Junior problems

J301. Let a and b be nonzero real numbers such that $ab \geq \frac{1}{a} + \frac{1}{b} + 3$. Prove that

$$ab \geq \left(\frac{1}{\sqrt[3]{a}} + \frac{1}{\sqrt[3]{b}}\right)^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, California, USA

Let $x := \sqrt[3]{ab}, y := -\dfrac{1}{\sqrt[3]{a}}, z := -\dfrac{1}{\sqrt[3]{b}}$; then, $xyz = 1$ and $ab \geq \dfrac{1}{a} + \dfrac{1}{b} + 3$ becomes equivalent with $x^3 + y^3 + z^3 \geq 3$. However,

$$(x + y + z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right) = x^3 + y^3 + z^3 - 3xyz = x^3 + y^3 + z^3 - 3 \geq 0$$

and

$$x^2 + y^2 + z^2 \geq xy + yz + zx \ ;$$

therefore,

$$x + y + z \geq 0,$$

which implies

$$x \geq -(y + z) \iff x^3 \geq (-y - z)^3 \iff ab \geq \left(\frac{1}{\sqrt[3]{a}} + \frac{1}{\sqrt[3]{b}}\right)^3.$$

Also solved by Polyahedra, Polk State College, USA, Himansu Mookherjee, Kolkata, India; Radouan Boukharfane, Morocco; Daniel Lasaosa, Pamplona, Spain; David Yang, Bergen County Academies, NJ, USA; Debojyoti Biswas, Kolkata, India; Eliott S. Kim, The Lawrenceville School, NJ, USA; Jaesung Son, Ridgewood, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Kevin Ren; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Woosung Jung, Korea International School, South Korea; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Joshua An, Washington University in St. Louis, MO, USA; Yeonjune Kang, Peddie School, Hightstown, NJ, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

J302. Given that the real numbers x, y, z satisfy $x + y + z = 0$ and

$$\frac{x^4}{2x^2 + yz} + \frac{y^4}{2y^2 + zx} + \frac{z^4}{2z^2 + xy} = 1,$$

determine, with proof, all possible values of $x^4 + y^4 + z^4$.

Proposed by Razvan Gelca, Texas Tech University, USA

Solution by Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco

Notice that $2x^2 + yz = x^2 + yz - x(y+z) = (x-y)(x-z)$ and $x^4 = \frac{1}{2}x^2(2x^2 + yz) - \frac{1}{2}x^2yz$, so

$$\frac{1}{2}(x^2 + y^2 + z^2) - \frac{xyz}{2}\left(\frac{x}{(x-y)(x-z)} + \frac{y}{(y-x)(y-z)} + \frac{z}{(z-x)(z-y)}\right) = 1.$$

It is clear that

$$\frac{x}{(x-y)(x-z)} + \frac{y}{(y-x)(y-z)} + \frac{z}{(z-x)(z-y)} = 0,$$

so

$$x^2 + y^2 + z^2 = 2$$

and

$$\left((x+y+z)^2 - (x^2 + y^2 + z^2)\right)^2 = 4(xy + yz + zx)^2 = 4(x^2y^2 + y^2z^2 + z^2x^2 + 2xyz(x+y+z)),$$

which it is equal to

$$4(x^2y^2 + y^2z^2 + z^2x^2) = 2\left((x^2 + y^2 + z^2)^2 - (x^4 + y^4 + z^4)\right)$$

Thus

$$4 = 2(4 - (x^4 + y^4 + z^4))$$

and the result follows.

Also solved by Polyahedra, Polk State College, USA, Himansu Mookherjee, Kolkata, India; Daniel Lasaosa, Pamplona, Spain; Radouan Boukharfane, Morocco; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Joshua An, Washington University in St. Louis, MO, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Adnan Ali, Mumbai, India; Arkady Alt, San Jose, California, USA; David Yang, Bergen County Academies, NJ, USA; Debojyoti Biswas, Kolkata, India; Eliott S. Kim, The Lawrenceville School, NJ, USA; Jaesung Son, Ridgewood, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănești, Romania; Woosung Jung, Korea International School, South Korea; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

J303. Let ABC be an equilateral triangle. Consider a diameter XY of the circle centered at C which passes through A and B such that lines AB and XY as well as lines AX and BY meet outside this circle. Let Z be the point of intersection of AX and BY. Prove that

$$AX \cdot XZ + BY \cdot YZ + 2CZ^2 = XZ \cdot YZ + 6AB^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Nikolaos Evgenidis, Gymnasium of Agia, Thessalia, Greece

Triangles ABZ and XYZ are similar, so it will be $YZ = 2AZ$, $XZ = 2BZ$ and $XY = 2AB$. Now, since $\angle YXZ + \angle XYZ + \angle BAC = \angle YXZ + \angle XYZ + \angle YZX$, it will be $\angle YZX = 60°$. It is also

$$2CZ^2 = ZX^2 + ZY^2 - \frac{XY^2}{2} \tag{1}$$

and

$$XY^2 = XZ^2 + YZ^2 - 2XZ \cdot YZ \cdot \cos 60° \Leftrightarrow XZ \cdot ZY = XZ^2 + YZ^2 - XY^2 \tag{2}$$

By (1) and (2) the given is written

$$AX \cdot XZ + BY \cdot YZ = XY^2.$$

But it is

$$AX \cdot XZ = XZ(XZ - AZ) = XZ^2 - AZ \cdot XZ = 4BZ^2 - AZ \cdot XZ$$

and

$$BY \cdot YZ = YZ(YZ - BZ) = YZ^2 - YZ \cdot BZ = 4AZ^2 - YZ \cdot BZ.$$

By (2), it is

$$XY^2 = 4BZ^2 + 4AZ^2 - 4BZ \cdot AZ.$$

So, it now suffices to prove

$$4BZ \cdot AZ = AZ \cdot XZ + BZ \cdot YZ(3).$$

It is known that $AZ \cdot XZ = BZ \cdot YZ$, which gives that (3) holds if

$$4BZ \cdot AZ = 2BZ \cdot YZ$$

that is true since $YZ = 2AZ$.

Also solved by Polyahedra, Polk State College, USA, Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănești, Romania; Daniel Lasaosa, Pamplona, Spain; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Joshua An, Washington University in St. Louis, MO, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Adnan Ali, Mumbai, India; Arkady Alt, San Jose, California, USA; David Yang, Bergen County Academies, NJ, USA; Himansu Mookherjee, Kolkata, India; Eliott S. Kim, The Lawrenceville School, NJ, USA; Kevin Ren; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Woosung Jung, Korea International School, South Korea; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

J304. Let a, b, c be real numbers such that $a+b+c=1$. Let M_1 be the maximum value of $a+\sqrt{b}+\sqrt[3]{c}$ and let M_2 be the maximum value of $a+\sqrt{b+\sqrt[3]{c}}$. Prove that $M_1 = M_2$ and find this value.

Proposed by Aaron Doman, University of California, Berkeley, USA

Solution by Daniel Lasaosa, Pamplona, Spain

By Lagrange's multiplier method, the maximum of $f(a,b,c) = a+\sqrt{b}+\sqrt[3]{c}$ when $a+b+c=1$ occurs for a real constant λ such that $1=\lambda$, $\frac{1}{2\sqrt{b}}=\lambda$, $\frac{1}{3\sqrt[3]{c^2}}=\lambda$, ie for $b=\frac{1}{4}$, $c=\frac{1}{3\sqrt{3}}$, and consequently $a=\frac{3}{4}-\frac{1}{3\sqrt{3}}$, for

$$M_1 = \frac{3}{4} - \frac{1}{3\sqrt{3}} + \frac{1}{2} + \frac{1}{\sqrt{3}} = \frac{5}{4} + \frac{2}{3\sqrt{3}}$$

Similarly, the maximum of $g(a,b,c) = a+\sqrt{b+\sqrt[3]{c}}$ occurs when $1=\lambda$, $\frac{1}{2\sqrt{b+\sqrt[3]{c}}}=\lambda$, and $\frac{1}{6\sqrt{b+\sqrt[3]{c}}\sqrt[3]{c^2}}=\lambda$. From the second relation we find $b+\sqrt[3]{c}=\frac{1}{4}$, which inserted in the third yields $c=\frac{1}{3\sqrt{3}}$, for $b=\frac{1}{4}-\frac{1}{\sqrt{3}}$, and consequently $a=\frac{3}{4}+\frac{2}{3\sqrt{3}}$, yielding

$$M_2 = \frac{3}{4} + \frac{2}{3\sqrt{3}} + \frac{1}{2} = \frac{5}{4} + \frac{2}{3\sqrt{3}} = M_1.$$

The conclusion follows.

Also solved by Polyahedra, Polk State College, USA, Radouan Boukharfane, Morocco; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Joshua An, Washington University in St. Louis, MO, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Arkady Alt, San Jose, California, USA; David Yang, Bergen County Academies, NJ, USA; Eliott S. Kim, The Lawrenceville School, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Kevin Ren; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Woosung Jung, Korea International School, South Korea; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

J305. Consider a triangle ABC with $\angle ABC = 30^\circ$. Suppose the length of the angle bisector from vertex B is twice the length of the angle bisector from vertex A. Find the measure of $\angle BAC$.

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Let ℓ_a, ℓ_b be the respective lengths of the angle bisectors from A, B. It is relatively well known (or can easily be found using Stewart's theorem and the bisector theorem) that

$$\ell_a = \frac{2bc}{b+c}\cos\frac{A}{2}, \qquad \ell_b = \frac{2ca}{c+a}\cos\frac{B}{2}.$$

It follows, after using the Sine Law and setting $C = 180^\circ - A - B$ and $B = 30^\circ$, that $\ell_b = 2\ell_a$ is equivalent to

$$\sin\frac{A}{2}(b+c) = 2\sin\frac{B}{2}(c+a),$$

$$\sin A\cos\left(60^\circ - \frac{A}{2}\right) = \sin A\cos\frac{B-C}{2} = 2\sin B\cos\frac{C-A}{2} = \cos\left(75^\circ - A\right).$$

In turn this can be further expressed as

$$\cos\left(75^\circ\right)\cos A = \sin A\left(\cos\left(60^\circ - \frac{A}{2}\right) - \cos\left(15^\circ\right)\right).$$

Now, if $A > 90^\circ$, $\cos A < 0$, hence $\cos\left(60^\circ - \frac{A}{2}\right) < \cos\left(15^\circ\right)$, yielding either $60^\circ - \frac{A}{2} > 15^\circ$, for $A < 90^\circ$, contradiction, or $60^\circ - \frac{A}{2} < -15^\circ$, for $A > 150^\circ = 180 - B$, absurd. On the other hand, if $A < 90^\circ$, then $\cos A > 0$, and $60^\circ - \frac{A}{2} < 15^\circ$, for $A > 90^\circ$, with contradiction again. It follows that $A = 90^\circ$, in which case clearly both sides are equal to $\cos\left(15^\circ\right)$, and consequently $\ell_b = 2\ell_a$.

Also solved by Polyahedra, Polk State College USA, Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Joshua An, Washington University in St. Louis, MO, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; David Yang, Bergen County Academies, NJ, USA; Himansu Mookherjee, Kolkata, India; Eliott S. Kim, The Lawrenceville School, NJ, USA; Jaesung Son, Ridgewood, NJ, USA; Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănești, Romania; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Woosung Jung, Korea International School, South Korea; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

J306. Let S be a nonempty set of positive real numbers such that for any a, b, c in S, the number $ab+bc+ca$ is rational. Prove that for any a and b in S, $\frac{a}{b}$ is a rational number.

Proposed by Bogdan Enescu, Buzau, Romania

Solution by Henry Ricardo, New York Math Circle

Suppose that $a, b, c \in S$. Replacing b by a—that is, taking the triplet (a, a, c)—we get $a^2+2ac = a(b+2c) \in \mathbf{Q}$. Now consider the triplet (b, b, c) to get $b^2 + 2bc = b(b + 2c) \in \mathbf{Q}$. Thus $\frac{a}{b} = \frac{a(b+2c)}{b(b+2c} \in \mathbf{Q}$.

Also solved by Polyahedra, Polk State College, USA, Daniel Lasaosa, Pamplona, Spain; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Joshua An, Washington University in St. Louis, MO, USA; Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Adnan Ali, Mumbai, India; Amedeo Sgueglia, Universitá degli studi di Padova, Italy; David Yang, Bergen County Academies, NJ, USA; Eliott S. Kim, The Lawrenceville School, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Kevin Ren; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănești, Romania; Woosung Jung, Korea International School, South Korea; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

Senior problems

S301. Let a, b, c be positive real numbers. Prove that

$$(a+b+c)(ab+bc+ca)(a^3+b^3+c^3) \leq (a^2+b^2+c^2)^3.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Mai Quốc Thắng, Ho Chi Minh City, Vietnam

Since the inequality is homogeneous, without loss of generality, we can assume that $p = a+b+c = 1$.

Let

$$q = ab+bc+ca > 0 \ ; \ r = abc > 0$$

We have

$$a^2+b^2+c^2 = 1-2q \ ; \ a^3+b^3+c^3 = 1-3q+3r \ ;$$

and

$$1 = (a+b+c)^2 \geq 3(ab+bc+ca) = 3q$$

The inequality can be rewritten as

$$q(1-3q+3r) \leq (1-2q)^3$$

Using $q^2 = (ab+bc+ca)^2 \geq 3abc(a+b+c) = 3r$, we have

$$q(1-3q+3r) = q-3q^2+3qr \leq q-3q^2+q^3$$

It suffices to prove

$$(1-2q)^3 \geq q-3q^2+q^3$$

Have

$$(1-2q)^3 - \left(q-3q^2+q^3\right) = (1-q)(3q-1)^2 \geq 0$$

The inequality is proved. Equality occurs if and only if $a = b = c$.

Also solved by Evgenidis Nikolaos, Gimnasium of Agia, Thessalia,Greece; Daniel Lasaosa, Pamplona, Spain; Yassine Hamdi, Lycée du Parc, Lyon, France; Li Zhou, Polk State College, Winter Haven, FL, USA; Joshua An, Washington University in St. Louis, MO, USA; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Woosung Jung, Korea International School, South Korea; Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănești, Romania; Sayak Mukherjee, Kolkata, India; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Henry Ricardo, New York Math Circle; Himansu Mookherjee, Kolkata, India; Arkady Alt, San Jose, California, USA; Philip Radoslavov Grozdanov, Yambol, Bulgaria; Radouan Boukharfane, Morocco; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Joe Hong, Seoul International School, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

S302. If triangle ABC has sidelengths a, b, c and triangle $A'B'C'$ has sidelengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$, prove that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \cos A' \cos B' \cos C'.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA

By the Law of Cosines

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \text{so that} \quad 1-\cos A = \frac{a^2-(b-c)^2}{2bc},$$

and

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \frac{\sqrt{a^2-(b-c)^2}}{2\sqrt{bc}}.$$

Similarly,

$$\sin\frac{B}{2} = \frac{\sqrt{b^2-(a-c)^2}}{2\sqrt{ac}} \quad \text{and} \quad \sin\frac{C}{2} = \frac{\sqrt{c^2-(a-b)^2}}{2\sqrt{ab}}.$$

Thus,

$$\begin{aligned}
\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= \frac{\sqrt{a^2-(b-c)^2}\cdot\sqrt{b^2-(a-c)^2}\cdot\sqrt{c^2-(a-b)^2}}{2\sqrt{bc}\cdot 2\sqrt{ac}\cdot 2\sqrt{ab}} \\
&= \frac{\sqrt{(b+c-a)^2}}{2\sqrt{bc}}\cdot\frac{\sqrt{(c+a-b)^2}}{2\sqrt{ac}}\cdot\frac{\sqrt{(a+b-c)^2}}{2\sqrt{ab}} \\
&= \frac{b+c-a}{2\sqrt{bc}}\cdot\frac{c+a-b}{2\sqrt{ac}}\cdot\frac{a+b-c}{2\sqrt{ab}} \\
&= \cos A' \cos B' \cos C'.
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Yassine Hamdi, Lycée du Parc, Lyon, France; Jaesung Son, Ridgewood, NJ, USA; Li Zhou, Polk State College, Winter Haven, FL, USA; Woosung Jung, Korea International School, South Korea; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Himansu Mookherjee, Kolkata, India; Debojyoti Biswas, Kolkata, India; Arkady Alt, San Jose, California, USA; Andrea Fanchini,Cantu',Italy; An Zhen-ping, Xianyang Normal University, Xianyang, Shaanxi, China; Adnan Ali, Mumbai, India; Philip Radoslavov Grozdanov, Yambol, Bulgaria; Joshua An, Washington University in St. Louis, MO, USA; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Joe Hong, Seoul International School, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

S303. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{n}{a_n}\right)$, for $n \geq 1$. Find $\lfloor a_{2014} \rfloor$.

Proposed by Marius Cavachi, Romania

Solution by Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco

By AM-GM we have for every $n \geq 1$

$$a_{n+1} = \frac{1}{2}(a_n + \frac{n}{a_n}) \geq \sqrt{n}$$

Hence,

$$a_{n+1} \leq \frac{1}{2}(a_n + \frac{n}{\sqrt{n-1}})$$

By induction it is straightforward from this that $\forall n \geq 2$

$$2^n a_{n+1} \leq a_1 + 2^{n-1} b_n + ... + 2b_3 + b_2$$

where

$$b_n = \frac{n}{\sqrt{n-1}}$$

and

$$b_{n+1} > b_n \Longleftrightarrow n^2 > n+1$$

so (b_n) is an increasing sequence and

$$a_{n+1} \leq \frac{1 + (2^{n-1} + ... + 1)b_n}{2^n} = \frac{1 + (2^n - 1)b_n}{2^n} < \frac{1}{2^n} + b_n$$

so

$$\sqrt{n} \leq a_{n+1} \leq \frac{1}{2^n} + \frac{n}{\sqrt{n-1}}$$

we apply this result for $n = 2013$ to obtain that

$$\lfloor a_{2014} \rfloor = 44$$

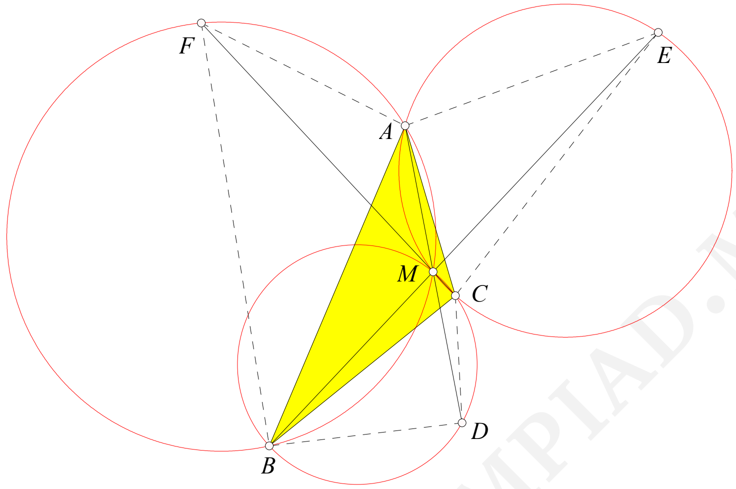
Also solved by Daniel Lasaosa, Pamplona, Spain; Woosung Jung, Korea International School, South Korea; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Henry Ricardo, New York Math Circle; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria; Arkady Alt, San Jose, California, USA; Li Zhou, Polk State College, Winter Haven, FL, USA; Joshua An, Washington University in St. Louis, MO, USA; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Radouan Boukharfane, Morocco; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Joe Hong, Seoul International School, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

S304. Let M be a point inside triangle ABC. Line AM intersects the circumcircle of triangle MBC for the second time at D. Similarly, line BM intersects the circumcircle of triangle MCA for the second time at E and line CM intersects the circumcircle of triangle MAB for the second time at F. Prove that

$$\frac{AD}{MD}+\frac{BE}{ME}+\frac{CF}{MF}\geq\frac{9}{2}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Li Zhou, Polk State College, USA



By Ptolemy's theorem, $MD\cdot BC = CM\cdot BD + BM\cdot CD$. By the law of sines,

$$\frac{BD}{BC}=\frac{\sin\angle AMB}{\sin\angle BMC},\quad \frac{CD}{BC}=\frac{\sin\angle CMA}{\sin\angle BMC}.$$

Hence $\frac{AM}{MD}=\frac{x}{y+z}$, where $x=AM\sin\angle BMC$, $y=BM\sin\angle CMA$, and $z=CM\sin\angle AMB$. Thus $\frac{AD}{MD}=\frac{AM}{MD}+1=\frac{x+y+z}{y+z}$. Therefore, by symmetry,

$$\begin{aligned}\frac{AD}{MD}+\frac{BE}{ME}+\frac{CF}{MF} &= (x+y+z)\left(\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right)\\ &\geq (x+y+z)\frac{9}{(y+z)+(z+x)+(x+y)}=\frac{9}{2},\end{aligned}$$

where the inequality is by virtue of the AM-HM inequality.

Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Joshua An, Washington University in St. Louis, MO, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Philip Radoslavov Grozdanov, Yambol, Bulgaria; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Woosung Jung, Korea International School, South Korea; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

S305. Solve in integers the following equation:

$$x^2 + y^2 + z^2 = 2(xy + yz + zx) + 1.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Pamplona, Spain

If two of the integers are equal (wlog $y = z$ because of the symmetry in the proposed equation), we need to solve $x(x - 4y) = 1$, for $x = \pm 1$ since it must divide 1, yielding in either case $y = z = 0$. Any other solution must have x, y, z all distinct.

If one of the integers is zero (wlog $z = 0$), the equation becomes $(x - y)^2 = 1$, with solution iff x, y are consecutive integers. Any other solution must have x, y, z all nonzero.

Since neither changing signs simultaneously to x, y, z, nor exchanging any two of them, changes the proposed equation, the previous arguments allow us to conclude that any solution must either be of the form $(x, y, z) = (k, k-1, 0)$ or one of its permutations, for any integer k, or must satisfy wlog $x > y > z > 0$, or must satisfy wlog $x > y > 0 > z$.

Case 1: If $x > y > 0 > z$, note that the equation rewrites

$$4xy + 1 = (x + y - z)^2 \geq (x + y + 1)^2 > (x + y)^2 + 1 > 4xy + 1,$$

the last inequality holding strictly because x, y are distinct. We have reached a contradiction, hence no solutions exist in this case.

Case 2: If $x > y > z > 0$, note that the equation rewrites as

$$(x + y - z - 1)(x + y - z + 1) = 4xy.$$

Now, $x + y - z - 1 < 2x$, or $x + y - z + 1 > 2y$, for $x > y + z - 1$, hence $x = y + z + \delta$, where $\delta \geq 0$ is a non-negative integer. Now,

$$\delta^2 = (x - y - z)^2 = 4yz + 1.$$

Assume now that y, z are any two positive integers such that $yz = \frac{n^2-1}{4}$ for some odd positive integer n, and take $x = y + z + n$. It follows that

$$x^2 + y^2 + z^2 - 2xy - 2zx = n^2 - 2yz = 2yz + 1,$$

and the equation is clearly satisfied. It follows that all solutions in this case are of the form

$$\left(y + z + \sqrt{4yz + 1}, y, z\right),$$

where y, z are positive integers such that $4yz + 1$ is a perfect square.

Restoring generality, all solutions are either a permutation of

$$(k, k - 1, 0),$$

where k takes any integral value, or a permutation of

$$\left(\frac{(2k + n)^2 - 1}{4k}, \frac{n^2 - 1}{4k}, k\right),$$

where n is any positive odd integral value, and k is any integral divisor (positive or negative) of $\frac{n^2-1}{4}$.

Also solved by Jaesung Son, Ridgewood, NJ, USA; Woosung Jung, Korea International School, South Korea; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Himansu Mookherjee, Kolkata, India; Debojyoti Biswas, Kolkata, India; Philip Radoslavov Grozdanov, Yambol, Bulgaria; Li Zhou, Polk State College, Winter Haven, FL, USA; Joshua An, Washington University in St. Louis, MO, USA; Radouan Boukharfane, Morocco; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Seong Kweon Hong, The Hotchkiss School, Lakeville, CT, USA; Joe Hong, Seoul International School, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

S306. Points M, N, K lie on sides BC, CA, AB of a triangle ABC, respectively and are different from its vertices. Triangle MNK is called beautiful if $\angle BAC = \angle KMN$ and $\angle ABC = \angle KNM$. If in triangle ABC there are two beautiful triangles with a common vertex, prove that triangle ABC is right.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Marius Stanean, Zalau, Romania

Suppose that there are two beautiful triangles (MNK and MN_1K_1) with a common vertex on the side BC. Is obviously that the two triangles are similar to the triangle ABC.

There is a spiral similarity (centered at M) swapping triangles $\triangle MNK \sim \triangle MN_1K_1$, i.e. there is another spiral similarity centered at M swapping triangles $\triangle MNN_1 \sim \triangle MKK_1$. Therefore $\angle MNN_1 = \angle MKK_1$ and $\angle MN_1N = \angle MK_1K$ hence $MNAK$ and MN_1AK_1 are cyclic quadrilaterals, so $\angle BAC + \angle NMK = 180^\circ \Longleftrightarrow \angle BAC = 90^\circ$.

Now, let N_0, K_0 the feet of the perpendiculars to AC, AB trough M. We have $\triangle MNN_0 \sim \triangle MKK_0$ and from this it follows that

$$\frac{MK_0}{MN_0} = \frac{MK}{MN} = \frac{AC}{AB} \Longleftrightarrow \frac{MK_0}{AC} = \frac{MN_0}{AB}$$

But $\dfrac{MK_0}{AC} = \dfrac{MB}{BC}$ and $\dfrac{MN_0}{AB} = \dfrac{MC}{BC}$, this means that M must be the middle of BC. In this situation, any right triangle in M, is a beautiful triangle.

Also solved by Daniel Lasaosa, Pamplona, Spain; Seung Hwan An, Taft School, Watertown, CT, USA; Joshua An, Washington University in St. Louis, MO, USA; Li Zhou, Polk State College, Winter Haven, FL, USA; Philip Radoslavov Grozdanov, Yambol, Bulgaria; Jaesung Son, Ridgewood, NJ, USA; Jin Hyup Hong; Shatlyk Mamedov, Dashoguz, Turkmenistan; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Woosung Jung, Korea International School, South Korea.

Undergraduate problems

U301. Let $x, y, z, t > 0$ such that $x \le 2$, $x + y \le 6$, $x + y + z \le 12$, and $x + y + z + t \le 24$. Prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \ge 1.$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

First solution by Brian Bradie, Christopher Newport University, Newport News, VA
Applying Jensen's inequality to the function $f(w) = 1/w$ for $w > 0$ yields

$$\begin{aligned}
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} &= \frac{1}{2} f\left(\frac{x}{2}\right) + \frac{1}{4} f\left(\frac{y}{4}\right) + \frac{1}{6} f\left(\frac{z}{6}\right) + \frac{1}{12} f\left(\frac{t}{12}\right) \\
&\ge f\left(\frac{x}{4} + \frac{y}{16} + \frac{z}{36} + \frac{t}{144}\right) \\
&= \frac{144}{36x + 9y + 4z + t} \\
&= \frac{144}{27x + 5(x+y) + 3(x+y+z) + (x+y+z+t)} \\
&\ge \frac{144}{27(2) + 5(6) + 3(12) + 24} = 1.
\end{aligned}$$

Second solution by Khakimboy Egamberganov, Tashkent, Uzbekistan
We have that $\frac{1}{x} \ge \frac{1}{2}$, $x + \frac{y}{2} + \frac{y}{2} \le 6$ and by Cauchy-Schwartz, $\frac{1}{x} + \frac{2}{y} + \frac{2}{y} \ge \frac{3}{2}$ and $\frac{1}{x} + \frac{4}{y} \ge \frac{3}{2}$. Similarly, we can $\frac{1}{x} + \frac{4}{y} + \frac{9}{z} \ge 3$ and $\frac{1}{x} + \frac{4}{y} + \frac{9}{z} + \frac{36}{t} \ge 6$. Hence

$$\frac{1}{x} \ge \frac{1}{2}, \qquad \frac{1}{x} + \frac{4}{y} \ge \frac{3}{2}, \qquad \frac{1}{x} + \frac{4}{y} + \frac{9}{z} \ge 3, \qquad \frac{1}{x} + \frac{4}{y} + \frac{9}{z} + \frac{36}{t} \ge 6.$$

So

$$36\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{z}\right) = 27\left(\frac{1}{x}\right) + 5\left(\frac{1}{x} + \frac{4}{y}\right) + 3\left(\frac{1}{x} + \frac{4}{y} + \frac{9}{z}\right) + \left(\frac{1}{x} + \frac{4}{y} + \frac{9}{z} + \frac{36}{t}\right) \ge 36$$

and

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \ge 1.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Woosung Jung, Korea International School, South Korea; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Kevin Ren; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Jaesung Son, Ridgewood, NJ, USA; Hyunseo Yang, Daecheong Middle School, Seoul, South Korea; Khakimboy Egamberganov, Tashkent, Uzbekistan; Mai Quốc Thắng, Ho Chi Minh City, Vietnam; Himansu Mookherjee, Kolkata, India; Arkady Alt, San Jose, California, USA; An Zhen-ping, Xianyang Normal University, Xianyang, Shaanxi, China; Adnan Ali, Mumbai, India; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Radouan Boukharfane, Morocco; Li Zhou, Polk State College, Winter Haven, FL, USA; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Joe Hong, Seoul International School, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

U302. Let a be a real number. Evaluate

$$a-\sqrt{a^2-\sqrt{a^4-\sqrt{a^8-\ldots}}}.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Li Zhou, Polk State College, USA

The problem's conclusion is flawed, because the expression is divergent for all $a \neq 0$. Indeed, if $a > 0$, then the sequence is

$$a, \quad a-\sqrt{a^2}=0, \quad a-\sqrt{a^2-\sqrt{a^4}}=a-\sqrt{a^2-a^2}=a, \quad \ldots,$$

that is, it alternates between a and 0. Likewise, if $a < 0$, then the sequence is

$$a, \quad a-\sqrt{a^2}=2a, \quad a-\sqrt{a^2-\sqrt{a^4}}=a-\sqrt{a^2-a^2}=a, \quad \ldots,$$

that is, it alternates between a and $2a$.

Also solved by Juan José Granier Torres, Universidad de Chile, Santiago, Chile; Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Li Zhou, Polk State College, Winter Haven, FL, USA; Radouan Boukharfane, Morocco; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria; Jaesung Son, Ridgewood, NJ, USA; Moubinool Omarjee, Lycée Henri IV, Paris France; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Woosung Jung, Korea International School, South Korea; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

U303. Let $p_1, p_2, \ldots, p_k$ be distinct primes and let $n = p_1 p_2 \cdots p_k$. For each function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, denote $P_f(n) = f(1)f(2)\cdots f(n)$.

(a) For how many functions f are n and $P_f(n)$ relatively prime?

(b) For how many functions f is $\gcd(n, P_f(n))$ a prime?

Proposed by Vladimir Cerbu and Mihai Piticari, Romania

Solution by Li Zhou, Polk State College, USA

(a) Let $[n] = \{1, 2, \ldots, n\}$, $A = \{a \in [n] : (a, n) = 1\}$, and F be the set of all such functions. Then $|A| = \phi(n) = (p_1 - 1)\cdots(p_k - 1)$, where ϕ is Euler's totient function. Now for each $f \in F$, $f(i) \in A$ for each $i \in [n]$. So $|F| = (\phi(n))^n$.

(b) Let $A_1 = \{a \in [n] : (a, n) = p_1\}$ and $F_1 = \{f \in F : (n, P_f(n)) = p_1\}$. Then $|A_1| = \phi(n/p_1) = (p_2 - 1)\cdots(p_k - 1)$. Now if $f \in F_1$, then $f(i) \in A \cup A_1$ for each $i \in [n]$, but not $f(i) \in A$ for *all* $i \in [n]$. Therefore,

$$|F_1| = |A \cup A_1|^n - |A|^n = (|A| + |A_1|)^n - |A|^n = (\phi(n))^n \left(\frac{p_1^n}{(p_1 - 1)^n} - 1 \right).$$

Hence the desired answer is

$$(\phi(n))^n \left(\sum_{i=1}^{k} \frac{p_i^n}{(p_i - 1)^n} - k \right).$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Woosung Jung, Korea International School, South Korea; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Jaesung Son, Ridgewood, NJ, USA; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Joe Hong, Seoul International School, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

U304. No solutions have yet been received.

U305. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that $a_1 + a_2 + \ldots + a_n < n^2$ for all $n \geq 1$. Prove that

$$\lim_{n\to\infty}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)=\infty.$$

Proposed by Mihai Piticari Campulung, Moldovenesc, Romania

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By rearranging the sequence if necessary, we may suppose that $(a_n)_{n\geq 1}$ is an increasing sequence. We argue by contradiction.

Let us suppose that $\lim\limits_{n\to\infty}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)<\infty$. This implies by comparison with the harmonic series that $\lim\limits_{n\to\infty}\frac{n}{a_n}=0$. Now, since the last limit exists, the following limit exists as well

$$\lim_{n\to\infty}\frac{n^2}{a_1+a_2+\ldots+a_n}=\lim_{n\to\infty}\frac{2n-1}{a_n}=0$$

by the Cezaro-Stolz criteria.

But this contradicts with the fact that $1\leq\frac{n^2}{a_1+a_2+\ldots+a_n}$.

Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Li Zhou, Polk State College, Winter Haven, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA; Salem Malikic, Simon Fraser University, Burnaby, BC, Canada; Radouan Boukharfane, Morocco; Khakimboy Egamberganov, Tashkent, Uzbekistan; Henry Ricardo, New York Math Circle; Jaesung Son, Ridgewood, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Juan José Granier Torres, Universidad de Chile, Santiago, Chile; Moubinool Omarjee, Lycée Henri IV, Paris France; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Stanescu Florin, School Serban Cioculescu, Gaesti, Dambovita, Romania; Woosung Jung, Korea International School, South Korea; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

U306. Let n be a natural number. Prove the identity

$$\pi = \sum_{k=1}^{n} \frac{2^{k+1}}{k\binom{2k}{k}} + \frac{4^{n+1}}{\binom{2n}{n}} \int_1^\infty \frac{1}{(1+x^2)^{n+1}} dx$$

and derive the estimate

$$\frac{2}{2^n\sqrt{n}} < \pi - \sum_{k=1}^{n} \frac{2^{k+1}}{k\binom{2k}{k}} < \frac{4}{2^n\sqrt{n}}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Li Zhou, Polk State College, USA

Let

$$P_n = \sum_{k=1}^{n} \frac{2^{k+1}}{k\binom{2k}{k}}, \quad I_n = \int_1^\infty \frac{1}{(1+x^2)^{n+1}} dx, \quad R_n = \frac{4^{n+1}}{\binom{2n}{n}} I_n.$$

Using the substitution $x = \tan\theta$ and integration by parts, we get

$$\begin{aligned} I_n &= \int_{\pi/4}^{\pi/2} \cos^{2n}\theta d\theta = \left[\cos^{2n-1}\theta\sin\theta\right]_{\pi/4}^{\pi/2} + (2n-1)\int_{\pi/4}^{\pi/2} \cos^{2n-2}\theta\sin^2\theta d\theta \\ &= -\frac{1}{2^n} + (2n-1)\left(I_{n-1} - I_n\right). \end{aligned}$$

Hence $1 + 2^{n+1}nI_n = 2^n(2n-1)I_{n-1}$ for all $n \geq 1$, with $I_0 = \frac{\pi}{4}$. Since $I_1 = -\frac{1}{4} + \frac{\pi}{8}$, it is easy to see that $P_1 + R_1 = \pi$. As an induction hypothesis, assume that $\pi = P_{n-1} + R_{n-1}$ for some $n \geq 2$. Then

$$\pi = P_{n-1} + R_{n-1} = P_{n-1} + \frac{4^n}{\binom{2n-2}{n-1}}\left(\frac{1}{2^n(2n-1)} + \frac{2^{n+1}n}{2^n(2n-1)}I_n\right) = P_n + R_n,$$

completing the induction.

To prove the estimate, we shall use the well-known Wallis' inequality

$$\frac{1}{\sqrt{\pi(n+1)}} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{1}{4^n}\binom{2n}{n} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}} < \frac{1}{\sqrt{3n+1}}.$$

First,

$$I_n = \frac{1}{2^n}\int_{\pi/4}^{\pi/2} (1+\cos 2\theta)^n d\theta = \frac{1}{2^{n+1}}\int_0^{\pi/2} (1-\cos z)^n dz.$$

Using the well-known refinement of Kober's inequality that $1 - \cos z > (2z/\pi)^2$ for $0 < z < \pi/2$, we get

$$2^{n+1}I_n > \left(\frac{2}{\pi}\right)^{2n}\int_0^{\pi/2} z^{2n} dz = \frac{\pi}{2(2n+1)} > \frac{3}{2(2n+1)}.$$

Thus

$$\pi - P_n = R_n > \frac{12\sqrt{3n+1}}{2^{n+2}(2n+1)} \geq \frac{2}{2^n\sqrt{n}},$$

where the last inequality follows from $\left(3\sqrt{(3n+1)n}\right)^2 - (4n+2)^2 = (11n+4)(n-1) \geq 0$.

Next, it is easy to check directly the upper bound of R_n for $n = 1, 2, 3$. Consider $n \geq 4$. Note that $1 - \cos z < \frac{3}{2\pi}z$ for $0 < z < \frac{\pi}{3}$ and $1 - \cos z < \frac{3}{\pi}z - \frac{1}{2}$ for $\frac{\pi}{3} < z < \frac{\pi}{2}$. Therefore

$$2^{n+1}I_n < \left(\frac{3}{2\pi}\right)^n\int_0^{\pi/3} z^n dz + \left(\frac{3}{\pi}\right)^n\int_{\pi/3}^{\pi/2}\left(z - \frac{\pi}{6}\right)^n dz = \frac{\pi}{3(n+1)}\left(1 + \frac{1}{2^{n+1}}\right) \leq \frac{11\pi}{32(n+1)}.$$

Hence

$$R_n < \frac{44\pi\sqrt{\pi(n+1)}}{32(n+1)2^{n+1}} < \frac{44}{2^{n+1}\sqrt{32(n+1)}} < \frac{4}{2^n\sqrt{n}},$$

where the second inequality follows from $\pi^3 < 32$ and the last inequality follows from $11 < 2\sqrt{32}$.

Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Radouan Boukharfane, Morocco; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Li Zhou, Polk State College, Winter Haven, FL, USA; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

Olympiad problems

O301. Let a, b, c, d be nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$\frac{a}{b+3} + \frac{b}{c+3} + \frac{c}{d+3} + \frac{d}{a+3} \leq 1.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Daniel Lasaosa, Pamplona, Spain and
We use in this solution the following
Lemma: For non-negative reals a, b, c, d, we have

$$\sqrt{a^2 + b^2 + c^2 + d^2}\left(a^2 + b^2 + c^2 + d^2\right) \geq 2(abc + bcd + cda + dab),$$

with equality iff $a = b = c = d$.
Proof: The proposed inequality is equivalent to

$$\left(a^2 + b^2 + c^2 + d^2\right)^3 \geq 4(abc + bcd + cda + dab)^2,$$

or to

$$a^6 + b^6 + c^6 + d^6 + 2\left(a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2\right) +$$
$$+3\left(a^4b^2 + a^2b^4 + b^4c^2 + b^2c^4 + c^4d^2 + c^2d^4 + d^4a^2 + d^2a^4 + a^4c^2 + a^2c^4 + b^4d^2 + b^2d^4\right) \geq$$
$$\geq 8\left(a^2b^2cd + b^2c^2da + c^2d^2ab + d^2a^2bc + a^2c^2bd + b^2d^2ca\right).$$

Note now that, by the AM-GM inequality, we have $a^4c^2 + b^4d^2 \geq 2a^2b^2cd$, $2a^6 + 2b^6 + c^6 + d^6 \geq 6a^2b^2cd$, and $a^2b^2c^2 + d^2a^2b^2 \geq 2a^2b^2cd$, with equality in all of them simultaneously, iff $a = b = c = d$. Using these inequalities and the result of permuting in them a, b, c, d, the Lemma follows. Note that in this particular problem, the Lemma results in $abc + bcd + cda + dab \leq 4$, with equality iff $a = b = c = d = 1$.

After some algebra, the inequality proposed in the problem statement is equivalent to

$$3(ab + bc + cd + da + ac + bd) + 2(abc + bcd + cda + dab) + abcd \leq 27.$$

Now, $ab+bc+cd+da \leq a^2+b^2+c^2+d^2$ because of the scalar product inequality applied to vectors (a, b, c, d) and (b, c, d, a), while $ac \leq \frac{a^2+c^2}{2}$ and $bd \leq \frac{b^2+d^2}{2}$ because of the AM-GM inequality, or

$$ab + bc + cd + da + ac + bd \leq 3\frac{a^2 + b^2 + c^2 + d^2}{2} = 6,$$

with equality iff $a = b = c = d = 1$. Moreover, by the QM-GM inequality, we find

$$1 = \sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}} \geq \sqrt[4]{abcd},$$

or $abcd \leq 1$, with equality iff $a = b = c = d = 1$. Finally, by the Lemma we have $abc + bcd + cda + dab \leq 4$. The conclusion follows, equality holds iff $a = b = c = d = 1$.

Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Woosung Jung, Korea International School, South Korea.

O302. Let ABC be an isosceles triangle with $AB = AC$ and let $M \in (BC)$ and $N \in (AC)$ such that $\angle BAM = \angle MNC$. Suppose that lines MN and AB intersect at P. Prove that the bisectors of angles BAM and BPM intersect at a point lying on line BC.

Proposed by Bogdan Enescu, Buzau, Romania

Solution by Marius Stanean, Zalau, Romania

Let A' reflection of A across BC. Because $\angle BAM = \angle MNC$ and $\angle ABM = \angle NCM$ we have $\triangle ABM \sim \triangle NCM$ and from this it follows that $\angle NMC = \angle AMB$. Therefore $\angle A'MB = \angle AMB = \angle NMC$ so A', M, N, P are collinear points.

Now in triangle $A'BP$, because BC is angle bisector of $\angle A'BP$ we deduce that the other two angle bisectors intersect on BC. Let X this point (i.e. the incenter of $\triangle A'BP$). But the fact that A' is the reflection of A with respect to BC, it follows that AX is angle bisector of $\angle BAM$.

Also solved by Mriganka Basu Roy Chowdhury, South Point High School, Kolkata, West Bengal, India; Daniel Lasaosa, Pamplona, Spain; Li Zhou, Polk State College, Winter Haven, FL, USA; Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănești, Romania; Saturnino Campo Ruiz, Salamanca, Spain; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Jaesung Son, Ridgewood, NJ, USA; Khakimboy Egamberganov, Tashkent, Uzbekistan; Himansu Mookherjee, Kolkata, India; Arkady Alt, San Jose, California, USA; Andrea Fanchini,Cantu',Italy; Adnan Ali, Mumbai, India; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Philip Radoslavov Grozdanov, Yambol, Bulgaria; Joe Hong, Seoul International School, South Korea; Seung Hwan An, Taft School, Watertown, CT, USA; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

O303. Let a, b, c be real numbers greater than 2 such that

$$\frac{1}{a^2-4}+\frac{1}{b^2-4}+\frac{1}{c^2-4}=\frac{1}{7}.$$

Prove that

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\leq\frac{3}{7}.$$

Proposed by Mihaly Bencze, Brasov, Romania

Solution by Mai Quốc Thắng, Ho Chi Minh City, Vietnam

With $t>2$, we have

$$\frac{1}{t+2}\leq\frac{9}{10\left(t^2-4\right)}+\frac{1}{10}$$

Because

$$\frac{9}{10\left(t^2-4\right)}+\frac{1}{10}-\frac{1}{t+2}=\frac{(t-5)^2}{10\,(t+2)\,(t-2)}\geq 0$$

With $a, b, c>2$, we have

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\leq\frac{9}{10}\cdot\left(\frac{1}{a^2-4}+\frac{1}{b^2-4}+\frac{1}{c^2-4}\right)+\frac{3}{10}=\frac{3}{7}$$

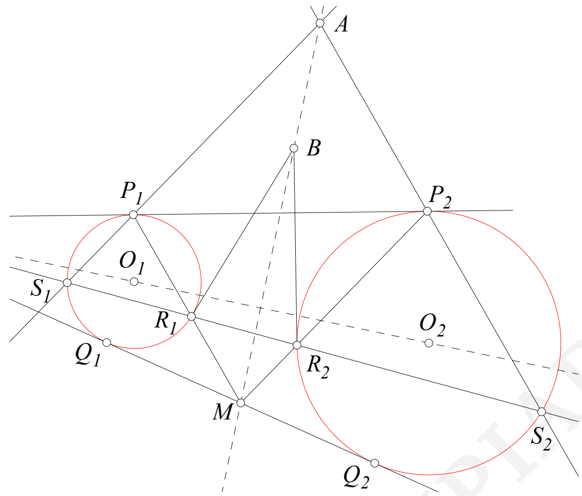
Equality holds clearly when $a=b=c=5$.

Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; Daniel Lasaosa, Pamplona, Spain; Joe Hong, Seoul International School, South Korea; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Li Zhou, Polk State College, Winter Haven, FL, USA; Philip Radoslavov Grozdanov, Yambol, Bulgaria; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Adnan Ali, Mumbai, India; An Zhen-ping, Xianyang Normal University, Xianyang, Shaanxi, China; Arkady Alt, San Jose, California, USA; Khakimboy Egamberganov, Tashkent, Uzbekistan; Jaesung Son, Ridgewood, NJ, USA; Jishnu Bose,Indian Statistical Institute, Kolkata, India; Shatlyk Mamedov, Dashoguz, Turkmenistan; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Salem Malikic, Simon Fraser University, Burnaby, BC, Canada; Woosung Jung, Korea International School, South Korea; Zarif Ibragimov, Samarkand, Uzbekistan; Yassine Hamdi, Lycée du Parc, Lyon, France; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

O304. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be non-intersecting circles centered at O_1 and O_2. One common external tangent of these circles touches $\mathcal{C}_i$ at P_i $(i = 1, 2)$. The other common external tangent touches $\mathcal{C}_i$ at Q_i $(i = 1, 2)$. Denote by M the midpoint of Q_1Q_2. Let P_iM intersect $\mathcal{C}_i$ at R_i and R_1R_2 intersect $\mathcal{C}_i$ again at S_i $(i = 1, 2)$. P_1S_1 intersects P_2S_2 at A. The tangent to $\mathcal{C}_1$ at R_1 and the tangent to $\mathcal{C}_2$ at R_2 intersect at B. Prove that $AB \perp O_1O_2$.

Proposed by Alex Anderson, UC Berkeley, USA

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA



By definition, $MP_1 \cdot MR_1 = MQ_1^2 = MQ_2^2 = MP_2 \cdot MR_2$. So P_1, P_2, R_2, R_1 are concyclic. Therefore $\angle AP_1P_2 = \angle P_1R_1S_1 = \angle P_1P_2R_2 = \angle P_2S_2R_2$. Likewise $\angle AP_2P_1 = \angle P_1S_1R_1$. Hence $\triangle AP_1P_2 \sim \triangle AS_2S_1$, and thus $AP_1 \cdot AS_1 = AP_2 \cdot AS_2$. Therefore AM is the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_2$, which implies that $AM \perp O_1O_2$. Moreover

$$\angle BR_1R_2 = \angle S_1P_1R_1 = \angle P_1AP_2 = \angle S_2P_2R_2 = \angle BR_2R_1,$$

so $BR_1 = BR_2$, that is, B is on the radical axis AM as well. This completes the proof.

Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; Joe Hong, Seoul International School, South Korea; Chakib Belgani, Youssoufia, Morocco and Mahmoud Ezzaki, Oujda, Morocco; Daniel Lasaosa, Pamplona, Spain; Mriganka Basu Roy Chowdhury, South Point High School, Kolkata, West Bengal, India; Peter C.Shim, Pingry School, Basking Ridge, NJ, USA; Marius Stanean, Zalau, Romania; Jin Hyup Hong, Great Neck South High School, New Hyde Park, NY, USA.

O305. Prove that for any positive integers m and a, there is a positive integer n such that $a^n + n$ is divisible by m.

Proposed by Alex Anderson, UC Berkeley, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Let $a = uv$, where all prime factors of u divide m, and all prime factors of v are coprime with m. If $v = 1$, then a^n is a multiple of m for all sufficiently large n, and it suffices to take n equal to a sufficiently large multiple of m. We will assume henceforth that there are prime factors of a which are coprime with m.

Lemma: Given a positive integer a, for all positive integer m, there exist arbitrarily large positive integers $n_0, n_1, n_2, \ldots, n_{m-1}$ such that

$$a^{n_i} + n_i \equiv i \pmod{m}.$$

Proof: If $m = 1$, the result is trivially true. If $m = 2$, it suffices to take n_0 arbitrarily large with the same parity as a, and n_1 arbitrarily large with opposite parity to a. Assume that the result is true for $1, 2, \ldots, m-1$, and let k be the periodicity of the remainders of a^n modulus m for sufficiently large n, ie k is the least positive integer such that $a^n \equiv a^{n+k} \pmod{m}$ for all sufficiently large n. This clearly exists because writing $a = uv$ as above, for all sufficiently large n the remainder of u^n modulus m' is constant, where m' is the product of all factors of m which are not coprime with a, and the remainders of u^n modulus $\frac{m}{m'}$, and of v^n modulus m, are periodic because of the Euler-Fermat theorem, which applies because u is coprime with $\frac{m}{m'}$, and v is coprime with m. Let d be the greatest common divider of m and k, or since $k \leq \varphi(m) < m$, where φ denotes Euler's totient function, then $d < m$, and by hypothesis of induction there exist arbitrarily large positive integers $n_0, n_1, \ldots, n_{d-1}$ such that, for each $i = 0, 1, \ldots, d-1$, we have

$$a^{n_i} + n_i \equiv i \pmod{d}.$$

Define $\mu = \frac{m}{d}$, clearly an integer, and consider the integers

$$a^{n_i} + n_i, \qquad a^{n_i+k} + n_i + k, \qquad a^{n_i+2k} + n_i + 2k, \qquad \ldots \qquad a^{n_i+(\mu-1)k} + n_i + (\mu-1)k,$$

which are clearly congruent modulus m with the integers

$$a^{n_i} + n_i, \qquad a^{n_i} + n_i + k, \qquad a^{n_i} + n_i + 2k, \qquad \ldots \qquad a^{n_i} + n_i + (\mu-1)k.$$

If we repeat this process for each $i = 0, 1, \ldots, d-1$, we generate $d\mu = m$ such integers. Assume that any two of them are congruent modulus m. If this happens for the same value of i, then δk is a multiple of m for some $0 < \delta \leq \mu - 1$. Since d is the greatest common divider of k, m, then $k\delta$ is a multiple of m iff $d\delta$ is a multiple of m, absurd since $0 < d\delta < d\mu = m$. Otherwise, there exist indices i, j and not necessarily distinct factors $c_i, c_j \in \{0, 1, \ldots, \mu-1\}$ such that

$$(a^{n_i} + n_i - a^{n_j} - n_j) + (c_i - c_j)k \equiv 0 \pmod{m}.$$

Now, $a^{n_i} + n_i$ and $a^{n_j} + n_j$ correspond to distinct remainders modulus d, or the first term in the LHS is not a multiple of d, but the second one is, since it contains a factor of k, yielding a contradiction. Since we have thus generated m integers with distinct remainders modulus m, each remainder modulus m appears exactly once in these integers. The Lemma follows.

The conclusion to the proposed problem clearly follows from taking $i = 0$ for the desired value of m in the Lemma.

Also solved by Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Radouan Boukharfane, Morocco; Li Zhou, Polk State College, Winter Haven, FL, USA; Khakimboy Egamberganov, Tashkent, Uzbekistan; Navid Safei,Student of Sharif,University of Technoogy in Policy Making of Science and Technology, Iran; Jishnu Bose,Indian Statistical Institute, Kolkata, India.

O306. Let ABC be a triangle with incircle γ and circumcircle Γ. Let Ω be the circle tangent to rays AB, AC, and to Γ externally, and let A' be the tangency point of Ω with Γ. Let the tangents from A' to γ intersect Γ again at B' and C'. Finally, let X be the tangency point of the chord $B'C'$ with γ. Prove that the circumcircle of triangle BXC is tangent to γ.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA

Let I be the center of γ, D the tangency point of γ with BC, and E the reflection of D across AI. Let $\mathcal{I}$ be the inversion in the circle with center A and radius $\sqrt{AB \cdot AC}$. Then $\mathcal{I}(\Gamma)$ is the line tangent to γ at E and $\mathcal{I}(\Omega) = \gamma$. So $\mathcal{I}(A') = E$. Now we need to prove a lemma.

Lemma. Suppose AI and $A'I$ intersect Γ again at M and M', respectively. Then MM' passes through the midpoint S of ID.

Proof. We use homogeneous barycentric coordinates and set $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, and $C = (0 : 0 : 1)$. Then with the usual notations, $I = (a : b : c)$, $D = (0 : s-c : s-b)$, and $D' = AA' \cap BC = \left(0 : \frac{b^2}{s-c} : \frac{c^2}{s-b}\right)$. Hence the equation of AD' is $c^2(s-c)y - b^2(s-b)z = 0$. Solving this with the equation of $\Gamma : a^2yz + b^2zx + c^2xy = 0$, we get $A' = \left(-a : \frac{b^2}{s-c} : \frac{c^2}{s-b}\right)$. Also, AI has equation $cy - bz = 0$. Solving this with the equation of Γ we get $M = \left(-\frac{a^2}{b+c} : b : c\right)$. Next, $A'I$ has equation

$$-\frac{bc(b-c)}{(s-b)(s-c)}x + \frac{ac(s-b+c)}{(s-a)(s-b)}y - \frac{ab(s-c+b)}{(s-a)(s-c)}z = 0.$$

Solving this with the equation of Γ we get $M' = \left(\frac{a^2}{(b-c)(s-a)} : \frac{b}{s-b+c} : -\frac{c}{s-c+b}\right)$. Finally, S has coordinates

$$\frac{1}{2}\left[\left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right) + \left(0, \frac{s-c}{a}, \frac{s-b}{a}\right)\right] = \left(a^2 : 2s(s-c) + ab : 2s(s-b) + ac\right),$$

so MS has equation

$$\left(b^2 - c^2\right)(s-a)x - a^2(s-b+c)y + a^2(s-c+b)z = 0.$$

Then it is easy to see this equation is satisfied by the coordinates of M'. This proves the lemma.

Now let O be the center of Γ and T the internal center of similitude of Γ and γ (X_{55} in Kimberling's Encyclopedia of Triangle Centers [ETC]). Then it is well known that the Gergonne point of $\triangle ABC$ (X_7 in ETC) is the isogonal conjugate (with respect to $\triangle ABC$) of T. Hence $T = IO \cap AA'$. Since the Gergonne point of $\triangle A'B'C'$ is also the isogonal conjugate (with respect to $\triangle A'B'C'$) of T, $A'X$ is the reflection of $A'A$ across $A'I$. Thus I is the incenter of $\triangle AQA'$, where $Q = AD \cap A'X$. Hence BC, $B'C'$, and IQ concur at a point P. Next, it is easy to see that M and M' are the centers of the circumcircles Σ and Σ' of $\triangle BCI$ and $\triangle B'C'I$, respectively. Note that P is the radical center of Γ, Σ, and Σ'. Finally, let R be the second intersection point of Σ and Σ'. Then MM' is the perpendicular bisector of IR, so the lemma implies that R is on DX as well. Therefore, P is the radical center of Γ, Σ, Σ', *and* γ, from which the required conclusion follows immediately.