Junior problems

J223. Let a and b be real numbers such that $\sin^3 a - \frac{4}{3}\cos^3 a \le b - \frac{1}{4}$. Prove that

$$\frac{3}{4}\sin a - \cos a \le b + \frac{1}{6}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, California, USA

Since

$$\sin^3 a - \frac{4}{3}\cos^3 a \le b - \frac{1}{4} \iff \sin^3 a - \frac{4}{3}\cos^3 a + \frac{5}{12} \le b + \frac{1}{6},$$

it suffices to prove that

$$\frac{3}{4}\sin a - \cos a \le \sin^3 a - \frac{4}{3}\cos^3 a + \frac{5}{12} \iff 4\sin a - 12\cos a - 12\sin^3 a + 16\cos^3 a \le 5.$$

But

$$9\sin a - 12\cos a - 12\sin^3 a + 16\cos^3 a = 3\left(3\sin a - 4\sin^3 a\right) + 4\left(4\cos^3 a - 3\cos a\right)$$

and this equals

$$3\sin 3a + 4\cos 3a = 5\left(\sin 3a \cdot \frac{3}{5} + \cos 3a \cdot \frac{4}{5}\right) = 5\sin\left(3a + \varphi\right) \le 5,$$

where

$$\cos\varphi = \frac{3}{5}, \sin\varphi = \frac{4}{5}.$$

(either that or simply by Cauchy Inequality write: $3\sin 3a + 4\cos 3a \le \sqrt{3^2+4^2} \cdot \sqrt{\sin^2 a + \cos^2 a} = 5$).

Also solved by Albert Stadler, Switzerland; Alessandro Ventullo, Milan, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Florin Stanescu, Cioculescu Serban High School, Gaesti, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Prithwijit De, Mumbai, India; Matteo Elia, Università di Roma "Tor Vergata", Roma, Italy.

J224. Consider 25 points inside the unit circle. Prove that among them there are two at most $\frac{1}{2}$ apart.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

We divide the given unit circle in to 24 parts following way.

$$X = \{(\rho, \varphi) : \ \rho \le \frac{1}{4}, 0 \le \varphi \le 2\pi\},$$

$$Y_i = \{(\rho, \varphi_i) : \ \frac{1}{4} \le \rho \le \frac{1}{2}, i\frac{\pi}{4} \le \varphi_i \le (i+1)\frac{\pi}{4}\}, \ i = 0, 1, \dots, 7,$$

$$Z_j = \{(\rho, \psi_i) : \ \frac{1}{2} \le \rho \le 1, j\frac{2\pi}{15} \le \psi_j \le (j+1)\frac{2\pi}{15}\}, \ j = 0, 1, \dots, 14.$$

It is easy to see that any part's diameter is less than $\frac{1}{2}$. Again by the Pigeonhole principle at least 2 points are in a part from the given 25 points. Therefore, distance of those 2 points is less than $\frac{1}{2}$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniel Lima, PECI, Rio de Janeiro, Brazil; Albert Stadler, Switzerland; G.R.A.20 Problem Solving Group, Roma, Italy.

J225. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)+abc \le \frac{1}{8}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)=\frac{(a+b+c)^4-(a^4+b^4+c^4)-6(ab+bc+ca)^2}{4},$$

or using that $a+b+c=1$, and hence $(ab+bc+ca)^2=a^2b^2+b^2c^2+c^2a^2+2abc$, the proposed inequality is equivalent to

$$(a^2+b^2+c^2)^2+4(ab+bc+ca)^2 \ge \frac{1}{2}.$$

Now, applying the AM-QM inequality to $a^2+b^2+c^2$ and $2(ab+bc+ca)$, we find that the LHS is at least $\frac{(a+b+c)^2}{2}=\frac{1}{2}$, with equality iff $a^2+b^2+c^2=2(ab+bc+ca)$, and since they add up to $(a+b+c)^2=1$, we find that equality holds in the proposed inequality iff

$$a+b+c=1, \qquad a^2+b^2+c^2=\frac{1}{2}.$$

There are infinitely many (a, b, c) that satisfy these conditions simultaneously, and can be found to be described by

$$a=\frac{1-c\pm\sqrt{c(2-3c)}}{2}, \qquad b=\frac{1-c\mp\sqrt{c(2-3c)}}{2},$$

for any $0 \le c \le \frac{2}{3}$.

Also solved by Arkady Alt, San Jose, California, USA; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Prithwijit De, Mumbai, India; Nicu Zlota, Focsani, Romania; Florin Stanescu, Cioculescu Serban High School, Gaesti, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Alessandro Ventullo, Milan, Italy; Albert Stadler, Switzerland; Bruno Nishimoto, PECI, Rio de Janeiro, Brazil.

J226. We are given $n \geq 4$ points in the plane, no three collinear. Denote by T_p the set of triangles with vertices in these points whose interior contains at least one of the other points. Prove that if $|T_p| \leq n-4$, then $|T_p| = 0$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lima Braga, PECI, Rio de Janeiro, Brazil

Suppose that $|T_p| > 0$ thus exists among the n points three points A, B, C and another point D inside the triangle ABC. Let l_a be the the half-line that starts in D is contained in DA and do not contain A. Similarly define l_b and l_c. That three half-lines will divide the plane in three disjoint regions. Take another of the n points, say X, if it is in the area delimited by l_a and l_b then the point D is inside XAB, analogous statements holds if X is in the other regions. Thus we can count one triangle in T_p for each one of the $n-4$ $X \notin \{A, B, C, D\}$, plus the triangle ABC it gives $|T_p| \geq 1 + n - 4 = n - 3$. Thus $|T_p| \leq n - 4$ implies $|T_p| = 0$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

J227. For a positive integer N let $r(N)$ be the number obtained by reversing the digits of N. Find all 3-digit numbers N such that $r^2(N) - N^2$ is the cube of a positive integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Francesco Bonesi, Matteo Elia, Lorenzo Luzzi and Alessio Podda. Università di Roma "Tor Vergata", Roma, Italy

We write n as $N = 100a + 10b + c$ with $a, b, c \in \{0, \ldots, 9\}$ and $a \geq 1$. Then

$$\begin{aligned} r(N)^2 - N^2 &= (100c + 10b + a)^2 - (100a + 10b + c)^2 \\ &= (100c + 10b + a - 100a - 10b - c)(100c + 10b + a + 100a + 10b + c) \\ &= 99(c - a)((c + a)101 + 20b). \end{aligned}$$

Since $99 = 11 \cdot 3^2$ and $0 < c - a < 9$ (the required cube should be positive), it follows that

$$(c + a)101 + 20b \equiv 2(c + a - b) \equiv 0 \pmod{11}$$

which implies that $b = c + a$ because a, b, c are digits. Therefore

$$r(N)^2 - N^2 = 11^3 \cdot 3^2 \cdot b \cdot (c - a)$$

and 3 should divide b or $c - a$. It suffices to check the following five cases:

i) if $b = 3$ then $N = 132$;

ii) if $b = 6$ then $N \in \{165, 264\}$

iii) if $b = 9$ then $N \in \{198, 297, 396, 495\}$

iv) if $c - a = 3$ then $N \in \{154, 275, 396\}$;

v) if $c - a = 6$ then $N = 187$.

It easy to verify that $N = 132$ are $N = 165$ are the only numbers with the required property.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo, Milan, Italy.

J228. Prove that a square of side 1 and a square of side 2 cannot fit inside a square of side less than 3 without overlapping.

Proposed by Roberto Bosch Cabrera, Florida, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

We will show that a square of side a and a square of side b cannot fit inside a square of side less than $a+b$ without overlapping.

Let us assume that a square $ABCD$ contains a square of side a and a square of side b without overlapping. We may also assume that the square of side a has one vertex on AB and another one on AD, whereas the square of side b has one vertex on BC and another one on CD.

Then the diagonal AC intersects the sides of the two squares in four points: P, P' and Q, Q'. Since the squares do not overlap, it follows that $|AP| \leq |AQ|$ which implies

$$|AB|\sqrt{2} = |AC| = |AQ| + |CQ| \geq |AP| + |CQ|.$$

Moreover, it is easy to see that

$$|AP| = \frac{1+\sin\theta\cos\theta}{\cos\theta+\sin\theta}\, a\sqrt{2} \quad \text{and} |CQ| = \frac{1+\sin\varphi\cos\varphi}{\cos\varphi+\sin\varphi}\, b\sqrt{2}$$

for some $\theta, \varphi \in [0, \pi/4]$. Now

$$1+\cos x\sin x - \cos x - \sin x = (1-\cos x)(1-\sin x) \geq 0$$

implies that $1+\cos x\sin x \geq \cos x + \sin x$ and

$$|AB|\sqrt{2} \geq |AP| + |CQ| \geq (a+b)\sqrt{2}.$$

Therefore $|AB| \geq a+b$.

Senior problems

S223. We define *magic numbers* as follows:

(i) all numbers from 0 to 9 are magic;

(ii) a number greater than 9 is magic if it is divisible by the number of its digits and the number obtained by deleting its final digit is also magic.

Find the greatest magic number.

Proposed by Roberto Bosch Cabrera, Florida, USA

No solutions have been received yet. However, Alessio Podda and Antonello Cirulli from Università di Roma "Tor Vergata", Roma, Italy managed to check via a computer software that there are no magic numbers with 26 digits and that there is only one with 25 digits: 3608528850368400786036725.

S224. Let a, b, c be real numbers greater than 2 such that

$$\frac{7-2a}{3a-6}+\frac{7-2b}{3b-6}+\frac{7-2c}{3c-6}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Since

$$\frac{7-2a}{3a-6}+\frac{7-2b}{3b-6}+\frac{7-2c}{3c-6}=-\frac{2}{3}+\frac{1}{a-2}-\frac{2}{3}+\frac{1}{b-2}-\frac{2}{3}+\frac{1}{c-2},$$

the given condition can be rewritten as

$$\frac{1}{a(a-2)}+\frac{1}{b(b-2)}+\frac{1}{c(c-2)}=1.$$

Suppose without loss of generality that $a\le b\le c$. If

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1$$

then $\dfrac{1}{a-2}+\dfrac{1}{b-2}+\dfrac{1}{c-2}>3$ and by Chebyshev's Inequality,

$$3<\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a-2}+\frac{1}{b-2}+\frac{1}{c-2}\right)\le 3,$$

contradiction.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Prithwijit De, Mumbai, India; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo, Milan, Italy.

S225. Let ABC be a triangle. Determine the points X on the median from vertex A for which the ratio $\frac{BX}{CX}$ is minimal or maximal.

Proposed by Roberto Bosch Cabrera, USA and Francisco Javier García Capitán, Spain

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

If ABC is isosceles at A, the median AM (where M is the midpoint of BC) is also the perpendicular bisector of BC, hence $BX = CX$ for any X on AM. Assume wlog henceforth that $b > c$. By the median theorem,

$$BX^2 + CX^2 = 2MX^2 + \frac{a^2}{2}.$$

Moreover, by Stewart's theorem,

$$BX^2 = \frac{MX \cdot AB^2 + AX \cdot BM^2}{AM} - AX \cdot MX = MX^2 + \frac{a^2}{4} - \frac{b^2 - c^2}{2AM}MX,$$

where $MX = 0$ if $X = M$, in which case clearly $BX = CX = \frac{a}{2}$, $MX > 0$ if $X \neq M$ is on ray MA, and $MX < 0$ if X in on line MA such that M is inside segment AX. By symmetry with respect to point M, we only need to treat the case where X is on ray MA and $MX > 0$; the extreme value of $\frac{BX}{CX}$ for X on this ray will equal to the extreme value of $\frac{CX'}{BX'}$ for X' on AM such that M is inside AX, where X' is clearly the symmetric of X with respect to M.

After exchanging B and C, it follows that

$$CX^2 - BX^2 = \frac{b^2 - c^2}{AM}MX.$$

Denote $u = \frac{BX^2}{CX^2}$, or

$$\frac{1-u}{1+u} = \frac{2(b^2 - c^2)}{AM} \cdot \frac{MX}{4MX^2 + a^2} \leq \frac{b^2 - c^2}{2a \cdot AM},$$

with equality iff $MX = \frac{a}{2}$. Therefore,

$$\frac{BX}{CX} \geq \sqrt{\frac{2a \cdot AM - b^2 + c^2}{2a \cdot AM + b^2 - c^2}},$$

with equality iff X is such that $MX = MB = MC$, ie iff X is the point where the circle with diameter BC intersects the median AM.

We conclude that the extrema of $\frac{BX}{CX}$ when $X \in AM$ occur for both points where the circle with diameter BC intersects AM, where one corresponds to a maximum and the other to a minimum, and with the maximum for X on the same half-plane as A with respect to BC iff $c > b$, and with X on the other half-plane iff $b > c$.

Also solved by Arkady Alt, San Jose, California, USA; Francesco Bonesi, Antonello Cirulli, Matteo Elia, and Alessio Podda, Università di Roma "Tor Vergata", Roma, Italy.

S226. Let x, y, z be pairwise distinct positive real numbers. Prove that

$$\frac{x+y}{(x-y)^2} + \frac{y+z}{(y-z)^2} + \frac{z+x}{(z-x)^2} \geq \frac{9}{x+y+z}.$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note that we may exchange x, y without altering the problem, or we may assume wlog $x > y > z > 0$. We may therefore define $u = x - y$, $v = y - z < y$, and the proposed inequality becomes

$$\frac{2y+u}{u^2} + \frac{2y-v}{v^2} + \frac{2y+u-v}{(u+v)^2} \geq \frac{9}{3y+u-v}.$$

After multiplying both sides by $u^2v^2(u+v)^2(3y+u-v)$ and rearranging terms, we find the equivalent inequality

$$6(u^2+uv+v^2)^2y^2 - (v-u)(u^2+uv+v^2)(2u^2-uv+2v^2)y \geq K,$$

where K depends on u, v, but not on y. Assume that we know the values of u, v, but not the value of y. Since the second derivative of the LHS is $12(u^2+uv+v^2) > 0$, the minimum of the LHS will be reached for the value of y for which the derivative of the LHS with respect to y is zero, ie, for

$$y = \frac{(v-u)(2u^2-uv+2v^2)}{12(u^2+uv+v^2)}.$$

Note that $z > 0$, or equivalently $v < y$, translates into

$$v(10u^2+13uv+10v^2) + u(2u^2-uv+2v^2) < 0,$$

clearly impossible since by the AM-GM inequality, $2u^2+2v^2 \geq 4uv > uv$. Therefore, the LHS does not have a minimum with respect to y, but an infimum that is never reached, and the value of this infimum occurs when $y = v$, or $z = 0$. Therefore, the proposed inequality will always hold, and strictly, if the following inequality holds for any $x > y > 0$:

$$\frac{x+y}{(x-y)^2} + \frac{1}{y} + \frac{1}{x} \geq \frac{9}{x+y}.$$

This inequality is equivalent, after multiplying both sides by $xy(x+y)(x-y)^2$, to

$$0 \leq x^4 - 8x^3y + 18x^2y^2 - 8xy^3 + y^4 = (x^2-4xy+y^2)^2,$$

clearly true, and with equality iff $x^2+y^2 = 4xy$. The conclusion follows. Equality never holds, but the LHS and RHS may be made as similar as desired, by choosing any two of x, y, z, wlog x, y, such that $x^2+y^2 = 4xy$, and by letting the third one tend to zero, in this case z.

Also solved by Arkady Alt, San Jose, California, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

S227. Let $\mathbb{N}^*$ be the set of positive integers. Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ such that

$$f(n+1) > \frac{f(n) + f(f(n))}{2}$$

for all n.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

Without losing in generality, we extend the function by letting $f(0) = 0$

i) We first prove that f is strictly increasing.

More precisely, we show by induction with respect to $n \geq 0$ that

$$f(0) < f(1) < f(2) < \cdots < f(n) < f(m) \quad \text{for any } m > n.$$

For $n = 0$ this is trivial because $f(0) = 0 < f(m)$ for $m > 0$. Assume that $n > 0$ and let $m_0 > n$ such that, $f(m_0) = \min_{k>n} f(k)$ (the minimum is attained because $f(\mathbb{N}^*) \subset \mathbb{N}^*$). Assume that $m_0 > n + 1$, then

$$\min(f(m_0 - 1), f(f(m_0 - 1))) \leq \frac{f(m_0 - 1) + f(f(m_0 - 1))}{2} < f(m_0)$$

which implies that $f(m_0 - 1) < f(m_0)$ or $f(f(m_0 - 1)) < f(m_0)$. The definition of m_0 implies that, $n \geq m_0 - 1 > n$ or $n \geq f(m_0 - 1) > f(n) \geq n$ which are contradictions. Hence $m_0 = n + 1$. Now if $m > n + 1$ then

$$f(0) < f(1) < f(2) < \cdots < f(n) < f(n+1) \leq \min(f(m-1), f(f(m-1))) < f(m).$$

ii) We prove that $f(m) < m + 2$ for all $m \geq 1$.

By i) we have that $f(m) \geq f(n) + m - n$ for $m > n \geq 0$. Therefore

$$\begin{aligned} f(f(m+1)) &= f(f(m) + f(m+1) - f(m)) \\ &\geq f(f(m)) + f(m+1) - f(m) \\ &> f(f(m)) + \frac{f(m) + f(f(m))}{2} - f(m) = \frac{3}{2} f(f(m)) - \frac{1}{2} f(m), \end{aligned}$$

and

$$\begin{aligned} f(m+2) &> \frac{f(m+1) + f(f(m+1))}{2} \\ &\geq \frac{f(m) + f(f(m)) + 3f(f(m)) - f(m)}{4} = f(f(m)). \end{aligned}$$

Since by i) f is strictly increasing, it follows that $f(m) < m + 2$. Thus, by ii) and by i) it is easy to verify that all the required functions are given by

$$f_0(n) = n, \; f_N(n) = \begin{cases} n & \text{if } n \in [1, N] \\ n+1 & \text{if } n \in [N+1, +\infty) \end{cases} \quad \text{for } N \geq 1, \text{ and } f_\infty(n) = n + 1.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S228. Given triangle ABC, let M, N, P be the midpoints of BC, CA, AB, respectively. Let D, E, F be the tangency points of the incircle $\omega(I, r)$ with sides BC, CA, AB, respectively. Let X be a point on the line AI such that $\frac{AI}{IX} = 2$, with I lying on the segment AX. Similarly, define Y and Z. Prove that

a) Lines MX, NY, PZ are parallel.

b) Lines DX, EY, FZ are concurrent on ω.

Proposed by Luiz Gonzalez, Maracaibo, Venezuela and Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

a) Let G be the centroid of ABC. It is well known that $\frac{AG}{GM} = 2$, or by Thales' theorem, $MX \parallel GI$, and similarly $NY, PZ \parallel GI$, or these three lines are parallel.

b) Denote by D', X' the respective symmetric points of D, X with respect to I. Clearly X' is the midpoint of IA, and I is the midpoint of DD'. Now, the exact trilinear coordinates of A, I, D are $(h_a, 0, 0)$, (r, r, r) and $(0, 2r\cos^2\frac{C}{2}, 2r\cos^2\frac{B}{2})$, where r is the inradius and $h_a = \frac{r(a+b+c)}{a}$ the length of the altitude from A. It follows that, in exact trilinear coordinates,

$$D' \equiv \left(2r, 2r\sin^2\frac{C}{2}, 2r\sin^2\frac{B}{2}\right), \qquad X' \equiv \left(\frac{2a+b+c}{a}\cdot\frac{r}{2}, \frac{r}{2}, \frac{r}{2}\right).$$

Every point in line $D'X'$ has therefore trilinear coordinates (α, β, γ) such that

$$\frac{a}{a+b+c}\left(\alpha\cos\frac{A}{2}\sin\frac{C-B}{2} - \beta\cos^2\frac{B}{2} + \gamma\cos^2\frac{C}{2}\right) + \left(\beta\sin^2\frac{B}{2} - \gamma\sin^2\frac{C}{2}\right) = 0.$$

The Feuerbach point $\mathcal{F}$ is known to have trilinear coordinates

$$\mathcal{F} \equiv \left(\sin^2\frac{B-C}{2}, \sin^2\frac{C-A}{2}, \sin^2\frac{A-B}{2}\right),$$

which substituted in the previous equation yield, after some algebra and use of Hero's formula, the partial results

$$\beta\sin^2\frac{B}{2} - \gamma\sin^2\frac{C}{2} = \sin^2\frac{C-A}{2}\cos^2\frac{A+C}{2} - \sin^2\frac{A-B}{2}\cos^2\frac{A+B}{2} =$$
$$= \frac{1}{4}\left((\sin C - \sin A)^2 - (\sin A - \sin B)^2\right) = \frac{(b-c)(2a-b-c)}{16R^2},$$

and

$$\alpha\cos\frac{A}{2}\sin\frac{B-C}{2} - \beta\cos^2\frac{B}{2} + \gamma\cos^2\frac{C}{2} =$$
$$= \sin^3\frac{B-C}{2}\sin\frac{B+C}{2} - \sin^2\frac{C-A}{2}\sin^2\frac{C+A}{2} + \sin^2\frac{A-B}{2}\sin^2\frac{A+B}{2} =$$
$$= \frac{(b-c)(a+b+c)(b+c-2a)}{16aR^2},$$

which when inserted clearly show that $D', X', \mathcal{F}$ are collinear. By symmetry with respect to I, and denoting by $\mathcal{F}'$ the symmetric of the Feuerbach point with respect to the incenter, we find that $D, X, \mathcal{F}'$ are collinear. Similarly, so ar $E, Y, \mathcal{F}'$ and $F, Z, \mathcal{F}'$. Hence, DX, EY, FZ meet at $\mathcal{F}'$, clearly on ω because $\mathcal{F}$ is on ω and $\mathcal{F}'$ is its symmetric with respect to I.

Also solved by Prithwijit De, HBCSE, Mumbai, India.

Undergraduate problems

U223. Let $(x_k)_{k\geq 1}$ be the positive roots of the equation $\tan x = x$. Prove that

$$\sum_{k=1}^{\infty} \frac{1}{{x_k}^2} = \frac{1}{10}.$$

Proposed by Roberto Bosch Cabrera, Florida, USA

First solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let us consider the entire complex function

$$f(z) = \sin(z) - z\cos(z).$$

It easy to show that its complex zeros are $(x_k)_{k\geq 1}$, $(-x_k)_{k\geq 1}$ and 0. These zeros are all simple with the exception of 0 whose order is 3. By using the Weierstrass factorization theorem, it can be shown that

$$f(z) = \frac{z^3}{3} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{x_k^2}\right).$$

On the other hand, by expanding $f(z)$ at 0 we have that

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} - z\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \frac{z^3}{3} \sum_{n=0}^{\infty} \frac{6(n+1)}{(2n+3)!}(-1)^n z^{2n}.$$

By comparing these two expressions we find that

$$S_n := \sum_{k_1<k_2<\cdots<k_n} \frac{1}{x_{k_1}^2 x_{k_2}^2 \cdots x_{k_n}^2} = \frac{6(n+1)}{(2n+3)!}.$$

In particular, for $n = 1$ we obtain that

$$S_1 = \sum_{k=1}^{\infty} \frac{1}{x_k^2} = \frac{6(2)}{5!} = \frac{1}{10}.$$

Second solution by John Mangual, UC Santa Barbara, California, USA

Since $\tan(-x) = -\tan x = -x$, every positive root has a corresponding negative root. Let x_k denote all of the roots

$$\sum_{x_k>0} \frac{1}{x_k^2} = \frac{1}{2}\sum_{x_k\neq 0} \frac{1}{x_k^2} = \frac{1}{10}$$

$$\sum_{x_k\neq 0} \frac{1}{x_k^2} = \frac{1}{5}$$

We can add over roots of functions without knowing what they are. If $p(x) = (x - r_1)\dots(x - r_n)$, using Vieté formulas.

$$-\frac{p'(0)}{p(0)} = \sum \frac{1}{r_i}$$

$$\frac{p''(0)}{2p(0)} = \sum_{i\neq j} \frac{1}{r_i r_j}$$

Then we can extract squares of the reciprocals using the identity, $(a + b)^2 - 2ab = a^2 + b^2$.

$$\frac{p''(0)}{2p(0)} + \frac{p'(0)}{p(0)} = \sum \frac{1}{r_i^2}$$

We can imagine the same thing holds true for a function with infinitely many roots. This can be justified using Weierstrass factorization.

If we let $p(x) = \tan x - x$, Taylor expansion shows this has a triple root at $x = 0$:

$$\tan x - x = \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^7)$$

Dividing by x^3 we retain the positive and negative roots, i.e. $p(x) = (\tan x - x)/x^3$. Then

$$p(0) = \frac{1}{3} \text{ and } p'(0) = 0 \text{ and } p''(0) = \frac{1}{15}$$

So our answer is $\sum_{r_k>0} \frac{1}{r_k^2} = \frac{1}{2}\sum_{r_k\neq 0} \frac{1}{r_k^2} = \frac{p''(0)}{2p(0)} = \frac{1}{10}$.

Also solved by Albert Stadler, Switzerland; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U224. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers satisfying $a_1 = a \neq 0$ and

$$a_{n+1} = \sqrt[3]{\frac{a_1^2}{1} + \frac{a_2^2}{2} + \ldots + \frac{a_n^2}{n}}, \quad n \geq 1$$

Find $\lim_{n\to\infty} (3a_n - \log n)$.

Proposed by Cezar Lupu, University of Pittsburgh, USA and Tudorel Lupu, Decebal High School, Constanta, Romania

Solutions by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia
Notice that

$$\begin{aligned} a_{n+1} = \sqrt[3]{\frac{a_1^2}{1} + \frac{a_2^2}{2} + \cdots + \frac{a_n^2}{n}} \quad &\Rightarrow \quad a_{n+1}^3 = \frac{a_1^2}{1} + \frac{a_2^2}{2} + \cdots + \frac{a_n^2}{n} \\ &\Rightarrow \quad a_{n+1}^3 = a_n^3 + \frac{a_n^2}{n} \\ &\Rightarrow \quad a_{n+1} = a_n \left(1 + \frac{1}{na_n}\right)^{\frac{1}{3}} \\ &\Rightarrow \quad a_{n+1} = a_n + \frac{1}{3n} + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Therefore, by the Stolz-Cesaro theorem

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{\log n} &= \lim_{n\to\infty} \frac{a_{n+1} - a_n}{\log(n+1) - \log n} \\ &= \lim_{n\to\infty} \left(\frac{1}{3n} + O\left(\frac{1}{n^2}\right)\right) \cdot \frac{1}{\log\left(1 + \frac{1}{n}\right)} \\ &= \lim_{n\to\infty} \left(\frac{1}{3} + O\left(\frac{1}{n}\right)\right) \cdot \frac{1}{\log\left(1 + \frac{1}{n}\right)^n} = \frac{1}{3}. \end{aligned}$$

Hence, we have

$$a_n = \frac{1}{3}\log n + O\left(\frac{\log n}{n}\right) \ \Rightarrow \ 3a_n - \log n = O\left(\frac{\log n}{n}\right),$$

which implies that

$$\lim_{n\to\infty} (3a_n - \log n) = 0.$$

Also solved by Moubinool Omarjee, Paris, France; Albert Stadler, Switzerland; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

U225. Find the maximal number of edges of the n-dimensional unit cube that are cut by a hyperplane.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

No solutions have been yet received.

U226. Let $\mathbf{x}$ and $\mathbf{y}$ be points on $\mathbb{S}^n$ which are randomly chosen from the uniform distribution on the unit n-sphere. Evaluate $\mathrm{E}\big[||\mathbf{x}-\mathbf{y}||^2\big]$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

No solutions have been yet received.

U227. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a-b) + f(b-c) + f(c-a) = 2f(a+b+c)$$

whenever a, b, c are real numbers such that $ab + bc + ca = 0$.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Taking $a = b = c = 0$ yields $3f(0) = 2f(0)$, or $f(0) = 0$. Taking $b = c = 0$ yields $f(a) + f(-a) = 2f(a)$, or $f(-a) = f(a)$ for any real a. In particular, this means that f is even, and hence its derivative is odd, or $f'(0) = 0$. For any pair a, b such that $a + b \neq 0$, take $c = -\frac{ab}{a+b}$ so that $ab + bc + ca = 0$, and note that

$$f(a-b) + f\left(\frac{b(2a+b)}{a+b}\right) + f\left(a + \frac{ab}{a+b}\right) = 2f\left(a + \frac{b^2}{a+b}\right).$$

Let now a be any nonzero real, $|b| < |a|$, and let $b \to 0$. Since f is differentiable, and using Landau notation, we have

$$f(a-b) = f(a) - bf'(a) + \frac{b^2}{2}f''(a) + O(b^3),$$

$$f\left(\frac{b(2a+b)}{a+b}\right) = \frac{b^2(2a+b)^2}{2(a+b)^2}f''(0) + O(b^3) = 2b^2 f''(0) + O(b^3),$$

$$f\left(a + \frac{ab}{a+b}\right) = f(a) + \frac{ab}{a+b}f'(a) + \frac{a^2b^2}{2(a+b)^2}f''(a) + O(b^3) =$$

$$= f(a) + bf'(a) - \frac{b^2}{a}f'(a) + \frac{b^2}{2}f''(a) + O(b^3),$$

$$f\left(a + \frac{b^2}{a+b}\right) = f(a) + \frac{b^2}{a+b}f'(a) + \frac{b^4}{(a+b)^2} + O(b^3) = f(a) + \frac{b^2}{a}f'(a) + O(b^3).$$

Inserting these results, it follows that, for any real x, we must have

$$xf''(x) - 3f'(x) = -2xf''(0),$$

$$\frac{d}{dx}\left(\frac{f'(x)}{x^3}\right) = \frac{x^3 f''(x) - 3x^2 f'(x)}{x^6} = -\frac{2}{x^3}f''(0) = f''(0)\frac{d}{dx}\left(\frac{1}{x^2}\right),$$

or for appropriately chosen integration constants C, D,

$$f'(x) = f''(0)x + 4Cx^3, \qquad f(x) = \frac{f''(0)}{2}x^2 + Cx^4 + D.$$

Now, $D = 0$ because $f(0) = 0$, and calling $f''(0) = 2B$, we find that necessarily f is of the form

$$f(x) = Bx^2 + Cx^4.$$

Now,

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 2(a+b+c)^2 - 6(ab+bc+ca),$$

while

$$(a-b)^4 + (b-c)^4 + (c-a)^4 = 2(a+b+c)^4 - 12(a^2+b^2+c^2)(ab+bc+ca) - 6(ab+bc+ca)^2,$$

and the functional equation is satisfied for any B, C whenever $ab + bc + ca = 0$, or $f(x) = Bx^2 + Cx^4$ is indeed a solution for any real-valued B, C.

U228. Let L/K be a separable algebraic extension of fields and let V, W and U be L-vector spaces. Furthermore, let $h : V \times W \mapsto U$ be a K-bilinear map satisfying

$$h(xa, xb) = x^2 h(a, b) \text{ for every } x \in L,\ a \in V, \text{ and } b \in W.$$

Prove that h is L-bilinear.

Proposed by Darij Grinberg, Massachussets Institute of Techonology, Romania

Solution by the author

We first prove the following lemma:

Lemma 1. Under the conditions of the problem, let $x \in L$, $a \in V$ and $b \in W$ be arbitrary. Let $\alpha = h(a, b)$ and $\beta = h(a, xb) - xh(a, b)$. Then, every positive $n \in \mathbb{N}$ satisfies

$$h(a, x^n b) = x^n \alpha + nx^{n-1}\beta. \tag{1}$$

Proof of Lemma 1. Let us prove that (1) holds for every positive $n \in \mathbb{N}$. We will prove this by strong induction over n:

Induction step[1]*:* Let $N \in \mathbb{N}$ be positive. Assume that (1) holds for every positive $n \in \mathbb{N}$ satisfying $n < N$. We must then prove that (1) holds for $n = N$.

The equality (1) holds for $n = 1$ //since

$$h\left(a, \underbrace{x^1}_{=x} b\right) = h(a, xb) = \underbrace{x}_{=x^1} \underbrace{h(a, b)}_{=\alpha} + \underbrace{h(a, xb) - xh(a, b)}_{\substack{=\beta=1\beta=1x^{1-1}\beta \\ (\text{since } 1x^{1-1}=1x^0=1 \text{ and thus } 1=1x^{1-1})}} = x\alpha + 1x^{1-1}\beta.$$

In other words, if $N = 1$, then (1) holds for $n = N$. Hence, if $N = 1$, the induction step is already completed. Thus, for the rest of the induction step, we can WLOG assume that $N \neq 1$. Assume this.

Since $N \in \mathbb{N}$ is positive, but $N \neq 1$, we must have $N \geq 2$. Thus, $N - 1$ lies in $\mathbb{N}$ and is positive. Consequently, (1) holds for $n = N - 1$ (since we assumed that (1) holds for every $n \in \mathbb{N}$ satisfying $n < N$). In other words, $h\left(a, x^{N-1}b\right) = x^{N-1}\alpha + (N-1)\, x^{(N-1)-1}\beta$. Now,

$$h\left(xa, \underbrace{x^N}_{=xx^{N-1}} b\right) = h\left(xa, xx^{N-1}b\right) = x^2 h\left(a, x^{N-1}b\right) \tag{2}$$

(by (1) applied to $x^{N-1}b$ instead of b).

It is easy to show that

$$h\left(xa, x^{N-1}b\right) = x^N \alpha + (N-2)\, x^{N-1}\beta \tag{3}$$

[2].

[1] A strong induction does not need an induction base.

[2] *Proof of (3).* We have $N \geq 2$. Thus, we must be in one of the following two cases:

Case 1: We have $N = 2$.

Case 2: We have $N > 2$.

Let us first consider Case 1. In this case, $N = 2$, so that $N - 1 = 1$, and thus $x^{N-1} = x^1 = x$, so that

$$h\left(xa, x^{N-1}b\right) = h(xa, xb) = x^2 \underbrace{h(a, b)}_{=\alpha} \qquad (\text{by } (\textbf{??}))$$
$$= x^2\alpha$$

But (1) (applied to $1+x$ and $x^{N-1}b$ instead of x and b) yields $h\left((1+x)a,(1+x)x^{N-1}b\right)=(1+x)^2 h\left(a,x^{N-1}b\right)$. Since

$$\begin{aligned}
& h\Bigg(\underbrace{(1+x)a}_{=a+xa},\underbrace{(1+x)x^{N-1}b}_{=x^{N-1}b+xx^{N-1}b}\Bigg) \\
&= h\Bigg(a+xa,x^{N-1}b+\underbrace{xx^{N-1}}_{=x^N}b\Bigg)=h\left(a+xa,x^{N-1}b+x^Nb\right) \\
&= h\left(a,x^{N-1}b\right)+h\left(a,x^Nb\right)+h\left(xa,x^{N-1}b\right)+\underbrace{h\left(xa,x^Nb\right)}_{\substack{=x^2h\left(a,x^{N-1}b\right)\\ \text{(by (2))}}} \qquad \text{(since } h \text{ is } K\text{-bilinear)} \\
&= h\left(a,x^{N-1}b\right)+h\left(a,x^Nb\right)+h\left(xa,x^{N-1}b\right)+x^2h\left(a,x^{N-1}b\right) \\
&= h\left(a,x^Nb\right)+h\left(a,x^{N-1}b\right)+x^2h\left(a,x^{N-1}b\right)+h\left(xa,x^{N-1}b\right),
\end{aligned}$$

this rewrites as

$$h\left(a,x^Nb\right)+h\left(a,x^{N-1}b\right)+x^2h\left(a,x^{N-1}b\right)+h\left(xa,x^{N-1}b\right)=(1+x)^2h\left(a,x^{N-1}b\right).$$

Compared with

$$\underbrace{x^N}_{\substack{=x^2\\ (\text{since } N=2)}}\alpha+\underbrace{(N-2)}_{\substack{=0\\ (\text{since } N=2)}}x^{N-1}\beta=x^2\alpha+0x^{N-1}\beta=x^2\alpha,$$

this yields $h\left(xa,x^{N-1}b\right)=x^N\alpha+(N-2)x^{N-1}\beta$. Thus, (3) is proven in Case 1.

Now, let us consider Case 2. In this case, $N>2$, so that $N-2$ is a positive element of $\mathbb{N}$. Consequently, (1) holds for $n=N-2$ (since we assumed that (1) holds for every $n\in\mathbb{N}$ satisfying $n<N$). In other words, $h\left(a,x^{N-2}b\right)=x^{N-2}\alpha+(N-2)x^{(N-2)-1}\beta$. Now,

$$\begin{aligned}
h\Bigg(xa,\underbrace{x^{N-1}}_{=xx^{N-2}}b\Bigg) &= h\left(xa,xx^{N-2}b\right)=x^2\underbrace{h\left(a,x^{N-2}b\right)}_{=x^{N-2}\alpha+(N-2)x^{(N-2)-1}\beta} \qquad \left(\text{by (\textbf{??}), applied to } x^{N-2}b \text{ instead of } b\right) \\
&= x^2\left(x^{N-2}\alpha+(N-2)x^{(N-2)-1}\beta\right) \\
&= \underbrace{x^2x^{N-2}}_{=x^{2+(N-2)}=x^N}\alpha+(N-2)\underbrace{x^2x^{(N-2)-1}}_{=x^{2+((N-2)-1)}=x^{N-1}}\beta=x^N\alpha+(N-2)x^{N-1}\beta.
\end{aligned}$$

Thus, (3) is proven in Case 2.

Thus, in each of the two cases 1 and 2, we have shown that (3) holds. Since Cases 1 and 2 are the only two possible cases, this shows that (3) always holds, qed.

In other words,

$$\begin{aligned}
h\left(a, x^N b\right) &= \underbrace{(1+x)^2 h\left(a, x^{N-1}b\right) - h\left(a, x^{N-1}b\right) - x^2 h\left(a, x^{N-1}b\right)}_{=\left((1+x)^2-1-x^2\right)h(a,x^{N-1}b)} - h\left(xa, x^{N-1}b\right) \\
&= \underbrace{\left((1+x)^2 - 1\right)}_{=2x} \underbrace{h\left(a, x^{N-1}b\right)}_{=x^{N-1}\alpha+(N-1)x^{(N-1)-1}\beta} - \underbrace{h\left(xa, x^{N-1}b\right)}_{\substack{=x^N\alpha+(N-2)x^{N-1}\beta \\ \text{(by (3))}}} \\
&= 2x\left(x^{N-1}\alpha + (N-1)\,x^{(N-1)-1}\beta\right) - \left(x^N\alpha + (N-2)\,x^{N-1}\beta\right) \\
&= 2 \underbrace{xx^{N-1}}_{=x^{1+(N-1)}=x^N} \alpha + 2\,(N-1) \underbrace{xx^{(N-1)-1}}_{=x^{1+((N-1)-1)}=x^{N-1}} \beta - x^N\alpha - (N-2)\,x^{N-1}\beta \\
&= 2x^N\alpha + 2\,(N-1)\,x^{N-1}\beta - x^N\alpha - (N-2)\,x^{N-1}\beta \\
&= \underbrace{\left(2x^N\alpha - x^N\alpha\right)}_{=(2-1)x^N\alpha} + \underbrace{\left(2\,(N-1)\,x^{N-1}\beta - (N-2)\,x^{N-1}\beta\right)}_{=(2(N-1)-(N-2))x^{N-1}\beta} \\
&= \underbrace{(2-1)}_{=1} x^N\alpha + \underbrace{(2\,(N-1) - (N-2))}_{=N} x^{N-1}\beta = x^N\alpha + Nx^{N-1}\beta.
\end{aligned}$$

In other words, (1) holds for $n = N$. This completes the induction step. Thus, the induction proof of (1) is done. In other words, Lemma 1 is proven.

Now, we will show:

> **Lemma 2.** Under the conditions of the problem, let $x \in L$, $a \in V$ and $b \in W$ be arbitrary. Then, $h(a, xb) = xh(a, b)$.

Proof of Lemma 2. Let $\alpha = h(a, b)$ and $\beta = h(a, xb) - xh(a, b)$. According to Lemma 1, every $n \in \mathbb{N}$ satisfies (1).

Since L is an algebraic extension of K, the element $x \in L$ has a minimal polynomial over K. Let $P \in K[X]$ be this minimal polynomial. Then, P is separable (since L is a separable extension of K, so that x is separable over K). In other words, $\gcd(P, P') = 1$ (where P' denotes the X-derivative of the polynomial P). Hence, no root of the polynomial P is simultaneously a root of P'. Thus, x is not a root of P' (because x is a root of the polynomial P (since P is the minimal polynomial of x)). In other words, $P'(x) \neq 0$.

Since $P \in K[X]$ is a polynomial over K, we can write P in the form $P = \sum\limits_{n=0}^{M} \lambda_n X^n$ for some $M \in \mathbb{N}$ and some elements λ_0, λ_1, ..., λ_M of K. Consider this M and these elements λ_0, λ_1, ..., λ_M.

Since $P = \sum\limits_{n=0}^{M} \lambda_n X^n$, we have $P' = \sum\limits_{n=1}^{M} n\lambda_n X^{n-1}$ (by the definition of the derivative of a polynomial), so that $P'(x) = \sum\limits_{n=1}^{M} n\lambda_n x^{n-1}$.

On the other hand, $0 = P(x) = \sum\limits_{n=0}^{M} \lambda_n x^n$ (since $P = \sum\limits_{n=0}^{M} \lambda_n X^n$), so that $0b = \sum\limits_{n=0}^{M} \lambda_n x^n b$. In other words,

$0 = \sum\limits_{n=0}^{M} \lambda_n x^n b$. Hence,

$$h(a,0) = h\left(a, \sum_{n=0}^{M} \lambda_n x^n b\right) = \sum_{n=0}^{M} \lambda_n h(a, x^n b) \qquad \text{(since } h \text{ is } K\text{-bilinear)}$$

$$= \lambda_0 h\left(a, \underbrace{x^0}_{=1} b\right) + \sum_{n=1}^{M} \lambda_n \underbrace{h(a, x^n b)}_{\substack{=x^n\alpha + nx^{n-1}\beta \\ \text{(by (1))}}} = \lambda_0 \underbrace{h(a,b)}_{\substack{=\alpha = x^0\alpha \\ \text{(since } x^0=1 \text{ and} \\ \text{thus } x^0\alpha=\alpha)}} + \sum_{n=1}^{M} \underbrace{\lambda_n\left(x^n\alpha + nx^{n-1}\beta\right)}_{=\lambda_n x^n\alpha + \lambda_n n x^{n-1}\beta}$$

$$= \lambda_0 x^0 \alpha + \underbrace{\sum_{n=1}^{M}\left(\lambda_n x^n \alpha + \lambda_n n x^{n-1}\beta\right)}_{=\left(\sum\limits_{n=1}^{M}\lambda_n x^n\right)\alpha + \left(\sum\limits_{n=1}^{M}\lambda_n n x^{n-1}\right)\beta} = \underbrace{\lambda_0 x^0\alpha + \left(\sum_{n=1}^{M}\lambda_n x^n\right)\alpha}_{=\left(\lambda_0 x^0 + \sum\limits_{n=1}^{M}\lambda_n x^n\right)\alpha} + \left(\sum_{n=1}^{M}\underbrace{\lambda_n n}_{=n\lambda_n} x^{n-1}\right)\beta$$

$$= \underbrace{\left(\lambda_0 x^0 + \sum_{n=1}^{M}\lambda_n x^n\right)}_{=\sum\limits_{n=0}^{M}\lambda_n x^n = 0}\alpha + \underbrace{\left(\sum_{n=1}^{M} n\lambda_n x^{n-1}\right)}_{=P'(x)}\beta = 0\alpha + P'(x)\cdot\beta = P'(x)\cdot\beta.$$

Since $h(a,0) = 0$ (because h is K-bilinear), this becomes $0 = P'(x)\cdot\beta$. Since $P'(x) \neq 0$, this yields $0 = \beta$ (since L is a field). Now, (1) (applied to $n = 1$) yields

$$h\left(a, x^1 b\right) = \underbrace{x^1}_{=x}\alpha + 1x^{1-1}\underbrace{\beta}_{=0} = x\underbrace{\alpha}_{=h(a,b)} + \underbrace{1x^{1-1}0}_{=0} = xh(a,b).$$

Since $x^1 = x$, this simplifies to $h(a, xb) = xh(a,b)$. This proves Lemma 2.

Notice that $h(a, b+b') = h(a,b) + h(a,b')$ for all $a \in V$, $b \in W$ and $b' \in W$ (since h is K-bilinear). This, combined with Lemma 2, yields that the map h is L-linear in its second variable. Similarly, the map h is L-linear in its first variable. Hence, the map h is L-linear in each of its two variables, i. e., an L-bilinear map.

Remarks

1) As the above solution shows, the problem can be generalized. Namely, the problem will still be valid if we replace "Let $L\diagup K$ be a separable algebraic extension of fields" by "Let K and L be commutative rings with 1 such that L is a K-algebra" and add the assumption that "For every $x \in L$ and every $u \in U$, there exists a polynomial $P \in K[X]$ such that $P(x) = 0$ and such that (if $P'(x)u = 0$ then $u = 0$)". (This assumption is what replaces the assumption that $L\diagup K$ be separable. It is used in our proof of Lemma 2.)

2) While this assumption looks like a reasonable replacement for separability in the case of K and L not (necessarily) being fields, there exists a better replacement: the notion of separable algebras. I don't know whether the problem still holds if L is just required to be a separable commutative K-algebra. I have not tried proving or disproving this. If you succeed at either, please let me know!

3) We can actually use our above problem to prove a known fact about separable algebraic field extensions:

> **Proposition.** Let $L\diagup K$ be a separable algebraic extension of fields. Let U be a L-vector space. Let $D : L \to U$ be a derivation[3] such that $D(K) = 0$. Then, $D = 0$.

[3] A *derivation* from L to U means a homomorphism $\delta : L \to U$ of abelian groups (not a priori required to be K-linear or L-linear) which satisfies

$$\left(\delta(xy) = \delta(x)\cdot y + x\cdot\delta(y) \qquad \text{for all } x \in L \text{ and } y \in L\right).$$

Proof of Proposition. Define a map $h : L \times L \to U$ by

$$\left(h\left(a,b\right) = aD\left(b\right) - bD\left(a\right) \qquad \text{for all } a \in L \text{ and } b \in L\right).$$

Then, any $a \in L$, $b \in L$ and $b' \in L$ satisfy

$$\begin{aligned} h\left(a, b+b'\right) &= a \underbrace{D\left(b+b'\right)}_{\substack{=D(b)+D(b') \\ \text{(since } D \text{ is a derivation and thus} \\ \text{a homomorphism of abelian groups)}}} - \underbrace{\left(b+b'\right) D\left(a\right)}_{=bD(a)+b'D(a)} \qquad \left(\text{by the definition of } h\left(a,b+b'\right)\right) \\ &= a\left(D\left(b\right)+D\left(b'\right)\right) - \left(bD\left(a\right)+b'D\left(a\right)\right) = aD\left(b\right)+aD\left(b'\right)-bD\left(a\right)-b'D\left(a\right) \\ &= \underbrace{\left(aD\left(b\right)-bD\left(a\right)\right)}_{\substack{=h(a,b) \\ \text{(since } h(a,b) \text{ was defined as } aD(b)-bD(a))}} + \underbrace{\left(aD\left(b'\right)-b'D\left(a\right)\right)}_{\substack{=h(a,b') \\ \text{(since } h(a,b') \text{ was defined as } aD(b')-b'D(a))}} \\ &= h\left(a,b\right)+h\left(a,b'\right). \end{aligned} \tag{4}$$

Also, any $a \in L$, $b \in L$ and $x \in K$ satisfy

$$\begin{aligned} h\left(a, xb\right) &= a \underbrace{D\left(xb\right)}_{\substack{=D(x)\cdot b+x\cdot D(b) \\ \text{(since } D \text{ is a derivation)}}} - xbD\left(a\right) \qquad \left(\text{by the definition of } h\left(a,xb\right)\right) \\ &= a\left(\underbrace{D\left(x\right)}_{\substack{=0 \\ \text{(since } x\in K \text{ and thus } D(x)\in D(K)=0)}} \cdot b + x\cdot D\left(b\right)\right) - xbD\left(a\right) \\ &= a\left(\underbrace{0\cdot b}_{=0} + x\cdot D\left(b\right)\right) - xbD\left(a\right) = ax\cdot D\left(b\right) - xbD\left(a\right) \\ &= x\underbrace{\left(aD\left(b\right)-bD\left(a\right)\right)}_{=h(a,b)} = xh\left(a,b\right). \end{aligned} \tag{5}$$

The map h is K-linear in its second variable (since any $a \in L$, $b \in L$ and $b' \in L$ satisfy (4), and since any $a \in L$, $b \in L$ and $x \in K$ satisfy (5)), and K-linear in its first variable (for similar reasons). Hence, the map h is K-bilinear.

For every $x \in L$, $a \in L$ and $b \in L$, we have

$$\begin{aligned} h\left(xa, xb\right) &= xa \underbrace{D\left(xb\right)}_{\substack{=D(x)\cdot b+x\cdot D(b) \\ \text{(since } D \text{ is a derivation)}}} - xb \underbrace{D\left(xa\right)}_{\substack{=D(x)\cdot a+x\cdot D(a) \\ \text{(since } D \text{ is a derivation)}}} \qquad \left(\text{by the definition of } h\left(xa,xb\right)\right) \\ &= xa\left(D\left(x\right)\cdot b + x\cdot D\left(b\right)\right) - xb\left(D\left(x\right)\cdot a + x\cdot D\left(a\right)\right) \\ &= \underbrace{xaD\left(x\right)\cdot b}_{=xD(x)\cdot ab} + \underbrace{xax\cdot D\left(b\right)}_{=x^2aD(b)} - \underbrace{xbD\left(x\right)\cdot a}_{=xD(x)\cdot ab} - \underbrace{xbx\cdot D\left(a\right)}_{=x^2bD(a)} \\ &= xD\left(x\right)\cdot ab + x^2aD\left(b\right) - xD\left(x\right)\cdot ab - x^2bD\left(a\right) \\ &= x^2aD\left(b\right) - x^2bD\left(a\right) = x^2\underbrace{\left(aD\left(b\right)-bD\left(a\right)\right)}_{=h(a,b)} = x^2h\left(a,b\right). \end{aligned}$$

Hence, our problem (applied to $V = L$ and $W = L$) yields that h is L-bilinear. Thus, every $x \in L$ satisfies $h\left(x\cdot 1, 1\right) = x \cdot h\left(1,1\right)$. But since

$$\begin{aligned} h\left(\underbrace{x\cdot 1}_{=x}, 1\right) &= h\left(x,1\right) = x \underbrace{D\left(1\right)}_{\substack{=0 \\ \text{(since } 1\in K \text{ and thus } D(1)\in D(K)=0)}} - \underbrace{1D\left(x\right)}_{=D(x)} \qquad \left(\text{by the definition of } h\left(x,1\right)\right) \\ &= -D\left(x\right) \end{aligned}$$

and

$$\begin{aligned} h(1,1) &= 1D(1) - 1D(1) && \text{(by the definition of } h(1,1)\text{)} \\ &= 0, \end{aligned}$$

this rewrites as $-D(x) = x \cdot 0$. Thus, every $x \in L$ satisfies $D(x) = -x \cdot 0 = 0$. In other words, $D = 0$. Proposition 30 is thus proven.

4) The condition that L/K be separable cannot be removed from the problem (without a proper replacement). In fact, if we let p be any prime, and consider the algebraic field extension $K = \mathbb{F}_p(T^p) \subseteq \mathbb{F}_p(T) = L$ (the classical example of a purely inseparable field extension) and let $V = L$, $U = L$ and $W = L$, then we can define an $\mathbb{F}_p$-bilinear map

$$h : V \times W \to U, \qquad \left(T^a, T^b\right) \mapsto (a-b)\, T^{a+b};$$

this map is K-bilinear but not L-bilinear, although it satisfies (1). [4]

Note that this counterexample is not as weird as it looks like; in fact, the form $h : V \times W \to U$ constructed in this counterexample can also be characterized as the map $L \times L \to L$, $(u, v) \mapsto -u\dfrac{d}{dT}v + v\dfrac{d}{dT}u$, so that it (up to sign) is an example of the same construction that we made in the proof of Proposition 30.

Using this construction, we can show a partial converse of the problem: If L/K is a *finitely generated but nonseparable* field extension, then there exists a K-bilinear map $h : L \times L \to L$ which satisfies (1) (for $V = L$ and $W = L$) without being L-bilinear. I don't know what can be said about non-finitely generated field extensions.

[4] Note that this map h is the Lie bracket of the infinite-dimensional Witt algebra over $\mathbb{F}_p$.

O223. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(y + f(x)) = f(x)f(y) + f(f(x)) + f(y) - xy \tag{6}$$

for all $x, y \in R$.

Proposed by Preudtanan Sriwongleang, Ramkamhaeng University, Thailand

Solution by Ajat Adriansyah ,Universitas Indonesia

For $y = 0$ we have $f(x)f(0) + f(0) = 0$, either $f(0) = 0$ or $f(x) = -1$ for all x, but the latter is not possible by (1) thus $f(0) = 0$. Replace y with $f(y)$ on (1), this yields

$$f(f(y) + f(x)) = f(x)f(f(y)) + f(f(x)) + f(f(y)) - xf(y) \tag{7}$$

and then we change the role of x and y, that is $f(f(y) + f(x)) = f(y)f(f(x)) + f(f(y)) + f(f(x)) - yf(x)$ and subtract this to equation (2), we have for $x \neq 0$.

$$\frac{f(f(x)) + x}{x} = \frac{f(f(y)) + y}{y} = k \Rightarrow f(f(x)) = kf(x) - x$$

and this equation is also satisfied for $x = 0$. Now equation (1) becomes

$$f(y + f(x)) = f(x)f(y) + kf(x) - x + f(y) - xy \tag{8}$$

Let $b = f(-1)$, substituting $y = -1$ to equation (3) we have

$$f(f(x) - 1) = (b + k)f(x) + b \tag{9}$$

Now substitute $y \to f(y) - 1$ in equation (3) and using equation (4) we have

$$\begin{aligned} f(f(x) + f(y) - 1) &= ((k + b)f(y) + b)f(x) + kf(x) - x + (k + b)f(y) + b - xf(y) + x \\ &= (k + b)[f(x)f(y) + f(x) + f(y)] + b - xf(y) \end{aligned}$$

Next we change the role of x and y in the above equation, we will get $f(f(x)+f(y)-1) = (k+b)[f(x)f(y)+f(x) + f(y)] + b - yf(x)$, and subtract both equations to get for $x, y \neq 0$

$$yf(x) = xf(y) \Rightarrow \frac{f(x)}{x} = \frac{f(y)}{y} = c \Rightarrow f(x) = cx$$

which is also true for $x = y = 0$. Now substituting $f(x) = cx$ to equation (1), we easily get $c = \pm 1$. Thus $f(x) = -x$ and $f(x) = x$ are the only solutions.

O224. Let a, b, c be positive real numbers. Prove that

$$\frac{3(a^3+b^3+c^3)}{2(a+b+c)(a^2+b^2+c^2)} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - 1 \leq \frac{3(a^3+b^3+c^3)}{2(a+b+c)(ab+bc+ca)}.$$

Proposed by Cezar Lupu, University of Pittsburgh, USA and Duc Huu Pham, Ballajura, Australia

Solution by Matteo Elia, Università di Roma "Tor Vergata", Roma, Italy

We define $S := [2,1,0] = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$. Then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - 1 = \frac{a^3+b^3+c^3+abc}{S+2abc}.$$

Since $(a+b+c)(a^2+b^2+c^2) = a^3+b^3+c^3+S$, the first inequality becomes

$$3(a^3+b^3+c^3)(S+2abc) \leq 2(a^3+b^3+c^3+S)(a^3+b^3+c^3+abc)$$

that is

$$([3,0,0]-[2,1,0])([3,0,0]-\frac{2}{3}[1,1,1]) = (2(a^3+b^3+c^3)-S)(2(a^3+b^3+c^3)-4abc) \geq 0$$

which holds by Muirhead's inequality.

By noting that $(a+b+c)(ab+bc+ca) = S+3abc$, the second inequality becomes

$$2(a^3+b^3+c^3+abc)(S+3abc) \leq 3(a^3+b^3+c^3)(S+2abc)$$

that is

$$[5,1,0]+[4,2,0]+[3,2,1] = S(a^3+b^3+c^3) \geq 2Sabc + 6a^2b^2c^2 = 2[3,2,1]+[2,2,2]$$

which holds by Muirhead's inequality.

Also solved by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Solution by Arkady Alt , San Jose, California, USA; Nicu Zlota, Focsani, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ajat Adriansyah ,Universitas Indonesia; Marin Sandu and Mihai Sandu, Bucuresti, Romania.

O225. For any prime $p > 3$, prove that

$$p\sum_{j=0}^{p-1}\frac{(-3)^j}{2j+1} = \left(\frac{p}{3}\right) \pmod{p^2}$$

where $\left(\frac{p}{3}\right)$ is the Legendre symbol.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

We have that

$$(1 \pm i\sqrt{3})^p = 2^p e^{\pm i\pi p/3} = 2^p(\cos(\pi p/3) \pm i\sin(\pi p/3)) = 2^{p-1}\left(1 \pm i\left(\frac{p}{3}\right)\sqrt{3}\right).$$

On the other hand

$$\begin{aligned}(1 \pm i\sqrt{3})^p = \sum_{k=0}^{p}\binom{p}{k}(\pm i\sqrt{3})^k &\equiv 1 \pm (i\sqrt{3})^p - p\sum_{k=1}^{p-1}\frac{(\mp i\sqrt{3})^k}{k} \\ &\equiv 1 \pm i\sqrt{3}(-3)^{(p-1)/2} - S_0 \pm i\sqrt{3}S_1 \pmod{p^2}\end{aligned}$$

where

$$S_0 = p\sum_{j=1}^{(p-1)/2}\frac{(-3)^j}{2j} \qquad \text{and} \qquad S_1 = p\sum_{j=0}^{(p-3)/2}\frac{(-3)^j}{2j+1}.$$

Hence

$$S_0 \equiv 1 - 2^{p-1} \pmod{p^2} \qquad \text{and} \qquad S_1 \equiv 2^{p-1}\left(\frac{p}{3}\right) - (-3)^{(p-1)/2} \pmod{p^2}.$$

Finally

$$\begin{aligned}p\sum_{j=0}^{p-1}\frac{(-3)^j}{2j+1} &= p\sum_{j=0}^{(p-3)/2}\frac{(-3)^j}{2j+1} + (-3)^{(p-1)/2} + (-3)^{(p-1)/2}p\sum_{j=1}^{(p-1)/2}\frac{(-3)^j}{p+2j} \\ &\equiv S_1 + (-3)^{(p-1)/2} + (-3)^{(p-1)/2}S_0 \\ &\equiv (2^{p-1}-1)\left(\left(\frac{p}{3}\right) - (-3)^{(p-1)/2}\right) + \left(\frac{p}{3}\right) \equiv \left(\frac{p}{3}\right) \pmod{p^2}\end{aligned}$$

because p divides $(2^{p-1}-1)$ and $\left((\frac{p}{3}) - (-3)^{(p-1)/2}\right)$.

O226. Let $n > 1$ be an odd integer and let $A_1 \ldots A_n$ be a regular polygon. Find the number of triangles $A_iA_jA_k$ up to a permutation that contain the center of the n-gon.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by G.R.A.20 Problem Solving Group, Roma, Italy

We enumerate first the triangles $A_1A_jA_k$ with $1 < j < k$ which are located on the opposite side of the diagonal $A_1A_{\frac{n+1}{2}}$ with respect to the center of the n-gon.

We choose $k \in \{3, 4, \ldots, \frac{n+1}{2}\}$ and then we take $j \in \{2, 3, \ldots, k-1\}$. Hence their number is

$$\sum_{k=3}^{\frac{n+1}{2}} (k-2) = \frac{(n-3)(n-1)}{8}.$$

By repeating the same argument for all the other vertices, we obtain the total number of triangles that does not contain the center of the n-gon:

$$\frac{n(n-3)(n-1)}{8}.$$

Therefore the number of triangles that contain the center of the n-gon is

$$t_n = \binom{n}{3} - \frac{n(n-3)(n-1)}{8} = \frac{n(n-1)}{2}\left(\frac{n-2}{3} - \frac{n-3}{4}\right) = \frac{1}{4}\binom{n+1}{3}.$$

The first terms of the sequence t_n are $1, 5, 14, 30, 55, 91$ for $n = 3, 5, 7, 9, 11, 13$.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Consider the circumcircle of the n-gon. Since n is odd, no right-angled triangle may be formed by any three of the vertices of the n-gon (no two vertices of the n-gon may be diametrally opposite). It follows that $\binom{n}{3}$ is the sum of all acute-angled triangles and all obtuse-angled triangles whose vertices are also vertices of the n-gon. Note also that the circumcircle of all these triangles is clearly also the center of the n-gon, and that it will be inside a triangle iff it is acute, and outside the triangle iff it is obtuse.

Let us find the number of obtuse triangles, and denote $m = \frac{n-1}{2}$. Clearly, if $A_iA_jA_k$ is obtuse, but not at A_i, then the diameter of its circumcircle through A_i leaves A_j, A_k on the same side. It follows that, using the $m = \frac{n-1}{2}$ vertices of the n-gon at each side of this diameter, we may form exactly $2\binom{m}{2} = m(m-1)$ obtuse triangles which are not obtuse at A_i. Since we may do this for each one of the n vertices of the n-gon, and thus each triangle would be counted twice (once for each one of its acute-angled vertices), we find that the total number of obtuse triangles whose vertices are vertices of the n-gon are $\frac{mn(m-1)}{2} = \frac{n(n-1)(n-3)}{8}$. The number of acute triangles is then

$$\binom{n}{3} - \frac{n(n-1)(n-3)}{8} = \frac{(n+1)n(n-1)}{24}.$$

This is also the number of triangles $A_iA_jA_k$ that contain the center of the n-gon. Note that this number is in fact an integer, since as n is odd, $n^2 \equiv 1 \pmod 8$, or 8 divides $n^2 - 1$, and since $n-1, n, n+1$ are consecutive integers, exactly one of them is a multiple of 3.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

O227. Decide whether there is a polynomial $f \in \mathbb{Q}[X]$ such that $f(\mathbb{Z}) \subset \mathbb{Z}$ and

a) there is no $g \in \mathbb{Z}[X]$ such that $f(\mathbb{Z}) = g(\mathbb{Z})$.

b) there is $h \in \mathbb{Z}[X, Y]$ such that $f(\mathbb{Z}) = h(\mathbb{Z} \times \mathbb{Z})$.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

No solutions have been yet received.

O228. Let Γ be an arbitrary circle in the plane of a given triangle ABC. Let $\mathcal{K}'_A$ and $\mathcal{K}''_A$ be the circles through B and C which are tangent to Γ at X' and X'', respectively. Similarly define $\mathcal{K}'_B$, $\mathcal{K}''_B$, $\mathcal{K}'_C$, $\mathcal{K}''_C$ and their tangency points with Γ, Y', Y'', Z', and Z'', respectively. Prove that the circumcircles of triangles $AX'X''$, $BY'Y''$, and $CZ'Z''$ are coaxal.

Proposed by Cosmin Pohoata, Princeton University, USA and Paul Yiu, Florida Atlantic University, USA

No solutions have been yet received.