1.$$

Therefore,

$$\sum_{i=1}^n|a_i||b_i|\le A^{1/p}B^{1/q}$$
$$=\left(\sum_{i=1}^n|a_i|^p\right)^{1/p}\left(\sum_{i=1}^n|b_i|^q\right)^{1/q}.$$

***Example 7.*** If $a, b, c, d > 0$ and

$$c^2+d^2=(a^2+b^2)^3,$$

then show that

$$\frac{a^3}{c}+\frac{b^3}{d}\ge 1.$$

**Solution 1.** Let

$$x_1=\sqrt{a^3/c},\quad x_2=\sqrt{b^3/d},$$
$$y_1=\sqrt{ac},\quad y_2=\sqrt{bd}.$$

By the Cauchy-Schwarz inequality,

$$\left(\frac{a^3}{c}+\frac{b^3}{d}\right)(ac+bd)$$
$$=\left(x_1^2+x_2^2\right)\left(y_1^2+y_2^2\right)$$
$$\ge(x_1y_1+x_2y_2)^2$$
$$=(a^2+b^2)^2$$
$$=\sqrt{(a^2+b^2)(c^2+d^2)}$$
$$\ge ac+bd.$$

Cancelling $ac + bd$ on both sides, we get the desired inequality.

**Solution 2.** Let

$$x=(a^3/c)^{2/3},\quad y=(b^3/d)^{2/3}.$$

By the $p = 3$, $q = 3/2$ case of Hölder's inequality,

$$a^2+b^2$$
$$=(c^{2/3})x+(d^{2/3})y$$
$$\le(c^2+d^2)^{1/3}(x^{3/2}+y^{3/2})^{2/3}$$

Cancelling $a^2+b^2=(c^2+d^2)^{1/3}$ on both sides, we get $1\le x^{3/2}+y^{3/2} = (a^3/c)+(b^3/d)$.

# Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *December 10, 2000.*

**Problem 111.** Is it possible to place 100 solid balls in space so that no two of them have a common interior point, and each of them touches at least one-third of the others? *(Source: 1997 Czech-Slovak Match)*

**Problem 112.** Find all positive integers ($x$, $n$) such that $x^n + 2^n + 1$ is divisor of $x^{n+1} + 2^{n+1} + 1$. (*Source: 1998 Romanian Math Olympiad*)

**Problem 113.** Let $a, b, c > 0$ and $abc \le 1$. Prove that

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \ge a + b + c.$$

(Hint: Consider the case $abc = 1$ first.)

**Problem 114.** (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) An infinite chessboard is given, with $n$ black squares and the remainder white. Let the collection of black squares be denoted by $G_0$. At each moment $t = 1, 2, 3, \ldots$, a simultaneous change of colour takes place throughout the board according to the following rule: every square gets the colour that dominates in the three square configuration consisting of the square itself, the square above and the square to the right. New collections of black squares $G_1$, $G_2$, $G_3$, … are so formed. Prove that $G_n$ is empty.

**Problem 115.** (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) Find the locus of the points $P$ in the plane of an equilateral triangle $ABC$ for which the triangle formed with lengths $PA$, $PB$ and $PC$ has constant area.

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## *Solutions*

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**Problem 106.** Find all positive integer ordered pairs ($a$, $b$) such that

$$\gcd(a,b) + \operatorname{lcm}(a,b) = a + b + 6,$$

where gcd stands for greatest common divisor (or highest common factor) and lcm stands for least common multiple.

***Solution.*** **CHAN An Jack** and **LAW Siu Lun Jack** (Mei Kei College, Form 6), **CHAN Chin Fei** (STFA Leung Kau Kui College), **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHAU Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **FUNG Wing Kiu Ricky** (La Salle College), **HUNG Chung Hei** (Pui Ching Middle School, Form 5), **KO Man Ho** (Wah Yan College, Kowloon, Form 7), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Ka Ho** (HKU, Year 1), **LEE Kevin** (La Salle College), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **MAK Hoi Kwan Calvin** (La Salle College), **OR Kin** (SKH Bishop Mok Sau Tseng Secondary School), **POON Wing Sze Jessica** (STFA Leung Kau Kui College, Form 7), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 6), **TONG Chin Fung** (SKH Lam Woo Memorial Secondary School, Form 6), **WONG Wing Hong** (La Salle College, Form 3) and **YEUNG Kai Shing** (La Salle College, Form 4).

Let $m = \gcd(a, b)$, then $a = mx$ and $b = my$ with $\gcd(x, y) = 1$. In that case, $\operatorname{lcm}(a, b) = mxy$. So the equation becomes $m + mxy = mx + my + 6$. This is equivalent to $m(x – 1)(y – 1) = 6$. Taking all possible positive integer factorizations of 6 and requiring $\gcd(x, y) = 1$, we have $(m, x, y) = (1, 2, 7)$, $(1, 7, 2)$, $(1, 3, 4)$, $(1, 4, 3)$, $(3, 2, 3)$ and $(3, 3, 2)$. Then $(a, b) = (2, 7)$, $(7, 2)$, $(3, 4)$, $(4, 3)$, $(6, 9)$ and $(9, 6)$. Each of these is easily checked to be a solution.

*Other recommended solvers:* **CHAN Kin Hang Andy** (Bishop Hall Jubilee School, Form 7) and **CHENG Kei Tsi Daniel** (La Salle College, Form 6).

**Problem 107.** For $a, b, c > 0$, if $abc = 1$, then show that

$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \ge \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.$$

***Solution 1.*** **CHAN Hiu Fai Philip** (STFA Leung Kau Kui College, Form 7), **LAW Ka Ho** (HKU, Year 1) and **TSUI Ka Ho Willie** (Hoi Ping Chamber of Commerce Secondary School, Form 7).

By the AM-GM inequality and the fact $abc = 1$, we get

$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \ge 2\left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ca}{b}} + \sqrt{\frac{ab}{c}}\right)$$

$$= \left(\sqrt{\frac{ca}{b}} + \sqrt{\frac{ab}{c}}\right) + \left(\sqrt{\frac{ab}{c}} + \sqrt{\frac{bc}{a}}\right) + \left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ca}{b}}\right) \ge 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \ge$$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} + 3\sqrt[6]{abc} = \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.$$

***Solution 2.*** **CHAN Kin Hang Andy** (Bishop Hall Jubliee School, Form 7), **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHAU Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Kei Tsi** (La Salle College, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **LAW Ka Ho** (HKU, Year 1) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Without loss of generality, assume $a \ge b \ge c$. Then $1/\sqrt{a} \le 1/\sqrt{b} \le 1/\sqrt{c}$. By the rearrangement inequality,

$$\frac{b}{\sqrt{a}} + \frac{c}{\sqrt{b}} + \frac{a}{\sqrt{c}} \ge \frac{a}{\sqrt{a}} + \frac{b}{\sqrt{b}} + \frac{c}{\sqrt{c}} = \sqrt{a} + \sqrt{b} + \sqrt{c}$$

Also, by the AM-GM inequality,

$$\frac{c}{\sqrt{a}} + \frac{a}{\sqrt{b}} + \frac{b}{\sqrt{c}} \ge 3.$$

Adding these two inequalities, we get the desired inequality.

*Generalization*: Professor Murray S. Klamkin (University of Alberta, Canada) sent in a solution, which proved a stronger inequality and later generalized it to $n$ variables. He made the sub-stitutions $x_1 = \sqrt{a}$, $x_2 = \sqrt{b}$, $x_3 = \sqrt{c}$ to get rid of square roots and let $S_m = x_1^m + x_2^m + x_3^m$ so that the inequality became

$$\frac{x_2^2 + x_3^2}{x_1} + \frac{x_3^2 + x_1^2}{x_2} + \frac{x_1^2 + x_2^2}{x_3} \ge S_1 + 3.$$

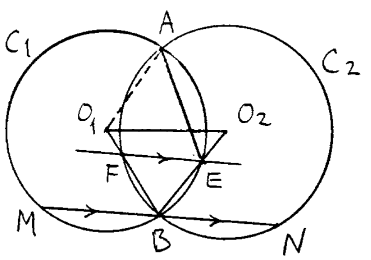
By the AM-GM inequality, $S_m \ge 3\sqrt[3]{x_1^m x_2^m x_3^m} = 3$. Since $S_2/3 \ge (S_1/3)^2 \ge S_1/3$ by the power mean inequality, we would get a stronger inequality by replacing $S_1 + 3$ by $2S_2$. Rearranging terms, this stronger inequality could be rewritten as $S_2(S_{-1} – 3) \ge S_1 – S_2$. Now the left side is nonnegative, but the right side is nonpositive. So the stronger inequality is true. If we replace 3 by $n$

and assume $x_1 \cdots x_n = 1$, then as above, we will get $S_m(S_{1-m} - n) \geq S_1 - S_m$ by the AM-GM and power mean inequalities. Expanding and regrouping terms, we get the stronger inequality in $n$ variables, namely

$$\sum_{i=1}^{n} \frac{S_m - x_i^m}{x_i^{m-1}} \geq (n-1)S_m.$$

*Other recommended solvers:* **CHAN Chin Fei** (STFA Leung Kau Kui College), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Hiu Fai** (Wah Yan College, Kowloon, Form 7), **LEE Kevin** (La Salle College, Form 5), **MAK Hoi Kwan Calvin** (La Salle College), **OR Kin** (SKH Bishop Mok Sau Tseng Secondary School) and **YEUNG Kai Shing** (La Salle College, Form 4).

**Problem 108.** Circles $C_1$ and $C_2$ with centers $O_1$ and $O_2$ (respectively) meet at points $A$, $B$. The radii $O_1B$ and $O_2B$ intersect $C_1$ and $C_2$ at $F$ and $E$. The line parallel to $EF$ through $B$ meets $C_1$ and $C_2$ at $M$ and $N$, respectively. Prove that $MN = AE + AF$. (*Source: 17th Iranian Mathematical Olympiad*)




***Solution.*** **YEUNG Kai Shing** (La Salle College, Form 4).

As the case $F = E = B$ would make the problem nonsensible, the radius $O_1B$ of $C_1$ can only intersect $C_2$, say at $F$. Then the radius $O_2B$ of $C_2$ intersect $C_1$ at $E$. Since $\Delta\ EO_1B$ and $\Delta\ FO_2B$ are isosceles, $\angle EO_1F = 180^\circ - 2\angle FBE = \angle EO_2F$. Thus, $E$, $O_2$, $O_1$, $F$ are concyclic. Then $\angle AEB = (360^\circ - \angle AO_1B)/2 = 180^\circ - \angle O_2O_1F = \angle O_2EF = \angle EBM$. So arc$AMB$ = arc$MAE$. Subtracting minor arc$AM$ from both sides, we get minor arc$MB$ = minor arc$AE$. So $MB = AE$. Similarly, $NB = AF$. Then $MN = MB + NB = AE + AF$.

*Other recommended solvers:* **Chan Kin Hang Andy** (Bishop Hall Jubilee School, Form 7), **CHAU Suk Ling** (Queen Elizabeth School, Form 6) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

**Problem 109**. Show that there exists an increasing sequence $a_1$, $a_2$, $a_3$, … of positive integers such that for every nonnegative integer $k$, the sequence $k + a_1$, $k + a_2$, $k + a_3$, … contains only finitely many prime numbers. (*Source: 1997 Math Olympiad of Czech and Slovak Republics*)

***Solution.*** **CHAU Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Kei Tsi** (La Salle College, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Hiu Fai** (Wah Yan College, Kowloon, Form 7), **LAW Ka Ho** (HKU, Year 1) and **YEUNG Kai Shing** (La Salle College, Form 4).

Let $a_n = n! + 2$. Then for every non-negative integer $k$, if $n \geq k + 2$, then $k + a_n$ is divisible by $k + 2$ and is greater than $k + 2$, hence not prime.

*Other commended solvers:* **CHAN Kin Hang Andy** (Bishop Hall Jubliee School, Form 7), **KO Man Ho** (Wah Yan College, Form 7), **LEE Kevin** (La Salle College, Form 5) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

**Problem 110.** In a park, 1000 trees have been placed in a square lattice. Determine the maximum number of trees that can be cut down so that from any stump, you cannot see any other stump. (Assume the trees have negligible radius compared to the distance between adjacent trees.) (*Source: 1997 German Mathematical Olympiad*)

***Solution.*** **CHAN Kin Hang Andy** (Bishop Hall Jubliee School, Form 7), **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **Chau Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Kei Tsi** (La Salle College, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **FUNG Wing Kiu Ricky** (La Salle College), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Ka Ho** (HKU, Year 1), **LEE Kevin** (La Salle College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **LYN Kwong To** and **KO Man Ho** (Wah Yan College, Kowloon, Form 7), **POON Wing Sze Jessica** (STFA Leung Kau Kui College, Form 7) and **YEUNG Kai Shing** (La Salle College, Form 4).

In every $2\times2$ subsquare, only one tree can be cut. So a maximum of 2500 trees can be cut down. Now let the trees be at $(x, y)$, where $x, y = 0, 1, 2, \ldots, 99$. If we cut down the 2500 trees at $(x, y)$ with both $x$ and $y$ even, then the condition will be satisfied. To see this, consider the stumps at $(x_1, y_1)$ and $(x_2, y_2)$ with $x_1$, $y_1$, $x_2$, $y_2$ even. The cases $x_1 = x_2$ or $y_1 = y_2$ are clear. Otherwise, write $(y_2 - y_1)/(x_2 - x_1) = m/n$ in lowest term. Then either $m$ or $n$ is odd and so the tree at $(x_1 + m, y_1 + n)$ will be between $(x_1, y_1)$ and $(x_2, y_2)$.

*Other recommended solvers:* **NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 7).

## Olympiad Corner

*(continued from page 1)*

**Problem 3.** ***(cont'd)***
Determine all values of $\lambda$ such that, for any point $M$ on the line and any initial position of the $n$ fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of $M$.

**Problem 4.** A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one a blue one, so that each contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

**Problem 5.** Determine whether or not there exists a positive integer $n$ such that $n$ is divisible by exactly 2000 different prime numbers, and $2^n + 1$ is divisible by $n$.

**Problem 6.** Let $AH_1$, $BH_2$, $CH_3$, be the altitudes of an acute-angled triangle ABC. The incircle of the triangle ABC touches the sides BC, CA, AB at $T_1$, $T_2$, $T_3$, respectively. Let the lines $\ell_1, \ell_2, \ell_3$ be the reflections of the lines $H_2H_3$, $H_3H_1$, $H_1H_2$ in the lines $T_2T_3$, $T_3T_1$, $T_1T_2$, respectively.

Prove that $\ell_1, \ell_2, \ell_3$ determine a triangle whose vertices lie on the incircle of the triangle ABC.

# Mathematical Excalibur

Volume 5, Number 5 — *November 2000 – December 2000*

## Olympiad Corner

*British Mathematical Olympiad, January 2000:*

*Time allowed: 3 hours 30 minutes*

**Problem 1.** Two intersecting circles $C_1$ and $C_2$ have a common tangent which touches $C_1$ at $P$ and $C_2$ at $Q$. The two circles intersect at $M$ and $N$, where $N$ is nearer to $PQ$ than $M$ is. The line $PN$ meets the circle $C_2$ again at $R$. Prove that $MQ$ bisects angle $PMR$.

**Problem 2.** Show that for every positive integer $n$,

$$121^n - 25^n + 1900^n - (-4)^n$$

is divisible by 2000.

**Problem 3.** Triangle $ABC$ has a right angle at $A$. Among all points $P$ on the perimeter of the triangle, find the position of $P$ such that

$$AP + BP + CP$$

is minimized.

**Problem 4.** For each positive integer $k$, define the sequence $\{a_n\}$ by

$$a_0 = 1 \quad \text{and} \quad a_n = kn + (-1)^n\, a_{n-1}$$

for each $n \geq 1$.

***(continued on page 4)***

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**On-line:** http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is *February 4, 2001*.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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# 五點求圓錐曲線

梁子傑
香港道教聯合會青松中學

我們知道，圓錐曲線是一些所謂二次形的曲線，即一條圓錐曲線會滿足以下的一般二次方程：$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$，其中 $A$、$B$ 及 $C$ 不會同時等於 0。假設 $A \neq 0$，那麼我們可以將上式除以 $A$，並化簡成以下模式：

$$x^2 + bxy + cy^2 + dx + ey + f = 0。$$

以上的方程給了我們一個啟示：就是五點能夠定出一個圓錐曲線。因為如果我們知道了五個不同點的坐標，我們可以將它們分別代入上面的方程中，從而得到一個有 5 個未知數（即 $b$、$c$、$d$、$e$ 和 $f$）和 5 條方程的方程組。祇要解出各未知數的答案，就可以知道該圓錐曲線的方程了。

不過，上述方法雖然明顯，但真正操作時又困難重重！這是由於有 5 個未知數的聯立方程卻不易解！而且我們在計算之初假設 $x^2$ 的係數非零，但萬一這假設不成立，我們就要改設 $B$ 或 $C$ 非零，並需要重新計算一次了。

幸好，我們可以通過「圓錐曲線族」的想法來解此問題。方法見下例：

**例：** 求穿過 $A(1, 0)$, $B(3, 1)$, $C(0, 3)$, $D(-4, -1)$, $E(-2, -3)$ 五點的圓錐曲線方程。

**解：** 利用兩點式，先求出以下各直線的方程：

$AB$：$\dfrac{y-0}{x-1} = \dfrac{1-0}{3-1}$，即 $x - 2y - 1 = 0$

$CD$：$\dfrac{y-3}{x-0} = \dfrac{-1-3}{-4-0}$，即 $x - y + 3 = 0$

$AC$：$\dfrac{y-0}{x-1} = \dfrac{3-0}{0-1}$，即 $3x + y - 3 = 0$

$BD$：$\dfrac{y-1}{x-3} = \dfrac{-1-1}{-4-3}$，即 $2x - 7y + 1 = 0$

然後將 $AB$ 和 $CD$ 的方程「相乘」，得一條圓錐曲線的方程：

$(x - 2y - 1)(x - y + 3) = 0$，即 $x^2 - 3xy + 2y^2 + 2x - 5y - 3 = 0$。

**注意：** 雖然上述的方程是一條二次形「曲線」，但實際上它是由兩條直線所組成的。同時，亦請大家留意，該曲線同時穿過 $A$、$B$、$C$ 和 $D$ 四點。

類似地，我們又將 $AC$ 和 $BD$「相乘」，得：

$(3x + y - 3)(2x - 7y + 1) = 0$，即 $6x^2 - 19xy - 7y^2 - 3x + 22y - 3 = 0$。

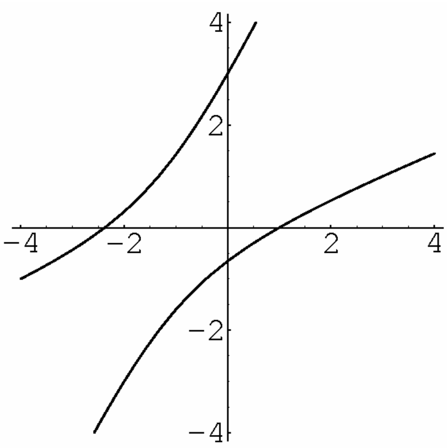
考慮圓錐曲線族：

$x^2 - 3xy + 2y^2 + 2x - 5y - 3 + k(6x^2 - 19xy - 7y^2 - 3x + 22y - 3) = 0$。很明顯，無論 $k$ 取任何數值，這圓錐曲線族都會同樣穿過 $A$、$B$、$C$ 和 $D$ 四點。

最後，將 $E$ 點的坐標代入曲線族中，得：$12 + k(-216) = 0$，即 $k = 1/18$，由此得所求的圓錐曲線方程為

$18(x^2 - 3xy + 2y^2 + 2x - 5y - 3) + (6x^2 - 19xy - 7y^2 - 3x + 22y - 3) = 0$，即

$24x^2 - 73xy + 29y^2 + 33x - 68y - 57 = 0$。



## Majorization Inequality

***Kin Y. Li***

The majorization inequality is a generalization of Jensen's inequality. While Jensen's inequality provides one extremum (either maximum or minimum) to a convex (or concave) expression, the majorization inequality can provide both in some cases as the examples below will show. In order to state this inequality, we first introduce the concept of majorization for ordered set of numbers. If

$$x_1 \ge x_2 \ge \cdots \ge x_n,$$
$$y_1 \ge y_2 \ge \cdots \ge y_n,$$
$$x_1 \ge y_1, \quad x_1 + x_2 \ge y_1 + y_2, \quad \ldots,$$
$$x_1 + \cdots + x_{n-1} \ge y_1 + \cdots + y_{n-1}$$

and

$$x_1 + \cdots + x_n = y_1 + \cdots + y_n,$$

then we say $(x_1, x_2, \ldots, x_n)$ *majorizes* $(y_1, y_2, \ldots, y_n)$ and write

$$(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n).$$

Now we are ready to state the inequality.

**Majorization Inequality.** *If the function f is convex on the interval I* = [*a*, *b*] *and*

$$(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n)$$

for $x_i, y_i \in I$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n)$$
$$\ge f(y_1) + f(y_2) + \cdots + f(y_n).$$

*For strictly convex functions, equality holds if and only if* $x_i = y_i$ *for i = 1, 2, …, n. The statements for concave functions can be obtained by reversing inequality signs.*

Next we will show that the majorization inequality implies Jensen's inequality. This follows from the observation that if $x_1 \ge x_2 \ge \cdots \ge x_n$, then $(x_1, x_2, \ldots, x_n) \succ$ $(x, x, \ldots, x)$, where $x$ is the arithmetic mean of $x_1, x_2, \ldots, x_n$. (Thus, applying the majorization inequality, we get Jensen's inequality.) For $k = 1, 2, \ldots, n - 1$, we have to show $x_1 + \cdots + x_k \ge kx$. Since

$$(n-k)(x_1 + \cdots + x_k)$$
$$\ge (n-k)kx_k \ge k(n-k)x_{k+1}$$
$$\ge k(x_{k+1} + \cdots x_n).$$

Adding $k(x_1 + \cdots + x_k)$ to the two extremes, we get

$$n(x_1 + \cdots + x_k) \ge k(x_1 + \cdots + x_n) = knx.$$

Therefore, $x_1 + \cdots + x_k \ge kx.$

***Example 1.*** For acute triangle ABC, show that

$$1 \le \cos A + \cos B + \cos C \le \frac{3}{2}$$

and determine when equality holds.

***Solution.*** Without loss of generality, assume $A \ge B \ge C$. Then $A \ge \pi/3$ and $C \le \pi/3$. Since $\pi/2 \ge A \ge \pi/3$ and

$$\pi \ge A + B (= \pi - C) \ge 2\pi/3,$$

we have $(\pi/2, \pi/2, 0) \succ (A, B, C) \succ (\pi/3, \pi/3, \pi/3)$. Since $f(x) = \cos x$ is strictly concave on $I = [0, \pi/2]$, by the majorization inequality,

$$1 = f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) + f(0)$$
$$\le f(A) + f(B) + f(C)$$
$$= \cos A + \cos B + \cos C$$
$$\le f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = \frac{3}{2}.$$

For the first inequality, equality cannot hold (as two of the angles cannot both be right angles). For the second inequality, equality holds if and only if the triangle is equilateral.

*Remarks.* This example illustrates the equilateral triangles and the degenerate case of two right angles are extreme cases for convex (or concave) sums.

***Example 2.*** Prove that if $a, b \ge 0$, then

$$\sqrt[3]{a + \sqrt[3]{a}} + \sqrt[3]{b + \sqrt[3]{b}} \le \sqrt[3]{a + \sqrt[3]{b}} + \sqrt[3]{b + \sqrt[3]{a}}.$$

(*Source: Math Horizons, Nov. 1995, Problem 36 of Problem Section, proposed by E.M. Kaye*)

***Solution.*** Without loss of generality, we may assume $b \ge a \ge 0$. Among the numbers

$$x_1 = b + \sqrt[3]{b}, \quad x_2 = b + \sqrt[3]{a},$$
$$x_3 = a + \sqrt[3]{b}, \quad x_4 = a + \sqrt[3]{a},$$

$x_1$ is the maximum and $x_4$ is the minimum. Since $x_1 + x_4 = x_2 + x_3$, we get $(x_1, x_4) \succ (x_2, x_3)$ or $(x_3, x_2)$ (depends on which of $x_2$ or $x_3$ is larger). Since $f(x) = \sqrt[3]{x}$ is concave on the interval $[0, \infty)$, so by the majorization inequality,

$$f(x_4) + f(x_1) \le f(x_3) + f(x_2),$$

which is the desired inequality.

***Example 3.*** Find the maximum of $a^{12} + b^{12} + c^{12}$ if $-1 \le a, b, c \le 1$ and $a + b + c = -1/2$.

***Solution.*** Note the continuous function $f(x) = x^{12}$ is convex on [–1, 1] since $f''(x) = 132x^{10} \ge 0$ on (–1, 1). If $1 \ge a \ge b \ge c \ge -1$ and

$$a + b + c = -\frac{1}{2},$$

then we get $(1, -1/2, -1) \succ (a, b, c)$. This is because $1 \ge a$ and

$$\frac{1}{2} = 1 - \frac{1}{2} \ge -c - \frac{1}{2} = a + b.$$

So by the majorization inequality,

$$a^{12} + b^{12} + c^{12}$$
$$= f(a) + f(a) + f(c)$$
$$\le f(1) + f\left(-\frac{1}{2}\right) + f(-1)$$
$$= 2 + \frac{1}{2^{12}}.$$

The maximum value $2 + (1/2^{12})$ is attained when $a = 1$, $b = -1/2$ and $c = -1$.

*Remarks.* The example above is a simplification of a problem in the 1997 Chinese Mathematical Olympiad.

***Example 4.*** (1999 IMO) Let $n$ be a fixed integer, with $n \ge 2$.

(a) Determine the least constant $C$ such that the inequality

$$\sum_{1 \le i < j \le n} x_i x_j \left(x_i^2 + x_j^2\right) \le C \left(\sum_{1 \le i \le n} x_i\right)^4$$

holds for all real numbers $x_1, x_2, \ldots, x_n \ge 0$.

(b) For this constant $C$, determine when equality holds.

***Solution.*** Consider the case $n = 2$ first. Let $x_1 = m + h$ and $x_2 = m - h$, then $m = (x_1 + x_2)/2$, $h = (x_1 - x_2)/2$ and

$$x_1 x_2 \left(x_1^2 + x_2^2\right) = 2\left(m^4 - h^4\right)$$
$$\le 2m^4 = \frac{1}{8}(x_1 + x_2)^4$$

with equality if and only if $h = 0$, i.e. $x_1 = x_2$.

***(continued on page 4)***

## Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *February 4, 2001.*

**Problem 116.** Show that the interior of a convex quadrilateral with area $A$ and perimeter $P$ contains a circle of radius $A/P$.

**Problem 117.** The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the other three lengths. Prove that two of the sides have the same length.

**Problem 118.** Let $R$ be the real numbers. Find all functions $f: R \to R$ such that for all real numbers $x$ and $y$,

$$f(xf(y) + x) = xy + f(x).$$

**Problem 119.** A circle with center $O$ is internally tangent to two circles inside it at points $S$ and $T$. Suppose the two circles inside intersect at $M$ and $N$ with $N$ closer to $ST$. Show that $OM \perp MN$ if and only if $S, N, T$ are collinear. (*Source: 1997 Chinese Senior High Math Competitiion*)

**Problem 120.** Twenty-eight integers are chosen from the interval [104, 208]. Show that there exist two of them having a common prime divisor.

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### *Solutions*

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**Problem 111.** Is it possible to place 100 solid balls in space so that no two of them have a common interior point, and each of them touches at least one-third of the others? *(Source: 1997 Czech-Slovak Match)*

***Solution 1.*** **LEE Kai Seng** (HKUST).

Take a smallest ball $B$ with center at $O$ and radius $r$. Any other ball touching $B$ at $x$ contains a smaller ball of radius $r$ also touching $B$ at $x$. Since these smaller balls are contained in the ball with center $O$ and radius $3r$, which has a volume 27 times the volume of $B$, there are at most 26 of these other balls touching $B$.

***Solution 2.*** **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Consider a smallest ball $S$ with center $O$ and radius $r$. Let $S_i$ and $S_j$ (with centers $O_i$ and $O_j$ and radii $r_i$ and $r_j$, respectively) be two other balls touching $S$ at $P_i$ and $P_j$, respectively. Since $r_i$, $r_j \geq$ r, we have $O_i\,O_j \geq r_i + r_j \geq r + r_i = O\,O_i$ and similarly $O_i\,O_j \geq O\,O_j$. So $O_i\,O_j$ is the longest side of $\Delta\, O\,O_i\,O_j$. Hence $\angle P_iOP_j = \angle O_iOO_j \geq 60^\circ$.

For ball $S_i$, consider the solid cone with vertex at $O$ obtained by rotating a $30^\circ$ angle about $OP_i$ as axis. Let $A_i$ be the part of this cone on the surface of $S$. Since $\angle P_iOP_j \geq 60^\circ$, the interiors of $A_i$ and $A_j$ do not intersect. Since the surface area of each $A_i$ is greater than $\pi(r\sin 30^\circ)^2 = \pi\, r^2/4$, which is 1/16 of the surface area of $S$, $S$ can touch at most 15 other balls. So the answer to the question is no.

*Other recommended solvers:* **CHENG Kei Tsi** (La Salle College, Form 6).

**Problem 112.** Find all positive integers ($x$, $n$) such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$. *(Source: 1998 Romanian Math Olympiad)*

***Solution.*** **CHENG Kei Tsi** (La Salle College, Form 6), **LEE Kevin** (La Salle College, Form 5) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

For $x = 1$, $2(1^n + 2^n + 1) > 1^{n+1} + 2^{n+1} + 1 > 1^n + 2^n + 1$. For $x = 2$, $2(2^n + 2^n + 1) > 2^{n+1} + 2^{n+1} + 1 > 2^n + 2^n + 1$. For $x = 3$, $3(3^n + 2^n + 1) > 3^{n+1} + 2^{n+1} + 1 > 2(3^n + 2^n + 1)$. So there are no solutions with $x = 1, 2, 3$.

For $x \geq 4$, if $n \geq 2$, then we get $x(x^n + 2^n + 1) > x^{n+1} + 2^{n+1} + 1$. Now

$$\begin{aligned} &x^{n+1} + 2^{n+1} + 1 \\ &= (x-1)(x^n + 2^n + 1) \\ &\quad + x^n - (2^n + 1)\,x + 3\cdot 2^n + 2 \\ &> (x-1)(x^n + 2^n + 1) \end{aligned}$$

because for $n = 2$, $x^n - (2^n + 1)x + 2^{n+1} = x^2 - 5x + 8 > 0$ and for $n \geq 3$, $x^n - (2^n + 1)x \geq x(4^{n-1} - 2^n - 1) > 0$. Hence only $n = 1$ and $x \geq 4$ are possible. In that case, $x^n + 2^n + 1 = x + 3$ is a divisor of $x^{n+1} + 2^{n+1} + 1 = x^2 + 5 = (x-3)(x+3) + 14$ if and only if $x + 3$ is a divisor of 14. Since $x + 3 \geq 7$, $x = 4$ or 11. So the solutions are $(x, y) = (4, 1)$ and $(11, 1)$.

**Problem 113.** Let $a, b, c > 0$ and $abc \leq 1$. Prove that

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq a + b + c\,.$$

***Solution.*** **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Since $abc \leq 1$, we get $1/(bc) \geq a$, $1/(ac) \geq b$ and $1/(ab) \geq c$. By the AM-GM inequality,

$$\frac{2a}{c} + \frac{c}{b} = \frac{a}{c} + \frac{a}{c} + \frac{c}{b} \geq 3\sqrt[3]{\frac{a^2}{bc}} \geq 3a\,.$$

Similarly, $2b/a + a/c \geq 3b$ and $2c/b + b/a \geq 3c$. Adding these and dividing by 3, we get the desired inequality.

Alternatively, let $x = \sqrt[9]{a^4b/c^2}$, $y = \sqrt[9]{c^4a/b^2}$ and $z = \sqrt[9]{b^4c/a^2}$. We have $a = x^2y$, $b = z^2x$, $c = y^2z$ and $xyz = \sqrt[3]{abc} \leq 1$. Using this and the re-arrangement inequality, we get

$$\begin{aligned} &\frac{a}{c} + \frac{b}{a} + \frac{c}{b} = \frac{x^2}{yz} + \frac{z^2}{xy} + \frac{y^2}{zx} \\ &\geq xyz\left(\frac{x^2}{yz} + \frac{z^2}{xy} + \frac{y^2}{zx}\right) = x^3 + y^3 + z^3 \\ &\geq x^2y + y^2z + z^2x = a + b + c\,. \end{aligned}$$

**Problem 114**. (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) An infinite chessboard is given, with $n$ black squares and the remainder white. Let the collection of black squares be denoted by $G_0$. At each moment $t = 1, 2, 3, \ldots$, a simultaneous change of colour takes place throughout the board according to the following rule: every square gets the colour that dominates in the three square configuration consisting of the square itself, the square above and the square to the right. New collections of black squares $G_1$, $G_2$, $G_3$, … are so formed. Prove that $G_n$ is empty.

***Solution.*** **LEE Kai Seng** (HKUST).

Call a rectangle (made up of squares on the chess board) *desirable* if with respect to its left-lower vertex as origin, every square in the first quadrant outside the rectangle is white. The most crucial fact is that knowing only the colouring of the squares in a desirable rectangle, we can determine their colourings at all later moments. Note that the smallest rectangle enclosing all black squares is a desirable rectangle. We will prove by induction that all squares of a desirable rectangle with at most *n* black squares will become white by *t* = *n*. The case *n* = 1 is clear. Suppose the cases *n* < *N* are true. Let *R* be a desirable rectangle with *N* black squares. Let $R_0$ be the smallest rectangle in *R* containing all *N* black squares, then $R_0$ is also desirable. Being smallest, the leftmost column and the bottom row of $R_0$ must contain some black squares. Now the rectangle obtained by deleting the left column of $R_0$ and the rectangle obtained by deleting the bottom row of $R_0$ are desirable and contain at most *n* - 1 black squares. So by *t* = *n* - 1, all their squares will become white. Finally the left bottom corner square of $R_0$ will be white by *t* = *n*.

*Comments:* This solution is essentially the same as the proposer's solution.

*Other commended solvers:* **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

**Problem 115.** (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) Find the locus of the points *P* in the plane of an equilateral triangle *ABC* for which the triangle formed with *PA, PB* and *PC* has constant area.

***Solution.*** **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Without loss of generality, assume *PA* ≥ *PB, PC*. Consider *P* outside the circumcircle of $\Delta ABC$ first. If *PA* is between *PB* and *PC*, then rotate $\Delta PAC$ about *A* by $60^\circ$ so that *C* goes to *B* and *P* goes to *P′*. Then $\Delta APP'$ is equilateral and the sides of $\Delta PBP'$ have length *PA, PB, PC*.

Let *O* be the circumcenter of $\Delta ABC$, *R* be the circumradius and $x = AB = AC = \sqrt{3}AO = \sqrt{3}R$. The area of $\Delta PBP'$ is the sum of the areas of $\Delta PAP'$, $\Delta PAB$, $\Delta P'AB$ (or $\Delta PAC$), which is

$$\frac{\sqrt{3}}{4}PA^2+\frac{1}{2}PA\cdot x\sin\angle PAB$$
$$+\frac{1}{2}PA\cdot x\sin PAC.$$

Now

$$\sin\angle PAB + \sin\angle PAC$$
$$= 2\sin 150^\circ\cos(\angle PAB-150^\circ)$$
$$= -\cos(\angle PAB+30^\circ)$$
$$=-\cos\angle PAO=\frac{PO^2-PA^2-R^2}{2PA\cdot R}.$$

Using these and simplifying, we get the area of $\Delta PBP'$ is $\sqrt{3}(PO^2-R^2)/4$.

If *PC* is between *PA* and *PB*, then rotate $\Delta PAC$ about *C* by $60^\circ$ so that *A* goes to *B* and *P* goes to *P′*. Similarly, the sides of $\Delta PBP'$ have length *PA, PB, PC* and the area is the same. The case *PB* is between *PA* and *PC* is also similar.

For the case *P* is inside the circumcircle of $\Delta ABC$, the area of the triangle with sidelengths *PA, PB, PC* can similarly computed to be $\sqrt{3}(R^2-PO^2)/4$. Therefore, the locus of *P* is the circle(s) with center *O* and radius $\sqrt{R^2\pm 4c/\sqrt{3}}$, where *c* is the constant area.

*Comments:* The proposer's solution only differed from the above solution in the details of computing areas.

## Olympiad Corner

***(continued from page 1)***

**Problem 4.** ***(cont'd)***

Determine all values of *k* for which 2000 is a term of the sequence.

**Problem 5.** The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy and Grumpy as a trio, and one of Bashful and Sneezy as a pair. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)
Suppose Snow White agreed to take part as well. In how many ways could the four teams then be formed?

## Majorization Inequality

***(continued from page 2)***

For the case *n* > 2, let $a_i = x_i/(x_1+\cdots+x_n)$ for *i* = 1, …, *n*, then $a_1+\cdots+a_n = 1$. In terms of $a_i$'s, the inequality to be proved becomes

$$\sum_{1\le i<j\le n} a_i a_j\left(a_i^2+a_j^2\right)\le C.$$

The left side can be expanded and regrouped to give

$$\sum_{i=1}^{n} a_i^3\left(a_1+\cdots+a_{i-1}+a_{i+1}+\cdots a_n\right)$$
$$=a_1^3(1-a_1)+\cdots+a_n^3(1-a_n).$$

Now *f*(*x*) = $x^3(1-x)=x^3-x^4$ = is strictly convex on $\left[0,\frac{1}{2}\right]$ because the second derivative is positive on $\left(0,\frac{1}{2}\right)$.

Since the inequality is symmetric in the $a_i$'s, we may assume $a_1\ge a_2\ge\cdots\ge a_n$.

If $a_1\le\frac{1}{2}$, then since

$$\left(\frac{1}{2},\frac{1}{2},0,...,0\right)\succ\left(a_1,a_2,...,a_n\right),$$

by the majorization inequality,

$$f(a_1)+f(a_2)+\cdots+f(a_n)$$
$$\le f\left(\frac{1}{2}\right)+f\left(\frac{1}{2}\right)+f(0)+\cdots+f(0)=\frac{1}{8}.$$

If $a_1>\frac{1}{2}$, then $1-a_1,a_2,...,a_n$ are in [0, $\frac{1}{2}$]. Since

$$(1-a_1,0,...,0)\succ(a_2,...,a_n),$$

by the majorization inequality and case *n* = 2, we have

$$f(a_1)+f(a_2)+\cdots+f(a_n)$$
$$\le f(a_1)+f(1-a_1)+f(0)+\cdots+f(0)$$
$$=f(a_1)+f(1-a_1)\le\frac{1}{8}.$$

Equality holds if and only if two of the variables are equal and the other $n-2$ variables all equal 0.