Junior problems

J307. Prove that for each positive integer n there is a perfect square whose sum of digits is equal to 4^n.

Proposed by Mihaly Bencze, Brasov, Romania

J308. Are there triples (p, q, r) of primes for which $(p^2 - 7)(q^2 - 7)(r^2 - 7)$ is a perfect square?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J309. Let n be an integer greater than 3 and let S be a set of n points in the plane that are not the vertices of a convex polygon and such that no three are collinear. Prove that there is a triangle with the vertices among these points having exactly one other point from S in its interior.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J310. Alice puts checkers in some cells of an 8×8 board such that:

a) there is at least one checker in any 2×1 or 1×2 rectangle.

b) there are at least two adjacent checkers in any 7×1 or 1×7 rectangle.

Find the least amount of checkers that Alice needs to satisfy both conditions.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

J311. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\frac{a(b^2+3)}{3c^2+1} + \frac{b(c^2+3)}{3a^2+1} + \frac{c(a^2+3)}{3b^2+1} \geq 3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J312. Let ABC be a triangle with circumcircle Γ and let P be a point in its interior. Let M be the midpoint of side BC and let lines AP, BP, CP intersect BC, CA, AB at X, Y, Z, respectively. Furthermore, let line YZ intersect Γ at points U and V. Prove that M, X, U, V are concyclic.

Proposed by Cosmin Pohoata, Princeton University, USA

Senior problems

S307. Let ABC be a triangle such that $\angle ABC - \angle ACB = 60^\circ$. Suppose that the length of the altitude from A is $\frac{1}{4}BC$. Find $\angle ABC$.

Proposed by Omer Cerrahoglu and Mircea Lascu, Romania

S308. Let n be a positive integer and let $\mathcal{G}_n$ be an $n \times n$ grid with the number 1 written in each of its unit squares. An operation consists of multiplying all entries of a column or all entries of a row by -1. Determine the number of distinct grids that can be obtained after applying a finite number of operations on $\mathcal{G}_n$.

Proposed by Marius Cavachi, Constanta, Romania

S309. Let $ABCD$ be a circumscriptible quadrilateral, which lies stricly inside a circle ω. Let ω_A be the circle outside of $ABCD$ that is tangent to AB, AD, and to ω at A'. Similarly, define B', C', D'. Prove that lines AA', BB', CC', DD' are concurrent.

Proposed by Khakimboy Egamberganov, Tashkent, Uzbekistan

S310. Let a, b, c be nonzero complex numbers such that $|a| = |b| = |c| = k$. Prove that

$$\sqrt{|-a+b+c|} + \sqrt{|a-b+c|} + \sqrt{|a+b-c|} \le 3\sqrt{k}.$$

Proposed by Marcel Chirita, Bucharest, Romania

S311. Let n be a positive integer. Prove that

$$\prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} (x+2j+1)^{\binom{n}{2j+1}} - \prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} (x+2j)^{\binom{n}{2j}}$$

is a polynomial of degree $2^{n-1} - n$, whose highest term's coefficient is $(n-1)!$.

Proposed by Albert Stadler, Herrliberg, Switzerland

S312. Let a, b, c, d be positive real numbers such that $a^3 + b^3 + c^3 + d^3 = 1$. Prove that

$$\frac{1}{1-bcd} + \frac{1}{1-cda} + \frac{1}{1-dab} + \frac{1}{1-abc} \le \frac{16}{3}.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Undergraduate problems

U307. Prove that any polynomial $f \in \mathbb{R}[X]$ can be written as a difference of increasing polynomials.

Proposed by Jishnu Bose, Calcutta, India

U308. Let a_1, b_1, c_1, a_2, b_2, c_2 be positive real numbers. Consider the functions $X(x,y)$ and $Y(x,y)$ which satisfy the system of functional equations

$$\frac{x}{X} = 1 + a_1x + b_1y + c_1Y,$$

$$\frac{y}{Y} = 1 + a_2x + b_2y + c_2X.$$

Prove that if $0 < x_1 \leq x_2$ and $0 < y_2 \leq y_1$, then $X(x_1,y_1) \leq X(x_2,y_2)$ and $Y(x_1,y_1) \geq Y(x_2,y_2)$.

Proposed by Razvan Gelca, Texas Tech University, USA

U309. Let $a_1,\ldots,a_n$ be positive real numbers such that $a_1+\ldots+a_n = 1$, $n \geq 2$. Prove that for every positive integer m,

$$\sum_{k=1}^{n} \frac{a_k^{m+1}}{1-a_k^m} \geq \frac{1}{n^m-1}.$$

Proposed by Titu Zvonaru, Comanesti and Neculai Stanciu, Romania

U310. Let $\mathcal{E}$ be an ellipse with foci F and G, and let P be a point in its exterior. Let A and B be the points where the tangents from P to $\mathcal{E}$ intersect $\mathcal{E}$, such that A is closer to F. Furthermore, let X be the intersection of AG with BF. Prove that XP bisects $\angle AXB$.

Proposed by Jishnu Bose, Calcutta, India

U311. Let $f : [0,1] \to [0,1]$ be a nondecreasing concave function such that $f(0) = 0$ and $f(1) = 1$. Prove that

$$\int_0^1 (f(x)f^{-1}(x))^2 dx \geq \frac{1}{12}.$$

Proposed by Marcel Chirita, Bucharest, Romania

U312. Let p be a prime and let R be a commutative ring with characteristic p. Prove that the sets $S_k = \{x \in R \mid x^p = k\}$, where $k \in \{1,\ldots,p\}$, have the same number of elements.

Proposed by Corneliu Manescu-Avram, Ploiesti, Romania

Olympiad problems

O307. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}\leq\frac{1}{abcd}.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

O308. Let ABC be a triangle and let X, Y be points in its plane such that

$$AX:BX:CX=AY:BY:CY.$$

Prove that the circumcenter of triangle ABC lies on the line XY.

Proposed by Cosmin Pohoata, USA and Josef Tkadlec, Czech Republic

O309. Determine the least real number μ such that

$$\mu(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})+\sqrt{a^2+b^2+c^2}\geq a+b+c$$

for all nonnegative real numbers a, b, c with $ab+bc+ca>0$. Find when equality holds.

Proposed by Albert Stadler, Herrliberg, Switzerland

O310. Let ABC be a triangle and let P be a point in its interior. Let X, Y, Z be the intersections of AP, BP, CP with sides BC, CA, AB, respectively. Prove that

$$\frac{XB}{XY}\cdot\frac{YC}{YZ}\cdot\frac{ZA}{ZX}\leq\frac{R}{2r}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O311. Let ABC be a triangle with circumcircle Γ centered at O. Let the tangents to Γ at vertices B and C intersect each other at X. Consider the circle $\mathcal{X}$ centered at X with radius XB, and let M be the point of intersection of the internal angle bisector of angle A with $\mathcal{X}$ such that M lies in the interior of triangle ABC. Denote by P the intersection of OM with the side BC and by E and F be the orthogonal projections of M on CA and AB, respectively. Prove that PE and FP are perpendicular.

Proposed by Cosmin Pohoata, Princeton University, USA

O312. Find all increasing bijections $f:(0,\infty)\to(0,\infty)$ satisfying

$$f(f(x))-3f(x)+2x=0$$

and for which there exists $x_0>0$ such that $f(x_0)=2x_0$.

Proposed by Razvan Gelca, Texas Tech University, USA