Junior problems

J313. Solve in real numbers the system of equations

$$x\left(y+z-x^3\right)=y\left(z+x-y^3\right)=z\left(x+y-z^3\right)=1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ashley Colopy, College at Brockport, SUNY

We can write the equations as

$$\begin{cases} y+z-x^3 & =\frac{1}{x} \\ z+x-y^3 & =\frac{1}{y} \\ x+y-z^3 & =\frac{1}{z} \end{cases} \iff \begin{cases} y+z & =\frac{1}{x}+x^3 \\ z+x & =\frac{1}{y}+y^3 \\ x+y & =\frac{1}{z}+z^3 \end{cases}$$

From the first equation we can see that $y+z$ has the same sign as x. Therefore there is another variable having the same sign as x. Repeating this for the other two equations we conclude that for each variable there is another one having the same sign as it. Thus either all three are negative or all three are positive.

We will solve the system assuming that all three variables are positive.

Using the AM - GM inequality we get

$$\begin{cases} y+z & =\frac{1}{x}+x^3 \geq 2\sqrt{x^3\frac{1}{x}}=2x \\ z+x & =\frac{1}{y}+y^3 \geq 2\sqrt{y^3\frac{1}{y}}=2y \\ x+y & =\frac{1}{z}+z^3 \geq 2\sqrt{z^3\frac{1}{z}}=2z \end{cases}$$

Adding all three inequalities we get

$$2x+2y+2z=\frac{1}{x}+x^3+\frac{1}{y}+y^3+\frac{1}{z}+z^3 \geq 2x+2y+2y.$$

Therefore all inequalities must be equalities and we get

$$\frac{1}{x}+x^3=2x$$

and similar equations for y and z.

The equation is equivalent to

$$x^4-2x^2+1=0 \iff (x^2-1)^2=0 \iff x=1.$$

Thus the solution of the system is $x=y=z=1$.

The negative case produces $x=y=z=-1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Peter C. Shim, The Pingry School, Basking Ridge, NJ; Seonmin Chung, Stuyvesant High School, NY; Adnan Ali, student at A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, Delhi, India; Arbër Avdullahu,Mehmet Akif College,Kosovo; Arkady Alt, San Jose, California, USA; Francesc Gispert Sánchez, CFIS, Universitat Politécnica de Catalunya, Barcelona, Spain; Joshua Noah Benabou, Manhasset High School, NY; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Evgenidis Nikolaos, M. N. Raptou High School, Larissa, Greece; Prithwijit De, HBCSE, Mumbai, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Shatlyk Mamedov, School Nr. 21, Dashoguz, Turkmenistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Jongyeob Lee, Stuyvesant High School, NY, USA; Michael Tang, Edina High School, MN, USA; Jaesung Son, Ridgewood, NJ, USA; Polyahedra, Polk State College, FL, USA; Kwon Il Ko, Cushing Academy, MA, USA.

J314. Alice was dreaming. In her dream, she thought that primes of the form $3k+1$ are weird. Then she thought it would be interesting to find a sequence of consecutive integers all of which are greater than 1 and which are not divisible by weird primes. She quickly found five consecutive numbers with this property:

$$8 = 2^3, \qquad 9 = 3^2, \qquad 10 = 2 \cdot 5, \qquad 11 = 11, \qquad 12 = 2^2 \cdot 3.$$

What is the length of the longest sequence she can find?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Adnan Ali, Student at A.E.C.S-4, Mumbai, India

First of all we note that we can have a sequence of maximum of 6 integers, as, in the sequence of any 7 integers, there is obviously one which is divisible by the prime $7 = 3(2) + 1$. Therefore, we show that the required number is 6. It suffices to see that

$$953 = 1 \cdot 953,\ 954 = 2 \cdot 3^2 \cdot 53,\ 955 = 5 \cdot 191,\ 956 = 2^2 \cdot 239,\ 957 = 3 \cdot 11 \cdot 29,\ 958 = 2 \cdot 479.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Peter C. Shim, The Pingry School, Basking Ridge, NJ; Seonmin Chung, Stuyvesant High School, NY; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Rebecca Buranich, College at Brockport, SUNY; Jongyeob Lee, Stuyvesant High School, NY, USA; Michael Tang, Edina High School, MN, USA; Jaesung Son, Ridgewood, NJ, USA; Polyahedra, Polk State College, FL, USA; Kwon Il Ko, Cushing Academy, MA, USA.

J315. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that

$$\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1} \geq \sqrt{5}+2.$$

Proposed by Cosmin Pohoata, Columbia University, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Denote $x=\sqrt{4a+1}-1$, $y=\sqrt{4b+1}-1$, $z=\sqrt{4c+1}-1$, and $s=x+y+z$. Note therefore that

$$s^2 \geq x^2+y^2+z^2 = 4(a+b+c)+3-2(x+1+y+1+z+1)+3 = 4-2s,$$

or

$$0 \leq \left(s+1+\sqrt{5}\right)\left(s+1-\sqrt{5}\right),$$

for $s \geq \sqrt{5}-1$, with equality iff $xy+yz+zx=0$, ie iff two of x, y, z are zero, or iff two of a, b, c are zero. It follows that

$$\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1} = s+3 \geq \sqrt{5}+2,$$

with equality iff (a, b, c) is a permutation of $(1, 0, 0)$.

Also solved by Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Peter C. Shim, The Pingry School, Basking Ridge, NJ; Seonmin Chung, Stuyvesant High School, NY; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, California, USA; Debojyoti Biswas, Kolkata, India; Erdenebayar Bayarmagnai, School Nr.11, Ulaanbaatar, Mongolia; Henry Ricardo, New York Math Circle; Marius Stanean, Zalau, Romania; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Salem Malikić, Simon Fraser University, Burnaby, BC, Canada; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Shatlyk Mamedov, School Nr. 21, Dashoguz, Turkmenistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Utsab Sarkar, Chennai Mathematical Institute, India; Vincent Huang, Schimelpfenig Middle School, Plano, TX; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; George Gavrilopoulos, High School of Nea Makri, Athens, Greece; An Zhen-ping, Mathematics Department, Xianyang Normal University, Xianyang, Shaanxi, China; Jongyeob Lee, Stuyvesant High School, NY, USA; Michael Tang, Edina High School, MN, USA; Jaesung Son, Ridgewood, NJ, USA; Polyahedra, Polk State College, FL, USA; Kwon Il Ko, Cushing Academy, MA, USA.

J316. Solve in prime numbers the equation

$$x^3 + y^3 + z^3 + u^3 + v^3 + w^3 = 53353.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Evgenidis Nikolaos, M.N. Raptou High School, Larissa, Greece
Lemma 1: Every cube of an integer is conguent with $1, 0, -1 \pmod 9$.
Lemma 2: Every cube of an integer is congruent with $1, 0, -1 \pmod 7$.

Both of these lemmas can be easily proved by examining all forms of an integer $\pmod 9$ and $\pmod 7$, correspondingly.

First, we observe that if all of x, y, z, u, w, v are odd primes, then their cubes' summary would be an even number, contradiction. So at least one of them will be equal to 2.

- if five of x, y, z, u, v, w are equal to 2 then the initial equation has no solutions.
- if three of x, y, z, u, v, w are equal to 2, let us say $w = v = u = 2$, then we have

  $$x^3 + y^3 + z^3 = 53329,$$

  which by Lemma 1 does not hold.

Suppose that only $w = 2$. Then it will be

$$x^3 + y^3 + z^3 + u^3 + v^3 = 53345 \tag{1}$$

By Lemma 1, we have that if any of the remaining x, y, z, u, v is not equal to 3, (1) does not hold, since $53345 \equiv 2 \pmod 9$, while LHS cannot be conguent with it.

Hence, let $v = 3$.

- if it is also $u = 3$, we obtain the equation

  $$x^3 + y^3 + z^3 = 53291.$$

  WLOG, suppose that $x \geq y \geq z$. This way, we get to the conclusion that $x = 29$, $x = 31$ and $x = 37$, for none of which we have a solution.

Back to the case only $v = 3$. This gives

$$x^3 + y^3 + z^3 + u^3 = 53318. \tag{2}$$

By Lemma 2, we can similarly conclude that if any of x, y, z, u is not equivalent to 7, we get to a contradiction since $53318 \equiv 6 \pmod 7$ and LHS of (2) cannot be congruent with it.

- if any of x, y, z equals 7 (apart from u that is taken), for instance $z = 7 \Leftrightarrow z^3 = 343$, then we have

  $$y^3 + z^3 = 52632.$$

  Now supposing $x \geq y$, we can find that either $x = 31$ or $x = 37$. In each case there is no solution in prime integers.

So, let $u = 7$ and we take that

$$x^3 + y^3 + z^3 = 52975.$$

Now suppose that

$$x \geq y \geq z. \tag{3}$$

Then, we have

$$3x^3 \geq 52975 \Leftrightarrow x \geq \lceil \frac{52975}{3} \rceil = 26$$

and

$$x \leq 38$$

since if $x \geq 38 \Leftrightarrow x^3 \geq 54872$, that is impossible.

We have taken that $26 \leq x \leq 38$. Consequently, $x = 29$, $x = 31$ and $x = 37$. By (3), we can drop out cases $x = 29$ and $x = 31$. For $x = 37$, we have

$$y^3 + z^3 = 2322.$$

Relation (3) implies that $y^3 \geq 1311$ and because y is an odd prime and $1311 \leq y^3 \leq 2322$, it must be $y = 13$ or $y = 11$. For $y = 11$ the given equation has no solution, while for $y = 13$ we get $z = 5$. Hence, the given equation has a solution $(x, y, z, u, v, w) = (37, 13, 5, 7, 3, 2)$ and all its permutations in prime integers.

Also solved by Daniel Lasaosa, Pamplona, Spain; Théo Lenoir, Institut Saint-Lô, Agneaux, France; Peter C. Shim, The Pingry School, Basking Ridge, NJ; Seonmin Chung, Stuyvesant High School, NY; William Kang, Bergen County Academies, Hackensack, NJ, USA; Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Milan, Italy; Evgenidis Nikolaos, M. N. Raptou High School, Larissa, Greece; Samantha Paradis, College at Brockport, SUNY; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Michael Tang, Edina High School, MN, USA; Jaesung Son, Ridgewood, NJ, USA; Polyahedra, Polk State College, FL, USA; Kwon Il Ko, Cushing Academy, MA, USA.

J317. In triangle ABC, the angle-bisector of angle A intersects line BC at D and the circumference of triangle ABC at E. The external angle-bisector of angle A intersects line BC at F and the circumference of triangle ABC at G. Prove that $DG \perp EF$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Adnan Ali, Student at A.E.C.S-4, Mumbai, India

Since $\angle BAC + \angle CAY = 180°, \angle DAC + \angle CAF = 90° = \angle GAE$. Hence, GE is the diameter of the circumcircle of $\triangle ABC$. Consider $\triangle FGE$ and let $BC \cap GE = Z$. Then, since $\angle BAE = \angle EAC$, E is the midpoint of $\widehat{BC}$. Similarly G is the midpoint of $\widehat{BAC}$. Hence $GE \perp BC \Rightarrow \angle CZE = 90°$. Thus $FZ \perp GE$, or that FZ is an altitude of $\triangle FGE$ through F. Similarly, we know that $\angle EAC + \angle CAF = 90°$. So, $EA \perp FG$. And so EA is the altitude of $\triangle FGE$ through E. One can see that $AE \cap FZ = D$. Hence to prove that $DG \perp EF$, it suffices to show that GX is an altitude of $\triangle FGE$ which passes through D where $X = GD \cap EF$. Since the altitudes of a triangle concur at a point, we conclude that the altitude through G also passes through D. And that forces $DG \perp EF$ and thus the result.

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J318. Determine the functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x-y) - xf(y) \leq 1 - x$ for all real numbers x and y.

Proposed by Marcel Chirita, Bucharest, Romania

Solution by Polyahedra, Polk State College, USA

Clearly, $f(x) \equiv 1$ is a solution. We show that it is the only one.

First, $f(0-(-x)) - 0f(-x) \leq 1 - 0$, so $f(x) \leq 1$ for all x. Next, $f(2x-x) - 2xf(x) \leq 1 - 2x$, that is,

$$(1-2x)[f(x)-1] \leq 0.$$

So $f(x) \geq 1$ for all $x > \frac{1}{2}$. Thus $f(x) = 1$ for all $x > \frac{1}{2}$.

Finally, if $x \leq \frac{1}{2}$ then $2 - x > 1$, so $f(2-x) = 1$. Since $f(2-x) - 2f(x) \leq 1 - 2$, $f(x) \geq 1$. Hence $f(x) = 1$ for all $x \leq \frac{1}{2}$ as well.

Also solved by Solution by Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Pamplona, Spain; Peter C. Shim, The Pingry School, Basking Ridge, NJ; Seonmin Chung, Stuyvesant High School, NY; William Kang, Bergen County Academies, Hackensack, NJ, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Arbër Avdullahu, Mehmet Akif College,Kosovo; Arbër Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Arkady Alt, San Jose, California, USA; Paul Revenant, Lycée Champollion, Grenoble, France; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Jongyeob Lee, Stuyvesant High School, NY, USA; Michael Tang, Edina High School, MN, USA; Jaesung Son, Ridgewood, NJ, USA; Kwon Il Ko, Cushing Academy, MA, USA.

Senior problems

S313. Let a, b, c be nonnegative real numbers such that $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$. Prove that

$$\sqrt{(a+b+1)(c+2)}+\sqrt{(b+c+1)(a+2)}+\sqrt{(c+a+1)(b+2)} \geq 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

$$\begin{aligned}
\sqrt{a}+\sqrt{b}+\sqrt{c}=3 &\Leftrightarrow 3-\sqrt{c}=\sqrt{a}+\sqrt{b} \\
&\Leftrightarrow 9-6\sqrt{c}+c=a+b+2\sqrt{ab} \\
&\Rightarrow 9-6\sqrt{c}+c \leq a+b+(a+b) \\
&\Leftrightarrow \frac{9}{2}-3\sqrt{c}+\frac{c}{2} \leq a+b \\
&\Leftrightarrow \frac{11}{2}-3\sqrt{c}+\frac{c}{2} \leq a+b+1.
\end{aligned}$$

Hence

$$\begin{aligned}
(a+b+1)(c+2) &\geq \left(\frac{11}{2}-3\sqrt{c}+\frac{c}{2}\right)(c+2) \\
&= 9+\left(\sqrt{c}-1\right)^2\left(\sqrt{c}-2\right)^2 \geq 9 \\
&\Rightarrow \sqrt{(a+b+1)(c+2)} \geq 3
\end{aligned}$$

Similarly,

$$\sqrt{(b+c+1)(a+2)} \geq 3$$
$$\sqrt{(c+a+1)(b+2)} \geq 3.$$

Adding all above inequality the desired inequality is proved. The equality holds only when $a=b=c=1$.

Second solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

$$\sum_{cyc}\sqrt{(a+b+1)(c+2)}=\sum_{cyc}\sqrt{(a+b+1)(1+1+c)}$$

Applying Cauchy-Schwartz's inequality,

$$\geq \sum_{cyc}(\sqrt{a}\cdot 1+\sqrt{b}\cdot 1+1\cdot\sqrt{c})=9.$$

Hence the inequality is proved.

The equality holds only when $a=b=c=1$.

Also solved by Li Zhou, Polk State College, USA; Daniel Lasaosa, Pamplona, Spain; William Kang, Bergen County Academies, Hackensack, NJ, USA; George Gavrilopoulos, High School of Nea Makri, Athens, Greece; An Zhen-ping, Mathematics Department, Xianyang Normal University, Xianyang, Shaanxi, China; Jongyeob Lee, Stuyvesant High School, NY, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Arbër Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Arkady Alt, San Jose, California, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Erdenebayar Bayarmagnai, School Nr.11, Ulaanbaatar, Mongolia; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Evgenidis Nikolaos, M. N. Raptou High School, Larissa, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Shatlyk Mamedov, School Nr. 21, Dashoguz, Turkmenistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Utsab Sarkar, Chennai Mathematical Institute, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

S314. Let p, q, x, y, z be real numbers satisfying

$$x^2y + y^2z + z^2x = p \qquad \text{and} \qquad xy^2 + yz^2 + zx^2 = q.$$

Evaluate $\left(x^3 - y^3\right)\left(y^3 - z^3\right)\left(z^3 - x^3\right)$ in terms of p and q.

Proposed by Marcel Chirita, Bucharest, Romania

Solution by Prithwijit De, HBCSE, Mumbai, India

Let ω be a non-real cube root of unity. We can write $(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = UVW$ where

$$U = (x - y)(y - z)(z - x) = q - p \tag{1}$$

$$V = (x - y\omega)(y - z\omega)(z - x\omega) = q\omega^2 - p\omega \tag{2}$$

and

$$W = (x - y\omega^2)(y - z\omega^2)(z - x\omega^2) = q\omega - p\omega^2. \tag{3}$$

Thus $UVW = (q - p)(q\omega^2 - p\omega)(q\omega - p\omega^2) = q^3 - p^3$. Therefore

$$(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = q^3 - p^3.$$

Also solved by Li Zhou, Polk State College, USA; Daniel Lasaosa, Pamplona, Spain; William Kang, Bergen County Academies, Hackensack, NJ, USA; Adnan Ali, student at A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, Delhi, India; Arbër Avdullahu, Mehmet Akif College, Kosovo; Arkady Alt, San Jose, California, USA; Bunyod Boltayev, Khorezm, Uzbekistan; Debojyoti Biswas, Kolkata, India; Erdenebayar Bayarmagnai, School Nr.11, Ulaanbaatar, Mongolia; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Navid Safei, University of Technoogy in Policy Making of Science and Technology, Iran; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Evgenidis Nikolaos, M. N. Raptou High School, Larissa, Greece; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Shatlyk Mamedov, School Nr. 21, Dashoguz, Turkmenistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania.

S315. Consider triangle ABC with inradius r. Let M and M' be two points inside the triangle such that $\angle MAB = \angle M'AC$ and $\angle MBA = \angle M'BC$. Denote by d_a, d_b, d_c and d'_a, d'_b, d'_c the distances from M and M' to the sides BC, CA, AB, respectively. Prove that

$$d_a d_b d_c d'_a d'_b d'_c \leq r^6.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan
We have that the points M and M' are isogonal points in the triangle ABC with $d_a \cdot d'_a = d_b \cdot d'_b = d_c \cdot d'_c$. Let

$$k = d_a \cdot d'_a = d_b \cdot d'_b = d_c \cdot d'_c$$

and

$$S_a = [BMC], \qquad S_b = [AMC], \qquad S_c = [AMB],$$
$$S'_a = [BM'C], \qquad S'_b = [AM'C], \qquad S'_c = [AM'B].$$

So we get that

$$k = \frac{4S_aS'_a}{a^2} = \frac{4S_bS'_b}{b^2} = \frac{4S_cS'_c}{c^2} = \left(\frac{2\sqrt{S_aS'_a} + 2\sqrt{S_bS'_b} + 2\sqrt{S_cS'_c}}{a+b+c}\right)^2$$

and by Cauchy-Schwartz inequality,

$$k \leq \frac{(2S_a + 2S_b + 2S_c)(2S'_a + 2S'_b + 2S'_c)}{(a+b+c)^2} = r^2.$$

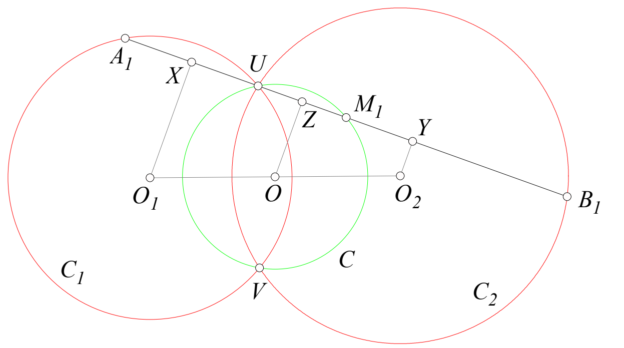
Hence $d_a d_b d_c d'_a d'_b d'_c = k^3 \leq r^6$ and the equality holds if and only if $M \equiv M' \equiv I$, where I is incenter of the triangle ABC.

Also solved by Li Zhou, Polk State College, USA; Daniel Lasaosa, Pamplona, Spain; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Marius Stanean, Zalau, Romania; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Prithwijit De, HBCSE, Mumbai, India.

S316. Circles $C_1(O_1, R_1)$ and $C_2(O_2, R_2)$ intersect in points U and V. Points A_1, A_2, A_3 lie on C_1 and points B_1, B_2, B_3 lie on C_2 such that A_1B_1, A_2B_2, A_3B_3 are passing through U. Denote by M_1, M_2, M_3 the midpoints of A_1B_1, A_2B_2, A_3B_3. Prove that $M_1M_2M_3V$ is a cyclic quadrilateral.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Li Zhou, Polk State College, USA



Let O, X, Y, Z be the midpoints of O_1O_2, A_1U, UB_1, UM_1, respectively. Then

$$\frac{X+Y}{2} = \frac{1}{2}\left(\frac{A_1+U}{2} + \frac{U+B_1}{2}\right) = \frac{1}{2}\left(\frac{A_1+B_1}{2} + U\right) = \frac{M_1+U}{2} = Z.$$

Hence $OZ \parallel O_1X$, and thus $OZ \perp UM_1$. Therefore, M_1 is on the circle C centered at O and of radius $OU = OV$. Likewise M_2 and M_3 are on C as well.

Second solution by Daniel Lasaosa, Pamplona, Spain
Lemma: Two secant circles (O_1, R_1) and (O_2, R_2) intersect at U, V. Let A, B be the respective points where a line through either U or V meets again both circles. Then, the midpoint M of AB satisfies $OM = OU = OV$, where O is the midpoint of O_1O_2.

Proof: Consider a cartesian coordinate system with center at O, such that $O_1 \equiv (-d, 0)$, $O_2 \equiv (d, 0)$, where $2d < R_1 + R_2$ because the circles are secant. The equations of the circles are then

$$(x+d)^2 + y^2 = {R_1}^2, \qquad (x-d)^2 + y^2 = {R_2}^2.$$

Let $U \equiv (\Delta, h)$ and $V \equiv (\Delta, -h)$, where $(\Delta + d)^2 + h^2 = {R_1}^2$ and $(\Delta - d)^2 + h^2 = {R_2}^2$. Any line through U has equation $y = h + m(x - \Delta)$, or its intersections with (O_1, R_1) satisfy

$$\left(m^2+1\right)x^2 + 2dx - 2m^2\Delta x + 2mhx + \left(m^2-1\right)\Delta^2 - 2d\Delta - 2mh\Delta = 0.$$

Since one of the roots, corresponding to U, is Δ, the other root equals the independent term divided by $(m^2+1)\Delta$, and substitution in the equation for the line through U yields

$$A \equiv \left(\frac{\left(m^2-1\right)\Delta - 2d - 2mh}{m^2+1}, -\frac{\left(m^2-1\right)h + 2m\Delta + 2md}{m^2+1}\right).$$

Similarly,

$$B \equiv \left(\frac{\left(m^2-1\right)\Delta + 2d - 2mh}{m^2+1}, -\frac{\left(m^2-1\right)h + 2m\Delta - 2md}{m^2+1}\right),$$

yielding

$$M \equiv \left(\frac{\left(m^2-1\right)\Delta - 2mh}{m^2+1}, -\frac{\left(m^2-1\right)h + 2m\Delta}{m^2+1}\right).$$

It follows that

$$OM^2 = \frac{\left(\left(m^2-1\right)^2 + 4m^2\right)\left(\Delta^2 + h^2\right)}{\left(m^2+1\right)^2} = \Delta^2 + h^2 = OU^2 = OV^2.$$

The Lemma follows, also when the line passes through V because of the symmetry with respect to line O_1O_2.

From the Lemma, it follows that the circle through U, V with center in the midpoint O of O_1O_2 passes also through M_1, M_2, M_3. The conclusion follows.

Also solved by Adnan Ali, A.E.C.S-4, Mumbai, India; Marius Stanean, Zalau, Romania; Prithwijit De, HBCSE, Mumbai, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.

S317. Let ABC be an acute triangle inscribed in a circle of radius 1. Prove that

$$\frac{\tan A}{\tan^3 B}+\frac{\tan B}{\tan^3 C}+\frac{\tan C}{\tan^3 A}\geq 4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, California, USA

Since $a=2\sin A, b=2\sin B, c=2\sin C$ and

$$(\tan A-\tan B)\left(\frac{1}{\tan^3 A}-\frac{1}{\tan^3 B}\right)=-\frac{(\tan A-\tan B)^2\left(\tan^2 A+\tan A\tan B+\tan^2 B\right)}{\tan^3 A\tan^3 B}\leq 0$$

then by Rearrangement Inequality

$$\sum\nolimits_{cyc}\tan A\cdot\frac{1}{\tan^3 B}\geq\sum\nolimits_{cyc}\tan A\cdot\frac{1}{\tan^3 A}=\sum\nolimits_{cyc}\frac{1}{\tan^2 A}=\sum\nolimits_{cyc}\cot^2 A=\sum\nolimits_{cyc}\left(\frac{1}{\sin^2 A}-1\right)=$$

$$\sum_{cyc}\frac{1}{\sin^2 A}-3=4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-3.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Marius Stanean, Zalau, Romania; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Prithwijit De, HBCSE, Mumbai, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Li Zhou, Polk State College, USA.

S318. Points $A_1, B_1, C_1, D_1, E_1, F_1$ are lying on the sides of AB, BC, CD, DE, EF, FA of a convex hexagon $ABCDEF$ such that

$$\frac{AA_1}{AB} = \frac{AF_1}{AF} = \frac{CC_1}{CD} = \frac{CB_1}{BC} = \frac{ED_1}{ED} = \frac{EE_1}{EF} = \lambda.$$

Prove that A_1D_1, B_1E_1, C_1F_1 are concurrent if and only if $\frac{[ACE]}{[BDF]} = \left(\frac{\lambda}{1-\lambda}\right)^2$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan

We have that

$$(1-\lambda)^2[ACE] - \lambda^2[BDF] = (1-\lambda)^2([OAE] + [OEC] + [OCA]) - \lambda^2([OBD] + [ODF] + [OFB]) \qquad (1).$$

Suppose that A_1D_1, B_1E_1, C_1F_1 are concurrent at one point O. Then, we get that

$$0 = [OA_1D_1] = (1-\lambda)[OA_1E] - \lambda[OA_1D] = (1-\lambda)\Big((1-\lambda)[OAE] - \lambda[OBE]\Big) - \lambda\Big(\lambda[OBD] - (1-\lambda)[OAD]\Big) =$$

$$= (1-\lambda)^2[OAE] - \lambda^2[OBD] + \lambda(1-\lambda)\Big([OAD] - [OBE]\Big),$$

so

$$(1-\lambda)^2[OAE] - \lambda^2[OBD] = \lambda(1-\lambda)\Big([OBE] - [OAD]\Big).$$

Analoguosly,

$$(1-\lambda)^2[OEC] - \lambda^2[OFB] = \lambda(1-\lambda)\Big([OFC] - [OBE]\Big)$$

$$(1-\lambda)^2[OCA] - \lambda^2[ODF] = \lambda(1-\lambda)\Big([OAD] - [OFC]\Big).$$

Thus, since (1) we have that $\frac{[ACE]}{[BDF]} = \left(\frac{\lambda}{1-\lambda}\right)^2$.

Suppose that $\frac{[ACE]}{[BDF]} = \left(\frac{\lambda}{1-\lambda}\right)^2$ and we will prove that A_1D_1, B_1E_1, C_1F_1 are concurrent. Since (1) we can find easily that there exists a point O such that

$$0 = (1-\lambda)^2[ACE] - \lambda^2[BDF] = [OA_1D_1] + [OB_1E_1] + [OC_1F_1].$$

So

$$[OA_1D_1] = [OB_1E_1] = [OC_1F_1]$$

and O be the intersection point of the lines A_1D_1, B_1E_1, C_1F_1.

Also solved by Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Li Zhou, Polk State College, USA.

U313. Let X and Y be nonnegative definite Hermitian matrices such that $X-Y$ is also nonnegative definite. Prove that $\operatorname{tr}\left(X^2\right) \geq \operatorname{tr}\left(Y^2\right)$.

Proposed by Radouan Boukharfane, Sidislimane, Morocco

Solution by Henry Ricardo, New York Math Circle

First we show that $\operatorname{tr}(XY) \geq 0$: For nonnegative definite Hermitian matrices X and Y, there exist matrices A and B such that $X = AA^*$ and $Y = BB^*$, giving us $\operatorname{tr}(XY) = \operatorname{tr}(AA^*BB^*) = \operatorname{tr}(A^*BB^*A) = \operatorname{tr}\left((A^*B)(A^*B)^*\right) \geq 0$.

Now we write

$$\operatorname{tr}(X^2) - \operatorname{tr}(Y^2) = \operatorname{tr}\left((X-Y)X\right) + \operatorname{tr}\left(Y(X-Y)\right) \geq 0,$$

which follows from the result proved above since $X-Y$ is nonnegative definite.

Also solved by Daniel Lasaosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Radouan Boukharfane, Sidislimane, Morocco; Alessandro Ventullo, Milan, Italy; Li Zhou, Polk State College, USA.

U314. Prove that for any positive integer k,

$$\lim_{n\to\infty}\left(\frac{1+\sqrt[n]{2}+\cdots+\sqrt[n]{k}}{k}\right)^n>\frac{k}{e},$$

where e is Euler constant.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Let $f(x)=\sqrt[n]{x}$. This function is clearly increasing (with first derivative $f(x)=\frac{x}{n}\sqrt[n]{x}$) and concave (with second derivative $f''(x)=-\frac{n-1}{n^2x^2}\sqrt[n]{x}$). It follows that, for $x\in[k-1,k]$, we have

$$f(x)\leq\sqrt[n]{k}-\frac{n-1}{n^2k^2}\sqrt[n]{k}(k-x),$$

with equality iff $x=k$, or

$$\frac{nk\sqrt[n]{k}}{n+1}=\int_0^k f(x)dx=\sum_{j=1}^k\int_{j-1}^j f(x)dx<\sum_{j=1}^k\sqrt[n]{j}-\sum_{j=1}^n\frac{n-1}{n^2j^2}\sqrt[n]{j}\int_{j-1}^j(j-x)dx=$$

$$=\sum_{j=1}^k\sqrt[n]{j}-\sum_{j=1}^k\frac{n-1}{2n^2j^2}\sqrt[n]{j},$$

or equivalently,

$$\left(\frac{1+\sqrt[n]{2}+\cdots+\sqrt[n]{k}}{k}\right)^n>\left(\frac{n\sqrt[n]{k}}{n+1}+\frac{n-1}{2n^2k}\sum_{j=1}^k\frac{\sqrt[n]{j}}{j^2}\right)^n>$$

$$>k\left(\frac{n}{n+1}\right)^n+\frac{n-1}{n+1}\frac{\sqrt[n]{k}}{2k}\sum_{j=1}^k\frac{\sqrt[n]{j}}{j^2}>k\left(\frac{n}{n+1}\right)^n+\frac{n-1}{n+1}\frac{\sqrt[n]{k}}{2k}\sum_{j=1}^k\frac{1}{j^2}.$$

The limit of the second term when $n\to\infty$ is clearly positive, since the factors outside the sum have limit $\frac{1}{2k}$, and the sum is positive and does not depend on n. It follows that

$$\lim_{n\to\infty}\left(\frac{1+\sqrt[n]{2}+\cdots+\sqrt[n]{k}}{k}\right)^n>k\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=k\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{k}{e}.$$

The conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Henry Ricardo, New York Math Circle; Arkady Alt , San Jose,California, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Alessandro Ventullo, Milan, Italy; Jaesung Son, Ridgewood, NJ, USA; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; Utsab Sarkar, Chennai Mathematical Institute, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan; William Kang, Bergen County Academies, Hackensack, NJ, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Li Zhou, Polk State College, USA.

U315. Let X and Y be complex matrices of the same order with $XY^2 - Y^2X = Y$. Prove that Y is nilpotent.

Proposed by Radouan Boukharfane, Sidislimane, Morocco

Solution by Li Zhou, Polk State College, USA

Let J be the Jordan canonical form of $Y_{n\times n}$. Then there is P such that $PYP^{-1} = J$. Let $Z = PXP^{-1}$, then the given condition becomes $ZJ^2 - J^2Z = J$. Let $J_1, \dots, J_k$ be the Jordan blocks of J corresponding to the eigenvalues $\lambda_1, \dots, \lambda_k$, not necessarily distinct. Then

$$J = \left[\begin{array}{c|c|c|c} J_1 & 0 & \cdots & 0 \\ \hline 0 & J_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & J_k \end{array}\right], \quad Z = \left[\begin{array}{c|c|c|c} Z_1 & * & \cdots & * \\ \hline * & Z_2 & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline * & * & \cdots & Z_k \end{array}\right],$$

where Z_i and J_i have the same order for each i. By block multiplication, the given condition implies $Z_iJ_i^2 - J_i^2Z_i = J_i$ for each i. Now let n_i be the order of J_i, then

$$n_i\lambda_i = tr(J_i) = tr(Z_iJ_i^2) - tr(J_i^2Z_i) = tr(J_i^2Z_i) - tr(J_i^2Z_i) = 0,$$

so $\lambda_i = 0$ for $1 \le i \le k$. Hence $J_i^{n_i} = 0$, thus $J^m = 0$, where $m = \max\{n_1, \dots, n_k\}$. Therefore, $Y^m = P^{-1}J^mP = 0$ as well.

Also solved by Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Alessandro Ventullo, Milan, Italy.

U316. The sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. For any natural number m define $v_2(m)$ as $v_2(m) = n$ if $2^n|m$ and $2^{n+1}|m$. Prove that there is exactly one positive number μ such that the equation

$$v_2([\mu n]!) = v_2(F_1F_2\cdots F_n)$$

is satisfied by the infinitely many positive integers n. determine that number μ.

Proposed by Albert Stadler, Herrliberg, Switzerland

First solution by by the proposer

We claim that the equation

$$v_2([\mu n]!) = v_2(F_1F_2\cdots F_n) \tag{1}$$

has infinitely many solutions in n if and only if $\mu = \frac{5}{6}$. We start with the following:

Lemma 1:

Let p be a prime. Let n be a natural number whose base p representation equals $n = \sum_{j=0}^{k} n_j p^j$, where $0 \leq n_j \leq p-1$, $0 \leq j \leq k$. For any natural number m define $v_p(m) = n$ if $p^n|m$ and $p^{n+1} \nmid m$. Then $v_p(n!) = \sum_{j\geq 1}\left[\dfrac{n}{p^j}\right] = \dfrac{n - n_0 - n_1 - n_2 - \cdots - n_k}{p-1}$.

Proof:

Among the integers $1, 2, \ldots, n$ there are exactly $\left[\frac{n}{p}\right]$ which are divisible by p, namely

$$p, 2p, ..., \left[\frac{n}{p}\right]p. \tag{2}$$

The integers between 1 and n which are divisible by p^2 (a subset of (2)) are

$$p^2, 2p^2, \ldots, \left[\frac{n}{p^2}\right]p^2,$$

which are $\left[\frac{n}{p^2}\right]$ in number, and so on. The number of integers between 1 and n, which are divisible by p^j but not by p^{j+1} is exactly $\left[\frac{n}{p^j}\right] - \left[\frac{n}{p^{j+1}}\right]$. Hence p divides $n!$ exactly

$$\sum_{j\geq 1} j\left(\left[\frac{n}{p^j}\right] - \left[\frac{n}{p^{j+1}}\right]\right) = \sum_{j\geq 1}\left[\frac{n}{p^j}\right]$$

times. We have

$$\sum_{j\geq 1}\left[\frac{n}{p^j}\right] = \sum_{j\geq 1}\frac{n}{p^j} - \sum_{j\geq 1}\left\{\frac{n}{p^j}\right\}$$

$$= \frac{n}{p-1} - \sum_{j\geq 1}\left\{\frac{n_0 + n_1p + n_2p^2 + \cdots + n_kp^k}{p^j}\right\} =$$

$$= \frac{n}{p-1} - \sum_{j=1}^{k}\frac{n_0 + n_1p + n_2p^2 + \cdots + n_{j-1}p^{j-1}}{p^j} - \sum_{j\geq k+1}\frac{n_0 + n_1p + n_2p^2 + \cdots + n_kp^k}{p^j} =$$

$$\frac{n}{p-1} - \sum_{j=1}^{k}\frac{1}{p^j}\sum_{i=1}^{j} n_{i-1}p^{i-1} - \frac{1}{p^k(p-1)}\sum_{i=0}^{k} n_ip^i = \frac{n}{p-1} - \sum_{i=1}^{k} n_{i-1}\sum_{j=i}^{k} p^{i-j-1} - \frac{1}{p^k(p-1)}\sum_{i=0}^{k} n_ip^i =$$

$$\frac{n}{p-1} - \frac{1}{p-1}\sum_{i=1}^{k} n_{i-1}(1 - p^{i-1-k}) - \frac{1}{p^k(p-1)}\sum_{i=0}^{k} n_ip^i =$$

$$= \frac{n - n_0 - n_1 - n_2 - \cdots - n_k}{p-1}.$$

This concludes the proof of Lemma 1.

Lemma 2:

$$v_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1,2 \mod 3 \\ 1, & \text{if } n \equiv 3 \mod 6 \\ 3, & \text{if } n \equiv 6 \mod 12 \\ v_2(n)+2, & \text{if } n \equiv 0 \mod 12 \end{cases}$$

Proof:

See Lemma 2 of Lengyel's paper
"The order of the Fibonacci and Lucas numbers"

Let $[\mu n] = \sum_i e_i 2^i$ and $\left[\dfrac{n}{12}\right] = \sum_i f_i 2^i$ be the binary representations of $[\mu n]$ and $\left[\frac{n}{12}\right]$ respectively. Then, by Lemma 1 and Lemma 2:

$$v_2([\mu n]!) = \sum_{k\geq 1}\left[\frac{[\mu n]}{2^k}\right] = \sum_i e_i(2^i - 1) = [\mu n] - \sum_i e_i,$$

$$v_2(F_1F_2\cdots F_n) = \sum_{k=1}^{n} v_2(F_k) = \sum_{0<6k+3\leq n} 1 + \sum_{0<12k+6\leq n} 3 + \sum_{0<12k\leq n} 2 + \sum_{0<12k\leq n} v_2(12k) =$$

$$= \sum_{0<6k+3\leq n} 1 + \sum_{0<12k+6\leq n} 3 + \sum_{0<12k\leq n} 4 + \sum_{0<k\leq \frac{n}{12}} v_2(k) = \left[\frac{n+3}{6}\right] + 3\left[\frac{n+6}{12}\right] + 4\left[\frac{n}{12}\right] + v_2\left(\left[\frac{n}{12}\right]!\right) =$$

$$\left[\frac{n}{6}+\frac{1}{2}\right] + 3\left[\frac{n}{12}+\frac{1}{2}\right] + 5\left[\frac{n}{12}\right] - \sum_i f_i.$$

We have $\sum_i e_i = O(\log n)$ and $\sum_i f_i = O(\log n)$. So, as n tends to infinity,

$$v_2([\mu n]!) = [\mu n] - \sum_i e_i = \mu n + O(\log n)$$

and

$$v_2(F_1F_2\cdots F_n) = \left[\frac{n}{6}+\frac{1}{2}\right] + 3\left[\frac{n}{12}+\frac{1}{2}\right] + 5\left[\frac{n}{12}\right] - \sum_i f_i = \frac{5n}{6} + O(\log n).$$

So a necessary condition that (1) has infinitely many solutions is $\mu = \dfrac{5}{6}$.

Consider the numbers $n = 81\cdot 2^k, k\geq 2$ Then $\left[\dfrac{5n}{6}\right] = 135\cdot 2^{k-1} = (2^7+2^2+2^1+2^0)2^{k-1}$, $v_2\left(\left[\dfrac{5n}{6}\right]!\right) = 135\cdot 2^{k-1} - 4$, $\left[\dfrac{n}{12}\right] = 27\cdot 2^{k-2} = (2^4+2^3+2^1+2^0)2^{k-2}$, $v_2\left(\left[\dfrac{n}{12}\right]!\right) = 27\cdot 2^{k-1} - 4$, $v_2(F_1F_2\cdots F_n) = \left[\dfrac{n}{6}+\dfrac{1}{2}\right] + 3\left[\dfrac{n}{12}+\dfrac{1}{2}\right] + 5\left[\dfrac{n}{12}\right] - \sum_i f_i = 27\cdot 2^{k-1} + 3\cdot 27\cdot 2^{k-2} + 5\cdot 27\cdot 2^{k-2} - 4 =$

$$135\cdot 2^{k-1} - 4 = v_2\left(\left[\frac{5n}{6}\right]!\right).$$

This concludes the proof.

Second solution by Li Zhou, Polk State College, USA
We show that $\mu = \frac{5}{6}$ and the equation is satisfied by $n = 3 \cdot 2^m + 17$ for all $m \geq 5$.

From the recurrence we see that F_n is odd for all n not divisible by 3. Note that $F_3 = 2$, $F_6 = 2^3$, and $F_{12} = 2^4 \cdot 9$. Assume q is an odd integer below. Then $(F_{3q}, F_6) = F_{(3q,6)} = F_3$ and $(F_{6q}, F_{12}) = F_{(6q,12)} = F_6$, so $v_2(F_{3q}) = 1$ and $v_2(F_{6q}) = 3$. Now $F_{2n} = F_n L_n$, where $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. By the well-known fact that $L_{3n} = \sum_{i=0}^{n} \binom{n}{i} 2^i L_i$, we get $L_{6k} \equiv L_0 \pmod 4$, that is, $v_2(L_{6k}) = 1$ for all $k \geq 0$. It is then easy to see by induction on m that $v_2(F_{6\cdot 2^m q}) = m + 3$ for all $m \geq 0$. Therefore,

$$\begin{aligned} v_2(F_1 \cdots F_n) &= \left\lceil \frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor \right\rceil + 3\left\lfloor \frac{n}{6} \right\rfloor + v_2\left(\left\lfloor \frac{n}{6} \right\rfloor !\right) = \left\lceil \frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor \right\rceil + 3\left\lfloor \frac{n}{6} \right\rfloor + \sum_{i=1}^{\infty} \left\lfloor \frac{\lfloor n/6 \rfloor}{2^i} \right\rfloor \\ &\sim \frac{n}{6} + \frac{n}{2} + \frac{n}{6} = \frac{5n}{6}, \quad n \to \infty. \end{aligned}$$

On the other hand,

$$v_2(\lfloor \mu n \rfloor !) = \sum_{i=1}^{\infty} \left\lfloor \frac{\lfloor \mu n \rfloor}{2^i} \right\rfloor \sim \mu n, \quad n \to \infty.$$

Hence, to satisfy the given equation for infinitely many n, μ must necessarily be $\frac{5}{6}$.

Finally, for all $n = 3 \cdot 2^m + 17$ with $m \geq 5$, the two formulas above become

$$v_2(F_1 \cdots F_n) = 2^{m-1} + 3 + 3\left(2^{m-1} + 2\right) + 2^{m-1} = 2^{m+1} + 2^{m-1} + 9,$$

$$\begin{aligned} v_2\left(\left\lfloor \frac{5n}{6} \right\rfloor !\right) &= \sum_{i=1}^{\infty} \left\lfloor \frac{5 \cdot 2^{m-1} + 14}{2^i} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \frac{2^{m+1} + 2^{m-1} + 14}{2^i} \right\rfloor \\ &= \left\lfloor \frac{14}{2} \right\rfloor + \left\lfloor \frac{14}{4} \right\rfloor + \left\lfloor \frac{14}{8} \right\rfloor + \sum_{i=0}^{m} 2^i + \sum_{i=0}^{m-2} 2^i \\ &= 11 + \left(2^{m+1} - 1\right) + \left(2^{m-1} - 1\right) = 2^{m+1} + 2^{m-1} + 9. \end{aligned}$$

U317. For any natural numbers s, t and p, prove that there is a number $M(s,t,p)$ such that every graph with a matching of size at least $M(s,t,p)$ contains either a clique K_s, an induced complete bipartite graph $K_{t,t}$ or an induced matching M_p. Does the result remain true if we replace the word "matching" by "path"?

Proposed by Cosmin Pohoata, Columbia University, USA

Solution by the proposer

The result is true for both "matching" and "path" cases. We'll only include a proof of it for "matching"; for the "path" proof, we refer to A. Atminas, V.V. Lozin, I. Razgon, *Linear time algorithm for computing a small complete biclique in graphs without long induced paths*, SWAT, 2012: 142-152.

The key part of the proof is the following analogue for bipartite graphs, which is easier to prove.

Lemma:.

For any natural numbers t and p, there is a natural number $N(t,p)$ such that every bipartite graph with a matching of size at least $N(t,p)$ contains either a complete bipartite graph $K_{t,t}$ or an induced matching M_p.

Proof:

For $p=1$ and arbitrary t, we can define $N(t,p)=1$. Now, for each fixed t, we prove the result by induction on p. Without loss of generality, we prove it only for values of the form $p=2^s$. Suppose we have shown the lemma for $p=2^s$ for some $s\geq 0$. Let us now show that it is sufficient to set $N(t,2p)=RB(t,RB(t,N(t,p)))$, where RB is the bipartite Ramsey number.

Consider a graph G with a matching of size at least $RB(t,RB(t,N(t,p)))$. Without loss of generality, we may assume that G contains no vertices outside of this matching. We also assume that G does not contain an induced $K_{t,t}$, since otherwise we are done. Then G must contain the bipartite complement of $K_{RB(t,N(t,p)),RB(t,N(t,p))}$ with vertex classes, say A and B. Now let C and D consist of the vertices matched to vertices in A and B respectively in the original matching in G.

Note that A, B, C, D are pairwise disjoint. Graphs $G[A\cup C]$ and $G[B\cup D]$ (induced on sets $A\cup C$ and $B\cup D$) now each contain a matching of size $RB(t,N(t,p))$. There are no edges between A and B, yet there might be edges between C and D. By our assumption, $G[C\cup D]$ is however $K_{t,t}$-free, therefore, by Ramsey's theorem, it must contain the bipartite complement of $K_{N(t,p),N(t,p)}$, with vertex sets $C'\subset C$, $D'\subset D$. Let $A'\subset A$ and $B'\subset B$ be the set of vertices matched to C' and D' respectively in the original matching in G. Now there are no edges in $G[A'\cup B']$ and none in $G[C'\cup D']$, but $G[A'\cup C']$ and $G[B'\cup D']$ both contain a matching of size $N(t,p)$. Since G is $K_{t,t}$-free, by the induction hypothesis, we conclude that they both contain an induced matching M_p. Putting these together we find that G contains an induced M_{2p}. This proves the Lemma.

Returning to the problem, define $M(s,t,p):=R(s,R(s,N(t,p)))$, where R is the classical Ramsey number and $N(t,p)$ is the N from the Lemma. Suppose that G is a $(K_s,K_{t,t})$-free graph with a matching of size $R(s,R(s,N(t,p)))$. Since G is K_s-free, by Ramsey's theorem, it must contain an independent set A of size $R(s,N(t,p))$. Let B be the set of vertices matched to A. Since $G[B]$ is K_s-free, it must contain an independent set B' of size $N(t,p)$. Let A' be the set of vertices matched to B'; the graph $H=G[A'\cup B']$ is a bipartite graph with a matching of size $N(t,p)$. By the above, since H is $K_{t,t}$-free, it thus follows that H contains an induced matching M_p. This completes the proof.

U318. Determine all possible values of $\sum_{k=1}^{\infty} \frac{(-1)^{q(k)}}{k^2}$, where $q(x)$ is a quadratic polynomial that assumes only integer values at integer places.

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Li Zhou, Polk State College, USA

It is well known that

$$q(x) = a\binom{x}{2} + b\binom{x}{1} + c\binom{x}{0} = \frac{1}{2}ax(x-1) + bx + c$$

for some $a, b, c \in \mathbb{Z}$ with $a \neq 0$. Note that for $k \in \mathbb{N}$,

$$q(k+4) = q(k) + 4ak + 6a + 4b \equiv q(k) \pmod{2}.$$

Now denote $\frac{1}{2}ax(x-1) + bx + c$ by $q_{a,b,c}(x)$, with $a, b, c \equiv 0$ or $1 \pmod 2$. Then

$$S_{0,0,0} = \sum_{k=1}^{\infty} \frac{(-1)^{q_{0,0,0}(k)}}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6},$$

$$S_{0,1,0} = \sum_{k=1}^{\infty} \frac{(-1)^{q_{0,1,0}(k)}}{k^2} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots = -\frac{\pi^2}{12},$$

$$\begin{aligned} S_{1,0,0} &= \sum_{k=1}^{\infty} \frac{(-1)^{q_{1,0,0}(k)}}{k^2} = \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{8^2} + \cdots \\ &= \left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right) - \frac{1}{4}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right) = G - \frac{\pi^2}{48}, \end{aligned}$$

where G is Catalan's constant, and

$$\begin{aligned} S_{1,1,0} &= \sum_{k=1}^{\infty} \frac{(-1)^{q_{1,1,0}(k)}}{k^2} = -\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} - \cdots \\ &= -\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right) - \frac{1}{4}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right) = -G - \frac{\pi^2}{48}. \end{aligned}$$

Finally, $S_{0,0,1} = -S_{0,0,0}$, $S_{0,1,1} = -S_{0,1,0}$, $S_{1,0,1} = -S_{1,0,0}$, and $S_{1,1,1} = -S_{1,1,0}$.

Also solved by Roberto Mastropietro and Emiliano Torti, University of Rome Tor Vergata", Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Pamplona, Spain.

Olympiad problems

O313. Find all positive integers n for which there are positive integers $a_0, a_1, \ldots, a_n$ such that $a_0 + a_1 + \cdots + a_n = 5(n-1)$ and

$$\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} = 2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

From the AM-HM Inequality,

$$\frac{5(n-1)}{n+1} \geq \frac{n+1}{2}.$$

So, $n^2 - 8n + 11 \leq 0 \Rightarrow 1 < 4 - \sqrt{5} \leq n \leq 4 + \sqrt{5} < 7$. Hence $2 \leq n \leq 6$. But it is easy to see that

$$\begin{aligned}
2+2+1 = 5, &\qquad \frac{1}{2}+\frac{1}{2}+\frac{1}{1} = 2, \qquad (\text{for } n=2)\\
3+3+3+1 = 10, &\qquad \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{1} = 2, \qquad (\text{for } n=3)\\
2+2+2+3+6 = 15, &\qquad \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{6} = 2, \qquad (\text{for } n=4)\\
3+3+3+2+3+6 = 20, &\qquad \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{2}+\frac{1}{3}+\frac{1}{6} = 2, \qquad (\text{for } n=5)\\
3+3+3+4+4+4+4 = 25, &\qquad \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = 2, \qquad (\text{for } n=6)
\end{aligned}$$

Hence $n \in \{2, 3, 4, 5, 6\}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; William Kang, Bergen County Academies, Hackensack, NJ, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Michael Tang, Edina High School, MN, USA; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Milan, Italy; Arbër Avdullahu, Mehmet Akif College, Kosovo; Arbër Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Erdenebayar Bayarmagnai, School Nr.11, Ulaanbaatar, Mongolia; Francesc Gispert Sánchez, CFIS, Universitat Politécnica de Catalunya, Barcelona, Spain; Henry Ricardo, New York Math Circle; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Shatlyk Mamedov, School Nr. 21, Dashoguz, Turkmenistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania Li Zhou, Polk State College, USA.

O314. Prove that every polynomial $p(x)$ with integer coefficients can be represented as a sum of cubes of several polynomials that return integer values for any integer x.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Navid Safei, University of Technology in Policy Making of Science and Technology, Iran
At first thanks to the following identity one can find that the polynomial x, could be written as sum of cubes of five integer valued polynomials as follows:

$$x = x^3 + \left(\frac{x^3 - x}{6}\right)^3 + \left(\frac{x^3 - x}{6}\right)^3 + \left(-1 - \frac{x^3 - x}{6}\right)^3 + \left(1 - \frac{x^3 - x}{6}\right)^3$$

It is obvious that the polynomial $\frac{x^3 - x}{6}$ is an integer-valued polynomial. By use of the following we can find that the polynomial x^2 , could be written as sum of cubes of integer-valued polynomials(just take place in the above identity x through x^2) . Now consider the polynomial $P(x)$ with integer coefficients then we can write the polynomial by the form $P(x) = P_1(x^3) + xP_2(x^3) + x^2P_3(x^3)$ where P_1, P_2, P_3 are polynomials with integer coefficients . it is obvious that these polynomials (i.e. P_1, P_2, P_3) could be written as the sum of cubes of polynomials with integer coefficients and thus integer-valued. Since polynomials x, x^2 could be written so , the statement was proved.

Also solved by Henri Godefroy, Stanislas Secondary School, Paris, France; Li Zhou, Polk State College, USA.

O315. Let a, b, c be positive real numbers. Prove that

$$\left(a^3+3b^2+5\right)\left(b^3+3c^2+5\right)\left(c^3+3a^2+5\right) \geq 27(a+b+c)^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy
The AGM yields $a^3+1+1 \geq 3a$ hence

$$\begin{aligned} (a^3+3b^2+5)(b^3+3c^2+5)(c^3+3a^2+5) &\geq \\ \geq (3a+3b^2+3)(3b+3c^2+3)(3c+3a^2+3) &= \\ = 27(a+b^2+1)(b+c^2+1)(c+a^2+1) \end{aligned}$$

we come to

$$(a+b^2+1)(b+c^2+1)(c+a^2+1) \geq (a+b+c)^3$$

Now Hölder comes in

$$(a+b^2+1)(1+b+c^2)(a^2+1+c) \geq \left(\sum_{\text{cyc}} a^{\frac{1}{3}} 1^{\frac{1}{3}} a^{\frac{2}{3}}\right)^3$$

whence the result.

Also solved by George Gavrilopoulos, High School of Nea Makri, Athens, Greece; An Zhen-ping, Mathematics Department, Xianyang Normal University, Xianyang, Shaanxi, China; Albert Stadler, Herrliberg, Switzerland; Erdenebayar Bayarmagnai, School Nr.11, Ulaanbaatar, Mongolia; Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Marius Stanean, Zalau, Romania; Farrukh Mukhammadiev, Academic Lyceum Nr.1 under the SamIES, Samarkand, Uzbekistan; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Li Zhou, Polk State College, USA.

O316. Prove that for all integers $k \geq 2$ there exists a power of 2 such that at least half of the last k digits are nines. For example, for $k = 2$ and $k = 3$ we have $2^{12} = \ldots 96$ and $2^{53} = \ldots 992$.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

Solution by Daniel Lasaosa, Pamplona, Spain

Claim: If $x \equiv -1 \pmod{5^k}$ for some integers x and $k \geq 1$, then $x^5 \equiv -1 \pmod{5^{k+1}}$.

Corollary: For every positive integer k, we have $2^{2 \cdot 5^{k-1}} \equiv -1 \pmod{5^k}$.

Proof: If $x \equiv -1 \pmod{5^k}$, an integer u exists such that $x = u \cdot 5^k - 1$, or

$$x^5 = \left(u^5 \cdot 5^{4k-1} - u^4 \cdot 5^{3k} + 2u^3 \cdot 5^{2k} - 2u^2 \cdot 5^k + u\right) 5^{k+1} - 1.$$

The Claim follows. Since $2^2 = 5 \cdot 1 - 1$, the result holds for $k = 1$, serving as the base case for induction over k, the step being guaranteed by the Claim. The Corollary follows.

By the Corollary, for every positive integer k, a positive integer u exists such that

$$2^{2 \cdot 5^{k-1}} = u5^k - 1, \qquad\qquad 0 < u10^k - 2^{2 \cdot 5^{k-1} + k} = 2^k.$$

Note that if $k = 2\ell \geq 2$ is an even positive integer, then $2^k < 10^\ell$, clearly true since it is equivalent to $\left(\frac{5}{2}\right)^\ell > 1$, whereas if $k = 2\ell + 1 \geq 3$ is an odd integer larger than 1, then $2^k < 10^\ell$, which is equivalent to $2^{\ell+1} < 5^\ell$, true for $\ell = 1$ (equivalent to $k = 3$), and clearly true for any $\ell \geq 1$ since the RHS is multiplied by 5 every time ℓ grows in one unit, the LHS being multiplied only by 2. It follows that, whatever the last ℓ digits are, the previous $k - \ell$ digits (which are clearly at least half of the last k digits) are 9's. The conclusion follows.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Albert Stadler, Herrliberg, Switzerland; Li Zhou, Polk State College, USA.

O317. Twelve scientists met at a math conference. It is known that every two scientists have a common friend among the rest of the people. Prove that there is a scientist who knows at least five people from the attendees of the conference.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

First solution by Li Zhou, Polk State College, USA

Number the scientists 1 through 12 and let $S = \{1, \ldots, 12\}$. For each $i \in S$, denote by F_i the set of friends of i. Assume that $|F_i| \le 4$ for all i.

(a) *Claim 1:* $|F_i| = 4$ for all i.

Indeed, if $|F_i| \le 3$ for some i, then

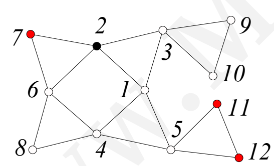
$$\left| \bigcup_{k \in F_i} (F_k \setminus \{i\}) \right| \le \sum_{k \in F_i} |F_k \setminus \{i\}| \le 3|F_i| \le 9 < 11,$$

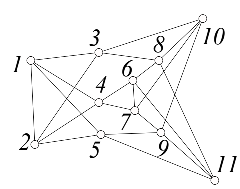
so there is $j \in S \setminus \{i\}$ such that i and j do not have a friend in common, a contradiction.

(b) *Claim 2:* If i and j are friends then $|F_i \cap F_j| = 1$.

Indeed, suppose that $F_i = \{j, k, l, m\}$ and $k, l \in F_j$ as well. Note that i and m must have a friend in common, so $|F_m \cap \{j, k, l\}| \ge 1$. Hence, $|(F_j \cup F_k \cup F_l \cup F_m) \setminus \{i, j, k, l, m\}| \le 6 < 7$, so there is $n \in S \setminus \{i, j, k, l, m\}$ such that i and n do not have a friend in common, a contradiction.

(c) Without loss of generality, assume now that $F_1 = \{2, 3, 4, 5\}$ with $F_1 \cap F_2 = \{3\}$ and $F_1 \cap F_4 = \{5\}$, as in the left graph below. For 1 to have a friend in common with each scientist in $\{6, \ldots, 12\}$, we must have $\{6, \ldots, 12\} \subseteq (F_2 \cup F_3 \cup F_4 \cup F_5)$. Now each of $2, 3, 4, 5$ needs to have two friends in $\{6, \ldots, 12\}$, so exactly one scientist in $\{6, \ldots, 12\}$ has two friends in $\{2, 3, 4, 5\}$ who are not friends. Without loss of generality, say 6 is a friend of 2 and 4, then we have the complete left graph below. However, for 2 to have a friend in common with each of 11 and 12, 7 has to be a friend of both 11 and 12, which contradicts Claim 2 since $\{5, 7\} \subseteq F_{11} \cap F_{12}$. This contradiction completes the proof.





(d) Finally we notice that it does require twelve scientists to draw the conclusion, as shown by the right graph above with eleven scientists.

Second solution by Michael Ma, Plano, TX

Let each of the scientists represent a vertex on a graph G. Now draw an edge between two vertices if the two corresponding scientists are friends. Now we know that every two vertices share a common neighbor. We will proceed now by contradiction. So assume that no vertex is of degree at least 5. Now say there exists a vertex with degree 3 or less. Then in G there are 11 other vertices that share a common neighbor. This common neighbor must be one of the 3 or less neighbors. Then by the pigeonhole principle there is a neighbor that has at least 44 edges to the other 11 vertices. Since this neighbor also has an edge to the original vertex, this vertex has degree at least 5. Contradiction.

So now every vertex has degree at least and at most and therefore exactly 4. Now take a vertex A. Say it's neighbors are B, C, D, E. Now the other 7 points have to share one of B, C, D, E with A. Say one of B, C, D, E is adjacent to 3 or more of these 7 points. WLOG say B. Then there are only 3 since the degree of B is 4. Then B must share a vertex with A. But none of these 3 points are neighbors with A and A is not a neighbor of itself. Contradiction.

So one of B, C, D, E is adjacent to at least 1 of the seven points and the other 3 are neighbors of exactly 2 of the 7 points. WLOG say E is adjacent to at least 1 and E is adjacent to F, D is adjacent to G, H, C is adjacent to I, L, and B is adjacent to J, K. Now E has to be adjacent with on of B, C, D. WLOG say D. Therefore since B and C cannot both be adjacent to E, B and C must be adjacent. Therefore E is adjacent to one of G, H, I, L, J, K. Assume E is connected to one of G, H. WLOG let it be H. Then J, K, I, L must share a neighbor with D. Since they are not adjacent to A or E, they must be adjacent to H or G. But H can only have one more neighbor and G only 2. Contradiction.

Hence E is connected to one of I, J, K, L. WLOG let it be L. Now since H shares a neighbor with D, G and H are adjacent. Similarly I, L and K, J are adjacent.

Now since L and E must share a common neighbor and L can- not be adjacent to A or D, its must be adjacent to F. Similarly, since K and J cannot be neighbors of L, since it's degree is $4, K$ and J are neighbors of F. Now since H and B share a common neighbor, H is connected to $K or J$. WLOG assume it's K. Similarly we get that J is adjacent to G. Now since H and C share a common neighbor, H and I are adjacent. Similarly, so are G and I. Now H and K share no common neighbors. Contradiction. So there is a vertex of degree at least 5.

Also solved by Francesc Gispert Sánchez, CFIS, Universitat Politécnica de Catalunya, Barcelona, Spain; Philip Radoslavov Grozdanov, Yambol, Bulgaria.

O318. Find all polynomials $f \in \mathbb{Z}[X]$ with the property that for any distinct primes p and q, $f(p)$ and $f(q)$ are relatively prime.

Proposed by Marius Cavachi, Constanta, Romania

Solution by Li Zhou, Polk State College, USA

For each degree $n \geq 0$, $f(x) = \pm x^n$ clearly have this property. We show that they are the only ones.

Suppose that $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is such a polynomial, with $n \geq 0$. Let p be any prime such that $p > |a_n| + \cdots + |a_0|$. If there is a prime $p_1 \neq p$ such that $p_1 | f(p)$, then by Dirichlet's theorem, there is a prime $q = p_1 u + p$ for some positive integer u. Hence $f(q) \equiv f(p) \equiv 0 \pmod{p_1}$, that is, $p_1 | \left(f(p), f(q)\right)$, a contradiction. Therefore $|f(p)|$ must be a power of p. Observe that

$$|f(p)| \leq |a_n| p^n + \cdots + |a_1| p + |a_0| \leq \left(|a_n| + \cdots + |a_0|\right) p^n < p^{n+1},$$

$$|f(p)| \geq |a_n| p^n - \left(|a_{n-1}| + \cdots + |a_0|\right) p^{n-1} \geq |a_n| p^n - (p-2) p^{n-1}.$$

If $|a_n| \geq 2$ then $|f(p)| > p^n$, so $f(p)$ cannot be a power of p. Thus $|a_n| = 1$. Then $|f(p)| > p^{n-1}$, so $|f(p)| = p^n$, which implies $f(p) - a_n p^n \in \{0, \pm 2p^n\}$. But

$$|f(p) - a_n p^n| = |a_{n-1} p^{n-1} + \cdots + a_1 p + a_0| \leq \left(|a_{n-1}| + \cdots + |a_0|\right) p^{n-1} < p^n,$$

so $f(p) - a_n p^n = 0$. This means that $f(x) - a_n x^n$ has all primes greater than $|a_n| + \cdots + |a_0|$ as zeros, thus it must be identically 0, that is, $f(x) = a_n x^n$ with $|a_n| = 1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Navid Safei, University of Technoogy in Policy Making of Science and Technology, Iran; Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan.