Mathematical Excalibur

Volume 19, Number 1 *April 2014 – August 2014*

Olympiad Corner

Below are the problems of the 2014 International Math Olympiad on July 8 and 9, 2014.

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \ldots + a_n}{n} \leq a_{n+1}.$$

Problem 2. Let $n \geq 2$ be an integer. Consider a $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90^\circ$, $\angle THC - \angle DTC = 90^\circ$. Prove that line BD is tangent to the circumcircle of triangle TSH.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 12, 2014***.

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IMO2014 and Beyond

Leung Tat-Wing

I write this article with three goals in mind: (1) to report on IMO 2014; (2) to give some idea how we can further train our team members and (3) finally and hopefully provide us some help of how to organize IMO 2016.

Itinerary The 55th International Mathematical Olympiad was held in Cape Town, South Africa from 3 July to 13 July, 2014. It took us 13 hours flying from Hong Kong to Johannesburg, waiting for a couple of hours, then another 2 hours' flight to Cape Town. Surely when compared with Argentina and Colombia, it was a much easier trip. Because we have to host IMO 2016, this year several observers (with leaders or deputy leaders) came with us. We have gathered a lot of information in this trip, which will help us tremendously in our preparation. This IMO was held when world cup matches were going on, and we were lucky that we still managed to watch several games, and at better times (6 pm or 10 pm). We missed only the final game, Germany vs Argentina, when we were exactly in our return flight, and I managed to get the result only when we got off the plane.

Weather in South Africa was nice. It was winter, and usually 20˚*C* during the day time and about 10˚*C* during the night. If it was raining, then it got a bit cooler. We first stayed in a hotel, right below a mountain, which I believe belongs to the Table Mountain range. The view, if I may say, is simply majestic. The city structure looks nice. It looks like a decent English town. The hotel is pretty normal and we stayed there for 6 days. Then we moved to the University of Cape Town (UCT) and stayed with the students. Our students arrived Cape Town three days after us, and they were stationed in dormitories of the University all the time. Though accommodation and food were not as good as in the hotel, I believe I can bear it. Only thing is, every entrance of a dormitory in the University is equipped with heavy iron gate and is watched by a security guard, which I found it a bit scary. This reminds me of the security issue in South Africa. Of course, it is a country with high unemployment rate (25%), high Gini index (6.3), and there are racial problems and other things.

Leaders spent three days to select the 6 problems from a shortlist of 30 problems, then refined the wordings and wrote the English version and other official versions. They discussed the marking schemes proposed by the Problem Group and coordinators, and approved the marking schemes. The students then arrived, and the next day leaders and contestants together participated in the Opening Ceremony, with leaders and contestants still separated so that they could not communicate during the Ceremony. Students then wrote the two 4.5 hour contests on the mornings of the next two days, while leaders had the time to do a bit of sight-seeing and the like. After the two contests, leaders were then moved to the University. After the two contests, students were free, then they had the chances to see further things. I knew my students got the chances to see the Cape of Good Hope, and took a cable-car to the top of the Table Mountain. Because I, as a leader, had to participate in the coordination process, had to miss both events. Coordination is a process in which the leader and deputy leader, plus two coordinators of the host county, come together to decide how many points are to be awarded to a particular problem submission of a student. Given that we had nice and detailed marking schemes, and the coordinators are generally very experienced, we encountered little trouble in deciding points. Then we had a final day excursion and the Closing Ceremony, on the same day. The next day we headed home.

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Problem Selection By the end of March 2014, the host country (South Africa) received 141 problem proposals from 43 countries. I don't know when the problem selection group started to work, but surely, it took them more than a month to select 30 shortlisted problems. Furthermore they modified them, supplied alternative solutions and comments, and prepared a booklet for Jury members to consider. Incidentally the problem group was composed of six international members. I was told, they managed to do something before they formally met in South Africa, and also after they left. The selected problems are of course of high quality. However I cannot say I am totally happy with the selection. Indeed I think the problems selected were rather skewed, there were 6 algebra problems, 9 combinatorics problems, 7 geometry problems and 8 number theory problems. Some algebra problems and number theory problems in fact have quite a bit of combinatorics flavor. Moreover, several hard combinatorics problems were simply too hard. The Jury worried very much if one of them was selected, no one would be able to solve it.

When the Jury members met, it was suggested that first we selected 2 out of 4 easy problems, with one problem from each of the topics algebra, combinatorics, geometry and number theory. Again 4 medium problems from the four topics were selected. When the two easy problems were chosen, the two medium problems from the other two categories were automatically selected. Then the hard problems (problem 3 and 6) were chosen arbitrary. The suggestion was adapted. Finally two easy problems of algebra and geometry were selected, and so were two medium problems on combinatorics and number theory. However I am not sure if the easy algebra problem is really an algebra problem, of course it involves some algebraic manipulations, but I think the result very much depends on the discrete structure of integers. It is not an inequality problem nor a functional equation problem anyway. The medium combinatorics problem concerns "holes" within a distribution of rooks in a checker board. The number theory problem again is not really number theory. There is no need for congruence or other number theory things. It basically involves merging or grouping of coins of different values, so it is more like a combinatorics problem. Finally a hard geometry problem and a hard combinatorics problem were selected. It is quite certain in these days two geometry problems are to be selected. Those are the problems contestants cannot easily quote high power theorems or use more specialized techniques. However due to the preference of leaders, in general there is no 3D geometry problems. In this contest, three problems are really of combinatorial flavor. So I think the new method of choosing problems does not guarantee a good distribution of problems. Concerning Problem 6, I have to say I don't like it and I have something more to say, but let's wait.

Coordination The process of coordination was done seriously and rigorously. After the six problems were selected by the Jury (composed of leaders from 101 countries), I believed the chief coordinator then instructed the six problem captains to write up detailed marking schemes, incorporating various solutions supplied by leaders. Each problem captain was responsible for only one specific problem, he knew essentially everything concerning that problem, originality, various solutions, etc. The marking schemes were then formally approved by the Jury. After the two contests, they scanned all the answers scripts of the students. We leaders then got back answer scripts of our students and tried to allocate suitable points for our contestants. A minor mishap was, the scanner could not scan marks of correcting fluid, and thus I was asked several times why were there correcting fluids found on my students' scripts. Luckily of course was, we did not add anything new.

Detailed schedules were given to us, so leaders knew when and where to go. The process of coordination was done formally within two days. I believe because of language issue and other reasons, coordinators were recruited internationally. They were composed of old time leaders, experienced problem solvers etc. Some we met more than 10 years' ago. They were very experienced and were able to spot errors made by students, whether an error is trivial (no point deducted), minor (1 or 2 points deducted) or major (at least 4 to 5 points deducted). I thank my deputy leader, Ching Tak Wing, our old-time trainee and IMO gold medalist, who helped us to go through the many convoluted arguments of our members. We were able to discuss (or argue) with our coordinators, to convince them that our members did do somethings of certain parts of a problem or so, and thus got few extra points. On the whole, I think our papers were fairly marked and the process of coordination was done well.

Results of our Students We got 4 silvers and 2 bronzes, ranked (unofficially) 18 out of 101 countries. Indeed 3 of our 4 silver medalists solved essentially 4 problems and the other silver medalist got 3 problems correct. Also our 2 bronze medalists essentially got 3 problems correct and were real close to silver. I don't think I can blame our students for not trying hard. Indeed they picked up a lot of techniques in these few years, learned (and are still learning) to face a problem fairly and squarely. I observed when they were doing problem 2 and 5 (medium problems), they had generated the habit of gathering data and information, using various grouping and simplification methods, induction and other techniques to solve them, even though their approaches were later found to be a bit clumsy and there were a few gaps (thus few points deducted). Because a lot of time were spent on problems 2 and 5, no one could do problems 3 and 6, thus no one could tackle the hard problems. Four of our six members were old-timers, and they are leaving us for universities. I think we need 2 to 3 years to have another group of members of this caliber.

Think of this issue the other way. If we want to keep our ranking, surely several silver and bronze medals are required. If we want to be ranked within the top 10 countries, for instance, we need two or three gold medals, and some silvers and bronzes. It depends on really what we want. For me, I think it is fine if we can produce a bunch of well-trained students, good and brave to face problems and are ready to pick up necessary skills and other things in the process. Getting a gold medal in an IMO is a process, is part of a training process, but not necessarily is an end, (not like getting a world cup).

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 12, 2014.***

Problem 446. If real numbers a and b satisfy $3^a+13^b=17^a$ and $5^a+7^b=11^b$, then prove that $a < b$.

Problem 447. For real numbers x, y, z, find all possible values of $\sin(x+y) + \sin(y+z) + \sin(z+x)$ if

$$\frac{\cos x+\cos y+\cos z}{\cos(x+y+z)}=\frac{\sin x+\sin y+\sin z}{\sin(x+y+z)}.$$

Problem 448. Prove that if s,t,u,v are integers such that $s^2-2t^2+5u^2-3v^2=2tv$, then $s = t = u = v = 0$.

Problem 449. Determine the smallest positive integer k such that no matter how $\{1,2,3,\ldots,k\}$ are partitioned into two sets, one of the two sets must contain two distinct elements m, n such that mn is divisible by $m+n$.

Problem 450. (*Proposed by Michel BATAILLE*) Let $A_1A_2A_3$ be a triangle with no right angle and O be its circumcenter. For $i = 1,2,3$, let the reflection of A_i with respect to O be A_i' and the reflection of O with respect to line $A_{i+1}A_{i+2}$ be O_i (subscripts are to be taken modulo 3). Prove that the circumcenters of the triangles OO_iA_i' ($i = 1,2,3$) are collinear.

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Solutions

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Problem 441. There are six circles on a plane such that the center of each circle lies outside of the five other circles. Prove there is no point on the plane lying inside all six circles.

Solution. **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **William FUNG**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Corneliu Mănescu-Avram** (Transportation High school, Ploieşti, Romania), **Math Activity Center** (Carmel Alison Lam Foundation Secondary School),

Assume there is a point P inside all six circles C_1, C_2, …, C_6 with centers O_1, O_2, …, O_6 and radii r_1, r_2, …, r_6 respectively. Then $O_iP < r_i$ for $i = 1,2,\ldots,6$. Connecting the six O_i to P, since the six angles about P sum to 360˚, there exists $\angle O_mPO_n \le 60˚$. Then in ΔO_mPO_n, either $O_mO_n \le O_mP < r_m$ or $O_mO_n \le O_nP < r_n$. This leads to either O_n is inside C_m or O_m is inside C_n, which is a contradiction.

Problem 442. Prove that if $n > 1$ is an integer, then n^5+n+1 has at least two distinct prime divisors.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Luke Minsuk KIM** (Stanford University) and **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4).

The case $n = 2$ is true as $n^5+n+1=5\times7$. For $n\ge3$, we have $n^5+n+1=(n^3-n^2+1)(n^2+n+1)$ and $n^3-n^2+1=(n^2+n+1)(n-2)+(n+3)$. Then $n^3-n^2+1 > n^2+n+1 > 1$. Assume n^5+n+1 is a power of some prime p. Then $n^3-n^2+1= p^s$ and $n^2+n+1=p^t$ with $s > t \ge 1$. Now

$$n+3=n^3-n^2+1-(n^2+n+1)(n-2)=p^s-p^t(n-2)$$

is a multiple of $p^t = n^2+n+1$. This leads to $n+3 \ge p^t=n^2+n+1$, i.e. $2 \ge n^2$, contradiction.

Other commended solvers: **Christian Pratama BUNAIDI** (SMA YPK Ketapang I, Indonesia), **CHAN Long Tin** (Cambridge University, Year 2), **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **Victorio Takahashi CHU** (Pontifícia Universidade Católica - São Paulo SP, Brazil), **Gabriel Cheuk Hung LOU**, **Corneliu Mănescu-Avram** (Transportation High school, Ploieşti, Romania), **Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Viet Nam), **Milan PAVIC** (Serbia), **Mamedov SHATLYK** (School of Young Physics and Mathematics No. 21, Dashoguz, Turkmenistan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 443. Each pair of n ($n\ge6$) people play a game resulting in either a win or a loss, but no draw. If among every five people, there is one person beating the other four and one losing to the other four, then prove that there exists one of the n people beating all the other n–1 people.

Solution. **Jon GLIMMS** (Vancouver, Canada).

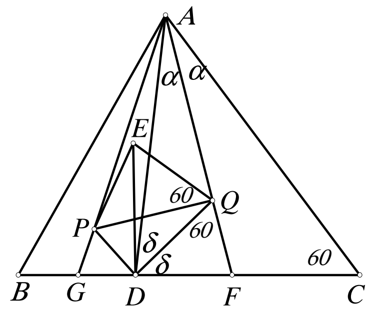
Assume no one beat all other n–1 people. Then the number of wins for each of the n people is 0,1,…, n–2. By the pigeonhole principle, there exist two people, say A and B with the same number of wins. Now, say A beat B. Due to same wins, there exists C such that A beat B, B beat C and C beat A.

Next add two other people to A, B, C. By given condition, one of these five lost to the other four. Observe that this one cannot be A, B, C, say it is D. Since $n\ge6$, ignoring D, we can add two other people to A, B, C. Again, by given condition, one of these five lost to the other four. Observe that this one cannot be A, B, C, D, say it is E. Then none of A,B,C,D,E beat the other four, contradicting the given condition.

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Problem 444. Let D be on side BC of equilateral triangle ABC. Let P and Q be the incenters of ΔABD and ΔACD respectively. Let E be the point so that ΔEPQ is equilateral and D, E are on opposite sides of line PQ. Prove that lines BC and DE are perpendicular.

Solution. **Jon GLIMMS** (Vancouver, Canada) and **T. W. LEE** (Alumni of New Method College).



We have $\angle QDP = \angle QDA+\angle PDA = ½(\angle CDA+\angle BDA)= 90˚$. Also, $\angle QDA =\angle QDC=90˚-\angle PDB$. To show $BC\perp DE$, i.e. $\angle PDE+\angle PDB=90˚$, it suffices to show $\angle QDA =\angle PDE$. This is the same as showing lines AD, ED

are symmetric with respect to the angle bisector of $\angle QDP$. For convenience, we refer to this condition by saying lines AD, ED are isogonal with respect to $\angle QDP$.

This will follow from the *isogonal conjugacy theorem* (see comments below) if we can show that (1) lines AQ, EQ are isogonal with respect to $\angle PQD$ and (2) lines AP, EP are isogonal with respect to $\angle DPQ$. For (1), we have $\angle AQD = 180^\circ - ½(\angle CAD + \angle CDA) = 120^\circ$. Let lines AQ, BC meet at F. Then $\angle FQD = 180^\circ - \angle AQD = 60^\circ = \angle EQP$ implies (1). For (2), similarly $\angle APD = 120^\circ$. Let lines AP, BC meet at G. Then $\angle GPD = 180^\circ - \angle APD = 60^\circ = \angle EPQ$ implies (2).

Comments: If we have (1) and (2), we can write down the two trigonometric forms of Ceva's theorem for points A and E with respect to $\Delta\ QDP$. Cancelling common factors in the two equations leads to

$$\frac{\sin\angle QDA}{\sin\angle ADP} = \frac{\sin\angle PDE}{\sin\angle EDQ}.$$

Then $\angle QDA = \angle PDE$ follows from $f(x) = \sin x / \sin(\angle QDP - x)$ is strictly increasing for $0 < x < \angle QDP$.

Other commended solvers: **CHAN Long Tin** (Cambridge University, Year 2) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Problem 445. For each positive integer n, prove there exists a polynomial $p(x)$ of degree n with integer coefficients such that $p(0)$, $p(1)$, …, $p(n)$ are distinct and each is of the form $2\times2014^k+3$ for some positive integer k.

Solution. **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Let $a = 2014$. Write $n!=n_1n_2$, where n_2 is the greatest divisor of $n!$ that is relatively prime to a. Then n_1 and a have the same prime divisors. By Euler's theorem, for $t=\varphi(n_2)$, we have $a^t \equiv 1 \pmod{n_2}$. For the polynomial

$$f(x) = \sum_{i=0}^{n} \frac{x(x-1)\cdots(x-i+1)}{i!}(a^t-1)^i$$

and $j=0,1,\ldots,n$, we have

$$f(j) = \sum_{i=0}^{j} \binom{j}{i}(a^t-1)^i = a^{tj}.$$

Let s be the maximum of the exponents appeared in the prime factorization of n_1. Then $a^s(a^t-1)/n!$ is a positive integer and $p(x)=2a^sf(x)+3$ is a polynomial of degree n with integer coefficients such that $p(j)=2a^{s+tj}+3$ for $j = 0,1,\ldots,n$.

Olympiad Corner

(***Continued from page 1***)

Problem 4. Points P and Q lie on side BC of acute-angled triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

Problem 5. For each positive integer n, the Bank of Cape Town issues coins of denomination $1/n$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most 99+½, prove that it is possible to split the collection into 100 or fewer groups, such that each group has total value at most 1.

Problem 6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite areas; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to color at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Notes: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

IMO2014 and Beyond

(***Continued from page 2***)

So far, about 10 Fields' medalists participated in the IMOs, but not everyone was a gold medalist (about half of them were). Even Terry Tao got bronze in his first year, then silver, then gold. Yes, of course I realize some administrators may think otherwise and have different ideas of what it means by sending a team to an IMO.

I heard many theories why we cannot produce even stronger team. Our students have to devote too much time on DSE, in particular SBA. We have no specialized schools, unlike Vietnam and Singapore. Our pool is too small, trainers are no good, training time are not enough, etc. All these are hard to rebuke (no counter-examples?), and not sure how to verify. They may well be so and so what can we do? Indeed in these few years we have strengthened our training process, more tests, asking our members to present and substantiating their views, etc. Indeed we received many suggestions from our former trainees.

We observed a few things by simply looking at the overall results. For instance, despite political trouble in the east, the Ukrainian team still did very good. They ranked 6 out of 101. The Israelites did as well as us (ranked 18). The Koreans, as usual, did very good, but not as formidable as last year. Indeed, Republic of Korea was ranked 7 and the Democratic Republic of Korea was ranked 14. During these 20 or so years, the North Koreans missed the contest altogether for 10 years, but during the times they were around, they did reasonably well. Although we were not as good as the populous countries like China (ranked 1), and USA (ranked 2). We did better than India (ranked 40) and Indonesia (ranked 30). This year we did slightly better than Thailand (ranked 22), the country to host IMO 2015. They have been good, and I was told they put a lot of money into the event and in training their team. We also did better than several traditionally strong countries such as Poland (ranked 28), Iran (ranked 21) and Bulgaria (ranked 37). Indeed Bulgaria has a long tradition of mathematical competitions, and their competition materials are often very well sought. As in the last few years, we still did not do as well as Singapore (ranked 8). However, when I looked closely at their results, I found their gold medalists were not really much better than our silver medalists and I think we can do as well? In short, it is very interesting by simply looking at the results of countries during the years, we may gather some ideas on how we should train our members in the future.

(to be continued)

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2014 Bulgarian National Math Olympiad on May 17-18, 2014.

Problem 1. *(Teodosi Vitanov, Emil Kolev)* Given is a circle k and a point A outside it. The segment BC is a diameter of k. Find the locus of the orthocenter of $\triangle ABC$, when BC is changing.

Problem 2. *(Nikolay Beluhov)* Consider a rectangular $n\times$m table where $n \geq 2$ and $m \geq 2$ are positive integers. Each cell is colored in one of the four colors: white, green, red or blue. Call such a coloring interesting if any 2×2 square contains every color exactly once. Find the number of interesting colorings.

Problem 3. *(Alexander Ivanov)* A real nonzero number is assigned to every point in space. It is known that for any tetrahedron τ the number written in the incenter equals the product of the four numbers written in the vertices of τ. Prove that all numbers equal 1.

Problem 4. *(Peter Boyvalenkov)* Find all prime numbers p and q such that

$$p^2 \mid q^3+1 \quad \text{and} \quad q^2 \mid p^6-1.$$

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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IMO2014 and Beyond (II)

Leung Tat-Wing

To discuss the IMO2014 problems, let's proceed from the easier problems to the harder problems.

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

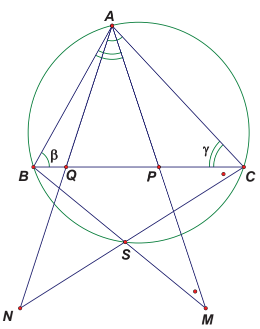
$$a_n < \frac{a_0 + a_1 + \ldots + a_n}{n} \leq a_{n+1}.$$

This problem is nice and easy. It gave us no problem. All of us got full scores in this problem. Nevertheless the problem is not entirely trivial, and indeed about 100 contestants scored nothing in this problem! First notice the middle term is not an arithmetical mean. Really during the question and answer period, some contestants did ask why the sequence doesn't start at index 1. Moreover the problem is not exactly an algebra problem, as it involves a strictly *increasing* sequence of *integers*. Try small cases, say $n = 1$. Then we need $a_1 < a_0+a_1$ sure, but not necessarily $a_0+a_1 \leq a_2$, why is that so? For $n = 2$, then we need $a_2 < (a_0+a_1+a_2)/2$, or $a_2 < a_0+a_1$, not necessarily true, but say when compared with the case of $n = 1$, if it is false, then $a_0+a_1 \leq a_2$ is true and we have an n satisfying the inequality! And the other side $a_0+a_1+a_2 \leq 2a_3$, why true again? If it is false, look at the left hand side for the case of $n = 3$. After several attempts, we really see what is going on. Indeed the inequality is equivalent to $na_n < a_0+a_1+\cdots+a_n \leq na_{n+1}$. The left hand inequality corresponds to $(a_0+a_1+\cdots+a_n) - na_n > 0$, while the right hand inequality corresponds to $(a_0+a_1+\cdots+a_n) - na_{n+1} \leq 0$, same as $(a_0+a_1+\cdots+a_{n+1}) - (n+1)a_{n+1} \leq 0$. Alas, if we define $d_n = (a_0+a_1+\cdots+a_n) - na_n$, then we just have to show there exists a *unique* n such that $d_n > 0 \geq d_{n+1}$! The proof is then complete if we can see (prove) d_n is a strictly *decreasing* sequence of *integers*. Not too bad.

Using induction, or other measures on the expression $(a_0+a_1+\cdots+a_n)/n$, our team members managed to solve the problem.

Problem 4. Points P and Q lie on side BC of acute-angled triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

This is the easiest problem in the competition, yet about 30 contestants did not get anything from it. Altogether more than 10 solutions were received, using synthetic geometry, coordinate geometry, complex numbers and the like. Some of us did it by coordinate geometry, setting the foot of A be (0,0), and coordinates $A(0,a)$, $B(b,0)$ and $C(c,0)$. Then get everything out of it via complicated calculations. But indeed if we can draw the picture properly, and do the angle tracings correctly, the problem is really not hard at all.



Indeed suppose BM and NC meet at S. Let $\angle ABC = \angle CAQ = \beta$ and $\angle ACB = \angle BAP = \gamma$, then $\triangle ABP \sim \triangle CAQ$. Hence

$$\frac{BP}{PM} = \frac{BP}{PA} = \frac{AQ}{QC} = \frac{QN}{QC}.$$

Also, $\angle NQC = \angle BQA = \angle APC = \angle BPM$. The last two statements imply $\triangle BPM \sim \triangle NQC$, hence $\angle BMP = \angle NCQ$. Then we also have $\triangle BPM \sim \triangle BSC$!

Finally, we have $\angle CSB = \angle MPB = \beta+\gamma = 180° - \angle ABC$. So $\angle CSB + \angle BAC = 180°$ and we are done.

Problem 2. Let $n \geq 2$ be an integer. Consider a $n\times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k\times k$ square which does not contain a rook on any of its k^2 unit squares.

All of us managed to give (basically) the correct answer ($\lceil \sqrt{n-1} \rceil$) and knew essentially how to tackle the question. There were gaps here and there and few points eventually deducted, but in my opinion, not really serious mistakes. Here n rooks are placed in a $n\times n$ board so that they are not attacking each other, and this time we ask for the largest possible gap (square with no rook). Of course the k^2 squares should be congruent to others and the "gap" square should be in one piece. Indeed several candidates had the same concern. This is really a classical chess board problem and I am not at all sure if the question was asked before somewhere.

First, given a $n\times n$ board with n rooks non-attacking (peaceful configuration). Suppose l is such that $l^2 < n$, then we can find a $l\times l$ square with no rook in it. Indeed there is a rook in the first *column*, consider the l consecutive *rows* starting with the row where the particular rook is placed. Now remove the first $n - l^2$ columns of this piece (hence at least one rook is removed). The remaining $l\times l^2$ piece can be decomposed into l $l\times l$ pieces of squares, but contain at most $l-1$ rooks, hence we have an empty $l\times l$ square.

Now we want to construct a peaceful configuration with largest possible square of size $\lceil \sqrt{n-1} \rceil \times \lceil \sqrt{n-1} \rceil$. Most of us see what the configuration should look like. We first let n be of the form l^2. Label the square with row i and column j as (i,j), with $0 \leq i \leq l-1$ and $0 \leq j \leq l-1$. The rooks are then placed on the positions $(il+j, jl+i)$, $0 \leq i,j \leq l-1$. One can easily check that any $l\times l$ square contains a rook.

Now comes where the most common gap lies. If $n < l^2$, we need to produce a peaceful configuration with no rook in any $l\times l$ square. The idea is of course to remove columns and rows from the previous construction. Only when (say) the top row and the leftmost column removed, *two* rooks may be removed, we have to put a rook back to an appropriate position (naturally where it should be) to return to a peaceful configuration!

(A 9×9 peaceful configuration with 2×2 squares as largest possible empty squares.)

								X
					X			
		X						
							X	
				X				
	X							
						X		
			X					
X								

Problem 5. For each positive integer n, the Bank of Cape Town issues coins of denomination $1/n$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99+½$, prove that it is possible to split the collection into 100 or fewer groups, such that each group has total value at most 1.

I am happy to see how our students handled this problem. In short, they used various grouping and induction techniques and tricks, and changed the problem to a format they can handle, thus solved the problems. Even though our arguments were sometimes rather unclear and convoluted, thus some points deducted because of gaps and other things, four of us essentially solved the problem. Indeed the main idea of solving the problem is by "merging" or "cleaning" the set of coins. Clearly if the process can still be completed after merging the coins, it can be done before merging!

Indeed the problem can be generalized as follows. Given coins of total value at most $N-½$, they can be split into N groups each of value at most 1. The problem then can be completed by the following steps.

(i) Two coins of values $1/(2k)$ may be merged into a coin of value $2\times 1/(2k)=1/2$, thus for every even number m, we may assume there is at most one coin of value $1/m$.

(ii) For every odd number m, there are at most $m-1$ coins of such value, otherwise they can be merged to form a coin of value 1 first.

(iii) Coins of value 1 must form a group of itself. Thus if there are d coins of value 1 in a group of N coins, we might as well consider a group of $N-d$ coins of values less than 1.

(iv) Now consider coins of values $1/(2k-1)$ and $1/(2k)$, with $k=1,2,\ldots,N$. We first place them into N groups according different values of k. In each group, the total value is at most

$$(2k-2)\times\frac{1}{2k-1}+\frac{1}{2k}=1-\frac{1}{2k-1}+\frac{1}{2k}<1.$$

The total value of all N groups is at most $N-½$. By taking average, there exists a group of total value at most

$$\frac{1}{N}(N-\frac{1}{2})=1-\frac{1}{2N}.$$

(v) All the remaining coins are of values less than $1/(2N)$. We may put them one by one into each group, as long as the value of each group does not exceed $1-1/(2N)$ and we are done!

The problem is meant to be a number theory problem, but is really more like a combinatorial problem. Our members managed to give different proofs to this problem and it is very nice. But indeed it is natural to consider coins of larger values (greedy method) first then consider coins of small values (a lot of them).

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90°$, $\angle THC - \angle DTC = 90°$. Prove that line BD is tangent to the circumcircle of triangle TSH.

In these few years, problems of this kind appear rather frequently. Proving a certain line is tangent to a certain (hidden) circle, or two (hidden) circles will touch each other, or the like, are generally not too easy. Still one should be able to handle them by first finding out some related geometric properties, and then obtain final results still by using only basic geometric properties and techniques.

Let us look at this problem. It is not easy to draw an accurate and nice picture, let alone proving it.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 20, 2014.***

Problem 451. Let P be an n-sided convex polygon on a plane and n>3. Prove that there exists a circle passing through three consecutive vertices of P such that every point of P is inside or on the circle.

Problem 452. Find the least positive real number r such that for all triangles with sides a,b,c, if $a \ge (b+c)/3$, then

$$c(a+b-c) \le r\,((a+b+c)^2+2c(a+c-b)).$$

Problem 453. Prove that there exist infinitely many pairs of relatively prime positive integers a,b with $a>b$ such that b^2-5 is divisible by a and a^2-5 is divisible by b.

Problem 454. Let Γ_1, Γ_2 be two circles with centers O_1, O_2 respectively. Let P be a point of intersection of Γ_1 and Γ_2. Let line AB be an external common tangent to Γ_1, Γ_2 with A on Γ_1, B on Γ_2 and A, B, P on the same side of line O_1O_2. There is a point C on segment O_1O_2 such that lines AC and BP are perpendicular. Prove that $\angle APC$=90°.

Problem 455. Let a_1, a_2, a_3, … be a permutation of the positive integers. Prove that there exist infinitely many positive integer n such that the greatest common divisor of a_n and a_{n+1} is at most $3n/4$.

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Solutions

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Problem 446. If real numbers a and b satisfy $3^a+13^b=17^a$ and $5^a+7^b=11^b$, then prove that $a < b$.

Solution. **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Elaine LAM** (Tsuen Wan Secondary School), **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam), **PANG Lok Wing, YAN Yin Wang** (United Christian College (Kowloon East), Teaching Staff) and **Simon YAU.**

If $a \ge b$, then $3^a+13^a \ge 3^a+13^b=17^a$. (*) Since 3/17<13/17<1, the function $f(x)$ =$(3/17)^x+(13/17)^x$ is strictly decreasing. By (*), $f(a) \ge 1> f(1)$. So $a < 1$.

Next, $5^b+7^b \le 5^a+7^b = 11^b$. (**) Since 5/11 < 7/11 < 1, the function $g(x) = (5/11)^x + (7/11)^x$ is strictly decreasing. By (**), $g(b) \le 1 < g(1)$. So b>1>a, contradiction.

Other commended solvers: **Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 447. For real numbers x, y, z, find all possible values of $\sin(x+y)$ + $\sin(y+z)$ + $\sin(z+x)$ if

$$\frac{\cos x+\cos y+\cos z}{\cos(x+y+z)}=\frac{\sin x+\sin y+\sin z}{\sin(x+y+z)}.$$

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **YAN Yin Wang** (United Christian College (Kowloon East), Teaching Staff), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Let S=$x+y+z$. Cross multiply and transfer all terms to one side. We get

$$\begin{aligned} 0 &= \sin S\cos x - \cos S\sin x + \sin S\cos y \\ &\quad - \cos S\sin y + \sin S\cos z - \cos S\sin z \\ &= \sin(S-x) + \sin(S-y) + \sin(S-z) \\ &= \sin(y+z) + \sin(z+x) + \sin(x+y). \end{aligned}$$

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Problem 448. Prove that if s,t,u,v are integers such that $s^2-2t^2+5u^2-3v^2=2tv$, then $s = t = u = v = 0$.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam), **YAN Yin Wang** (United Christian College (Kowloon East), Teaching Staff), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Assume s,t,u,v are not all zeros. By cancelling all common factors of $s,t,u,v,$ we may assume they are relatively prime. We can rewrite the equation as

$$2(s^2+5u^2) = (2t+v)^2+5v^2. \qquad (\dagger)$$

For $0 \le x, y \le 4$, we have $2x^2 \equiv y^2 \pmod 5$ if and only if $x \equiv y \equiv 0 \pmod 5$. (‡) So $s^2+5u^2 \equiv 2t+v \equiv 0 \pmod 5$, which implies $s = 5m$ and $2t+v = 5n$ for some integers m,n. Substituting these into (†), we get $2(5m^2+u^2)=5n^2+v^2$. By (‡), u, v are divisible by 5. Then s,t,u,v are divisible by 5, contradicting they are relatively prime. So s,t,u,v are all zeros.

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India),

Problem 449. Determine the smallest positive integer k such that no matter how {1,2,3,…,k} are partitioned into two sets, one of the two sets must contain two distinct elements m, n such that mn is divisible by $m+n$.

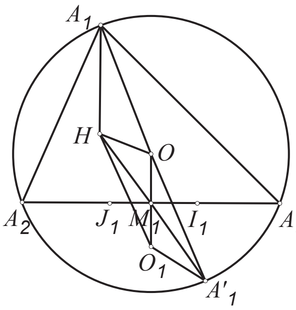
Solution. **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Call distinct positive integers m,n a <u>*good*</u> pair if mn is divisible by $m+n$. Collect all good pairs with $m,n \le 40$. We will try to separate m,n first. Let A ={1,2,3,5,8, 10, 12, 13, 14, 18, 19, 21, 22, 23, 30, 31,32,33,34} and B = {4, 6, 7, 9, 11, 15, 16, 17, 20, 24, 25, 26, 27, 28, 29, 35, 36, 37, 38, 39}. Each of A and B do not contain any good pair. For $1 \le k \le 39$, we can remove integers greater than k from A and B to get 2 disjoint subsets of {1, 2, …, k} with no good pair in each subset.

For k=40, put 6, 12, 24, 40, 10, 15 and 30 around a circle. Notice any two consecutive terms in this circle is a good pair. No matter how we divide {1,2,…,40} into 2 disjoint subsets, one of the subsets will contain at least 4 of 7 numbers in the circle. So there will be a good pair in that subset. Therefore, 40 is the desired least integer.

Other commended solvers: **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam).

Problem 450. (*Proposed by Michel BATAILLE*) Let $A_1A_2A_3$ be a triangle with no right angle and O be its circumcenter. For i = 1,2,3, let the reflection of A_i with respect to O be A_i' and the reflection of O with respect to line $A_{i+1}A_{i+2}$ be O_i (subscripts are to be taken modulo 3). Prove that the circumcenters of the triangles OO_iA_i' (i = 1,2,3) are collinear.



Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4).

Notice that O_1 is the reflection of O with respect to the midpoint M_1 of A_2A_3. By the nine point circle theorem (see Math Excalibur, *vol*.3, *no* 1, *p*,1), AH, OM_1 are parallel and their lengths are 2:1. Now A_1O=OA_1'. So, in $\triangle A_1A_1'H$, M_1 is the midpoint of $A_1'H$, i.e. H is the reflection of A_1' with respect to M_1.

Let I_1 be the circumcenter of $\triangle OO_1A_1'$. Then I_1 lies on the perpendicular bisector A_2A_3 of OO_1. Reflect I_1 with respect to M_1 to J_1. Then J_1 also lies on A_2A_3. With respect to M_1, J_1 is the circumcenter of the reflection of $\triangle OO_1A_1'$, i.e. $\triangle OO_1H$. So, J_1 also lies on the perpendicular bisector of OH.

Define I_2, I_3, J_2, J_3 similarly. As J_2, J_3 also lie on the perpendicular bisector of OH by a similar proof, J_1, J_2, J_3 are collinear. Then by Menelaus' theorem,

$$\frac{A_2J_1}{J_1A_3}\cdot\frac{A_3J_2}{J_2A_1}\cdot\frac{A_1J_3}{J_3A_2}=-1.$$

As A_3I_1/I_1A_2=A_2J_1/J_1A_3 (due to I_1, J_1 are reflection of the midpoint of A_2A_3) and similarly for I_2, J_2, I_3, J_3, we have

$$\frac{A_3I_1}{I_1A_2}\cdot\frac{A_1I_2}{I_2A_3}\cdot\frac{A_2I_3}{I_3A_1}=-1.$$

By the converse of Menelaus' Theorem, I_1, I_2, I_3 are collinear as desired.

Other commended solvers: **Andrea FANCHINI** (Cantú, Italy), **Corneliu Mănescu-Avram** (Transportation High school, Ploieşti, Romania), **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam), **Samiron SADHUKHAN** (Kendriya Vidyalaya, Barrackpore, Kolkata, India), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Olympiad Corner

(Continued from page 1)

Problem 5. *(Nikolay Nikolov)* Find all functions f : $\mathbb{Q}^+ \to \mathbb{R}^+$ such that

$f(xy) = f(x+y)(f(x)+f(y))$ for any $x,y \in \mathbb{Q}^+$.

Problem 6. *(Nikolay Beluhov)* The quadrilateral $ABCD$ is inscribed in the circle k. The lines AC and BD meet in E and the lines AD and BC meet in F. Show that the line through the incenters of ΔABE and ΔABF and the line through the incenters of ΔCDE and ΔCDF meet on k.

IMO2014 and Beyond (II)

(Continued from page 2)

First, let the line passing through C and is perpendicular to SC meets AB at Q. Then $\angle SQC$=90°−$\angle BSC$=180°−$\angle SHC$. So C, H, S, Q are concyclic. Moreover SQ is a diameter of this circle, thus the circumcenter K of SHC lies on AB. Likewise, circumcenter L of the circle CHT lies on AD. To show the circumcircle of the triangle SHT is tangent to BD, it suffices to show the perpendicular bisectors of HS and HT meet at AH. But the two perpendicular bisectors coincide with the angle bisectors of AKH and ALH, thus by the bisector theorem, it suffices to show AK/KH=AL/LH. Let M be the midpoint of CH, then B,C,M,K are concyclic, L,C,M,D are concyclic. By the sine law, AK/AL= sin$\angle ALK$ / sin$\angle AKL$ = $(DM/CL)/(BM/CK)$ = CK/CL = KH/LH.

Problem 6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite areas; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to color at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

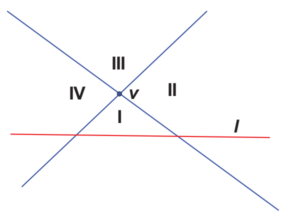
Notes: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

I have to admit that I don't like this problem at all. Indeed it was meant to be an "open end" problem, that students may produce different results with different degrees of difficulty. But when I first saw the problem, I thought we should give an algorithm, say a greedy algorithm, or other heuristic that gives good pattern (with as many blue colored lines as possible), and then analyze the pattern and give an estimate. Not so. (I guess I have become kind of intuitionist.) I doubt if there was any algorithmic solution anyway. Indeed in the official solution, a best possible solution is assumed, surely it exists, but we were not told how to get there.

Let me reproduce a part of the proof as follows. Given a set of n lines colored blue and red, and the lines colored blue is as large as possible (maximality argument), so that every finite region still has at least one boundary line colored red. Assume k lines are colored blue. Call a vertex which is the intersection of two blue lines *blue* as well, so there are ${}_kC_2$ blue vertices.

Now take any red line l, using the maximality argument, there exists at least one region with this red line l as the *only* red side, (for if all regions have two or more red lines, surely we can change one more red line to blue). In this region there is at least one blue vertex v since any finite region has at least three lines. We then associate the blue vertex with the red line. Now finally every blue vertex v belongs to four regions, (some may be unbounded), hence it may be associated with at most four red lines. Therefore the total number of red lines is at most $4{}_kC_2$=$2k(k-1)$.

On the other hand, there are $n-k$ red lines, thus, $n-k \le 2k(k-1)$. Solving for n, we get $n \le 2k^2-k \le 2k^2$. Hence, $k \ge \lceil \sqrt{n/2}\, \rceil$ and we get an estimate on the number of blue lines!



By putting some weights on the blue vertices, or by refining local analysis, one may get the stronger result $k \ge \sqrt{n}$.

Mathematical Excalibur

Volume 19, Number 3 *November 2014 – January 2015*

Olympiad Corner

Below are the problems of the IMO2015 Hong Kong Team Selection Test 2 held on 25th October, 2014.

Problem 1. Assume the dimensions of an answer sheet to be 297 mm by 210 mm. Suppose that your pen leaks and makes some non-intersecting ink stains on the answer sheet. It turns out that the area of each ink stain does not exceed 1 mm^2. Moreover, any line parallel to an edge of the answer sheet intersects at most one ink stain. Prove that the total area of the ink stains is at most 253.5 mm^2. (You may assume a stain is a connected piece.)

Problem 2. Let $\{a_n\}$ be a sequence of positive integers. It is given that $a_1=1$, and for every $n\geq 1$, a_{n+1} is the smallest positive integer greater than a_n which satisfies the following condition: for any integers i, j, k, with $1 \leq i, j, k \leq n+1$, $a_i+a_j \neq 3a_k$. Find a_{2015}.

Problem 3. Let ABC be an equilateral triangle, and let D be a point on AB between A and B. Next, let E be a point on AC with DE parallel to BC. Further, let F be the midpoint of CD and G the circumcentre of $\triangle ADE$. Determine the interior angles of $\triangle BFG$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 31, 2015***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

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Variations and Generalisations to the Rearrangement Inequality

Law Ka Ho

A. The rearrangement inequality

In *Math Excalibur*, vol. 4, no. 3, we can find the following

Theorem 1 (Rearrangement inequality) Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two increasing sequences of real numbers. Then amongst all **random sums** of the form

$$a_1 b_{\sigma_1} + a_2 b_{\sigma_2} + \cdots + a_n b_{\sigma_n},$$

where $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is a permutation of $(1,2,\ldots,n)$,

- the greatest is the **direct sum** $a_1b_1+a_2b_2+\cdots+a_nb_n$;
- the smallest is the **reverse sum** $a_1b_n+a_2b_{n-1}+\cdots+a_nb_1$.

A well-known corollary of the rearrangement inequality is the following

Theorem 2 (Chebyshev's inequality) With the same setting in Theorem 1, the quantity

$$\frac{(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n)}{n}$$

lies between the direct sum and the reverse sum, again with equality if and only if at least one of the two sequences is constant.

B. A variation --- from 'sum' to 'product'

The different 'sums' in the rearrangement inequality are in fact 'sums of products'. For this reason we shall from now on call them **P-sums**, to remind ourselves that we take products and then sum them up. Naturally, we ask what happens if we look at 'product of sums' (**S-products**) instead.

A little trial *suggests* that, opposite to the case of P-sums, the direct S-product is minimum while the reverse S-product is maximum. For example we may take the sequences $1\leq 2\leq 3\leq 4$ and $5\leq 6\leq 7\leq 8$. The direct S-product of these sequences is $(1+5)(2+6)(3+7)(4+8) = 5760$ and the reverse S-product of the sequences is $(1+8)(2+7)(3+6)(4+5) = 6561$. We can also check some random S-products, e.g we have $(1+6)(2+5)(3+8)(3+7) = 5929$ and $(1+6)(2+7)(3+8)(4+5) = 6237$.

But then a little further thought shows that this is not quite right. For instance we may take $1\leq 2\leq 3\leq 4$ and $-5\leq -2 \leq 1\leq 2$ and end up with a reverse S-product $(1+2)(2+1)[3+(-2)][4+(-5)]$, which is negative. Yet, some random S-products, such as $[1+(-2)](2+2)(3+1)[4+(-5)]$, can be positive.

It turns out that we have to require the variables to be non-negative for the result to hold.

Theorem 3 (Rearrangement inequality for S-products) Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two increasing sequences of non-negative real numbers. Then amongst all random S-products of the form

$$(a_1+b_{\sigma_1})(a_2+b_{\sigma_2})\cdots(a_n+b_{\sigma_n})$$

where $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is a permutation of $(1,2,\ldots,n)$,

- the smallest is the direct S-product $(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)$;
- the greatest is the reverse S-product $(a_1+b_n)(a_2+b_{n-1})\cdots(a_n+b_1)$.

Proof Take any random S-product

$$(a_1+b_{\sigma_1})(a_2+b_{\sigma_2})\cdots(a_n+b_{\sigma_n})$$

which is not the direct S-product. Then there exists $i<j$ such that $b_{\sigma_i} > b_{\sigma_j}$.

Let's see what happens if we swap σ_i and σ_j. In that case only two terms are changed. Consider the two products

$$P_1 = (a_i+b_{\sigma_i})(a_j+b_{\sigma_j}) \text{ and}$$

$$P_2 = (a_i+b_{\sigma_j})(a_j+b_{\sigma_i}).$$

(continued on page 2)

After expanding, cancelling and factoring, we have

$$P_2 - P_1 = (a_i - a_j)(b_{\sigma_j} - b_{\sigma_i}),$$

which is non-positive since $a_i - a_j \le 0$ and $b_{\sigma_i} > b_{\sigma_j}$. So $P_2 \ge P_1$. This means swapping σ_i and σ_j leads to a larger (or equal) S-product. It follows that the direct S-product is the minimum amongst all random S-products. In a similar manner we can prove that the reverse S-product is the maximum.

Example 4 (IMO 1966) In the interior of sides BC, CA, AB of ΔABC, points K, L, M respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of ΔABC.

Solution Let a, b, c denote the lengths of the sides opposite A, B, C respectively. Let also a_1 and a_2 denote the lengths of the two segments after the side with length a is cut into two parts by the point K (i.e. $BK = a_1$ and $KC = a_2$), and similarly for b_1, b_2, c_1, c_2. The six variables a_1, a_2, b_1, b_2, c_1, c_2 can be ordered to form an increasing sequence. By the rearrangement inequality for S-products, the direct S-product

$$(a_1+a_1)(a_2+a_2)(b_1+b_1)(b_2+b_2)(c_1+c_1)(c_2+c_2)$$
$$= 64a_1a_2b_1b_2c_1c_2$$

is less than or equal to the random S-product

$$(a_1+a_2)(a_2+a_1)(b_1+b_2)(b_2+b_1)(c_1+c_2)(c_2+c_1)$$
$$= a^2b^2c^2.$$

Let S denote the area of ΔABC. If triangles AML, BKM, CLK all have areas greater than $S/4$, then using the above result we have

$$\begin{aligned}\left(\frac{S}{4}\right)^3 &< \left(\frac{1}{2}c_1b_2\sin A\right)\left(\frac{1}{2}c_2a_1\sin B\right)\left(\frac{1}{2}a_2b_1\sin C\right)\\ &\le \frac{a^2b^2c^2}{8\cdot 64}\cdot\sin A\sin B\sin C\\ &= \frac{1}{64}\left(\frac{1}{2}ab\sin C\right)\left(\frac{1}{2}bc\sin A\right)\left(\frac{1}{2}ca\sin B\right)\\ &= \left(\frac{S}{4}\right)^3\end{aligned}$$

which is a contradiction.

Example 5 (IMO 1984) Prove that

$$0 \le xy + yz + zx - 2xyz \le 7/27,$$

where x, y and z are non-negative real numbers for which $x+y+z=1$.

Solution The left-hand inequality is pretty easy. We have

$$\begin{aligned}&xy+yz+zx-2xyz\\ &=(xy-xyz)+(yz-xyz)+(zx-xyz)+xyz\\ &=xy(1-z)+yz(1-x)+zx(1-y)+xyz\\ &=xy(x+y)+yz(y+z)+zx(z+x)+xyz\ge 0.\end{aligned}$$

For the right-hand inequality, it is well-known that

$$\begin{aligned}1 &= (x+y+z)^2\\ &= x^2+y^2+z^2+xy+yz+zx\\ &\ge 3(xy+yz+zx)\end{aligned}$$

and so $xy + yz + zx \le 1/3$. By the rearrangement inequality for S-products, we have

$$\begin{aligned}&(1-2x)(1-2y)(1-2z)\\ &\le\left(\frac{1-2x}{2}+\frac{1-2y}{2}\right)\left(\frac{1-2y}{2}+\frac{1-2z}{2}\right)\left(\frac{1-2z}{2}+\frac{1-2x}{2}\right)\\ &=zxy\ .\end{aligned}$$

(The rearrangement inequality for S-products applies only if the three terms on the left hand side are non-negative. However, if this is not true then exactly one of them is negative and the result therefore still holds.) Expanding gives

$$1-2(x+y+z)+4(xy+yz+zx)-8xyz\le xyz$$

or $9xyz \ge 4(xy+yz+zx)-1$. From this, we have

$$\begin{aligned}&xy+yz+zx-2xyz\\ &\le xy+yz+zx-2\left(\frac{4(xy+yz+zx)-1}{9}\right)\\ &=\frac{xy+yz+zx+2}{9}\le\frac{2\frac{1}{3}}{9}=\frac{7}{27}.\end{aligned}$$

C. A generalisation — from two sequences to more

Another natural direction of generalising the rearrangement inequality (for P-sums) is to consider the case in which there are more than two sequences. This time we need two subscripts to index the terms, one for the index of the sequence and one for the index of a particular term of a sequence. Again, we need to restrict ourselves to sequences of non-negative numbers (for both P-sums and S-products), otherwise one can easily construct counter- examples. Also, note that there is no such thing as 'reverse P-sum/S-product' when there are more than two sequences.

Theorem 6 (Rearrangement inequality for multiple sequences) Suppose there are m increasing sequences (each with n terms) of non-negative numbers, say, $a_{i1} \le a_{i2} \le \cdots \le a_{in}$, where $i = 1, 2, \ldots, m$. Then

- the direct P-sum $\sum_{j=1}^{n} a_{1j}a_{2j}\cdots a_{mj}$ is greater than or equal to any other random P-sum of the form $\sum_{j=1}^{n} a_{1\sigma_{1j}}a_{2\sigma_{2j}}\cdots a_{m\sigma_{mj}}$;
- the direct S-product $\prod_{j=1}^{n}(a_{1j}+a_{2j}+\cdots+a_{mj})$ is smaller than or equal to any other random S-product of the form $\prod_{j=1}^{n}(a_{1\sigma_{1j}}+a_{2\sigma_{2j}}+\cdots+a_{m\sigma_{mj}})$.

Here $(\sigma_{i1},\sigma_{i2},\ldots,\sigma_{in})$ is a permutation of $(1,2,\ldots,n)$ for $i = 1,2, \ldots, m$.

Remarks.

(1) Theorem 6 is sometimes known as '微微對偶不等式' in Chinese.

(2) A less clumsy way to express Theorem 6 is to use matrices. With the above m sequences we may form the matrices

$$A=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\text{ and}$$

$$B=\begin{pmatrix}a_{1\sigma_{11}} & a_{1\sigma_{12}} & \cdots & a_{1\sigma_{1n}}\\ a_{2\sigma_{21}} & a_{2\sigma_{22}} & \cdots & a_{2\sigma_{2n}}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m\sigma_{m1}} & a_{m\sigma_{m2}} & \cdots & a_{m\sigma_{mn}}\end{pmatrix}.$$

Here each row of A is in ascending order (corresponding to one of the m increasing sequences) while each row of B is a permutation of the terms in the corresponding row of A (corresponding to a permutation of the corresponding sequence). Then Theorem 6 says

- the sum of column products (P-sum) in A is greater than or equal to that in B;
- the product of column sums (S-product) in A is less than or equal to that in B.

(3) The proof of Theorem 6 is essentially the same as that of Theorem 3, and is therefore omitted.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***January 31, 2015.***

Problem 456. Suppose $x_1, x_2, \ldots, x_n$ are non-negative and their sum is 1. Prove that there exists a permutation σ of $\{1,2,\cdots,n\}$ such that

$$x_{\sigma(1)}x_{\sigma(2)}+x_{\sigma(2)}x_{\sigma(3)}+\cdots+x_{\sigma(n)}x_{\sigma(1)} \le 1/n.$$

Problem 457. Prove that for each n = 1,2,3,…, there exist integers a, b such that if integers x, y are relatively prime, then $\sqrt{(a-x)^2+(b-y)^2} > n.$

Problem 458. Nonempty sets A_1, A_2, A_3 form a partition of $\{1,2,\ldots,n\}$. If $x+y=z$ have no solution with x in A_i, y in A_j, z in A_k and $\{i,j,k\}=\{1,2,3\}$, then prove that A_1, A_2, A_3 cannot have the same number of elements.

Problem 459. H is the orthocenter of acute $\triangle ABC$. D,E,F are midpoints of sides BC, CA, AB respectively. Inside $\triangle ABC$, a circle with center H meets DE at P,Q, EF at R,S, FD at T,U. Prove that $CP=CQ=AR=AS=BT=BU$.

Problem 460. If $x,y,z > 0$ and $x+y+z+2 = xyz$, then prove that

$$x+y+z+6 \ge 2\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right).$$

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Solutions

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Problem 451. Let P be an n-sided convex polygon on a plane and n>3. Prove that there exists a circle passing through three consecutive vertices of P such that every point of P is inside or on the circle.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India) and **T.W. LEE** (Alumni of New Method College).

Let R_{XYZ} denote the radius of the circle through vertices X,Y,Z of P. Let circle Γ through vertices A,B,C of P be one with maximal radius. Without loss of generality, we may assume $\angle ABC$ and $\angle ACB < 90°$. If there is a vertex D of P outside Γ, let AD meet Γ at E. Then $\angle ADC < \angle AEC = \angle ABC$. By the extended sine law

$$R_{ADC} = \frac{AC}{2\sin\angle ADC} > \frac{AC}{2\sin\angle ABC} = R_{ABC},$$

contradicting maximality of Γ. So all vertices of P is on or inside Γ.

Let F be the vertex of P next to A (toward C). If F is inside Γ, then $AFCB$ is convex and $\angle AFC+\angle ABC > 180°$. Hence $0° < 180°-\angle AFC < \angle ABC < 90°$. Then

$$R_{AFC} = \frac{AC}{2\sin\angle AFC} > \frac{AC}{2\sin\angle ABC} = R_{ABC},$$

contradiction. So F is on Γ. Similarly, the vertex of P next to A (toward B) is on Γ.

Problem 452. Find the least positive real number r such that for all triangles with sides a,b,c, if $a \ge (b+c)/3$, then

$$c(a+b-c) \le r\,((a+b+c)^2+2c(a+c-b)).$$

Solution. **Jon GLIMMS** and **Samiron SADHUKHAN** (Kendriya Vidyalaya, India).

Let $I = a+b-c$. Then $a \ge (b+c)/3$ implies $a-b \ge -(a+b-c)/2 = -I/2$ (*)

Using $a+b+c=I+2c$, (*) and the *AM-GM* inequality, we have

$$\begin{aligned} J &= \frac{(a+b+c)^2+2c(a+c-b)}{2c(a+b-c)} \\ &= \frac{I^2+4cI+4c^2}{2cI}+\frac{a+c-b}{I} \\ &= \frac{I}{2c}+2+\frac{3c}{I}+\frac{a-b}{I} \\ &\ge \frac{3}{2}+\frac{I}{2c}+\frac{3c}{I} \ge \frac{3}{2}+2\sqrt{\frac{3}{2}}. \end{aligned}$$

Equality hold if $a = (b+c)/3$ and $I^2=6c^2$, i.e. $a:b:c = 2+\sqrt{6} : 2+3\sqrt{6} : 4$. The least r such that $1/(2J) \le r$ is $(\sqrt{24}-3)/15$.

Problem 453. Prove that there exist infinitely many pairs of relatively prime positive integers a,b with $a>b$ such that b^2-5 is divisible by a and a^2-5 is divisible by b.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM) and **Samiron SADHUKHAN** (Kendriya Vidyalaya, India).

Note $(a,b) = (11,4)$ is a solution. From any solution (a,b) with $a>b\ge4$, we get $a^2-5=bc$ and $b^2-5=ad$ for some positive integers c and d. Now we show (c,a) is another such solution. First $bc = a^2-5 > a^2-a = a(a-1) \ge ab$ implies $c>a$. If a prime p divides $\gcd(a,c)$, then $a^2-5=bc$ and $b^2-5=ad$ imply $b^2=ad+5=ad+a^2-bc$ is divisible by p. Since $\gcd(a,b)=1$, we get $\gcd(c,a)=1$. Using $\gcd(a,b)=1$ and $a(a+d)=a^2+b^2-5 = b(b+c)$, we see a divides $b+c$. Then a divides $(b+c)(c-b) + (b^2-5) = c^2-5$. So there are infinitely many solutions.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **O Kin Chit** (G. T. (Ellen Yeung College), **WONG Yat** (G. T. (Ellen Yeung) College), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 454. Let Γ_1, Γ_2 be two circles with centers O_1, O_2 respectively. Let P be a point of intersection of Γ_1 and Γ_2. Let line AB be an external common tangent to Γ_1, Γ_2 with A on Γ_1, B on Γ_2 and A, B, P on the same side of line O_1O_2. There is a point C on segment O_1O_2 such that lines AC and BP are perpendicular. Prove that $\angle APC=90°$.

Solution. **Serik JUMAGULOV** (Karaganda State University, Qaragandy City, Kazakhstan).

Other than P, let the circles also meet at Q. If $PQ \cap AB = M$, then M is the midpoint of AB as $MA^2 = MP\times MQ = MB^2$. Let $PQ \cap O_1O_2 = K$, $BP\cap AC=N$ and AL be a diameter of the circle with center O_1. Since $PQ \perp O_1O_2$ and $BN\perp AC$, $PNCK$ is cyclic. Now $\angle PBM = 90°-\angle NAB = \angle CAO_1$ and $\angle BPM = \angle KPN = \angle ACO_1$. So $\Delta ACO_1 \sim \Delta BPM$. Then $AC/BP = AO_1/BM = AL/BA$. So $\Delta ACL \sim \Delta BPA$. Then $\angle ALP = \angle BAP = \angle ALC$. So L,C,P are collinear. As AL is a diameter, $\angle APC = 90°$.

Other commended solvers: **Andrea FANCHINI** (Cantú, Italy), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 455. Let a_1, a_2, a_3, … be a permutation of the positive integers. Prove that there exist infinitely many positive integer n such that the greatest common divisor of a_n and a_{n+1} is at most $3n/4$.

Solution. **Jon GLIMMS** and **Samiron SADHUKHAN** (Kendriya Vidyalaya, India).

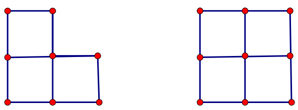
Assume that there exists N such that for all $n\ge N$, $\gcd(a_n,a_{n+1})>3n/4$. Then for all $n \ge 4N$, $a_n\ge \gcd(a_n,a_{n+1}) > 3n/4 \ge 3N$. Since a_1, a_2, a_3, … is a permutation of the positive integers, we see $\{1,2,\cdots,3N\}$ is a subset of $\{a_1, a_2,\cdots, a_{4N-1}\}$. Now the intersection of $\{1,2,\cdots,3N\}$ and $\{a_{2N}, a_{2N+1},\cdots,a_{4N-1}\}$ has at least $3N-(2N-1)$

= N+1 elements. By the pigeonhole principle, there exists k such that $2N \le k < 4N-1$ and $a_k,\ a_{k+1} \le 3N$. Then $\gcd(a_k,a_{k+1}) \le ½\max\{a_k,\ a_{k+1}\} \le 3N/2 \le 3k/4$, contradiction.

Olympiad Corner

(Continued from page 1)

Problem 4. A 11×11 grid is to be covered completely without overlapping by some 2×2 squares and L-shapes each composed of three unit cells. Determine the smallest number of L-shapes used. (Each shape must cover some grids entirely and cannot be placed outside the 11×11 grid. The L-shapes may be reflected or rotated when placed on the grid.)



Variations and Generalisations

(Continued from page 2)

Example 7 Let $x_1,\ x_2,\ \ldots,\ x_n$ be non-negative real numbers whose sum is at most 1/2. Show that $(1-x_1)(1-x_2)\cdots(1-x_n) \ge 1/2$.

Solution Form the n×n matrix

$$A = \begin{pmatrix} 1-x_1 & 1 & \cdots & 1 \\ 1-x_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1-x_n & 1 & \cdots & 1 \end{pmatrix}$$

whose rows are in ascending order. Consider the matrix

$$B = \begin{pmatrix} 1-x_1 & 1 & \cdots & 1 \\ 1 & 1-x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-x_n \end{pmatrix}$$

in which each row is a permutation of the terms in the corresponding row of A. By the rearrangement inequality for multiple sequences, the P-sum in A is greater than the P-sum in B, i.e.

$$(1-x_1)(1-x_2)\cdots(1-x_n) + n - 1 \ge (1-x_1) + (1-x_2) + \cdots + (1-x_n).$$

It follows that

$$(1-x_1)(1-x_2)\cdots(1-x_n) \ge 1 - (x_1 + x_2 + \cdots + x_n) \ge 1-1/2 = 1/2.$$

Example 8 Let $x_1,\ x_2,\ \ldots,\ x_n$ be positive real numbers with sum 1. Show that

$$\frac{x_1x_2\cdots x_n}{(1-x_1)(1-x_2)\cdots(1-x_n)} \le \frac{1}{(n-1)^n}.$$

Solution Without loss of generality assume $x_1 \le x_2 \le \cdots \le x_n$. Form the $(n-1)\times n$ matrix

$$A = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

whose rows are in ascending order. The S-product of A is thus $(n-1)^n x_1x_2\cdots x_n$. Now the matrix B given by

$$B = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_1 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_n & \cdots & x_{n-2} \end{pmatrix}$$

has the property that each of its rows is a permutation of the terms in the corresponding row of A. Furthermore, since $x_1,\ x_2,\ \ldots,\ x_n$ have sum 1, the S-product of B is equal to $(1-x_1)(1-x_2)\cdots(1-x_n)$. By the rearrangement inequality for multiple sequences, we have $(n-1)^n x_1x_2\cdots x_n \le (1-x_1)(1-x_2)\cdots(1-x_n)$.

D. Proofs of some classic inequalities

The rearrangement inequality for multiple sequences can be used to prove a number of classic inequalities. We look at some such examples in this final section.

Theorem 9 (Bernoulli inequality)
For real numbers $x_1,\ x_2,\ \ldots,\ x_n$, where either all are non-negative or all are negative but not less than –1, we have

$$\prod_{i=1}^{n}(1+x_i) \ge 1 + \sum_{i=1}^{n} x_i.$$

Proof Without loss of generality assume $x_1 \le x_2 \le \cdots \le x_n$. Suppose $x_1,\ x_2,\ \ldots,\ x_n$ are all non-negative. Consider the n×n matrices

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1+x_1 \\ 1 & 1 & \cdots & 1+x_2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+x_n \end{pmatrix} \quad \text{and}$$

$$B = \begin{pmatrix} 1+x_1 & 1 & \cdots & 1 \\ 1 & 1+x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+x_n \end{pmatrix}.$$

Then A and B satisfy the properties stated in Theorem 6. Thus the P-sum in A is greater than or equal to that in B,

i.e. $n-1+\prod_{i=1}^{n}(1+x_i) \ge \sum_{i=1}^{n}(1+x_i)$.

This gives $\prod_{i=1}^{n}(1+x_i) \ge 1 + \sum_{i=1}^{n} x_i$.

The proof in the latter case (in which $x_1,\ x_2,\ \ldots,\ x_n$ are negative but not less than –1) is essentially the same; just move the rightmost column of A to the leftmost.

Theorem 10 (Generalised Chebyshev's inequality) For m increasing sequences (each with n terms) of non-negative real numbers, say, $a_{i1} \le a_{i2} \le \cdots \le a_{in}$, where i=1,2,…, m,

the direct P-sum $\sum_{j=1}^{n} a_{1j}a_{2j}\cdots a_{mj}$ is

greater than or equal to

$$\frac{1}{n^{m-1}}\prod_{i=1}^{m}\left(a_{i1}+a_{i2}+\cdots+a_{in}\right).$$

Proof Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Now we can randomly form a matrix B as follows. The first row of B is the same as that of A. Each other row of B is obtained by shifting the corresponding row of A to the right by k places, where k is randomly chosen from 0, 1, 2, …,$n-1$. (For instance, if k=1, then the second row of B will be $(a_{2n},\ a_{21},\ \cdots,\ a_{2\,n-1}$.) Thus a total of n^{m-1} different possible B's can be formed. Each of them has a P-sum less than or equal to that of A, according to Theorem 6. The sum of all the P-sums for these n^{m-1} is precisely

$$\prod_{i=1}^{m}\left(a_{i1}+a_{i2}+\cdots+a_{in}\right),$$

which should therefore be less than or equal to n^{m-1} times the P-sum of A, i.e. n^{m-1} times the direct P-sum. This gives us the desired result.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the Team Selection Test 1 for the Dutch IMO team held in June, 2014.

Problem 1. Determine all pairs (a,b) of positive integers satisfying

$$a^2+b \mid a^2b+a \text{ and } b^2-a \mid ab^2+b.$$

Problem 2. Let $\triangle ABC$ be a triangle. Let M be the midpoint of BC and let D be a point on the interior of side AB. The intersection of AM and CD is called E. Suppose that $|AD|=|DE|$. Prove that $|AB|=|CE|$.

Problem 3. Let a, b and c be rational numbers for which $a+bc$, $b+ac$ and $a+b$ are all non-zero and for which we have

$$\frac{1}{a+bc}+\frac{1}{b+ac}=\frac{1}{a+b}.$$

Prove that $\sqrt{(c-3)(c+1)}$ is rational.

Problem 4. Let $\triangle ABC$ be a triangle with $|AC|=2|AB|$ and let O be its circumcenter. Let D be the intersection of the angle bisector of $\angle A$ and BC. Let E be the orthogonal projection of O on AD and let $F\neq D$ be a point on AD satisfying $|CD|=|CF|$. Prove that $\angle EBF=\angle ECF$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 10, 2015***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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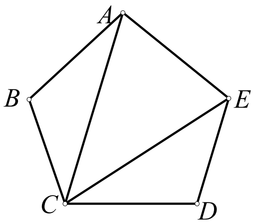
Polygonal Problems

Kin Yin Li

In geometry textbooks, we often come across problems about triangles and quadrilaterals. In this article we will present some problems about n-sided polygons with $n > 4$. This type of problem appears every few years in math olympiads of many countries.

<u>*Example 1.*</u> Prove that if $ABCDE$ is a convex pentagon with all sides equal and $\angle A\geq\angle B\geq\angle C\geq\angle D\geq\angle E$, then it is a regular pentagon.

<u>*Solution.*</u>



Since

$$AC=2AB\sin\frac{\angle B}{2}\geq 2CD\sin\frac{\angle D}{2}=CE,$$

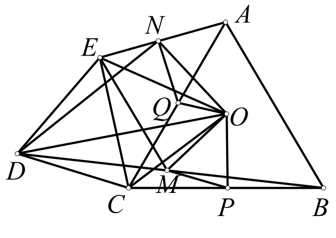
we get $\angle AEC\geq\angle EAC$. Next,

$$\angle EAC=\angle A-\frac{180^\circ-\angle B}{2}=\angle A+\frac{\angle B}{2}-90^\circ$$
$$\geq\angle E+\frac{\angle D}{2}-90^\circ=\angle E-\frac{180^\circ-\angle D}{2}$$
$$=\angle AEC$$

Hence, $\angle EAC=\angle AEC$. Then equality holds everywhere above so that $\angle A=\angle E$ and we are done.

<u>*Example 2.*</u> (*Bulgaria*, 1979) In convex pentagon $ABCDE$, $\triangle ABC$ and $\triangle CDE$ are equilateral. Prove that if O is the center of $\triangle ABC$ and M, N are midpoints of BD, AE respectively, then $\triangle OME\sim\triangle OND$.

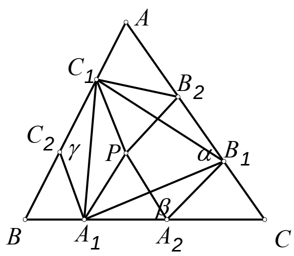
<u>*Solution.*</u>



Let P, Q be the midpoints of BC, AC respectively. Observe that $\angle COP=60^\circ$, $OC=2OP$, $PM\|CD$, $\angle DCE=60^\circ$ and $EC=DC=2MP$. Then rotating about O by 60° clockwise and follow by doubling distance from O, we see $\triangle OPM$ goes to $\triangle OCE$. Hence $\angle EOM=\angle COP=60^\circ$ and $OE=2OM$. Similarly we can rotate about O by 60° counterclockwise and double distance from O to bring $\triangle OQN$ to $\triangle OCD$. Then $\angle DON=60^\circ$, $OD=2ON$ and so $\triangle OME\sim\triangle OND$.

<u>*Example 3.*</u> (*IMO* 2005) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, so that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that A_1B_2, B_1C_2 and C_1A_2 are concurrent.

<u>*Solution.*</u>



Let P be the point inside $\triangle ABC$ such that $\triangle A_1A_2P$ is equilateral. Observe that $A_1P\|C_1C_2$ and $A_1P=C_1C_2$. So $A_1PC_1C_2$ is a rhombus. Similarly, $B_1PB_2B_1$ is a rhombus. So $\triangle C_1B_2P$ is equilateral. Let $\alpha=\angle B_2B_1A_2$, $\beta=\angle B_1A_2A_1$ and $\gamma=\angle C_1C_2A_1$. Then α and β are external angles of $\triangle CB_1A_2$ with $\angle C=60^\circ$. So $\alpha+\beta=240^\circ$. Now $\angle B_2PA_2=\alpha$ and $\angle C_1PA_1=\gamma$. So $\alpha+\gamma=360^\circ-(\angle C_1PB_2+\angle A_1PA_2)=240^\circ$. So $\beta=\gamma$. Similarly, $\angle C_1B_2B_1=\beta$. Hence, $\triangle A_1A_2B_1$, $\triangle B_1B_2C_1$ and $\triangle C_1C_2A_1$ are congruent, which implies $\triangle A_1B_1C_1$ is equilateral. Since sides of $A_1A_2B_1B_2C_1C_2$ have equal lengths, lines A_1B_2, B_1C_2 and C_1A_2 are the perpendicular bisectors of the sides of $\triangle A_1B_1C_1$ and the result follows.

(continued on page 2)

Example 4. (*Czechoslovakia*, 1974) Prove that if a circumscribed hexagon $ABCDEF$ satisfies

$$AB=BC,\ CD=DE \text{ and } EF=FA,$$

then the area of $\triangle ACE$ is less than or equal to the area of $\triangle BDF$.

Solution. Let O be the circumcenter of hexagon $ABCDEF$ and R be the radius of the circumcircle. Let

$$\alpha=\angle CAE,\beta=\angle AEC,\gamma=\angle ACE.$$

From the given conditions on the sides, we get

$$\angle AOB=\angle BOC=\beta,$$
$$\angle COD=\angle DOE=\alpha,$$
$$\angle EOF=\angle FOA=\gamma.$$

Let $[XYZ]$ denote the area of $\triangle XYZ$. We have

$$\begin{aligned}[ACE]&=\frac{EC\cdot CA\cdot AE}{4R}\\&=\frac{2R\sin\alpha\cdot 2R\sin\beta\cdot 2R\sin\gamma}{4R}\\&=2R^2\sin\alpha\sin\beta\sin\gamma.\end{aligned}$$

Similarly,

$$[BDF]=2R^2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}.$$

Now for positive α, β, γ satisfying $\alpha+\beta+\gamma=180^\circ$, we have

$$\begin{aligned}&\sin^2\alpha\sin^2\beta\sin^2\gamma\\&=(\sin\alpha\sin\beta)(\sin\gamma\sin\alpha)(\sin\beta\sin\gamma)\\&=\prod_{cyc}\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}\\&\le\prod_{cyc}\frac{1-\cos(\alpha+\beta)}{2}\\&=\sin^2\frac{\alpha+\beta}{2}\sin^2\frac{\beta+\gamma}{2}\sin^2\frac{\gamma+\alpha}{2}.\end{aligned}$$

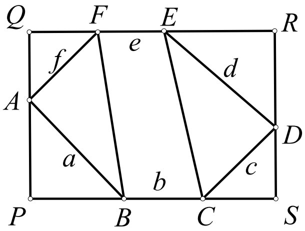
Therefore, $[ACE]\le[BDF]$.

Example 5. (*IMO* 1996) Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF and CD is parallel to FA. Let R_A, R_C, R_E be the circumradii of triangles FAB, BCD, DEF respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A+R_C+R_E\ge\frac{P}{2}.$$

Solution. Let a, b, c, d, e, f denote the lengths of the sides AB, BC, CD, DE, EF, FA respectively. By the parallel conditions, we have $\angle A=\angle D$, $\angle B=\angle E$, $\angle C=\angle F$.

Consider rectangle $PQRS$ such that A is on PQ; F,E are on QR; D is on RS and B,C are on SP.



We have $BF\ge PQ=SR$. So $2BF\ge PQ+SR$, which is the same as

$$2BF\ge(a\sin B+f\sin C)+(c\sin C+d\sin B).$$

Similarly,

$$2BD\ge(c\sin A+b\sin B)+(c\sin B+f\sin A),$$
$$2DF\ge(c\sin C+d\sin A)+(a\sin A+b\sin C).$$

Next, by the extended sine law,

$$R_A=\frac{BF}{2\sin A},R_C=\frac{BD}{2\sin C},R_E=\frac{DE}{2\sin E}.$$

Then using the inequalities and equations above, we have

$$\begin{aligned}&R_A+R_C+R_E\\&\ge\frac{a}{4}\left(\frac{\sin B}{\sin A}+\frac{\sin A}{\sin B}\right)+\cdots+\frac{f}{4}\left(\frac{\sin A}{\sin F}+\frac{\sin F}{\sin A}\right)\\&\ge\frac{a+b+c+d+e+f}{2}=\frac{P}{2}.\end{aligned}$$

Example 6. (*Great Britain*, 1988) Let four consecutive vertices A, B, C, D of a regular polygon satisfy

$$\frac{1}{AB}=\frac{1}{AC}+\frac{1}{AD}.$$

Determine the number of sides of the polygon.

Solution. Let the circumcircle of the polygon have center O and radius R. Let $\alpha=\angle AOB$, then $0<3\alpha=\angle AOD<360^\circ$. So $0<\alpha<120^\circ$. Also, from

$$AB=2R\sin\frac{\alpha}{2},\qquad AC=2R\sin\alpha,$$
$$AD=2R\sin\frac{3\alpha}{2},$$

we get

$$\frac{1}{\sin\frac{\alpha}{2}}=\frac{1}{\sin\alpha}+\frac{1}{\sin\frac{3\alpha}{2}}.$$

Clearing denominators, we have

$$\begin{aligned}0&=\sin\alpha\sin\frac{3\alpha}{2}-\left(\sin\alpha+\sin\frac{3\alpha}{2}\right)\sin\frac{\alpha}{2}\\&=\frac{1}{2}\left(\cos\frac{\alpha}{2}-\cos\frac{5\alpha}{2}\right)-\frac{1}{2}\left(\cos\frac{\alpha}{2}-\cos\frac{3\alpha}{2}\right)\\&\qquad-\frac{1}{2}(\cos\alpha-\cos2\alpha)\\&=\frac{1}{2}\left(\left(\cos\frac{3\alpha}{2}+\cos2\alpha\right)-\left(\cos\alpha+\cos\frac{5\alpha}{2}\right)\right)\\&=\cos\frac{7\alpha}{4}\left(\cos\frac{\alpha}{4}-\cos\frac{3\alpha}{4}\right)\\&=2\cos\frac{7\alpha}{4}\sin\frac{\alpha}{4}\sin\frac{\alpha}{2}.\end{aligned}$$

Then $7\alpha/4=90^\circ$, that is $\alpha=360^\circ/7$. So the polygon has 7 sides.

Example 7. (*Austria*, 1973) Prove that if the angles of a convex octagon are all equal and the ratio of all pairs of adjacent sides is rational, then each pair of opposite sides has equal length.

Solution. Without loss of generality, we may assume the sides of such a polygon $A_1A_2\ldots A_8$ are rational (since the conclusion is the same for octagons similar to such an octagon). Now the sum of all angles of the octagon is $6\times180^\circ$. Hence each angle is 45°.

Let v_n be the vector from A_n to A_{n+1} for $n=1,2,\ldots,8$ (with $A_9=A_1$). Then the angle between v_n and v_{n+1} at the origin is 45°. Observe that the sum of these vectors is zero since we start at A_1 and traverse the octagon once to return to A_1.

Let i and j be a pair of unit vectors perpendicular to each other at the origin. By rotation, we may assume v_1 is a vector in the i direction and v_3 is in the j direction. Then $v_1+v_5=xi$ and $v_3+v_7=yj$ for some rational x and y. Also,

$$v_2+v_4+v_6+v_8=r\sqrt{2}i\pm r\sqrt{2}j$$

for some rational r. Then

$$(x+r\sqrt{2})i+(y\pm r\sqrt{2})j=\sum_{n=1}^{8}v_n=0.$$

Since, x and r are rational, we must have $x=r=0$. That is, $v_5=-v_1$. By rotating the i,j vectors by 45°, similarly we get $v_6=-v_2$. Then also $v_7=-v_3$ and $v_8=-v_4$. The result follows.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 10, 2015.***

Problem 461. Inside rectangle $ABCD$, there is a circle. Points W, X, Y, Z are on the circle such that lines AW, BX, CY, DZ are tangent to the circle. If AW=3, BX=4, CY=5, then find DZ with proof.

Problem 462. For all $x_1, x_2, \ldots, x_n \geq 0$, let $x_{n+1} = x_1$, then prove that

$$\sum_{k=1}^{n}\sqrt{\frac{1}{(x_k+1)^2}+\frac{x_{k+1}^2}{(x_{k+1}+1)^2}} \geq \frac{n}{\sqrt{2}}.$$

Problem 463. Let S be a set with 20 elements. N 2-element subsets of S are chosen with no two of these subsets equal. Find the least number N such that among any 3 elements in S, there exist 2 of them belong to one of the N chosen subsets.

Problem 464. Determine all positive integers n such that for n, there exists an integer m with 2^n-1 divides m^2+289.

Problem 465. Points A, E, D, C, F, B lie on a circle Γ in clockwise order. Rays AD, BC, the tangents to Γ at E and at F pass through P. Chord EF meets chords AD and BC at M and N respectively. Prove that lines AB, CD, EF are concurrent.

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Solutions

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Problem 456. Suppose $x_1, x_2, \ldots, x_n$ are non-negative and their sum is 1. Prove that there exists a permutation σ of $\{1,2,\cdots,n\}$ such that

$x_{\sigma(1)}x_{\sigma(2)}+x_{\sigma(2)}x_{\sigma(3)}+\cdots+x_{\sigma(n)}x_{\sigma(1)} \leq 1/n.$

Solution. **CHAN Long Tin** (Cambridge University, Year 3), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Samiron SADHUKHAN** (Kendriya Vidyalaya, India) and **WONG Yat** (G. T. (Ellen Yeung) College).

Assume the contrary is true. Let $\sigma(n+1) = \sigma(1)$ for all permutations σ. For $1 \leq i < j \leq n$, the terms x_ix_j and x_jx_i appear a total of $2n(n-2)!$ times in

$$\sum_{n\in S_n}\sum_{k=1}^{n} x_{\sigma(k)}x_{\sigma(k+1)}.$$

So, we have

$$\begin{aligned}\frac{n!}{n} &< \sum_{\sigma\in S_n}\sum_{k=1}^{n} x_{\sigma(k)}x_{\sigma(k+1)}\\ &= 2n(n-2)!\sum_{1\leq i<j\leq n} x_ix_j\\ &= n(n-2)!\left[\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}x_i^2\right]\\ &= n(n-2)!\left(1-\sum_{i=1}^{n}x_i^2\right).\end{aligned}$$

This simplifies to (*) $\sum_{i=1}^{n}x_i^2 < \frac{1}{n}$. However, by the Cauchy-Schwarz inequality,

$$n\sum_{i=1}^{n}x_i^2 = \sum_{i=1}^{n}1^2\sum_{i=1}^{n}x_i^2 \geq \left(\sum_{i=1}^{n}x_i\right)^2 = 1,$$

which contradicts (*).

Problem 457. Prove that for each n = 1,2,3,…, there exist integers a, b such that if integers x, y are relatively prime, then $\sqrt{(a-x)^2+(b-y)^2} > n.$

Solution. **Samiron SADHUKHAN** (Kendriya Vidyalaya, India) and **WONG Yat** (G. T. (Ellen Yeung) College).

There are $(2n+1)^2$ ordered pairs (r,s) of integers satisfying $|r|, |s| \leq n$. Assign a <u>distinct</u> prime number $p_{r,s}$ to each such (r,s). By the Chinese remainder theorem, there exist integers a,b such that for all integers r; s satisfying $|r|, |s| \leq n$, we have $a \equiv r \pmod{p_{r,s}}$ and $b \equiv s \pmod{p_{r,s}}$.

Let integers x, y be relatively prime. Assume (x,y) has distance at most n from (a,b). Then $|a-x| \leq n$ and $|b-y| \leq n$. Let $a-x=r$ and $b-y=s$. Then $x=a-r$ and $y=b-s$ are multiples of $p_{r,s}$, contradicting $\gcd(x,y) = 1$. Therefore,

$$\sqrt{(a-x)^2+(b-y)^2} > n.$$

Problem 458. Nonempty sets A_1, A_2, A_3 form a partition of $\{1,2,\ldots,n\}$. If $x+y=z$ have no solution with x in A_i, y in A_j, z in A_k and $\{i,j,k\}=\{1,2,3\}$, then prove that A_1, A_2, A_3 cannot have the same number of elements.

Solution. **Oliver GEUPEL** (Brühl, NRW, Germany) and **John GLIMMS**.

Without loss of generality, say $1\in A_1$ and the smallest element in $A_2 \cup A_3$ is $b\in A_2$. Let the elements in A_3 be $c_1, c_2, \ldots, c_k$ in increasing order.

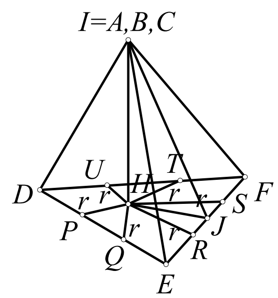
Assume $c_{i+1}-c_i=1$ for some i. Then take i to be the smallest possible. Since $b\in A_2$, the equations $(c_i-b)+b=c_i$ and $(c_i-b+1)+b=c_{i+1}$ imply c_i-b and c_i-b+1 are both not in A_1.

Since $1\in A_1$ and $(c_i-b)+1= c_i-b+1$, so either c_i-b+1 and c_i-b both are in A_2 or both are in A_3. Since i is smallest such that $c_{i+1}-c_i=1$, so c_i-b+1 and c_i-b cannot be in A_3. However, c_i-b+1 and c_i-b in A_2, $b-1$ in A_1 (by property of b) and $(b-1)+(c_i-b+1)=c_i$ lead to contradiction. So $c_{i+1}-c_i \geq 2$ for all i.

Finally, since $1+(c_i-1)=c_i$, we get $c_i-1 \notin B$. Hence $c_i-1 \in A$. Then A_1 contains 1, c_1-1, c_2-1, ... , c_k-1. Therefore, A_1 has more elements than A_3.

Problem 459. H is the orthocenter of acute $\triangle ABC$. D,E,F are midpoints of sides BC, CA, AB respectively. Inside $\triangle ABC$, a circle with center H meets DE at P,Q, EF at R,S, FD at T,U. Prove that $CP=CQ=AR=AS=BT=BU$.

Solution. **John GLIMMS.**



Let lines AH and FE meet at J. From $AH\perp BC$ and $BC \parallel FE$, we get FE is perpendicular to AJ and HJ. By folding along DE, EF and FD, we can make a tetrahedron having $\triangle DEF$ as the base and points A, B, C meet at a point I. Then FE is perpendicular to IJ and HJ. So FE is perpendicular to the plane through I,J,H. Then $FE\perp IH$. Similarly, $DE\perp IH$. Then the plane through D,E,F is perpendicular to IH. By Pythagoras' theorem, $IH^2+r^2 = CP^2 = CQ^2 = AR^2 = AS^2 = BT^2 = BV^2$, where r is the radius of the circle.

Other commended solvers: **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Andrea FANCHINI**

(Cantú, Italy), **William FUNG**, **Oliver GEUPEL** (Brühl, NRW, Germany), **MANOLOUDIS Apostolis** (4 High School of Korydallos, Piraeus, Greece), **Samiron SADHUKHAN** (Kendriya Vidyalaya, India), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 460. If $x,y,z > 0$ and $x+y+z+2 = xyz$, then prove that

$$x+y+z+6 \ge 2\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right).$$

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **CHAN Long Tin** (Cambridge University, Year 3), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Oliver GEUPEL** (Brühl, NRW, Germany), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Vijaya Prasad NULLARI** (Retired Principal, AP Educational Service, India), **Nicuşor ZLOTA** ("Traian Vuia" Technical College,Focşani, Romania) and **Titu ZVONARU** (Comăneşti, Romania).

Let

$$a=\frac{1}{1+x}, b=\frac{1}{1+y}, c=\frac{1}{1+z}.$$

Using $x+y+z+2 = xyz$, we get $a+b+c = 1$. Then $x = (1-a)/a = (b+c)/a$ and similarly $y=(c+a)/b$ and $z=(a+b)/c$. By the AM-GM inequality, we have

$$\begin{aligned} & x+y+z+6 \\ &= \sum_{cyc}\frac{b+c}{a}+6 \\ &= \sum_{cyc}\left(\frac{c+a}{c}+\frac{a+b}{b}\right) \\ &\ge 2\sum_{cyc}\sqrt{\frac{(c+a)(a+b)}{bc}} \\ &= 2\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right). \end{aligned}$$

Other commended solvers: **Paolo PERFETTI** (Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **WONG Yat** (G. T. (Ellen Yeung) College).

Olympiad Corner

(*Continued from page 1*)

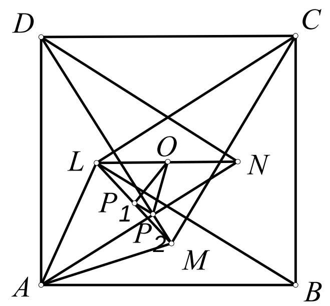
Problem 5. On each of the 2014^2 squares of a 2014×2014-board a light bulb is put. Light bulbs can be either on or off. In the starting situation a number of light bulbs are on. A move consists of choosing a row or column in which at least 1007 light bulbs are on and changing the state of all 2014 light bulbs in this row or column (from on to off or from off to on). Find the smallest non-negative integer k such that from each starting situation there is a finite sequence of moves to a situation in which at most k light bulbs are on.

Polygonal Problems

(*Continued from page 2*)

<u>*Example 8.*</u> (*IMO* 1997) Equilateral triangles *ABK, BCL, CDM, DAN* are constructed inside the square *ABCD*. Prove that the midpoints of the four segments *KL, LM, MN, NK* and the midpints of the eight segments *AK, BK, BL, CL, CM, DM, DN, AN* are the twelve vertices of a regular dodecagon.

<u>*Solution.*</u>



Let us denote the midpoints of segments *LM, AN, BL, MN, BK, CM, NK, CL, DN, KL, DM, AK* by P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, respectively. To prove the dodecagon

$$P_1P_2P_3P_4P_5P_6P_7P_8P_9P_{10}P_{11}P_{12}$$

is regular, we observe that $BL=BA$ and $\angle ABL=30°$. Then $\angle BAL=75°$. Similarly $\angle DAM=75°$. So

$$\angle LAM=\angle BAL+\angle DAM-\angle BAD=60°.$$

Along with $AL=AM$, we see triangle ALM is equilateral.

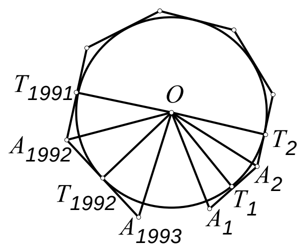
Looking at triangles OLM and ALN, we get $OP_1=½LM$, $OP_2=½AL$ and $OP_2 \parallel AL$. Hence, $OP_1=OP_2$, $\angle P_1OP_2=\angle P_1AL = 30°$, $\angle P_2OM = \angle DAL=15°$ and $\angle P_2OP_3 = 2\angle P_2OM = 30°$. By symmetry, we can conclude that the dodecagon is regular.

<u>*Example 9.*</u> (*IMO* 1992, *Shortlisted Problem from India*) Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:

(i) its sides lengths are 1,2,3,…,1992 in some order;

(ii) the polygon is circumscribable about a circle.

<u>*Solution.*</u> For a positive number r, let us draw a circle of radius r and let us draw a polygonal path $A_1A_2\ldots A_{1993}$ such that for $i=1$ to 1992, side A_iA_{i+1} is tangent to the circle at a point T_i and $T_{1992}A_{1993} = A_1T_1$, $T_1A_2 = A_2T_2$, … , $T_{1991}A_{1992} = A_{1992}T_{1992}$.



To achieve condition (i), we need A_1A_2, A_2A_3, …, $A_{1992}A_{1993}$ to be a permutation of 1, 2, …, 1992. This can be done as follow:

If $i\equiv 1 \pmod 4$, then let $A_iT_i=1/2$.
If $i\equiv 3 \pmod 4$, then let $A_iT_i=3/2$.
If $i\equiv 0,2 \pmod 4$, then let $A_iT_i=i-3/2$.

We can check that the lengths of A_iA_{i+1} for $i=1$ to 1992 are 1, 2, 4, 3, 5, 6, 8, 7,…, 1989,1990,1992,1991.

To achieve condition (ii), we define a function

$$f(r)=\sum_{i=1}^{1992}\angle A_iOA_{i+1} = 2\sum_{i=1}^{1992}\arctan\frac{A_iT_i}{r}.$$

Observe that $f(r)$ is a continuous function on $(0,\infty)$. As r tends to 0, $f(r)$ tends to infinity and as r tends to infinity, $f(r)$ tends to 0. By the intermediate value theorem, there exists r such that $f(r) = 2\pi$. Then $A_{1993}=A_1$ and $A_1A_2\ldots A_{1992}$ is a desired polygon.

We remark that if 1992 is replaced by other positive integers of the form $4k$, then there are such $4k$-sided polygon.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2015 Canadian Mathematical Olympiad held in January 28, 2015.

Notation: If V and W are two points, then VW denotes the line segment with endpoints V and W as well as the length of this segment.

Problem 1. Let $\mathbb{N} = \{1,2,3,\ldots\}$ denote the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2+n$ for every positive integer n.

Problem 2. Let ABC be an acute-angled triangle with altitudes AD, BE and CF. Let H be the orthocenter, that is, the point where the altitudes meet. Prove that

$$\frac{AB\cdot AC+BC\cdot BA+CA\cdot CB}{AH\cdot AD+BH\cdot BE+CH\cdot CF}\le 2.$$

Problem 3. On a $(4n+2)\times(4n+2)$ square grid, a turtle can move between squares sharing a side. The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 27, 2015***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Tournament of the Towns

Kin Yin Li

In 1980, Kiev, Moscow and Riga participated in a mathematical problem solving contest for high school students, later called the Tournament of the Towns. At present thousands of high school students from dozens of cities all over the world participate in this contest. In this article, we present some very interesting math problems from this contest. At the end of the article, there are some information on where interested readers can find past problems and solutions of this contest.

Here are some examples we enjoy.

Example 1. (*Junior Questions, Spring 1981, proposed by A. Andjans*) Each of 64 friends simultaneously learns one different item of news. They begin to phone one another to tell them their news. Each conversation last exactly one hour, during which time it is possible for two friends to tell each other all of their news. What is the minimum number of hours needed in order for all of the friends to know all the news?

Solution. More generally, suppose there are 2^n friends. After n rounds, the most anyone can learn are 2^n pieces of gossip. Hence n rounds are necessary. We now prove by induction on n that n rounds are also sufficient. For n=1, the result is trivial. Suppose the result holds up to $n-1$ for some $n\ge 2$. Consider the next case with 2^n friends. Have them call each other impairs in the first round. After this, divide them into two groups, each containing one member from each pair who had exchanged gossip. Each group has 2^{n-1} friends who know all the gossip among them. By the induction hypothesis, $n-1$ rounds are sufficient for everyone within each group to learn everything. This completes the induction argument. In particular, with 64 friends, 6 rounds are both necessary and sufficient.

Example 2. (*Senior Questions, Spring 1983, proposed by A. Andjans*) There are K boys placed around a circle. Each of them has an even number of sweets. At a command each boy gives half of his sweets to the boy on his right. If, after that, any boy has an odd number of sweets, someone outside the circle gives him one more sweet to make the number even. This procedure can be repeated indefinitely. Prove that there will be a time at which all boys have the same number of sweets.

Solution. Suppose initially the maximum number of sweets a boy has is $2m$, and the minimum is $2n$. We may as well assume that $m>n$. After a round of exchange and possible augmentation, we claim that the most any boy can have is $2m$ sweets. This is because he could have kept at most m sweets, and received m more in the exchange, but will not be augmented if he already has $2m$ sweets.

On the other hand, at least one boy who had $2n$ sweets will have more than that, because as long as $m>n$, one of these boys will get more than he gives away. It follows that while the maximum cannot increase, the minimum must increase until all have the same number of sweets.

Example 3. (*Junior Questions, Autumn 1984*) Six musicians gathered at a chamber music festival. At each scheduled concert some of these musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled in order to enable each musician to listen, as a member of the audience, to all the other musicians?

(continued on page 2)

<u>*Solution*</u>. Let the musicians be *A, B, C, D, E* and *F*. Suppose there are only three concerts. Since each of the six must perform at least once, at least one concert must feature two or more musicians. Say both *A* and *B* perform in the first concert. They must still perform for each other. Say *A* performs in the second concert for *B* and *B* in the third for *A*. Now *C*, *D*, *E* and *F* must all perform in the second concert, since it is the only time *B* is in the audience. Similarly, they must all perform in the third. The first concert alone is not enough to allow *C*, *D*, *E* and *F* to perform for one another. Hence we need at least four concerts. This is sufficient, as we may have *A*, *B* and *C* in the first, *A*, *D* and *E* in the second, *B*, *D* and *F* in the third and *C*, *E* and *F* in the fourth.

<u>*Example 4.*</u> (*Junior Questions, Autumn 1984, proposed by V. G. Ilichev*) On the Island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two different chameleons of different colours meet, they both simultaneously change colour to the third colour (eg. If a grey and a brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be the same colour?

<u>*Solution*</u>. In this case the numbers of chameleons of each colour at the start have remainders of 0, 1 and 2 when divided by three. Each "meeting" maintains such a situation (not necessarily in any order) as two of these remainders must either be reduced by 1 (or increased by 2) while the other must be increased by 2 (or reduced by 1). Thus at least two colours are present at any stage, guaranteeing the possibility of obtaining all of the three colours in fact by future meetings.

<u>*Note.*</u> The only way of getting chameleons to be of the same colour would be getting an equal number of two colours first. This would mean getting two with the same remainder on division by three. This would have been possible if we had started with, say 15 of each colour. From this position we can obtain sets with remainders equal to {0,0,0}, {1,1,1} and {2,2,2}.

<u>*Example 5.*</u> (*Junior Questions, Spring 1985, proposed by S. Fomin*) There are 68 coins, each coin having a different weight that that of each other. Show how to find the heaviest and lightest coin in 100 weighings on a balance beam.

<u>*Solution 1*</u>. First divide into 34 pairs and perform 34 weighings, each time identifying the heavier and lighter coins. Put all the heavier coins into one group and the lighter coins into another. Divide the group with heavier coins into 17 pairs, and perform 17 weighings on these to identify the 17 heavier coins. Continue this process with the group of heavier coins each time. If there is an odd number of coins at any stage, the odd coin out must be carried over to the following stage. There will be a total of 17+8+4+2+1+1=3 such weighings required for identifying the heaviest coin.

A similar 33 weighings of the lighter group will identify the lightest coin. The total number of weighing is thus 34+33+33=100, as required.

<u>*Solution 2*</u>. More generally, we show that $3n-2$ weighings are sufficient for $2n$ coins. We first divide the coins into n pairs, and use n weighings to sort them out into a "heavy" pile and a "light" pile. The heaviest coin is among the n coins in the "heavy" pile. Each weighing eliminates 1 coin. Since there are n coins, $n-1$ weighings are necessary and sufficient. Similarly, $n-1$ weighings will locate the lightest coin in the "light" pile. Thus the task can be accomplished in $3n-2$ weighings.

<u>*Example 6.*</u> (*Junior Questions, Spring 1987, proposed by D. Fomin*) A certain number of cubes are painted in six colours, each cube having six faces of different colours (the colours in different cubes may be arranged differently). The cubes are placed on a table so as to form a rectangle. We are allowed to take out any column of cubes, rotate it (as a whole) along its long axis and place it in a rectangle. A similar operation with rows is also allowed. Can we always make the rectangle monochromatic (i.e. such that the top faces of all the cubes are the same colour) by means of such operations?

<u>*Solution*</u>. The task can always be accomplished, and we can select the top colour in advance, say red. By fixing a cube, we mean bringing its red face to the top. Given a rectangular block, we fix one cube at a time, from left to right, and from front to back.

Suppose that the cube in the i-th row and the j-th column is the next to be fixed. Suppose that we need to rotate the i-th row. In order not to unfix the first $j-1$ cubes of this row, we rotate each of the first $j-1$ columns so that all red faces are to the left. They remain to the left when the i-th row is rotated. We can now refix the first $j-1$ columns.

Similarly, if we need to rotate the j-th column, we can go through an analogous three-step process.

<u>*Example 7.*</u> (*Senior Questions, Autumn 1987, proposed by A. Andjans*) A certain town is represented as an infinite plane, which is divided by straight lines into squares. The lines are streets, while the squares are blocks. Along a certain street there stands a policeman on each 100^{th} intersection. Somewhere in the town there is a bandit, whose position and speed are unknown, but he can move only along the streets. The aim of the police is to see the bandit. Does there exist an algorithm available to the police to enable them to achieve their aim?

<u>*Solution.*</u> We assume that (a) there is no limit to how far a policeman can see along the street he is on; (b) there is no overall time limit, and (c) if the bandit is ever on the same street as a policeman he will be seen.

Let i, j and k denote integers, let the North-South streets be $x=i$ for all i, the East-West streets $y=j$ for all j and suppose the k-th policeman is at $(100k,0)$.

For all even k the k-th policeman remains stationary throughout. This traps the bandit in the infinite strip between $x=200k$ and $x=200(k+1)$ for some k, say k^*.

All other policemen first travel along $y=0$ towards $(0,0)$ until they reach the first cross street $x=s$ for which there is a policeman on every street $x=i$ for i between 0 and s. Police are to travel at regulation speed, say one block per minute, but nevertheless there will come a time, dependent only on k^*, when every street $x=i$ on the k^* strip will be policed.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***August 27, 2015.***

Problem 466. Let k be an integer greater than 1. If k+2 integers are chosen among 1,2,3,…,3k, then there exist two of these integers m,n such that k<|m−n|<2k.

Problem 467. Let p be a prime number and q be a positive integer. Take any pq consecutive integers. Among these integers, remove all multiples of p. Let M be the product of the remaining integers. Determine the remainder when M is divided by p in terms of q.

Problem 468. Let $ABCD$ be a cyclic quadrilateral satisfying $BC>AD$ and $CD>AB$. E, F are points on chords BC, CD respectively and M is the midpoint of EF. If $BE=AD$ and $DF=AB$, then prove that $BM\perp DM$.

Problem 469. Let m be an integer greater than 4. On the plane, if m points satisfy no three of them are collinear and every four of them are the vertices of a convex quadrilateral, then prove that all m of the points are the vertices of a m-sided convex polygon.

Problem 470. If a, b, c>0, then prove that

$$\frac{a}{b(a^2+2b^2)}+\frac{b}{c(b^2+2c^2)}+\frac{c}{a(c^2+2a^2)} \ge \frac{3}{ab+bc+ca}.$$

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Solutions

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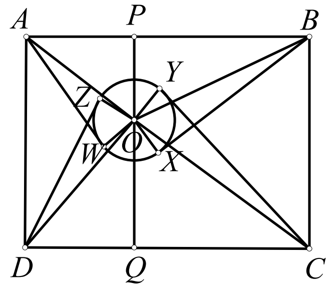
Problem 461. Inside rectangle $ABCD$, there is a circle. Points W, X, Y, Z are on the circle such that lines AW, BX, CY, DZ are tangent to the circle. If AW=3, BX=4, CY=5, then find DZ with proof.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Andrea FANCHINI** (Cantú, Italy), **William FUNG**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Jon GLIMMS**, **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, India), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).



Let r be the radius of the circle. By Pythagoras' theorem, we have

$$r^2 = AW^2-AO^2 = BX^2-BO^2 = CY^2-CO^2 = DZ^2-DO^2. \quad (*)$$

Let P,Q be the feet of perpendiculars from O to AB, CD respectively. Then

$$AO^2-BO^2 = (AP^2+PO^2)-(BP^2+PO^2) = (DQ^2+QO^2)-(CQ^2+QO^2) = DO^2-CO^2.$$

Using (*), we get $AW^2-BX^2=AO^2-BO^2=DO^2-CO^2=DZ^2-CY^2$. Then

$$DZ=\sqrt{AW^2-BX^2+CY^2}=3\sqrt{2}.$$

Problem 462. For all $x_1, x_2, \ldots, x_n \ge 0$, let $x_{n+1}=x_1$, then prove that

$$\sum_{k=1}^{n}\sqrt{\frac{1}{(x_k+1)^2}+\frac{x_{k+1}^2}{(x_{k+1}+1)^2}} \ge \frac{n}{\sqrt{2}}.$$

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **DHRUV Nevatia** (10th Standard, Ramanujan Academy, Nashik, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **MAMEDOV Shatlyk** (School of Young Physics and Maths N 21, Dashogus, Turkmenistan), **Corneliu MĂNESCU- AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Toshihiro SHIMIZU** (Kawasaki, Japan), **WADAH Ali** (Ben Badis College, Algeria), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

By squaring both sides or RMS-AM inequality, we have for all $a,b \ge 0$,

$$\sqrt{a^2+b^2} \ge \frac{a+b}{\sqrt{2}}.$$

Applying this, we get

$$\sum_{k=1}^{n}\sqrt{\frac{1}{(x_k+1)^2}+\frac{x_{k+1}^2}{(x_{k+1}+1)^2}} \ge \sum_{k=1}^{n}\frac{1}{\sqrt{2}}\left(\frac{1}{x_k+1}+(1-\frac{1}{x_{k+1}+1})\right)=\frac{n}{\sqrt{2}}.$$

Problem 463. Let S be a set with 20 elements. N 2-element subsets of S are chosen with no two of these subsets equal. Find the least number N such that among any 3 elements in S, there exist 2 of them belong to one of the N chosen subsets.

Solution. **Jon GLIMMS**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let $x\in S$ be contained in k of the N 2-elements subsets of S, where k is least among the elements of S.

Let $x_1, x_2, \ldots, x_k$ be the other elements in k of the N 2-element subsets with x. As k is least, so each of the x_i's is also contained in at least k of the N 2-element subsets of S.

Also, there are m=19−k elements $w_1, w_2, \ldots, w_m \in S$ not in any of the N 2-element subsets of S with x. For every pair w_r, w_s of these, $\{w_r, w_s\}$ is one of these N 2-element subsets of S (otherwise, no two of x, w_r, w_s form one of the N 2-element subsets). Then

$$N\geq\binom{k+1}{2}+\binom{19-k}{2}=(k-9)^2+90\geq 90.$$

To get the least case of N=90, we divide the 20 elements into two groups of 10 elements. Then take all 2-element subsets of each of the two groups to get 45+45=90 2-element subsets of S.

Problem 464. Determine all positive integers n such that for n, there exists an integer m with 2^n-1 divides m^2+289.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **PANG Lok Wing** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

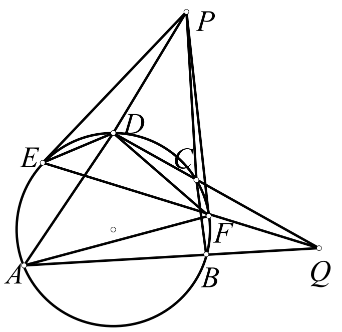
The case n=1 is a solution. For n>1, we first show if a prime q of the form $4k$+3 divides a^2+b^2, then q divides a and b. Assume gcd(q,a)=1. Let $c=a^{q-2}$. Then by Fermat's little theorem, $ac=a^{q-1}\equiv 1$ (mod q). As $q|a^2+b^2$, so $b^2\equiv -a^2$ (mod q). Then $(bc)^2\equiv -(ac)^2\equiv\ -1$ (mod q) and $(bc)^{q-1}=(bc)^{2(2k+1)}\equiv = -1$ (mod q), contradicting Fermat's little theorem. So q divides a (and b similarly).

If n>1, then $2^n-1\equiv 3$ (mod 4). Hence 2^n-1 has a prime divisor $q\equiv 3$ (mod 4). By the fact above, q divides m^2+289 implies q divides m and 17. Then q=17 $\not\equiv 3$ (mod 4), contradiction.

Problem 465. Points A, E, D, C, F, B lie on a circle Γ in clockwise order. Rays AD, BC, the tangents to Γ at E and at F pass through P. Chord EF meets chords AD and BC at M and N respectively. Prove that lines AB, CD, EF are concurrent.

Comments. A number of solvers pointed out if lines AB, CD are parallel, then by symmetry lines AB, CD, EF are all parallel. So below, we present solutions for the case when lines AB and CD intersect at a point.

***Solution* 1. Jon GLIMMS.**



Let lines AB, CD meet at Q. We have

(1) $\angle AFE=\angle ADE=180^\circ-\angle PDE$,
(2) $\angle EFD=\angle PED$,
(3) $\angle FDQ=\angle PFC$,
(4) $\angle QAF=\angle FCB=180^\circ-\angle PCF$.
(5) $\angle DAQ=\angle DCP$,
(6) $\angle QDA=180^\circ-\angle PDC$.

Then

$$\frac{\sin\angle AFE}{\sin\angle EFD}=\frac{\sin\angle PDE}{\sin\angle PED}=\frac{PE}{PD},$$
$$\frac{\sin\angle FDQ}{\sin\angle QAF}=\frac{\sin\angle PFC}{\sin\angle PCF}=\frac{PC}{PF},$$
$$\frac{\sin\angle DAQ}{\sin\angle QDA}=\frac{\sin\angle DCP}{\sin\angle PDC}=\frac{PD}{PC}.$$

Multiplying these and using PE=PF, we have

$$\frac{\sin\angle AFE}{\sin\angle EFD}\cdot\frac{\sin\angle FDQ}{\sin\angle QAF}\cdot\frac{\sin\angle DAQ}{\sin\angle QDA}$$
$$=\frac{PE}{PD}\cdot\frac{PC}{PF}\cdot\frac{PD}{PC}=1.$$

Applying the converse of the trigonometric form of Ceva's theorem to $\triangle ADF$ and point Q, we get lines AB, CD, EF are concurrent at Q.

***Solution* 2. Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India) and **William FUNG**.

Since the tangents to Γ at E and at F intersect at P, line EF is the polar of P. Since lines AD, BC intersect at P, the polar of P (that is, line EF) passes through the intersection of lines AB and CD.

Other commended solvers: **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Olympiad Corner

(*Continued from page 1*)

Problem 3 (Cont'd). In terms of n, what is the largest positive integer k such that there must be a row or a column that the turtle has entered at least k distinct times?

Problem 4. Let ABC be an acute-angled triangle with circumcenter O. Let Γ be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that $BP\cdot CQ$=$AP\cdot AQ$. Prove that Γ is tangent to the circumcircle of triangle BOC.

Problem 5. Let p be a prime number for which $(p-1)/2$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1+\sqrt{2p}$ positive integers n such that n<p and p divides $a^n+b^n+c^n$.

Tournament of the Towns

(*Continued from page 2*)

When this happens the bandit will be trapped on some street $y=j^*$, on a single block between $x=i^*$ and $x=i^*+1$ for some i^*.

For each k, as soon as all streets on the k-th strip are policed, one of the policemen travels north and another travels south. For $k=k^*$ this will inevitably reveal the bandit.

After reading these examples, should anyone want to read more, below are websites, which books on this contest can be ordered or problems and solutions of the recent Tournament of the Towns can be found.

www.amtt.com.au/ProductList.php?pager=1&startpage=1

www.artofproblemsolving.com/community/c3239_tournament_of_towns

www.math.toronto.edu/oz/turgor/archives.php