Junior problems

J319. Let $0 = a_0 < a_1 < \ldots < a_n < a_{n+1} = 1$ such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\frac{a_1}{a_2 - a_0} + \frac{a_2}{a_3 - a_1} + \cdots + \frac{a_n}{a_{n+1} - a_{n-1}} \geq \frac{1}{a_n}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J320. Find all positive integers n for which $2014^n + 11^n$ is a perfect square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J321. Let x, y, z be positive real numbers such that $xyz(x + y + z) = 3$. Prove that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{54}{(x+y+z)^2} \geq 9\,.$$

Proposed by Marius Stânean, Zalau, Romania

J322. Let ABC be a triangle with centroid G. The parallel lines through a point P situated in the plane of the triangle to the medians $AA^{'}$, $BB^{'}$, $CC^{'}$ intersect lines BC, CA, AB at A_1, B_1, C_1, respectively. Prove that

$$A^{'}A_1 + B^{'}B_1 + C^{'}C_1 \geq \frac{3}{2}PG.$$

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, Romania

J323. In triangle ABC,

$$\sin A + \sin B + \sin C = \frac{\sqrt{5} - 1}{2}.$$

Prove that $\max(A, B, C) > 162°$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J324. Let ABC be a triangle and let X, Y, Z be the reflections of A, B, C in the opposite sides. Let X_b, X_c be the orthogonal projections of X on AC, AB, Y_c, Y_a the orthogonal projections of Y on BA, BC, and Z_a, Z_b the orthogonal projections of Z on CB, CA, respectively. Prove that X_b, X_c, Y_c, Y_a, Z_a, Z_b are concyclic.

Proposed by Cosmin Pohoață, Columbia University, USA

Senior problems

S319. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that for any positive real number t,

$$(at^2+bt+c)(bt^2+ct+a)(ct^2+at+b) \geq t^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S320. Let ABC be a triangle with circumcenter O and incenter I. Let D, E, F be the tangency points of the incircle with BC, CA, AB, respectively. Prove that line OI is perpendicular to angle bisector of $\angle EDF$ if and only if $\angle BAC = 60^\circ$.

Proposed by Marius Stânean, Zalau, Romania

S321. Let x be a real number such that $x^m(x+1)$ and $x^n(x+1)$ are rational for some relatively prime positive integers m and n. Prove that x is rational.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

S322. Let $ABCD$ be a cyclic quadrilateral. Points E and F lie on the sides AB and BC, respectively, such that $\angle BFE = 2\angle BDE$. Prove that

$$\frac{EF}{AE} = \frac{FC}{AE} + \frac{CD}{AD}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

S323. Solve in positive integers the equation

$$x+y+(x-y)^2 = xy.$$

Proposed by Neculai Stanciu and Titu Zvonaru, Romania

S324. Find all functions $f: S \to S$ satisfying

$$f(x)f(y)+f(x)+f(y) = f(xy)+f(x+y)$$

for all $x, y \in S$ when (i) $S = \mathbb{Z}$; (ii) $S = \mathbb{R}$.

Proposed by Prasanna Ramakrishnan, Port of Spain, Trinidad and Tobago

Undergraduate problems

U319. Let A, B, C be the measures (in radians) of the angles of a triangle with circumradius R and inradius r. Prove that

$$\frac{A}{B}+\frac{B}{C}+\frac{C}{A}\leq\frac{2R}{r}-1.$$

Proposed by Nermin Hodžić, Bosnia and Herzegovina and Salem Malikić, Canada

U320. Evaluate

$$\sum_{n\geq 0}\frac{2^n}{2^{2^n}+1}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

U321. Consider the sequence of polynomials $(P_s)_{s\geq 1}$ defined by

$$P_{k+1}(x)=(x^a-1)P_k^{'}(x)-(k+1)P_k(x), k=1,2,\ldots,$$

where $P_1(x)=x^{a-1}$ and a is an integer greater than 1.

1. Find the degree of P_k.
2. Determine $P_k(0)$

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, Romania

U322. Evaluate

$$\sum_{n=1}^{\infty}\frac{16n^2-12n+1}{n(4n-2)!}.$$

Proposed by Titu Andreescu, USA and Oleg Mushkarov, Bulgaria

U323. Let X and Y be independent random variables following a uniform distribution

$$p_X(x)=\begin{cases}1 & 0<x<1,\\ 0 & \text{otherwise.}\end{cases}$$

What is the probability that inequality $X^2+Y^2\geq 3XY$ is true?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U324. Let $f:[0,1]\to\mathbb{R}$ be a differentiable function such that $f(1)=0$. Prove that there is $c\in(0,1)$ such that $|f(c)|\leq|f'(c)|$.

Proposed by Marius Cavachi, Constanta, Romania

Olympiad problems

O319. Let $f(x)$ and $g(x)$ be arbitrary functions defined for all $x \in \mathbb{R}$. Prove that there is a function $h(x)$ such that $(f(x)+h(x))^{2014}+(g(x)+h(x))^{2014}$ is an even function for all $x \in \mathbb{R}$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O320. Let n be a positive integer and let $0 < y_i \le x_i < 1$ for $1 \le i \le n$. Prove that

$$\frac{1-x_1\cdots x_n}{1-y_1\cdots y_n} \le \frac{1-x_1}{1-y_1}+\cdots+\frac{1-x_n}{1-y_n}.$$

Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

O321. Each of the diagonals AD, BE, CF of the convex hexagon $ABCDEF$ divide its area in half. Prove that

$$AB^2+CD^2+EF^2=BC^2+DE^2+FA^2.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O322. Let ABC be a triangle with circumcircle Γ and let M be the midpoint of arc BC not containing A. Lines ℓ_b and ℓ_c passing through B and C, respectively, are parallel to AM and meet Γ at $P \neq B$ and $Q \neq C$. Line PQ intersects AB and AC at X and Y, respectively, and the circumcircle of AXY intersects AM again at N.
Prove that the perpendicular bisectors of BC, XY, and MN are concurrent.

Proposed by Prasanna Ramakrishnan, Port of Spain, Trinidad and Tobago

O323. Prove that the sequence $2^{2^1}+1, 2^{2^2}+1, \ldots 2^{2^n}+1, \ldots$ and an arbitrary infinite increasing arithmetic sequence have either infinitely many terms in common or at most one term in common.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O324. Let a, b, c, d be nonnegative real numbers such that $a^3+b^3+c^3+d^3+abcd=5$. Prove that

$$abc+bcd+cda+dab-abcd \le 3.$$

Proposed by An Zhen-ping, Xianyang Normal University, China