## Junior problems

J385. If the equalities

$$2(a+b)-6c-3(d+e)=6$$
$$3(a+b)-2c+6(d+e)=2$$
$$6(a+b)+3c-2(d+e)=-3$$

hold simultaneously, evaluate $a^2-b^2+c^2-d^2+e^2$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia*

Given equations are equivalent to the following equations:

$$\left.\begin{aligned} 2(a+b)-6c-3(d+e)&=6\\ 3(a+b)-2c+6(d+e)&=2\\ 6(a+b)+3c-2(d+e)&=-3 \end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} 2(a+b)-6c-3(d+e)&=6\\ 7c+10.5(d+e)&=-7\\ -22.5(d+e)&=0 \end{aligned}\right\}$$

Hence we get,

$$\begin{aligned} d+e=0,\ c=-1,\ a+b=0 &\Leftrightarrow a^2=b^2,\ d^2=e^2,\ c^2=1\\ &\Leftrightarrow a^2-b^2+c^2-d^2+e^2=1. \end{aligned}$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Alessandro Ventullo, Milan, Italy; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; Prithwijit De, HBCSE, Mumbai, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, New Delhi, India; Andrianna Boutsikou, High School Of Nea Makri, Athens, Greece; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; David E. Manes, Oneonta, NY, USA; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Orgilerdene Erdenebaatar, National University of Mongolia, Mongolia; Jennifer Johannes, College at Brockport, SUNY, NY, USA; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Robert Bosch, USA and Jorge Erick, Brazil; Tamoghno Kandar; Polyahedra, Polk State College, FL, USA; Dolsan Zheksheev, Karakol High School, Kyrgyzstan; Bekjol Joldubai, Kadamcay High School, Kyrgyzstan; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA; A.S.Arun Srinivaas, Chennai, India; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA.*

J386. Find all real solutions to the system of equations

$$x + yzt = y + ztx = z + txy = t + xyz = 2.$$

*Proposed by Mohamad Kouroshi, Tehran, Iran*

*Solution by Alessandro Ventullo, Milan, Italy*

Subtracting the second equation to the first equation, the third equation to the second equation, the fourth equation to the third equation and the first equation to the fourth equation, we obtain

$$\begin{aligned} (x-y)(1-zt) &= 0 \\ (y-z)(1-tx) &= 0 \\ (z-t)(1-xy) &= 0 \\ (t-x)(1-yz) &= 0. \end{aligned}$$

We have four cases.

**(i)** $x - y = 0$ and $z - t = 0$, i.e. $x = y$ and $z = t$. Substituting these values into the second and the fourth equation, we get $(x - z)(1 - zx) = 0$. If $x = z$, then $x + x^3 = 2$, which gives $x = y = z = t = 1$. If $zx = 1$, then $y + t = 2$, i.e. $x + z = 2$, which gives $x = 1, z = 1$, so $x = y = z = t = 1$.

**(ii)** $x - y = 0$ and $1 - xy = 0$, i.e. $x = y$ and $xy = 1$. We obtain $x = y = \pm 1$. If $x = y = 1$, then $zt = 1$ and $z + t = 2$, which gives $z = t = 1$. If $x = y = -1$, then $zt = -3$ and $z + t = 2$, which gives $z = 3, t = -1$ or $z = -1, t = 3$.

**(iii)** $1 - zt = 0$ and $z - t = 0$, i.e. $z = t$ and $zt = 1$. We obtain $z = t = \pm 1$. If $z = t = 1$, then $xy = 1$ and $x + y = 2$, which gives $x = y = 1$. If $z = t = -1$, then $xy = -3$ and $x + y = 2$, which gives $x = 3, y = -1$ or $x = -1, y = 3$.

**(iv)** $1 - zt = 0$ and $1 - xy = 0$, i.e. $zt = 1$ and $xy = 1$. We obtain $x + y = 2$ and $z + t = 2$, which gives $x = y = z = t = 1$.

In conclusion, the real solutions to the given system of equations are

$$(x, y, z, t) \in \{(1, 1, 1, 1), (1, 1, 3, -1), (1, 1, -1, 3), (3, -1, 1, 1), (-1, 3, 1, 1)\}.$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Dolsan Zheksheev, Karakol High School, Kyrgyzstan; Bekjol Joldubai, Kadamcay High School, Kyrgyzstan; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; David E. Manes, Oneonta, NY, USA; A.S.Arun Srinivaas, Chennai, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Andrianna Boutsikou, High School Of Nea Makri, Athens, Greece; Erdenebayar Bayarmagnai, National University of Mongolia, Mongolia; Robert Bosch, USA; Albert Stadler, Herrliberg, Switzerland; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA.*

J387. Find all digits $a, b, c, x, y, z$ for which $\overline{abc}$, $\overline{xyz}$, and $\overline{abcxyz}$ are all perfect squares (no leading zeros allowed).

*Proposed by Adrian Andreescu, Dallas, Texas*

*Solution by Polyahedra, Polk State College, USA*

Let $m^2 = \overline{abc}$, $n^2 = \overline{xyz}$, and $k^2 = \overline{abcxyz}$. Then $1000m^2 + n^2 = k^2$, with $10 \le m, n \le 31$ and $317 \le k \le 999$. There are three cases.

Case I: Suppose that $n$ and $k$ are not divisible by 5. Since $5^3 | (k^2 - n^2)$, we must have either $5^3 | (k - n)$ or $5^3 | (k + n)$. Thus $k = 125q \pm n$, where $3 \le q \le 8$. Then $8m^2 = 125q^2 \pm 2qn$, so $q$ must be even, that is, $q \in \{4, 6, 8\}$.

If $q = 4$, then $m^2 = 250 \pm n$. But $250 + n \in [261, 281]$ which contains no perfect square; and $250 - n \in [219, 239] \setminus \{225\}$ which contains no perfect square either.

If $q = 6$, then $2m^2 = 125 \cdot 9 \pm 3n$. So $m = 3b$, $n = 3a$, and $2b^2 = 125 \pm a$. But $\frac{125+a}{2} \in [65, 67]$ which contains no perfect square; and $\frac{125-a}{2} \in [58, 60]$ which contains no perfect square either.

If $q = 8$, then $m^2 = 1000 - 2n$. So $m = 2d$, $n = 2c$, and $d^2 = 250 - c \in [236, 244]$ which contains no perfect square.

Case II: Suppose that $5|n$ but 25 does not divide $n$. Then $5|k$ but 25 does not divide $k$. Write $n = 5a$ and $k = 5b$, with $a \in \{2, 3, 4, 6\}$. Then $40m^2 + a^2 = b^2$.

If $a = 2$, then $b = 2c$ and $c^2 - 10m^2 = 1$. This Pell's equation has fundamental solution $(c_1, m_1) = (19, 6)$ and all solutions $(c_i, m_i)$ given by $c_i + m_i\sqrt{10} = (19 + 6\sqrt{10})^i$. Hence no such $m_i \in [10, 31]$.

If $a = 3$, then $b^2 - 10(2m)^2 = 9$. This Pell's equation has all solutions generated by three distinct sets of fundamental solutions $(b, 2m) = (7, 2)$, $(13, 4)$, and $(57, 18)$. Since $(7 + 2\sqrt{10})(19 + 6\sqrt{10}) = 253 + 80\sqrt{10}$, we have either $2m \le 18$ or $2m \ge 80$, thus no solution $m \in [10, 31]$.

If $a = 4$, then $b = 4d$ and $m = 2e$. So $d^2 - 10e^2 = 1$, thus $(d, e) = (19, 6)$, and $(m, n, k) = (12, 20, 380)$.

If $a = 6$, then $b = 2f$ and $f^2 - 10m^2 = 9$. So $(f, m) = (57, 18)$ and $(m, n, k) = (18, 30, 570)$.

Case III: Finally, consider $25|n$. Then $n = 25$, $k = 25a$, and $m = 5b$. So $a^2 - 10(2b)^2 = 1$. Hence $(a, 2b) = (19, 6)$ and $(m, n, k) = (15, 25, 475)$.

In conclusion, there are three solutions: 144400, 324900, and 225625.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Alessandro Ventullo, Milan, Italy; David E. Manes, Oneonta, NY, USA; Albert Stadler, Herrliberg, Switzerland; Joel Schlosberg, Bayside, NY, USA; Robert Bosch, USA.*

J388. Let $ABCD$ be a cyclic quadrilateral with $AB = AD$. Points $M$ and $N$ are taken on sides $CD$ and $BC$, respectively, such that $DM + BN = MN$. Prove that the circumcenter of triangle $AMN$ lies on segment $AC$.

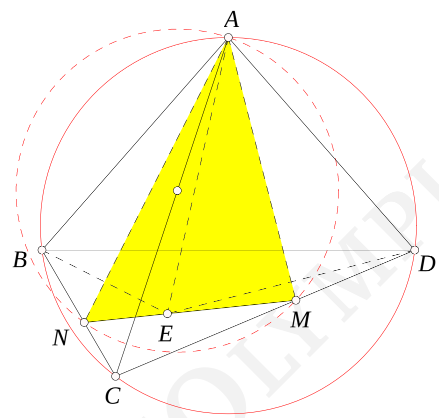
*Proposed by Hayk Sedrakyan, Paris, France*

*Solution by Polyahedra, Polk State College, USA*

As in the figure, locate $E$ on $MN$ such that $EM = MD$. Then $BN = NE$. So $\angle ABE + \angle ADE = \angle AEB + \angle AED$. If $\angle ABE > \angle AEB$, then $\angle ADE < \angle AED$, which imply that $AB < AE < AD$, a contradiction. Likewise, we cannot have $\angle ABE < \angle AEB$. Hence $AB = AE = AD$, thus $\triangle ABN \cong \triangle AEN$ and $\triangle AEM \cong \triangle ADM$. Therefore,

$$\begin{aligned} \angle ANM &= \angle ANB = \pi - \angle ABN - \angle BAN = \pi - \angle ABD - \angle CAD - \angle BAN \\ &= \frac{\pi}{2} + \frac{1}{2}\angle BAD - \angle CAM - \angle MAD - \angle BAN = \frac{\pi}{2} - \angle CAM, \end{aligned}$$

from which the conclusion follows.




*Also solved by Daniel Lasaosa, Pamplona, Spain; Dolsan Zheksheev, Karakol High School, Kyrgyzstan; Bekjol Joldubai, Kadamcay High School, Kyrgyzstan; Andrianna Boutsikou, High School Of Nea Makri, Athens, Greece; Erdenebayar Bayarmagnai,National University of Mongolia, Mongolia; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Robert Bosch, USA.*

J389. Solve in real numbers the system of equations

$$\left(x^2-y+1\right)\left(y^2-x+1\right)=2\left[\left(x^2-y\right)^2+\left(y^2-x\right)^2\right]=4.$$

*Proposed by Alessandro Ventullo, Milan, Italy*

*Solution by Robert Bosch, USA*

Let $x^2-y=a, y^2-x=b$. The system becomes

$$\begin{aligned} a^2+b^2 &= 2, \\ ab+a+b &= 3. \end{aligned}$$

After multiply by 2 the second equation and adding we obtain $(a+b)^2+2(a+b)-8=0$. So $a+b=-4$ or $a+b=2$. In the first case $a^2+\dfrac{49}{a^2}=2$. This equation is equivalent to $a^4-2a^2+49=0$, that can be considered a quadratic on $a^2$, its discriminant is negative, thus there are not real solutions. Now, if $a+b=2$, the equation to be solved is $a^2+\dfrac{1}{a^2}=2$, or $(a^2-1)^2=0$, so $a=\pm 1$. Hence the pairs $(a,b)$ to consider are $(1,1)$ and $(-1,-1)$. Anyways, $x^2-y=y^2-x$, or $(x-y)(x+y+1)=0$, so $x=y$ or $x+y=-1$. If $x=y$, the resulting equation is $x^2-x-1=0$, so $x=y=\dfrac{1\pm\sqrt{5}}{2}$. If $x+y=-1$, and $a=b=1$, we obtain $x^2+y^2=1$, and after plugging the value $y=-x-1$, the quadratic $2x^2+2x=0$, and $x=0, y=-1$ or $x=-1, y=0$. Finally if $a=b=-1$, then $x^2+y^2=-3<0$, impossible for real numbers. The solutions $(x,y)$ to the original system are

$$\left(\frac{1\pm\sqrt{5}}{2},\frac{1\pm\sqrt{5}}{2}\right),$$
$$(0,-1),$$
$$(-1,0).$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Bekjol Joldubai, Kadamcay High School, Kyrgyzstan; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA; A.S.Arun Srinivaas, Chennai, India; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; Prithwijit De, HBCSE, Mumbai, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; David E. Manes, Oneonta, NY, USA; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Erdenebayar Bayarmagnai, National University of Mongolia, Mongolia; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Vincelot Ravoson, Lycée Henri IV, France, Paris; Wada Ali, Ben Badis College, Algeria.*

J390. Let $ABC$ be a triangle. Points $D, D'$ lie on side $BC$, points $E, E'$ lie on side $AC$ and points $F, F'$ lie on side $AB$ such that $AD = AD' = BE = BE' = CF = CF'$. Prove that if $AD, BE, CF$ are concurrent, then so are $AD', BE', CF'$.

*Proposed by Josef Tkadlec, Vienna, Austria*

*Solution by Robert Bosch, USA and Jorge Erick, Brazil*

Let $P$ be the feet of altitude from $A$, then $PD = PD'$ and

$$\begin{aligned} BD \cdot BD' &= (BP - PD)(BP + PD), \\ &= BP^2 - PD^2, \\ &= BP^2 + AP^2 - AD^2, \\ &= AB^2 - AD^2, \\ &= AB^2 - BE^2. \end{aligned}$$

By the same argument $AB^2 - BE^2 = AE \cdot AE'$. In a similar way we obtain

$$\begin{aligned} CE \cdot CE' &= BF \cdot BF', \\ AF \cdot AF' &= CD \cdot CD'. \end{aligned}$$

Therefore

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = \frac{BD \cdot BD'}{AE \cdot AE'} \cdot \frac{CE \cdot CE'}{BF \cdot BF'} \cdot \frac{AF \cdot AF'}{CD \cdot CD'} = 1.$$

The conclusion is

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \Leftrightarrow \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = 1.$$

Thus by Ceva's theorem the result follows.

*Also solved by Daniel Lasaosa, Pamplona, Spain.*

## Senior problems

S385. Let $a, b, c$ be positive real numbers. Prove that

$$\frac{1}{a^3+8abc}+\frac{1}{b^3+8abc}+\frac{1}{c^3+8abc}\le\frac{1}{3abc}.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*Solution by Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece*
Since $abc>0$ the given inequality is equivalent to

$$\frac{abc}{a^3+8abc}+\frac{abc}{b^3+8abc}+\frac{abc}{c^3+8abc}\le\frac{1}{3}.$$

Then, we can rewrite the inequality as follows:

$$\frac{1}{\frac{a^3+8abc}{abc}}+\frac{1}{\frac{b^3+8abc}{abc}}+\frac{1}{\frac{c^3+8abc}{abc}}\le\frac{1}{3}\Leftrightarrow\frac{1}{\frac{a^2}{bc}+8}+\frac{1}{\frac{b^2}{ca}+8}+\frac{1}{\frac{c^2}{ab}+8}\le\frac{1}{3}.$$

Now, we set $\frac{a^2}{bc}=x, \frac{b^2}{ca}=y, \frac{c^2}{ab}=z$ with $xyz=1$.
Therefore, it suffices to prove that

$$\frac{1}{x+8}+\frac{1}{y+8}+\frac{1}{z+8}\le\frac{1}{3}.$$

Doing the maths we transform the above mentioned inequality to

$$3[(xy+yz+zx)+16(x+y+z)+192]\le xyz+8(xy+yz+zx)+64(x+y+z)+512$$

or equivalently because $xyz=1$,

$$5(xy+yz+zx)+16(x+y+z)\ge 63(1).$$

By AM-GM inequality, we obtain that $xy+yz+zx\ge 3\sqrt[3]{(xyz^2)}=3$ and $x+y+z\ge 3\sqrt[3]{xyz}=3$. Hence, (1) holds.
Equality holds if and only if $x=y=z=1\Leftrightarrow a=b=c$.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA; A.S.Arun Srinivaas, Chennai, India; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Alessandro Ventullo, Milan, Italy; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; Arkady Alt, San Jose, CA, USA; Henry Ricardo, New York Math Circle, NY, USA; Prithwijit De, HBCSE, Mumbai, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Andrianna Boutsikou, High School Of Nea Makri, Athens, Greece; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; David E. Manes, Oneonta, NY, USA; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Erdenebayar Bayarmagnai, National University of Mongolia, Mongolia; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Robert Bosch, USA.*

S386. Evaluate

$$\frac{\cos\frac{\pi}{4}}{2}+\frac{\cos\frac{2\pi}{4}}{2^2}+\cdots+\frac{\cos\frac{n\pi}{4}}{2^n}$$

*Proposed by Mohamad Kouroshi, Tehran, Iran*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Let $\rho=e^{\frac{i\pi}{4}}$ and let $\Re\{z\}$ denote the real part of complex number $z$. Clearly, $\cos\frac{k\pi}{4}=\Re\left\{\rho^k\right\}$, or the sum that we need to evaluate rewrites as

$$\sum_{k=1}^{n}\Re\left\{\frac{\rho^k}{2^k}\right\}=\Re\left\{\frac{\frac{\rho}{2}-\frac{\rho^n}{2^n}}{1-\frac{\rho}{2}}\right\}=\Re\left\{\frac{\rho\left(2-\rho^*\right)-\frac{\rho^n\left(2-\rho^*\right)}{2^{n-1}}}{\left(2-\rho\right)\left(2-\rho^*\right)}\right\}.$$

Now,

$$\left(2-\rho\right)\left(2-\rho^*\right)=4-2\left(\rho+\rho^*\right)+|\rho|^2=4-4\cos\frac{\pi}{4}+1=5-2\sqrt{2},$$

$$\Re\left\{\rho\left(2-\rho^*\right)\right\}=2\Re\{\rho\}-|\rho|^2=\sqrt{2}-1,$$

$$\Re\left\{\rho^n\left(2-\rho^*\right)\right\}=2\Re\left\{\rho^n\right\}-|\rho|^2\Re\left\{\rho^{n-1}\right\}=2\cos\frac{n\pi}{4}-\cos\frac{(n-1)\pi}{4},$$

or the proposed sum equals

$$\frac{\sqrt{2}-1-\frac{1}{2^{n-2}}\cos\frac{n\pi}{4}+\frac{1}{2^{n-1}}\cos\frac{(n-1)\pi}{4}}{5-2\sqrt{2}}.$$

*Also solved by Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA.*

S387. Find all nonnegative real numbers $k$ such that

$$\sum a(a-b)(a-kb) \geq 0$$

for all nonnegative numbers $a, b, c$.

*Proposed by Mehtaab Sawhney, Commack High School, New York, USA*

*Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India*

We show that $k = k_0$ is the maximum possible value of $k$, where $k_0 = \frac{-1+(3+\sqrt{2})(\sqrt{2\sqrt{2}-1})}{2}$. Assume that the inequality holds for all nonnegative reals $a, b, c$ for $k \leq k_0$. Then let $c = 0$ and W.L.O.G. let $a > b$. Then the inequality is $a^3 + b^3 - a^2b \geq kab(a-b)$. Let $x = a/b > 1$, then the inequality is same as $k \leq x + \dfrac{1}{x(x-1)}$. Let $f(x) = x + \dfrac{1}{x(x-1)}$, then by routine calculus, we get $f_{\min} = \frac{-1+(3+\sqrt{2})(\sqrt{2\sqrt{2}-1})}{2}$ for $x = \frac{(\sqrt{2}+1)+\sqrt{2\sqrt{2}-1}}{2}$. Hence $k \leq f_{\min} = \frac{-1+(3+\sqrt{2})(\sqrt{2\sqrt{2}-1})}{2} = k_0$.

Now to complete the proof we note that the inequality rearranges to

$$(a^3+b^3+c^3) + k(ab^2+bc^2+ca^2) \geq (k+1)(a^2b+b^2c+c^2a)$$

$$\Leftrightarrow (b+c)(b-c)^2 + (c+a)(c-a)^2 + (a+b)(a-b)^2 \geq (2k+1)(a-b)(b-c)(a-c) \tag{1}$$

Denote the left and right hand sides of (1) by $\mathcal{L}(a,b,c)$ and $\mathcal{R}(a,b,c)$ respectively. W.L.O.G assume that $a \geq b \geq c$ and let $a_1 = a - c \geq 0$, $b_1 = b - c \geq 0$, $c_1 = 0$, then $\mathcal{L}(a,b,c) \geq \mathcal{L}(a_1,b_1,c_1)$ and $\mathcal{R}(a,b,c) = \mathcal{R}(a_1,b_1,c_1)$. Thus, to prove (1), it suffices to prove that

$$\mathcal{L}(a_1,b_1,c_1) \geq \mathcal{R}(a_1,b_1,c_1),$$

which is the same as $a_1^3 + b_1^3 - a_1^2 b_1 \geq k a_1 b_1 (a_1 - b_1)$. But we have already proved this at the start. Thus, the inequality holds true for all $a, b, c \geq 0$ for all $k \leq k_0$, where $k_0 = \frac{-1+(3+\sqrt{2})(\sqrt{2\sqrt{2}-1})}{2}$.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Robert Bosch, USA and Jorge Erick, Brazil.*

S388. Let $a, b, c$ be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{11a-6}{c}+\frac{11b-6}{a}+\frac{1ac-6}{b}\leq\frac{15}{abc}.$$

*Proposed by Marius Stănean, Zalău, România*

*Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*
The given inequality may be written equivalently as

$$\sum_{cyclic} ab(11a-6)\leq 15.$$

by the rearrangement inequality

$$\sum_{cyclic} ab(11a-6)\leq\sum_{cyclic} a^2(11a-6)=\sum_{cyclic}(11a^3-6a^2)=\sum_{cyclic}\left(11(a^2)^{3/2}-6(a^2)\right).$$

Since function $11x^{3/2}-6x$ is convex, then

$$\sum_{cyclic}\left(11(a^2)^{3/2}-6(a^2)\right)\leq 3\left(11\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2}-6\left(\frac{a^2+b^2+c^2}{3}\right)\right)=15.$$

*Also solved by Arkady Alt, San Jose, CA, USA.*

S389. Let $n$ be a positive integer. Prove that for any integers $a_1, a_2, \ldots, a_{2n+1}$ there is a rearrangement $b_1, b_2, \ldots, b_{2n+1}$ such that $2^n n!$ divides

$$(b_1 - b_2)(b_3 - b_4) \cdots (b_{2n-1} - b_{2n}).$$

*Proposed by Cristinel Mortici, Valahia University, Târgovişte, România*

*Solution by Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece*

Let $S = \{a_1, a_2, ..., a_{2n+1}\}$ be the set of the statement. Then, it is obvious that $S = \{b_1, b_2, ..., b_{2n+1}\}$ because $b_1, b_2, ..., b_{2n+1}$ is a rearrangement of the elements $a_1, a_2, ..., a_{2n+1}$.

Define a sequence of sets $S_{2k+1} = \{b_1, b_2, ..., b_{2k+1}\}$ for $1 \leq k \leq n$ and $n \in \mathbb{N}$.

Observe that there are $2k$ different residues $\mod (2k)$. Since $S_{2k+1}$ has $2k+1$ elements, applying the Pigeonhole Principle, we deduce that there are at least 2 elements of this set , say $b_{2k-1}, b_{2k}$, that have the same residue $mod(2k)$.

Then, $2k \mid (b_{2k-1} - b_{2k})$ $(*)$.

Hence, by $(*)$ we obtain the following relations

$$2n \mid (b_{2n-1} - b_{2n}) \quad \text{for} \quad k = n$$

$$2(n-1) \mid (b_{2n-3} - b_{2n-2}) \quad \text{for} \quad k = n-1$$

$$2(n-2) \mid (b_{2n-5} - b_{2n-4}) \quad \text{for} \quad k = n-2$$

$$\cdots\cdots\cdots$$

$$2 \cdot 2 \mid (b_{3-1} - b_4) \quad \text{for} \quad k = 2$$

$$2 \mid (b_1 - b_2) \quad \text{for} \quad k = 1.$$

Finally, we can conclude that

$$2 \cdot (2 \cdot 2) \cdots [2(n-2)] \cdot [2(n-1)] \cdot (2n) \mid (b_1 - b_2)(b_3 - b_4) \cdots (b_{2n-1} - b_{2n})$$

which is obviously equivalent to

$$2^n n! \mid (b_1 - b_2)(b_3 - b_4) \cdots (b_{2n-1} - b_{2n})$$

as we desired.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Bekjol Joldubai, Kadamcay High School, Kyrgyzstan; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Bekjol Joldubai, Kadamcay High School, Kyrgyzstan; Joel Schlosberg, Bayside, NY, USA; Alessandro Ventullo, Milan, Italy; Robert Bosch, USA and Jorge Erick, Brazil.*

S390. Let $ABC$ be a triangle and $G$ be its centroid. Lines $AG, BG, CG$ meet the circumcircle of triangle $ABC$ at $A_1, B_1, C_1$, respectively. Prove that

$$\sqrt{a^2+b^2+c^2} \le GA_1 + GB_1 + GC_1 \le 2R + \frac{1}{6}\left(\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c}\right).$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia*

Let $AA_1 \cap BC = A_2$, $BB_1 \cap CA = B_2$, $CC_1 \cap AC = C_2$. By the Power of the Pointing theorem, we get:

$$AA_2 \cdot A_2A_1 = BA_2 \cdot A_2C \Leftrightarrow m_a(GA - \frac{1}{3}m_a) = \frac{a^2}{4}$$
$$\Leftrightarrow GA_1 = \frac{1}{3}m_a + \frac{a^2}{4m_a} = \frac{4m_a^2 + a^2}{12m_a} + \frac{1}{6}\cdot\frac{a^2}{m_a}.$$

Similarly, we get

$$GB_1 = \frac{1}{3}m_b + \frac{b^2}{4m_b}, \quad GC_1 = \frac{1}{3}m_c + \frac{c^2}{4m_c}.$$

Hence we get

$$GA_1 + GB_1 + GC_1 = \frac{1}{12}\left(\frac{4m_a^2+a^2}{m_a} + \frac{4m_b^2+b^2}{m_b} + \frac{4m_c^2+c^2}{m_c}\right) + \frac{1}{6}\left(\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c}\right).$$

From the other hand,

$$2R \ge AA_1 = AA_2 + A_2A_1 = m_a + \frac{a^2}{4m_a} = \frac{4m_a^2 + a^2}{4m_a}$$

or

$$\frac{4m_a^2 + a^2}{m_a} \le 8R.$$

Similarly, we have

$$\frac{4m_b^2 + b^2}{m_b} \le 8R, \quad \frac{4m_c^2 + c^2}{m_c} \le 8R.$$

Thus we have,

$$GA_1 + GB_1 + GC_1 \le \frac{1}{12}(8R + 8R + 8R) + \frac{1}{6}\left(\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c}\right)$$
$$= 2R + \frac{1}{6}\left(\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c}\right).$$

Hence right side inequality is proved. Use the median length formula, we get

$$4m_a^2 = 2b^2 + 2c^2 - a^2, \quad 4m_b^2 = 2c^2 + 2a^2 - b^2, \quad 4m_c^2 = 2a^2 + 2b^2 - c^2 :$$

$$GA_1 + GB_1 + GC_1 = \frac{1}{12}\left(\frac{2b^2+2c^2}{m_a} + \frac{2c^2+2a^2}{m_b} + \frac{2a^2+2b^2}{m_c}\right) + \frac{1}{6}\left(\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c}\right)$$
$$= \frac{1}{6}(a^2+b^2+c^2)\left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}\right).$$

Applying AM-GM inequality three times and using following formula, we get

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2):$$

$$\begin{aligned}
GA_1+GB_1+GC_1 &\geq \frac{1}{6}(a^2+b^2+c^2)\cdot 3\cdot\sqrt[3]{\frac{1}{m_a}\cdot\frac{1}{m_b}\cdot\frac{1}{m_c}}\\
&\geq \frac{1}{2}(a^2+b^2+c^2)\cdot\frac{1}{\sqrt[3]{m_am_bm_c}}\\
&\geq \frac{1}{2}(a^2+b^2+c^2)\cdot\frac{3}{m_a+m_b+m_c}\\
&\geq \frac{1}{2}(a^2+b^2+c^2)\cdot\frac{3}{\sqrt{3(m_a^2+m_b^2+m_c^2)}}\\
&= \frac{3}{2}\cdot\frac{2}{3}\cdot\sqrt{a^2+b^2+c^2}=\sqrt{a^2+b^2+c^2}.
\end{aligned}$$

LHS is proved. Equality holds only when $a=b=c$.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Robert Bosch, USA; Nikos Kalapodis, Patras, Greece; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA.*

U385. Evaluate

$$\lim_{n\to\infty}\sqrt{n}\left(\sqrt{\frac{(n+1)^n}{n^{n-1}}}-\sqrt{\frac{n^{n-1}}{(n-1)^{n-2}}}\right).$$

*Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Note first that, since $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$, we then have

$$\ln\left(\frac{(n+1)^n}{n^{n-1}}\right)=\ln(n)+n\ln\left(1+\frac{1}{n}\right)=\ln(n)+1-\frac{1}{2n}+\frac{1}{3n^2}-\dots,$$

or

$$\frac{(n+1)^n}{n^{n-1}}=en\cdot\exp\left(-\frac{1}{2n}\right)\cdot\exp\left(\frac{1}{3n^2}\right)\dots=e\left(n-\frac{1}{2}+\frac{11}{24n}\right)+O\left(\frac{1}{n^2}\right),$$

where Landau notation has been used. Similarly,

$$\frac{n^{n-1}}{(n-1)^{n-2}}=e\left(n-\frac{3}{2}+\frac{11}{24(n-1)}\right)+O\left(\frac{1}{(n-1)^2}\right)=e\left(n-\frac{3}{2}+\frac{11}{24n}\right)+O\left(\frac{1}{n^2}\right).$$

Now,

$$n-\frac{1}{2}+\frac{11}{24n}+O\left(\frac{1}{n^2}\right)=\left(\sqrt{n}-\frac{1}{4\sqrt{n}}+O\left(\frac{1}{n\sqrt{n}}\right)\right)^2,$$

or

$$\sqrt{\frac{(n+1)^n}{n^{n-1}}}=\sqrt{en}-\frac{\sqrt{e}}{4\sqrt{n}}+O\left(\frac{1}{n\sqrt{n}}\right).$$

Similarly,

$$\sqrt{\frac{n^{n-1}}{(n-1)^{n-2}}}=\sqrt{en}-\frac{3\sqrt{e}}{4\sqrt{n}}+O\left(\frac{1}{n\sqrt{n}}\right),$$

or

$$\sqrt{n}\left(\sqrt{\frac{(n+1)^n}{n^{n-1}}}-\sqrt{\frac{n^{n-1}}{(n-1)^{n-2}}}\right)=\frac{\sqrt{e}}{2}+O\left(\frac{1}{n}\right),$$

whose limit when $n\to\infty$ is clearly $\frac{\sqrt{e}}{2}$, and we are done.

*Also solved by Arkady Alt, San Jose, CA, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Erdenebayar Bayarmagnai, National University of Mongolia, Mongolia; Moubinool Omarjee, Lycée Henri IV, Paris, France; Robert Bosch, USA.*

U386. Given a convex quadrilateral $ABCD$, denote by $S_A, S_B, S_C, S_D$ the area of triangles $BCD, CDA, DAB, ABC$, respectively. Determine the point $P$ in the plane of the quadrilateral such that

$$S_A \cdot \overrightarrow{PA} + S_B \cdot \overrightarrow{PB} + S_C \cdot \overrightarrow{PC} + S_D \cdot \overrightarrow{PD} = 0.$$

*Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Note first that the area of $ABCD$ is $S = S_A + S_C = S_B + S_D$. Note also that, denoting by $O$ the point where the diagonals $AC$ and $BD$ meet, and by $h_A, h_C$ the respective distances from $A, C$ to $BD$, we have that the sine of the angle formed by the diagonals is $\frac{h_A}{|OA|} = \frac{h_C}{|OC|}$, or $h_A \cdot \overrightarrow{OC} + h_C \cdot \overrightarrow{OA} = 0$ becase $\overrightarrow{OA}, \overrightarrow{OC}$ are vectors in opposite directions on line $AC$. Note finally that $2S_A = BD \cdot h_C$ and $2S_C = 2BD \cdot h_A$, or $S_C \cdot \overrightarrow{OC} + S_A \cdot \overrightarrow{OA} = 0$, hence for any point $P$ on the plane,

$$S_A \cdot \overrightarrow{PA} + S_C \cdot \overrightarrow{PC} = -S \cdot \overrightarrow{OP}.$$

Similarly,

$$S_B \cdot \overrightarrow{PB} + S_D \cdot \overrightarrow{PD} = -S \cdot \overrightarrow{OP},$$

or for any point $P$ in the plane, we have

$$S_A \cdot \overrightarrow{PA} + S_B \cdot \overrightarrow{PB} + S_C \cdot \overrightarrow{PC} + S_D \cdot \overrightarrow{PD} = -2S\overrightarrow{OP},$$

where $S > 0$ because $ABCD$ is convex, hence $P$ is the intersection of diagonals $AC$ and $BD$, and no other point in the plane may satisfy the condition given in the problem statement. The conclusion follows.

*Also solved by Adnan Ali, A.E.C.S-4, Mumbai, India.*

U387. A polynomial with complex coefficients is called special if all its roots lie on the unit circle. Is any complex polynomial the sum of two special polynomials?

*Proposed by Gabriel Dospinescu, Ecole Normale Supérieure, Lyon, France*

*Solution by Robert Bosch, USA*

The answer is yes. We understand (on the unit circle) as inside or on the unit disk. The idea is to use Rouché's theorem. Let's see, let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

this polynomial can be written as

$$\begin{aligned} p(z) &= (az^{n+1}) + (-az^{n+1} + a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0), \\ &= (f(z)) + (g(z)), \end{aligned}$$

where

$$|a| > |a_n| + |a_{n-1}| + \cdots + |a_0|.$$

Clearly the polynomial $f(z)$ has all its roots inside the unit disk, we shall prove $g(z)$ is special too. By Rouché's theorem it's enough to verify the following inequality

$$|p(z)| < |-f(z)|,$$

on the curve $\gamma : |z| = 1$, since

$$g(z) = p(z) + (-f(z)).$$

Finally

$$|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| < |az^{n+1}|,$$

is true by Triangle Inequality and by the condition imposed on the constant $a$.

*Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India.*

U388. Evaluate

$$\sum_{n=1}^{\infty} \frac{\sin^{2n}\theta + \cos^{2n}\theta}{n^2}.$$

*Proposed by Li Zhou, Polk State College, USA*

*Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia*

1) Let $\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$ or $\theta = \pi m, m \in \mathbb{Z}$. Then we have

$$\sum_{n=1}^{\infty} \frac{\sin^{2n}\theta + \cos^{2n}\theta}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2) Let $\theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$ and $\theta \neq \pi m, m \in \mathbb{Z}$.
Substitute that $\sin^2\theta = t$, then we have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sin^{2n}\theta + \cos^{2n}\theta}{n^2} &= \sum_{n=1}^{\infty} \frac{t^n + (1-t)^n}{n^2} \\
&= -\left(\int_0^t \frac{\ln(1-z)}{z} dz + \int_0^{1-t} \frac{\ln(1-z)}{z} dz\right) \\
&= Li_2(t) + Li_2(1-t) = \frac{\pi^2}{6} - \ln t \ln(1-t).
\end{aligned}$$

Where $Li_2(t)$ is Dilogarithm function and

$$Li_2(t) + Li_2(1-t) = \frac{\pi^2}{6} - \ln t \ln(1-t)$$

equality is Landen's formula. Hence we have:

$$\sum_{n=1}^{\infty} \frac{\sin^{2n}\theta + \cos^{2n}\theta}{n^2} = \frac{\pi^2}{6} - \ln\left(\sin^2\theta\right)\ln\left(1-\sin^2\theta\right).$$

*Also solved by Arkady Alt, San Jose, CA, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Robert Bosch, USA and Jorge Erick, Brazil.*

U389. Let $P$ be a nonconstant polynomial whose zeros $x_1, x_2, ..., x_n$ are all real. Prove that

$$\exp\left(\int_a^b \frac{P'''(x)P(x)}{P'(x)}dx\right) < \left|\frac{P(a)^2P'(b)^3}{P'(a)^3P(b)^2}\right|,$$

whenever $a < b < \min(x_1, x_2, ..., x_n)$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by the author*

Use the identity

$$\frac{1}{x-x_1} + \frac{1}{x-x_2} + ... + \frac{1}{x-x_n} = \frac{P'(x)}{P(x)}.$$

Take the derivative:

$$\frac{P''(x)P(x) - P'(x)^2}{P(x)^2} = \sum -\frac{1}{(x-x_k)^2}$$

Take the derivative one more time:

$$\frac{(P'''(x)P(x) + P''(x)P'(x) - 2P'(x)P''(x))P(x)^2 - (P''(x)P(x) - P'(x)^2)2P(x)P'(x)}{P(x)^4} = \sum \frac{2}{(x-x_k)^3} < 0$$

for $a \le x \le b < \min(x_1, ..., x_n)$.

Then

$$P'''(x)P(x)^3 + P''(x)P'(x)P(x)^2 - 2P''(x)P'(x)P(x)^2 - 2P''(x)P'(x)P(x)^2 + 2P'(x)^3P(x) < 0,$$

which is equivalent to

$$P'''(x)P(x)^3 - 3P''(x)P'(x)P(x)^2 + 2P'(x)^3P(x) \le 0.$$

Dividing by $P(x)^2P'(x)^2$ yields

$$\frac{P'''(x)P(x)}{P'(x)^2} \le \frac{3P''(x)}{P'(x)} - \frac{2P'(x)}{P(x)}$$

.

By integration,

$$\int_a^b \frac{P'''(x)P(x)}{P'(x)^2} \le 3\left|\ln P'(b) - \ln P'(a)\right| - 2\left|\ln P(b) - \ln P(a)\right| = \frac{\ln P(a)^2P'(b)^3}{P'(a)^3P(b)^2},$$

hence

$$\exp\left(\int_a^b \frac{P'''(x)P(x)}{P'(x)^2}\right) \le \left|\frac{P(a)^2P'(b)^3}{P'(a)^3P(b)^2}\right|,$$

as desired.

U390. Prove that there is a unique representation of $\sin(\pi z)$ as a series of the form

$$\sin(\pi z) = \sum_{k=1}^{\infty} a_k z^k (1-z)^k$$

that converges for all complex numbers $z$, wherein the coefficients $a_k$ are real number satisfying $|a_k| \le c \cdot \frac{\pi^{2k}}{(2k)!}$ for some absolute constant $c$.

*Proposed by Albert Stadler, Herrliberg, Switzerland*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Note first that if such an expression exists, it must be unique. Indeed, let $j$ be the lowest value of $k$ such that $a_k$ is different in both expression, or the difference between both expressions would be a polynomial with nonzero coefficient for $z^j$, thus nonzero.

Define now $v = z(1-z)$, or

$$\sin(\pi z) = \sin\left(\pi\left(z-\frac{1}{2}\right)+\frac{\pi}{2}\right) = \cos\left(\pi\left(z-\frac{1}{2}\right)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}\left(z-\frac{1}{2}\right)^{2n}}{(2n)!} =$$

$$= \sum_{n=0}^{\infty} \frac{\pi^{2n}\left(v-\frac{1}{4}\right)^n}{(2n)!},$$

where we have used that $\left(z-\frac{1}{2}\right)^2 = z^2 - z + \frac{1}{4} = \frac{1}{4} - v$. Note therefore that the coefficient $a_k$ of $v^k = z^k(1-z)^k$ is

$$a_k = \sum_{n=k}^{\infty} \frac{\pi^{2n}}{(2n)!}\binom{n}{k}\left(-\frac{1}{4}\right)^{n-k} = \frac{\pi^{2k}}{(2k)!}\sum_{d=0}^{\infty}\frac{(-1)^d \pi^{2d}(2k)!}{4^d(2k+2d)!}\binom{k+d}{k} =$$

$$= \frac{\pi^{2k}}{(2k)!}\sum_{d=0}^{\infty}\frac{(-2)^d(2k-1)!!}{(2k+2d-1)!!}\frac{1}{d!}\left(\frac{\pi}{4}\right)^{2d},$$

where we have defined $d = n-k$, and $(2m-1)!! = (2m-1)(2m-3)\cdots 3\cdot 1$. Note now first that, since the series is unique and $\sin(\pi z) = 0$, we must have $a_0 = 0$, or we will henceforth obviate $k = 0$. Note next that for all $k \ge 1$, we have $2 < 2k+1, 2k+3, \ldots, 2k+2d-1$, or $2^d(2k-1)!! < (2k+2d-1)!!$, hence

$$|a_k| \le \frac{\pi^{2k}}{(2k)!}\sum_{d=0}^{\infty}\frac{2^d(2k-1)!!}{(2k+2d-1)!!}\frac{1}{d!}\left(\frac{\pi}{4}\right)^{2d} < \frac{\pi^{2k}}{(2k)!}\sum_{d=0}^{\infty}\frac{1}{d!}\left(\frac{\pi}{4}\right)^{2d} = \exp\left(\frac{\pi^2}{16}\right).$$

The conclusion follows, and it suffices to take $c = \exp\left(\frac{\pi^2}{16}\right)$.

O385. Let $f(x,y)=\frac{x^3-y^3}{6}+3xy+48$. Let $m$ and $n$ be odd integers such that $|f(m,n)|\le mn+37$. Evaluate $f(m,n)$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by Robert Bosch, USA and Jorge Erick, Brazil*

If $\alpha=\dfrac{m-n}{2}$, then $\alpha$ is integer and $m=n+2\alpha$. Consider the following identity:

$$\frac{m^3-n^3}{6}+kmn=(\alpha+k)\left[(n+\alpha)^2+\frac{1}{3}(\alpha-2k)^2\right]-\frac{4}{3}k^3,$$

due to

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca),$$

for $a=m, b=-n, c=2k$. Thus

$$\begin{aligned} f(m,n)-(mn+37) &= (\alpha+2)\left[(n+\alpha)^2+\frac{1}{3}(\alpha-4)^2\right]+\frac{1}{3}\le 0\Rightarrow \alpha+2<0,\\ f(m,n)+mn+37 &= (\alpha+4)\left[(n+\alpha)^2+\frac{1}{3}(\alpha-8)^2\right]-\frac{1}{3}\ge 0\Rightarrow \alpha+4>0. \end{aligned}$$

We conclude that $\alpha=-3$ and

$$f(m,n)=(\alpha+3)\left[(n+\alpha)^2+\frac{1}{3}(\alpha-6)^2\right]+12=f(-5,1)=12.$$

*Also solved by Alessandro Ventullo, Milan, Italy; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Prithwijit De, HBCSE, Mumbai, India.*

O386. Find all pairs $(m, n)$ of positive integers such that $3^m - 2^n$ is a perfect square.

*Proposed by Alessandro Ventullo, Milan, Italy*

*Solution by David E. Manes, Oneonta, NY, USA*
By direct computation if $m \leq 4$, then $(1,1), (2,3), (3,1)$ and $(4,5)$ are solutions. We will show that there are no others.

Assume $m \geq 5$. If $n = 1$, then a simple induction argument shows that $3^m - 2^n$ is always strictly between two consecutive squares. Therefore, $3^m - 2$ is not a perfect square. Thus, $n \geq 2$. Assume $3^m - 2^n = k^2$ for some integer $k$ and the integer $m$ is odd. Then $3^m \equiv 3 \mod 4$ and $2^n \equiv 0 \mod 4$ imply $3^m - 2^n \equiv 3 \mod 4$, a contradiction since $k^2 \equiv 0$ or $1 \mod 4$. Therefore, $m$ is even so that $m = 2r$ for some integer $r$. Then $3^{2r} - k^2 = 2^n$. Factoring, we get $(3^r + k)(3^r - k) = 2^n$ so that $3^r + k = 2^s$ and $3^r - k = 2^t$ for some integers $s, t, s > t$ and $s + t + n$. Adding the two equations, one obtains $2 \cdot 3^r = 2^t(2^{s-t} + 1$. Therefore $3^r = 2^{t-1}(2^{s-t} + 1)$ implies $t = 1$. Therefore, $3^r - 2^{s-1} = 1$, an equation that has solutions only if $r = 1$ or 2 in which case $m \leq 4$, a contradiction. Hence, the only solutions are the ones claimed above.

*Also solved by Adnan Ali, A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Prithwijit De, HBCSE, Mumbai, India; Robert Bosch, USA and Jorge Erick, Brazil; Minh Pham Hoang, High School for the Gifted, Vietnam National University, Ho Chi Minh City, Vietnam.*

O387. Are there integers n for which $3^{6n-3}+3^{3n-1}+1$ is a perfect cube?

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Note that

$$\left(3^{2n-1}+1\right)^3=3^{6n-3}+3^{4n-1}+3^{2n}+1>3^{6n-3}+3^{3n-1}+1>3^{6n-3}=\left(3^{2n-1}\right)^3,$$

or since $3^{6n-3}+3^{3n-1}+1$ is always strictly between two consecutive perfect cubes, it can never be a perfect cube itself, and we are done.

*Also solved by Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Minh Pham Hoang, High School for the Gifted, Vietnam National University, Ho Chi Minh City, Vietnam; Alessandro Ventullo, Milan, Italy; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Adnan Ali, A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arpon Basu, AECS-4, Mumbai, India; Bekjol Joldubai, Kadamcay High School, Kyrgyzstan; David E. Manes, Oneonta, NY, USA; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Robert Bosch, USA.*

O388. Prove that in any triangle $ABC$ with area $S$,

$$\frac{m_a m_b m_c \left(m_a+m_b+m_c\right)}{\sqrt{{m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2}} \geq 2S.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*Solution by Daniel Lasaosa, Pamplona, Spain*

We prove a stronger result, namely that the inequality also holds with $3S$ in the RHS instead of $2S$. Note first that, since ${m_a}^2 = \frac{2b^2+2c^2-a^2}{4}$ and similarly for $m_b, m_c$, we have

$${m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2 = \frac{9\left(a^2b^2+b^2c^2+c^2a^2\right)}{16},$$

$${m_a}^4+{m_b}^4+{m_c}^4 = \frac{9\left(a^4+b^4+c^4\right)}{16},$$

or using Heron's formula,

$$9S^2 = \frac{18\left(a^2b^2+b^2c^2+c^2a^2\right)-9\left(a^4+b^4+c^4\right)}{16} =$$

$$= 2\left({m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2\right)-\left({m_a}^4+{m_b}^4+{m_c}^4\right).$$

Squaring both sides and multiplying by ${m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2$, the proposed inequality is equivalent to

$${m_a}^2{m_b}^2{m_c}^2\left(m_a+m_b+m_c\right)^2+\left({m_a}^4+{m_b}^4+{m_c}^4\right)\left({m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2\right) \geq$$

$$\geq 2\left({m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2\right)^2,$$

or after some algebra, to

$${m_a}^2{m_b}^2\left(\left(m_a+m_b\right)^2-{m_c}^2\right)\left(m_a-m_b\right)^2+{m_b}^2{m_c}^2\left(\left(m_b+m_c\right)^2-{m_a}^2\right)\left(m_b-m_c\right)^2+$$

$$+{m_c}^2{m_a}^2\left(\left(m_c+m_a\right)^2-{m_b}^2\right)\left(m_c-m_a\right)^2 \geq 0.$$

Now, it is well known that

$$\left(m_a+m_b\right)^2-{m_c}^2 = \left(m_a+m_b+m_c\right)\left(m_a+m_b-m_c\right) \geq 0,$$

since the medians of triangle $ABC$ are the sides of a triangle whose area is $\frac{3}{4}$ the area of $ABC$. Or, all terms in the LHS are non-negative, being simultaneoulsy zero iff $m_a = m_b = m_c$. The conclusion follows, equality holds iff $ABC$ is equilateral.

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O389. Let $a, b, c$ be positive real numbers such that $abc = 1$. Prove that

$$\frac{a^2(b+c)}{b^2+c^2} + \frac{b^2(c+a)}{c^2+a^2} + \frac{c^2(a+b)}{a^+b^2} \geq \sqrt{3(a+b+c)}.$$

*Proposed by Bazarbaev Sardar, National University of Uzbekistan, Uzbekistan*

*Solution by Erdenebayar Bayarmagnai, National University of Mongolia, Mongolia*

$$\frac{a^2(b+c)}{b^2+c^2} - a + \frac{b^2(c+a)}{c^2+a^2} - b + \frac{c^2(a+b)}{a^2+b^2} - c = \frac{ab(a-b)+ac(a-c)}{b^2+c^2} + \frac{bc(b-c)+ba(b-a)}{c^2+a^2} + \frac{ca(c-a)+cb(c-b)}{a^2+b^2} =$$

$$= \left(\frac{ab(a-b)}{b^2+c^2} - \frac{ab(a-b)}{c^2+a^2}\right) + \left(\frac{ac(a-c)}{b^2+c^2} - \frac{ac(a-c)}{a^2+b^2}\right) + \left(\frac{bc(b-c)}{c^2+a^2} - \frac{bc(b-c)}{a^2+b^2}\right) =$$

$$= \frac{ab(a-b)(a^2-b^2)}{(b^2+c^2)(c^2+a^2)} + \frac{ac(a-c)(a^2-c^2)}{(b^2+c^2)(a^2+b^2)} + \frac{bc(b-c)(b^2-c^2)}{(c^2+a^2)(a^2+b^2)} =$$

$$= \frac{ab(a-b)^2(a+b)}{(b^2+c^2)(c^2+a^2)} + \frac{ac(a-c)^2(a+c)}{(b^2+c^2)(a^2+b^2)} + \frac{bc(b-c)^2(b+c)}{(c^2+a^2)(a^2+b^2)} \geq 0 \ \Rightarrow$$

$$\Rightarrow \ \frac{a^2(b+c)}{b^2+c^2} + \frac{b^2(c+a)}{c^2+a^2} + \frac{c^2(a+b)}{a^2+b^2} \geq (a+b+c) = \sqrt{(a+b+c)(a+b+c)} \geq \sqrt{3\sqrt[3]{abc}(a+b+c)} = \sqrt{3(a+b+c)}$$

This equality holds only when $a = b = c = 1$

*Also solved by Albert Stadler, Herrliberg, Switzerland; Emil Gasimov, Baku Istek Lyceum, Azerbaijan; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Pham Ngoc Khanh, Hanoi National University of Education, Vietnam; Robert Bosch, USA.*

O390. Let $p > 2$ be a prime. Find the number of $4p$ element subsets of the set $\{1, 2, \cdots, 6p\}$ for which the sum of the elements is divisble by $2p$.

*Proposed by Vlad Matei, University of Wisconsin, Madison, USA*

*Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India*

We shall be using the Root of Unity Filter to solve the problem, so we introduce it first:

**Theorem 1 (Root of Unity Filter):** Define $\varepsilon = e^{2\pi i/n}$ for a positive integer $n$. For any polynomial $F(x) = f_0 + f_1x + f_2x^2 + \cdots$ (where $f_k = 0$ if $k > \deg F$), the sum $f_0 + f_n + f_{2n} + \cdots$ is given by

$$f_0 + f_n + f_{2n} + \cdots = \frac{1}{n}\big(F(1) + (\varepsilon) + F(\varepsilon^2) + \cdots + F(\varepsilon^{n-1})\big).$$

**Proof:** We use a property of the sum $s_k = 1 + \varepsilon^k + \varepsilon^{2k} + \cdots + \varepsilon^{(n-1)k}$. If $n|k$, then $\varepsilon^k = 1$ and so $s_k = n$, else $\varepsilon^k \neq 1$ and so $s_k = \frac{1-\varepsilon^{nk}}{1-\varepsilon^k} = 0$. Thus

$$F(1) + (\varepsilon) + F(\varepsilon^2) + \cdots + F(\varepsilon^{n-1}) = f_0s_0 + f_1s_1 + f_2s_2 + \cdots = n(f_0 + f_n + f_{2n} + \cdots).$$

$\square$

Now, to start with, we define a generating function $G(x, y)$ as

$$G(x, y) = \textstyle\sum_{n,k\geq 0} g_{n,k}x^ny^k$$

where $g_{n,k}$ is the number of $k$-element subsets of $\{1, 2, \cdots, 6p\}$ having a sum $n$. So, the answer required by the problem is nothing but $A := g_{2p,4p} + g_{4p,4p} + g_{6p,4p} + \cdots$. Next we observe that if a number $m$ is not in a subset, it doesn't affect the size or sum of the subset, while if it is present in the subset, it increases the sum by $m$ and size by 1. Thus $G$ must contain the term $(1 + x^my)$ and so

$$G(x, y) = (1 + xy)(1 + x^2y)\cdots(1 + x^{6p}y).$$

To get the value of $A$ we must extract two types of terms from $G$: $y^{4p}$ and powers of $x^{2p}$. We can do the latter using Theorem 1 and so we perform it first. Define $\varepsilon = e^{2\pi i/2p}$. Then the filter tells us that

$$\sum_{\substack{n,k\geq 0\\ 2p|n}} g_{n,k}y^k = \frac{1}{2p}\Big(G(1, y) + G(\varepsilon^k, y) + G(\varepsilon^{2k}, y) + \cdots + G(\varepsilon^{(2p-1)k}, y)\Big), \qquad (2)$$

so we need to calculate $G(\varepsilon^k, y)$ for $0 \leq k \leq 2p - 1$. For $k = 0$, we have $G(1, y) = (1 + y)^{6p}$. For $k = 2\ell$, where $1 \leq \ell \leq p - 1$, $\varepsilon^{2\ell} = \gamma^\ell$, where $\gamma = e^{2\pi i/p}$. Since $\gcd(\ell, p) = 1$ the set $\{\ell, 2\ell, \cdots, p\ell\}$ is a complete residue set modulo $p$. Thus we have

$$\begin{aligned} G(\varepsilon^{2\ell}, y) = G(\gamma^\ell, y) &= (1 + \gamma^\ell y)(1 + \gamma^{2\ell}y)\cdots(1 + \gamma^{6p\ell}y) \\ &= \Big((1 + \gamma^\ell y)(1 + \gamma^{2\ell}y)\cdots(1 + \gamma^{p\ell}y)\Big)^6 \\ &= \big((1 + \gamma y)(1 + \gamma^2y)\cdots(1 + \gamma^py)\big)^6 \\ &= (1 + y^p)^6. \end{aligned}$$

For $k=p$, $\varepsilon=-1$, and so we have $G(-1,y)=(1-y^2)^{3p}$. Now for $1\le k\le 2p-1$ ($2\nmid k$ and $k\neq p$) $\gcd(2p,k)=1$ and so the set $\{k,2k,\cdots,2pk\}$ is a complete residue set modulo $2p$. Thus

$$\begin{aligned}
G(\varepsilon^k,y) &= (1+\varepsilon^k y)(1+\varepsilon^{2k}y)\cdots(1+\varepsilon^{6pk}y)\\
&= \Big((1+\varepsilon^k y)(1+\varepsilon^{2k}y)\cdots(1+\varepsilon^{2pk}y)\Big)^3\\
&= \big((1+\varepsilon y)(1+\varepsilon^2 y)\cdots(1+\varepsilon^{2p}y)\big)^3\\
&= (1-y^{2p})^3.
\end{aligned}$$

Putting all these values back in (1) gives

$$\sum_{\substack{n,k\ge 0\\ 2p\mid n}} g_{n,k}y^k=\frac{1}{2p}\left((1+y)^{6p}+(1-y^2)^{3p}+(p-1)(1+y^p)^6+(p-1)(1-y^{2p})^3\right)$$

,

from which we can easily extract the coefficient of $y^{4p}$. Thus our answer (i.e. $A$) is

$$\frac{1}{2p}\Big(\tbinom{6p}{4p}+\tbinom{3p}{2p}+(p-1)\tbinom{6}{4}+(p-1)\tbinom{3}{2}\Big)=\frac{1}{2p}\Big(\tbinom{6p}{4p}+\tbinom{3p}{2p}+18p-18\Big).$$