# A Note on Power of a Point

**Abstract**

In this article we present an efficient metric criterion (based on the power of a point) for perpendicularity, when one of the lines is joining centers of two circles.

**Definition.** Let $\omega$ be a circle centered at $O$ with radius $r$. Then to each point $P$ in the plane of $\omega$ we can assign a number $p(P, \omega) = OP^2 - r^2$. This number is called the power of $P$ with respect to $\omega$.

We assume the reader is familiar with the basic properties of the power of a point including the existence of the radical axis. In this paper we further develop the concept of the radical axis and show that the locus of points for which the difference of powers with respect to some two given circles remains constant is a line parallel to the corresponding radical axis. This is a key lemma in all of the following problems.

**Lemma.** *Let $\omega_1$, $\omega_2$ be circles centered at $O_1$, $O_2$, respectively ($O_1 \neq O_2$). Then line $AB$ is perpendicular to $O_1O_2$ if and only if*

$$p(A, \omega_1) - p(A, \omega_2) = p(B, \omega_1) - p(B, \omega_2).$$

*Proof.* Rewriting the condition from the statement using the very definition of the power of a point we observe it is equivalent to

$$AO_1^2 - AO_2^2 = BO_1^2 - BO_2^2.$$

Let $A'$, $B'$ be the projections of $A$, $B$ onto line $O_1O_2$. Then by the *Pythagorean theorem* the above reduces to

$$A'O_1^2 - A'O_2^2 + (AA'^2 - AA'^2) = B'O_1^2 - B'O_2^2 + (BB'^2 - BB'^2),$$

which clearly holds if $A' = B'$ or, in other words, if $AB$ is perpendicular to $O_1O_2$.

For the "only if" part, it is clear that the value of $A'O_1^2 - A'O_2^2 = (A'O_1 + A'O_2)(A'O_1 - A'O_2)$ as $A'$ moves along the line $O_1O_2$ is strictly monotonic on segment $O_1O_2$ and simple computation shows that it's monotonic on both rays as well. Thus it is monotonic on the entire line. Details are left for the reader. $\square$

This lemma enables us to reduce a geometric problem to a metric relation. Once we can express everything in terms of independent variables, the problem becomes clearer. Moreover, the amount of computation can often be reduced if we make use of symmetry.

That's enough for the introduction, now let's solve some problems!

**Notation.** In a triangle $ABC$, denote by $a$, $b$, $c$ the lengths of respective sides, let $s = \frac{a+b+c}{2}$, let $r$ be the inradius and denote by $\omega, \omega_i, \omega_a, \omega_b, \omega_c$ the circumcircle, the incircle and the corresponding excircles. Finally, let

$$x = \frac{-a+b+c}{2}, \quad y = \frac{a-b+c}{2}, \quad z = \frac{a+b-c}{2}.$$

**Problem 1.** *Let $BC$ be the longest side of a scalene triangle $ABC$. Point $K$ on the ray $CA$ satisfies $KC = BC$. Similarly, point $L$ on the ray $BA$ satisfies $BL = BC$. Prove that $KL$ is perpendicular to $OI$ where $O$ and $I$ denote the circumcenter and the incenter of $ABC$, respectively.*

*Proof.* Denote by $D$, $E$, $F$ the points of tangency of the incircle with sides $BC$, $CA$, $AB$, respectively. Keeping the above mentioned Lemma in mind, we want to express the power of $K$ with respect to $\omega$ and $\omega_i$ in terms of $a, b, c$ or $x, y, z$. Choosing the second option, straightforward computation gives

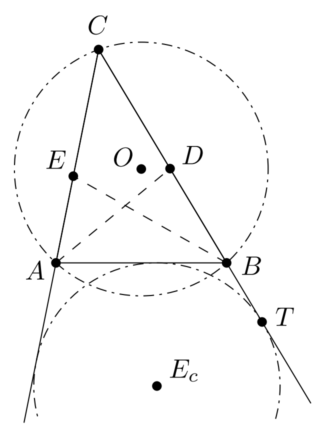
$$p(K,\omega) - p(K,\omega_i) = KA \cdot KC - KE^2 = (y-x)\cdot(y+z) - BD^2 = \\ = (y^2 + yz - xy - xz) - y^2 = yz - x(y+z),$$

which is symmetric with respect to $y$ and $z$ thus it is also symmetric with respect to $b$ and $c$. Using this symmetry and the stated Lemma we can conclude the proof. $\square$

**Problem 2.** *Let $ABC$ be a triangle, $O$ its circumcenter and $E_c$ its $C$–excenter. Let $D$, $E$ be the feet of angle bisectors of $\angle A$, $\angle B$ respectively. Show that $DE$ is perpendicular to $OE_c$.*

*Proof.* Due to the *Angle Bisector theorem* the lengths $DB$, $DC$ are computable in terms of $a, b, c$. Therefore the power of $D$ with respect to circumcircle of $ABC$ is simply

$$p(D,\omega) = -\frac{ab}{b+c} \cdot \frac{ac}{b+c} = -\frac{a^2bc}{(b+c)^2}.$$




Determining the power of $D$ with respect to the $C$–excircle is not much harder. If $T$ denotes point of tangency of the $C$–excircle and $BC$ then by equal tangents $CT = s$, so

$$p(D,\omega_c) = DT^2 = (CT - CD)^2 = \left(\frac{a+b+c}{2} - \frac{ab}{b+c}\right)^2 =$$

$$= \left(\frac{(b+c)^2 + a(c-b)}{2(b+c)}\right)^2 = \frac{(b+c)^4 + 2a(c-b)\cdot(b+c)^2 + a^2(c-b)^2}{4(b+c)^2}.$$

Looking at the difference we obtain

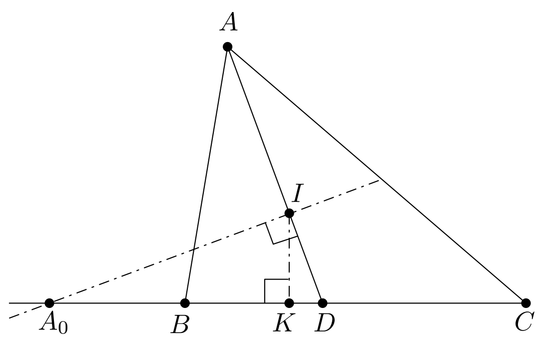
$$\begin{aligned}4(b+c)^2\big(p(D,\omega_c) - p(D,\omega)\big) &= (b+c)^4 + 2a(c-b)(b+c)^2 + a^2b^2 - 2a^2bc + a^2c^2 + 4a^2bc = \\ &= (b+c)^2\left((b+c)^2 + 2a(c-b) + a^2\right) = (b+c)^2\left(c^2 + 2c(a+b) + (a-b)^2\right),\end{aligned}$$

which is (after dividing by $(b+c)^2$) symmetric in $a, b$. This again suffices to finish the proof. □

**Problem 3.** *In a scalene triangle $ABC$ with incenter $I$ and circumcenter $O$ let the line passing through $I$ perpendicular to $AI$ intersect side $BC$ at $A_0$. Points $B_0$, $C_0$ are defined similarly. Prove that $A_0$, $B_0$, $C_0$ lie on a line perpendicular to $IO$.*

*Proof.* Let $AI$ intersect $BC$ at $D$ and denote by $K, L, M$ the points of tangency of the incircle $\omega$ with sides $BC, CA, AB$ respectively.

Assume WLOG $b > c$. Note that $\angle BID = \frac{\alpha}{2} + \frac{\beta}{2} < 90°$, thus $A_0$ lies outside segment $BC$ so that $A_0B < A_0C$.




By *Angle Bisector theorem* we have

$$BD = \frac{ac}{b+c} = \frac{(y+z)(x+y)}{s+x}.$$

But then

$$DK = BD - BK = \frac{(y+z)(x+y)}{s+x} - y = \frac{x(z-y)}{s+x}.$$

Now we observe $\triangle A_0KI \sim \triangle IKD$ (AA) and so $\frac{A_0K}{IK} = \frac{IK}{DK}$. From this point on it is pure computation, as $A_0K = \frac{r^2}{DK}$ and from *Heron's formula* we know $r^2 = \frac{xyz}{s}$. We calculate distances

$$A_0K = \frac{yz(s+x)}{s(z-y)}, \quad A_0B = \frac{y(zx+sy)}{s(z-y)}, \quad A_0C = \frac{z(xy+sz)}{s(z-y)}$$

and now we can express

$$p(A_0,\omega) - p(A_0,\omega_i) = A_0B \cdot A_0C - A_0K^2 = \frac{yz\big((zx+sy)(xy+sz) - yz(s+x)^2\big)}{s^2(y-z)^2} = \frac{xyz}{s}.$$

The last expression is symmetric in $x, y, z$ so $A_0, B_0, C_0$ are indeed collinear on a line perpendicular to $OI$. $\square$

In the following problem the Lemma doesn't seem applicable. The trick is to use inversion first.

**Problem 4** (APMO 2010)**.** *Let $ABC$ be an acute angled triangle satisfying the conditions $AB > BC$ and $AC > BC$. Denote by $O$ and $H$ the circumcenter and orthocenter, respectively, of the triangle. Suppose that the circumcircle of the triangle $AHC$ intersects line $AB$ at $M$ different from $A$, and similarly the circumcircle of the triangle $AHB$ intersects line $AC$ at $N$ different from $A$. Prove that the circumcenter of the triangle $MNH$ lies on line $OH$.*

*Proof.* First note that as $\triangle ABC$ is acute, points $M, N$ lie on its perimeter. Now we invert around $H$. Denote by $A', B', C', M', N'$ images of the corresponding points and let $O_1$ be the circumcenter of $\triangle A'B'C'$. Recall that the circumcircles of triangles $ABH, BCH, CAH$ all have equal radii as they are reflections of the circumcircle of $\triangle ABC$ over the triangle's sides. Thus their images are lines that all have the same distance from $H$. So $H$ is the incenter of $\triangle A'B'C'$ (that's the key!). We construct point $M'$ as the intersection of $A'C'$ and the circumcircle of $\triangle A'B'H$. As $M$ was inside segment $AB$, $M'$ lies on arc $A'B'$ that does not contain $H$, so it lies outside segment $A'C'$ and by the same argument $N'$ lies outside segment $A'B'$.

The image of circle $MNH$ is line $M'N'$, so we need to prove $M'N' \perp OH$ (consider symmetry w.r.t. line $OH$) or $M'N' \perp O_1H$ as $O, O_1, H$ are collinear. Now denote by $\alpha', \beta', \gamma'$ the corresponding angles in $\triangle A'B'C'$ and use the circle $A'HB'M'$ to get

$$\angle B'M'C' = 180^\circ - \angle A'HB' = \frac{\alpha'}{2} + \frac{\beta'}{2} = \angle M'B'H + \frac{\beta'}{2} = \angle M'B'C'.$$

So $C'M' = B'C'$ and similarly $B'N' = B'C'$ and what we are left to prove is exactly Problem 1. $\square$

# References


[1] Pohoata, C., *Problem S53*, Mathematical Reflections **3**, (2007).

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[4] Coxeter H.S.M., Greitzer S.L.,*Geometry revisited*, MAA, 1967.



Michal Rolínek and Josef Tkadlec

Prague, Czech Republic