Junior problems

J301. Let a and b be nonzero real numbers such that $ab \geq \frac{1}{a} + \frac{1}{b} + 3$. Prove that

$$ab \geq \left(\frac{1}{\sqrt[3]{a}} + \frac{1}{\sqrt[3]{b}}\right)^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J302. Given that the real numbers x, y, z satisfy $x + y + z = 0$ and

$$\frac{x^4}{2x^2 + yz} + \frac{y^4}{2y^2 + zx} + \frac{z^4}{2z^2 + xy} = 1,$$

determine, with proof, all possible values of $x^4 + y^4 + z^4$.

Proposed by Razvan Gelca, Texas Tech University, USA

J303. Let ABC be an equilateral triangle. Consider a diameter XY of the circle centered at C which passes through A and B such that lines AB and XY as well as lines AX and BY meet outside this circle. Let Z be the point of intersection of AX and BY. Prove that

$$AX \cdot XZ + BY \cdot YZ + 2CZ^2 = XZ \cdot YZ + 6AB^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J304. Let a, b, c be real numbers such that $a + b + c = 1$. Let M_1 be the maximum value of $a + \sqrt{b} + \sqrt[3]{c}$ and let M_2 be the maximum value of $a + \sqrt{b + \sqrt[3]{c}}$. Prove that $M_1 = M_2$ and find this value.

Proposed by Aaron Doman, University of California, Berkeley, USA

J305. Consider a triangle ABC with $\angle ABC = 30^\circ$. Suppose the length of the angle bisector from vertex B is twice the length of the angle bisector from vertex A. Find the measure of $\angle BAC$.

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

J306. Let S be a nonempty set of positive real numbers such that for any a, b, c in S, the number $ab + bc + ca$ is rational. Prove that for any a and b in S, $\frac{a}{b}$ is a rational number.

Proposed by Bogdan Enescu, Buzau, Romania

Senior problems

S301. Let a, b, c be positive real numbers. Prove that

$$(a+b+c)(ab+bc+ca)(a^3+b^3+c^3) \le (a^2+b^2+c^2)^3.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S302. If triangle ABC has sidelengths a, b, c and triangle $A'B'C'$ has sidelengths $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, prove that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \cos A' \cos B' \cos C'.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S303. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{n}{a_n}\right)$, for $n \ge 1$. Find $\lfloor a_{2014} \rfloor$.

Proposed by Marius Cavachi, Romania

S304. Let M be a point inside triangle ABC. Line AM intersects the circumcircle of triangle MBC for the second time at D. Similarly, line BM intersects the circumcircle of triangle MCA for the second time at E and line CM intersects the circumcircle of triangle MAB for the second time at F. Prove that

$$\frac{AD}{MD} + \frac{BE}{ME} + \frac{CF}{MF} \ge \frac{9}{2}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

S305. Solve in integers the following equation:

$$x^2 + y^2 + z^2 = 2(xy + yz + zx) + 1.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S306. Points M, N, K lie on sides BC, CA, AB of a triangle ABC, respectively and are different from its vertices. Triangle MNK is called *beautiful* if $\angle BAC = \angle KMN$ and $\angle ABC = \angle KNM$. If in triangle ABC there are two beautiful triangles with a common vertex, prove that triangle ABC is right.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Undergraduate problems

U301. Let x, y, z, $t > 0$ such that $x \leq 2$, $x+y \leq 6$, $x+y+z \leq 12$, and $x+y+z+t \leq 24$. Prove that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t} \geq 1.$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

U302. Let a be a real number. Evaluate

$$a-\sqrt{a^2-\sqrt{a^4-\sqrt{a^8-\dots}}}\,.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

U303. Let p_1, p_2, $\dots$, p_k be distinct primes and let $n = p_1 p_2 \dots p_k$. For each function $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$, denote $P_f(n) = f(1)f(2)\dots f(n)$.

(a) For how many functions f are n and $P_f(n)$ are relatively prime?

(b) For how many functions f is $\gcd(n, P_f(n))$ a prime?

Proposed by Vladimir Cerbu and Mihai Piticari, Romania

U304. In a finite graph G, we call a subset S of the set of vertices a *dominating* set if the following conditions are satisfied

(i) the subgraph induced by S is connected,

(ii) every vertex of G is either in S or is adjacent to a vertex of S.

Given a positive integer k, find the maximum number of edges of G such that there are no dominating sets of size k or less.

Proposed by Cosmin Pohoata, Princeton University, USA

U305. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that $a_1 + a_2 + \dots + a_n < n^2$ for all $n \geq 1$. Prove that

$$\lim_{n\to\infty}\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)=\infty.$$

Proposed by Mihai Piticari Campulung, Moldovenesc, Romania

U306. Let n be a natural number. Prove the identity

$$\pi=\sum_{k=1}^{n}\frac{2^{k+1}}{k\binom{2k}{k}}+\frac{4^{n+1}}{\binom{2n}{n}}\int_1^\infty \frac{1}{(1+x^2)^{n+1}}\,dx$$

and derive the estimate

$$\frac{2}{2^n\sqrt{n}}<\pi-\sum_{k=1}^{n}\frac{2^{k+1}}{k\binom{2k}{k}}<\frac{4}{2^n\sqrt{n}}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Olympiad problems

O301. Let a, b, c, d be nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$. Prove that

$$\frac{a}{b+3}+\frac{b}{c+3}+\frac{c}{d+3}+\frac{d}{a+3}\le 1.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

O302. Let ABC be an isosceles triangle with $AB=AC$ and let $M\in(BC)$ and $N\in(AC)$ such that $\angle BAM=\angle MNC$. Suppose that lines MN and AB intersect at P. Prove that the bisectors of angles BAM and BPM intersect at a point lying on line BC.

Proposed by Bogdan Enescu, Buzau, Romania

O303. Let a, b, c be real numbers greater than 2 such that

$$\frac{1}{a^2-4}+\frac{1}{b^2-4}+\frac{1}{c^2-4}=\frac{1}{7}.$$

Prove that

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\le\frac{3}{7}.$$

Proposed by Mihaly Bencze, Brasov, Romania

O304. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be non-intersecting circles centered at O_1 and O_2. One common external tangent of these circles touches $\mathcal{C}_i$ at $P_i\,(i=1,2)$. The other common external tangent touches $\mathcal{C}_i$ at $Q_i\,(i=1,2)$. Denote by M the midpoint of Q_1Q_2. Let P_iM intersect $\mathcal{C}_i$ at R_i and R_1R_2 intersect $\mathcal{C}_i$ again at $S_i\,(i=1,2)$. P_1S_1 intersects P_2S_2 at A. The tangent to $\mathcal{C}_1$ at R_1 and the tangent to $\mathcal{C}_2$ at R_2 intersect at B. Prove that $AB\perp O_1O_2$.

Proposed by Alex Anderson, UC Berkeley, USA

O305. Prove that for any positive integers m and a, there is a positive integer n such that a^n+n is divisible by m.

Proposed by Gregory Galperin, Eastern Illinois University, USA

O306. Let ABC be a triangle with incircle γ and circumcircle Γ. Let Ω be the circle tangent to rays AB, AC, and to Γ externally, and let A' be the tangency point of Ω with Γ. Let the tangents from A' to γ intersect Γ again at B' and C'. Finally, let X be the tangency point of the chord $B'C'$ with γ. Prove that the circumcircle of triangle BXC is tangent to γ.

Proposed by Cosmin Pohoata, Princeton University, USA