Junior problems

J259. Among all triples of real numbers (x, y, z) which lie on a unit sphere $x^2 + y^2 + z^2 = 1$ find a triple which maximizes $\min(|x-y|, |y-z|, |z-x|)$.

Proposed by Arkady Alt , San Jose, California, USA

J260. Solve in integers the equation

$$x^4 - y^3 = 111.$$

Proposed by José Hernández Santiago, Oaxaca, México

J261. Let $A_1 \dots A_n$ be a polygon inscribed in a circle with center O and radius R. Find the locus of points M on the circumference such that

$$A_1M^2 + \cdots + A_nM^2 = 2nR^2.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J262. Find all positive integers m, n such that $\binom{m+1}{n} = \binom{n}{m+1}$.

Proposed by Roberto Bosch Cabrera, Havana, Cuba.

J263. The n-th pentagonal number is given by the formula $p_n = \frac{n(3n-1)}{2}$. Prove that there are infinitely many pentagonal numbers that can be written as a sum of two perfect squares of positive integers.

Proposed by José Hernández Santiago, Oaxaca, México

J264. In triangle ABC, $2\angle A = 3\angle B$. Prove that

$$(a^2 - b^2)(a^2 + ac - b^2) = b^2c^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior problems

S259. Let a, b, c, d, e be integers such that

$$a(b+c)+b(c+d)+c(d+e)+d(e+a)+e(a+b)=0.$$

Prove that $a+b+c+d+e$ divides $a^5+b^5+c^5+d^5+e^5-5abcde$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S260. Let $m<n$ be positive integers and let $x_1, x_2, ..., x_n$ be positive real numbers. If A is a subset of $\{1,2,...,n\}$, define $s_A=\sum_{i\in A} x_i$ and $A^c=\{i\in\{1,2,\ldots,n\} | i\notin A\}$. Prove that

$$\sum_{|A|=m} \frac{s_A}{s_{A^c}} \geq \frac{m}{n-m}\binom{n}{m},$$

where the sum is taken over all m-element subsets A of $\{1,2,\ldots,n\}$.

Proposed by Mircea Becheanu, University of Bucharest, Romania

S261. Let ABC be a triangle with circumcircle Γ and let $\mathcal{K}$ be the circle simultaneously tangent to AB, AC and Γ, internally. Let X be a point on the circumcircle of ABC and let Y, Z be the intersections of Γ with the tangents from X with respect to $\mathcal{K}$. As X varies on Γ, what is the locus of the incenters of triangles XYZ?

Proposed by Cosmin Pohoata, Princeton University, USA

S262. Let a, b, c be the sides of a triangle and let m_a, m_b, m_c be the lengths of its medians. Prove that

$$a^2+b^2+c^2-ab-bc-ca \leq 4\left(m_a^2+m_b^2+m_c^2-m_am_b-m_bm_c-m_cm_a\right).$$

Proposed by Arkady Alt , San Jose, California, USA

S263. Prove that for all $n\geq 2$ and all $1\leq i\leq n$ we have

$$\sum_{j=1}^{n}(-1)^{n-j}\frac{\binom{n+j}{n}\binom{n}{n-j}}{i+j}=1.$$

Proposed by Marcel Chirita, Bucharest, Romania

S264. Let a, b, c, x, y, z be positive real numbers such that $ab+bc+ca=xy+yz+zx=1$. Prove that

$$a(y+z)+b(z+x)+c(x+y)\geq 2.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Undergraduate problems

U259. Compute

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n(n+a)}\right)^{n^3}}{\left(1+\frac{1}{n(n+b)}\right)^{n^3}}.$$

Proposed by Arkady Alt , San Jose, California, USA

U260. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ which are derivatives and satisfy

$$f^2 \in \int f(x)dx.$$

Proposed by Mihai Piticari, "Dragos Voda" National College, Romania

U261. Let $T_n(x)$ be the sequence of Chebyshev polynomials of the first kind, defined by $T_0(x)=0$, $T_1(x)=x$, and

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x)$$

for $n \geq 1$. Prove that for all $x \geq 1$ and all positive integers n

$$x \leq \sqrt[n]{T_n(x)} \leq 1 + n(x-1).$$

Proposed by Arkady Alt , San Jose, California, USA

U262. Let a and b be positive real numbers. Find $\lim_{n\to\infty} \sqrt[n]{\prod_{i=1}^{n}\left(a+\frac{b}{i}\right)}$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U263. Let $n \geq 2$ be an integer. A general $n \times n$ magic square is a matrix $A \in M_n(\mathbf{R})$ such that the sum of the elements in each row of A is the same. Prove that the set of $n \times n$ general magic squares is an $\mathbf{R}$-vector space and find its dimension.

Proposed by Cosmin Pohoata, Princeton University, USA

U264. Let A be a finite ring such that $1+1=0$. Prove that the equations $x^2=0$ and $x^2=1$ have the same number of solutions in A.

Proposed by Mihai Piticari, "Dragos Voda" National College, Romania

Olympiad problems

O259. Solve in integers the equation $x^5+15xy+y^5=1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O260. Let p be a positive real number. Define a sequence $(a_n)_{n\geq 1}$ by $a_1=0$ and

$$a_n=\left\lfloor \frac{n+1}{2}\right\rfloor^p+a_{\lfloor \frac{n}{2}\rfloor}$$

for $n\geq 2$. Find the minimum of $\frac{a_n}{n^p-1}$ over all positive integers n.

Proposed by Arkady Alt , San Jose, California, USA

O261. Find all positive integers n for which

$$\sigma(n)-\phi(n)\leq 4\sqrt{n},$$

where $\sigma(n)$ is the sum of positive divisors of n and ϕ is Euler's totient function.

Proposed by Albert Stadler Buchenrain, Herrliberg, Switzerland

O262. Let $n\geq 3$ be an integer. Consider a convex n-gon $A_1\dots A_n$ for which there is a point P in its interior such that $\angle A_iPA_{i+1}=\frac{2\pi}{n}$ for all $i\in[1,n-1]$. Prove that P is the point which minimizes the sum of distances to the vertices of the n-gon.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O263. A tournament T with a linear order $<$ on its vertices is called an *ordered tournament* and is denoted by $(T,<)$. If $(T,<)$ and $(T',<')$ are ordered tournaments, say $(T,<)$ is induced from $(T',<')$ if there is a map $f:V(T)\to V(T')$ satisfying

(i) $f(u)<' f(v)$ if and only if $u<v$;

(ii) $\overrightarrow{f(u)f(v)}\in E(T')$ if and only if $\overrightarrow{uv}\in E(T)$.

Prove that for any ordered tournament $(T,<)$, there exists a tournament T' such that, for every ordering $<'$ of T', $(T,<)$ is induced from $(T',<')$.

Proposed by Cosmin Pohoata, Princeton University, USA

O264. Let $p>3$ be a prime. Prove that $2^{p-1}\equiv 1 \pmod{p^2}$ if and only if the numerator of

$$\frac{1}{2}+\frac{1}{3}\left(1+\frac{1}{2}\right)+\cdots+\frac{1}{\frac{p-1}{2}}\left(1+\frac{1}{2}+\cdots+\frac{1}{\frac{p-3}{2}}\right)$$

is a multiple of p.

Proposed by Gabriel Dospinescu, Lyon, France