Junior problems

J235. In the equality $\sqrt{ABCDEF} = DEF$, different letters represent different digits. Find the six-digit number $ABCDEF$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

From the given equality, we conclude that $ABCDEF^2 - DEF \equiv 0 \pmod{1000}$, and since $ABCDEF \equiv DEF \pmod{1000}$, we have $DEF^2 - DEF = DEF(DEF-1) \equiv 0 \pmod{1000}$. Since DEF and $DEF-1$ are coprime and $1000 = 2^3 \cdot 5^3$, one of these two numbers must be odd and divisibile by 5^3, while the other must be divisible by 2^3. Since DEF and $DEF-1$ are three digit numbers, $DEF \in \{125, 375, 625, 875\}$ or $DEF - 1 \in \{125, 375, 625, 875\}$. In the first case, $DEF - 1$ is divisible by 8 if and only if $DEF = 625$; in the second case DEF is divisible by 8 if and only if $DEF - 1 = 375$. So, $DEF = 625, 376$, but

$$625^2 = 390625 \qquad 376^2 = 141376,$$

hence the only number which satisfies the required conditions is 625.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Kwan Chung Hang, Hong Kong, People's Republic of China; Radouan Boukharfane, Polytechnique de Montral, Canada; Pascal Reisert, Mathematical Institute, Munich, Germany; Prithwijit De, HBCSE, Mumbai, India; Zolbayar Shagdar, Orchlon Cambridge International School, Ulaanbaatar, Mongolia; Alessio Podda, Università di Roma "Tor Vergata", Roma, Italy.

J236. Let ABC be a triangle and let $ABRS$ and $ACXY$ be the two squares constructed on sides AB and AC which are directed towards the exterior of the triangle. If U is the circumcenter of triangle SAY, prove that the line AU is the A-symmedian of triangle ABC.

Proposed by Cosmin Pohoata, Princeton University, USA

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $V = BC \cap AU$, and denote $\beta = \angle ASY$, $\gamma = \angle AYS$. Clearly, AUY is isosceles at U with $\angle AUY = 2\angle ASY = 2\beta$, or $\angle YAU = 90^\circ - \beta$, and since $\angle CAY = 90^\circ$, then $\angle CAV = 180^\circ - \angle CAY - \angle YAU = \beta$. Similarly, $\angle BAV = \gamma$. Moreover, applying the Sine Law to triangle AYS together with $AY = AC = b$ and $AS = AB = c$ yields $b \sin\gamma = c \sin\beta$. Finally, applying the Sine Law to triangles BAV and CAV together with $\angle BVA + \angle CVA = 180^\circ$ yields

$$\frac{c\sin\gamma}{BV} = \sin\angle BVA = \sin\angle CVA = \frac{b\sin\beta}{CV},$$

or equivalently,

$$\frac{BV}{CV} = \frac{c\sin\gamma}{b\sin\beta} = \frac{c^2}{b^2},$$

clearly equivalent to AV (and hence AU) being the A-symmedian in triangle ABC. The conclusion follows.

Second solution by the author

If $\delta(P, \ell)$ denotes the distance from point P to line ℓ, notice that

$$\frac{\delta(U, AB)}{\delta(U, AC)} = \frac{\frac{AB}{2}}{\frac{AC}{2}} = \frac{AB}{AC},$$

since U lies on the perpendicular bisectors of the sides AS and AY of squares $ABRS$ and $ACXY$. This implies that AU is the A-symmedian of triangle ABC.

Also solved by Ercole Suppa, Teramo, Italy; Radouan Boukharfane, Polytechnique de Montral, Canada; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Zolbayar Shagdar, Orchlon Cambridge International School, Ulaanbaatar, Mongolia.

J237. Prove that the diameter of the incircle of a triangle ABC is equal to $\frac{AB-BC+CA}{\sqrt{3}}$ if and only if $\angle BAC = 60^\circ$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Let a, b, c be the sidelengths, s the semiperimeter, and r the inradius of triangle ABC. Accordingly, we rewrite the condition $2r = \frac{AB-BC+CA}{\sqrt{3}}$ as $\frac{r}{s-a} = \frac{1}{\sqrt{3}}$, which, by to the well-known formula $\tan\frac{A}{2} = \frac{r}{s-a}$, becomes $\tan\frac{A}{2} = \frac{1}{\sqrt{3}}$. This is equivalent with $\angle A = 60^\circ$, as claimed.

Also solved by Arkady Alt , San Jose, California, USA; Ioan Viorel Codreanu, Satulung, Maramure, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Radouan Boukharfane, Polytechnique de Montral, Canada; Sayan Das, Kolkata, India; Alessandro Ventullo, Milan, Italy; Prithwijit De, HBCSE, Mumbai, India; Ercole Suppa, Teramo, Italy; Zolbayar Shagdar, Orchlon Cambridge International School, Ulaanbaatar, Mongolia; Kwan Chung Hang, Hong Kong, People's Republic of China.

J238. Given a real number $\alpha \in (0,1)$, prove that there is a positive integer N such that for any N points in the plane, no three collinear, there is a triangle with one its angles greater than $\alpha\pi$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

By the Erdös-Szekeres Theorem, for any integer $n > 2$ there exists an integer $f(n)$ such that any set of at least $f(n)$ points in the plane, no three collinear, contains a subset of n points that are the vertices of a convex n-gon. Since the sum of the n internal angles of a convex n-gon is equal to $\pi(n-2)$, it follows that the internal angle at one of its vertices, say P, is at least $\pi(n-2)/n$. Now take $n \geq 2/(1-\alpha)$ and $N \geq f(n)$ then $\pi(n-2)/n \geq \pi\alpha$ and the triangle with vertices P and the two adjacent ones has the required property.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

J239. Let a and b be real numbers so that $2a^2 + 3ab + 2b^2 \leq 7$. Prove that

$$\max\{2a + b, 2b + a\} \leq 4.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Assume, by the way of contradiction, that $\max\{2a + b, 2b + a\} > 4$. This means that at least one of the numbers $2a + b$, $2b + a$ is greater than 4. Assume $2a + b > 4$ without loss of generality. Then $b > 4 - 2a$. This inequality implies that

$$\begin{aligned} 2a^2 + 3ab + 2b^2 - 7 &> 2a^2 + 3a\,(4 - 2a) + 2\,(4 - 2a)^2 - 7 \\ &= 4a^2 - 20a + 28 \\ &= (2a - 5)^2 + 3 \\ &> 0 \end{aligned}$$

This gives the desired contradiction because $2a^2 + 3ab + 2b^2 \leq 7$ by hypothesis.

Also solved by Marin Sandu and Mihai Sandu, Bucharest, Romania; Problem Solving group of Qafqaz University, Baku, Azerbaijan; Perfetti Paolo, Università degli studi di Tor Vergata Roma, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa and Simone Coccia, Teramo, Italy; Arkady Alt , San Jose, California; Alessandro Ventullo, Milan, Italy; Radouan Boukharfane, Polytechnique de Montral, Canada; Zolbayar Shagdar, Orchlon Cambridge International School, Ulaanbaatar, Mongolia; Kwan Chung Hang, Hong Kong, People's Republic of China.

J240. Let ABC be an acute triangle with orthocenter H. Points H_a, H_b, and H_c in its interior satisfy

$$\angle BH_aC = 180^\circ - \angle A, \qquad \angle CH_aA = 180^\circ - \angle C, \qquad \angle AH_aB = 180^\circ - \angle B,$$

$$\angle CH_bA = 180^\circ - \angle B, \qquad \angle AH_bB = 180^\circ - \angle A, \qquad \angle BH_bC = 180^\circ - \angle C,$$

$$\angle AH_cB = 180^\circ - \angle C, \qquad \angle BH_cC = 180^\circ - \angle B, \qquad \angle CH_cA = 180^\circ - \angle A,$$

Prove that the points H, H_a, H_b, H_c are concyclic.

Proposed by Michal Rolinek, Charles University, Czech Republic

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote by N_a, N_b, N_c the midpoints of AH, BH, CH, and by O_a, O_b, O_c the circumcenters of BHC, CHA, AHB respectively. Since $\angle HBC = 90^\circ - C$ and $\angle HCB = 90^\circ - B$, it follows that $\angle BHC = B + C = 180^\circ - A = \angle BH_aC$, or B, C, H, H_a are concyclic, and by the Sine Law, the radius of the circle through them is equal to the circumradius R of ABC. Moreover, O_b, O_c are on the perpendicular bisector of AH, and $O_bA = O_bH = O_cA = O_cH = R$, hence AO_bHO_c is a rhombus with center N_a, or N_a is the midpoint of O_bO_c. All these results clearly apply also when A, B, C are cyclically permuted.

Note now that $\angle BH_aH = \angle BCH = 90^\circ - B$, while $\angle BH_aA = 180^\circ - B$, or $\angle AH_aH = \angle BH_aA - \angle BH_aH = 90^\circ$. It follows that H_a is on the circle with diameter AH, or H_a is the second point of intersection of this circle with the circumcircle of BHC (the other one being H). Since HH_a is a common chord of these two circles, their centers O_a, N_a lie on the perpendicular bisector of HH_a. Denoting by P the centroid of $O_aO_bO_c$, clearly P is on the median O_aN_a, which is also the perpendicular bisector of HH_a, hence $PH = PH_a$, and similarly $PH = PH_b$ and $PH = PH_c$. Therefore P is the center of a circle through H, H_a, H_b, H_c. The conclusion follows.

Senior problems

S235. Solve the equation

$$\frac{8}{\{x\}} = \frac{9}{x} + \frac{10}{[x]},$$

where $[x]$ and $\{x\}$ denote the greatest integer less or equal than x and the fractional part of x, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $m = [x]$, $y = \{x\}$, where clearly m is an integer, $0 \leq y < 1$, and $x = m + y$. Note that if either $\{x\} = 0$ or $m = 0$, at least one of the sides of the equation is not well defined. We may therefore accept (or not) a trivial solution $x = 0$ where both sides of the given equation are infinite, and any real solution where both sides of the proposed equation are defined require m, x nonzero. We will find only these latter solutions, for which we may multiply both sides of the proposed equation by $x[x]\{x\}$. Clearly,

$$8m^2 = 8x[x] - 8[x]\{x\} = [x]\{x\} + 10x\{x\} = 11my + 10y^2 < 11m + 10,$$

or equivalently $0 > 8m^2 - 11m - 10 = (8m + 5)(m - 2)$, for $2 > m > -\frac{5}{8}$. Since m is an integer, this leaves the only nonzero solution $m = 1$, for $0 = 10y^2 + 11y - 8 = (2y - 1)(5y + 8)$, and since $y = -\frac{8}{5}$ does not satisfy $0 < y < 1$, it must be $y = \frac{1}{2}$. It follows that the only nonzero solution, ie the only solution for which both sides of the proposed equation are well-defined real numbers, is $x = m + y = 1 + \frac{1}{2} = \frac{3}{2}$.

Also solved by Florin Stanescu Serban Cioculescu School, Gaesti, Dambovita, Romania; Ioan Viorel Codreanu, Satulung, Maramure, Romania; Arkady Alt , San Jose, California, USA; Alessandro Ventullo, Milan, Italy; Zolbayar Shagdar, Orchlon Cambridge International School, Ulaanbaatar, Mongolia; Alessio Podda, Università di Roma "Tor Vergata", Roma, Italy; Sayan Das, Kolkata, India; Prithwijit De, HBCSE, Mumbai, India; Radouan Boukharfane, Polytechnique de Montral, Canada; Kwan Chung Hang, Hong Kong, People's Republic of China; Albert Stadler, Switzerland.

S236. Consider all cyclic quadrilaterals $ABCD$ inscribed in a given circle ω for which AB always passes through a given point K and whose diagonals intersect at a given point P. Prove that CD also passes through some fixed point.

Proposed by Josef Tkadlec, Charles University, Czech Republic

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let L be the point where KP intersects CD. Clearly, triangles APB and DPC are similar because $ABCD$ is cyclic. Moreover, $\angle LPC = \angle KPA$ and $\angle LPD = \angle KPB$. Denote by L' be the point where the symmetric of PK with respect to the angle bisector of $\angle APB$ intersects AP. After some simple trigonometry,

$$\frac{AL'}{BL'} = \frac{AP^2}{BP^2} \cdot \frac{BK}{AK},$$

or using Stewart's theorem,

$$\frac{AB}{BL'} = \frac{BK \cdot AP^2 + AK \cdot BP^2}{AK \cdot BP^2} = \frac{AB(PK^2 + AK \cdot BK)}{AK \cdot BP^2},$$

$$BL' = \frac{AK \cdot BP^2}{PK^2 + AK \cdot BK},$$

and similarly

$$AL' = \frac{BK \cdot AP^2}{PK^2 + AK \cdot BK},$$

or using again Stewart's theorem,

$$PL' = \sqrt{\frac{AL' \cdot BP^2 + BL' \cdot AP^2}{AB} - AL' \cdot BL'} = \frac{AP \cdot BP \cdot PK}{PK^2 + AK \cdot BK}.$$

Note that $\frac{PK}{PK^2+AK \cdot BK}$ is a constant, since P, K are fixed, whereas $AK \cdot BK$ is the power of the fixed point K with respect to the fixed circle ω.

Now, triangles APL' and DPL are similar, and so are BPL' and CPL, with the same proportionality constant as APB and DPC. It follows that a real constant $\rho = \frac{DP}{AP} = \frac{CP}{BP} = \frac{PL}{PL'}$ exists, or

$$PL = \rho \cdot \frac{AP \cdot BP \cdot PK}{PK^2 + AK \cdot BK} = \sqrt{AP \cdot CP \cdot BP \cdot DP} \cdot \frac{PK}{PK^2 + AK \cdot BK},$$

clearly constant since $AP \cdot CP = BP \cdot DP$ is the power of the fixed point P with respect to the fixed circle ω, whereas the second factor in the RHS is constant, as proved earlier. Therefore, there exists a fixed point L, on fixed line PK such that P is inside segment KL, and at a fixed distance PL from fixed point P, so that CD always passes through L. The conclusion follows.

Second solution by Cosmin Pohoata, Princeton University, USA

Let the fixed point through which AB always passes be P and let Q be the intersection of the diagonals of $ABCD$. Consider the inversion with center Q and power the power of point of Q with respect to the circumcircle Γ of $ABCD$. Then, the lines CD are mapped into the circles PAB, which all meet again on the line PQ at the point T so that $PT \cdot PQ$ is the power of P with respect to Γ. This point is thus fixed, and we are done.

S237. Harry Potter, in one of his journeys, stumbled upon a magic beads string. To achieve his goal, he must take out all the beads on this string. It is known that he can only remove one bead at a time, from the left part of the string. The string contains beads of 7 different colors labeled $1, 2, \ldots, 7$ and it is under the following spell: whenever Harry removes the first bead from the left, after each bead of color $1 \leq i \leq 6$ left on the string, new beads of colors $i+1, \ldots, 7$ will pop-up in this order. For example, if on the string we have the colours $1, 4, 3, 7$, after Harry takes out the first bead, we will have $4, 5, 6, 7, 3, 4, 5, 6, 7, 7$. Does Harry have any chance to complete his task regardless the beads string he starts with?

Proposed by Catalin Turcas, University of Warwick, United Kingdom

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let $P(m, r)$ be the proposition that if the minimal color of the initial string is m and there are $r \geq 1$ occurrences of such color in the string then Harry can take out all the beads on this string in a finite number of moves. The proposition $P(7, r)$ is certainly true because the task can be completed in r moves. Now consider a string with $r \geq 1$ beads of color 6 and some beads of color 7. The first bead 6 from the left separates two strings (possibly empty), s and s' which stay on the left and on the right respectively. The string s, where all beads are of color 7, can be eliminated in a finite number of moves because $P(7, r')$ is true, then we can eliminate the bead of color 6, and what remains is a finite string s'' which could be different from s' but, by construction, it has the same number of 6 that is $r-1$. So if $r > 1$ then $P(6, r)$ is true as soon as $P(6, r-1)$ is true and $P(6, 1)$ is true soon as $P(7, r')$ is true. Hence inductively we can prove that $P(6, r)$ is true for any $r \geq 1$. The argument can be extended in a similar way to the the other colors and finally we find that $P(1, r)$ is true for any $r \geq 1$. Therefore Harry is able to complete his task regardless the beads string he starts with.

Also solved by Pascal Reisert, Mathematical Institute, Munich, Germany; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S238. Let ABC be a triangle with incenter I and let D, E, F be the tangency points of the incircle with sides BC, CA, AB, respectively. Let M be the midpoint of the arc BC of the circumcircle which contains vertex A. Furthermore, let P and Q be the midpoints of segments DE and DF. Prove that MI bisects the segment PQ.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that if $\angle B = \angle C$, $A = M$, and by the symmetry around the internal bisector MI of A, which is also the perpendicular bisector of BC, it follows that MI is the perpendicular bisector of PQ. We will assume henceforth that wlog $\angle B > \angle C$.

By symmetry around the angle bisector of $\angle B$, BI is the perpendicular bisector of DF, or $BI \perp DF$, and since $DI \perp BC$, triangles IQD and IDB are similar, or $IQ \cdot IB = ID^2 = r^2$, where r is the inradius of ABC. Similarly, $IP \cdot IC = r^2$, or IPQ and IBC are similar.

Now, $PQ \parallel EF$ because P, Q are the midpoints of DE, DF, and by symmetry around the angle bisector of A, AI is the perpendicular bisector of EF, or $AI \perp PQ$. It follows by similarity that IM passes through the midpoint of PQ iff the angle formed by IM and IA is the same as the angle formed by ID and IA', where A' is the midpoint of BC. Now, since AI intersects the midpoint N of the arc BC of the circumcircle which does not contain A, hence at the point N such that MN is a diameter, it follows that $\angle MAI = 90° = \angle IDA'$, or IM passes through the midpoint of PQ iff AIM and DIA' are similar, hence iff $\frac{IA}{AM} = \frac{ID}{DA'}$.

Now, $ID = r = IA \sin \frac{A}{2}$. Moreover, AI forms with BC an angle $180° - B - \frac{A}{2} = 90° - \frac{B-C}{2}$, or $\angle ANM = \frac{B-C}{2}$ because MN is the perpendicular bisector of BC, or by the Sine Law, $AM = 2R \sin \frac{B-C}{2}$, where R is the circumradius of ABC. We conclude that the proposed result is equivalent to

$$DA' = 2R \sin \frac{B-C}{2} \sin \frac{A}{2} = 2R \sin \frac{B-C}{2} \cos \frac{B+C}{2} = R \sin B - R \sin C =$$

$$= \frac{b-c}{2} = \frac{a}{2} - \frac{c+a-b}{2} = BA' - BD,$$

clearly true. The conclusion follows.

Also solved by Andr Macieira Braga Costa, Belo Horizonte, Minas Gerais, Brazil.

S239. Solve in nonnegative integers the equation

$$2(x^3+y^3+z^3)=3(x+y+z)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy
Let $s=x+y+z$, then by the AM-GM inequality, we see that

$$\frac{s^2}{2}=\frac{x^3+y^3+z^3}{3}\geq\left(\frac{s}{3}\right)^3$$

which implies $0\leq s\leq 27/2=13.5$. By checking (x,y,z) in the finite set $[0,13]^3$, it is easy to verify that, up to a permutation, the solutions are: $(0.0,0)$, $(0,3,3)$ and $(3,4,5)$.

S240. Let ABC be a triangle with circumcircle Γ and let M, N, P be points on the sides BC, CA, AB, respectively. Let A', B', C' be the intersections of AM, BN, CP with Γ different from the vertices of the triangle. Prove that

$$\frac{MA}{MA'} + \frac{NB}{NB'} + \frac{PC}{PC'} \geq 4\left(2 - \frac{r}{R}\right)^2,$$

where R and r are the circumradius and the inradius of triangle ABC.

Proposed by Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $u = \frac{BM}{BC}$, or the power of M with respect to Γ is $AM \cdot MA' = BM \cdot CM = u(1-u)a^2$. Therefore, $\frac{MA}{MA'} = \frac{MA^2}{u(1-u)a^2}$. But using Stewart's theorem,

$$MA^2 = \frac{BM \cdot CA^2 + CM \cdot BA^2}{BC} - BM \cdot CM = ub^2 + (1-u)c^2 - u(1-u)a^2,$$

or

$$\frac{MA}{MA'} + 1 = \frac{ub^2 + (1-u)c^2}{u(1-u)a^2} = \frac{(b+c)^2}{a^2} + \frac{(ub - (1-u)c)^2}{u(1-u)a^2} \geq \frac{(b+c)^2}{a^2},$$

with equality iff $ub = (1-u)c$. Therefore, after some algebra,

$$\frac{MA}{MA'} + \frac{NB}{NB'} + \frac{PC}{PC'} \geq 9 + \sum_{\text{cyc}} \frac{b^2c^2 + 2a^2(ab+bc+ca)}{a^2b^2c^2}(b-c)^2,$$

with equality iff AM, BN, CP are the internal bisectors of the angles of ABC.

Using well-known relations for the area S of ABC, one can find

$$\frac{r}{R} = \frac{4S^2}{abcs} = \frac{4(s-a)(s-b)(s-c)}{abc},$$

or after some algebra,

$$4\left(2 - \frac{r}{R}\right)^2 - 9 = \left(\frac{(s-a)(b-c)^2 + (s-b)(c-a)^2 + (s-c)(a-b)^2}{abc} + 3\right)^2 - 9 =$$

$$= \sum_{\text{cyc}} \frac{6abc(s-a) + (s-a)^2(b-c)^2}{a^2b^2c^2}(b-c)^2 + 2\sum_{\text{cyc}} \frac{(s-b)(s-c)(c-a)^2(a-b)^2}{a^2b^2c^2}.$$

The proposed problem is then, after some algebra, equivalent to showing that

$$\sum_{\text{cyc}} \left(2abc(a+6b+6c) + 2(a^2-2bc)^2 + 32S^2 + ab(b-c)^2\right)(b-c)^2 +$$

$$+ \sum_{\text{cyc}} (a(b+c-a) + (b+c)(b+c+a) + 22bc)(c-a)^2(a-b)^2 \geq 0.$$

Clearly all terms in both sums are non-negative, being zero iff $a = b = c$. The conclusion follows, equality holds iff ABC is equilateral and AM, BN, CP are the internal bisectors of its angles.

Also solved by Arkady Alt , San Jose, California, USA.

U235. Let $a > b$ be positive real numbers and let n be a positive integer. Prove that

$$\frac{\left(a^{n+1}-b^{n+1}\right)^{n-1}}{\left(a^n-b^n\right)^n} > \frac{n}{(n+1)^2} \cdot \frac{e}{a-b},$$

where e is the Euler number.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Campos Salas, Universidad de Costa Rica

The inequality is homogeneous in a, b, so we can start by assuming (without loss of generality) that $b = 1$. Now, let

$$f(a) = \log\left(\frac{(a-1)(a^{n+1}-1)^{n-1}}{(a^n-1)^n}\right) = \log\left(\frac{(a^n+\cdots+1)^{n-1}}{(a^{n-1}+\cdots+1)^n}\right),$$

for $a \neq 1$ and $f(1) = \log((n+1)^{n-1}/n^n)$. Then f is differentiable in $\mathbb{R}^+$. For $a \neq 1$ we have that

$$\begin{aligned} f'(a) &= \frac{1}{a-1} + \frac{(n^2-1)a^n}{a^{n+1}-1} - \frac{n^2a^{n-1}}{a^n-1} \\ &= \frac{(a^n-1)(a^n+\cdots+1) - a^{2n} - (n^2-1)a^n + n^2a^{n-1}}{(a^{n+1}-1)(a^n-1)} \\ &= \frac{(a^n-1)(a^{n-1}+\cdots+1) - n^2a^{n-1}(a-1)}{(a^{n+1}-1)(a^n-1)} \\ &= \frac{(a-1)((a^{n-1}+\cdots+1)^2 - n^2a^{n-1})}{(a^{n+1}-1)(a^n-1)}. \end{aligned}$$

All the factors in $f'(a)$ are positive in $(1, \infty)$, therefore f is strictly increasing in $[1, \infty)$. It follows that for $a > 1$ we have that

$$\frac{(a^{n+1}-1)^{n-1}}{(a^n-1)^n} > \frac{(n+1)^{n-1}}{(a-1)n^n}.$$

It remains to prove that

$$\frac{(n+1)^{n-1}}{n^n} \geq \frac{en}{(n+1)^2},$$

which is equivalent to $((n+1)/n)^{n+1} > e$. This follows from the well-known inequality $e^x \geq 1 + x$ for $x = -1/(n+1)$. This completes the proof.

Also solved by Albert Stadler, Switzerland; Arkady Alt , San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland; Radouan Boukharfane, Polytechnique de Montral, Canada.

U236. Let $f(X)$ be an irreducible polynomial in $\mathbb{Z}[X]$. Prove that $f(XY)$ is irreducible in $\mathbb{Z}[X,Y]$.

Proposed by Mircea Becheanu, University of Bucharest, Romania

Solution by Cosmin Pohoata, Princeton University, USA

Using basic commutative algebra results, we can see this as follows. The ring of integers $\mathbb{Z}$ is a UFD, thus so is $\mathbb{Z}[X]$; hence, since f is irreducible in $\mathbb{Z}[X]$, it is prime there, and so the quotient $\mathbb{Z}[X]/f$ is a domain. Hence, $(\mathbb{Z}[X]/f)[Y]$ is a domain, and therefore, $Z[X,Y]/f$ is a domain. Consequently, f is prime in $\mathbb{Z}[X,Y]$, and since $\mathbb{Z}[X,Y]$ is a UFD, it follows that f is irreducible in $\mathbb{Z}[X,Y]$, as claimed.

Also solved by Radouan Boukharfane, Polytechnique de Montral, Canada; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessio Podda, Università di Roma "Tor Vergata", Roma, Italy, Alessandro Ventullo, Milan, Italy.

U237. Let $\mathcal{H}$ be a hyperbola with foci A and B and center O. Let P be an arbitrary point on $\mathcal{H}$ and let the tangent of $\mathcal{H}$ through P cut its asymptotes at M and N. Prove that $PA + PB = OM + ON$.

Proposed by Luis Gonzalez, Maracaibo, Venezuela

Solution by Alessandro Ventullo, Milan, Italy

Let $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, so that $O = (0,0)$. By symmetry, suppose that P is on the first quadrant. Then $P = \left(k, \dfrac{b}{a}\sqrt{k^2 - a^2}\right)$, where $k \geq a$ is an arbitrary real number. The equation of the tangent of $\mathcal{H}$ at P is $\dfrac{kx}{a^2} - \dfrac{\frac{b}{a}\sqrt{k^2-a^2}y}{b^2} = 1$, so intersecting this line with the asymptotes $y = \pm\dfrac{b}{a}x$ we get the points

$$M = \left(k + \sqrt{k^2 - a^2}, \frac{b}{a}(k + \sqrt{k^2 - a^2})\right), \qquad N = \left(k - \sqrt{k^2 - a^2}, -\frac{b}{a}(k - \sqrt{k^2 - a^2})\right).$$

Then

$$OM + ON = \frac{2k}{a}\sqrt{a^2 + b^2} = 2ke,$$

where $e = \sqrt{a^2 + b^2}/a$ is the eccentricity of the hyperbola. Since the foci have coordinates $A = (-c, 0), B = (c, 0)$ where $c = \sqrt{a^2 + b^2}$, we have

$$\begin{aligned}
PA + PB &= \sqrt{(k+c)^2 + \frac{b^2}{a^2}(k^2 - a^2)} + \sqrt{(k-c)^2 + \frac{b^2}{a^2}(k^2 - a^2)} \\
&= \frac{4kc}{\sqrt{(k+c)^2 + \frac{b^2}{a^2}(k^2 - a^2)} - \sqrt{(k-c)^2 + \frac{b^2}{a^2}(k^2 - a^2)}} \\
&= \frac{2kc}{a} \\
&= 2ke
\end{aligned}$$

and the statement follows.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U238. Let X be a random variable with median $m = 0$, mean μ_X, and variance σ_X^2. Denote by $\sigma_{|X|}^2$ the variance of the random variable $|X|$. Prove that

$$|\mu_X| \le \sigma_{|X|}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Since we may exchange X by $-X$ without altering the problem, we may assume wlog that $\mu_X \ge 0$. Denote by $f_X(x)$ the probability density function (PDF) of X. Since $m = 0$, we have

$$\int_{-\infty}^{0} f_X(x)dx = \int_{0}^{+\infty} f_X(x)dx = \frac{1}{2}.$$

Define random variables Y, Z, with respective PDF's $f_Y(y), f_Z(z)$, such that $f_Y(y) = 2f_X(y)$ for $y \ge 0$, $f_Z(z) = 2f_X(-z)$ for $z \ge 0$, and $f_Y(-x) = f_Z(x) = 0$ for all $x > 0$. Clearly,

$$\mu_X = \int_{-\infty}^{+\infty} x f_X(x)dx = \frac{1}{2}\int_{-\infty}^{0} x f_Z(-x)dx + \frac{1}{2}\int_{0}^{+\infty} x f_Y(x)dx = \frac{\mu_Y - \mu_Z}{2},$$

whereas $f_{|X|}(x) = f_Y(x) + f_Z(x)$, or $\mu_{|X|} = \frac{\mu_Y + \mu_Z}{2}$. Moreover,

$$\sigma_X^2 = \int_{-\infty}^{+\infty} x^2 f_X(x)dx - {\mu_X}^2 =$$

$$= \frac{1}{2}\int_{-\infty}^{0} x^2 f_Z(-x)dx + \frac{1}{2}\int_{0}^{+\infty} x^2 f_Y(x)dx - \left(\frac{\mu_Y - \mu_Z}{2}\right)^2 = \frac{\sigma_Y^2 + \sigma_Z^2}{2} + {\mu_{|X|}}^2,$$

whereas

$$\sigma_{|X|}^2 = \int_{-\infty}^{+\infty} x^2 f_{|X|}(x)dx - {\mu_{|X|}}^2 =$$

$$= \frac{1}{2}\int_{-\infty}^{0} x^2 f_Z(-x)dx + \frac{1}{2}\int_{0}^{+\infty} x^2 f_Y(x)dx - \left(\frac{\mu_Y + \mu_Z}{2}\right)^2 = \frac{\sigma_Y^2 + \sigma_Z^2}{2} + {\mu_X}^2 =$$

$$= \sigma_X^2 - \mu_Y \cdot \mu_Z$$

Assume now that $|\mu_X| > \sigma_{|X|}$, then

$$\sigma_X^2 - \mu_Y \cdot \mu_Z < \left(\frac{\mu_Y - \mu_Z}{2}\right)^2, \qquad {\mu_{|X|}}^2 > \sigma_X^2 = \frac{\sigma_Y^2 + \sigma_Z^2}{2} + {\mu_{|X|}}^2,$$

contradiction. The conclusion follows. Note that equality in the proposed inequality is only possible if $\sigma_Y^2 + \sigma_Z^2 = 0$, impossible unless X is a discrete random variable that takes values $-a, b$, with $a, b \ge 0$, each one with probability $\frac{1}{2}$, hence $Y = b$ and $Z = -a$. Note that, in this case, $\mu_X = \frac{b-a}{2}$, $\mu_{|X|} = \sigma_X = \frac{b+a}{2}$ and $\sigma_{|X|} = |\mu_X| = \frac{|b-a|}{2}$.

Second solution by Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland

The median minimizes the function $f(a) = E|X - a|$ and hence: $E|X - 0| \le E|X - \mu_X|$. Moreover, using the inequality $(EY)^2 \le EY^2$:

$$(E|X|)^2 \le (E|X - \mu_X|)^2 \le E(X - \mu_X)^2 = \sigma_X^2 = EX^2 - (EX)^2$$

Hence: $\mu_X^2 = (EX)^2 \le EX^2 - (E|X|)^2 = \sigma_{|X|}^2$, which yields us $|\mu_X| \le \sigma_{|X|}$.

U239. Let ABC be a triangle and let P be a point in plane, not lying on the circumcircle Γ of ABC. Let AP, BP, CP intersect again Γ at points X, Y, Z, respectively. Let the tangents from X to the incircle of ABC meet the sideline BC at A_1 and A_2; similarly, define B_1, B_2 and C_1, C_2. Prove that the points A_1, A_2, B_1, B_2, C_1, C_2 lie on a conic.

Proposed by Cosmin Pohoata, Princeton University, USA

No solutions have been received yet.

U240. Let $A \in M_n(\mathbb{Z})$ and let $(a_n)_{n\geq 0}$ be defined by $a_0 = 1$ and

$$a_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} a_{n-j} \mathrm{tr}\left(A^{j+1}\right), \qquad n \geq 0.$$

Prove that all terms of the sequence $(a_n)_{n\geq 0}$ are integers.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

To avoid confusion, we will assume that $A \in M_m(\mathbb{Z})$. Let $r_1, r_2, \ldots, r_m$ be the eigenvalues of A, and to lighten the notation, let $t_k = \mathrm{tr}\left(A^k\right)$. Clearly, the r_i's are the roots of the characteristic polynomial $p(x) = x^m - s_1 x^{m-1} + s_2 x^{m-2} - \cdots + (-1)^m s_m$. Now, $p(x)$ clearly has integral coefficients since it is the determinant of $xI_m - A$, where A has integral coefficients, and s_k is the sum of all possible products of k distinct elements in $\{r_1, r_2, \ldots, r_m\}$. Moreover, it is well known that A^k has eigenvalues ${r_i}^k$, whereas the trace of a matrix equals the sum of its eigenvalues, or $t_k = {r_1}^k + {r_2}^k + \cdots + {r_m}^k$ is also integral for all positive integer k. We first prove the following

Claim 1: For all $n \geq 0$,

$$a_n = \sum r_1^{u_1} r_2^{u_2} \ldots r_m^{u_m},$$

where the sum is taken over all possible sets of non-negative integer u_i's such that $u_1 + u_2 + \cdots + u_m = n$.

Proof 1: For $n = 0$, clearly $u_1 = u_2 = \cdots = u_m = 0$ is the only possible set of u_i's yielding exactly one term in the sum with value 1, while $a_0 = 1$, or the result is true. For $n = 1$, the only possible sets of u_i's are $u_i = 1$ for $i \in \{1, 2, \ldots, m\}$, and simultaneously $u_j = 0$ for $j \neq i$, or the sum is $r_1 + r_2 + \cdots + r_m = t_1$, while $a_1 = \frac{1}{1} a_0 t_1 = t_1$, or the result is again true for $n = 1$.

Assume that the result is true for $0, 1, 2, \ldots, n$. Note that a_{n-j} is homogeneous polynomial of degree $n - j$ in $r_1, r_2, \ldots, r_m$, while t_{j+1} is a homogeneous polynomial of degree $j + 1$, or $a_{n-j} t_{j+1}$ is a homogeneous polynomial of degree $n + 1$ in $r_1, r_2, \ldots, r_m$. Now, every term in this polynomial must be the product of a factor ${r_i}^{j+1}$ from t_{j+1} for $n \geq j \geq 0$, and a factor $r_1^{v_1} r_2^{v_2} \ldots r_m^{v_m}$ from a_{n-j} with non-negative integers v_i such that $0 \leq v_1 + v_2 + \cdots + v_m = n - j \leq n$, clearly yielding all possible products of the form $r_1^{u_1} r_2^{u_2} \ldots r_m^{u_m}$ for non-negative integer u_i's such that $u_1 + u_2 + \cdots + u_m = n$. Now, for each one of these products, there are exactly $n + 1$ terms of the form $r_i^{w_i}$ that appear in it, namely u_i terms of the form $r_i^{w_i}$, with exponents $w_i = 1, 2, \ldots, u_i$ and for each $i \in \{1, 2, \ldots, m\}$, yielding exactly $u_1 + u_2 + \cdots + u_m = n + 1$ such terms. For each one of these terms, there is a corresponding factor in a_{n+1-w_i} such that the product of both is $r_1^{u_1} r_2^{u_2} \ldots r_m^{u_m}$ with coefficient 1. Since there are $n + 1$ such products, and a factor of $\frac{1}{n+1}$ that multiplies the sum of the $a_{n-j} t_{j+1}$, the claim follows by induction.

Claim 2: For all $n \geq 1$,

$$a_n = s_1 a_{n-1} - s_2 a_{n-2} + \ldots,$$

where the sum carries out until $-(-1)^n s_n a_0$ if $n \leq m$, and until $-(-1)^m s_m a_{n-m}$ if $n \geq m$.

Proof 2: It clearly suffices to show that the proposed form of a_n in this claim, equals the form of a_n in the RHS of Claim 1. Now, by the Claim 1, $s_k a_{n-k}$ is a homogeneous polynomial of degree n in $r_1, r_2, \ldots, r_m$, while reciprocally, every term of the form $r_1^{u_1} r_2^{u_2} \ldots r_m^{u_m}$ with $u_1 + u_2 + \cdots + u_m = n$, may be written as the product of one of the terms in s_k, multiplied by one of the terms in a_{n-k}, for all $k \leq m$. Moreover, if exactly p out of the m u_i's are nonzero, there are exactly $\binom{p}{k}$ terms in s_k for any $k \leq p$, for which a term in a_{n-k} can be found such that the product of both is the corresponding term $r_1^{u_1} r_2^{u_2} \ldots r_m^{u_m}$; indeed, there are $\binom{p}{k}$ terms in s_k that are products of k of the p r_i's with positive exponent, and clearly $r_1^{u_1} r_2^{u_2} \ldots r_m^{u_m}$ divided by this product is a product of the form $r_1^{v_1} r_2^{v_2} \ldots r_m^{v_m}$ with non-negative integers $v_1, v_2, \ldots, v_m$ such that $v_1 + v_2 + \cdots + v_m = n - k$, hence a term in a_{n-k}. It follows that the coefficient of $r_1^{u_1} r_2^{u_2} \dots r_m^{u_m}$ in $s_1 a_{n-1} - s_2 a_{n-2} + \dots$ is

$$\binom{p}{1} - \binom{p}{2} + \cdots - (-1)^p \binom{p}{p} = 1 - (1-1)^p = 1,$$

since $n = u_1 + u_2 + \cdots + u_m \geq p$, because p is the number of the u_i's that are positive integers. The Claim 2 follows.

Clearly $a_0 = 1$ and $a_1 = t_1 = r_1 + r_2 + \cdots + r_m$ are integers. Since the $s_1, s_2, \dots, s_m$ are also integers, if $a_0, a_1, \dots, a_{n-1}$ are integers, then so is a_n, because by the Claim 2 it is a sum of products of integers. The conclusion follows by trivial induction.

Also solved by Moubinool Omarjee, Lycée Henri IV, Paris, France; Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland; Pascal Reisert, Mathematical Institute, Munich, Germany.

Olympiad problems

O235. Solve in integers the equation
$$xy - 7\sqrt{x^2+y^2} = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Clearly, $xy \geq 1$. By symmetry, we reduce to $x, y > 0$. Rewriting the equation in the form $(xy-1)^2 = 49(x^2+y^2)$, we put $xy = 7t+1, t \geq 0$, so that $x^2+y^2 = t^2$. Then
$$(x+y)^2 = x^2+y^2+2xy = t^2+2(7t+1) = (t+7)^2 - 47,$$
which gives
$$(t+7-x-y)(t+7+x+y) = 47.$$
So, we must solve the systems of equations
$$\begin{cases} t+7-x-y &= 1 \\ t+7+x+y &= 47 \end{cases} \qquad \begin{cases} t+7-x-y &= -47 \\ t+7+x+y &= -1 \end{cases}$$
Solving the first, we get $t = 17$ and $x+y = 23$; solving the second we get $t = -31 < 0$, i.e. no solution. So we have $x+y = 23, xy = 120$, which gives $x = 15, y = 8$ and $x = 8, y = 15$. In conclusion, the solutions are
$$(15,8),(8,15),(-15,-8),(-8,-15).$$

Also solved by Florin Stanescu Serban Cioculescu School, Gaesti, Dambovita, Romania; Albert Stadler, Switzerland; Ioan Viorel Codreanu, Satulung, Maramure, Romania; Sayan Das, Kolkata, India; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Arkady Alt, San Jose, California, USA; Radouan Boukharfane, Polytechnique de Montral, Canada; Kwan Chung Hang, Hong Kong, People's Republic of China.

O236. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}+\frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)}\geq\frac{3(a+b+c)}{2(ab+bc+ca)}.$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy
First, note that the inequality can be rewritten as

$$\sum_{\text{cyc}}\frac{a^4}{a^3(b+c)^2}+\frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)}\geq\frac{3(a+b+c)}{2(ab+bc+ca)}.$$

Now, notice that Cauchy-Schwarz yields

$$\sum_{\text{cyc}}\frac{a^4}{a^3(b+c)^2}\geq\frac{(a^2+b^2+c^2)^2}{\sum_{cyc}a^3(b+c)^2},\text{ thus we want }\frac{(a^2+b^2+c^2)^2}{\sum_{cyc}a^3(b+c)^2}+\frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)}\geq\frac{3(a+b+c)}{2(ab+bc+ca)}.$$

After getting rid of the denominators, this just rewrites as

$$\sum_{\text{cyc}}(3a^4b^2c+2a^6b)\geq\sum_{\text{cyc}}(a^3b^2c^2+a^3b^3c+2a^5bc+a^5b^2),$$

which follows from AM-GM. Indeed, notice that $a^6b+a^4bc^2\geq 2a^5bc$ yields

$$\sum_{\text{cyc}}(2a^4b^2c+a^6b)\geq\sum_{\text{cyc}}(a^3b^2c^2+a^3b^3c+a^5b^2)$$

and $\frac{4a^6b+ab^6}{5}\geq a^5bc$ yields

$$\sum_{\text{cyc}}2a^4b^2c\geq\sum_{\text{cyc}}(a^3b^2c^2+a^3b^3c)$$

Hence, since

$$a^4b^2c+b^4a^2c\geq 2a^3b^3c\quad\text{and}\quad\frac{a^4b^2c+a^4b^2c+ab^2c^4}{3}\geq a^3b^2c^2,$$

we get precisely what we want.

Also solved by Marin Sandu and Mihai Sandu, Bucharest, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Campos Salas, Universidad de Costa Rica; Radouan Boukharfane, Polytechnique de Montral; Kwan Chung Hang, Hong Kong, People's Republic of China; Albert Stadler, Switzerland; Arkady Alt, San Jose, California, USA.

O237. Let x, y, z be positive real numbers such that

$$\left(2x^4+3y^4\right)\left(2y^4+3z^4\right)\left(2z^4+3x^4\right) \le (3x+2y)(3y+2z)(3z+2x).$$

Prove that $xyz \le 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Campos Salas, Universidad de Costa Rica
We'll prove the contrapositive, namely, that if $xyz \ge 1$ then

$$(2x^4+3y^4)(2y^4+3z^4)(2z^4+3x^4) \ge (3x+2y)(3y+2z)(3z+2x).$$

We would like to obtain an inequality of the form

$$(2x^4+3y^4)^a(2y^4+3z^4)^b(2z^4+3x^4)^c \ge x^\alpha y^\beta z^\gamma(3x+2y).$$

Assuming $a+b+c=1$ and applying Hölder's inequality we have that

$$(2x^4+3y^4)^a(2y^4+3z^4)^b(2z^4+3x^4)^c \ge 3x^{4c}y^{4a}z^{4b}+2x^{4a}y^{4b}z^{4c}.$$

In order to obtain an expression as before we can take $b=c$ such that

$$3x^{4c}y^{4a}z^{4b}+2x^{4a}y^{4b}z^{4c} = 3x^{4b}y^{4a}z^{4b}+2x^{4a}y^{4b}z^{4b} = 3x^{4a}y^{4a}z^{4b}(3x^{4(b-a)}+2y^{4(b-a)}).$$

Setting $b-a=1/4$ we get that $a=1/6$ and $b=c=5/12$. From this we conclude that

$$(2x^4+3y^4)^{1/6}(2y^4+3z^4)^{5/12}(2z^4+3x^4)^{5/12} \ge x^{2/3}y^{2/3}z^{5/3}(3x+2y).$$

Multiplying this with the other two expressions we obtain the result.

Also solved by Albert Stadler, Switzerland; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Arkady Alt, San Jose, California, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Radouan Boukharfane, Polytechnique de Montral.

O238. Consider real numbers $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$. It is known that for every real number X there is a pair (a_i, b_i) such that $a_i X + b_i \geq 0$. Prove that there are indices $i, j \in \{1, 2, \ldots, n\}$ such that each real number X satisfies at least one of the inequalities $a_i X + b_i \geq 0$, $a_j X + b_j \geq 0$.

Proposed by Andrei Ciupan, Harvard University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume that $a_k = 0$ for some pair (a_k, b_k). Now, if $b_k \geq 0$, then $a_k X + b_k = b_k \geq 0$ for all real X, and taking $i = k$ and any j, we have the desired indexes, whereas if $b_k < 0$, then $a_k X + b_k = b_k < 0$ for all real X, and pair (a_k, b_k) is completely useless, or it can be removed from the list of pairs. We thus need to consider only cases where all a_i are nonzero. Since we will assume henceforth that $a_i \neq 0$, we may define $c_i = -\frac{b_i}{a_i}$ for each $i = 1, 2, \ldots, n$.

We will say that a pair *covers* real X iff $a_i X + b_i \geq 0$. If $a_i > 0$, we will say that the pair (a_i, b_i) is *positive*, and (a_i, b_i) clearly covers exactly all reals X such that $X \geq -\frac{b_i}{a_i} = c_i$, whereas if $a_i < 0$, we will say that the pair (a_j, b_j) is *negative*, and (a_j, b_j) clearly covers exactly all reals X such that $X \leq -\frac{b_j}{a_j} = c_j$. In other words, positive pair (a_i, b_i) covers all reals in $[c_i, \infty)$ and negative pair (a_j, b_j) covers all reals in $(-\infty, c_j]$. Define now $C_+ = \min\{c_i\}$ for all positive pairs (a_i, b_i), and $C_- = \max\{c_j\}$ for all negative pairs (a_j, b_j). Clearly, indices i, j exist such that (a_i, b_i) covers $[C_+, \infty)$ and (a_j, b_j) covers $(-\infty, C_-]$.

Assume that $C_+ > C_-$, taking any real $X \in (C_-, C_+)$, for any positive pair (a_i, b_i), we have that $X < C_+$ is outside $[c_i, \infty)$, and $a_i X + b_i < 0$, whereas for any negative pair (a_j, b_j), we have that $X > C_-$ is outside $(-\infty, c_j]$, and $a_j X + b_j < 0$. This contradicts the conditions given in the problem statement, hence $C_- \geq C_+$, or any real X is either at most C_- (hence covered by negative pair (a_j, b_j)) or at least C_+ (hence covered by positive pair (a_i, b_i)), or both (hence covered by both pairs). The conclusion follows.

Note that this solution also provides the way to find such pairs; it suffices to take the positive pair (a_i, b_i) with smallest value of c_i, and the negative pair (a_j, b_j) with largest value of c_j.

O239. Let ABC be a triangle and let D, E, F be the tangency points of its incircle with the sides BC, CA, AB, respectively. Let U be the second intersection of AD with the circumcircle $\mathcal{C}$ of triangle ABC and let X be the tangency point of the A-mixtilinear incircle with $\mathcal{C}$. Furthermore, let V, W be the midpoints of segments DE and DF. Prove that VW, UX, BC are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Lemma: In trilinear coordinates, the point where the A-mixtilinear incircle and the circumcircle are tangent, is the second point where line $c(s-b)\beta = b(s-c)\gamma$ intersects the circumcircle of ABC, the first one clearly being $A \equiv (1,0,0)$.

Proof: The circumcircle has equation $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$. It is relatively well known (or can be found using trilinear coordinates) that the radius of the A-mixtilinear incircle is $\frac{2r}{1+\cos A}$, r being the inradius of ABC. By similarity between incircle and A-mixtilinear incircle, if the latter touches sides AB, AC at points P, Q respectively, we find that $AP = AQ = \frac{b+c-a}{1+\cos A}$, for $BP = a\frac{1-\cos C}{1+\cos A}$ and $CQ = a\frac{1-\cos B}{1+\cos A}$. Thus, after some trigonometry,

$$P \equiv \left(\sin\frac{C}{2}\cos\frac{B}{2}, \cos\frac{A}{2}, 0\right), \qquad Q \equiv \left(\sin\frac{B}{2}\cos\frac{C}{2}, 0, \cos\frac{A}{2}\right).$$

Therefore, the system of equations formed by the equation of the A-mixtilinear incircle,

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta - (\ell\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0,$$

where ℓ, m, n are real parameters, and either one of equations $\beta = 0$, $\gamma = 0$, must have exactly one solution, respectively Q, P. After some algebra, it follows that

$$m = \left(\frac{\sin\frac{C}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}}\right)^2 \frac{a\ell}{b}, \qquad n = \left(\frac{\sin\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}\right)^2 \frac{a\ell}{c}.$$

Therefore, the point where the circumcircle and the A-mixtilinear incircle are tangent, satisfies $\alpha + \frac{m}{\ell}\beta + \frac{n}{\ell}\gamma = 0$, and simultaneously $(b\gamma + c\beta)\alpha = -a\beta\gamma$, or after some algebra,

$$\sin\frac{C}{2}\cos\frac{B}{2}c\beta = \pm\sin\frac{B}{2}\cos\frac{C}{2}b\gamma,$$

where we must clearly pick the solution such that β, γ have the same sign since the A-mixtilinear incircle touches the circumcircle in arc BC that does not contain A. The Lemma is then equivalent to

$$\sin\frac{C}{2}\cos\frac{B}{2}(s-c) = \sin\frac{B}{2}\cos\frac{C}{2}(s-b),$$

or since $s - b = 4R\cos\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}$, and similarly for $s - c$, the Lemma clearly follows.

Since $BD = s - b$, $CD = s - c$, in exact trilinear coordinates we have $D \equiv (0, (s-c)\sin C, (s-b)\sin B)$, and similarly for E, F, or in trilinear (not necessarily exact) coordinates,

$$V \equiv (c(s-c), c(s-c), a(s-a) + b(s-b)),$$

$$W \equiv (b(s-b), c(s-c) + a(s-a), b(s-b)).$$

It follows that point $Y = BC \cap VW$ has trilinear coordinates $(0, \beta, \gamma)$ such that the vectors formed by these three sets of trilinear coordinates are linearly dependent, or after some algebra, $Y \equiv \left(0, c(s-c)^2, -b(s-b)^2\right)$.

Now, all points in line AD with (not necessarily exact) trilinear coordinates (α, β, γ) clearly satisfy $(s-b)b\beta = (s-c)c\gamma$, whereas the equation of the circumcircle of ABC is $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$, or after some algebra U has (not necessarily exact) trilinear coordinates

$$U \equiv (-abc(s-b)(s-c), c(s-c)L, b(s-b)L),$$

where $L = b^2(s-b) + c^2(s-c)$. Therefore, line UY has equation

$$\left(b^2(s-b) + c^2(s-c)\right)\alpha + b(s-b)^2\beta + c(s-c)^2\gamma = 0.$$

Now, the circumcircle of ABC has equation $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$, or the intersecion of UY and the circumcircle happens when

$$0 = bc(s-b)^2\beta^2 + bc(s-c)^2\gamma^2 - (b^2+c^2)(s-b)(s-c)\beta\gamma =$$

$$= \left(b(s-b)\beta - c(s-c)\gamma\right)\left(c(s-b)\beta - b(s-c)\gamma\right),$$

where the first factor clearly corresponds to U, or the second point Z where UY intersects the circumcircle of ABC satisfies $c(s-b)\beta = b(s-c)\gamma$. Since Z is on this line and on the circumcircle of ABC, and does not coincide with A, the conclusion directly follows from the Lemma.

O240. Let m and n be positive integers and let $x = (x_1, \ldots, x_m)$ be a vector of positive real numbers such that $\sum_{i=1}^m x_i = 1$. Consider the set Y, defined as

$$Y = \left\{ y = (y_1, \ldots, y_m) | y_i \in \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}, \sum_{i=1}^m y_i = 1 \right\}.$$

Prove that there is $y^* = (y_1^*, \ldots, y_m^*) \in Y$ such that

$$\sum_{i=1}^m |y_i^* - x_i| \leq \frac{m}{2n}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Pascal Reisert, Mathematical Institute, Munich, Germany

Let $e_i = (0, \ldots, 0, 1, 0, \ldots 0)$ be the standard basis in $\mathbb{R}^m$. By definition

$$x \in X := \left\{ \tilde{x} \in \mathbb{R}^m | \tilde{x} = \sum_{i=1}^m r_i e_i, \sum_{i=1}^m r_i = 1, r_i \geq 0 \right\}$$
$$y \in Y := \left\{ \tilde{y} \in \mathbb{R}^m | \tilde{y} = \sum_{i=1}^m s_i e_i, \sum_{i=1}^m s_i = 1, s_i \in \frac{1}{n} \cdot \mathbb{N} \right\}.$$

Hence Y is a lattice in X and every $x = \sum_{i=1}^m r_i e_i \in X$ lies in a regular $(m-1)$-simplex $\tilde{\Delta}$. Therefore it will be enough to prove that every point in $\tilde{\Delta}$ has distance less or equal $\frac{m}{2n}$ to at least one of the vertexes. Since we measure only distances within $\tilde{\Delta}$, and since every norm, in particular the $\|\cdot\|_1$-norm, is homogeneous, every $\tilde{\Delta}$ is isometric (by point symmetry at one of the vertexes and translations) to a shrunken standard $(m-1)$-simplex Δ. We will prove the claim for the simplex Δ with vertexes $v_i = \frac{1}{n} \cdot e_i, 1 \leq i \leq m$. Denote by $V = \{v_1, \ldots, v_m\}$ the set of vertexes.
The center of Δ is

$$M = \left(\frac{1}{mn}, \frac{1}{mn}, \ldots, \frac{1}{mn} \right)$$

and

$$\operatorname{dist}(V, M) = \operatorname{dist}(v_i, M) = \frac{2(m-1)}{nm}, \quad \forall 1 \leq i \leq m.$$

But M is the point of Δ furthest away from V: Take a point $p = M + \sum_{i=1}^m r_i' e_i \in \Delta, \sum_{i=1}^m r_i' = 0$ and assume w. l. o. g. that $r_1' > 0$. Then

$$\begin{aligned}
\operatorname{dist}(p, V) \leq \operatorname{dist}(p, v_1) &= \left| r_1' + \frac{1}{mn} - \frac{1}{n} \right| + \sum_{i=2}^m \left(\frac{1}{mn} + r_i' \right) \\
&= -r_1' + \frac{m-1}{mn} + \sum_{i=2}^m \left(\frac{1}{mn} + r_i' \right) = -2r_1' + \operatorname{dist}(M, V) \\
\Rightarrow \operatorname{dist}(p, V) < \operatorname{dist}(M, V),
\end{aligned}$$

where we used $p \in \Delta$ and $m > 0$ to get rid of the norm.
Hence we will always find a $y^* = M$ with M the center of the to x corresponding simplex and $\operatorname{dist}(x, y^*) \leq \frac{2(m-1)}{nm} \leq \frac{m}{2n}$ for $m \geq 1$. For $m > 2$ the last inequality is strict.[1]

Remark: We get the claimed estimate directly by expanding our lattice to $\mathbb{R}^m$. Then the center of a lattice $m-$cube has the claimed distance from the lattice. Of course than we have to check, that the minimal (or any small enough) distance is already obtained on Y.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

[1] Multiply both sides by $2mn$: $m^2 - 4m + 4 = (m-2)^2 \geq 0$.