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Some Properties of Inversions in Alpha Plane

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Abstract. In this paper, the authors introduce inversion which is also valid in the alpha plane geometry, and give some properties such as cross ratio, harmonic conjugates with respect to inversion in the alpha plane geometry.

1. Introduction

If one wants to measure the distance between two points on a plane, then one can use frequently Euclidean distance which is defined as the length of segment between these points. Although it is the most popular distance function, it is not practical when we measure the distance which we actually move in the real world. So taxicab distance and Chinese checkers distance were introduced. Taxicab and Chinese checkers distance functions are similar to moving with a car or Chinese chess in the real world. Later, Tian [13] gave a family of metrics, α*-metric* (*alpha metric*) for $\alpha \in [0, \pi/4]$, which includes the taxicab and Chinese checkers metrics as special cases. Then, some authors developed and studied on these topics (see [5, 8, 10]). For example, Gelişgen and Kaya extended the α-distance to three and *n* dimensional spaces in [6] and [7], respectively. Afterwards, Colakoğlu [4] extended the α-metric for $\alpha \in [0, \pi/2)$. According to the latter, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points in $\mathbb{R}^2$, then for each $\alpha \in [0, \pi/2)$ and $\lambda(\alpha) = (\sec\alpha - \tan\alpha)$, the α*-distance* between *P* and *Q* is

$$d_\alpha(P, Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + \lambda(\alpha)\min\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Obviously, there are infinitely many different distance functions depending on values of α. But we suppose that values of α are initially determined and fixed unless otherwise stated.

Alpha plane geometry is a non-Euclidean geometry, and also a Minkowski geometry. Here, the linear structure is the same as the Euclidean one but distance is not *uniform* in all directions ([14]). That is, α-plane is almost the same as Euclidean plane since the points are the same, the lines are the same, and the angles are measured in the same way. Since the α-plane geometry has a different distance function it seems interesting to study the α-analog of the topics that include the concepts of distance in the Euclidean geometry.

One of the concepts which include notation of distance is an inversion. There are two kinds of transformations which are their own inverses. However, a new

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transformation also is its own inverse. This transformation is an inversion in a circle. As it has been stated in [9], this particular transformation was probably first introduced by Apollonius of Perga (225 BCE – 190 BCE). The systematic investigation of inversions began with Jakob Steiner (1796-1863) in the 1820s. During the following decades, many physicists and mathematicians independently rediscovered inversions, proving the properties that were most useful for their particular applications. For example, William Thomson used inversions to calculate the effect of a point charge on a nearby conductor made of two intersecting planes. In 1855, August F. Möbius gave the first comprehensive treatment of inversions, and Mario Pieri developed the subject axiomatically and systematically in *New Principles of the Geometry of Inversions, memoirs* I *and* II in the early 1910s, proving all of the known results as its own geometry independent of Euclidean geometry (For more detail see [9, 11]).

Since inversions have attracted attention of scientists from past to present, there are a lot of studies about them. Many scientists studied and also are studying different aspects of this concept. In [3, 12], the authors investigated the inversions with respect to the central conics in real Euclidean plane. The inversions with respect to taxicab circle was studied in detail in [1, 10].

In this paper, the authors introduce an inversion which is also valid in the alpha plane geometry, and give some properties such as cross ratio, harmonic conjugates with respect to inversion in the alpha plane geometry.

2. Preliminaries about alpha plane and some properties of alpha circular inversions

In this section, some basic concepts are briefly reviewed from [5] without proof. When one considers the d_α-metric, it is shown that the shortest path between the points P_1 and P_2 is the line segment which is parallel to a coordinate axis and a line segment making the α angle with the other coordinate axis. Thus, the shortest distance d_α between P_1 and P_2 is the sum of the Euclidean lengths of such two line segments.

Proposition 1. *Every Euclidean translation preserves distance in alpha plane. So each of them is an isometry of* $\mathbb{R}^2_\alpha$.

Proposition 2. *Let* d_E *and* ℓ *denote the Euclidean distance function and a line through the points* P_1 *and* P_2 *in the analytical plane. If* ℓ *has slope* m*; then* $d_\alpha(P_1,P_2) = \frac{M}{\sqrt{1+m^2}} d_E(P_1,P_2)$ *where* $M = \begin{cases} 1+\lambda(\alpha)|m|, & if \ |m| \leq 1 \\ \lambda(\alpha)+|m|, & if \ |m| \geq 1 \end{cases}$.

Proposition 2 states that d_α-distance along any line is some positive constant multiple of Euclidean distance along the same line.

Corollary 3. *Let* P_1, P_2, *and* X *be three collinear points in* $\mathbb{R}^2$. *Then* $d_E(P_1,X) = d_E(P_2,X)$ *if and only if* $d_\alpha(P_1,X) = d_\alpha(P_2,X)$.

Corollary 4. *Let* P_1, P_2, *and* X *be three distinct collinear points in* $\mathbb{R}^2$. *Then* $d_E(P_1,X)/d_E(P_2,X) = d_\alpha(P_1,X)/d_\alpha(P_2,X)$.

That is, the ratios of the Euclidean and d_α-distances along a line are the same. Notice that the latter corollary gives us the validity of the theorems of Menelaus and Ceva in $\mathbb{R}^2_\alpha$.

As it has been stated in [2] and [10], in the Euclidean plane an inversion in a circle of radius r is a mapping in which a point P and its image $P^{\shortmid}$ are on a ray emanating from the center O of the circle such that $d(O,P)d(O,P^{\shortmid}) = r^2$. This mapping is conformal.

Clearly if $P^{\shortmid}$ is the inverse of P, then P is the inverse of $P^{\shortmid}$. Note also that if P is in the interior of $\mathcal{C}$, $P^{\shortmid}$ is exterior to $\mathcal{C}$; and viceversa. So the interior of $\mathcal{C}$ except for $\mathcal{O}$ is mapped to the exterior and the exterior to the interior. $\mathcal{C}$ itself is left pointwise fixed. O has no image, and no point of the plane is mapped to O. However, points close to O are mapped to points far from O, and points far from O mapped to points close to O. Thus adjoining one "ideal point", or "point at infinity", to the Euclidean plane, we can include O in the domain and range of an inversion.

Now in $\mathbb{R}^2_\alpha$, the definition of inversion with respect to an α-circle can be given as follows:

Definition. Let $\mathcal{C}$ be an α-circle centered at a point O with radius r in $\mathbb{R}^2_\alpha$, and let P_∞ be the ideal point adjoining one to the alpha plane. In $\mathbb{R}^2_\alpha$ the *alpha circular inversion* with respect to $\mathcal{C}$ is the function such that

$$I_\alpha\left(O,r\right):\ \mathbb{R}^2_\alpha\cup\{P_\infty\}\to\mathbb{R}^2_\alpha\cup\{P_\infty\}$$

defined by $I_\alpha\left(O,r\right)\left(O\right)=P_\infty$, $I_\alpha\left(O,r\right)\left(P_\infty\right)=O$, and $I_\alpha\left(O,r\right)\left(P\right)=P^{\shortmid}$ for $P\neq O,\ P'$ where $P^{\shortmid}$ is on the ray $\overrightarrow{OP}$ and $d_\alpha(O,P)d_\alpha(O,P^{\shortmid})=r^2$. The point $P^{\shortmid}$ is called the *alpha circular inverse* of P in $\mathcal{C}$; $\mathcal{C}$ is said to be *the circle of inversion*, and O is called *the center of inversion*.

The following lemma states that an inversion in a circle is a transformation of the plane that points outside the circle get mapped to points inside the circle and vice versa.

Lemma 5. *Let $\mathcal{C}$ be an alpha circle with respect to the center O in the alpha inversion $I_\alpha\left(O,r\right)$. If the point P is in the interior of $\mathcal{C}$, then the point $P^{\shortmid}$ is exterior to $\mathcal{C}$, and conversely.*

Proof. Suppose that the point P is in the interior of $\mathcal{C}$. So $d_\alpha(O,P)<r$. Since $P^{\shortmid}=I_\alpha\left(O,r\right)$, $d_\alpha\left(O,P\right)\cdot d_\alpha\left(O,P^{\shortmid}\right)=r^2$. Then

$$r^2=d_\alpha\left(O,P\right)\cdot d_\alpha\left(O,P^{\shortmid}\right)<r\cdot d_\alpha\left(O,P^{\shortmid}\right)\ \text{and}\ \ d_\alpha\left(O,P^{\shortmid}\right)>r.$$

That is, the point $P^{\shortmid}$ is in the exterior of $\mathcal{C}$. $\square$

The next proposition gives a relation getting for coordinates of $P^{\shortmid}$ in terms of coordinates of P.

Proposition 6. *Let $I_\alpha(O,r)$ be an alpha circular inversion with respect to an alpha circle $\mathcal{C}$ centered at origin and the radius r in $\mathbb{R}^2_\alpha$. If $P=(x,y)$ and $P^\prime=(x^\prime,y^\prime)$ are inverse points according to the alpha circular inversion, then*

$$x^\prime=\frac{r^2x}{\left(\max\{|x|,|y|\}+\lambda(\alpha)\min\{|x|,|y|\}\right)^2}\ ,$$
$$y^\prime=\frac{r^2y}{\left(\max\{|x|,|y|\}+\lambda(\alpha)\min\{|x|,|y|\}\right)^2}.$$

Proof. The α-circle $\mathcal{C}$ with the center origin and the radius r consists of the points which satisfies the equation $\max\{|x|,|y|\}+\lambda(\alpha)\min\{|x|,|y|\}=r$. Let $P=(x,y)$ and $P^\prime=(x^\prime,y^\prime)$ are inverse points with respect to $\mathcal{C}$. Since the points O, P and $P^\prime$ are collinear and the rays $\overrightarrow{OP}\ and\ \overrightarrow{OP^\prime}$ are same direction, $\overrightarrow{OP^\prime}=k\overrightarrow{OP}$ for $k\in\mathbb{R}^+$. Since $d_\alpha(O,P)\cdot d_\alpha(O,P^\prime)=r^2$, it is obtained that

$$k=\frac{r^2}{\left(\max\{|x|,|y|\}+\lambda(\alpha)\min\{|x|,|y|\}\right)^2}.$$

Obviously the required results are obtained by substituting the value of k in $(x^\prime,y^\prime)=(kx,ky)$. □

The following corollary immediately is given by using the fact that all translations are isometries of alpha plane.

Corollary 7. *Let $I_\alpha(O,r)$ be an alpha circular inversion with respect to an alpha circle $\mathcal{C}$ centered at $O=(a,b)$ and the radius r. If $P=(x,y)$ is a point of $\mathbb{R}^2_\alpha$, then $P^\prime=(x^\prime,y^\prime)$ is obtained by*

$$x^\prime=a+\frac{r^2(x-a)}{\left(\max\{|x-a|,|y-b|\}+\lambda(\alpha)\min\{|x-a|,|y-b|\}\right)^2}\ ,$$
$$y^\prime=b+\frac{r^2(y-b)}{\left(\max\{|x-a|,|y-b|\}+\lambda(\alpha)\min\{|x-a|,|y-b|\}\right)^2}.$$

Now the following useful properties are well known in Euclidean plane:

i. Lines passing through the center of inversion map into themselves.
ii. Circles with center of inversion map to circles with center of inversion.
iii. Circles not passing through the center of inversion map into circles that do not pass through the center of inversion.
iv. Lines not through the center of inversion map into circles through the center of inversion and conversely.

Unfortunately all of these properties are not valid in the alpha plane. The following theorem state that whether which one of these properties are satisfied or not. Since one can easily give an example for properties which do not satisfy and one can easily prove the satisfying properties by using definition of alpha circular inversion, the next theorem is given without proof.

Theorem 8.

i. *The alpha circular inversion $I_\alpha(O,r)$ maps the lines passing through O onto themselves.*
ii. *The alpha circular inversion $I_\alpha(O,r)$ maps the alpha circles with the center O onto the alpha circles with the center O.*
iii. *The alpha circular inversion $I_\alpha(O,r)$ does not map the alpha circles not through O onto any alpha circles.*
iv. *The alpha circular inversion $I_\alpha(O,r)$ does not map the lines not containing the center of the alpha circular inversion circle onto alpha circles the center O.*
v. *The alpha circular inversion $I_\alpha(O,r)$ does not map the alpha circles containing the center of the inversion circle onto straight lines not containing O.*

3. The cross ratio and harmonic conjugates in $\mathbb{R}^2_\alpha$

The next propositions will be used to show preserving the cross ratio under alpha circular inversion.

Proposition 9. *Let $\mathcal{C}$ be an α-circle of inversion with center O and radius r, and let P, Q, and O be any three distinct collinear points in $\mathbb{R}^2_\alpha$. If P, P', and Q, Q' are pairs of inverse points, then $d_\alpha(P',\ Q') = \dfrac{r^2 d_\alpha(P,Q)}{d_\alpha(O,P)\,d_\alpha(O,Q)}$.*

Proof. Firstly suppose that O, P, Q are collinear. It is follows from definition of alpha circular inversion that $d_\alpha(O,P)\cdot d_\alpha(O,P') = d_\alpha(O,Q)\cdot d_\alpha(O,Q') = r^2$. By using Corollary 7, one can get

$$\begin{aligned} d_\alpha\left(P',Q'\right) &= \left|d_\alpha\left(O,P'\right) - d_\alpha\left(O,Q'\right)\right| \\ &= \left|\frac{r^2}{d_\alpha(O,P)} - \frac{r^2}{d_\alpha(O,Q)}\right| \\ &= \frac{r^2 d_\alpha(P,Q)}{d_\alpha(O,P)\,d_\alpha(O,Q)}. \end{aligned}$$

□

If P, Q, and O are not collinear, then the equality in Proposition 9 is not valid in $\mathbb{R}^2_\alpha$. For example, for $O=(0,0)$, $P=(1,2)$, $Q=(0,1)$ and $r=3$, the inversion $I_\alpha(O,r)$ maps P and Q into $P' = \left(\frac{9}{(2+\lambda(\alpha))^2}, \frac{18}{(2+\lambda(\alpha))^2}\right)$ and $Q' =$

$\left(0, \dfrac{9}{(2+\lambda(\alpha))^2}\right)$, respectively. One can easily see that

$$\begin{aligned} d_\alpha(P,Q) &= 1+\lambda(\alpha), \\ d_\alpha\left(P^{\prime},\ Q^{\prime}\right) &= \frac{9}{(2+\lambda(\alpha))^2}(1+\lambda(\alpha)), \\ d_\alpha(O,P) &= 2+\lambda(\alpha), \text{ and} \\ d_\alpha(O,Q) &= 1. \end{aligned}$$

So the equality in Proposition 9 obviously is not valid in $\mathbb{R}^2_\alpha$. But the following proposition shows that the equality in Proposition 9 is satisfied under such conditions.

Proposition 10. *Let $\mathcal{C}$ be an α-circle of inversion with center O and radius r, and let P, Q and O be any three distinct non-collinear points in $\mathbb{R}^2_\alpha$. If P, $P^{\prime}$, and Q, $Q^{\prime}$ are pairs of inverse points and P, Q lie on the lines with slope $\{0,\infty\}$ or $\{-1,1\}$ passing through the origin, then $d_\alpha(P^{\prime},Q^{\prime}) = \dfrac{r^2 d_\alpha(P,Q)}{d_\alpha(O,P)\, d_\alpha(O,Q)}$.*

Proof. Since $P=(p,0)$ and $Q=(0,q)$ map into $P^{\prime}=\left(\frac{r^2}{p},0\right)$, $Q^{\prime}=\left(0,\frac{r^2}{q}\right)$ or $P=(p,p)$ and $Q=(q,-q)$ map into

$$\begin{aligned} P^{\prime} &= \left(\frac{r^2}{p(1+\lambda(\alpha))^2}, \frac{r^2}{p(1+\lambda(\alpha))^2}\right), \\ Q^{\prime} &= \left(\frac{r^2}{q(1+\lambda(\alpha))^2}, \frac{-r^2}{q(1+\lambda(\alpha))^2}\right), \end{aligned}$$

one can easily show that

$$d_\alpha\left(P^{\prime},Q^{\prime}\right) = \frac{r^2 d_\alpha(P,Q)}{d_\alpha(O,P)\, d_\alpha(O,Q)}.$$

□

Let $d_\alpha[P,Q]$ denote the alpha directed distance from P to Q along a line in the alpha plane. If the ray with initial point P containing Q has the positive direction of orientation $d_\alpha[P,Q] = d_\alpha(P,Q)$, and if the ray has the opposite direction $d_\alpha[P,Q] = -d_\alpha(P,Q)$.

Now let P, Q, R and S be four distinct points on oriented line in the alpha plane. Then their *alpha cross ratio* $(PQ,RS)_\alpha$ is defined by

$$(PQ,RS)_\alpha = \frac{d_\alpha[P,R]\, d_\alpha[Q,S]}{d_\alpha[P,S]\, d_\alpha[Q,R]}.$$

Note that the alpha cross ratio is positive if both R and S are between P and Q or if neither R nor S is between P and Q, whereas the cross ratio is negative. If the pairs $\{P,Q\}$ and $\{R,S\}$ seperate each other. Also an alpha circular inversion with respect to $\mathcal{C}$ centered at origin which is different P, Q, R and S preserve the alpha cross ratio.

Theorem 11. *The alpha circular inversion preserve the alpha cross ratio.*

Proof. Suppose that P, Q, R and S be four collinear points in the alpha plane. Consider the alpha circular inversion $I_\alpha(O,r)$. Let $I_\alpha(O,r)$ map P, Q, R and S into $P^{\shortmid}$, $Q^{\shortmid}$, $R^{\shortmid}$ and $S^{\shortmid}$, respectively. First note that the alpha circular inversion preserves the seperation or non-seperation of the pairs P, Q and R, S, and also it reverses the α-directed distance from the point P to the point Q along a line *l* to α-directed distance from the point $Q^{\shortmid}$ to the point $P^{\shortmid}$. The required result follows from Proposition 9:

$$\begin{aligned}
(P^{\shortmid}Q^{\shortmid}, R^{\shortmid}S^{\shortmid})_{\boldsymbol{\alpha}} &= \frac{d_\alpha(P^{\shortmid},R^{\shortmid})\,d_\alpha(Q^{\shortmid},S^{\shortmid})}{d_\alpha(P^{\shortmid},S^{\shortmid})\,d_\alpha(Q^{\shortmid},R^{\shortmid})} \\
&= \frac{\dfrac{r^2 d_\alpha(P,R)}{d_\alpha(O,P)\,d_\alpha(O,R)}\,\dfrac{r^2 d_\alpha(Q,S)}{d_\alpha(O,Q)\,d_\alpha(O,S)}}{\dfrac{r^2 d_\alpha(P,S)}{d_\alpha(O,P)\,d_\alpha(O,S)}\,\dfrac{r^2 d_\alpha(Q,R)}{d_\alpha(O,Q)\,d_\alpha(O,R)}} \\
&= \frac{d_\alpha(P,R)\,d_\alpha(Q,S)}{d_\alpha(P,S)\,d_\alpha(Q,R)} \\
&= (PQ,RS)_\alpha .
\end{aligned}$$

□

Let l be a line in $\mathbb{R}^2_\alpha$. Suppose that P, Q, R and S are four points on l. It is called that P, Q, R and S form a *harmonic set* if $(PQ,RS)_\alpha = -1$, and it is denoted by $H(PQ,RS)_\alpha$. That is, any pair R and S on l for which

$$\frac{d_\alpha[P,R]\,d_\alpha[S,Q]}{d_\alpha[P,S]\,d_\alpha[Q,R]} = -1$$

is said to divide P and Q harmonically. The points R and S are called *alpha harmonic conjugates* with respect to P and Q.

Theorem 12. *Let $\mathcal{C}$ be an alpha circle with the center O, and line segment $[PQ]$ a diameter of $\mathcal{C}$ in $\mathbb{R}^2_\alpha$. Let R and S be distinct points of the ray $\overrightarrow{OP}$, which divide the segment $[PQ]$ internally and externally. Then R and S are alpha harmonic conjugates with respect to P and Q if and only if R and S are inverse points with respect to the alpha circular inversion $I_\alpha(O,r)$.*

Proof. Let R and S are alpha harmonic conjugates with respect to *P* and *Q*. Then

$$(PQ,RS)_\alpha = -1 \Rightarrow \frac{d_\alpha[PR]\,d_\alpha[QS]}{d_\alpha[PS]\,d_\alpha[QR]} = -1.$$

Since *R* divides the line segment $[PQ]$ internally and *R* is on the ray $\overrightarrow{OQ}$,

$$d(R,Q) = r - d(O,R) \quad \text{and} \quad d(P,R) = r + d(O,R).$$

Since S divides the line segment $[PQ]$ externally and S is on the ray $\overrightarrow{OQ}$,

$$d(P,S) = \; d(O,S) + r \quad \text{and} \quad d(Q,S) = \; d(O-S) - r.$$

Hence

$$\frac{(r+d_\alpha(O,R))(d_\alpha(O,S)-r)}{(r+d_\alpha(O,S))(d_\alpha(O,R)-r)}=-1$$
$$\Rightarrow\ (r+d_\alpha(O,R))(d_\alpha(O,S)-r)=(r+d_\alpha(O,S))(r-d_\alpha(O,R)).$$

Simplifying the last equality, $d_\alpha(O,R)\cdot d_\alpha(O,S)=r^2$ is obtained. Therefore R and S are alpha inverse points with respect to the alpha circular inversion $I_\alpha(O,r)$.

Conversely, if R and S are alpha inverse points with respect to the alpha circular inversion $I_\alpha(O,r)$ the proof is similar. □

4. Concluding remarks

The study of inversion in the non-Euclidean planes suggest interesting and challenging problems. For example, in [1, 10], the authors investigated some properties of circular inversion in the taxicab plane, and we investigated some properties of circular inversion in the alpha plane. The obtaining results include of getting results for taxicab case since alpha distance include the taxicab and Chinese checkers distance as special cases. Moreover, we think that this topic could provoke further development by interested readers or their students.

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Integer Sequences and Circle Chains Inside a Hyperbola

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Abstract. In this paper we derive formulas for inscribing, inside a branch of a generic hyperbola, a chain of mutually tangent circles; moreover, we establish conditions to relate the chain of circles to certain integer sequences.

1. Introduction

Let us consider a branch of hyperbola having axis coincident with the x-axis and described by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad x > 0 \tag{1}$$

where a and b are arbitrary positive real numbers.

Let us inscribe inside the hyperbola a chain of circles tangent to the hyperbola itself and mutually tangent between them. See an example in Figure 1.

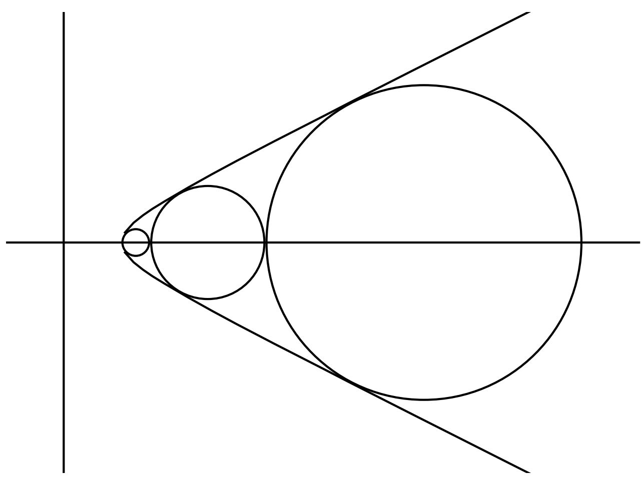

Figure 1. Circle chain inside a branch of hyperbola

In Figure 1, we have shown, for simplicity, only the right branch of the hyperbola. Nevertheless, a symmetrical circle chain can be drawn inside the left branch.

In the following, the formulas and the results that will be presented are valid for the right branch. But they can be immediately extended to the left branch by only changing x into $-x$.

By considering Figure 1, one can make the following remarks:

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- The generic n-th circle of the chain is tangent the previous $(n-1)$-th, to the $(n+1)$-th one, and to the hyperbola.
- All the centers of the circles lie on the x-axis; thus the generic n-th circle having radius r_n has center coordinates given by $(X_n,\ 0)$ with $n = 0,\ 1,\ \ldots$.
- The first circle (the smaller one identified by index 0) is tangent to the hyperbola at its vertex having coordinates $(a, 0)$; therefore, one has:

$$X_0 = a + r_0. \tag{2}$$

- Due to the mutual tangency between two consecutive circles, one can write:

$$X_n - X_{n-1} = r_n + r_{n-1}. \tag{3}$$

2. Center and radius of a generic circle of the chain

The generic n-th circle of the chain has equation:

$$(x - X_n)^2 + y^2 = r_n^2. \tag{4}$$

In order to impose the tangency condition between the n-th circle and the hyperbola, one has to start from the system:

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} & = 1, \\ (x - X_n)^2 + y^2 & = r_n^2. \end{cases} \tag{5}$$

From the equations in system (5):

$$(a^2 + b^2)x^2 - 2a^2 X_n x + (a^2 X_n^2 - a^2 b^2 - a^2 r_n^2) = 0. \tag{6}$$

The tangency condition between the hyperbola and the circles of the chain requests that the discriminant Δ of equation (6) is zero i.e.:

$$\frac{\Delta}{4} = a^2(a^2 b^2 + a^2 r_n^2 - b^2 X_n^2 + b^4 + b^2 r_n^2) = 0, \tag{7}$$

which yields

$$X_n^2 = \left(1 + \frac{a^2}{b^2}\right)(r_n^2 + b^2). \tag{8}$$

From (8), one can also write:

$$X_{n-1}^2 = \left(1 + \frac{a^2}{b^2}\right)(r_{n-1}^2 + b^2). \tag{9}$$

By subtracting (9) from (8) and taking into account (3) one gets:

$$X_n + X_{n-1} = \left(1 + \frac{a^2}{b^2}\right)(r_n - r_{n-1}). \tag{10}$$

Equations (3) and (10), after some algebraical steps, can be rewritten in recursive form as:

$$\begin{cases} X_n & = \left(2\frac{b^2}{a^2} + 1\right) X_{n-1} + 2\left(\frac{b^2}{a^2} + 1\right) r_{n-1}, \\ r_n & = 2\frac{b^2}{a^2} X_{n-1} + \left(2\frac{b^2}{a^2} + 1\right) r_{n-1} \end{cases} \tag{11}$$

or in matrix form

$$\begin{bmatrix} X_n \\ r_n \end{bmatrix} = \begin{bmatrix} 2\frac{b^2}{a^2}+1 & 2\frac{b^2}{a^2}+2 \\ 2\frac{b^2}{a^2} & 2\frac{b^2}{a^2}+1 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ r_{n-1} \end{bmatrix} \tag{12}$$

that allows to express X_n and r_n in terms of X_0 and r_0 by means of the following relation:

$$\begin{bmatrix} X_n \\ r_n \end{bmatrix} = \begin{bmatrix} 2\frac{b^2}{a^2}+1 & 2\frac{b^2}{a^2}+2 \\ 2\frac{b^2}{a^2} & 2\frac{b^2}{a^2}+1 \end{bmatrix}^n \begin{bmatrix} X_0 \\ r_0 \end{bmatrix} \tag{13}$$

Finally, by means of (2) and (8) it possible to write an explicit expression for X_0 and r_0 in function of the hyperbola parameters a and b:

$$\begin{bmatrix} X_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} \frac{b^2+a^2}{a} \\ \frac{b^2}{a} \end{bmatrix} \tag{14}$$

3. Integer sequences associated with circle chains

In this paragraph, we want to establish possible connections between the circle chains and certain integer sequences. To this aim, it is useful to introduce the new variables:

$$\widetilde{X_n} = \frac{X_n}{X_0}, \qquad \widetilde{r_n} = \frac{r_n}{r_0}$$

so that, by remembering equation (14), equation (13) becomes:

$$\begin{bmatrix} \widetilde{X_n} \\ \widetilde{r_n} \end{bmatrix} = \begin{bmatrix} 2\frac{b^2}{a^2}+1 & 2\frac{b^2}{a^2} \\ 2\frac{b^2}{a^2}+2 & 2\frac{b^2}{a^2}+1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{15}$$

From (15), one can generate two sequences $\{\widetilde{X_n}\}$ and $\{\widetilde{r_n}\}$ that both depend on the ratio b/a.

Now, one may pose the question: is it possible to find values for the ratio b/a so that $\{\widetilde{X_n}\}$ and $\{\widetilde{r_n}\}$ are integer sequences? The answer is affirmative as stated by the following theorem:

Theorem. *If the ratio b/a is given by*
(CASE A): $\frac{b}{a} = k$, $k = 1,\ 2,\ \ldots$ (16a)
or
(CASE B): $\frac{b}{a} = \frac{2k+1}{2}$, $k = 0,\ 1, \ldots,$ (16b)
then $\{\widetilde{X_n}\}$ and $\{\widetilde{r_n}\}$ are integer sequences.

Proof. (Case A) From (16a) we have that the ratio b/a is an integer and the elements of the 2×2 matrix in (15) are integers; therefore, also any generic n-th power of the matrix will be composed by only integers. Hence, from (15) one can conclude that, for any value of n, both $\widetilde{X_n}$ and $\widetilde{r_n}$ are integers.

(Case B) The 2×2 matrix in (15), that we name $[M(k)]$, in this case, becomes:

$$[M(k)] = \begin{bmatrix} \frac{4k^2+4k+5}{2} & \frac{4k^2+4k+1}{2} \\ \frac{4k^2+4k+9}{2} & \frac{4k^2+4k+5}{2} \end{bmatrix} \tag{17}$$

First of all, we demonstrate by induction that the n-th power of $[M(k)]$ can be written in the following form:

$$[M(k)]^n = \begin{bmatrix} \frac{P_{2n}(k)}{2} & \frac{(4k^2+4k+1)Q_{2n-2}(k)}{2} \\ \frac{(4k^2+4k+9)Q_{2n-2}(k)}{2} & \frac{P_{2n}(k)}{2} \end{bmatrix} \tag{18}$$

where $P_{2n}(k)$ and $Q_{2n-2}(k)$ are two polynomial functions of the integer variable k of degrees $2n$ and $2n-2$ respectively, both generating odd integers. Moreover, a further relation between them holds:

$$\frac{P_{2n}(k) + Q_{2n-2}(k)}{2} = \text{odd integer} \tag{19}$$

Equations (18) and (19) represent the inductive hypothesis $H(n)$.

For $n = 1$, one immediately notes, from (17), that the inductive hypothesis is true; in this case $P_2 = 4k^2 + 4k + 5$ and $Q_0 = 1$ and also (19) is verified.

Now we demonstrate that if $H(n)$ is true, then $H(n+1)$ too is true.

The starting point is the following relation:

$$[M(k)]^{n+1} = \begin{bmatrix} \frac{4k^2+4k+5}{2} & \frac{4k^2+4k+1}{2} \\ \frac{4k^2+4k+9}{2} & \frac{4k^2+4k+5}{2} \end{bmatrix} \begin{bmatrix} \frac{P_{2n}(k)}{2} & \frac{(4k^2+4k+1)Q_{2n-2}(k)}{2} \\ \frac{(4k^2+4k+9)Q_{2n-2}(k)}{2} & \frac{P_{2n}(k)}{2} \end{bmatrix} \tag{20}$$

From (20) and after some algebraical steps, one obtains:

$$[M(k)]^{n+1} = \begin{bmatrix} \frac{(4k^2+4k+5)P_{2n}(k)+(4k^2+4k+1)(4k^2+4k+9)Q_{2n-2}(k)}{4} & \frac{(4k^2+4k+1)\left[(4k^2+4k+5)Q_{2n-2}(k)+P_{2n}(k)\right]}{4} \\ \hline \frac{(4k^2+4k+9)\left[(4k^2+4k+5)Q_{2n-2}(k)+P_{2n}(k)\right]}{4} & \frac{(4k^2+4k+5)P_{2n}(k)+(4k^2+4k+1)(4k^2+4k+9)Q_{2n-2}(k)}{4} \end{bmatrix} \tag{21}$$

that can be written as:

$$[M(k)]^{n+1} = \begin{bmatrix} \frac{P_{2(n+1)}(k)}{4} & \frac{(4k^2+4k+1)Q_{2(n+1)-2}(k)}{4} \\ \frac{(4k^2+4k+9)Q_{2(n+1)-2}(k)}{4} & \frac{P_{2(n+1)}(k)}{4} \end{bmatrix} \tag{22}$$

where:

$$P_{2(n+1)}(k) = \frac{(4k^2+4k+5)P_{2n}(k) + (4k^2+4k+1)(4k^2+4k+9)Q_{2n-2}(k)}{2}, \tag{23}$$

$$Q_{2(n+1)-2}(k) = \frac{(4k^2+4k+5)Q_{2n-2}(k) + P_{2n}(k)}{2}. \tag{24}$$

Furthermore, equation (23), after some algebraical steps, can be written as:

$$\begin{aligned} P_{2(n+1)}(k) = {} & 2(k^2+k+1)P_{2n}(k) + 2(4k^2+4k+1)(k^2+k+2)Q_{2n-2}(k) \\ & + 2(k^2+k)Q_{2n-2}(k) + \frac{P_{2n}(k)+Q_{2n-2}(k)}{2}. \end{aligned} \tag{25}$$

By taking into account (19), one immediately notices that (25) is an odd integer.

In an analogous way, one can write:

$$Q_{2(n+1)-2}(k) = 2(k^2+k+1)Q_{2n-2}(k) + \frac{P_{2n}(k)+Q_{2n-2}(k)}{2}. \tag{26}$$

By taking into account (26), one immediately notices that (26) is an odd integer. Finally, by adding (25) and (26), and dividing by 2, one gets:

$$\begin{aligned}\frac{P_{2(n+1)}(k)+Q_{2(n+1)-2}(k)}{2} = \; & \frac{P_{2n}(k)+Q_{2n-2}(k)}{2}(2k^2+2k+3) \\ & + 2(2k^4+4k^3+3k^2+9k+1)Q_{2n-2}(k)\end{aligned} \tag{27}$$

By remembering (19) one has that the first addend in (27) is an odd integer. Conversely the second addend in (27) is an even integer. Thus (27) is an odd integer.

This concludes the demonstration by induction; therefore, equation (18) is true for each $n \geq 1$.

Now, by remembering (18) and (15) , we can write:

$$\begin{bmatrix} \widetilde{X_n} \\ \widetilde{r_n} \end{bmatrix} = [M(k)]^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{P_{2n}(k)}{2} & \frac{(4k^2+4k+1)Q_{2n-2}(k)}{2} \\ \frac{(4k^2+4k+9)Q_{2n-2}(k)}{2} & \frac{P_{2n}(k)}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{28}$$

so that

$$\begin{bmatrix} \widetilde{X_n} \\ \widetilde{r_n} \end{bmatrix} = \begin{bmatrix} \frac{P_{2n}(k)+(4k^2+4k+1)Q_{2n-2}(k)}{2} \\ \frac{(4k^2+4k+9)Q_{2n-2}(k)+P_{2n}(k)}{2} \end{bmatrix} \tag{29}$$

On the right hand side of (29), both numerators are sums of two odd integers. Therefore, each of them is an even integer. Consequently both $\widetilde{X_n}$ and $\widetilde{r_n}$ are integers for every integer n. This concludes the proof. □

4. Integer sequences classified in OEIS

In the previous paragraph, we have shown that if equations (16a) or (16b) hold, then the sequences $\{\widetilde{X_n}\}$ and $\{\widetilde{r_n}\}$ are composed by integer numbers. By varying the value of the parameter b/a one can generate an infinite number of integer sequences. A certain number of them are classified in OEIS (On-Line Encyclopedia of Integer Sequences) [1]. The results, we found, are shown in Table I.

Table I: Integer sequences associated with circle chains
and classified in OEIS

Ratio b/a	$\{\widetilde{X_n}\}$	$\{\widetilde{r_n}\}$
$1/2$	$A001519$	$A002878$
1	$A001653$	$A002315$
$3/2$	$A078922$	$A097783$
2	$A007805$	$A049629$
$5/2$	$A097835$	$A097834$
3	$A097315$	$A097314$
$7/2$	$A097838$	$A097837$
4	$A078988$	$A078989$
$9/2$	$A097841$	$A097840$
5	$A097727$	$A097726$
$11/2$	$A097843$	$A097842$
6	$A097730$	$A097729$
$13/2$	$A098244$	$A097845$
7	$A097733$	$A097732$

Ratio b/a	$\{\widetilde{X_n}\}$	$\{\widetilde{r_n}\}$
$15/2$	$A098247$	$A098246$
8	$A097736$	$A097735$
$17/2$	$A098250$	$A098249$
9	$A097739$	$A097738$
$19/2$	$A098253$	$A098252$
10	$A097742$	$A097741$
$21/2$	$A098256$	$A098255$
11	$A097767$	$A097766$
$23/2$	$A098259$	$A098258$
12	$A097770$	$A097769$
$25/2$	$A098262$	$A098261$
13	$A097773$	$A097772$
$27/2$	$A098292$	$A098291$
14	$A097776$	$A097775$

5. Examples

We show now some examples of integer sequences that can be obtained for different values of the parameter b/a.

Example 1. If $b/a = 1/2$, one gets the two following sequences:
$\{\widetilde{X_n}\} = \{1,\ 2,\ 5,\ 13,\ 34,\ 89,\ \ldots\}$ that is classified in OEIS as $A001519$;
$\{\widetilde{r_n}\} = \{1,\ 4,\ 11,\ 29,\ 76,\ 199,\ \ldots\}$ that is classified in OEIS as $A002878$.

It is interesting to note that $\{\widetilde{X_n}\}$ is composed by a bisection of Fibonacci numbers i.e. F_{2n-1} while $\{\widetilde{r_n}\}$ is composed by a bisection of Lucas numbers $\{L_{2n}\}$.

Example 2. If $b/a = 1$ one gets the two following sequences:
$\{\widetilde{X_n}\} = \{1,\ 5,\ 29,\ 169,\ 985,\ 5741,\ \ldots\}$ that is classified in OEIS as $A001653$;
$\{\widetilde{r_n}\} = \{1,\ 7,\ 41,\ 239,\ 1393,\ 8119,\ \ldots\}$ that is classified in OEIS as $A002315$.

Reference.

[1] N. J. A. Sloane (editor), *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`.

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Adjugate Points and Adjugate Triangle

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Abstract. In this paper we introduce two new notions in triangle geometry: the adjugate points and the adjugate triangle of a point with respect to a given triangle. These notions are used to better characterize the anticomlementary triangles.

1. Preliminaries

Consider a triangle ABC with lengths of sides $BC = a$, $CA = b$, and $AB = c$. Denote by s its semiperimeter and Δ its area. It is well known that the circumradius, inradius, and exradii are given by

$$R = \frac{abc}{4\Delta}, \quad r = \frac{\Delta}{s}, \quad r_a = \frac{\Delta}{s-a}, \quad r_b = \frac{\Delta}{s-b}, \quad r_c = \frac{\Delta}{s-c}.$$

We work with barycentric coordinates, absolute and homogeneous, of points with reference to the triangle. Every *finite* point P is given by its absolute barycentric coordinates $(x_P,\ y_P,\ z_P)$ with $x_P + y_P + z_P = 1$. It is more convenient to work with homogeneous barycentric coordinates. Thus, the same point P is also given by $P = (x:\ y:\ z)$, where $x : y : z = x_P :\ y_P :\ z_P$.

To express the coordinates more succinctly, we also make use of the following notations:

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

These satisfy the following relations, where $S = 2\Delta$, and for convenience, we write S_{BC} for S_BS_C etc.

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$$\begin{aligned}
&S_A = S\cot A,\ S_B = S\cot B,\ S_C = S\cot C;\\
&S_{BC} + S_{CA} + S_{AB} = S^2,\\
&a^2S_A + b^2S_B + c^2S_C = 2S^2,\\
&a^2S_A + S_{BC} = b^2S_B + S_{CA} = c^2S_C + S_{AB} = S^2;\\
&bc + ca + ab = s^2 + r^2 + 4Rr,\\
&a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr),\\
&a^3 + b^3 + c^3 = 2s(s^2 - 3r^2 - 6Rr).
\end{aligned}$$

2. Adjugate points and adjugate triangles

Suppose M is a center of triangle ABC, and let $f(a,b,c)$ be a center function for M (given for example by the barycentric coordinates of M):

$$M = (f(a,b,c) :\ f(b,c,a) :\ f(c,a,b)).$$

Definition. The points

$$\begin{aligned}
M_a &= \ (f(-a,b,c) : f(b,c,-a) : f(c,-a,b)),\\
M_b &= \ (f(a,-b,c) : f(-b,c,a) : f(c,a,-b)),\\
M_c &= \ (f(a,b,-c) : f(b,-c,a) : f(-c,a,b))
\end{aligned}$$

are called the *adjugate points*, and $\Delta M_aM_bM_c$ the *adjugate triangle* of M with respect to the reference triangle ABC.

Example 1. The adjugate points of the incenter $I = (a : b : c)$ are the centers of excircles, i.e. the points

$$I_a = (-a : b : c), \quad I_b = (a : -b : c), \quad I_c = (a : b : -c).$$

The adjugate triangle of the incenter is the excentral triangle $I_aI_bI_c$ of ABC.

Let X, Y, Z be the tangency points of incircle with the sides of triangle ABC:

$$X = (0 : s-c : s-b), \quad Y = (s-c : 0 : s-a), \quad Z = (s-b : s-a : 0).$$

Similarly we denote the points of tangency of the A-excircle, B-excircle, C-excircle respectively, with the sides of triangle ABC:

$$\begin{array}{lll}
X_a = (0 : s-b : s-c), & Y_a = (-(s-b) : 0 : s), & Z_a = (-(s-c) : s : 0);\\
X_b = (0 : -(s-a) : s), & Y_b = (s-a : 0 : s-c), & Z_b = (s : -(s-c) : 0);\\
X_c = (0 : s : -(s-a)), & Y_c = (s : 0 : -(s-b)), & Z_c = (s-a : s-b : 0).
\end{array}$$

Theorem 1. *Each of the triplets of lines (AX, BY, CZ), $(AX_a,\ BY_a,\ CZ_a)$, $(AX_b,\ BY_b,\ CZ_b)$, $(AX_c,\ BY_c,\ CZ_c)$ are concurrent. Equivalently, ABC is perspective with each of the triangles XYZ, $X_aY_aZ_a$, $X_bY_bZ_b$, and $X_cY_cZ_c$.* See Figure 1.

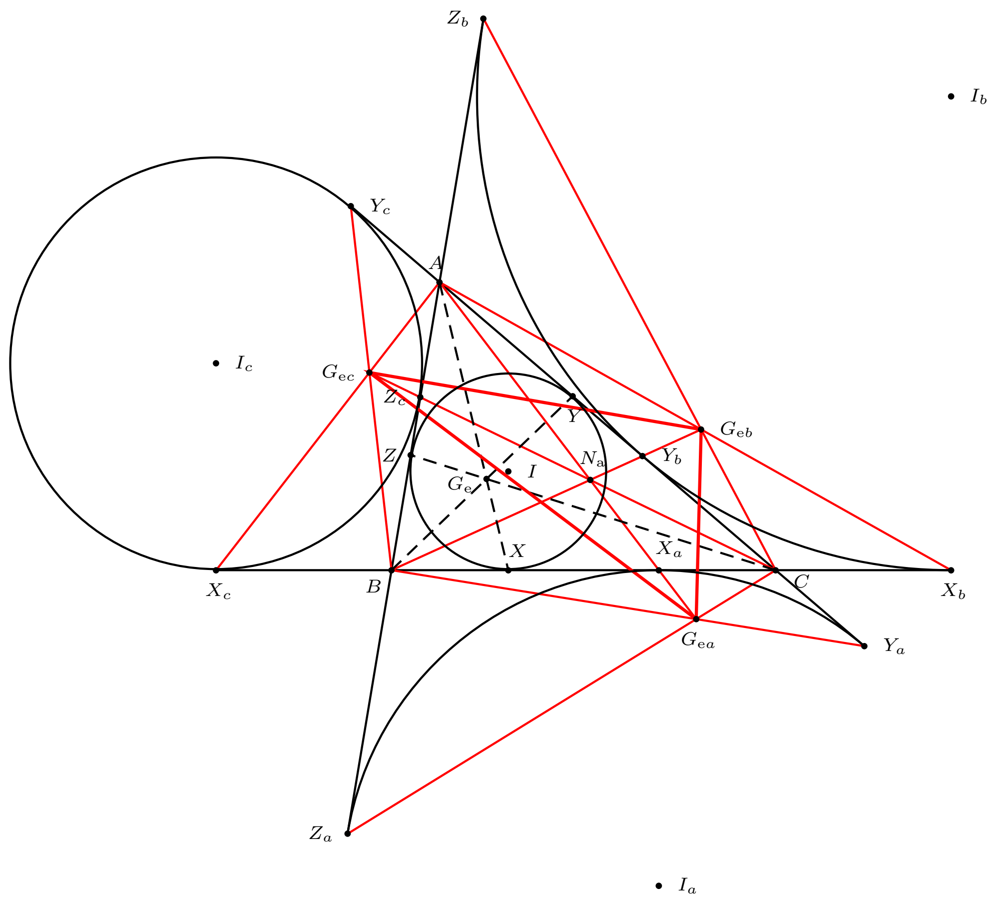

Figure 1. The Gergonne point and its adjugate triangle

Proof. We begin with the equations of the lines:

$AX:\ -(s-b)y+(s-c)z=0,$ $BY:\ (s-a)x-(s-c)z=0,$ $CZ:\ -(s-a)x+(s-b)y=0;$	$AX_a:\ (s-c)y-(s-b)z=0,$ $BY_a:\ (s-b)x+sz=0,$ $CZ_a:\ sx+(s-c)y=0;$
$AX_b:\ sy+(s-a)z=0,$ $BY_b:\ (s-c)x-(s-a)z=0,$ $CZ_b:\ (s-c)x+sy=0;$	$AX_c:\ (s-a)y+sz=0,$ $BY_c:\ (s-c)z+sz=0,$ $CZ_c:\ -(s-b)x+(s-a)y=0.$

It is well known that AX, BY, CZ are concurrent at the Gergonne point

$$G_{\mathrm{e}} = \left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right).$$

With the above equations, it is easy to verify that the concurrency of each of the triples:

Triple of lines		Point of concurrency
$AX_a \cap BY_a \cap CZ_a$		$G_{\text{ea}} = \left(-\frac{1}{s} :\ \frac{1}{s-c} :\ \frac{1}{s-b}\right)$
$AX_b \cap BY_b \cap CZ_b$		$G_{\text{eb}} = \left(\frac{1}{s-c} :\ -\frac{1}{s} :\ \frac{1}{s-a}\right)$
$AX_c \cap BY_c \cap CZ_c$		$G_{\text{ec}} = \left(\frac{1}{s-b} :\ \frac{1}{s-a} :\ -\frac{1}{s}\right)$

This completes the proof of the theorem. The points G_{ea}, G_{eb}, G_{ec} are the adjugate points of the Gergonne point Ge. □

Theorem 2. *Each of the triplets of lines* (AX_a, BY_b, CZ_c), $(AX,\ BY_c,\ CZ_b)$, $(AX_c,\ BY,\ CZ_a)$, $(AX_b,\ BY_a,\ CZ)$ *are concurrent. Equivalently,* ABC *is perspective with each of the triangles* $X_aY_bZ_c$, XY_cZ_b, X_cYZ_a, *and* X_bY_aZ. See Figure 2.

Proof. Again, we begin with the equations of the lines:

$AX_a :\ (s-c)y - (s-b)z = 0,$ $BY_b :\ (s-c)x - (s-a)z = 0,$ $CZ_c :\ -(s-b)x + (s-a)y = 0;$		$AX :\ -(s-b)y + (s-c)z = 0,$ $BY_c :\ (s-b)x + sz = 0,$ $CZ_b :\ (s-c)x + sy = 0;$
$AX_c :\ (s-a)y + sz = 0,$ $BY :\ (s-a)x - (s-c)z = 0,$ $CZ_a :\ sx + (s-c)y = 0;$		$AX_b :\ sy + (s-a)z = 0,$ $BY_a :\ sx + (s-b)z = 0,$ $CZ :\ -(s-a)x + (s-b)y = 0.$

It is well known that AX_a, BY_b, CZ_c are concurrent at the Nagel point

$$N_{\text{a}} = (s-a :\ s-b :\ s-c).$$

The other three points of concurrency are the adjugate points of the Nagel point N_{a}:

Triple of lines		Point of concurrency
$AX \cap BY_c \cap CZ_b$		$N_{\text{aa}} = (-s :\ s-c :\ s-b)$
$AX_c \cap BY \cap CZ_a$		$N_{\text{ab}} = (s-c :\ -s :\ s-a)$
$AX_b \cap BY_a \cap CZ$		$N_{\text{ac}} = (s-b :\ s-a :\ -s)$

This completes the proof of the theorem. The points N_{aa}, N_{ab}, N_{ac} are the adjugate points of the Nagel point Na. □

Theorem 3. *The triangle* ABC *and the adjugate triangle* $G_{\text{ea}}G_{\text{eb}}G_{\text{ec}}$ *of the Gergonne point are perspective, and the perspector is the Nagel point.* See Figure 1.

Proof. The line AG_{ea}, BG_{eb}, CG_{ec} have equations $\frac{y}{s-b} - \frac{z}{s-c} = 0$, $\frac{z}{s-c} - \frac{x}{s-a} = 0$, and $\frac{x}{s-a} - \frac{y}{s-b} = 0$. They are concurrent at $(s-a :\ s-b :\ s-c)$, the Nagel point. □

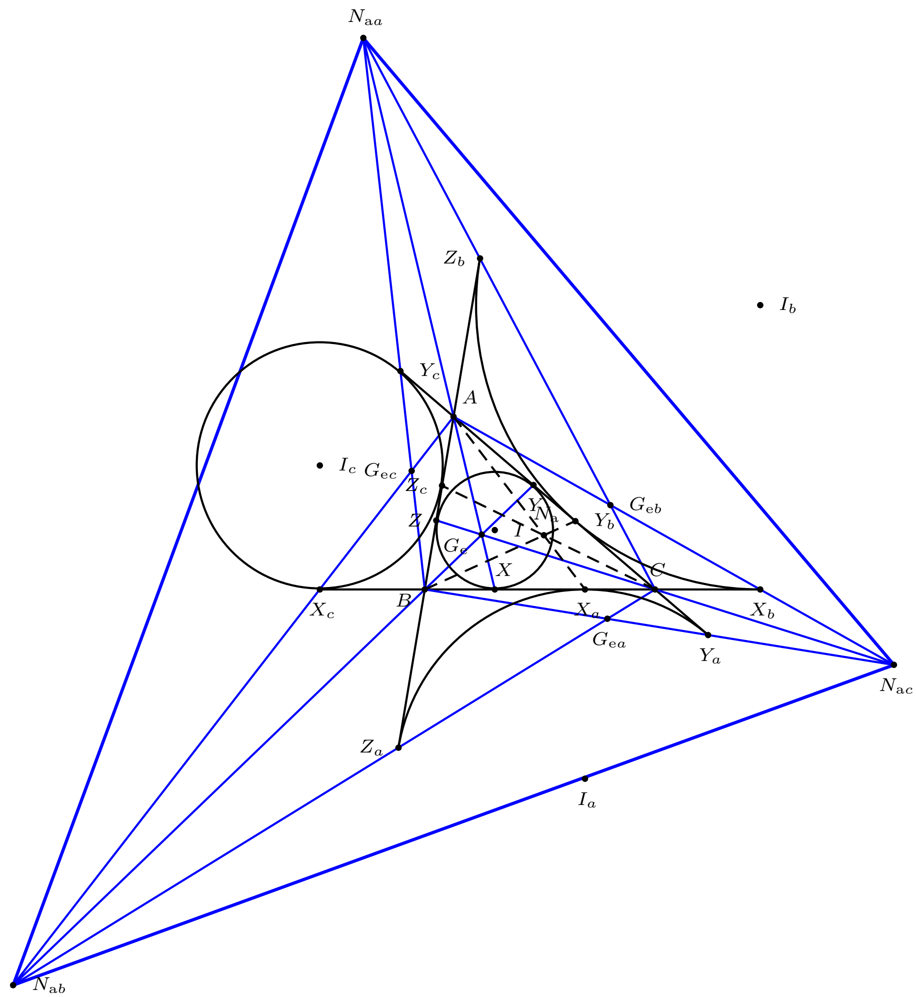

Figure 2. The Nagel point and its adjugate triangle

Theorem 4. *The triangle ABC and the adjugate triangle $N_{aa}N_{ab}N_{ac}$ of the Nagel point are perspective, and the perspector is the Gergonne point.*

Proof. The line AN_{aa}, BN_{ab}, CN_{ac} have equations $(s-b)y-(s-c)z=0$, $(s-c)z-(s-a)x=0$, and $(s-a)x-(s-b)y=0$. They are concurrent at $\left(\frac{1}{s-a}:\frac{1}{s-b}:\frac{1}{s-c}\right)$, the Gergonne point. See Figure 2. $\square$

Example 2. The incircle is internally tangent to the nine-point circle. Its point of tangency is the Feuerbach point F of the triangle ABC:

$$F=((b-c)^2(s-a):(c-a)^2(s-b):(a-b)^2(s-c)).$$

The nine-point circle is also tangent externally to the excircles at the following points:

$$\begin{aligned}F_a &= (-(b-c)^2 s : (c+a)^2(s-c) : (a+b)^2(s-b)),\\ F_b &= ((b+c)^2(s-c) : -(c-a)^2 s : (a+b)^2(s-a)),\\ F_c &= ((b+c)^2(s-b) : (c+a)^2(s-a) : -(a-b)^2 s).\end{aligned}$$

Since the coordinates of F can be rewritten as

$$((b-c)^2(b+c-a) : (c-a)^2(c+a-b) : (a-b)^2(a+b-c)),$$

its A-adjugate point is

$$\begin{aligned}&((b-c)^2(b+c-(-a)) : (c-(-a))^2(c+(-a)-b)\\ &\qquad : ((-a)-b)^2((-a)+b-c))\\ =\ &((b-c)^2(a+b+c) : -(c+a)^2(a+b-c) : -(a+b)^2(c+a-b))\\ =\ &(-(b-c)^2 s : (c+a)^2(s-c) : (a+b)^2(s-b)).\end{aligned}$$

This is the point F_a above; similarly for the B- and C-adjugate points. Therefore, the Feuerbach triangle $F_aF_bF_c$ is the adjugate triangle of the Feuerbach point. See Figure 3.

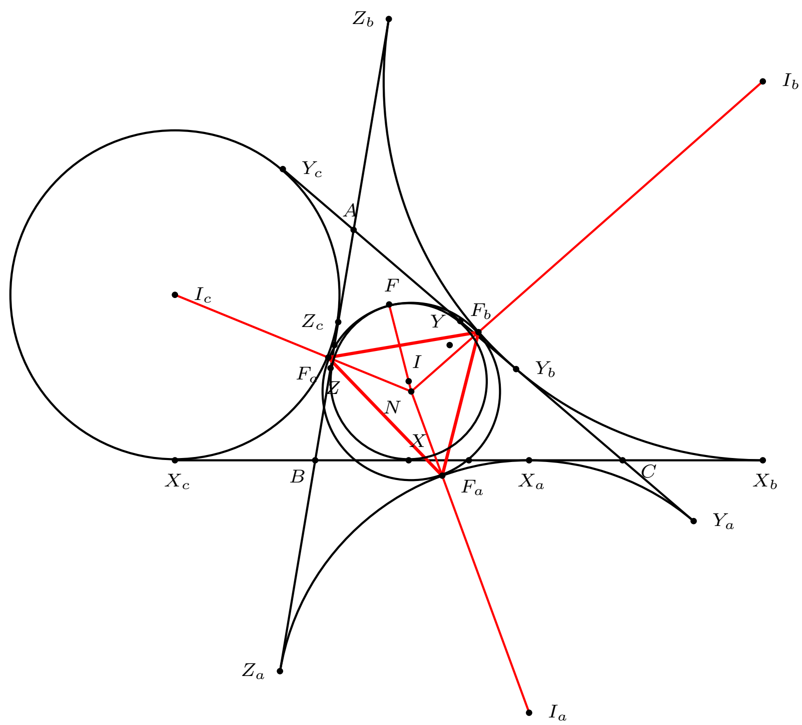

Figure 3. The Feuerbach point and its adjugate triangle

3. Anticomplementary triangle

The anticomplementary triangle is the triangle $A'B'C'$ which has the given triangle ABC as its medial triangle. Consider the homothety h with center in triangle centroid G and ratio -2, i.e. $M'G = 2GM$, $G \in (MM')$. This homothety $\mathsf{h} = \mathsf{h}(G, -2)$ transforms the reference triangle ABC into the anticomplementary triangle $A'B'C'$. Indeed, if (x_M, y_M, z_M) are the absolute barycentric coordinates of M then

$$\mathsf{h}(M) = M' = 3G - 2M = (1 - 2x_M,\ 1 - 2y_M,\ 1 - 2z_M).$$

Therefore,

$$\mathsf{h}(A) = A' = (-1, 1, 1), \quad \mathsf{h}(B) = B' = (1, -1, 1), \quad \mathsf{h}(C) = C' = (1, 1, -1).$$

It is easy to verify that the points A, B, C are the midpoints of segments $B'C'$, $C'A'$, $A'B'$ respectively.

For $\mathsf{h} = \mathsf{h}(G, -2)$, we shall call $M' = \mathsf{h}(M)$ the *anticomplement* of M. In homogeneous barycentric coordinates, if $M = (x :\ y :\ z)$, then

$$M' = (-x + y + z :\ x - y + z :\ x + y - z).$$

Here are the coordinates of some common triangle centers and their complements. (The notations follow [2] and [3]).

M	M'
$I = (a : b : c)$	$N_{\mathrm{a}} = (s - a :\ s - b :\ s - c)$
$O = (a^2 S_A :\ b^2 S_B :\ c^2 S_C)$	$H = (S_{BC} :\ S_{CA} :\ S_{AB})$
$H = (S_{BC} :\ S_{CA} :\ S_{AB})$	$X(20) = (S^2 - 2S_{BC} :\ S^2 - 2S_{CA} :\ S^2 - 2S_{AB})$
$N = (S^2 + S_{BC} :\ S^2 + S_{CA} :\ S^2 + S_{AB})$	$O = (a^2 S_A :\ b^2 S_B :\ c^2 S_C)$
$K = (a^2 :\ b^2 :\ c^2)$	$X(69) = (S_A :\ S_B :\ S_C)$
$X(9) = (a(s - a) :\ b(s - b) :\ c(s - c))$	$G_{\mathrm{e}} = \left(\frac{1}{s-a} :\ \frac{1}{s-b} :\ \frac{1}{s-c}\right)$
$S_{\mathrm{p}} = (b + c :\ c + a :\ a + b)$	$I = (a : b : c)$

Theorem 5. *The Euler line of the anticomplementary triangle $A'B'C'$ coincide with the Euler line of the reference triangle ABC .*

Proof. The Euler line is determined by the circumcenter, the orthocenter and the centroid of a triangle. Since $O' = H$, the Euler lines of $A'B'C'$ and ABC coincide. □

Lemma 6 (see [1])**.** *The anticomplement of the Feuerbach point F is the triangle center*

$$X(100) = \left(\frac{a}{b - c} :\ \frac{b}{c - a} :\ \frac{c}{a - b}\right).$$

Proof. In homogeneous barycentric coordinates,

$$F = ((b - c)^2(s - a) :\ (c - a)^2(s - b) :\ (a - b)^2(s - c)).$$

The A-coordinate of its anticomplement is

$$\begin{aligned}
&-(b-c)^2(s-a)+(c-a)^2(s-b)+(a-b)^2(s-c)\\
=&\ (-(b-c)^2+(c-a)^2+(a-b)^2)s-(-a(b-c)^2+b(c-a)^2+c(a-b)^2)\\
=&\ (2a^2-2ab-2ac+2bc)s-(a^2(b+c)+a(-(b-c)^2-2bc-2bc)+bc^2+b^2c)\\
=&\ 2(a-b)(a-c)s-a^2(b+c)+a(b+c)^2-bc(b+c)\\
=&\ 2(a-b)(a-c)s-(b+c)(a^2-a(b+c)+bc)\\
=&\ 2(a-b)(a-c)s-(b+c)(a-b)(a-c)\\
=&\ (2s-(b+c))(a-b)(a-c)\\
=&\ a(a-b)(a-c).
\end{aligned}$$

Similarly, the B- and C-coordinates are $b(b-c)(b-a)$ and $c(c-a)(c-b)$. Therefore,

$$F'=(a(a–b)(a–c):b(b–c)(b–a):\ c(c–a)(c–b)=\left(\frac{a}{b-c}:\ \frac{b}{c-a}:\ \frac{c}{a-b}\right).$$

□

Lemma 7. *The anticomplement of F_a is the point*

$$F_a'=(-a(a+b)(a+c):\ b(b-c)(b+a):\ c(c-b)(c+a)).$$

Proof. From

$$F_a=\ (-(b-c)^2s:\ (c+a)^2(s-c):\ (a+b)^2(s-b)),$$

the A-coordinate of F_a' is

$$\begin{aligned}
&(b-c)^2s+(c+a)^2(s-c)+(a+b)^2(s-b)\\
=&\ ((b-c)^2+(c+a)^2+(a+b)^2)s-(c(c+a)^2+b(a+b)^2)\\
=&\ 2(a^2+b^2+c^2-bc+ca+ab)s-(a^2(b+c)+2a(b^2+c^2)+b^3+c^3)\\
=&\ (a+b+c)(a^2+a(b+c)+(b^2-bc+c^2))-(a^2(b+c)+2a(b^2+c^2)+(b^3+c^3))\\
=&\ a^3+a^2(b+c)+a(b^2-bc+c^2)+a^2(b+c)+a(b+c)^2+(b^3+c^3)\\
&-a^2(b+c)-2a(b^2+c^2)-(b^3+c^3)\\
=&\ a^3+a^2(b+c)+abc\\
=&\ a(a+b)(a+c).
\end{aligned}$$

The B-coordinate of the anticomplement of F_a is

$$\begin{aligned}
&-(b-c)^2 s-(c+a)^2(s-c)+(a+b)^2(s-b)\\
=&\ -(b-c)^2 s-(c+a)^2(s-c)-(a+b)^2(s-b)+2(a+b)^2(s-b)\\
=&\ -a(a+b)(a+c)+2(a+b)^2(s-b)\\
=&\ (a+b)[-a(a+c)+2(s-b)(a+b)]\\
=&\ (a+b)[-a(a+c)+(c+a-b)(a+b)]\\
=&\ (a+b)[-a^2-ac+a^2-b^2+ac+bc]\\
=&\ -b(a+b)(b-c).
\end{aligned}$$

Similarly, the C-coordinate is $-c(a+c)(c-b)$. Therefore,

$$F'_a=(-a(a+b)(a+c):\ b(b-c)(b+a):\ c(c-b)(c+a)).$$

□

Similarly the anticomplements of F_b and F_c are

$$\begin{aligned}
F'_b=&\ (a(a-c)(a+b):\ -b(b+c)(b+a):\ c(c-a)(c+b)),\\
F'_c=&\ (a(a-b)(a+c):\ b(b-a)(b+c):\ -c(c+a)(c+b)).
\end{aligned}$$

Lemma 8. *The points F', F'_a, F'_b, F'_c are on the circumcircle $O(R)$.*

Proof. These points are the anticomplements of F, F_a, F_b, F_c, which are on the nine-point circle. Since the anticomplement of the nine-point circle is the circumcle, the result follows. □

Theorem 9. (a) *The anticomplements of the excenters I_a, I_b, I_c are the adjugate points N_{aa}, N_{ab}, N_{ac} of the Nagel point.*

(b) *The triplets of points $(F,\ N_{\rm a},\ O)$, $(F'_a,\ N_{aa},\ O)$, $(F'_b,\ N_{\rm ab},\ O)$, $(F'_c,\ N_{\rm ac},\ O)$ are collinear, and $FN \parallel OF'$, $F_aN \parallel OF'_a$, $F_bN \parallel OF'_b$, $F_cN \parallel OF'_c$.*

(c) *The circle $N_{\rm a}(2r)$ is the incircle; the circles $N_{\rm aa}(2r_a)$, $N_{\rm ab}(2r_b)$, $N_{\rm ac}(2r_c)$ are the excircles of the anticomplementary triangle $A'B'C'$.*

(d) *The incircle $N_{\rm a}(2r)$, and respectively the excircles $N_{\rm aa}(2r_a)$, $N_{ab}(2r_b)$, $N_{\rm ac}(2r_c)$ and the nine-point circle $O(R)$ of the anticomplementary triangle $A'B'C'$ are tangent at the anticomplements of Feuerbach point $F'\equiv X(100)$ and F'_a, F'_b, F'_c.*

Proof. (a) Since $I_a=(-a:\ b:c)$, we have:

$$\begin{aligned}
I'_a=&\ (-(-a)+b+c:\ -a-b+c:\ -a+b-c)\\
=&\ (a+b+c:\ -(a+b-c):\ -(c+a-b))\\
=&\ (-s:\ s-c:\ s-b)\\
=&\ N_{\rm aa}.
\end{aligned}$$

Similarly, $I'_b=N_{\rm a}b$ and $I'_c=N_{\rm ac}$.

(b) The Feuerbach point F, the incenter I, and the nine-point center N, are collinear. Since the homothety preserve the collinearity, the points F', $N_{\rm a}$, and

O as anticomplements of F, I, N are collinear, too. The points F_a', N_{aa}, O are collinear since they are the images under h of the collinear points F_a, I_a, N. Since $\frac{OG}{GN} = 2 = \frac{F'G}{GF}$, the lines FN and OF' are parallel.

(c) Denote by $D(P, L)$ the distance from the point P to the straight line L. The equations of the sidelines of $A'B'C'$ are:

$$B'C' : \ y + z = 0, \quad C'A' : \ z + x = 0, \quad A'B' : \ x + y = 0.$$

We prove that

$$D(N_{\mathrm{a}},\ B'C') = D(N_{\mathrm{a}},\ C'A') \ = \ D(N_{\mathrm{a}},\ A'B') = 2r$$

and

$$D(N_{\mathrm{aa}},\ B'C') \ = \ D(N_{\mathrm{aa}},\ C'A') \ = \ D(N_{\mathrm{aa}},\ A'B') \ = 2r_a.$$

Indeed,

$$D(N_{\mathrm{a}},\ B'C') = \ S\frac{\left|\frac{s-b}{s} + \frac{s-c}{s}\right|}{\sqrt{b^2+c^2-2S_A}} = \frac{S}{s} = \frac{2sr}{s} = 2r,$$

$$D(N_{\mathrm{aa}},\ B'C') = \ S\frac{\left|-\frac{s-c}{s-a} - \frac{s-b}{s-a}\right|}{\sqrt{b^2+c^2-2S_A}} = \frac{S}{s-a} = \frac{2(s-a)r_a}{s-a} = 2r_a.$$

(d)

$$\begin{aligned}
&(ON_{\mathrm{a}})^2 \\
= \ & R^2 - \frac{1}{s^2}\left[a^2(s-b)(s-c) + b^2(s-c)(s-a) + c^2(s-a)(s-b)\right] \\
= \ & R^2 - \frac{1}{s^2}\left[-s^2(a^2+b^2+c^2) + s(a^3+b^3+c^3) + abc\cdot 2s\right] \\
= \ & R^2 - \frac{1}{s^2}\left[-2s^2(s^2-r^2-4Rr) + 2s^2(s^2-3r^2-6Rr) + 4Rsr\cdot 2s\right] \\
= \ & R^2 - 2(-2r^2+2Rr) \\
= \ & (R-2r)^2;
\end{aligned}$$

i.e. $ON_{\mathrm{a}} = R - 2r$.

Similarly, $ON_{\mathrm{aa}} = R + 2r_a$, $ON_{\mathrm{ab}} = R + 2r_b$, and $ON_{\mathrm{ac}} = R + 2r_c$. □

Remarks. (1) The excentral triangle of the anticomplementary triangle $A'B'C'$ is the adjugate triangle $N_{\mathrm{aa}}N_{\mathrm{ab}}N_{\mathrm{ac}}$ of the Nagel point.

(2) Since homotheties preserve parallelism, the sides of the excentral triangles $I_aI_bI_c$ and $N_{\mathrm{aa}}N_{\mathrm{ab}}N_{\mathrm{ac}}$ are parallel, i.e.

$$I_bI_c \parallel N_{\mathrm{ab}}N_{\mathrm{ac}}, \quad I_cI_a \parallel N_{\mathrm{ac}}N_{\mathrm{aa}}, \quad I_aI_b \parallel N_{\mathrm{aa}}N_{\mathrm{ab}},$$

and

$$N_{\mathrm{ab}}N_{\mathrm{ac}} = 2I_bI_c, \quad N_{\mathrm{ac}}N_{\mathrm{aa}} = 2I_cI_a, \quad N_{\mathrm{aa}}N_{\mathrm{ab}} = 2I_aI_b.$$

Furthermore, we have

$$AI_a \parallel A'N_{\mathrm{aa}}, \quad BI_b \parallel B'N_{\mathrm{ab}}, \quad CI_c \parallel C'N_{\mathrm{ac}}.$$

Corollary 10. *The anticomplementary triangle is the orthic triangle of the adjugate triangle $N_{\rm aa}N_{\rm ab}N_{\rm ac}$ of the Nagel point.*

Theorem 11. *The Feuerbach triangle of the anticomplementary triangle $A'B'C'$ is the adjugate triangle of $X(100)$, the anticomplement of the Feuerbach point F.*

Proof. A center function of F' is $f(a,b,c) = a(a-b)(a-c)$. The points F'_a, F'_b, F'_c are the adjugate points of $X(100)$. Indeed,

$$\begin{aligned} F'_a = &\ (-a(a+b)(a+c) : b(b-c)(b+a) : c(c+a)(c-b)) \\ = &\ (f(-a,b,c) : f(b,c,-a) : f(c,-a,b)). \end{aligned}$$

□

Theorem 12. *The adjugate triangles $G_{\rm ea}G_{\rm eb}G_{\rm ec}$ and $N_{\rm aa}N_{\rm ab}N_{\rm ac}$ of the Gergonne and Nagel points are perspective, and the perspector is the symmedian point of the anticomplementary triangle $A'B'C'$, i.e. the point $K' = X(69)$.*

Proof. It is easy to verify that the lines through the corresponding pairs of points have equations

$$\begin{aligned} a(b-c)sx + b(c+a)(s-c)y - c(a+b)(s-b)z &= 0, \\ -a(b+c)(s-c)x \quad + b(c-a)sy + c(a+b)(s-a)z &= 0, \\ a(b+c)(s-b)x - b(c+a)(s-a)y \quad + c(a-b)sz &= 0, \end{aligned}$$

and that each of these lines contains the point

$$X(69) = (b^2+c^2-a^2 :\ c^2+a^2-b^2 :\ a^2+b^2-c^2).$$

□

Theorem 13. *The Gergonne point $G_{\rm e}$, the symmedian point of the anticomplementary triangle K', and the Nagel point $N_{\rm a}$ are collinear.*

Proof. The line containing these points has equation

$$a(b-c)(s-a)x + b(c-a)(s-b)y + c(a-b)(s-c)z = 0.$$

□

Theorem 14. *The pairs of perspective triangles*

$$(ABC,\ G_{\rm ea}G_{\rm eb}G_{\rm ec}) \quad (ABC,\ N_{\rm aa}N_{\rm ab}N_{\rm ac}) \quad (G_{\rm ea}G_{\rm eb}G_{\rm ec},\ N_{\rm aa}N_{\rm ab}N_{\rm ac})$$

have a common perspectrix, which is the trilinear polar of the isotomic conjugate of incenter with equation $ax + by + cz = 0$.

Proof. (a) The sidelines of triangle $G_{\rm ea}G_{\rm eb}G_{\rm ec}$ have equations

$$\begin{aligned} G_{\rm eb}G_{\rm ec} : &\quad -(b+c)(s-b)(s-c)x + bs(s-a)y + cs(s-a)z = 0, \\ G_{\rm ec}G_{\rm ea} : &\quad as(s-b)x - (c+a)(s-c)(s-a)y + cs(s-b)z = 0, \\ G_{\rm ea}G_{\rm eb} : &\quad as(s-c)x + bs(s-c)y - (a+b)(s-a)(s-b)z = 0. \end{aligned}$$

These lines intersect BC, CA, AB respectively at the three points

$$(0 :\ c :\ -b), \quad (-c :\ 0 :\ a), \quad (b :\ -a :\ 0)$$

collinear on the line $ax + by + cz = 0$. This shows that ABC and $G_{\mathrm{ea}}G_{\mathrm{eb}}G_{\mathrm{ec}}$ are perspective with perspectrix the trilinear polar of $\left(\frac{1}{a} : \frac{1}{b} : \frac{1}{c}\right)$.

(b) The sidelines of triangle $N_{\mathrm{aa}}N_{\mathrm{ab}}N_{\mathrm{ac}}$ have equations

$$\begin{aligned} N_{\mathrm{ab}}N_{\mathrm{ac}} : &\qquad (b+c)x + by + cz = 0, \\ N_{\mathrm{ac}}N_{\mathrm{aa}} : &\qquad ax + (c+a)y + cz = 0, \\ N_{\mathrm{aa}}N_{\mathrm{ab}} : &\qquad ax + by + (a+b)z = 0. \end{aligned}$$

These lines intersect BC, CA, AB respectively at the same three points

$$(0 : c : -b), \quad (-c : 0 : a), \quad (b : -a : 0)$$

on the line $ax + by + cz = 0$, showing that ABC and $N_{\mathrm{aa}}N_{\mathrm{ab}}N_{\mathrm{ac}}$ are perspective with the same perspectrix.

(c) For the triangles $G_{\mathrm{ea}}G_{\mathrm{eb}}G_{\mathrm{ec}}$ and $N_{\mathrm{aa}}N_{\mathrm{ab}}N_{\mathrm{ac}}$, we have

$$\begin{aligned} G_{\mathrm{eb}}G_{\mathrm{ec}} \cap N_{\mathrm{ab}}N_{\mathrm{ac}} &= (0 : c : -b), \\ G_{\mathrm{ec}}G_{\mathrm{ea}} \cap N_{\mathrm{ac}}N_{\mathrm{aa}} &= (-c : 0 : a), \\ G_{\mathrm{ea}}G_{\mathrm{eb}} \cap N_{\mathrm{aa}}N_{\mathrm{ab}} &= (b : -a : 0). \end{aligned}$$

Again, the corresponding lines intersect at the same three collinear points on $ax + by + cz = 0$.

This shows that the three pairs of perspective triangles have the same perspectrix. □

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Heptagonal Triangle and Trigonometric Identities

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Abstract. We will study the trigonometric identities for heptagonal triangles. Let $a < b < c$ be the heptagonal triangle's sides and let R be the circumradius. We will prove the following:

$$2b^2 - a^2 = \sqrt{7}bR, \quad 2c^2 - b^2 = \sqrt{7}cR, \quad 2a^2 - c^2 = -\sqrt{7}aR.$$

We will also prove the following trigonometric formula:

$$4\sin\frac{2k\pi}{7} - \tan\frac{k\pi}{7} = \begin{cases} \sqrt{7} & \text{for } k = 1, 2, 4, \\ -\sqrt{7} & \text{for } k = 3, 5, 6., \end{cases}$$

1. Introduction

In this paper, for convenience, let $\theta = \pi/7$. A heptagonal triangle is an obtuse scalene triangle whose vertexes coincide with the first, second, and fourth vertexes of a regular heptagon. Its angles have measures $\theta, 2\theta, 4\theta$. Let $a < b < c$ be the heptagonal triangle's sides and let R be the circumradius. We will prove the following:

Theorem 1. *With above notations, we have*

$$2b^2 - a^2 = \sqrt{7}bR, \quad 2c^2 - b^2 = \sqrt{7}cR, \quad 2a^2 - c^2 = -\sqrt{7}aR.$$

This result is a corollary of the following identities:

Theorem 2.

$$4\sin\frac{2k\pi}{7} - \tan\frac{k\pi}{7} = \begin{cases} \sqrt{7} & \text{for } k = 1, 2, 4, \\ -\sqrt{7} & \text{for } k = 3, 5, 6. \end{cases}$$

The purpose of this paper is to prove our results. In later sections, we will also show how to use our methods to prove some known identities which are sums of mixed powers of sine values.

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2. Sums of sine powers

We start with the following theorem which can be proved easily from trigonometric identities from [1, 5].

Theorem 3. (1) $\{\sin 2\theta, \sin 4\theta, \sin 8\theta\}$ *are the roots of*

$$x^3 - \frac{\sqrt{7}}{2}x^2 + \frac{\sqrt{7}}{8} = 0.$$

(2) $\tan\theta \tan 2\theta \tan 4\theta = \tan\theta + \tan 2\theta + \tan 4\theta = -\sqrt{7}.$
(3) $\sec 2\theta + \sec 4\theta + \sec 8\theta = -4.$

Definition. For an integer n, let

$$S(n) = \sin^n 2\theta + \sin^n 4\theta + \sin^n 8\theta.$$

Proposition 4. $S(n)$ *satisfies the recurrence relation:*

$$S(n) = \frac{\sqrt{7}}{2}S(n-1) - \frac{\sqrt{7}}{8}S(n-3).$$

? *Proof.* This follows easily from Theorem 1 □

Now using recurrence relation we can compute $S(n)$ for any integer n. In the following, we will only show a few terms which will be used in later applications.

Example 1. With above notations, the values of $S(n)$ for $n = 1, \ldots,$ 20 are as follows.

n	0	1	2	3	4	5	6
$S(n)$	3	$\frac{\sqrt{7}}{2}$	$\frac{7}{2^2}$	$\frac{\sqrt{7}}{2}$	$\frac{7\cdot 3}{2^4}$	$\frac{7\sqrt{7}}{2^4}$	$\frac{7\cdot 5}{2^5}$
$S(-n)$	3	0	2^3	$-\frac{2^3\cdot 3\sqrt{7}}{7}$	2^5	$-\frac{2^5\cdot 5\sqrt{7}}{7}$	$\frac{2^6\cdot 17}{7}$

n	7	8	9	10	11	12	13
$S(n)$	$\frac{7^2\sqrt{7}}{2^7}$	$\frac{7^2\cdot 5}{2^8}$	$\frac{7\cdot 25\sqrt{7}}{2^9}$	$\frac{7^2\cdot 9}{2^9}$	$\frac{7^2\cdot 13\sqrt{7}}{2^{11}}$	$\frac{7^2\cdot 33}{2^{11}}$	$\frac{7^2\cdot 3\sqrt{7}}{2^9}$
$S(-n)$	$-2^7\sqrt{7}$	$\frac{2^9\cdot 11}{7}$	$-\frac{2^{10}\cdot 33\sqrt{7}}{7^2}$	$\frac{2^{10}\cdot 29}{7}$	$-\frac{2^{14}\cdot 11\sqrt{7}}{7^2}$	$\frac{2^{12}\cdot 269}{7^2}$	$-\frac{2^{13}\cdot 117\sqrt{7}}{7^2}$

n	14	15	16	17	18	19	20
$S(n)$	$\frac{7^4\cdot 5}{2^{14}}$	$\frac{7^2\cdot 179\sqrt{7}}{2^{15}}$	$\frac{7^3\cdot 131}{2^{16}}$	$\frac{7^3\cdot 3\sqrt{7}}{2^{12}}$	$\frac{7^3\cdot 493}{2^{18}}$	$\frac{7^3\cdot 181\sqrt{7}}{2^{18}}$	$\frac{7^5\cdot 19}{2^{19}}$
$S(-n)$	$\frac{2^{14}\cdot 51}{7}$	$-\frac{2^{21}\cdot 17\sqrt{7}}{7^3}$	$\frac{2^{17}\cdot 237}{7^2}$	$-\frac{2^{17}\cdot 1445\sqrt{7}}{7^3}$	$\frac{2^{19}\cdot 2203}{7^3}$	$-\frac{2^{19}\cdot 1919\sqrt{7}}{7^3}$	$\frac{2^{20}\cdot 5851}{7^3}$

3. Lemmas

Definition. For integers, m, n, let

$$W(m,n) = \sin^m 2\theta \sin^n 4\theta + \sin^m 4\theta \sin^n 8\theta + \sin^m 8\theta \sin^n 2\theta$$

and let

$$P = \sin 2\theta \sin 4\theta \sin 8\theta = -\frac{\sqrt{7}}{8}.$$

Lemma 5.

$$\begin{aligned} W(m.n)+W(n,m) =&\ S(m)S(n)-S(m+n), \\ W(m.n)W(n,m) =&\ P^{m+n}S(-(m+n))+P^mS(2n-m)+P^nS(2m-n). \end{aligned}$$

Proof. This can be proved easily using simple algebra. □

Remark. Note that if $m \neq n$, $W(m,n)$ is not a symmetrical polynomial in $\{\sin 2\theta, \sin 4\theta, \sin 8\theta\}$ and in general, it is not easy to compute $W(m,n)$ directly. Here is our approach. Using Lemma 5, we can compute $W(m.n)+W(n,m)$ and $W(m.n)W(n,m)$ in terms of $S(n)$ and P. Then we solve a quadratic equation

$$x^2-(W(m.n)+W(n,m))x+W(m.n)W(n,m)=0.$$

and use approximate values to identify the solutions.

Lemma 6.

$$W(2,3)=\frac{7\sqrt{7}}{32}.$$

Proof. By Lemma 5,

$$\begin{aligned} W(2,3)+W(3,2) =&\ S(2)S(3)-S(5)=\frac{7\sqrt{7}}{16}, \\ W(2,3)W(3,2) =&\ P^5S(-5)+P^2S(4)+P^3S(1)=\frac{343}{1024}. \end{aligned}$$

Solving the quadratic equation

$$t^2-\frac{7\sqrt{7}}{16}t+\frac{343}{1024}=0,$$

we have

$$t=\left\{\frac{7\sqrt{7}}{32},\frac{7\sqrt{7}}{32}\right\}.$$

This proves this lemma. □

Definition. For convenience let

$$\begin{aligned} R =&\ \sin 2\theta\sin 4\theta\tan 8\theta+\sin 4\theta\sin 8\theta\tan 2\theta+\sin 8\theta\sin 2\theta\tan 4\theta, \\ U =&\ \sin 2\theta\sin 4\theta\tan 4\theta+\sin 4\theta\sin 8\theta\tan 8\theta+\sin 8\theta\sin 2\theta\tan 2\theta, \\ V =&\ \sin 2\theta\sin 8\theta\tan 8\theta+\sin 4\theta\sin 2\theta\tan 2\theta+\sin 8\theta\sin 4\theta\tan 4\theta, \\ X =&\ \sin^2 2\theta\tan 2\theta+\sin^2 4\theta\tan 4\theta+\sin^2 8\theta\tan 8\theta, \\ Y =&\ \sin^2 2\theta\tan 8\theta+\sin^2 4\theta\tan 2\theta+\sin^2 8\theta\tan 4\theta, \\ Z =&\ \sin^2 2\theta\tan 4\theta+\sin^2 4\theta\tan 8\theta+\sin^2 8\theta\tan 2\theta. \end{aligned}$$

Lemma 7.

$$R=\frac{\sqrt{7}}{2},\quad V=\sqrt{7},\quad U=-\frac{3\sqrt{7}}{2},\quad X=-\frac{5\sqrt{7}}{4}.$$

Proof. With above notations,

$$\begin{aligned}
R = &\ \sin 2\theta \sin 4\theta \tan 8\theta + \sin 4\theta \sin 8\theta \tan 2\theta + \sin 8\theta \sin 2\theta \tan 4\theta \\
= &\ (\sin 2\theta \sin 4\theta \sin 8\theta)(\frac{1}{\cos 2\theta} + \frac{1}{\cos 4\theta} + \frac{1}{\cos 8\theta}) \\
= &\ (\sin 2\theta \sin 4\theta \sin 8\theta)(\sec 2\theta + \sec 4\theta + \sec 8\theta) \\
= &\ \frac{\sqrt{7}}{2} \\
V = &\ \sin 2\theta \sin 4\theta \tan 2\theta + \sin 4\theta \sin 8\theta \tan 4\theta + \sin 8\theta \sin 2\theta \tan 8\theta \\
= &\ 2(\sin^3 2\theta + \sin^3 4\theta + \sin^3 8\theta) \\
= &\ 2S(3) \\
= &\ \sqrt{7}; \\
R + U + V = &\ (\sin 2\theta \sin 4\theta + \sin 4\theta \sin 8\theta + \sin 8\theta \sin 2\theta) \\
&\ \cdot (\tan 2\theta + \tan 4\theta + \tan 8\theta) \\
= &\ 0; \\
U = &\ -R - V \\
= &\ -\frac{3\sqrt{7}}{2}; \\
X = &\ \sin^2 2\theta \tan 2\theta + \sin^2 4\theta \tan 4\theta + \sin^2 8\theta \tan 8\theta. \\
= &\ (1 - \cos^2 2\theta) \tan 2\theta + (1 - \cos^2 4\theta) \tan 4\theta + (1 - \cos^2 8\theta) \tan 8\theta \\
= &\ (\tan 2\theta + \tan 4\theta + \tan 8\theta) - \frac{1}{2} S(1) \\
= &\ -\frac{5\sqrt{7}}{4}.
\end{aligned}$$

$\square$

Lemma 8.

$$Y = \frac{\sqrt{7}}{4}, \qquad Z = -\frac{3\sqrt{7}}{4}.$$

Proof.

$$\begin{aligned}
Y + Z = &\ (\sin^2 2\theta + \sin^2 4\theta + \sin^2 8\theta)(\tan 2\theta + \tan 4\theta + \tan 8\theta) - X \\
= &\ S(2)(\tan 2\theta + \tan 4\theta + \tan 8\theta)) + \frac{5\sqrt{7}}{4} \\
= &\ -\frac{\sqrt{7}}{2}.
\end{aligned}$$

Next, we compute

$$\begin{aligned}
& 4Y - Z \\
= & \ (4\sin^2 4\theta - \sin^2 8\theta)\tan 2\theta + (4\sin^2 8\theta - \sin^2 2\theta)\tan 4\theta \\
& + (4\sin^2 2\theta - \sin^8\theta)\tan 8\theta \\
= & \ (4\sin^2 4\theta - 4\sin^2 4\theta\cos^2 4\theta)\tan 2\theta + (4\sin^2 8\theta - 4\sin^2 8\theta\cos^2 8\theta)\tan 4\theta \\
& + (4\sin^2 2\theta - 4\sin^2 2\theta\cos^2 2\theta)\tan 8\theta \\
= & \ 4\sin^2 4\theta(1-\cos^2 4\theta)\tan 2\theta + 4\sin^2 8\theta(1-\cos^2 8\theta)\tan 4\theta \\
& + 4\sin^2 2\theta(1-\cos^2 2\theta)\tan 8\theta \\
= & \ 4(\sin^4 4\theta\tan 2\theta + \sin^4 8\theta\tan 4\theta + \sin^4 2\theta\tan 8\theta) \\
= & \ 4(2\sin 2\theta\cos 2\theta\tan 2\theta\sin^3 4\theta + 2\sin 4\theta\cos 4\theta\tan 4\theta\sin^3 8\theta \\
& + 2\sin 8\theta\cos 8\theta\tan 8\theta\sin^3 2\theta) \\
= & \ 8(\sin^2 2\theta\sin^3 4\theta + \sin^2 4\theta\sin^3 8\theta + \sin^2 8\theta\sin^3 2\theta) \\
= & \ 8W(2,3) \\
= & \ \frac{7\sqrt{7}}{4}.
\end{aligned}$$

It follows that

$$Y = \frac{\sqrt{7}}{4}, \qquad Z = -\frac{3\sqrt{7}}{4}.$$

□

4. Main theorem

Proposition 9. *With above notations,*

$$\begin{aligned}
\tan 2\theta = & -2\sin 2\theta + 2\sin 4\theta - 2\sin 8\theta, \\
\tan 4\theta = & -2\sin 2\theta - 2\sin 4\theta + 2\sin 8\theta, \\
\tan 8\theta = & \ 2\sin 2\theta - 2\sin 4\theta - 2\sin 8\theta.
\end{aligned}$$

Proof. First we consider a system of linear equations:

$$\begin{aligned}
\tan 2\theta = & \ x\sin 2\theta + y\sin 4\theta + z\sin 8\theta, \\
\tan 4\theta = & \ x\sin 4\theta + y\sin 8\theta + z\sin 2\theta, \\
\tan 8\theta = & \ x\sin 8\theta + y\sin 2\theta + z\sin 4\theta.
\end{aligned}$$

Note that by adding up the equations

$$x + y + z = \frac{\tan 2\theta + \tan 4\theta + \tan 8\theta}{\sin 2\theta + \sin 4\theta + \sin 8\theta} = -2.$$

Then

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}$$

where

$$\begin{aligned}\Delta =\ & DET\begin{bmatrix}\sin 2\theta & \sin 4\theta & \sin 8\theta\\ \sin 4\theta & \sin 8\theta & \sin 2\theta\\ \sin 8\theta & \sin 2\theta & \sin 4\theta\end{bmatrix},\\ \Delta_x =\ & DET\begin{bmatrix}\tan 2\theta & \sin 4\theta & \sin 8\theta\\ \tan 4\theta & \sin 8\theta & \sin 2\theta\\ \tan 8\theta & \sin 2\theta & \sin 4\theta\end{bmatrix},\\ \Delta_y =\ & DET\begin{bmatrix}\sin 2\theta & \tan 2\theta & \sin 8\theta\\ \sin 4\theta & \tan 4\theta & \sin 2\theta\\ \sin 8\theta & \tan 8\theta & \sin 4\theta\end{bmatrix},\\ \Delta_z =\ & DET\begin{bmatrix}\sin 2\theta & \sin 4\theta & \tan 2\theta\\ \sin 4\theta & \sin 8\theta & \tan 4\theta\\ \sin 8\theta & \sin 2\theta & \tan 8\theta\end{bmatrix}.\end{aligned}$$

Then by expanding the determinants,

$$\begin{aligned}\Delta =\ & 3P - S(3) = -\frac{7\sqrt{7}}{8};\\ \Delta_y =\ & U - Y = \frac{-3\sqrt{7}}{2} - \frac{\sqrt{7}}{4} = -\frac{7\sqrt{7}}{2},\\ y =\ & \frac{\Delta_y}{\Delta} = 2;\\ \Delta_z =\ & V - Z = \sqrt{7} + \frac{3\sqrt{7}}{4} = \frac{7\sqrt{7}}{2},\\ z =\ & \frac{\Delta_z}{\Delta} = -2.\end{aligned}$$

Finally,

$$x = -2 - y - z = -2.$$

□

Now we can prove Theorem 2.

Proof. By Proposition 9

$$\begin{aligned}\tan 2\theta &= -2\sin 2\theta + 2\sin 4\theta - 2\sin 8\theta\\ &=\ 4\sin 4\theta - 2(\sin 2\theta + 2\sin 4\theta + 2\sin 8\theta)\\ &=\ 4\sin 4\theta - \sqrt{7}.\end{aligned}$$

Similarly, we can prove other identities. □

5. The heptagonal triangle

The heptagonal triangle and trigonometric identities for angles $\theta, 2\theta, 4\theta$ of the heptagonal triangle have been studied in [1, 5]. We will use some results from [1, 5].

Proposition 10. *With above notations, we have*

(1) $\frac{a}{\sin\theta} = \frac{b}{\sin 2\theta} = \frac{c}{\sin 4\theta} = 2R$;
(2) $\cos\theta = \frac{b}{2a}, \quad \cos 2\theta = \frac{c}{2b}, \quad \cos 4\theta = -\frac{a}{2c}$;
(3) $b^2 - a^2 = ca, \quad c^2 - b^2 = ab, \quad a^2 - c^2 = -bc$.

Now we prove Theorem 1 as follows.

Proof. By Theorem 2 and Proposition 10

$$\sin 2\theta = \frac{b}{2R}, \quad \sin\theta = \frac{a}{2R}, \quad \cos\theta = \frac{b}{2a}.$$

It follows that

$$4\sin 2\theta - \tan\theta = \frac{2b}{R} - \frac{a^2}{bR}.$$

$$2b^2 - a^2 = \sqrt{7}bR.$$

Similarly, we can prove other identities. □

6. More trigonometric identities

Proposition 11. *With above notations,*

(1) $\sin^3 2\theta \sin 4\theta + \sin^3 4\theta \sin 8\theta + \sin^3 8\theta \sin 2\theta = 0,$
(2) $\sin 2\theta \sin^3 4\theta + \sin 4\theta \sin^3 8\theta + \sin 8\theta \sin^3 2\theta = \frac{7}{2^4},$
(3) $\sin^4 2\theta \sin 4\theta + \sin^4 4\theta \sin 8\theta + \sin^4 8\theta \sin 2\theta = 0,$
(4) $\sin 2\theta \sin^4 4\theta + \sin 4\theta \sin^4 8\theta + \sin 8\theta \sin^4 2\theta = \frac{7\sqrt{7}}{2^5},$
(5) $\sin^{11} 2\theta \sin^3 4\theta + \sin^{11} 4\theta \sin^3 8\theta + \sin^{11} 8\theta \sin^3 2\theta = 0,$
(6) $\sin^3 2\theta \sin^{11} 4\theta + \sin^3 4\theta \sin^{11} 8\theta + \sin^3 8\theta \sin^{11} 2\theta = \frac{7^3 \cdot 17}{2^{14}}.$

Proof. To prove equations (1) and (2), by Lemma 5,

$$\begin{aligned}
W(3,1) + W(1,3) =&\ S(3)S(1) - S(4) \\
=&\ \frac{\sqrt{7}}{2} \cdot \frac{\sqrt{7}}{2} - \frac{7 \cdot 3}{2^4} \\
=&\ \frac{7}{2^4}; \\
W(3,1)W(1,3) =&\ P^4 S(-4) + P^3 S(-1) + P^1 S(5) \\
=&\ \frac{7^2}{2^{12}} \cdot 2^5 - \frac{7\sqrt{7}}{2^9} \cdot 0 - \frac{\sqrt{7}}{2^3} \cdot \frac{7\sqrt{7}}{2^4} \\
=&\ 0.
\end{aligned}$$

Then solving the quadratic equation

$$x^2 - \frac{7}{16}x = 0,$$

we have roots $\{0, \frac{7}{16}\}$.

Using approximate values, we can conclude that

$$W(3,1) = 0, \quad W(1,3) = \frac{7}{16}.$$

Similarly, to prove equations (3), (4), we compute

$$\begin{aligned}
W(4,1) + W(1,4) =& \ S(4)S(1) - S(5) \\
=& \ \frac{7 \cdot 3}{2^4} \cdot \frac{\sqrt{7}}{2} - \frac{7\sqrt{7}}{2^4} \\
=& \ \frac{7\sqrt{7}}{2^5}; \\
W(4,1)W(1,4) =& \ P^5 S(-5) + P^4 S(-2) + P^1 S(7) \\
=& \ \frac{-7^2\sqrt{7}}{2^{15}} \cdot \frac{-2^5 \cdot 5\sqrt{7}}{7} + \frac{7^2}{2^{12}} \cdot 2^3 + \frac{-\sqrt{7}}{2^3} \cdot \frac{7^2\sqrt{7}}{2^7} \\
=& \ \frac{7^2 \cdot 5}{2^{10}} + \frac{7^2}{2^9} + \frac{-7^3}{2^{10}} = 0.
\end{aligned}$$

Similarly by solving the quadratic equation

$$x^2 - \frac{7\sqrt{7}}{2^5}x = 0$$

and numerical computations, we have that

$$W(4,1) = 0, \quad W(1,4) = \frac{7\sqrt{7}}{2^5}.$$

Similarly, to prove equations (5) and (6), we compute

$$\begin{aligned}
W(11,3) + W(3,11) =& \ S(11)S(3) - S(14) \\
=& \ \frac{7^2 \cdot 13\sqrt{7}}{2^{11}} \cdot \frac{\sqrt{7}}{2} - \frac{7^4 \cdot 5}{2^{14}} \\
=& \ \frac{7^3 \cdot 17}{2^{14}}; \\
W(11,3)W(3,11) =& \ P^{14} S(-14) + P^{11} S(-5) + P^3 S(19) \\
=& \ \frac{7^7}{2^{42}} \cdot \frac{2^{14} \cdot 51}{7} + \frac{-7^5\sqrt{7}}{2^{33}} \cdot \frac{-2^5 \cdot 5\sqrt{7}}{7} + \frac{-7\sqrt{7}}{2^9} \cdot \frac{7^3 \cdot 181\sqrt{7}}{2^{18}} \\
=& \ \frac{7^6 \cdot 51}{2^{28}} + \frac{7^5 \cdot 5}{2^{28}} + \frac{-7^5 \cdot 181}{2^{27}} \\
=& \ 0.
\end{aligned}$$

Similarly by solving the quadratic equation

$$x^2 - \frac{7^3 \cdot 17}{2^{14}} x = 0$$

and numerical computations, we have that

$$W(11,3) = 0, \quad W(3,11) = \frac{7^3 \cdot 17}{2^{14}}.$$

□

To further illustrate our method, we will prove the following identities.

Proposition 12.

$$\frac{\sin 2\theta}{\sin^4 4\theta} + \frac{\sin 4\theta}{\sin^4 8\theta} + \frac{\sin 8\theta}{\sin^4 2\theta} = \frac{72\sqrt{7}}{7},$$
$$\frac{\sin 4\theta}{\sin^4 2\theta} + \frac{\sin 8\theta}{\sin^4 4\theta} + \frac{\sin 2\theta}{\sin^4 8\theta} = \frac{64\sqrt{7}}{7}.$$

Proof.

$$\begin{aligned}
W(1,-4) + W(-4,1) = &\ S(1)S(-4) - S(-3) = \frac{\sqrt{7}}{2} \cdot 2^5 + \frac{2^3 \cdot 3\sqrt{7}}{7} = \frac{2^3 \cdot 17\sqrt{7}}{7}. \\
W(1,-4)W(-4,1) = &\ P^{-3}S(3) + P^1 S(-9) + P^{-4}S(6) \\
= &\ \frac{-2^9\sqrt{7}}{7^2} \cdot \frac{\sqrt{7}}{2} + \frac{-\sqrt{7}}{2^3} \cdot \frac{-2^{10} \cdot 33\sqrt{7}}{7^2} + \frac{2^{12}}{7^2} \cdot \frac{7 \cdot 5}{2^5} \\
= &\ \frac{-2^8}{7} + \frac{2^7 \cdot 33}{7} + \frac{2^7 \cdot 5}{7} \\
= &\ \frac{2^9 \cdot 9}{7}.
\end{aligned}$$

Solving the quadratic equation

$$x^2 + \frac{136\sqrt{7}}{7} x - \frac{4068}{7} = 0$$

we have solutions

$$\{x = \frac{64\sqrt{7}}{7}, \frac{72\sqrt{7}}{7}\}.$$

□

7. The heptagonal triangle again

Theorem 13. *With above notations, we have*

(1) $a^3b - b^3c + c^3a = 0$.
(2) $a^4b + b^4c - c^4a = 0$.
(3) $a^{11}b^3 - b^{11}c^3 + c^{11}a^3 = 0$.
(4) $a^3c + b^3a - c^3b = -7R^4$.
(5) $a^4c - b^4a + c^4b = 7\sqrt{7}R^5$.
(6) $a^{11}c^3 + b^{11}a^3 - c^{11}b^3 = -7^3 \cdot 17R^{14}$.

Proof. By [1],

$$a = 2R\sin\theta, b = 2R\sin 2\theta, c = 2R\sin 4\theta.$$

Then Theorem 13 follows easily from Proposition 11. □

8. Other related results

Remark. In [2, 3, 4] there are similar trigonometric identities which were proved as corollaries of theta function identities.

$$\frac{\sin 2\theta}{\sin\theta} - \frac{\sin 3\theta}{\sin 2\theta} + \frac{\sin\theta}{\sin 3\theta} = 1,$$
$$\frac{\sin\theta}{\sin 2\theta} - \frac{\sin 2\theta}{\sin 3\theta} + \frac{\sin 3\theta}{\sin\theta} = 2,$$
$$\frac{\sin^2\theta}{\sin 3\theta} - \frac{\sin^2 2\theta}{\sin\theta} + \frac{\sin^2 3\theta}{\sin 2\theta} = 0,$$
$$\frac{\sin 2\theta}{\sin^4\theta} - \frac{\sin\theta}{\sin^4 3\theta} + \frac{\sin 3\theta}{\sin^4 2\theta} = \frac{64\sqrt{7}}{7},$$
$$\frac{\sin^4 3\theta}{\sin\theta} - \frac{\sin^4\theta}{\sin 2\theta} - \frac{\sin^4 2\theta}{\sin 3\theta} = \frac{5\sqrt{7}}{8},$$
$$\frac{\sin^7 2\theta}{\sin^7\theta} - \frac{\sin^7 3\theta}{\sin^7 2\theta} + \frac{\sin^7\theta}{\sin^7 3\theta} = 57,$$
$$\frac{\sin^7\theta}{\sin^7 2\theta} - \frac{\sin^7 2\theta}{\sin^7 3\theta} + \frac{\sin^7 3\theta}{\sin^7\theta} = 289,$$
$$\frac{\sin^3 3\theta}{\sin^6\theta} - \frac{\sin^3\theta}{\sin^6 2\theta} + \frac{\sin^3 2\theta}{\sin^6 3\theta} = \frac{368}{\sqrt{7}},$$
$$\frac{\sin 2\theta}{\sin^2 3\theta} - \frac{\sin\theta}{\sin^2 2\theta} + \frac{\sin 3\theta}{\sin^2\theta} = 2\sqrt{7},$$
$$\csc^7\theta - \csc^7 2\theta - \csc^7 3\theta = 2^7\sqrt{7}.$$

All those identities can be easily proved using our method.

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The Radical Axis of the Circumcircle and Incircle of a Bicentric Quadrilateral

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Abstract. We present an unusual method to identify the radical axis of the circumcircle and incircle of a bicentric quadrilateral. Along the way, we demonstrate a number of interesting properties of the configuration.

1. Bicentric Quadrilateral

Let $ABCD$ be a bicentric quadrilteral (see Figure 1). We employ the following definitions:

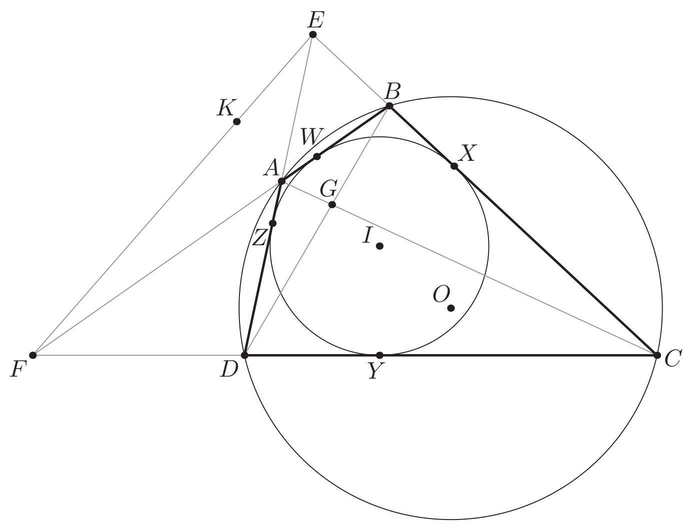

Figure 1. The bicentric quadrilateral configuration.

- E, F, and G are the intersections of $\overline{AD}$ and $\overline{BC}$, $\overline{AB}$ and $\overline{CD}$, and $\overline{AC}$ and $\overline{BD}$, respectively.
- I and O are the incenter and circumcenter of $ABCD$, respectively.
- K is the Miquel Point of $ABCD$. (The *Miquel Point* is the common intersection of the circumcircles of $\triangle ECD, \triangle FBC, \triangle EBA$, and $\triangle FAD$.)
- W, X, Y, and Z are the points at which the incircle meets sides $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$, respectively.
- ω and Ω are the incircle and circumcircle of $ABCD$, respectively.

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Next, we present a number of well-known facts about this configuration, which will be used throughout the rest of the paper.

Proposition 1. *In bicentric quadrilateral $ABCD$,*

(a) *W, G, Y and Z, G, X are collinear, with $\overline{WGY} \perp \overline{ZGX}$.*

(b) *Points O, G, K are collinear with $\overline{OGK} \perp \overline{EF}$.*

(c) *Points I, G, K are collinear; hence, O, I, G, and K are collinear, and $\overline{OIGK} \perp \overline{EF}$.*

(d) *$\angle EIF$ is a right angle.*

Proof. (a) The collinearities follow from applications of Brianchon's Theorem on the two degenerate hexagons $AWBCYD$ and $ABXCDZ$, which gives that G is also the intersection of $\overline{WGY}$ and $\overline{ZGX}$. That $\overline{WGY} \perp \overline{ZGX}$ has been shown in [2].

(b) In fact, K is the inverse of G with respect to Ω ([3], Theorem 10.12), so O, G, K collinear is immediate. Meanwhile, by Brokard's Theorem, $\overline{EF}$ is the polar of G with respect to Ω, immediately implying $\overline{OGK} \perp \overline{EF}$.

(c) It has been shown that $\frac{AZ}{ZD} = \frac{BX}{XC}$ in [4]; hence, as K is the center of the spiral similarity mapping $\overline{AB}$ to $\overline{DC}$, it also maps $\overline{AB}$ to $\overline{ZX}$. Thus, by [6], it follows that $EKZX$ is a cyclic quadrilateral; furthermore, as $\angle IZE = \angle IXE = 90^\circ$, we know that E, K, Z, I, and X are concyclic.

In a similar manner we may show that F, K, W, I, and Y are concyclic. Now, inversion about ω maps the circumcircle of $WIYFK$ to $\overline{WY}$ and the circumcircle of $XIZKE$ to $\overline{XZ}$; thus K maps to G, and the fact is proven.

(d) This follows immediately from $\overline{EI} \perp \overline{ZX}$ and $\overline{FI} \perp \overline{WY}$ and see (a). □

2. Radical Axis

We will show the following result:

Theorem 2. *The radical axis of Ω and ω is the G-midline in $\triangle EFG$.*

Let Γ_E be the E-mixtilinear incircle of $\triangle EDC$—that is, the circle tangent to sides $\overline{ED}$ and $\overline{EC}$, and internally tangent to the circumcircle of $\triangle EDC$.

Theorem 3. *The radical center of Γ_E, ω, and Ω is the midpoint of $\overline{FG}$, point P.*

Proof. First, suppose that Γ_E meets $\overline{ED}$ and $\overline{EC}$ at Z_1 and X_1, respectively; it is well-known ([1], [5]) that I lies on $\overline{Z_1X_1}$.

Now, as $\overline{ZZ_1}$ and $\overline{XX_1}$ are the common external tangent segments between ω and Γ_E, it follows that the radical axis of Γ_E and ω is the line passing through the midpoints of $\overline{ZZ_1}$ and $\overline{XX_1}$. Then it must follow that this radical axis passes through P.

Next, we will show that P lies on the radical axis of Ω and Γ_E. Let (EF) denote the circle with diameter $\overline{EF}$. We begin with a lemma:

Lemma 4. *(EF) is orthogonal to both Ω and Γ_E.*

Proof. We begin by noting that as K was defined to be the Miquel Point, $KADF$ is cyclic; thus inversion about E swaps the pairs $\{K, F\}, \{A, D\}$, and $\{B, C\}$.

Note also that EIF is a right triangle (see Proposition 1(d)) with K as the foot of the I-altitude, so $EK \cdot EF = EI^2$; thus I maps to itself under this inversion.

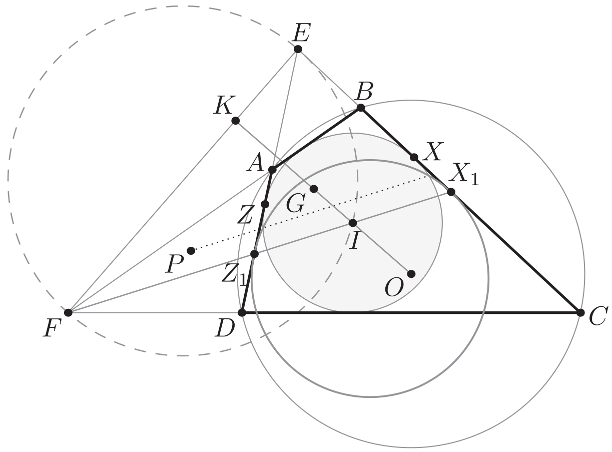

Figure 2. P lies on the radical axis of Γ_E and ω. (EF) is orthogonal to Ω and Γ_E.

As EIZ_1 is a right triangle with Z as the foot of the I-altitude, it follows that the same inversion swaps $\{Z, Z_1\}$, and similarly that it swaps $\{X, X_1\}$. Thus Γ_E swaps with ω and (EF) swaps with $\overline{OIGK}$; it follows that (EF) and Γ_E are indeed orthogonal, because $\overline{OIGK}$ passes through the center of ω.

Also, as Ω remains invariant under the inversion and $\overline{OIGK}$ passes through its center too, Ω and (EF) are also orthogonal. The lemma is proven. □

Let N be the center of (EF); or, that is, the midpoint of $\overline{EF}$. It follows from our lemma that N has equal power with respect to Γ_E and Ω; thus N lies on the radical axis of Γ_E and Ω.

We will now prove one more lemma:

Lemma 5. *If O_E is the center of Γ_E, then O, O_E, and F are collinear.*

Proof. We will show, equivalently, that $\overline{EG}$ is the polar of F with respect to both Γ_E and Ω.

It is obvious that $\overline{EG}$ is the polar of F with respect to Ω: that follows immediately from Brokard's Theorem applied to cyclic quadrilateral $ABCD$.

We now observe that F lies on $\overline{Z_1X_1}$ as $\overline{IF} \perp \overline{IE}$ and $\overline{Z_1IX_1} \perp \overline{EI}$. Thus E lies on the polar of F with respect to Γ_E.

To show that G lies on this polar, we will show equivalently that $(\overline{EF}, \overline{EG}; \overline{EZ_1}, \overline{EX_1})$ is a harmonic bundle. Suppose that $\overline{ZGX}$ meets $\overline{EF}$ again at J; we want to show that $(J, G; Z, X)$ is harmonic, because then we are done.

Observe that $\angle GKJ = 90°$; also note that I is the midpoint of arc $\widehat{ZX}$ not containing E on the circumcircle of $EKZIX$. Thus $\overline{KG}$ bisects $\angle ZKX$; by a well-known condition ([3], Lemma 9.18), this implies that $(J, G; Z, X)$ is indeed harmonic. Thus we have shown that $\overline{EG}$ is the polar of F with respect to both Γ_E and Ω.

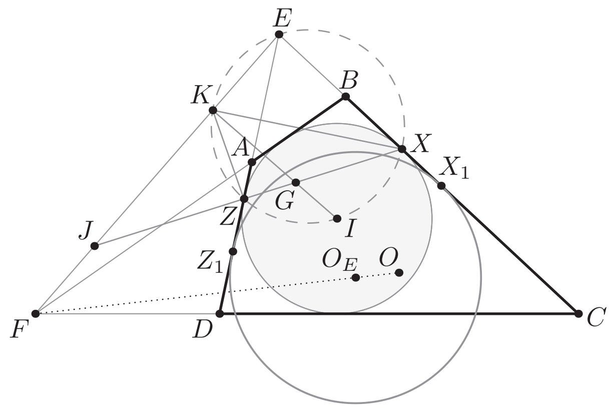

Figure 3. O, O_E and F are collinear.

Finally, this implies that $\overline{EG} \perp \overline{FO}$ and $\overline{EG} \perp \overline{FO_E}$; thus O, O_E, and F are collinear, as desired. □

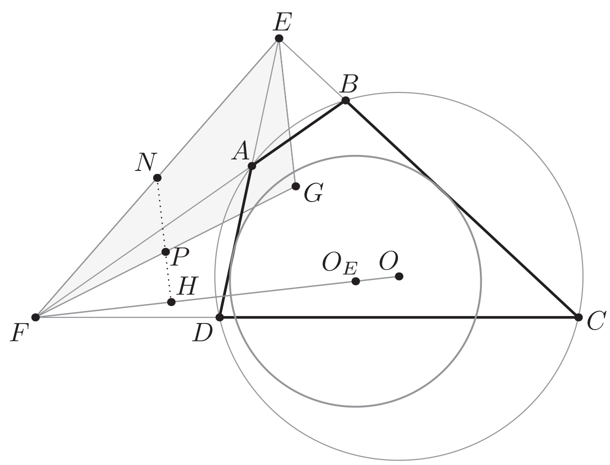

Figure 4. P lies on the radical axis of Γ_E, Ω.

We know that the radical axis of two circles is a line perpendicular to the line joining their centers. Thus, as we already know N to lie on the radical axis of Γ_E and Ω, it follows that the radical axis is the line through N perpendicular to $\overline{OO_EF}$, which is just the line through N parallel to $\overline{EG}$. This line obviously passes through point P, as $\overline{HN}$ is actually the F-midline in $\triangle EFG$.

Thus we have shown that P lies on the radical axis of Γ_E and Ω and of Γ_E and ω; it follows indeed that P is the radical center of Γ_E, ω, and Ω, as desired. □

Having proven this, we may conclude a similar theorem, by symmetry. Let Γ_F be the F-mixtilinear incircle of $\triangle FBC$.

Theorem 6. *The radical center of* Γ_F, ω, *and* Ω *is the midpoint of* $\overline{EG}$, *point* Q.

Now, we are finally ready to show Theorem 2.

Proof. From Theorem 3 and Theorem 6 we see that P and Q both lie on the radical axis of Ω and ω. Thus $\overline{PQ}$—indeed the G-midline in $\triangle EFG$—is the radical axis of Ω and ω, as desired. □

It is worth noting that every vertex of the medial triangle of $\triangle EFG$ is a radical center, as N is also the radical center of Γ_E, Γ_F, and Ω.

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Division of an Angle into Equal Parts and Construction of Regular Polygons by Multi-Fold Origami

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Abstract. This article analyses geometric constructions by origami when up to n simultaneous folds may be done at each step. It shows that any arbitrary angle can be m-sected if the largest prime factor of m is $p \leq n+2$. Also, the regular m-gon can be constructed if the largest prime factor of $\phi(m)$ is $q \leq n+2$, where ϕ is Euler's totient function.

1. Introduction

Two classic construction problems of plane geometry are the division of an arbitrary angle into equal parts and t he construction of regular polygons [14]. It is well known that the use of straight edge and compass allows for the bisection of angles and the constructions of regular m-gons if $m = 2^a p_1 p_2 \cdots p_k$, where $a, k \geq 0$ and each p_i is a distinct odd prime of the form $p_i = 2^{b_i} + 1$. It is also known that origami extends the constructions by allowing for the trisection of angles and the constructions of regular m-gons if $m = 2^{a_1} 3^{a_2} p_1 p_2 \cdots p_k$, where $a_1, a_2, k \geq 0$ and each p_i is a distinct prime of the form $p_i = 2^{b_{i,1}} 3^{b_{i,2}} + 1 > 3$ [1].

Standard origami constructions are performed by a sequence of elementary single-fold operations, one at a time. Each elementary operation solves a set of specific incidences constraints between given points and lines and their folded images [1, 2, 8]. A total of eight elementary operations may be defined and stated as in Table 1 [12]. The operations can solve arbitrary cubic equations [3, 7], and therefore they can be applied to related construction problems such as the duplication of the cube [15] and those mentioned above [3, 4, 5].

The range of origami constructions may be extended further by using multi-fold operations, in which up to n simultaneous folds may be performed at each step [2], instead of single folds. In the case of $n = 2$, the set of possible elementary operations increases to 209 or more (the exact number has still not been determined). It has been shown that 2-fold origami allows for the geometric solution of arbitrary septic equations [9], quintisection of an angle [10] and construction of the regular hendecagon [13].

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Table 1. Single-fold operations [12]. $\mathcal{O}$ denotes the medium in which folds are performed; e.g., a sheet of paper, fabric, plastic, metal or any other foldable material.

#	Operation
1	Given two distinct points P and Q, fold $\mathcal{O}$ to place P onto Q.
2	Given two distinct lines r and s, fold $\mathcal{O}$ to align r and s.
3	Fold along a given a line r.
4	Given two distinct points P and Q, fold $\mathcal{O}$ along a line passing through P and Q.
5	Given a line r and a point P, fold $\mathcal{O}$ along a line passing through P to reflect r onto itself.
6	Given a line r, a point P not on r and a point Q, fold $\mathcal{O}$ along a line passing through Q to place P onto r.
7	Given two lines r and s, a point P not on r and a point Q not on s, where r and s are distinct or P and Q are distinct, fold $\mathcal{O}$ to place P onto r, and Q onto s.
8	Given two lines r and s, and a point P not on r, fold $\mathcal{O}$ to place P onto r, and to reflect s onto itself.

Thus, the purpose of this article is to analyze the general case of n-fold origami with arbitrary $n \geq 1$ and determine what angle divisions and regular polygons can be obtained.

2. Single- and multi-fold origami

An n-fold elementary operation is the resolution of a minimal set of incidence constraints between given points, lines, and their folded images, that defines a finite number of sets of n fold lines [2]. For the case of $n = 1$, all possible elementary operations are those listed in Table 1. An example of operation for $n = 2$ is illustrated in Fig. 1.

Any number of n_i-fold operations, $i = 1, 2, \ldots, k$, may be gather together and considered as a unique n-fold operation, with $n = \sum_{i=1}^{k} n_i$. Thus, we define n-fold origami as the construction tool consisting of all the k-fold elementary operations, with $1 \leq k \leq n$.

The medium on which all folds are performed is assumed to be an infinite Euclidean plane. Points are referred by their Cartesian xy-coordinates or by identifying them as complex numbers, as convenient. A point or complex number is said to be *n-fold constructible* iff it can be constructed starting from numbers 0 and 1 and applying a sequence of n-fold operations. It has been shown that the set of constructible numbers in $\mathbb{C}$ by single-fold origami is the smallest subfield of $\mathbb{C}$ that is closed under square roots, cube roots and complex conjugation [1]. An immediate corollary is that the field $\mathbb{Q}$ of rational numbers is n-fold constructible, for any $n \geq 1$.

The present analysis is based on the following version of a previous theorem by Alperin and Lang [2].

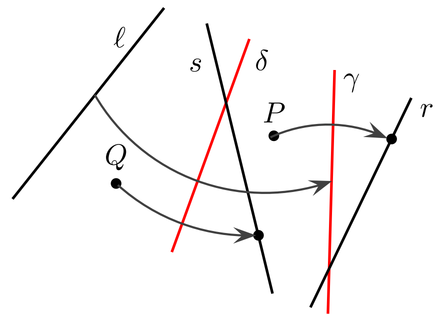

Figure 1. A two-fold operation [13]. Given two points P and Q and three lines ℓ, r, s, simultaneously fold along a line γ to place P onto r, and along a line δ to place Q onto s and to align ℓ and γ.

Theorem 1. *The real roots of any mth-degree polynomial with n-fold constructible coefficients are n-fold constructible if $m \leq n + 2$.*

Proof. The real roots of any mth-degree polynomial may be obtained by Lill 's method [11, 7, 17]. It consists of defining first a right-angle path from and origin O to a terminus T, where the lengths and directions of the path's segments are given by the non-zero coefficients of the polynomial. Next, a second right-angle path with m segments between O and T is constructed by folding, and this construction demands the execution of $m - 2$ simultaneous folds, if $m \geq 3$, or a single fold, if $m \leq 3$. The first intersection (from O) between both paths is the sought solution.

Details of the method may be found in the cited references. An example for solving $x^5 - a = 0$ is shown in Fig. 2. □

It must be noted that the roots of $5th$- and $7th$-degree polynomials may be obtained by 2-fold origami, instead of the 3- and 5-fold origami, respectively, predicted by the above theorem [16, 9]. Therefore, Theorem 1 only posses a sufficient condition on the number of simultaneous folds required.

3. Angle section

Let us consider first the case of division into any prime number of parts.

Lemma 2. *Any angle may be divided into p equal parts by n-fold origami if p is a prime and $p \leq n + 2$.*

Proof. Let ℓ be a line forming an angle θ with the x-axis on the plane. Then, point $P(\cos\theta, 0)$ may be constructed as shown in Fig. 3.

Consider next the multiple angle identity

$$\cos(p\alpha) = T_p(\cos\alpha) \tag{1}$$

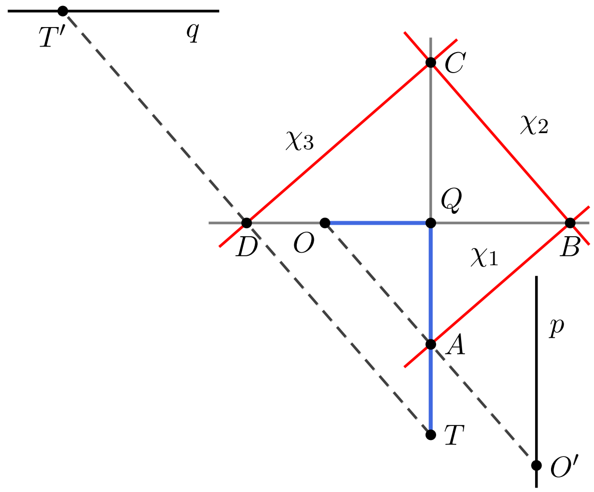

Figure 2. Geometrical solution of $x^5 - a = 0$ by 3-fold origami. Set perpendicular segments $\overline{OQ}$ and $\overline{QT}$ with respective lengths 1 and a, line p parallel to $\overline{QT}$ at a distance of 1, and line q parallel to $\overline{OQ}$ at a distance of a. Next, construct Lill's path $\overline{OA}$, $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DT}$ by performing three simultaneous folds: fold χ_1 places point O onto line p, fold χ_2 is perpendicular to χ_1 and passes through the intersection of χ_1 with the direction line of $\overline{OQ}$ (point B), and fold χ_3 is perpendicular to χ_2, passes through the intersection of χ_2 with the direction line of $\overline{QT}$ (point C), and places point T onto line q. Point A is at the intersection of χ_1 with the direction line of QT, and the length of $\overline{QA}$ is $\sqrt[5]{a}$.

where T_p is the pth Chebyshev polynomial of the first kind, defined by

$$T_0(x) = 1, \tag{2}$$

$$T_1(x) = x, \tag{3}$$

$$T_{p+1}(x) = 2xT_p(x) - T_{p-1}(x). \tag{4}$$

Letting $\theta = p\alpha$, then Eq. (1) is a pth-degree polynomial equation on $x = \cos(\theta/p)$ with integer (constructible) coefficients. According to Theorem 1, the equation may be solved by $p-2$-fold origami, if $p \geq 3$, or single-fold origami, if $p \leq 3$. Then, a line ℓ' forming an angle θ/p may be constructed from $\cos(\theta/p)$ by reversing the procedure in Fig. 3. $\square$

The lemma is easily extended to the general case of division into an arbitrary number of parts.

Theorem 3. *Any angle may be divided into $m \geq 2$ equal parts by n-fold origami if the largest prime factor p of m satisfies $p \leq n + 2$.*

Proof. Let $m = p_1 p_2 \cdots p_k$, where each p_i is a prime and $p_i \leq n + 2$. Then, the theorem is proved by induction over k and applying Lemma 2. $\square$

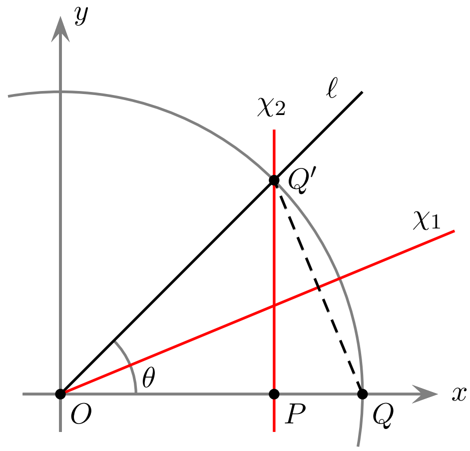

Figure 3. Construction for Lemma 2. Given points $O(0,0)$, $Q(1,0)$, and line ℓ forming an angle θ with $\overline{OQ}$: (1) fold along a line (χ_1) to place ℓ onto $\overline{OQ}$, and next (2) fold along a perpendicular (χ_2) to $\overline{OQ}$ passing through Q'. The intersection of $\overline{OQ}$ and χ_2 is $P = (\cos\theta,\, 0)$.

Again, we remark that the above theorem only posses a sufficient condition on the number of multiple folds required. For $m = 5$, it predicts $n = 3$; however, a solution using only 2-fold origami has been published [10].

Example 1. Any angle may be divided into 11 equal parts by 9-fold origami.

4. Regular polygons

The analysis follows similar steps to previous treatments on geometric constructions by single-fold origami and other tools [6, 18, 19].

Consider an m-gon ($m \geq 3$) circumscribed in a circle with radius 1 and centered at the origin in the complex plane. Its vertices are given by the mth-roots of unity, which are the solutions of $z^m - 1 = 0$.

Let us recall that an mth root of unity is primitive if it is not a kth root of unity for $k < m$. The primitive mth roots are solutions of the mth cyclotomic polynomial

$$\Phi_m(z) = \prod_{\substack{1\leq k\leq m \\ \gcd(k,m)=1}} \left(z - e^{2i\pi k/m}\right). \tag{5}$$

This polynomial has degree $\phi(m)$, where ϕ is Euler's totient function; i.e., $\phi(m)$ is the number of positive integers $k \leq m$ that are coprime to m. A property of any mth primitive root ξ_m is that all the m distinct roots may be obtained as ξ_m^k, for $k = 0, 1, \ldots, m-1$. This property provides a convenient way to construct the regular m-gon.

Lemma 4. *The regular m-gon is n-fold constructible if a primitive mth root of unity is n-fold constructible.*

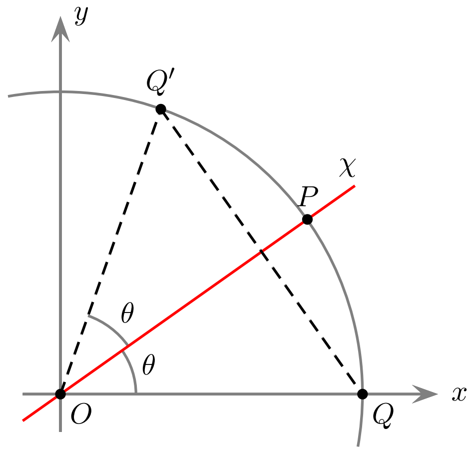

Figure 4. Given $O = (0, 0)$, $Q = (1, 0)$ and $P = (\cos\theta, \sin\theta)$, a fold along line χ passing through O and Q places Q on $Q' = (\cos 2\theta, \sin 2\theta)$.

Proof. Let $\xi_m = e^{i\theta}$ be a primitive mth root of unity. Then, $\xi_m^k = e^{ik\theta}$ and therefore all roots may be constructed from ξ_m by applying rotations of an angle θ around the origin. The rotations may be performed by single-fold origami, as shown in Fig. 4. Once all the roots have been constructed, segments connecting consecutive roots may be created by single folds. □

Next, we state a sufficient condition for the n-fold constructability of a number $\alpha \in \mathbb{C}$.

Lemma 5. *A number $\alpha \in \mathbb{C}$ is n-fold constructible if there is a field tower $\mathbb{Q} = F_0 \subseteq F1 \subseteq \cdots \subseteq F_{k-1} \subseteq F_k \subset \mathbb{C}$, such that $\alpha \in F_k$ and $[F_j : F_{j-1}] \in \{2, 3, \ldots, n+2\}$for each $j = 1, 2, \ldots, k$.*

Proof. The theorem is proved by induction over k. If $k = 0$, then $\alpha \in F_0 = \mathbb{Q}$ is constructible by single-fold origami [1], and therefore is n-fold constructible for any $n \geq 1$.

Next, assume that F_{k-1} is n-fold constructible. Let $\alpha \in F_k$, then α is a root of a minimal polynomial p with coefficients in F_{k-1}, and its degree divides $[F_k : F_{k-1}]$. If α is real, then it may be constructed by n-fold origami (Theorem 1). If not, then its complex conjugate $\bar{\alpha}$ is also a root of p. The real and imaginary parts of α, $\Re(\alpha) = (\alpha + \bar{\alpha})/2$ and $\Im(\alpha) = (\alpha - \bar{\alpha})/2$, respectively, are in F_k and therefore they are real roots of minimal polynomials $p_{\Re}$ and $p_{\Im}$ with coefficients in F_{k-1}. Again, the degrees of both $p_{\Re}$ and $p_{\Im}$ divide $[F_k : F_{k-1}]$ and hence $\Re(\alpha)$ and $\Im(\alpha)$ are n-fold origami constructible. □

Using the above lemmas, we finally obtain a sufficient condition for the constructability of the regular m-gon.

Theorem 6. *The regular m-gon is n-fold constructible if the largest prime factor p of $\phi(m)$ satisfies $p \leq n + 2$.*

Proof. Let $\phi(m) = p_1 p_2 \cdots p_k$, where each p_i is a prime and $p_i \leq n+2$, and ξ_m be a primitive mth root of unity. The Galois group Γ of the extension $\mathbb{Q}(\xi_m) : \mathbb{Q}$ is abelian and has order $\phi(m)$ [18]. Therefore, it has a series of normal subgroups $1 = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_r = \Gamma$ where each factor $\Gamma_{j+1}/|\Gamma_j$ is abelian and has order p_i for some $1 \leq i \leq k$. By the Galois correspondence, there is a field tower $\mathbb{Q}(\xi_m) = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_r = \mathbb{Q}$ such that $[K_j : K_{j+1}] = p_i$. Thus, by Lemma 5, ξ_m is n-fold constructible, and by Lemma 4, the m-gon is n-fold constructible. □

Example 2. The totient of 199 is $\phi(199) = 2 \cdot 3^2 \cdot 11$. Therefore, the regular 199-gon may be constructed by 9-fold origami.

5. Final comments

Gleason [6] noted that any regular m-gon may be constructed if, in addition to straight edge and compass, a tool to p-sect any angle is available for every prime factor p of $\phi(m)$. The above results match his conclusion: if n-fold origami can p-sect any angle for every prime factor p of $\phi(m)$, then, by Lemma 2, the largest prime factor is $p_{\max} \leq n+2$. By Theorem 6, the m-gon can be construted.

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Some Archimedean Circles in an Arbelos

Le Viet An and Emmanuel A. J. García

Abstract. We construct six circles congruent to the Archimedean twin circles in the arbelos.

For a point C on the segment AB, let AC, BC and AB be semicircles constructed on the same side. The area bounded by the three semicircles is called an arbelos. The perpendicular passing through C divides the arbelos into two curvilinear triangles with congruent incircles. Circles congruent to those circles are said to be Archimedean. Let a and b be the radii of semicircles AC and BC, respectively. The common radius of Archimedean circles is given by $\frac{ab}{a+b}$. In this paper we introduce some Archimedean circles, which we hope to be new. More examples of Archimedean circles can be found in [1] and [2].

Theorem 1. *In an arbelos, from E, draw a tangent line to semicircle AC, at F. Construct G similarly. Let H be the intersection of tangent lines EF and DG. Let circle (H, CH) cut semicircles AC and BC at I and J, respectively. Call I' the orthogonal projection of I onto AB. Construct J' similarly. Let S be the center of circle bounded by circle (A, AJ'), circle (B, BI'), and semicircle AC. Construct circle centered at T similarly. Then, circles centered at S and T are Archimedean twins (See Figure 1).*

Proof. Let CI cut DH at M. Let CJ cut EH at N. Notice that $CDIH$ is a kite, then, CI is perpendicular to DH and, as a consequence, CM is parallel to $EG = b$. As $\triangle CDM \sim \triangle EDG$, we have

$$\frac{b}{\frac{CI}{2}} = \frac{a+b}{a}$$

$$CI = \frac{2ab}{a+b}.$$

Remark. Notice that circle with diameter CI is Archimedean. This is Archimedean circle O_{50_a} in [2].

Notice also that $\triangle CII' \sim \triangle EDG$, then,

$$\frac{CI'}{b} = \frac{CI}{a+b} = \frac{\frac{2ab}{a+b}}{a+b} = \frac{2ab}{(a+b)^2}$$

$$CI' = \frac{2ab^2}{(a+b)^2}.$$

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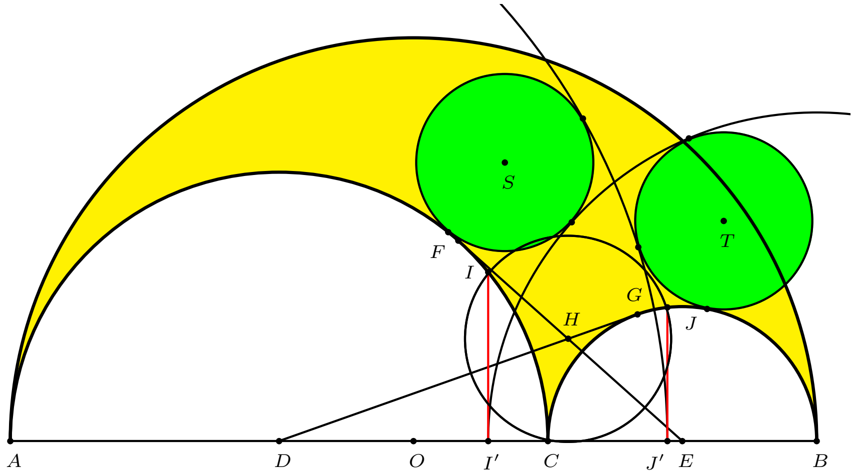

Figure 1. Circles with centers S and T are Archimedean twins

Similarly, notice that $\triangle CEN \sim \triangle DEF$ (See Figure 2), then,

$$\frac{a}{CN} = \frac{a+b}{b}$$

$$CN = \frac{ab}{a+b}.$$

Notice that $\triangle CEN \sim \triangle CJJ'$, then,

$$\frac{CJ'}{\frac{ab}{a+b}} = \frac{\frac{2ab}{a+b}}{b} = \frac{2a}{a+b}$$

$$CJ' = \frac{2a^2b}{(a+b)^2}.$$

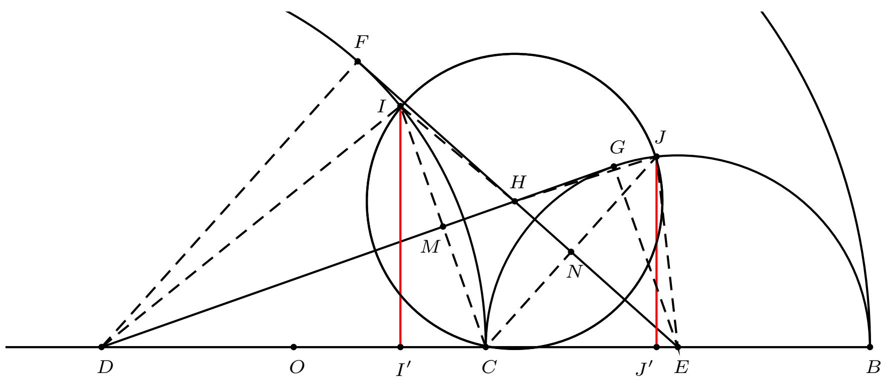

Figure 2. $\triangle CEN \sim \triangle DEF$ and $\triangle CEN \sim \triangle CJJ'$

Focusing on $\triangle ASB$ and cevian DS, if we call x the radius of circle centered at S, from the Stewart's theorem we have

$$\left[2a+\frac{2a^2b}{(a+b)^2}-x\right]^2(a+2b)+\left[2b+\frac{2ab^2}{(a+b)^2}+x\right]^2a=(2a+2b)[(a+x)^2+a(a+2b)].$$

Expanding, solving for x and simplifying we get

$$x=\frac{ab}{a+b}.$$

□

A similar reasoning goes for circle centered at T.

Theorem 2. *In an arbelos, let semicircle DE cut semicircles AC and BC in F and G, respectively. Let H be on chord DF such that $\angle CHF = 90°$. Similarly, construct I. Then, circles with radii HF and IG are Archimedean (See Figure 3).*

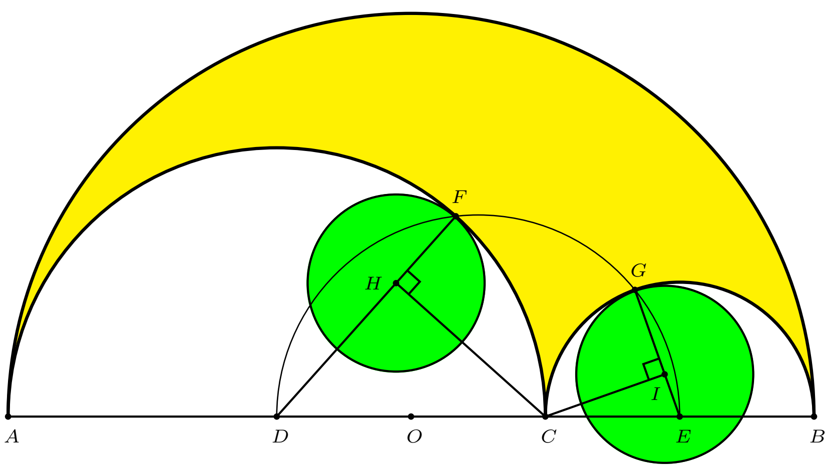

Figure 3. Circles with radii HF and IG are Archimedean twins.

Proof. Because of Thales's theorem, $DF = a$ is perpendicular to EF. As a consequence, $\triangle CDH \sim \triangle EDF$. It follows

$$\frac{a}{DH}=\frac{a+b}{a}$$

$$DH=\frac{a^2}{a+b}$$

$$FH=a-\frac{a^2}{a+b}=\frac{ab}{a+b}.$$

□

A similar reasoning must show the congruency for circle with radius GI.

Theorem 3. *In an arbelos, let the circle with center at B and radius BC meet semicircle AB in F. Similarly, construct I, on semicircle AB. Let AF intersect semicircle AC in G. Let G' be the orthogonal projection of G onto AB. The circle centered at R is bounded by semicircle AB, circle (A, AC) and the line GG'. Similarly, construct the circle centered at S. Then, the circles centered at R and S are Archimedean (See Figure 4).*

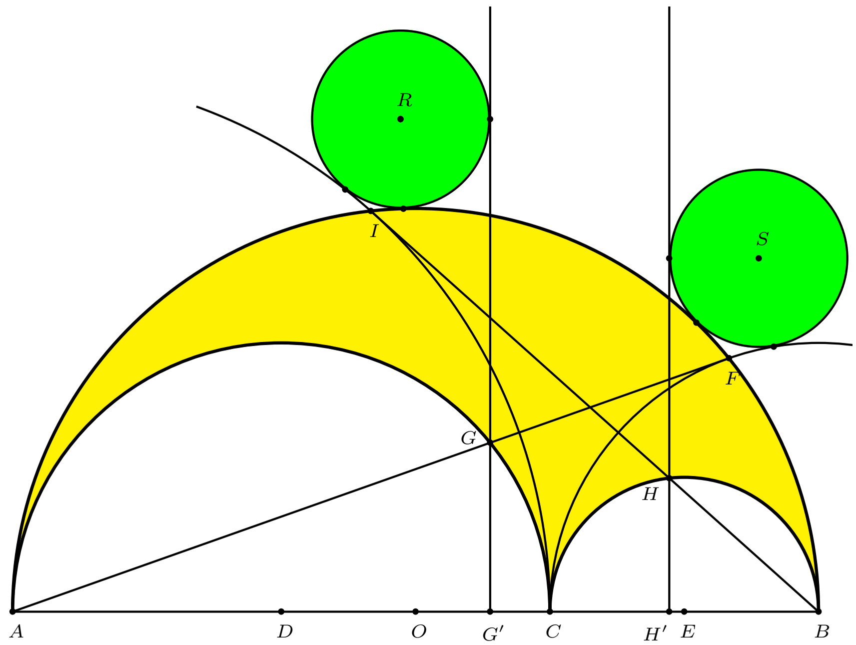

Figure 4. Circles with centers R and S are Archimedean twins.

Proof. Notice that because of Thales's theorem $\angle AGC = \angle AFB = 90°$. Therefore, $\triangle AGC \sim \triangle AFB$. Thus, we have

$$\frac{GC}{2b} = \frac{2a}{2a+2b}$$

$$GC = \frac{2ab}{a+b}.$$

Remark. Notice that the circle with radius $\frac{GC}{2}$ is Archimedean.

$$AF = \sqrt{(2a+2b)^2 - 4b^2} = 2\sqrt{a^2+2ab}.$$

$$\frac{AF}{AG} = \frac{2a+2b}{2a}$$

$$AG = \frac{2a\sqrt{a^2+2ab}}{a+b}.$$

$$GC = \sqrt{4a^2 - AG^2} = \sqrt{4a^2 - \frac{4a^2(a^2+2ab)}{(a+b)^2}}$$

$$GC = \frac{2a\sqrt{4ab+b^2}}{a+b}.$$

As $\triangle AGG' \sim \triangle AGC$, it follows that

$$\frac{AG'}{AG} = \frac{AG}{AC}$$

$$AG' = \frac{2a(a^2+2ab)}{(a+b)^2}.$$

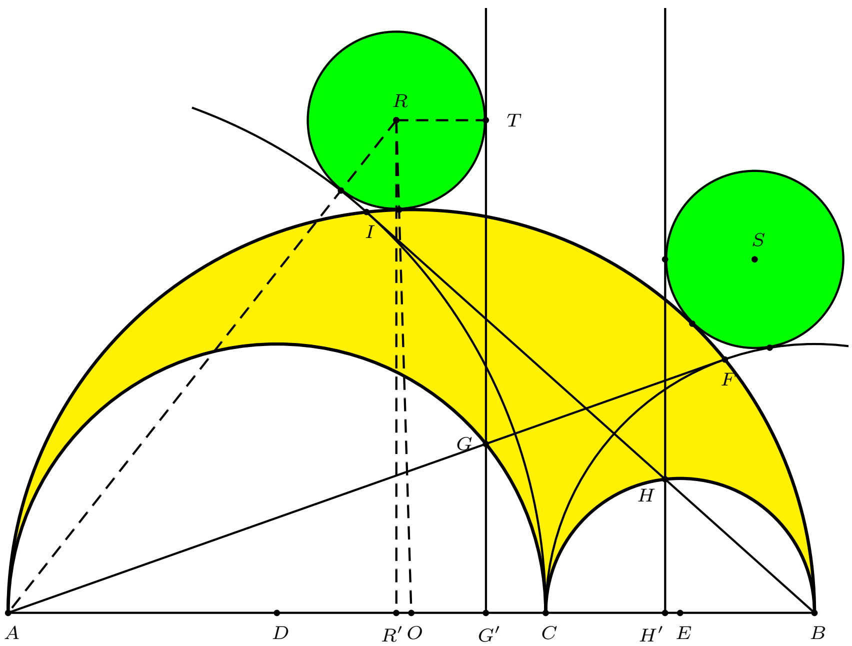

Figure 5. R' is the orthogonal projection of R onto AB.

Let R' be the orthogonal projection of R onto AB. If we call x the radius of circle centered at R, by the Pythagorean theorem (See Figure 5),

$$RR' = \sqrt{(2a+x)^2 - (AG'-x)^2}.$$

Focusing on triangle $\triangle ORR'$, by the Pythagorean theorem,

$$[AO - (AG'-x)]^2 + RR'^2 = OR^2.$$

If we replace the segments by their expressions in terms of a and b, we have the following equation

$$\left[(a+b) - \frac{2a(a^2+2ab)}{(a+b)^2} + x\right]^2 + (2a+x)^2 - \left[\frac{2a(a^2+2ab)}{(a+b)^2} - x\right]^2 = (a+b+x)^2.$$

Expanding, solving for x and simplifying we get

$$x = \frac{ab}{a+b}.$$

$\square$

A similar reasoning goes for circle centered at S.

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A Remark on Archimedean Incircles of an Isosceles Triangle

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Abstract. We generalize several Archimedean circles of the arbelos, which are the incircles of an isosceles triangles.

1. Introduction

We consider an arbelos configuration formed by three circles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB (see Figure 1). Let a and b be the radii of α and β, respectively. Circles of radius $r_{\mathrm{A}} = ab/(a+b)$ are said to be Archimedean. In [3], a special Archimedean circle is considered, which is the incircle of an isosceles triangle formed by a point lying outside of the circle γ and the two points of tangency of γ from the point. Similar Archimedean circles are also considered in [2]. In this paper we generalize those circles.

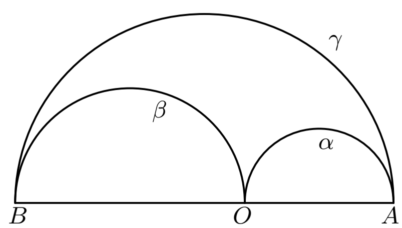

Figure 1.

2. Circles generated by a point and a circle

In this section we generalize the Archimedean circles in [2, 3].

Theorem 1. *Let δ be a circle of radius d. For a point E lying outside of δ, let F and G be the points of tangency of the tangents of δ from E and $e = |ES|$, where S is the closest point on δ to E. Then the following statements hold.*
(i) *The point S coincides with the incenter of the triangle EFG.*
(ii) *The inradius of the triangle EFG equals $de/(d+e)$.*

Proof. Assume that D is the center of δ, M is the midpoint of FS, and T is the midpoint of FG (see Figure 2). Since the triangles DMF and FTS are similar,

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$2\angle SFT = \angle FDS$, while $2\angle SFE = \angle FDS$ by the inscribed angle theorem. Hence FS bisects $\angle EFG$. Therefore S is the incenter of EFG. This proves (i). If x is the inradius of EFG, $x/e = |DF|/|DE| = d/(d+e)$. This implies $x = de/(d+e)$. □

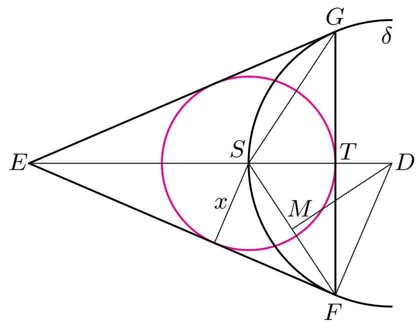

Figure 2.

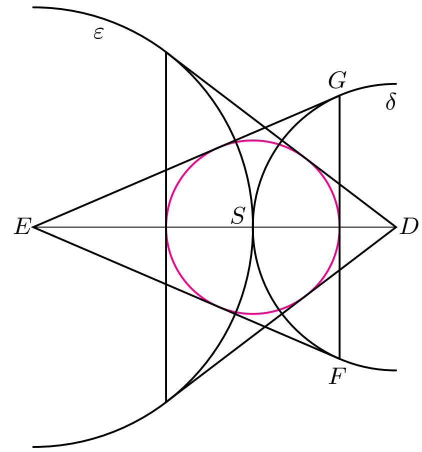

Figure 3.

Let ε be the circle of center E passing through S in Theorem 1. As the part (ii) shows, the inradius of EFG is symmetric in d and e. Hence the same circle is also constructed from the point D and the circle ε (see Figure 3).

Definition. If E is a point lying outside of a circle δ and the tangents of δ from D touches δ at points F and G, then we call the incircle of EFG the circle generated by D and δ. Also we say that the circle is generated by the circles δ and ε, where ε is the circle of center E touching δ externally.

Now we consider the arbelos. Let O_a, O_b and O_c be the centers of the circles α, β and γ, respectively.

Corollary 2. *The following statement hold.*
(i) *The circle generated by a point P and the circle α (resp. β) is Archimedean if and only if $|PO_a| = a + b$ (resp. $|PO_b| = a + b$).*
(ii) *The circle generated by a point P and the circle γ is Archimedean if and only if $|PO_c| = a + b + d$, where $d = ab(a + b)/(a^2 + ab + b^2)$.*

Proof. The part (i) is obvious by Theorem 1(ii). Solving the equation $(a+b)x/(a+b+x) = r_{\mathrm{A}}$ for x, we get $x = d$. This proves (ii). □

Notice that d is the inradius of the arbelos. The two Archimedean circles generated by a point and one of the circles α and β given in [2] are obtained in a special case in the event of Corollary 2(i). Also the Archimedean circle generated by a point and the circle γ given in [3] is obtained in a special case in the event of Corollary 2(ii). Corollary 2(i) gives an interesting special case (see Figure 4).

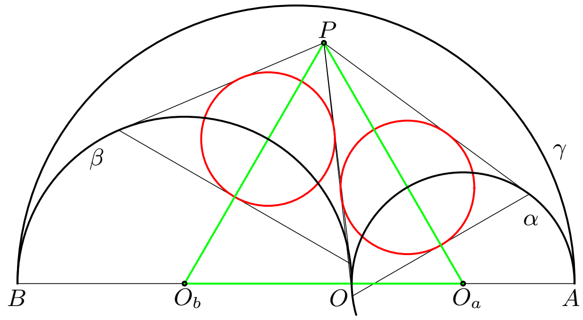

Figure 4.

Corollary 3. *If P is a point lying on the circle of center O_a (resp. O_b) congruent to γ, then the circle generated by P and α (resp. β) is Archimedean. If PO_aO_b is an equilateral triangle, then the circles generated by P and each of α and β are Archimedean.*

The Archimedean circle generated by α and β can be found in [1], which is denoted by W_8.

References

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