Junior problems

J241. Determine all positive integers that can be represented as

$$\frac{ab+bc+ca}{a+b+c+\min(a,b,c)}$$

for some positive integers a, b, c.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Giulio Calimici, Francesco De Sclavis, Andrea Fiacco, Michele Ricciardi, Emiliano Torti, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy

Let n be a positive integer. For $a = b = n$ and $c = 2n$, we have that

$$\frac{ab+bc+ca}{a+b+c+\min(a,b,c)} = \frac{n^2+2n^2+2n^2}{n+n+2n+\min(n,n,2n)} = \frac{5n^2}{5n} = n.$$

Hence, all positive integers can be represented in this way.

Also solved by Armend Sh. Shabani, University of Prishtina, Republic of Kosova; Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Roberto Bosch Cabrera, Havana, Cuba.

J242. Let ABC be a triangle and let D, E, F be the feet of the altitudes from A, B, C to the sides BC, CA, AB, respectively. Let X, Y, Z be the midpoints of segments EF, FD, DE and let x, y, z be the perpendiculars from X, Y, Z to BC, CA, AB, respectively. Prove that the lines x, y, z are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

First solution by Roberto Bosch Cabrera, Havana, Cuba
The lines x, y, z are concurrent if and only if $XB^2 - BZ^2 + ZA^2 - AY^2 + YC^2 - CX^2 = 0$ by Carnot's theorem. Using the Law of Cosines we obtain:

$$XB^2 = BF^2 + \frac{FE^2}{4} + 2BF \cdot FE \cos C$$
$$BZ^2 = BD^2 + \frac{DE^2}{4} + 2BD \cdot DE \cos A$$
$$ZA^2 = AE^2 + \frac{DE^2}{4} + 2AE \cdot DE \cos B$$
$$AY^2 = AF^2 + \frac{FD^2}{4} + 2AF \cdot FD \cos C$$
$$YC^2 = CD^2 + \frac{FD^2}{4} + 2CD \cdot FD \cos A$$
$$CX^2 = CE^2 + \frac{FE^2}{4} + 2CE \cdot FE \cos B$$

but we have that $BF = a\cos B$, $BD = c\cos B$, $AE = c\cos A$, $AF = b\cos A$, $CD = b\cos C$, $CE = a\cos C$, $FE = a\cos A$, $DE = c\cos C$, $FD = b\cos B$. So substituying all that we need to prove is

$$a^2\cos^2 B + b^2\cos^2 C + c^2\cos A = a^2\cos^2 C + b^2\cos^2 A + c^2\cos^2 B$$

and now using that $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$ we obtain the conclusion.

Second solution by the author Note that the points A, B, C are the excenters of triangle DEF. It is very simple to verify that the midpoint X of EF has equal powers with respect to the $E-$ and $F-$excircles. Thus, the perpendicular from X to BC is precisely the radical axis of the two excircles. It then follows that the lines x, y, z are concurrent at the radical center of the excircles of triangle DEF.

Also solved by Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Saturnino Campo Ruiz, Salamanca, Spain.

J243. Let a, b, c be real numbers such that

$$\left(-\frac{a}{2}+\frac{b}{3}+\frac{c}{6}\right)^3+\left(\frac{a}{3}+\frac{b}{6}-\frac{c}{2}\right)^3+\left(\frac{a}{6}-\frac{b}{2}+\frac{c}{3}\right)^3=\frac{1}{8}.$$

Prove that

$$(a-3b+2c)(2a+b-3c)(-3a+2b+c)=9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, California, USA

Notice that

$$\left(-\frac{a}{2}+\frac{b}{3}+\frac{c}{6}\right)^3+\left(\frac{a}{3}+\frac{b}{6}-\frac{c}{2}\right)^3+\left(\frac{a}{6}-\frac{b}{2}+\frac{c}{3}\right)^3=\frac{1}{8} \text{ iff } x^3+y^3+z^3=27,$$

where

$$x:=(-3a+2b+c)^3,\ y:=(2a+b-3c)^3,\ z:=(a-3b+2c)^3.$$

Since for any reals x, y, z, we know that

$$x^3+y^3+z^3-3xyz=(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right),$$

it follows in our case, where we know $x+y+z=0$, that $x^3+y^3+z^3-3xyz=0$, i.e. $xyz=9$. This is precisely what we want.

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Roberto Bosch Cabrera, Havana, Cuba; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Alessandro Ventullo, Milan, Italy; Zolbayar Shagdar, Orchlon International School, Ulaanbaatar, Mongolia; Sayan Das, Kolkata, India; Giulio Calimici, Francesco De Sclavis, Tommaso Gianiorio, Michele Ricciardi, Emiliano Torti, Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy; Prithwijit De, HBCSE, Mumbai, India; Stanescu Florin, Serban Cioculescu School, Gaesti, Dambovita, Romania; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Armend Sh. Shabani, University of Prishtina, Republic of Kosova.

J244. Let a and b be positive real numbers. Prove that

$$1 \leq \frac{\sqrt[n]{a^n + b^n}}{\sqrt[n+1]{a^{n+1} + b^{n+1}}} \leq \sqrt[n(n+1)]{2}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The second inequality can be written as

$$\sqrt[n]{\frac{a^n + b^n}{2}} \leq \sqrt[n+1]{\frac{a^{n+1} + b^{n+1}}{2}},$$

clearly true by the power mean inequality, and with equality iff $a = b$.

The first inequality is clearly equivalent to proving that

$$(1 + x^n)^{n+1} \geq \left(1 + x^{n+1}\right)^n,$$

where since a, b may be interchanged without altering the problem, we have defined wlog $1 \geq x = \frac{b}{a} > 0$. Using Newton's binomial formula and rearranging terms, this is in turn equivalent to

$$0 \leq x^{n(n+1)} + \sum_{k=1}^{n} x^{kn} \left(\binom{n}{k-1} + \binom{n}{k}(1 - x^k) \right),$$

where the RHS is clearly positive, since all terms are non-negative, and all are zero simultaneously iff $x = 0$, which is impossible.

The conclusion follows, equality holds in the second equality iff $a = b$, and although the first inequality is strict, we can have the middle term approach 1 arbitrarily close as we take $\frac{b}{a} \to 0$.

Also solved by Stanescu Florin, Serban Cioculescu School, Gaesti, Dambovita, Romania; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Arkady Alt, San Jose, California, USA; Alessandro Ventullo, Milan, Italy; Francesco De Sclavis, Università di Roma "Tor Vergata", Roma, Italy; Sayan Das, Kolkata, India; Roberto Bosch Cabrera, Havana, Cuba.

J245. Find all triples (x, y, z) of positive real numbers satifying simultaneously the inequalities $x + y + z - 2xyz \leq 1$ and

$$xy + yz + zx + \frac{1}{xyz} \leq 4.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Alessandro Ventullo, Milan, Italy

Using the HM-AM Inequality, from the first inequality we have

$$\frac{9xyz}{xy + yz + zx} \leq x + y + z \leq 1 + 2xyz,$$

which gives $\frac{9xyz}{2xyz + 1} \leq xy + yz + zx$. From the second inequality, we have

$$\frac{9xyz}{2xyz + 1} + \frac{1}{xyz} \leq 4.$$

Putting $t = xyz > 0$ and clearing the denominators, we obtain

$$9t^2 + 2t + 1 \leq 4t(2t + 1),$$

i.e. $(t - 1)^2 \leq 0$, which gives $t = 1$. So, the first inequality is $x + y + z \leq 3xyz$, but by AM-GM Inequality $3xyz \leq x + y + z$, which implies $x + y + z = 3xyz$. The equality holds if and only if $x = y = z = 1$, so the only triple which satisfies the two inequalities is $(1, 1, 1)$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; G.R.A.20 Problem Solving Group, Roma, Italy; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Arkady Alt, San Jose, California; Roberto Bosch Cabrera, Havana, Cuba; Sayan Das, Kolkata, India; Zolbayar Shagdar, Orchlon International School, Ulaanbaatar, Mongolia.

J246. Let ABC be a triangle with circumcircle Γ and let P be a point on the side BC. Let Ω be the circle tangent to BC at P and to Γ internally. Let τ be the length of the tangents from A to Ω and let U and V be the intersections of Γ with the circle centered at A and radius τ. Prove that UV is tangent to Γ.

Proposed by Cosmin Pohoata, Princeton University, USA

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let T be the point where Ω and Γ touch, and let $T' = UV \cap AT$. Since $AU = AV = \tau$, applying Stewart's theorem to cevian AT' in triangle AUV yields $AT'^2 = \tau^2 - UT' \cdot VT'$. But $UT' \cdot VT' = AT' \cdot TT'$ is the power of T' with respect to Γ, or $AT \cdot AT' = \tau^2$ is the power of A with respect to Ω, hence T' is on Ω.

Let now W be the intersection of UV and the common tangent to Γ and Ω at T. Applying Stewart's theorem to cevian WT' in triangle WTA, we find

$$WT'^2 = \frac{TT' \cdot WA^2 + AT' \cdot WT^2}{AT} - AT' \cdot TT',$$

or $WT' = WT$ iff $WA^2 = WT^2 + \tau^2$. But $WA^2 - \tau^2$ is the power of W with respect to the circle with center A and radius τ, or since UV is a common chord of this circle and Γ, $WT' = WT$ is equivalent to WT^2 being the power of W with respect to Γ, clearly true since WT is tangent to Γ at T.

Finally, W is the radical center of Ω, Γ, and the circle with center A and radius τ, since it is on the common tangent at T of the first two, and on a common chord of the last two. It follows that $WT^2 = WT'^2$ is the power of W with respect to Ω, and since T' is on Ω, WT' is tangent to Ω at T'. The conclusion follows.

Second solution by Roberto Bosch Cabrera, Havana, Cuba.

Notice that if we apply an inversion with center A and power τ the image of circle Ω is Ω and the image of circle Γ is a line tangent to Ω passing by U, V since these points are fixed. Hence the segment UV is tangent to Ω.

Also solved by Saturnino Campo Ruiz, Salamanca, Spain; Mehdi Mikael Trense, Montpellier, France.

Senior problems

S241. Let p and q be odd primes such that $\frac{p^3-q^3}{3} \geq 2pq+3$. Prove that

$$\frac{p^3-q^3}{4} \geq 3pq+16.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Clearly, $p > q$, so $p-2 \geq q$. It must be $p-2 \neq q$, otherwise we would have

$$p^3-q^3 = p^3-(p-2)^3 = 6p^2-12p+8 = 6pq+8,$$

which contradicts the given inequality. So $p-4 \geq q$ and

$$p^3-q^3 \geq p^3-(p-4)^3 = 12p^2-48p+64 = 12p(p-4)+64 \geq 12pq+64,$$

which gives the conclusion.

Also solved by Albert Stadler, Switzerland; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Armend Sh. Shabani, University of Prishtina, Republic of Kosova; Prithwijit De, HBCSE, Mumbai, India; Zolbayar Shagdar, Orchlon International School, Ulaanbaatar, Mongolia; Roberto Bosch Cabrera, Havana, Cuba; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Giulio Calimici, Daniele Fakhoury, Emiliano Torti, Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy.

S242. Let ABC be a triangle and denote by K, L, M the midpoints of the arcs BC, CA, AB, respectively (the ones not containing the vertices of the triangle). Show that the perimeter of the hexagon $AMBKCL$ is greater than or equal to $4(R+r)$.

Proposed by Michal Rolinek, Charles University, Czech Republic

Solution by Roberto Bosch Cabrera, Havana, Cuba

Let O be the circumcenter of triangle ABC. We have that $\angle AOB = 2C$ and $\angle AOM = \angle BOM = C$. If we apply Law of Sines in triangle AOM then $\frac{AM}{\sin C} = \frac{R}{\cos \frac{C}{2}}$, hence $AM = MB = 2R \sin \frac{C}{2}$. By analogy $BK = KC = 2R \sin \frac{A}{2}$ and $CL = LA = 2R \sin \frac{B}{2}$. It follows that the perimeter of the hexagon $AMBKCL$ is given by $P(AMBKCL) = 4R\left(\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}\right)$. We need to prove that $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \geq 1 + \frac{r}{R} = \cos A + \cos B + \cos C$. Nonetheless

$$\begin{aligned}
\cos A + \cos B &= 2 \sin \frac{C}{2} \cos \frac{A-B}{2} \leq 2 \sin \frac{C}{2} \\
\cos B + \cos C &= 2 \sin \frac{A}{2} \cos \frac{B-C}{2} \leq 2 \sin \frac{A}{2} \\
\cos C + \cos A &= 2 \sin \frac{B}{2} \cos \frac{C-A}{2} \leq 2 \sin \frac{B}{2},
\end{aligned}$$

thus by adding these relations we get precisely what we want. The equality holds if and only if triangle ABC is equilateral.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Mehdi Mikael Trense, Montpellier, France.

S243. A group of boys and girls went to a dance party. It is known that for every pair of boys, there are exactly two girls who danced with both of them; and for every pair of girls there are exactly two boys who danced with both of them. Prove that the numbers of girls and boys are equal.

Proposed by Iurie Boreico, Stanford University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume that there were n boys and m girls at the dance party. Denote by $b_1, b_2, \ldots, b_N$ all the possible pairs of boys, of which there are exactly $N = \binom{n}{2}$. Similarly, denote by $g_1, g_2, \ldots, g_M$ all the possible pairs of girls, of which there are exatly $M = \binom{m}{2}$. We will say that b_i and g_j are related iff both boys in b_i danced with both girls in g_j. If b_i is related to g_j and g_k for $j \neq k$, then there are at least three girls (if g_j, g_k have a girl in common), possibly four (if g_j, g_k are disjoint) with which both boys in b_i danced, contradiction. On the other hand, since both boys in b_i danced with two girls, b_i is related to the pair g_j formed by these two girls. Therefore, each pair of boys b_i is related to exactly one pair of girls g_j. Assume that two distinct pairs of boys b_i, b_k with $i \neq k$ are related to the same pair of girls g_j. Then, both girls in g_j danced with at least three boys (if b_i, b_k have one boy in common), possibly four (if b_i, b_k are disjoint) boys, contradiction. On the other hand, for each pair of girls g_j, there are exactly two boys that danced with both of them, and the pair b_i formed by both, is related to g_j. We have thus proved that b_i being related to g_j is a one-to-one correspondence between the N pairs of boys and the M pairs of girls, or $N = M$, and $(m-n)(m+n+1) = 0$. It follows that $m = n$. This completes the proof.

Also solved by Francesco De Sclavis, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy; Alessandro Ventullo, Milan, Italy.

S244. Let ABC be an acute-angled triangle and let τ be the inradius of its orthic triangle. Prove that

$$r \geq \sqrt{R\tau},$$

where r and R are the inradius and circumradius of triangle ABC.

Proposed by Luis Gonzalez, Maracaibo, Venezuela

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain
Claim: In any triangle ABC, we have $\cos A + \cos B + \cos C \leq \frac{3}{2}$, with equality iff ABC is equilateral.
Proof: The inequality may be written as

$$4 \sin \frac{A}{2} \cos \frac{B-C}{2} \leq 1 + 4 \sin^2 \frac{A}{2},$$

clearly true since the RHS is, by the AM-GM inequality, at least $4 \sin \frac{A}{2}$, and the LHS is at most $4 \sin \frac{A}{2}$, with equality in both cases iff $\sin \frac{A}{2} = \frac{1}{2}$ and $\cos \frac{B-C}{2} = 1$, ie iff ABC is equilateral. The Claim follows.

Denote by D, E, F the respective feet of the altitudes from A, B, C. It is well known (or easily found) that $DE = c \cos C$, $EF = a \cos A$ and $FD = b \cos B$, and moreover $\angle DEF = 180° - 2\angle B$, $\angle EFD = 180° - 2\angle C$ and $\angle FDE = 180° - 2\angle A$. The perimeter of the orthic triangle is then

$$a \cos A + b \cos B + c \cos C = \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{2abc} = \frac{2S}{R},$$

where S, R are the area and circumradius of ABC, and both Heron's formula and the well-known relation $S = \frac{abc}{4R}$ have been used. The area of the orthic triangle is easily found to be $2S \cos A \cos B \cos C$, or

$$\tau = 2R \cos A \cos B \cos C,$$

while it is well known (or easily found, for example comparing different expressions for the area of ABC) that $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$. Using that $2 \sin^2 x = 1 - \cos(2x)$, and assuming wlog that $A \leq B \leq C$, or $2 \cos A \geq 1$, we may rewrite the proposed inequality as

$$(1 - \cos A)(1 - \cos B - \cos C) \geq \cos B \cos C(2 \cos A - 1).$$

Now, $1 - \cos B - \cos C \geq \frac{2 \cos A - 1}{2}$ by the Claim, or it suffices to show that

$$1 \geq \cos A + 2 \cos B \cos C = \cos(B - C),$$

clearly true and with equality iff $B = C$. Now, when $\cos B \cos C = \frac{1 - \cos A}{2}$, the proposed inequality is equivalent to the Claim. The conclusion follows, equality holds iff ABC is equilateral, in which case $R = 2r = 4\tau$.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Arkady Alt, San Jose, California, USA; Sayan Das, Kolkata, India.

S245. Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Prove that

$$\frac{1}{(a-1)(b-1)(c-1)}+\frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Christopher Wiriawan, Surya Institute, Jakarta, Indonesia

Notice that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ is equivalent to $ab+bc+ca=abc$. Hence,

$$\frac{1}{(a-1)(b-1)(c-1)}+\frac{8}{(a+1)(b+1)(c+1)}=\frac{1}{a+b+c-1}+\frac{8}{2abc+a+b+c+1}$$

Now, by AM-HM, we have that

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=3, \text{ thus } a+b+c \geq 9.$$

Also, by GM-HM,

$$(abc)^{\frac{1}{3}} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=3, \text{ hence } abc \geq 27.$$

It therefore follows that

$$\frac{1}{a+b+c-1}+\frac{8}{2abc+a+b+c+1} \leq \frac{1}{9-1}+\frac{8}{2 \cdot 27+9+1}=2 \cdot \frac{1}{8}=\frac{1}{4},$$

which proves our inequality.

Also solved by Albert Stadler, Switzerland; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Arkady Alt , San Jose ,California, USA; Prithwijit De, HBCSE, Mumbai, India; Sayan Das, Kolkata, India; Zolbayar Shagdar, Orchlon International School, Ulaanbaatar, Mongolia; Roberto Bosch Cabrera, Havana, Cuba; Andrea Fiacco, and Francesco De Sclavis, Università di Roma "Tor Vergata", Roma, Italy; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Mehdi Mikael Trense, Montpellier, France.

S246. Let ABC be a triangle with circumcircle Ω and let X, Y, Z be points on the sides BC, CA, AB, respectively. Let α, β, γ be the circles tangent to Ω that are also tangent to BC, CA, AB at points X, Y, Z, respectively. Let X', Y', Z' be the tangency points of Ω with α, β, γ. Prove that the lines AX, BY, CZ are concurrent if and only if the lines AX', BY', CZ' are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Consider the homothety of center X' that brings α into Ω, which clearly exists because α, Ω are tangent at X'. Clearly, lines $X'B, X'C$ meet α at points X_b, X_c such that $\frac{X'X_b}{X'B} = \frac{X'X_c}{X'C} = \rho$ is the scaling factor of the homothety. Since the powers of B, C with respect to α are respectively $BX^2 = BX_b \cdot BX'$ and $CX^2 = CX_c \cdot CX'$, we have

$$\frac{BX'}{BX} = \sqrt{\frac{BX'}{BX_b}} = \sqrt{\frac{1}{1-\rho}} = \sqrt{\frac{CX'}{CX_c}} = \frac{CX'}{CX}.$$

Now, denote by X'' the point where AX' meets BC. Clearly, triangles $BX'X''$ and ACX'' are similar, or $\frac{CX''}{X'X''} = \frac{AC}{BX'}$. Analogously, triangles BAX'' and $X'CX''$ are also similar, or $\frac{BX''}{X'X''} = \frac{AB}{CX'}$, for

$$\frac{BX''}{CX''} = \frac{AB}{AC} \cdot \frac{BX'}{CX'},$$

and similarly for the cyclic permutations of the vertices of ABC. Defining Y'', Z'' analogously to X'', we conclude that

$$\frac{BX''}{CX''} \cdot \frac{CY''}{AY''} \cdot \frac{AZ''}{BZ''} = \frac{BX'}{CX'} \cdot \frac{CY'}{AY'} \cdot \frac{AZ'}{BZ'}.$$

The conclusion follows by Menelaus' theorem.

Also solved by Roberto Bosch Cabrera, Havana, Cuba.

Undergraduate problems

U241. Let $a > b$ be positive real numbers. Prove that

$$c_n = \frac{\sqrt[n+1]{a^{n+1} - b^{n+1}}}{\sqrt[n]{a^n - b^n}}$$

is a decreasing sequence and find its limit.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Arkady Alt, San Jose, California, USA

Since $\dfrac{\sqrt[n+1]{a^{n+1} - b^{n+1}}}{\sqrt[n]{a^n - b^n}} = \dfrac{\sqrt[n+1]{1 - \left(\dfrac{b}{a}\right)^{n+1}}}{\sqrt[n]{1 - \left(\dfrac{b}{a}\right)^{n}}}$ and $\dfrac{b}{a} < 1$, we then find that

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{a^{n+1} - b^{n+1}}}{\sqrt[n]{a^n - b^n}} = \frac{\lim\limits_{n\to\infty} \sqrt[n+1]{1 - \left(\dfrac{b}{a}\right)^{n+1}}}{\lim\limits_{n\to\infty} \sqrt[n]{1 - \left(\dfrac{b}{a}\right)^{n}}} = 1.$$

Now, let $t := \dfrac{a}{b}$. We have $t > 1$ and we note that $c_n > c_{n+1}$ is equivalent with

$$\frac{\sqrt[n+1]{t^{n+1} - 1}}{\sqrt[n]{t^n - 1}} > \frac{\sqrt[n+2]{t^{n+2} - 1}}{\sqrt[n+1]{t^{n+1} - 1}} \text{ i.e. } \left(t^{n+1} - 1\right)^{\frac{2}{n+1}} \geq \left(t^n - 1\right)^{\frac{1}{n}} \left(t^{n+2} - 1\right)^{\frac{1}{n+2}},$$

which rewrites as $\left(t^{n+1} - 1\right)^2 \geq \left(t^n - 1\right)^{\frac{n+1}{n}} \left(t^{n+2} - 1\right)^{\frac{n+1}{n+2}}$.

However, $\left(t^{n+1} - 1\right)^2 \geq \left(t^n - 1\right)\left(t^{n+2} - 1\right)$, since this is just the same thing as saying $t^n + t^{n+2} \geq 2t^{n+1}$, which is obviously true as $(t-1)^2 \geq 0$. Thus, it suffices to prove that

$$\left(t^n - 1\right)\left(t^{n+2} - 1\right) \geq \left(t^n - 1\right)^{\frac{n+1}{n}} \left(t^{n+2} - 1\right)^{\frac{n+1}{n+2}},$$

or equivalently $\left(t^{n+2} - 1\right)^{\frac{1}{n+2}} \geq \left(t^n - 1\right)^{\frac{1}{n}}$. But now, since $\dfrac{t^{n+1} - 1}{t^n - 1} \geq t$ iff $t \geq 1$, we have that $\left(\dfrac{t^{n+1} - 1}{t^n - 1}\right)^n \geq t^n > t^n - 1$, which gives $\left(t^{n+1} - 1\right)^n > \left(t^n - 1\right)^{n+1}$, so $\left(t^{n+1} - 1\right)^{\frac{1}{n+1}} > \left(t^n - 1\right)^{\frac{1}{n}}$, and therefore, $\left(t^{n+2} - 1\right)^{\frac{1}{n+2}} \geq \left(t^n - 1\right)^{\frac{1}{n}}$. This completes the proof.

Also solved by G.R.A.20 Problem Solving Group, Roma, Italy; Albert Stadler,Switzerland; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Alessandro Ventullo, Milan, Italy.

U242. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following condition: whenever $x+y+z = 2k\pi$ is an integer multiple multiple of 2π,

$$f^2(x) + f^2(y) + f^2(z) - 2f(x)f(y)f(z) = 1.$$

Proposed by Iurie Boreico, Stanford University, USA

Solution by Andrea Fiacco, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy

We claim that

$$f(x) = \cos(Nx) \quad \text{or} \quad f(x) = \cos(Nx - 2\pi/3) \quad \text{for some } N \in \mathbb{Z}.$$

If $x = y = z = 0$ then $3f^2(0) - 2f^3(0) = 1$ which implies that $f(0) = 1$ or $f(0) = -1/2$. Now, assume that $f(0) = 1$ and that f is not identically 1, then we will show that $f(x) = \cos(Nx)$ for some positive integer N (the other case, where $f(0) = -1/2$, can be processed in a similar way). We divide the proof in several steps:

i) $f(-t) = f(t)$ for any $t \in \mathbb{R}$.

If $x = t$, $y = -t$ and $z = 0$ then $f^2(t)+f^2(-t)+f^2(0)-2f(t)f(-t)f(0) = 1$, that is $(f(t)-f(-t))^2 = 0$.

ii) $f(t) = f(t + 2\pi)$ for any $t \in \mathbb{R}$.

If $x = t$, $y = -t - 2\pi$, $z = 0$ then $f^2(t) + f^2(-(t + 2\pi)) + f^2(0) - 2f(t)f(-(t + 2\pi))f(0) = 1$, that is, by i), $(f(t) - f(t + 2\pi))^2 = 0$.

iii) $G = f^{-1}(1)$ is a closed infinite additive subgroup of $\mathbb{R}$ different from $\mathbb{R}$, which implies that G has some least positive element $a = 2\pi/N$ for some positive integer N, and $G = \{na : n \in \mathbb{Z}\}$.

If $f(x) = 1$ and $f(y) = 1$ then for $z = -x - y$ we have that $2 + f^2(-(x + y)) - 2f(-(x + y)) = 1$, that is, by i), $(f(x + y) - 1)^2 = 0$. Moreover, the group G is closed because f is continuous and it is infinite because $2\pi n \in G$ for all $n \in \mathbb{Z}$. Thus G has some least positive element $a = 2\pi/N$ where N is a positive integer.

iv) $f^2(t/2) = (f(t) + 1)/2$ for any $t \in \mathbb{R}$.

If $t \in \mathbb{R} \setminus G$ then $x = t/2$, $y = t/2$, $z = -t$ then $2f^2(t/2) + f^2(-t) - 2f^2(t/2)f(-t) = 1$, that is, by i), $(1 - f(t))(2f^2(t/2) - f(t) - 1) = 0$ and $f^2(t/2) = (f(t) + 1)/2$. By the continuity of f, it holds in $\overline{\mathbb{R} \setminus G} = \mathbb{R}$.

v) $|f(t)| \leq 1$ for any $t \in \mathbb{R}$.

Since $|f|$ is continuous and periodic, it attains the maximum value at some point t_0. If $|f(t_0| > 1$, then , by iv), $f(2t_0) = 2f^2(t_0) - 1 > |f(t_0)|$ which is a contradiction.

vi) $f(a/2^m) = \cos(2\pi/2^m)$ for any non-negative integer m.

It holds for $m = 1$. Moreover, assuming that it holds up to n, then, by iv),

$$f^2((t_0/2^{m+1})) = \frac{f((t_0/2^m)) + 1}{2} = \frac{\cos(2\pi/2^m) + 1}{2} = \cos^2(2\pi/2^{m+1})$$

which implies that $f((t_0/2^{m+1})) = \cos(2\pi/2^{m+1})$.

vii) $f(na/2^m) = \cos(2\pi n/2^m)$ for any non-negative integers m, n.

It holds for $n = 0, 1$. Suppose it holds up to n, then, by letting $x = (n+1)a/2^m$, $y = -na/2^m$, and $z = -a/2^m$, we have that

$$\begin{aligned} f^2\left(\frac{(n+1)a}{2^m}\right) &+ \cos^2\left(\frac{2\pi n}{2^m}\right) + \cos^2\left(\frac{2\pi}{2^m}\right) \\ &= 2f\left(\frac{(n+1)a}{2^m}\right)\cos^2\left(\frac{2\pi n}{2^m}\right)\cos^2\left(\frac{2\pi}{2^m}\right) + 1. \end{aligned}$$

Moreover

$$\begin{aligned} \cos^2\left(\frac{2\pi(n+1)}{2^m}\right) &+ \cos^2\left(\frac{2\pi n}{2^m}\right) + \cos^2\left(\frac{2\pi}{2^m}\right) \\ &= 2\cos\left(\frac{2\pi(n+1)}{2^m}\right)\cos^2\left(\frac{2\pi n}{2^m}\right)\cos^2\left(\frac{2\pi}{2^m}\right) + 1. \end{aligned}$$

Therefore, by subtracting these two equations, we obtain

$$\left(f\left(\frac{(n+1)a}{2^m}\right) - \cos^2\left(\frac{2\pi(n+1)}{2^m}\right)\right)^2 = 0.$$

viii) $f(x) = \cos(Nx)$ for some positive integer N.

The set of rational numbers $n/2^m$ for any non-negative integers m, n is dense in $\mathbb{R}^+$, hence by continuity and by i), $f(x) = \cos(2\pi x/a)$ for all $x \in \mathbb{R}$.

U243. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function such that $f'(a) = f'(b) = 0$ and with the property that there is a real valued function g for which $g(f'(x)) = f(x)$ for all x in $\mathbb{R}$. Prove that f is constant.

Proposed by Mihai Piticari and Sorin Radulescu, Bucharest, Romania

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

To have $f'(a)$ and $f'(b)$, the domain of f must include both a and b; so we define $f\colon [a, b] \to \mathbb{R}$. However, note that

$$x \neq y \ \wedge \ f'(x) = f'(y) \implies f(x) = f(y)$$

because otherwise, $g(x)$ is undefined as a function. By Rolle's theorem there is a point $c \in (a, b)$ such that $f'(c) = 0$ and supposing that $f(c) \neq 0$, we would have $f'(a) = f'(c) = 0$ but $f(a) \neq f(c)$. This contradiction can be avoided if we suppose that $f(c) = 0$. The same happens with the minimum point that exists, say, at $x = d$. Thus we are forced to say $f(d) = f(c) = 0$, which implies that $f(x)$ is constantly equal to zero.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Arkady Alt , San Jose ,California, USA; Alessandro Ventullo, Milan, Italy; Harun Immanuel, ITS Surabaya; Giulio Calimici, Francesco De Sclavis, Andrea Fiacco, Michele Ricciardi, Emiliano Torti, Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy.

U244. Jimmy is analyzing a random variable X with infinite mean and variance. He needs to come up with a tentative mean and variance for his approximation model. Jimmy knows that X has symmetric distribution around 0. He decides that his tentative mean is going to be $\mu_X^* = 0$ and his tentative variance is going to be

$$(\sigma_X^*)^2 = E_{|X|\le 1}\left[X^2\right] + \exp\left(E_{|X|\ge 1}\left[\ln\left(X^2\right)\right]\right).$$

Prove that the tentative variance that Jimmy came up with for a standard Cauchy random variable $C(0,1)$ is

$$(\sigma_X^*)^2 = 2 - \frac{\pi}{2} + \exp\left(\frac{2\pi^2}{3}\right).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Claim: Catalan's constant K can be expressed as

$$K = -\int_0^1 \frac{\ln(x)dx}{x^2+1} = -\int_{-\infty}^0 \frac{ydy}{2\cosh(y)} = \int_0^{+\infty} \frac{ydy}{2\cosh(y)} = \int_1^{+\infty} \frac{\ln(x)dx}{x^2+1}.$$

Proof: The second and fourth equalities are a consequence of performing the variable exchange $x = e^y$, and the third equality is a consequence of the fact that $\cosh(y)$ has even parity, and y has odd parity. It remains thus to prove only that K equals any of the other terms. Consider function

$$f(x) = \mathrm{Li}_2(-ie^{-x}) - \mathrm{Li}_2(ie^{-x}) + x\log\left(\frac{1-ie^{-x}}{1+ie^{-x}}\right),$$

where $\mathrm{Li}_n(x)$ is the polylogarithm function. Since $\frac{d\mathrm{Li}_n(x)}{dx} = \frac{\mathrm{Li}_{n-1}(x)}{x}$ and $\mathrm{Li}_1(x) = -\ln(1-x)$, we have after some calculations $\frac{df(x)}{dx} = \frac{ix}{\cosh(x)}$, or

$$\int_0^{+\infty} \frac{ydy}{2\cosh(y)} = \frac{i}{2}f(0) - \frac{i}{2}\lim_{x\to\infty} f(x) = \frac{i}{2}\left(\mathrm{Li}_2(-i) - \mathrm{Li}_2(i)\right) - \frac{i}{2}\lim_{x\to\infty} x\log\left(\frac{1-ie^{-x}}{1+ie^{-x}}\right).$$

Now, by the definition of the polylogarithm function and of Catalan's constant K,

$$\mathrm{Li}_2(-i) - \mathrm{Li}_2(i) = \sum_{k=1}^{\infty} \frac{(-i)^k - i^k}{k^2} = -2i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = -2iK,$$

while using that $\ln(1+x) = O(x)$ for x with small absolute value, where Landau notation has been used, we have

$$\lim_{x\to\infty} x\log\left(\frac{1-ie^{-x}}{1+ie^{-x}}\right) = \lim_{x\to\infty} xO\left(e^{-x}\right) = 0.$$

The Claim follows.

A random variable $C(0,1)$ has probability density funcion $\frac{1}{\pi(x^2+1)}$. Clearly

$$\int_0^1 \frac{x^2dx}{x^2+1} = 1 - \int_0^1 \frac{dx}{x^2+1} = 1 - \arctan(1) + \arctan(0) = 1 - \frac{\pi}{4},$$

or using that the distribution of $C(0,1)$ is symmetric around 0, and using the Claim, it follows that

$$(\sigma_X^*)^2 = \frac{2}{\pi}\int_0^1 \frac{x^2dx}{x^2+1} + \exp\left(\frac{4}{\pi}\int_1^{\infty} \frac{\ln(x)dx}{x^2+1}\right) = \frac{2}{\pi} - \frac{1}{2} + \exp\left(\frac{4K}{\pi}\right),$$

where K is Catalan's constant.

U245. Let K be a finite field of characteristic $p > 2$ and let $a \in K - \{0\}$. Let f be any polynomial over K. Prove that the following statements are equivalent:

i) $f(x) = f(x+a)$;

ii) there is $g \in K[x]$ such that $f(x) = g(x^p - a^{p-1}x)$.

Proposed by Mihai Piticari and Sorin Radulescu, Bucharest, Romania

Solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

Let $s(x) = x^p - a^{p-1}x$, then

$$s(x+a) = (x+a)^p - a^{p-1}(x+a) = x^p + a^p - a^{p-1}x - a^p = x^p - a^{p-1}x = s(x),$$

where we used the fact that $\mathrm{char}(K) = p$. Note that $s(0) = 0$, so $s(na) = 0$, for all $n \in \{0, ..., p-1\}$ and

$$s(x) = \prod_{n=0}^{p-1}(x - na).$$

It follows immediately that ii) implies i): for any $g \in K[x]$,

$$f(x+a) = g(s(x+a)) = g(s(x)) = f(x).$$

Now, assume that the converse is false. Let $f \in K[x]$ with minimal degree such that $f(x) = f(x+a)$, and f is not a polynomial in $s(x)$ (in particular it is not constant), with coefficients in K. Let $h(x) = f(x) - f(0)$, then it satisfies $h(x) = h(x+a)$, and $h(0) = 0$, so $h(na) = 0$, for all $n \in \{0, ..., p-1\}$. This means that $s(x)$ divides $h(x)$ and $h(x) = s(x)q(x)$ for some $q \in K[x]$. Then

$$s(x)q(x+a) = s(x+a)q(x+a) = f(x+a) = f(x) = s(x)q(x),$$

which yields $q(x) = q(x+a)$. But $\deg(q) < \deg(f)$, and by the minimality of f, there exists $g \in K[x]$, such that $q(x) = g(x^p - a^{p-1}x)$. Therefore

$$f(x) = h(x) + f(0) = (x^p - a^{p-1}x)g(x^p - a^{p-1}x) + f(0),$$

and the right-hand side is a polynomial in $K[x]$ evaluated in $x^p - a^{p-1}x$, so we have a contradiction.

U246. Find all pairs m, n of positive integers for which there is a non-abelian group G such that the maps $x \to x^m$ and $x \to x^n$ are endomorphisms of G.

Proposed by Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

We will prove that there is a non-abelian group G such that the maps $x \to x^m$ and $x \to x^n$ are endomorphisms of G if and only if $\gcd(m(m-1), n(n-1)) \neq 2$.

Let p be a prime and consider the non-abelian group

$$U_p = \left\{ \begin{bmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} : r, s, t \in \mathbb{Z}_p \right\}$$

It is easy to see that $x^a = e$ for all $x \in U_p$ with $a = p$ if p is an odd prime and $a = 4$ if $p = 2$. Hence, for any $k \in \mathbb{Z}$, $x \to x^{ka} = e$ and $x \to x^{ka+1} = x$ are endomorphism of U_p.

If $m = 1$ and $n = 1$ then any non-abelian group G will work.

Assume that $d = \gcd(m(m-1), n(n-1)) > 2$. If $p|d$ with p odd prime then $m, n \in \{kp+1, kp : k \in \mathbb{Z}\}$ and one can take $G = U_p$. Otherwise $4|d$ which implies that $m, n \in \{4k+1, 4k : k \in \mathbb{Z}\}$ and one can take $G = U_2$.

Now we prove that $\gcd(m(m-1), n(n-1)) = 2$ then G is abelian. Let

$$A = \{a \in \mathbb{Z} : x \to x^a \text{ is an endomorphism of } G \text{ and } \forall x \in G,\, x^a \in Z(G)\}$$

where $Z(G)$ is the center of G. Note that $0 \in A$.

1) If $a \in A$ then $-a \in A$.

   If $x^a \in Z(G)$ then $x^{-a} \in Z(G)$. Moreover

$$(xy)^{-a} = ((xy)^{-1})^a = (y^{-1}x^{-1})^a = (y^{-1})^a(x^{-1})^a = (x^{-1})^a(y^{-1})^a = x^{-a}y^{-a}.$$

2) If $a, b \in A$ then $a + b \in A$.

   If $x^a, x^b \in Z(G)$ then $x^{a+b} \in Z(G)$. Moreover

$$(xy)^{a+b} = (xy)^a(xy)^b = x^a y^a x^b y^b = x^a x^b y^a y^b = x^{a+b}y^{a+b}.$$

3) $n(n-1) \in A$ and $m(m-1) \in A$.

   We prove the first claim (the other is similar).

   3.1) $x \to x^{1-n}$ is an endomorphism of G.

$$(x^{-1})^n(y^{-1})^n = (x^{-1}y^{-1})^n = x^{-1}(y^{-1}x^{-1})^{n-1}y^{-1}$$

   which implies

$$(x)^{1-n}(y)^{1-n} = (x^{-1})^{n-1}(y^{-1})^{n-1} = (y^{-1}x^{-1})^{n-1} = (xy)^{1-n}$$

   3.2) $x \to x^{n(1-n)}$ is an endomorphism of G by composition of $x \to x^n$ and $x \to x^{1-n}$.

3.3) $\forall x, y \in G$, x^n commutes with y^{1-n}

$$yx^n y^n y^{-1} = y(xy)^n y^{-1} = (yx)^n = y^n x^n$$

which implies that $y^{1-n}x^n = x^n y^{1-n}$.

3.4) $\forall x \in G$, $x^{n(1-n)} \in Z(G)$

By 3.3), $x^{n(1-n)}$ commutes with y^n and y^{1-n}. Thus $x^{n(1-n)}$ commutes also with $y^n y^{1-n} = y$.

3.5) By 3.2) and 3.4), $n(1-n) \in A$ and, by 1), $n(n-1) \in A$

4) G is abelian.

Since $\gcd(m(m-1), n(n-1)) = 2$, there exist $p, q \in \mathbb{Z}$ such that $pm(m-1) + q(n(n-1) = 2$. Therefore, by 1), 2), 3), we have that $2 \in A$. So $x \to x^2$ is an endomorphism of G, which means that for all $x, y \in G$ $xyxy = (xy)^2 = x^2y^2 = xxyy$, that is $yx = xy$.

Olympiad problems

O241. Let a and b be real numbers such that $3 \leq a^2 + ab + b^2 \leq 6$. Prove that $2 \leq a^4 + b^4 \leq 72$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $S = (a+b)^2$ and $D = (a-b)^2$. The problem is then equivalent to showing that, if $12 \leq 3S + D \leq 24$ where S, D are non-negative reals, then $16 \leq S^2 + 6SD + D^2 \leq 576$. Now,

$$S^2 + 6SD + D^2 = \frac{(3S+D)^2}{9} + \frac{8D}{9}(3S+D) + \frac{16DS}{3} \geq 16 + \frac{16DS}{3} + \frac{32D}{3} \geq 16,$$

with equality iff $D = 0$ and $S = 4$, ie iff $a = b = \pm 1$. At the same time,

$$S^2 + 6SD + D^2 \leq (3S+D)^2 - 8S^2 \leq 576 - 8S^2 \leq 576,$$

with equality iff $S = 0$ and $D = 24$, ie iff $a = -b = \pm\sqrt{6}$.

Also solved by Francesco De Sclavis, Università di Roma "Tor Vergata", Roma, Italy; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Switzerland; Arkady Alt , San Jose ,California, USA; Corneliu Manescu-Avram, Transportation High School, Ploiesti, Romania; Sayan Das, Kolkata, India; Roberto Bosch Cabrera, Havana, Cuba; Christopher Wiriawan, Surya Institute, Jakarta, Indonesia; Prithwijit De, HBCSE, Mumbai, India; Stanescu Florin, Serban Cioculescu School , Gaesti, Dambovita, Romania; Mehdi Mikael Trense, Montpellier, France; Cyril Letrouit, Lycee Jean-Baptiste Say, Paris, France.

O242. Let $n \geq 3$ be an odd integer. Consider a regular n-gon $\mathcal{A} = A_1A_2..., A_n$. Find the locus of points, P, inside $\mathcal{A}$ such that

$$\angle PA_1A_2 + \angle PA_2A_3 + \cdots + \angle PA_nA_1 = 90^\circ + 180^\circ \cdot k$$

for some integer k, where the angles are directed.

Proposed by Alex Anderson, University of California, Berkeley, USA

Solution by the author

We can set $A_k = z^k$ where $z = e^{\frac{2\pi i}{n}}$ and $P = p$ to represent the points in the complex plane. Then

$$\frac{p - z^k}{z^{k+1} - z^k} = r \cdot e^{\angle PA_kA_{k+1}}$$

where r is a real number. Conjugating both sides and dividing that new relation by the old one gives us:

$$\frac{\frac{p-z^k}{z^{k+1}-z^k}}{\overline{\frac{p-z^k}{z^{k+1}-z^k}}} = e^{2\angle PA_kA_{k+1}}$$

Multiply the relations to get

$$\frac{R}{\overline{R}} = \exp\left(\sum_{k=1}^{n} 2\angle PA_kA_{k+1}\right)$$

where

$$R = \prod_{k=1}^{n} \frac{p - z^k}{z^{k+1} - z^k}$$

We can easily evaluate R:

$$R = \frac{p^n - 1}{(z-1)^n \cdot z^{1+2+...n}} = \frac{p^n - 1}{(z-1)^n \cdot z^{(n)(n-1)/2}}$$

Then

$$\frac{R}{\overline{R}} = \frac{p^n - 1}{\overline{p}^n - 1} \cdot \frac{(\overline{z} - 1)^n}{(z-1)^n} \cdot \frac{\overline{z}^{(n)(n-1)/2}}{z^{(n)(n-1)/2}}$$

Using the fact that $\overline{z} = z^{-1}$ and that $z^n = 1$, we obtain

$$\frac{R}{\overline{R}} = \frac{p^n - 1}{\overline{p}^n - 1} \cdot (-1)^n$$

So we obtain the important relation that

$$\frac{p^n - 1}{\overline{p}^n - 1} \cdot (-1)^n = \exp\left(\sum_{k=1}^{n} 2\angle PA_kA_{k+1}\right)$$

Now the rest of the problem is straightforward. $\theta = 90 + 180 \cdot k$ for some integer k iff $e^{2i\theta} = -1$. Therefore, P is in the locus iff

$$\frac{p^n - 1}{\overline{p}^n - 1} = 1$$

because n is odd. Simplifying, we get $p = z^i\overline{p}$ where $i = 1, 2, \ldots, n$. Therefore the locus is easily seen to be the perpendicular bisectors of the sides of $\mathcal{A}$.

O243. Let m, n be positive integers with $n > m$. Prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+n-2k}{n-1} = \binom{n}{m+1}.$$

Proposed by Iurie Boreico, Stanford University, USA

Solution by Kostas Tsouvalas, Lemnos, Greece and Anastasios Kotronis, Athens, Greece

By Cauchy's theorem we have that $\binom{n}{m} = \frac{1}{2\pi i}\int_R \frac{(z+1)^n}{z^{m+1}}\,dz$, where R is any circle surrounding the origin.

Now:

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^n\binom{n}{k}\binom{m+n-2k}{n-1} &= \frac{1}{2\pi i}\sum_{k=0}^{n}(-1)^n\binom{n}{k}\int_R \frac{(z+1)^{m+n-2k}}{z^n}\,dz \\
&= \frac{1}{2\pi i}\int_R \frac{(z+1)^{m+n}}{z^n}\sum_{k=0}^{n}(-1)^n\binom{n}{k}\frac{1}{(z+1)^{2k}}\,dz \\
&= \frac{1}{2\pi i}\int_R \frac{(z+1)^{m+n}}{z^n}\left(1-\frac{1}{(z+1)^2}\right)^n dz \\
&= \frac{1}{2\pi i}\int_R \frac{(z+2)^n}{(z+1)^{n-m}}\,dz \\
{}^{1} &= \mathrm{Res}_{z=-1}\frac{(z+2)^n}{(z+1)^{n-m}} \\
&= \lim_{z\to-1}\frac{1}{(n-m-1)!}\frac{d^{n-m-1}}{dz^{n-m-1}}\left((z+2)^n\right) \\
&= \binom{n}{m+1}
\end{aligned}$$

Also solved by Konstantinos Tsouvalas, University of Athens, Greece, Athens; Angel Plaza , University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Switzerland; Giulio Calimici, and Emiliano Torti, Università di Roma "Tor Vergata", Roma, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O244. Let ABC be a triangle and let D, E, F be the tangency points of the incircle with BC, CA, AB, respectively. Let EF meet the circumcircle Γ of ABC at X and Y. Furthermore, let T be the second intersection of the circumcircle of DXY with the incircle. Prove that AT passes through the tangency point A' of the A-mixtilinear incircle with Γ.

Proposed by Sammy Luo, North Carolina School of Science and Mathematics and Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote by O_a the center of the mixtilinear incircle. It is well known that the mixtilinear inradius is $\rho_a = \frac{r}{\cos^2 \frac{A}{2}}$, or the exact trilinear coordinates of O_a are

$$O_a \equiv \left(\rho_a \frac{1 + \cos A - \cos B - \cos C}{2}, \rho_a, \rho_a\right).$$

Consider now the second intersection Z', other than A, of line $c(s-b)\beta = b(s-c)\gamma$ with the circumcircle of ABC. Since the equation of the latter is $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$, the coordinates of Z' satisfy $c(s-b)\beta = b(s-c)\gamma = -bc\alpha$, or in non-exact trilinear coordinates, after some algebra we find

$$Z' \equiv \left(-\frac{r}{2R}, 1 - \cos B, 1 - \cos C\right).$$

We can furthermore verify after some algebra that $(1 + \cos A - \cos B - \cos C, 2, 2)$, $(\cos A, \cos B, \cos C)$, and $\left(-\frac{r}{2R}, 1 - \cos B, 1 - \cos C\right)$ are linearly dependent, or Z' is on the circumcirle, on line OO_a, and on the sector determined by rays AB, AC, ie $Z' = A'$ is the point where the mixtilinear incircle touches the circumcircle. Indeed, this linearly dependence is equivalent to $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$, true by Carnot's theorem. The problem then reduces to proving that the intersection Z of line $c(s-b)\beta = b(s-c)\gamma$ and the incircle, futhest from A, lies on the circumcircle of DXY, where clearly Z is the result of scaling Z' with respect to A and a scaling factor $\cos^2 \frac{A}{2}$ (ie, the same scaling that transforms the A-mixtilinear incircle into the incircle).

Now, using that $c(s-c) = 4Rr \cos^2 \frac{C}{2}$, we may rewrite the equation for the incircle in trilinear coordinates as

$$a^2(s-a)^2\alpha^2 + b^2(s-b)^2\beta^2 + c^2(s-c)^2\gamma^2 =$$
$$= 2bc(s-b)(s-c)\beta\gamma + 2ca(s-c)(s-a)\gamma\alpha + 2ab(s-a)(s-b)\alpha\beta.$$

After some algebra, the intersection with line $c(s-b)\beta = b(s-c)\gamma$ closest to side BC (therefore further from A) satisfies

$$abc(s-a)\alpha = (b-c)^2 c(s-b)\beta = (b-c)^2 b(s-c)\gamma.$$

For α, β, γ to be exact trilinear coordinates, we need $a\alpha + b\beta + c\gamma = 2S$, or equivalently, in exact trilinear coordinates

$$Z \equiv \left(\frac{(b-c)^2(s-b)(s-c)}{2R\left(a(s-a) + (b-c)^2\right)}, \frac{ab(s-c)(s-a)}{2R\left(a(s-a) + (b-c)^2\right)}, \frac{ca(s-a)(s-b)}{2R\left(a(s-a) + (b-c)^2\right)}\right).$$

It now suffices to show that this point is concyclic with D, X, Y.

Now, given four points with exact trilinear coordinates (α, β, γ) with respect to a reference triangle ABC, it is well known that they are concyclic iff the four 4-vectors of the form $(a\beta\gamma + b\gamma\alpha + c\alpha\gamma, \alpha, \beta, \gamma)$ are collinear. Denote by $\vec{u}, \vec{v}$ the respective 4-vectors for D, Z. A linear combination with not both zero coefficients ρ, κ such that $\rho\vec{u} + \kappa\vec{v} \equiv (0, \alpha', \beta', \gamma')$. Clearly, $a\beta\gamma + b\gamma\alpha + c\alpha\gamma = 0$ for X, Y because they are both on the circumcircle, and since they are also both on line EF, it suffices to show that $(\alpha', \beta', \gamma')$ satisfies the equation for line EF.

Note that, in exact trilinear coordinates, $D \equiv (0, (s-c)\sin C, (s-b)\sin B)$, and similarly for E, F. It follows that the 4 vector for D is

$$\vec{u} \equiv (a(s-b)(s-c)\sin B \sin C, 0, (s-c)\sin C, (s-b)\sin B),$$

while the equation for line EF is

$$EF \equiv a(s-a)\alpha = b(s-b)\beta + c(s-c)\gamma.$$

The 4-vector for Z is found from previous results after a few calculations and using Heron's formula:

$$\vec{v} \equiv \left(\frac{arS^2}{KR}, \frac{(b-c)^2(s-b)(s-c)}{2KR}, \frac{ab(s-c)(s-a)}{2KR}, \frac{ca(s-a)(s-b)}{2KR} \right),$$

where $K = a(s-a) + (b-c)^2$. It follows that $\kappa = KRbc$, $\rho = -4R^2rs(s-a) = -abcR(s-a)$ yields the desired linear combination, for which

$$(\alpha', \beta', \gamma') \equiv \left(K' \frac{(b-c)(s-b)(s-c)}{a}, (s-c)(s-a), -(s-a)(s-b) \right)$$

where $K' = \frac{abc(b-c)}{2}$. It then suffices to show that

$$(s-a)(b-c)(s-b)(s-c) = b(s-b)(s-c)(s-a) - c(s-c)(s-a)(s-b),$$

clearly true. The conclusion follows.

Also solved by Nguyen Dang Qua, Hanoi University Of Science, Hanoi, Vietnam.

O245. Prove that in a $\left(1+\sqrt{2}\right)\times\left(1+\sqrt{2}\right)$ square we cannot fit five 1×1 squares without overlapping, but we can fit them in a $\left(2+\frac{1}{\sqrt{2}}\right)\times\left(2+\frac{1}{\sqrt{2}}\right)$ square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

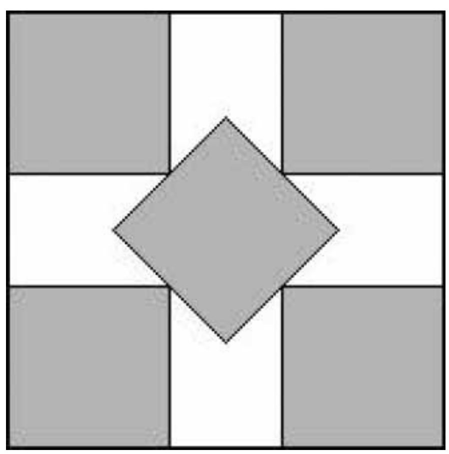
Solution by Tommaso Gianiorio, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy
We first show that in the square $Q=[0,1+\sqrt{2}]^2$, we cannot fit five 1×1 squares without overlapping. Our main claim is the following: if a 1×1 square U is inside Q then at least one of these four points belongs to U:

$$p_1=\left(\frac{2+\sqrt{2}}{4},\frac{2+3\sqrt{2}}{4}\right),\qquad p_2=\left(\frac{2+3\sqrt{2}}{4},\frac{2+3\sqrt{2}}{4}\right),$$
$$p_3=\left(\frac{2+\sqrt{2}}{4},\frac{2+\sqrt{2}}{4}\right),\qquad p_4=\left(\frac{2+\sqrt{2}}{4},\frac{2+3\sqrt{2}}{4}\right).$$

Indeed, let c be the center of U, then its distance from the sides of Q is greater or equal to $1/2$, which implies that $c\in[1/2,1/2+\sqrt{2}]^2\subset\cup_{i=1}^4 B(p_i,1/2)$. Since U contains the closed disc $B(c,1/2)$, it contains at least one of the four points P_i. Actually, it is easy to verify that it is strictly inside U with only one exception, that is when the midpoints of the sides of U are just P_1,P_2,P_3,P_4.

Now, if we assume that there are five 1×1 squares inside Q, then we have two cases. If all five squares has at least one of the four points P_1,P_2,P_3,P_4 in their interior, then, by the box principle, at least two of them overlap. Otherwise, one of the five squares is in the exceptional position and it overlaps with the other four squares necessarily.

On the other hand, five 1×1 squares can be placed without overlapping in a $(2+\frac{1}{\sqrt{2}})\times(2+\frac{1}{\sqrt{2}})$ square in the following way



Also solved by Mehdi Mikael Trense, Montpellier, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O246. Let P be a point inside or on the boundary of a convex polygon $A_1, A_2, \ldots, A_n$. Prove that the maximum value of $f(P) = \sum_{i=1}^{n} |P - A_i|$ is achieved when P is a vertex $A_1, A_2, \ldots, A_n$.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let $\mathcal{C}$ be the set of points inside or on the boundary of the convex polygon. We first note that the map $P \to f(P)$ is convex: if $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ then, for any $P, Q \in \mathcal{C}$, we have that $\alpha P + \beta Q \in \mathcal{C}$ and

$$\begin{aligned} f(\alpha P + \beta Q) &= \sum_{i=1}^{n} |\alpha(P - A_i) + \beta(Q - A_i)| \\ &\leq \alpha \sum_{i=1}^{n} |P - A_i| + \beta \sum_{i=1}^{n} |P - A_i| = \alpha f(P) + \beta f(Q). \end{aligned}$$

Moreover, for any point P in C is a convex combination of the vertices $A_1, A_2, \ldots, A_n$, that is, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and

$$P = \sum_{i=1}^{n} \alpha_i A_i.$$

Hence, by the convexity of f,

$$f(P) = f(\sum_{i=1}^{n} \alpha_i A_i) \leq \sum_{i=1}^{n} \alpha_i f(A_i) \leq \sum_{i=1}^{n} \alpha_i \max_{i=1,\ldots,n} f(A_i) = \max_{i=1,\ldots,n} f(A_i),$$

which means that the maximum value of $f(P)$ is attained when P is a vertex of $\mathcal{C}$.

Also solved by Arkady Alt, San Jose, California, USA; Roberto Bosch Cabrera, Havana, Cuba; Mehdi Mikael Trense, Montpellier, France.