Mathematical Excalibur

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Olympiad Corner

Below was Slovenia's Selection Examinations for the IMO 2005.

First Selection Examination

Problem 1. Let M be the intersection of diagonals AC and BD of the convex quadrilateral $ABCD$. The bisector of angle ACD meets the ray BA at the point K. Prove that if $MA\cdot MC + MA\cdot CD = MB\cdot MD$, then $\angle BKC = \angle BDC$.

Problem 2. Let R_+ be the set of all positive real numbers. Find all functions $f: R_+\to R_+$ such that $x^2(f(x)+f(y)) = (x+y)f(f(x)y)$ holds for any positive real numbers x and y.

Problem 3. Find all pairs of positive integers (m, n) such that the numbers m^2-4n and n^2-4m are perfect squares.

Second Selection Examination

Problem 1. How many sequences of 2005 terms are there such that the following three conditions hold:

(a) no sequence has three consecutive terms equal to each other,
(b) every term of every sequence is equal to 1 or −1, and

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 16, 2006***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Muirhead's Inequality

Lau Chi Hin

Muirhead's inequality is an important generalization of the AM-GM inequality. It is a powerful tool for solving inequality problem. First we give a definition which is a generalization of arithmetic and geometric means.

Definition. Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$. The *p-mean* of $x_1, x_2, \ldots, x_n$ is defined by

$$[p] = \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)}^{p_1} x_{\sigma(2)}^{p_2}\cdots x_{\sigma(n)}^{p_n},$$

where S_n is the set of all permutations of $\{1,2,\ldots, n\}$. (The summation sign means to sum $n!$ terms, one term for each permutation σ in S_n.)

For example, $[(1,0,\ldots,0)] = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the arithmetic mean of $x_1, x_2, \ldots, x_n$ and $[(1/n,1/n,\ldots,1/n)] = x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n}$ is their geometric mean.

Next we introduce the concept of majorization in $\mathbb{R}^n$. Let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ satisfy conditions

1. $p_1 \ge p_2 \ge \cdots \ge p_n$ and $q_1 \ge q_2 \ge \cdots \ge q_n$,
2. $p_1 \ge q_1$, $p_1+p_2 \ge q_1+q_2$, …, $p_1+p_2+\cdots+p_{n-1} \ge q_1+q_2+\cdots+q_{n-1}$ and
3. $p_1+p_2+\cdots+p_n = q_1+q_2+\cdots+q_n$.

Then we say $(p_1, p_2, \ldots, p_n)$ *majorizes* $(q_1, q_2, \ldots, q_n)$ and write

$$(p_1, p_2, \ldots, p_n) \succ (q_1, q_2, \ldots, q_n).$$

Theorem (Muirhead's Inequality). Let $x_1, x_2, \ldots, x_n$ be positive real numbers and $p, q \in \mathbb{R}^n$. If $p \succ q$, then $[p] \ge [q]$. Furthermore, for $p \ne q$, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Since $(1,0,\ldots,0) \succ (1/n,1/n,\ldots,1/n)$, AM-GM inequality is a consequence.

Example 1. For any $a, b, c > 0$, prove that

$$(a+b)(b+c)(c+a) \ge 8abc.$$

Solution. Expanding both sides, the desired inequality is

$$a^2b+a^2c+b^2c+b^2a+c^2a+c^2b \ge 6abc.$$

This is equivalent to $[(2,1,0)] \ge [(1,1,1)]$, which is true by Muirhead's inequality since $(2,1,0) \succ (1,1,1)$.

For the next example, we would like to point out a useful trick. When the product of $x_1, x_2, \ldots, x_n$ is 1, we have

$$[(p_1, p_2, \ldots, p_n)] = [(p_1-r, p_2-r, \ldots, p_n-r)]$$

for any real number r.

Example 2. (*IMO 1995*) For any $a, b, c > 0$ with $abc = 1$, prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is

$$\begin{aligned}&2(a^4b^4+b^4c^4+c^4a^4)\\&+2(a^4b^3c+a^4c^3b+b^4c^3a+b^4a^3c+c^4a^3b\\&+c^4b^3a)+2(a^3b^3c^2+b^3c^3a^2+c^3a^3b^2)\\&\ge 3(a^5b^4c^3+a^5c^4b^3+b^5c^4a^3+b^5a^4c^3\\&+c^5a^4b^3+c^5b^4a^3)+6a^4b^4c^4.\end{aligned}$$

This is equivalent to $[(4,4,0)]+2[(4,3,1)] + [(3,3,2)] \ge 3[(5,4,3)] + [(4,4,4)]$. Note $4+4+0 = 4+3+1 = 3+3+2 = 8$, but $5+4+3 = 4+4+4 = 12$. So we can set $r = 4/3$ and use the trick above to get $[(5,4,3)] = [(11/3,8/3,5/3)]$ and also $[(4,4,4)] = [(8/3,8/3,8/3)]$.

Observe that $(4,4,0) \succ (11/3,8/3,5/3)$, $(4,3,1) \succ (11/3,8/3,5/3)$ and $(3,3,2) \succ (8/3,8/3,8/3)$. So applying Muirhead's inequality to these three majorizations and adding the inequalities, we get the desired inequality.

Example 3. (*1990 IMO Shortlisted Problem*) For any $x, y, z > 0$ with $xyz = 1$, prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)}\geq\frac{3}{4}.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is

$$4(x^4+y^4+z^4+x^3+y^3+z^3)$$
$$\geq 3(1+x+y+z+xy+yz+zx+xyz).$$

This is equivalent to $4[(4,0,0)] + 4[(3,0,0)] \geq [(0,0,0)] + 3[(1,0,0)] + 3[(1,1,0)] + [(1,1,1)]$.

For this, we apply Muirhead's inequality and the trick as follow:

$[(4,0,0)] \geq [(4/3,4/3,4/3)] = [(0,0,0)]$,
$3[(4,0,0)] \geq 3[(2,1,1)] = 3[(1,0,0)]$,
$3[(3,0,0)] \geq 3[(4/3,4/3,1/3)] = 3[(1,1,0)]$
and $[(3,0,0)] \geq [(1,1,1)]$.

Adding these, we get the desired inequality.

Remark. For the following example, we will modify the trick above. In case $xyz \geq 1$, we have

$$[(p_1, p_2, p_3)] \geq [(p_1-r, p_2-r, p_3-r)]$$

for every $r \geq 0$. Also, we will use the following

Fact. For $p, q \in \mathbb{R}^n$, we have

$$\frac{[p]+[q]}{2}\geq\left[\frac{p+q}{2}\right].$$

This is because by the AM-GM inequality,

$$\frac{x_{\sigma(1)}^{p_1}\cdots x_{\sigma(n)}^{p_n}+x_{\sigma(1)}^{q_1}\cdots x_{\sigma(n)}^{q_n}}{2}$$
$$\geq x_{\sigma(1)}^{(p_1+q_1)/2}\cdots x_{\sigma(n)}^{(p_n+q_n)/2}.$$

Summing over $\sigma\in S_n$ and dividing by $n!$, we get the inequality.

Example 4. (*2005 IMO*) For any $x, y, z > 0$ with $xyz \geq 1$, prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0.$$

Solution. Multiplying by the common denominator and expanding both sides, the desired inequality is equivalent to $[(9,0,0)]+4[(7,5,0)]+[(5,2,2)]+[(5,5,5)] \geq [(6,0,0)] + [(5,5,2)] + 2[(5,4,0)] + 2[(4,2,0)] + [(2,2,2)]$.

To prove this, we note that

(1) $[(9,0,0)] \geq [(7,1,1)] \geq [(6,0,0)]$

(2) $[(7,5,0)] \geq [(5,5,2)]$

(3) $2[(7,5,0)] \geq 2[(6,5,1)] \geq 2[(5,4,0)]$

(4) $[(7,5,0)] + [(5,2,2)] \geq 2[(6,7/2,1)]$
$\geq 2[(9/2,2,-1/2)] \geq 2[(4,2,0)]$

(5) $[(5,5,5)] \geq [(2,2,2)]$,

where (1) and (3) are by Muirhead's inequality and the remark, (2) is by Muirhead's inequality, (4) is by the fact, Muirhead's inequality and the remark and (5) is by the remark.

Considering the sum of the leftmost parts of these inequalities is greater than or equal to the sum of the rightmost parts of these inequalities, we get the desired inequalities.

Alternate Solution. Since

$$\frac{x^5-x^2}{x^5+y^2+z^2}-\frac{x^5-x^2}{x^3(x^2+y^2+z^2)}$$
$$=\frac{(x^3-1)^2(y^2+z^2)}{x(x^2+y^2+z^2)(x^5+y^2+z^2)}\geq 0,$$

we have

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}$$
$$\geq\frac{x^5-x^2}{x^3(x^2+y^2+z^2)}+\frac{y^5-y^2}{y^3(y^2+z^2+x^2)}+\frac{z^5-z^2}{z^3(z^2+x^2+y^2)}$$
$$\geq\frac{1}{x^2+y^2+z^2}\left(x^2-\frac{1}{x}+y^2-\frac{1}{y}+z^2-\frac{1}{z}\right)$$
$$\geq\frac{1}{x^2+y^2+z^2}(x^2+y^2+z^2-yz-zx-xy)$$
$$=\frac{(x-y)^2+(y-z)^2+(z-x)^2}{2(x^2+y^2+z^2)}\geq 0.$$

Proofs of Muirhead's Inequality

Kin Yin Li

Let $p \succ q$ and $p \neq q$. From $i = 1$ to n, the first nonzero $p_i - q_i$ is positive by condition 2 of majorization. Then there is a negative $p_i - q_i$ later by condition 3. It follows that there are $j < k$ such that $p_j > q_j$, $p_k < q_k$ and $p_i = q_i$ for any possible i between j, k.

Let $b = (p_j+p_k)/2$, $d = (p_j-p_k)/2$ so that $[b-d,b+d] = [p_k, p_j] \supset [q_k, q_j]$. Let c be the maximum of $|q_j-b|$ and $|q_k-b|$, then $0 \leq c < d$. Let $r = (r_1,\ldots,r_n)$ be defined by $r_i = p_i$ except $r_j = b + c$ and $r_k = b - c$. By the definition of c, either $r_j = q_j$ or $r_k = q_k$. Also, by the definitions of b, c, d, we get $p \succ r$, $p \neq r$ and $r \succ q$. Now

$$n!([p]-[r])=\sum_{\sigma\in S_n}x_\sigma\left(x_{\sigma(j)}^{p_j}x_{\sigma(k)}^{p_k}-x_{\sigma(j)}^{r_j}x_{\sigma(k)}^{r_k}\right)$$
$$=\sum_{\sigma\in S_n}x_\sigma\left(u^{b+d}v^{b-d}-u^{b+c}v^{b-c}\right),$$

where x_σ is the product of $x_{\sigma(i)}^{p_i}$ for $i \neq j, k$ and $u = x_{\sigma(j)}$, $v = x_{\sigma(k)}$. For each permutation σ, there is a permutation ρ such that $\sigma(i) = \rho(i)$ for $i \neq j, k$ and $\sigma(j) = \rho(k)$, $\sigma(k) = \rho(j)$. In the above sum, if we pair the terms for σ and ρ, then $x_\sigma = x_\rho$ and combining the parenthetical factors for the σ and ρ terms, we have

$$(u^{b+d}v^{b-d}-u^{b+c}v^{b-c})+(v^{b+d}u^{b-d}-v^{b+c}u^{b-c})$$
$$=u^{b-d}v^{b-d}(u^{d+c}-v^{d+c})(u^{d-c}-v^{d-c})\geq 0.$$

So the above sum is nonnegative. Then $[p] \geq [r]$. Equality holds if and only if $u = v$ for all pairs of σ and ρ, which yields $x_1 = x_2 = \cdots = x_n$. Finally we recall r has at least one more coordinate in agreement with q than p. So repeating this process finitely many times, we will eventually get the case $r = q$. Then we are done.

Next, <u>*for the advanced readers*</u>, we will outline a longer proof, which tells more of the story. It is consisted of two steps. The first step is to observe that if $c_1, c_2, \ldots, c_k \geq 0$ with sum equals 1 and $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$, then

$$\sum_{i=1}^{k}c_i[v_i]\geq\left[\sum_{i=1}^{k}c_iv_i\right].$$

This follows by using the weighted AM-GM inequality instead in the proof of the fact above. (For the statement of the weighted AM-GM inequality, see *Mathematical Excalibur*, *vol.* 5, *no.* 4, *p.* 2, remark in column 1).

The second step is the difficult step of showing $p \succ q$ implies there exist nonnegative numbers $c_1, c_2, \ldots, c_{n!}$ with sum equals 1 such that

$$q=\sum_{i=1}^{n!}c_iP_i,$$

where $P_1, P_2, \ldots, P_{n!} \in \mathbb{R}^n$ whose coordinates are the $n!$ permutations of the coordinates of p. Muirhead's inequality follows immediately by applying the first step and observing that $[P_i]=[p]$ for $i=1,2,\ldots, n!$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***April 16, 2006.***

Problem 246. A spy plane is flying at the speed of 1000 kilometers per hour along a circle with center A and radius 10 kilometers. A rocket is fired from A at the same speed as the spy plane such that it is always on the radius from A to the spy plane. Prove such a path for the rocket exists and find how long it takes for the rocket to hit the spy plane. (*Source: 1965 Soviet Union Math Olympiad*)

Problem 247. (*a*) Find all possible positive integers $k \ge 3$ such that there are k positive integers, every two of them are not relatively prime, but every three of them are relatively prime.

(*b*) Determine with proof if there exists an infinite sequence of positive integers satisfying the conditions in (*a*) above.
(*Source: 2003 Belarussian Math Olympiad*)

Problem 248. Let $ABCD$ be a convex quadrilateral such that line CD is tangent to the circle with side AB as diameter. Prove that line AB is tangent to the circle with side CD as diameter if and only if lines BC and AD are parallel.

Problem 249. For a positive integer n, if $a_1,\cdots, a_n$, b_1, $\cdots$, b_n are in [1,2] and $a_1^2+\cdots+a_n^2=b_1^2+\cdots+b_n^2$, then prove that

$$\frac{a_1^3}{b_1}+\cdots+\frac{a_n^3}{b_n}\le\frac{17}{10}(a_1^2+\cdots+a_n^2).$$

Problem 250. Prove that every region with a convex polygon boundary cannot be dissected into finitely many regions with nonconvex quadrilateral boundaries.

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Solutions

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Problem 241. Determine the smallest possible value of

$$S = a_1\cdot a_2\cdot a_3+b_1\cdot b_2\cdot b_3+c_1\cdot c_2\cdot c_3,$$

if a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 is a permutation of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. (*Source: 2002 Belarussian Math. Olympiad*)

Solution. **CHAN Ka Lok** (STFA Leung Kau Kui College), **CHAN Tsz Lung** (HKU Math PG Year 1), **G.R.A. 20 Math Problem Group** (Roma, Italy), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, OR, USA, teacher), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **Problem Solving Group @ Miniforum** and **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School).

The idea is to get the 3 terms as close as possible. We have $214 = 70 + 72 + 72 = 2\cdot5\cdot7 + 1\cdot8\cdot9 + 3\cdot4\cdot6$. By the AM-GM inequality, $S \ge 3(9!)^{1/3}$. Now $9! = 70\cdot72\cdot72 > 70\cdot73\cdot71 > 71^3$. So $S > 3\cdot71 = 213$. Therefore, 214 is the answer.

Problem 242. Prove that for every positive integer n, 7 is a divisor of 3^n+n^3 if and only if 7 is a divisor of 3^nn^3+1. (*Source: 1995 Bulgarian Winter Math Competition*)

Solution. **CHAN Tsz Lung** (HKU Math PG Year 1), **G.R.A. 20 Math Problem Group** (Roma, Italy), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, OR, USA, teacher), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **Problem Solving Group @ Miniforum**, **Tak Wai Alan WONG** (Markham, ON, Canada) and **YUNG Fai**.

Note $3^n \not\equiv 0 \pmod 7$. If $n \not\equiv 0 \pmod 7$, then $n^3 \equiv 1$ or $-1 \pmod 7$. So 7 is a divisor of 3^n+n^3 if and only if $-3^n \equiv n^3 \equiv 1 \pmod 7$ or $-3^n \equiv n^3 \equiv -1 \pmod 7$ if and only if 7 is a divisor of 3^nn^3+1.

Commended solvers: **CHAN Ka Lok** (STFA Leung Kau Kui College), **LAM Shek Kin** (TWGHs Lui Yun Choy Memorial College) and **WONG Kai Cheuk** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 243. Let R^+ be the set of all positive real numbers. Prove that there is no function $f: R^+ \to R^+$ such that

$$(f(x))^2 \ge f(x+y)(f(x)+y)$$

for arbitrary positive real numbers x and y. (*Source: 1998 Bulgarian Math Olympiad*)

Solution. **José Luis DíAZ-BARRERO**, (Universitat Politècnica de Catalunya, Barcelona, Spain).

Assume there is such a function. We rewrite the inequality as

$$f(x)-f(x+y)\ge\frac{f(x)y}{f(x)+y}.$$

Note the right side is positive. This implies $f(x)$ is a strictly decreasing.

First we prove that $f(x) - f(x+1) \ge 1/2$ for $x > 0$. Fix $x > 0$ and choose a natural number n such that $n \ge 1 / f(x+1)$. When $k = 0, 1, \ldots, n-1$, we obtain

$$f(x+\frac{k}{n})-f(x+\frac{k+1}{n})$$

$$\ge\frac{f(x+\frac{k}{n})\frac{1}{n}}{f(x+\frac{k}{n})+\frac{1}{n}}\ge\frac{1}{2n}.$$

Adding the above inequalities, we get $f(x) - f(x+1) \ge 1/2$.

Let m be a positive integer such that $m \ge 2f(x)$. Then

$$f(x)-f(x+m)=\sum_{i=1}^{m}(f(x+i-1)-f(x+i))$$

$$\ge m/2 \ge f(x).$$

So $f(x+m) \le 0$, a contradiction.

Commended solvers: **Problem Solving Group @ Miniforum**.

Problem 244. An infinite set S of coplanar points is given, such that every three of them are not collinear and every two of them are not nearer than 1cm from each other. Does there exist any division of S into two disjoint infinite subsets R and B such that inside every triangle with vertices in R is at least one point of B and inside every triangle with vertices in B is at least one point of R? Give a proof to your answer. (*Source: 2002 Albanian Math Olympiad*)

Solution.(Official Solution)

Assume that such a division exists and let M_1 be a point of R. Then take four points M_2, M_3, M_4, M_5 different from M_1, which are the nearest points to M_1 in R. Let r be the largest distance between M_1 and each of these four points. Let H be the convex hull of these five points. Then the interior of H lies inside the circle of radius r centered at M_1, but all other points of R is outside or on the circle. Hence the interior of H does not contain any other point of R.

Below we will say two triangles are *disjoint* if their interiors do not intersect. There are 3 possible cases:

(a) H is a pentagon. Then H may be divided into three disjoint triangles with vertices in R, each of them containing a point of B inside. The triangle with these points of B as vertices would contain another point of R, which would be in H. This is impossible.

(b) H is a quadrilateral. Then one of the M_i is inside H and the other M_j, M_k, M_l, M_m are at its vertices, say clockwise. The four disjoint triangles $M_iM_jM_k$, $M_iM_kM_l$, $M_iM_lM_m$, $M_iM_mM_i$ induce four points of B, which can be used to form two disjoint triangles with vertices in B which would contain two points in R. So H would then contain another point of R inside, other than M_i, which is impossible.

(c) H is a triangle. Then it contains inside it two points M_i, M_j. One of the three disjoint triangles $M_iM_kM_l$, $M_iM_lM_m$, $M_iM_mM_k$ will contain M_j. Then we can break that triangle into three smaller triangles using M_j. This makes five disjoint triangles with vertices in R, each having one point of B inside. With these five points of B, three disjoint triangles with vertices in B can be made so that each one of them having one point of R. Then H contains another point of R, different from M_1, M_2, M_3, M_4, M_5, which is impossible.

Problem 245. $ABCD$ is a concave quadrilateral such that $\angle BAD = \angle ABC = \angle CDA = 45^\circ$. Prove that $AC = BD$.

Solution. **CHAN Tsz Lung** (HKU Math PG Year 1), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **Problem Solving Group @ Miniforum**, **WONG Kai Cheuk** (Carmel Divine Grace Foundation Secondary School, Form 6), **WONG Man Kit** (Carmel Divine Grace Foundation Secondary School, Form 6) and **WONG Tsun Yu** (St. Mark's School, Form 6).

Let line BC meet AD at E, then $\angle BEA = 180^\circ - \angle ABC - \angle BAD = 90^\circ$. Note ΔAEB and ΔCED are 45°-90°-45° triangles. So $AE = BE$ and $CE = DE$. Then $\Delta AEC \cong \Delta BED$. So $AC = BD$.

Commended solvers: **CHAN Ka Lok** (STFA Leung Kau Kui College), **CHAN Pak Woon** (HKU Math UG Year 1), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YUEN Wah Kong** (St. Joan of Arc Secondary School).

Olympiad Corner

(continued from page 1)

Problem 1. (Cont.)

(c) the sum of all terms of every sequence is at least 666?

Problem 2. Let O be the center of the circumcircle of the acute-angled triangle ABC, for which $\angle CBA < \angle ACB$ holds. The line AO intersects the side BC at the point D. Denote by E and F the centers of the circumcircles of triangles ABD and ACD respectively. Let G and H be two points on the rays BA and CA such that $AG=AC$ and $AH=AB$, and the point A lies between B and G as well as between C and H. Prove the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

Problem 3. Let a, b and c be positive numbers such that $ab + bc + ca = 1$. Prove the inequality

$$3\sqrt[3]{\frac{1}{abc} + 6(a+b+c)} \le \frac{\sqrt[3]{3}}{abc}.$$

Proofs of Muirhead's Inequality

(continued from page 2)

For the proof of the second step, we follow the approach in J. Michael Steele's book *The Cauchy-Schwarz Master Class*, MAA-Cambridge, 2004. For a $n\times n$ matrix M, we will denote its entry in the j-th row, k-th column by M_{jk}. Let us introduce the term *permutation matrix* for $\sigma \in S_n$ to mean the $n\times n$ matrix $M(\sigma)$ with $M(\sigma)_{jk} = 1$ if $\sigma(j)=k$ and $M(\sigma)_{jk} = 0$ otherwise. Also, introduce the term *doubly stochastic matrix* to mean a square matrix whose entries are nonnegative real numbers and the sum of the entries in every row and every column is equal to one. The proof of the second step follows from two results:

Hardy-Littlewood-Polya's Theorem. If $p \succ q$, then there is a $n\times n$ doubly stochastic matrix D such that $q = Dp$, where we write p and q as column matrices.

Birkhoff's Theorem. For every doubly stochastic matrix D, there exist nonnegative numbers $c(\sigma)$ with sum equals 1 such that

$$D = \sum_{\sigma\in S_n} c(\sigma) M(\sigma).$$

Granting these results, for P_i's in the second step, we can just let $P_i = M(\sigma_i)p$.

Hardy-Littlewood-Polya's theorem can be proved by introducing r as in the first proof. Following the idea of Hardy-Littlewood-Polya, we take T to be the matrix with

$$T_{jj} = \frac{d+c}{2d} = T_{kk},\quad T_{jk} = \frac{d-c}{2d} = T_{kj},$$

all other entries on the main diagonal equal 1 and all other entries of the matrix equal 0. We can check T is doubly stochastic and $r = Tp$. Then we repeat until $r = q$.

Birkhoff's theorem can be proved by induction on the number N of positive entries of D using Hall's theorem (see *Mathematical Excalibur*, *vol.* 1, *no.* 5, *p.* 2). Note $N \ge n$. If $N = n$, then the positive entries are all 1's and D is a permutation matrix already. Next for $N > n$, suppose the result is true for all doubly stochastic matrices with less than N positive entries. Let D have exactly N positive entries. For $j = 1,\ldots, n$, let W_j be the set of k such that $D_{jk} > 0$. We need a *system of distinct representatives* (SDR) for $W_1,\ldots,W_n$. To get this, we check the condition in Hall's theorem. For every collection $W_{j_1},\ldots,W_{j_m}$, note m is the sum of all positive entries in column $j_1,\ldots,j_m$ of D. This is less than or equal to the sum of all positive entries in those columns that have at least one positive entry among row $j_1,\ldots,j_m$. This latter sum is the number of such columns and is also the number of elements in the union of $W_{j_1},\ldots,W_{j_m}$.

So the condition in Hall's theorem is satisfied and there is a SDR for $W_1,\ldots, W_n$. Let $\sigma(i)$ be the representative in W_i, then $\sigma \in S_n$. Let $c(\sigma)$ be the minimum of $D_{1\sigma(1)},\ldots,D_{n\sigma(n)}$. If $c(\sigma) = 1$, then D is a permutation matrix. Otherwise, let

$$D' = (1 - c(\sigma))^{-1}(D - c(\sigma)\, M(\sigma)).$$

Then $D = c(\sigma)\, M(\sigma) + (1 - c(\sigma))\, D'$ and D' is a double stochastic matrix with at least one less positive entries than D. So we may apply the cases less than N to D' and thus, D has the required sum.

Mathematical Excalibur

Volume 11, Number 2 — *April 2006 – May 2006*

Olympiad Corner

Below was the Find Round of the 36th Austrian Math Olympiad 2005.

Part 1 (May 30, 2005)

Problem 1. Show that an infinite number of multiples of 2005 exist, in which each of the 10 digits 0,1,2,…,9 occurs the same number of times, not counting leading zeros.

Problem 2. For how many integer values of a with $|a| \le 2005$ does the system of equations $x^2 = y + a$, $y^2 = x + a$ have integer solutions?

Problem 3. We are given real numbers a, b and c and define s_n as the sum $s_n = a^n + b^n + c^n$ of their n-th powers for non-negative integers n. It is known that $s_1 = 2$, $s_2 = 6$ and $s_3 = 14$ hold. Show that

$$|\, s_n^2 - s_{n-1} \cdot s_{n+1} \,| = 8$$

holds for all integers $n > 1$.

Problem 4. We are given two equilateral triangles ABC and PQR with parallel sides, "one pointing up" and "one pointing down." The common area of the triangles' interior is a hexagon. Show that the lines joining opposite corners of this hexagon are concurrent.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 16, 2006***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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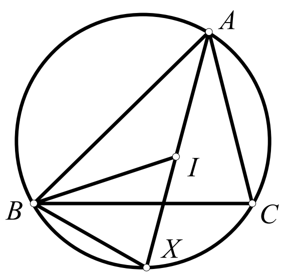
Angle Bisectors Bisect Arcs

Kin Y. Li

In general, angle bisectors of a triangle do not bisect the sides opposite the angles. However, ***angle bisectors always bisect the arcs opposite the angles on the circumcircle of the triangle!*** In math competitions, this fact is very useful for problems concerning angle bisectors or incenters of a triangle *involving the circumcircle*. Recall that the *incenter* of a triangle is the point where the three angle bisectors concur.

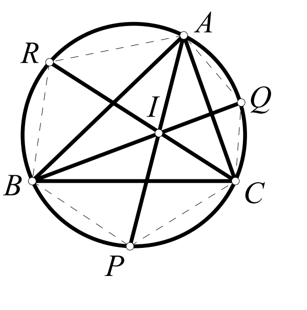
Theorem. Suppose the angle bisector of $\angle BAC$ intersect the circumcircle of ΔABC at $X \ne A$. Let I be a point on the line segment AX. Then I is the incenter of ΔABC if and only if $XI = XB = XC$.



Proof. Note $\angle BAX = \angle CAX = \angle CBX$. So $XB = XC$. Then

$$\begin{aligned} & I \text{ is the incenter of } \Delta ABC \\ \Leftrightarrow\ & \angle CBI = \angle ABI \\ \Leftrightarrow\ & \angle IBX - \angle CBX = \angle BIX - \angle BAX \\ \Leftrightarrow\ & \angle IBX = \angle BIX \\ \Leftrightarrow\ & XI = XB = XC. \end{aligned}$$

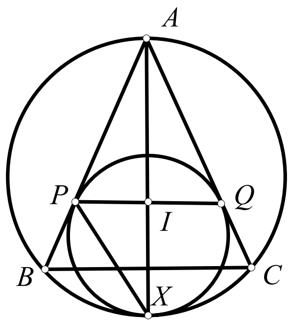
Example 1. (*1982 Australian Math Olympiad*) Let ABC be a triangle, and let the internal bisector of the angle A meet the circumcircle again at P. Define Q and R similarly. Prove that $AP + BQ + CR > AB + BC + CA$.



Solution. Let I be the incenter of ΔABC. By the theorem, we have $2IR = AR + BR > AB$ and similarly $2IP > BC$, $2IQ > CA$. Also $AI + BI > AB$, $BI + CI > BC$ and $CI + AI > CA$. Adding all these inequalities together, we get

$$2(AP + BQ + CR) > 2(AB + BC + CA).$$

Example 2. (*1978 IMO*) In ABC, $AB = AC$. A circle is tangent internally to the circumcircle of ABC and also to the sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of ΔABC.



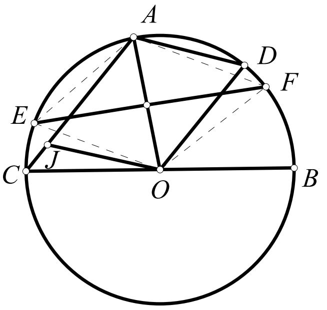
Solution. Let I be the midpoint of line segment PQ and X be the intersection of the angle bisector of $\angle BAC$ with the arc BC not containing A.

By symmetry, AX is a diameter of the circumcircle of ΔABC and X is the midpoint of the arc PXQ on the inside circle, which implies PX bisects $\angle QPB$. Now $\angle ABX = 90^\circ = \angle PIX$ so that X, I, P, B are concyclic. Then

$$\angle IBX = \angle IPX = \angle BPX = \angle BIX.$$

So $XI = XB$. By the theorem, I is the incenter of ΔABC.

Example 3. (*2002 IMO*) Let BC be a diameter of the circle Γ with center O. Let A be a point on Γ such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incenter of the triangle CEF.

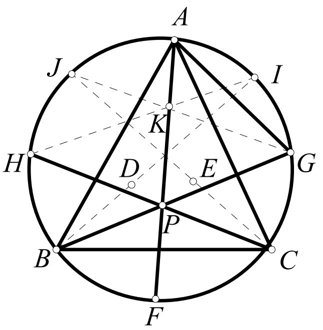


Solution. The condition $\angle AOB < 120^\circ$ ensures I is inside ΔCEF (when $\angle AOB$ increases to 120°, I will coincide with C). Now radius OA and chord EF are perpendicular and bisect each other. So $EOFA$ is a rhombus. Hence A is the midpoint of arc EAF. Then CA bisects $\angle ECF$. Since $OA = OC$, $\angle AOD = 1/2\angle AOB = \angle OAC$. Then DO is parallel to AJ. Hence $ODAJ$ is a parallelogram. Then $AJ = DO = EO = AE$. By the theorem, J is the incenter of ΔCEF.

Example 4. (*1996 IMO*) Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.



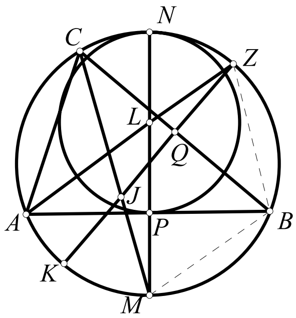
Solution. Let lines AP, BP, CP intersect the circumcircle of ΔABC again at F, G, H respectively. Now

$$\angle APB - \angle ACB = \angle FPG - \angle AGB = \angle FAG.$$

Similarly, $\angle APC - \angle ABC = \angle FAH$. So AF bisects $\angle HAG$. Let K be the incenter of ΔHAG. Then K is on AF and lines HK, GK pass through the midpoints I, J of minor arcs AG, AH respectively. Note lines BD, CE also pass through I, J as they bisect $\angle ABP$, $\angle ACP$ respectively.

Applying Pascal's theorem (see *vol*.10, *no*. 3 of Math Excalibur) to B, G, J, C, H, I on the circumcircle, we see that $P=BG\cap CH$, $K=GJ\cap HI$ and $BI\cap CJ= BD\cap CE$ are collinear. Hence, $BD\cap CE$ is on line PK, which is the same as line AP.

Example 5. (*2006 APMO*) Let A, B be two distinct points on a given circle O and let P be the midpoint of line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let ℓ be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of ℓ and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.



Solution. Let the perpendicular to AB through P intersect circle O at N and M with N and C on the same side of line AB. By symmetry, segment NP is a diameter of the circle of O_1 and its midpoint L is the center of O_1. Let line AL intersect circle O again at Z. Let line ZQ intersect line CM at J and circle O again at K.

Since AB and AC are tangent to circle O_1, AL bisects $\angle CAB$ so that Z is the midpoint of arc BC. Since Q is the midpoint of segment BC, $\angle ZQB = 90^\circ = \angle LPA$ and $\angle JQC = 90^\circ = \angle MPB$. Next

$$\angle ZBQ = \angle ZBC = \angle ZAC = \angle LAP.$$

So ΔZQB, ΔLPA are similar. Since M is the midpoint of arc AMB,

$$\angle JCQ = \angle MCB = \angle MCA = \angle MBP.$$

So ΔJQC, ΔMPB are similar.

By the intersecting chord theorem, $AP \cdot BP = NP \cdot MP = 2LP \cdot MP$. Using the similar triangles above, we have

$$\frac{1}{2} = \frac{LP \cdot MP}{AP \cdot BP} = \frac{ZQ \cdot JQ}{BQ \cdot CQ}.$$

By the intersecting chord theorem, $KQ \cdot ZQ = BQ \cdot CQ$ so that

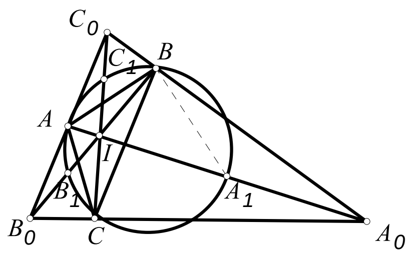
$$KQ = (BQ \cdot CQ)/ZQ = 2JQ.$$

This implies J is the midpoint of KQ. Hence the circle with center J and diameter KQ is tangent to circle O at K and tangent to BC at Q. Since J is on the bisector of $\angle BCA$, this circle is also tangent to AC. So this circle is O_2.

Example 6. (*1989 IMO*) In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:

(i) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$,

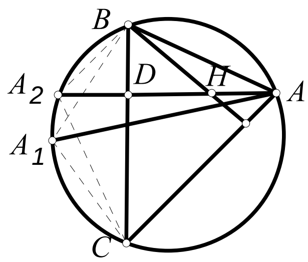
(ii) the area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.



Solution. (i) Let I be the incenter of ΔABC. Since internal angle bisector and external angle bisector are perpendicular, we have $\angle B_0BA_0 = 90^\circ$. By the theorem, $A_1I = A_1B$. So A_1 must be the midpoint of the hypotenuse A_0I of right triangle IBA_0. So the area of ΔBIA_0 is twice the area of ΔBIA_1.

Cutting the hexagon $AC_1BA_1CB_1$ into six triangles with common vertex I and applying a similar area fact like the last statement to each of the six triangles, we get the conclusion of (i).

(ii) Using (i), we only need to show the area of hexagon $AC_1BA_1CB_1$ is at least twice the area of ΔABC.



(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 16, 2006.***

Problem 251. Determine with proof the largest number x such that a cubical gift of side x can be wrapped completely by folding a unit square of wrapping paper (without cutting).

Problem 252. Find all polynomials $f(x)$ with integer coefficients such that for every positive integer n, $2^n - 1$ is divisible by $f(n)$.

Problem 253. Suppose the bisector of $\angle BAC$ intersect the arc opposite the angle on the circumcircle of ΔABC at A_1. Let B_1 and C_1 be defined similarly. Prove that the area of $\Delta A_1B_1C_1$ is at least the area of ΔABC.

Problem 254. Prove that if $a, b, c > 0$, then

$$\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2 \geq 4\sqrt{3abc\,(a+b+c)}.$$

Problem 255. Twelve drama groups are to do a series of performances (with some groups possibly making repeated performances) in seven days. Each group is to see every other group's performance at least once in one of its day-offs.

Find with proof the minimum total number of performances by these groups.

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Solutions

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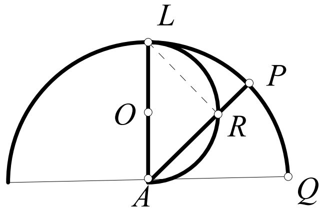
Problem 246. A spy plane is flying at the speed of 1000 kilometers per hour along a circle with center A and radius 10 kilometers. A rocket is fired from A at the same speed as the spy plane such that it is always on the radius from A to the spy plane. Prove such a path for the rocket exists and find how long it takes for the rocket to hit the spy plane. (*Source: 1965 Soviet Union Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher), **G.R.A. 20 Math Problem Group** (Roma, Italy) and **Alex O Kin-Chit** (STFA Cheng Yu Tung Secondary School).



Let the spy plane be at Q when the rocket was fired. Let L be the point on the circle obtained by rotating Q by 90° in the forward direction of motion with respect to the center A. Consider the semicircle with diameter AL on the same side of line AL as Q. We will show the path from A to L along the semicircle satisfies the conditions.

For any point P on the arc QL, let the radius AP intersect the semicircle at R. Let O be the midpoint of AL. Since

$$\angle QAP = \angle RLA = 1/2 \angle ROA$$

and $AL = 2AO$, the length of arc AR is the same as the length of arc QP. So the conditions are satisfied.
Finally, the rocket will hit the spy plane at L after $5\pi/1000$ hour it was fired.

Comments: One solver guessed the path should be a curve and decided to try a circular arc to start the problem. The other solvers derived the equation of the path by a differential equation as follows: using polar coordinates, since the spy plane has a constant angular velocity of 1000/10 = 100 rad/sec, so at time t, the spy plane is at $(10, 100t)$ and the rocket is at $(r(t), \theta(t))$. Since the rocket and the spy plane are on the same radius, so $\theta(t) = 100t$. Now they have the same speed, so

$$(r'(t))^2 + (r(t)\theta'(t))^2 = 10^6.$$

Then

$$\frac{r'(t)}{\sqrt{100 - r(t)^2}} = 100.$$

Integrating both sides from 0 to t, we get the equation $r = 10\sin(100t) = 10\sin\theta$, which describes the path above.

Problem 247. (a) Find all possible positive integers $k \geq 3$ such that there are k positive integers, every two of them are not relatively prime, but every three of them are relatively prime.

(b) Determine with proof if there exists an infinite sequence of positive integers satisfying the conditions in (a) above.

(*Source: 2003 Belarussian Math Olympiad*)

Solution. **G.R.A. 20 Math Problem Group** (Roma, Italy) and **YUNG Fai**.

(a) We shall prove by induction that the conditions are true for every positive integer $k \geq 3$.

For $k = 3$, the numbers 6, 10, 15 satisfy the conditions. Assume it is true for some $k \geq 3$ with the numbers being $a_1, a_2, \ldots, a_k$. Let $p_1, p_2, \ldots, p_k$ be distinct prime numbers such that each p_i is greater than $a_1a_2\ldots a_k$. For $I = 1$ to k, let $b_i = a_ip_i$ and let $b_{k+1} = p_1p_2\ldots p_k$. Then

$gcd(b_i, b_j) = gcd(a_i, a_j) > 1$ for $1 \leq i < j \leq k$,

$gcd(b_i, b_{k+1}) = p_i > 1$ for $1 \leq i \leq k$,

$gcd(b_h, b_i, b_j) = gcd(a_h, a_i, a_j) = 1$ for $1 \leq h \leq i < j \leq k$ and

$gcd(b_i, b_j, b_{k+1}) = 1$ for $1 \leq i < j \leq k$,

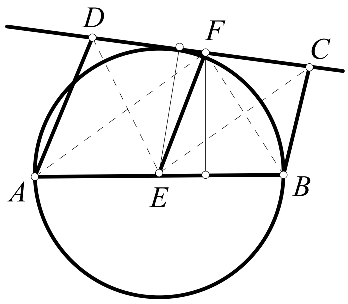
completing the induction.

(b) Assume there are infinitely many positive integers $a_1, a_2, a_3, \ldots$ satisfying the conditions in (a). Let a_1 have exactly m prime divisors. For $i = 2$ to $m + 2$, since each of the $m + 1$ numbers $gcd(a_1, a_i)$ is divisible by one of these m primes, by the pigeonhole principle, there are i, j with $2 \leq i < j \leq m + 2$ such that $gcd(a_1, a_i)$ and $gcd(a_1, a_j)$ are divisible by the same prime. Then $gcd(a_1, a_i, a_j) > 1$, a contradiction.

Commended solvers: **CHAN Nga Yi** (Carmel Divine Grace Foundation Secondary School, Form 6) and **CHAN Yat Sing** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 248. Let $ABCD$ be a convex quadrilateral such that line CD is tangent to the circle with side AB as diameter. Prove that line AB is tangent to the circle with side CD as diameter if and only if lines BC and AD are parallel.

Solution. **Jeff CHEN** (Virginia, USA) and **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).



Let E be the midpoints of AB. Since CD is tangent to the circle, the distance from E to line CD is $h_1 = AB/2$. Let F be the midpoint of CD and let h_2 be the distance from F to line AB. Observe that the areas of ΔCEF and ΔDEF = $CD{\cdot}AB/8$. Now

line AB is tangent to the circle with side CD as diameter
$\Leftrightarrow$ h_2=$CD/2$
$\Leftrightarrow$ areas of ΔAEF, ΔBEF, ΔCEF and ΔDEF are equal to $AB{\cdot}CD/8$
$\Leftrightarrow$ $AD \| EF$, $BC \| EF$
$\Leftrightarrow$ $AD \| BC$.

Problem 249. For a positive integer n, if $a_1, \cdots, a_n, b_1, \cdots, b_n$ are in [1,2] and $a_1^2+\cdots+a_n^2 = b_1^2+\cdots+b_n^2$, then prove that

$$\frac{a_1^3}{b_1}+\cdots+\frac{a_n^3}{b_n} \le \frac{17}{10}(a_1^2+\cdots+a_n^2).$$

Solution. **Jeff CHEN** (Virginia, USA).

For x, y in [1,2], we have

$$\begin{aligned} & 1/2 \le x/y \le 2 \\ \Leftrightarrow\quad & y/2 \le x \le 2y \\ \Leftrightarrow\quad & (y/2-x)(2y-x) \le 0 \\ \Leftrightarrow\quad & x^2+y^2 \le 5xy/2. \end{aligned}$$

Let $x = a_i$ and $y = b_i$, then $a_i^2 + b_i^2 \le 5a_ib_i/2$. Summing and manipulating, we get

$$-\sum_{i=1}^n a_ib_i \le -\frac{2}{5}\sum_{i=1}^n (a_i^2+b_i^2) = -\frac{4}{5}\sum_{i=1}^n a_i^2.$$

Let $x = (a_i^3/b_i)^{1/2}$ and $y = (a_ib_i)^{1/2}$. Then $x/y = a_i/b_i$ in [1,2]. So $a_i^3/b_i + a_ib_i \le 5a_i^2/2$.
Summing, we get

$$\sum_{i=1}^n \frac{a_i^3}{b_i} + \sum_{i=1}^n a_ib_i \le \frac{5}{2}\sum_{i=1}^n a_i^2.$$

Adding the two displayed inequalities, we get

$$\frac{a_1^3}{b_1}+\cdots+\frac{a_n^3}{b_n} \le \frac{17}{10}(a_1^2+\cdots+a_n^2).$$

Problem 250. Prove that every region with a convex polygon boundary cannot be dissected into finitely many regions with nonconvex quadrilateral boundaries.

Solution. **YUNG Fai.**

Assume the contrary that there is a dissection of the region into nonconvex quadrilateral $R_1, R_2, \ldots, R_n$. For a nonconvex quadrilateral R_i, there is a vertex where the angle is $\theta_i > 180^\circ$, which we refer to as the large vertex of the quadrilateral. The three other vertices, where the angles are less than 180° will be referred to as small vertices.

Since the boundary of the region is a convex polygon, all the large vertices are in the interior of the region. At a large vertex, one angle is $\theta_i > 180^\circ$, while the remaining angles are angles of small vertices of some of the quadrilaterals and add up to $360^\circ - \theta_i$. Now

$$\sum_{i=1}^n (360^\circ - \theta_i)$$

accounts for all the angles associated with all the small vertices. This is a contradiction since this will leave no more angles from the quadrilaterals to form the angles of the region.

Olympiad Corner

(continued from page 1)

Part 2, Day 1 (June 8, 2005)

Problem 1. Determine all triples of positive integers (a,b,c), such that $a + b + c$ is the least common multiple of a, b and c.

Problem 2. Let a, b, c, d be positive real numbers. Prove

$$\frac{a+b+c+d}{abcd} \le \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3}.$$

Problem 3. In an acute-angled triangle ABC, circle k_1 with diameter AC and k_2 with diameter BC are drawn. Let E be the foot of B on AC and F be the foot of A on BC. Furthermore, let L and N be the points in which the line BE intersects with k_1 (with L lying on the segment BE) and K and M be the points in which the line AF intersects with k_2 (with K on the segment AF). Prove that $KLMN$ is a cyclic quadrilateral.

Part 2, Day 2 (June 9, 2005)

Problem 4. The function f is defined for all integers {0, 1, 2, …, 2005}, assuming non-negative integer values in each case. Furthermore, the following conditions are fulfilled for all values of x for which the function is defined:

$$f(2x+1) = f(2x),\quad f(3x+1) = f(3x) \quad \text{and} \quad f(5x+1) = f(5x).$$

How many different values can the function assume at most?

Problem 5. Determine all sextuples (a,b,c,d,e,f) of real numbers, such that the following system of equations is fulfilled:

$$\begin{aligned} &4a=(b+c+d+e)^4,\ 4b=(c+d+e+f)^4,\\ &4c=(d+e+f+a)^4,\ 4d=(e+f+a+b)^4,\\ &4e=(f+a+b+c)^4,\ 4f=(a+b+c+d)^4. \end{aligned}$$

Problem 6. Let Q be a point in the interior of a cube. Prove that an infinite number of lines passing through Q exists, such that Q is the mid-point of the line-segment joining the two points P and R in which the line and the cube intersect.

Angle Bisectors Bisect Arcs

(continued from page 2)

Let H be the orthocenter of ΔABC. Let line AH intersect BC at D and the circumcircle of ΔABC again at A_2. Note

$$\begin{aligned} \angle A_2BC &= \angle A_2AC \\ &= \angle DAC \\ &= 90^\circ - \angle ACD \\ &= \angle HBC. \end{aligned}$$

Similarly, we have $\angle A_2CB = \angle HCB$. Then $\Delta BA_2C \cong \Delta BHC$. Since A_1 is the midpoint of arc BA_1C, it is at least as far from chord BC as A_2. So the area of ΔBA_1C is at least the area of ΔBA_2C. Then the area of quadrilateral BA_1CH is at least twice the area of ΔBHC.

Cutting hexagon $AC_1BA_1CB_1$ into three quadrilaterals with common vertex H and comparing with cutting ΔABC into three triangles with common vertex H in terms of areas, we get the conclusion of (ii).

Remarks. In the solution of (ii), we saw the orthocenter H of ΔABC has the property that $\Delta BA_2C \cong \Delta BHC$ (hence, also $HD = A_2D$). These are useful facts for problems related to the orthocenters involving the circumcircles.

Mathematical Excalibur

Volume 11, Number 3 *June 2006 – October 2006*

Olympiad Corner

The following were the problems of the IMO 2006.

Day 1 (July 12, 2006)

Problem 1. Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Problem 2. Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Problem 3. Determine the least real number M such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b and c.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 25, 2006***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Summation by Parts

Kin Y. Li

In calculus, we have a formula called <u>*integration by parts*</u>

$$\int_s^t f(x)g(x)dx = F(t)g(t) - F(s)g(s) - \int_s^t F(x)g'(x)dx,$$

where $F(x)$ is an anti-derivative of $f(x)$. There is a discrete version of this formula for series. It is called <u>*summation by parts*</u>, which asserts

$$\sum_{k=1}^n a_k b_k = A_n b_n - \sum_{k=1}^{n-1} A_k(b_{k+1} - b_k),$$

where $A_k = a_1+a_2+\cdots+a_k$. This formula follows easily by observing that $a_1 = A_1$ and for $k > 1$, $a_k = A_k - A_{k-1}$ so that

$$\sum_{k=1}^n a_k b_k = A_1 b_1 + (A_2 - A_1)b_2 + \cdots + (A_n - A_{n-1})b_n$$
$$= A_n b_n - A_1(b_2 - b_1) - \cdots - A_{n-1}(b_n - b_{n-1})$$
$$= A_n b_n - \sum_{k=1}^{n-1} A_k(b_{k+1} - b_k).$$

From this identity, we can easily obtain some famous inequalities.

<u>Abel's Inequality.</u> Let $m \leq \sum_{i=1}^k a_i \leq M$ for $k = 1,2,\ldots,n$ and $b_1 \geq b_2 \geq \cdots \geq b_n > 0$. Then

$$b_1 m \leq \sum_{k=1}^n a_k b_k \leq b_1 M.$$

<u>Proof.</u> Let $A_k = a_1+a_2+\cdots+a_k$. Applying summation by parts, we have

$$\sum_{k=1}^n a_k b_k = A_n b_n + \sum_{k=1}^{n-1} A_k(b_k - b_{k+1}).$$

The right side is at least

$$mb_n + \sum_{k=1}^{n-1} m(b_k - b_{k+1}) = mb_1$$

and at most

$$Mb_n + \sum_{k=1}^{n-1} M(b_k - b_{k+1}) = Mb_1.$$

<u>K. L. Chung's Inequality.</u> Suppose $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1,2,\ldots,n$. Then

$$\sum_{i=1}^n a_i^2 \leq \sum_{i=1}^n b_i^2.$$

<u>Proof.</u> Applying summation by parts and Cauchy-Schwarz' inequality, we have

$$\sum_{i=1}^n a_i^2 = \left(\sum_{i=1}^n a_i\right)a_n + \sum_{k=1}^{n-1}\left(\sum_{i=1}^k a_i\right)(a_k - a_{k+1})$$
$$\leq \left(\sum_{i=1}^n b_i\right)a_n + \sum_{k=1}^{n-1}\left(\sum_{i=1}^k b_i\right)(a_k - a_{k+1})$$
$$= \sum_{i=1}^n a_i b_i$$
$$\leq \left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2}.$$

Squaring and simplifying, we get

$$\sum_{i=1}^n a_i^2 \leq \sum_{i=1}^n b_i^2.$$

Below we will do some more examples to illustrate the usefulness of the summation by parts formula.

<u>Example 1.</u> (*1978 IMO*) Let n be a positive integer and $a_1, a_2, \cdots, a_n$ be a sequence of distinct positive integers. Prove that

$$\sum_{k=1}^n \frac{a_k}{k^2} \geq \sum_{k=1}^n \frac{1}{k}.$$

<u>Solution.</u> Since the a_i's are distinct positive integers, $A_k = a_1 + a_2 + \cdots + a_k$ is at least $1 + 2 + \cdots + k = k(k+1)/2$.

Applying summation by parts, we have

$$\sum_{k=1}^n \frac{a_k}{k^2} = \frac{A_n}{n^2} + \sum_{k=1}^{n-1} A_k\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right)$$
$$\geq \frac{n(n+1)/2}{n^2} + \sum_{k=1}^{n-1}\frac{k(k+1)}{2}\frac{(2k+1)}{k^2(k+1)^2}$$
$$= \frac{1}{2}\left(1 + \frac{1}{n} + \sum_{k=1}^{n-1}\frac{2k+1}{k(k+1)}\right)$$
$$= \frac{1}{2}\left(1 + \frac{1}{n} + \sum_{k=1}^{n-1}\left(\frac{1}{k} + \frac{1}{k+1}\right)\right)$$
$$= \frac{1}{2}\left(\sum_{k=1}^{n}\frac{1}{k} + \sum_{k=0}^{n-1}\frac{1}{k+1}\right)$$
$$= \sum_{k=1}^n \frac{1}{k}.$$

Example 2. (*1982 USAMO*) If x is a positive real number and n is a positive integer, then prove that

$$[nx] \ge \frac{[x]}{1} + \frac{[2x]}{2} + \frac{[3x]}{3} + \cdots + \frac{[nx]}{n},$$

where $[t]$ denotes the greatest integer less than or equal to t.

Solution. Let $a_k = [kx]/k$. Then

$$A_k = \sum_{i=1}^{k} \frac{[ix]}{i}.$$

In terms of A_k, we are to prove $[nx] \ge A_n$. The case $n = 1$ is easy. Suppose the cases 1 to $n - 1$ are true. Applying summation by parts, we have

$$\sum_{k=1}^{n}[kx] = \sum_{k=1}^{n} a_k k = A_n n - \sum_{k=1}^{n-1} A_k.$$

Using this and the inductive hypothesis,

$$\begin{aligned} A_n n &= \sum_{k=1}^{n}[kx] + \sum_{k=1}^{n-1} A_k \\ &\le \sum_{k=1}^{n}[kx] + \sum_{k=1}^{n-1}[kx] \\ &= [nx] + \sum_{k=1}^{n-1}\big([kx] + [(n-k)x]\big) \\ &\le [nx] + \sum_{k=1}^{n-1}[kx + (n-k)x] \\ &= n[nx], \end{aligned}$$

which yields case n.

Example 3. Consider a polygonal line $P_0P_1P_2\ldots P_n$ such that $\angle P_0P_1P_2 = \angle P_1P_2P_3 = \cdots = \angle P_{n-2}P_{n-1}P_n$, all measure in counterclockwise direction. If $P_0P_1 > P_1P_2 > \cdots > P_{n-1}P_n$, show that P_0 and P_n cannot coincide.

Solution. Let a_k be the length of $P_{k-1}P_k$. Consider the complex plane. Each P_k corresponds to a complex number. We may set $P_0 = 0$ and $P_1 = a_1$. Let $\theta = \angle P_0P_1P_2$ and $z = -\cos\theta + i\sin\theta$, then $P_n = a_1 + a_2 z + \cdots + a_n z^{n-1}$. Applying summation by parts, we get

$$\begin{aligned} P_n &= (a_1 - a_2) + (a_2 - a_3)(1+z) + \cdots \\ &\quad + a_n(1 + z + \cdots + z^{n-1}). \end{aligned}$$

If $\theta = 0$, then $z = 1$ and $P_n > 0$. If $\theta \ne 0$, then assume $P_n = 0$. We get $P_n(1-z) = 0$, which implies

$$\begin{aligned} &(a_1 - a_2)(1-z) + (a_2 - a_3)(1-z^2) + \cdots \\ &\quad + a_n(1-z^n) = 0. \end{aligned}$$

Then

$$\begin{aligned} &(a_1 - a_2) + (a_2 - a_3) + \cdots + a_n = \\ &(a_1 - a_2)z + (a_2 - a_3)z^2 + \cdots + a_n z^n. \end{aligned}$$

However, since $|z| = 1$ and $z \ne 1$, by the triangle inequality,

$$\begin{aligned} &|(a_1 - a_2)z + (a_2 - a_3)z^2 + \cdots + a_n z^n| \\ &< |(a_1 - a_2)z| + |(a_2 - a_3)z^2| + \cdots + |a_n z^n| \\ &= (a_1 - a_2) + (a_2 - a_3) + \cdots + a_n, \end{aligned}$$

which is a contradiction to the last displayed equation. So $P_n \ne 0 = P_0$.

Example 4. Show that the series $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ converges.

Solution. Let $a_k = \sin k$ and $b_k = 1/k$. Using the identity

$$\sin m \sin\frac{1}{2} = \frac{\cos(m - \frac12) - \cos(m + \frac12)}{2},$$

we get

$$A_k = \sin 1 + \cdots + \sin k = \frac{\cos\frac12 - \cos(k + \frac12)}{2\sin\frac12}.$$

Then $|A_k| \le 1/(\sin \frac12)$ and hence

$$\lim_{n\to\infty} A_n b_n = 0.$$

Applying summation by parts, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\sin k}{k} &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k b_k \\ &= \lim_{n\to\infty}\Big(A_n b_n - \sum_{k=1}^{n-1} A_k(b_{k+1} - b_k)\Big) \\ &= \sum_{k=1}^{\infty} A_k\left(\frac{1}{k} - \frac{1}{k+1}\right). \end{aligned}$$

Since

$$\sum_{k=1}^{\infty}\left|A_k\left(\frac1k - \frac{1}{k+1}\right)\right| \le \frac{1}{\sin\frac12}\sum_{k=1}^{\infty}\left(\frac1k - \frac{1}{k+1}\right) = \frac{1}{\sin\frac12},$$

so $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ converges.

Example 5. Let $a_1 \ge a_2 \ge \cdots \ge a_n$ with $a_1 \ne a_n$, $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} |x_i| = 1$. Find the least number m such that

$$\left|\sum_{i=1}^{n} a_i x_i\right| \le m(a_1 - a_n)$$

always holds.

Solution. Let $S_i = x_1 + x_2 + \cdots + x_i$. Let

$$p = \sum_{x_i > 0} x_i, \quad q = -\sum_{x_i < 0} x_i.$$

Then $p - q = 0$ and $p + q = 1$. So $p = q = \frac12$. Thus, $-\frac12 \le S_k \le \frac12$ for $k = 1, 2, \cdots, n$.

Applying summation by parts, we get

$$\begin{aligned} \left|\sum_{i=1}^{n} a_i x_i\right| &= \left|S_n a_n - \sum_{k=1}^{n-1} S_k(a_{k+1} - a_k)\right| \\ &\le \sum_{k=1}^{n-1} |S_k|(a_k - a_{k+1}) \\ &\le \sum_{k=1}^{n-1} \frac12 (a_k - a_{k+1}) \\ &= \frac12 (a_1 - a_n). \end{aligned}$$

When $x_1 = 1/2$, $x_n = -1/2$ and all other $x_i = 0$, we have equality. So the least such m is 1/2.

Example 6. Prove that for all real numbers $a_1, a_2, \ldots, a_n$, there is an integer m among $1, 2, \ldots, n$ such that if

$$0 \le \theta_n \le \theta_{n-1} \le \cdots \le \theta_1 \le \frac{\pi}{2},$$

then $\left|\sum_{i=1}^{n} a_i \sin\theta_i\right| \le \left|\sum_{i=1}^{m} a_i\right|.$

Solution. Let $A_i = a_1 + a_2 + \cdots + a_i$ and $b_i = \sin\theta_i$, then $1 \ge b_1 \ge b_2 \ge \cdots \ge b_n \ge 0$. Next let $|A_m|$ be the maximum among $|A_1|, |A_2|, \ldots, |A_n|$. With $a_{n+1} = b_{n+1} = 0$, we apply summation by parts to get

$$\begin{aligned} \left|\sum_{i=1}^{n} a_i \sin\theta_i\right| &= \left|\sum_{i=1}^{n+1} a_i b_i\right| \\ &= \left|\sum_{i=1}^{n} A_i(b_{i+1} - b_i)\right| \\ &\le \sum_{i=1}^{n} |A_m|(b_i - b_{i+1}) \\ &= |A_m| b_1 \\ &\le |A_m|. \end{aligned}$$

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***November 25, 2006.***

Problem 256. Show that there is a rational number q such that

$$\sin 1^\circ \sin 2^\circ \cdots \sin 89^\circ \sin 90^\circ = q\sqrt{10}.$$

Problem 257. Let $n > 1$ be an integer. Prove that there is a unique positive integer $A < n^2$ such that $[n^2/A] + 1$ is divisible by n, where $[x]$ denotes the greatest integer less than or equal to x. (*Source: 1993 Jiangsu Math Contest*)

Problem 258. (*Due to Mihaiela Vizental and Alfred Eckstein, Arad, Romaina*) Show that if A, B, C are in the interval $(0, \pi/2)$, then

$$f(A,B,C) + f(B,C,A) + f(C,A,B) \ge 3,$$

where $f(x,y,z) = \dfrac{4\sin x + 3\sin y + 2\sin z}{2\sin x + 3\sin y + 4\sin z}$.

Problem 259. Let AD, BE, CF be the altitudes of acute triangle ABC. Through D, draw a line parallel to line EF intersecting line AB at R and line AC at Q. Let P be the intersection of lines EF and CB. Prove that the circumcircle of $\triangle PQR$ passes through the midpoint M of side BC.
(*Source: 1994 Hubei Math Contest*)

Problem 260. In a class of 30 students, number the students 1, 2, …, 30 from best to worst ability (no two with the same ability). Every student has the same number of friends in the class, where friendships are mutual. Call a student <u>*good*</u> if his ability is better than more than half of his friends. Determine the maximum possible number of good students in this class.
(*Source: 1998 Hubei Math Contest*)

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Solutions

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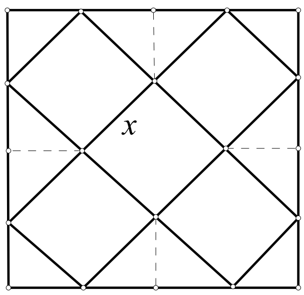
Problem 251. Determine with proof the largest number x such that a cubical gift of side x can be wrapped completely by folding a unit square of wrapping paper (without cutting).

Solution. **CHAN Tsz Lung** (Math, HKU) and **Jeff CHEN** (Virginia, USA).

Let A and B be two points inside or on the unit square such that the line segment AB has length d. After folding, the distance between A and B <u>along the surface of the cube</u> will be at most d because the line segment AB on the unit square after folding will provide one path between the two points along the surface of the cube, which may or may not be the shortest possible.

In the case A is the center of the unit square and B is the point opposite to A on the surface of the cube with respect to the center of the cube, then the distance along the surface of the cube between them is at least $2x$. Hence, $2x \le d \le \sqrt{2}/2$. Therefore, $x \le \sqrt{2}/4$.

The maximum $x = \sqrt{2}/4$ is attainable can be seen by considering the following picture of the unit square.



Commended solvers: **Alex O Kin-Chit** (STFA Cheng Yu Tung Secondary School) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7).

Problem 252. Find all polynomials $f(x)$ with integer coefficients such that for every positive integer n, $2^n - 1$ is divisible by $f(n)$.

Solution. **Jeff CHEN** (Virginia, USA) and **G.R.A. 20 Math Problem Group** (Roma, Italy).

We will prove that the only such polynomials $f(x)$ are the constant polynomials 1 and −1.

Assume $f(x)$ is such a polynomial and $|f(n)| \ne 1$ for some $n > 1$. Let p be a prime which divides $f(n)$, then p also divides $f(n+kp)$ for every integer k. Therefore, p divides $2^{n+kp}-1$ for all integers $k \ge 0$.

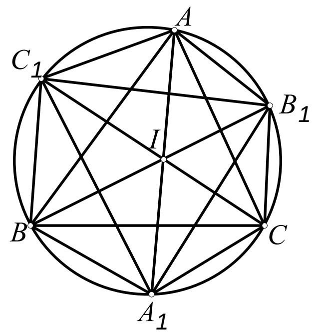
When $k = 0$, p divides $2^n - 1$, which implies $2^n \equiv 1 \pmod p$. By Fermat's little theorem, $2^p \equiv 2 \pmod p$. Finally, when $k = 1$, we get

$$1 \equiv 2^{n+p} = 2^n\, 2^p \equiv 1\cdot 2 = 2 \pmod p$$

implying p divides $2 - 1 = 1$, which is a contradiction.

Problem 253. Suppose the bisector of $\angle BAC$ intersect the arc opposite the angle on the circumcircle of ΔABC at A_1. Let B_1 and C_1 be defined similarly. Prove that the area of $\Delta A_1B_1C_1$ is at least the area of ΔABC.

Solution. **CHAN Tsz Lung** (Math, HKU), **Jeff CHEN** (Virginia, USA) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7).



By a well-known property of the incenter I (see page 1 of <u>Mathematical Excalibur</u>, vol. 11, no. 2), we have $AC_1 = C_1I$ and $AB_1 = B_1I$. Hence, $\Delta AC_1B_1 \cong \Delta IC_1B_1$. Similarly, $\Delta BA_1C_1 \cong \Delta IA_1C_1$ and $\Delta CB_1A_1 \cong \Delta IB_1A_1$. Letting $[\cdots]$ denote area, we have

$$[AB_1CA_1BC_1] = 2[A_1B_1C_1].$$

If ΔABC is not acute, say $\angle BAC$ is not acute, then

$$[ABC] \le \frac{1}{2}[ABA_1C]$$

$$\le \frac{1}{2}[AB_1CA_1BC_1] = [A_1B_1C_1].$$

Otherwise, ΔABC is acute and we can apply the fact that

$$[ABC] \le \frac{1}{2}[AB_1CA_1BC_1] = [A_1B_1C_1]$$

(see example 6 on page 2 of <u>Mathematical Excalibur</u>, vol. 11, no. 2).

Commended solvers: **Samuel Liló Abdalla** (Brazil) and **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).

Problem 254. Prove that if a, b, $c > 0$, then

$$\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2 \geq 4\sqrt{3abc\,(a+b+c)}.$$

Solution 1. **José Luis Díaz-Barrero** (Universitat Politècnica de Catalunya, Barcelona, Spain) and **G.R.A. 20 Math Problem Group** (Roma, Italy).

Dividing both sides by $\sqrt{abc(a+b+c)}$, the inequality is equivalent to

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{a+b+c}}+\frac{(\sqrt{a+b+c})^3}{\sqrt{abc}} \geq 4\sqrt{3}.$$

By the AM-GM inequality,

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \geq 3(\sqrt{abc})^{1/3}.$$

Therefore, it suffices to show

$$\frac{3(\sqrt{abc})^{1/3}}{\sqrt{a+b+c}}+\frac{(\sqrt{a+b+c})^3}{\sqrt{abc}}=\frac{3}{t}+t^3 \geq 4\sqrt{3},$$

where again by the AM-GM inequality,

$$t=\frac{\sqrt{a+b+c}}{(\sqrt{abc})^{1/3}}=\sqrt{\frac{a+b+c}{(abc)^{1/3}}} \geq \sqrt{3}.$$

By the AM-GM inequality a third time,

$$\frac{3}{t}+t^3=\frac{3}{t}+\frac{t^3}{3}+\frac{t^3}{3}+\frac{t^3}{3} \geq \frac{4t^2}{\sqrt{3}} \geq 4\sqrt{3}.$$

Solution 2. **Alex O Kin-Chit** (STFA Cheng Yu Tung Secondary School).

By the AM-GM inequality, we have

$$a+b+c \geq 3(abc)^{1/3} \qquad (1)$$

and $$\sqrt{a}+\sqrt{b}+\sqrt{c} \geq 3(abc)^{1/6}. \qquad (2)$$

Applying (2), (1), the AM-GM inequality and (1) in that order below, we have

$$\begin{aligned}&\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})+(a+b+c)^2\\ &\geq 3(abc)^{2/3}+3(abc)^{1/3}(a+b+c)\\ &\geq 4\left(3(abc)^{2/3}(abc)(a+b+c)^3\right)^{1/4}\\ &\geq 4\left(3(abc)^{2/3}(abc)3(abc)^{1/3}(a+b+c)^2\right)^{1/4}\\ &= 4\sqrt{3abc(a+b+c)}.\end{aligned}$$

Commended solvers: **Samuel Liló Abdalla** (Brazil), **CHAN Tsz Lung** (Math, HKU), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7).

Problem 255. Twelve drama groups are to do a series of performances (with some groups possibly making repeated performances) in seven days. Each group is to see every other group's performance at least once in one of its day-offs.

Find with proof the minimum total number of performances by these groups.

Solution. **CHAN Tsz Lung** (Math, HKU).

Here are three important observations:

(1) Each group perform at least once.

(2) If more than one groups perform on the same day, then each of these groups will have to perform on another day so the other groups can see its performance in their day-offs.

(3) If a group performs exactly once, on the day it performs, it is the only group performing.

We will show the minimum number of performances is 22. The following performance schedule shows the case 22 is possible.

Day 1: Group 1
Day 2: Group 2
Day 3: Groups 3, 4, 5, 6
Day 4: Groups 7, 8, 9, 3
Day 5: Groups 10, 11, 4, 7
Day 6: Groups 12, 5, 8, 10
Day 7: Groups 6, 9, 11, 12.

Assume it is possible to do at most 21 performances. Let k groups perform exactly once, then $k + 2(12 - k) \leq 21$ will imply $k \geq 3$.

<u>Case 1: Exactly 3 groups perform exactly once</u>, say group 1 on day 1, group 2 on day 2 and group 3 on day 3.

(a) If at least 4 groups perform on one of the remaining 4 days, say groups 4, 5, 6, 7 on day 4, then by (2), each of them has to perform on one of the remaining 3 days. By the pigeonhole principle, two of groups 4, 5, 6, 7 will perform on the same day again later, say groups 4 and 5 perform on day 5. Then they will have to perform separately on the last 2 days for the other to see. Then groups 1, 2, 3 once each, groups 4, 5 thrice each and groups 6, 7, …, 12 twice each at least , resulting in at least

$$3 + 2 \times 3 + 7 \times 2 = 23$$

performances, contradiction.

(b) If at most 3 groups perform on each of the remaining 4 days, then there are at most $3 \times 4 = 12$ slots for performances. However, each of groups 4 to 12 has to perform at least twice, yielding at least $9 \times 2 = 18$ (> 12) performances, contradiction.

<u>Case 2: More than 3 groups perform exactly once, say k groups with $k > 3$.</u> By argument similar to case 1(a), we see at most 3 groups can perform on each of the remaining $7 - k$ days (meaning at most $3(7 - k)$ performance slots). Again, the remaining $12 - k$ groups have to perform at least twice, yielding $2(12 - k) \leq 3(7 - k)$, which implies $k \leq -3$, contradiction.

Commended solvers: **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7) and **Raúl A. SIMON** (Santiago, Chile).

Comments: This was a problem in the 1994 Chinese IMO team training tests. In the Chinese literature, there is a solution using the famous Sperner's theorem which asserts that for a set with n elements, the number of subsets so that no two with one contains the other is at most $\binom{n}{[n/2]}$. We hope to present this solution in a future article.

Olympiad Corner

(continued from page 1)

<u>Day 2 (July 13, 2006)</u>

Problem 4. Determine all pairs (x,y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problem 5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\cdots P(P(x))\cdots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Problem 6. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Mathematical Excalibur

Volume 11, Number 4 *November 2006 – December 2006*

Olympiad Corner

The 9th China Hong Kong Math Olympiad was held on Dec. 2, 2006. The following were the problems.

Problem 1. Let M be a subset of $\{1, 2, \ldots, 2006\}$ with the following property: For any three elements x, y and z ($x < y < z$) of M, $x + y$ does not divide z. Determine the largest possible size of M. Justify your claim.

Problem 2. For a positive integer k, let $f_1(k)$ be the square of the sum of the digits of k. (For example $f_1(123) = (1+2+3)^2 = 36$.) Let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{2007}(2^{2006})$. Justify your claim.

Problem 3. A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O. Let E be the intersection of diagonals AC and BD. If P is a point inside $ABCD$ such that

$$\angle PAB + \angle PCB = \angle PBC + \angle PDC = 90^\circ,$$

prove that O, P and E are collinear.

Problem 4. Let a_1, a_2, a_3,… be a sequence of positive numbers. If there exists a positive number M such that for every $n = 1, 2, 3, \ldots$,

$$a_1^2 + a_2^2 + \ldots + a_n^2 < M a_{n+1}^2,$$

then prove that there exists a positive number M' such that for every $n = 1, 2, 3, \ldots$,

$$a_1 + a_2 + \ldots + a_n < M' a_{n+1}.$$

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 25, 2007***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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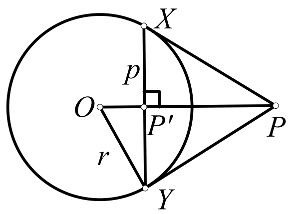
Pole and Polar

Kin Y. Li

Let C be a circle with center O and radius r. Recall the inversion with respect to C (see <u>*Mathematical Excalibur*</u>, *vol*. 9, *no*. 2, *p*.1) sends every point $P \neq O$ in the same plane as C to the image point P' on the ray $\overrightarrow{OP}$ such that $OP \cdot OP' = r^2$. The <u>*polar*</u> of P is the line p perpendicular to the line OP at P'. Conversely, for any line p not passing through O, the <u>*pole*</u> of p is the point P whose polar is p. The function sending P to p is called the <u>*pole-polar transformation*</u> (or <u>*reciprocation*</u>) with respect to O and r (or with respect to C).



Following are some useful facts:

(1) If P is outside C, then recall P' is found by drawing tangents from P to C, say tangent at X and Y. Then $P' = OP \cap XY$, where $\cap$ denotes intersection. By symmetry, $OP \perp XY$. So the polar p of P is the line XY.

Conversely, for distinct points X, Y on C, the pole of the line XY is the intersection of the tangents at X and Y. Also, it is the point P on the perpendicular bisector of XY such that O, X, P, Y are concyclic since $\angle OXP = 90^\circ = \angle OYP$.

(2) (<u>***La Hire's Theorem***</u>) Let x and y be the polars of X and Y, respectively. Then X is on line $y \Leftrightarrow Y$ is on line x.

<u>*Proof.*</u> Let X', Y' be the images of X, Y for the inversion with respect to C. Then $OX \cdot OX' = r^2 = OY \cdot OY'$ implies X, X', Y, Y' are concyclic. Now

$$\begin{aligned} X \text{ is on } y &\Leftrightarrow \angle XY'Y = 90^\circ \\ &\Leftrightarrow \angle XX'Y = 90^\circ \\ &\Leftrightarrow Y \text{ is on } x. \end{aligned}$$

(3) Let x,y,z be the polars of distinct points X,Y,Z respectively. Then $Z = x \cap y \Leftrightarrow z = XY$.

<u>*Proof.*</u> By La Hire's theorem, Z on $x \cap y$ $\Leftrightarrow$ X on z and Y on z $\Leftrightarrow$ $z = XY$.

(4) Let W, X, Y, Z be on C. The polar p of $P = XY \cap WZ$ is the line through $Q = WX \cap ZY$ and $R = XZ \cap YW$.

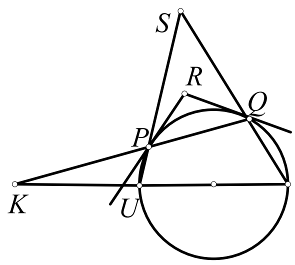
<u>*Proof.*</u> Let S, T be the poles of $s = XY$, $t = WZ$ respectively. Then $P = s \cap t$. By fact (3), $S = x \cap y$, $T = w \cap z$ and $p = ST$. For hexagon $WXXZYY$, we have

$$Q = WX \cap ZY, \quad S = XX \cap YY, \quad R = XZ \cap YW,$$

where XX denotes the tangent line at X. By Pascal's theorem (see <u>*Mathematical Excalibur*</u>, *vol*. 10, *no*. 3, *p*.1), Q,S,R are collinear. Similarly, considering the hexagon $XWWYZZ$, we see Q,T,R are collinear. Therefore, $p = ST = QR$.

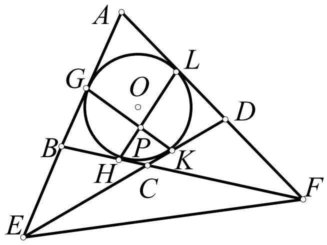
Next we will present some examples using the pole-polar transformation.

<u>***Example 1.***</u> Let UV be a diameter of a semicircle. P,Q are two points on the semicircle with $UP<UQ$. The tangents to the semicircle at P and Q meet at R. If $S = UP \cap VQ$, then prove that $RS \perp UV$.



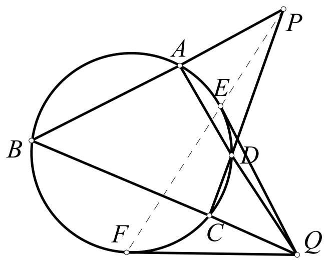
<u>***Solution (due to CHENG Kei Tsi).***</u> Let $K = PQ \cap UV$. With respect to the circle, by fact (4), the polar of K passes through $UP \cap VQ = S$. Since the tangents to the semicircle at P and Q meet at R, by fact (1), the polar of R is PQ. Since K is on line PQ, which is the polar of R, by La Hire's theorem, R is on the polar of K. So the polar of K is the line RS. As K is on the diameter UV extended, by the definition of polar, we get $RS \perp UV$.

Example 2. Quadrilateral $ABCD$ has an inscribed circle Γ with sides AB, BC, CD, DA tangent to Γ at G, H, K, L respectively. Let $AB\cap CD = E$, $AD\cap BC = F$ and $GK\cap HL = P$. If O is the center of Γ, then prove that $OP\perp EF$.



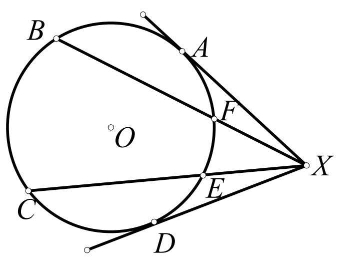
Solution. Consider the pole-polar transformation with respect to the inscribed circle. By fact (1), the polars of E, F are lines GK, HL respectively. Since $GK\cap HL = P$, by fact (3), the polar of P is line EF. By the definition of polar, we get $OP\perp EF$.

Example 3. (*1997 Chinese Math Olympiad*) Let $ABCD$ be a cyclic quadrilateral. Let $AB\cap CD = P$ and $AD\cap BC = Q$. Let the tangents from Q meet the circumcircle of $ABCD$ at E and F. Prove that P, E, F are collinear.



Solution. Consider the pole-polar transformation with respect to the circumcircle of $ABCD$. Since $P = AB\cap CD$, by fact (4), the polar of P passes through $AD\cap BC = Q$. By La Hire's theorem, P is on the polar of Q, which by fact (1), is the line EF.

Example 4. (*1998 Austrian-Polish Math Olympiad*) Distinct points A, B, C, D, E, F lie on a circle in that order. The tangents to the circle at the points A and D, the lines BF and CE are concurrent. Prove that the lines AD, BC, EF are either parallel or concurrent.

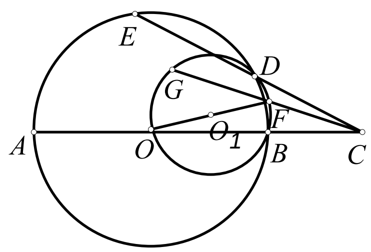


Solution. Let O be the center of the circle and $X = AA\cap DD\cap BF\cap CE$.

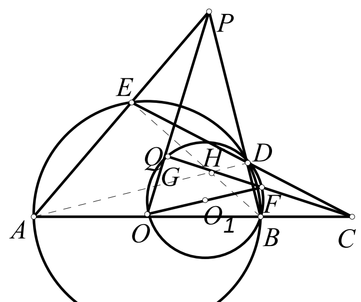
If $BC||EF$, then by symmetry, lines BC and EF are perpendicular to line OX. Since $AD\perp OX$, we get $BC||EF||AD$.

If lines BC, EF intersect, then by fact (4), the polar of $X = CE\cap BF$ passes through $BC\cap EF$. Since the tangents at A and D intersect at X, by fact (1), the polar of X is line AD. Therefore, AD, BC and EF are concurrent in this case.

Example 5. (*2006 China Western Math Olympiad*) As in the figure below, AB is a diameter of a circle with center O. C is a point on AB extended. A line through C cuts the circle with center O at D, E. OF is a diameter of the circumcircle of ΔBOD with center O_1. Line CF intersect the circumcircle again at G. Prove that O,A,E,G are concyclic.



Solution (due to WONG Chiu Wai). Let $AE\cap BD=P$. By fact (4), the polar of P with respect to the circle having center O is the line through $BA\cap DE = C$ and $AD\cap EB = H$. Then $OP\perp CH$. Let $Q = OP\cap CH$.



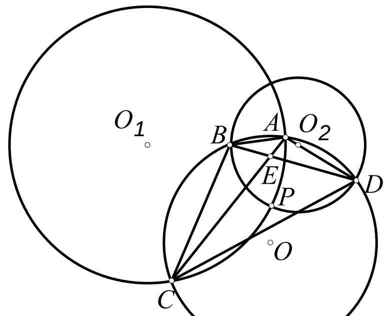
We claim $Q = G$. Once this shown, we will have $P = BD \cap OG$. Then $PE\cdot PA = PD\cdot PB = PG\cdot PO$, which implies O, A, E, G are concyclic.

To show $Q = G$, note that $\angle PQH$, $\angle PDH$ and $\angle PEH$ are 90°, which implies P, E, Q, H, D are concyclic. Then $\angle PQD = \angle PED = \angle DBO$, which implies Q, D, B, O are concyclic. Therefore, $Q = G$ since they are both the point of intersection (other than O) of the circumcircle of ΔBOD and the circle with diameter OC.

Example 6. (*2006 China Hong Kong Math Olympiad*) A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O. Let E be the intersection of diagonals AC and BD. If P is a point inside $ABCD$ such that

$$\angle PAB+\angle PCB=\angle PBC+\angle PDC=90^\circ,$$

prove that O, P and E are collinear.



Solution (due to WONG Chiu Wai). Let Γ, Γ_1, Γ_2 be the circumcircles of quadrilateral $ABCD$, ΔPAC, ΔPBD with centers O, O_1, O_2 respectively. We first show that the polar of O_1 with respect to Γ is line AC. Since OO_1 is the perpendicular bisector of AC, by fact (1), all we need to show is that

$$\angle AOC + \angle AO_1C = 180^\circ.$$

For this, note

$$\begin{aligned}&\angle APC\\ &= 360^\circ - (\angle PAB + \angle PCB + \angle ABC)\\ &= 270^\circ - \angle ABC\\ &= 90^\circ + \angle ADC\end{aligned}$$

and so

$$\begin{aligned}\angle AO_1C &= 2(180^\circ - \angle APC)\\ &= 2(90^\circ - \angle ADC)\\ &= 180^\circ - 2\angle ADC\\ &= 180^\circ - \angle AOC.\end{aligned}$$

Similarly, the polar of O_2 with respect to Γ is line BD. By fact (3), since $E = AC\cap BD$, the polar of E with respect to Γ is line O_1O_2. So $OE\perp O_1O_2$.

(Next we will consider radical axis and radical center, see <u>*Mathematical Excalibur*</u>, *vol*. 4, *no*. 3, *p*. 2.) Among Γ, Γ_1, Γ_2, two of the pairwise radical axes are lines AC and BD. This implies E is the radical center. Since Γ_1, Γ_2 intersect at P, so PE is the radical axis of Γ_1, Γ_2, which implies $PE\perp O_1O_2$. Combining with $OE\perp O_1O_2$ proved above, we see O, P and E are collinear.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***January 25, 2007.***

Problem 261. Prove that among any 13 consecutive positive integers, one of them has sum of its digits (in base 10) divisible by 7.

Problem 262. Let O be the center of the circumcircle of ΔABC and let AD be a diameter. Let the tangent at D to the circumcircle intersect line BC at P. Let line PO intersect lines AC, AB at M, N respectively. Prove that $OM = ON$.

Problem 263. For positive integers m, n, consider a $(2m+1)\times(2n+1)$ table, where in each cell, there is exactly one ant. At a certain moment, every ant moves to a horizontal or vertical neighboring cell. Prove that after that moment, there exists a cell with no ant.

Problem 264. For a prime number $p > 3$ and arbitrary integers a, b, prove that $ab^p - ba^p$ is divisible by $6p$.

Problem 265. Determine (with proof) the maximum of

$$\sum_{j=1}^{n}(x_j^4 - x_j^5),$$

where $x_1, x_2, \ldots, x_n$ are nonnegative real numbers whose sum is 1.

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Solutions

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Problem 256. Show that there is a rational number q such that

$$\sin 1^\circ \sin 2^\circ \cdots \sin 89^\circ \sin 90^\circ = q\sqrt{10}.$$

Solution 1. **Jeff CHEN** (Virginia, USA), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher), **G.R.A. 20 Math Problem Group** (Roma, Italy) and **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

Let $\omega = e^{2\pi i/180}$. Then

$$P(z) = \sum_{n=0}^{179} z^n = \prod_{k=1}^{179}(z - \omega^k).$$

Using $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i} = \dfrac{e^{2ix} - 1}{2ie^{ix}}$, we have

$$\prod_{k=1}^{90} \sin k^\circ = \prod_{k=1}^{90} \frac{\omega^k - 1}{2i\omega^{k/2}}.$$

Also, $\displaystyle\prod_{k=1}^{90} \sin k^\circ = \prod_{k=91}^{179} \sin k^\circ = \prod_{k=91}^{179} \frac{\omega^k - 1}{2i\omega^{k/2}}.$

Then

$$\left|\prod_{k=1}^{90} \sin k^\circ\right|^2 = \prod_{k=1}^{179} \frac{|\omega^k - 1|}{2} = \frac{|P(1)|}{2^{179}} = \frac{90}{2^{178}}.$$

Therefore, $\displaystyle\prod_{k=1}^{90} \sin k^\circ = \frac{3}{2^{89}}\sqrt{10}.$

Solution 2. **Jeff CHEN** (Virginia, USA), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).

Let S be the left-handed side. Note

$$\begin{aligned}\sin 3\theta &= \sin\theta\cos 2\theta + \cos\theta \sin 2\theta\\ &= \sin\theta(\cos^2\theta - \sin^2\theta + 2\cos^2\theta)\\ &= 4\sin\theta\left(\frac{3}{4}\cos^2\theta - \frac{1}{4}\sin^2\theta\right)\\ &= 4\sin\theta\sin(60^\circ - \theta)\sin(60^\circ + \theta).\end{aligned}$$

So, $\sin\theta\sin(60^\circ - \theta)\sin(60^\circ + \theta) = \dfrac{\sin 3\theta}{4}.$

Using this, we have

$$\begin{aligned}S &= \sin 30^\circ \sin 60^\circ \prod_{n=1}^{29} \sin n^\circ \sin(60^\circ - n^\circ)\sin(60^\circ + n^\circ)\\ &= \frac{\sqrt{3}}{4^{30}} \sin 3^\circ \sin 6^\circ \sin 9^\circ \cdots \sin 87^\circ\\ &= \frac{\sqrt{3}}{4^{30}} \sin 30^\circ \sin 60^\circ \prod_{m=1}^{9} \sin 3m^\circ \sin(60^\circ - 3m^\circ)\sin(60^\circ + 3m^\circ)\\ &= \frac{3}{4^{40}} \sin 9^\circ \sin 18^\circ \sin 27^\circ \cdots \sin 81^\circ\\ &= \frac{3}{4^{40}} \sin 9^\circ \cos 9^\circ \sin 18^\circ \cos 18^\circ \sin 27^\circ \cos 27^\circ \sin 36^\circ \cos 36^\circ \sin 45^\circ\\ &= \frac{3\sqrt{2}}{2^{85}} \sin 18^\circ \sin 36^\circ \sin 54^\circ \sin 72^\circ\\ &= \frac{3\sqrt{2}}{2^{85}} \sin 18^\circ \cos 18^\circ \sin 36^\circ \cos 36^\circ\\ &= \frac{3\sqrt{2}}{2^{87}} \sin 36^\circ \sin 72^\circ\\ &= \frac{3\sqrt{2}}{2^{87}} \frac{\sqrt{10-2\sqrt{5}}}{4} \frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{3}{2^{89}}\sqrt{10}.\end{aligned}$$

Problem 257. Let $n > 1$ be an integer. Prove that there is a unique positive integer $A < n^2$ such that $[n^2/A] + 1$ is divisible by n, where $[x]$ denotes the greatest integer less than or equal to x. (*Source: 1993 Jiangsu Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **G.R.A. 20 Math Problem Group** (Roma, Italy) and **Fai YUNG**.

We claim the unique number is $A = n+1$. If $n = 2$, then $1 \le A < n^2 = 4$ and only $A = 3$ works. If $n > 2$, then $[n^2/A]+1$ divisible by n implies $\dfrac{n^2}{A} + 1 \ge \left[\dfrac{n^2}{A}\right] + 1 \ge n.$ This leads to

$$A \le \frac{n^2}{n-1} = n + 1 + \frac{1}{n-1}.$$ So $A \le n + 1$.

The case $A = n + 1$ works because

$$\left[\frac{n^2}{n+1}\right] + 1 = (n-1) + 1 = n.$$

The case $A = n$ does not work because $[n^2/n] + 1 = n + 1$ is not divisible by n when $n > 1$.

For $0 < A < n$, assume $[n^2/A]+1 = kn$ for some positive integer k. This leads to

$$kn - 1 = \left[\frac{n^2}{A}\right] \le \frac{n^2}{A} < \left[\frac{n^2}{A}\right] + 1 = kn,$$

which implies $n < kA \le (n^2+A)/n < n+1$. This is a contradiction as kA is an integer and cannot be strictly between n and $n + 1$.

Problem 258. (*Due to Mihaiela Vizental and Alfred Eckstein, Arad, Romaina*) Show that if A, B, C are in the interval $(0, \pi/2)$, then

$$f(A,B,C) + f(B,C,A) + f(C,A,B) \ge 3,$$

where

$$f(x,y,z) = \frac{4\sin x + 3\sin y + 2\sin z}{2\sin x + 3\sin y + 4\sin z}.$$

Solution. **Samuel Liló Abdalla** (Brazil), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher) and **Fai YUNG**.

Note

$$f(x,y,z) + 1 = \frac{6\sin x + 6\sin y + 6\sin z}{2\sin x + 3\sin y + 4\sin z}.$$

For a, b, $c > 0$, by the *AM-HM* inequality, we have

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 9.$$

Multiplying by $\dfrac{2}{3}$ on both sides, we get

$$(a+b+c)\frac{2}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 6. \quad (*)$$

Let $r = \sin A$, $s = \sin B$, $t = \sin C$, $a = 1/(2r + 3s + 4t)$, $b = 1/(2s + 3t + 4r)$ and

$c = 1/(2t + 3r + 4s)$. Then

$$\frac{2}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=6r+6s+6t.$$

Using (*), we get

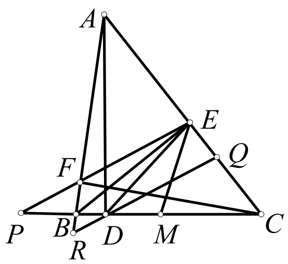
$$f(A,B,C)+f(B,C,A)+f(C,A,B)+3$$
$$=\frac{6r+6s+6t}{2r+3s+4t}+\frac{6r+6s+6t}{2s+3t+4r}+\frac{6r+6s+6t}{2t+3r+4s}$$
$$\geq 6.$$

The result follows.

Problem 259. Let *AD, BE, CF* be the altitudes of acute triangle *ABC*. Through *D*, draw a line parallel to line *EF* intersecting line *AB* at *R* and line *AC* at *Q*. Let *P* be the intersection of lines *EF* and *CB*. Prove that the circumcircle of ΔPQR passes through the midpoint *M* of side *BC*.

(*Source: 1994 Hubei Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA).



Observe that

(1) $\angle BFC = 90^\circ = \angle BEC$ implies *B, F, E, C* concyclic;

(2) $\angle AEB = 90^\circ = \angle ADB$ implies *A,B,D,E* concyclic.

By (1), we have $\angle ACB = \angle AFE$. From $EF||QR$, we get $\angle AFE = \angle ARQ$. So $\angle ACB = \angle ARQ$. Then *B, Q, R, C* are concyclic. By the intersecting chord theorem,

$$RD\cdot QD=BD\cdot CD \qquad (*)$$

Since $\angle BEC = 90^\circ$ and *M* is the midpoint of *BC*, we get $MB = ME$ and $\angle EBM = \angle BEM$. Now

$$\angle EBM = \angle EPM + \angle BEP$$
$$\angle BEM = \angle DEM + \angle BED.$$

By (1) and (2), $\angle BEP = \angle BCF = 90^\circ - \angle ABC = \angle BAD = \angle BED$. So $\angle EPM = \angle DEM$. Then right triangles *EPM* and *DEM* are similar. We have $ME/MP = MD/ME$ and so

$$MB^2=ME^2=MD\cdot MP=MD(MD+PD)$$
$$=MD^2+MD\cdot PD.$$

Then $MD\cdot PD = MB^2- MD^2$
$= (MB-MD)(MB+MD)$
$= BD(MC+MD)$
$= BD\cdot CD.$

Using (*), we get $RD\cdot QD=MD\cdot PD$. By the converse of the intersecting chord theorem, *P, Q, R, M* are concyclic.

Commended solvers: **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).

Problem 260. In a class of 30 students, number the students 1, 2, …, 30 from best to worst ability (no two with the same ability). Every student has the same number of friends in the class, where friendships are mutual. Call a student <u>*good*</u> if his ability is better than more than half of his friends. Determine the maximum possible number of good students in this class.
(*Source: 1998 Hubei Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA) and **Fai YUNG**.

Suppose each student has *m* friends and *n* is the maximum number of good students. There are 15*m* pairs of friendship.

For *m* odd, $m = 2k - 1$ for some positive integer *k*. For $j = 1, 2, \ldots, k$, student *j* has at least $(2k-j) \geq k > m/2$ worse friends, hence student *j* is good. For the other $n - k$ good students, every one of them has at least *k* worse friends. Then

$$\sum_{j=1}^{k}(2k-j)+(n-k)k \leq 15(2k-1).$$

Solving for *n*, we get

$$n \leq 30.5-\left(\frac{15}{k}+\frac{k}{2}\right)\leq 30.5-\sqrt{30}<26.$$

For *m* even, $m = 2k$ for some positive integer *k*. For $j = 1, 2, \ldots, k$, student *j* has at least $(2k + 1 - j) > k = m/2$ worse friends, hence student *j* is good. For the other $n - k$ good students, every one of them has at least $k + 1$ worse friends. Then

$$\sum_{j=1}^{k}(2k+1-j)+(n-k)(k+1) \leq 15\cdot 2k.$$

Solving for *n*, we get

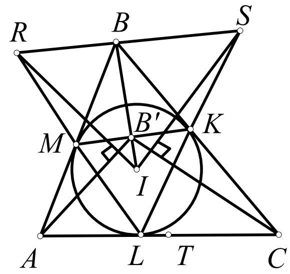
$$n \leq 31.5-\left(\frac{31}{k+1}+\frac{k+1}{2}\right)\leq 31.5-\sqrt{62}<24.$$

Therefore, $n \leq 25$. For an example of $n = 25$, in the odd case, we need to take $k = 5$ (so $m = 9$). Consider the 6×5 matrix *M* with $M_{ij}= 5(i-1)+j$. For M_{1j}, let his friends be M_{6j}, M_{1k} and M_{2k} for all $k \neq j$. For M_{ij} with $1 < i < 6$, let his friends be M_{6j}, $M_{(i-1)k}$ and $M_{(i+1)k}$ for all $k \neq j$. For M_{6j}, let his friends be M_{ij} and M_{5k} for all $i < 6$ and $k \neq j$. It is easy to check 1 to 25 are good.

Pole and Polar

(continued from page 2)

<u>***Example 7.***</u> (*1998 IMO*) Let *I* be the incenter of triangle *ABC*. Let the incircle of *ABC* touch the sides *BC*, *CA* and *AB* at *K, L* and *M* respectively. The line through *B* parallel to *MK* meets the lines *LM* and *LK* at *R* and *S* respectively. Prove that angle *RIS* is acute.



<u>***Solution.***</u> Consider the pole-polar transformation with respect to the incircle. Due to tangency, the polars of *B, K, L, M* are lines *MK, BC, CA, AB* respectively. Observe that *B* is sent to $B' = IB\cap MK$ under the inversion with respect to the incircle. Since *B'* is on line *MK*, which is the polar of *B*, by La Hire's theorem, *B* is on the polar of *B'*. Since $MK||RS$, so the polar of *B'* is line *RS*. Since *R,B,S* are collinear, their polars concur at *B'*.

Next, since the polars of *K, L* intersect at *C* and since *L, K, S* are collinear, their polars concur at *C*. Then the polar of *S* is *B'C*. By the definition of polar, we get $IS\perp B'C$. By a similar reasoning, we also get $IR\perp B'A$. Then $\angle RIS = 180^\circ - \angle AB'C$.

To finish, we will show *B'* is inside the circle with diameter *AC*, which implies $\angle AB'C > 90^\circ$ and hence $\angle RIS < 90^\circ$. Let *T* be the midpoint of *AC*. Then

$$2\overrightarrow{B'T} = \overrightarrow{B'C} + \overrightarrow{B'A}$$
$$= (\overrightarrow{B'K} + \overrightarrow{KC}) + (\overrightarrow{B'M} + \overrightarrow{MA})$$
$$= \overrightarrow{KC} + \overrightarrow{MA}.$$

Since $\overrightarrow{KC}$ and $\overrightarrow{MA}$ are nonparallel,

$$B'T < \frac{KC+MA}{2} = \frac{CL+AL}{2} = \frac{AC}{2}.$$

Therefore, *B'* is inside the circle with diameter *AC*.

Mathematical Excalibur

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Olympiad Corner

Below are the 2006 British Math Olympiad (Round 2) problems.

Problem 1. Find the minimum possible value of x^2+y^2 given that x and y are real numbers satisfying $xy(x^2-y^2)=x^2+y^2$ and $x\neq 0$.

Problem 2. Let x and y be positive integers with no prime factors larger than 5. Find all such x and y which satisfy $x^2-y^2=2^k$ for some non-negative integer k.

Problem 3. Let ABC be a triangle with $AC > AB$. The point X lies on the side BA extended through A, and the point Y lies on the side CA in such a way that $BX = CA$ and $CY = BA$. The line XY meets the perpendicular bisector of side BC at P. Show that $\angle BPC+\angle BAC=180^\circ$.

Problem 4. An exam consisting of six questions is sat by 2006 children. Each question is marked right or wrong. Any three children have right answers to at least five of the six questions between them. Let N be the total number of right answers achieved by all the children (i.e. the total number of questions solved by child 1 + the total solved by child 2 + ··· + the total solved by child 2006). Find the least possible value of N.

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

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Difference Operator

Kin Y. Li

Let h be a nonzero real number and $f(x)$ be a function. When $f(x+h)$ and $f(x)$ are real numbers, we call

$$\Delta_h f(x)=f(x+h)-f(x)$$

the *first difference of f at x with step h*. For functions f, g and real number c, we have

$$\Delta_h(f+g)(x)=\Delta_h f(x)+\Delta_h g(x) \text{ and } \Delta_h(cf)(x)=c\Delta_h f(x).$$

Also, $\Delta_h^0 f(x)$ or $I\,f(x)$ stands for $f(x)$.

For any integer $n\geq 1$, we define the *n-th difference* by $\Delta_h^n f(x)=\Delta_h(\Delta_h^{n-1}f)(x)$.

For example,

$$\Delta_h^2 f(x)=f(x+2h)-2f(x+h)+f(x),$$

$$\Delta_h^3 f(x)=f(x+3h)-3f(x+2h)+3f(x+h)-f(x).$$

By induction, we can check that

$$\Delta_h^n f(x)=\sum_{k=0}^{n}(-1)^{n-k}C_n^k f(x+kh), \qquad (\alpha)$$

where $C_n^0=1$ and for $k>0$,

$$C_n^k=\binom{n}{k}=\frac{n(n-1)\cdots(n-k+1)}{k!}.$$

(*Note*: for these formulas, we may even let n be a real number!!!)

If h=1, we simply write Δ and omit the subscript h. For example, in case of a sequence $\{x_n\}$, we have $\Delta x_n=x_{n+1}-x_n$.

Facts.(1) For function $f(x)$, n=0,1,2,…,

$$f(x+n)=\sum_{k=0}^{n}C_n^k\Delta^k f(x);$$

in particular, if $\Delta^m f(n)$ is a nonzero constant for every positive integer n, then $f(n)=\sum_{k=0}^{m}C_n^k\Delta^k f(0)$;

(2) if $P(x)=ax^n+\cdots$ is a polynomial of degree n, then for all x,

$\Delta_h^n P(x)=an!h^n$ and $\Delta_h^m P(x)=0$ for $m>n$.

Let k be a positive integer. As a function of x, C_x^k has the properties:

(a) $C_x^{k-1}+C_x^k=C_{x+1}^k$ (so $\Delta C_x^k=C_x^{k-1}$);

(b) for $0\leq r\leq k$, $\Delta^r C_x^k=C_x^{k-r}$;

for $r>k$, $\Delta^r C_x^k=0$;

(c) $C_1^k+C_2^k+\cdots+C_n^k=C_{n+1}^{k+1}$ (just add C_1^{k+1}=0 to the left and apply (a) repeatedly).

Similar to fact (1), if $f(x)$ is a degree m polynomial, then

$$f(x)=\sum_{k=0}^{m}C_x^k\Delta^k f(0). \qquad (\beta)$$

(This is because both sides are degree m polynomials and from property (b), the k-th differences at 0 are the same for k = 0 to m, which implies the values of both sides at 0, 1, 2, …, m are the same.)

Example 1. Sum $S_n=1^4+2^4+\cdots+n^4$ in terms of n.

Solution. Let $f(x)=x^4$. By (β) and (c),

$$S_n=\sum_{j=1}^{n}f(j)=\sum_{j=1}^{n}\sum_{k=0}^{4}C_j^k\Delta^k f(0)$$

$$=\sum_{k=0}^{4}\Big(\sum_{j=1}^{n}C_j^k\Big)\Delta^k f(0)=\sum_{k=0}^{4}C_{n+1}^{k+1}\Delta^k f(0).$$

x :	0	1	2	3	4
$f(x)$:	0	1	16	81	256
$\Delta f(x)$:	1	15	65	175	
$\Delta^2 f(x)$:	14	50	110		
$\Delta^3 f(x)$:	36	60			
$\Delta^4 f(x)$:	24				

Therefore,

$$S_n=\binom{n+1}{2}+14\binom{n+1}{3}+36\binom{n+1}{4}+24\binom{n+1}{5}$$

$$=n(n+1)(2n+1)(3n^2+3n-1)/30.$$

Example 2. (*2000 Chinese IMO Team Selection Test*) Given positive integers k, m, n satisfying $1\leq k\leq m\leq n$. Find

$$\sum_{i=0}^{n}(-1)^i\frac{1}{n+k+i}\frac{(m+n+i)!}{i!(n-i)!(m+i)!}.$$

Solution. Define

$$g(x)=\frac{(x+m+1)(x+m+2)\cdots(x+m+n)}{x+n+k}.$$

From $1\leq k\leq m\leq n$, we see $m+1\leq n+k\leq m+n$. So $g(x)$ is a polynomial of degree $n-1$. By fact (2) and formula (α),

$$0=(-1)^n\Delta^n g(0)=\sum_{i=0}^{n}(-1)^i C_n^i g(i)$$

$$=\sum_{i=0}^{n}(-1)^i\frac{n!}{i!(n-i)!}\frac{(m+n+i)!}{(m+i)!}\frac{1}{n+k+i}.$$

The required sum is $(-1)^n\,\Delta^n g(0)/n! = 0$.

Example 3. (*1949 Putnam Exam*) The sequence $x_0, x_1, x_2, \ldots$ is defined by the conditions $x_0 = a$, $x_1 = b$ and for $n \ge 1$,

$$x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n},$$

where a and b are given numbers. Express $\lim_{n\to\infty} x_n$ in terms of a and b.

Solution. The recurrence relation can be written as

$$\Delta x_n = -\frac{\Delta x_{n-1}}{2n}.$$

Repeating this $n-2$ times, we get

$$\Delta x_n = \left(-\frac{1}{2}\right)^n \frac{1}{n!}\Delta x_1 = \left(-\frac{1}{2}\right)^n \frac{1}{n!}(b-a).$$

Then

$$x_n = x_0 + \sum_{i=0}^{n-1}\Delta x_i = a + (b-a)\sum_{i=0}^{n-1}\left(-\frac{1}{2}\right)^i \frac{1}{i!}.$$

Using the fact

$$e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!},$$

we get $\lim_{n\to\infty} x_n = a + (b-a)e^{-1/2}$.

Example 4. (*2004 Chinese Math Olympiad*) Given a positive integer c, let $x_1, x_2, x_3, \ldots$ satisfy $x_1 = c$ and

$$x_n = x_{n-1} + \left[\frac{2x_{n-1} - (n+2)}{n}\right] + 1$$

for $n = 2,3,\ldots$, where $[x]$ is the greatest integer less than or equal to x. Find a general formula of x_n in terms of n.

Solution. First tabulate some values.

	x_1	x_2	x_3	x_4	x_5	x_6
c=1	1	1	1	1	1	1
c=2	2	3	4	5	6	7
c=3	3	5	7	10	13	17
c=4	4	7	11	16	22	29
c=5	5	9	14	20	27	35
c=6	6	11	17	25	34	45
c=7	7	13	21	31	43	57

Next tabulate first differences in each column.

column 1: 1,1,1,1,1,1,…
column 2: 2,2,2,2,2,2,…
column 3: 3,3,4,3,3,4,…
column 4: 4,5,6,4,5,6,…
column 5: 5,7,9,5,7,9,…
column 6: 6,10,12,6,10,12,…

We suspect they are periodic with period 3. Let $x(c,n)$ be the value of x_n for the sequence with $x_1 = c$. For rows 1 and 2, the first differences seem to be constant and for row 4, the second differences seem to be constant. Using fact (1) and induction, we get

$$x(1,n) = 1, \quad x(2,n) = n + 1 \qquad \text{(i)}$$

and $x(4,n) = (n^2+3n+4)/2$ for all n. Now

$$x(4,n) - x(1,n) = \frac{(n+1)(n+2)}{2}.$$

To check the column difference periodicity, we claim that for a fixed c,

$$x(c+3, n) = x(c, n) + (n+1)(n+2)/2.$$

If $n = 1$, then $x(c+3,1) = c+3 = x(c,1) + 3$ and so case $n = 1$ is true. Suppose the case $n-1$ is true. By the recurrence relation, $x(c+3,n)$ equals

$$x(c+3,n-1) + \left[\frac{2x(c+3,n-1)-(n+2)}{n}\right] + 1.$$

From the case $n-1$, we get $x(c+3, n-1) = x(c, n-1) + n(n+1)/2$. Using this, the displayed expression simplifies to

$$x(c,n-1) + \left[\frac{2x(c,n-1)-(n+2)}{n}\right] + \frac{n^2+3n+4}{2},$$

which is $x(c, n) + (n+1)(n+2)/2$ by the recurrence relation. This completes the induction for the claim.

Now the claim implies

$$x(c,n) = x(d,n) + \left(\frac{c-d}{3}\right)\frac{(n+1)(n+2)}{2}, \qquad \text{(ii)}$$

where $d = 1,2$ or 3 subject to $c \equiv d \pmod 3$. Since $x(1,n)$ and $x(2,n)$ are known, all we need to find is $x(3,n)$.

For the case $c = 3$, studying $x_1,x_3,x_5,\ldots$ and $x_2,x_4,x_6,\ldots$ separately, we can see that the second differences of these sequences seem to be constant. Using fact (1) and induction, we get

$x(3,n) = (n^2+4n+7)/4$ if n is odd and
$x(3,n) = (n^2+4n+8)/4$ if n is even. (iii)

Formula (ii) along with formulas (i) and (iii) provided the required answer for the problem.

Example 5. Let $g(x)$ be a polynomial of degree n with real coefficients. If $a \ge 3$, then prove that one of the numbers $|1-g(0)|, |a-g(1)|, |a^2-g(2)|, \ldots, |a^{n+1}-g(n+1)|$ is at least 1.

Solution. Let $f(x) = a^x - g(x)$. We have

$$\Delta a^x = a^{x+1} - a^x = (a-1)a^x,$$
$$\Delta^2 a^x = (a-1)\Delta a^x = (a-1)^2 a^x,$$
$$\ldots,$$
$$\Delta^{n+1} a^x = (a-1)^{n+1}a^x.$$

In particular, $\Delta^{n+1} a^0 = (a-1)^{n+1}$. Now

$$\Delta^{n+1} f(0) = \Delta^{n+1} a^0 - \Delta^{n+1} g(0) = (a-1)^{n+1}.$$

Since $a \ge 3$, we get $2^{n+1} \ge \Delta^{n+1} f(0)$. Assume $|a^k - g(k)| < 1$ for $k = 0,1,\ldots,n+1$. Then

$$\Delta^{n+1} f(0) = \sum_{k=0}^{n+1}(-1)^{n+1-k} C_{n+1}^k \left(a^k - g(k)\right)$$

$$< \sum_{k=0}^{n+1} C_{n+1}^k = 2^{n+1},$$

which is a contradiction.

Example 6. (*1984 USAMO*) Let $P(x)$ be a polynomial of degree $3n$ such that

$$P(0) = P(3) = \cdots = P(3n) = 2,$$
$$P(1) = P(4) = \cdots = P(3n-2) = 1,$$
$$P(2) = P(5) = \cdots = P(3n-1) = 0.$$

If $P(3n+1)$, then find n.

Solution. By fact (2) and (α),

$$0 = (-1)^{n+1}\Delta^{3n+1}P(0) = \sum_{k=0}^{3n+1}(-1)^k C_{3n+1}^k P(3n+1-k)$$

$$= 730 + 2\sum_{j=0}^{n}(-1)^{3j+1}C_{3n+1}^{3j+1} + \sum_{j=1}^{n}(-1)^{3j}C_{3n+1}^{3j}.$$

We can write this as $2a+b = -3^6$, where

$$a = \sum_{j=0}^{n}(-1)^{3j+1}C_{3n+1}^{3j+1}, \quad b = \sum_{j=0}^{n}(-1)^{3j}C_{3n+1}^{3j}.$$

To find a and b, we consider the cube root of unity $\omega = e^{2\pi i/3}$, the binomial expansion of $f(x) = (1-x)^{3n+1}$ and let

$$c = \sum_{j=1}^{n}(-1)^{3j-1}C_{3n+1}^{3j-1}.$$

Now $0 = f(1) = b-a-c$, $f(\omega) = b - a\omega - c\omega^2$ and $f(\omega^2) = b - a\omega^2 - c\omega$. Solving, we see

$$a = -\left(f(1) + \omega^2 f(\omega) + \omega f(\omega^2)\right)/3$$
$$= 2\left(\sqrt{3}\right)^{3n-1}\cos\frac{3n-1}{6}\pi$$

and $b = \left(f(1) + f(\omega) + f(\omega^2)\right)/3$

$$= 2\left(\sqrt{3}\right)^{3n-1}\cos\frac{3n+1}{6}\pi.$$

Studying the equation $2a + b = -3^6$, we find that it has no solution when n is odd and one solution when n is even, namely when $n = 4$.

Example 7. (*1980 Putnam Exam*) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \ge 0$?

Solution. Among all sequences satisfying $u_{n+1} = 2u_n - n^2$ for all $n \ge 0$, the difference v_n of any two such sequences will satisfy $v_{n+1} = 2v_n$ for all $n \ge 0$. Then $v_n = 2^n v_0$ for all $n \ge 0$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***March 25, 2007.***

Problem 266. Let

$$N = 1+10+10^2+\cdots+10^{1997}.$$

Determine the 1000th digit after the decimal point of the square root of N in base 10.

Problem 267. For any integer a, set

$$n_a = 101a - 100\cdot 2^a.$$

Show that for $0 \le a, b, c, d \le 99$, if

$$n_a + n_b \equiv n_c + n_d \pmod{10100},$$

then $\{a,b\} = \{c,d\}$.

Problem 268. In triangle ABC, $\angle ABC = \angle ACB = 40^\circ$. Points P and Q are inside the triangle such that $\angle PAB = \angle QAC = 20^\circ$ and $\angle PCB = \angle QCA = 10^\circ$. Must B, P, Q be collinear? Give a proof.

Problem 269. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$, $a_{n+1} = f(a_n)$ for all $n \ge 0$. Prove that if there exists a positive integer m for which $a_m = 0$, then either $a_1 = 0$ or $a_2 = 0$.

Problem 270. The distance between any two of the points A, B, C, D on a plane is at most 1. Find the minimum of the radius of a circle that can cover these four points.

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Solutions

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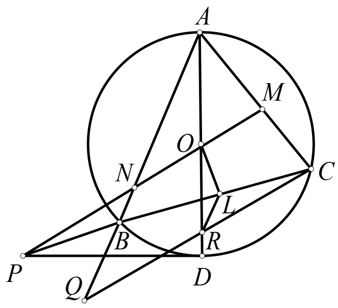
Problem 261. Prove that among any 13 consecutive positive integers, one of them has sum of its digits (in base 10) divisible by 7.

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Raffles Junior College, Singapore), **G.R.A. 20 Math Problem Group** (Roma, Italy), **Naoki S. D. LING**, **Anna Ying PUN** (HKU, Math, Year 1), **Simon YAU Chi Keung**, **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Consider the tens digits of the 13 consecutive positive integers. By the pigeonhole principle, there are at least $[13/2] + 1 = 7$ of them with the same tens digit. The sums of digits for these 7 numbers are consecutive. Hence, one of the sums of digits is divisible by 7.

Problem 262. Let O be the center of the circumcircle of ΔABC and let AD be a diameter. Let the tangent at D to the circumcircle intersect line BC at P. Let line PO intersect lines AC, AB at M, N respectively. Prove that $OM=ON$.

Solution 1. **Jeff CHEN** (Virginia, USA).



We may assume B is between P and C (otherwise interchange B and C, then N and M). Through C, draw a line parallel to line MN and intersect line AN at Q. Let line AO intersect line CQ at R. Since $MN \| CQ$, triangles AMN and ACQ are similar. To show $OM = ON$, it suffices to show $RC = RQ$.

Let L be the midpoint of BC. We will show $LR \parallel BQ$ (which implies $RC = RQ$).

Now $\angle OLP = \angle OLB = 90^\circ = \angle ODP$, which implies O, P, D, L are concyclic. Then $\angle ODL = \angle OPL$. From $OP \parallel RC$, we get $\angle RDL = \angle RCL$, which implies L,R,D,C are concyclic. Then

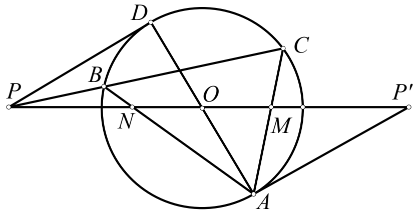
$$\angle RLC = 180^\circ - \angle RDC = 180^\circ - \angle ADC = 180^\circ - \angle ABC = \angle QBC.$$

Therefore, $LR \parallel BQ$ as claimed.

Solution 2. **CHEUNG Wang Chi** (Raffles Junior College, Singapore).

Set O as the origin and line MN as the x-axis.

Let P' be the reflection of P with respect to O. Then the coordinates of P and P' are of the form $(p,0)$ and $(-p,0)$.



The equation of the circumcircle as a conic section is of the form $x^2+y^2-r^2=0$. The equation of the pair of lines AP' and BC as a (degenerate) conic section is

$$\big(y - m(x+p)\big)\big(y - n(x-p)\big) = 0,$$

where m is the slope of line AP' and n is the slope of line BC. Since these two conic sections intersect at A, B, C, so the equation of the pair of lines AB and AC as a (degenerate) conic section is of the form

$$x^2 + y^2 - r^2 = \lambda\big(y - m(x+p)\big)\big(y - n(x-p)\big),$$

for some real number λ. When we set $y = 0$, we see the x-coordinates of M and N satisfies $x^2 - r^2 = \lambda mn(x^2 - p^2)$, whose roots are some positive number and its negative. Therefore, $OM = ON$.

Commended solvers: **Courtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher) and **Anna Ying PUN** (HKU, Math, Year 1).

Problem 263. For positive integers m, n, consider a $(2m+1)\times(2n+1)$ table, where in each cell, there is exactly one ant. At a certain moment, every ant moves to a horizontal or vertical neighboring cell. Prove that after that moment, there exists a cell with no ant.

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Raffles Junior College, Singapore), **G.R.A. 20 Math Problem Group** (Roma, Italy), **Naoki S. D. LING**, **Anna Ying PUN** (HKU, Math, Year 1), **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Assign the value $(-1)^{i+j}$ to the cell in the i-th row, j-th column of the table. Then two horizontal or vertical neighboring cells will have values of opposite sign. Since $2m+1$ and $2n+1$ are odd, there is exactly one more cell with negative values than cells with positive values. Before the moment, there is one more ant in cells with negative values than ants in cells with positive values. After the moment, two of the ants from cells with negative values will occupy a common cell with a positive value. Then there exists a cell with no ant.

Problem 264. For a prime number $p > 3$ and arbitrary integers a, b, prove that $ab^p - ba^p$ is divisible by $6p$.

Solution. **Samuel Liló ABDALLA** (São Paulo, Brazil), **Claudio ARCONCHER** (Jundiaí, Brazil), **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Raffles Junior College, Singapore), **G.R.A. 20 Math Problem Group** (Roma, Italy), **HO Ka Fai** (Carmel Divine Grace Foundation Secondary School, Form 6), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **Anna Ying PUN** (HKU, Math, Year 1), **Simon YAU Chi Keung**, **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Observe that

$ab^p - ba^p = ab[(b^{p-1}-1)-(a^{p-1}-1)].$

For $q = 2$, 3 or p, if a or b is divisible by q, then the right side is divisible by q.

Otherwise, a and b are relatively prime to q. Now $p - 1$ is divisible by $q - 1$, which is 1, 2 or $p - 1$. By Fermat's little theorem, both $a^{q-1}, b^{q-1} \equiv 1 \pmod q$. So $a^{p-1}, b^{p-1} \equiv 1 \pmod q$. Hence, the bracket factor above is divisible by q. Thus $ab^p - ba^p$ is divisible by 2, 3 and p. Therefore, it is divisible by $6p$.

Problem 265. Determine (with proof) the maximum of

$$\sum_{j=1}^{n}(x_j^4 - x_j^5),$$

where $x_1, x_2, \ldots, x_n$ are nonnegative real numbers whose sum is 1.
(*Source: 1999 Chinese IMO Team Selection Test*)

Solution. **Jeff CHEN** (Virginia, USA), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **Anna Ying PUN** (HKU, Math, Year 1) and **YIM Wing Yin** (HKU, Year 1).

Let $f(x) = x^4 - x^5 = x^4(1-x)$. Since $f''(x) = 4x^2(3-5x)$, we see that $f(x)$ is strictly convex on [0, 3/5]. Suppose $n \ge 3$. Without loss of generality, we may assume $x_1 \ge x_2 \ge \cdots \ge x_n$. If $x_1 \le 3/5$, then since

$$\left(\frac{3}{5}, \frac{2}{5}, 0, \cdots, 0\right) \succ (x_1, x_2, \cdots, x_n),$$

by the majorization inequality (see <u>Math Excalibur</u>, *vol.* 5, *no.* 5, *pp.* 2,4),

$$\sum_{i=1}^{n} f(x_i) \le f\left(\frac{3}{5}\right) + f\left(\frac{2}{5}\right).$$

If $x_1 > 3/5$, then $1 - x_1, x_2, \ldots, x_n$ are in [0, 2/5]. Since

$$(1-x_1, 0, \ldots, 0) \succ (x_2, \ldots, x_n),$$

by the majorization inequality,

$$\sum_{i=1}^{n} f(x_i) \le f(x_1) + f(1-x_1).$$

Thus the problem is reduced to the case $n = 2$. So now consider nonnegative a, b with $a + b = 1$. We have

$$\begin{aligned} f(a)+f(b) &= a^4(1-a) + b^4(1-b) \\ &= a^4b + b^4a = ab(a^3+b^3) \\ &= ab[(a+b)^3 - 3ab(a+b)] \\ &= 3ab(1-3ab)/3 \\ &\le 1/12 \end{aligned}$$

by the *AM-GM* inequality. Equality case holds when $ab = 1/6$ in addition to $a + b = 1$, for example when

$$(a,b) = \left(\frac{3+\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6}\right).$$

Therefore, the maximum is 1/12.

Difference Operator

(continued from page 2)

Next we look for a particular solution of $u_{n+1} = 2u_n - n^2$ for all $n \ge 0$. Observe that $n^2 = u_n - (u_{n+1} - u_n) = (I - \Delta)u_n$. From the sum of geometric series, we guess

$$u_n = (I-\Delta)^{-1}n^2 = (I + \Delta + \Delta^2 + \cdots)n^2$$

$$= n^2 + (2n+1) + 2 = n^2 + 2n + 3$$

should work. Indeed, this is true since $(n+1)^2 + 2(n+1) + 3 = 2(n^2 + 2n + 3) - n^2$.

Combining, we see that the general solution to $u_{n+1} = 2u_n - n^2$ for all $n \ge 0$ is

$u_n = n^2 + 2n + 3 + 2^n v_0$ for any real v_0.

Finally, to have $u_0 = a$, we must choose $v_0 = a - 3$. Hence, the sequence we seek is

$u_n = n^2 + 2n + 3 + 2^n(a-3)$ for all $n \ge 0$.

Since $$\lim_{n\to\infty} \frac{2^n}{n^2+2n+3} = +\infty,$$

u_n will be negative for large n if $a - 3 < 0$. Conversely, if $a - 3 \ge 0$, then all $u_n > 0$. Therefore, the answer is $a \ge 3$.

<u>***Example 8.***</u> (*1971 Putnam Exam*) Let c be a real number such that n^c is an integer for every positive integer n. Show that c is a non-negative integer.

<u>***Solution.***</u> 2^c is an integer implies $c \ge 0$.

Next we will do the case c is between 0 and 1 using the mean value theorem. This motivates and clarifies the general argument for the case $c \ge 1$. Assume $0 \le c < 1$. Then choose a positive integer $n > c^{1/(1-c)}$. Applying the mean value theorem to $f(x) = x^c$ on $[n, n+1]$, we know there exists a number w between n and $n + 1$ such that $\Delta n^c = (n+1)^c - n^c = cw^{c-1}$. On the left side, we have an integer, but on the right side, since $w > n > c^{1/(1-c)}$, we have $0 \le cw^{c-1} < 1$. Hence, $c = 0$.

For $c \ge 1$, let us mention there is an extension of the mean value theorem, which asserts that if f is continuous on $[a,b]$, k-times differentiable on (a,b), $0 < h \le (b-a)/k$ and $x + kh \le b$, then there exists a number v such that $a < v < b$ and

$$\frac{\Delta_h^k f(x)}{h^k} = f^{(k)}(v).$$

Taking this for the moment, we will finish the argument as follows. Let k be the integer such that $k-1 \le c < k$. Choose an integer n so large that

$$c(c-1)(c-2)\cdots(c-k+1)n^{c-k} < 1.$$

Applying the extension of the mean value theorem mentioned above to $f(x) = x^c$ on $[n, n+k]$, there is a number v between n and $n + k$ such that

$$\Delta^k n^c = c(c-1)(c-2)\cdots(c-k+1)v^{c-k}.$$

Again, the left side is an integer, but the right side is in the interval [0, 1). Therefore, both sides are 0 and $c = k-1$.

Now the extension of the mean value theorem can be proved by doing math induction on k. The case $k = 1$ is the mean value theorem. Next, suppose the extension is true for the case $k-1$. Let $0 < h \le (b-a)/k$. On $[a, b-h]$, define

$$g(x) = \frac{\Delta_h f(x)}{h} = \frac{f(x+h) - f(x)}{h}.$$

Applying the case $k-1$ to $g(x)$, we know there exists a number v_0 such that $a < v_0 < b - h$ and

$$\frac{\Delta_h^k f(x)}{h^k} = \frac{\Delta_h^{k-1} g(x)}{h^{k-1}} = g^{(k-1)}(v_0).$$

By the mean value theorem, there exists h_0 such that $0 < h_0 < h$ and

$$g^{(k-1)}(v_0) = \frac{f^{(k-1)}(v_0+h) - f^{(k-1)}(v_0)}{h}$$

$$= f^{(k)}(v_0 + h_0).$$

Finally, $v = v_0 + h_0$ is between a and b.