Junior problems

J235. In the equality $\sqrt{ABCDEF} = DEF$, different letters represent different digits. Find the six-digit number $ABCDEF$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J236. Let ABC be a triangle and let $ABRS$ and $ACXY$ be the two squares constructed on sides AB and AC which are directed towards the exterior of the triangle. If U is the circumcenter of triangle SAY, prove that the line AU is the A-symmedian of triangle ABC.

Proposed by Cosmin Pohoata, Princeton University, USA

J237. Prove that the diameter of the incircle of a triangle ABC is equal to $\frac{AB-BC+CA}{\sqrt{3}}$ if and only if $\angle BAC = 60^\circ$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J238. Given a real number $\alpha \in (0,1)$, prove that there is a positive integer N such that for any N points in the plane, no three collinear, there is a triangle with one its angles greater than $\pi\alpha$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J239. Let a and b be real numbers so that $2a^2 + 3ab + 2b^2 \le 7$. Prove that

$$\max\{2a+b, 2b+a\} \le 4.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J240. Let ABC be an acute triangle with orthocenter H. Points H_a, H_b, and H_c in its interior satisfy

$$\begin{array}{lllllllll}
\angle BH_aC & = & 180^\circ - \angle A, & \angle CH_aA & = & 180^\circ - \angle C, & \angle AH_aB & = & 180^\circ - \angle B, \\
\angle CH_bA & = & 180^\circ - \angle B, & \angle AH_bB & = & 180^\circ - \angle A, & \angle BH_bC & = & 180^\circ - \angle C, \\
\angle AH_cB & = & 180^\circ - \angle C, & \angle BH_cC & = & 180^\circ - \angle B, & \angle CH_cA & = & 180^\circ - \angle A.
\end{array}$$

Prove that the points H, H_a, H_b, H_c are concyclic.

Proposed by Michal Rolinek, Charles University, Czech Republic

Senior problems

S235. Solve the equation

$$\frac{8}{\{x\}} = \frac{9}{x} + \frac{10}{[x]},$$

where $[x]$ and $\{x\}$ denote the greatest integer less or equal than x and the fractional part of x, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S236. Consider all cyclic quadrilaterals $ABCD$ inscribed in a given circle ω for which AB always passes through a given point K and whose diagonals intersect at a given point P. Prove that CD also passes through some fixed point.

Proposed by Josef Tkadlec, Charles University, Czech Republic

S237. Harry Potter, in one of his journeys, stumbled upon a magic beads string. To achieve his goal, he must take out all the beads on this string. It is known that he can only remove one bead at a time, from the left part of the string. The string contains beads of 7 different colors labeled $1, 2, \ldots, 7$ and it is under the following spell: whenever Harry removes the first bead from the left, after each bead of color $1 \leq i \leq 6$ left on the string, new beads of colors $i + 1, \ldots, 7$ will pop-up in this order. For example, if on the string we have the colours $1, 4, 3, 7$, after Harry takes out the first bead, we will have $4, 5, 6, 7, 3, 4, 5, 6, 7, 7$. Does Harry have any chance to complete his task regardless the beads string he starts with?

Proposed by Catalin Turcas, University of Warwick, United Kingdom

S238. Let ABC be a triangle with incenter I and let D, E, F be the tangency points of the incircle with sides BC, CA, AB, respectively. Let M be the midpoint of the arc BC of the circumcircle which contains vertex A. Furthermore, let P and Q be the midpoints of segments DE and DF. Prove that MI bisects the segment PQ.

Proposed by Cosmin Pohoata, Princeton University, USA

S239. Solve in integers the equation

$$2(x^3 + y^3 + z^3) = 3(x + y + z)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S240. Let ABC be a triangle with circumcircle Γ and let M, N, P be points on the sides BC, CA, AB, respectively. Let A', B', C' be the intersections of AM, BN, CP with Γ different from the vertices of the triangle. Prove that

$$\frac{MA}{MA'} + \frac{MB}{MB'} + \frac{MC}{MC'} \geq 4\left(2 - \frac{r}{R}\right)^2,$$

where R and r are the circumradius and the inradius of triangle ABC.

Proposed by Marius Stanean, Zalau, Romania

Undergraduate problems

U235. Let $a > b$ be positive real numbers and let n be a positive integer. Prove that

$$\frac{(a^{n+1}-b^{n+1})^{n-1}}{(a^n-b^n)^n} > \frac{n}{(n+1)^2}\cdot\frac{e}{a-b},$$

where e is the Euler number.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U236. Let $f(X)$ be an irreducible polynomial in $\mathbb{Z}[X]$. Prove that $f(XY)$ is irreducible in $\mathbb{Z}[X,Y]$.

Proposed by Mircea Becheanu, University of Bucharest, Romania

U237. Let $\mathcal{H}$ be a hyperbola with foci A and B and center O. Let P be an arbitrary point on $\mathcal{H}$ and let the tangent of $\mathcal{H}$ through P cut its asymptotes at M and N. Prove that $PA+PB=OM+ON$.

Proposed by Luis Gonzalez, Maracaibo, Venezuela

U238. Let X be a random variable with median $m=0$, mean μ_X, and variance σ_X^2. Denote by $\sigma_{|X|}^2$ the variance of the random variable $|X|$. Prove that

$$|\mu_X| \le \sigma_{|X|}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U239. Let ABC be a triangle and let P be a point in its plane, not lying on the circumcircle Γ of triangle ABC. Let AP, BP, CP intersect Γ again at points X, Y, Z, respectively. Let the tangents from X to the incircle of ABC meet BC at A_1 and A_2; similarly, define B_1, B_2 and C_1, C_2. Prove that points A_1, A_2, B_1, B_2, C_1, C_2 lie on a conic.

Proposed by Cosmin Pohoata, Princeton University, USA

U240. Let $A \in M_n(\mathbb{Z})$ and let $(a_n)_{n\ge 0}$ be defined by $a_0=1$ and

$$a_{n+1}=\frac{1}{n+1}\sum_{j=0}^{n} a_{n-j}\,\mathrm{tr}(A^{j+1}),\ n\ge 0.$$

Prove that all terms of the sequence $(a_n)_{n\ge 0}$ are integers.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Olympiad problems

O235. Solve in integers the equation

$$xy - 7\sqrt{x^2 + y^2} = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O236. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} + \frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)} \geq \frac{3(a+b+c)}{2(ab+bc+ca)}.$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

O237. Let x, y, z be positive real numbers such that

$$(2x^4 + 3y^4)(2y^4 + 3z^4)(2z^4 + 3x^4) \leq (3x + 2y)(3y + 2z)(3z + 2x).$$

Prove that $xyz \leq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O238. Consider real numbers $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$. It is known that for every real number X there is a pair (a_i, b_i) such that $a_i X + b_i \geq 0$. Prove that there are indices $i, j \in \{1, 2, \ldots n\}$ such that each real number X satisfies at least one of the inequalities $a_i X + b_i \geq 0$, $a_j X + b_j \geq 0$.

Proposed by Andrei Ciupan, Harvard University, USA

O239. Let ABC be a triangle and let D, E, F be the tangency points of its incircle with the sides BC, CA, AB, respectively. Let U be the second intersection of AD with the circumcircle $\mathcal{C}$ of triangle ABC and let X be the tangency point of the A-mixtilinear incircle with the $\mathcal{C}$. Furthermore, let V, W be the midpoints of segments DE and DF. Prove that VW, UX, BC are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

O240. Let m and n be positive integers and let $x = (x_1, \ldots, x_m)$ be a vector of positive real numbers such that $\sum_{i=1}^{m} x_i = 1$. Consider the set Y, defined as

$$Y = \left\{ y = (y_1, \ldots, y_m) \mid y_i \in \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}, \sum_{i=1}^{m} y_i = 1 \right\}.$$

Prove that there is $y^* = (y_1^*, \ldots, y_m^*) \in Y$ such that

$$\sum_{i=1}^{m} |y_i^* - x_i| \leq \frac{m}{2n}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA