The Monge-D'Alembert Circle Theorem

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Among numerous beautiful theorems in geometry some stand out for their simplicity and broad applicability in various problems, where it is often hard to obtain the same result with equal elegance using other techniques. One of the results that fits this description very well is the *Monge-D'Alembert circle theorem*, named after the renowned French geometers Gaspard Monge and Jean-le-Rond D'Alembert. This theorem states that the pairwise *exsimilicenters* (or external centers of similitude) of three distinct circles, all lying in the same plane, are collinear. The exsimilicenter (insimilicenter) of two circles $\mathcal{C}_1(x_1, r_1)$ and $\mathcal{C}_1(x_1, r_1)$ with centers $X_1(x_1)$, $X_2(x_2)$ given in Cartesian coordinates is defined by

$$S_e(\mathcal{C}_1, \mathcal{C}_2) = \frac{r_1 x_2 - r_2 x_1}{r_2 - r_1} \qquad \left(S_i(\mathcal{C}_1, \mathcal{C}_2) = \frac{r_1 x_2 + r_2 x_1}{r_2 + r_1} \right).$$

This means that S_e and S_i both lie on the line X_1X_2 such that $S_eX_1/S_eX_2 = r_1/r_2$ and $S_iX_1/S_iX_2 = -\, r_1/r_2$. Note that for two non-intersecting circles, none of which is inside the other, their exsimilicenter (insimilicenter) is the intersection point of the common external (internal) tangents. If the two circles are internally (externally) tangent to each other, then their exsimilicenter (insimilicenter) coincides with their corresponding tangency point.

The Monge-D'Alembert theorem can be proved using Desargues' theorem.

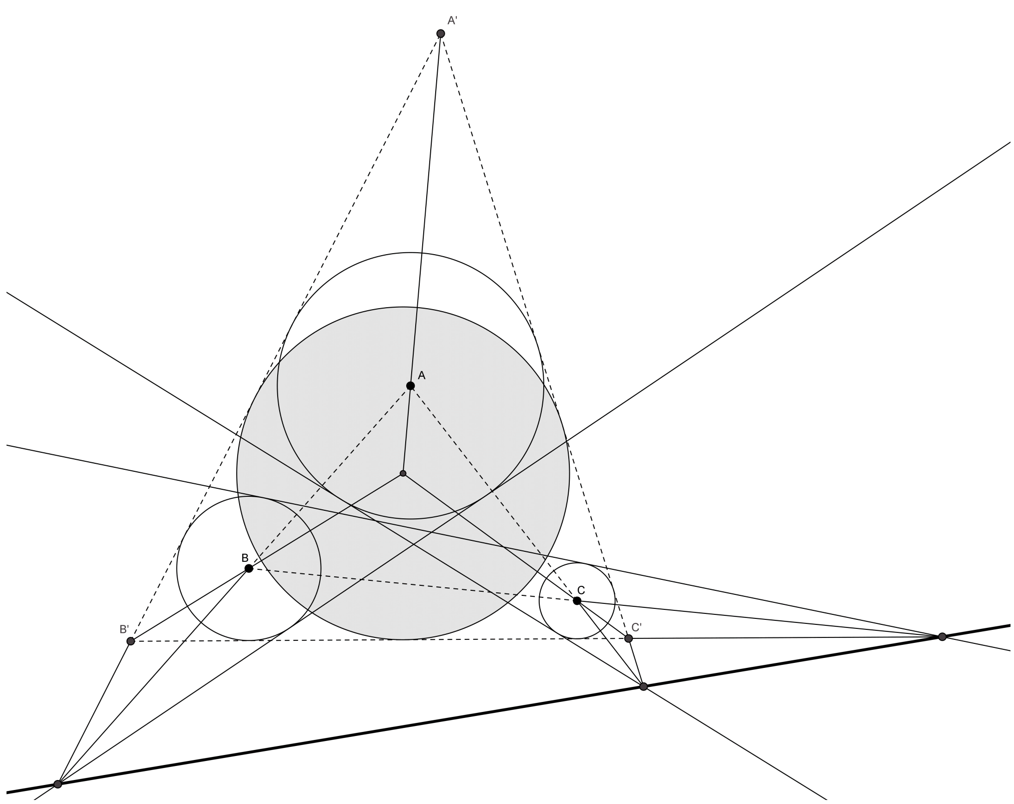

FIGURE 1.

Proof. Denote the centers of the three circles by A, B, C, and let A', B', C' be the intersections of the pairwise common tangents disjoint to triangle ABC (see FIGURE 1). In $A'B'C'$ the lines AA', BB', CC' serve as the internal bisectors of angles A', B', and C', respectively. Therefore, the three lines concur at the incenter of $A'B'C'$. This shows that the two triangles ABC and $A'B'C'$ are perspective from a point. By Desargues' theorem, they are also perspective from a line. This means that the three exsimilicenters are collinear.

Note that this proof can be adapted to show the following variation (of course, we could have used Menelaus' theorem as well):

COROLLARY 1. Two of the insimilicenters determined by three distinct circles, all lying in the same plane, are collinear with the exsimilincenter of the last pair of circles.

The Monge-D'Alembert circle theorem first appeared in Monge's *Géométrie Déscriptive, Lecons donné aux écoles normales l'an de la république* in 1798. Though D'Alembert didn't publish anything related to this subject, it is often said that he had inspired Monge in his research. Moreover, *their* theorem was subsequently extended by the same Monge to a 3-dimensional case.

- The six centers of similitude of three coplanar circles lie by threes on four straight lines.
- The vertices of the six common tangents cones of three spheres, taken in pairs, lie by threes on four straight lines.
- Given any four spheres in space fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane. If we draw the six other tangent cones, then their vertices lie by threes in planes with threes of the first group.

We will prove a part of the second theorem, where we only consider the externally tangent cones. The other proofs are similar and we leave them as exercises for the reader.

Proof. Consider the three spheres S_A, S_B, S_C with the circles $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ as equators. The point P_1 is the vertex of the $smallest$ cone containing S_A and S_B, which implies that it is in all planes tangent to both S_A and S_B that do not pass between the spheres. In particular it lies on the line ρ determined by the intersection of the two planes tangent to all three spheres and not passing between any of them. Similarly the other two points P_2 and P_3 lie on this line ρ.

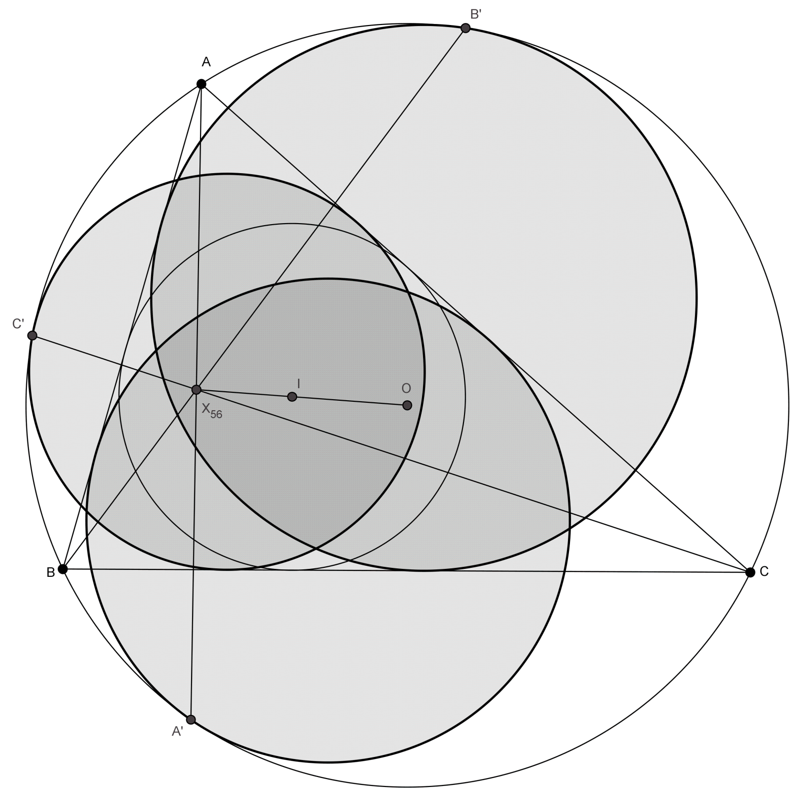

FIGURE 2.

We continue below with some applications of this beautiful theorem. Let us begin with the so-called *mixtilinear incircles* of a triangle. This term was introduced by L. Bankoff [2] when naming the three circles each tangent to two sides and to the circumcircle internally. Geometric constructions, properties and relations between them and the *mixtilinear excircles* (defined as the 3 circles each tangent to two sides and externally to the circumcircle) can be found in [2], [5], [6]. Here, we focus just on P. Yiu's main result from [6], which is also illustrated in FIGURE 2.

THEOREM 2 (Yiu). The three lines each joining a vertex to the point of contact of the circumcircle with the respective mixtilinear incircle are concurrent at the exsimilicenter of the circumcircle and incircle.

Proof. Denote by $\mathcal{K}_A, \mathcal{K}_B, \mathcal{K}_C$ the mixtilinear incircles in the angles A, B, and C, respectively, of triangle ABC, with circumcircle (O) and incircle (I). Let A', B', C' be the tangency points of these three circles with (O). By the Monge-D'Alembert theorem, the exsimilicenters A (of $\mathcal{K}_A$ and (I)), A' (of $\mathcal{K}_A$ and (O)) and X_{56} (of (O) and (I)) - this notation is according to [3] - are collinear. Similarly, the points B, B', X_{56} are collinear and so are C, C', X_{56}. Therefore, the lines AA', BB', CC' are concurrent at the exsimilicenter of (O) and (I). □

We are now ready to prove a result which was given as the last problem in an Iranian Team Selection Test for the International Mathematical Olympiad from 2002.

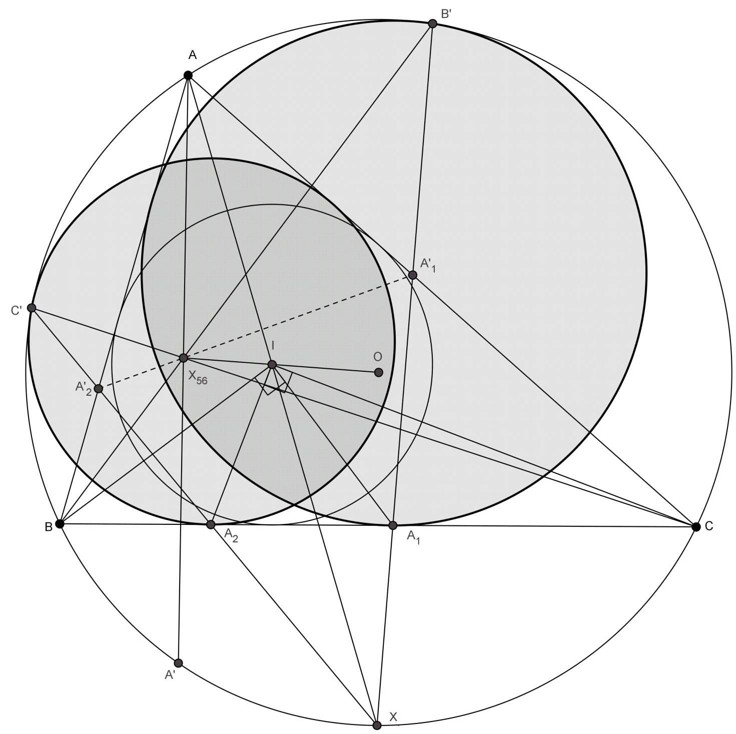

FIGURE 3.

EXERCISE 3. Let I be the incenter of a given triangle ABC and let A_1, A_2 be the intersections of the sideline BC with the perpendiculars in I to IB, IC, respectively (see FIGURE 3). Denote by X second intersection of AI with the circumcircle of ABC, and let A_1', A_2' be the intersections of $A'A_1$, $A'A_2$ with the sidelines CA, and AB, respectively. Analogously, define B_1', B_2', C_1', C_2'. Then, the lines $A_1'A_2'$, $B_1'B_2'$ and $C_1'C_2'$ are concurrent.

Proof. We begin with an auxiliary result which despite of its simplicity has been proposed as a problem for the IMO (International Mathematical Olympiad) in 1993.

LEMMA 4. In the configuration described in the proof of Theorem 2 (see also FIGURE 2), where V, W are the contact points of the circle $\mathcal{K}_a$ with the sidelines CA, AB, respectively, the incenter I of triangle ABC is the midpoint of segment VW.

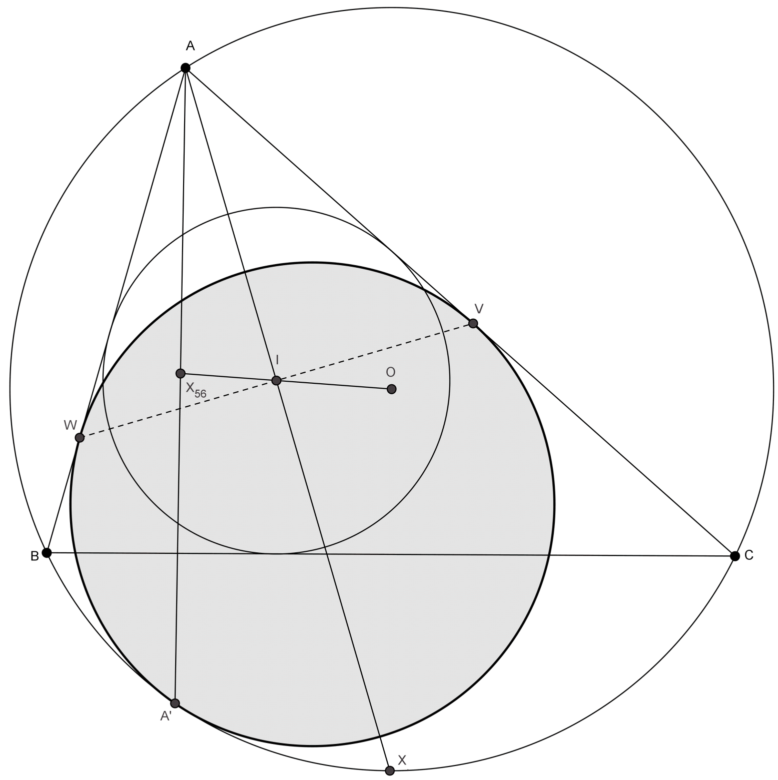

FIGURE 4.

Several proofs are at hand here. For example, see the discussion from [7]. We shall present one which makes use of Pascal's theorem, but the reader is encouraged to try alternative approaches.

Proof of Lemma 4. Denote by Y, Z the intersections of the internal bisectors of the angles ABC and BCA with the circumcircle of ABC, different from the vertices of the triangle. By Corollary 1, the exsimilicenter A' of (O) and $\mathcal{K}_a$, the insimilicenter W of $\mathcal{K}_a$ and the degenerated circle AB and the insimilicenter Z of the degenerated circle AB and (O) are collinear. Similarly, we establish that the points A', V, Y are collinear, and in this case, according to Pascal's theorem in the inscribed hexagon $ABYA'ZC$, the points V, I and W lie on a same line. Combining this with the fact that AI is the perpendicular bisector of the segment VW, yields that I is the midpoint of VW. This proves Lemma 4.

Returning now to the proof of Exercise 3, we see that A_1, A_2 are the tangency points of the sideline BC with the circles $\mathcal{K}_b$ and $\mathcal{K}_c$. Thus the points B', A_1, X and C', A_2, X are respectively collinear, and therefore from Pascal's theorem in the inscribed hexagon $ABB'XC'C$, we get that the intersection point of BB' and CC' lies on the line $A_1'A_2'$. But from Theorem 2, this point is X_{56}, the exsimilicenter (O) and (I). In this case, we can analogously say that X_{56} lies on the lines $B_1'B_2'$ and $C_1'C_2'$. This completes the proof of Exercise 3. $\square$

The next remarkable result concerns the *Malfatti circles* of a triangle, the three circles located inside the triangle, mutually tangent to each other, and each tangent to two sides of the triangle. See

FIGURE 5. Gian Francesco Malfatti posed the problem of constructing the above three circles, and he conjectured that this was the solution to the so called Malfatti problem. This problem consists of finding three non-overlapping circles within a right triangle so that their total area is maximized. His conjecture was incorrect, since some examples are known where the Malfatti circles are not the best solution.

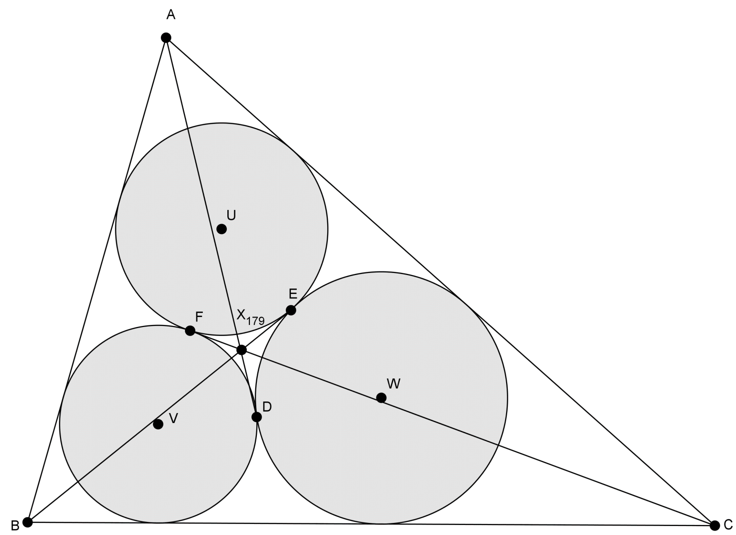

FIGURE 5.

EXERCISE 5. Let $\Gamma_A, \Gamma_B, \Gamma_C$ be the three Malfatti circles. As can be seen in FIGURE 5, we denote by D the tangency point of Γ_B and Γ_C, by E the tangency point of Γ_C and Γ_A, and by F the tangency point of Γ_A and Γ_B. Then, the lines $AD,\ BE,\ CF$ are concurrent.

Proof. Let U, V and W be the centers of the circles Γ_A, Γ_B and Γ_C respectively. The pairs of lines EF and VW, FD and WU, and DE and UV meet at X, Y and Z respectively (one or all three points may be at infinity, but the argument below works in those cases too). Menelaus' Theorem applied to triangle UVW and transversals EFX, FDY and DEZ yields that X, Y and Z are the intersection points of the common external tangents to Γ_B and Γ_C, Γ_C and Γ_A, and Γ_A and Γ_B respectively. By the Monge-D'Alembert circle theorem, these points are collinear. Consequently, according to Desargues' theorem, the lines AD, BE and CF are concurrent. □

Though given recently as Problem 2 in a Romanian IMO Team Selection Test from 2007, this result is rather known in literature. The common point of AD, BE and CF appears as X_{179} in [3], with trilinear coordinates

$$\left(\sec^4 \frac{A}{4} \ :\ \sec^4 \frac{B}{4} \ :\ \sec^4 \frac{A}{4}\right)$$

computed by Peter Yff, and is named the *first Ajima-Malfatti point*.

Now we will turn our attention to a proposal of Poland for the 48th edition of the IMO, hosted by Vietnam in 2007. This problem was sent by Waldemar Pompe and appeared as problem $G8$ on the IMO Shortlist, which was also the last geometry problem. Again, a very elegant solution can be given, using the Monge-D'Alembert circle theorem.

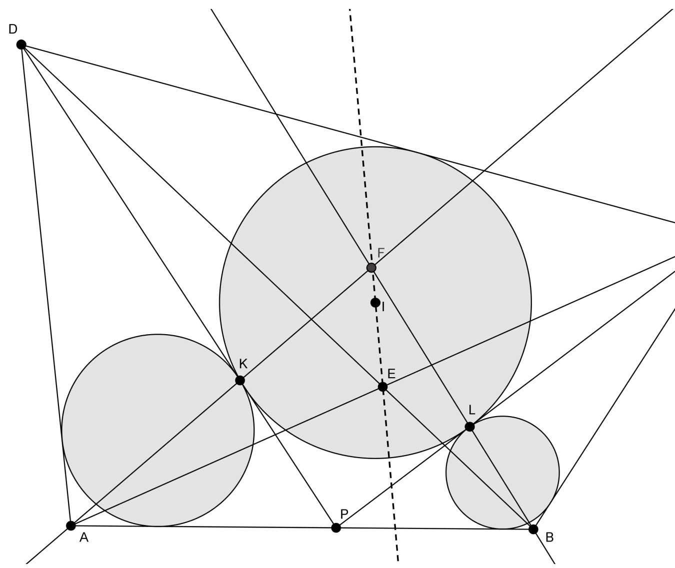

FIGURE 6.

EXERCISE 6. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Then, the points E, I, and F are collinear.

Proof. Consider the circle $\Gamma(O)$ tangent to the lines AB, BC, AD of the quadrilateral $ABCD$ (this circle exists, according to Apollonius' problem) and denote by ρ_1, ρ_2, ρ_p the incircles of triangles APD, BPC and CPD respectively.

Since A is the exsimilicenter of ρ_1 and Γ, K is the insimilicenter of ρ_1 and ρ_p, from Corollary 1, we know that the line AK intersects OI at the insimilicenter F of Γ and ρ_p. Analogously, we get that the line BK intersects OI at the same insimilicenter of Γ and ρ_p, and thus it remains to prove that E lies on the line OI.

Following Pithot's theorem, it is easy to see now that the quadrilaterals $APCD$ and $PBCD$ are circumscribed. Denote by ω_a and ω_b their incircles. Now A is the exsimilicenter of ω_a and Γ, C is the exsimilicenter of ω_a and ρ_p, and thus, by the Monge-D'Alembert circle theorem, the line AC intersects the line OI at the exsimilicenter of the circles ρ_p and Γ. Similarly, since B is the exsimilcenter of ω_b and Γ, D is the exsimilcenter of ρ_p and ω_b, the line BD intersects OI at the exsimilicenter of Γ and ρ_p. In this case, we conclude that E, F are the insimilicenter and exsimilicenter of the circles Γ and ρ_p, respectively. This completes the proof of Exercise 6, and moreover it shows that the cross-ratio (O, E, I, F) is -1. $\square$

As a final application of the Monge-D'Alembert circle theorem, we chose last year's most difficult IMO problem. Some people may disagree on calling it 'the most difficult problem', but by far no other problem was solved by so few students during the contest. It was proposed by Vladimir Shmarov, and selected as problem 6 in the contest. The problem was solved by an exceptionally low number of students. Only 12 out of the 535 contestants managed to give perfect solutions.

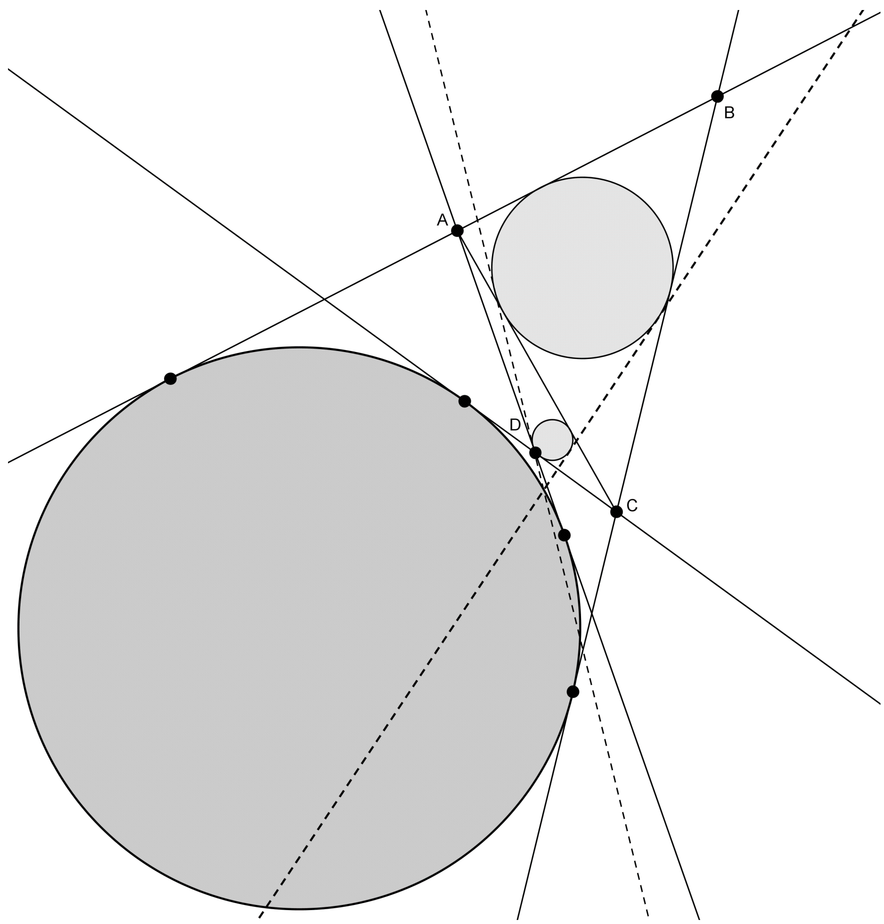

FIGURE 7.

EXERCISE 7. Let $ABCD$ be a convex quadrilateral with $|BA|$ different from $|BC|$. Denote the incircles of triangles ABC and ADC by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to ray BA beyond A and to ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to k_1 ad k_2 intersect on k.

Proof. Let us first establish the following preliminary result, which will turn out to be crucial in our proof.

LEMMA 8. In the configuration described above, let us further denote by E, F the tangency points of k_1, k_2 with AC, and let E', F' be their antipodes in the circles k_1, and k_2, respectively. Then, the points B, E' and F are collinear.

Proof of Lemma 8. Notice that B is the exsimilincenter of the circles k, k_1 and D is the insimilicenter of k and k_2. Corollary 1 now says that U, the intersection point of AC and BD, is the insimilincenter of k_1 and k_2.

Let P, Q, R, S be the tangency points of k with the lines AB, CD, DA, BC, respectively. If Y is the intersection point of AB, CD, and if Z is the intersection point of BC, DA, by Brianchon's theorem in the (degenerated) circumscribed hexagon $YQDZSB$, we know that YZ, QS, BD are concurrent. On the other hand, from the same Brianchon theorem, applied this time to the (degenerated) circumscribed hexagon $YDRZBP$, the lines YZ, PR, BD concur at one point, and thus we get that the lines $YZ,\ BD,\ QS,\ PR$ are all concurrent; denote their common point by V.

Because BD, AC meet at U, and AB, CD meet at Y, the pencil of lines $(ZB,\ ZU,\ ZA,\ ZY)$ is harmonic; so, by intersecting it with the transversal BD, we get that the points U, V divide the segment BD harmonically. If O is the center of k, we can see now that the pencil of lines $(OB,\ OU,\ OD,\ OV)$ is a harmonic, and so after projecting the intersection points of this pencil with the line through the centers of $k_1,\ k_2$ onto the line AC, we obtain that the points E, F divide UW harmonically, where by W we have denoted the intersection of AC and OV. (Note that $OV \perp AC$ since V is the pole of AC with respect to k.)

Furthermore, let k_3, k_4 be the excircles of ΔADC and ΔABC, respectively, and call E_r, F_r their tangency points with AC. Proceeding as above, we deduce that E_r and F_r divide the same segment UW harmonically, and combining this with $|E_rF_r| = |EF|$ (because $E_r,\ F_r$ are the reflections of $E,\ F$ in the midpoint of AC), and with the fact that U lies on the interior of both $[EF]$ and $[E_rF_r]$, yields $E \equiv E_r$ and $F \equiv F_r$. In this case, according to the Monge-D'Alembert theorem applied to k_1, k_4 and the degenerated circle AC, the points B, E' and $F_r \equiv F$ are collinear. This proves Lemma 8.

Turning back our attention to the main objective (the proof of Exercise 7), let V' be the intersection -the one closer to AC- of the line OV with k. Then, by the Monge-D'Alembert theorem applied to k, k_1 and the degenerated circle AC, we get that the points V', E', B are collinear, and thus from Lemma 8, we get that V' lies on the line $E'F$. Similarly, we now obtain the collinearity of E, F', V', and thus we have established that the external center of similitude V' of k_1 and k_2 is the point of intersection of $E'F$ and EF' (note that this point is unique, since $|BA| \neq |BC|$), which lies on k according to its definition. This completes the proof of Exercise 7. $\square$

References

[1] R. C. Archibald, Centers of similitude of circles and certain theorems attributed to Monge. Were they known to the Greeks?, *Amer. Math. Monthly*, Vol. 22, No. 1. (Dec.,1915), pp. 6-12.

[2] L. Bankoff, A Mixtilinear Adventure, *Crux Math.* **9** (1983) 2-7.

[3] C. Kimberling, Encyclopedia of Triangle Centers, available at
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[4] K. L. Nguyen, J. C. Salazar, On Mixtilinear Incircles and Excircles, *Forum Geom.* 6 (2006), 1-16.

[5] C. Pohoata, Geometric Constructions of the Mixtilinear Incircles, *Crux Math.*, to appear.

[6] P. Yiu, Mixtilinear Incircles, *Amer. Math. Monthly*, Vol. 106, No. 10. (Dec.,1999), pp. 952-955.

[7] http://www.mathlinks.ro/Forum/viewtopic.php?p=196786#196786.

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