More on an Extension of Fagnano's Problem

Nguyen Minh Ha

Abstract: In this article we extend Fagnano's problem on triangles of minimal perimeter which inscribe a given acute triangle.

Key Words: Fagnano's inequality, outer product, same direction, opposite direction.
MSC 2020: 51M04, 51M16.

1. Introduction

In 1775, the Italian mathematician G. Fagnano stated this famous problem [1].

Problem 1. For a given acute triangle ABC determine the inscribed triangle XYZ of minimal perimeter.

The original solution to Problem 1 was given by G. Fagnano himself, using calculus.
The second proof was given by L. Fejer, using reflection [1].
In 2004, N. M. Ha proposed another proof of Problem 1, using the scalar product of two vectors [2].

The answer to Problem 1 is that the perimeter of triangle XYZ is minimal if and only if X, Y and Z coincide with D, E, and F, respectively, where D, E and F are the orthogonal projections of the orthocenter H of the triangle ABC onto BC, CA and AB, respectively.

In this paper, we generalize Problem 1 by changing "acute triangle ABC" to "(x,y,z)-acute triangle ABC" and by changing the expression $YZ+ZX+XY$ (the perimeter of triangle XYZ) to $xYZ+yZX+zXY$, where x, y, z are positive numbers.

Definition 1. Given are a triangle ABC and a triple (x, y, z) of positive real numbers. If there is a point O inside triangle ABC such that $\frac{AO}{x}=\frac{BO}{y}=\frac{CO}{z}$, then triangle ABC is called (x,y,z)-acute triangle.

The point O in Definition 1 is unique (if it exists) and is the intersection inside triangle ABC of two of the three Apollonius circles (ω_a), (ω_b), (ω_c):

$$(\omega_a)=\left\{M/\frac{MB}{MC}=\frac{y}{z}\right\};(\omega_b)=\left\{M/\frac{MC}{MA}=\frac{z}{x}\right\};(\omega_c)=\left\{M/\frac{MA}{MB}=\frac{x}{y}\right\}.$$

Definition 2. The point O in Definition 1 is called (x, y, z) - circumcenter of the (x,y,z) - acute triangle ABC

Definition 3. The point H, which is the isogonal conjugate point with the point O in Definition 2, is called (x, y, z) - orthocenter of (x, y, z) - acute triangle ABC.

If $x=y=z$ then the (x, y, z) - acute triangle is just an acute triangle, the (x, y, z)-circumcenter is just the circumcenter, and the (x, y, z) - orthocenter is just the orthocenter.

We present some examples of (x,y,z) - acute triangle, (x, y, z) - circumcenter and (x, y, z) - orthocenter.

Example 1. Let ABC be an acute triangle. Denote by O and H the circumcenter and the orthocenter of this triangle, respectively. If we put $x = \cos A$, $y = \cos B$, $z = \cos C$, then triangle ABC is (x,y,z)-acute triangle and points H and O are respectively the (x, y, z)-circumcenter, and the (x, y, z) - orthocenter of this triangle.

Example 2 . Let ABC be an acute triangle and let I be its incenter. Set $x = \sqrt{\frac{p-a}{a}}$, $y = \sqrt{\frac{p-b}{b}}$, $z = \sqrt{\frac{p-c}{c}}$. Then, triangle ABC is a (x,y,z) - acute triangle and the point I is both the (x, y, z) - circumcenter and the (x, y, z) - orthocenter of this triangle.

Example 3. Let ABC be a triangle. Denote by G and L the centroid and the Lemoine's point of this triangle, respectively. If we put $x = \sqrt{2(b^2+c^2)-a^2}$, $y = \sqrt{2(c^2+a^2)-b^2}$, $z = \sqrt{2(a^2+b^2)-c^2}$, then triangle ABC is a (x,y,z) - acute triangle and points G and L are respectively the (x, y, z) - circumcenter and the (x, y, z) - orthocenter. If we put $x = \sqrt{\frac{2(b^2+c^2)}{a^2}-1}$, $y = \sqrt{\frac{2(c^2+a^2)}{b^2}-1}$, $z = \sqrt{\frac{2(a^2+b^2)}{c^2}-1}$, then triangle ABC is a (x,y,z)-acute triangle, points L and G are the (x, y, z) - circumcenter and the (x, y, z) - orthocenter, respectively.

Using Definition 1, Problem 1 can be generalized as follows:

Problem 2. Given a triple (x, y, z) of positive real numbers and a (x, y, z) - acute triangle ABC, determine the inscribed triangle XYZ such that the expression $xYZ + yZX + zXY$ is minimal.

Later on, we will prove that the expression $xYZ + yZX + zXY$ is smallest if and only if X, Y and Z coincide with D, E and F, where D, E,and F are the orthogonal projections of the (x, y, z) - orthocenter H of triangle ABC onto BC, CA and AB, respectively.

Based on the result of Problem 2, we have a short proof for the following problem which was proposed and solved by T. Q. Hung and N. T. T. Duong [3].

Problem 3. Given are a triangle ABC and a point P inside the triangle. Let R_a, R_b and R_c be the radii of the circumcircles of triangles PBC, PCA and PAB, respectively. Determine the inscribed triangle XYZ such that the expression $\frac{YZ}{R_a} + \frac{ZY}{R_b} + \frac{XY}{R_c}$ is minimal.

Notation:

- $A, B/\Delta$ means that points A and B are on the same half plane defined by line Δ.

- $A/\Delta/B$ means that points A and B are on different half planes defined by line Δ.
- $\vec{a}\uparrow\uparrow\vec{b}$ means that vectors $\vec{a}$ and $\vec{b}$ are in the same direction [4, 5].
- $\vec{a}\uparrow\downarrow\vec{b}$ c means that vectors $\vec{a}$ and $\vec{b}$ are in opposite direction [4, 5].
- $\Delta XYZ\uparrow\uparrow\Delta UVW$ means that triangles XYZ and UVW are in the same direction [4, 5].
- $\Delta XYZ\uparrow\downarrow\Delta UVW$ means that triangles XYZ and UVW are in opposite direction [4, 5].
- $S[XYZ]$ means the signed area of triangle XYZ.
- $\vec{a}\wedge\vec{b}$ means the outer product of vectors $\vec{a}$ and $\vec{b}$.
- $(\vec{a},\vec{b})$ means the oriented angle between vectors $\vec{a}$ and $\vec{b}$ [4, 5].
- (a,b) means the oriented angle between lines a and b [4, 5].

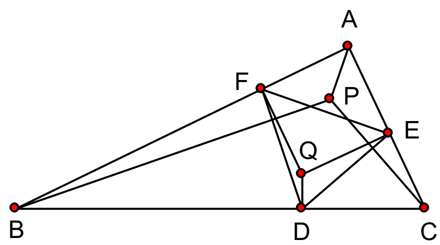
To simplify, we assume that triangle ABC always has positive orientation (The direction of the rotation is opposite to the direction of clockwise rotation).

1. Solution for Problem 2

We need two lemmas.

Lemma 1. Given are a triangle ABC and points P and Q (these points are not necessarily lying inside the triangle). If D, E and F are the orthogonal projections of Q onto BC, CA and AB, respectively, then, P and Q are isogonal conjugate points with respect to the triangle ABC if and only if AP, BP and CP are perpendicular with EF, FD and DE, respectively.

The proof of Lemma 1 is familiar and simple, so it will not be shown here (f.1).



(Figure 1)

Lemma 2. Given are a triangle ABC and a point O inside that triangle. Let H be the isogonal conjugate point with O. Let D, E and F be the orthogona projections of O onto lines BC, CA and AB, respectively. Then

$$(\overrightarrow{EF},\overrightarrow{AO})\equiv(\overrightarrow{FD},\overrightarrow{BO})\equiv(\overrightarrow{DE},\overrightarrow{CO})\equiv\frac{\pi}{2}(mod\,2\pi).$$

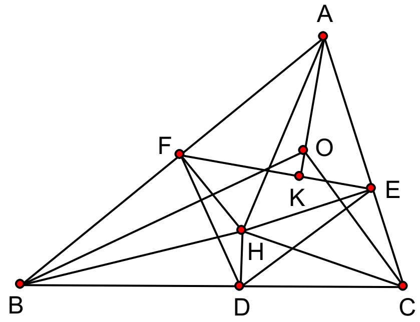
Proof.

Since O and H are isogonal conjugate points with respect to triangle ABC, by using Lemma 1, we see that AO, BO and CO are perpendicular to EF, FD and DE, respectively.

There are two cases:

Case 1. OAB, OAC, OBC, OBA, OCA, OCB are acute angles (f.2).

It is obvious that D, E and F are on segments BC, CA and AB, respectively.
Let K be the intersection of AO and EF.
It is easy to prove that $K, F / EA$; $F, B / EA$; $E, C / AB$.
Hence $\Delta KEA \uparrow\uparrow \Delta FEA$; $\Delta FEA \uparrow\uparrow \Delta BEA \uparrow\uparrow \Delta EAB$; $\Delta EAB \uparrow\uparrow \Delta CAB \uparrow\uparrow \Delta ABC$.
Therefore $\Delta KEA \uparrow\uparrow \Delta ABC$.
So ΔKEA has positive orientation.



.   (Figure 2)

Since $AK \perp EF$, we have $(\overrightarrow{KE}, \overrightarrow{KA}) \equiv \frac{\pi}{2} (mod\, \pi)$.

Note that $\overrightarrow{EF} \uparrow\downarrow \overrightarrow{KE}$; $\overrightarrow{AO} \uparrow\downarrow \overrightarrow{KA}$, hence $(\overrightarrow{EF}, \overrightarrow{AO}) \equiv \frac{\pi}{2} (mod\, \pi)$.

Similarly, $(\overrightarrow{FD}, \overrightarrow{BO}) \equiv \frac{\pi}{2} (mod\, 2\pi)$; $(\overrightarrow{DE}, \overrightarrow{CO}) \equiv \frac{\pi}{2} (mod\, 2\pi)$.

Thus $(\overrightarrow{EF}, \overrightarrow{AO}) \equiv (\overrightarrow{FD}, \overrightarrow{BO}) \equiv (\overrightarrow{DE}, \overrightarrow{CO}) \equiv \frac{\pi}{2} (mod\, 2\pi)$.

Case 2. There is only one non-acute angle among OAB, OAC, OBC, OBA, OCA, OCB.

Without any loss of the generality, we may assume that OAB is non-acute (f.3).
It is obvious that F and D are on segments AB and BC, respectively, and that E is on the opposite ray of the ray AC
Let K be the intersection of AO and EF.
It is easy to prove that $K / EA / F$; $F, B / EA$; $E / AB / C$.
Hence $\Delta KEA \uparrow\downarrow \Delta FEA$; $\Delta FEA \uparrow\uparrow \Delta BEA \uparrow\uparrow \Delta EAB$; $\Delta EAB \uparrow\downarrow \Delta CAB \uparrow\uparrow \Delta ABC$.
So $\Delta KEA \uparrow\uparrow \Delta ABC$.
Therefore ΔKEA has positive orientation.

Since $AK \perp EF$, we have $(\overrightarrow{KE}, \overrightarrow{KA}) \equiv \frac{\pi}{2} (mod\, \pi)$.

Note that $\overrightarrow{EF} \uparrow\uparrow \overrightarrow{KE}$; $\overrightarrow{AO} \uparrow\uparrow \overrightarrow{KA}$, so $(\overrightarrow{EF}, \overrightarrow{AO}) \equiv \frac{\pi}{2} (mod\, \pi)$.

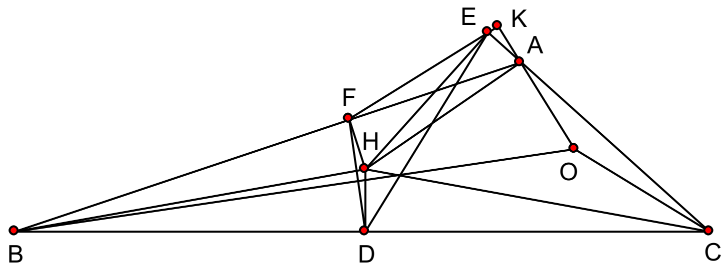
Since F and D are on segments AB and BC, respectively, as in Case 1,

$$(\overrightarrow{FD}, \overrightarrow{BO}) \equiv \frac{\pi}{2} (mod\, 2\pi).$$

Since D is on segment BC and E is on the opposite ray of AC, as in Case 1,

$$(\overrightarrow{DE}, \overrightarrow{CO}) \equiv \frac{\pi}{2} (mod\, 2\pi).$$

Thus $(\overrightarrow{EF},\overrightarrow{AO})\equiv(\overrightarrow{FD},\overrightarrow{BO})\equiv(\overrightarrow{DE},\overrightarrow{CO})\equiv\frac{\pi}{2}(mod\,2\pi)$.



(Figure 3) □

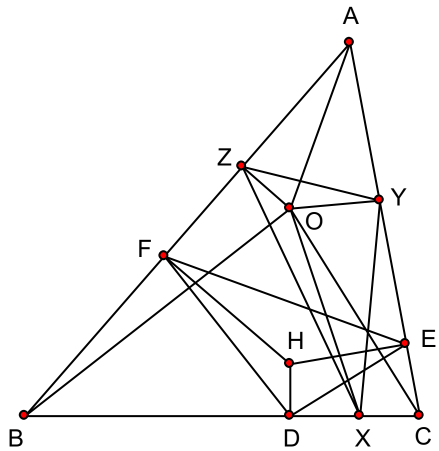
Back to Problem 2.

We use notation mentioned above (f.4, f.5).

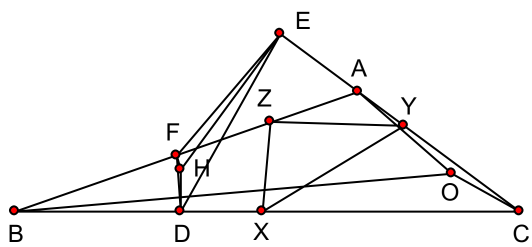
Since O is (x, y, z)-circumcenter of the triangle ABC, there is a positive number k such that $\frac{AO}{x}=\frac{BO}{y}=\frac{CO}{z}=k$.

Since O is inside triangle ABC, H and O are isogonal conjugate with respect to the triangle ABC, by using Lemma 2, we have:

$$(\overrightarrow{EF},\overrightarrow{AO})\equiv(\overrightarrow{FD},\overrightarrow{BO})\equiv(\overrightarrow{DE},\overrightarrow{CO})\equiv\frac{\pi}{2}(mod\,2\pi).$$



(Figure 4)



(Figure 5)

Hence

$$xYZ+yZX+zXY$$
$$=\frac{AO}{k}.YZ+\frac{BO}{k}.ZX+\frac{CO}{k}.XY$$
$$=\frac{2}{k}(\frac{1}{2}YZ.AO+\frac{1}{2}ZX.BO+\frac{1}{2}XY.CO)$$

$$\geq \frac{2}{k}\left(\frac{1}{2}YZ.AO\sin(\overrightarrow{YZ},\overrightarrow{AO})+\frac{1}{2}ZX.BO\sin(\overrightarrow{ZX},\overrightarrow{BO})+\frac{1}{2}XY.CO\sin(\overrightarrow{XY},\overrightarrow{CO})\right)$$
$$=\frac{2}{k}\left(\frac{1}{2}\overrightarrow{YZ}\wedge\overrightarrow{AO}+\frac{1}{2}\overrightarrow{ZX}\wedge\overrightarrow{BO}+\frac{1}{2}\overrightarrow{XY}\wedge\overrightarrow{CO}\right)$$
$$=\frac{2}{k}\left(\frac{1}{2}(\overrightarrow{AZ}-\overrightarrow{AY})\wedge\overrightarrow{AO}+\frac{1}{2}(\overrightarrow{BX}-\overrightarrow{BZ})\wedge\overrightarrow{BO}+\frac{1}{2}(\overrightarrow{CY}-\overrightarrow{CX})\wedge\overrightarrow{CO}\right)$$
$$=\frac{2}{k}\left(\frac{1}{2}\overrightarrow{AZ}\wedge\overrightarrow{AO}-\frac{1}{2}\overrightarrow{AY}\wedge\overrightarrow{AO}+\frac{1}{2}\overrightarrow{BX}\wedge\overrightarrow{BO}-\frac{1}{2}\overrightarrow{BZ}\wedge\overrightarrow{BO}+\frac{1}{2}\overrightarrow{CY}\wedge\overrightarrow{CO}-\frac{1}{2}\overrightarrow{CX}\wedge\overrightarrow{CO}\right)$$
$$=\frac{2}{k}(S[AZO]-S[AYO]+S[BXO]-S[BZO]+S[CYO]-S[CXO])$$
$$=\frac{2}{k}(S[OBX]+S[OXC]+S[OCY]+S[OYA]+S[OAZ]+S[OZB])$$
$$=\frac{2}{k}(S[OBC]+S[OCA]+S[OAB])$$
$$=\frac{2}{k}(S[OBD]+S[ODC]+S[OCE]+S[OEA]+S[OAF]+S[OFB])$$
$$=\frac{2}{k}(S[AFO]-S[AEO]+S[BDO]-S[BFO]+S[CEO]-S[CDO])$$
$$=\frac{2}{k}\left(\frac{1}{2}\overrightarrow{AF}\wedge\overrightarrow{AO}-\frac{1}{2}\overrightarrow{AE}\wedge\overrightarrow{AO}+\frac{1}{2}\overrightarrow{BD}\wedge\overrightarrow{BO}-\frac{1}{2}\overrightarrow{BF}\wedge\overrightarrow{BO}+\frac{1}{2}\overrightarrow{CE}\wedge\overrightarrow{CO}-\frac{1}{2}\overrightarrow{CD}\wedge\overrightarrow{CO}\right)$$
$$=\frac{1}{k}((\overrightarrow{AF}-\overrightarrow{AE})\wedge\overrightarrow{AO}+(\overrightarrow{BD}-\overrightarrow{BF})\wedge\overrightarrow{BO}+(\overrightarrow{AE}-\overrightarrow{AD})\wedge\overrightarrow{CO})$$
$$=\frac{1}{k}(\overrightarrow{EF}\wedge\overrightarrow{AO}+\overrightarrow{FD}\wedge\overrightarrow{BO}+\overrightarrow{DE}\wedge\overrightarrow{CO})$$
$$=\frac{1}{k}(EF.AO\sin(\overrightarrow{EF},\overrightarrow{AO})+FD.BO\sin(\overrightarrow{FD},\overrightarrow{BO})+DE.CO\sin(\overrightarrow{DE},\overrightarrow{CO}))$$
$$=\frac{AO}{k}.EF+\frac{BO}{k}.FD+\frac{CO}{k}.DE$$
$$=xEF+yFD+zDE.$$

Therefore, the following statements are equivalent:

1. $xYZ+yZX+zXY=xEF+yFD+zDE.$
2. $(\overrightarrow{YZ},\overrightarrow{AO})\equiv(\overrightarrow{ZX},\overrightarrow{BO})\equiv(\overrightarrow{XY},\overrightarrow{CO})\equiv\frac{\pi}{2}(mod\,2\pi).$
3. $(\overrightarrow{YZ},\overrightarrow{AO})\equiv(\overrightarrow{EF},\overrightarrow{AO})\,(mod\,2\pi)$;
   $(\overrightarrow{ZX},\overrightarrow{BO})\equiv(\overrightarrow{FD},\overrightarrow{BO})\,(mod\,2\pi)$;
   $(\overrightarrow{XY},\overrightarrow{CO})\equiv(\overrightarrow{DE},\overrightarrow{CO})\,(mod\,2\pi).$
4. $\overrightarrow{YZ}\uparrow\uparrow\overrightarrow{EF}$; $\overrightarrow{ZX}\uparrow\uparrow\overrightarrow{FD}$; $\overrightarrow{DE}\uparrow\uparrow\overrightarrow{XY}.$
5. X, Y and Z coincide with D, E and F, respectively.

In summary, the expression $xYZ+yZX+zXY$ is minimal if and only if X, Y and Z are the orthogonal projections of (x, y, z)-orthocenter H of the triangle ABC onto BC, CA and AB, respectively.

□

Solution for Problem 3

We need a lemma.

Lemma 3. Let P and Q be isogonal conjugates with respect to triangle ABC. Let R_b and R_c be the radii of the circumcircles of triangles PCA and PAB, respectively. Then

$$QB.R_b = QC.R_c.$$

Proof.

Let (K_b) and (K_c) be the circumcircles of triangles PCA and PAB, respectively. Let E and F be the second intersections of BP, CP and (K_b), (K_c), respectively (f. 6).

It is obvious that $(BQ,BA) \equiv (BC,BP) \equiv (BC,BE)(mod\ \pi)$;

$$(AB,AQ) \equiv (AP,AC) \equiv (EP,EC) \equiv (EB,EC)(mod\ \pi).$$

Therefore, triangles BQA and BCE are same-orientation similar.

Similarly, triangles CQA and CBF are same-orientation similar.

Since $\left(K_cA,K_cK_b\right) \equiv \frac{1}{2}\left(\overrightarrow{K_cA},\overrightarrow{K_cP}\right) \equiv (BA,BP) \equiv (BA,BE)(mod\ \pi)$;

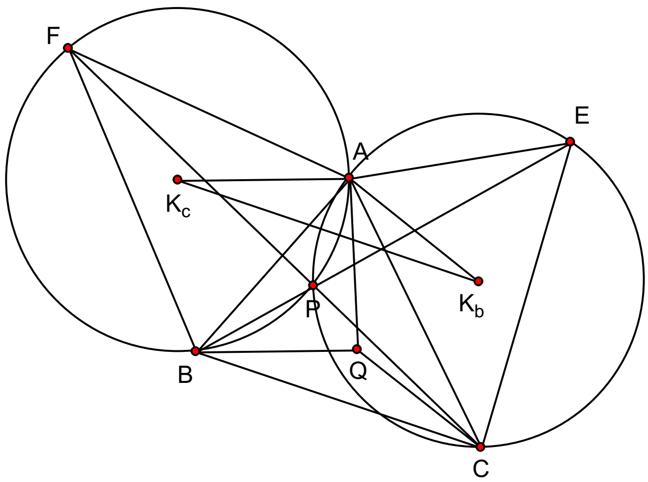
$$\left(K_bA,K_bK_c\right) \equiv \frac{1}{2}\left(\overrightarrow{K_bA},\overrightarrow{K_bP}\right) \equiv (EA,EP) \equiv (EA,EB)(mod\ \pi).$$

Therefore, triangles ABE and AK_cK_b are same-orientation similar.

Similarly, triangles AFC and AK_cK_b are same-orientation similar.

Therefore, triangles ABE and AFC are same-orientation similar.

Hence triangles ABF and AEC are same-orientation similar.



(Figure 6)

So $\dfrac{BQ}{CQ} = \dfrac{BQ}{AQ}.\dfrac{AQ}{CQ} = \dfrac{CB}{CE}.\dfrac{BF}{BC} = \dfrac{FB}{CE} = \dfrac{AB}{AE} = \dfrac{AK_c}{AK_b} = \dfrac{R_c}{R_c}.$

Hence $BQ.R_b = CQ.R_b.$ □

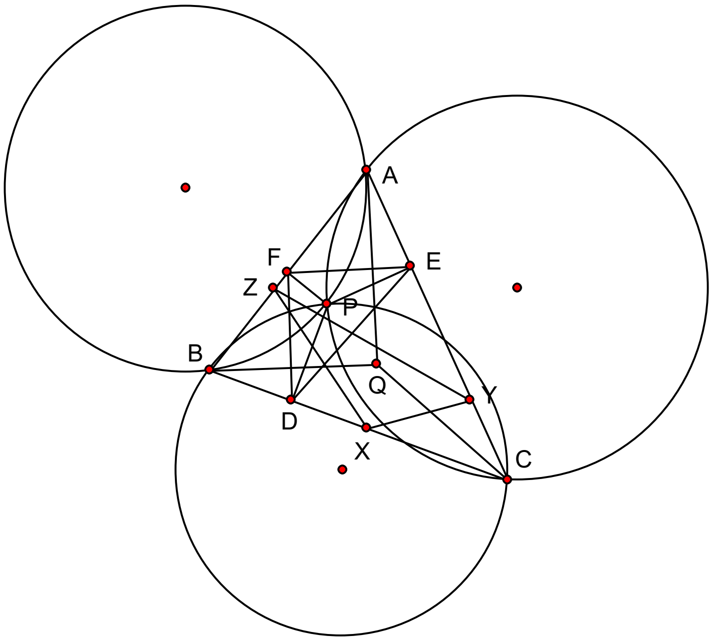
The proof mentioned above can be found in [6].

Now, back to Problem 3.

Let Q be the isogonal conjugate of P with respect to triangle ABC. Let D, E and F be the orthogonal projections of P onto BC, CA and AB, respectively (f.7).

Using Lemma 3, we have $\dfrac{AQ}{\frac{1}{R_a}} = \dfrac{BQ}{\frac{1}{R_b}} = \dfrac{CQ}{\frac{1}{R_b}}$.

Hence, using the solution of Problem 2, the expression $\dfrac{YZ}{R_a} + \dfrac{ZY}{R_b} + \dfrac{XY}{R_c}$ is minimal if and only if X, Y and Z coincide with D, E and F, respectively.



(Figure.7) □

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HA NOI UNIVERSITY OF EDUCATION

HA NOI, VIET NAM
Email address: minhha27255@yahoo.com