Junior problems

J199. Prove that there are infinitely many pairs (p, q) of primes such that $p^6 + q^4$ has two positive divisors whose difference is $4pq$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, USA

Let $p = 2$ and let q be odd prime number. Then

$$p^6 + q^4 = 64 + q^4 = \left(8 + 4q + q^2\right)\left(8 - 4q + q^2\right)$$

and we have

$$\left(8 + 4q + q^2\right) - \left(8 - 4q + q^2\right) = 8q = 4pq.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Prithwijit De, HBCSE, Mumbai, India; Roberto Bosch Cabrera, Havana, Cuba; Tigran Hakobyan, Armenia.

J200. Let x, y, z be positive real numbers with $x \leq 2, \ y \leq 3$ and $x+y+z=11$. Prove that $xyz \leq 36$.

Proposed by Mircea Lascu, Zalau, and Marius Stanean, Zalau, Romania

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

By AM-GM inequality, we have

$$\begin{aligned} xyz = \frac{1}{6}\cdot(3x)\cdot(2y)\cdot z &\leq \frac{1}{6}\left(\frac{3x+2y+z}{3}\right)^3 \\ &= \frac{1}{6}\left(\frac{2x+y+11}{3}\right)^3 \\ &\leq \frac{1}{6}\left(\frac{2\cdot 2+3+11}{3}\right)^3 \\ &= 36. \end{aligned}$$

We have equality if and only if $x=2, y=3, z=6$.

Also solved by Arkady Alt, San Jose, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Omran Kouba, Damascus, Syria; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Prithwijit De, HBCSE, Mumbai, India; Roberto Bosch Cabrera, Havana, Cuba; Salem Malikic, Sarajevo, Bosnia and Herzegovina; Tigran Hakobyan, Armenia; Christopher Wiriawan, Jakarta, Indonesia.

J201. Let ABC be an isosceles triangle with $AB = AC$. Point D lies on side AC such that $\angle CBD = 3\angle ABD$. If

$$\frac{1}{AB} + \frac{1}{BD} = \frac{1}{BC},$$

find $\angle A$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Roberto Bosch Cabrera, Florida, USA

Let $\angle ABD = x$, and note that we have $\angle CBD = 3x$ and $\angle ACB = 4x$. By the Law of Sines in triangles ABC and BDC, we obtain that $\frac{AB}{\sin 4x} = \frac{BC}{\sin 8x}$ and $\frac{BD}{\sin 4x} = \frac{BC}{\sin 7x}$, respectively. Thus, the relation $\frac{1}{AB} + \frac{1}{BD} = \frac{1}{BC}$ is equivalent to $\sin 8x + \sin 7x = \sin 4x$, where we can consider x to lie in $(0, \frac{\pi}{8})$ since $\angle A = \pi - 8x > 0$. Now, using that $\sin 4x - \sin 8x = -2\sin(2x)\cos 6x$ the equation can be rewritten as $\sin 7x = (-2\cos 6x)\sin(2x)$, and so we spot the solution $x = \frac{\pi}{9}$. We will finish by proving that it is unique on this interval. We split this discussion into two cases. First, suppose $x > \frac{\pi}{9}$ is some other solution, and observe that $6x > \frac{6\pi}{9}$, so $-2\cos 6x > 1$; hence $\sin 7x > \sin 2x$. But $2x \in (0, \frac{\pi}{4})$ and $7x \in (\frac{7\pi}{9}, \frac{7\pi}{8})$, so $\sin 7x = \sin(\pi - 7x)$ yields $\pi - 7x > 2x$, i.e. $x < \frac{\pi}{9}$, which is a contradiction. Now, suppose $x < \frac{\pi}{9}$ is a solution, and see that $\sin 7x < \sin 2x$; if $7x \leq \frac{\pi}{2}$, then $7x < 2x$, and we got a contradiction, whereas if $7x > \frac{\pi}{2}$, then $\pi - 7x < 2x$, i.e .$x > \frac{\pi}{9}$, again contradiction. We conclude that $x = \frac{\pi}{9}$ is the only solution, and so $\angle A = \pi - \frac{8\pi}{9} = \frac{\pi}{9}$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Omran Kouba, Damascus, Syria; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.

J202. Let ABC be a triangle with incenter I and let A_1, B_1, C_1 be the symmetric points of I with respect to the midpoints of sides BC, CA, AB. If I_a, I_b, I_c denote the excenters corresponding to sides BC, CA, AB, respectively, prove that lines I_aA_1, I_bB_1, I_cC_1 are concurent.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

First solution by Cosmin Pohoata, Princeton University, USA

Consider the homothety $\mathcal{H}(I, 1/2)$ with pole I and ratio $1/2$. It is well-known that the midpoints of the segments II_a, II_b, II_c are in fact the midpoints of the arcs BC, CA, AB not containing the vertices of the triangle; therefore, the lines I_aA_1, I_bB_1, I_cC_1 are concurrent since the mediators of the triangle ABC are concurrent.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

In exact trilinear coordinates, $I \equiv (r, r, r)$, r being the inradius, and the midpoint of BC is $M_a \equiv \left(0, \frac{h_b}{2}, \frac{h_c}{2}\right)$, where h_a, h_b, h_c are the respective lengths of the altitudes from A, B, C. Thus $A_1 \equiv (-r, h_b - r, h_c - r)$. Denoting by r_a, r_b, r_c the exradii corresponding to sides BC, CA, AB, clearly $I_a \equiv (-r_a, r_a, r_a)$, or in trilinear coordinates (α, β, γ), the points on line I_aA_1 satisfy

$$0 = \begin{vmatrix} \alpha & \beta & \gamma \\ -1 & 1 & 1 \\ -r & h_b - r & h_c - r \end{vmatrix} = \begin{vmatrix} \alpha + \beta & \alpha + \gamma \\ h_b - 2r & h_c - 2r \end{vmatrix},$$

or by cyclic permutations, the point that satisfies $(h_a - 2r)(\beta + \gamma) = (h_b - 2r)(\gamma + \alpha) = (h_c - 2r)(\alpha + \beta)$ is simultaneously on lines I_aA_1, I_bB_1, I_cC_1. The conclusion follows.

Also solved by Felipe Ignacio Arbulu Lopez, Instituto Nacional General Jose Migue Carrera, Santiago, Chile; Roberto Bosch Cabrera, Florida, USA.

J203. Let $ABCD$ be a trapezoid ($AB||CD$) with acute angles at vertices A and B. Line BC and the tangent lines from A and E to the circle of center D tangent to AB are concurrent at F. Prove that AC bisects the segment EF if and only if $AF + EF = AB$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Roberto Bosch Cabrera, Florida, USA

We begin with the following preliminary result:

Lemma. Let ABC be an arbitrary triangle with incenter I centroid G. Then, the lines IG and BC are parallel if and only if $b + c = 2a$.

Proof. Let l_a, m_a be the bisector and median from A, respectively. We have that IG is parallel to BC if and only if $\frac{AI}{l_a} = \frac{AG}{m_a}$. However, $AI^2 = (s-a)^2 + r^2$ by Pythagoras theorem, where s denotes the semiperimeter and r the inradius of triangle ABC. Besides, we have that

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} \quad \text{and} \quad l_a^2 = \frac{bc(a+b+c)(b+c-a)}{(b+c)^2},$$

so using $\frac{AG}{m_a} = \frac{2}{3}$, we obtain

$$9\left[(s-a)^2 + \frac{(s-a)(s-b)(s-c)}{s}\right] = \frac{4bc(a+b+c)(b+c-a)}{(b+c)^2},$$

which after several simple algebraic manipulations becomes $(b+c-2a)(2a+5b+5c) = 0$, i.e. $b + c = 2a$.

Returning to the original problem we have that $AF + EF = 2AE$ and note that G is the midpoint of EF if and only if C is the centroid of triangle AEF, so by applying the Lemma for this triangle, we arrive to desired conclusion.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain

J204. Give a straightedge and compass construction of a triangle ABC starting with its incenter I, the foot of the altitude from A, and the midpoint of the side BC.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Cosmin Pohoata, Princeton University, USA

We are given the foot of the altitude from A and the midpoint M of BC, so the line determined by the two is precisely the line BC. Now, knowing the incenter we can draw the perpendicular from I to BC and get the incircle of ABC. Recall the well-known fact that if D is the tangency point of the incircle with the side BC, and D' the antipode of D with respect to the incircle, then the points A, D', X are collinear, where X is the tangency point of the A-excircle with BC. However, in order to use this, we need to find X. This is not a problem whatsoever, since $MD = MX$; so we have the construction of X as the reflection of D in M. Now, just draw the lines XD' and the altitude from A (which we can draw since we have the foot of the altitude on BC and the line BC); they intersect at the vertex A. Afterwards, just take the tangents from A to the incircle and intersect them with BC; this will give us the vertices B and C. Hence our construction is complete.

Also solved by Arkady Alt, San Jose, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Raul A. Simon, Chile.

Senior problems

S199. In triangle ABC let BB' and CC' be the angle bisectors of $\angle B$ and $\angle C$. Prove that

$$B'C' \geq \frac{2bc}{(a+b)(a+c)}\left[(a+b+c)\sin\frac{A}{2} - \frac{a}{2}\right].$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Campos Salas, Costa Rica

First, note that by Ptolemy's inequality for quadrilateral $BCB'C'$,

$$B'C' \geq \frac{BB' \cdot CC' - BC' \cdot CB'}{BC}.$$

Now, $BB' = \frac{2ac}{a+c}\cos\frac{B}{2}$ and that $\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{a}\sin\frac{A}{2}$, and these imply that

$$BB' \cdot CC' = \frac{4a^2bc}{(a+b)(a+c)}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{2abc(a+b+c)}{(a+b)(a+c)}\sin\frac{A}{2}.$$

In addition, $BC' = \frac{ac}{a+b}$ and this yields $BC' \cdot CB' = \frac{a^2bc}{(a+b)(a+c)}$; therefore by just plugging in these identities in the inequality above, the desired conclusion follows.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Evangelos Mouroukos, Agrinio, Greece; Roberto Bosch Cabrera, Florida, USA.

S200. On each vertex of the regular hexagon $A_1A_2A_3A_4A_5A_6$ we place a rod. On each rod we have ai rings, where ai corresponds to the vertex A_i. Taking a ring from any three adjacent rods we can create chains of three rings. What is the maximum number of such chains that we can create?

Proposed by Arkady Alt, San Jose, USA

Solution by Roberto Bosch Cabrera, Florida, USA

If by adjacent rods, we understand consecutive rods, we have the following possibilities for three adjacent vertices: $A_1A_2A_3, A_2A_3A_4, A_3A_4A_5, A_4A_5A_6, A_5A_6A_1, A_6A_1A_2$. Hence, the maximum number of chains is

$$M := a_1a_2a_3 + a_2a_3a_4 + a_3a_4a_5 + a_4a_5a_6 + a_5a_6a_1 + a_6a_1a_2.$$

S201. Prove that in any triangle,

$$r_a \le 4R\sin^3\left(\frac{A}{3}+\frac{\pi}{6}\right).$$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

Solution by Prithwijit De, HBCSE, Mumbai, India

We know that $r_a = R(1-\cos A+\cos B+\cos C)$. Also,

$$4R\sin^3\left(\frac{A}{3}+\frac{\pi}{6}\right) = R\left(3\sin\left(\frac{A}{3}+\frac{\pi}{6}\right)-\cos A\right).$$

Therefore, to establish the inequality we need to show that

$$1+\cos B+\cos C \le 3\sin\left(\frac{A}{3}+\frac{\pi}{6}\right). \qquad (*)$$

Now, in order to get (*), observe that it is enough to prove the inequality for acute angles B and C, since if one of them is obtuse or right, its cosine is nonpositive. On the other hand, $\cos x$ is concave on the interval $\left(0,\frac{\pi}{2}\right)$, thus by Jensen's inequality for concave functions,

$$\frac{1+\cos B+\cos C}{3} \le \cos\left(\frac{0+B+C}{3}\right) = \sin\left(\frac{A}{3}+\frac{\pi}{6}\right). \qquad (**)$$

This proves our inequality, as from (**) it follows that $r_a \le 4R\sin^3\left(\frac{A}{3}+\frac{\pi}{6}\right)$.

Also solved by Arkady Alt, San Jose, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Evangelos Mouroukos, Agrinio, Greece; Omran Kouba, Damascus, Syria; Roberto Bosch Cabrera, Florida, USA.

S202. Let a and b be integers such that $a^2m - b^2n = a - b$, where m and n are consecutive integers Prove that $\gcd(a, b) = \sqrt{|a - b|}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Omran Kouba, Damascus, Syria

Let $\delta = \gcd(a, b)$ and write $a = \delta a'$ and $b = \delta b'$ with $\gcd(a', b') = 1$. From $a^2m - b^2n = a - b$ we get $(ma - 1)a' = (nb - 1)b'$, so $a'|(nb - 1)b'$ and $\gcd(a', b') = 1$. This implies that $a'|(nb - 1)$, and consequently there exists an integer λ such that $nb - 1 = \lambda a'$ and $ma - 1 = \lambda b'$ or,

$$n\delta b' - \lambda a' = 1, \quad \text{and} \quad m\delta a' - \lambda b' = 1 \tag{1}$$

In particular, $\gcd(\delta, \lambda) = 1$. Now subtracting the two equalities in (1) we find that

$$(n\delta + \lambda)b' = (m\delta + \lambda)a',$$

and again, since $a'|(n\delta + \lambda)b'$ and $\gcd(a', b') = 1$, we conclude that $a'|(n\delta + \lambda)$ and that there exists an integer μ such that

$$n\delta + \lambda = \mu a', \qquad \text{and} \qquad m\delta + \lambda = \mu b'. \tag{2}$$

On the other hand, recalling that m and n are consecutive integers we get from (2) that

$$\delta = \varepsilon\mu(a' - b'), \qquad \text{with } \varepsilon = n - m \in \{-1, +1\}, \tag{3}$$

Whereas from (2) we have $\lambda = \mu(a' - n\varepsilon(a' - b'))$. So, μ is a common divisor of λ and δ which are coprime, hence $\mu \in \{-1, +1\}$. Finally, multiplying both sides of (3) by δ and using the fact that $|\varepsilon\mu| = 1$, we conclude that $\delta^2 = |a - b|$, which is the desired conclusion.

Also solved by Arkady Alt, San Jose, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Roberto Bosch Cabrera, Florida, USA; Salem Malikic, Sarajevo, Bosnia and Herzegovina; Tigran Hakobyan, Armenia.

S203. Let ABC be a triangle, and P a point not lying on its sides. Call XYZ the cevian triangle of P with respect to ABC and consider the points Y_a, Z_a of intersection BC with the parallel lines to AX through Y and Z, respectively. Prove that AX, YZ_a, Y_aZ concur in a point Q that satisfies the cross-ratio

$$(AXPQ) = \frac{AP}{AX}.$$

Proposed by Francisco Javier Garcia Capitan, Spain

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Triangles ACX and YCY_a are clearly similar, hence $\frac{XY_a}{AY} = \frac{CX}{AC}$ and $\frac{YY_a}{CY} = \frac{AX}{AC}$. Triangles ABX and ZBZ_a are also similar, hence $\frac{XZ_a}{AZ} = \frac{BX}{AB}$ and $\frac{ZZ_a}{BZ} = \frac{AX}{AB}$. Define $Q = AX \cap YZ_a$ and $Q' = AX \cap ZY_a$, or triangles QXZ_a and YY_aZ_a are similar yielding $\frac{QX}{ZZ_a} = \frac{XY_a}{Y_aZ_a}$, and triangles $Q'XY_a$ and ZZ_aY_a are also similar, for $\frac{Q'X}{YY_a} = \frac{XZ_a}{Y_aZ_a}$. It follows that $Q = Q'$ iff $XZ_a \cdot YY_a = XY_a \cdot ZZ_a$, or equivalently iff $AY \cdot BZ \cdot CX = AZ \cdot BX \cdot CY$, ie by Ceva's theorem iff AX, BY, CZ concur. Since they do (at P), it follows that Q is indeed the point where AX, YZ_a, ZY_a concur.

Note that $(AXPQ) = \frac{AP}{AX}$ is by definition equivalent to $AX \cdot QX = AQ \cdot PX$. Applying Menelaus' theorem to triangle APY and lines BXC, and triangle ACX and line Z_aQY, we find

$$\frac{AX}{PX} = \frac{AC \cdot BY}{PB \cdot CY}, \qquad \frac{AQ}{QX} = \frac{AY \cdot CZ_a}{CY \cdot XZ_a}.$$

Writing $CZ_a = CX + XZ_a$, and inserting $XZ_a = \frac{ZX \cdot BX}{AB}$, we find after some algebra that the proposed result is equivalent to

$$AY \cdot BZ \cdot CX = AY \cdot CX(AB - AZ) = BX \cdot AZ(AC - AY) = AZ \cdot BX \cdot CY,$$

clearly true again by Ceva's theorem since AX, BY, CZ concur.

Also solved by Roberto Bosch Cabrera, Florida, USA;

S204. Find all positive integers k and n such that $k^n - 1$ and n are divisible by precisely the same primes.

Proposed by Tigran Hakobyan, Yerevan, Armenia

Solution by Alessandro Ventullo, Milan, Italy

We immediately see that if $k = 1$, then $k^n - 1$ would be divisible by every prime p, but since n is a positive integer, n is only divisible by a finite number of primes. So, we have $k > 1$. If $n = 1$, then $k - 1 \geq 1$, so we obtain $k - 1 = 1$, i.e, $k = 2$. If $n = 2$, then $k^2 - 1 = (k-1)(k+1)$ and we want

$$(k-1)(k+1) = 2^m,$$

with m a positive integer. Therefore $k = 3$ and $m = 1$. Now, suppose that $n > 2$. We have

$$n = \prod_{i=1}^{h} p_i^{\alpha_i}, \qquad k = 1 + \prod_{i=1}^{h} p_i^{\beta_i},$$

with α_i, β_i positive integers and p_i prime for every $1 \leq i \leq h$. Suppose $\alpha_i \neq \beta_i$ for every $1 \leq i \leq h$ and put $a = \prod_{i=1}^{h} p_i^{\beta_i}$. So

$$\begin{aligned}
k^n - 1 \quad &= (1+a)^n - 1 \\
&= a\left(a^{n-1} + na^{n-2} + \ldots + \frac{n(n-1)}{2}a + n\right) \\
&= a(ab+n) \\
&= \prod_{i=1}^{h} p_i^{\beta_i + \min\{\alpha_i,\beta_i\}} \left(\prod_{i=1}^{h} p_i^{\beta_i - \min\{\alpha_i,\beta_i\}} b + \prod_{i=1}^{h} p_i^{\alpha_i - \min\{\alpha_i,\beta_i\}}\right)
\end{aligned}$$

where $b = \left(a^{n-2} + na^{n-3} + \ldots + \frac{n(n-1)}{2}\right)$. It's easy to see that only one between $\prod_{i=1}^{h} p_i^{\beta_i - \min\{\alpha_i,\beta_i\}}$ and $\prod_{i=1}^{h} p_i^{\alpha_i - \min\{\alpha_i,\beta_i\}}$ is divisible by p_i for every $1 \leq i \leq h$, so

$$\left(\prod_{i=1}^{h} p_i^{\beta_i - \min\{\alpha_i,\beta_i\}} b + \prod_{i=1}^{h} p_i^{\alpha_i - \min\{\alpha_i,\beta_i\}}\right)$$

is not divisible by any of the p_i and then is divisible by another prime p^* which doesn't divide n, contradiction. So $\alpha_j = \beta_j$ for some j. Suppose that n is odd. By the same argument,

$$\left(\prod_{i=1}^{h} p_i^{\beta_i - \min\{\alpha_i,\beta_i\}} b + \prod_{i=1}^{h} p_i^{\alpha_i - \min\{\alpha_i,\beta_i\}}\right)$$

is not divisible by any of the p_i with $i \neq j$ and for p_j we have $a = p_j^{\alpha_j} a_1$ and $n = p_j^{\alpha_j} n_1$, where a_1 and n_1 are not divisible by p_j. Then

$$a\left(ab+n\right) = ap_j^{\alpha_j}\left(a_1 b + n_1\right),$$

but since b is divisible by p_j, $(a_1 b + n_1)$ is not divisible by p_j and therefore $a(ab+n)$ is not divisible by any of the prime p_i for every $1 \leq i \leq h$. So n is even and a must be even, i.e., $a = 2a_1$ with a_1 a positive integer and

$$\begin{aligned}
(1+a)^n - 1 \quad &= [(1+a)^{n/2} - 1][(1+a)^{n/2} + 1] \\
&= [(1+a)^{n/2} - 1]\left[a^{n/2} + \frac{n}{2} a^{n/2-1} + \ldots + \frac{n}{2} a + 2\right] \\
&= 2[(1+a)^{n/2} - 1][2^{n/2-1} a_1^{n/2} + 2^{n/2-2} a_1^{n/2-1} + \ldots + a_1 + 1] \\
&= 2c[(1+a)^{n/2} - 1]
\end{aligned}$$

where $c = [2^{n/2-1} a_1^{n/2} + 2^{n/2-2} a_1^{n/2-1} + \ldots + a_1 + 1]$. Since c is not divisible by a_1, any prime which divides a_1 (and a) doesn't divide c and then c is not divisibile by any prime which divides n and thus is divisible by another prime which doesn't divide n. This contradiction shows that the only positive integers which satisfy the required conditions are $n = 1, k = 2$ and $n = 2, k = 3$.

Also solved by Albert Stadler, Switzerland.

Undergraduate problems

U199. Prove that in any triangle ABC,

$$3\sqrt{3} \leq \cot\frac{A+B}{4} + \cot\frac{B+C}{4} + \cot\frac{C+A}{4} \leq \frac{3\sqrt{3}}{2} + \frac{s}{2r},$$

where s and r denote the semiperimeter and the inradius of triangle ABC, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Since the function $f(t) = \cot t$ is convex in $(0; \frac{\pi}{2})$, by Jensen's inequality,

$$\begin{aligned} \cot\frac{A+B}{4} + \cot\frac{B+C}{4} + \cot\frac{C+A}{4} &\geq 3 \cdot \cot\frac{(A+B)+(B+C)+(C+A)}{12} \\ &= 3\cot\frac{\pi}{6} = 3\sqrt{3}. \end{aligned}$$

So the inequality from the left hand side is proven.

For the other one, choose $f(t) = \cot t, x = \frac{A}{2}, y = \frac{B}{2}, z = \frac{C}{2}$ in Popoviciu's inequality, and get

$$\begin{aligned} &\frac{2}{3}(\cot\frac{A+B}{4} + \cot\frac{B+C}{4} + \cot\frac{C+A}{4}) \\ &\qquad \leq \frac{1}{3}(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}) + \cot\frac{A+B+C}{6} \\ &\qquad = \frac{1}{3}(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}) + \cot\frac{\pi}{6} \\ &\qquad = \frac{1}{3}(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}) + \sqrt{3}. \end{aligned}$$

It follows that

$$\cot\frac{A+B}{4} + \cot\frac{B+C}{4} + \cot\frac{C+A}{4} \leq \frac{1}{2}(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}) + \frac{3\sqrt{3}}{2},$$

which via the well-known

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s}{r}$$

gives us our desired inequality.

Also solved by Daniel Campos Salas, Costa Rica; Roberto Bosch Cabrera, Florida, USA.

U200. Let p be an odd prime and let n be an integer greater than 1. Find all integers k for which there exists an $n \times n$ matrix A of rank k such that $A + A^2 + \cdots + A^p = 0$.

Proposed by Gabriel Dospinescu, Ecole Polythehnique, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain;

It should be specified in the problem statement that A must be a matrix with real entries, otherwise the problem becomes trivial, and a matrix of any rank $k \in \{0, 1, \dots, n\}$ may be found; it suffices to take a diagonal matrix whose main diagonal are k p-th roots of unity other than 1, and $n - k$ zeros.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A, and let P be the invertible matrix such that $P^{-1}AP = T$ is triangular. Clearly, T and A have the same rank since multiplication by an invertible matrix preserves the rank. Now,

$$0 = P^{-1}0P = T + T^2 + \cdots + T^p,$$

where the RHS is clearly a triangular matrix whose main diagonal contains the terms of the form

$$\lambda_i + {\lambda_i}^2 + \cdots + {\lambda_i}^p = 0.$$

Note that $\lambda_i \neq 1$ since otherwise this result is not true, or we may multiply both sides by $\lambda_i - 1$, yielding $\lambda_i \left({\lambda_i}^p - 1\right) = 0$, ie, either $\lambda_i = 0$, or it is a p-th root of unity other than 1. If A has real entries, its characteristic polynomial has real coefficients, and its nonzero roots can be grouped in complex conjugate pairs, ie, A has an even number of nonzero eigenvalues, thus an even rank. Let now λ be a p-th root of unity other than 1, clearly a second degree polynomial with real coefficients exists such that λ and its complex conjugate λ^* are its roots; a matrix with real entries can then be found such that its eigenvalues are λ and λ^*. Repeat ℓ times this 2×2 matrix along the diagonal of A, and fill the rest of A with zeros, and a matrix A satisfying the conditions of the problem is found (since A thus formed is clearly diagonalizable, and the diagonal of $T + T^2 + \cdots + T^p$ is clearly filled with 0's). Since this can be done for any $0 \leq \ell \leq \frac{n}{2}$, we conclude that the integers k that we are looking for, are all non-negative even integers that do not exceed n.

U201. Evaluate

$$\sum_{n=2}^{\infty} \frac{3n^2-1}{(n^3-n)^2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Moubinool Omarjee, Paris, France

Since

$$\frac{3k^2-1}{(k^3-k)^2} = -\frac{1}{2k^2} + \frac{1}{2(k-1)^2} - \frac{1}{2k^2} + \frac{1}{2(k+1)^2},$$

we have that

$$\sum_{k=2}^{n} \frac{3k^2-1}{(k^3-k)^2} = \frac{1}{2} - \frac{1}{2n^2} + \frac{1}{2(n+1)^2} - \frac{1}{8},$$

thus the limit is $\frac{3}{8}$.

Also solved by Alessandro Ventullo, Milan, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Anastasios Kotronis, Athens, Greece; Arkady Alt, San Jose, USA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Emanuele Tron, Pinerolo, Italy; Evangelos Mouroukos, Agrinio, Greece; G. C. Greubel, Newport News, USA; Gerhardt Hinkle, Springfield, USA; N.J. Buitrago A., Universidad Nacional, Colombia; Jedrzej Garnek, Adam Mickiewicz University, Poznan, Poland; Lorenzo Pascali, Universita di Roma La Sapienza, Roma, Italy; Omran Kouba, Damascus, Syria; Raul A. Simon, Chile; Roberto Bosch Cabrera, Florida, USA; Anastasios Kotronis, Athens, Greece; Tigran Hakobyan, Armenia.

U202. The interval $(0,1]$ is divided into N equal intervals $(\frac{i-1}{N}, \frac{i}{N}]$, where $i \in [1, n]$. An interval is called *special* if it contains at least one number from the set $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{N}\}$. Find a good approximation for the number of special intervals.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Jedrzej Garnek, Adam Mickiewicz University, Poznan, Poland

Denote the number of special intervals by a_N. We will not only approximate a_N, but also give the exact formula. The inetrval $\left(\frac{i-1}{N}, \frac{i}{N}\right]$ is special if and only if there exists at least one $k \in \{1, 2, \dots, N\}$ such that

$$\frac{i-1}{N} < \frac{1}{k} \leq \frac{i}{N},$$

i.e. if and only if $\lceil \frac{N}{k} \rceil = i$. Thus, the problem is equivalent to finding how many different values does $\lceil \frac{N}{k} \rceil$ take for $k = 1, 2, \dots, N$.

To answer this, let us first note that the sequence $\left\{\lceil \frac{N}{t} \rceil\right\}_{t \geq 1}$ is non-increasing. Next, consider the function $f(x) = x^2 - x - N$, which is strictly increasing for $x \geq 1$ and tends to $+\infty$ as $x \to +\infty$. Let t be the largest positive integer for which $f(t) < 0$ (i.e. $f(t+k) \geq 0$ for $k \geq 1$). Then $t(t-1) < N \leq t(t+1)$, so for $2 \leq i \leq t$ we have

$$\frac{N}{i-1} - \frac{N}{i} = \frac{N}{i(i-1)} > 1,$$

so $\lceil \frac{N}{i-1} \rceil > \lceil \frac{N}{i} \rceil$, and thus we get $\lceil \frac{N}{1} \rceil > \lceil \frac{N}{2} \rceil > \ldots > \lceil \frac{N}{t} \rceil$. If $i > t$, then

$$\frac{N}{i-1} - \frac{N}{i} = \frac{N}{i(i-1)} \leq 1,$$

hence we have that $\lceil \frac{N}{i-1} \rceil \in \left\{\lceil \frac{N}{i} \rceil, \lceil \frac{N}{i} \rceil + 1\right\}$. Denote $\lceil \frac{N}{t} \rceil$ by s. Obviously, in the non-increasing sequence of positive integers $\lceil \frac{N}{t} \rceil, \lceil \frac{N}{t+1} \rceil, \dots, \lceil \frac{N}{N} \rceil = 1$ every two consecutive terms differ by at most 1, so

$$\left\{\lceil \frac{N}{t} \rceil, \lceil \frac{N}{t+1} \rceil, \dots, \lceil \frac{N}{N} \rceil\right\} = \{1, 2, \dots, s\}.$$

It follows that $a_N = (t-1) + s = t + \lceil \frac{N}{t} \rceil - 1$.

Now, let's try to simply this expression by proving the formula $t + \lceil \frac{N}{t} \rceil - 1 = \lceil 2\sqrt{n} \rceil - 1$. We have two cases.

If $N \leq t^2$, then $t^2 - t < N \leq t^2$, so $\lceil \frac{N}{t} \rceil = t$. Moreover, $2\sqrt{N} \leq 2t$ and $2t - 1 = \sqrt{4t^2 - 4t + 1} < \sqrt{4(t^2 - t + 1)} \leq 2\sqrt{N}$, so we have that $\lceil 2\sqrt{N} \rceil = 2t$ and $t + \lceil \frac{N}{t} \rceil = 2t - 1 = \lceil 2\sqrt{N} \rceil - 1$.

Otherwise, if $N > t^2$, then $t^2 < N \leq t^2 + t$, so $\lceil \frac{N}{t} \rceil = t + 1$ and $2t < 2\sqrt{N} < \sqrt{4t^2 + 4t} < \sqrt{4t^2 + 4t + 1} = 2t + 1$, so $\lceil 2\sqrt{N} \rceil = 2t + 1$. It follows that $t + \lceil \frac{N}{t} \rceil - 1 = 2t = \lceil 2\sqrt{N} \rceil - 1$, hence we get the same answer. This settles our discussion.

Also solved by Emanuele Tron, Pinerolo, Italy; Raul A. Simon, Chile.

U203. Let P be a polynomial of degree 5, with real coefficients, all whose zeros are real. Prove that for each real number a that is not a zero of P or P' there is a real number b such that

$$b^2P(a) + 4bP'(a) + 5P''(a) = 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We will prove the following more general result: let P be an n-th degree polynomial ($n \geq 2$) with real coefficients and only real zeroes, and let C, D be real numbers such that $nC^2 \geq 4(n-1)D \geq (n-1)C^2$. Then, there is at least one real number b such that

$$b^2P(a) + CbP'(a) + DP''(a) = 0.$$

The real b is unique if and only if $nC^2 = 4(n-1)D$ and P has all zeros equal, otherwise exactly two real b's exist. Clearly, the originally proposed problem is a particular case with $n = 5$, $C = 4$ and $D = 5$, and since $5C^2 = 80 = 16D$, equality holds in the proposed problem if and only if P has five equal real zeros.

Let z_i, $i = 1, 2, \ldots, n$ be the zeros of P, or $P'(a) = SP(a)$, $P''(a) = 2TP(a)$, where we have defined

$$S = \sum_{i=1}^{n} u_i, \qquad T = \sum_{1\leq i<j\leq n} u_iu_j, \qquad u_i = \frac{1}{a - z_i},$$

and each u_i is a real number because a is not a zero of P, hence S, T are also real. Therefore, for any real C, D, the quadratic equation in b,

$$b^2P(a) + CbP'(a) + DP''(a) = 0,$$

has real solutions if and only if the discriminant $\Delta \geq 0$, where

$$\Delta = C^2\left(P'(a)\right)^2 - 4DP(a)P''(a) = \left(C^2S^2 - 8DT\right)(P(a))^2,$$

or since a is not a zero of P, if and only if $C^2S^2 - 8DT \geq 0$. Now, note that

$$\begin{aligned} C^2S^2 - 8DT &= C^2\sum_{i=1}^{n} u_i^2 - (4D - C^2)\sum_{1\leq i<j\leq n} 2u_iu_j \\ &= \left(nC^2 - 4(n-1)D\right)\sum_{i=1}^{n} u_i^2 + (4D - C^2)\sum_{1\leq i<j\leq n}(u_i - u_j)^2 \\ &\geq 0, \end{aligned}$$

with equality if and only if $nC^2 = 4(n-1)D$, and simultaneously all u_i are equal; this last condition is clearly equivalent to all zeros of P being equal.

Remark: Note that the condition $P'(a) \neq 0$ is not necessary; indeed, note that if $4(n-1)D \geq (n-1)C^2$, then $D \geq 0$, while when $S = 0$, then $2T = S^2 - (u_1^2 + \cdots + u_n^2) < 0$, or

$2DT < 0$, and $b = \pm\sqrt{-2DT}$ is indeed always real and a solution of $b^2P(a)+CbP'(a)+DP''(a) = 0$ when $P'(a) = 0$.

Second solution by Evangelos Mouroukos, Agrinio, Greece

Let $a \in \mathbb{R}$ be such that $P(a) \neq 0$. It suffices to show that the discriminant $D = 16[P'(a)]^2 - 20P(a)P''(a)$ of the quadratic equation $x^2P(a) + 4xP'(a) + 5P''(a) = 0$ is nonnegative, or, equivalently, that

$$4\big[P'(a)\big]^2 \geq 5P(a)P''(a).$$

More generally, we show that if $n \geq 2$ is an integer, P is a polynomial of degree n with real coefficients and real roots and $a \in \mathbb{R}$ is such that $P(a) \neq 0$, then we have

$$(n-1)\left[P'(a)\right]^2 \geq nP(a)P''(a). \tag{1}$$

Let $P(x) = c(x-r_1)(x-r_2)\cdots(x-r_n)$, where $c, r_1, r_2, \ldots r_n \in \mathbb{R}$ and $c \neq 0$. If $a \in \mathbb{R}$ is such that $P(a) \neq 0$, then it is easy to see that

$$\frac{P'(a)}{P(a)} = \sum_{i=1}^{n} \frac{1}{a-r_i} \tag{2}$$

and

$$\frac{P''(a)P(a) - [P'(a)]^2}{[P(a)]^2} = -\sum_{i=1}^{n} \frac{1}{(a-r_i)^2}.$$

Hence,

$$\frac{P''(a)}{P(a)} = \left[\frac{P'(a)}{P(a)}\right]^2 - \sum_{i=1}^{n} \frac{1}{(a-r_i)^2}$$

or, equivalently,

$$\frac{P''(a)}{P(a)} = \left(\sum_{i=1}^{n} \frac{1}{a-r_i}\right)^2 - \sum_{i=1}^{n} \frac{1}{(a-r_i)^2}. \tag{3}$$

We apply the *Cauchy-Schwarz inequality* to get

$$\left(\sum_{i=1}^{n} \frac{1}{a-r_i}\right)^2 \leq \left(\sum_{i=1}^{n} 1^2\right)\left(\sum_{i=1}^{n} \frac{1}{(a-r_i)^2}\right) = n\sum_{i=1}^{n} \frac{1}{(a-r_i)^2}.$$

The above inequality can be written equivalently as

$$(n-1)\left(\sum_{i=1}^{n} \frac{1}{a-r_i}\right)^2 \geq n\left(\sum_{i=1}^{n} \frac{1}{a-r_i}\right)^2 - n\sum_{i=1}^{n} \frac{1}{(a-r_i)^2},$$

or, using (2) and (3),

$$(n-1)\left(\frac{P'(a)}{P(a)}\right)^2 \geq n\frac{P''(a)}{P(a)},$$

which is equivalent to (1).

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, USA; Daniel Campos Salas, Costa Rica; Jedrzej Garnek, Adam Mickiewicz University, Poznan, Poland; Omran Kouba, Damascus, Syria; Roberto Bosch Cabrera, Florida, USA.

U204. Let $A_1A_2\ldots A_n$ be a convex polygon and let P be a point in its interior. Prove that

$$\min_{i\in\{1,2,\ldots,n\}} \angle PA_iA_{i+1} \le \frac{\pi}{2} - \frac{\pi}{n}.$$

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

We argue by contradiction. Accordingly, suppose that

$$\min_{\{i\in 1,2,\ldots,n\}} \angle PA_iA_{i+1} > \frac{\pi}{2} - \frac{\pi}{n}.$$

Then, if we denote the angle $\angle PA_iA_{i+1}$ by α_i for all $i = 1,2,\ldots,n$, then

$$\begin{aligned} d_i &= PA_i \cdot \sin\angle PA_iA_{i+1} = PA_i \cdot \sin\alpha_i \\ &> PA_i \cdot \sin(\frac{\pi}{2} - \frac{\pi}{n}) = PA_i \cdot \cos\frac{\pi}{n}, \end{aligned}$$

for all $i = 1,2,\ldots,n$, where d_i is distance between P and the side A_iA_{i+1}.
Summing up alll these inequalities, it follows that

$$d_1 + d_2 + \cdots + d_n > \cos\frac{\pi}{n}(PA_1 + PA_2 + \cdots + PA_n),$$

which contradicts the general version of the Erdös-Mordell inequality from H. C. Lenhard, *Arch. Math.*, 12 (1961), pp. 311-314. This proves our problem.

Olympiad problems

O199. Prove that an acute triangle with $\angle A = 20^\circ$ and side-lengths a, b, c satisfying

$$\sqrt[3]{a^3+b^3+c^3-3abc} = \min(b,c)$$

is isosceles.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by El Alami Mohamed

First, note that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{\sin 160^\circ}{8\sin 20^\circ} = \frac{1}{8}$. Let $\alpha = \frac{\sin 80^\circ}{\sin 20^\circ}$; note that the previous identity yields $\alpha = 4\cos 20^\circ \cos 40^\circ = \frac{1}{2\cos 80^\circ}$, thus, since $\alpha = 4\cos 20^\circ \cos 40^\circ = 1 + 2\cos 20^\circ$, it follows that $\alpha - 3 = -2(1-\cos 20^\circ) = -4\sin^2 10^\circ$, i.e. $\alpha^2(\alpha-3) = -1$, that is $\alpha^3 - 3\alpha^2 + 1 = 0$. At this point, we assume without loss of generality that $b \geq c$. Accordingly, the equality from the hypothesis can be rewritten as $a^3 - 3abc + b^3 = 0$, which after dividing both sides by a^3 becomes $x^3 - 3xy + 1 = 0$, where $x = \frac{b}{a}$ and $y = \frac{c}{a}$. Now, since $b \geq c$ we have $x \geq y$ and $\angle B \geq \angle C$, so $90^\circ \geq \angle B \geq 80^\circ \geq \angle C$ (as the angles $\angle B$ and $\angle C$ both add up to 160°. Hence, by the Law of Sines, $x \geq \frac{\sin 80^\circ}{\sin 20^\circ} = \alpha$. On the other hand, $\alpha > 2$ and $x \longmapsto x^3 - 3x^2 + 1$ is strictly increasing in $]2, +\infty[$; therefore $0 = x^3 - 3xy + 1 \geq x^3 - 3x^2 + 1 \geq \alpha^3 - 3\alpha^2 + 1 = 0$, and so it follows that $x = y = \alpha$. This proves that ABC is isosceles, as desired.

Also solved by Emanuele Tron, Pinerolo, Italy; Omran Kouba, Damascus, Syria; Roberto Bosch Cabrera, Florida, USA.

O200. Determine all primes that do not have a multiple in the sequence $a_n = 2^n n^2 + 1,\ n \geq 1$.

Proposed by Andrei Ciupan, Harvard University, USA

Solution by Daniel Campos Salas, Costa Rica

We will prove that 2 and the primes congruent to -1 modulo 8 are all the primes that do not have a multiple in the sequence.

First, note that it is clear that 2 satisfies the statement. Next, consider the primes of the form $4k+1$. Let p be such a prime and recall that in this case there is a positive integer q, $q < p$, such that $p | q^2 + 1$ (for example, check $q = \left(\frac{p-1}{2}\right)!$). Take $n = (p-1)(p-q)$; by Fermat's little theorem,

$$2^n n^2 + 1 \equiv n^2 + 1 \equiv q^2 + 1 \equiv 0 \pmod{p},$$

and this proves the claim in this case.

Furthermore, let p be a prime of the form $8k+3$, for some positive integer k. In this case, -1 and 2 are not quadratic residues modulo p, which implies that -2 is a quadratic residue modulo p, i.e. there exists a positive integer q, $q < p$ such that $p | q^2 + 2$. Now, take $n = (p-1)(p-q-1) - 1$, and see that Fermat's little theorem gives

$$2^n n^2 + 1 \equiv 2^{p-2} q^2 + 1 \equiv 0 \pmod{p},$$

which proves the claim in this second case.

Finally, consider a prime $p = 8k - 1$ for some positive integer k, and note that here 2 is a quadratic residue, whereas -1 is not a quadratic residue. This clearly implies that the congruence $2^n n^2 + 1 \equiv 0$ does not have solution, thus yielding our conclusion.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; El Alami Mohamed; Emanuele Tron, Pinerolo, Italy; Felipe Ignacio Arbulu Lopez, Instituto Nacional General Jose Migue Carrera, Santiago, Chile; Jedrzej Garnek, Adam Mickiewicz University, Poznan, Poland; Roberto Bosch Cabrera, Florida, USA; Tigran Hakobyan, Armenia.

O201. Let ABC be a triangle with circumcenter O, and let perpendiculars at B, C to BC, CA intersect the sidelines CA, AB at E, F, respectively. Proe that the perpendiculars to OB and OC at F and E, respectively intersect at a point L lying on the altitude AD, satisfying $DL = LA \sin^2 A$.

Proposed by Francisco Javier Garcia Capitan, Spain

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We begin by proving the following preliminary result:

Claim. Let ABC be a triangle with circumcircle O, and draw lines through B, C intersecting at X, such that $\angle ABX = \pi - B$, $\angle ACX = \pi - C$ (ie, BX, CX are the symmetric of AB, AC with respect to the external bisectors of B, C, respectively). Then, A, O, X are collinear, and $AX = \frac{b \sin C}{\cos A} = \frac{c \sin B}{\cos A}$.

Proof. The following proof is valid for acute triangles. However, using signed angles and distances it is easily extended to any triangle. Clearly, $\angle CBX = \pi - 2B$, $\angle BCX = \pi - 2C$, and $\angle BXC = \pi - 2A$. By the Sine Law, $BX = \frac{c \cos C}{\cos A}$, while by the Cosine Law, and using that $\angle ABX = \pi - B$, we have after some algebra

$$AX^2 = AB^2 + BX^2 - 2AB \cdot BX \cos \angle ABX = \left(\frac{c \sin B}{\cos A} \right)^2 .$$

Again by the Sine Law,

$$\sin(\angle BAX) = \frac{BX \sin(\pi - B)}{AX} = \cos C = \sin\left(\frac{\pi}{2} - C \right).$$

Exchanging B, C we analogously obtain the symmetric results. The claim follows, since $\angle BAO = \frac{\pi}{2} - C$ and $\angle CAO = \frac{\pi}{2} - B$.

Returning to the original problem note first that since $BE \perp AF$ and $CF \perp AE$, then EB, FC are the respective altitudes from E, F in triangle AEF. It also follows that B, C are on the circle with diameter EF, or ABC and AEF are similar. Since $AB = AE \cos A$, it follows that triangle AEF is the result of the following transformation: reflect ABC on the internal bisector of angle A, then perform a scaling with factor $\frac{1}{\cos A}$ and center A. Note that, since the altitude from one vertex and the line through that vertex and the circumcircle, are symmetric with respect to the internal bisector of the corresponding angle, then the altitude from A in ABC is also the line through A and the circumcircle of AEF. Moreover, the perpendicular to OB at F clearly forms an angle equal to $\frac{\pi}{2} + \angle ABO = \pi - C$ with respect to AF. By the Claim, the point L is clearly on the line through A and the circumcircle of AEF, hence on the altitude from A in ABC, and by the relation of similarity between ABC and AEF, at a distance from A equal to

$$AL = \frac{1}{\cos A} \frac{b \sin C}{\cos A} = \frac{AD}{\cos^2 A}.$$

This yields $DL = AL - AD = AL \sin^2 A$, which is precisely what we wanted to prove.

Also solved by El Alami Mohamed; Roberto Bosch Cabrera, Florida, USA.

O202. Find all pairs (x, y) of positive integers for which there is a nonnegative integer z such that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)=1+\left(\frac{2}{3}\right)^{z}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

No solution has been received yet.

O203. Let M be an arbitrary point on the circumcircle of triangle ABC and let the tangents from this point to the incircle of the triangle meet the sideline BC at X_1, and X_2. Prove that the second intersection of the circumcircle of triangle MX_1X_2 with the circumcircle of ABC (different from M) coincides with the tangency point of the circumcircle with mixtilinear incircle in angle A (As usual, the $A-$mixtrilinear incircle names the circle tanget to AB, AC and to the circumcircle of ABC internally.).

Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ

Solution by Vladimir Zajic, New York, USA

Assume without loss of generality that M lies on the same side of line BC as the vertex A. In this case, we denote by (I) the common incircle of triangles ABC, and MX_1X_2 with radius r, and let D, E, F, Y_1, Y_2 be the tangency points of (I) with the sidelines BC, CA, AB, MX_1, MX_2.

Consider the inversion Ψ with center I and power r^2, which takes the vertices A, B, C, M, X_1, X_2 to the midpoints A', B', C', M', X_1', X_2' of the sides EF, FD, DE, Y_2Y_1, Y_1D, DY_2 of the intouch triangles DEF, and DY_2Y_1. The circumcircles (O), (P) of triangles ABC, and MX_1X_2 become the circumcircles (O'), (P') of the triangles $A'B'C'$, $M'X_1'X_2'$, which, because they coincide with the nine-point circles of two triangles of the same circumcircle, are congruent and have the common radius $\frac{r}{2}$. Since the sidelines BC, CA, AB, MX_1, MX_2 are tangent to the inversion circle (I), their images under Ψ are the congruent circles $\Gamma_a, \Gamma_b, \Gamma_c, \Omega_1, \Omega_2$ with diameters ID, IE, IF, IX_1, IX_2, respectively. The congruent circles $(O'), \Gamma_b, \Gamma_c$ with radii $\frac{r}{2}$ meet at point A'. Thus, a circle (A', r) with center A' and radius r is tangent to all three at points diametrically opposite to A'.

The internal angle-bisector AI of the angle $\angle A$ passing through the inversion center I is carried into itself. The mixtilinear incircle (K_a) and the mixtilinear excircle (L_a) of the triangle $\triangle ABC$ in the angle A are the only two circles centered on AI and simultanously tangent to CA, AB, and (O). Since the inversion center I is the similarity center of a circle and its inversion image, only the images $(K_a'), (L_a')$ of (K_a), (L_a) are centered on AI and tangent to Γ_b, Γ_c, (O'). The mixtilinear excircle (L_a), lying outside of the circumcircle (O) and outside of the inversion circle (I), has both intersections with AI on the ray $(IL_a$. Since the inversion in (I) has positive power r^2, its image (L_a') also has both intersections with AI on the ray $(IL_a$ and it is centered on the ray $(IL_a$. It cannot be identical with the circle (A', r) centered on the opposite ray $(IA$. Therefore, the image of the the mixtilinear incircle (K_a) in angle A under Ψ is the circle (A', r). Furthermore, the inversive image of the tangency point Z of the circles (K_a), and (O) is the tangency point Z' of (A', r) and (O'), the antipode of A' with respect to the circumcircle (O').

Let now A_0, B_0, C_0, O_1, O_2 be the centers of the congruent circles Γ_a, Γ_b, Γ_c, Ω_1, Ω_2. Since Γ_a is the reflection of (O') in $B'C'$ and since O' and I are isogonal conjugated with respect to triangle $A'B'C'$, the quadrilateral $B'Z'C'I$ is a parallelogram, and therefore, its diagonals $B'C'$, IZ' cut each other at half at the midpoint of segment $B'C'$. It now follows that the quadrilateral $IA_0Z'O'$ is also a parallelogram, and thus, $IA_0 = O'Z' = \frac{r}{2}$.

Now since $A_0I = A_0X_1' = O_1I = O_1X_1' = \frac{r}{2}$, the quadrilateral $A_0IO_1X_1'$ is a rhombus, and since the segments O_1X_1', IA_0, $O'Z'$ are parallel and congruent, the quadrilateral $O_1X_1'Z'O'$ is a parallelogram. In conclusion, the triangles $PX_1'Z'$, and $M'O_1O'$ are congruent and $P'Z' = M'O' = \frac{r}{2}$. Hence, Z' lies on the circle (P'), which means that the second intersection of the circumcircle (P) of triangle MX_1X_2 with the circumcircle (O) of ABC (different from M) coincides with the tangency point Z of the mixtilinear incircle in angle A with the circumcircle (O). $\square$

Note that the proposed problem ensures a geometric constructions of the mixtilinear incircles.

Construction. Let M be an arbitrary point on the circumcircle of a triangle ABC and let the tangents from this point to the incircle of the triangle intersect the sideline BC at point X_1 and X_2. Denote by Z the second intersection of the circumcircles of triangles ABC and MX_1X_2, and let K_a be the intersection of the lines AI and ZO, where I, O are the incenter, and circumcenter, respectively. The circle centered at K_a with radius K_aZ is the mixtilinear incircle in angle A, being simultanously tangent to AB, AC, and internally to (O).

Also solved by El Alami Mohamed

O204. Alice and Bob play the following game: Alice has a pawn in the upper-left unit cell of a $(2m+1) \times (2n+1)$ board that she wants to move to the low-right cell after a infinite number of steps. At each step she is allowed to move the pawn to an adjacent square, while Bob chooses a particular cell to "block", but still so that Alice still has a path from her current position to the low-right corner via adjacent cells. What is the maximum number of moves Bob can force Alice to make?

Proposed by Radu Bumbacea, Bucharest, Romania

No solution has been received yet.