Junior problems

J289. Let a be a real number such that $0 \le a < 1$. Prove that

$$\left\lfloor a\left(1+\left\lfloor \frac{1}{1-a}\right\rfloor\right)\right\rfloor + 1 = \left\lfloor \frac{1}{1-a}\right\rfloor.$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Since $0 \le a < 1$, then $0 < 1-a \le 1$.

If $k \in \mathbb{N}$ such that $\frac{1}{1+k} < 1-a \le \frac{1}{k}$, then $k \le \frac{1}{1-a} < k+1$, and $\frac{k-1}{k} \le a < \frac{k}{k+1}$, so $\left\lfloor \frac{1}{1-a}\right\rfloor = k$.

On the other hand, for the left-hand side of the proposed identity we have

$$\begin{aligned}\left\lfloor a\left(1+\left\lfloor \frac{1}{1-a}\right\rfloor\right)\right\rfloor + 1 &= \lfloor a(1+k)\rfloor + 1\\ &= k-1+1 = k.\end{aligned}$$

Also solved by Archisman Gupta, RKMV, Agartala, Tripura, India; Joshua Benabou, Manhasset High School, NY, USA; Daniel Lasaosa, Pamplona, Navarra, Spain; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Mathematical Group Galaktika shqiptare, Albania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Alessandro Ventullo, Milan, Italy; Polyahedra, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Viet Quoc Hoang, University of Auckland, New Zealand.

J290. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$\sqrt[3]{13a^3+14b^3}+\sqrt[3]{13b^3+14c^3}+\sqrt[3]{13c^3+14a^3} \geq 3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Shohruh Ibragimov, Lyceum Nr.2 under the SamIES, Samarkand, Uzbekistan
From Hölder's inequality we easily obtain $(13a^3+14b^3)(13+14)(13+14) \geq (13a+14b)^3$ =>

$$\sqrt[3]{13a^3+14b^3} \geq \frac{13a+14b}{9} \tag{1}$$

Similarly, we have

$$\sqrt[3]{13b^3+14c^3} \geq \frac{13b+14c}{9}, \sqrt[3]{13c^3+14a^3} \geq \frac{13c+14a}{9} \tag{2}$$

From (1) and (2) we get

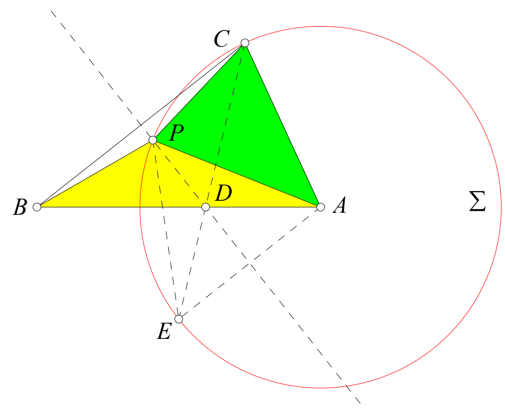
$$\sqrt[3]{13a^3+14b^3}+\sqrt[3]{13b^3+14c^3}+\sqrt[3]{13c^3+14a^3} \geq \frac{27(a+b+c)}{9}=3$$

Also solved by Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Daniel Lasaosa, Pamplona, Navarra, Spain; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, California, USA; An Zhen-ping, Xianyang Normal University, China; Viet Quoc Hoang, University of Auckland, New Zealand; Ioan Viorel Codreanu, Satulung, Maramures, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Polyahedra, Polk State College, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Sayak Mukherjee, Kolkata, India; Sayan Das, Indian Statistical Institute, Kolkata, India; Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănești, Romania; Alessandro Ventullo, Milan, Italy; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece.

J291. Let ABC be a triangle such that $\angle BCA = 2\angle ABC$ and let P be a point in its interior such that $PA = AC$ and $PB = PC$. Evaluate the ratio of areas of triangles PAB and PAC.

Proposed by Panagiote Ligouras, Noci, Italy

Solution by Polyahedra, Polk State College, USA



Let Σ be the circle with center A and radius AC. Suppose that the bisector of $\angle ACB$ intersects AB at D and Σ at E. Then PD is the perpendicular bisector of BC. Since $\angle AEC = \angle ACE = \angle BCE$, $EA \parallel BC$. Thus $\angle BAE = \angle ABC$, so PD is the perpendicular bisector of AE as well. Hence $\triangle APE$ is equilateral. Therefore, $\angle PCE = \frac{1}{2}\angle PAE = 30^\circ$, $\angle APC = 30^\circ + B$, and $\angle APB = 60^\circ + \angle EPB = 60^\circ + \angle APC = 90^\circ + B$. Finally, let $[\cdot]$ denote area, then

$$\frac{[PAB]}{[PAC]} = \frac{\sin\angle APB}{\sin\angle APC} = \frac{\cos B}{\sin(30^\circ + B)} = \frac{2}{1+\sqrt{3}\tan B}.$$

Also solved by Andrea Fanchini, Cantú, Italy; Daniel Lasaosa, Pamplona, Navarra, Spain; Arkady Alt, San Jose, California, USA.

J292. Find the least real number k such that for every positive real numbers x, y, z, the following inequality holds:

$$\prod_{\text{cyc}} (2xy + yz + zx) \leq k(x + y + z)^6.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

Solution by Albert Stadler, Herrliberg, Switzerland

We claim that $k = \dfrac{64}{729}$.

Let $x = y = z = 1 \Rightarrow$

$$\frac{\prod_{cyc} (2xy + yz + zx)}{(x + y + z)^6} = \frac{64}{729} \leq k$$

By the Cauchy-Schwarz inequality , $xy + yz + zx \leq x^2 + y^2 + z^2$.

So $3(xy + yz + zx) \leq 2xy + 2yz + 2zx + x^2 + y^2 + z^2 = (x + y + z)^2$.

By AM-GM:

$$\prod_{cyc}(2xy + yz + zx) \leq \left(\frac{\sum_{cyc}(2xy + yz + zx)}{3}\right)^3 = \left(\frac{4\sum_{cyc} xy}{3}\right)^3 \leq \left(\frac{4(x + y + z)^2}{9}\right)^3 = \frac{64}{729}(x + y + z)^6,$$

Which proves that $k \leq \dfrac{64}{729}$ and the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Alessandro Ventullo, Milan, Italy; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Shohruh Ibragimov, Lyceum Nr.2 under the SamIES, Samarkand, Uzbekistan; Sayan Das, Indian Statistical Institute, Kolkata, India; Sayak Mukherjee, Kolkata, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sun Mengyue Lansheng, Fudan Middle School, Shanghai, China; Arkady Alt, San Jose, California, USA; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Polyahedra, Polk State College, FL, USA; Jan Jurka, Brno, Czech Republic.

J293. Find all positive integers x, y, z such that

$$\left(x+y^2+z^2\right)^2-8xyz=1.$$

Proposed by Aaron Doman, University of California, Berkeley , USA

Solution by Alessandro Ventullo, Milan, Italy
We rewrite the equation as

$$x^2+2x(y^2+z^2-4yz)+(y^2+z^2)^2-1=0.$$

Since x must be a positive integer, the discriminant of this quadratic equation in x must be non-negative, i.e.

$$(y^2+z^2-4yz)^2-(y^2+z^2)^2+1\geq 0,$$

which is equivalent to

$$-8yz(y-z)^2+1\geq 0,$$

which gives $yz(y-z)^2\leq 1/8$. Since y and z are positive integers, it follows that

$$yz(y-z)^2=0,$$

so $y-z=0$, i.e. $y=z$. The given equation becomes $(x-2y^2)^2=1$, which yields $x=2y^2\pm 1$. Therefore, all the positive integer solutions to the given equation are

$$(2n^2-1,n,n),(2n^2+1,n,n),\qquad n\in\mathbb{Z}^+.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Navarra, Spain; Sima Sharifi, College at Brockport, SUNY, USA; Sayan Das, Indian Statistical Institute, Kolkata, India; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Viet Quoc Hoang, University of Auckland, New Zealand; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Polyahedra, Polk State College, FL, USA.

J294. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$1 \leq (a^2-a+1)(b^2-b+1)(c^2-c+1) \leq 7.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

Denote $p=abc$ and $s=ab+bc+ca$, where clearly $0 \leq p \leq 1$, with $p=0$ iff at least one of a, b, c is zero, and $p=1$ iff $a=b=c=1$ by the AM-GM inequality, while $0 \leq s \leq 3$, with $s=0$ iff two out of a, b, c are zero, and $s=3$ iff $a=b=c=1$ by the scalar product inequality. Note that

$$(a^2-a+1)(b^2-b+1)(c^2-c+1) = p^2+s^2-ps-4s-p+7.$$

The lower bound then rewrites as

$$(2+p-s)^2+2(1-p) \geq p(3-s).$$

Now, $9p = 3abc(a+b+c) \leq (ab+bc+ca)^2 = s^2$, or $9(1-p) \geq (3+s)(3-s)$, or it suffices to show that

$$(2+p-s)^2 + \frac{6+2s-9p}{9}(3-s) \geq 0.$$

Assume that $9p > 6$, or $p > \frac{2}{3}$, hence by the AM-GM inequality, $s \geq 3\sqrt[3]{p^2} \geq \sqrt[3]{12} > 2$, or $6+2s > 10 > 9p$. It follows that both terms in the LHS are non-negative, being zero iff $s=3$ and simultaneously $s=p+2$, for $p=1$.

On the other hand, the upper bound rewrites as

$$p(1-p)+s(3-s)+s+ps \geq 0,$$

clearly true because $s, ps, p(1-p), s(3-s)$ are all non-negative. Note that equality requires $s=0$, and since a, b, c are non-negative, this requires at least two out of a, b, c to be zero, resulting in $p=0$.

The conclusion follows, equality holds in the lower bound iff $a=b=c=1$, and in the upper bound iff (a, b, c) is a permutation of $(3, 0, 0)$.

Also solved by Polyahedra, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Viet Quoc Hoang, University of Auckland, New Zealand; Arkady Alt, San Jose, California, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Shivang jindal, Jaipur, Rajasthan, India; Albert Stadler, Herrliberg, Switzerland.

Senior problems

S289. Let x, y, z be positive real numbers such that $x \leq 4$, $y \leq 9$ and $x + y + z = 49$. Prove that

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} \geq 1.$$

Proposed by Marius Stanean, Zalau, Romania

Solution by Li Zhou, Polk State College, FL, USA

Applying Jensen's inequality to the convex function $f(t) = 1/\sqrt{t}$ for $t > 0$, we get

$$\begin{aligned}
\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} &= \frac{1}{2} f\left(\frac{x}{4}\right) + \frac{1}{3} f\left(\frac{y}{9}\right) + \frac{1}{6} f\left(\frac{z}{36}\right) \geq f\left(\frac{1}{2} \cdot \frac{x}{4} + \frac{1}{3} \cdot \frac{y}{9} + \frac{1}{6} \cdot \frac{z}{36}\right) \\
&= \frac{\sqrt{216}}{\sqrt{27x + 8y + z}} = \frac{\sqrt{216}}{\sqrt{26x + 7y + 49}} \geq \frac{\sqrt{216}}{\sqrt{26(4) + 7(9) + 49}} = 1.
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Sima Sharifi, College at Brockport, SUNY, USA; Arkady Alt, San Jose, California, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Shohruh Ibragimov, Lyceum Nr.2 under the SamIES, Samarkand, Uzbekistan; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sun Mengyue , Lansheng Fudan Middle School, Shanghai, China.

S290. Prove that there is no integer n for which

$$\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}=\left(\frac{4}{5}\right)^2$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Alessandro Ventullo, Milan, Italy

Let p be the greatest prime number such that $p \le n < 2p$. Then the given equality can be written as

$$5^2 k = (4 \cdot n!)^2,$$

where $k = \sum_{i=2}^{n} \frac{(n!)^2}{i^2}$. Observe that $k \equiv \frac{(n!)^2}{p^2} \not\equiv 0 \pmod{p}$. Since $p|5^2k$ and p does not divide k, it follows that $p|5^2$, i.e. $p = 5$. So, $n \in \{5, 6, 7, 8, 9\}$. An easy check shows that none of these values satisfies the equality.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Yassine Hamdi, Lycée du Parc, Lyon, France; Sima Sharifi, College at Brockport, SUNY, USA; Li Zhou, Polk State College, FL, USA; Arghya Datta, Hooghly Collegiate School, Kolkata, India; Albert Stadler, Herrliberg, Switzerland.

S291. Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that

$$\left(2a^2 - 3ab + 2b^2\right)\left(2b^2 - 3bc + 2c^2\right)\left(2c^2 - 3ca + 2a^2\right) \geq \frac{5}{3}\left(a^2 + b^2 + c^2\right) - 4.$$

Proposed by Titu Andreescu, USA and Marius Stanean, Romania

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain
Note that

$$27\left(2a^2 - 3ab + 2b^2\right)\left(2b^2 - 3bc + 2c^2\right)\left(2c^2 - 3ca + 2a^2\right) + 27 \cdot 4 - 45\left(a^2 + b^2 + c^2\right) =$$

$$= 27\left(2a^2 - 3ab + 2b^2\right)\left(2b^2 - 3bc + 2c^2\right)\left(2c^2 - 3ca + 2a^2\right) + 4\left(ab + bc + ca\right)^3 -$$

$$-5\left(a^2 + b^2 + c^2\right)\left(ab + bc + ca\right)^2 =$$

$$= 211(a^4b^2 + b^4c^2 + c^4a^2 + a^2b^4 + b^2c^4 + c^2a^4) - 320(a^3b^3 + b^3c^3 + c^3a^3) -$$

$$-334abc(a^3 + b^3 + c^3) + 164abc(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2) - 288a^2b^2c^2,$$

or it suffices to show that this last expression is non-negative. Note that it can be rearranged as

$$\sum_{\text{cyc}}\left(3c^4 + 160a^2b^2 + 48c^2(a+b)^2 - 164abc(a+b)\right)(a-b)^2 =$$

$$= \sum_{\text{cyc}}\left(3c^4 + 5c^2(a+b)^2 + 39(bc + ca - 2ab)^2 + 4(bc + ca - ab)^2\right)(a-b)^2,$$

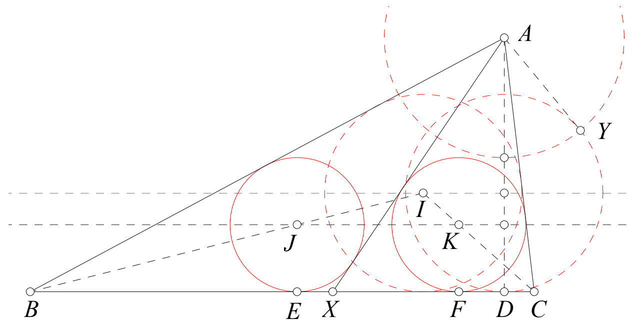
clearly non-negative, and being zero iff $a = b = c$. The conclusion follows.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Arkady Alt, San Jose, California, USA.

S292. Given triangle ABC, prove that there exists X on the side BC such that the inradii of triangles AXB and AXC are equal and find a ruler and compass construction.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Li Zhou, Polk State College, USA



As usual, let a, b, c, s, and r be the sides BC, CA, AB, semiperimeter, and inradius of $\triangle ABC$. Let I, J, K be the incenters of triangles ABC, ABX, AXC, and D, E, F be the feet of perpendiculars from A, J, K onto BC. Let $h = AD$, $r_1 = JE$, and $r_2 = KF$. As X moves from B to C, r_1 increases from 0 to r while r_2 decreases from r to 0. Hence, there exists X such that $r_1 = r_2 = t$. Now $\frac{t}{r} = \frac{BE}{s-b}$, so $BE = \frac{t(s-b)}{r}$. Likewise, $FC = \frac{t(s-c)}{r}$. Let $[\cdot]$ denote area. Then

$$\begin{aligned} \frac{1}{2}ah &= [ABC] = [ABX] + [AXC] = t(c + EX) + t(b + XF) \\ &= t(c + b + a - BE - FC) = \frac{t}{r}(2rs - ta) = \frac{t}{r}(ah - at). \end{aligned}$$

Hence, $t^2 - ht + \frac{1}{2}hr = 0$, which yields $t = \frac{1}{2}\left(h - \sqrt{h(h-2r)}\right)$. This suggests an easy construction:

Construct the length $2r$ on AD. Then construct the length $AY = \sqrt{h(h-2r)}$. Taking away the length AY from AD gives us $h - \sqrt{h(h-2r)}$. Halving this yields t, as in the figure.

Also solved by Titu Zvonaru, Comănești, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania; Arkady Alt, San Jose, California, USA.

S293. Let a, b, c be distinct real numbers and let n be a positive integer. Find all nonzero complex numbers z such that

$$az^n + b\bar{z} + \frac{c}{z} = bz^n + c\bar{z} + \frac{a}{z} = cz^n + a\bar{z} + \frac{b}{z}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

Since $z \neq 0$, we may multiply by z all terms in the proposed equation, or

$$az^{n+1} + b|z|^2 + c = bz^{n+1} + c|z|^2 + a = cz^{n+1} + a|z|^2 + b,$$

yielding

$$(a-b)z^{n+1} + (b-c)|z|^2 + (c-a) = (b-c)z^{n+1} + (c-a)|z|^2 + (a-b) = 0,$$

and eliminating the terms with z^{n+1}, we obtain

$$|z|^2 = \frac{(a-b)^2 - (b-c)(c-a)}{(b-c)^2 - (a-b)(c-a)} = \frac{a^2+b^2+c^2-ab-bc-ca}{a^2+b^2+c^2-ab-bc-ca} = 1,$$

since $a^2+b^2+c^2 > ab+bc+ca$ because of the Cauchy-Schwarz Inequality, a, b, c being distinct. Inserting this result in the original equation and rearranging terms, we obtain

$$a\left(z^{n+1}-1\right) = b\left(z^{n+1}-1\right) = c\left(z^{n+1}-1\right),$$

or z must be one of the $n+1$-th roots of unity. For any one of those $n+1$ roots, we have $\bar{z} = \frac{1}{z} = z^n$, and all $n+1$-th roots of unity are clearly solutions of the proposed equation.

Also solved by Arkady Alt, San Jose, California, USA; Moubinool Omarjee Lycée Henri IV, Paris, France; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; Li Zhou, Polk State College, FL, USA.

S294. Let $s(n)$ be the sum of digits of n^2+1. Define the sequence $(a_n)_{n\geq 0}$ by $a_{n+1}=s(a_n)$, with a_0 an arbitrary positive integer. Prove that there is n_0 such that $a_{n+3}=a_n$ for all $n\geq n_0$.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

Solution by Alessandro Ventullo, Milan, Italy

We have to prove that the given sequence is 3-periodic. Since $f(5)=8, f(8)=11$ and $f(11)=5$, it suffices to prove that for every positive integer a_0 there exists some $n\in\mathbb{N}$ such that $a_n\in\{5,8,11\}$. Let m be the number of digits of a_0. We prove the statement by induction on m. For $m\leq 2$ we proceed by a direct check. If $a_0\in\{5,8,11\}$ there is nothing to prove. If a_0 is a two-digit number, then $a_0^2\leq 10000$, so $a_1\leq 37$ and we reduce to analyze the cases for $a_0\leq 37$.

(i) If $a_0\in\{2,7,20\}$, then $a_1=5$. If $a_0\in\{1,10,26,28\}$, then $a_1\in\{2,20\}$, so $a_2=5$. Finally, if $a_0\in\{3,6,9,12,15,18,27,30,33\}$, then $a_3=5$.

(ii) If $a_0\in\{4,13,23,32\}$, then $a_1=8$.

(iii) If $a_0\in\{17,19,21,35,37\}$, then $a_1=11$. If $a_0\in\{14,22,24,31,36\}$, then $a_1\in\{17,19\}$, so $a_2=11$. Finally, if $a_0\in\{16,25,29,34\}$, then $a_3=11$.

Thus, we have proved that if a_0 is a one or two-digit number, then $a_n\in\{5,8,11\}$ for some $n\in\mathbb{N}$, i.e. the sequence is 3-periodic. Let $m\geq 2$ and suppose that the statement is true for all $k\leq m$. Let a_0 be an $(m+1)$-digit number. Then, $10^m\leq a_0<10^{m+1}$ which implies $10^{2m}\leq a_0^2<10^{2(m+1)}$. Hence,

$$a_1=f(a_0)\leq 9\cdot 2(m+1)+1<10^m,$$

and by the induction hypothesis, the sequence $(a_n)_{n\geq 1}$ is 3-periodic, which implies that the sequence $(a_n)_{n\geq 0}$ is 3-periodic, as we wanted to prove.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Arghya Datta, Hooghly Collegiate School, Kolkata, India; Li Zhou, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U289. Let $a \geq 1$ be such that $(\lfloor a^n \rfloor)^{\frac{1}{n}} \in \mathbb{Z}$ for all sufficiently large integers n. Prove that $a \in \mathbb{Z}$.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Li Zhou, Polk State College, FL, USA

For the purpose of contradiction, suppose that $a = m+h$ for some positive integer m and some real $h \in (0,1)$. Then for all $n \geq \frac{1}{h}$,

$$m^n + 1 \leq m^n + nh < a^n < (m+1)^n,$$

and thus $m^n < \lfloor a^n \rfloor < (m+1)^n$. Hence, $m < (\lfloor a^n \rfloor)^{\frac{1}{n}} < m+1$ for all sufficiently large n, a desired contradiction.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Pamplona, Navarra, Spain; Alessandro Ventullo, Milan, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U290. Prove that there are infinitely many triples of primes (p_{n-1}, p_n, p_{n+1}) such that $\frac{1}{2}(p_{n+1}+p_{n-1}) \leq p_n$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Assume for the sake of contradiction that there is an integer n_0 such that for all $n > n_0$,

$$p_{n-1} + p_{n+1} > 2p_n$$

that is, $d_n > d_{n-1} \geq 1$ where $d_n = p_{n+1} - p_n$. Hence, for $k \geq 1$,

$$d_{n_0+k-1} \geq d_{n_0+k-2} + 1 \geq \cdots \geq d_{n_0} + k - 1 \geq k$$

and

$$p_{n_0+k} = d_{n_0+k-1} + d_{n_0+k-2} + \cdots + d_{n_0} + p_{n_0} > k + (k-1) + \cdots + 1 = \frac{k(k+1)}{2} > \frac{k^2}{2}.$$

On the other hand, by the Prime Number Theorem, $\pi(n) \sim n/\ln(n)$ and since $x/\ln(x)$ is strictly increasing for $x \geq e$, it follows that

$$1 \leftarrow \frac{\pi(p_{n_0+k})\ln(p_{n_0+k})}{p_{n_0+k}} = \frac{(n_0+k)\ln(p_{n_0+k})}{p_{n_0+k}} < \frac{(n_0+k)\ln(k^2/2)}{k^2/2} \to 0$$

which yields a contradiction.

Also solved by Julien Portier, Francois 1er, France; Daniel Lasaosa, Pamplona, Navarra, Spain; Arpan Sadhukhan, Indian Statistical Institute, Kolkata; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Li Zhou, Polk State College, FL, USA.

U291. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded function and let $\mathcal{S}$ be the set of all increasing maps $\varphi\colon \mathbb{R} \to \mathbb{R}$. Prove that there is a unique function g in $\mathcal{S}$ satisfying the conditions

a) $f(x) \le g(x)$ for all $x \in \mathbb{R}$.

b) If $h \in \mathcal{S}$ and $f(x) \le h(x)$ for all $x \in \mathbb{R}$ then $g(x) \le h(x)$ for all $x \in \mathbb{R}$.

Proposed by Marius Cavachi, Constanta, Romania

Solution by Arkady Alt, San Jose, California, USA

a) Since f is bounded then for any $x \in \mathbb{R}$ set $G(x) := \{f(t) \mid t \in \mathbb{R} \text{ and } t \le x\}$ is bounded. Therefore for any $x \in \mathbb{R}$ we can define $g(x) := \sup G(x)$ and, obviously, that function $g(x)$ defined by such way satisfy to condition (a).

b) Let now $h \in S$ and $f(x) \le h(x)$ for all $x \in \mathbb{R}$.
Since $f(t) \le h(t)$ for any $t \le x$ then $g(x) = \sup\limits_{t \le x} f(t) \le \sup\limits_{t \le x} h(t) = h(x)$ (since h is increasing in $t \in (-\infty, x]$).

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy.

U292. Let r be a positive real number. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^r x} dx.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Robinson Higuita, Universidad de Antioquia , Colombia

We denote by

$$I(r) := \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^r(x)} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^r x}{\sin^r x + \cos^r x} dx.$$

If we make the substitution $x = \frac{\pi}{2} - y$, we have

$$I(r) = \int_{\frac{\pi}{2}}^0 \frac{\sin^r(\frac{\pi}{2}-y)}{\sin^r(\frac{\pi}{2}-y)+\cos^r(\frac{\pi}{2}-y)}(-dy) = \int_0^{\frac{\pi}{2}} \frac{\cos^r y}{\cos^r y + \sin^r y}.$$

Therefore

$$\begin{aligned} 2I(r) &= \int_0^{\frac{\pi}{2}} \frac{\sin^r x}{\sin^r x + \cos^r x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos^r y}{\cos^r y + \sin^r y} \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin^r y + \cos^r y}{\cos^r y + \sin^r y} dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}. \end{aligned}$$

Thus

$$I(r) = \frac{\pi}{4}.$$

We note that the hypothesis on r is not important.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Albert Stadler, Herrliberg, Switzerland; Sayak Mukherjee, Kolkata, India; Samin Riasat, Dhaka, Bangladesh; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Antoine Barré and Dmitry Chernyak, Lycée Stanislas, Paris, France; Arkady Alt, San Jose, California, USA; Moubinool Omarjee Lycée Henri IV, Paris, France; Li Zhou, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U293. Let $f : (0, \infty) \to \mathbb{R}$ be a bounded continuous function and let $\alpha \in [0, 1)$. Suppose there exist real numbers $a_0, ..., a_k$, whith $k \geq 2$, so that $\sum_{p=0}^{k} a_p = 0$ and

$$\lim_{x\to\infty} x^{\alpha} \left| \sum_{p=0}^{k} a_p f(x+p) \right| = a.$$

Prove that $a = 0$.

Proposed by Marcel Chirita, Bucharest, Romania

Solution by the author

Let us denote $M = sup|f(x)|$ and $g(x) = \sum_{p=0}^{k} a_p f(x+p)$.
$g : (0, \infty) \to \mathbb{R}$ is a continuous bounded function.

Assume that $a > 0$.

$$\text{Then there is an } N > 0, \text{ such that } x^{\alpha}|g(x)| > a, \text{ for } \forall x > N \tag{1}$$

From (1) we conclude that function g has the same sign $\forall x > N$, since it is continuous, and therefore has the intermediate value property.
First, suppose that function g is positive on the interval (N, ∞).

$$\text{From (1) we have } |g(x)| > \frac{a}{x^{\alpha}} \tag{2}$$

Denoting $n_1 = [N] + 1$ and $x = n_1, n_1 + 1, n_1 + 2, ...$ and summing up yields to

$$\sum_{n\geq n_1} |g(k)| > a \sum_{n\geq n_1} \frac{1}{n^{\alpha}}.$$

Since the series $\sum_{n\geq 1} \frac{1}{n^{\alpha}}$ is divergent for $\alpha \in (0, 1)$ it follows that $\sum_{n\geq n_1} \frac{1}{n^{\alpha}}$ is divergent as well

$$\Rightarrow \sum_{n\geq n_1} |g(k)| = \infty.$$

Now, let $T = max\{|a_0|, |a_1|, |a_2|, ..., |a_k|\}$. Taking into consideration positiveness of g on (N, ∞) and the same values of n we get

$$S_n = \sum_{n\geq n1} |g(k)| = \sum_{n\geq n1} |a_p f(n+p)| = |\sum_{n\geq n_1} a_p f(n+p)| =$$

$|a_k f(n+k) + (a_k + a_{k-1}) f(n+k-1) + ... + (a_k + a_{k-1} + ... + a_1) f(n+1) + (a_{k-1} + a_{k-2} + ... + a_0) f(n_1 + k-1) + (a_{k-2} + a_{k-3} + ... + a_0) f(n_1 + k - 2) + ... + (a_1 + a_0) f(n_1 + 1) + a_0 f(n_1)| \leq$

$$|a_k|M + |a_k + a_{k-1}|M + ... + |a_k + a_{k-1} + ... + a_1|M+$$

$|a_{k-1} + a_{k-2} + ... + a_0|M + |a_{k-2} + a_{k-3} + ... + a_0|M + ... + |a_1 + a_0|M + |a_0|M \leq MT + 2MT + ... + kMT + kMT + ... + 2MT + MT = k(k+1)MT$ and the series converges.

Therefore, we have reached a contradiction $\Rightarrow a = 0$. Similarly to prove for $g(x) \leq 0$.

U294. Let $p_1, p_2, \ldots, p_n$ be pairwise distinct prime numbers. Prove that

$$\mathbb{Q}\left(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}\right) = \mathbb{Q}\left(\sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_n}\right).$$

Proposed by Marius Cavachi, Constanta, Romania

Solution by Alessandro Ventullo, Milan, Italy

Clearly, $\mathbb{Q}(\sqrt{p_1}+\sqrt{p_2}+\ldots+\sqrt{p_n})$ is a subfield of $\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$. Observe that $\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$ is Galois over $\mathbb{Q}$, since it is the splitting field of the polynomial $(x^2-\sqrt{p_1})\cdots(x^2-\sqrt{p_n})$. Every automorphism σ is completely determined by its action on $\sqrt{p_1}, \ldots, \sqrt{p_n}$, which must be mapped to $\pm\sqrt{p_1}, \ldots, \pm\sqrt{p_n}$, respectively. Therefore, $\mathrm{Gal}(\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})/\mathbb{Q})$ is the group generated by $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, where σ_i is the automorphism defined by

$$\sigma_i(\sqrt{p_j}) = \begin{cases} -\sqrt{p_j} & \text{if } i = j \\ \sqrt{p_j} & \text{if } i \neq j. \end{cases}$$

Clearly, the only automorphism that fixes $\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$ is the identity. Moreover, it's easy to see that the only automorphism that fixes the element $\sqrt{p_1} + \ldots + \sqrt{p_n}$ is the identity, which means that the only automorphism that fixes $\mathbb{Q}(\sqrt{p_1} + \ldots + \sqrt{p_n})$ is the identity. Hence, by the Fundamental Theorem of Galois Theory, $\mathbb{Q}(\sqrt{p_1} + \sqrt{p_2} + \ldots + \sqrt{p_n}) = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain.

O289. Let a, b, x, y be positive real numbers such that $x^2 - x + 1 = a^2$, $y^2 + y + 1 = b^2$, and $(2x-1)(2y+1) = 2ab + 3$. Prove that $x + y = ab$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Khakimboy Egamberganov, S.H.Sirojiddinov lyceum, Tashkent, Uzbekistan

From the conditions $(2x-1)^2 = 4a^2 - 3$ and $(2y+1)^2 = 4b^2 - 3$ and

$$(2ab+3)^2 = (2x-1)^2(2y+1)^2 = (4a^2-3)(4b^2-3),$$

$$a^2b^2 - ab - a^2 - b^2 = 0 \qquad (1).$$

We have $(2x-1)^2 + (2y+1)^2 = 4(a^2+b^2) - 6$, and since $(2x-1)(2y+1) = 2ab+3$, we get that

$$(2(x+y))^2 = 4(a^2+b^2) - 6 + (4ab+6),$$

$$(x+y)^2 = a^2 + b^2 + ab \qquad (2).$$

Since (1), (2), we get that $(x+y)^2 = a^2b^2$ and

$$x + y = ab.$$

and we are done.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Moubinool Omarjee Lycée Henri IV, Paris, France; Shohruh Ibragimov, Lyceum Nr.2 under the SamIES, Samarkand, Uzbekistan; Daniel Văcaru, Piteşti, Romania; Alessandro Ventullo, Milan, Italy; Li Zhou, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, California, USA; Ayoub Hafid, Elaraki School, Morroco.

O290. Let Ω_1 and Ω_2 be the two circles in the plane of triangle ABC. Let α_1, α_2 be the circles through A that are tangent to both Ω_1 and Ω_2. Similarly, define β_1, β_2 for B and γ_1, γ_2 for C. Let A_1 be the second intersection of circles α_1 and α_2. Similarly, define B_1 and C_1. Prove that the lines AA_1, BB_1, CC_1 are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

Note that we must assume that Ω_1, Ω_2 have different radii. Otherwise, the figure clearly has symmetry around the perpendicular bisector of the segment joining their centers, and lines AA_1, BB_1, CC_1 would be parallel to the line joining their centers (which could be considered equivalent to AA_1, BB_1, CC_1 meeting at infinity). We will assume in the rest of the problem that Ω_1, Ω_2 have distinct radii, hence a homothety with center O and scaling factor ρ with $|\rho| \neq 1$ can be found, which transforms Ω_1 into Ω_2.

Claim 1: Let Ω_1, Ω_2 be two circles with different radii, and let ω be a circle simultaneously tangent to both, respectively at points T_1, T_2. Then, T_1, T_2 and the center O of the homothety that transforms Ω_1 into Ω_2 are collinear.

Proof 1: Let O_1, O_2 be the respective centers, and R_1, R_2 the radii, of Ω_1, Ω_2. Let O', r be the center and radius of ω. Consider triangle O_1O_2O', where T_1, T_2 are clearly inside segments O_1O' and O_2O'. Let P be the second point where T_1T_2 intersects O_1O_2, where by Menelaus' theorem, we have

$$\frac{O_1P}{PO_2} = \frac{T_2O'}{O_2T_2} \cdot \frac{T_1O_1}{O'T_1} = \frac{r}{R_2} \cdot \frac{R_1}{r} = \frac{R_1}{R_2},$$

or indeed P is a center of a homothety that transforms O_1 into O_2 with scaling factor with absolute value $\frac{R_1}{R_2}$, hence $P = O$. The Claim 1 follows.

Claim 2: The power of O with respect to ω is invariant for all the possible circles ω which are simultaneously tangent to Ω_1, Ω_2.

Proof 2: Since T_1, T_2 are points on ω which are collinear with O, the power of O with respect to ω is $OT_1 \cdot OT_2$. Let P_1, P_2 be the respective powers of O with respect to Ω_1, Ω_2, which clearly satisfy $P_2 = \rho^2 P_1$. Consider now points T_1', resulting from applying the homothety to T_1. Clearly, T_1' is collinear with O, T_1, hence on line OT_2; at the same time, $T_1 \in \Omega_1$, or $T_1' \in \Omega_2$ by construction, hence T_1', T_2 are the two points where a line through O intersects T_2, or $OT_1' \cdot OT_2 = P_2$. But since $OT_1' = |\rho| OT_1$, we have $OT_1 \cdot OT_2 = \frac{P_2}{|\rho|} = |\rho| P_1$, independently on the choice of ω. The Claim 2 follows.

By the Claim 2, the power of O with respect to α_1, α_2 is the same, or since AA_1 is their radical axis, then A, A_1, O are collinear, or line AA_1 passes through O. Similarly, BB_1, CC_1 pass through O. The conclusion follows.

Also solved by Saturnino Campo Ruiz, Salamanca, Spain; Shohruh Ibragimov, Lyceum Nr.2 under the SamIES, Samarkand, Uzbekistan; Khakimboy Egamberganov, S.H.Sirojiddinov lyceum, Tashkent, Uzbekistan; Alessandro Pacanowski, PECI, Rio de Janeiro, Brazil.

O291. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{\sqrt{4a^2+ab+4b^2}}+\frac{b^2}{\sqrt{4b^2+bc+4c^2}}+\frac{a^2}{\sqrt{4c^2+ca+4a^2}}\geq\frac{a+b+c}{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Marius Stanean

From Hölder's Inequality, we have

$$\left(\sum_{cyc}\frac{a^2}{\sqrt{4a^2+ab+4b^2}}\right)^2\left(\sum_{cyc}a^2(4a^2+ab+4b^2)\right)\geq(a^2+b^2+c^2)^3. \tag{1}$$

But

$$\begin{aligned}\sum_{cyc}a^2(4a^2+ab+4b^2)&=4(a^4+b^4+c^4)+a^3b+b^3c+c^3a+4(a^2b^2+b^2c^2+c^2a^2)\\&=4(a^2+b^2+c^2)^2+a^3b+b^3c+c^3a-4(a^2b^2+b^2c^2+c^2a^2)\\&\leq\frac{13(a^2+b^2+c^2)^2-12(a^2b^2+b^2c^2+c^2a^2)}{4},\end{aligned}$$

where the last line follows from Cartoaje's Inequality,

$$(a^2+b^2+c^2)^2\geq 3(a^3b+b^3c+c^3a).$$

Hence, considering (1), it follows that it is sufficient to prove that

$$27(a^2+b^2+c^2)^3\geq[13(a^2+b^2+c^2)^2-12(a^2b^2+b^2c^2+c^2a^2)](a+b+c)^2$$

$$9(a^2+b^2+c^2)^2[3(a^2+b^2+c^2)-(a+b+c)^2]-4(a+b+c)^2[(a^2+b^2+c^2)^2-3(a^2b^2+b^2c^2+c^2a^2)]\geq 0$$

$$9(a^2+b^2+c^2)^2[(a-b)^2+(a-c)(b-c)]-2(a+b+c)^2[(a+b)^2(a-b)^2+(a+c)(b+c)(a-c)(b-c)]\geq 0$$

$$[9(a^2+b^2+c^2)^2-2(a+b+c)^2(a+b)^2](a-b)^2+[9(a^2+b^2+c^2)^2-2(a+b+c)^2(a+c)(b+c)](a-c)(b-c)\geq 0.$$

Without loss of generality, we may assume $a\leq b\leq c$. Then it suffices to show that

$$9(a^2+b^2+c^2)^2\geq 2(a+b+c)^2(a+c)(b+c),$$

but this is true because

$$3(a^2+b^2+c^2)\geq(a+b+c)^2$$

by Cauchy-Schwarz and

$$\begin{aligned}&3(a^2+b^2+c^2)-2(a+c)(b+c)\geq 0\\\iff&c^2-2(a+b)c+3(a^2+b^2)-2ab\geq 0\\\iff&(c-a-b)^2+2(a-b)^2\geq 0,\end{aligned}$$

so we are done.

Second solution by the author
We will begin by finding x, y such that

$$\frac{a^2}{\sqrt{4a^2+ab+4b^2}} \geq xa + yb$$

for all $a, b > 0$. Letting $a = b = 1$, the "best" such x, y will satisfy $x + y = \frac{1}{3}$. Note that the inequality is homogeneous, so letting $t = \frac{a}{b}$, we have

$$\frac{t^2}{\sqrt{4t^2+t+4}} \geq xt + y = x(t-1) + \frac{1}{3}. \tag{1}$$

The inequality clearly holds if the RHS is negative. Otherwise, squaring and multiplying both sides by 9 yields

$$\frac{9t^4}{4t^2+t+4} \geq [3x(t-1)+1]^2$$
$$9t^4 - (4t^2+t+4)[3x(t-1)+1]^2 \geq 0.$$

Considering the LHS as a function of t, say $f(t)$, we want it to have a double root at $t = 1$. This means

$$f'(1) = 27 - 54x = 0,$$

so $x = \frac{1}{2}$ and $y = -\frac{1}{6}$. This implies

$$f(t) = \frac{1}{4}(15t-4)(t-1)^2.$$

Clearly, f is positive for $t > \frac{4}{15}$. For $t \leq \frac{4}{15}$, (1) becomes

$$\frac{3t^2}{\sqrt{4t^2+4t+4}} \geq \frac{3t-1}{2},$$

which must be true since the RHS is negative. Thus,

$$\frac{a^2}{\sqrt{4a^2+ab+4b^2}} \geq \frac{3a-b}{6}$$

for all $a, b > 0$. Adding up the similar inequalities with a and c and b and c yields

$$\begin{aligned}\sum_{cyc} \frac{a^2}{\sqrt{4a^2+ab+4b^2}} &\geq \sum_{cyc} \frac{3a-b}{6} \\ &= \frac{a+b+c}{3},\end{aligned}$$

as desired.

Also solved by Arkady Alt, San Jose, California, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Batzolis, Mandoulides High School, Thessaloniki, Greece; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.

O292. For each positive integer k let

$$T_k = \sum_{j=1}^{k} \frac{1}{j2^j}.$$

Find all prime numbers p for which

$$\sum_{k=1}^{p-2} \frac{T_k}{k+1} \equiv 0 \pmod{p}.$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Lyon

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

It is known that the following identity holds

$$\sum_{1\le j\le k\le n} \binom{n}{k} \frac{(-1)^k(1-x)^j}{jk} = \sum_{k=1}^{n} \frac{x^k}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2}.$$

Let $p > 2$ be a prime (for $p = 2$ the congruence trivially holds).
Then, $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$ and the above identity imply

$$\begin{aligned}
\sum_{k=1}^{p-2} \frac{T_k}{k+1} &= \sum_{j=1}^{p-2} \frac{1}{j2^j} \sum_{k=j}^{p-2} \frac{1}{k+1} = \sum_{1\le j<k\le p-1} \frac{(1/2)^j}{jk} = \sum_{1\le j\le k\le p-1} \frac{(1/2)^j}{jk} - \sum_{k=1}^{p-1} \frac{(1/2)^k}{k^2} \\
&\equiv \sum_{1\le j\le k\le p-1} \binom{p-1}{k} \frac{(-1)^k(1-1/2)^j}{jk} - \sum_{k=1}^{p-1} \frac{(1/2)^k}{k^2} \pmod{p} \\
&= \sum_{k=1}^{p-1} \frac{(1/2)^k}{k^2} - \sum_{k=1}^{p-1} \frac{1}{k^2} - \sum_{k=1}^{p-1} \frac{(1/2)^k}{k^2} = -\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \begin{cases} 1 & \text{if } p = 3, \\ 0 & \text{if } p > 3. \end{cases} \pmod{p}.
\end{aligned}$$

O293. Let x, y, z be positive real numbers and let $t^2 = \frac{xyz}{\max(x,y,z)}$. Prove that

$$4\left(x^3+y^3+z^3+xyz\right)^2 \ge \left(x^2+y^2+z^2+t^2\right)^3.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy
The inequality is implied by

$$4(x^3+y^3+z^3+xyz)^2 \ge \left(x^2+y^2+z^2+\frac{\frac{(a+b+c)^3}{27}}{\frac{a+b+c}{3}}\right)^3$$

Now define

$$a+b+c = 3u, \quad ab+bc+ca = 3v^2, \quad abc = w^3$$

We have

$$a^2+b^2+c^2 = 9u^2-6v^2, \quad a^3+b^3+c^3 = 27u^3-27uv^2+3w^3$$

The inequality reads as

$$4(27u^3-27uv^2+4w^3)^2 \ge \left((9u^2-6v^2)+u^2\right)^3$$

that is

$$64w^6+864w^3(-uv^2+u^3)+1836u^2v^4+1916u^6-4032u^4v^2+216v^6 \ge 0$$

This a convex parabola in w^3 whose minimum has abscissa negative. Since $w^3 \ge 0$, it follows that the parabola is nonnegative if and only it is nonnegative its value for $w^3 = 0$. The theory states that, fixed the values of (u, v), the minimum of w^3 occurs for $a = 0$ or $a = b$.

If $a = 0$ we have

$$479(b^6+c^6)+1156b^3c^3 \ge 780(b^4c^2+b^2c^4)+150(c^5b+cb^5)$$

which implied by

$$479(b^6+c^6)+1156b^3c^3 \ge 930(c^5b+cb^5)$$

This follows by AM-GM since

$$\frac{479}{2}b^6+\frac{479}{2}c^6+578b^3c^3 \ge 3\sqrt[3]{\frac{(479)^2 578}{4}}b^5c > 963b^5c$$

and the same with (c, b) in place of (b, c).

If $a = b$ we have

$$(254b^4+1972b^3c+525b^2c^2+658c^3b+479c^4)(b-c)^2 \ge 0$$

and the proof is complete.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Daniel Lasaosa, Pamplona, Navarra, Spain; Shohruh Ibragimov, Lyceum Nr.2 under the SamIES, Samarkand, Uzbekistan; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Khakimboy Egamberganov, S.H.Sirojiddinov lyceum, Tashkent, Uzbekistan; Arkady Alt, San Jose, California, USA.

O294. Let ABC be a triangle with orthocenter H and let D, E, F be the feet of the altitudes from A, B and C. Let X, Y, Z be the reflections of D, E, F across EF, FD, and DE, respectively. Prove that the circumcircles of triangles HAX, HBY, HCZ share a common point, other than H.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

Claim: Let ABC be any triangle, I its incenter, I_a its excenter opposite vertex A, and A' the symmetric of A with respect to side BC. Define similarly I_b, I_c and B', C'. Then, the circles through I, I_a, A', through I, I_b, B' and through I, I_c, C', pass through a second common point other than I.

Proof 1: In exact trilinear coordinates, $I \equiv (r, r, r)$, $I_a \equiv (-r_a, r_a, r_a)$ and $A' \equiv (-h_a, 2h_a \cos C, 2h_a \cos B)$, or using non-exact trilinear coordinates, we have

$$I \equiv (1, 1, 1), \qquad I_a \equiv (-1, 1, 1), \qquad A' \equiv (-1, 2\cos C, 2\cos B).$$

The equation of a circle in trilinear coordinates is given by

$$(\ell\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) + k(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0,$$

where substitution of the coordinates of the three given points yields $k = -(\ell + m + n)$ when applied to the incenter, and consequently $m + n = 0$ and $k = -\ell$ when applied to the excenter, yielding finally

$$m = -n = \ell \frac{1 - 2\cos A}{2(\cos C - \cos B)}$$

when applied to A'. Indeed, the circle equation

$$\left(\alpha + \frac{(1 - 2\cos A)(\beta - \gamma)}{2(\cos C - \cos B)}\right) = \frac{a\beta\gamma + b\gamma\alpha + c\alpha\beta}{a\alpha + b\beta + c\gamma}$$

is easily checked to be satisfied by I, I_a, A'. Analogous equations may be found for the circles through I, I_b, B' and I, I_c, C'. The intersection of the circles through I, I_b, B' and I, I_c, C' must clearly satisfy

$$\beta + \frac{(1 - 2\cos B)(\gamma - \alpha)}{2(\cos A - \cos C)} = \gamma + \frac{(1 - 2\cos C)(\alpha - \beta)}{2(\cos B - \cos A)},$$

or

$$\frac{1 + 2\cos A - 2\cos B - 2\cos C}{(\cos A - \cos B)(\cos A - \cos C)}\alpha + \frac{1 + 2\cos B - 2\cos C - 2\cos A}{(\cos B - \cos A)(\cos B - \cos C)})\beta +$$
$$+\frac{1 + 2\cos C - 2\cos B - 2\cos A}{(\cos C - \cos A)(\cos C - \cos B)}\gamma = 0.$$

This line equation represents the radical axis of these two circles, and is clearly satified by I, and since it is invariant under cyclic permutation of A, B, C and α, β, γ, it is also therefore the equation of the radical axes of the other two pairs of circles, which intersect the three circles at I, and at another point. The Claim follows.

Consider triangle DEF. Clearly, X, Y, Z are the reflections of its vertices with respect to its sides, H is its incenter, and A, B, C its excenters. The conclusion follows by direct application of the Claim.

Also solved by Khakimboy Egamberganov, S.H.Sirojiddinov lyceum, Tashkent, Uzbekistan; Sebastiano Mosca, Pescara, Italy