Mathematical Excalibur

Volume 6, Number 1 January 2001 – March 2001

Olympiad Corner

17th Balkan Mathematical Olympiad, 3-9 May 2000:

Time allowed: 4 hours 30 minutes

Problem 1. Find all the functions f : $\mathbf{R} \to \mathbf{R}$ with the property:

$$f(xf(x)+f(y))=(f(x))^2+y,$$

for any real numbers x and y.

Problem 2. Let ABC be a nonisosceles acute triangle and E be an interior point of the median AD, $D \in (BC)$. The point F is the orthogonal projection of the point E on the straight line BC. Let M be an interior point of the segment EF, N and P be the orthogonal projections of the point M on the straight lines AC and AB, respectively. Prove that the two straight lines containing the bisectrices of the angles PMN and PEN have no common point.

Problem 3. Find the maximum number of rectangles of the dimensions $1\times 10\sqrt{2}$, which is possible to cut off from a rectangle of the dimensions 50×90, by using cuts parallel to the edges of the initial rectangle.

(continued on page 2)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

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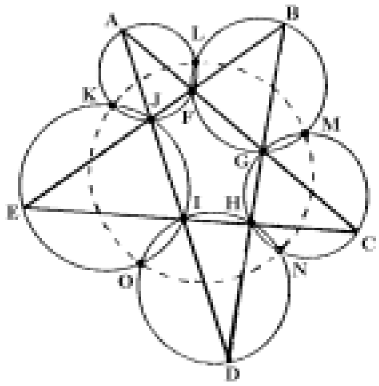
Concyclic Problems

Kin Y. Li

Near Christmas last year, I came across two beautiful geometry problems. I was informed of the first problem by a reporter, who was covering President Jiang Zemin's visit to Macau. While talking to students and teachers, the President posed the following problem.

For any pentagram ABCDE obtained by extending the sides of a pentagon FGHIJ, prove that neighboring pairs of the circumcircles of ΔAJF , BFG, CGH, DHI, EIJ intersect at 5 concyclic points K, L, M, N, O as in the figure.



The second problem came a week later. I read it in the Problems Section of the November issue of the *American Mathematical Monthly*. It was proposed by Floor van Lamoen, Goes, The Netherlands. Here is the problem.

A triangle is divided by its three medians into 6 smaller triangles. Show that the circumcenters of these smaller triangles lie on a circle.

To get the readers appreciating these problems, here I will say, *stop reading, try to work out these problems and come back to compare your solutions with those given below!*

Here is a guided tour of the solutions. The first step in enjoying geometry problems is to draw accurate pictures with compass and ruler!

Now we look at ways of getting solutions to these problems. Both are concyclic problems with more than 4 points. Generally, to do this, we show the points are concyclic four at a time. For example, in the first problem, if we can show *K, L, M, N* are concyclic, then by similar reasons, *L, M, N, O* will also be concyclic so that all five points lie on the circle passing through *L, M, N*.

There are two common ways of showing 4 points are concyclic. One way is to show the sum of two opposite angles of the quadrilateral with the 4 points as vertices is 180°. Another way is to use the converse of the intersecting chord theorem, which asserts that if lines WX and YZ intersect at P and $PW\cdot PX$ = $PY\cdot PZ$, then W, X, Y, Z are concyclic. (The equation implies ΔPWY , PZX are similar. Then $\angle PWY$ = $\angle PZX$ and the conclusion follows.)

For the first problem, as the points *K, L, M, N, O* are on the circumcirles, checking the sum of opposite angles equal 180° is likely to be easier as we can use the theorem about angles on the same segment to move the angles. To show *K, L, M, N* are concyclic, we consider showing $\angle LMN$ + $\angle LKN = 180^\circ$. Since the sides of $\angle LMN$ are in two circumcircles, it may be wise to break it into two angles *LMG* and *GMN*. Then the strategy is to *change* these to other angles *closer* to $\angle LKN$.

Now $\angle LMG = 180^\circ - \angle LFG = \angle LFA$ = $\angle LKA$. (So far, we are on track. We *bounced* $\angle LMG$ to $\angle LKA$, which shares a side with $\angle LKN$.) Next, $\angle GMN$ = $\angle GCN$ = $\angle ACN$. Putting these together, we have

$$\begin{aligned}&\angle LMN+\angle LKN\\ &=\angle LKA+\angle ACN+\angle LKN\\ &=\angle AKN+\angle ACN .\end{aligned}$$

Now if we can only show *A, K, N, C* are concyclic, then we will get 180° for the displayed equations above and we will finish. However, life is not that easy. This turned out to be the hard part. If you draw the circle through *A, C, N*, then you see it goes through *K* as expected and surprisingly, it also goes through another point, *I*. With this discovery, there is new hope. Consider the arc through *B, I, O*. On the two sides of this arc, you can see there are *corresponding point pairs* (*A, C*), (*K, N*), (*J, H*), (*F, G*). So to show *A, K, N, C* are concyclic, we can first try to show *N* is on the circle through *A, C, I*, then in that argument, if we interchange *A* with *C, K*, with *N* and so on, we should also get *K* is on the circle through *C, A, I*. Then *A, K, N, C* (and *I*) will be concyclic and we will finish.

Wishful thinking like this sometimes works! Here are the details:

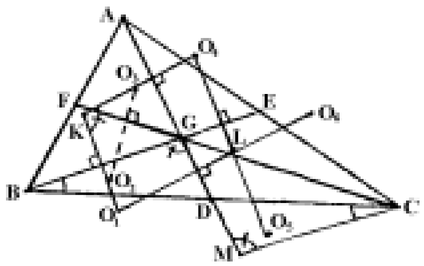
$$\angle ACN = \angle GCN \ = 180^\circ - \angle GHN$$

$$= \angle NHD = \angle NID \ = 180^\circ - \angle AIN\ .$$

So *N* is on the circle *A, C, I*. *Interchanging letters*, we get *similarly K* is on circle *C, A, I*. So *A, K, N, C* (and *I*) are concyclic. Therefore, *K, L, M, N, O* are indeed concyclic.

(*History*. My friend C.J. Lam did a search on the electronic database JSTOR and came across an article titled *A Chain of Circles Associated with the 5-Line* by J.W. Clawson published in the *American Mathematical Monthly*, volume 61, number 3 (March 1954), pages 161-166. There the problem was attributed to the nineteenth century geometer Miquel, who published the result in Liouville's Journal de Mathematiques, volume 3 (1838), pages 485-487. In that paper, Miquel proved his famous theorem that for four pairwise intersecting lines, taking three of the lines at a time and forming the circles through the three intersecting points, the four circles will always meet at a common point, which nowadays are referred to as the *Miquel point*. The first problem was then deduced as a corollary of this Miquel theorem.)

For the second problem, as the 6 circumcenters of the smaller triangles are not on any circles that we can see immediately, so we may try to use the converse of the intersecting chord theorem. For a triangle *ABC*, let *G, D, E, F* be the centroid, the midpoints of sides *BC, CA, AB*, respectively. Let O_1, O_2, O_3, O_4, O_5, O_6 be the circumcenters of triangles *DBG, BFG, FAG, AEG, ECG, CDG*, respectively.



Well, should we draw the 6 circumcircles? It would make the picture complicated. The circles do not seem to be helpful at this *early* stage. We give up on drawing the circles, but the circumcenters are important. So at least we should locate them. To locate the circumcenter of ΔFAG, for example, which two sides do we draw perpendicular bisectors? Sides *AG* and *FG* are the choices because they are also the sides of the other small triangles, so we can save some work later. Trying this out, we discover these perpendicular bisectors produce many parallel lines and parallelograms!

Since circumcenters are on perpendicular bisectors of chords, lines $O_3\,O_4$, $O_6\,O_1$ are perpendicular bisectors of *AG, GD*, respectively. So they are perpendicular to line *AD* and are $\frac{1}{2}$ *AD* units apart. Similarly, the two lines $O_1\,O_2$, $O_4\,O_5$ are perpendicular to line *BE* and are $\frac{1}{2}$ *BE* units apart. Aiming in showing O_1, O_2, O_3, O_4 are concyclic by the converse of the intersecting chord theorem, let *K* be the intersection of lines $O_1\,O_2$, $O_3\,O_4$ and *L* be the intersection of the lines $O_4\,O_5$, $O_6\,O_1$. Since the area of the parallelogram $KO_4\,LO_1$ is

$$\frac{1}{2}AD\cdot KO_4 = \frac{1}{2}BE\cdot KO_1,$$

we get KO_1/KO_4 = *AD/BE*.

Now that we get ratio of KO_1 and KO_4, we should examine KO_2 and KO_3. Trying to understand ΔKO_2O_3, we first find its angles. Since $KO_2 \perp BG$, $O_2O_3 \perp FG$ and $KO_3 \perp AG$, we see that $\angle KO_2O_3 = \angle BGF$ and $\angle KO_3O_2 = \angle FGA$. Then $\angle O_2KO_3 = \angle DGB$. At this point, you can see the angles of ΔKO_2O_3 equal the three angles with vertices at *G* on the left side of segment *AD*.

Now we try to put these three angles together in another way to form another triangle. Let *M* be the point on line *AG* such that *MC* is parallel to *BG*. Since $\angle MCG = \angle BGF$, $\angle MGC = \angle FGA$ (and $\angle GMC = \angle BGD$,) we see ΔKO_2O_3, *MCG* are similar.

The sides of ΔMCG are easy to compute in term of *AD, BE, CF*. As *AD* and *BE* occurred in the ratio of KO_1 and KO_4, this is just what we need! Observe that ΔMCD, *GBD* are congruent since $\angle MCD = \angle GBD$ (by *MC* parallel to *GB*), *CD = BD* and $\angle MDC = \angle GDB$. So

$$MG = 2GD = \frac{2}{3}AD,$$

$$MC = GB = \frac{2}{3}BE$$

(and $CG = \frac{2}{3}CF$. Incidentally, this means the three medians of a triangle can be put together to form a triangle! Actually, this is well-known and was the reason we considered ΔMCG .) We have $KO_3\,/\,KO_2$ = *MG/MC* = *AD/BE* = $KO_1\,/\,KO_4$.

So $KO_1\cdot KO_2 = KO_3\cdot KO_4$, which implies O_1, O_2, O_3, O_4 are concyclic. Similarly, we see that O_2, O_3, O_4, O_5 concyclic (using the parallelogram formed by the lines O_1O_2, O_4O_5, O_2O_3, O_5O_6 instead) and O_3, O_4, O_5, O_6 are concyclic.

Olympiad Corner

(continued from page 1)

Problem 4. We say that a positive integer *r* is a *power*, if it has the form $r = t^s$ where *t* and *s* are integers, $t \geq 2$, $s \geq 2$. Show that for any positive integer *n* there exists a set *A* of positive integers, which satisfies the conditions:

1. *A* has *n* elements;
2. any element of *A* is a power;
3. for any r_1, r_2, …, r_k $(2 \leq k \leq n)$ from *A* the number $\dfrac{r_1 + r_2 + \cdots + r_k}{k}$ is a power.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***April 15, 2001***.

Problem 121. Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1.
(Two integers are *relative prime* if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus 37 = 22 + 15 represents the number 37 in the desired way.) (*Source: Second Bay Area Mathematical Olympaid*)

Problem 122. Prove that the product of the lengths of the three angle bisectors of a triangle is less than the product of the lengths of the three sides. (*Source: 1957 Shanghai Junior High School Math Competition*)

Problem 123. Show that every convex quadrilateral with area 1 can be covered by some triangle of area at most 2. (*Source: 1989 Wuhu City Math Competition*)

Problem 124. Find the least integer n such that among every n distinct numbers $a_1, a_2, \ldots, a_n$, chosen from [1, 1000], there always exist a_i, a_j such that

$$0 < a_i - a_j < 1 + 3\sqrt[3]{a_i a_j}\,.$$

(*Source: 1990 Chinese Team Training Test*)

Problem 125. Prove that
$\tan^2 1^\circ + \tan^2 3^\circ + \tan^2 5^\circ + \cdots + \tan^2 89^\circ$
is an integer.

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Solutions

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Problem 116. Show that the interior of a convex quadrilateral with area A and perimeter P contains a circle of radius A/P.

Solution 1. **CHAO Khek Lun** (St. Paul's College, Form 6).

Draw four rectangles on the sides of the quadrilateral and each has height A/P pointing inward. The sum of the areas of the rectangles is A. Since at least one interior angle of the quadrilateral is less than 180°, at least two of the rectangles will overlap. So the union of the four rectangular regions does not cover the interior of the quadrilateral. For any point in the interior of the quadrilateral not covered by the rectangles, the distance between the point and any side of the quadrilateral is greater than A/P. So we can draw a desired circle with that point as center.

Solution 2. **CHUNG Tat Chi** (Queen Elizabeth School, Form 4) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Let $BCDE$ be a quadrilateral with area A and perimeter P. One of the diagonal, say BD is inside the quadrilateral. Then either ΔBCD or ΔBED will have an area greater than or equal to $A/2$. Suppose this is ΔBCD. Then $BCDE$ contains the incircle of ΔBCD, which has a radius of

$$\begin{aligned} &\frac{2[BCD]}{BC+CD+DB} \\ &> \frac{2[BCD]}{BC+CD+DE+EB} \\ &\ge \frac{A}{P}, \end{aligned}$$

where the brackets denote area. Hence, it contains a circle of radius A/P.
Comment: Both solutions do not need the convexity assumption.

Problem 117. The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the other three lengths. Prove that two of the sides have the same length.

Solution. **CHAO Khek Lun** (St. Paul's College, Form 6) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Suppose the sides are a, b, c, d with $a < b < c < d$. Since $d < a + b + c < 3d$ and d divides $a + b + c$, we have $a + b + c = 2d$. Now each of a, b, c divides $a + b + c + d = 3d$. Let $x = 3d/a$, $y = 3d/b$ and $z = 3d/c$. Then $a < b < c < d$ implies $x > y > z > 3$. So $z \ge 4$, $y \ge 5$, $x \ge 6$. Then

$$2d = a + b + c \le \frac{3d}{6} + \frac{3d}{5} + \frac{3d}{4} < 2d,$$

a contradiction. Therefore, two of the sides are equal.

Problem 118. Let R be the real numbers. Find all functions $f: R \to R$ such that for all real numbers x and y,

$$f(xf(y) + x) = xy + f(x).$$

Solution 1. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Putting $x = 1$, $y = -1 - f(1)$ and letting $a = f(y) + 1$, we get

$$f(a) = f(f(y) + 1) = y + f(1) = -1.$$

Putting $y = a$ and letting $b = f(0)$, we get

$$b = f(xf(a) + x) = ax + f(x),$$

so $f(x) = -ax + b$. Putting this into the equation, we have

$$a^2xy - abx - ax + b = xy - ax + b.$$

Equating coefficients, we get $a = \pm 1$ and $b = 0$, so $f(x) = x$ or $f(x) = -x$. We can easily check both are solutions.

Solution 2. **LEE Kai Seng** (HKUST).

Setting $x = 1$, we get

$$f(f(y)+1) = y + f(1).$$

For every real number a, let $y = a - f(1)$, then $f(f(y) + 1) = a$ and f is surjective. In particular, there is b such that $f(b) = -1$. Also, if $f(c) = f(d)$, then

$$\begin{aligned} c + f(1) &= f(f(c)+1) \\ &= f(f(d)+1) \\ &= d + f(1). \end{aligned}$$

So $c = d$ and f is injective. Taking $x = 1$, $y = 0$, we get $f(f(0)+1) = f(1)$. Since f is injective, we get $f(0) = 0$.
For $x \ne 0$, let $y = -f(x)/x$, then

$$f(xf(y) + x) = 0 = f(0).$$

By injectivity, we get $xf(y) + x = 0$. Then

$$f(-f(x)/x) = f(y) = -1 = f(b)$$

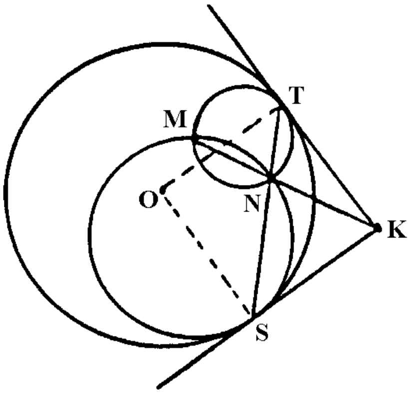
and so $-f(x)/x = b$ for every $x \ne 0$. That is, $f(x) = -bx$. Putting this into the given equation, we find $f(x) = x$ or $f(x) = -x$, which are checked to be solutions.

Other commended solvers: **CHAO Khek Lun** (St. Paul's College, Form 6) and **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 6).

Problem 119. A circle with center O is internally tangent to two circles inside it at points S and T. Suppose the two circles inside intersect at M and N with N

closer to ST. Show that $OM \perp MN$ if and only if S, N, T are collinear. (*Source: 1997 Chinese Senior High Math Competition*)

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).



Consider the tangent lines at S and at T. (Suppose they are parallel, then S, O, T will be collinear so that M and N will be equidistant from ST, contradicting N is closer to ST.) Let the tangent lines meet at K, then $\angle OSK = 90^\circ = \angle OTK$ implies O, S, K, T lie on a circle with diameter OK. Also, $KS^2 = KT^2$ implies K is on the *radical axis MN* of the two inside circles. So M, N, K are collinear.

If S, N, T are collinear, then $\angle SMT$ = $\angle SMN + \angle TMN$ = $\angle NSK + \angle KTN$ = $180^\circ - \angle SKT$. So M, S, K, T, O are concyclic. Then $\angle OMN$ = $\angle OMK$ = $\angle OSK$ = 90°.

Conversely, if $OM \perp MN$, then $\angle OMK$ = $90^\circ = \angle OSK$ implies M, S, K, T, O are concyclic. Then

$$\begin{aligned}\angle SKT &= 180^\circ - \angle SMT\\ &= 180^\circ - \angle SMN - \angle TMN\\ &= 180^\circ - \angle NSK - \angle KTN.\end{aligned}$$

Thus, $\angle TNS = 360^\circ - \angle NSK - \angle SKT$ - $\angle KTN = 180^\circ$. Therefore, S, N, T are collinear.

Comments: For the meaning of *radical axis*, we refer the readers to pages 2 and 4 of *Math Excalibur, vol. 4, no. 3* and the corrections on page 4 of *Math Excalibur, vol. 4, no. 4*.

Other commended solvers: **CHAO Khek Lun** (St. Paul's College, Form 6).

Problem 120. Twenty-eight integers are chosen from the interval [104, 208]. Show that there exist two of them having a common prime divisor.

Solution 1. **CHAO Khek Lun** (St. Paul's College, Form 6), **CHAU Suk Ling** (Queen Elizabeth School, Form 6) and **CHUNG Tat Chi** (Queen Elizabeth School, Form 4).

Applying the inclusion-exclusion principle, we see there are 82 integers on [104, 208] that are divisible by 2, 3, 5 or 7. There remain 23 other integers on the interval. If 28 integers are chosen from the interval, at least 28 – 23 = 5 are among the 82 integers that are divisible by 2, 3, 5 or 7. So there will exist two that are both divisible by 2, 3, 5 or 7.

Solution 2. **CHAN Yun Hung** (Carmel Divine Grace Foundation Secondary School, Form 4), **KWOK Sze Ming** (Queen Elizabeth School, Form 5), **LAM Shek Ming** (La Salle College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **WONG Tak Wai Alan** (University of Toronto) and **WONG Wing Hong** (La Salle College, Form 3).

There are 19 prime numbers on the interval. The remaining 86 integers on the interval are all divisible by at least one of the prime numbers 2, 3, 5, 7, 11 and 13 since 13 is the largest prime less than or equal to $\sqrt{208}$. So every number on the interval is a multiple of one of these 25 primes. Hence, among any 26 integers on the interval at least two will have a common prime divisor.

A Proof of the Majorization Inequality

Kin Y. Li

Quite a few readers would like to see a proof of the majorization inequality, which was discussed in the last issue of the *Mathematical Excalibur*. Below we will present a proof. We will first make one observation.

Lemma. Let $a < c < b$ and f be convex on an interval I with a, b, c on I. Then the following are true:

$$\frac{f(c)-f(a)}{c-a} \le \frac{f(b)-f(a)}{b-a}$$

and

$$\frac{f(b)-f(c)}{b-c} \le \frac{f(b)-f(a)}{b-a}.$$

Proof. Since $a < c < b$, we have $c = (1 - t)a + tb$ for some $t \in (0, 1)$. Solving for t, we get $t = (c - a)/(b - a)$. Since f is convex on I,

$$f(c) \le (1-t)\,f(a) + t\,f(b)$$

$$= \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b),$$

which is what we will get if we solve for $f(c)$ in the two inequalities in the statement of the lemma.

In brief the lemma asserts that the slopes of chords are increasing as the chords are moving to the right. Now we are ready to proof the majorization inequality.

Suppose

$$(x_1, x_2, ..., x_n) \succ (y_1, y_2, ..., y_n).$$

Since $x_i \ge x_{i+1}$ and $y_i \ge y_{i+1}$ for i = 1, 2, …, n – 1, it follows from the lemma that the slopes

$$m_i = \frac{f(x_i)-f(y_i)}{x_i - y_i}$$

satisfy $m_i \ge m_{i+1}$ for 1 ≤ i ≤ n – 1. (For example, if $y_{i+1} \le y_i \le x_{i+1} \le x_i$, then applying the lemma twice, we get

$$\begin{aligned}m_{i+1} &= \frac{f(x_{i+1})-f(y_{i+1})}{x_{i+1}-y_{i+1}}\\ &\le \frac{f(x_{i+1})-f(y_i)}{x_{i+1}-y_i}\\ &\le \frac{f(x_i)-f(y_i)}{x_i-y_i} = m_i\end{aligned}$$

and similarly for the other ways y_{i+1}, y_i, x_{i+1}, x_i are distributed.)

For k = 1, 2, …, n, let

$$X_k = x_1 + x_2 + \cdots + x_k$$

and

$$Y_k = y_1 + y_2 + \cdots + y_k.$$

Since $X_k \ge Y_k$ for k = 1, 2, …, n – 1 and $X_n = Y_n$, we get

$$\sum_{k=1}^{n} (X_k - Y_k)(m_k - m_{k+1}) \ge 0,$$

where we set $m_{n+1} = 0$ for convenience. Expanding the sum, grouping the terms involving the same m_k 's and letting $X_0 = 0 = Y_0$, we get

$$\sum_{k=1}^{n} (X_k - X_{k-1} - Y_k + Y_{k-1}) m_k \ge 0,$$

which is the same as

$$\sum_{k=1}^{n} (x_k - y_k) m_k \ge 0.$$

Since $(x_k - y_k)m_k = f(x_k) - f(y_k)$, we get

$$\sum_{k=1}^{n} (f(x_k) - f(y_k)) m_k \ge 0.$$

Transferring the $f(y_k)$ terms to the right, we get the majorization inequality.

Mathematical Excalibur

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Olympiad Corner

The 2000 Canadian Mathematical Olympiad

Problem 1. At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners).

Problem 2. A *permutation* of the integers 1901, 1902, …, 2000 is a sequence a_1, a_2, …, a_{100} in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

$$s_1 = a_1,\ s_2 = a_1 + a_2,$$
$$s_3 = a_1 + a_2 + a_3, ..., s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is *June 30, 2001*.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Base n Representations

Kin Y. Li

When we write down a number, it is understood that the number is written in base 10. We learn many interesting facts at a very young age. Some of these can be easily explained in terms of base 10 representation of a number. Here is an example.

Example 1. Show that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. How about divisibility by 11?

Solution. Let $M = d_m 10^m + \cdots + d_1 10 + d_0$, where d_i = 0, 1, 2, …, 9. The binomial theorem tells us $10^k = (9+1)^k = 9N_k + 1$. So

$$M = d_m(9N_m + 1) + \cdots + d_1(9+1) + d_0$$
$$= 9(d_m N_m + \cdots + d_1) + (d_m + \cdots + d_1 + d_0).$$

Therefore, M is a multiple of 9 if and only if $d_m + \cdots + d_1 + d_0$ is a multiple of 9.

Similarly, we have $10^k = 11N_k' + (-1)^k$. So M is divisible by 11 if and only if $(-1)^m d_m + \cdots - d_1 + d_0$ is divisible by 11.

Remarks. In fact, we can also see that the remainder when M is divided by 9 is the same as the remainder when the sum of the digits of M is divided by 9. Recall the notation $a \equiv b \pmod{c}$ means a and b have the same remainder when divided by c. So we have $M \equiv d_m + \cdots + d_1 + d_0 \pmod 9$.

The following is an IMO problem that can be solved using the above remarks.

Example 2. (*1975 IMO*) Let A be the sum of the decimal digits of 4444^{4444}, and B be the sum of the decimal digits of A. Find the sum of the decimal digits of B.

Solution. Since $4444^{4444} < (10^5)^{4444} = 10^{22220}$, so $A < 22220 \times 9 = 199980$. Then $B < 1 + 9 \times 5 = 46$ and the sum of the decimal digits of B is at most 3+9=12. Now $4444 \equiv 7 \pmod 9$ and $7^3 = 343 \equiv 1 \pmod 9$ imply $4444^3 \equiv 1 \pmod 9$. Then $4444^{4444} = (4444^3)^{1481} 4444 \equiv 7 \pmod 9$. By the remarks above, A, B and the sum of the decimal digits of B also have remainder 7 when divided by 9. So the sum of the decimal digits of B being at most 12 must be 7.

Although base 10 representations are common, numbers expressed in other bases are sometimes useful in solving problems, for example, base 2 is common. Here are a few examples using other bases.

Example 3. (*A Magic Trick*) A magician asks you to look at four cards. On the first card are the numbers 1, 3, 5, 7, 9, 11, 13, 15; on the second card are the numbers 2, 3, 6, 7, 10, 11, 14, 15; on the third card are the numbers 4, 5, 6, 7, 12, 13, 14, 15; on the fourth card are the numbers 8, 9, 10, 11, 12, 13, 14, 15. He then asks you to pick a number you saw in one of these cards and hand him all the cards that have that number on them. Instantly he knows the number. Why?

Solution. For n = 1, 2, 3, 4, the numbers on the n-th card have the common feature that their n-th digits from the end in base 2 representation are equal to 1. So you are handing the base 2 representation of your number to the magician. As the numbers are less than 2^4, he gets your number easily.

Remarks. A variation of this problem is the following. A positive integer less than 2^4 is picked at random. What is the least number of yes-no questions you can ask that always allow you to know the number? Four questions are enough as you can ask if each of the four digits of the number in base 2 is 1 or not. Three questions are not enough as there are 15 numbers and three questions can only provide $2^3 = 8$ different yes-no combinations.

Example 4. (*Bachet's Weight Problem*) Give a set of distinct integral weights that allowed you to measure any object having weight $n = 1, 2, 3, \ldots, 40$ on a balance. Can you do it with a set of no more than four distinct integral weights?

Solution. Since the numbers 1 to 40 in base 2 have at most 6 digits, we can do it with the set 1, 2, 4, 8, 16, 32. To get a set with fewer weights, we observe that we can put weights from this set on both sides of the balance! Consider the set of weights 1, 3, 9, 27. For example to determine an object with weight 2, we can put it with a weight of 1 on one side to balance a weight of 3 on the other side. Note the sum of 1, 3, 9, 27 is 40. For any integer *n* between 1 and 40, we can write it in base 3. If the digit 2 appears, change it to $3-1$ so that *n* can be written as a *unique* sum and difference of 1, 3, 9, 27. For example, 22 = $2\cdot 9$ + 3 + 1 = (3 – 1)9 + 3 + 1 = 27 – 9 + 3 + 1 suggests we put the weights of 22 with 9 on one side and the weights of 27, 3, 1 on the other side.

Example 5. (*1983 IMO*) Can you choose 1983 pairwise distinct nonnegative integers less than 10^5 such that no three are in arithmetic progression?

Solution. Start with 0, 1 and at each step add the *smallest* integer which is not in arithmetic progression with any two preceding terms. We get 0, 1, 3, 4, 9, 10, 12, 13, 27, 18, … . In base 3, this sequence is

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, \ldots$$

(Note this sequence is the nonnegative integers in base 2.) Since 1982 in base 2 is 11110111110, so switching this from base 3 to base 10, we get the 1983th term of the sequence is $87843 < 10^5$. To see this sequence works, suppose *x*, *y*, *z* with *x* < *y* < *z* are three terms of the sequence in arithmetic progression. Consider the rightmost digit in base 3 where *x* differs from *y*, then that digit for *z* is a 2, a contradiction.

Example 6. Let $[r]$ be the greatest integer less than or equal to *r*. Solve the equation

$$[x]+[2x]+[4x]+[8x]$$
$$+[16x]+[32x]=12345.$$

Solution. If *x* is a solution, then since $r-1<[r]\le r$, we have 63*x* – 6 < 12345 ≤ 63*x*. It follows that $195 < x < 196$. Now write the number *x* in base 2 as $11000011.abcde\ldots$, where the digits $a, b, c, d, e, \ldots$ are 0 or 1. Substituting this into the equation, we will get 12285 + 31*a* + 15*b* + 7*c* + 3*d* + *e* = 12345. Then 31*a* + 15*b* + 7*c* + 3*d* + *e* = 60, which is impossible as the left side is at most 31 + 15 + 7 + 3 + 1 = 57. Therefore, the equation has no solution.

Example 7. (*Proposed by Romania for 1985 IMO*) Show that the sequence $\{a_n\}$ defined by $a_n = [n\sqrt{2}]$ for *n* = 1, 2, 3, … (where the brackets denote the greatest integer function) contains an infinite number of integral powers of 2.

Solution. Write $\sqrt{2}$ in base 2 as $b_0.b_1b_2b_3\ldots$, where each $b_i = 0$ or 1. Since $\sqrt{2}$ is irrational, there are infinitely many $b_k = 1$. If $b_k = 1$, then in base 2, $2^{k-1}\sqrt{2} = b_0\cdots b_{k-1}.b_k\cdots$. Let $m = [2^{k-1}\sqrt{2}]$, then

$$2^{k-1}\sqrt{2}-1<[2^{k-1}\sqrt{2}]=m<2^{k-1}\sqrt{2}-\frac{1}{2}.$$

Multiplying by $\sqrt{2}$ and adding $\sqrt{2}$, we get $2^k < (m+1)\sqrt{2} < 2^k + \frac{\sqrt{2}}{2}$. Then $[(m+1)\sqrt{2}] = 2^k$.

Example 8. (*American Mathematical Monthly, Problem 2486*) Let *p* be an odd prime number. For any positive integer *k*, show that there exists a positive integer *m* such that the rightmost *k* digits of m^2, when expressed in the base *p*, are all 1's.

Solution. We prove by induction on *k*. For $k = 1$, take $m = 1$. Next, suppose m^2 in base *p*, ends in *k* 1's, i.e.

$$m^2 = 1 + p + \cdots + p^{k-1} + (ap^k + \cdots).$$

This implies *m* is not divisible by *p*. Let gcd stand for greatest common divisor (or highest common factor). Then gcd(*m*, *p*) = 1. Now

$$(m+cp^k)^2 = m^2 + 2mcp^k + c^2p^{2k}$$
$$= 1 + p + \cdots + p^{k-1} + (a+2mc)p^k + \cdots.$$

Since $\gcd(2m, p) = 1$, there is a positive integer *c* such that $(2m)c \equiv 1 - a \pmod{p}$. This implies $a + 2mc$ is of the form $1 + Np$ and so $(m+cp^k)^2$ will end in at least $(k+1)$ 1's as required.

Example 9. Determine which binomial coefficients $C_r^n = \dfrac{n!}{r!(n-r)!}$ are odd.

Solution. We remark that modulo arithmetic may be extended to polynomials with integer coefficients. For example, $(1+x)^2 = 1 + 2x + x^2 \equiv 1 + x^2$ (mod 2). If $n = a_m + \cdots + a_1$, where the a_i 's are distinct powers of 2. We have $(1+x)^{2^k} \equiv 1 + x^{2^k}$ (mod 2) by induction on *k* and so

$$(1+x)^n \equiv (1+x^{a_m})\cdots(1+x^{a_1}) \pmod 2.$$

The binomial coefficient C_r^n is odd if and only if the coefficient of x^r in $(1+x^{a_m})\cdots(1+x^{a_1})$ is 1, which is equivalent to *r* being 0 or a sum of one or more of the a_i 's. For example, if $n = 21 = 16 + 4 + 1$, then C_r^n is odd for *r* = 0, 1, 4, 5, 16, 17, 20, 21 only.

Example 10. (*1996 USAMO*) Determine (with proof) whether there is a subset *X* of the integers with the following property: for any integer *n* there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

This is a difficult problem. Here we will try to lead the reader to a solution. For a problem that we cannot solve, we can try to change it to an easier problem. How about changing the problem to positive integers, instead of integers? At least we do not have to worry about negative integers. That is still not too obvious how to proceed. So can we change it to an even simpler problem? How about changing 2 to 10?

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***June 30, 2001***.

Problem 126. Prove that every integer can be expressed in the form $x^2 + y^2 - 5z^2$, where *x, y, z* are integers.

Problem 127. For positive real numbers *a, b, c* with $a + b + c = abc$, show that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}} \le \frac{3}{2},$$

and determine when equality occurs. (*Source: 1998 South Korean Math Olympiad*)

Problem 128. Let *M* be a point on segment *AB*. Let *AMCD*, *BEHM* be squares on the same side of *AB*. Let the circumcircles of these squares intersect at *M* and *N*. Show that *B, N, C* are collinear and *H* is the orthocenter of ΔABC. (*Source: 1979 Henan Province Math Competition*)

Problem 129. If *f(x)* is a polynomial of degree 2*m*+1 with integral coefficients for which there are 2*m*+1 integers $k_1, k_2, \ldots, k_{2m+1}$ such that $f(k_i) = 1$ for $i = 1, 2, \cdots, 2m+1$, prove that *f(x)* is not the product of two nonconstant polynomials with integral coefficients.

Problem 130. Prove that for each positive integer *n*, there exists a circle in the *xy*-plane which contains exactly *n* lattice points in its interior, where a *lattice point* is a point with integral coordinates. (*Source: H. Steinhaus, Zadanie 498, Matematyka 10 (1957), p. 58*)

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Solutions

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Problem 121. Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1.

(Two integers are *relative prime* if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus 37 = 22 + 15 represents the number 37 in the desired way.) (*Source: Second Bay Area Mathematical Olympaid*)

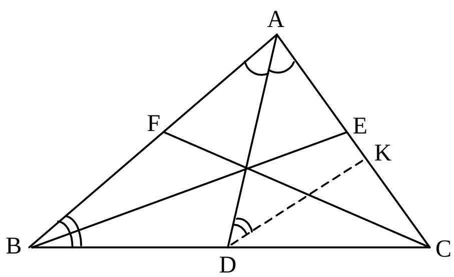
Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 5), **CHONG Fan Fei** (Queen's College, Form 4), **CHUNG Tat Chi** (Queen Elizabeth School, Form 4), **LAW Siu Lun** (Ming Kei College, Form 6), **NG Cheuk Chi** (Tsuen Wan Public Ho Chuen Yiu Memorial College), **WONG Wing Hong** (La Salle College, Form 3) & **YEUNG Kai Sing** (La Salle College, Form 4).

For an integer $n \ge 7$, *n* is either of the form $2j + 1$ ($j > 2$) or $4k$($k > 1$) or $4k + 2$($k > 1$). If $n = 2j + 1$, then *j* and $j + 1$ are relatively prime and $n = j + (j + 1)$. If $n = 4k$, then $2k - 1$ (>1) and $2k + 1$ are relatively prime and $n = (2k - 1) + (2k + 1)$. If $n = 4k + 2$, then $2k - 1$ and $2k + 3$ are relatively prime and $n = (2k - 1) + (2k + 1)$.

Other commended solvers: **HON Chin Wing** (Pui Ching Middle School, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 6) & **WONG Tak Wai Alan** (University of Toronto).

Problem 122. Prove that the product of the lengths of the three angle bisectors of a triangle is less than the product of the lengths of the three sides. (*Source: 1957 Shanghai Junior High School Math Competition*).

Solution. **YEUNG Kai Sing** (La Salle College, Form 4).

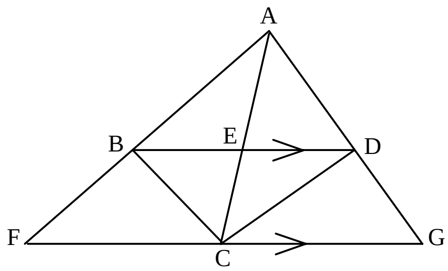


Let *AD*, *BE* and *CF* be the angle bisectors of ΔABC, where *D* is on *BC*, *E* is on *CA* and *F* is on *AB*. Since $\angle ADC = \angle ABD + \angle BAD > \angle ABD$, there is a point *K* on *CA* such that $\angle ADK = \angle ABD$. Then ΔABD is similar to ΔADK. So $AB/AD = AD/AK$. Then $AD^2 = AB \cdot AK < AB \cdot CA$. Similarly, $BE^2 < BC \cdot AB$ and $CF^2 < CA \cdot BC$. Multiplying these in-equalities and taking square roots, we get $AD \cdot BE \cdot CF < AB \cdot BC \cdot CA$.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 5), **HON Chin Wing** (Pui Ching Middle School, Form 6) & **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Problem 123. Show that every convex quadrilateral with area 1 can be covered by some triangle of area at most 2. (*Source: 1989 Wuhu City Math Competition*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 4) & **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).



Let *ABCD* be a convex quadrilateral with area 1. Let *AC* meet *BD* at *E*. Without loss of generality, suppose $AE \ge EC$. Construct ΔAFG, where lines *AB* and *AD* meet the line parallel to *BD* through *C* at *F* and *G* respectively. Then ΔABE is similar to ΔAFC. Now $AE \ge EC$ implies $AB \ge BF$. Let $[XY \cdots Z]$ denote the area of polygon $XY \cdots Z$, then $[ABC] \ge [FBC]$. Similarly, $[ADC] \ge [GDC]$. Since $[ABC] + [ADC] = [ABCD] = 1$, so $[AFG] = [ABCD] + [FBC] + [GDC] \le 2[ABCD] = 2$ and ΔAFG covers *ABCD*.

Problem 124. Find the least integer *n* such that among every n distinct numbers $a_1, a_2, \ldots, a_n$, chosen from [1,1000], there always exist a_i, a_j such that

$$0 < a_i - a_j < 1 + 3\sqrt[3]{a_i a_j}.$$

(*Source: 1990 Chinese Team Training Test*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 4) & **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

For $n \le 10$, let $a_i = i^3$ $(i = 1, 2, \cdots, n)$. Then the inequality cannot hold since $0 < i^3 - j^3$ implies $i - j \ge 1$ and so $i^3 - j^3 = (i-j)^3 + 3ij(i-j) \ge 1 + 3ij$. For n = 11, divide [1,1000] into intervals $[k^3+1, (k+1)^3]$ for k = 0, 1, …, 9. By pigeonhole principle, among any 11 distinct numbers $a_1, a_2, \cdots, a_{11}$ in [1, 1000], there always exist a_i, a_j, say $a_i > a_j$, in the same interval. Let $x = \sqrt[3]{a_i}$ and $y = \sqrt[3]{a_j}$, then $0 < x - y < 1$ and $0 < a_i - a_j = x^3 - y^3 = (x-y)^3 + 3xy(x-y) < 1 + 3xy = 1 + 3\sqrt[3]{a_i a_j}$.

Other commended solvers: **NG Cheuk Chi** (Tsuen Wan Public Ho Chuen Yiu Memorial College), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 6), **WONG Wing Hong** (La Salle College, Form 3) & **YEUNG Kai Sing** (La Salle College, Form 4).

Problem 125. Prove that

$$\tan^2 1^\circ + \tan^2 3^\circ + \tan^2 5^\circ + \cdots + \tan^2 89^\circ$$

is an integer.

Solution. **CHAO Khek Lun** (St. Paul's College, Form 6).

For $\theta = 1^\circ, 3^\circ, 5^\circ, \ldots, 89^\circ$, we have $\cos\theta \ne 0$ and $\cos 90\theta = 0$. By de Moivre's theorem, $\cos 90\theta + i\sin 90\theta = (\cos\theta + i\sin\theta)^{90}$. Taking the real part of both sides, we get

$$0 = \sum_{k=0}^{45} (-1)^k C_{2k}^{90} \cos^{90-2k}\theta \sin^{2k}\theta.$$

Dividing by $\cos^{90}\theta$ on both sides and letting $x = \tan^2\theta$, we get

$$0 = \sum_{k=0}^{45} (-1)^k C_{2k}^{90} x^k.$$

So $\tan^2 1^\circ, \tan^2 3, \tan^2 5^\circ, \ldots, \tan^2 89^\circ$ are the 45 roots of this equation. Therefore, their sum is $C_{88}^{90} = 4005$.

Olympiad Corner

(continued from page 1)

How many of these permutations will have no terms of the sequence s_1, …, s_{100} divisible by three?

Problem 3. Let A = $(a_1, a_2, \cdots, a_{2000})$ be a sequence of integers each lying in the interval [-1000, 1000]. Suppose that the entries in A sum to 1. Show that some nonempty subsequence of A sums to zero.

Problem 4. Let $ABCD$ be a convex quadrilateral with

$$\angle CBD = 2\angle ADB,$$
$$\angle ABD = 2\angle CDB$$

and $AB = CB.$

Prove that $AD = CD.$

Problem 5. Suppose that the real numbers $a_1, a_2, \cdots, a_{100}$ satisfy

$$a_1 \ge a_2 \ge \cdots \ge a_{100} \ge 0,$$
$$a_1 + a_2 \le 100$$

and $a_3 + a_4 + \cdots + a_{100} \le 100.$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \cdots, a_{100}$ which achieve this maximum.

Base n Representations

(continued from page 2)

Now try an example, say n = 12345. We can write n in more than one ways in the form $a + 10b$. Remember we want a, b to be unique in the set X. Now for b in X, $10b$ will *shift* the digits of b to the left one space and fill the last digit with a 0. Now we can try writing n = 12345 = 10305 + 10(204). So if we take X to be the positive integers whose even position digits from the end are 0, then the problem will be solved for n = a + 10b. How about $n = a + 2b$? If the reader examines the reasoning in the case $a + 10b$, it is easy to see the success comes from separating the digits and observing that multiplying by 10 is a shifting operation in base 10. So for $a + 2b$, we take X to be the set of positive integers whose base 2 even position digits from the end are 0, then the problem is solved for positive integers.

How about the original problem with *integers*? It is tempting to let X be the set of positive or negative integers whose base 2 even position digits from the end are 0. It does not work as the example 1 + 2·1 = 3 = 5 + 2 (-1) shows uniqueness fails. Now what other ways can we describe the set X we used in the last paragraph? Note it is also the set of positive integers whose base 4 representations have only digits 0 or 1. How can we take care of uniqueness and negative integers at the same time? One idea that comes close is the Bachet weights.

The brilliant idea in the official solution of the 1996 USAMO is do things in base (-4). That is, show every integer has a unique representation as $\sum_{i=0}^{k} c_i(-4)^i$, where each c_i = 0, 1, 2 or 3 and $c_k \ne 0$. Then let X be the set of integers whose base (-4) representations have only $c_i = 0$ or 1 will solve the problem.

To show that an integer n has a base (-4) representation, find an integer m such that $4^0 + 4^2 + \cdots + 4^{2m} \ge n$ and express

$$n + 3\ (4^1 + 4^3 + \cdots + 4^{2m-1})$$

in base 4 as $\sum_{i=0}^{2m} b_i 4^i$. Now set $c_{2i} = b_{2i}$ and $c_{2i-1} = 3 - b_{2i-1}$. Then $n = \sum_{i=0}^{2m} c_i(-4)^i$.

To show the uniqueness of base (-4) representation of n, suppose n has two distinct representations with digits c_i's and d_i's. Let j be the smallest integer such that $c_j \ne d_j$. Then

$$0 = n - n = \sum_{i=j}^{k} (c_i - d_i)(-4)^i$$

would have a nonzero remainder when divided by 4^{j+1}, a contradiction.

Mathematical Excalibur

Volume 6, Number 3 *June 2001 – October 2001*

Olympiad Corner

The 42nd International Mathematical Olympiad, Washington DC, USA, 8-9 July 2001

Problem 1. Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A. Suppose that $\angle BCA \ge \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.

Problem 2. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \ge 1$$

for all positive real numbers a, b and c.

Problem 3. Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 10, 2001***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Pell's Equation (I)

Kin Y. Li

Let d be a positive integer that is not a square. The equation $x^2 - dy^2 = 1$ with variables x, y over integers is called ***Pell's equation***. It was Euler who attributed the equation to John Pell (1611-1685), although Brahmagupta (7th century), Bhaskara (12th century) and Fermat had studied the equation in details earlier.

A solution (x, y) of Pell's equation is called *positive* if both x and y are positive integers. Hence, positive solutions correspond to the lattice points in the first quadrant that lie on the hyperbola $x^2 - dy^2 = 1$. A positive solution (x_1, y_1) is called the *least positive solution* (or *fundamental solution*) if it satisfies $x_1 < x$ and $y_1 < y$ for every other positive solution (x, y). (As the hyperbola $x^2 - dy^2 = 1$ is strictly increasing in the first quadrant, the conditions for being least are the same as requiring $x_1 + y_1\sqrt{d} < x + y\sqrt{d}$.)

Theorem. *Pell's equation $x^2 - dy^2 = 1$ has infinitely many positive solutions. If (x_1, y_1) is the least positive solution, then for $n = 1, 2, 3, \ldots$, define*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n.$$

The pairs (x_n, y_n) are all the positive solutions of the Pell's equation. The x_n's and y_n's are strictly increasing to infinity and satisfy the recurrence relations $x_{n+2} = 2x_1x_{n+1} - x_n$ and $y_{n+2} = 2x_1y_{n+1} - y_n$.

We will comment on the proof. The least positive solution is obtained by writing $\sqrt{d}$ as a simple continued fraction. It turns out

$$\sqrt{d} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

where $a_0 = [\sqrt{d}]$ and $a_1, a_2, \ldots$ is a periodic positive integer sequence. The continued fraction will be denoted by $\langle a_0, a_1, a_2, \ldots\rangle$. The *k-th convergent* of $\langle a_0, a_1, a_2, \ldots\rangle$ is the number $\frac{p_k}{q_k} = \langle a_0, a_1, a_2, \ldots, a_k\rangle$ with p_k, q_k relatively prime. Let $a_1, a_2, \ldots, a_m$ be the period for $\sqrt{d}$. The least positive solution of Pell's equation turns out to be

$$(x_1, y_1) = \begin{cases} (p_{m-1}, q_{m-1}) & \text{if } m \text{ is even} \\ (p_{2m-1}, q_{2m-1}) & \text{if } m \text{ is odd} \end{cases}.$$

For example, $\sqrt{3} = \langle 1, 1, 2, 1, 2, \ldots\rangle$ and so $m = 2$, then $\langle 1, 1\rangle = \frac{2}{1}$. We check $2^2 - 3\cdot 1^2 = 1$ and clearly, $(2,1)$ is the least positive solution of $x^2 - 3y^2 = 1$. Next, $\sqrt{2} = \langle 1, 2, 2, \ldots\rangle$ and so $m = 1$, then $\langle 1, 2\rangle = \frac{3}{2}$. We check $3^2 - 2\cdot 2^2 = 1$ and again clearly $(3, 2)$ is the least positive solution of $x^2 - 2y^2 = 1$.

Next, if there is a positive solution (x, y) such that $x_n + y_n\sqrt{d} < x + y\sqrt{d} < x_{n+1} + y_{n+1}\sqrt{d}$, then consider $u + v\sqrt{d} = (x + y\sqrt{d})/(x_n + y_n\sqrt{d})$. We will get $u + v\sqrt{d} < x_1 + y_1\sqrt{d}$ and $u - v\sqrt{d} = (x - y\sqrt{d})/(x_n - y_n\sqrt{d})$ so that $u^2 - dv^2 = (u - v\sqrt{d})(u + v\sqrt{d}) = 1$, con-tradicting (x_1, y_1) being the least positive solution.

To obtain the recurrence relations, note that

$$(x_1+y_1\sqrt{d})^2 = x_1^2+dy_1^2+2x_1y_1\sqrt{d}$$
$$=2x_1^2-1+2x_1y_1\sqrt{d}$$
$$=2x_1(x_1+y_1\sqrt{d})-1.$$

So

$$x_{n+2}+y_{n+2}\sqrt{d}$$
$$=(x_1+y_1\sqrt{d})^2(x_1+y_1\sqrt{d})^n$$
$$=2x_1(x_1+y_1\sqrt{d})^{n+1}-(x_1+y_1\sqrt{d})^n$$
$$=2x_1x_{n+1}-x_n+(2x_1y_{n+1}-y_n)\sqrt{d}.$$

The related equation $x^2-dy^2=-1$ may not have a solution, for example, $x^2-3y^2=-1$ cannot hold as $x^2-3y^2\equiv x^2+y^2\neq -1$ (mod 4). However, if d is a prime and $d\equiv 1$ (mod 4), then a theorem of Lagrange asserts that it will have a solution. In general, if $x^2-dy^2=-1$ has a least positive solution (x_1,y_1), then all its positive solutions are pairs (x, y), where $x+y\sqrt{d}=(x_1+y_1\sqrt{d})^{2n-1}$ for some positive integer n.

In passing, we remark that some k-th convergent numbers are special. If the length m of the period for $\sqrt{d}$ is even, then $x^2-dy^2=1$ has $(x_n,y_n)=(p_{nm-1},q_{nm-1})$ as all its positive solutions, but $x^2-dy^2=-1$ has no integer solution. If m is odd, then $x^2-dy^2=1$ has (p_{jm-1},y_{jm-1}) with j even as all its positive solutions and $x^2-dy^2=-1$ has (p_{jm-1},q_{jm-1}) with j odd as all its positive solutions.

Example 1. Prove that there are infinitely many triples of consecutive integers each of which is a sum of two squares.

Solution. The first such triple is $8=2^2+2^2$, $9=3^2+0^2$, $10=3^2+1^2$, which suggests we consider triples x^2-1, x^2, x^2+1. Since $x^2-2y^2=1$ has infinitely many positive solutions (x, y), we see that $x^2-1=y^2+y^2$, $x^2=x^2+0^2$ and x^2+1 satisfy the requirement and there are infinitely many such triples.

Example 2. Find all triangles whose sides are consecutive integers and areas are also integers.

Solution. Let the sides be $z-1, z, z+1$. Then the semiperimeter $s=\dfrac{3z}{2}$ and the area is $A=\dfrac{z\sqrt{3(z^2-4)}}{4}$. If A is an integer, then z cannot be odd, say $z = 2x$, and $z^2-4=3\omega^2$. So $4x^2-4=3\omega^2$, which implies ω is even, say $\omega=2y$. Then $x^2-3y^2=1$, which has $(x_1,y_1)=(2, 1)$ as the least positive solution. So all positive solutions are (x_n,y_n), where $x_n+y_n\sqrt{3}=(2+\sqrt{3})^n$. Now $x_n-y_n\sqrt{3}=(2-\sqrt{3})^n$. Hence,

$$x_n=\frac{(2+\sqrt{3})^n+(2-\sqrt{3})^n}{2}$$

and

$$y_n=\frac{(2+\sqrt{3})^n-(2-\sqrt{3})^n}{2\sqrt{3}}.$$

The sides of the triangles are $2x_n-1$, $2x_n, 2x_n+1$ and the areas are $A=3x_ny_n$.

Example 3. Find all positive integers k, m such that $k < m$ and

$$1+2+\cdots+k=(k+1)+(k+2)+\cdots+m.$$

Solution. Adding $1+2+\cdots+k$ to both sides, we get $2k(k+1)=m(m+1)$, which can be rewritten as $(2m+1)^2-2(2k+1)^2=-1$. Now the equation $x^2-2y^2=-1$ has $(1,1)$ as its least positive solution. So its positive solutions are pairs $x_n+y_n\sqrt{2}=(1+\sqrt{2})^{2n-1}$. Then

$$x_n=\frac{(1+\sqrt{2})^{2n-1}+(1-\sqrt{2})^{2n-1}}{2}$$

and

$$y_n=\frac{(1+\sqrt{2})^{2n-1}-(1-\sqrt{2})^{2n-1}}{2\sqrt{2}}.$$

Since $x^2-2y^2=-1$ implies x is odd, so x is of the form $2m+1$. Then $y^2=2m^2+m+1$ implies y is odd, so y is of the form $2k+1$. Then $(k,m)=\left(\dfrac{y_n-1}{2},\ \dfrac{x_n-1}{2}\right)$ with $n = 2, 3, 4, \ldots$ are all the solutions.

Example 4. Prove that there are infinitely many positive integers n such that n^2+1 divides $n!$.

Solution. The equation $x^2-5y^2=-1$ has $(2,1)$ as the least positive solution. So it has infinitely many positive solutions. Consider those solutions with $y>5$. Then $5<y<2y\le x$ as $4y^2\le 5y^2-1=x^2$. So $2(x^2+1)=5\cdot y\cdot 2y$ divides $x!$, which is more than we want.

Example 5. For the sequence $a_n=\left[\sqrt{n^2+(n+1)^2}\right]$, prove that there are infinitely many n's such that $a_n-a_{n+1}>1$ and $a_{n+1}-a_n=1$.

Solution. First consider the case $n^2+(n+1)^2=y^2$, which can be rewritten as $(2n+1)^2-2y^2=-1$. As in example 3 above, $x^2-2y^2=-1$ has infinitely many positive solutions and each x is odd, say $x=2n+1$ for some n. For these n's, $a_n=y$ and $a_{n-1}=\left[\sqrt{(n-1)^2+n^2}\right]=\left[\sqrt{y^2-4n}\right]$. The equation $y^2=n^2+(n+1)^2$ implies $n>2$ and $a_{n-1}\le\sqrt{y^2-4n}<y-1=a_n-1$. So $a_n-a_{n-1}>1$ for these n's. Also, for these n's, $a_{n+1}=\left[\sqrt{(n+1)^2+(n+2)^2}\right]=\left[\sqrt{y^2+4n+4}\right]$.

As $n < y < 2n + 1$, we easily get $y+1<\sqrt{y^2+4n+4}<y+2$. So $a_{n+1}-a_n=(y+1)-y=1$.

Example 6. (American Math Monthly E2606, proposed by R.S. Luthar) Show that there are infinitely many integers n such that $2n + 1$ and $3n + 1$ are perfect squares, and that such n must be multiples of 40.

Solution. Consider $2n+1=u^2$ and $3n+1=v^2$. On one hand, $u^2+v^2\equiv 2$ (mod 5) implies u^2, $v^2\equiv 1$ (mod 5), which means n is a multiple of 5.

On the other hand, we have $3u^2-2v^2=1$. Setting $u = x + 2y$ and $v = x + 3y$, the equation becomes $x^2-6y^2=1$. It has infinitely many positive solutions. Since $3u^2-2v^2=1$, u is odd, say $u = 2k + 1$. Then $n=2k^2+2k$ is even. Since $3n+1=v^2$, so v is odd, say $v = 4m\pm 1$. Then $3n=16m^2\pm 8m$, which implies n is also a multiple of 8.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***November 10, 2001***.

Problem 131. Find the greatest common divisor (or highest common factor) of the numbers $n^n - n$ for n = 3, 5, 7, … .

Problem 132. Points D, E, F are chosen on sides AB, BC, CA of ΔABC, respectively, so that $DE = BE$ and $FE = CE$. Prove that the center of the circumcircle of ΔADF lies on the angle bisector of $\angle DEF$. (*Source: 1989 USSR Math Olympiad*)

Problem 133. (a) Are there real numbers a and b such that $a+b$ is rational and a^n+b^n is irrational for every integer $n \ge 2$? (b) Are there real numbers a and b such that $a+b$ is irrational and a^n+b^n is rational for every integer $n \ge 2$? (*Source: 1989 USSR Math Olympiad*)

Problem 134. Ivan and Peter alternatively write down 0 or 1 from left to right until each of them has written 2001 digits. Peter is a winner if the number, interpreted as in base 2, is not the sum of two perfect squares. Prove that Peter has a winning strategy. (*Source: 2001 Bulgarian Winter Math Competition*)

Problem 135. Show that for $n \ge 2$, if $a_1, a_2, ..., a_n > 0$, then

$$(a_1^3+1)(a_2^3+1)\cdots(a_n^3+1) \ge$$
$$(a_1^2 a_2+1)(a_2^2 a_3+1)\cdots(a_n^2 a_1+1).$$

(*Source: 7th Czech-Slovak-Polish Match*)

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Solutions

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Problem 126. Prove that every integer can be expressed in the form $x^2+y^2-5z^2$, where x, y, z are integers.

Solution. **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **CHENG Man Chuen** (CUHK, Math Major, Year 1), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **FOK Chi Kwong** (Yuen Long Merchants Association Secondary School, Form 5), **IP Ivan** (St. Joseph's College, Form 6), **KOO Koopa** (Boston College, Sophomore), **LAM Shek Ming Sherman** (La Salle College, Form 6), **LAU Wai Shun** (Tsuen Wan Public Ho Chuen Yiu Memorial College, Form 6), **LEE Kevin** (La Salle College, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC IA Wong Tai Shan Memorial College), **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **YEUNG Kai Sing** (La Salle College, Form 5) and **YUNG Po Lam** (CUHK, Math Major, Year 2).

For n odd, say $n = 2k + 1$, we have $(2k)^2+(k+1)^2-5k^2 = 2k+1 = n$. For n even, say $n = 2k$, we have $(2k-1)^2 + (k-2)^2 - 5(k-1)^2 = 2k = n$.

Problem 127. For positive real numbers a, b, c with $a + b + c = abc$, show that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}} \le \frac{3}{2},$$

and determine when equality occurs. (*Source: 1998 South Korean Math Olympiad*)

Solution. **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **KOO Koopa** (Boston College, Sophomore), **LEE Kevin** (La Salle College, Form 6) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Let $A = \tan^{-1} a$, $B = \tan^{-1} b$, $C = \tan^{-1} c$. Since $a, b, c > 0$, we have $0 < A, B, C < \frac{\pi}{2}$. Now $a + b + c = abc$ is the same as $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. Then

$$\tan C = \frac{-(\tan A+\tan B)}{1-\tan A\tan B} = \tan(\pi - A - B)$$

which implies $A + B + C = \pi$. In terms of A, B, C the inequality to be proved is $\cos A + \cos B + \cos C \le \frac{3}{2}$, which follows by applying Jensen's inequality to $f(x) = \cos x$ on $(0, \frac{\pi}{2})$.

Other commended solvers: **CHENG Man Chuen** (CUHK, Math Major, Year 1), **IP Ivan** (St. Joseph's College, Form 6), **LAM Shek Ming Sherman** (La Salle College, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College), **TSUI Ka Ho** (Hoi Ping Chamber of Commerce Secondary School, Form 7), **WONG Wing Hong** (La Salle College, Form 4) and **YEUNG Kai Sing** (La Salle College, Form 5).

Problem 128. Let M be a point on segment AB. Let $AMCD$, $BEHM$ be squares on the same side of AB. Let the circumcircles of these squares intersect at M and N. Show that B, N, C are collinear and H is the orthocenter of ΔABC. (*Source: 1979 Henan Province Math Competition*)

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College) and **YUNG Po Lam** (CUHK, Math Major, Year 2).

Since $\angle BNM = \angle BHM = 45^\circ = \angle CDM = \angle CDM$, it follows B, N, C are collinear. Next, $CH \perp AB$. Also, $BH \perp ME$ and $ME \| AC$ imply $BH \perp AC$. So H is the orthocenter of ΔABC.

Other commended solvers: **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **CHENG Man Chuen** (CUHK, Math Major, Year 1), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **IP Ivan** (St. Joseph's College, Form 6), **KWOK Sze Ming** (Queen Elizabeth School, Form 6), **LAM Shek Ming Sherman** (La Salle College, Form 6), **Lee Kevin** (La Salle College, Form 6), **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **WONG Wing Hong** (La Salle College, Form 4) and **YEUNG Kai Sing** (La Salle College, Form 5).

Problem 129. If $f(x)$ is a polynomial of degree $2m + 1$ with integral coefficients for which there are $2m + 1$ integers $k_1, k_2, \ldots, k_{2m+1}$ such that $f(k_i) = 1$ for $i = 1, 2, \ldots, 2m + 1$, prove that $f(x)$ is not the product of two nonconstant polynomials with integral coefficients.

Solution. **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **CHENG Man Chuen** (CUHK, Math Major, Year 1), **IP Ivan** (St. Joseph's College, Form 6), **KOO Koopa** (Boston College, Sophomore), **LAM Shek Ming Sherman** (La Salle College, Form 6), **LEE Kevin** (La Salle College, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form

7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College), **YEUNG Kai Sing** (La Salle College, Form 5) and **YUNG Po Lam** (CUHK, Math Major, Year 2).

Suppose f is the product of two non-constant polynomials with integral co-efficients, say $f = PQ$. Since $1 = f(k_i) = P(k_i)\,Q(k_i)$ and $P(k_i)$, $Q(k_i)$ are integers, so either both are 1 or both are –1. As there are $2m + 1$ k_i's, either $P(k_i) = Q(k_i) = 1$ for at least $m + 1$ k_i's or $P(k_i) = Q(k_i) = -1$ for at least $m + 1$ k_i's. Since deg f = $2m$ + 1, one of deg P or deg Q is at most m. This forces P or Q to be a constant polynomial, a contradiction.

Other commended solvers: **NG Cheuk Chi** (Tsuen Wan Public Ho Chuen Yiu Memorial College) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Problem 130. Prove that for each positive integer n, there exists a circle in the xy-plane which contains exactly n lattice points in its interior, where a *lattice point* is a point with integral coordinates. (*Source: H. Steinhaus, Zadanie 498, Matematyka 10 (1957), p. 58*)

Solution. **CHENG Man Chuen** (CUHK, Math Major, Year 1) and **IP Ivan** (St. Joseph's College, Form 6).

Let $P = \left(\sqrt{2}, \dfrac{1}{3}\right)$. Suppose lattice points $(x_0, y_0), (x_1, y_1)$ are the same distance from P. Then $\left(x_0 - \sqrt{2}\right)^2 + \left(y_0 - \dfrac{1}{3}\right)^2 = \left(x_1 - \sqrt{2}\right)^2 + \left(y_1 - \dfrac{1}{3}\right)^2$. Moving the x terms to the left, the y terms to the right and factoring, we get

$$(x_0 - x_1)\left(x_0 + x_1 - 2\sqrt{2}\right)$$

$$= (y_0 - y_1)\left(y_0 + y_1 - \frac{2}{3}\right).$$

As the right side is rational and $\sqrt{2}$ is irrational, we must have $x_0 = x_1$. Then the left side is 0, which forces $y_1 = y_0$ since $y_1 + y_0$ is integer. So the lattice points are the same.

Now consider the circle with center at P and radius r. As r increases from 0 to infinity, the number of lattice points inside the circle increase from 0 to infinity. As the last paragraph shows, the increase cannot jump by 2 or more. So the statement is true.

Other commended solvers: **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **KOO Koopa** (Boston College, Sophomore), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College), **NG Ka Chun** (Queen Elizabeth School, Form 7) and **YEUNG Kai Sing** (La Salle College, Form 4).

Olympiad Corner

(continued from page 1)

Problem 4. Let n be an odd integer greater than 1, let $k_1, k_2, ..., k_n$ be given integers. For each of the n! permutations $a = (a_1, a_2, ..., a_n)$ of 1, 2, …, n, let

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Prove that there are two permutations b and c, $b \neq c$, such that n! is a divisor of $S(b) - S(c)$.

Problem 5. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

Problem 6. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that $ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime.

Pell's Equation (I)

(continued from page 2)

Example 7. Prove that the only positive integral solution of $5^a - 3^b = 2$ is $a = b = 1$.

Solution. Clearly, if a or b is 1, then the other one is 1, too. Suppose (a, b) is a solution with both $a, b > 1$. Considering (mod 4), we have $1 - (-1)^b \equiv 2$ (mod 4), which implies b is odd. Considering (mod 3), we have $(-1)^a \equiv 2$ (mod 3), which implies a is odd.

Setting $x = 3^b + 1$ and $y = 3^{(b-1)/2}\,5^{(a-1)/2}$, we get $15y^2 = 3^b5^a = 3^b(3^b + 2) = (3^b + 1)^2 - 1 = x^2 - 1$. So (x, y) is a positive solution of $x^2 - 15y^2 = 1$. The least positive solution is $(4,1)$. Then $(x, y) = (x_n, y_n)$ for some positive integer n, where $x_n + y_n\sqrt{15} = (4 + \sqrt{15})^n$. After examining the first few y_n's, we observe that y_{3k} are the only terms that are divisible by 3. However, they also seem to be divisible by 7, hence cannot be of the form 3^c5^d.

To confirm this, we use the recurrence relations on y_n. Since $y_1 = 1$, $y_2 = 8$ and $y_{n+2} = 8y_{n+1} - y_n$, taking y_n (mod 3), we get the sequence 1, 2, 0, 1, 2, 0… and taking y_n (mod 7), we get 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, ….

Therefore, no $y = y_n$ is of the form 3^c5^d and $a, b > 1$ cannot be solution to $5^a - 3^b = 2$.

Example 8. Show that the equation $a^2 + b^3 = c^4$ has infinitely many solutions.

Solution. We will use the identity

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

which is a standard exercise of mathematical induction. From the identity, we get $\left(\dfrac{(n-1)n}{2}\right)^2 + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$ for $n > 1$. All we need to do now is to show there are infinitely many positive integers n such that $n(n + 1)/2 = k^2$ for some positive integers k. Then $(a, b, c) = ((n - 1)n/2, n, k)$ solves the problem.

Now $n(n + 1)/2 = k^2$ can be rewritten as $(2n+1)^2 - 2(2k)^2 = 1$. We know $x^2 - 2y^2 = 1$ has infinitely many positive solutions. For any such (x, y), clearly x is odd, say $x = 2m + 1$. They $y^2 = 2m^2 + 2m$ implies y is even. So any such (x, y) is of the form $(2n+1, 2k)$. Therefore, there are infinitely many such n.

Mathematical Excalibur

Volume 6, Number 4 *October 2001 – December 2001*

Olympiad Corner

The 18th Balkan Mathematical Olympiad, Belgrade, Yugoslavia, 5 May 2001

Problem 1. Let n be a positive integer. Show that if a and b are integers greater then 1 such that $2^n - 1 = ab$, then the number $ab-(a-b)-1$ is of the form $k \cdot 2^{2m}$, where k is odd and m is a positive integer.

Problem 2. Prove that if a convex pentagon satisfies the following conditions:

(1) all interior angles are congruent; and

(2) the lengths of all sides are rational numbers,

then it is a regular pentagon.

Problem 3. Let a, b, c be positive real numbers such that $a+b+c \geq abc$. Prove that

$$a^2+b^2+c^2 \geq \sqrt{3}abc.$$

Problem 4. A cube of dimensions $3 \times 3 \times 3$ is divided into 27 congruent unit cubical cells.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***15 January 2001***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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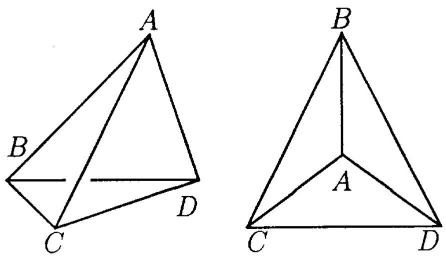
六個頂點的多面體

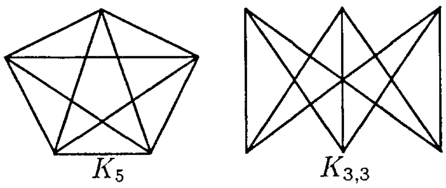
吳主居 **(Richard Travis NG)**

圖一左顯示一個四面體，用邊作輪廓，如將底面擴張，然後把其他的邊壓下去，可得到一個平面圖，各邊祇在頂點處相交，如圖一右所示。



（插圖一）

非平面圖不能是多面體的輪廓，最基本的非平面圖有兩個。第一個有五個頂點，兩兩相連，稱為 K_5，見圖二左。第二個有六個頂點，分為兩組，各有三個，同組的互不相連，不同組的則兩兩相連，稱為 $K_{3,3}$，見圖二右。



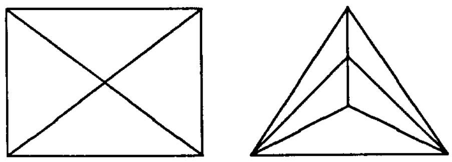
（插圖二）

一個頂點所在邊上的數量，稱為它的度數，一個代表多面體的平面圖，每個頂點的度數，都不能少於 3，所以任何多面體，都不少於四個頂點。假如它祇有四個頂點，它們的度數必定是 (3，3，3，3)，唯一的可能就是圖一的四面體。

假如一個多面體有五個頂點，看來它們的度數可能會是：

(3，3，3，3，3)，(3，3，3，3，4)，(3，3，3，4，4)，(3，3，4，4，4)，(3，4，4，4，4) 或 (4，4，4，4，4)。

不過很快便會發現，左面那三組是不可能的，因為各頂點度數之和，必定是邊數的雙倍，不可能是奇數。右面第一組是個四邊形為底的金字塔，見圖三左，第二組是個三角形為底的雙金字塔，見圖三右。最後一組是 K_5，不是平面圖，不能代表多面體。

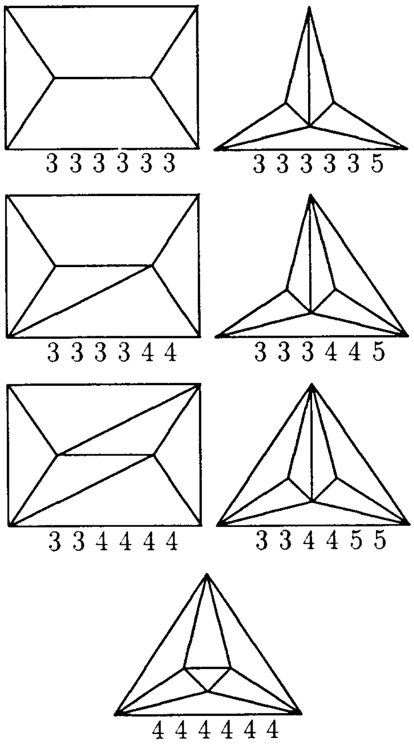


（插圖三）

六個頂點的多面體，有多少個呢？每個頂點的度數，都是 3,4 或 5，有下列可能:

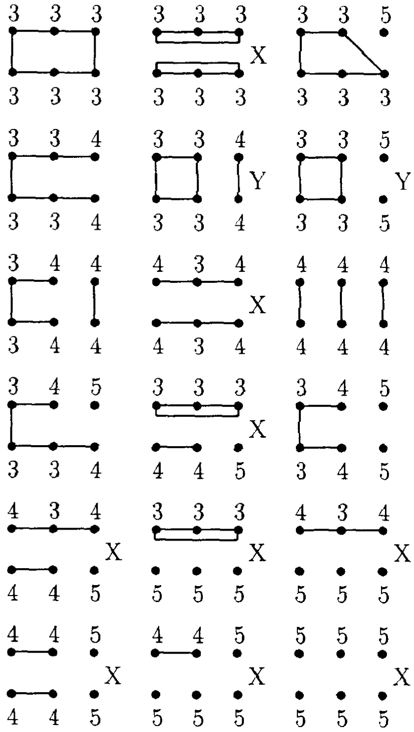
(3，3，3，3，3，3)，(3，3，3，3，3，5)，(3，3，3，3，4，4)，(3，3，3，3，5，5)，(3，3，3，4，4，5)，(3，3，4，4，4，4)，(3，3，3，5，5，5)，(3，3，4，4，5，5)，(3，4，4，4，4，5)，(4，4，4，4，4，4)，(3，3，5，5，5，5)，(3，4，4，5，5，5)，(4，4，4，4，5，5)，(3，5，5，5，5，5)，(4，4，5，5，5，5)或(5，5，5，5，5，5)。

這十六組其中四組，有兩種表達方式，所以共有二十種情況，我們發現有七個不同的多面體，見圖四。



（插圖四）

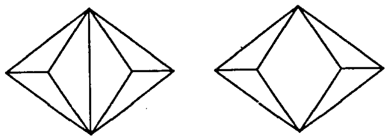
最後證明，就祇有這七種。我們先試劃這些圖，因為頂點太多兩兩相連，祇劃出缺了的邊比較容易，亦立即發現 (3，3，5，5，5，5) 和 (3，5，5，5，5，5) 這兩組是不能成立的，其餘十八種在圖五列出。



（插圖五）

沒做記號那七組，代表我們那七種多面體，用 X 做記號的，都含有 $K_{3,3}$ 在內，所以不可能是平面圖。用 Y 做記號的，雖然它們都是平面圖，但不能代表多面體。

先看圖六左的 (3，3，3，3，5，5)，它僅能代表兩個共邊的四面體，不是一個多面體。再看圖六右 (3，3，3，3，4，4) 的第二種情況，兩個度數為 4 的頂點互不相連，兩個四邊形的面，有兩個不相鄰的公共頂點，這也是不可能的。



（插圖六）

Remarks by Professor Andy Liu

(University of Alberta, Canada)

Polyhedra with Six Vertices is the work of Richard Travis Ng, currently a Grade 12 student at Archbishop MacDonald High School in Edmonton, Canada. The result is equivalent to that in John McClellan's *The Hexahedra Problem* (Recreational Mathematics Magazine, 4, 1961, 34-40), which counts the number of polyhedra with six faces. The problem is also featured in Martin Gardner's "New Mathematical Dviersions" (Mathematical Association of America, 1995, 224-225 and 233). However, the proof in this article is much simpler.

The 2001 Hong Kong IMO team with Professor Andrew Wiles at Washington, DC taken on July 13, 2001. From left to right, *Leung Wai Ying, Yu Hok Pun, Ko Man Ho, Professor Wiles, Cheng Kei Tsi, Chan Kin Hang, Chao Khek Lun*.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***15 January 2001.***

Problem 136. For a triangle *ABC*, if sin *A*, sin *B*, sin *C* are rational, prove that cos *A*, cos *B*, cos *C* must also be rational.
If cos *A*, cos *B*, cos *C* are rational, must at least one of sin *A*, sin *B*, sin *C* be rational?

Problem 137. Prove that for every positive integer *n*,

$$(\sqrt{3}+\sqrt{2})^{1/n}+(\sqrt{3}-\sqrt{2})^{1/n}$$

is irrational.

Problem 138. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) If $a+b$ and $a-b$ are relatively prime integers, find the greatest common divisor (or the highest common factor) of $2a+(1+2a)(a^2-b^2)$ and $2a(a^2+2a-b^2)(a^2-b^2)$.

Problem 139. Let a line intersect a pair of concentric circles at points *A*, *B*, *C*, *D* in that order. Let *E* be on the outer circle and *F* be on the inner circle such that chords *AE* and *BF* are parallel. Let *G* and *H* be points on chords *BF* and *AE* that are the feet of perpendiculars from *C* to *BF* and from *D* to *AE*, respectively. Prove that *EH* = *FG*. (*Source: 1958 Shanghai City Math Competition*)

Problem 140. A convex pentagon has five equal sides. Prove that the interior of the five circles with the five sides as diameters do not cover the interior of the pentagon.

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Solutions

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Problem 131. Find the greatest common divisor (or highest common factor) of the numbers n^n-n for *n* = 3, 5, 7, … .

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **Jack LAU Wai Shun** (Tsuen Wan Public Ho Chuen Yiu Memorial College, Form 6), **LEE Tsun Man Clement** (St. Paul's College, Form 3), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **Boris YIM Shing Yik** (Wah Yan College, Kowloon) and **YUEN Ka Wai** (Carmel Divine Grace Foundation Secondary School, Form 6).

Since the smallest number is $3^3-3=24$, the greatest common divisor is at most 24. For $n=2k+1$,

$$n^n-n=n\left((n^2)^k-1\right)$$

$$=(n-1)n(n+1)(n^{2k-2}+\cdots+1).$$

Now one of $n-1,\ n,\ n+1$ is divisible by 3. Also, $(n-1)(n+1)=4k(k+1)$ is divisible by 8. So n^n-n is divisible by 24. Therefore, the greatest common divisor is 24.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHAU Suk Ling** (Queen Elizabeth School, Form 7), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School), **KWOK Sze Ming** (Queen Elizabeth School, Form 6), **LAW Siu Lun** (CCC Ming Kei College, Form 7), **Antonio LEI Iat Fong** and **Alvin LEE Kar Wai** (Colchester Royal Grammar School, England), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **Campion LOONG** (STFA Leung Kau Kui College, Form 6), **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Ho Chung** (Queen's College, Form 3), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 7), **TSOI Hung Ming** (SKH Lam Woo Memorial Secondary School, Form 7), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **Tak Wai Alan WONG** (University of Toronto, Canada), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6), **WONG Wing Hong** (La Salle College, Form 4) and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

Problem 132. Points *D*, *E*, *F* are chosen on sides *AB*, *BC*, *CA* of ΔABC, respectively, so that *DE* = *BE* and *FE* = *CE*. Prove that the center of the circumcircle of ΔADF lies on the angle bisector of $\angle DEF$. (*Source: 1989 USSR Math Olympiad*)

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHAU Suk Ling** (Queen Elizabeth School, Form 7), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **FOK Chi Kwong** (Yuen Long Merchants Association Secondary School, Form 5), **KWOK Sze Ming** (Queen Elizabeth School, Form 6), **KWONG Tin Yan** (True Light Girls' College, Form 6), **Antonio LEI Iat Fong** and **Alvin LEE Kar Wai** (Colchester Royal Grammar School, England), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **SIU Ho Chung** (Queen's College, Form 3), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

Let *O* be the circumcenter of ΔADF and α, β, γ be the measures of angles *A*, *B*, *C* of ΔABC. Then $\angle DOF = 2\alpha$ and $180^\circ - \angle DEF = \angle BED + \angle CEF = 360^\circ - 2\beta - 2\gamma = 2\alpha = \angle DOF$. So *ODEF* is a cyclic quadrilateral. Since *OD* = *OF*, $\angle DEO = \angle FEO$. So *O* is on the angle bisector of $\angle DEF$.

Other commended solvers: **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TSOI Hung Ming** (SKH Lam Woo Memorial Secondary School, Form 7) and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

Problem 133. (a) Are there real numbers *a* and *b* such that $a+b$ is rational and a^n+b^n is irrational for every integer $n \ge 2$? (b) Are there real numbers *a* and *b* such that $a+b$ is irrational and a^n+b^n is rational for every integer $n \ge 2$? (*Source: 1989 USSR Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

(a) Let $a = \sqrt{2}+1$ and $b = -\sqrt{2}$. Then $a+b=1$ is rational. For an integer $n \ge 2$, from the binomial theorem, since binomial coefficients are positive integers, we get

$$(\sqrt{2}+1)^n = r_n\sqrt{2}+s_n,$$

where r_n, s_n are positive integers. For every positive integer k, we have $a^{2k} + b^{2k} = r_{2k}\sqrt{2} + s_{2k} + 2^k$ and $a^{2k+1} + b^{2k+1} = (r_{2k+1}-2^k)\sqrt{2} + s_{2k+1}$. Since

$$r_{2k+1} \ge 2^k + C_2^{2k+1}2^{k-1} > 2^k,$$

a^n+b^n is irrational for $n \ge 2$.

(b) Suppose such a and b exist. Then neither of them can be zero from cases n = 2 and 3. Now

$$(a^2+b^2)^2 = (a^4+b^4)+2a^2b^2$$

implies a^2b^2 is rational, but then

$$(a^2+b^2)(a^3+b^3)$$
$$=(a^5+b^5)+a^2b^2(a+b)$$

will imply $a+b$ is rational, which is a contradiction.

Other commended solvers: **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **TSUI Chun Wa** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 134. Ivan and Peter alternatively write down 0 or 1 from left to right until each of them has written 2001 digits. Peter is a winner if the number, interpreted as in base 2, is not the sum of two perfect squares. Prove that Peter has a winning strategy. (*Source: 2001 Bulgarian Winter Math Competition*)

Solution. (*Official Solution)*

Peter may use the following strategy: he plans to write three 1's and 1998 0's, until Ivan begins to write a 1. Once Ivan writes his first 1, then Peter will switch to follow Ivan exactly from that point to the end.

If Peter succeeded to write three 1's and 1998 0's, then Ivan wrote only 0's and the number formed would be 21×4^{1998}. This is not the sum of two perfect squares since 21 is not the sum of two perfect squares.

If Ivan wrote a 1 at some point, then Peter's strategy would cause the number to have an even number of 0's on the right preceded by two 1's. Hence, the number would be of the form $(4n+3)4^m$. This kind of numbers are also not the sums of two perfect squares, otherwise we have integers x, y such that

$$x^2+y^2=(4n+3)4^m,$$

which implies x, y are both even if m is a positive integer. Keep on canceling 2 from both x and y. Then at the end, we will get $4n$ + 3 as a sum of two perfect squares, which is impossible by checking the sum of odd and even perfect squares.

Other commended solvers: **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Problem 135. Show that for $n \ge 2$, if $a_1, a_2, ..., a_n > 0$, then

$$(a_1^3+1)(a_2^3+1)\cdots(a_n^3+1) \ge$$
$$(a_1^2a_2+1)(a_2^2a_3+1)\cdots(a_n^2a_1+1).$$

(*Source: 7th Czech-Slovak-Polish Match*)

Solution 1. **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School), **FOK Chi Kwong** (Yuen Long Merchants Association Secondary School, Form 5) and **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

First we shall prove that

$$\left(a_1^3+1\right)^2\left(a_2^3+1\right) \ge \left(a_1^2a_2+1\right)^3.$$

By expansion, this is the same as

$$a_1^6a_2^3+2a_1^3a_2^3+a_2^3+a_1^6+2a_1^3+1$$
$$\ge a_1^6a_2^3+3a_1^4a_2^2+3a_1^2a_2+1.$$

This follows by regrouping and factoring to get

$$a_1^3(a_1-a_2)^2(a_1+2a_2)$$
$$+\ (a_1-a_2)^2(2a_1+a_2)\ge 0$$

or from

$$a_2^3+2a_1^3 \ge 3\left(a_2^3a_1^3a_1^3\right)^{1/3} = 3a_1^2a_2,$$
$$2a_1^3a_2^3+a_1^6 \ge 3\left(a_1^{12}a_2^6\right)^{1/3} = 3a_1^4a_2^2,$$

by the AM-GM inequality. Similarly, we get

$$\left(a_i^3+1\right)^2\left(a_{i+1}^3+1\right) \ge \left(a_i^2a_{i+1}+1\right)^3$$

for i = 2, 3, …, n with $a_{n+1} = a_1$. Multiplying these inequalities and taking cube root, we get the desired inequality.

Solution 2. **Murray KLAMKIN** (University of Alberta, Canada) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Let $a_{n+1} = a_1$. For i = 1, 2, …, n, by Hölder's inequality, we have

$$(a_i^3+1)^{2/3}(a_{i+1}^3+1)^{1/3}$$
$$\ge (a_i^3)^{2/3}(a_{i+1}^3)^{1/3}+1.$$

Multiplying these n inequalities, we get the desired inequality.

Comments: For the statement and proof of Hölder's inequality, we refer the readers to vol. 5, no. 4, page 2 of *Math Excalibur*.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **Antonio LEI Iat Fong** and **Alvin LEE Kar Wai** (Colchester Royal Grammar School, England), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TSOI Hung Ming** (SKH Lam Woo Memorial Secondary School, Form 7), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

Olympiad Corner

(continued from page 1)

One of these cells is empty and the others are filled with unit cubes labeled in an arbitrary manner with numbers 1, 2, …, 26. An admissible move is the moving of a unit cube into an adjacent empty cell. Is there a finite sequence of admissible moves after which the unit cube labeled with k and the unit cube labeled with 27 – k are interchanged, for each k = 1, 2, …, 13? (Two cells are said to be adjacent if they share a common face.)

Mathematical Excalibur

Volume 6, Number 5 *January 2002 – February 2002*

Olympiad Corner

The 10th Winter Camp, Taipei, Taiwan, February 14, 2001.

Problem 1. Determine all integers a and b which satisfy that

$$a^{13}+b^{90}=b^{2001}.$$

Problem 2. Let $\langle a_n \rangle$ be sequence of real numbers satisfying the recurrence relation

$$a_1 = k,\ \ a_{n+1} = \left[\sqrt{2}a_n\right],\ \ n = 1, 2, \ldots .$$

where $[x]$ denotes the largest number which is less or equal than x. Find all positive integers k for which three exist three consecutive terms a_{i-1}, a_i, a_{i+1} satisfy $2a_i = a_{i-1} + a_{i+1}$.

Problem 3. A real number r is said to be *attainable* if there is a triple of positive real numbers (a, b, c) such that a, b, c are not the lengths of any triangle and satisfy the inequality

$$rabc > a^2b + b^2c + c^2a.$$

(a) Determine whether or not $\frac{7}{2}$ is *attainable*.

(b) Find all positive integer n such that n is *attainable*.

(continued on page 4)

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On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 23, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Vector Geometry

Kin Y. Li

A *vector* $\overrightarrow{XY}$ is an object having a magnitude (the length XY) and a direction (from X to Y). Vectors are very useful in solving certain types of geometry problems. First, we will mention some basic concepts related to vectors. Two vectors are considered the same if and only if they have the same magnitudes and directions. A vector $\overrightarrow{OX}$ from the origin O to a point X is called a *position vector*. For convenience, often a position vector $\overrightarrow{OX}$ will simply be denoted by X, when the position of the origin is understood, so that the vector $\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX}$ will simply be $Y - X$. The length of the position vector $\overrightarrow{OX} = X$ will be denoted by $|X|$. We have the triangle inequality $|X+Y| \le |X| + |Y|$, with equality if and only if $X = tY$ for some $t \ge 0$. Also, $|cX| = |c||X|$ for number c.

For a point P on the line XY, in terms of position vectors, $P = tX + (1 - t)Y$ for some real number t. If P is on the segment XY, then $t = PY/XY \in [0, 1]$.

Next, we will present some examples showing how vectors can be used to solve geometry problems.

Example 1. (*1980 Leningrad High School Math Olympiad*) Call a segment in a convex quadrilateral a *midline* if it joins the midpoints of opposite sides. Show that if the sum of the midlines of a quadrilateral is equal to its semiperimeter, then the quadrilateral is a parallelogram.

Solution. Let $ABCD$ be such a convex quadrilateral. Set the origin at A. The sum of the lengths of the midlines is

$$\frac{|B+C-D|+|D+C-B|}{2}$$

and the semiperimeter is

$$\frac{|B|+|C-D|+|D|+|C-B|}{2}.$$

So

$$|B+C-D|+|D+C-B| = |B|+|C-D|+|D|+|C-B|$$

By triangle inequality, $|B|+|C-D| \ge |B+C-D|$, with equality if and only if $B = t(C-D)$ (or $AB \| CD$). Similarly, $|D|+|C-B| \ge |D+C-B|$, with equality if and only if $AD \| BC$. For the equation to be true, both triangle inequalities must be equalities. In that case, $ABCD$ is a parallelogram.

Example 2. (*Crux Problem 2333*) D and E are points on sides AC and AB of triangle ABC, respectively. Also, DE is not parallel to CB. Suppose F and G are points of BC and ED, respectively, such that $BF : FC = EG : GD = BE : CD$. Show that GF is parallel to the angle bisector of $\angle BAC$.

Solution. Set the origin at A. Then $E = pB$ and $D = qC$ for some $p, q \in (0, 1)$. Let $t = \frac{BF}{FC}$, then $F = \frac{tC+B}{t+1}$ and $G = \frac{tD+E}{t+1} = \frac{tqC+pB}{t+1}$.

Since $BE = tCD$, so $(1-p)|B| = t(1-q)|C|$. Thus,

$$F - G = \frac{t(1-q)}{t+1}C + \frac{1-p}{t+1}B$$
$$= \frac{(1-p)|B|}{t+1}\left(\frac{C}{|C|}+\frac{B}{|B|}\right).$$

This is parallel to $\frac{C}{|C|}+\frac{B}{|B|}$, which is in the direction of the angle bisector of $\angle BAC$.

* *

The *dot product* of two vectors X and Y is the number $X \cdot Y = |X||Y| \cos\theta$, where θ is the angle between the vectors. Dot product has the following properties:

(1) $X\cdot Y = Y\cdot X, (X+Y)\cdot Z = X\cdot Z + Y\cdot Z$ and $(cX)\cdot Y = c(X\cdot Y)$.

(2) $|X|^2 = X\cdot X,\ |X\cdot Y| \le |X||Y|$ and $OX \perp OY$ if and only if $X\cdot Y = 0$.

Example 3. (*1975 USAMO*) Let A, B, C, D denote four points in space and AB the distance between A and B, and so on. Show that

$$AC^2+BD^2+AD^2+BC^2 \ge AB^2+CD^2.$$

Solution. Set the origin at A. The inequality to be proved is

$$C\cdot C+(B-D)\cdot(B-D) + D\cdot D + (B-C)\cdot(B-C) \ge B\cdot B+(C-D)\cdot(C-D).$$

After expansion and regrouping, this is the same as $(B-C-D)\cdot(B-C-D) \ge 0$, with equality if and only if $B-C = D = D-A$, i.e. is $BCAD$ is a parallelogram.

* *

For a triangle ABC, the position vectors of its centroid is

$$G=\frac{A+B+C}{3}.$$

If we take the circumcenter O as the origin, then the position of the orthocenter is $H = A + B + C$ as $\overrightarrow{OH} = 3\overrightarrow{OG}$. Now for the incenter I, let a, b, c be the side lengths and AI intersect BC at D. Since $BD{:}CD = c{:}b$ and $DI{:}AI = \frac{ca}{b+c}{:}c = a{:}b+c$, so $D = \frac{bB+cC}{b+c}$ and $I=\frac{aA+bB+cC}{a+b+c}$.

Example 4. (2^{nd} *Balkan Math Olympiad*) Let O be the center of the circle through the points A, B, C, and let D be the midpoint of AB. Let E be the centroid of triangle ACD. Prove that the line CD is perpendicular to line OE if and only if $AB = AC$.

Solution. Set the origin at O. Then

$$D=\frac{A+B}{2},$$

$$E=\frac{A+C+D}{3}=\frac{3A+B+2C}{6},$$

$$D-C=\frac{A+B-2C}{2}.$$

Hence $CD \perp OE$ if and only if $(A+B-2C)\cdot(3A+B+2C)=0$. Since $A\cdot A = B\cdot B = C\cdot C$, this is equivalent to $A\cdot(B-C) = A\cdot B - A\cdot C = 0$, which is the same as $OA \perp BC$, i.e. $AB = AC$.

Example 5. (*1990 IMO Usused Problem, Proposed by France*) Given ΔABC with no side equal to another side, let G, I and H be its centroid, incenter and orthocenter, respectively. Prove that $\angle GIH > 90^\circ$.

Solution. Set the origin at the circumcenter. Then

$$H = A+B+C,\quad G=\frac{A+B+C}{3},$$

$$I=\frac{aA+bB+cC}{a+b+c}.$$

We need to show $(G-I)\cdot(H-I) = G\cdot H + I\cdot I - I\cdot(G+H) < 0$. Now $A\cdot A = B\cdot B = C\cdot C = R^2$ and $2B\cdot C = B\cdot B + C\cdot C - (B-C)\cdot(B-C) = 2R^2-a^2, \ldots$. Hence,

$$G\cdot H=\frac{(A+B+C)\cdot(A+B+C)}{3} = 3R^2-\frac{a^2+b^2+c^2}{3},$$

$$I\cdot I=\frac{(aA+bB+cC)\cdot(aA+bB+cC)}{(a+b+c)^2} = R^2-\frac{abc}{a+b+c},$$

$$I\cdot(G+H)=\frac{4(aA+bB+cC)\cdot(A+B+C)}{3(a+b+c)} = 4R^2-\frac{2[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{3(a+b+c)}.$$

Thus, it is equivalent to proving $(a+b+c)(a^2+b^2+c^2)+3abc > 2[a^2(b+c)+b^2(c+a)+c^2(a+b)]$, which after expansion and regrouping will become $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) > 0$. To obtain this inequality, without loss of generality, assume $a \ge b \ge c$. Then $a(a-b)(a-c) \ge b(a-b)(b-c)$ so that the sum of the first two terms is nonnegative. As the third term is also nonnegative, the above inequality is true.

* *

The *cross product* of two vectors X and Y is a vector $X\times Y$ having magnitude $|X||Y| \sin\theta$, where θ is the angle between the vectors, and direction perpendicular to the plane of X and Y satisfying the right hand rule. Cross product has the following properties:

(1) $X\times Y = -Y\times X$, $(X+Y)\times Z = X\times Z + Y\times Z$ and $(cX)\times Y = c(X\times Y)$.

(2) $\frac{|X\times Y|}{2}$ is the area of triangle XOY. When $X, Y \ne O$, $X\times Y = 0$ if and only if X, O, Y are collinear.

Example 6. (*1984 Annual Greek High School Competition*) Let $A_1\ A_2\ A_3\ A_4\ A_5\ A_6$ be a convex hexagon having its opposite sides parallel. Prove that triangles $A_1\ A_3\ A_5$ and $A_2\ A_4\ A_6$ have equal areas.

Solution. Set the origin at any point. As the opposite sides are parallel, $(A_1-A_2)\times(A_4-A_5)=0$, $(A_3-A_2)\times(A_5-A_6)=0$ and $(A_3-A_4)\times(A_6-A_1)=0$. Expanding these equations and adding them, we get $A_1\times A_3+A_3\times A_5+A_5\times A_1 = A_2\times A_4+A_4\times A_6+A_6\times A_2$. Now

$$[A_1\ A_3\ A_5]=\frac{|(A_1-A_3)\times(A_1-A_5)|}{2} = \frac{|A_1\times A_3+A_3+A_5+A_5\times A_1|}{2}.$$

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***March 23, 2002***.

Problem 141. Ninety-eight points are given on a circle. Maria and José take turns drawing a segment between two of the points which have not yet been joined by a segment. The game ends when each point has been used as the endpoint of a segment at least once. The winner is the player who draws the last segment. If José goes first, who has a winning strategy? (*Source: 1998 Iberoamerican Math Olympiad*)

Problem 142. $ABCD$ is a quadrilateral with $AB \parallel CD$. P and Q are on sides AD and BC respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Prove that P and Q are equal distance from the intersection point of the diagonals of the quadrilateral. (*Source: 1994 Russian Math Olympiad, Final Round*)

Problem 143. Solve the equation $\cos\cos\cos\cos x = \sin\sin\sin\sin x$. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Problem 144. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all (non-degenerate) triangles ABC with consecutive integer sides a, b, c and such that $C = 2A$.

Problem 145. Determine all natural numbers $k > 1$ such that, for some distinct natural numbers m and n, the numbers $k^m + 1$ and $k^n + 1$ can be obtained from each other by reversing the order of the digits in their decimal representations. (*Source: 1992 CIS Math Olympiad*)

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Solutions
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Problem 136. For a triangle ABC, if $\sin A$, $\sin B$, $\sin C$ are rational, prove that $\cos A$, $\cos B$, $\cos C$ must also be rational. If $\cos A$, $\cos B$, $\cos C$ are rational, must at least one of $\sin A$, $\sin B$, $\sin C$ be rational?

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **LO Chi Fai** (STFA Leung Kau Kui College, Form 6), **WONG Tak Wai Alan** (University of Toronto), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

If $\sin A$, $\sin B$, $\sin C$ are rational, then by cosine law and sine law,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}-\frac{a}{b}\frac{a}{c}\right)$$

$$= \frac{1}{2}\left(\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B}-\frac{\sin A}{\sin B}\frac{\sin A}{\sin C}\right)$$

is rational. Similarly, $\cos B$ and $\cos C$ are rational. In the case of an equilateral triangle, $\cos A = \cos B = \cos C = \cos 60^\circ = \frac{1}{2}$ is rational, but $\sin A = \sin B = \sin C = \sin 60^\circ = \frac{\sqrt{3}}{2}$ is irrational.

Other commended solvers: **LEE Tsun Man Clement** (St. Paul's College, Form 3), **LOONG King Pan Campion** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **TANG Chun Pong** (La Salle College, Form 4).

Problem 137. Prove that for every positive integer n,

$$(\sqrt{3}+\sqrt{2})^{1/n}+(\sqrt{3}-\sqrt{2})^{1/n}$$

is irrational.

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Let $x = (\sqrt{3}+\sqrt{2})^{1/n}$. Since $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right) = 1$, $x^{-1} = (\sqrt{3}-\sqrt{2})^{1/n}$. If $x + x^{-1}$ is rational, then $x^2 + x^{-2} = (x + x^{-1})^2 - 2$ is also rational. Since

$$x^{k+1}+x^{-(k+1)} = (x+x^{-1})\left(x^k+x^{-k}\right) - \left(x^{k-1}+x^{-(k-1)}\right),$$

by math induction, $x^n + x^{-n} = 2\sqrt{3}$ would be rational, a contradiction. Therefore, $x + x^{-1}$ is irrational.

Other commended solvers: **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

Problem 138. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) If $a + b$ and $a - b$ are relatively prime integers, find the greatest common divisor (or the highest common factor) of $2a + (1 + 2a)(a^2 - b^2)$ and $2a(a^2 + 2a - b^2)(a^2 - b^2)$.

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Let (r, s) denote the greatest common divisor (or highest common factor) of r and s. If $(r, s) = 1$, then for any prime p dividing rs, either p divides r or p divides s, but not both. In particular p does not divide $r + s$. So $(r + s, rs) = 1$. Let $x = a + b$ and $y = a - b$. Then

$$\begin{aligned} &2a + (1 + 2a)(a^2 - b^2)\\ &= x + y + (1 + x + y)xy\\ &= (x + y + xy) + (x + y)xy \end{aligned}$$

and

$$\begin{aligned} &2a(a^2 + 2a - b^2)(a^2 - b^2)\\ &= (x + y)(xy + x + y)xy. \end{aligned}$$

Now $(x, y) = 1$ implies $(x + y, xy) = 1$. Repeating this twice, we get

$$(x + y + xy, (x + y)xy) = 1$$

and

$$((x + y + xy + (x + y)xy, (x + y + xy)(x + y)xy) = 1.$$

So the answer to the problem is 1.

Other commended solvers: **LEE Tsun Man Clement** (St. Paul's College, Form 3), **POON Yiu Keung** (HKUST, Math Major, Year 1), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TANG Chun Pong** (La Salle College, Form 4), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 4).

Problem 139. Let a line intersect a pair of concentric circles at points A, B, C, D in that order. Let E be on the outer circle and F be on the inner circle such that chords AE and BF are parallel. Let

G and H be points on chords BF and AE that are the feet of perpendiculars from C to BF and from D to AE, respectively. Prove that $EH = FG$. (*Source: 1958 Shanghai City Math Competition*)

Solution. **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

Let M be the midpoint of BC (and AD). Since $\angle DHA = 90^\circ$, $\angle ADH = \angle DHM$. Since $BF \parallel AE$, $\angle BAE = \angle FEA$ by symmetry with respect to the diameter perpendicular to BF and AE. Now $\angle FEA = \angle BAE = 90^\circ - \angle ADH = 90^\circ - \angle DHM = \angle AHG$. So $EF \parallel HG$. Since $EH \parallel FG$ also, $EFGH$ is a parallelogram. Therefore, $EH = FG$.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7).

Problem 140. A convex pentagon has five equal sides. Prove that the interior of the five circles with the five sides as diameters do not cover the interior of the pentagon.

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Let the pentagon be $A_1A_2A_3A_4A_5$ and $2r$ be the common length of the sides. Let M_{ij} be the midpoint of A_iA_j and C_i be the circle with diameter A_iA_{i+1} for i = 1, 2, 3, 4, 5 (with $A_6 = A_1$). Since $540 - 3\cdot 60 = 2\cdot 180$ and $\angle A_i <$ 180°, there are at least 3 interior angles (in particular, two adjacent angles) greater than 60°. So we may suppose $\angle A_1, \angle A_2 > 60^\circ$. Since $A_3A_4 = A_5A_4$, we get $A_4M_{35} \perp A_3A_5$. Then M_{35} is on C_3, C_4 and the points on the ray from A_4 to M_{35} lying beyond M_{35} is outside C_3, C_4.

Next, since $\angle A_1 > 60^\circ$ and $A_1A_2 = A_1A_5$, A_2A_5 is the longest side of $\Delta A_1A_2A_5$. By the midpoint theorem, $M_{23}M_{35} = \dfrac{A_2A_5}{2} > \dfrac{A_1A_2}{2} = r$ so that M_{35} is outside C_2. Similarly, M_{35} is outside C_5. If M_{35} is not outside C_1, then $A_2M_{35} < A_1A_2 = A_2A_3$ and $\angle A_1M_{35}A_2 \geq 90^\circ$. Since $A_3M_{35} < A_3A_4 = A_2A_3$ also, A_2A_3 must be the longest side of $\Delta A_2A_3M_{35}$. Then $\angle A_2M_{35}A_3 > 60^\circ$. Similarly, $\angle A_1M_{35}A_5 > 60^\circ$. Then, we have $\angle A_1M_{35}A_2 < 60^\circ$, a contradiction. So M_{35} is outside C_1, too.

For i = 1, 2, 5 let $d_i = M_{35}M_{i,i+1} - r > 0$. Let d be the distance from M_{35} to the intersection point of the pentagon with the ray from A_4 to M_{35} lying beyond M_{35}. Choose a point X beyond M_{35} on the ray from A_4 to M_{35} with $XM_{35} < d, d_1, d_2$ and d_5. Then X is inside the pentagon and is outside C_3, C_4. Also, for $i = 1, 2, 5$,

$$XM_{i,i+1} > M_{35}M_{i,i+1} - XM_{35}$$
$$= r + d_i - XM_{35} > r$$

so that X is outside C_1, C_2, C_5.

Comments: The point M_{35} is enough for the solution as it is not in the interior of the 5 circles. The point X is better as it is not even on any of the circles.

Olympiad Corner

(continued from page 1)

Problem 4. Let O be the center of excircle of ΔABC touching the side BC internally. Let M be the midpoint of AC, P the intersection point of MO and BC. Prove that $AB = BP$, if $\angle BAC = 2\angle ACB$.

Problem 5. Given that 21 regular pentagons P_1, P_2, …, P_{21} are such that for any $k \in \{1, 2, 3, \ldots, 20\}$, all the vertices of P_{k+1} are the midpoints of the sides of P_k. Let S be the set of the vertices of $P_1, P_2, \ldots, P_{21}$. Determine the largest positive integer n for which there always exist four points A, B, C, D from S such that they are the vertices of an isosceles trapezoid and with the same color if we use n kinds of different colors to paint the element of S.

Vector Geometry

(continued from page 2)

Similarly,

$$[A_2A_4A_6] = \frac{|A_2 \times A_4 + A_4 \times A_6 + A_6 \times A_2|}{2}.$$

So $[A_1A_3A_5] = [A_2A_4A_6]$.

Example 7. (*1996 Balkan Math Olympiad*) Let $ABCDE$ be a convex pentagon and let M, N, P, Q, R be the midpoints of sides AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR, DM have a common point, show that this point also lies on EN.

Solution. Set the origin at the commom point. Since, A, P and the origin are collinear,

$$0 = A \times P = A \times \left(\frac{C+D}{2}\right) = \frac{A\times C + A\times D}{2}.$$

So $A\times C = D\times A$. Similarly, $B\times D = E\times B$, $C\times E = A\times C$, $D\times A = B\times D$. Then $E\times B = C\times E$. So $E\times N = E\times \left(\dfrac{B+C}{2}\right) = 0$, which implies E, N and the origin are collinear.

Example 8. (*16th Austrian Math Olympiad*) A line interesects the sides (or sides produced) BC, CA, AB of triangle ABC in the points A_1, B_1, C_1, respectively. The points A_2, B_2, C_2 are symmetric to A_1, B_1, C_1 with respect to the midpoints of BC, CA, AB, respectively. Prove that A_2, B_2, C_2 are collinear.

Solution. Set the origin at a vertex, say C. Then $A_1 = c_1B, B_1 = c_2A, C_1 = A + c_3(B - A)$ for some constants c_1, c_2, c_3. Since A_1, B_1, C_1, are collinear,

$$0 = (B_1 - A_1)\times(C_1 - A_1)$$
$$= (c_1 - c_1c_2 - c_1c_3 + c_2c_3)A\times B.$$

Since

$$A_2 = B - A_1 = (1 - c_1)B,$$
$$B_2 = A - B_1 = (1 - c_2)A$$

and

$$C_2 = (A + B) - C_1 = c_3A + (1 - c_3)B,$$

so A_2, B_2, C_2, are collinear if and only if

$$0 = (B_2 - A_2)\times(C_2 - A_2)$$
$$= (c_1 - c_1c_2 - c_1c_3 + c_2c_3)A\times B,$$

which is true.