Junior problems

J295. Let a, b, c be positive integers such that $(a-b)^2 + (b-c)^2 + (c-a)^2 = 6abc$. Prove that $a^3 + b^3 + c^3 + 1$ is not divisible by $a + b + c + 1$.

Proposed by Mihaly Bencze, Brasov, Romania

J296. Several positive integers are written on a board. At each step, we can pick any two numbers u and v, where $u \geq v$, and replace them with $u + v$ and $u - v$. Prove that after a finite number of steps we can never obtain the initial set of numbers.

Proposed by Marius Cavachi, Constanta, Romania

J297. Let a, b, c be digits in base $x \geq 4$. Prove that

$$\frac{\overline{ab}}{\overline{ba}} + \frac{\overline{bc}}{\overline{cb}} + \frac{\overline{ca}}{\overline{ac}} \geq 3,$$

where all numbers are written in base x.

Proposed by Titu Zvonaru, Comanesti and Neculai Stanciu, Buzau, Romania

J298. Consider a right angle $\angle BAC$ and circles $\omega_1, \omega_2, \omega_3, \omega_4$ passing through A. The centers of circles ω_1 and ω_2 lie on ray AB and the centers of circles ω_3 and ω_4 lie on ray AC. Prove that the four points of intersection, other than A, of the four circles are concyclic.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

J299. Prove that no matter how we choose n numbers from the set $\{1, 2, \ldots, 2n\}$, one of them will be a square-free integer.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J300. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{\sqrt{2a^2 + 16ab + 7b^2} + c} + \frac{c+a}{\sqrt{2b^2 + 16bc + 7c^2} + b} + \frac{a+b}{\sqrt{2c^2 + 16ca + 7a^2} + b} \geq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior problems

S295. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\sum_{cyc} \frac{(a+\sqrt{b})^2}{\sqrt{a^2-ab+b^2}} \leq 12.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S296. A ball in Vienna is attended by n ladies (some of which are wearing red dresses) and m gentlemen. Some ladies and some gentlemen are acquainted. Dancing floor is occupied by acquainted mixed pairs. At some point during the night, all the present gentlemen were seen on the dancing floor. At some other time, all the ladies wearing red dresses were on the dancing floor. Show that at some point there could be all gentlemen and all red-dressed ladies on the dancing floor.

Proposed by Michal Rolinek, Institute of Science and Technology, Vienna

S297. Let ABC be a triangle and let A_1, A_2, B_1, B_2, C_1, C_2 be points that trisect segments BC, CA, AB, respectively. Cevians $AA_1, AA_2, BB_1, BB_2, CC_1, CC_2$ intersect each other at the vertices of a convex hexagon that does not have any intersection points inside it. Prove that if the hexagon is cyclic then our triangle is equilateral.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S298. Prove the following identity

$$\sum_{k=0}^{a}(-1)^{a-k}\binom{a}{k}\binom{b+k}{c} = \binom{b}{c-a}.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

S299. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let P an arbitrary point in its plane. If $\{E\} = PD \cap AC$, $\{F\} = PC \cap BD$, $\{M\} = PD \cap AF$ and $\{N\} = PC \cap BE$, prove that $MN \parallel AB$.

Proposed by Mihai Miculita and Marius Stanean, Romania

S300. Let x, y, z be positive numbers and $a, b > 0$ such that $a+b=1$. Prove that

$$(x+y)^3(y+z)^3 \geq 64abxy^2z\,(ax+y+bz)^2.$$

Proposed by Marius Stanean, Zalau, Romania

Undergraduate problems

U295. Let a be a real number such that $(\lfloor na \rfloor)_{n\geq 1}$ is an arithmetic sequence. Prove that a is an integer.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

U296. Let a and b be real nonzero numbers and let $z_0 \in \mathbb{C}\backslash\mathbb{R}$ be a root to the equation $z^{n+1} + az + nb = 0$, where n is a positive integer. Prove that $|z_0| \geq \sqrt[n+1]{b}$.

Proposed by Mihaly Bencze, Brasov, Romania

U297. Let $a_0 = 0, a_1 = 2$, and $a_{n+1} = \sqrt{2 - \frac{a_{n-1}}{a_n}}$ for $n \geq 0$. Find $\lim_{n\to\infty} 2^n a_n$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U298. Determine all pairs (m, n) of positive integers such that the polynomial

$$f = (X + Y)^2(mXY + n) + 1$$

is irreducible in $\mathbb{Z}[X, Y]$.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

U299. Let ABC be a triangle with incircle ω and let A_0, B_0, C_0 be points outside ω. Tangents from A_0 to ω intersect BC at A_1 and A_2. Points B_1, B_2 and C_1, C_2 are defined similarly. Prove that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a conic if and only if triangle ABC and $A_0B_0C_0$ are perspective.

Proposed by Luis Gonzalez, Maracaibo, Venezuela

U300. Let $f : [a, b] \to [a, b]$ be a function having lateral limits in every point. If

$$\lim_{t\to x^-} f(t) \leq \lim_{t\to x^+} f(t)$$

for all $x \in [a, b]$, prove that there is an $x_0 \in [a, b]$ such that $\lim_{t\to x_0} f(t) = x_0$.

Proposed by Dan Marinescu and Mihai Piticari, Romania

Olympiad problems

O295. Let a, b, c, x, y, z be positive real numbers such that $x+y+z=1$ and

$$2ab+2bc+2ca > a^2+b^2+c^2.$$

Prove that

$$a\,(x+3yz)+b(y+3xz)+c\,(z+3xy) \leq \frac{2}{3}(a+b+c).$$

Proposed by Arkady Alt, San Jose, California, USA

O296. Let m be a positive integer. Prove that $\phi(n)$ divides mn, only for finitely many square-free integers n, where ϕ is Euler's totient function.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O297. Cells of an 11×11 square are colored in n colors. It is known that the number of cells of each color is greater than 6 and less than 14. Prove that one can find a row and a column whose cells are colored in at least four different colors.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O298. Let n be a square-free positive integer. Find the number of functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $f(1)f(2)\cdots f(n)$ divides n.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

O299. Let a, b, c be positive real integers such that $a^2+b^2+c^2+abc=4$.

$$\sqrt{1-abc}(3-a-b-c) \geq |(a-1)(b-1)(c-1)|.$$

Proposed by Marius Stanean, Zalau, Romania

O300. Let ABC be a triangle with circumcircle Γ and incircle ω. Let D, E, F be the tangency points of ω with BC, CA, AB, respectively, let Q be the second intersection of AD with Γ, and let the T be the intersection of the tangents at B and C with respect to Γ. Furthermore, let QT intersect Γ for the second time at R. Prove that AR, EF, BC are concurrent.

Proposed by Faraz Masroor, Gulliver Preparatory, Florida, USA