Junior problems

J295. Let a, b, c be positive integers such that $(a-b)^2+(b-c)^2+(c-a)^2 = 6abc$. Prove that $a^3+b^3+c^3+1$ is not divisible by $a+b+c+1$.

Proposed by Mihaly Bencze, Brasov, Romania

Solution by Raffaella Rodoquino and Alessandro Ventullo, Milan, Italy

Clearly,

$$(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2-ab-bc-ac)$$

which implies

$$a^2+b^2+c^2-ab-bc-ac = 3abc.$$

Now, using the identity

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

we obtain that

$$a^3+b^3+c^3-3abc = 3abc(a+b+c),$$

or

$$a^3+b^3+c^3 = 3abc(a+b+c+1)$$

which means that $a+b+c+1$ divides $a^3+b^3+c^3$. If $a+b+c+1$ divides $a^3+b^3+c^3+1$, then $a+b+c+1$ divides $(a^3+b^3+c^3+1)-(a^3+b^3+c^3) = 1$, which is impossible, since $a+b+c+1 > 1$.
Thus, $a^3+b^3+c^3+1$ is not divisible by $a+b+c+1$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Jin Hwan An, Seoul International School, Seoul, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA; Nikolaos Evgenidis, student, High-School of Agia,Thessalia, Greece; Sun Mengyue, Lansheng Fudan Middle School, Shanghai, China; Amedeo Sgueglia, Universitá degli studi di Padova, Italy; Arber Avdullahu, Mehmet Akif College, Kosovo; Arghya Datta, Hooghly Collegiate School, Kolkata, India; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Debojyoti Biswas, Uttarpara Govt. High School, West Bengal, India; Dimitris Oikonomou, 2nd Hight School, Nauplio, Greece; Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Ilyes Hamdi, Lycée Voltaire, Doha, Qatar; Joshua Benabou, Manhasset High School, NY, USA; Titu Zvonaru, Comanesti and Neculai Stanciu, Buzau, Romania; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Prithwijit De, HBCSE, Mumbai, India; Yeongwoo Hong,Seoul International School, South Korea; Polyahedra, Polk State College, FL, USA; Jishnu Bose, Uttarpara Govt. High School, Kolkata, India; Jeong Ho Ha, Ross School, East Hampton, NY, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

J296. Several positive integers are written on a board. At each step, we can pick any two numbers u and v, where $u \geq v$, and replace them with $u+v$ and $u-v$. Prove that after a finite number of steps we can never obtain the initial set of numbers.

Proposed by Marius Cavachi, Constanta, Romania

Solution by Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan

Let S_0 be the sum of all numbers on board and S_i be the sum of all numbers on board after the i-step. $u \geq v$ yields to $(u+v)+(u-v) \geq u+v$ and S_n is a non-decreasing sequence, $S_i \leq S_{i+1}$.

If $u > v$ are the selected numbers at the $(i+1)$-step, then $S_{i+1} > S_i$ and we can't obtain the initial set of numbers anymore. (1)

Thus, at the 1-step we need to consider equal numbers $u = v$. Hence, after the step, the board has one 0, all other numbers are positive integers. Due to our result in (1) we can't pick u together with 0. Therefore, always 0 has to be put on the board, but the numbers of the initial set are positive integers $\Rightarrow$ contradiction.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, Buzau, Romania; Yeongwoo Hong, Seoul International School, South Korea; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Debojyoti Biswas, Uttarpara Govt. High School, West Bengal, India; Arber Avdullahu, Mehmet Akif College, Kosovo; Alessandro Ventullo, Milan, Italy; Dmitry Chernyak, Lycée Stanislas, Paris, France; Jeong Ho Ha, Ross School, East Hampton, NY, USA; Jishnu Bose, Uttarpara Govt. High School, Kolkata, India; Polyahedra, Polk State College, FL, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Jin Hwan An, Seoul International School, South Korea.

J297. Let a, b, c be digits in base $x \geq 4$. Prove that

$$\frac{\overline{ab}}{\overline{ba}} + \frac{\overline{bc}}{\overline{cb}} + \frac{\overline{ca}}{\overline{ac}} \geq 3,$$

where all numbers are written in base x.

Proposed by Titu Zvonaru, Comanesti and Neculai Stanciu, Buzau, Romania

Solution by Arkady Alt, San Jose, California, USA

Note that $\dfrac{\overline{ab}}{\overline{ba}} + \dfrac{\overline{bc}}{\overline{cb}} + \dfrac{\overline{ca}}{\overline{ac}} \geq 3$ can be written as $\dfrac{ax+b}{bx+a} + \dfrac{bx+c}{cx+b} + \dfrac{cx+a}{ax+c} \geq 3$.

Since by Cauchy Inequality

$$\sum_{cyc} \frac{ax+b}{bx+a} = \sum_{cyc} \frac{(ax+b)^2}{(ax+b)(bx+a)} \geq \frac{\left(\sum_{cyc}(ax+b)\right)^2}{\sum_{cyc}(ax+b)(bx+a)} =$$

$$\frac{(a+b+c)^2(x+1)^2}{\sum_{cyc}\left(abx^2+\left(a^2+b^2\right)x+ab\right)} = \frac{(a+b+c)^2(x+1)^2}{(ab+bc+ca)x^2+2\left(a^2+b^2+c^2\right)x+ab+bc+ca}$$

it suffice to prove inequality

$$(a+b+c)^2(x+1)^2 \geq 3\left((ab+bc+ca)x^2+2\left(a^2+b^2+c^2\right)x+ab+bc+ca\right).$$

We have

$$(a+b+c)^2(x+1)^2 - 3\left((ab+bc+ca)x^2+2\left(a^2+b^2+c^2\right)x+ab+bc+ca\right) =$$

$\left(a^2+b^2+c^2-ab-bc-ca\right)\left(x^2-4x+1\right) \geq 0$ because $a^2+b^2+c^2 \geq ab+bc+ca$ and

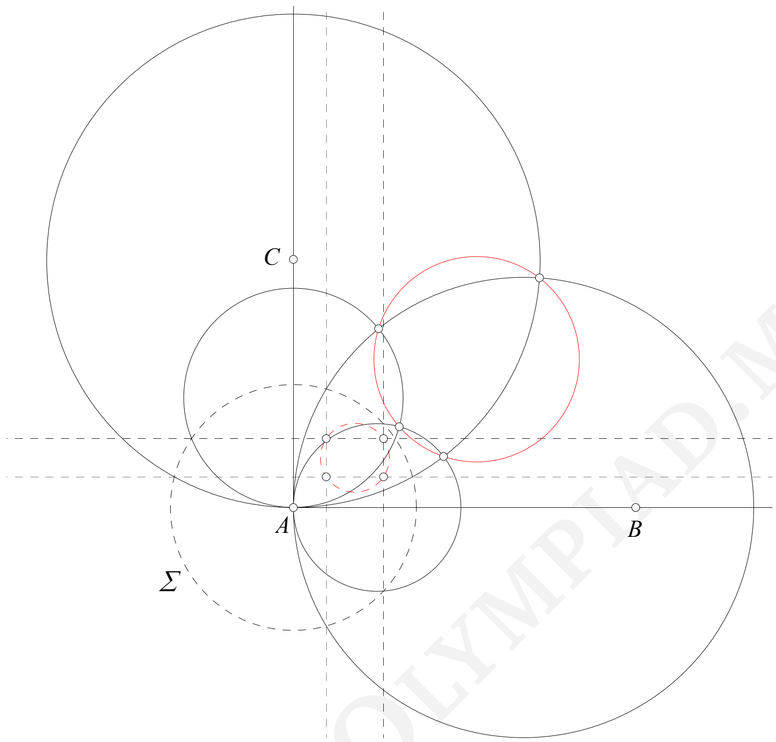
$$x^2-4x+1 = x(x-4)+1 \geq 1.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Yeongwoo Hong,Seoul International School, South Korea; Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Jeong Ho Ha, Ross School, East Hampton, NY, USA; Polyahedra, Polk State College, FL, USA; Jin Hwan An, Seoul International School, Seoul, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA.

J298. Consider a right angle $\angle BAC$ and circles $\omega_1, \omega_2, \omega_3, \omega_4$ passing through A. The centers of circles ω_1 and ω_2 lie on ray AB and the centers of circles ω_3 and ω_4 lie on ray AC. Prove that the four points of intersection, other than A, of the four circles are concyclic.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Polyahedra, Polk State College, USA



Let σ be the inversion in a circle Σ with center A. Then $\sigma(\omega_1)$ and $\sigma(\omega_2)$ are lines perpendicular to AB and $\sigma(\omega_3)$ and $\sigma(\omega_4)$ are lines perpendicular to AC. So the intersection points of these four lines form a rectangle and thus lie on a circle Ω'. Hence the other four intersection points of $\omega_1, \omega_2, \omega_3, \omega_4$ lie on the circle $\Omega = \sigma(\Omega')$.

Also solved by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, Buzau, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Dimitris Oikonomou, 2nd Hight School, Nauplio, Greece; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Dmitry Chernyak, Lycée Stanislas, Paris, France; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Polyahedra, Polk State College, USA; Jin Hwan An, Seoul International School, Seoul, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA.

J299. Prove that no matter how we choose n numbers from the set $\{1, 2, \ldots, 2n\}$, one of them will be a square-free integer.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by José Hernández Santiago, México

It suffices to show that, for every $n \in \mathbb{N}$, the number of natural numbers in the interval $[1, 2n]$ which are not square-free is less than n. Since the thesis of the problem is trivially true for $n = 1$, in what follows we suppose that $n > 1$.

Let us denote by $\overline{Q}_{2n}$ the set of natural numbers in the interval $[1, 2n]$ which are not square-free. Besides, if $p \in [1, 2n]$ is a prime a number, let us denote by $\mathcal{M}(p^2)$ the set of multiples of p^2 which belong to the interval $[1, 2n]$. So, if q is the greatest prime number in $[1, \sqrt{2n}]$ then

$$\overline{Q}_{2n} = \mathcal{M}(2^2) \cup \mathcal{M}(3^2) \cup \mathcal{M}(5^2) \cup \ldots \cup \mathcal{M}(q^2). \tag{1}$$

Since any natural number N has $\left\lfloor \frac{2n}{N} \right\rfloor$ multiples in the interval $[1, 2n]$, we obtain from (1) that

$$\begin{aligned} |\overline{Q}_{2n}| &\leq \left\lfloor \frac{2n}{2^2} \right\rfloor + \left\lfloor \frac{2n}{3^2} \right\rfloor + \left\lfloor \frac{2n}{5^2} \right\rfloor + \ldots + \left\lfloor \frac{2n}{q^2} \right\rfloor \\ &\leq \frac{2n}{2^2} + \frac{2n}{3^2} + \frac{2n}{5^2} + \ldots + \frac{2n}{q^2} \\ &= 2n \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{q^2} \right) \\ &< 2n \sum_p \frac{1}{p^2} \end{aligned} \tag{2}$$

where $\sum_p \frac{1}{p^2}$ is the series of the squared reciprocals of the prime numbers. This series is convergent and the well-known fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ implies that

$$\begin{aligned} \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &> 1 + \sum_p \frac{1}{p^2} + \sum_{n=2}^{\infty} \frac{1}{(2n)^2} \\ &= 1 + \sum_p \frac{1}{p^2} + \frac{1}{4} \left(\frac{\pi^2}{6} - 1 \right) \\ &= \frac{3}{4} + \frac{\pi^2}{24} + \sum_p \frac{1}{p^2}. \end{aligned} \tag{3}$$

From (2) and (3) we conclude that

$$|\overline{Q}_{2n}| < 2n \left(\frac{\pi^2}{8} - \frac{3}{4} \right) = n \left(\frac{\pi^2 - 6}{4} \right) < n$$

which was what we desired to establish.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Titu Zvonaru, Comanesti and Neculai Stanciu, Buzau, Romania; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Yeongwoo Hong,Seoul International School, South Korea; Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Arkady Alt, San Jose, California, USA; Arghya Datta, Hooghly Collegiate School, Kolkata, India; Arber Avdullahu, Mehmet Akif College, Kosovo; Alessandro Ventullo, Milan, Italy; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Jeong Ho Ha, Ross School, East Hampton, NY, USA; Polyahedra, Polk State College, FL, USA; Jin Hwan An, Seoul International School, Seoul, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA.

J300. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{\sqrt{2a^2+16ab+7b^2}+c}+\frac{c+a}{\sqrt{2b^2+16bc+7c^2}+a}+\frac{a+b}{\sqrt{2c^2+16ca+7a^2}+b}\geq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Sayan Das, Indian Statistical Institute, Kolkata
Note that for positive real x, y we have

$$\sqrt{2x^2+16xy+7y^2}\leq 2x+3y$$

since on squaring it is equivalent to $2(x-y)^2 \geq 0$. Therefore we have

$$\sum_{cyc}\frac{b+c}{\sqrt{2a^2+16ab+7b^2}+c}\geq\sum_{cyc}\frac{b+c}{2a+3b+c}$$

Again by Cauchy-Schwarz inequality we have

$$\sum_{cyc}\frac{b+c}{2a+3b+c}=\sum_{cyc}\frac{(b+c)^2}{2ab+2ac+4bc+3b^2+c^2}\quad\geq\frac{(2(a+b+c))^2}{4\sum_{cyc}a^2+8\sum_{cyc}ab}=1$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, Buzau, Romania; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Yeongwoo Hong,Seoul International School, South Korea; Jishnu Bose, Uttarpara Govt. High School, Kolkata, India; Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Arkady Alt, San Jose, California, USA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; An Zhen-ping, Xianyang Normal University, Xianyang, Shaanxi, China; Sun Mengyue, Lansheng Fudan Middle School, Shanghai, China; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Jeong Ho Ha, Ross School, East Hampton, NY, USA; Polyahedra, Polk State College, FL, USA; Jin Hwan An, Seoul International School, Seoul, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA.

Senior problems

S295. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\sum_{\text{cyc}} \frac{(a+\sqrt{b})^2}{\sqrt{a^2-ab+b^2}} \leq 12.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy
We know that $4(a^2+b^2-ab) \geq (a+b)^2$ since $(a-b)^2 \geq 0$. Consequently,

$$\sum_{\text{cyc}} 2\frac{(a+\sqrt{b})^2}{a+b} \leq 12$$

or

$$\sum_{\text{cyc}} 2\frac{a^2+2a\sqrt{b}+b}{a+b} \leq 12$$

Moreover,

$$\frac{a^2}{a+b} = a - \frac{ab}{a+b}, \qquad \frac{b}{a+b} = 1 - \frac{a}{a+b}$$

and then we come to

$$2(a+b+c)+6+4\sum_{\text{cyc}} \frac{a\sqrt{b}}{a+b} = 12+4\sum_{\text{cyc}} \frac{a\sqrt{b}}{a+b} \leq 12 + \sum_{\text{cyc}} 2\frac{ab+a}{a+b}$$

which is actually

$$2\sum_{\text{cyc}} \frac{a\sqrt{b}}{a+b} \leq \sum_{\text{cyc}} \frac{ab+a}{a+b}$$

The proof concludes by observing that

$$ab+a \geq 2a\sqrt{b}$$

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Arber Avdullahu, Mehmet Akif College, Kosovo; Arkady Alt, San Jose, California, USA; Mai Quoc Thang, Ho Chi Minh City, Vietnam; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Peter Tirtowijoyo Young, SMAK St. Louis 1 Surabaya, Indonesia; Sayak Mukherjee, Kolkata, India; Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, Buzau, Romania; Philip Radoslavov Grozdanov, Yambol, Bulgaria.

S296. A ball in Vienna is attended by n ladies (some of which are wearing red dresses) and m gentlemen. Some ladies and some gentlemen are acquainted. Dancing floor is occupied by acquainted mixed pairs. At some point during the night, all the present gentlemen were seen on the dancing floor. At some other time, all the ladies wearing red dresses were on the dancing floor. Show that at some point there could be all gentlemen and all red-dressed ladies on the dancing floor.

Proposed by Michal Rolinek, Institute of Science and Technology, Vienna

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let r be the number of ladies in red dresses. Clearly $r \leq m$, since otherwise not all ladies could ever be seen on the dancing floor simultaneously. If $r = m$, then all gentlemen must have been on the dancing floor when all the ladies in red dresses were on it. Otherwise, assume that $r < m$, and we will find an algorithm to simultaneously pair all r ladies in red dresses, and all m gentlemen, so that the resulting m pairs containing the $m + r$ people can be simultaneously on the dancing floor. Pair first all gentlemen with the ladies they were dancing with, when all m of them where on the dancing floor. If all ladies with red dresses are paired we are finished, otherwise pick a random lady with a red dress which is not paired, and pair her with the gentleman she was dancing with when all r ladies with red dresses were on the dancing floor. A new lady is left unpaired; if she does not wear a red dress, the number of unpaired ladies with red dresses has decreased; otherwise proceed with this new lady with a red dress as with the previous one. The process of pairing and unpairing ladies with red dresses cannot continue indefinitely, because 1) there is a finite number of them, and 2) the process cannot be cyclic since once a lady in a red dress is paired with a gentleman after being unpaired, he is the gentleman who was dancing with her when all r ladies with red dresses were on the dancing floor, and no other lady but her will be paired with him ever again. Note that at no point in the process any gentleman is left unpaired, and the number of unpaired ladies with red dresses decreases. The conclusion follows.

S297. Let ABC be a triangle and let $A_1, A_2, B_1, B_2, C_1, C_2$ be points that trisect segments BC, CA, AB, respectively. Cevians $AA_1, AA_2, BB_1, BB_2, CC_1, CC_2$ intersect each other at the vertices of a convex hexagon that does not have any intersection points inside it. Prove that if the hexagon is cyclic then our triangle is equilateral.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AC} = \vec{v}$. Wlog, we have

$$\overrightarrow{AC_1} = \frac{\vec{u}}{3}, \qquad \overrightarrow{AC_2} = \frac{2\vec{u}}{3}, \qquad \overrightarrow{AA_1} = \frac{2\vec{u}+\vec{v}}{3},$$

$$\overrightarrow{AA_2} = \frac{\vec{u}+2\vec{v}}{3}, \qquad \overrightarrow{AB_1} = \frac{2\vec{v}}{3}, \qquad \overrightarrow{AB_2} = \frac{\vec{v}}{3}.$$

Note that the vertices of the hexagon are $AA_1 \cap BB_2$, $AA_1 \cap CC_2$, $BB_1 \cap CC_2$, $BB_1 \cap AA_2$, $CC_1 \cap AA_2$ and $CC_1 \cap BB_2$. For instance, segment AA_1 is the set of points X such that $\overrightarrow{AX} = \rho\overrightarrow{AA_1}$ for $\rho \in [0,1]$, while segment BB_2 is the set of points Y such that $\overrightarrow{AY} = \kappa\overrightarrow{AB} + (1-\kappa)\overrightarrow{AB_2}$ for $\kappa \in [0,1]$. It follows that $P_{A,B} = AA_1 \cap BB_2$ occurs for $\rho = \frac{3}{5}$, $\kappa = \frac{2}{5}$, or $\overrightarrow{AP_{A,B}} = \frac{2\vec{u}+\vec{v}}{5}$. Similarly, we can define $P_{A,C} = AA_1 \cap CC_2$, $P_{B,C} = BB_1 \cap CC_2$, $P_{B,A} = BB_1 \cap AA_2$, $P_{C,A} = CC_1 \cap AA_2$ and $P_{C,B} = CC_1 \cap BB_2$, obtaining after some algebra

$$\overrightarrow{AP_{A,C}} = \frac{2\vec{u}+\vec{v}}{4}, \qquad \overrightarrow{AP_{B,C}} = \frac{2\vec{u}+2\vec{v}}{5}, \qquad \overrightarrow{AP_{B,A}} = \frac{\vec{u}+2\vec{v}}{4},$$

$$\overrightarrow{AP_{C,A}} = \frac{\vec{u}+2\vec{v}}{5}, \qquad \overrightarrow{AP_{C,B}} = \frac{\vec{u}+\vec{v}}{4}.$$

Now, convex quadrilateral $P_{C,B}P_{B,C}P_{B,A}P_{C,A}$ is cyclic iff the sine of the angle between $\overrightarrow{P_{C,B}P_{B,C}}$ and $\overrightarrow{P_{C,B}P_{C,A}}$ equals the sine between $\overrightarrow{P_{B,A}P_{C,A}}$ and $\overrightarrow{P_{B,A}P_{B,C}}$, or using the vector product and dropping irrelevant multiplicative constant factors, we must have

$$\frac{(\vec{u}+\vec{v}) \times (3\vec{v}-\vec{u})}{|\vec{u}+\vec{v}| \cdot |3\vec{v}-\vec{u}|} = \frac{(-\vec{u}-2\vec{v}) \times (3\vec{u}-2\vec{v})}{|-\vec{u}-2\vec{v}| \cdot |3\vec{u}-2\vec{v}|}$$

Note now that both numerators equal $4\vec{u} \times \vec{v}$ and $8\vec{u} \times \vec{v}$ respectively, or this result is equivalent to

$$(c^2+4b^2+4k)(9c^2+4b^2-12k) = |-\vec{u}-2\vec{v}|^2 \cdot |3\vec{u}-2\vec{v}|^2 = 4\,|\vec{u}+\vec{v}|^2 \cdot |3\vec{v}-\vec{u}|^2 =$$
$$= 4(c^2+b^2+2k)(9b^2+c^2-6k),$$

where we have used that $\vec{u} \cdot \vec{u} = c^2$, $\vec{v} \cdot \vec{v} = b^2$ and we have denoted $k = \vec{u} \cdot \vec{v}$. It follows that

$$k = \vec{u} \cdot \vec{v} = \frac{4b^4 - c^4}{8(c^2-2b^2)}.$$

Inverting the role of $\vec{u}, \vec{v}$ and b, c, we conclude that quadrilateral $P_{C,B}P_{B,C}P_{A,C}P_{A,B}$ is cyclic iff

$$k = \vec{u} \cdot \vec{v} = \frac{4c^4 - b^4}{8(b^2-2c^2)},$$

or for both quadrilaterals to be simultaneously cyclic, it is necessary that

$$0 = 2b^6 - 7b^4c^2 + 7b^2c^4 - 2c^6 = \left(b^2-c^2\right)\left(2b^4 - 5b^2c^2 + 2c^4\right).$$

In either case, it is not hard to see that $\cos A = \frac{k}{bc}$ must be negative, or ABC must be obtuse at A. By cyclic symmetry, it follows that ABC must be obtuse at each angle, absurd. Therefore, the hexagon can never be cyclic. Indeed, drawing an equilateral triangle and forming the described hexagon, it is not hard to check that it is NOT cyclic, even when ABC is equilateral.

Also solved by Peter Tirtowijoyo Young, SMAK St. Louis 1 Surabaya, Indonesia.

S298. Prove the following identity

$$\sum_{k=0}^{a}(-1)^{a-k}\binom{a}{k}\binom{b+k}{c}=\binom{b}{c-a}.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

First solution by G.R.A.20 Problem Solving Group, Roma, Italy

Given $a \geq 0$, let

$$F(b,c)=\sum_{k=0}^{a}(-1)^{a-k}\binom{a}{k}\binom{b+k}{c}.$$

For $c \geq 0$, for $b = c - a$ the identity follows from the fact that for $k > n$, $\binom{n}{k} = 0$:

$$\begin{aligned} F(c-a,c) &= \sum_{k=0}^{a}(-1)^{a-k}\binom{a}{k}\binom{c-a+k}{c} \\ &= (-1)^{a-a}\binom{a}{a}\binom{c-a+a}{c} = 1 = \binom{c-a}{c-a}. \end{aligned}$$

Now we prove the full identity by induction on $b \geq c - a$.

$$\begin{aligned} F(b+1,c) &= \sum_{k=0}^{a}(-1)^{a-k}\binom{a}{k}\binom{b+1+k}{c} \\ &= \sum_{k=0}^{a}(-1)^{a-k}\binom{a}{k}\left(\binom{b+k}{c-1}+\binom{b+k}{c}\right) \\ &= F(b,c-1)+F(b,c) = \binom{b}{c-1-a}+\binom{b}{c-a} = \binom{b+1}{c-a}. \end{aligned}$$

Second solution by Albert Stadler, Herrliberg, Switzerland

We express the binomial coefficients in terms of complex integrals, using Cauchy's theorem:

$$\binom{a}{k} = \frac{1}{2\pi i} = \oint_{|u|=3} \frac{(1+u)^a}{u^{k+1}} du, \binom{b+k}{c} = \frac{1}{2\pi i} \oint_{|v|=1} \frac{(1+v)^{b+k}}{v^{c+1}} dv$$

So,

$$\sum_{k=0}^{a} (-1)^{a-k} \binom{a}{k} \binom{b+k}{c} = \sum_{k=0}^{\infty} (-1)^{a-k} \binom{a}{k} \binom{b+k}{c} =$$

$$\sum_{k=0}^{\infty} (-1)^{a-k} \frac{1}{2\pi i} \oint_{|u|=3} \frac{(1+u)^a}{u^{k+1}} du \frac{1}{2\pi i} \oint_{|v|=1} \frac{(1+v)^{b+k}}{v^{c+1}} dv =$$

$$(-1)^a \frac{1}{2\pi i} \oint_{|u|=3} \frac{(1+u)^a}{u} \frac{1}{2\pi i} \oint_{|v|=1} \frac{(1+v)^b}{v^{c+1}} \cdot \frac{1}{1+\dfrac{1+v}{u}} dv du =$$

$$(-1)^a \frac{1}{2\pi i} \oint_{|v|=1} \frac{(1+v)^b}{v^{c+1}} \cdot \frac{1}{2\pi i} \oint_{|u|=3} \frac{(1+u)^a}{u+v+1} du dv =$$

$$(-1)^a \frac{1}{2\pi i} \oint_{|v|=1} \frac{(1+v)^b (-v)^a}{v^{c+1}} dv = \frac{1}{2\pi i} \oint_{|v|=1} \frac{(1+v)^b}{v^{c-a+1}} dv = \binom{b}{c-a}.$$

Remark: We have used the residue theorem to deduce that:

$$\frac{1}{2\pi i} \oint_{|u|=3} \frac{(1+u)^a}{u+v+1} du = (-v)^a.$$

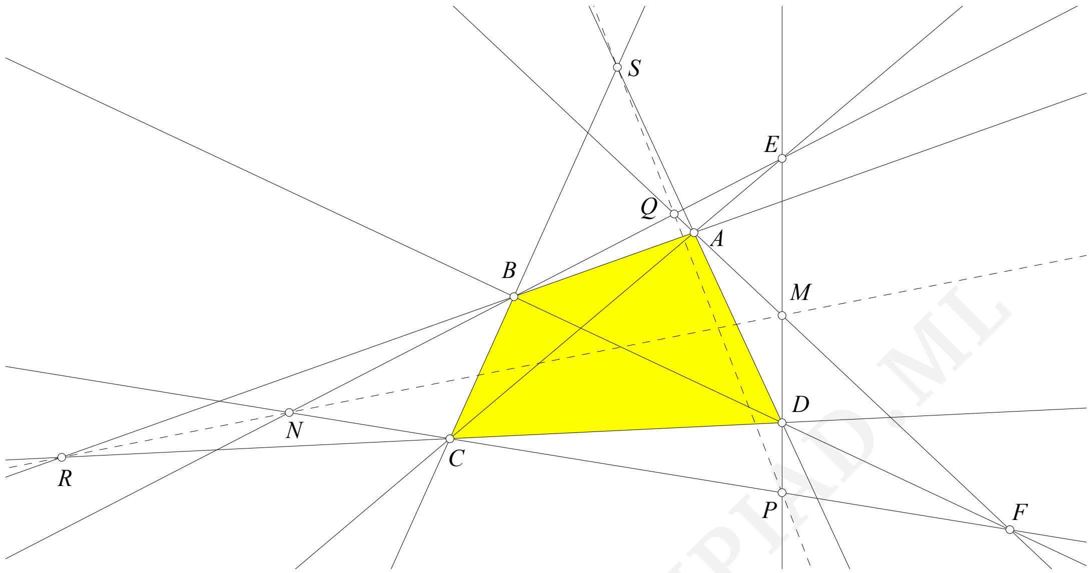
Also solved by G.R.A.20 Problem Solving Group, Roma, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; G. C. Greubel, Newport News, VA, USA; Dimitris Oikonomou, 2nd Hight School, Nauplio, Greece; Arkady Alt, San Jose, California, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

S299. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let P an arbitrary point in its plane. If $\{E\} = PD \cap AC$, $\{F\} = PC \cap BD$, $\{M\} = PD \cap AF$ and $\{N\} = PC \cap BE$, prove that $MN \parallel AB$.

Proposed by Mihai Miculita and Marius Stanean, Romania

Solution by Li Zhou, Polk State College, FL, USA

We prove the more general result that if $ABCD$ is a quadrilateral then AB, CD and MN are concurrent.



Let $Q = AF \cap BE$ and $R = AB \cap MN$. Let $f : X \mapsto X'$ be a projective transformation such that $P'M'Q'N'$ is a square. Then $A'C' \parallel M'P'$ and $B'D' \parallel N'P'$. Hence, by Menelaus' theorem,

$$-\frac{M'R'}{R'N'} = \frac{B'Q'}{N'B'} \cdot \frac{A'M'}{Q'A'} = \frac{D'M'}{P'D'} \cdot \frac{C'P'}{N'C'},$$

which implies that $R = CD \cap MN$ as well.

Also solved by Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, Buzau, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S300. Let x, y, z be positive real numbers and $a, b > 0$ such that $a + b = 1$. Prove that

$$(x+y)^3(y+z)^3 \geq 64abxy^2z(ax+y+bz)^2$$

Proposed by Marius Stanean, Zalau, Romania

Solution by Li Zhou, Polk State College, FL, USA

Let $A, B \in (0, \pi/2)$ such that $\tan^2 A = x/y$ and $\tan^2 B = z/y$. Then the claimed inequality becomes

$$\sec^6 A \sec^6 B \geq 64ab \tan^2 A \tan^2 B \left(a \sec^2 A + b \sec^2 B\right)^2,$$

which is equivalent to

$$1 \geq 8\sqrt{ab} \sin A \sin B \left(a \cos^2 B + b \cos^2 A\right).$$

Now by the AM-GM inequality,

$$\begin{aligned} 8\sqrt{ab} \sin A \sin B \left(a \cos^2 B + b \cos^2 A\right) &\leq \left(2\sqrt{ab} \sin A \sin B + a \cos^2 B + b \cos^2 A\right)^2 \\ &= \left[1 - \left(\sqrt{a} \sin B - \sqrt{b} \sin A\right)^2\right]^2 \leq 1, \end{aligned}$$

completing the proof.

Undergraduate problems

U295. Let a be a real number such that $(\lfloor na \rfloor)_{n\geq 1}$ is an arithmetic sequence. Prove that a is an integer.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Corneliu Manescu-Avram, Transportation High School, Ploiesti, Romania

We have $a = \lfloor a \rfloor + \{a\}$ and $\lfloor na \rfloor = n\lfloor a \rfloor + \lfloor n\{a\} \rfloor$, so we can suppose without loss of generality, that $0 \leq a < 1$, since the first terms form an arithmetic sequence.

It remains to prove that $f(n) = \lfloor (n+1)a \rfloor - \lfloor na \rfloor$ is a constant only for $a = 0$. Indeed, for $a \neq 0$ there is a positive integer m such that $\frac{1}{m+1} \leq a < \frac{1}{m}$, which implies $\lfloor (m-1)a \rfloor = \lfloor ma \rfloor = 0$, $\lfloor (m+1)a \rfloor = 1$, so that $f(m-1) = 0$ and $f(m) = 1$.

For $a \neq 0$ that function f is nonconstant, which ends the proof.

Also solved by Philip Radoslavov Grozdanov, Yambol, Bulgaria; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Alessandro Ventullo, Milan, Italy; Arghya Datta, Hooghly Collegiate School, Kolkata, India; Arkady Alt, San Jose, California, USA; Jishnu Bose, Uttarpara Govt. High School, Kolkata, India; Sayak Mukherjee, Kolkata, India.

U296. Let a and b be real nonzero numbers and let $z_0 \in \mathbb{C} \setminus \mathbb{R}$ be a root to the equation $z^{n+1} + az + nb = 0$, where n is a positive integer. Prove that $|z_0| \geq \sqrt[n+1]{b}$.

Proposed by Mihaly Bencze, Brasov, Romania

Solution by Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan

Let $z_0 = |z_0|(\cos\alpha + i\sin\alpha) = |z_0|e^{i\alpha}$, where $\sin\alpha \neq 0$.

Since the given equation we have

$$|z_0|^{n+1}\cos(n+1)\alpha + a|z_0|\cos\alpha + nb = 0$$

and

$$|z_0|^{n+1}\sin(n+1)\alpha + a|z_0|\sin\alpha = 0.$$

So

$$|z_0|^{n+1}\sin n\alpha = nb\sin\alpha$$

and since $\sin\alpha \neq 0$ we get that $\sin n\alpha \neq 0$ and

$$|z_0|^{n+1} = \frac{nb\sin\alpha}{\sin n\alpha} \qquad (1).$$

We can use by induction and will prove that $|\sin n\alpha| \leq n|\sin\alpha|$ and since $\sin\alpha \neq 0$, $|\sin n\alpha| < n|\sin\alpha|$. Hence see (1) and we have

$$|z_0|^{n+1} = |b|\frac{n|\sin\alpha|}{|\sin n\alpha|} \geq |b|.$$

This gives us, $|z_0| \geq \sqrt[n+1]{|b|} \geq \sqrt[n+1]{b}$ and we are done.

Also solved by Corneliu-Mănescu-Avram, Transportation High School, Ploiesti, Romania.

U297. Let $a_0 = 0$, $a_1 = 2$, and $a_{n+1} = \sqrt{2 - \frac{a_{n-1}}{a_n}}$ for $n \geq 1$. Find $\lim_{n\to\infty} 2^n a_n$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
First, we will show, by induction on n, that

$$a_n = 2\sin\frac{\pi}{2^n}.$$

Note

$$2\sin\frac{\pi}{2^0} = 2\sin 0 = 0 = a_0 \quad \text{and} \quad 2\sin\frac{\pi}{2^1} = 2\sin\frac{\pi}{2} = 2\cdot 1 = 2 = a_1.$$

Then

$$\begin{aligned} a_{n+1} &= \sqrt{2 - \frac{a_{n-1}}{a_n}} = \sqrt{2 - \frac{2\sin\frac{\pi}{2^{n-1}}}{2\sin\frac{\pi}{2^n}}} \\ &= \sqrt{2 - 2\cos\frac{\pi}{2^n}} = \sqrt{4\sin^2\frac{\pi}{2^{n+1}}} = 2\sin\frac{\pi}{2^{n+1}}, \end{aligned}$$

as required. Finally,

$$\lim_{n\to\infty} 2^n a_n = \lim_{n\to\infty} 2^{n+1}\sin\frac{\pi}{2^n} = \lim_{n\to\infty}\left(2\pi\cdot\frac{\sin\frac{\pi}{2^n}}{\frac{\pi}{2^n}}\right) = 2\pi.$$

Also solved by Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee, Lycée Henri IV, Paris, France; Moubinool Omarjee, Lycée Henri IV, Paris, France; G. C. Greubel, Newport News, VA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Arkady Alt, San Jose, California, USA; Alessandro Ventullo, Milan, Italy; Dmitry Chernyak, Lycée Stanislas, Paris, France; Li Zhou, Polk State College, Winter Haven, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; N.J. Buitrago A., Universidade de São Paulo, São Paulo, SP, Brazil.

U298. Determine all pairs (m, n) of positive integers such that the polynomial

$$f = (X+Y)^2(mXY+n)+1$$

is irreducible in $\mathbb{Z}[X,Y]$.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

When $m=0$, we have $f(X,Y) = nX^2+2nXY+nY^2+1$. Assume that his polynomial is reducible, or since its degree is 2, it must be the product of first degree polynomials, ie integers a,b,c and u,v,w exist such that

$$nX^2+2nXY+nY^2+1 = (aX+bY+c)(uX+vY+w) = \\ auX^2+(av+bu)XY+bvY^2+(aw+cu)X+(bw+cv)Y+cw.$$

Since $cw=1$ for c,w integers, and we may simultaneously change the signs of a,b,c,u,v,w without altering the problem, we may assume wlog that $c=w=1$, yielding $u=-a$, $v=-b$, for $a^2=b^2=ab=-n$, or the polynomial is reducible when $m=0$ iff $n=-k^2$ for some nonzero integer k, in which case

$$f(X,Y) = -k^2X^2-2k^2XY-k^2Y^2+1 = (1+kX+kY)(1-kX-kY).$$

For f to be reducible and not of this form, m must be nonzero, which we will assume to hold for during the rest of this solution.

Assume now that m is nonzero, or the degree of f is 4. If f is reducible, then every factoring polynomial that is not symmetric on X,Y (because f is symmetric on X,Y), must be paired with another one, such that one becomes the other after exchanging X,Y. Since the highest degree terms are X^3Y, X^2Y^2, XY^3 but not X^4 or Y^4, then X^2, Y^2 cannot appear in the same polynomial unless it is symmetric on X,Y, and X^3, Y^3 cannot appear in any polynomial. We conclude that all possible ways to factor f are (or can be reduced to) the following:

$$(aX^2+bXY+cX+dY+e)(aY^2+bXY+dX+cY+e) = \\ ab(X^3Y+XY^3)+(a^2+b^2)X^2Y^2+ad(X^3+Y^3)+(ac+bc+bd)(X^2Y+XY^2)+ \\ (ae+cd)(X^2+Y^2)+(c^2+d^2+2be)XY+(c+d)e(X+Y)+e^2,$$

or

$$(aX^2+aY^2+bXY+cX+cY+d)(uXY+vX+vY+w) = \\ au(X^3Y+XY^3)+buX^2Y^2+av(X^3+Y^3)+(av+bv+cu)(X^2Y+XY^2)+ \\ (aw+cv)(X^2+Y^2)+(bw+2cv+du)XY+(cw+dv)(X+Y)+dw,$$

or

$$(aX^2Y+aXY^2+bX^2+bY^2+cXY+dX+dY+e)(uX+uY+v) = \\ au(X^3Y+XY^3)+2auX^2Y^2+bu(X^3+Y^3)+(av+bu+cu)(X^2Y+XY^2)+ \\ (bv+du)(X^2+Y^2)+(cv+2du)XY+(dv+eu)(X+Y)+ve,$$

and there can be no other, since the product of two polynomials in X,Y, one of which is the result of exchanging X,Y in the other, becomes a symmetric polynomial in X,Y.

We analyze case by case, identifying the coefficients of corresponding terms and noting that

$$f(X,Y) = m(X^3Y + XY^3) + 2mX^2Y^2 + n(X^2 + Y^2) + 2nXY + 1$$

Case 1: We must have $e^2 = 1$, or since we may exchange signs in all coefficients without altering the problem, we may assume wlog that $e = 1$, in which case $d = -c$, $a = n + c^2$ and $b = 2n - 2c^2$. It follows that $0 = ac + bc + bd = ac = cn + c^3$, yielding either $c = 0$ or $n = -c^2$. In the first case, we have $a = n$, $b = 2n$, which result in $m = ab = 2n^2$ and $2m = a^2 + b^2 = 5n^2$, in contradiction with $m \neq 0$. Therefore, f cannot be expressed as the product of two polynomials in the first form.

Case 2: We must have $dw = 1$, or again wlog $d = w = 1$, yielding $v = -c$. If $c \neq 0$, then $a = 0$ for $m = 0$, in contradiction with $m \neq 0$, or $c = v = 0$, for $a = n$ and $b + u = 2n$. Moreover, $bu = 2m = 2au$, or since $m \neq 0$, we have $b = 2a = 2n$, for $u = 0$, contradiction. Therefore, f cannot be expressed as the product of two polynomials in the second form.

Case 3: Note that if $u = 0$, then $m = 0$, contradiction, or $u \neq 0$ yielding $b = 0$, for $du = n$ and $cv = 0$. At the same time, $ve = 1$, or again $v = e = 1$, yielding $c = 0$, and now $a = 0$, or $m = 0$, contradiction. Therefore, f cannot be expressed either as the product of two polynomials in the third and last form.

We conclude that f is irreducible for any nonzero m, and for $m = 0$ as long as n is not of the form $-k^2$ for some nonzero integer k.

Also solved by Jishnu Bose, Uttarpara Govt. High School, Kolkata, India.

U299. Let ABC be a triangle with incircle ω and let A_0, B_0, C_0 be points outside ω. Tangents from A_0 to ω intersect BC at A_1 and A_2. Points B_1, B_2 and C_1, C_2 are defined similarly. Prove that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a conic if and only if triangle ABC and $A_0B_0C_0$ are perspective.

Proposed by Luis Gonzalez, Maracaibo, Venezuela

No solutions have been received yet.

U300. Let $f : [a, b] \to [a, b]$ be a function having lateral limits in every point. If

$$\lim_{t\to x^-} f(t) \leq \lim_{t\to x^+} f(t)$$

for all $x \in [a, b]$, prove that there is an $x_0 \in [a, b]$ such that $\lim_{t\to x_0} f(t) = x_0$.

Proposed by Dan Marinescu and Mihai Piticari, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain
Denote

$$g(x) = \lim_{t\to x^+} f(t) - x, \qquad h(x) = \lim_{t\to x^-} f(t) - x.$$

Note that g is defined over $[a, b)$, and h over $(a, b]$. The proposed result is equivalent to the following: x_0 exists such that either $x_0 = a$ and $g(a) = 0$, $x_0 = b$ and $h(b) = 0$, or $x_0 \in (a, b)$ and $g(x_0) = h(x_0) = 0$. Note also that $g(x) \geq h(x)$ at every point in (a, b), where $g(x) = h(x)$ iff f is continuous at x, and that there are finitely many discontinuity points in (a, b); if there would be infinitely many, there would be a cumulation point $x \in [a, b]$ in whose vecinity there would be infinitely many discontinuities, and at least one of the lateral limits at such x would not exist. Clearly, g, h are continuous wherever f is continuous. Note finally that, since f takes values in $[a, b]$, we have $g(a) \geq 0$ and $h(b) \leq 0$.

Assume that the proposed result is false, or $g(a) > 0$ and $h(b) < 0$. Let $a < x_1 < x_2 < \cdots < x_n < b$ be the finitely many discontinuity points in (a, b). Since f is continuous in (a, x_1), and $g(x) = h(x) \neq 0$ in (a, x_1), we have $h(x_1) > 0$. But $g(x_1) > h(x_1)$ because x_1 is a discontinuity point, or $g(x_1) > 0$. For all $i = 1, 2, \ldots, n-1$, if $g(x_i) > 0$, and since f is continuous in (x_i, x_{i+1}), we have again $g(x_{i+1}) > h(x_{i+1}) > 0$, or after trivial induction $g(x_n) > 0$, and $h(b) > 0$. But $h(b) < 0$, or we have reached a contradiction, hence the proposed result follows.

Also solved by Sayak Mukherjee, Kolkata, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

O295. Let a, b, c, x, y, z be positive real numbers such that $x + y + z = 1$ and

$$2ab + 2bc + 2ca > a^2 + b^2 + c^2.$$

Prove that

$$a(x + 3yz) + b(y + 3xz) + c(z + 3xy) \leq \frac{2}{3}(a + b + c).$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Li Zhou, Polk State College, USA
Using $x + y + z = 1$ we get

$$\begin{aligned} B &= 2(a + b + c) - 3\left[a(x + 3yz) + b(y + 3zx) + c(z + 3xy)\right] \\ &= 2(a + b + c)(x + y + z)^2 - 3\sum_{cyc} a\left[x(x + y + z) + 3yz\right] \\ &= 2au^2 + avw + 2bv^2 + bwu + 2cw^2 + cuv, \end{aligned}$$

where $u = y - z$, $v = z - x$, and $w = x - y$. Replacing w by $-(u + v)$, we obtain further

$$B = (2c + 2a - b)u^2 + (5c - a - b)uv + (2b + 2c - a)v^2.$$

Now the discriminant

$$(5c - a - b)^2 - 4(2c + 2a - b)(2b + 2c - a) = 9(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca) < 0.$$

Also, $(2c + 2a - b) + (2b + 2c - a) = a + b + 4c > 0$. So both $2c + 2a - b > 0$ and $2b + 2c - a > 0$. Therefore, $B \geq 0$, completing the proof.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

O296. Let m be a positive integer. Prove that $\phi(n)$ divides mn, only for finitely many square-free integers n, where ϕ is Euler's totient function.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Li Zhou, Polk State College, USA

Let $a \geq 0$ be the largest power of 2 dividing m. Suppose that $n = p_1 p_2 \cdots p_k$ is such a square-free integer, with primes $p_k > \cdots > p_1 \geq 2$. Then $\phi(n) = (p_1 - 1) \cdots (p_k - 1)$. Since $p_2 - 1, \ldots, p_k - 1$ are even, $k - 1 \leq a + 1$, i.e., $k \leq a + 2$. Next, we show inductively the claim that $p_i \leq (m+1)^{2^{i-1}}$ for $1 \leq i \leq k$. Indeed, any prime factor of $p_1 - 1$ is smaller than p_1, so $p_1 - 1$ must divide m, thus $p_1 \leq m + 1$. Now assume that the claim is true for all i up to some $j < k$. Again, all prime factors of $p_{j+1} - 1$ are smaller than p_{j+1}, so $p_{j+1} - 1$ must divide $mp_1 \cdots p_j$. Hence,

$$p_{j+1} \leq mp_1 \cdot p_j + 1 \leq m(m+1)(m+1)^2 \cdots (m+1)^{2^{j-1}} + 1 \leq (m+1)^{2^j},$$

completing the induction. Therefore, the total number of such n is no greater than

$$(m+1)(m+1)^2(m+1)^4 \cdots (m+1)^{2^{a+1}} = (m+1)^{2^{a+2}-1}.$$

Also solved by Dimitris Oikonomou, 2nd Hight School, Nauplio, Greece; Arpan Sadhukhan, Indian Statistical Institute, Kolkata, India; Jishnu Bose, Uttarpara Govt. High School, Kolkata, India.

O297. Cells of an 11×11 square are colored in n colors. It is known that the number of cells of each color is greater than 6 and less than 14. Prove that one can find a row and a column whose cells are colored in at least four different colors.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Dimitris Oikonomou, 2nd Hight School, Nauplio, Greece

First of all, name each color with one of the the numbers $1, 2, \dots, n$. Next, name the cells as follows:

1	2	...	11
12	13	...	21
...	...	...	...
111	112	...	121

Now we will the bound n. Let d_i be the number of cells in the matrix colored with the i-th color. Obviously from the condition $7 \leq d_i \leq 13$.

Now let $A = (a_{i,j})$ be a $n \times 121$ $(0,1)$-matrix. The rows represent the colors and the columns represent the cells. Let $a_{i,j} = 1$ if the j-th cell is colored by the i-th color, otherwise $a_{i,j} = 0$. Now since the sum of the elements of the matrix is the same if we count it either by columns or rows and each column has exactly one 1 we have:

$$\sum_{i=1}^{n} d_i = 121 \Rightarrow 7n \leq 121 \leq 13n \Rightarrow 10 \leq n \leq 17.$$

Now we will use the probabilistic method in order to solve the problem. Construct a 21- regular hyper-graph $\mathcal{H}$ with 121 vertices, representing the cells of the matrix and the hyper-edges representing each combination of a row and a line. Color each vertex of $\mathcal{H}$ with one of the n colors as it was in the matrix(for example 1-st row and the 1-st column is represented by the line which pass from the vertices $(1, 2, \dots, 11, 12, 23, 34, \dots, 111)$ and if the i-th cell is colored with the j-th color then the i-th vertex will be colored with the j-th color as well.)

At this point we will use *Lovasz Local Lemma(LLL)* which states:

Let $A_1, A_2, ..., A_k$ be a series of events such that each event occurs with probability at most p and such that each event is independent of all the other events except for at most d of them. If $ep(d+1) \leq 1$ then there is a nonzero probability that none of the events occurs.

Let A_f be the event that the hyper-edge f is painted with at most 3 colors. Obviously

$$p = \mathbb{P}(A_f) \leq \frac{\binom{n}{3} 3^{21}}{n^{21}} \leq \frac{\binom{17}{3} 3^{21}}{10^{21}}.$$

Moreover, each event A_f is clearly mutually independent of all the other events $A_(f^{'})$ for all hyper-edges f' that do not intersect f. Now since any edge f of such an $\mathcal{H}$ contains 21 vertices, each of which is incident with 21 edges (including f), it follows that f intersects at most $d = 21(21-1)$ other edges.

Now we have $ep(d+1) \le \dfrac{421 \cdot e \cdot \binom{17}{3} \cdot 3^{21}}{10^{21}} < \dfrac{8983769744864520}{10^{21}} < 1$, after some tedious calculations using the obvious inequality $e < 3$. So according to the Lovasz Local Lemma there is a nonzero probability that none of the A_f occurs which means that there exists a hyper-edge with at least 4 different colors or one can find a row and a column whose cells are colored in at least four different colors.

Remark: Using exactly the same argument we can prove that there is a row and a column with at least *5* different colors. Indeed, we have $p' \le \dfrac{\binom{17}{4}4^{21}}{10^21}$, $d+1 = 421$,and, using a calculator

$$ep'(d+1) \le \frac{421 \cdot e \cdot \binom{17}{4} \cdot 4^{21}}{10^{21}} \le \frac{11978801069077172091240802754973}{10^{21}} < 1.$$

Note that for 6 different colors this method fails.

O298. Let n be a square-free positive integer. Find the number of functions $f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ such that $f(1)f(2)\cdots f(n)$ divides n.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by G.R.A.20 Problem Solving Group, Roma, Italy
We solve the problem for any positive integer n with prime factorization

$$p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}.$$

Any function f with the required property is uniquely determined by

$$f(j)=p_1^{x(1,j)}p_2^{x(2,j)}\cdots p_r^{x(r,j)}$$

where $x(i,j)$ are non-negative integers for $i=1,\ldots,r$ and $j=1,\ldots,n$, such that

$$x(i,1)+\cdots+x(i,n)\le\alpha_i,\qquad \text{for } i=1,\ldots,r.$$

The number of solutions of the above inequality is $\binom{n+\alpha_i}{\alpha_i}$, hence the number of such functions is

$$\prod_{i=1}^{r}\binom{n+\alpha_i}{\alpha_i}.$$

If n is square-free then this number is $(n+1)^r$.

Also solved by Khakimboy Egamberganov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Sayak Mukherjee, Kolkata, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Jishnu Bose, Uttarpara Govt. High School, Kolkata, India.

O299. Let a, b, c be positive real integers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$\sqrt{1-abc}(3-a-b-c) \geq |(a-1)(b-1)(c-1)|\,.$$

Proposed by Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that if $a, b, c \geq 1$, then $a^2 + b^2 + c^2 + abc \geq 4$, with equality iff $a = b = c = 1$, and similarly if $a, b, c \leq 1$. Note also that $a, b, c < 2$, since if wlog $a \geq 2$, then $a^2 \geq 4$, and $b^2 + c^2 + abc > 0$, in contradiction with $a^2 + b^2 + c^2 + abc = 4$. Note further that if one of a, b, c equals 1 (wlog $c = 1$ by symmetry in the variables), then the proposed inequality rewrites as

$$\sqrt{1-ab}(2-a-b) \geq 0,$$

with condition $3 = a^2 + b^2 + ab$. By the AM-GM, it follows that $3 = a^2 + b^2 + ab \geq 3ab$, or $ab = 1 - \delta$ for some $0 < \delta \leq 1$, and $a + b = \sqrt{3 + ab} = \sqrt{4 - \delta} \leq 2$, with equality iff $\delta = 0$ and equivalently $a = b = 1$. It follows that whenever one of a, b, c equals 1, the proposed result holds, with equality iff $a = b = c = 1$. Any case where none of a, b, c equals 1 falls necessarily under one of the following two cases:

Case 1: Wlog by symmetry in the variables, $a > 1 > b \geq c$. Let us denote $a = 1 + \delta$ where $0 < \delta < 1$, and $d = b - c$ where $0 \leq d < 1$. Then, the condition rewrites as $(1 - bc - \delta)(3 + \delta) = d^2 \geq 0$, or $bc \leq 1 - \delta$ with equality iff $b = c$, for $1 - abc \geq 1 - (1 + \delta)(1 - \delta) = \delta^2$, and $\sqrt{1 - abc} \geq \delta = a - 1$, or it suffices to show that $3 - a - b - c \geq (1 - b)(1 - c)$, or equivalently, $bc \leq 2 - a = 1 - \delta$, which we have already proved. The proposed inequality holds in this case, with equality iff $bc = 1 - \delta$, ie iff $b = c = \sqrt{2 - a}$. Direct substitution shows that indeed $a^2 + b^2 + c^2 + abc = a^2 + 4 - 2a + a(2 - a) = 4$, while $\sqrt{1 - abc} = a - 1$ and $3 - a - b - c = \left(1 - \sqrt{2 - a}\right)^2 = (1 - b)(1 - c)$.

Case 2: Wlog by symmetry in the variables, $a < 1 < b \leq c$. Denoting $a = 1 - \delta$ where $0 < \delta < 1$, and $d = c - b$ where $0 \leq d < 1$, the condition rewrites as $(3 - \delta)bc = (1 + \delta)(3 - \delta) - d^2$, or $bc \leq 1 + \delta$ with equality iff $b = c$, for $\sqrt{1 - abc} \geq \delta = 1 - a$, and again it suffices to show that $3 - a - b - c \geq (b - 1)(c - 1)$, or $bc \leq 2 - a = 1 - \delta$, already proved. Analogously as in case 1, it is readily shown that equality holds iff $b = c = \sqrt{2 - a}$ by direct substitution in the condition and in the inequality.

The conclusion follows, equality holds iff (a, b, c) is a permutation of $(k, k, 2 - k^2)$ for any real $0 < k < \sqrt{2}$. In the case $k = 1$, we revert to the trivial equality case $a = b = c = 1$.

O300. Let ABC be a triangle with circumcircle Γ and incircle ω. Let D, E, F be the tangency points of ω with BC, CA, AB, respectively, let Q be the second intersection of AD with Γ, and let T be the intersection of the tangents at B and C with respect to Γ. Furthermore, let QT intersect Γ for the second time at R. Prove that AR, EF, BC are concurrent.

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Let EF meet BC at X. Since lines AD, BE, CF are concurrent, the quadruplet (X, B, D, C) is a harmonic division; hence pencil (AX, AB, AD, AC) is harmonic too. By intersecting it with Γ, it follows that $AX \cap \Gamma$, B, Q, C are the vertices of a harmonic quadrilateral. But the tangents at B and C with respect to Γ meet at T, which lies on the line QR, therefore $RBQC$ is a harmonic quadrilateral. It thus follows that R is the second intersection of AX with Γ. This proves that the lines AR, EF, BC are concurrent, as claimed.

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