Junior problems

J205. Find the greatest n-digit number $a_1a_2\dots a_n$ with the following properties:

i) all its digits are different from zero and distinct;

ii) for each $k = 2,\dots,n-1$, $\frac{1}{a_{k-1}}, \frac{1}{a_k}, \frac{1}{a_{k+1}}$ is either an arithmetic sequence or a geometric sequence.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Alessandro Ventullo, Milan, Italy

The two properties imply that $1 \le n \le 9$, $1 \le a_k \le 9$, $a_i \ne a_j$ if $i \ne j$ and

$$\text{(i) } a_k(a_{k-1}+a_{k+1}) = 2a_{k-1}a_{k+1} \qquad \text{or} \qquad \text{(ii) } a_{k-1}a_{k+1} = a_k^2.$$

for each $k = 2,\dots,8$. We now turn to the analysis of each case.

(i) For $a_k = 2,4,6,8$, we obtain the four equations

$$a_{k+1} = a_{k-1}(a_{k+1}-1) \tag{1}$$

$$2a_{k+1} = a_{k-1}(a_{k+1}-2) \tag{2}$$

$$3a_{k+1} = a_{k-1}(a_{k+1}-3) \tag{3}$$

$$4a_{k+1} = a_{k-1}(a_{k+1}-4) \tag{4}$$

(1) has no solutions since $a_{k+1}-1$ can't divide a_{k+1}.
(2) implies that $a_{k+1}-2$ divides $2a_{k+1}$ and since $2a_{k+1} = 2(a_{k+1}-2)+4$, $a_{k+1}-2$ must divide 4, i.e. $a_{k+1} = 3,6$ and by symmetry $a_{k-1} = 6,3$ respectively.
(3) implies that $a_{k+1}-3$ divides $3a_{k+1}$ and since $3a_{k+1} = 3(a_{k+1}-3)+9$, $a_{k+1}-3$ must divide 9, so $a_{k+1} = 4,6$ and $a_{k-1} = 12,6$ respectively, so there are no solutions.
(4) implies that $a_{k+1}-4$ divides $4a_{k+1}$ and since $4a_{k+1} = 4(a_{k+1}-4)+16$, $a_{k+1}-4$ must divide 16, so $a_{k+1} = 5,6,8$ and $a_{k-1} = 20,12,8$ respectively, so there are no solutions.

Now, let's see what happens if $a_{k-1}+a_{k+1} = 8,10,12$ (note that we can immediately discard $a_{k-1}+a_{k+1} = 4,6,14,16$ because for any a_{k-1}, a_{k+1} the LHS would be divisible by 3, by 4 or by 7 and the RHS not). We obtain

$$a_{k-1}+a_{k+1} = 8, \qquad 4a_k = a_{k-1}a_{k+1} \tag{5}$$

$$a_{k-1}+a_{k+1} = 10, \qquad 5a_k = a_{k-1}a_{k+1} \tag{6}$$

$$a_{k-1}+a_{k+1} = 12, \qquad 6a_k = a_{k-1}a_{k+1} \tag{7}$$

(5) implies that $a_{k-1} = 6,2$ and by symmetry $a_{k+1} = 2,6$ respectively, so $a_k = 3$.
(6) has no solutions since the LHS of the second equation is divisible by 5 and the RHS not.
(7) has no solutions since the LHS of the second equation is divisible by 6 and the RHS not.

(ii) Since $a_{k-1} \ne a_{k+1}$, it is easy to see that the only solutions are

$$a_k = 2, \qquad a_{k-1} = 1,4, \qquad a_{k+1} = 4,1$$

$$a_k = 3, \qquad a_{k-1} = 1,9, \qquad a_{k+1} = 9,1$$

$$a_k = 4, \qquad a_{k-1} = 2,8, \qquad a_{k+1} = 8,2$$

$$a_k = 6, \qquad a_{k-1} = 4, 9, \qquad a_{k+1} = 9, 4.$$

In conclusion, we have the strings

$$a_{k-1}a_k a_{k+1} = \{643, 346, 632, 236, 124, 421, 139, 931, 248, 842, 469, 964\}.$$

Juxtaposing the strings, we see that the greatest number we can construct is 9643.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

It is not hard to calculate all possible sequences. Having in mind that $\frac{1}{a_{k-1}}, \frac{1}{a_k}, \frac{1}{a_{k+1}}$ are a geometric sequence iff $a_{k-1}a_{k+1} = {a_k}^2$ is a perfect square, we find the only possible geometric sequences (except for inversion) to be $1, 2, 4$; $1, 3, 9$; $2, 4, 8$; $4, 6, 9$.

For the sequence to be arithmetic, note that $a_{k-1}+a_{k+1}$ divides $2a_{k-1}a_{k+1}$, or it divides $2a_{k-1}^2$. Assume wlog that $a_{k+1} < a_{k-1}$, and by sheer trial and error we find that this is true when $(a_{k-1}, a_{k+1}) = (2, 6), (3, 6)$, for sequences (unique except for inversion) $2, 3, 6$ and $3, 4, 6$.

Note now that $5, 7$ do not appear in any sequence. Note also that $1, 8, 9$ do not appear in the middle of any sequence, or at most two of them appear in the number as a_1 and a_n, and the number has at most 6 digits, these being $2, 3, 4, 6$ for the middle digits, and the first and last digits being two out of $1, 8, 9$. Again by sheer trial and error, we notice that the longest numbers starting (or ending) by 1 are 1248 and 139 (or 8421 and 931), and that the longest number starting (or ending) by 8 is 8421 (or 1248). The longest number starting by 96 is 9643, while any other number starting by 9 has at most 3 digits. Therefore, the number that we are looking for is 9643.

J206. Let A, B, C, X, Y, Z be points in the plane. Prove that the circumcircles of triangles AYZ, BZX, CXY are concurrent if and only if the circumcircles of triangles XBC, YCA, ZAB are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

By the symmetry of the configuration, note that it suffices to show only one implication, as the other becomes equivalent after exchanging the roles of A, B, C and X, Y, Z. It is therefore enough to show that if the circumcircles of ABZ, BCX, CAY concur, then so do the circumcircles of XYC, YZA, ZXB.

Assume that the circumcircles of ABZ, BCX, CAY concur at point P, and define Q as the second point of intersection (other than Y), where the circumcircles of XYC and YZA meet. It suffices to show that $BZQX$ is cyclic, or equivalently, that $\angle BZQ + \angle BXQ = 180°$. Now, note that

$$\angle BZQ = \angle BZA + \angle AZQ = 180° - \angle BPA + \angle AYQ,$$

and

$$\angle BXQ = 360° - \angle BXC - \angle CXQ = 180° - \angle BPC + \angle CYQ,$$

or

$$\angle BZQ + \angle BXQ = \angle CPA + \angle AYC = 180°,$$

where we have used that $APXC$, $CPAY$, $BPAZ$, $CXQY$ and $ZAQY$ are cyclic. The conclusion follows.

Note. Although the above proof is configuration dependent (more precisely the angle computation), an easy alteration using angles modulo 180○ takes care of the situation in all its generality.

Second solution by the author

We make the same observation that it suffices to show one direction, and we suppose that the circumcircles of triangles ABZ, BCX, CAY are concurrent at a point P. Consider the inversion $\Psi(P, r^2)$ with pole P and power r^2, where r is arbitrary. This maps the points A, B, C to the points $\Psi(A)$, $\Psi(B)$, $\Psi(C)$ and X, Y, Z to the points $\Psi(X)$, $\Psi(Y)$, $Psi(Z)$ that lie on the sides $\Psi(B)\Psi(C)$, $\Psi(C)\Psi(A)$, $\Psi(A)Psi(B)$ of triangle $\Psi(A)\Psi(B)\Psi(C)$ respecctively. The conclusion that the circumcircles of triangle XYC, YZA, ZXB then follows by Miquel's theorem in the inverted diagram.

J207. Find the greatest number of the form $2^a \cdot 5^b + 1$, with a and b nonnegative integers, that divides a number all whose digits are distinct.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

We use the fact that $2^{29} = 536870912$ is a nine-digit number all whose digits are distinct. Then $2^{29} \cdot 10 + 4 = 5368709124$ and we claim that the answer to the problem is $2^{28} \cdot 5 + 1 = 1342177281$. It suffices to verify that:

a) $2^{28} \cdot 8 + 1 = 2^{31} + 1 = 2147483649$ does not have pairwise distinct digits, nor does any of its multiples: multiplying it by a factor greater than 4 will generate a number with more than 10 digits;

b) $2^{28} \cdot 16 + 1 = 2^{32} + 1 = 4294967297$ does not have pairwise distinct digits, nor does any of its multiples: multiplying it by a factor greater than 2 will generate a number with more than 10 digits;

c) $2^{26} \cdot 5^2 + 1 = 6710886401$ does not have distinct digits either and any of its multiples would have more than 10 digits, so again not all digits could be distinct;

d) $5^{13} + 1 < 1342177281$, $5^{14} + 1 = 6103515626$ does not have distinct digits, and any of its multiples would have more than 10 digits;

e) $2 \cdot 5^{13} + 1 = 2441406251$ and $4 \cdot 5^{13} + 1 = 4882812501$;

f) any number of the form $2^a \cdot 5^b + 1$ with $a > 2$ and $b > 2$ ends in 001, so if it is greater than $1,342,177,281$ any of its 10-digit multiples will have a repeating 0 among the last three digits.

This completes the proof.

J208. Let K be the symmedian point of triangle ABC and let R be its circumradius. Prove that

$$AK + BK + CK \leq 3R.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology,USA

Matteo Elia, Università di Roma "Tor Vergata", Roma, Italy.

Given the symmedian point, it is known that

$$\frac{|A'B|}{|A'C|} = \frac{c^2}{b^2} \quad \text{and} \quad \frac{|AK|}{|A'K|} = \frac{b^2+c^2}{a^2}$$

where $a = |BC|$, $b = |AC|$, $c = |AB|$, and A' is the intersection of the cevian through A and K. These relations allow us to write the distance of K from the vertex A in terms of the side lengths

$$|AK| = \frac{bc\sqrt{2(b^2+c^2)-a^2}}{a^2+b^2+c^2}.$$

Hence, by the Cauchy-Schwarz inequality

$$\begin{aligned}|AK|+|BK|+|CK| &= \frac{bc\cdot\sqrt{2(b^2+c^2)-a^2}+ac\cdot\sqrt{2(a^2+c^2)-b^2}+ab\cdot\sqrt{2(a^2+b^2)-c^2}}{a^2+b^2+c^2} \\ &\leq \sqrt{\frac{3(b^2c^2+a^2c^2+a^2b^2)}{a^2+b^2+c^2}}.\end{aligned}$$

Moreover $\sqrt{a^2+b^2+c^2} \leq 3R$, so it suffices to prove that

$$3(b^2c^2+a^2c^2+a^2b^2) \leq (a^2+b^2+c^2)^2$$

which is certainly true because it is equivalent to

$$(a^2-b^2)^2+(a^2-c^2)^2+(b^2-c^2)^2 \geq 0.$$

Also solved by Arkady Alt,San Jose ,California, USA, Daniel Lasaosa, Universidad Pública de Navarra, Spain; Titu Zvonaru, Comanesti, Romania; Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria

J209. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{(b+c)^5}{a}+\frac{(c+a)^5}{b}+\frac{(a+b)^5}{c}\geq\frac{32}{9}\left(ab+bc+ca\right).$$

Proposed by Marius Stanean and Mircea Lascu, Zalau, Romania

Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy.

Proof The inequality is equivalent to

$$\sum_{\text{cyc}}\frac{(b+c)^6}{a(b+c)}\geq\frac{32}{9}(ab+bc+ca)$$

The Cauchy-Schwarz inequality yields

$$\sum_{\text{cyc}}\frac{(b+c)^6}{a(b+c)}\geq\frac{((a+b)^3+(b+c)^3+(c+a)^3)^2}{a(b+c)+b(c+a)+c(a+b)}.$$

Hence it is enough to prove the following inequality

$$(a+b)^3+(b+c)^3+(c+a)^3\geq\frac{8}{3}(ab+bc+ca).$$

But the convexity of the map $x\to x^3$ and the hypothesis show that

$$(a+b)^3+(b+c)^3+(c+a)^3\geq\frac{8}{9}.$$

On the other hand, the inequality $(a+b+c)^2\geq 3(ab+bc+ca)$ implies that $ab+bc+ca\leq\frac{1}{3}$. Combining these two results yields the desired estimate.

Also solved by Arkady Alt, San Jose, California, USA; G.R.A.20 Problem Solving Group, Roma, Italy;Daniel Lasaosa, Universidad Publica de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Rosangela Matias de Oliveira, Parnamirim, Brazil; Titu Zvonaru, Comanesti, Romania; Neculai Stanciu, George Emil Palade Secondary school, Buzau, Romania; Prasanna Ramakrishan, Trinidad and Tobago; Ajat Adriansyah, Indonesia; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria

J210. Let P and Q be points in the plane of triangle ABC such that the sets of distances from P and Q to the vertices of triangle coincide, i.e. $(AP, BP, CP) = (AQ, BQ, CQ)$. Prove that the perpendicular bisector of PQ passes through a fixed point.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ivan Borsenco, Massachusetts Institute of Technology, USA

We prove that perpendicular bisector passes through the centroid G, that is $PG = QG$. Using Lagrange identity we have

$$AP^2 + BP^2 + CP^2 = PG^2 + AG^2 + BG^2 + CG^2$$

and

$$AQ^2 + BQ^2 + CQ^2 = QG^2 + AG^2 + BG^2 + CG^2.$$

Thus $PG = QG$ and we are done. It is an interesting question to find the locus of pairs of points (P, Q) for which $AP = BQ$, $BP = CQ$, $CP = AQ$, because this is the only case for which equality $(AP, BP, CP) = (AQ, BQ, CQ)$ can hold in a scalene triangle.

Senior problems

S205. Let $C_0(O, R)$ be a circle and let I be a point located at a distance $d < R$ from O. Consider two circles $C_1(I, r_1)$ and $C_2(I, r_2)$ such that there is a triangle inscribed in C_0 and circumscribed about C_1 and there is a quadrilateral inscribed in C_0 and circumscribed about C_2. Prove that $1 < \frac{r_2}{r_1} \le \sqrt{2}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

It is well-known that $d^2 = OI^2 = R^2 - 2Rr_1$, or $r_1 = \frac{R^2-d^2}{2R} = \frac{(R+d)(R-d)}{2R}$, while it is less known (but still relatively well-known) that

$$\frac{1}{(R-d)^2} + \frac{1}{(R+d)^2} = \frac{1}{{r_2}^2}.$$

This is called Fuss' theorem; see for example R. A. Johnson, *Advanced Geometry* for a proof. Therefore,

$$\frac{r_1^2}{r_2^2} = \frac{(R+d)^2}{4R^2} + \frac{(R-d)^2}{4R^2} = \frac{R^2+d^2}{2R^2}.$$

Clearly $2R^2 > R^2 + d^2 \ge R^2$, with equality iff $d = 0$. The conclusion follows, equality holds iff $I = O$.

S206. Find all integers $n \geq 2$ having a prime divisor p such that $n-1$ is divisible by the exponent of p in $n!$.

Proposed by Tigran Hakobyan, Yerevan, Armenia

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Write $n = kp$ for some positive integer k and let α be the exponent of p in $n!$. By Legendre's formula we have

$$\alpha = \sum_{j \geq 1} \left[\frac{n}{p^j} \right] = \frac{n - s_p(n)}{p-1}.$$

Here $s_p(n)$ is the sum of digits of n when written in base p. The first equality implies that $k \leq \alpha < \frac{n}{p-1}$. We deduce that

$$p > \frac{n-1}{\alpha} > \frac{(kp-1)(p-1)}{n} = p - 1 - \frac{p-1}{kp} > p - 2.$$

Since by assumption $\frac{n-1}{\alpha}$ is an integer, we must have $n - 1 = (p-1)\alpha = n - s_p(n)$. Hence $s_p(n) = 1$ and n is a power of p. The converse being clear, it follows that the answer to the problem is: all prime powers.

Also solved by Alessandro Ventullo, Milan, Italy; Prasanna Ramakrishan, Trinidad and Tobago

S207. Let a, b, c be distinct nonzero real numbers such that $ab+bc+ca=3$ and $a+b+c \neq abc+\frac{2}{abc}$. Prove that

$$\left(\sum_{cyc} \frac{a(b-c)}{bc-1}\right) \cdot \left(\sum_{cyc} \frac{bc-1}{a(b-c)}\right)$$

is the square of an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA.

We have

$$a+b+c \neq abc+\frac{2}{abc} \iff a^2b^2c^2 - abc\,(a+b+c)+2 \neq 0 \iff (ab-1)\,(bc-1)\,(ca-1) \neq 0.$$

By denoting $x = bc-1, y = ca-1, z = ab-1$, it suffices to show that for nonzero real numbers x, y, z such that $x+y+z=0$,

$$\left(\sum_{cyc} \frac{z-y}{x}\right) \cdot \left(\sum_{cyc} \frac{x}{z-y}\right)$$

is the square of an integer. We will prove that this quantity is equal to 9.

Note that

$$\sum_{cyc} \frac{z-y}{x} = -\frac{(y-x)\,(z-y)\,(x-z)}{xyz}$$

and

$$\sum_{cyc} \frac{x}{z-y} = \frac{1}{(y-x)\,(z-y)\,(x-z)} \sum_{cyc} x\,(y-x)\,(x-z)$$

so that

$$\left(\sum_{cyc} \frac{z-y}{x}\right) \cdot \left(\sum_{cyc} \frac{x}{z-y}\right) = \frac{\sum_{cyc} x\,(x-y)\,(x-z)}{xyz}.$$

Next,

$$\begin{aligned}\sum_{cyc} x\,(x-y)\,(x-z) &= \sum_{cyc} x\left(x^2-xy-xz+yz\right)\\ &= \sum_{cyc} x^3 - \sum_{cyc} x^2(y+z) + 3xyz.\end{aligned}$$

and the hypothesis $x+y+z=0$ yields $\sum_{cyc} x^3 = 3xyz$, $\sum_{cyc} x^2(y+z) = -\sum_{cyc} x^3 = -3xyz$. All in all,

$$\left(\sum_{cyc} \frac{z-y}{x}\right) \cdot \left(\sum_{cyc} \frac{x}{z-y}\right) = 9.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ajat Adriansyah, Indonesia; Alessandro Ventullo,Milan,Italy; Titu Zvonaru, Comanesti, Romania; Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania

S208. Let $f \in \mathbb{Z}[X]$ be such that $f(1) + f(2) + \cdots + f(n)$ is a perfect square for all positive integers n. Prove that there exists a positive integer k and a polynomial $g \in \mathbb{Z}[X]$ with $g(0) = 0$ and $k^2 f(X) = g^2(X) - g^2(X-1)$.

Proposed by Vlad Matei, University of Cambridge, United Kingdom

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We begin with a preliminary result.

Lemma: Given $f \in \mathbf{Z}[X]$, there exists a polynomial $p \in \mathbf{Q}[X]$ such that $p(0) = 0$ and $p(X) - p(X-1) = f(X)$.

Proof. Write $f(X) = a_u X^u + a_{u-1} X^{u-1} + \cdots + a_0$, where $a_u \neq 0$, and let us look for $p(X) = b_{u+1} X^{u+1} + b_u X^u + \cdots + b_1 X$. The condition $p(X) - p(X-1) = f(X)$ is equivalent (after expanding $(X-1)^j$ using the binomial formula) to the system of equations

$$\sum_{v=t+1}^{u+1} \binom{v}{t} b_v (-1)^{v-t} = -a_t.$$

for $t = 0, 1, \ldots, u$. This linear system with unknowns b_i has rational coefficients and it can be solved successively, starting with the last equation (for $t = u$), then going on to the next-to-last equation and so on.

Let us choose p as in the previous lemma and observe that $f(1) + f(2) + \ldots + f(n) = p(n)$ for all positive integers n. Hence $p(n)$ is a perfect square for all $n \geq 1$. It is well-known (but nontrivial) that this forces the existence of a polynomial $h \in \mathbf{Q}[X]$ such that $p = h^2$. There exists a positive integer k such that $g := k \cdot h \in \mathbf{Z}[X]$. By construction, we have $k^2 f(X) = g^2(X) - g^2(X-1)$ and $g(0) = 0$, so the problem is solved.

Also solved by Andrea Del Monaco, Università di Roma "Tor Vergata", Roma, Italy

S209. Let a, b, c be the sidelengths, s the semiperimeter, r the inradius and R the circumradius of a triangle ABC. Prove that

$$\frac{sr}{R}\left(1+\frac{R-2r}{4R+r}\right) \leq \frac{(s-b)(s-c)}{a}+\frac{(s-c)(s-a)}{b}+\frac{(s-a)(s-b)}{c}.$$

Proposed by Darij Grinberg, Massachusetts Institute of Technology, USA, and Cosmin Pohoata, Princeton University, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Let x, y, z be positive real numbers such that $a = y+z$, $b = z+x$ and $c = x+y$. Let $P = xy+yz+zx$. Note that

$$(x+y)(y+z)(z+x) = sP - xyz$$

and

$$\frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)} = \frac{(x+y)(y+z)(z+x)}{4xyz}.$$

Hence the desired inequality can also be written

$$s\left(1+\frac{sP-9xyz}{4sP}\right) \leq \sum \frac{xy}{x+y} \cdot \frac{(x+y)(y+z)(z+x)}{4xyz}$$

$$\Leftrightarrow 4s + \frac{sP-9xyz}{P} \leq \sum \frac{(x+z)(y+z)}{z}$$

$$\Leftrightarrow 5s - \frac{9xyz}{P} \leq 3s + \sum \frac{xy}{z} \Leftrightarrow 2\sum \frac{1}{xy} - \sum \frac{1}{x^2} \leq \frac{9}{xy+yz+zx}.$$

But this is simply Schur's inequality, written in the form

$$2(uv+vw+uw) - (u^2+v^2+w^2) \leq \frac{9uvw}{u+v+w},$$

applied to $u = \frac{1}{x}$, $v = \frac{1}{y}$ and $w = \frac{1}{z}$.

Also solved by Pham Huy Hoang,High School for Gifted students,Hanoi University of Science,Hanoi National University; Arkady Alt , San Jose ,California, USA; Titu Zvonaru, Comanesti, Romania; Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain

S210. Let p be an odd prime and let $F(x) = \sum_{k=0}^{p-1} \binom{2k}{k} x^k$. Prove that for all $x \in \mathbb{Z}$,

$$(-1)^{\frac{p-1}{2}} F(x) \equiv F\left(\frac{1}{16} - x\right) \pmod{p^2}.$$

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France.

Solution proposed by G.R.A.20 Problem Solving Group, Roma, Italy.

Using generating functions, one can easily check that the following identity holds for any real numbers s and t and any positive integer n

$$f(s,t) := \sum_{k=0}^{n} \binom{2k}{k}\binom{n+k}{2k}(s-t)^{n-k}t^k = \sum_{k=0}^{n} \binom{n}{k}^2 s^{n-k}t^k.$$

Also, the function f is symmetric, i.e.

$$f(t,s) = \sum_{k=0}^{n} \binom{n}{k}^2 t^{n-k}s^k = \sum_{j=0}^{n} \binom{n}{n-j}^2 t^j s^{n-j} = f(s,t).$$

Now, let $n = (p-1)/2$, and note that

$$\binom{n+k}{2k} = \frac{\prod_{j=1}^{k}(p^2-(2j-1)^2)}{4^k(2k)!} \equiv \frac{\prod_{j=1}^{k}(2j-1)^2}{(-4)^k(2k)!} = \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}.$$

Since $f(1-16x,-16x) = f(-16x,1-16x)$, it follows that

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 x^k \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \left(\frac{1}{16}-x\right)^k \pmod{p^2}.$$

Consequently, the statement is fully proved by observing that p divides $\binom{2k}{k}$ for $(p-1)/2 < k < p$.

Undergraduate problems

U205. Let E be a vector space with norms $||\cdot||_1$ and $||\cdot||_2$. Decide if $\min\left(||\cdot||_1, ||\cdot||_2\right)$ is a norm.

Proposed by Roberto Bosch Cabrera, Florida, USA

Solution by Antonello Cirulli, Università di Roma "Tor Vergata", Roma, Italy.
The answer is that $\min(\|\cdot\|_1, \|\cdot\|_2)$ is not always a norm. To see this, consider $E = \mathbb{R}^2$ with

$$\|(x,y)\|_1 = 2|x| + |y| \quad \text{and} \quad \|(x,y)\|_2 = |x| + 2|y|.$$

We argue that $|||(x,y)||| := \min(\|(x,y)\|_1, \|(x,y)\|_2)$ is not a norm by observing that it does not satisfy the triangle inequality, as:

$$|||(1,2)||| + |||(2,1)||| = \min(4,5) + \min(5,4) = 8 < 9 = \min(9,9) = |||(1,2) + (2,1)|||.$$

Actually, more counterexamples can be found by observing that the open unit balls

$$B_1 := \{v \in E : \|v\|_1 < 1\} \quad \text{and} \quad B_2 := \{v \in E : \|v\|_2 < 1\}$$

are convex, while the set $\{v \in E : \min(\|v\|_1, \|v\|_2) < 1\} = B_1 \cup B_2$ could be non-convex.

Also solved by Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Alessandro Ventullo,Milan,Italy

U206. Prove that there is precisely one group with 30 elements and 8 automorphisms.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Alessio Podda, Università di Roma "Tor Vergata", Roma, Italy.

Let G be a group such that $|G| = 30$, and let us consider the group of internal automorphisms $\mathrm{Inn}(G)$ that satisfies

$$\mathrm{Aut}(G) \geq \mathrm{Inn}(G) \cong G/Z(G)$$

where $Z(G)$ is the center of G.

Since $|Z(G)|$ divides $|G| = 30$ and $|\mathrm{Inn}(G)| = |G|/|Z(G)|$ divides $|\mathrm{Aut}(G)| = 8$, we have that the possible values of $|Z(G)|$ are 15 or 30. Let us assume that $|Z(G)| = 15$, then there are two left cosets, namely $Z(G)$ and $aZ(G) = G - Z(G)$ where $a \in G - Z(G)$. Let $b \in aZ(G)$ and let $z \in Z(G)$ such that $b = az$. Since z commutes with a,

$$ab = a(az) = (az)a = ba$$

which implies that $a \in Z(G)$ which is a contradiction. Therefore $|Z(G)| = 30$ and G is abelian.

Now $|G| = 30 = 2 \cdot 3 \cdot 5$ and, by the fundamental theorem of finite abelian groups, G can be expressed as the direct sum of cyclic subgroups of prime-power order:

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{30}.$$

Note that $|\mathrm{Aut}(\mathbb{Z}_{30})| = \varphi(30) = 8$.
Note. It can be shown that the other three groups with 30 elements have more than 8 automorphisms.

Also solved by Moubinool Omarjee, Paris, France;Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain;Jose Hernandez Santiago, Oaxaca, Mexico.

U207. Let $n \geq 3$ be an odd integer. Evaluate

$$\sum_{k=1}^{\frac{n-1}{2}} \sec \frac{2k\pi}{n}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria

We will prove that

$$\sum_{k=1}^{\frac{n-1}{2}} \sec \frac{2k\pi}{n} = \begin{cases} \frac{n-1}{2}, & \text{if } n \equiv 1 \mod 4 \\ -\frac{n+1}{2}, & \text{if } n \equiv 3 \mod 4 \end{cases}$$

Let T_n denote Chebychev's polynomial of the first kind of degree n, which is defined by the formula $T_n(\cos\theta) = \cos(n\theta)$. Since $T_n'(\cos\theta) = n\sin(n\theta)/\sin\theta$ we conclude that $\{\cos(k\pi/n) : 1 \leq k \leq n-1\}$ are the $n-1$ distinct zeros of T_n', which is then of degree $n-1$. This proves that there exists a constant λ such that $T_n'(X) = \lambda \prod_{1\leq k<n}(X - \cos(k\pi/n))$ and consequently

$$\frac{T_n''(X)}{T_n'(X)} = \sum_{k=1}^{n-1} \frac{1}{X - \cos(k\pi/n)}.$$

Noting that $\cos\frac{k\pi}{n} = \cos\frac{(n-k)\pi}{n}$, we see that

$$\frac{T_n''(X)}{T_n'(X)} = \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{1}{X - \cos(k\pi/n)} + \frac{1}{X + \cos(k\pi/n)}\right) = \sum_{k=1}^{n-1} \frac{X}{X^2 - \cos^2(k\pi/n)},$$

so

$$\frac{T_n''(X)}{T_n'(X)} = \sum_{k=1}^{n-1} \frac{2X}{2X^2 - 1 - \cos(2k\pi/n)}$$

and by substituting $X = \cos\theta$, we get that

$$\frac{T_n''(\cos\theta)}{T_n'(\cos\theta)} = \sum_{k=1}^{n-1} \frac{2\cos\theta}{\cos(2\theta) - \cos(2k\pi/n)}$$

On the other hand from $T_n'(\cos\theta) = n\sin(n\theta)/\sin\theta$ we see that

$$-(\sin\theta)\frac{T_n''(\cos\theta)}{T_n'(\cos\theta)} = n\cot(n\theta) - \cot\theta.$$

Therefore, we conclude that

$$\sum_{k=1}^{n-1} \frac{1}{\cos(2\theta) - \cos(2k\pi/n)} = \frac{1}{2\sin^2\theta} - \frac{n\cot(n\theta)}{\sin(2\theta)},$$

and this is equivalent to, for odd n, to

$$\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{\cos(2k\pi/n) - \cos(2\theta)} = \frac{n\cot(n\theta)}{2\sin(2\theta)} - \frac{1}{4\sin^2\theta}$$

In particular, taking $\theta = \pi/4$ we obtain

$$\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{\cos(2k\pi/n)} = \frac{n\cot(n\pi/4) - 1}{2} = \frac{n(-1)^{(n-1)/2} - 1}{2}$$

which is the desired conclusion.

Also solved by Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Anastasios Kotronis, Athens, Greece; Jedrzej Garnek, Adam Mickiewicz University, Poznan, Poland

U208. Let X and Y be standard Cauchy random variables $C(0,1)$. Prove that the probability density function of random variable $Z = X^2 + Y^2$ is given by

$$f_Z(t) = \frac{2}{\pi} \cdot \frac{1}{(t+2)\sqrt{t+1}}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The probability density function for a Cauchy variable X $C(0,1)$ is

$$f_X(t) = \frac{1}{\pi} \cdot \frac{1}{t^2+1}.$$

Therefore, the probability that $|X| < \sqrt{u}$ is

$$\int_{-\sqrt{u}}^{\sqrt{u}} f_X(t)dt = \frac{1}{\pi}\int_{-\sqrt{u}}^{\sqrt{u}} \frac{dt}{t^2+1} = \frac{\arctan(\sqrt{u}) - \arctan(-\sqrt{u})}{\pi} = \frac{2\arctan(\sqrt{u})}{\pi}.$$

Or equivalently, the probability density function for X^2 is

$$f_{X^2}(u) = \frac{d}{du}\left(\frac{2\arctan(\sqrt{u})}{\pi}\right) = \frac{1}{\pi(1+u)\sqrt{u}}$$

for $u \geq 0$, and 0 for $u < 0$. Therefore,

$$\begin{aligned}
f_Z(t) &= \int_0^t f_{X^2}(u) f_{Y^2}(t-u)du \\
&= \int_0^t \frac{du}{\pi^2(1+u)(1+t-u)\sqrt{u}\sqrt{t-u}} \\
&= \frac{2}{\pi^2}\int_0^{\frac{t}{2}} \frac{du}{(1+u)(1+t-u)\sqrt{u}\sqrt{t-u}} \\
&= \frac{4}{\pi^2}\int_0^{\frac{t}{2}} \frac{dv}{(1+t+v^2)\sqrt{t^2-4v^2}} \\
&= \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}} \frac{d\alpha}{4+4t+t^2\cos^2\alpha},
\end{aligned}$$

where we have used the symmetry of the original integrand around $u = \frac{t}{2}$, and performed the variable changes $v = \sqrt{u(t-u)}$ and $v = \frac{t}{2}\sin\alpha$. Now, it is easily provable that for any positive reals A, B, we have

$$\frac{d}{d\alpha}\left(\frac{1}{\sqrt{A^2+AB}}\arctan\left(\frac{\sqrt{A}\tan\alpha}{\sqrt{A+B}}\right)\right) = \frac{1}{A+B\cos^2\alpha},$$

or taking limits for $\alpha = 0$ and $\alpha = \frac{\pi}{2}$, the definite integral would be

$$\int_0^{\frac{\pi}{2}} \frac{d\alpha}{A+B\cos^2\alpha} = \frac{\pi}{2\sqrt{A^2+AB}},$$

hence

$$f_Z(t) = \frac{4}{\pi}\frac{1}{\sqrt{(4+4t)^2+(4+4t)t^2}} = \frac{2}{\pi}\frac{1}{(t+2)\sqrt{t+1}}.$$

The conclusion follows.

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; G.R.A.20 Problem Solving Group, Roma, Italy; Jedrzej Garnek, Adam Mickiewicz University, Poznan, Poland; Lataianu Bogdan, Saskatoon, Canada

U209. Let $r \geq 2$ be a positive integer and G be an $(r-1)$-edge-connected r-regular graph with an even number of vertices. Prove that for every edge e of the graph there is a perfect matching of G containing e.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by the author

First, it is to be mentioned that the result also appears in J. Plesnik, Connectivity of regular graphs and the existence of 1-factors, *Mat. asopis Sloven. Akad. Vied*, 22 (1972), pp. 310318, as Igor Rivin kindly pointed out on recent thread on MathOverflow (see: http://mathoverflow.net/questions/78905/a-k-1-edge-connected-k-regular-graph-is-matching-covered). Also, the solution Plesnik gives is more clever than my "brute-force" approach, so I will choose to follow his ideas instead.

Consequently, we begin with a simple preliminary result.

Lemma. Let G be a r-regular graph ($r \geq 2$) and $S \subset V(G)$. Then for each component C of $G - S$ we have that

$$|E_G(C, S)| \equiv r \mod 2,$$

where $E_G(C, S)$ denotes the set of edges of G with one end in C and another end in S.

In particular, if G is an $r-1$-edge-connected r-regular graph, then $|E_G(C, S)| \geq r - 1$, and by parity $|E_G(C, S)| \geq r$.

Proof of Lemma. Since $|C|$ is odd, we have

$$r \equiv r|C| = \sum_{u \in V(C)} \deg_G(u) = |E_G(C, S)| + 2|E(C)| \equiv |E_G(C, S)| \mod 2.$$

Moreover, if G is an $r-1$-edge-connected r-regular graph, then $|E_G(C, S)| \geq r - 1$, and by parity this inequality actually yields $|E_G(C, S)| \geq r$. This proves our Lemma.

Now, returning to the problem, we will proceed by contradiction. That is, suppose that there exists an edge $e = xy$ such that there is no perfect matching containing e. Then, the graph $G - \{x, y\}$ has no perfect matching. From Tutte's matching theorem this means that there exists $S' \subset V(G) - \{x, y\}$ such that

$$\mathrm{odd}(G - \{x, y\} - S') > |S'|.$$

Let $S = S' \cup \{x, y\}$. Then, $\mathrm{odd}(G - S) \geq |S'| + 2 = |S|$ by parity. Furthermore, let $C_1, \ldots, C_m$ be the odd components of $G - S$, where we let $m = \mathrm{odd}(G - S)$. Then, by the Lemma we have just proven, we have that $|E_G(C_i, S)| \geq r$. Hence,

$$|F(S)| \leq r|S| - 2 \leq rm - 2, \quad \text{(since } e = xy \text{ is in the induced subgraph of } G \text{ by the vertex set } S\text{)}$$

where $F(S)$ is the set of edges with exactly one end-vertex in the set S and $|F(s)| \geq |E_G(C_1 \cup \ldots \cup C_m, S)| \geq rm$.

This is obviously a contradiction with the above, so for every edge e of G, G has a perfect matching containing e. This completes the proof.

U210. A graph G arises from G_1 and G_2 by pasting them along S if G has induced subgraphs G_1, G_2 with $G = G_1 \cup G_2$ and S is such that $S = G_1 \cap G_2$. A graph is called chordal if it can be constructed recursively by pasting along complete subgraphs, starting from complete subgraphs. For a graph $G(V, E)$ define its Hilbert polynomial $H_G(x)$ to be

$$H_G(x) = 1 + Vx + Ex^2 + c(K_3)x^3 + c(K_4)x^4 + \ldots + c(K_{\omega(G)})x^{\omega(G)},$$

where $c(K_i)$ is the number of $i - cliques$ in G and $\omega(G)$ is the clique number of G. Prove that $H_G(-1) = 0$ if and only if G is chordal or a tree.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

No solutions have been yet received.

Olympiad problems

O205. Find all n such that each number containing n 1's and one 3 is prime.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

No solutions have been yet received.

O206. Let $D \in BC$ be the foot of the A-symmedian of triangle ABC with centroid G. The circle passing through A and tangent to BC at D intersects sides AB and AC at E and F, respectively. If $3AD^2 = AB^2 + AC^2$, prove that G lies on EF.

Proposed by Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The result is true iff ABC is isosceles at A, in which case $\sqrt{3}a = 2b = 2c$, D is the midpoint of BC, and the power of B, C with respect to the circle defined in the problem statement is $BE \cdot AB = \frac{BC^2}{4}$, or $BE = CF = \frac{b}{3} = \frac{c}{3}$, and by Thales theorem EF intersects the median AD at a point G' such that $\frac{G'D}{AD} = \frac{1}{3}$, ie, $G' = G$. We will now show that, if $3AD^2 = AB^2 + AC^2$ and E, G, F are collinear, then $b = c$.

It is well known (or easily provable) that the A-symmedian bisects side BC in two segments with lengths in ratio $\frac{BD}{CD} = \frac{c^2}{b^2}$, or $BD = \frac{c^2 a}{b^2+c^2}$ and $CD = \frac{b^2 a}{b^2+c^2}$. Hence, by Stewart's theorem,

$$AD^2 = \frac{BD \cdot AC^2 + CD \cdot AB^2}{BC} - BD \cdot CD = \frac{b^2c^2(2b^2 + 2c^2 - a^2)}{(b^2 + c^2)^2}.$$

Therefore, $3AD^2 = AB^2 + AC^2$ iff

$$3a^2b^2c^2 = 6b^2c^2(b^2 + c^2) - (b^2 + c^2)^3.$$

Denote $u = \frac{AE}{AB}$, $v = \frac{AF}{AC}$, and apply Menelaus' theorem to triangle ABN, where N is the midpoint of AC. E, G, F are collinear iff

$$1 = \frac{AE}{EB} \cdot \frac{BG}{GN} \cdot \frac{NF}{FA} = 2\frac{u}{1-u}\frac{AF - CF}{2AF} = \frac{2uv - u}{v - uv},$$

ie E, G, F are collinear iff $3uv = u + v$. Moreover, the power of B with respect to the circle defined in the problem statement is clearly $BD^2 = BA \cdot BE$, or $u = \frac{AB^2 - BD^2}{AB^2}$; similarly, $v = \frac{AC^2 - CD^2}{AC^2}$, ie, E, G, F are collinear iff

$$2b^2BD^2 + 2c^2CD^2 = b^2c^2 + 3BD^2CD^2,$$

or equivalently, iff

$$3b^2c^2a^4 - 2a^2(b^2 + c^2)^3 + (b^2 + c^2)^4 = 0.$$

Therefore, if $3AD^2 = AB^2 + AC^2$ and E, G, F are collinear, the following equality holds:

$$(b + c)^2 \left(b^4 - b^2c^2 + c^4\right)(b - c)^2 = 0.$$

Now, by the AM-GM inequality, and since $b, c > 0$, the first two factors may not be zero, or necessarily $b = c$.

We finish with a counterexample of the problem statement when ABC is not isosceles at A. Take $\angle A = 60°$, or $\cos A = \frac{1}{2}$, and $\frac{b^3}{c^3} = \frac{3+\sqrt{5}}{2}$, for example $b = \sqrt[3]{\frac{(\sqrt{5}+1)^2}{4}}$, $c = 1$, and $a = \sqrt{b^2 - b + 1}$. Clearly, $b > a > c$, and $a + c = \sqrt{(b-1)^2 + b} + 1 > b - 1 + 1 = b$, or such a triangle exists. Inserting the previous values yields $3AD^2 = b^2 + c^2$, while $u + v \neq 3uv$, or $3AD^2 = AB^2 + AC^2$, while G is not on EF.

O207. Define a sequence $(x_n)_{n\geq 1}$ of rational numbers by $x_1 = x_2 = x_3 = 1$ and $x_n x_{n-3} = {x_{n-1}}^2 + x_{n-1}x_{n-2} + {x_{n-2}}^2$ for all $n \geq 4$. Prove that x_n is an integer for every positive integer n.

Proposed by Darij Grinberg, Massachusetts Institute of Technology, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

First, note that a simple calculation yields $x_4 = 3$. Next, we see that for $n \geq 4$ we get that

$$x_n x_{n-3} = x_{n-1}^2 + x_{n-1}x_{n-2} + x_{n-2}^2, \text{ i.e. } x_{n+1}x_{n-2} = x_n^2 + x_n x_{n-1} + x_{n-1}^2.$$

It follows that

$$\begin{aligned}
& x_{n+1}x_{n-2} - x_n x_{n-3} = x_n^2 + x_n x_{n-1} - x_{n-1}x_{n-2} - x_{n-2}^2 \\
\Rightarrow\ & x_{n-2}(x_{n+1} + x_{n-1} + x_{n-2}) = x_n(x_n + x_{n-1} + x_{n-3}) \\
\Rightarrow\ & x_{n-2}(x_{n+1} + x_n + x_{n-1} + x_{n-2}) = x_n(x_n + x_{n-1} + x_{n-2} + x_{n-3}) \\
\Rightarrow\ & \frac{x_{n+1} + x_n + x_{n-1} + x_{n-2}}{x_n x_{n-1}} = \frac{x_n + x_{n-1} + x_{n-2} + x_{n-3}}{x_{n-1}x_{n-2}} \\
\Rightarrow\ & \frac{x_{n+1} + x_n + x_{n-1} + x_{n-2}}{x_n x_{n-1}} = \cdots = \frac{x_4 + x_3 + x_2 + x_1}{x_3 x_2} = \frac{3+1+1+1}{1 \cdot 1} = 6 \\
\Rightarrow\ & x_{n+1} = 6x_n x_{n-1} - x_n - x_{n-1} - x_{n-2}.
\end{aligned}$$

We conclude that $x_{n+1} = 6x_n x_{n-1} - x_n - x_{n-1} - x_{n-2}$, and so, since $x_1 = x_2 = x_3 = 1$, we infer that x_n is an integer for all positive integers n. This completes the proof.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria

O208. Let $z_1, z_2, ..., z_n$ be complex numbers such that $z_1^k + z_2^k + ... + z_n^k$ is the k-th power of a rational number for all $k > 2011$. Prove that at most one of the numbers z_i is nonzero.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France and Octav Dragoi, ICHB, Bucharest, Romania

Solution by the authors

Replacing z_i by z_i^{2012}, we may assume that $z_1^k + z_2^k + ... + z_n^k = x_k^k$ for some sequence of rational numbers $(x_k)_{k\geq 1}$. By deleting those z_i equal to 0, we may assume that all z_i are nonzero. We will prove that $n = 1$. By considering only $k = 1, 2, ..., n$ and using Newton's formulae, we deduce that the elementary symmetric sums in the z_i's are rational numbers and so all z_i's are algebraic numbers. By multiplying all z_i by a suitable positive integer N, we may assume that all z_i's are algebraic integers. Let $K = \mathbb{Q}(z_1, z_2, ..., z_n)$, a number field. Let p be a prime number not dividing any of the norms $N_{K/\mathbb{Q}}(z_i)$ and let $\wp$ be a prime ideal of O_K (ring of algebraic integers of K) such that $p \in \wp$. Let $F = O_K/\wp$, a finite field with, say, q elements. Then for any $x \in O_K$ we have $x^{q-1} \equiv 0 \pmod{\wp}$ or $x^{q-1} \equiv 1 \pmod{\wp}$ (the first happens if and only if $x \in \wp$). Now, $z_1^{q-1} + ... + z_n^{q-1} \equiv n \pmod{\wp}$ (as $z_i \notin \wp$ for all i, by the choice of p). Since by assumption $z_1^{q-1} + ... + z_n^{q-1}$ is of the form x^{q-1}, we obtain $n \equiv 0 \pmod{\wp}$ or $n \equiv 1 \pmod{\wp}$. The first condition implies $p|n$, while the second one implies $p|n-1$. We deduce that for all but finitely many primes p we have $p|n$ or $p|n-1$. The result follows.

O209. Let P be a point on the side BC of triangle ABC with circumcircle Γ, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the circles internally tangent to Γ and also to AP, BP, and AP, CP, respectively. If I is the incenter of triangle ABC and M is the midpoint of the arc BC of Γ not containing the vertex A, prove that the radical axis of $\mathcal{T}_1$ and $\mathcal{T}_2$ is the line determined by M and the midpoint of the segment IP.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by the author

To prove that the midpoint M of the arc BC of Γ lies on the radical axis of the Thebault circles it suffices to show that the points D, E, X, Y are concyclic, where D, E and X, Y are the tangency points of $\mathcal{T}_1$ and $\mathcal{T}_2$ with BC and Γ, respectively (D and X are on $\mathcal{T}_1$). Indeed, by Archimedes' Lemma, since BC is a chord of Γ tangent to both $\mathcal{T}_1$ and $\mathcal{T}_2$ at D and E, respectively, the lines XD and YE intersect at M, so the noting that $MD \cdot MX = ME \cdot MY$ is the power of point of M with respect to the circumcircle of $DEXY$ establishes that M is on the radical axis τ of $\mathcal{T}_1$ and $\mathcal{T}_2$. Now, the concyclity of X, D, E, and Y is just a matter of angle-chasing. One can easily note that $\angle DXY = \angle MXY = \angle MAY = \angle MAB + \angle CAY = \angle CEY$.

For the second part that the midpoint J of segment IP lies on τ we recall the following preliminary result also known as Sawayama's Lemma (or the Sawayama-Thebault theorem for it proved to be a key step in the first proof of Thebault's theorem). We state it following J. L. Ayme, Sawayama and Thbault's Theorem, *Forum Geometricorum*, 3 (2003), pp. 225-229, article in which one can find a neat synthetic proof.

Lemma. Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. Let P be the center of the circle $\mathcal{C}_1$ which touches DC, DA at E, F and the circumcircle $\mathcal{C}_2$ of ABC at K. Then the chord of contact EF passes through the incenter I of triangle ABC.

Returning to our diagram and keeping in mind the different notations, we now consider the homothecy $\mathcal{H}(I, 2)$ of center I and factor 2. The Lemma above now gives our desired conclusion (for the fact that I lies on the polar of P wrt. each of the Thebault circles implies that P lies on the polar of I wrt. the two circles). This completes the proof.

O210. Suppose that the set of positive integers is partitioned into a set of sequences $(L_{n,i})_{i\geq 1}$ such that $L_{n,i}$ divides $L_{n,i+1}$ for all positive integers n and i. Prove that for all positive integers t, there are infinitely many n such that $\omega(L_{n,1}) = t$, where for a positive integer a, decomposed into primes as $a = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_r}^{\alpha_r}$, $\omega(a) = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

Proposed by Radu Bumbacea, Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Enumerate the primes in increasing order, $p_1 = 2, p_2 = 3, \dots$. Fix a positive integer t and let A_m be the set of all numbers of the form $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_m^{\alpha_m}$, such that $\omega(n) = t$. The cardinality of A_m is the number of solutions of the equation $\alpha_1 + \ldots + \alpha_m = t$ in nonnegative integers. Associating to a solution of the equation the $m-1$-tuple $(\alpha_1 + 1, \alpha_1 + \alpha_2 + 2, \ldots, \alpha_1 + \ldots + \alpha_{m-1} + m - 1)$ we obtain a bijection between the number of solutions of the equation and the subsets with $m-1$ elements of $\{1, 2, \ldots, m+t-1\}$. Hence $|A_m| = \binom{m+t-1}{t}$. Let $A_{<m}$ be the set of numbers $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_m^{\alpha_m}$ such that $\omega(n) < t$. Using a similar argument, we obtain that $|A_{<m}| = \binom{m-1+t}{t-1}$. Hence $\lim_{m\to\infty} |A_m| - |A_{<m}| = \infty$.

Note that any element of A_m equals $L_{n,k}$ for some positive integers n, k, and if $k \neq 1$, then $L_{n,1} \in A_{<m}$. Using this observation and the hypothesis, it follows that the number of elements in A_m that occupy the first position in a sequence is at least $|A_m| - |A_{<m}|$. The conclusion follows from the first paragraph of the solution.