Junior problems

J253. Prove that if $a, b, c > 0$ satisfy $abc = 1$, then

$$\frac{1}{ab+a+2} + \frac{1}{bc+b+2} + \frac{1}{ca+c+2} \leq \frac{3}{4}.$$

Proposed by Marcel Chirita, Bucharest, Romania

J254. Solve the following equation $F_{a_1} + F_{a_2} + \cdots + F_{a_k} = F_{a_1+a_2+\cdots+a_k}$, where F_i is the ith Fibonacci number.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

J255. Consider five points in a plane at a distance of at least 1 from each other. Prove that there exists two points located at a distance of at least $\frac{\sqrt{5}+1}{2}$ from each other.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J256. Evaluate

$$1^2 2! + 2^2 3! + \cdots + n^2(n+1)!$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J257. Let BC be a fixed chord of the circle ω and let A vary on the major arc BC of ω.

a) Show that the mirror images of H over the A-angle bisector also run along a circle (possibly with zero radius).

b) Show that the projections of H on the A-angle bisector run along a circle.

Proposed by Michal Rolinek, Charles University, Czech Republic

J258. Let x, y, z be positive real numbers such that $x \leq 1$, $y \leq 2$ and $x + y + z = 6$. Prove that

$$(x+1)(y+1)(z+1) \geq 4xyz.$$

Proposed by Marius Stanean, Zalau, Romania

Senior problems

S253. Solve in positive real numbers the system of equations:

$$\begin{aligned}(2x)^{2013}+(2y)^{2013}+(2z)^{2013} &= 3\\ xy+yz+zx+2xyz &= 1.\end{aligned}$$

Proposed by Roberto Bosch Cabrera, Havana, Cuba

S254. Let G be a graph with n vertices, where $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. Half of the edges of G are colored in red and half of the edges are colored in blue. Denote by $T_{r,r,b}$ the number of triangles with exactly two red edges and by $T_{b,b,r}$ the number of triangles with exactly two blue edges. Prove that if the number of red monochromatic triangles is equal to the number of blue monochromatic triangles, then $T_{r,r,b} = T_{b,b,r}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S255. Solve in real numbers the equation

$$2^x+2^{-x}+3^x+3^{-x}+\left(\frac{2}{3}\right)^x+\left(\frac{2}{3}\right)^{-x}=9x^4-7x^2+6.$$

Proposed by Mihaly Bencze, Brasov, Romania

S256. Two congruent circles ω_1 and ω_2 are both tangent to line ℓ from the same side. Common internal tangent k of ω_1 and ω_2 intersects ℓ at B. Points D and E are chosen on k (and on the same side of ℓ as the circles). Draw a tangent from D to ω_1 and from E to ω_2 (other than k in both cases) and denote their intersection with ℓ as A and C, respectively. Show that if A, B, and C lie on ℓ in this order and satisfy $BE/BD = BA/BC$ then the line joining the B-excenters of $\triangle ABD$ and $\triangle EBC$ is perpendicular to k.

Proposed by Michal Rolinek, Charles University, Czech Republic

S257. Find all complex numbers z for which

$$(z-z^2)(1-z+z^2)^2=\frac{1}{7}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S258. Which rational numbers can be written as the sum of the inverses of finitely many pairwise distinct triangular numbers?

Proposed by Gabriel Dospinescu, Lyon, France

Undergraduate problems

U253. Evaluate

$$\sum_{n>1} \frac{3n^2+1}{(n^3-n)^3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U254. Let A be an $n \times n$ matrix with entries $a_{i,j} \in \mathbb{R}$. Let σ_i^2 and τ_j^2 be the variance of entries in row i and column j, respectively. Denote by σ_A^2 the variance of all entries in A and let $B = \frac{1}{n}\sum_{i=1}^n \sigma_i^2$ and $C = \frac{1}{n}\sum_{j=1}^n \tau_j^2$. Prove that

$$\max(B, C) \leq \sigma_A^2 \leq B + C.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U255. Let S_n be the group of permutations of $\{1, 2, \ldots, n\}$. If $d > 1$ is an integer, let H_d be the set of those $\sigma \in S_n$ for which there are $k \geq 1$ and $\sigma_1, \ldots, \sigma_k \in S_n$ with $\sigma = \sigma_1^d \cdots \sigma_k^d$. Find H_2 and H_3.

Proposed by Mihai Piticari and Sorin Radulescu, Romania

U256. Let G be a finite group and H be a subgroup of G which has index p, for some prime p. Suppose that the order of H and $p-1$ are relatively prime. Prove that H is normal.

Proposed by Cosmin Pohoata, Princeton University, USA

U257. a) Let p and q be distinct primes and let G be a non-commutative group with pq elements. Prove that the center of G is trivial.

b) Let p, q, r be pairwise distinct primes and let G be a non-commutative group with pqr elements. Prove that the number of elements of the center of G is either 1 or a prime number.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

U258. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ having the intermediate value property and satisfying $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Proposed by Marius and Sorin Radulescu, Bucharest, Romania

Olympiad

O253. Find the least multiple of 2013 for which the system of equations

$$(x^2+y^2)(y^2+z^2)(z^2+x^2)=x^6+y^6+z^6+4n^2, \quad xyz=n$$

is solvable in integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O254. Consider two parallelograms that intersect exactly in eight points. Prove that the common area of these parallelograms is greater than or equal to half of the area of one of them.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O255. A set of positive integers is called *good* if $ab+1$ is a square for every pair of distinct elements a and b of the set. A *good* set S is called *maximal* if $S \cup \{n\}$ is not *good* for every positive integer n.
a) Show that there are not *maximal* sets with cardinality $1, 2, 3$.
b) Show that we can find infinitely many *maximal* sets with cardinality 4.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

O256. Let $A_1 \ldots A_n$ be a regular polygon and let M be a point inside it. Prove that

$$\sin \angle A_1MA_2 + \sin \angle A_2MA_3 + \cdots + \sin \angle A_nMA_1 > \sin \frac{2\pi}{n} + (n-2)\sin \frac{\pi}{n}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O257. Let p be an odd prime and let $N_k = \sum_{j=1}^{\frac{p-1}{2}} \tan^{2k} \frac{j\pi}{p}$. Prove the following statements:

(a) N_k is an integer.

(b) N_k is divisible by $p^{1+\lfloor 2\frac{k-1}{p-1} \rfloor}$.

(c) $N_{\frac{p-1}{2}k}$ is not divisible by p^{k+1}.

Proposed by Albert Stadler, Herrliberg, Switzerland

O258. Let $\sigma(n)$ be the sum of all positive divisors of n. Prove that for all $n > 1$,

$$\sum_{k=0}^{n-1} (-1)^k (2k+1) \sigma\left(\frac{n^2+n}{2} - \frac{k^2+k}{2}\right) = (-1)^{n-1} \frac{n(n+1)(2n+1)}{6}.$$

Proposed by Gabriel Dospinescu, Lyon, France