Mathematical Excalibur

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Olympiad Corner

The Fifth Hong Kong (China) Mathematical Olympiad was held on December 21, 2002. The problems are as follow.

Problem 1. Two circles intersect at points A and B. Through the point B a straight line is drawn, intersecting the first circle at K and the second circle at M. A line parallel to AM is tangent to the first circle at Q. The line AQ intersects the second circle again at R.

(a) Prove that the tangent to the second circle at R is parallel to AK.

(b) Prove that these two tangents are concurrent with KM.

Problem 2. Let $n \geq 3$ be an integer. In a conference there are n mathematicians. Every pair of mathematicians communicate in one of the n official languages of the conference. For any three different official languages, there exist three mathematicians who communicate with each other in these three languages. Determine all n for which this is possible. Justify your claim.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2003***.

For individual subscription for the next five issues for the 02-03 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Functional Equations

Kin Y. Li

A *functional equation* is an equation whose variables are ranging over functions. Hence, we are seeking all possible functions satisfying the equation. We will let $\mathbb{Z}$ denote the set of all integers, $\mathbb{Z}^+$ or $\mathbb{N}$ denote the positive integers, $\mathbb{N}_0$ denote the nonnegative integers, $\mathbb{Q}$ denote the rational numbers, $\mathbb{R}$ denote the real numbers, $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{C}$ denote the complex numbers.

In simple cases, a functional equation can be solved by introducing some substitutions to yield more information or additional equations.

Example 1. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1-x) = 2x - x^4$$

for all $x \in \mathbb{R}$.

Solution. Replacing x by $1 - x$, we have

$$(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4.$$

Since $f(1-x) = 2x - x^4 - x^2 f(x)$ by the given equation, substituting this into the last equation and solving for $f(x)$, we get $f(x) = 1 - x^2$.

Check: For $f(x) = 1 - x^2$,

$$x^2 f(x) + f(1-x) = x^2(1-x^2) + (1-(1-x)^2) = 2x - x^4.$$

For certain types of functional equations, a standard approach to solving the problem is to determine some special values (such as $f(0)$ or $f(1)$), then inductively determine $f(n)$ for $n \in \mathbb{N}_0$, follow by the values $f(1/n)$ and use density to find $f(x)$ for all $x \in \mathbb{R}$. The following are examples of such approach.

Example 2. Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that the *Cauchy equation*

$$f(x+y) = f(x) + f(y)$$

holds for all $x, y \in \mathbb{Q}$.

Solution. *Step 1* Taking $x = 0 = y$, we get $f(0) = f(0) + f(0) + f(0)$, which implies $f(0) = 0$.

Step 2 We will prove $f(kx) = k f(x)$ for $k \in \mathbb{N}$, $x \in \mathbb{Q}$ by induction. This is true for $k = 1$. Assume this is true for k. Taking $y = kx$, we get

$$f((k+1)x) = f(x+kx) = f(x) + f(kx) = f(x) + k f(x) = (k+1) f(x).$$

Step 3 Taking $y = -x$, we get

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x),$$

which implies $f(-x) = -f(x)$. So

$$f(-kx) = -f(kx) = -k f(x) \text{ for } k \in \mathbb{N}.$$

Therefore, $f(kx) = k f(x)$ for $k \in \mathbb{Z}$, $x \in \mathbb{Q}$.

Step 4 Taking $x = 1/k$, we get

$$f(1) = f(k(1/k)) = k f(1/k),$$

which implies $f(1/k) = (1/k) f(1)$.

Step 5 For $m \in \mathbb{Z}$, $n \in \mathbb{N}$,

$$f(m/n) = m f(1/n) = (m/n) f(1).$$

Therefore, $f(x) = cx$ with $c = f(1)$.

Check: For $f(x) = cx$ with $c \in \mathbb{Q}$,

$$f(x+y) = c(x+y) = cx + cy = f(x) + f(y).$$

In dealing with functions on $\mathbb{R}$, after finding the function on $\mathbb{Q}$, we can often finish the problem by using the following fact.

Density of Rational Numbers *For every real number x, there are rational numbers $p_1, p_2, p_3, \ldots$ increase to x and there are rational numbers $q_1, q_2, q_3, \ldots$ decrease to x.*

This can be easily seen from the decimal representation of real numbers. For example, the number $\pi = 3.1415\ldots$ is the limits of 3, 31/10, 314/100, 3141/1000, 31415/10000, … and also 4, 32/10, 315/100, 3142/1000, 31416/10000, ….

(In passing, we remark that there is a similar fact with rational numbers replaced by irrational numbers.)

Example 3. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in \mathbb{R}$ and $f(x)\ge 0$ for $x\ge 0$.

Solution. Step 1 By example 2, we have $f(x)=xf(1)$ for $x\in\mathbb{Q}$.

Step 2 If $x\ge y$, then $x-y\ge 0$. So

$$f(x)=f((x-y)+y)=f(x-y)+f(y)\ge f(y).$$

Hence, f is increasing.

Step 3 If $x\in\mathbb{R}$, then by the density of rational numbers, there are rational p_n, q_n such that $p_n\le x\le q_n$, the p_n's increase to x and the q_n's decrease to x. So by step 2, $p_n f(1)=f(p_n)\le f(x)\le f(q_n)=q_n f(1)$. Taking limits, the sandwich theorem gives $f(x)=xf(1)$ for all x. Therefore, $f(x)=cx$ with $c\ge 0$. The checking is as in example 2.

Remarks. (1) In example 3, if we replace the condition that "$f(x)\ge 0$ for $x\ge 0$" by "f is monotone", then the answer is essentially the same, namely $f(x)=cx$ with $c=f(1)$. Also if the condition that "$f(x)\ge 0$ for $x\ge 0$" is replaced by "f is continuous at 0", then steps 2 and 3 in example 3 are not necessary. We can take rational p_n's increase to x and take limit of $p_n f(1)=f(p_n)=f(p_n-x)+f(x)$ to get $xf(1)=f(x)$ since p_n-x increases to 0.

(2) The Cauchy equation $f(x+y)=f(x)+f(y)$ for all $x, y\in\mathbb{R}$ has noncontinuous solutions (in particular, solutions not of the form $f(x)=cx$). This requires the concept of a *Hamel basis* of the vector space $\mathbb{R}$ over $\mathbb{Q}$ from linear algebra.

The following are some useful facts related to the Cauchy equation.

Fact 1. *Let $A=\mathbb{R}$, $[0,\infty)$ or $(0,\infty)$. If $f:A\to\mathbb{R}$ satisfies $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$ for all $x, y\in A$, then either $f(x)=0$ for all $x\in A$ or $f(x)=x$ for all $x\in A$.*

Proof. By example 2, we have $f(x)=f(1)\,x$ for all $x\in\mathbb{Q}$. If $f(1)=0$, then $f(x)=f(x\cdot 1)=f(x)f(1)=0$ for all $x\in A$.

Otherwise, we have $f(1)\ne 0$. Since $f(1)=f(1)f(1)$, we get $f(1)=1$. Then $f(x)=x$ for all $x\in A\cap\mathbb{Q}$.

If $y\ge 0$, then $f(y)=f(y^{1/2})^2\ge 0$ and

$$f(x+y)=f(x)+f(y)\ge f(x),$$

which implies f is increasing. Now for any $x\in A\setminus\mathbb{Q}$, by the density of rational numbers, there are $p_n, q_n\in\mathbb{Q}$ such that $p_n<x<q_n$, the p_n's increase to x and the q_n's decrease to x. As f is increasing, we have $p_n=f(p_n)\le f(x)\le f(q_n)=q_n$. Taking limits, the sandwich theorem gives $f(x)=x$ for all $x\in A$.

Fact 2. *If a function $f:(0,\infty)\to\mathbb{R}$ satisfies $f(xy)=f(x)f(y)$ for all $x, y>0$ and f is monotone, then either $f(x)=0$ for all $x>0$ or there exists c such that $f(x)=x^c$ for all $x>0$.*

Proof. For $x>0$, $f(x)=f(x^{1/2})^2\ge 0$. Also $f(1)=f(1)f(1)$ implies $f(1)=0$ or 1. If $f(1)=0$, then $f(x)=f(x)f(1)=0$ for all $x>0$. If $f(1)=1$, then $f(x)>0$ for all $x>0$ (since $f(x)=0$ implies $f(1)=f(x(1/x))=f(x)f(1/x)=0$, which would lead to a contradiction).

Define $g:\mathbb{R}\to\mathbb{R}$ by $g(w)=\ln f(e^w)$. Then

$$\begin{aligned}g(x+y)&=\ln f(e^{x+y})=\ln f(e^x e^y)\\&=\ln f(e^x)f(e^y)\\&=\ln f(e^x)+\ln f(e^y)\\&=g(x)+g(y).\end{aligned}$$

Since f is monotone, it follows that g is also monotone. Then $g(w)=cw$ for all w. Therefore, $f(x)=x^c$ for all $x>0$.

As an application of these facts, we look at the following example.

Example 4. (*2002 IMO*) Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that

$$\begin{aligned}&(f(x)+f(z))(f(y)+f(t))\\=\;&f(xy-zt)+f(xt+yz)\end{aligned}$$

for all x, y, z, t in $\mathbb{R}$.

Solution. (*Due to Yu Hok Pun, 2002 Hong Kong IMO team member, gold medalist*) Suppose $f(x)=c$ for all x. Then the equation implies $4c^2=2c$. So c can only be 0 or 1/2. Reversing steps, we can also check $f(x)=0$ for all x or $f(x)=1/2$ for all x are solutions.

Suppose the equation is satisfied by a nonconstant function f. Setting $x=0$ and $z=0$, we get $2f(0)\,(f(y)+f(t))=2f(0)$, which implies $f(0)=0$ or $f(y)+f(t)=1$ for all y, t. In the latter case, setting $y=t$, we get the constant function $f(y)=1/2$ for all y. Hence we may assume $f(0)=0$.

Setting $y=1$, $z=0$, $t=0$, we get $f(x)f(1)=f(x)$. Since $f(x)$ is not the zero function, $f(1)=1$. Setting $z=0$, $t=0$, we get $f(x)f(y)=f(xy)$ for all x,y. In particular, $f(w)=f(w^{1/2})^2\ge 0$ for $w>0$.

Setting $x=0$, $y=1$ and $t=1$, we have $2f(1)f(z)=f(-z)+f(z)$, which implies $f(z)=f(-z)$ for all z. So f is even.

Define the function $g:(0,\infty)\to\mathbb{R}$ by $g(w)=f(w^{1/2})\ge 0$. Then for all $x,y>0$,

$$\begin{aligned}g(xy)&=f((xy)^{1/2})=f(x^{1/2}y^{1/2})\\&=f(x^{1/2})f(y^{1/2})=g(x)\,g(y).\end{aligned}$$

Next f is even implies $g(x^2)=f(x)$ for all x. Setting $z=y$, $t=x$ in the given equation, we get

$$\begin{aligned}(g(x^2)+g(y^2))^2&=g((x^2+y^2)^2)\\&=g(x^2+y^2)^2\end{aligned}$$

for all x,y. Taking square roots and letting $a=x^2$, $b=y^2$, we get $g(a)+g(b)=g(a+b)$ for all $a, b>0$.

By fact 1, we have $g(w)=w$ for all $w>0$. Since $f(0)=0$ and f is even, it follows $f(x)=g(x^2)=x^2$ for all x.

Check: If $f(x)=x^2$, then the equation reduces to

$$(x^2+z^2)(y^2+t^2)=(xy-zt)^2+(xt+yz)^2,$$

which is a well known identity and can easily be checked by expansion or seen from $|p|^2|q|^2=|pq|^2$, where $p=x+iz$, $q=y+it\in\mathbb{C}$.

The concept of fixed point of a function is another useful idea in solving some functional equations. Its definition is very simple. We say w is a *fixed point* of a function f if and only if w is in the domain of f and $f(w)=w$. Having information on the fixed points of functions often help to solve certain types of functional equations as the following examples will show.

Example 5. (*1983 IMO*) Determine all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(xf(y))=yf(x)$ for all $x, y\in\mathbb{R}^+$ and as $x\to+\infty$, $f(x)\to 0$.

Solution. Step 1 Taking $x=1=y$, we get $f(f(1))=f(1)$. Taking $x=1$ and $y=f(1)$, we get $f(f(f(1)))=f(1)^2$. Then $f(1)^2=f(f(f(1)))=f(f(1))=f(1)$, which implies $f(1)=1$. So 1 is a fixed point of f.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***February 28, 2003***.

Problem 171. (*Proposed by Ha Duy Hung, Hanoi University of Education, Hanoi City, Vietnam*) Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\left[\frac{c}{ab}\right]-\left[\frac{c}{a}\right]\left[\frac{c}{b}\right]\le c\min\left\{\frac{1}{a},\frac{1}{b}\right\}.$$

Problem 172. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all positive integers such that they are equal to the square of the sum of their digits in base 10 representation.

Problem 173. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 3/2 times any other group.

Problem 174. Let M be a point inside acute triangle ABC. Let A', B', C' be the mirror images of M with respect to BC, CA, AB, respectively. Determine (with proof) all points M such that A, B, C, A', B', C' are concyclic.

Problem 175. A regular polygon with n sides is divided into n isosceles triangles by segments joining its center to the vertices. Initially, $n+1$ frogs are placed inside the triangles. At every second, there are two frogs in some common triangle jumping into the interior of the two neighboring triangles (one frog into each neighbor). Prove that after some time, at every second, there are at least $[(n+1)/2]$ triangles, each containing at least one frog.

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Solutions

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In the last issue, problems 166, 167 and 169 were stated incorrectly. They are revised as problems 171, 172, 173, respectively. As the problems became easy due to the mistakes, we received many solutions. Regretfully we do not have the space to print the names and affiliations of all solvers. We would like to apologize for this.

Problem 166. Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$c\,[a/b]-[c/a]\,[c/b]\ \le\ c\min\{1/a, 1/b\}$.

Solution. Over 30 solvers disproved the inequality by providing different counter-examples, such as $(a, b, c) = (3, 2, 1)$.

Problem 167. Find all positive integers such that they are equal to the sum of their digits in base 10 representation.

Solution. Over 30 solvers sent in solutions similar to the following. For a positive integer N with digits $a_n, \ldots, a_0$ (from left to right), we have

$$\begin{aligned} N &= a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_0 \\ &\ge a_n + a_{n-1} + \cdots + a_0 \end{aligned}$$

because $10^k > 1$ for $k > 0$. So equality holds if and only if $a_n = a_{n-1} = \cdots = a_1 = 0$. Hence, $N = 1, 2, \ldots, 9$ are the only solutions.

Problem 168. Let AB and CD be nonintersecting chords of a circle and let K be a point on CD. Construct (with straightedge and compass) a point P on the circle such that K is the midpoint of the part of segment CD lying inside triangle ABP. (*Source: 1997 Hungarian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7)

Draw the midpoint M of AB. If $AB \parallel CD$, then draw ray MK to intersect the circle at P. Let AP, BP intersect CD at Q,R, respectively. Since $AB \parallel QR$, $\Delta ABP \sim \Delta QRP$. Then M being the midpoint of AB will imply K is the midpoint of QR.

If AB intersects CD at E, then draw the circumcircle of EMK meeting the original circle at S and S'. Draw the circumcircle of BES meeting CD at R. Draw the circumcircle of AES meeting CD at Q. Let AQ, BR meet at P. Since $\angle PBS = \angle RBS = \angle RES = \angle QES = \angle QAS = \angle PAS$, P is on the original circle.

Next, $\angle SMB = \angle SME = \angle SKE = \angle SKR$ and $\angle SBM = 180^\circ - \angle SBE = 180^\circ - \angle SRE = \angle SRK$ imply $\Delta SMB \sim \Delta SKR$ and $MB/KR = BS/RS$. Replacing M by A and K by Q, similarly $\Delta SAB \sim \Delta SQR$ and $AB/QR = BS/RS$. Since $AB = 2MB$, we get $QR = 2KR$. So K is the midpoint of QR.

Problem 169. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 11/2 times any other group.

Solution. Almost all solvers used the following argument. Let m and M be the weights of the lightest and heaviest apple(s). Then $3m \ge M$. If the problem is false, then there are two groups A and B with weights w_A and w_B such that $(11/2)\,w_B < w_A$. Since $4m \le w_B$ and $w_A \le 4M$, we get $(11/2)4m < 4M$ implying $3m \le (11/2)m < M$, a contradiction.

Problem 170. (*Proposed by Abderrahim Ouardini, Nice, France*) For any (nondegenerate) triangle with sides a, b, c, let $\sum' h(a, b, c)$ denote the sum $h(a, b, c) + h(b, c, a) + h(c, a, b)$. Let $f(a, b, c) = \sum' \left(a/(b+c-a)\right)^2$ and $g(a, b, c) = \sum' j(a, b, c)$, where $j(a,b,c) = (b+c-a)/\sqrt{(c+a-b)(a+b-c)}$. Show that $f(a, b, c) \ge \max\{3, g(a, b, c)\}$ and determine when equality occurs. (Here $\max\{x,y\}$ denotes the maximum of x and y.)

Solution. **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 6), **D. Kipp JOHNSON** (Valley Catholic High School, Beaverton, Oregon, USA), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **TAM Choi Nang Julian** (SKH Lam Kau Mow Secondary School) and **WONG Wing Hong** (La Salle College, Form 5).

Let $x = b + c - a$, $y = c + a - b$ and $z = a + b - c$. Then $a = (y+z)/2$, $b = (z+x)/2$ and $c = (x+y)/2$.

Substituting these and using the *AM-GM* inequality, the rearrangement inequality and the *AM-GM* inequality again, we find

$$\begin{aligned} &f(a,b,c) \\ &= \left(\frac{y+z}{2x}\right)^2 + \left(\frac{z+x}{2y}\right)^2 + \left(\frac{x+y}{2z}\right)^2 \\ &\ge \left(\frac{\sqrt{yz}}{x}\right)^2 + \left(\frac{\sqrt{zx}}{y}\right)^2 + \left(\frac{\sqrt{xy}}{z}\right)^2 \end{aligned}$$

$$\geq \frac{\sqrt{yz}\sqrt{zx}}{xy}+\frac{\sqrt{zx}\sqrt{xy}}{yz}+\frac{\sqrt{xy}\sqrt{yz}}{zx}$$

$$=\frac{x}{\sqrt{yz}}+\frac{y}{\sqrt{zx}}+\frac{z}{\sqrt{xy}}=g(a,b,c)$$

$$\geq 3\sqrt[3]{\frac{xyz}{\sqrt{yz}\sqrt{zx}\sqrt{xy}}}=3.$$

So $f(a,b,c) \geq g(a,b,c) = \max\{3, g(a,b,c)\}$ with equality if and only if $x = y = z$, which is the same as $a = b = c$.

Olympiad Corner

(continued from page 1)

Problem 3. If $a \geq b \geq c \geq 0$ and $a + b + c = 3$, then prove that $ab^2 + bc^2 + ca^2 \leq 27/8$ and determine the equality case(s).

Problem 4. Let p be an odd prime such that $p \equiv 1 \pmod 4$. Evaluate (with reason)

$$\sum_{k=1}^{\frac{p-1}{2}}\left\{\frac{k^2}{p}\right\},$$

where $\{x\} = x - [x]$, $[x]$ being the greatest integer not exceeding x.

Functional Equations

(continued from page 2)

<u>*Step 2*</u> Taking $y = x$, we get $f(xf(x)) = xf(x)$. So $w = xf(x)$ is a fixed point of f for every $x \in \mathbb{R}^+$.

<u>*Step 3*</u> Suppose f has a fixed point $x > 1$. By step 2, $xf(x) = x^2$ is also a fixed point, $x^2 f(x^2) = x^4$ is also a fixed point and so on. So the x^m's are fixed points for every m that is a power of 2. Since $x > 1$, for m ranging over the powers of 2, we have $x^m \to \infty$, but $f(x^m) = x^m \to \infty$, not to 0. This contradicts the given property. Hence, f cannot have any fixed point $x > 1$.

<u>*Step 4*</u> Suppose f has a fixed point x in the interval (0,1). Then

$$1 = f((1/x)\,x) = f((1/x)f(x)) = xf(1/x),$$

which implies $f(1/x) = 1/x$. This will lead to f having a fixed point $1/x > 1$, contradicting step 3. Hence, f cannot have any fixed point x in (0,1).

<u>*Step 5*</u> Steps 1, 3, 4 showed the only fixed point of f is 1. By step 2, we get $xf(x) = 1$ for all $x \in \mathbb{R}^+$. Therefore, $f(x) = 1/x$ for all $x \in \mathbb{R}^+$.

<u>*Check:*</u> For $f(x) = 1/x$, $f(xf(y)) = f(x/y) = y/x = yf(x)$. As $x \to \infty$, $f(x) = 1/x \to 0$.

Example 6. (1996 IMO) Find all functions $f: \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all $m, n \in \mathbb{N}_0$.

Solution. <u>*Step 1*</u> Taking $m = 0 = n$, we get $f(f(0)) = f(f(0)) + f(0)$, which implies $f(0) = 0$. Taking $m = 0$, we get $f(f(n)) = f(n)$, i.e. $f(n)$ is a fixed point of f for every $n \in \mathbb{N}_0$. Also the equation becomes

$$f(m + f(n)) = f(m) + f(n).$$

<u>*Step 2*</u> If w is a fixed point of f, then we will show kw is a fixed point of f for all $k \in \mathbb{N}_0$. The cases $k = 0, 1$ are known. If kw is a fixed point, then $f((k+1)w) = f(kw + w) = f(kw) + f(w) = kw + w = (k+1)w$ and so $(k+1)w$ is also a fixed point.

<u>*Step 3*</u> If 0 is the only fixed point of f, then $f(n) = 0$ for all $n \in \mathbb{N}_0$ by step 1. Obviously, the zero function is a solution.

Otherwise, f has a least fixed point $w > 0$. We will show the only fixed points are kw, $k \in \mathbb{N}_0$. Suppose x is a fixed point. By the division algorithm, $x = kw + r$, where $0 \leq r < w$. We have

$$\begin{aligned} x &= f(x) = f(r + kw) = f(r + f(kw)) \\ &= f(r) + f(kw) = f(r) + kw. \end{aligned}$$

So $f(r) = x - kw = r$. Since w is the least positive fixed point, $r = 0$ and $x = kw$.

Since $f(n)$ is a fixed point for all $n \in \mathbb{N}_0$ by step 1, $f(n) = c_n w$ for some $c_n \in \mathbb{N}_0$. We have $c_0 = 0$.

<u>*Step 4*</u> For $n \in \mathbb{N}_0$, by the division algorithm, $n = kw + r$, $0 \leq r < w$. We have

$$\begin{aligned} f(n) &= f(r + kw) = f(r + f(kw)) \\ &= f(r) + f(kw) = c_r w + kw \\ &= (c_r + k)\,w = (c_r + [n/w])\,w. \end{aligned}$$

<u>*Check:*</u> For each $w > 0$, let $c_0 = 0$ and let $c_1, \ldots, c_{w-1} \in \mathbb{N}_0$ be arbitrary. The function $f(n) = (c_r + [n/w])w$, where r is the remainder of n divided by w, (and the zero function) are all the solutions. Write $m = kw + r$ and $n = lw + s$ with $0 \leq r, s < w$. Then

$$\begin{aligned} f(m + f(n)) &= f(r + kw + (c_s + l)\,w) \\ &= c_r w + kw + c_s w + lw \\ &= f(f(m)) + f(n). \end{aligned}$$

Other than the fixed point concept, in solving functional equations, the injectivity and surjectivity of the functions also provide crucial informations.

Example 7. (*1987 IMO*) Prove that there is no function $f: \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(f(n)) = n + 1987$.

Solution. Suppose there is such a function f. Then f is injective because $f(a) = f(b)$ implies

$$a = f(f(a)) - 1987 = f(f(b)) - 1987 = b.$$

Suppose $f(n)$ misses exactly k distinct values $c_1, \ldots, c_k$ in $\mathbb{N}_0$, i.e. $f(n) \neq c_1, \ldots, c_k$ for all $n \in \mathbb{N}_0$. Then $f(f(n))$ misses the $2k$ distinct values $c_1, \ldots, c_k$ and $f(c_1), \ldots, f(c_k)$ in $\mathbb{N}_0$. (The $f(c_j)$'s are distinct because f is injective.) Now if $w \neq c_1, \ldots, c_k, f(c_1), \ldots, f(c_k)$, then there is $m \in \mathbb{N}_0$ such that $f(m) = w$. Since $w \neq f(c_j)$, $m \neq c_j$, so there is $n \in \mathbb{N}_0$ such that $f(n) = m$, then $f(f(n)) = w$. This shows $f(f(n))$ misses only the $2k$ values $c_1, \ldots, c_k, f(c_1), \ldots, f(c_k)$ and no others. Since $n + 1987$ misses the 1987 values 0, 1, …, 1986 and $2k \neq 1987$, this is a contradiction.

Example 8. (*1999 IMO*) Determine all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

Solution. Let $c = f(0)$. Setting $x = y = 0$, we get $f(-c) = f(c) + c - 1$. So $c \neq 0$. Let A be the range of f, then for $x = f(y) \in A$, we get $c = f(0) = f(x) + x^2 + f(x) - 1$. Solving for $f(x)$, this gives $f(x) = (c + 1 - x^2)/2$.

Next, if we set $y = 0$, we get

$$\begin{aligned} &\{f(x - c) - f(x) : x \in \mathbb{R}\} \\ &= \{cx + f(c) - 1 : x \in \mathbb{R}\} = \mathbb{R} \end{aligned}$$

because $c \neq 0$. Then $A - A = \{y_1 - y_2 : y_1, y_2 \in A\} = \mathbb{R}$.

Now for an arbitrary $x \in \mathbb{R}$, let $y_1, y_2 \in A$ be such that $y_1 - y_2 = x$. Then

$$\begin{aligned} f(x) &= f(y_1 - y_2) = f(y_2) + y_1 y_2 + f(y_1) - 1 \\ &= (c+1-y_2^2)/2 + y_1 y_2 + (c+1-y_1^2)/2 - 1 \\ &= c - (y_1 - y_2)^2/2 = c - x^2/2. \end{aligned}$$

However, for $x \in A$, $f(x) = (c + 1 - x^2)/2$. So $c = 1$. Therefore, $f(x) = 1 - x^2/2$ for all $x \in \mathbb{R}$.

<u>*Check:*</u> For $f(x) = 1 - x^2/2$, both sides equal $1/2 + y^2/2 - y^4/8 + x - xy^2/2 - x^2/2$.

Mathematical Excalibur

Volume 8, Number 2 *April 2003 – May 2003*

Olympiad Corner

The Final Round of the 51st Czech and Slovak Mathematical Olympiad was held on April 7-10, 2002. Here are the problems.

Problem 1. Solve the system

$$(4x)_5 + 7y = 14,$$
$$(2y)_5 - (3x)_7 = 74,$$

in the domain of the integers, where $(n)_k$ stands for the multiple of the number k closest to the number n.

Problem 2. Consider an arbitrary equilateral triangle KLM, whose vertices K, L and M lie on the sides AB, BC and CD, respectively, of a given square $ABCD$. Find the locus of the midpoints of the sides KL of all such triangles KLM.

Problem 3. Show that a given natural number A is the square of a natural number if and only if for any natural number n, at least one of the differences

$$(A+1)^2 - A,\ (A+2)^2 - A,$$
$$(A+3)^2 - A,\ \ldots,\ (A+n)^2 - A$$

is divisible by n.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 26, 2003***.

For individual subscription for the next five issues for the 02-03 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Countability

Kin Y. Li

Consider the following two questions:

(1) Is there a nonconstant polynomial with integer coefficients which has every prime number as a root?
(2) Is every real number a root of some nonconstant polynomial with integer coefficients?

The first question can be solved easily. Since the set of roots of a nonconstant polynomial is finite and the set of prime numbers is infinite, the roots cannot contain all the primes. So the first question has a negative answer.

However, for the second question, both the set of real numbers and the set of roots of nonconstant polynomials with integer coefficients are infinite. So we cannot answer this question as quickly as the first one.

In number theory, a number is said to be *algebraic* if it is a root of a nonconstant polynomial with integer coefficients, otherwise it is said to be *transcendental*. So the second question asks if every real number is algebraic.

Let's think about the second question. For every rational number a/b, it is clearly the root of the polynomial $P(x) = bx - a$. How about irrational numbers? For numbers of the form $\sqrt[n]{a/b}$, it is a root of the polynomial $P(x) = b\,x^n - a$. To some young readers, at this point they may think, perhaps the second question has a positive answer. We should do more checking before coming to any conclusion. How about π and e? Well, they are hard to check. Are there any other irrational number we can check?

Recall $\cos(3\theta)=4\cos^3\theta-3\cos\theta$. So setting $\theta=20^\circ$, we get $1/2 = 4\cos^3 20^\circ - 3\cos 20^\circ$. It follows that $\cos 20^\circ$ is a root of the polynomial $P(x) = 8x^3 - 6x - 1$. With this, we seem to have one more piece of evidence to think the second question has a positive answer.

So it is somewhat surprising to learn that the second question turns out to have a negative answer. In fact, it is known that π and e are not roots of nonconstant polynomials with integer coefficients, i.e. they are transcendental. Historically, the second question was answered before knowing π and e were transcendental. In 1844, Joseph Liouville proved for the first time that transcendental numbers exist, using continued fractions. In 1873, Charles Hermite showed e was transcendental. In 1882, Ferdinand von Lindemann generalized Hermite's argument to show π was also transcendental. Nowadays we know almost all real numbers are transcendental. This was proved by Georg Cantor in 1874. We would like to present Cantor's countability theory used to answer the question as it can be applied to many similar questions.

Let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{Z}$ the set of all integers, $\mathbb{Q}$ the set of all rational numbers and $\mathbb{R}$ the set of all real numbers.

Recall a *bijection* is a function f: $A\to B$ such that for every b in B, there is exactly one a in A satisfying $f(a) = b$. Thus, f provides a way to *correspond* the elements of A with those of B in a *one-to-one* manner.

We say a set S is *countable* if and only if S is a finite set or there exists a bijection $f: \mathbb{N}\to S$. For an infinite set, since such a bijection is a one-to-one correspondence between the positive integers and the elements of S, we have

$$1\leftrightarrow s_1,\ 2\leftrightarrow s_2,\ 3\leftrightarrow s_3,\ 4\leftrightarrow s_4,\ \ldots$$

and so the elements of S can be listed *orderly* as $s_1, s_2, s_3, \ldots$ without repetition or omission. Conversely, any such list of the elements of a set is equivalent to showing the set is countable since assigning $f(1) = s_1$, $f(2) = s_2$, $f(3) = s_3$, … readily provide a bijection.

Certainly, $\mathbb{N}$ is countable as the identity function $f: \mathbb{N}\to\mathbb{N}$ defined by $f(n) = n$ is a bijection. This provides the usual listing of $\mathbb{N}$ as 1, 2, 3, 4, 5, 6, …. Next, for $\mathbb{Z}$, the usual listing would be

…, –4, –3, –2, –1, 0, 1, 2, 3, 4, ….

However, to be in a one-to-one correspondence with $\mathbb{N}$, there should be a first element, followed by a second element, etc. So we can try listing $\mathbb{Z}$ as

0, 1, –1, 2, –2, 3, –3, 4, –4, ….

From this we can construct a bijection $g: \mathbb{N}\to\mathbb{Z}$, namely define g as follow:

$$g(n) = (1 - n) / 2 \text{ if } n \text{ is odd}$$

and

$$g(n) = n / 2 \text{ if } n \text{ is even.}$$

For $\mathbb{Q}$, there is no usual listing. So how do we proceed? Well, let's consider listing the set of all positive rational numbers $\mathbb{Q}^+$ first. Here is a table of $\mathbb{Q}^+$.

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

In the m-th row, the numerator is m and in the n-th column, the denominator is n.

Consider the southwest-to-northeast diagonals. The first one has 1/1, the second one has 2/1 and 1/2, the third one has 3/1, 2/2, 1/3, etc. We can list $\mathbb{Q}^+$ by writing down the numbers on these diagonals one after the other. However, this will repeat numbers, for example, 1/1 and 2/2 are the same. So to avoid repetitions, we will write down only numbers whose numerators and denominators are relatively prime. This will not omit any positive rational numbers because we can cancel common factors in the numerator and denominator of a positive rational number to arrive at a number in the table that we will not skip. Here is the list we will get for $\mathbb{Q}^+$:

1/1, 2/1, 1/2, 3/1, 1/3, 4/1, 3/2, 2/3, 1/4, 5/1, 1/5, 6/1, 5/2, 4/3, 3/4, … .

Once we have a listing of $\mathbb{Q}^+$, we can list $\mathbb{Q}$ as we did for $\mathbb{Z}$ from $\mathbb{N}$, i.e.

0, 1/1, –1/1, 2/1, –2/1, 1/2, –1/2, 3/1, –3/1, 1/3, –1/3, 4/1, –4/1, 3/2, ….

This shows $\mathbb{Q}$ is countable, although the bijection behind this listing is difficult to write down.

If a bijection $h: \mathbb{N} \to \mathbb{Q}$ is desired, then we can do the following. Define $h(1) = 0$. For an integer $n > 1$, write down the prime factorization of $g(n)$, where g is the function above. Suppose

$$g(n) = \pm\, 2^a\, 3^b\, 5^c\, 7^d \ldots.$$

Then we define

$$h(n) = \pm\, 2^{g(a+1)}\, 3^{g(b+1)}\, 5^{g(c+1)}\, 7^{g(d+1)} \ldots$$

with $g(n)$, $h(n)$ taking the same sign.

Next, how about $\mathbb{R}$? This is interesting. It turns out $\mathbb{R}$ is <u>*uncountable*</u> (i.e. not countable). To explain this, consider the function $u: (0,1) \to\mathbb{R}$ defined by $u(x) = \tan \pi(x–1/2)$. It has an inverse function $v(x) = 1/2 + (\text{Arctan}\, x)/\pi$. So both u and v are bijections. Now assume there is a bijection $f: \mathbb{N}\to \mathbb{R}$. Then $F = v\circ f: \mathbb{N}\to (0,1)$ is also a bijection. Now we write the decimal representations of $F(1)$, $F(2)$, $F(3)$, $F(4)$, $F(5)$, … in a table.

$$\begin{aligned}
F(1) &= 0.a_{11}a_{12}a_{13}a_{14}\ldots\\
F(2) &= 0.a_{21}a_{22}a_{23}a_{24}\ldots\\
F(3) &= 0.a_{31}a_{32}a_{33}a_{34}\ldots\\
F(4) &= 0.a_{41}a_{42}a_{43}a_{44}\ldots\\
F(5) &= 0.a_{51}a_{52}a_{53}a_{54}\ldots\\
F(6) &= 0.a_{61}a_{62}a_{63}a_{64}\ldots
\end{aligned}$$

Consider the number

$$r = 0.\, b_1b_2b_3b_4b_5b_6\ldots,$$

where the digit $b_n = 2$ if $a_{nn} = 1$ and $b_n = 1$ if $a_{nn} \neq 1$. Then $F(n) \neq r$ for all n because $a_{nn} \neq b_n$. This contadicts F is a bijection. Thus, no bijection $f: \mathbb{N}\to\mathbb{R}$ can exist. Therefore, (0,1) and $\mathbb{R}$ are both uncountable.

We remark that the above argument shows no matter how the elements of (0,1) are listed, there will always be numbers omitted. The number r above is one such number.

So some sets are countable and some sets are uncountable.

For more complicated sets, we will use the following theorems to determine if they are countable or not.

<u>Theorem 1.</u> Let A be a subset of B. If B is countable, then A is countable.

<u>Theorem 2.</u> If for every integer n, S_n is a countable set, then their union is countable.

For the next theorem, we introduce some terminologies first. An object of the form $(x_1,\ldots,x_n)$ is called an <u>*ordered n-tuple*</u>. For sets $T_1, T_2, \ldots, T_n$, the <u>*Cartesian product*</u> $T_1\times\cdots\times T_n$ of these sets is the set of all ordered n-tuples $(x_1,\ldots,x_n)$, where each x_i is an element of T_i for $i = 1,\ldots, n$.

<u>Theorem 3.</u> If $T_1, T_2, \ldots, T_n$ are countable sets, then their Cartesian product is also countable.

We will give some brief explanations for these theorems. For theorem 1, if A is finite, then A is countable. So suppose A is infinite, then B is infinite. Since B is countable, we can list B as $b_1, b_2, b_3, \ldots$ without repetition or omission. Removing the elements b_i that are not in A, we get a list for A without repetition or omission.

For theorem 2, let us list the elements of S_n without repetition or omission in the n-th row of a table. (If S_n is finite, then the row contains finitely many elements.) Now we can list the union of these sets by writing down the diagonal elements as we have done for the positive rational numbers. To avoid repetition, we will not write the element if it has appeared before. Also, if some rows are finite, it is possible that as we go diagonally, we may get to a "hole". Then we simply skip over the hole and go on.

For theorem 3, we use mathematical induction. The case $n = 1$ is trivial. For the case $n = 2$, let $a_1, a_2, a_3, \ldots$ be a list of the elements of T_1 and $b_1, b_2, b_3, \ldots$ be a list of the elements of T_2 without repetition or omission. Draw a table with (a_i, b_j) in the i-th row and j-th column. Listing the diagonal elements as for the positive rational numbers, we get a list for $T_1\times T_2$ without repetition or omission. This takes care the case $n = 2$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***April 26, 2003***.

Problem 176. (*Proposed by Achilleas PavlosPorfyriadis,AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that the fraction

$$\frac{m(n+1)+1}{m(n+1)-n}$$

is irreducible for all positive integers m and n.

Problem 177. A locust, a grasshopper and a cricket are sitting in a long, straight ditch, the locust on the left and the cricket on the right side of the grasshopper. From time to time one of them leaps over one of its neighbors in the ditch. Is it possible that they will be sitting in their original order in the ditch after 1999 jumps?

Problem 178. Prove that if $x < y$, then there exist integers m and n such that

$$x < m + n\sqrt{2} < y.$$

Problem 179. Prove that in any triangle, a line passing through the incenter cuts the perimeter of the triangle in half if and only if it cuts the area of the triangle in half.

Problem 180. There are $n \geq 4$ points in the plane such that the distance between any two of them is an integer. Prove that at least 1/6 of the distances between them are divisible by 3.

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Solutions

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Problem 171. (*Proposed by Ha Duy Hung, Hanoi University of Education, Hanoi City, Vietnam*) Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\left[\frac{c}{ab}\right]-\left[\frac{c}{a}\right]\left[\frac{c}{b}\right] \leq c \min\left\{\frac{1}{a},\frac{1}{b}\right\}.$$

Solution. **LEE Man Fui** (STFA Leung Kau Kui College, Form 6) and **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6).

Since the inequality is symmetric in a and b, without loss of generality, we may assume $a \geq b$. For every x, $bx \geq b[x]$. Since $b[x]$ is an integer, we get $[bx] \geq b[x]$. Let $x = c/(ab)$. We have

$$\begin{aligned} &c[c/(ab)] - [c/a][c/b] \\ &= c[x] - [bx][c/b] \\ &\leq (c/b)[bx] - [bx][c/b] \\ &= [bx]\left((c/b) - [c/b] \right) \\ &< bx \cdot 1 = c/a = c \min\{1/a, 1/b\}. \end{aligned}$$

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **Rooney TANG Chong Man** (Hong Kong Chinese Women's Club College, Form 5).

Problem 172. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all positive integers such that they are equal to the square of the sum of their digits in base 10 representation.

Solution. **D. Kipp JOHNSON** (Valley Catholic High School, Beaverton, Oregon, USA), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and and **WONG Wing Hong** (La Salle College, Form 5).

Suppose there is such an integer n and it has k digits. Then $10^{k-1} \leq n \leq (9k)^2$. However, for $k \geq 5$, we have

$$(9k)^2 = 81k^2 < (5^4/2)2^k \leq (5^{k-1}/2)2^k = 10^{k-1}.$$

So $k \leq 4$. Then $n \leq 36^2$. Since n is a perfect square, we check $1^2, 2^2, \ldots, 36^2$ and find only 1 and $9^2 = 81$ work.

Other commended solvers: **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6) and **Rooney TANG Chong Man** (Hong Kong Chinese Women's Club College, Form 5).

Problem 173. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 3/2 times any other group. (*Source: 1997 Russian Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **D. Kipp JOHNSON** (Valley Catholic High School, Beaverton, Oregon, USA).

Let $a_1, a_2, \ldots, a_{300}$ be the weights of the apples in increasing order. For $j = 1, 2, \ldots, 75$, let the j-th group consist of the apples with weights $a_j, a_{75+j}, a_{150+j}, a_{225+j}$. Note the weights of the groups are increasing. Then the ratio of the weights of any two groups is at most

$$\begin{aligned} &\frac{a_{75}+a_{150}+a_{225}+a_{300}}{a_1+a_{76}+a_{151}+a_{226}} \\ &\leq \frac{a_{76}+a_{151}+a_{226}+3a_1}{a_1+a_{76}+a_{151}+a_{226}} \\ &= 1+\frac{2}{1+(a_{76}+a_{151}+a_{226})/a_1}. \end{aligned}$$

Since $3 \leq (a_{76}+a_{151}+a_{226})/a_1 \leq 9$, so the ratio of groups is at most $1+2/(1+3)=3/2$.

Other commended solvers: **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **Terry CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6).

Problem 174. Let M be a point inside acute triangle ABC. Let A', B', C' be the mirror images of M with respect to BC, CA, AB, respectively. Determine (with proof) all points M such that A, B, C, A', B', C' are concyclic.

Solution. **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

For such M, note the points around the circle are in the order A, B', C, A', B, C'. Now $\angle ACC' = \angle ABC'$ as they are subtended by chord AC'. Also, $AB'=AC'$ because they both equal to AM by symmetry. So $\angle ABC' = \angle ACB'$ as they are subtended by chords AC' and AB' respectively. By symmetry, we also have $\angle ACB' = \angle ACM$. Therefore, $\angle ACC' = \angle ACM$ and so C, M, C' are collinear. Similarly, A, M, A' are collinear. Then $CM \perp AB$ and $AM \perp BC$. So M is the orthocenter of ΔABC.

Conversely, if M is the orthocenter, then $\angle ACB' = \angle ACM = 90° - \angle BAC = \angle ABB'$, which implies A, B, C, B' are concyclic. Similarly, A' and C' are on the circumcircle of ΔABC.

Other commended solvers: **CHEUNG**

Yun Kuen (Hong Kong Chinese Women's Club College, Form 5), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 5).

Problem 175. A regular polygon with n sides is divided into n isosceles triangles by segments joining its center to the vertices. Initially, $n + 1$ frogs are placed inside the triangles. At every second, there are two frogs in some common triangle jumping into the interior of the two neighboring triangles (one frog into each neighbor). Prove that after some time, at every second, there are at least $[\,(n + 1) / 2\,]$ triangles, each containing at least one frog. (*Source: 1993 Jiangsu Province Math Olympiad*)

Solution. (*Official Solution*)

By the pigeonhole principle, the process will go on forever. Suppose there is a triangle that never contains any frog. Label that triangle number 1. Then label the other triangles in the clockwise direction numbers 2 to n. For each frog in a triangle, label the frog the number of the triangle. Let S be the sum of the squares of all frog numbers. On one hand, $S \le (n + 1)\, n^2$. On the other hand, since triangle 1 never contains any frog, then at every second, some two terms of S will change from $i^2 + i^2$ to $(i+1)^2 + (i-1)^2 = 2i^2 + 2$ with $i < n$. Hence, S will keep on increasing , which contradicts $S \le (n+1)\, n^2$. Thus, after some time T, every triangle will eventually contain some frog at least once.

By the jumping rule, for any pair of triangles sharing a common side, if one of them contains a frog at some second, then at least one of them will contain a frog from then on. If n is even, then after time T, the n triangles can be divided into $n / 2 = [\,(n + 1) / 2]$ pairs, each pair shares a common side and at least one of the triangles in the pair has a frog. If n is odd, then after time T, we may remove one of the triangles with a frog and divide the rest into $(n - 1)/2$ pairs. Then there will exist $1 + (n-1)/2 = [\,(n+1)/2]$ triangles, each contains at least one frog.

Other commended solvers: **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 4. Find all pairs of real numbers a, b for which the equation in the domain of the real numbers

$$\frac{ax^2 - 24x + b}{x^2 - 1} = x$$

has two solutions and the sum of them equals 12.

Problem 5. A triangle KLM is given in the plane together with a point A lying on the half-line opposite to KL. Construct a rectangle $ABCD$ whose vertices B, C and D lie on the lines KM, KL and LM, respectively. (We allow the rectangle to be a square.)

Problem 6. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying for all $x, y \in \mathbb{R}^+$ the equality

$$f(x f(y)) = f(xy) + x.$$

Countability

(continued from page 2)

Assume the case $n = k$ is true. For k+1 countable sets $T_1, \ldots, T_k, T_{k+1}$, we apply the case $n = k$ to conclude $T_1 \times \cdots \times T_k$ is countable. Then $(T_1 \times \cdots \times T_k) \times T_{k+1}$ is countable by the case $n = 2$.

We should remark that for theorem 1, if C is uncountable and B is countable, then C cannot be a subset of B. As for theorem 2, it is also true for finitely many set $S_1, \ldots, S_n$ because we can set $S_{n+1}, S_{n+2}, \ldots$ all equal to S_1, then the union of $S_1, \ldots, S_n$ is the same as the union of $S_1, \ldots, S_n, S_{n+1}, S_{n+2}, \ldots$. However, for theorem 3, it only works for finitely many sets. Although it is possible to define ordered infinite tuples, the statement is not true for the case of infinitely many sets.

Now we go back to answer question 2 stated in the beginning of this article. We have already seen that $C = \mathbb{R}$ is uncountable. To see question 2 has a negative answer, it is enough to show the set B of all algebraic numbers is countable. By the remark for theorem 1, we can conclude that $C = \mathbb{R}$ cannot be a subset of B. Hence, there exists at least one real number which is not a root of any nonconstant polynomial with integer coefficients.

To show B is countable, we will first show the set D of all nonconstant polynomials with integer coefficients is countable.

Observe that every nonconstant polynomial is of degree n for some positive integer n. Let D_n be the set of all polynomials of degree n with integer coefficients. Let $\mathbb{Z}'$ denote the set of all nonzero integers. Since $\mathbb{Z}'$ is a subset of $\mathbb{Z}$, $\mathbb{Z}'$ is countable by theorem 1 (or simply deleting 0 from a list of $\mathbb{Z}$ without repetition or omission).

Note every polynomial of degree n is of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \text{ (with } a_n \ne 0),$$

which is uniquely determined by its coefficients. Hence, if we define the function $w : \mathbb{Z}' \times \mathbb{Z} \times \cdots \times \mathbb{Z} \to D_n$ by

$$w(a_n, a_{n-1}, \ldots, a_0) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

then w is a bijection. By theorem 3, $\mathbb{Z}' \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ is countable. So there is a bijection $q : \mathbb{N} \to \mathbb{Z}' \times \mathbb{Z} \times \cdots \times \mathbb{Z}$. Then $w \circ q : \mathbb{N} \to D_n$ is also a bijection. Hence, D_n is countable for every positive integer n. Since D is the union of $D_1, D_2, D_3, \ldots$, by theorem 2, D is countable.

Finally, let $P_1, P_2, P_3, \ldots$ be a list of all the elements of D. For every n, let R_n be the set of all roots of P_n, which is finite by the fundamental theorem of algebra. Hence R_n is countable. Since B is the union of $R_1, R_2, R_3, \ldots$, by theorem 2, B is countable and we are done.

Historically, the countability concept was created by Cantor when he proved the rational numbers were countable in 1873. Then he showed algebraic numbers were also countable a little later. Finally in December 1873, he showed real numbers were uncountable and wrote up the results in a paper, which appeared in print in 1874. It was this paper of Cantor that also introduced the one-to-one correspondence concept into mathematics for the first time!

Mathematical Excalibur

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Olympiad Corner

The XV Asia Pacific Mathematics Olympiad took place on March 2003. The time allowed was 4 hours. No calculators were to be used. Here are the problems.

Problem 1. Let a, b, c, d, e, f be real numbers such that the polynomial

$$P(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \ldots, 8$. Determine all possible values of f.

Problem 2. Suppose $ABCD$ is a square piece of cardboard with side length a. On a plane are two parallel lines ℓ_1 and ℓ_2, which are also a units apart. The square $ABCD$ is placed on the plane so that sides AB and AD intersect ℓ_1 at E and F respectively. Also, sides CB and CD intersect ℓ_2 at G and H respectively. Let the perimeters of ΔAEF and ΔCGH be m_1 and m_2 respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 10, 2003***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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容斥原則和 Turan 定理

梁達榮

設 A 為有限集，以 $|A|$ 表示它含元素的個數。如果有兩個有限集 A 和 B，以 $A\cup B$ 表示 A 和 B 的并集（它包含屬於 A 或 B 的元素），以 $A\cap B$ 表示 A 和 B 的交集（它包含同時屬於 A 和 B 的元素）。眾所周知，如果 A 和 B 之間沒有共同元素，則 $|A\cup B|=|A|+|B|$，但是如果 A 和 B 之間有共同元素 x，當數算 A 元素的數目時，x 被算了一次，但數算 B 元素的數目時，x 又再被算了一次。為了抵消這樣的重覆，在計算 $|A\cup B|$ 時，我們要減去重覆數算的次數，即 $|A\cap B|$。因此 $|A\cup B|=|A|+|B|-|A\cap B|$。

對於三個集的并集 $A\cup B\cup C$，我們可以先數算 A，B 和 C 的個數，相加起來，發覺是太大了，必須減去一些交集的個數，現在 A，B 和 C 中任兩個集的交集可以是 $A\cap B$，$A\cap C$ 和 $B\cap C$，當我們減去這些交集的元素個數時，發覺又變得太少了，最後我們還要加上三個集的交集的元素個數，最後得 $|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$。

一般來說，如果有 n 個有限集 A_1, $A_2,\ldots,A_n$，則 $|A_1\cup A_2\cup\cdots\cup A_n| = \sum_{i=1}^{n}|A_i| - \sum_{1\le i\prec j\le n}|A_i\cap A_j| + \sum_{1\le i\prec j\prec k\le n}|A_i\cap A_j\cap A_k| - \cdots + (-1)^{n-1}|A_1\cap A_2\cap\cdots\cap A_n|$ 等式中右邊第一個和式代表 A_1 至 A_n 各集元素個數的總和，第二個和式代表任何兩個集的交集元素個數的總和，餘此類推，直到考慮 $A_1, A_2,\cdots$ 至 A_n 的交集為止。

上面的等式，一般稱為容斥原則 (Inclusion-Exclusion Principle)，其命意義相當明顯。證明可以採歸納法，但也可以利用二項式定理加以證明。過程大概如下。設 x 屬於 $A_1\cup A_2\cup\cdots\cup A_n$，則 x 屬於其中 k 個 A_i，$(k\ge 1)$，為方便計，設 x 屬於 $A_1, A_2,\ldots, A_k$，但不屬於 $A_{k+1},\ldots,A_n$。這樣的話，x 在 $A_1\cup A_2\cup\cdots\cup A_n$ 的"貢獻"為 1。在右邊第一個和式中，x 的"貢獻"為 $k=C_1^k$。在第二個和式中，由於 x 在 $A_1, A_2,\ldots, A_k$ 中出現，則 x 在它們任兩個集的交集中出現，但不在其他兩個集的交集中出現，因此，x 在第二個和式中的"貢獻"為 C_2^k。這樣分析下去，我們發覺 x 在右邊的"貢獻"總和是 $C_1^k - C_2^k + C_3^k - \cdots + (-1)^{k+1}C_k^k = 1-(1-1)^k = 1$。

留意我們用到了二項式定理，由於 x 在兩邊的貢獻相等，我們獲得了容斥原則成立的證明。

再者二項式系數有以下的性質。C_m^k 在 $m\le\frac{k}{2}$ 時遞增，在 $m\ge\frac{k}{2}$ 時遞減。(例如 $k=5$，有 $C_0^5<C_1^5<C_2^5=C_3^5>C_4^5>C_5^5$，$C_m^5$ 在 $m=2, 3$ 時取最大值，$k=6$，$C_0^6<C_1^6<C_2^6<C_3^6>C_4^6>C_5^6>C_6^6$，$C_m^6$ 在 $m=3$ 時最大值。) 利用這個關係，讀者可以証明，如果在容斥原則的右邊，略去一個正項及它以後各項，則式的左邊大於右邊，這是因為 x 對於右邊的貢獻非正，或者被略去的貢獻非負。同理，如果在容斥原則的右邊略去一個負項及它以後各項，則式的左邊變為小於右邊。這是一個有用的估計。

容斥原則作為數算集的大小的用途上時常出現，應用廣泛。

例一： 這是一個經典的題目，將 1, 2, …, n 重新安排次序，得到一個排列，如果沒有一個數字在原先的位置上，則稱之為亂序，(例如，4321 是一個亂序，4213 不是)，現在問，有多少個亂序？

解答: 顯而易見，所有的排列數目是 $n! = n\times(n-1)\times\cdots\times 1$。但如果直接找尋亂序的數目，卻不是很容易。因此我們定義 A_i 為 i 在正確位置的排列，$1 \le i \le n$。易見 $|A_i| = (n-1)!$，同理 $|A_i \cap A_j| = (n-2)!$，此處 $i \ne j$，等等。因此

$$\begin{aligned}
&|A_1 \cup A_2 \cup \cdots \cup A_n| \\
&= \sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i \cap A_j| \\
&+ \sum_{1\le i<j<k\le n}|A_i \cap A_j \cap A_k| - \cdots \\
&+ (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n| \\
&= n(n-1)! - C_2^n(n-2)! \\
&+ C_3^n(n-3)! - \cdots + (-1)^{n-1}1 \\
&= n! - \tfrac{n!}{2!} + \tfrac{n!}{3!} - \cdots + (-1)^{n-1}\tfrac{n!}{n!}
\end{aligned}$$

最後，亂序的數目是

$$\begin{aligned}
&n! - |A_1 \cup A_2 \cup \cdots \cup A_n| \\
&= n!\left(\tfrac{1}{2!} - \tfrac{1}{3!} + \cdots + (-1)^n \tfrac{1}{n!}\right)
\end{aligned}$$ 。

例二： (IMO 1991) 設 $S = \{1, 2, \ldots, 280\}$。求最小的自然數 n，使得 S 的每一個 n 元子集都含有 5 個兩兩互素的數。

解答: 首先利用容斥原則求得 $n \ge 217$。設 A_1, A_2, A_3, A_4 是 S 中分別為 2, 3, 5, 7 的倍數的集，則

$$\begin{aligned}
&|A_1| = 140, |A_2| = 93, |A_3| = 56, |A_4| = 40, \\
&|A_1 \cap A_2| = 46, |A_1 \cap A_3| = 28, \\
&|A_1 \cap A_4| = 20, |A_2 \cap A_3| = 18, \\
&|A_2 \cap A_4| = 13, |A_3 \cap A_4| = 8, \\
&|A_1 \cap A_2 \cap A_3| = 9, |A_1 \cap A_2 \cap A_4| = 6, \\
&|A_1 \cap A_3 \cap A_4| = 4, |A_2 \cap A_3 \cap A_4| = 2, \\
&|A_1 \cap A_2 \cap A_3 \cap A_4| = 1
\end{aligned}$$

因此

$$\begin{aligned}
&|A_1 \cap A_2 \cap A_3 \cap A_4| = 140 + 93 \\
&+ 56 + 40 - 46 - 28 - 20 - 18 - 13 \\
&- 8 + 9 + 6 + 4 + 2 - 1 = 216
\end{aligned}$$

對於這個 216 元的集，任取 5 個數，必有兩個同時屬於 A_1, A_2, A_3 或 A_4，因此不互素。按題意，所以必須有 $n \ge 217$。現在要證明 S 中任一 217 元集必有 5 個互素的數，方法是要構造適當的"抽屜"。其中一個比較簡潔的構造是這樣的。設 A 是 S 的一個子集，並且 $|A| \ge 217$。定義

$B_1 = \{1$ 和 S 中的素數$\}$，$|B_1| = 60$，

$B_2 = \{2^2, 3^2, 5^2, 7^2, 11^2, 13^2\}, |B_2| = 6,$

$B_3 = \{2\times131, 3\times89, 5\times53, 7\times37, 11\times23, 13\times19\}, |B_3| = 6,$

$B_4 = \{2\times127, 3\times87, 5\times47, 7\times31, 11\times19, 13\times17\}, |B_4| = 6,$

$B_5 = \{2\times113, 3\times79, 5\times43, 7\times27, 11\times17\}, |B_5| = 5,$

$B_6 = \{2\times109, 3\times73, 5\times41, 7\times23, 11\times13\}, |B_6| = 5.$

易見 B_1 至 B_6 互不相交，並且 $|B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6| = 88$。去掉這 88 個數，$S$ 中尚有 $280 - 88 = 192$ 個數。現在 A 最小有 217 個元素，$217 - 192 = 25$，即是說 A 中最小有 25 個元屬於 B_1 至 B_6。易見，不可能每個 B_i 只含 A 中 4 個或以下的元素，即是說最少有 5 個或以上的元素屬於同一個 B_i，因此互素。注意這裏我們用到另一個原則：抽屜原則。

例三： (1989 IMO) 設 n 是正整數。我們說集$\{1, 2, 3, \ldots, 2n\}$的一個排列 $(x_1, x_2, \ldots, x_{2n})$ 具有性質 P，如果在$\{1, 2, 3, \ldots, 2n-1\}$中至少有一個 i，使得 $|x_i - x_{i+1}| = n$ 成立。證明具有性質 P 的排列比不具有性質 P 的排列多。

解答: 留意如果 $|x_i - x_{i+1}| = n$，其中一個 x_i 或 x_{i+1} 必小於 $n+1$。因此對於 $k = 1, 2, \ldots, n$, 定義 A_k 為 k 與 $k+n$ 相鄰的排列的組合，易見 $|A_k| = 2\times(2n-1)!$。(這是因為 k 與 $k+n$ 并合在一起，但位置可以互相交換，想像它是一個"數"，而另外有 $2n-2$ 個數，這$(2n-2)+1$ 個數位置隨意。) 同時 $|A_k \cap A_h| = 2^2\times(2n-2)!$, $1 \le k < h \le n$, (k 與 $k+n$ 合在一起成為一個"數" h 與 $h+n$ 合在一起成為一個"數"。)因此具性質 P 的排列的數目

$$\begin{aligned}
&|A_1 \cup A_2 \cup \cdots \cup A_n| \ge \sum_{k=1}^{n}|A_k| \\
&- \sum_{1\le k<h\le n}|A_k \cap A_h| \\
&= 2\times(2n-1)!\times n - C_2^n \times 2^2 \times (2n-2)! \\
&= 2n\times(2n-2)!\times n = (2n)!\times\tfrac{n}{2n-1} > (2n)!\times\tfrac{1}{2}
\end{aligned}$$

這個數目超過$(2n)!$的一半，因此具性質 P 的排列比不具性質 P 的排列多。

(這一個問題，當年被視為一個難題，但如果看到它與容斥原則的關係，就變得很容易了。)

例四：設 n 和 k 為正整數， $n > 3$, $\frac{n}{2} < k < n$。平面上有 n 個點，其中任意三點不共線，如果其中每個點至少與其它 k 個點用線連結，則連結的線段中至少有三條圍成一個三角形。

解答: 因為 $n > 3, k > \frac{n}{2}$，則 $k \ge 2$，所以 n 個點中必中兩個點 v_1 和 v_2 相連結。考慮餘下的點，設與 v_1 相連結的點集為 A，與 v_2 相連結的點集為 B，則 $|A| \ge k-1, |B| \ge k-1$。另外

$$\begin{aligned}
&n-2 \ge |A\cup B| = |A| + |B| - |A\cap B| \\
&\ge 2k-2-|A\cap B|
\end{aligned}$$

即 $|A\cap B| \ge 2k-n > 0$。因此，存在點 v_3 與 v_1 和 v_2 相連結，構成一個三角形。

例五： 一次會議有 1990 位數學家參加，其中每人最少有 1327 位合作者。証明，可以找到 4 位數學家，他們中每兩人都合作過。

証明: 將數學家考慮為一個點集，曾經合作過的連結起來，得到一個圖。如上例，v_1 互 v_2 曾合作過，所以連結起來，餘下的，設 A 為和 v_1 合作過的點集，B 為和 v_2 合作過的點集，則 $|A| \ge 1326, |B| \ge 1326$，同樣，$|A\cup B| \le 1990 - 2 = 1998$，因此

$$\begin{aligned}
&|A\cap B| = |A| + |B| - |A\cup B| \\
&\ge 2\times1326 - 1998 = 664 > 0
\end{aligned}$$

即是說，可以找到數學家 v_3，與 v_1 和 v_2 都合作過。設 C 為除 v_1 和 v_2 以外，與 v_3 合作過的數學家，即 $|C| \ge 1325$。同時

$1998 \ge |(A\cap B)\cup C| = |A\cap B| + |C| - |A\cap B\cap C|$ 即

$$\begin{aligned}
&|A\cap B\cap C| \ge |A\cap B| + |C| - 1988 \\
&\ge 664 + 1325 - 1988 = 1 > 0
\end{aligned}$$ 。

因此 $A\cap B\cap C$ 非空，取 $v_4 \in A\cap B\cap C$，則 v_1, v_2, v_3, v_4 都曾經合作過。

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***August 10, 2003***.

Problem 181. (*Proposed by Achilleas PavlosPorfyriadis, AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that in a convex polygon, there cannot be two sides with no common vertex, each of which is longer than the longest diagonal.

Problem 182. Let a_0, a_1, a_2, … be a sequence of real numbers such that

$$a_{n+1} \geq a_n^2 + 1/5 \quad \text{for all } n \geq 0.$$

Prove that $\sqrt{a_{n+5}} \geq a_{n-5}^2$ for all $n \geq 5$.

Problem 183. Do there exist 10 distinct integers, the sum of any 9 of which is a perfect square?

Problem 184. Let $ABCD$ be a rhombus with $\angle B = 60^\circ$. M is a point inside ΔADC such that $\angle AMC = 120^\circ$. Let lines BA and CM intersect at P and lines BC and AM intersect at Q. Prove that D lies on the line PQ.

Problem 185. Given a circle of n lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided one also changes the state of every d-th bulb after it (where d is a divisor of n and is less than n), provided that all n/d bulbs were originally in the same state as one another. For what values of n is it possible to turn all the bulbs on by making a sequence of moves of this kind?

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Solutions

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Problem 176. (*Proposed by Achilleas PavlosPorfyriadis,AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that the fraction

$$\frac{m(n+1)+1}{m(n+1)-n}$$

is irreducible for all positive integers m and n.

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School), **Anderson TORRES** (Colegio Etapa, Brazil, 3rd Grade) and **Alan T. W. WONG** (Markham, ON, Canada).

If the fraction is reducible, then $m(n+1)+1$ and $m(n+1)-n$ are both divisible by a common factor $d>1$. So their difference $n+1$ is also divisible by d. This would lead to

$$1 = (m(n+1)+1) - m(n+1)$$

divisible by d, a contradiction.

Other commended solvers: **CHEUNG Tin** (STFA Leung Kau Kui College, Form 4), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **Alexandre THIERY** (Pothier High School, Orleans, France), **Michael A. VEVE** (Argon Engineering Associates, Inc., Virginia, USA) and **Maria ZABAR** (Trieste College, Trieste, Italy).

Problem 177. A locust, a grasshopper and a cricket are sitting in a long, straight ditch, the locust on the left and the cricket on the right side of the grasshopper. From time to time one of them leaps over one of its neighbors in the ditch. Is it possible that they will be sitting in their original order in the ditch after 1999 jumps?

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA), **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **Anderson TORRES** (Colegio Etapa, Brazil, 3rd Grade).

Let L, G, C denote the locust, grasshopper, cricket, respectively. There are 6 orders:

$$LCG, CGL, GLC, CLG, GCL, LGC.$$

Let LCG, CGL, GLC be put in one group and CLG, GCL, LGC be put in another group. Note after one leap, an order in one group will become an order in the other group. Since 1999 is odd, the order LGC originally will change after 1999 leaps.

Problem 178. Prove that if $x < y$, then there exist integers m and n such that

$$x < m + n\sqrt{2} < y.$$

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Note $0 < \sqrt{2} - 1 < 1$. For a positive integer

$$k > \frac{\log(b-a)}{\log(\sqrt{2}-1)},$$

we get $0 < (\sqrt{2}-1)^k < b - a$. By the binomial expansion,

$$x = (\sqrt{2}-1)^k = p + q\sqrt{2}$$

for some integers p and q. Next, there is an integer r such that

$$r - 1 \leq \frac{a - [a]}{x} < r.$$

Then a is in the interval

$$I = [[a] + (r-1)x, [a] + rx).$$

Since the length of I is $x < b - a$, we get

$$a < [a] + rx = ([a] + rp) + rq\sqrt{2} < b.$$

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA), **Alexandre THIERY** (Pothier High School, Orleans, France) and **Anderson TORRES** (Colegio Etapa, Brazil, 3rd Grade).

Problem 179. Prove that in any triangle, a line passing through the incenter cuts the perimeter of the triangle in half if and only if it cuts the area of the triangle in half.

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School), and **Alexandre THIERY** (Pothier High School, Orleans, France).

Let ABC be the triangle, s be its semiperimeter and r be its inradius. Without loss of generality, we may assume the line passing through the incenter cuts AB and AC at P and Q respectively. (If the line passes through a vertex of ΔABC, we may let $Q = C$.)

Let $[XYZ]$ denote the area of ΔXYZ. The line cuts the perimeter of ΔABC in half if and only if $AP + AQ = s$, which is equivalent to

$$\begin{aligned}[APQ] &= [API] + [AQI] \\ &= (r\cdot AP)/2 + (r\cdot AQ)/2 \\ &= rs/2 = [ABC]/2.\end{aligned}$$

i.e. the line cuts the area of ΔABC in half.

Problem 180. There are $n \ge 4$ points in the plane such that the distance between any two of them is an integer. Prove that at least 1/6 of the distances between them are divisible by 3.

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

We will first show that for any 4 of the points, there is a pair with distance divisible by 3. Assume A, B, C, D are 4 of the points such that no distance between any pair of them is divisible by 3. Since $x \equiv 1$ or 2 (mod 3) implies $x^2 \equiv 1$ (mod 3), $AB^2, AC^2, AD^2, BC^2, BD^2$ and CD^2 are all congruent to 1 (mod 3).

Without loss of generality, we may assume that $\angle ACD = \alpha + \beta$, where $\alpha = \angle ACB$ and $\beta = \angle BCD$. By the cosine law,

$$AD^2 = AC^2 + CD^2 - 2AC\cdot CD\cos\angle ACD.$$

Now

$$\begin{aligned}\cos\angle ACD &= \cos(\alpha+\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

By cosine law, we have

$$\cos\alpha = \frac{AC^2 + BC^2 - AB^2}{2AC\cdot BC} \text{ and}$$

$$\cos\beta = \frac{BC^2 + CD^2 - BD^2}{2BC\cdot CD}.$$

Using $\sin x = \sqrt{1-\cos^2 x}$, we can also find $\sin\alpha$ and $\sin\beta$. Then

$$\begin{aligned}2BC^2\cdot AD^2 &= 2BC^2(AC^2 + CD^2) \\ &\quad - (2AC\cdot BC)(2BC\cdot CD)\cos\angle ACD \\ &= P + Q,\end{aligned}$$

where

$$\begin{aligned}P &= 2BC^2(AC^2 + CD^2) \\ &\quad - (AC^2+BC^2-AB^2)(BC^2+CD^2-BD^2)\end{aligned}$$

and

$$\begin{aligned}Q^2 &= (4AC^2\cdot BC^2 - (AC^2 + BC^2 - AB^2)^2) \\ &\quad \times (4BC^2\cdot CD^2 - (BC^2 + CD^2 - BD^2)^2).\end{aligned}$$

However, $2BC^2 \cdot AD^2 \equiv 2 \pmod 3$, $P \equiv 0 \pmod 3$ and $Q \equiv 0 \pmod 3$. This lead to a contradiction.

For $n \ge 4$, there are C_4^n groups of 4 points. By the reasoning above, each of these groups has a pair of points with distance divisible by 3. This pair of points is in a total of C_2^{n-2} groups. Since $C_4^n / C_2^{n-1} = \frac{1}{6}C_2^n$, the result follows.

Olympiad Corner

(continued from page 1)

Problem 3. Let $k \ge 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \ge 3k/4$. Let n be a composite integer. Prove:

(a) if $n = 2p_k$, then n does not divide $(n-k)!$;

(b) if $n > 2p_k$, then n divides $(n-k)!$.

Problem 4. Let a, b, c be the sides of a triangle, with $a + b + c = 1$, and let $n \ge 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}.$$

Problem 5. Given two positive integers m and n, find the smallest positive integer k such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people.

容斥原則和 Turan 定理

(continued from page 2)

套用圖論的語言，例四和例五的意義正如，給定一個 n 點的圖，最少有多少條線，才可以保證有一個三角形 (K_3) 或一個 K_4（四點的圖，任兩點都相連），或者換另一種說法，設有一個 n 點的圖沒有三角形，則該圖最多有多少條線段，等等。這一範圍的圖論稱為極端圖論。最先的結果是這樣的：

__Mantel 定理 (1907)__: 設 n 點的簡單圖不含 K_3，則其邊數最大值為 $\left[\frac{n^2}{4}\right]$。

（此處 $[x]$ 是小於或等於 x 的最大整數。在例四中，邊數和多於 $\binom{n}{2} \times n \times \frac{1}{2} > \left[\frac{n^2}{4}\right]$，因此結果立即成立。）

比較精緻的命題是這樣的。

__定理__： 如果 n 點的圖有 q 條邊，則圖至少有 $\dfrac{4q(q-\frac{n^2}{4})}{3n}$ 個三角形。

__例六__： 在圓周上有 21 個點，由其中二點引伸至圓心所成的圓心角度，最多有 110 個大於 120°。

__解答__： 如果兩點與圓心形成的圓心角度大於 120°，則將兩點連結起來，得到一個圖，這個圖沒有三角形，因此邊數最多有 $\left[\frac{21^2}{4}\right] = \left[\frac{441}{4}\right] = 110$ 條，或者最多有 110 個引伸出來的圓心角度大於 120°。

如上所說，定義 K_p 是一個 p 個點的完全圖，即 p 個點任兩點都相連，對於一個 n 點的圖 G，如果沒有包含 K_p，則 G 最多有多少條邊呢?

__Turan 定理(1941)__: 如果一個 n 點的圖 G 不含 K_p，則該圖最多有 $\frac{p-2}{2(p-1)}n^2 - \frac{r(p-1-r)}{2(p-1)}$ 條邊，其中 r 是由 $n = k(p-1) + r$, $0 \le r < p - 1$ 所定義的。如 Mantel 定理的情況，這個定理是極端圖論的一個起點。

Paul Turan (1910-1976) 猶太裔匈牙利人，當他在考慮這一類問題時，還是被關在一個集中營內的呢！

Mathematical Excalibur

Volume 8, Number 4 *August 2003 – October 2003*

Olympiad Corner

The 2003 International Mathematical Olympiad took place on July 2003 in Japan. Here are the problems.

Problem 1. Let A be a subset of the set $S = \{1, 2, \ldots, 1000000\}$ containing exactly 101 elements. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ such that the sets

$$a_j = \{x + t_j \mid x \in A\} \text{ for } j = 1, 2, \ldots, 100$$

are pairwise disjoint.

Problem 2. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

Problem 3. A convex hexagon is given in which any two opposite sides have the following property: the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal. (A convex hexagon $ABCDEF$ has three pairs of opposite sides: AB and DE, BC and EF, CD and FA.)

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 30, 2003***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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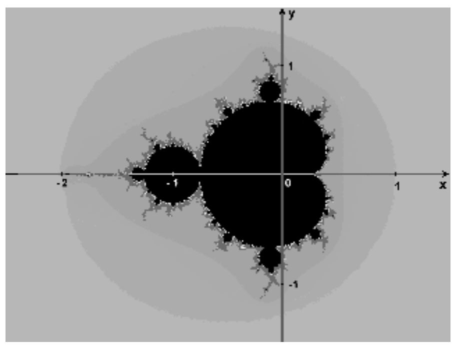
利用 GW-BASIC 繪畫曼德勃羅集的方法

梁子傑老師
香港道教聯合會青松中學

已知一個複數 c_0，並由此定義一個複數數列 $\{c_n\}$，使 $c_{n+1} = c_n^2 + c_0$，其中 $n = 0, 1, 2, \ldots\ldots$。如果這個數列有界，即可以找到一個正實數 M，使對於一切的 n，$|c_n| < M$，那麼 c_0 便屬於曼德勃羅集（Mandelbrot Set）之內。



可以將以上定義寫成一個 GW-BASIC 程序（對不起！我本人始終都是喜歡最簡單的電腦語言，而且我認為將 GW-BASIC 程序翻譯成其他電腦語言亦不難），方法如下：

```
10  LEFT = 150 : TOP = 380 :
    W = 360 : M = .833
20  R = 2.64 : S = 2 * R / W
30  RECEN = 0 : IMCEN = 0
40  SCREEN 9 : CLS
50  FOR Y = 0 TO W
60  FOR X = 0 TO W
70  REC = S * (X – W / 2) + RECEN :
    IMC = S * (Y – W / 2) + IMCEN
80  RE = REC : IM = IMC
90  RE2 = RE * RE : IM2 = IM * IM :
    J = 0
100 WHILE RE2 + IM2 <= 256 AND
    J < 15
110 IM = 2 * RE * IM + IMC
120 RE = RE2 – IM2 + REC
130 RE2 = RE * RE :
    IM2 = IM * IM : J = J + 1
140 WEND
150 IF J < 3 THEN GOTO 220
160 IF J >= 3 AND J < 6 THEN
    COLOR 14 : REM YELLOW
170 IF J >= 6 AND J < 9 THEN
    COLOR 1 : REM BLUE
180 IF J >= 9 AND J < 12 THEN
    COLOR 2 : REM GREEN
190 IF J >= 12 AND J < 15 THEN
    COLOR 15 : REM WHITE
200 IF J >= 15 THEN
    COLOR 12 : REM RED
210 PSET (X + LEFT, (TOP – Y)* M)
220 NEXT X
230 NEXT Y
240 COLOR 15 : REM WHITE
250 LINE (LEFT, (TOP – W / 2) * M)
    – (W + LEFT, (TOP – W / 2) * M)
260 LINE (W / 2 + LEFT, (TOP – W)
    * M) – (W / 2 + LEFT, TOP * M)
270 END
```

以下是這程序的解釋：

W 紀錄在電腦畫面上將要畫出圖形的大小。現將 W 設定為 360（見第 10 行），表示打算在電腦畫面上一個 360×360 的方格內畫出曼德勃羅集（見第 50 及 60 行）。

LEFT 是繪圖時左邊的起點，TOP 是圖的最低的起點（見第 210、250 及 260 行）。注意：在 GW-BASIC 中，畫面坐標是由上至下排列的，並非像一般的理解，將坐標由下至上排，因此要以 "TOP – Y" 的方法將常用的坐標轉換成電腦的坐標。

由於電腦畫面上的一點並非正方形，橫向和縱向的大小並不一樣，故引入 M（$= \frac{5}{6}$）來調節長闊比（見第 10、210、250 及 260 行）。

留意 W 衹是「畫面上」的大小，並非曼德勃羅集內每一個複數點的實際坐標，故需要作出轉換。R 是實際的數值（見第 20 行），即繪畫的範圍實軸由 –R 畫至 +R，同時虛軸亦由 –R 畫至 +R。S 計算 W 與 R 之間的比例，並應用於後面的計算之中（見第 20 及 70 行）。

RECEN 和 IMCEN 用來定出中心點的位置，現在以 (0, 0) 為中心 (見第 30 行)。我們可以通過更改 R、RECEN 和 IMCEN 的值來移動或放大曼德勃羅集。

第 40 行選擇繪圖的模式及清除舊有的畫面。

程序的第 50 及 60 行定出畫面上的坐標 X 和 Y，然後在第 70 行計算出對應複數 c_0 的實值和虛值。

注意：若 $c_0 = a_0 + b_0\,i$，$c_n = a_n + b_n\,i$，則

$$\begin{aligned} c_{n+1} &= c_n^{\,2} + c_0 \\ &= (a_n + b_n\,i)^2 + (a_0 + b_0\,i) \\ &= a_n^{\,2} - b_n^{\,2} + 2a_nb_n\,i + a_0 + b_0\,i \\ &= (a_n^{\,2} - b_n^{\,2} + a_0) \\ &\quad + (2a_nb_n + b_0)i\text{。} \end{aligned}$$

所以 c_{n+1} 的實部等於 $a_n^{\,2} - b_n^{\,2} + a_0$，而虛部則等於 $2a_nb_n + b_0$。

將以上的計算化成程序，得第 110 及 120 行。REC 和 IMC 分別是 c_0 的實值和虛值。RE 和 IM 分別是 c_n 的實值和虛值。RE2 和 IM2 分別是 c_n 的實值和虛值的平方。

J 用來紀錄第 100 至 140 行的循環的次數。第 100 行亦同時計算 c_n 模的平方。若模的平方大於 256 或者循環次數多於 15，循環將會終止。這時候，J 的數值越大，表示該數列較「收斂」，即經過多次計算後，c_n 的模仍不會變得很大。第 150 至 200 行以顏色將收斂情況分類，紅色表示最「收斂」的複數，其次是白色，跟著是綠色、藍色和黃色，而最快擴散的部分以黑色表示。第 210 行以先前選定的顏色畫出該點。

曼德勃羅集繪畫完成後，以白色畫出橫軸及縱軸 (見第 240 至 260 行)，以供參考。程序亦在此結束。

執行本程序所須的時間，要視乎電腦的速度，以現時一般的電腦而言，整個程序應該可以 1 分鐘左右完成。

<u>參考書目</u>

Heinz-Otto Peitgen, Hartmut Jürgens and Dietmar Saupe (1992) *Fractals for the Classroom Part Two: Introduction to Fractals and Chaos*. NCTM, Springer-Verlag.

IMO 2003

T. W. Leung

The 44th International Mathematical Olympiad (IMO) was held in Tokyo, Japan during the period 7 - 19 July 2003. Because Hong Kong was declared cleared from SARS on June 23, our team was able to leave for Japan as scheduled. The Hong Kong Team was composed as follows.

Chung Tat Chi (Queen Elizabeth School)
Kwok Tsz Chiu (Yuen Long Merchants Assn. Sec. School)
Lau Wai Shun (T. W. Public Ho Chuen Yiu Memorial College)
Siu Tsz Hang (STFA Leung Kau Kui College)
Yeung Kai Sing (La Salle College)
Yu Hok Pun (SKH Bishop Baker Secondary School)
Leung Tat Wing (Leader)
Leung Chit Wan (Deputy Leader)

Two former Hong Kong Team members, Poon Wai Hoi and Law Ka Ho, paid us a visit in Japan during this period.

The contestants took two 4.5 Hours contests on the mornings of July 13 and 14. Each contest consisted of three questions, hence contest 1 composed of Problem 1 to 3, contest 2 Problem 4 to 6. In each contest usually the easier problems come first and harder ones come later. After normal coordination procedures and Jury meetings cutoff scores for gold, silver and bronze medals were decided. This year the cutoff scores for gold, silver and bronze medals were 29, 19 and 13 respectively. Our team managed to win two silvers, two bronzes and one honorable mention. (Silver: Kwok Tsz Chiu and Yu Hok Pun, Bronze: Siu Tsz Hang and Yeung Kai Sing, Honorable Mention: Chung Tat Chi, he got a full score of 7 on one question, which accounted for his honorable mention, and his total score is 1 point short of bronze). Among all contestants three managed to obtain a perfect score of 42 on all six questions. One contestant was from China and the other two from Vietnam.

The Organizing Committee did not give official total scores for individual countries, but it is a tradition that scores between countries were compared. This year the top five teams were Bulgaria, China, U.S.A., Vietnam and Russia respectively. The Bulgarian contestants did extremely well on the two hard questions, Problem 3 and 6. Many people found it surprising. On the other hand, despite going through war in 1960s Vietnam has been strong all along. Perhaps they have participated in IMOs for a long time and have a very good Russian tradition.

Among 82 teams, we ranked unofficially 26. We were ahead of Greece, Spain, New Zealand and Singapore, for instance. Both New Zealand and we got our first gold last year. But this year the performance of the New Zealand Team was a bit disappointing. On the other hand, we were behind Canada, Australia, Thailand and U.K.. Australia has been doing well in the last few years, but this year the team was just 1 point ahead of us. Thailand has been able to do quite well in these few years.

IMO 2004 will be held in Greece, IMO 2005 in Mexico, IMO 2006 in Slovenia. IMO 2007 will be held in Vietnam, the site was decided during this IMO in Japan.

For the reader who will try out the IMO problems this year, here are some comments on Problem 3, the hardest problem in the first day of the competitions.

Problem 3. A convex hexagon is given in which any two opposite sides have the following property: the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal. (A convex hexagon *ABCDEF* has three pairs of opposite sides: *AB* and *DE*, *BC* and *EF*, *CD* and *FA*.)

The problem is hard mainly because one does not know how to connect the given condition with that of the interior angles. Perhaps hexagons are not as rigid as triangles. It also reminded me of No. 5, IMO 1996, another hard problem of polygons.

The main idea is as follows. Given a hexagon *ABCDEF*, connect *AD*, *BE* and *CF* to form the diagonals. From the given condition of the hexagon, it can be proved that the triangles formed by the diagonals and the sides are actually equilateral triangles. Hence the interior angles of the hexagons are 120°. Good luck.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***November 30, 2003***.

Problem 186. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Let α, β, γ be complex numbers such that

$$\alpha+\beta+\gamma=1,$$
$$\alpha^2+\beta^2+\gamma^2=3,$$
$$\alpha^3+\beta^3+\gamma^3=7.$$

Determine the value of $\alpha^{21}+\beta^{21}+\gamma^{21}$.

Problem 187. Define $f(n)=n!$. Let

$$a=0.f(1)f(2)f(3)\ldots.$$

In other words, to obtain the decimal representation of a write the numbers $f(1)$, $f(2)$, $f(3)$, … in base 10 in a row. Is a rational? Give a proof.

Problem 188. The line S is tangent to the circumcircle of acute triangle ABC at B. Let K be the projection of the orthocenter of triangle ABC onto line S (i.e. K is the foot of perpendicular from the orthocenter of triangle ABC to S). Let L be the midpoint of side AC. Show that triangle BKL is isosceles.

Problem 189. $2n+1$ segments are marked on a line. Each of the segments intersects at least n other segments. Prove that one of these segments intersect all other segments.

Problem 190. (*Due to Abderrahim Ouardini*) For nonnegative integer n, let $\lfloor x\rfloor$ be the greatest integer less than or equal to x and

$$f(n)=\left[\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right]-\left[\sqrt{9n+1}\right].$$

Find the range of f and for each p in the range, find all nonnegative integers n such that $f(n)=p$.

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Solutions

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Problem 181. (*Proposed by Achilleas PavlosPorfyriadis, AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that in a convex polygon, there cannot be two sides with no common vertex, each of which is longer than the longest diagonal.

Proposer's Solution.

Suppose a convex polygon has two sides, say AB and CD, which are longer than the longest diagonal, where A, B, C, D are distinct vertices and A, C are on opposite side of line BD. Since AC, BD are diagonals of the polygon, we have $AB > AC$ and $CD > BD$. Hence,

$$AB + CD > AC + BD.$$

By convexity, the intersection O of diagonals AC and BD is on these diagonals. By triangle inequality, we have

$$AO + BO > AB \text{ and } CO + DO > CD.$$

So $AC + BD > AB + CD$, a contradiction.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **John PANAGEAS** (Kaisari High School, Athens, Greece), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (CUHK, Math Major, Year 1) and **YAU Chi Keung** (CNC Memorial College, Form 6).

Problem 182. Let a_0, a_1, a_2, … be a sequence of real numbers such that

$$a_{n+1} \ge a_n^2 + 1/5 \quad \text{for all } n \ge 0.$$

Prove that $\sqrt{a_{n+5}} \ge a_{n-5}^2$ for all $n \ge 5$. (*Source: 2001 USA Team Selection Test*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5) and **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School).

Adding $a_{n+1}-a_n^2 \ge 1/5$ for nonnegative integers $n = k, k+1, k+2, k+3, k+4$, we get

$$a_{k+5}-\sum_{n=k+1}^{k+4}(a_n^2-a_n)-a_k^2 \ge 1.$$

Observe that

$$x^2-x+1/4=(x-1/2)^2 \ge 0$$

implies $1/4 \ge -(x^2-x)$. Applying this to the inequality above and simplifying, we easily get $a_{k+5} \ge a_k^2$ for nonnegative integer k. Then $a_{k+10} \ge a_{k+5}^2 \ge a_k^4$ for nonnegative integer k. Taking square root, we get the desired inequality.

Other commended solvers: **POON Ming Fung** (STFA Leung Kau Kui College, Form 6) and **SIU Tsz Hang** (CUHK, Math Major, Year 1).

Problem 183. Do there exist 10 distinct integers, the sum of any 9 of which is a perfect square? (*Source: 1999 Russian Math Olympiad*)

Solution. **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **SIU Tsz Hang** (CUHK, Math Major, Year 1).

Let $a_1, a_2, \ldots, a_{10}$ be distinct integers and S be their sum. For $i = 1, 2, \ldots, 10$, we would like to have $S - a_i = k_i^2$ for some integer k_i. Let T be the sum of $k_1^2, \ldots, k_{10}^2$. Adding the 10 equations, we get $9S = T$. Then $a_i = S - (S - a_i) = (T/9) - k_i^2$. So all we need to do is to choose integers $k_1, k_2, \ldots, k_{10}$ so that T is divisible by 9. For example, taking $k_i = 3i$ for $i = 1, \ldots, 10$, we get 376, 349, 304, 241, 160, 61, –56, –191, –344, –515 for $a_1, \ldots, a_{10}$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5).

Problem 184. Let $ABCD$ be a rhombus with $\angle B = 60^\circ$. M is a point inside ΔADC such that $\angle AMC = 120^\circ$. Let lines BA and CM intersect at P and lines BC and AM intersect at Q. Prove that D lies on the line PQ. (*Source: 2002 Belarussian Math Olympiad*)

Solution. **John PANAGEAS** (Kaisari High School, Athens, Greece), and **POON Ming Fung** (STFA Leung Kau Kui College, Form 6).

Since $ABCD$ is a rhombus and $\angle ABC = 60^\circ$, we see $\angle ADC$, $\angle DAC$, $\angle DCA$, $\angle PAD$ and $\angle DCQ$ are all 60°.

Now

$$\angle CAM + \angle MCA = 180^\circ - \angle AMC = 60^\circ$$

and

$$\angle DCM + \angle MCA = \angle DCA = 60^\circ$$

imply $\angle CAM = \angle DCM$.

Since $AB \parallel CD$, we get

$$\angle APC = \angle DCM = \angle CAQ.$$

Also, $\angle PAC = 120^\circ = \angle ACQ$. Hence ΔAPC and ΔCAQ are similar. So $PA/AC = AC/CQ$.

Since $AC = AD = DC$, so $PA/AD = DC/CQ$. As $\angle PAD = 60^\circ = \angle DCQ$, so ΔPAD and ΔDCQ are similar. Then

$$\angle PDA + \angle ADC + \angle CDQ$$
$$= \angle PDA + \angle PAD + \angle APD = 180^\circ.$$

Therefore, P, D, Q are collinear.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Tsz Hang** (CUHK, Math Major, Year 1), **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School).

Problem 185. Given a circle of n lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided one also changes the state of every d-th bulb after it (where d is a divisor of n and is less than n), provided that all n/d bulbs were originally in the same state as one another. For what values of n is it possible to turn all the bulbs on by making a sequence of moves of this kind?

Solution.

Let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$ and the lights be at 1, $\omega, \omega^2, \ldots, \omega^{n-1}$ with the one at 1 on initially. If d is a divisor of n that is less than n and the lights at

$$\omega^a, \omega^{a+d}, \omega^{a+2d}, \cdots, \omega^{a+(n-d)}$$

have the same state, then we can change the state of these n/d lights. Note their sum is a geometric series equal to

$$\omega^a(1-\omega^n)/(1-\omega^d) = 0.$$

So if we add up the numbers corresponding to the lights that are on before and after a move, it will remain the same. Since in the beginning this number is 1, it will never be

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$$

Therefore, all the lights can never be on at the same time.

Comments: This problem was due to Professor James Propp, University of Wisconsin, Madison (see his website http://www.math.wisc.edu/~propp/) and was selected from page 141 of the highly recommended book by Paul Zeitz titled *The Art and Craft of Problem Solving*, published by Wiley.

Olympiad Corner

(continued from page 1)

Problem 4. Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB respectively. Show that $PQ = QR$ if and only if the bisector of $\angle ABC$ and $\angle ADC$ meet on AC.

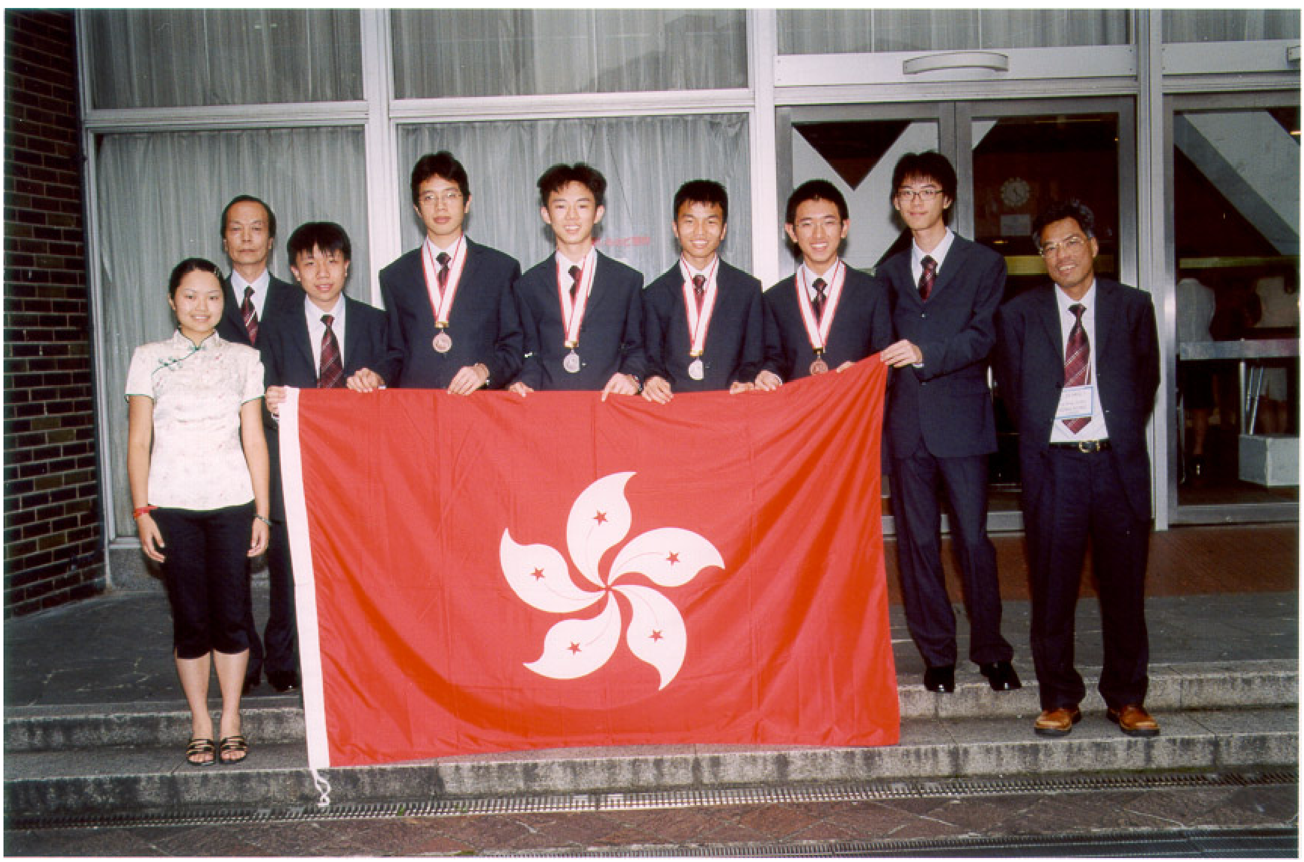
Problem 5. Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be real numbers with $x_1 \le x_2 \le \cdots \le x_n$.

(a) Prove that

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$

(b) Show that equality holds if and only if $x_1, x_2, \ldots, x_n$ is an arithmetic sequence.

Problem 6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.



The 2003 Hong Kong IMO team from left to right: Wei Fei Fei (Guide), Leung Chit Wan (Deputy Leader), Chung Tat Chi, Siu Tsz Hang, Kwok Tsz Chiu, Yu Hok Pun, Yeung Kai Sing, Lau Wai Shun, Leung Tat Wing (Leader).

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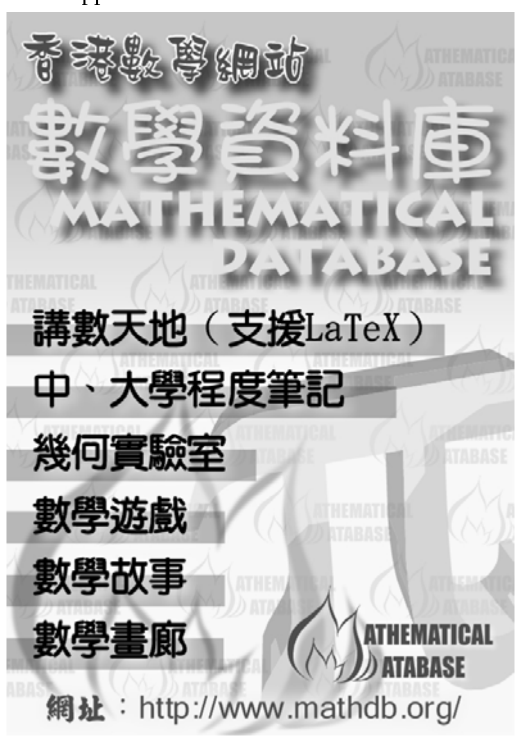

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後記：如果各位有興趣爲數學資料庫出一分力，我們無任歡迎。請即與數學資料庫 (webmaster@mathdb.org) 聯絡。

Mathematical Excalibur

Volume 8, Number 5 *November 2003 – December 2003*

Olympiad Corner

The 2003 USA Mathematical Olympiad took place on May 1. Here are the problems.

Problem 1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

Problem 2. A convex polygon P in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygons P are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Problem 3. Let $n \neq 0$. For every sequence of integers $A = a_0, a_1, a_2, \ldots, a_n$ satisfying $0 \leq a_i \leq i$, for $i = 0, \ldots, n$, define another sequence $t(A) = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$ by setting $t(a_i)$ to be the number of terms in the sequence A that precede the terms a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

(continued on page 4)

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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集與子集族

梁達榮

眾所周知，如果 S 是一個含 n 個元素的集，則它有 2^n 個子集，(包含空集及 S 本身)。不過如果選取子集的條件有所限制，例如子集只能有最多 k 個元素，或者所選取的兩個子集都必須相交（或不相交）等，則所能選取的子集必相應減少。

反過來說，如果 S 含一固定數目的子集，而這些子集又適合某些條件，則 n 的值不可能太小，又或者可以推到這些子集必須含有一些共同元素等。

這一類問題，泛稱集與子集族的問題，已經有很多有趣的成果。另外這些問題很能考驗學生的分析能力，並且需要的數學知識較少，所以在數學比賽中亦經常出現。先舉一個較簡單的例子。

<u>例一</u>：(蘇聯數學競賽 1965) 有一個委員會共舉行了 40 次會議，每次會議共有 10 人參加。並且每 2 個委員最多共一起參加同一會議 1 次。試證該委會員組成人數必多於 60 人。

<u>證明</u>：每一個會議有 10 人參加，因此共有 $C_2^{10} = 45$ “對”委員。按條件每一對委員不會在其他會議中出現，即 40 個會議共產生 $40 \times 45 = 1{,}800$ 不同的委員對。

如果該委員會有 n 人，則有 $C_2^n = \dfrac{n(n-1)}{2}$ 不同的對。所以必有 $1800 \leq \dfrac{n(n-1)}{2}$，解之即得 $n > 60$。

如果所有委員組成一個集，則每一個會議參加的 10 人就是這個集的一個子集。按條件即是說任意這樣的兩個子集的交集最多包含一個元素。現在這樣的子集共 40 個，可以推斷這個集不可能太小、考慮這些問題，一個可能的策略是，尋找一個適當的觀察量，再用兩個不同的角度估計這個量，在這個例子中我們考慮的是共同參加同一會議的委員對。

<u>例一的另一證明</u>：我們也可以從以下的角度考慮此一問題。因為有 40 次會議，每次有 10 人參加，所以共 400 “人次”參加這些會議。假設這個委員會的總人數不多於 60 人，因為 $400/60 \approx 6.67$，則其中 1 人必參加 7 個或以上的會議。但是按照條件，參加這 7 個或以上會議的其他委員都不可能相同，因此共有 $7 \times 9 = 63$ 或以上不同的委員，矛盾！（留意在這裏用到鴿巢原理。）

有時候這一類問題可以另外的形式出現：

<u>例二</u>：(奧地利－波蘭數學競賽 1978) 有 1978 個集，每集含 40 個元素，並且任兩集剛好有 1 個共同元素。試證這 1978 個集必含有 1 個共同元素。

<u>證明</u>：設 A 為其中一個集，考慮其他 1977 個集，每一個集與 A 都有一個共同元素。由於 $1977/40 \approx 49.43$，即是說，A 中必有一個元素 x 在另外 50 個集 $A_1, A_2, \ldots, A_{50}$ 內，且因條件所限，x 是 $A_1, A_2, \ldots, A_{50}$ 的惟一公共元。

考慮另外一個集 B，如果 x 不在 B 內，由於 B 和 $A_1, A_2, ..., A_{50}$ 都相交，且由條件所限，相交的元素都不同，則 B 最少有 51 個元素，這是不可能的。所以 x 在 B 內，且 B 是任意的，所以 x 在任一個集內，證畢。

這個結果可以這樣推廣，且證明完全相似：設有 n 個集，每一個集有 k 個元素，任意兩集剛好有一個共同元素。如果 $n > k^2 - k + 1$，則這 n 個集有一個共同元素。

考慮一個較為困難的例子:

例三：(俄羅斯數學競賽 1996) 由 1600 個議員組成 16000 個委員會，每個委員會由 80 個委員組成。試證明：一定存在兩個委員會，它們之間至少有 4 個相同的議員。

證明：這一次我們不考慮每一個委員會組成委員的對，反過來考慮每一個議員所參加委員會形成的對。設議員 1, 2, … , 1600 分別參加了 $k_1, k_2, \ldots, k_{1600}$ 個委員會，則總共有 $C_2^{k_1} + C_2^{k_2} + \ldots + C_2^{k_{1600}}$ 個委員會對。如果委員會的數目是 N, 則 $k_1 + k_2 + \ldots + k_{1600} = 80N$，(在題中 $N = 16000$，且每個委員會由 80 人組成。) 現在試圖估計這些委員會對

$$C_2^{k_1} + C_2^{k_2} + \cdots + C_2^{k_{1600}}$$

$$= \frac{\sum_{i=1}^{1600} k_i^2 - \sum_{i=1}^{1600} k_i}{2}$$

$$\geq \frac{(\sum_{i=1}^{1600} k_i)^2}{3200} - \frac{(\sum_{i=1}^{1600} k_i)}{2}$$

$$= \frac{(80N)^2}{3200} - \frac{80N}{2}$$

$$= 2N^2 - 40N = 2N(N-20)。$$

如果任兩個委員會最多有 3 個共同議員，則最多有

$$3C_2^N = \frac{3N(N-1)}{2}$$

個委員會對。因此

$$2N(N-20) \leq \frac{3}{2}N(N-1)。$$

即 $N \leq 77$，與 $N = 16, 000$ 矛盾。

(留意在估計中用到 Cauchy-Schwarz Inequality。) 無獨有偶，我們有以下的例子:

例四：(IMO1998) 在一次比賽中，有 m 個比賽員和 n 個評判，其中 $n \geq 3$ 是一個奇數。每一個評判對每一個比賽員進行評審為合格或不合格。如果任一對評判最多對 k 個比賽員的評審一致，試證明

$$\frac{k}{m} \geq \frac{n-1}{2n}。$$

證明: 題目已經提醒我們，我們考慮的是評判所成的“對”，這些”對”評判員對某些比賽員的決定一致。對於比賽員 i，$1 \leq i \leq m$，如果有 x_i 個評判認為他合格，y_i 個評判認為他不合格，則評判一致的對是

$$C_2^{x_i} + C_2^{y_i}$$

$$= \frac{(x_i^2 + y_i^2) - (x_i + y_i)}{2}$$

$$\geq \frac{(x_i + y_i)^2}{4} - \frac{(x_i + y_i)}{2}$$

$$= \frac{1}{4}n^2 - \frac{n}{2} = \frac{1}{4}\left[(n-1)^2 - 1\right]。$$

因為 n 是奇數，而 $C_2^{x_i} + C_2^{y_i}$ 是整數，因為 $C_2^{x_i} + C_2^{y_i}$ 最少是 $\frac{1}{4}(n-1)^2$。現在因為有 n 個評判，而任一對評判最多對 k 個比賽員意見一致，因為一致的評判最多是 kC_2^n。所以

$$kC_2^n \geq \sum_{i=1}^{m}\left[C_2^{x_i} + C_2^{y_i}\right] \geq \frac{m(n-1)^2}{4}$$

，化簡結果即為所求。

現在考慮一個形式略為不同的題目。我們的對象是一些長為 n 的數列，這些數列只包括 0 或 1，兩個這樣的數列的“距離”定義為對應位置數字不同的個數。例如 1101011 和 1011000 為兩個長為 7 的數列，它們在位置 2,3,6,7 的數字不同，因此它們的距離是 4。用集的言語來描述是，有 7 個元素 1,2,3,4,5,6 和 7 的一個集，數列一在位置 1,2,4,6,7 非零，因此可想像是包括 1,2,4,6,7 的一個子集，數列二是包括 1,3 和 4 的子集，屬於數列一或數列二，但不同時屬於兩個數列的子集包括 2,3,6,7，稱為兩個子集的對稱差，而“距離”正好是對稱差所含元素的數目。現在可以考慮的是，給定 n 和距離的限制,，這樣的數列最多是多少。

例五：有 m 個包括 0 或 1，長為 n 的數列，如果任兩個數列間的距離最少為 d，試證明

$$m \leq \frac{2d}{2d-n}。$$

證明：現在要考慮的是任兩個數列中“相異對”的數目，因為有 C_2^m 對數列，而任一對數列的“相異對”或“距離最少是 d，因此總距離最少是 dC_2^m。將這些數列排起來成為 m 個橫行，每一直行 j, $1 \leq j \leq n$ 就對應著那些數列的 j 位置。如果 j 直行有 x_j 個 "0"，則有 $m - x_j$ 個 "1"，因此相異對有 $x_j(m - x_j)$ 個。觀察到

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***February 28, 2004***.

Problem 191. Solve the equation

$$x^3-3x=\sqrt{x+2}.$$

Problem 192. Inside a triangle *ABC*, there is a point *P* satisfies $\angle PAB = \angle PBC = \angle PCA = \varphi$. If the angles of the triangle are denoted by α, β and γ, prove that

$$\frac{1}{\sin^2\varphi}=\frac{1}{\sin^2\alpha}+\frac{1}{\sin^2\beta}+\frac{1}{\sin^2\gamma}.$$

Problem 193. Is there any perfect square, which has the same number of positive divisors of the form $3k + 1$ as of the form $3k + 2$? Give a proof of your answer.

Problem 194. (*Due to Achilleas Pavlos PORFYRIADIS, American College of Thessaloniki "Anatolia", Thessaloniki, Greece*) A circle with center *O* is internally tangent to two circles inside it, with centers O_1 and O_2, at points *S* and *T* respectively. Suppose the two circles inside intersect at points *M*, *N* with *N* closer to *ST*. Show that *S*, *N*, *T* are collinear if and only if $SO_1/OO_1 = OO_2/TO_2$.

Problem 195. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Given *n* ($n > 3$) points on a plane, no three of them are collinear, *x* pairs of these points are connected by line segments. Prove that if

$$x \ge \frac{n(n-1)(n-2)+3}{3(n-2)},$$

then there is at least one triangle having these line segments as edges.

Find all possible values of integers $n > 3$ such that $\frac{n(n-1)(n-2)+3}{3(n-2)}$ is an integer and the minimum number of line segments guaranteeing a triangle in the above situation is this integer.

****

Solutions

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Problem 186. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Let α, β, γ be complex numbers such that

$$\alpha+\beta+\gamma=1,$$
$$\alpha^2+\beta^2+\gamma^2=3,$$
$$\alpha^3+\beta^3+\gamma^3=7.$$

Determine the value of $\alpha^{21}+\beta^{21}+\gamma^{21}$.

Solution. **Helder Oliveira de CASTRO** (Colegio Objetivo, 3rd Grade, Sao Paulo, Brazil), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 7), **FOK Kai Tung** (Yan Chai Hospital No. 2 Secondary School, Form 7), **FUNG Chui Ying** (True Light Girls' College, Form 6), **Murray KLAMKIN** (University of Alberta, Edmonton, Canada), **LOK Kin Leung** (Tuen Mun Catholic Secondary School, Form 6), **SIU Ho Chung** (Queen's College, Form 5), **YAU Chi Keung** (CNC Memorial College, Form 7) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Using the given equations and the identities

$$(\alpha+\beta+\gamma)^2=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha),$$
$$(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$$
$$=\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma,$$

we get $\alpha\beta+\beta\gamma+\gamma\alpha=-1$ and $\alpha\beta\gamma=1$. These imply α, β, γ are the roots of $f(x) = x^3-x^2-x-1=0$. Let $S_n=\alpha^n+\beta^n+\gamma^n$, then $S_1 = 1$, $S_2 = 3$, $S_3 = 7$ and for $n > 0$,

$$S_{n+3}-S_{n+2}-S_{n+1}-S_n$$
$$=\alpha^n f(\alpha)-\beta^n f(\beta)-\gamma^n f(\gamma)=0.$$

Using this recurrence relation, we find $S_4=11$, $S_5=21$, … , $S_{21}=361109$.

Problem 187. Define $f(n) = n!$. Let

$$a = 0.f(1)f(2)f(3)\ldots.$$

In other words, to obtain the decimal representation of a write the numbers $f(1)$, $f(2)$, $f(3)$, … in base 10 in a row. Is *a* rational? Give a proof. (*Source: Israeli Math Olympiad*)

Solution. **Helder Oliveira de CASTRO** (Colegio Objetivo, 3rd Grade, Sao Paulo, Brazil), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6), **Murray KLAMKIN** (University of Alberta, Edmonton, Canada) and **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Assume *a* is rational. Then its decimal representation will eventually be periodic. Suppose the period has *k* digits. Then for every $n > 10^k$, $f(n)$ is nonzero and ends in at least *k* zeros, which imply the period cannot have *k* digits. We got a contradiction.

Problem 188. The line *S* is tangent to the circumcircle of acute triangle *ABC* at *B*. Let *K* be the projection of the orthocenter of triangle *ABC* onto line *S* (i.e. *K* is the foot of perpendicular from the orthocenter of triangle *ABC* to *S*). Let *L* be the midpoint of side *AC*. Show that triangle *BKL* is isosceles. (*Source: 2000 Saint Petersburg City Math Olympiad*)

Solution. **SIU Ho Chung** (Queen's College, Form 5).

Let *O*, *G* and *H* be the circumcenter, centroid and orthocenter of triangle *ABC* respectively. Let *T* and *R* be the projections of *G* and *L* onto line *S*. From the Euler line theorem (cf. *Math Excalibur, vol.* 3, *no.* 1, *p.*1), we know that *O*, *G*, *H* are collinear, *G* is between *O* and *H* and 2 *OG* = *GH*. Then *T* is between *B* and *K* and 2 *BT* = *TK*.

Also, *G* is on the median *BL* and 2 *LG* = *BG*. So *T* is between *B* and *R* and 2 *RT* = *BT*. Then 2 *BR* = 2 (*BT* + *RT*) = *TK* + *TB* = *BK*. So *BR* = *RK*. Since *LR* is perpendicular to line *S*, by Pythagorean theorem, *BL*=*LK*.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6) and **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Problem 189. $2n + 1$ segments are marked on a line. Each of the segments intersects at least *n* other segments. Prove that one of these segments intersect all other segments. (*Source 2000 Russian Math Olympiad*)

Solution. **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

We imagine the segments on the line as intervals on the real axis. Going from left to right, let I_i be the i-th segment we meet with $i = 1, 2, \ldots, 2n+1$. Let I_i^l and I_i^r be the left and right endpoints of I_i respectively. Now I_1 contains $I_2^l, \ldots, I_{n+1}^l$. Similarly, I_2 which already intersects I_1 must contain $I_3^l, \ldots, I_{n+1}^l$ and so on. Therefore the segments $I_1, I_2, \ldots, I_{n+1}$ intersect each other.

Next let I_k^r be the rightmost endpoint among $I_1^r, I_2^r, \ldots, I_{n+1}^r$ $(1 \le k \le n+1)$. For each of the n remaining intervals $I_{n+2}, I_{n+3}, \ldots, I_{2n+1}$, it must intersect at least one of $I_1, I_2, \ldots, I_{n+1}$ since it has to intersect at least n intervals. This means for every $j \ge n+2$, there is at least one $m \le n+1$ such that $I_j^l \le I_m^r \le I_k^r$, then I_k intersects I_j and hence every interval.

Problem 190. (*Due to Abderrahim Ouardini*) For nonnegative integer n, let $[x]$ be the greatest integer less than or equal to x and

$$f(n) = \left[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}\right] - \left[\sqrt{9n+1}\right].$$

Find the range of f and for each p in the range, find all nonnegative integers n such that $f(n) = p$.

Combined Solution by the Proposer and **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6).

For positive integer n, we claim that

$$\sqrt{9n+8} < g(n) < \sqrt{9n+9},$$

where

$$g(n) = \sqrt{n} + \sqrt{n+1} + \sqrt{n+2}.$$

This follows from

$$g(n)^2 = 3n + 3 + 2(\sqrt{n(n+1)} + \sqrt{(n+1)(n+2)} + \sqrt{(n+2)n})$$

and the following readily verified inequalities for positive integer n,

$$(n+0.4)^2 < n(n+1) < (n+0.5)^2,$$

$$(n+1.4)^2 < (n+1)(n+2) < (n+1.5)^2$$

and $(n+0.7)^2 < (n+2)n < (n+1)^2$. The claim implies the range of f is a subset of nonnegative integers.

Suppose there is a positive integer n such that $f(n) \ge 2$. Then

$$\sqrt{9n+9} > [g(n)] > 1 + \sqrt{9n+1}.$$

Squaring the two extremes and comparing, we see this is false for $n > 1$. Since $f(0) = 1$ and $f(1) = 1$, we have $f(n) = 0$ or 1 for all nonnegative integers n.

Next observe that

$$\sqrt{9n+8} < [g(n)] < \sqrt{9n+9}$$

is impossible by squaring all expressions. So $[g(n)] = [\sqrt{9n+8}]$.

Now $f(n) = 1$ if and only if $p = [g(n)]$ satisfies $[\sqrt{9n+1}] = p - 1$, i.e.

$$\sqrt{9n+1} < p \le \sqrt{9n+8}.$$

Considering squares (mod 9), we see that $p^2 = 9n + 4$ or $9n + 7$.

If $p^2 = 9n+4$, then $p = 9k+2$ or $9k+7$. In the former case, $n = 9k^2 + 4k$ and $(9k+1)^2 \le 9n+1 = 81k^2 + 36k + 1 < (9k+2)^2$ so that $[\sqrt{9n+1}] = 9k+1 = p-1$. In the latter case, $n = 9k^2 + 14k + 5$ and $(9k+6)^2 \le 9n+1 = 81k^2 + 126k + 46 < (9k+7)^2$ so that $[\sqrt{9n+1}] = 9k+6 = p-1$.

If $p^2 = 9n+7$, then $p = 9k+4$ or $9k+5$. In the former case, $n = 9k^2 + 8k + 1$ and $(9k+3)^2 \le 9n+1 = 81k^2 + 72k + 10 < (9k+4)^2$ so that $[\sqrt{9n+1}] = 9k+3 = p-1$. In the latter case, $n = 9k^2 + 10k + 2$ and $(9k+4)^2 \le 9n+1 = 81k^2 + 90k + 19 < (9k+5)^2$ so that $[\sqrt{9n+1}] = 9k+4 = p-1$.

Therefore, $f(n) = 1$ if and only if n is of the form $9k^2 + 4k$ or $9k^2 + 14k + 5$ or $9k^2 + 8k + 1$ or $9k^2 + 10k + 2$.

Olympiad Corner

(continued from page 1)

Problem 4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Rays BA and ED intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Problem 5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

Problem 6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

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(continued from page 2)

$$x_j(m-x_j) \le \left[\frac{x_j+(m-x_j)}{2}\right]^2 = \frac{m^2}{4}$$

，因此

$$dc_2^m \le \sum_{j=1}^n x_j(m-x_j) \le \sum_{j=1}^n \frac{m^2}{4} = \frac{nm^2}{4}\text{。}$$

化簡即得 $m \le \dfrac{2d}{2d-n}$ 。

例如 $n = 7$，$d = 4$，得 $\dfrac{2d}{2d-n} = 8$ ，

所以不可能構造 9 個長為 7，而相互間最少距離為 4 的數列。(讀者可試圖構造 8 個這樣的數列。) 這個例子實際上是編碼理論一個結果的特殊情況，這個結果一般稱為 Plotkin 限 (Plotkin Bound)。

集和子集族還有許多有趣的結果，有待研究和討論。

Inclusion-Exclusion Principle and Turan's Theorem

LEUNG Tat-Wing

For a finite set A, the cardinality $|A|$ denote its number of elements. If there are two finite sets A and B, let $A \cup B$ denote the union of A and B. (It includes the elements in A or B.) Let $A \cap B$ denote the intersection of A and B. (It includes the elements in both A and B.) Everybody knows that if A and B do not have any common element, then $|A \cup B| = |A| + |B|$. However, if A and B have a common element x, then in counting the elements of A, x will be counted once, but in counting the elements of B, x will be counted once more. In order to avoid such repetition, in computing $|A \cup B|$, we have to subtract the number of repetitions, namely $|A \cap B|$. So $|A \cup B| = |A| + |B| - |A \cap B|$.

For the union $A \cup B \cup C$ of three sets, we can first compute the cardinalities of A, B and C. Adding them, we find it is too big. So we have to subtract the cardinalities of some intersections. Now the intersection of any two of A, B and C can be $A \cap B, A \cap C$ or $B \cap C$. When we subtract the number of elements in these intersections, we find it becomes too small. Finally we have to add the number of elements in the intersection of the three sets. At the end $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

In general, if we have n finite sets $A_1, A_2, \ldots, A_n$, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$, where the first sum on the right side is the total of the cardinalities of A_1 to A_n, the second sum is the total of the cardinalities of the intersection of every two sets and so on until we get to the intersection of $A_1, A_2, \ldots, A_n$.

The equation above is generally called the Inclusion-Exclusion Principle, whose name is obvious. It can be proved by mathematical induction. Moreover it can also be proved by binomial theorem like the following. For x belongs to $A_1 \cup A_2 \cup \cdots \cup A_n$, let x belongs to k A_i $(k \ge 1)$, say for convenience, x belongs to $A_1, A_2, \ldots, A_k$, but does not belong to $A_{k+1}, \ldots, A_n$. Then the "contribution" of x in $A_1 \cup A_2 \cdots \cup A_n$ is 1. In the first sum on right side, the "contribution" of x is $k = C_1^k$. In the second sum, as x appears in $A_1, A_2, \ldots, A_k$, x will appear in the intersesection of every two of them. So the "contribution" in the second sum is C_2^k. Analyzing these further, we will find the sum of all "contributions" of x on the right side is $C_1^k - C_2^k + C_3^k - \cdots + (-1)^{k+1} C_k^k = 1 - (1-1)^k = 1$. Note we have used the binomial theorem. As the contribution of x on both sides are equal, we have obtained a proof of the Inclusion-Exclusion Principle.

Furthermore the binomial coefficients have the following properties. When $m \le \frac{k}{2}$, C_m^k increases. When $k \ge \frac{k}{2}$, it decreases. (For example, when $k = 5$, we have $C_0^5 < C_1^5 < C_2^5 = C_3^5 > C_4^5 > C_5^5$, C_m^5 reaches maximum when $k = 2, 3$. When $k = 6$, $C_0^6 < C_1^6 < C_2^6 < C_3^6 > C_4^6 > C_5^6 > C_6^6$, C_m^6 reaches maximum when $k = 3$.) Using this relation, the reader can prove that if in the right side of the inclusion-exclusion formula, deleting a positive term and all other terms that follow it, the left side will become greater than the right side. This is because the contribution by x on the right will become nonpositive. Or the contribution in the deleted terms is nonnegative. Similarly, if in the right side of the inclusion-exclusion formula, deleting a negative term and all other terms that follow it, the left side will become less than the right side. This is a useful estimate.

The use of the Inclusion-Exclusion Principle in computing the sizes of sets appears often, with a wide range of applications.

Example 1: This is a classical problem. Take a rearrangement of the numbers $1, 2, \ldots, n$. If no number occupied the same position as before, then we say it is a derangement. (For example, 4321 is a derangement, but 4213 is not.) Now, how many derangements are there?

Solution: Obviously, there are $n! = n \times (n-1) \times \cdots \times 1$ rearrangements. If we try to find the number of derangement directly, this is not easy. So for $1 \le i \le n$, we define A_i to be the set of rearrangements having i in the correct position. It is easy to see that $|A_i| = (n-1)!$, similarly, $|A_i \cap A_j| = (n-2)!$, here $i \ne j$, and so on. Hence

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

$$= n(n-1)! - C_2^n(n-2)! + C_3^n(n-3)! - \cdots + (-1)^{n-1}1 = n! - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1}\frac{n!}{n!}.$$

Finally, the number of derangements is $n! - |A_1 \cup A_2 \cup \cdots \cup A_n| = n!(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!})$.

Example 2: (IMO 1991) Let $S = \{1, 2, \ldots, 280\}$. Find the smallest natural number n such that every n element subset of S contains 5 pairwise relatively prime numbers.

Solution: First we use the inclusion-exclusion principle to get $n \geq 217$. Let A_1, A_2, A_3, A_4 be the subsets of S containing multiples of $2, 3, 5, 7$, respectively. Then $|A_1| = 140, |A_2| = 93, |A_3| = 56, |A_4| = 40, |A_1 \cap A_2| = 46, |A_1 \cap A_3| = 28, |A_1 \cap A_4| = 20, |A_2 \cap A_3| = 18, |A_2 \cap A_4| = 13, |A_3 \cap A_4| = 8, |A_1 \cap A_2 \cap A_3| = 9, |A_1 \cap A_2 \cap A_4| = 6, |A_1 \cap A_3 \cap A_4| = 4, |A_2 \cap A_3 \cap A_4| = 2, |A_1 \cap A_2 \cap A_3 \cap A_4| = 1$. So $|A_1 \cap A_2 \cap A_3 \cap A_4| = 140 + 93 + 56 + 40 - 46 - 28 - 20 - 18 - 13 - 8 + 9 + 6 + 4 + 2 - 1 = 216$. For this 216 element set, among any 5 numbers, there must be two both belong to A_1, A_2, A_3 or A_4, hence not relatively prime. According to the problem, we must have $n \geq 217$.

Now we prove that every 217 element subset of S must have 5 pairwise relatively prime numbers. The idea is to construct proper "pigeonholes". Here is an elegant construction. Let A be a subset of S with $|A| \geq 217$. Define $B_1 = \{1 \text{ or prime numbers in } S\}$, $|B_1| = 60, B_2 = \{2^2, 3^2, 5^2, 7^2, 11^2, 13^2\}, |B_2| = 6, B_3 = \{2\times 131, 3\times 89, 5\times 53, 7\times 37, 11\times 23, 13\times 19\}, |B_3| = 6, B_4 = \{2\times 127, 3\times 87, 5\times 47, 7\times 31, 11\times 19, 13\times 17\}, |B_4| = 6, B_5 = \{2\times 113, 3\times 79, 5\times 43, 7\times 27, 11\times 17\}, |B_5| = 5, B_6 = \{2\times 109, 3\times 73, 5\times 41, 7\times 23, 11\times 13\}, |B_6| = 5$. It is easy to see B_1 and B_6 are disjoint. Also $|B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6| = 88$. Removing these 88 numbers, S still has $280 - 88 = 192$ numbers. Now A has at least 217 elements, $217 - 192 = 25$, that is, there are at least 25 elements in A that belong to B_1 to B_6. Obviously it cannot be that every B_i only contains 4 or less elements of A. That is, there are at least 5 elements of A belong to the same B_i, hence are relatively prime. Notwe we have used another principle: pigeonhole principle.

Example 3: (1989 IMO) Let n be a positive integer. We say a permutation $(x_1, x_2, \ldots, x_{2n})$ of $\{1, 2, \ldots, 2n\}$ has property P if and only if there is at least one i in $\{1, 2, \ldots, 2n-1\}$ such that $|x_i - x_{i+1}| = n$ holds. Prove that there are more permutations with property P than those permutations without property P.

Solution: Note if $|x_i - x_{i+1}| = n$, then one of x_i or x_{i+1} must be less than $n+1$. For $k = 1, 2, \ldots, n$, define A_k to be the set of all permutations having k and $k+n$ next to each other. It is easy to see that $|A_k| = 2 \times (2n-1)!$. (This is because k and $k+n$ are grouped together, their positions may be interchanged, think of them as one "number", there are $2n-2$ others, and so $(2n-2)+1$ positions for any number.) Also $|A_k \cap A_h| = 2^2 \times (2n-2)!, 1 \leq k < h \leq n$, ($k$ and $k+n$ are grouped as one "number" and h and $h+n$ are grouped as one "number".) So the number of permutations with property P is

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \geq \sum_{k=1}^{n} |A_k| - \sum_{1 \leq k < h \leq n} |A_k \cap A_h| = 2 \times (2n-1)! \times n - C_2^n \times 2^2 \times (2n-2)!$$
$$= 2n \times (2n-2)! \times n = (2n)! \times \frac{n}{2n-1} > (2n)! \times \frac{1}{2}.$$

This number is more than half of $(2n)!$. So the permutations with property P is more than those permutations without property P. (Years ago this problem was considered difficult, but with inclusion-exclusion relations, this becomes easy.)

Example 4: Let n and k be positive integers, $n > 3$, $\frac{n}{2} < k < n$. There are n points on the plane, every three of them are not collinear. If every point is connected to at least k other points by segments, then there are three segments forming a triangle.

Solution: Since $n > 3, k > \frac{n}{2}$, so $k \geq 2$. Hence among the n points, there are two points v_1 and v_2 that are connected by a segments. Consider the remaining points. Let A be the set of points connected to v_1 and B be the set of points connected to v_2, then $|A| \geq k-1, |B| \geq k-1$. Also,

$$n - 2 \geq |A \cup B| = |A| + |B| - |A \cap B| \geq 2k - 2 - |A \cap B|,$$

that is $|A \cap B| \geq 2k - n > 0$. So there exists a point v_3 connected to v_1 and v_2 forming a triangle.

Example 5: There was 1990 mathematicians participated in a meeting. Every one of them has collaborated with at least 1327 others. Prove that we can find 4 mathematicians, every pair of them collaborated with each other.

Solution: Consider the mathematicians as points of a set. Connect pairs that collaborated with an edge to yield a graph. As the above example, for v_1 and v_2 that collaborated, they are connected. For the remaining points, let A be the set of collaborators of v_1 and B be the set of collaborators of v_2. Then $|A| \geq 1326, |B| \geq 1326$. Similarly,

$$|A \cap B| = |A| + |B| - |A \cup B| \geq 2 \times 1326 - 1998 = 664 > 0,$$

that is, we can find a mathematician v_3 that collaborated with v_1 and v_2. Let C be the set of mathematicians that collaborated with v_3 excluding v_1 and v_3. That is $|C| \geq 1325$. Also

$$1998 \geq |(A \cap B) \cup C| = |A \cap B| + |C| - |A \cap B \cap C|$$

that is $|A \cap B \cap C| \geq |A \cap B| + |C| - 1988 \geq 664 + 1325 - 1998 = 1 > 0$. So $A \cap B \cap C$ is nonempty. Take $v_4 \in A \cap B \cap C$. Then v_1, v_2, v_3, v_4 collaborated.

In graph theory terminologies, examples 4 and 5 can be interpreted as giving a graph of n vertices, to determine the least number of edges that will gaurantee the existence of a triangle (K_3) or a K_4 (a subgraph with four vertices, every two vertices are connected by an edge). Putting it in another way, for a graph of n vertices with no triangle, to determine the maximum number of its edges. This area of graph theory is called extremal graph theory. The first result is the following:

Mantel's Theorem (1907): For a simple graph with n vertices containing no K_3, the maximum number of edges is $[\frac{n^2}{4}]$.

(Here $[x]$ is the greatest integer less than or equal to x. In example four, the number of edges is greater than $\left(\frac{n}{2}\right) \times n \times \frac{1}{2} > \left[\frac{n^2}{4}\right]$, then the result follows immediately.)

A more delicate result is the following:

Theorem: If a graph with n vertices has q edges, then the graph has at least $\dfrac{4q(q - \frac{n^2}{4})}{3n}$ triangles.

Example 6: There are 21 points on a circle. Among the angles formed by extending pairs of points to the center, there are at most 110 of these are greater than $120°$.

Solution: If the angle formed by extending two points to the center is greater than $120°$, then connect these two points by an edge. This yields a graph. The graph has no triangles. So the number of edges is at most $\left[\frac{21^2}{4}\right] = \left[\frac{441}{4}\right] = 110$, or there can be at most 110 such angles greater than $120°$.

As above, define K_p to be a p vertices complete graph, that is any two of the p vertices are connected by an edge. For a graph G with n vertices, if it does not contain any K_p, then what is the maximum number of edges G can have?

Turan's Theorem (1941): If a graph G with n vertices does not contain any K_p, then that graph has at most $\dfrac{p-2}{2(p-1)}n^2 - \dfrac{r(p-1-r}{2(p-1)}$ edges, where r is defined by $n = k(p-1) + r, 0 \leq r < p-1$. As in the situation of Mantel's theorem, this is a starting point of extremal graph theory.

Paul Turan (1910-1976) was a Jewish Hungarian. At the time he was considering these kinds of problems, he was in a concentration camp!

Using GW-BASIC for Drawing Mandelbrot Sets

Mr. LEUNG Chi Kit

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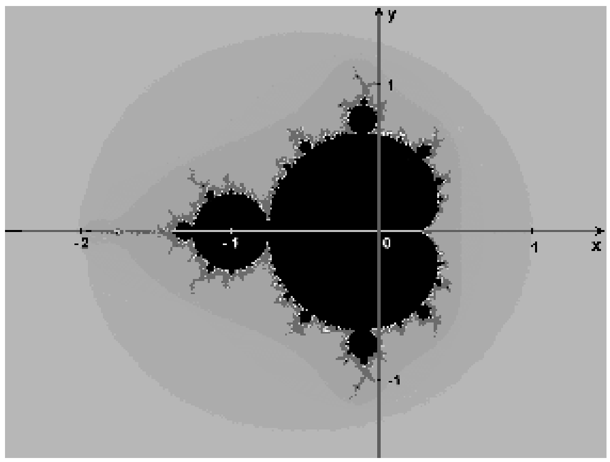
For a given complex number c_0, define the sequence of complex numbers { c_n }by $c_{n+1} = c_n^2 + c_0$ for $n = 0 , 1 , 2 , ….$ If the sequence is bounded, that is, we can find a real number M such that for every n, $| c_n | < M$, then c_0 is said to belong to the Mandelbrot set.



We can use the definition above to write a GW-BASIC program（Sorry！I always like to use the simplest computer language and I think it is not difficult to translate a GW-BASIC program to other computer language）as follow：

```
10      LEFT = 150 : TOP = 380 : W = 360 : M = .833
20      R = 2.64 : S = 2 * R / W
30      RECEN = 0 : IMCEN = 0
40      SCREEN 9 : CLS
50      FOR Y = 0 TO W
60              FOR X = 0 TO W
70                      REC = S * (X – W / 2) + RECEN : IMC = S * (Y – W / 2) + IMCEN
80                      RE = REC : IM = IMC
90                      RE2 = RE * RE : IM2 = IM * IM : J = 0
100                     WHILE RE2 + IM2 <= 256 AND J < 15
110                             IM = 2 * RE * IM + IMC
120                             RE = RE2 – IM2 + REC
130                             RE2 = RE * RE : IM2 = IM * IM : J = J + 1
140                     WEND
150                     IF J < 3 THEN GOTO 220
160                     IF J >= 3 AND J < 6 THEN COLOR 14 : REM YELLOW
```

```
170             IF J >= 6 AND J < 9 THEN COLOR 1 : REM BLUE
180             IF J >= 9 AND J < 12 THEN COLOR 2 : REM GREEN
190             IF J >= 12 AND J < 15 THEN COLOR 15 : REM WHITE
200             IF J >= 15 THEN COLOR 12 : REM RED
210             PSET (X + LEFT, (TOP – Y) * M)
220         NEXT X
230     NEXT Y
240     COLOR 15 : REM WHITE
250     LINE (LEFT, (TOP – W / 2) * M) – (W + LEFT, (TOP – W / 2) * M)
260     LINE (W / 2 + LEFT, (TOP – W) * M) – (W / 2 + LEFT, TOP * M)
270     END
```

The following explains the program：

W sets the size of the picture to be drawn on the computer screen. Initially W is set to 360（see line 10). This means we plan to draw the Mandelbrot set in a 360×360 square in the computer screen (see lines 50 and 60).

LEFT is the leftmost position of the picture on the screen, TOP is the lowest position of the picture (see lines 210, 250 and 260). Caution: in GW-BASIC, the coordinates of the computer screen go from top to bottom unlike our usual convention of going from bottom to top. So we have to use "TOP – Y" to convert the usual coordinate system to the computer screen coordinate system.

Since a pixel on the computer screen is not a square, so the horizontal and vertical sizes are not the same, hence we introduce M（$= \frac{5}{6}$）to adjust the length-to-width ratio（see lines 10, 210, 250 and 260).

Note W is only the size on the screen and not the actual coordinates of the complex numbers in the Mandelbrot set. So W needs to be transformed. R is the actual value（see line 20). That is, for the range of the picture, the real axis goes from –R to R and the imaginary axis also goes from –R to R. S computes the ratio of W and R and is used in later computations（see lines 20 and 70).

RECEN and IMCEN are used to locate the position of the center. The center is initialized to (0 , 0)（see line 30). By changing the value of R, RECEN or IMCEN, we can move or dilate the Mandelbrot set.

Line 40 chooses the format of the picture and erase the old screen.

Lines 50 and 60 of the program set the ranges of X and Y. Then line 70 computes the real and imaginary parts of the corresponding c_0.

Observe that if $c_0 = a_0 + b_0 i$，$c_n = a_n + b_n I$, then

$$\begin{aligned} c_{n+1} &= c_n^2 + c_0 \\ &= (a_n + b_n i)^2 + (a_0 + b_0 i) \\ &= a_n^2 - b_n^2 + 2a_nb_n i + a_0 + b_0 i \\ &= (a_n^2 - b_n^2 + a_0) + (2a_nb_n + b_0)i. \end{aligned}$$

So the real part of c_{n+1} is $a_n^2 - b_n^2 + a_0$ and the imaginary part is $2a_nb_n + b_0$.

Converting these computations to codes yield lines 110 and 120. REC and IMC are the real and imaginary parts of c_0 respectively. RE and IM are the real and imaginary parts of c_n respectively. RE2 and IM2 are the squares of the real and imaginary parts of c_n respectively.

J is a counter for running the loops in lines 100 and 140. Line 100 also computes the square of the modulus of c_n. If the square of the modulus is greater than 256 or the loop has been executed 15 times, then we terminate the loop. Consequently, the larger the value of J is, the closer the sequence will tend to "converge". That is, after many computations, the modulus of c_n still does not get big. Lines 150 and 200use colors to classify the rate of convergences. Red indicates the complex numbers with fastest converegence, then comes white, green, blue and yellow. The region with the fastest divergence is indicated in black. Line 210 chooses the color for the point.

After drawing the Mandelbrot set, we draw the horizontal and vertical axes in white（see lines 240 and 260) for reference. This concludes the program.

The running time for the program depends on the speed of the computer. For the present computers, the whole program can finish in about a minute.

<u>Reference</u>

Heinz-Otto Peitgen, Hartmut Jürgens and Dietmar Saupe (1992) *Fractals for the Classroom Part Two: Introduction to Fractals and Chaos.* NCTM, Springer-Verlag.

Sets and Subsets

T. W. Leung

It is well known that a set with n elements contains altogether 2^n subsets, (including the empty set and itself as subsets). However if we put restrictions on the choices of subsets, say each subset may contain at most k elements, or any two chosen subsets may have non-trivial (or trivial) intersection, then our choices will be more limited. Conversely if we can select from the subsets of a set with n elements a certain number of subsets, and these subsets satisfy some pre-defined properties, then n probably cannot be too small. Or perhaps we can deduce any two of these subsets must have non-trivial intersection, etc.

There have already been many results relating sets and subsets. On the other hand, as problems of this kind may test students' analytical ability, and the required knowledge in solving this kind of problems is usually minimal, they appear in mathematical contests quite frequently.

Example 1: (Soviet Union Mathematical Olympiad 1965) A committee had 40 meetings, with 10 committee members attended each meeting. Every two committee members attended a meeting together at most once. Show that there are more than 60 members in the committee.

Solution: Every meeting was attended by 10 members. Thus there were $C_2^{10} = 45$ pairs of committee members. According to the condition given each pair would not appear in another meeting, hence for 40 meetings, there were $40 \times 45 = 1800$ distinct pairs of committee members. Now the committee had n members and there were $C_2^n = \dfrac{n(n-1)}{2}$ distinct pairs. We must have $1800 \le \dfrac{n(n-1)}{2}$, solve to get $n > 60$. □

Using the set language, the committee members form a set. Members attending a particular meeting is a subset of this set. The condition implies the intersection of any two of these subsets contains at most 1 element. Now we have altogether 40 such subsets, and we can deduce that the size of the committee cannot be too small. One strategy is solving this type of problems is to look for a suitable observable quantity, then investigate this quantity using different perspectives. In this case the quantity we look for is the number of "pairs" of committee members attending a same meeting.

Alternative Solution of Example 1: The problem may be tackled as follows. Each committee was consisted of 10 members, hence altogether $40 \times 10 = 400$ "persons" attended the meetings. Suppose the committee had at most 60 members, since $\dfrac{400}{60} \approx 6.67$, we must have 1 member in the committee who attended at least 7 meetings. But from the condition given, the members (except this fellow) attending these 7 or more meeting must not be the same. Thus there are at least $7 \times 9 = 63$ members, a contradiction! (Note implicitly we have used the pigeon hole principle.) □

Example 2: (Austrian-Polish Mathematics Competition 1978) There are given 1978 sets, each containing 40 elements. Every two sets have exactly one element in common. Prove that all 1978 sets have a common element.

Solution: Suppose A is one of the sets. Consider the other 1977 sets, each set has a common element with A. Since $\dfrac{1977}{40} \approx 49.43$, there exists $x \in A$ which also belongs to 50 other sets $A_1, A_2, \cdots, A_{50}$ say, and x is the only common element of these sets. Consider now another set B. If $x \notin B$, as B intersects non-trivially with $A_1, A_2, \cdots, A_{50}$ and the intersecting elements cannot be common, hence B contains at least 50 elements, a contradiction. Thus we must have $x \in B$. B is arbitrary, hence x is in every other subset. The proof is complete. □

The result can be extended and the proof is entirely analogous: Suppose there are *n* sets, with each set containing *k* elements. Every two sets contain exactly one common element. If $n > k^2 - k + 1$, then the *n* sets contain a common element.

We now consider a more difficult example.

Example: (Soviet Union Mathematical Olympiad 1996) 16000 committees were formed with members coming from 1600 councillors. Each committee consisted of exactly 80 members. Show that there exist two committees, and they had at least 4 common members.

Solution: In this problem we do not consider pairs of members in each committee. Instead we consider from the perspective an individual councillor, and see how many pairs of committees he joined. Suppose councillors 1, 2, …, 1600 joined respectively $k_1, k_2, \cdots, k_{1600}$ committees. Hence altogether there are $C_2^{k_1} + C_2^{k_2} + ... + C_2^{k_{1600}}$ "committee pairs". Now there are N committees, ($N = 16000$), then $k_1 + k_2 + \cdots + k_{1600} = 80N$. We estimate these committee pairs

$$C_2^{k_1} + C_2^{k_2} + ... + C_2^{k_{1600}} = \frac{k_1(k_1-1)}{2} + \frac{k_2(k_2-1)}{2} + ... + \frac{k_{1600}(k_{1600}-1)}{2}$$

$$= \frac{k_1^2 + k_2^2 + ... + k_{1600}^2}{2} - \frac{k_1 + k_2 + ... + k_{1600}}{2} \geq \frac{1}{2}\frac{(k_1 + k_2 + ... + k_{1600})^2}{1600} - \frac{k_1 + k_2 + ... + k_{1600}}{2}$$

$$= \frac{1}{2}\frac{\left(80N\right)^2}{1600} - \frac{80N}{2} = 2N^2 - 40N = 2N\left(N - 20\right)$$

Suppose now every two committee have at most three common members, then can only be at most $3C_2^N = \frac{3N(N-1)}{2}$ committee pairs. Hence $2N(N-20) \leq \frac{3}{2}N(N-1)$, or $N \leq 77$, contradicting $N = 16000$.□

(Note that in the estimate we have made use of the Cauchy-Schwarz Inequality.)

Curiously enough, we found a similar example.

Example 4: (International Mathematical Olympiad 1998) In a contest there are *m* candidates and *n* judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most *k* candidates. Prove that $\frac{k}{m} \geq \frac{n-1}{2n}$.

Solution: The problem has reminded us to consider pairs of judges who agree on particular candidates. Consider candidate *i*, $1 \leq i \leq m$, suppose x_i judges consider him pass and y_i judges think he fail. Then the number of pairs of judges that agree on this candidate is

$$C_2^{x_i} + C_2^{y_i} = \frac{x_i\left(x_i - 1\right)}{2} + \frac{y_i\left(y_i - 1\right)}{2} = \frac{x_i^2 + y_i^2}{2} - \frac{x_i + y_i}{2} \geq \frac{1}{2}\frac{\left(x_i + y_i\right)^2}{2} - \frac{x_i + y_i}{2}$$

$$= \frac{1}{4}n^2 - \frac{n}{2} = \frac{1}{4}\left[\left(n-1\right)^2 - 1\right].$$

Now n is odd and $C_2^{x_i}+C_2^{y_i}$ is an integer, hence $C_2^{x_i}+C_2^{y_i}$ is at least $\frac{1}{4}(n-1)^2$. Also there are n judges and any pair of judges agree on at most k candidates, there are at most kC_2^n pairs that agree on certain candidates. Thus $kC_2^n \geq \sum_{i=1}^{m}\left[C_2^{x_i}+C_2^{y_i}\right] \geq \frac{m(n-1)^2}{4}$, simplify to obtain the desired result. □

We now consider a problem of slightly different flavor. Consider sequences of length n and such that each sequence contains only 0s or 1s. Define the "distance" between two such sequences as the number of positions they differ. For example 1101011 and 1011000 are two sequences of length 7, and they differ on positions 2, 3, 6 and 7, hence the distance between them is 4. Using set language, we say that a set contains 7 elements 1, 2, 3, 4, 5, 6 and 7, as sequence 1 are non-zero at positions 1, 2, 4, 6, and 7, and we may consider sequence 1 as a subset containing elements 1, 2, 4, 6 and 7. Likewise sequence 2 is a subset containing 1, 3 and 4. Elements belong to sequence 1 or sequence 2, but not both, in this case consist of 2, 3, 6 and 7, together form a subset, called the "symmetric difference" of the two subsets. One can verifies easily that the distance between the sequences corresponds precisely to the number of elements in the symmetric difference. Now our concern is, given n and a restriction on the distances between the sequences, at most how many sequences we can get?

<u>Example 5</u>: Given m sequences of length n and the sequences contain only 0s and 1s. Suppose the distance between any two such sequences is at least d, then $m \leq \frac{2d}{2d-n}$.

<u>Solution</u>: Consider any two sequences and the corresponding pairs that differ. First we note that there are C_2^m pairs of sequences. For any pair of sequences, the number of pairs that differ or "distance" between them is at least d. Hence the sum of total distances between all possible pairs of sequences is at least dC_2^m. Now we list all of these m sequences as m rows. Then each column j, $1 \leq j \leq n$, correspond to the j^{th} position of these m sequence. If the j^{th} column contains x_j 0s, then it will contain $m-x_j$ 1s. This column will contribute $x_j(m-x_j)$ pairs of positions that the sequences differ. Observe

$$x_j\left(m-x_j\right) \leq \left[\frac{x_j+\left(m-x_j\right)}{2}\right]^2 = \frac{m^2}{4},$$

thus $dC_2^m \leq \sum_{j=1}^{n} x_j\left(m-x_j\right) \leq \sum_{j=1}^{n}\frac{m^2}{4} = \frac{nm^2}{4}$. Simplify to get $m \leq \frac{2d}{2d-n}$. □

For instance, if $n=7$, $d=4$, then $\frac{2d}{2d-n}=8$. It is impossible to construct 9 sequences if the minimum distance between any pair is 4. (Our readers perhaps would like to construct 8 such sequences.) In fact this result is a special case of a theorem in Coding Theory, usually referred as a Plotkin Bound.

There are many interesting results concerning sets and subsets for us to investigate.