Mathematical Excalibur

Volume 16, Number 1 *May 2011*

Olympiad Corner

Below are the problems of the 2011 Asia Pacific Math Olympiad, which was held in March 2011.

Problem 1. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers a^2+b+c, b^2+c+a, c^2+a+b to be perfect squares.

Problem 2. Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_iA_jA_k$ can take where i, j, k are distinct integers between 1 and 5.

Problem 3. Let ABC be an acute triangle with $\angle BAC=30°$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2, respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2, respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90°$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***June 25, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Harmonic Series (II)

Leung Tat-Wing

As usual, for integers a, b, n (with $n > 0$), we write $a\equiv b$ (mod n) to mean $a-b$ is divisible by n. If *b≠0 and n are relatively prime* (i.e. they have no common prime divisor), then $0, b, 2b, \ldots, (n-1)b$ are distinct (mod n) because for $0 \le s < r < n$, $rb \equiv sb$ (mod n) implies $(r-s)b = kn$. Since b, n have no common prime divisor, this means b divides k. Then $0 < (k/b)n = r-s < n$, contradicting $b \le k$. Hence, there is a unique r among $1, \ldots, n-1$ such that $rb\equiv 1$ (mod n). We will denote this r as b^{-1} or $1/b$ (mod n). Further, we can extend (mod n) to fractions by defining $a/b \equiv ab^{-1}$ (mod n). We can easily check that the usual properties of fractions holds in mod n arithmetic.

Next, we will introduce Wolstenholme's theorem, which is an important relation concerning harmonic series.

Theorem (Wolstenholme): For a prime $p \ge 5$,

$$H(p-1)=1+\frac{1}{2}+\ldots+\frac{1}{p-1}\equiv 0 \pmod{p^2}.$$

(More precisely, for a prime $p \ge 5$, if

$$H(p-1)=1+\frac{1}{2}+\ldots+\frac{1}{p-1}=\frac{a}{b},$$

then $p^2 \mid a$.)

Example We have

$$H(10)=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}=\frac{7381}{2520}$$

and $11^2 \mid 7381$.

First proof We have

$$H(p-1)=1+\frac{1}{2}+\cdots+\frac{1}{p-1}$$

$$=\sum_{n=1}^{(p-1)/2}\left(\frac{1}{n}+\frac{1}{p-n}\right)=p\sum_{n=1}^{(p-1)/2}\frac{1}{n(p-n)}.$$

So we need to prove

$$\sum_{n=1}^{(p-1)/2}\frac{1}{n(p-n)}\equiv 0 \pmod{p}.$$

Now $\displaystyle\sum_{n=1}^{(p-1)/2}\frac{1}{n(p-n)}\equiv -\sum_{n=1}^{(p-1)/2}\frac{1}{n^2} \pmod{p}.$

Since every $1/n^2$ is congruent to exactly one of the numbers $1^2, 2^2, \ldots, [(p-1)/2]^2$ (mod p) and $1/n^2$ are all distinct for $n=1,2,\ldots,(p-1)/2$, we have when $p \ge 5$,

$$\sum_{n=1}^{(p-1)/2}\frac{1}{n^2}\equiv\sum_{k=1}^{(p-1)/2}k^2=\frac{(p^2-1)p}{24}\equiv 0 \pmod{p}.$$

Wolstenholme's theorem follows.

Second proof (using polynomials mod p) We use a theorem of Lagrange, which says if $f(x) = c_0 + c_1x + \cdots + c_nx^n$ is a polynomial of degree n, with integer coefficients, and if $f(x) \equiv 0$ (mod p) has more than n solutions, where p is prime, then every coefficient of $f(x)$ is divisible by p. The proof is not hard. It can be done basically by induction and the division algorithm mod p. The statement is false if p is not prime. For instance, $x^2 - 1 \equiv 0$ (mod 8) has 4 solutions. Here is the other proof.

From Fermat's Little theorem, $x^{p-1} \equiv 1$ (mod p) has $1, 2, \ldots, p-1$ as solutions. Thus $x^{p-1}-1 \equiv (x-1)(x-2) \cdots (x-p+1)$ (mod p). Let

$$(x-1)(x-2)\cdots(x-p+1)$$
$$=x^{p-1}-s_1x^{p-2}+\cdots-s_{p-2}x+s_{p-1}. \quad (*)$$

By Wilson's theorem, $s_{p-1}=(p-1)!\equiv -1$ (mod p). Thus

$$0\equiv s_1x^{p-2}+\cdots-s_{p-2}x \pmod{p}.$$

The formula is true for every integer x. By Lagrange's theorem, p divides each of $s_1, s_2, \ldots, s_{p-2}$. Putting $x = p$ in (*), we get $(p-1)!=p^{p-1}-s_1p^{p-2}+\cdots-s_{p-2}p+s_{p-1}$. Canceling out $(p-1)!$ and dividing both sides by p, we get

$$0=p^{p-2}-s_1p^{p-3}+\cdots+s_{p-3}p-s_{p-2}.$$

As $p \ge 5$, each of the terms is congruent to 0 (mod p^2). Hence, we have $s_{p-2}\equiv 0$ (mod p^2). Finally,

$$s_{p-2}=(p-1)!(1+\frac{1}{2}+\ldots+\frac{1}{p-1})=(p-1)!\frac{a}{b}.$$

This proves Wolstenholme's theorem.

Using Wolstenholme's theorem and setting $x = kp$ in (*), we get

$$\begin{aligned}&(kp-1)(kp-2)\cdots(kp-p+1)\\ &= (kp)^{p-1}-s_1(kp)^{p-2}+\cdots\\ &\qquad +s_{p-3}(kp)^2-s_{p-2}kp+s_{p-1}\\ &\equiv s_{p-3}(kp)^2 - s_{p-2}kp + s_{p-1}\\ &\equiv (p-1)! \pmod{p^3}.\end{aligned}$$

Upon dividing by $(p-1)!$, we have

$$\binom{kp-1}{p-1} \equiv 1 \pmod{p^3}, \quad k = 1,2,\ldots.$$

This result may in fact be taken as the statement of Wolstenholme's theorem.

Here are a few further remarks. Wolstenholme's theorem on the congruence of harmonic series is related to the Bernoulli numbers B_n. For instance, we have

$$\binom{kp-1}{p-1} \equiv 1-\frac{1}{3}(k^2-k)p^3B_{p-3} \pmod{p^4},$$

which is usually called *Glaisher's congruence*. These numbers are related to Fermat's Last Theorem. It is known that for any prime $p \ge 5$,

$$\binom{kp-1}{p-1} \equiv 1 \pmod{p^3}.$$

Are there primes satisfying

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}?$$

These primes are called *Wolstenholme primes*. (So far, we only know 16843 and 2124679 are such primes). In another direction, one can ask if there exist composite numbers n such that

$$\binom{kn-1}{n-1} \equiv 1 \pmod{n^3}?$$

All these are very classical questions.

Example 10 (APMO 2006): Let $p \ge 5$ be a prime and let r be the number of ways of placing p checkers on a $p\times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5.

Solution Observe that

$$r = \binom{p^2}{p} - p = p\left(\frac{(p^2-1)\cdots(p^2-(p-1))}{(p-1)!} - 1\right).$$

Hence it suffices to show that

$$(p^2-1)(p^2-2)\cdots(p^2-(p-1)) - (p-1)! \equiv 0 \pmod{p^4} \qquad (1)$$

Now let

$$\begin{aligned}f(x) &= (x-1)(x-2)\cdots(x-(p-1))\\ &= x^{p-1} + s_1x^{p-2} + \cdots + s_{p-2}x + s_{p-1}. \quad (2)\end{aligned}$$

Thus the first congruence relation is the same as $f(p^2) - (p-1)! \equiv 0 \pmod{p^4}$. Therefore it suffices to show that $s_{p-2}\,p^2 \equiv 0 \pmod{p^4}$ or $s_{p-2} \equiv 0 \pmod{p^2}$, which is exactly Wolstenholme's theorem.

Example 11 (Putnam 1996): Let p be a prime number greater than 3 and $k = [2p/3]$. Show that

$$\binom{p}{1}+\binom{p}{2}+\ldots+\binom{p}{k} \equiv 0 \pmod{p^2}$$

For example,

$$\binom{7}{1}+\binom{7}{2}+\binom{7}{3}+\binom{7}{4} = 98 \equiv 0 \pmod{7^2}.$$

Solution Recall

$$\binom{p}{i} = \frac{p(p-1)\ldots(p-i+1)}{1\cdot 2\cdot\ldots i}.$$

This is a multiple of p if $1 \le i \le p-1$. Modulo p, the right side after divided by p is congruent to

$$\frac{(-1)\cdots(-(i-1))}{1\cdot 2\cdots i} = (-1)^{i-1}\frac{1}{i}.$$

Hence, to prove the congruence, it suffices to show

$$1-\frac{1}{2}+\frac{1}{3}-\cdots+(-1)^{k-1}\frac{1}{k} \equiv 0 \pmod{p}.$$

Now observe that

$$-\frac{1}{2i} \equiv \frac{1}{2i}+\frac{1}{p-i} \pmod{p}.$$

This allows us to replace the sum by

$$1+\frac{1}{2}+\ldots+\frac{1}{p-1} \equiv 0 \pmod{p},$$

which is Wolstenholme's theorem.

We can also give a more detailed proof as follow. Let

$$H(n) = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

and

$$P(n) = 1-\frac{1}{2}+\ldots+(-1)^{n-1}\frac{1}{n}.$$

Then the problem is reduced to showing that for any $p > 3$, p divides the numerator of $P([2p/3])$. First we note that p divides the numerator of $H(p-1)$ because

$$\begin{aligned}&2H(p-1)\\ &= (1+\frac{1}{p-1})+(\frac{1}{2}+\frac{1}{p-2})+\cdots+(\frac{1}{p-1}+1)\\ &= \frac{p}{p-1}+\frac{p}{2(p-2)}+\cdots+\frac{p}{p-1} \equiv 0 \pmod{p}.\end{aligned}$$

Next we have two cases.

<u>Case 1</u> ($p = 3n+1$) Then $[2p/3] = 2n$. So we must show p divides the numerator of $P(2n)$. Now

$$\begin{aligned}&H(3n)-P(2n)\\ &= 2(1+\frac{1}{2}+\ldots+\frac{1}{2n})+(\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots+\frac{1}{3n})\\ &= (1+\frac{1}{2}+\ldots+\frac{1}{n})+(\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots+\frac{1}{3n})\\ &= (1+\frac{1}{p-1})+(2+\frac{1}{p-2})+\ldots+(\frac{1}{n}+\frac{1}{p-n})\\ &= \frac{p}{p-1}+\frac{p}{2(p-2)}+\ldots+\frac{p}{n(p-n)}.\end{aligned}$$

So p divides the numerators of both $H(3n)$ and $H(3n) - P(2n)$, hence also the numerator of $P(2n)$.

<u>Case 2</u> ($p = 3n+2$) Then $[2p/3] = 2n+1$. So we must show p divides the numerator of $P(2n+1)$. Now

$$\begin{aligned}&H(3n+1)-P(2n+1)\\ &= 2(1+\frac{1}{2}+\ldots+\frac{1}{2n})+(\frac{1}{2n+2}+\frac{1}{2n+2}+\ldots+\frac{1}{3n+1})\\ &= (1+\frac{1}{2}+\ldots+\frac{1}{n})+(\frac{1}{2n+2}+\frac{1}{2n+2}+\ldots+\frac{1}{3n+1})\\ &= (1+\frac{1}{p-1})+(2+\frac{1}{p-2})+\ldots+(\frac{1}{n}+\frac{1}{p-n})\\ &= \frac{p}{p-1}+\frac{p}{2(p-2)}+\ldots+\frac{p}{n(p-n)}.\end{aligned}$$

So, p divides the numerator of $H(3n+1)-P(2n+1)$, and hence $P(2n+1)$.

Example 12: Let $p \ge 5$ be a prime, show that if

$$1+\frac{1}{2}+\ldots+\frac{1}{p} = \frac{a}{b},$$

then $p^4 | ap - b$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***June 25, 2011.***

Problem 371. Let $a_1, a_2, a_3, \ldots$ be a sequence of nonnegative rational numbers such that $a_m+a_n=a_{mn}$ for all positive integers m, n. Prove that there exist two terms that are equal.

Problem 372. (*Proposed by Terence ZHU)* For all $a,b,c>0$ and $abc=1$, prove that

$$\frac{1}{a(a+1)+ab(ab+1)}+\frac{1}{b(b+1)+bc(bc+1)}$$

$$+\frac{1}{c(c+1)+ca(ca+1)}\geq\frac{3}{4}.$$

Problem 373. Let x and y be the sums of positive integers $x_1,x_2,\ldots,x_{99}$ and $y_1,y_2,\ldots,y_{99}$ respectively. Prove that there exists a 50 element subset S of $\{1,2,\ldots,99\}$ such that the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

Problem 374. O is the circumcenter of acute ΔABC and T is the circumcenter of ΔAOC. Let M be the midpoint of side AC. On sides AB and BC, there are points D and E respectively such that $\angle BDM=\angle BEM=\angle ABC$. Prove that $BT\perp DE$.

Problem 375. Find (with proof) all odd integers $n>1$ such that if a, b are divisors of n and are relatively prime, then $a+b-1$ is also a divisor of n.

****

Solutions

****

Problem 366. Let n be a positive integer in base 10. For i=1,2,…,9, let $a(i)$ be the number of digits of n that equal i. Prove that

$$2^{a(1)}3^{a(2)}\cdots 9^{a(8)}10^{a(9)}\leq n+1$$

and determine all equality cases.

Solution. **LAU Chun Ting** (St. Paul's Co-educational College, Form 2).

Let $f(n)=2^{a(1)}3^{a(2)}\cdots 9^{a(8)}10^{a(9)}$. If n is a number with one digit, then $f(n)=n+1$. Suppose all numbers A with k digits satisfy the given inequality $f(A)\leq A+1$. For any $(k+1)$ digit number, it is of the form $10A+B$, where A is a k digit number and $0\leq B\leq 9$. We have

$$\begin{aligned}f(10A+B) &= (B+1)f(A)\leq(B+1)(A+1)\\ &= (B+1)A+B+1\leq 10A+B+1.\end{aligned}$$

Equality holds if and only if $f(A)=A+1$ and $B=9$. By induction, the inequality holds for all positive integers n and equality holds if and only if all but the leftmost digits of n are 9's.

Other commended solvers: **CHAN Long Tin** (Diocesan Boys' School), **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School), **Gordon MAN Siu Hang** (CCC Ming Yin College) and **YUNG Fai**.

Problem 367. For n = 1,2,3,…, let x_n and y_n be positive real numbers such that

$$x_{n+2}=x_n+x_{n+1}^2$$

and

$$y_{n+2}=y_n^2+y_{n+1}.$$

If x_1, x_2, y_1, y_2 are all greater than 1, then prove that there exists a positive integer N such that for all $n>N$, we have $x_n>y_n$.

Solution. **LAU Chun Ting** (St. Paul's Co-educational College, Form 2) and **Gordon MAN Siu Hang** (CCC Ming Yin College).

Since x_1, x_2, y_1, y_2 are all greater than 1, by induction, we can get $x_{n+1}>x_n^2>1$ and $y_{n+1}>1+y_n>n$ for $n\geq 2$. Then $x_{n+2}=x_n+x_{n+1}^2>x_{n+1}^2>x_n^4$ and $y_{n+2}=y_n^2+y_{n+1}=y_n^2+y_{n-1}^2+y_n<3y_n^2<y_n^3$ for all $n\geq 4$.

Hence, $\log x_{n+2}>4\log x_n$ and $\log y_{n+2}<3\log y_n$. So for $n\geq 4$,

$$\frac{\log x_{n+2}}{\log y_{n+2}}>\frac{4}{3}\left(\frac{\log x_n}{\log y_n}\right). \qquad (*)$$

As $4/3>1$, by taking logarithm, we can solve for a positive integer k satisfying the inequality

$$\left(\frac{4}{3}\right)^k\min\left\{\frac{\log x_4}{\log y_4},\frac{\log x_5}{\log y_5}\right\}>1.$$

Let $N=2k+3$. If $n>N$, then either $n=2m+4$ or $n=2m+5$ for some integer $m\geq k$.

Applying (*) m times, we have

$$\frac{\log x_n}{\log y_n}>\left(\frac{4}{3}\right)^m\min\left\{\frac{\log x_4}{\log y_4},\frac{\log x_5}{\log y_5}\right\}>1.$$

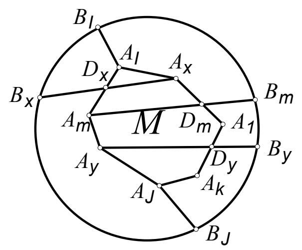
This implies $x_n>y_n$.

Other commended solvers: **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School) and **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam).

Problem 368. Let C be a circle, $A_1, A_2, \ldots, A_n$ be distinct points inside C and $B_1, B_2, \ldots, B_n$ be distinct points on C such that no two of the segments $A_1B_1, A_2B_2, \ldots, A_nB_n$ intersect. A grasshopper can jump from A_r to A_s if the line segment A_rA_s does not intersect any line segment A_tB_t ($t\neq r,s$). Prove that after a certain number of jumps, the grasshopper can jump from any A_u to any A_v.

Solution. **William PENG**.

The cases $n=1$ or 2 are clear. Suppose $n\geq 3$. By reordering the pairs A_i, B_i, we may suppose the convex hull of $A_1, A_2, \ldots, A_n$ is the polygonal region M with vertices $A_1, A_2, \ldots, A_k$ ($k\leq n$). For $1\leq m\leq k$, if every A_mB_m intersects M only at A_m, then the n-th case follows by removing two pairs of A_m, B_m separately and applying case $n-1$.



Otherwise, there exists a segment A_mB_m intersecting M at more than 1 point. Let it intersect the perimeter of M again at D_m. Since A_iB_i's do not intersect, so A_jD_j's (being subsets of A_iB_i's) do not intersect. In particular, D_m is not a vertex of M.

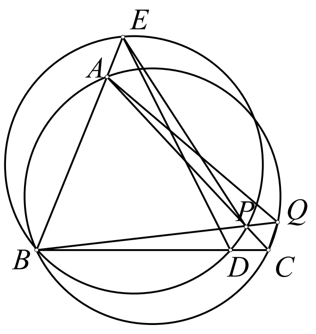
Now A_mD_m divides the perimeter of M into two parts. Moving from A_m to D_m clockwise on the perimeter of M, there are points A_x, D_x such that there is no D_w between them. As D_x is not a vertex, there is a vertex A_I between A_x and D_x. Then A_IB_I only intersect M at A_I. Also, moving from A_m to D_m anti-clockwise on the perimeter of M, there is A_J such that A_JB_J only intersects M at A_J. Then A_IB_I and A_JB_J do not intersect any diagonal of M with endpoints different from A_I and A_J.

Removing A_I, B_I and applying case $n-1$, the grasshopper can jump between any two of the points $A_1,\ldots,A_{I-1}, A_{I+1},\ldots,A_n$. Also, removing A_J, B_J and applying case $n-1$, the grasshopper can jump between any two of the points $A_1,\ldots,A_{J-1}, A_{J+1},\ldots,A_n$. Using these two cases, we see the grasshopper can jump from any A_u to any A_v via A_t ($t\neq I,J$).

Other commended solvers: **T. h. G**.

Problem 369. ABC is a triangle with $BC > CA > AB$. D is a point on side BC and E is a point on ray BA beyond A so that $BD=BE=CA$. Let P be a point on side AC such that E, B, D, P are concyclic. Let Q be the intersection point of ray BP and the circumcircle of $\triangle ABC$ different from B. Prove that $AQ+CQ=BP$.

Solution. **CHAN Long Tin** (Diocesan Boys' School), **Giorgos KALANTZIS** (Demenica's Public High School, Patras, Greece) and **LAU Chun Ting** (St. Paul's Co-educational College, Form 2).



Since A,B,C,Q are concyclic and E,P,D,B are concyclic, we have

$$\angle AQC = 180°-\angle ABC = \angle EPD$$

and

$$\angle PED = \angle PBD = \angle QAC.$$

Hence, $\triangle AQC$ and $\triangle EPD$ are similar. So we have $AQ/AC=PE/DE$ and $CQ/AC = PD/DE$. Cross-multiplying and adding these two equations, we get

$$(AQ+CQ)\times DE = (PE+PD)\times AC. \quad (*)$$

For cyclic quadrilateral $EPDB$, by the Ptolemy theorem, we have

$$\begin{aligned} BP\times DE &= PE\times BD+PD\times BE \\ &= (PE+PD)\times AC \qquad (**) \end{aligned}$$

Comparing (*) and (**), we have $AQ+CQ=BP$.

Other commended solvers: **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School).

Problem 370. On the coordinate plane, at every lattice point (x,y) (these are points where x, y are integers), there is a light. At time $t = 0$, exactly one light is turned on. For $n = 1, 2, 3, \ldots$, at time $t = n$, every light at a lattice point is turned on if it is at a distance 2005 from a light that was on at time $t = n - 1$. Prove that every light at a lattice point will eventually be turned on at some time.

Solution. **LAU Chun Ting** (St. Paul's Co-educational College, Form 2), **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School), **Gordon MAN Siu Hang** (CCC Ming Yin College) and **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy).

We may assume the light that was turned on at $t = 0$ was at the origin.

Let $z = 2005 = 5\times401 = (2^2+1^2)(20^2+1^2) = |(2+i)(20+i)|^2 = |41+22i|^2 = 41^2+22^2$. Let $x = 41^2-22^2=1037$ and $y = 2\times41\times22 = 1716$. Then $x^2+y^2 = z^2$.

By the Euclidean algorithm, we get $\gcd(1037, 1716) = 1$. By eliminating the remainders in the calculations, we get $84\times1716-139\times1037=1$.

Let V_1, V_2, V_3, V_4, V_5 be the vectors from the origin to (2005,0), (1037, 1716), (1037, −1716), (1716, 1037), (1716, −1037) respectively. We have $V_2 + V_3 = (2\times1037,0)$ and $V_4 + V_5 = (2\times1716,0)$. Then we can get $(1,0) = 1003[84(V_4+V_5)-139(V_2+V_3)]-V_1$.

So, from the origin, following these vector movements, we can get to the point (1,0). Similarly, we can get to the point (0,1). As $(a,b) = a(1,0) + b(0,1)$, we can get to any lattice point.

Olympiad Corner

(continued from page 1)

Problem 4. Let n be a fixed positive odd integer. Take $m+2$ distinct points P_0, P_1, …, P_{m+1} (where m is a non-negative integer) on the coordinate plane in such a way that the following 3 conditions are satisfied:

(1) $P_0=(0,1)$, $P_{m+1}=(n+1,n)$, and for each integer i, $1 \le i \le m$, both x- and y-coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).

(2) For each integer i, $0 \le i \le m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.

(3) For each pair i,j with $0 \le i < j \le m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 point.

Determine the maximum possible value that m can take.

Problem 5. Find all functions $f:\mathbb{R}\to\mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following 2 conditions:

(1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.

(2) For every pair of real numbers x and y, $f(xf(y)) + yf(x) = xf(y) + f(xy)$ is satisfied.

Harmonic Series (II)

(continued from page 2)

<u>**Solution**</u> By Wolstenholme's theorem,

$$p^2 \left| (p-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{p-1}\right)\right..$$

So,

$$1+\frac{1}{2}+\ldots+\frac{1}{p-1} = p^2\frac{x}{y},$$

where x, y are integers with y not divisible by p. So we have

$$\frac{a}{b}-\frac{1}{p} = p^2\frac{x}{y},$$

which implies $ap-b = p^3bx/y$. Finally,

$$\frac{a}{b} = \frac{2\cdot3\cdots p+1\cdot3\cdot4\cdots p+\cdots+1\cdot2\cdots(p-1)}{p!}$$

and the numerator of the right side is of the form $mp+(p-1)!$. Hence, it is not divisible by p. So $p \mid b$ and $p^4 \mid p^3bx/y = ap-b$.

<u>**Example 13**</u>: Let p be an odd prime, then prove that

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\ldots+(-1)^{p-2}\frac{1}{(p-1)^2} \equiv 0 \pmod{p}.$$

<u>**Solution**</u> The proof is not hard. Indeed,

$$\begin{aligned} &\sum_{k=1}^{p-1}(-1)^{k-1}\frac{1}{k^2} \\ &= -\sum_{k=1}^{(p-1)/2}\left((-1)^k\frac{1}{k^2}+(-1)^{p-k}\frac{1}{(p-k)^2}\right) \\ &\equiv -\sum_{k=1}^{(p-1)/2}\left((-1)^k\frac{1}{k^2}+(-1)^{1-k}\frac{1}{(-k)^2}\right) = 0 \pmod{p}. \end{aligned}$$

Mathematical Excalibur

Volume 16, Number 2 June - October 2011

Olympiad Corner

Below are the problems of the 28th Balkan Math Olympiad, which was held in May 6, 2011. Time allowed was 4½ hours.

Problem 1. Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. The midpoints of AB and CD are F and G respectively, and ℓ is the line through G parallel to AB. The feet of the perpendiculars from E onto the lines ℓ and CD are H and K, respectively. Prove that the lines EF and HK are perpendicular.

Problem 2. Given real numbers x, y, z such that $x+y+z = 0$, show that

$$\frac{x(x+2)}{2x^2+1}+\frac{y(y+2)}{2y^2+1}+\frac{z(z+2)}{2z^2+1}\geq 0.$$

When does equality hold?

Problem 3. Let S be a finite set of positive integers which has the following property: if x is a member of S, then so are all positive divisors of x. A non-empty subset T of S is *good* if whenever $x, y \in T$ and $x < y$, the ration y/x is a power of a prime number.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 10, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

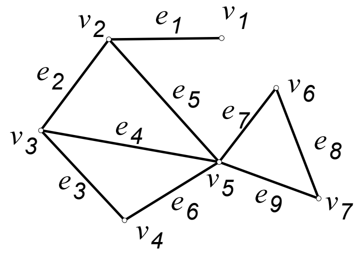
Fax: (852) 2358 1643
Email: makyli@ust.hk

Euler's Planar Graph Formula

Kin Y. Li

A *graph* G is consisted of a *nonempty* set $V(G)$ (its elements are called *vertices*) and a set $E(G)$ (its elements are called *edges*), where an edge is to be thought of as a continuous curve joining a vertex u in $V(G)$ to a vertex v in $V(G)$. A graph G is *finite* if and only if $V(G)$ is a finite set. It is *simple* if and only if each edge in $E(G)$ joins some pair of distinct vertices in $V(G)$ and no other edge joins the same pair. *In this article, all graphs are understood to be finite and simple*.

A graph is *connected* if and only if for every pair of distinct vertices a, b, there is a sequence of edges $e_1, e_2, \ldots, e_n$ such that for i from 1 to n, edge e_i joins v_i and v_{i+1} with $v_1 = a$ and $v_{n+1} = b$. A graph is *planar* if and only if it can be drawn on a plane with no pair of edges intersect at any point other than a vertex of the graph. A planar graph divides the plane into regions (bounded by edges) called *faces*.



In the graph above, there are 7 vertices (labeled v_1 to v_7), 9 edges (labeled e_1 to e_9) and 4 faces (the 3 triangular regions and the outside region bounded by e_1, e_5, e_7, e_8, e_9, e_6, e_3, e_2, e_1). The following theorem due to Euler relates the number of vertices, the number of edges and the number of regions for a connected planar graph and is the key tool in solving some interesting problems.

Euler's Theorem on Planar Graphs

Let V, E, F denote the number of vertices, the number of edges, the number of faces respectively for a connected planar (finite simple) graph. Then $V - E + F = 2$, which we will called *Euler's formula*.

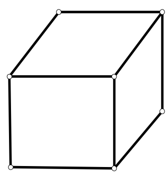
We will *sketch* the usual mathematical induction proof on E. If $E = 0$, then since $V(G)$ is nonempty and G is connected, we have $V = 1$ and $F = 1$. So $V-E+F=2$. Also, if $E=1$, then $V = 2$, $F=1$ and again the formula is true.

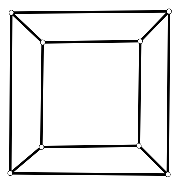
Suppose the cases $E < k$ are true. For the case $E=k$, either there is a cycle (that is a sequence of edges $e_1, e_2, \ldots, e_n$ such that for i from 1 to n, edge e_i joins v_i and v_{i+1} with $v_1 = v_{n+1}$) or no cycle.

In the former case, removing e_n will result in a connected graph with E decreases by 1, V stays the same and F decreases by 1 (since the two regions sharing e_n in their boundaries will become one). The formula still holds.

In the latter case, we call these graphs *trees*. It can be proved that they satisfy $E=V-1$ and $F=1$ (which implies Euler's formula). Basically, removing any edge will split such a graph into two connected graphs with each having no cycle. This observation would allow us to do the induction on E.

Before presenting some examples, we remark that Euler's formula also applies to convex polyhedrons. These are the boundary surfaces of three dimensional convex solids obtained by intersecting finitely many (half-spaces on certain sides of) planes. For example, take the surface of a cube, $V=8$, $E=12$, $F=6$ so that $V - E + F = 2$. For any convex polyhedron, we can obtain a connected planar graph by choosing a face as base, stretching the base sufficiently big and taking a top view projection onto the plane containing the base. The following is a cube and a planar graph for its boundary surface.





Example 1. There are $n > 3$ points on a circle. Each pair of them is connected by a chord such that no three of these chords intersect at the same point inside the circle. Find the number of regions formed inside the circle.

Solution. Removing the n arcs on the circle, we get a *simple* connected planar graph, where the vertices are the n points on the circle and the intersection points inside the circle. For every 4 of the n points, we can draw two chords intersecting at a point inside the circle. So the number of vertices is $V = n + {}_nC_4$.

Since there are $n-1$ edges incident with each of the n points on the circle, 4 edges incident with every intersection point inside the circle and each edge is counted twice, so the number of edges is $E = (n(n-1)+4\ {}_nC_4)/2$.

By Euler's formula, the number of faces for this graph is $F = 2 - V + E$. Excluding the outside face and adding the n regions having the n arcs as boundary, the number of regions inside the circle is $F - 1 + n = n + 1 - V + E = 1 + {}_nC_4 + n(n-1)/2$.

For the next few examples, we define the degree of a vertex v in a graph to be the number of edges meeting at v. Below $d(v)$ will denote the degree of v. *The sum of degrees of all vertices equals twice the number of edges since each edge is counted twice at its two endponts.*

Example 2. A square region is partitioned into n convex polygonal regions. Find the maximal number of edges in the figure.

Solution. Let V, E, F be the number of vertices, edges, faces respectively in the graph. Euler's formula yields

$$n+1 = F = 2-V+E \quad \text{or} \quad V = E + 1 - n.$$

Let A, B, C, D be the vertices of the square, then $t = d(A) + d(B) + d(C) + d(D) \geq 8$ as each term is at least 2.

Let W be the set of vertices inside the square. For any v in W, we have $d(v) \geq 3$ since angles of convex polygons are less than 180°. Let s be the sum of $d(v)$ for all v in W. Since there are $V-4$ vertices in W, we have $s \geq 3(V-4)$.

Now summing degree of all vertices, we get $s + t = 2E$. Then

$$2E-8 \geq 2E - t = s \geq 3(V-4) = 3(E-3-n),$$

which simplifies to $E \leq 3n+1$.

Finally, the case $E = 3n+1$ is possible by partitioning the square region into n rectangles using $n - 1$ line segments parallel to a side of the square. So the maximum possible value of E is $3n+1$.

Example 3. (*2000 Belarussian Math Olympiad*) In a convex polyhedron with m triangular faces (and possibly faces of other shapes), exactly four edges meet at each vertex. Find the minimum possible value of m.

Solution. Let V, E, F be the number of vertices, edges, faces respectively on such a polyhedron. Since each vertex is met by 4 distinct edges, summing all degrees, we have $2E = 4V$.

Next, summing the number of edges in the F faces and observing that each edge is counted twice on the 2 faces sharing it, we get $2E \geq 3m+4(F-m)$.

By Euler's formula, we have

$$2 = V - E + F = (E/2) - E + F = F - E/2,$$

which implies

$$4F - 8 = 2E \geq 3m + 4(F-m).$$

This simplifies to $m \geq 8$. A regular octahedron is an example of the case $m = 8$. So the minimum possible m is 8.

Example 4. (*1985 IMO proposal by Federal Republic of Germany*) Let M be the set of edge-lengths of an octahedron whose faces are congruent quadrilaterals. Prove that M has at most three elements.

Solution. The octahedron has $(4\times 8)/2=16$ edges. By Euler's formula, it has $V = 2 + E - F = 2 + 16 - 8 = 10$ vertices.

Next, let n_i be the number of vertices v with $d(v) = i$. Then, counting vertices and edges respectively in terms of n_i's, we have

$$V = n_3 + n_4 + n_5 + \cdots = 10$$

and

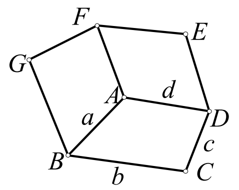
$$2E = 3n_3 + 4n_4 + 5n_5 + \cdots = 2\times 16.$$

Eliminating n_3, we get

$$n_4 + 2n_5 + 3n_6 + \cdots = 2.$$

Hence, $n_4 \leq 2$, $n_5 \leq 1$ and $n_i = 0$ for $i \geq 6$. Then $n_3 = 10 - n_4 - n_5 > 0$.

Let A be a vertex with degree 3. Assume M has 4 distinct elements a, b, c, d. Then the 3 faces about A are like the figure below, where we may take $AB = a$, $BC = b$, $CD = c$ and $DA = d$.



Since $ABCD$ and $ABGF$ are congruent, so $AF = b$ or d. Also, since $ABCD$ and $AFED$ are congruent, so $AF = a$ or c. Hence, two of a, b, c, d must be equal, contradiction. Therefore, M has at most 3 elements.

Example 5. Let n be a positive integer. A convex polyhedron has $10n$ faces. Prove that n of the faces have the same number of edges.

Solution. Let V be the number of vertices of this polyhedron. For the $10n$ faces, let these faces be polygons with $a_1, a_2, \ldots, a_{10n}$ sides respectively, where the a_i's are arranged in ascending order . Then the number of edges of the polyhedron is $E = (a_1 + a_2 + \cdots + a_{10n})/2$. By Euler's formula, we have

$$V - \frac{a_1 + a_2 + \cdots + a_n}{2} + 10n = 2. \quad (*)$$

Also, since the degree of every vertex is at least 3, we get

$$a_1+a_2+\cdots+a_{10n} \geq 3V. \quad (**)$$

Using (*) and (**), we can eliminate V and solve for $a_1+a_2+\cdots+a_{10n}$ to get

$$a_1+a_2+\cdots+a_{10n} \leq 60n - 12. \quad (***)$$

Assume no n faces have equal number of edges. Then we have $a_1, a_2, \ldots, a_{n-1} \geq 3$, $a_n, a_{n+1}, \ldots, a_{2n-2} \geq 4$ and so on. This leads to

$$\begin{aligned} & a_1 + a_2 + \cdots + a_{10n} \\ \geq\ & (3 + 4 + \cdots + 12)(n-1) + 13\times 10 \\ =\ & 75n + 55. \end{aligned}$$

Comparing with (***), we get $75n + 55 \leq 60n - 12$, which is false for n.

Example 6. (*1975 Kiev Math Olympiad and 1987 East German Math Olympiad*) An arrowhead is drawn on every edge of a convex polyhedron H such that at every vertex, there are at least one arrowhead pointing toward the vertex and another arrowhead pointing away from the vertex. Prove that there exist at least two faces of H, the arrowheads on each of its boundary form a (clockwise or counterclockwise) cycle.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 10, 2011.***

Problem 376. A polynomial is <u>monic</u> if the coefficient of its greatest degree term is 1. Prove that there exists a monic polynomial $f(x)$ with integer coefficients such that for every prime p, $f(x) \equiv 0 \pmod{p}$ has solutions in integers, but $f(x) = 0$ has no solution in integers.

Problem 377. Let n be a positive integers. For i=1,2,…,n, let z_i and w_i be complex numbers such that for all 2^n choices of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ equal to ± 1, we have

$$\left|\sum_{i=1}^{n}\varepsilon_i z_i\right| \le \left|\sum_{i=1}^{n}\varepsilon_i w_i\right|.$$

Prove that $\sum_{i=1}^{n}|z_i|^2 \le \sum_{i=1}^{n}|w_i|^2$.

Problem 378. Prove that for every positive integers m and n, there exists a positive integer k such that $2^k - m$ has at least n distinct positive prime divisors.

Problem 379. Let ℓ be a line on the plane of ΔABC such that ℓ does not intersect the triangle and none of the lines AB, BC, CA is perpendicular to ℓ.

Let A', B' C' be the feet of the perpendiculars from A, B, C to ℓ respectively. Let A'', B'', C'' be the feet of the perpendiculars from A', B', C' to lines BC, CA, AB respectively.

Prove that lines $A'A''$, $B'B''$, $C'C''$ are concurrent.

Problem 380. Let $S = \{1,2,\ldots,2000\}$. If A and B are subsets of S, then let $|A|$ and $|B|$ denote the number of elements in A and in B respectively. Suppose the product of $|A|$ and $|B|$ is at least 3999. Then prove that sets $A-A$ and $B-B$ contain at least one common element, where $X-X$ denotes $\{s-t : s, t \in X \text{ and } s \ne t\}$.

****

Solutions

****

Problem 371. Let $a_1, a_2, a_3, \ldots$ be a sequence of nonnegative rational numbers such that $a_m + a_n = a_{mn}$ for all positive integers m, n. Prove that there exists two terms that are equal.

Solution. **U. BATZORIG** (National University of Mongolia), **CHUNG Kwan** (King's College) and **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School).

Let p and q be distinct primes. If a_p and a_q are zeros, then we are done. Otherwise, consider

$$m = p^{Na_q} \quad and \quad n = q^{Na_p},$$

where N is a positive integer that makes both Na_q and Na_p integers. Obviously, we have $m \ne n$ and

$$a_m = (Na_q)a_p = (Na_p)a_q = a_n.$$

Other commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil).

Problem 372. (*Proposed by Terence ZHU)* For all $a,b,c > 0$ and abc=1, prove that

$$\frac{1}{a(a+1)+ab(ab+1)} + \frac{1}{b(b+1)+bc(bc+1)} + \frac{1}{c(c+1)+ca(ca+1)} \ge \frac{3}{4}.$$

Solution. **V. ADIYASUREN** (National University of Mongolia) and **B. SANCHIR** (Mathematics Institute of the National University of Mongolia), **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School) and **Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

Substituting $a = z/y$, $b = x/z$, $c = y/x$ (say by choosing x=ab=1/c, y=1, z=a) into the inequality and simplifying, we get

$$\sum_{cyc} f(x,y,z) \ge \frac{3}{4},$$

where

$$f(x,y,z) = \frac{y^2}{z(z+y)+x(x+y)} \quad \text{and}$$

$$\sum_{cyc} f(x,y,z) = f(x,y,z) + f(y,z,x) + f(z,x,y).$$

Let $g(x,y,z) = y^2(z^2 + zy + x^2 + xy)$. By the Cauchy-Schwarz inequality, we have

$$\sum_{cyc} f(x,y,z) \sum_{cyc} g(x,y,z) \ge \left(\sum_{cyc} y^2\right)^2.$$

So it is enough to prove

$$\left(\sum_{cyc} y^2\right)^2 \Big/ \left(\sum_{cyc} g(x,y,z)\right) \ge \frac{3}{4}. \quad (*)$$

Expanding and factorizing, we get

$$4\left(\sum_{cyc} y^2\right)^2 - 3\left(\sum_{cyc} g(x,y,z)\right)$$

$$= 4\sum_{cyc} y^4 + 2\sum_{cyc} x^2y^2 - 3\sum_{cyc} xy(x^2+y^2)$$

$$= 3\sum_{cyc}(x-y)^2(x^2+y^2) + \sum_{cyc}(x^2-y^2)^2 \ge 0.$$

This implies (*), which implies the desired inequality.

Other commended solvers: **CHUNG Kwan** (King's College), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Problem 373. Let x and y be the sums of positive integers $x_1, x_2, \ldots, x_{99}$ and $y_1, y_2, \ldots, y_{99}$ respectively. Prove that there exists a 50 element subset S of $\{1,2,\ldots,99\}$ such that the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

Solution. **William Peng** and **Jeff Peng**.

Arrange the numbers $x_1, x_2, \ldots, x_{99}$ in descending order, say $x_{n(1)} \ge x_{n(2)} \ge \cdots \ge x_{n(99)}$ so that

$$\{n(1), n(2), \ldots, n(99)\} = \{1,2,\ldots,99\}.$$

Let $A = \{n(2), n(4), \ldots, n(98)\}$ and $B = \{n(3), n(5), \ldots, n(99)\}$. We have

$$x_{n(1)} + \sum_{j\in B} x_j > \sum_{i\in A} x_i \ge \sum_{j\in B} x_j.$$

If $\sum_{i\in A} y_i \ge \sum_{j\in B} y_j$, then let $S = A \cup \{n(1)\}$.

Now S has 50 elements. Also,

$$\sum_{i\in S} x_i > \sum_{i\in A} x_i \ge \sum_{j\in B} x_j$$

and

$$\sum_{i\in S} y_i > \sum_{i\in A} y_i \ge \sum_{j\in B} y_j.$$

So the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

If $\sum_{i\in A} y_i < \sum_{j\in B} y_j$, then let $S = B\cup\{n(1)\}$.

Again S has 50 elements. Now

$$\sum_{i\in S} x_i = x_{n(1)} + \sum_{j\in B} x_j > \sum_{i\in A} x_i \geq \sum_{j\in B} x_j$$

and

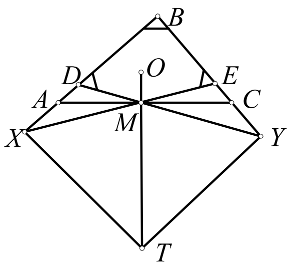
$$\sum_{i\in S} y_i > \sum_{j\in B} y_j > \sum_{i\in A} y_i .$$

So the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

Other commended solvers: **U. BATZORIG** (National University of Mongolia) and **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School),

Problem 374. O is the circumcenter of acute ΔABC and T is the circumcenter of ΔAOC. Let M be the midpoint of side AC. On sides AB and BC, there are points D and E respectively such that $\angle BDM = \angle BEM = \angle ABC$. Prove that $BT \perp DE$.

***Solution*.** **William Peng** and **Jeff Peng**.



By the exterior angle theorem, $\angle ABC = \angle BDM > \angle BAM$ and also $\angle ABC = \angle BEM > \angle BCM$. So $\angle ABC$ is the largest angle in ΔABC. Then we have $60° < \angle ABC < 90°$. This implies O is on the same side of line AC as B. Then T will be on the opposite side of line AC as O. Also, O, M, T are on the perpendicular bisector of line AC.

Let X be the intersection of lines AB and ME. Let Y be the intersection of lines CB and MD. Now

$$\begin{aligned}\angle DXE &= 180° - \angle XBE - \angle BEX\\ &= 180° - 2\angle ABC\end{aligned}$$

and similarly $\angle EYD = 180° - 2\angle ABC$. So $\angle DXE = \angle EYD$, which implies D, X, Y, E are concyclic.

Next, since T is the circumcenter of ΔAOC, so

$$\begin{aligned}\angle ATM &= \angle ATO = 2\angle ACO\\ &= 2(90° - \angle BXE)\\ &= 180° - 2\angle ABC\\ &= \angle BXE = \angle AXM.\end{aligned}$$

This implies A, M, T, X are concyclic. So $\angle AXT = 180° - \angle AMT = 90°$. Similarly, $\angle CYT = 90°$. Then $\angle BXT = \angle BYT$, which implies B, X, T, Y are concyclic. So

$$\angle TBY = \angle TXY = 90° - \angle BXY. \quad (*)$$

Since D, X, Y, E are concyclic,

$$\begin{aligned}&\angle BED + \angle TBE\\ =&\ \angle BXY + \angle TBY\\ =&\ 90° \qquad \text{by } (*),\end{aligned}$$

which implies $BT \perp DE$.

Other commended solvers: **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School),

Problem 375. Find (with proof) all odd integers $n > 1$ such that if a, b are divisors of n and are relatively prime, then $a+b-1$ is also a divisor of n.

Solution. **U. BATZORIG** (National University of Mongolia), **William Peng** and **Jeff Peng**.

For such odd n, let p be its <u>least</u> prime divisor. Then $n = p^m a$, where m is the exponent of p in the prime factorization of n. We will show $a = 1$.

Assume $a > 1$. Then every prime divisors of a is at least $p+2$. Also $c = a+p-1$ ($> p$) is a divisor of n. Since

$$\gcd(c,a) = \gcd(c-a,a) = \gcd(p-1,a) = 1,$$

this implies $c = p^r$ with $r \geq 2$. Then $d = a+p^2-1$ ($> p^2$) is also a divisor of n. Similarly,

$$\gcd(d,a) = \gcd(d-a,a) = \gcd(p^2-1,a) = 1.$$

So $d = p^s$ with $s \geq 3$. Finally, $p^r - p = c - p = a - 1 = d - p^2$, which is divisible by p^2, while $p^r - p$ is not. Therefore, $a = 1$.

It is easy to check all $n = p^m$ with p an odd prime and m a positive integer indeed satisfy the condition.

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) A non-empty subset T of S is <u>bad</u> if whenever $x, y \in T$ and $x < y$, the ration y/x is not a power of a prime number. We agree that a singleton subset of S is both good and bad. Let k be the largest possible size of a good subset of S. Prove that k is also the smallest number of pairwise-disjoint bad subsets whose union is S.

Problem 4. Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of one of these two triangles is at least 3/2.

Euler's Planar Graph Formula

(continued from page 2)

<u>*Solution.*</u> Call $\{a,b\}$ a <u>*hook*</u> if a, b are two consecutive edges on the boundary of some face of H. Call a hook $\{a,b\}$ <u>*traversible*</u> if the arrowheads on a and b are both counterclockwise or both clockwise.

Note every hook is part of the boundary of a unique face. Let E be the number of edges on H and h be the number of hooks on H. As each edge on H is a part of 4 hooks, we get $h = 2E$.

Next at every vertex v, $d(v) \geq 3$. By the given condition on the vertices, there must be at least 2 traversible hooks through every vertex. Let V be the number of vertices on H, then there are at least $2V$ traversible hooks on H.

Let h_+ and h_- be the number of traversible and non-traversible hooks respectively on H. Then $h_+ \geq 2V$.

In every face where the boundary arrowheads do not form a cycle, there are at least two changes in directions on the boundary, which result in at least two non-traversible hooks. Let F be the number of faces on H. Let f_+ be the number of faces the boundary arrowheads form cycles. Let $f_- = F - f_+$. Then $h_- \geq 2f_-$.

By Euler's formula, $V - E + F = 2$. Then

$$\begin{aligned}2f_+ &= 2F - 2f_-\\ &= (4 + 2E - 2V) - 2f_-\\ &\geq 4 + h - h_+ - 2f_-\\ &= 4 + h_- - 2f_- \geq 4,\end{aligned}$$

which implies $f_+ \geq 2$. This gives the desired conclusion.

Mathematical Excalibur

Volume 16, Number 3 *November 2011 -January 2012*

Olympiad Corner

Below are the problems of the 2011 International Math Olympiad.

Problem 1. Given any set $A=\{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1+a_2+a_3+a_4$ by s_A. Let n_A denote the number of pairs (i,j) with $1\le i<j\le 4$ for which a_i+a_j divides s_A. Find the sets A of four distinct positive integers which achieve the largest possible value of n_A.

Problem 2. Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to S. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely.

Show that we can choose a point P in S and a line ℓ going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Remarks on IMO 2011

Leung Tat-Wing

The 52nd IMO was held in Amsterdam, Netherlands, on 12-24, July, 2011. Contestants took two 4½ hour exams during the mornings of July 18 and 19. Each exam was consisted of 3 problems of varying degree of difficulty. The problems were first shortlisted by the host country, selected from problems submitted earlier by various countries. Leaders from 101 countries then picked the 2011 IMO problems (see *Olympiad Corner*). Traditionally an easy pair was selected (Problems 1 and 4), then a hard pair (Problems 3 and 6), with Problem 6 usually selected as the "anchor problem", and finally the intermediate pair (Problems 2 and 5). I would like to discuss first the problems selected, aim to provide something extra besides those which were provided by the solutions. However I would discuss the problems by slightly different grouping.

Problems 1 and 4

First the easy pair, problems 1 and 4. The problem selection committee thought that both problems were quite easy. It was nice to select one as a problem of the contest. But if both problems were selected, then the paper would be too easy (or even disastrous). Indeed eventually both problems were selected. But it was not enough for anyone to get a bronze medal even if he could solve both problems (earning 14 points) as the cut-off for bronze was 16.

In my opinion problem 1 is the easier of the pair. Indeed we may without loss of generality assume $a_1 < a_2 < a_3 < a_4$. So if the sum of one pair of the a_i's divides s_A, then it will also divide the sum of the other pair. But clearly a bigger pair cannot divide a smaller pair, so it is impossible that $a_3 + a_4$ dividing $a_1 + a_2$, nor is it possible that $a_2 + a_4$ dividing $a_1 + a_3$. Therefore the maximum possible value of n_A can only be 4. To achieve this, it suffices to consider divisibility conditions among the other pairs.

Now as we need $a_1 + a_4$ dividing $a_2 + a_3$ and also $a_2 + a_3$ dividing $a_1 + a_4$, we must have $a_1 + a_4 = a_2 + a_3$. Putting $a_4 = a_2+a_3-a_1$ into the equations $a_3 + a_4 = m(a_1 + a_2)$ and $a_2 + a_4 = n(a_1 + a_3)$ with $m > n > 1$, we eventually get $(m,n) = (3,2)$ or $(4,2)$. Finally we get $(a_1, a_2, a_3, a_4) = (k, 5k, 7k, 11k)$ or $(k, 11k, 19k, 29k)$, where k is a positive integer. As the derivation of the answers is rather straight-forward, it does not pose any serious difficulty.

For problem 4, it is really quite easy if one notes the proper recurrence relation. Indeed the weights $2^0, 2^1, 2^2, \ldots, 2^{n-1}$ form a "super-increasing sequence", any weight is heavier than the sum of all lighter weights. Denote by $f(n)$ the number of ways of placing the weights. We consider first how to place the lightest weight (weight 1). Indeed if it is placed in the first move, then it has to be in the left pan. However if it is placed in the second to the last move, then it really doesn't matter where it goes, using the "super-increasing property". Hence altogether there are $2n-1$ possibilities of placing the weight of weight 1. Now placing the weights $2^1, 2^2, \ldots, 2^{n-1}$ clearly is the same as placing the weights $2^0, 2^1, \ldots, 2^{n-2}$. There are $f(n-1)$ ways of doing this. Thus we establish the recurrence relation $f(n) = (2n-1)f(n-1)$. Using $f(1) = 1$, by induction, we get

$$f(n) = (2n-1)(2n-3)(2n-5)\cdots 1.$$

The problem becomes a mere exercise of recurrence relation if one notices how to place the lightest weight (minimum principle).

It is slightly harder if we consider how to place the heaviest weight. Indeed if the heaviest weight is to be placed in the i^{th} move, then it has to be placed in the left pan. There are $\binom{n-1}{i-1}$ ways of choosing the previous $i-1$ weights and there are $f(i-1)$ ways of placing them. After the heaviest weight is placed, it doesn't matter how to place the other weights, and there are $(n-i)!\times 2^{n-i}$ ways of placing the remaining weights. Thus

$$f(n)=\sum_{i=1}^{n}\binom{n-1}{i-1}f(i-1)(n-i)!2^{n-i}.$$

Replacing n by $n-1$ and by comparing the two expressions we again get $f(n) = (2n-1)f(n-1)$. We have no serious difficulty with this problem.

Problems 3 and 5

In my opinion both problems 3 and 5 were of similar flavor. Both were "functional equation" type of problems. Problem 3 was slightly more involved and problem 5 more number theoretic. One can of course put in many values and obtain some equalities or inequalities. But the important thing is to substitute some suitable values so that one can derive important relevant properties that can solve the problem.

In problem 5, indeed the condition $f(m-n) \mid (f(m) - f(n))$ (*) poses very serious restrictions on the image of $f(x)$. Putting n=0, one gets $f(m) \mid (f(m) - f(0))$, thus $f(m) \mid f(0)$. Since $f(0)$ can only have finitely many factors, the image of $f(x)$ must be finite. Putting m=0, one gets $f(-n) \mid f(n)$, and by interchanging n and $-n$, one gets $f(n) = f(-n)$. Now $f(n) \mid (f(2n) - f(n))$, hence $f(n) \mid f(2n)$, and by induction $f(n) \mid f(mn)$. Put $n = 1$ into the relation. One gets $f(1) \mid f(m)$. The image of $f(x)$ is therefore a finite sequence $f(1) = a_1 < a_2 < \cdots < a_k = f(0)$. One needs to show $a_i \mid a_{i+1}$. To complete the proof, one needs to analyze the sequence more carefully, say one may proceed by induction on k. But personally I like the following argument. Let $f(x) = a_i$ and $f(y) = a_{i+1}$. We have $f(x-y) \mid (f(y) - f(x)) < f(y)$ and $f(y) - f(x)$ is positive, hence $f(x - y)$ is in the image of $f(x)$ and therefore $f(x-y) \le a_i = f(x)$. Now if $f(x-y) < f(x)$, then $f(x) - f(x - y) > 0$. Thus $f(y) = f(x-(x-y)) \mid (f(x) - f(x - y))$.
In this case the right-hand side is positive. We have $f(y) \le f(x) - f(x-y)) < f(x) < f(y)$, a contradiction. So we have $f(x-y) = f(x)$. Thus $f(x) \mid f(y)$ as needed.

It seems that Problem 3 is more involved. However, by making useful and clever substitutions, it is possible to solve the problem in a relatively easy way. The following solution comes from one of our team members. Put $y = z-x$ into the original equation $f(x+y) \le yf(x) + f(f(x))$, one gets $f(z) \le z f(x) - xf(x) + f(f(x))$. By letting $z = f(k)$ in the derived inequality one gets $f(f(k)) \le f(k)f(x) - xf(x) + f(f(x))$.
Interchanging k and x one then gets $f(f(x)) \le f(k)f(x) - kf(k) + f(f(k))$. Hence

$$\begin{aligned} f(x+y) &\le y f(x) + f(f(x)) \\ &\le f(x)f(k) - kf(k) + f(f(k)). \end{aligned}$$

Letting $y = f(k) - x$ in the inequality, we get

$$\begin{aligned} f(f(k)) \le\ & f(k)f(x) - xf(x) + \\ & f(k)f(x) - kf(k) + f(f(k)) \end{aligned}$$

or $0 \le 2f(k)f(x) - xf(x) - kf(k)$. Finally letting $k = 2f(x)$ and simplifying, we arrive at the important and essential (hidden) inequality $0 \le -xf(x)$. This means for $x > 0$, $f(x) \le 0$, and for $x < 0$, $f(x) \ge 0$. But if there is an $x_0 < 0$ such that $f(x_0) > 0$, then putting $x = x_0$ and $y = 0$ into the original equation, we gets $0 < f(x_0) \le f(f(x_0))$. However if $f(x_0) > 0$, then $f(f(x_0)) \le 0$, hence a contradiction. This means for all $x < 0$, $f(x) = 0$. Finally one has to prove $f(0) = 0$. We suppose first $f(0) > 0$. Put $x = 0$ and $y < 0$ sufficiently small into the original equation, one gets $f(y) < 0$, a contradiction. Suppose $f(0) < 0$. Take $x, y < 0$. We get

$$\begin{aligned} 0 = f(x+y) &\le y f(x) + f(f(x)) \\ &= y f(x) + f(0) = f(0) < 0, \end{aligned}$$

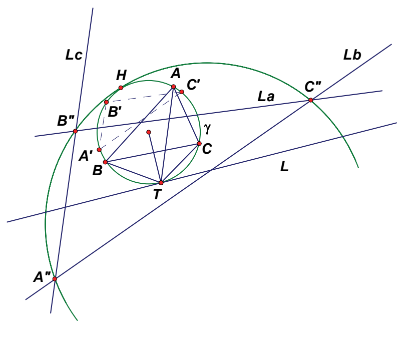
again contradiction! This implies $f(0) = 0$.

Problem 2

To me, problem 2 was one of a kind. The problem was considered as "intermediate" and should not be too hard. However at the end only 21 out of 564 contestants scored full marks. It was essentially a problem of computational geometry. We know that if there is a line that goes through two or more of the points and such that all other points are on the line or only on one side of the points, then by repeatedly turning angles as indicated in the problem, the convex hull of the point set will be constructed (so-called Jarvis' march). Therefore some points may be missed. So in order to solve the problem, we cannot start from the "boundary". Thus it is natural that we start from the "center", or a line going through a point that separates the other points into equal halves (or differ by one). Indeed this idea is correct. The hard part is how to substantiate the argument. Many contestants found it hard. Induction argument does not work because adding or deleting one point may change the entire route. The proposer gives the following "continuity argument". We consider only the case that there are an odd number of points on the plane. Let l be a line that goes through one of the points and that separates the other points into two equal halves. Note that such line clearly exists. Color one half-plane determined by the line orange (for Netherlands) and the other half-plane blue. The color of the plane changes accordingly while the line is turning. Note also that when the line moves to another pivot, the number of points on the two sides remain the same, except when two points are on the line during the change of pivots. So consider what happen when the line turns 180°, (turning while changing pivots). The line will go through the same original starting point. Only the colors of the two sides of the line interchange! This means all the points have been visited at least once! A slightly modified argument works for the case there are an even number of points on the plane.

Problem 6

This was the most difficult problem of the contest (the anchor problem), only 6 out of more than 564 contestants solved the problem. Curiously these solvers were not necessarily from the strongest teams. The problem is hard and beautiful, and I feel that it may be a known problem because it is so nice. However, I am not able to find any further detail. It is not convenient to reproduce the full solution here. But I still want to discuss the main idea used in the first official solution briefly.



From $\triangle ABC$ and the tangent line L at T, we produce the reflecting lines L_a, L_b, and L_c. The reflecting lines meet at A'', B'' and C'' respectively. Now from A, we draw a circle of radius AT, meeting the circumcircle γ of ABC at A'. Likewise we have BT=BB' and CT=CC' (see the figure).

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 28, 2012.***

Problem 381. Let k be a positive integer. There are 2^k balls divided into a number of piles. For every two piles A and B with p and q balls respectively, if $p \ge q$, then we may transfer q balls from pile A to pile B. Prove that it is always possible to make finitely many such transfers so as to have all the balls end up in one pile.

Problem 382. Let $v_0 = 0$, $v_1 = 1$ and

$$v_{n+1} = 8v_n - v_{n-1} \quad \text{for } n = 1,2,3,\ldots.$$

Prove that v_n is divisible by 3 if and only if v_n is divisible by 7.

Problem 383. Let O and I be the circumcenter and incenter of $\triangle ABC$ respectively. If $AB \neq AC$, points D, E are midpoints of AB, AC respectively and $BC=(AB+AC)/2$, then prove that the line OI and the bisector of $\angle CAB$ are perpendicular.

Problem 384. For all positive real numbers a,b,c satisfying $a + b + c = 3$, prove that

$$\frac{a^2+3b^2}{ab^2(4-ab)}+\frac{b^2+3c^2}{bc^2(4-bc)}+\frac{c^2+3a^2}{ca^2(4-ca)}\ge 4.$$

Problem 385. To prepare for the IMO, in everyday of the next 11 weeks, Jack will solve at least one problem. If every week he can solve at most 12 problems, then prove that for some positive integer n, there are n consecutive days in which he can solve a total of 21 problems.

****

Solutions

****

Problem 376. A polynomial is *monic* if the coefficient of its greatest degree term is 1. Prove that there exists a monic polynomial $f(x)$ with integer coefficients such that for every prime p, $f(x) \equiv 0 \pmod p$ has solutions in integers, but $f(x) = 0$ has no solution in integers.

Solution. **Alumni 2011** (Carmel Alison Lam Foundation Secondary School), **Maxim BOGDAN** ("Mihai Eminescu" National College, Botosani, Romania), **Koopa KOO** and **Andy LOO** (St. Paul's Co-educational College).

Let $f(x)=(x^2-2)(x^2-3)(x^2-6)$. Then $f(x) = 0$ has no solution in integers. For $p = 2$ or 3, $f(6) \equiv 0 \pmod p$. For a prime $p > 3$, if there exists x such that $x^2 \equiv 2$ or $3 \pmod p$, then $f(x) \equiv 0 \pmod p$ has solutions in integers. Otherwise, from Euler's criterion, it follows that there will be x such that $x^2 \equiv 6 \pmod p$ and again $f(x) \equiv 0 \pmod p$ has solutions in integers.

Comments: For readers not familiar with Euler's criterion, we will give a bit more details. For c relatively prime to a prime p, by Fermat's little theorem, we have

$$(c^{(p-1)/2}-1)(c^{(p-1)/2}+1) = c^{p-1}-1 \equiv 0 \pmod p,$$

which implies $c^{(p-1)/2} \equiv 1$ or $-1 \pmod p$.

If there exists x such that $x^2 \equiv c \pmod p$, then $c^{(p-1)/2} \equiv x^{p-1} \equiv 1 \pmod p$. Conversely, if $c^{(p-1)/2} \equiv 1 \pmod p$, then there is x such that $x^2 \equiv c \pmod p$. [This is because there is a primitive root $g \pmod p$ (see *vol.* 15, *no.* 1, *p.* 1 of Math Excalibur), so we get $c \equiv g^i \pmod p$ for some positive integer i, then $g^{i(p-1)/2} \equiv 1 \pmod p$. Since g is a primitive root $\pmod p$, so $i(p-1)/2$ is a multiple of $p-1$, then i must be even, hence $c \equiv (g^{i/2})^2 \pmod p$.] In above, if 2 and 3 are not squares $\pmod p$, then $6^{(p-1)/2}=2^{(p-1)/2}3^{(p-1)/2} \equiv (-1)^2 = 1 \pmod p$, hence 6 is a square $\pmod p$.

Problem 377. Let n be a positive integer. For $i=1,2,\ldots,n$, let z_i and w_i be complex numbers such that for all 2^n choices of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ equal to ± 1, we have

$$\left|\sum_{i=1}^n \varepsilon_i z_i\right| \le \left|\sum_{i=1}^n \varepsilon_i w_i\right|.$$

Prove that $\displaystyle\sum_{i=1}^n |z_i|^2 \le \sum_{i=1}^n |w_i|^2$.

Solution. **William PENG** and **Jeff PENG** (Dallas,Texas, USA).

The case $n = 1$ is clear. Next, recall the parallelogram law $|a+b|^2+|a-b|^2=2|a|^2+2|b|^2$, which follows from adding the + and − cases of the identity

$$(a \pm b)(\bar{a} \pm \bar{b}) = a\bar{a} \pm a\bar{b} \pm b\bar{a} + b\bar{b}.$$

For $n = 2$, we have

$$|z_1+z_2| \le |w_1+w_2| \text{ and } |z_1-z_2| \le |w_1-w_2|.$$

Squaring both sides of these inequalities, adding them and applying the parallelogram law, we get the desired inequality. Next assume the case $n=k$ holds. Then for the $n=k+1$ case, we use the 2^k choices with $\varepsilon_1 = \varepsilon_2$ to get from the $n=k$ case that

$$\begin{aligned} &|z_1+z_2|^2+|z_3|^2+\cdots+|z_{k+1}|^2 \\ \le\ &|w_1+w_2|^2+|w_3|^2+\cdots+|w_{k+1}|^2. \end{aligned}$$

Similarly, using the other 2^k choices with $\varepsilon_1 = -\varepsilon_2$, we get

$$\begin{aligned} &|z_1-z_2|^2+|z_3|^2+\cdots+|z_{k+1}|^2 \\ \le\ &|w_1-w_2|^2+|w_3|^2+\cdots+|w_{k+1}|^2. \end{aligned}$$

Adding the last two inequalities and applying the parallelogram law, we get the $n=k+1$ case.

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School),**Maxim BOGDAN** ("Mihai Eminescu" National College, Botosani, Romania), **O Kin Chit, Alex** (G.T.(Ellen Yeung) College) and **Mohammad Reza SATOURI** (Bushehr, Iran).

Problem 378. Prove that for all positive integers m and n, there exists a positive integer k such that $2^k - m$ has at least n distinct positive prime divisors.

Solution. **William PENG** and **Jeff PENG**(Dallas,Texas, USA).

For the case m is odd, we will prove the result by inducting on n. If $n=1$, then just choose k large so that the odd number $2^k - m$ is greater than 1. Next assume there exists a positive integer k such that $j = 2^k - m$ has at least n distinct positive prime divisors. Let $s= k+\varphi(j^2)$, where $\varphi(j^2)$ is the number of positive integers at most j^2 that are relatively prime to j^2. Since j is odd, by Euler's theorem,

$$2^s - m \equiv 2^k \times 1 - m = j \pmod{j^2}.$$

Then $2^s - m$ is of the form $j+tj^2$ for some positive integer t. Hence it is divisible by j and $(2^s - m)/j$ is relatively prime to j. Therefore, $2^s - m$ has at least $n+1$ distinct prime divisors.

For the case m is even, write $m=2^i r$, where i is a nonnegative integer and r is odd. Then as proved above there is k such that $2^k - r$ has at least n distinct prime divisors and so is $2^{i+k} - m$.

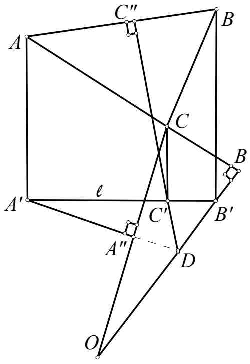
Other commended solvers: **Maxim BOGDAN** ("Mihai Eminescu" National College, Botosani, Romania)

Problem 379. Let ℓ be a line on the plane of ΔABC such that ℓ does not intersect the triangle and none of the lines *AB, BC, CA* is perpendicular to ℓ.

Let *A', B', C'* be the feet of the perpendiculars from *A, B, C* to ℓ respectively. Let *A", B", C"* be the feet of the perpendiculars from *A', B', C'* to lines *BC, CA, AB* respectively.

Prove that lines *A'A", B'B", C'C"* are concurrent.

Solution. **William PENG** and **Jeff PENG** (Dallas, Texas, USA) and **ZOLBAYAR Shagdar** (9th Grade, Orchlon Cambridge International School, Mongolia).



Let lines *B'B"* and *C'C"* intersect at *D*. To show line *A'A"* also contains *D*, since $\angle CA"A' = 90°$, it suffices to show $\angle CA"D = 90°$.

Let lines *BC* and *B'B"* intersect at *O*. We claim that $\Delta DOA"$ is similar to $\Delta COB"$. (Since $\angle OB"C = 90°$, the claim will imply $\angle OA"D = 90°$, which is the same as $\angle CA"D = 90°$.)

For the claim, first note $\angle AC"D = 90° = \angle AB"D$, which implies *A,C",B",D* are concyclic. So $\angle C"AB" = \angle B"DC"$. Next, $\angle BC"D = 90° = \angle DA"B$ implies *B,C",A",D* are concyclic. So $\angle C"BA" = \angle A"DC"$. Then

$$\begin{aligned}\angle ODA" &= 180° - (\angle A"DC" + \angle B"DC")\\ &= 180° - (\angle C"BA" + \angle C"AB")\\ &= \angle ACB\\ &= \angle OCB".\end{aligned}$$

This along with $\angle DOA" = \angle COB"$ yield the claim and we are done.

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School) and **Maxim BOGDAN** ("Mihai Eminescu" National College, Botosani, Romania).

Problem 380. Let $S = \{1,2,\ldots,2000\}$. If *A* and *B* are subsets of *S*, then let $|A|$ and $|B|$ denote the number of elements in *A* and in *B* respectively. Suppose the product of $|A|$ and $|B|$ is at least 3999. Then prove that sets $A-A$ and $B-B$ contain at least one common element, where $X-X$ denotes $\{s-t : s, t \in X \text{ and } s \neq t\}$.
(*Source*: 2000 *Hungarian-Israeli Math Competition*)

Solution. **Maxim BOGDAN** ("Mihai Eminescu" National College, Botosani, Romania) and **William PENG** and **Jeff PENG** (Dallas,Texas, USA).

Note that the set $T=\{(a,b): a\in A \text{ and } b\in B\}$ has $|A|\times|B| \geq 3999$ elements. Also, the set $W=\{a+b: a\in A \text{ and } b\in B\}$ is a subset of $\{2,3,\ldots 4000\}$. If $W = \{2,3,\ldots,4000\}$, then 2 and 4000 in *W* imply sets *A* and *B* both contain 1 and 2000. This leads to $A-A$ and $B-B$ both contain 1999.

If $W\neq\{2,3,\ldots 4000\}$, then *W* has less than 3999 elements. By the pigeonhole principle, there would exist $(a,b) \neq (a',b')$ in *T* such that $a+b=a'+b'$. This leads to $a-a'=b'-b$ in both $A-A$ and $B-B$.

Olympiad Corner

(continued from page 1)

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq y f(x) + f(f(x))$$

for all real numbers *x* and *y*. Prove that $f(x) = 0$ for all $x \leq 0$.

Problem 4. Let $n > 0$ be an integer. We are given a balance and *n* weights of weigh $2^0, 2^1, \ldots, 2^{n-1}$. We are to place each of the *n* weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all the weights have been placed.

Determine the number of ways in which this can be done.

Problem 5. Let *f* be a function from the set of integers to the set of positive integers. Suppose that, for any two integers *m* and *n*, the difference $f(m)-f(n)$ is divisible by $f(m-n)$. Prove that, for all integers *m* and *n* with $f(m)\leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

Problem 6. Let *ABC* be an acute triangle with circumcircle γ. Let *L* be a tangent line to γ, and let L_a, L_b and L_c be the line obtained by reflecting *L* in the lines *BC, CA* and *AB*, respectively. Show that the circumcircle of the triangle determined by the lines L_a, L_b and L_c is tangent to the circle γ.

Remarks on IMO 2011

(continued from page 2)

The essential point is to observe that *A"B"C"* is in fact homothetic to *A'B'C'*, with the homothetic center at *H*, a point on γ, i.e. *A"B"C"* is an expansion of *A'B'C'* at *H* by a constant centre. This implies the circumcircle of *A"B"C"* is tangent to γ at *H*.

A lot of discussions were conducted concerning changing the format of the Jury system during the IMO. At present the leaders assemble to choose six problems from the short-listed problems. There are issues concerning security and also financial matter (to house the leaders in an obscure place far away from the contestants can be costly). Many contestants need good results to obtain scholarships and enter good universities and the leaders have incentive for their own good to obtain good results for their teams. For me I am inclined to let the Jury system remains as such. The main reason is simply the law of large numbers, a better paper may be produced if more people are involved. Indeed both the Problem Selection Group and the leaders may make mistakes. But we get a better chance to produce a better paper after detailed discussion. In my opinion we generally produce a more balanced paper. The discussion is still going on. Perhaps some changes are unavoidable, for better or for worse.

Here are some remarks concerning the performance of the teams. We keep our standard or perhaps slightly better than the last few years. I am glad that some of our team members are able to solve the harder problems. Although the Chinese team is still ranked first (unofficially), they are not far better than the other strong teams (USA, Russia, etc). In particular, the third rank performance of the Singaporean team this time is really amazing.

Mathematical Excalibur

Volume 16, Number 4 *February 2012*

Olympiad Corner

Below are the problems of the 2011-2012 British Math Olympiad Round 1 held on 2 December 2011.

Problem 1. Find all (positive or negative) integers n for which $n^2+20n+11$ is a perfect square.

Problem 2. Consider the numbers 1, 2, ⋯, n. Find, in terms of n, the largest integer t such that these numbers can be arranged in a row so that all consecutive terms differ by at least t.

Problem 3. Consider a circle S. The point P lies outside S and a line is drawn through P, cutting S at distinct points X and Y. Circles S_1 and S_2 are drawn through P which are tangent to S at X and Y respectively. Prove that the difference of the radii of S_1 and S_2 is independent of the positions of P, X and Y.

Problem 4. Initially there are m balls in one bag, and n in the other, where m, $n > 0$. Two different operations are allowed:

a) Remove an equal number of balls from each bag;
b) Double the number of balls in one bag.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 28, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Zsigmondy's Theorem

Andy Loo (St. Paul's Co-educational College)

In recent years, a couple of "hard" number theoretic problems in the IMO turn out to be solvable by simple applications of deep theorems. For instances, IMO 2003 Problem 6 and IMO 2008 Problem 3 are straight forward corollaries of the Chebotarev density theorem and a theorem of Deshouillers and Iwaniec respectively. In this article we look at yet another mighty theorem, which was discovered by the Austro-Hungarian mathematician Karl Zsigmondy in 1882 and which can be used to tackle many Olympiad problems at ease.

Zsigmondy's theorem

First part: If a, b and n are positive integers with $a>b$, gcd(a, b)=1 and $n≥2$, then $a^n–b^n$ has at least one prime factor that does not divide $a^k–b^k$ for all positive integers $k<n$, with the exceptions of:

i) $2^6–1^6$ and

ii) n=2 and $a+b$ is a power of 2.

Second part: If a, b and n are positive integers with $a>b$ and $n≥2$, then a^n+b^n has at least one prime factor that does not divide a^k+b^k for all positive integers $k<n$, with the exception of 2^3+1^3.

The proof of this theorem is omitted due to limited space. Interested readers may refer to [2].

To see its power, let us look at how short solutions can be obtained using Zsigmondy's theorem to problems of various types.

Example 1 (Japanese MO 2011). Find all quintuples of positive integers (a,n,p,q,r) such that

$$a^n–1 = (a^p–1)(a^q–1)(a^r–1).$$

Solution. If $a≥3$ and $n≥3$, then by Zsigmondy's theorem, $a^n–1$ has a prime factor that does not divide $a^p–1$, $a^q–1$ and $a^r–1$ (plainly $n>p,q,r$), so there is no solution. The remaining cases ($a < 3$ or $n < 3$) are easy exercises for the readers.

Example 2 (IMO Shortlist 2000). Find all triplets of positive integers (a,m,n) such that $a^m+1|(a+1)^n$.

Solution. Note that (a,m,n)=(2,3,n) with $n ≥ 2$ are solutions. For $a > 1$, $m ≥ 2$ and $(a,m) ≠ (2,3)$, by Zsigmondy's theorem, a^m+1 has a prime factor that does not divide $a+1$, and hence does not divide $(a+1)^n$, so there is no solution. The cases (a =1 or m =1) lead to easy solutions.

Example 3 (Math Olympiad Summer Program 2001) Find all quadruples of positive integers (x,r,p,n) such that p is a prime, $n,r>1$ and $x^r–1 = p^n$.

Solution. If $x^r–1$ has a prime factor that does not divide $x–1$, then since $x^r–1$ is divisible by $x–1$, we deduce that $x^r–1$ has at least two distinct prime factors, a contradiction *unless* (by Zsigmondy's theorem) we have the exceptional cases x=2, r=6 and r=2, $x+1$ is a power of 2. The former does not work. For the latter, obviously p=2 since it must be even. Let $x+1=2^y$. Then

$$2^n = x^2–1 = (x+1)(x–1) = 2^y(2^y–2).$$

It follows that y=2 (hence x=3) and n=3.

Example 4 (Czech-Slovak Match 1996). Find all positive integral solutions to $p^x–y^p=1$, where p is a prime.

Solution. The equation can be rewritten as $p^x = y^p+1$. Now y = 1 leads to (p,x) = (2,1) and (y,p) = (2,3) leads to x = 2. For $y > 1$ and $p≠3$, by Zsigmondy's theorem, y^p+1 has a prime factor that does not divide $y+1$. Since y^p+1 is divisible by $y+1$, it follows that y^p+1 has at least two prime factors, a contradiction.

Remark. Alternatively, the results of Examples 3 and 4 follow from Catalan's conjecture (proven in 2002), which guarantees that the only positive integral solution to the equation $x^a–y^b=1$ with $x,y,a,b>1$ is x=3, a=2, y=2, b=3.

Example 5 (Polish MO 2010 Round 1). Let p and q be prime numbers with $q>p>2$. Prove that $2^{pq}-1$ has at least three distinct prime factors.

Solution. Note that 2^p-1 and 2^q-1 divide $2^{pq}-1$. By Zsigmondy's theorem, $2^{pq}-1$ has a prime factor p_1 that does not divide 2^p-1 and 2^q-1. Moreover, 2^q-1 has a prime factor p_2 that does not divide 2^p-1. Finally, 2^p-1 has a prime factor p_3.

The next example illustrates a more involved technique of applying Zsigmondy's theorem to solve a class of Diophantine equations.

Example 6 (Balkan MO 2009). Solve the equation $5^x - 3^y = z^2$ in positive integers.

Solution. By considering (mod 3), we see that x must be even. Let $x=2w$. Then $3^y = 5^{2w}-z^2 = (5^w-z)(5^w+z)$. Note that

$$\begin{aligned}(5^w-z,5^w+z) &= (5^w-z,2z)\\ &= (5^w-z,z)\\ &= (5^w,z) = 1,\end{aligned}$$

so $5^w - z = 1$ and $5^w + z = 3^a$ for some positive integer $a \geq 2$. Adding, $2(5^w) = 3^a + 1$. For $a = 2$, we have $w = 1$, corresponding to the solution $x = 2$, $y = 2$ and $z = 4$. For $a \geq 3$, by Zsigmondy's theorem, 3^a+1 has a prime factor p that does not divide $3^2 + 1 = 10$, which implies $p\neq 2$ or 5, so there is no solution in this case.

Example 7. Find all positive integral solutions to $p^a-1=2^n(p-1)$, where p is a prime.

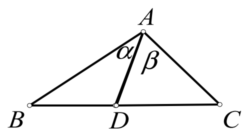
Solution. The case $p=2$ is trivial. Assume p is odd. If a is not a prime, let $a=uv$. Then p^u-1 has a prime factor that does not divide $p-1$. Since p^u-1 divides $p^a-1=2^n(p-1)$, this prime factor of p^u-1 must be 2. But by Zsigmondy's theorem, p^a-1 has a prime factor that does not divide p^u-1 and $p-1$, a contradiction to the equation. So a is a prime. The case $a=2$ yields $p=2^n-1$, i.e. the Mersenne primes. If a is an odd number, then by Zsigmondy's theorem again, $p^a-1=2^n(p-1)$ has a prime factor that does not divide $p-1$; this prime factor must be 2. However, 2 divides $p-1$, a contradiction.

(continued on page 4)

A Geometry Theorem

Kin Y. Li

The following is a not so well known, but useful theorem.



Subtended Angle Theorem. D is a point inside $\angle BAC$ (<180°). Let $\alpha=\angle BAD$ and $\beta=\angle CAD$. D is on side BC if and only if

$$\frac{\sin(\alpha+\beta)}{AD}=\frac{\sin\alpha}{AC}+\frac{\sin\beta}{AB} \qquad (*).$$

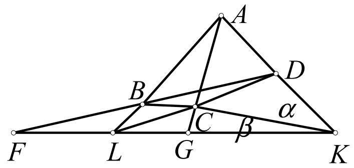
Proof. Note D is on segment BC if and only if the area of $\triangle ABC$ is the sum of the areas of $\triangle ABD$ and $\triangle ACD$. This is

$$\frac{AB\cdot AC\sin(\alpha+\beta)}{2}=\frac{AB\cdot AD\sin\alpha}{2}+\frac{AC\cdot AD\sin\beta}{2}.$$

Multiplying by $2/(AB\cdot AC\cdot AD)$ yields (*).

Below, we will write $PQ\cap RS=X$ to mean lines PQ and RS intersect at point X.

Example 1. Let $AD\cap BC=K$, $AB\cap CD=L$, $BD\cap KL=F$ and $AC\cap KL=G$. Prove that $1/KL = ½(1/KF +1/KG)$.



Solution. Applying the subtended angle theorem to $\triangle KAL$, $\triangle KDL$, $\triangle KDF$ and $\triangle KAG$, we get

$$\frac{\sin(\alpha+\beta)}{KB}=\frac{\sin\alpha}{KL}+\frac{\sin\beta}{KA},\ \frac{\sin(\alpha+\beta)}{KC}=\frac{\sin\alpha}{KL}+\frac{\sin\beta}{KD}$$

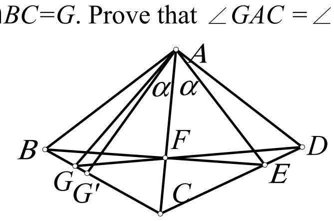
$$\frac{\sin(\alpha+\beta)}{KB}=\frac{\sin\alpha}{KF}+\frac{\sin\beta}{KD},\ \frac{\sin(\alpha+\beta)}{KC}=\frac{\sin\alpha}{KG}+\frac{\sin\beta}{KA}$$

Call these (1), (2), (3), (4) respectively. Doing (1)+(2)−(3)−(4), we get

$$0=\frac{2\sin\alpha}{KL}-\frac{\sin\alpha}{KF}-\frac{\sin\alpha}{KG},$$

which implies the desired equation.

Example 2. (*1999 Chinese National Math Competition*) In the convex quadrilateral $ABCD$, diagonal AC bisects $\angle BAD$. Let E be on side CD such that $BE\cap AC=F$ and $DF\cap BC=G$. Prove that $\angle GAC = \angle EAC$.



Solution. Let $\angle BAC=\angle DAC=\theta$ and G' be on segment BC such that $\angle G'AC = \angle EAC=\alpha$. We will show G', F, D are collinear, which implies $G'=G$. Applying the subtended angle theorem to $\triangle ABE$, $\triangle ABC$ and $\triangle ACD$ respectively, we get

$$(1)\ \frac{\sin(\theta+\alpha)}{AF}=\frac{\sin\alpha}{AB}+\frac{\sin\theta}{AE},$$

$$(2)\ \frac{\sin\theta}{AG'}=\frac{\sin\alpha}{AB}+\frac{\sin(\theta-\alpha)}{AC}\ \text{and}$$

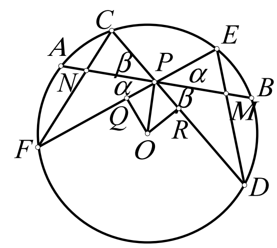
$$(3)\ \frac{\sin\theta}{AE}=\frac{\sin\alpha}{AD}+\frac{\sin(\theta-\alpha)}{AC}.$$

Doing (1)−(2)+(3), we get

$$\frac{\sin(\theta+\alpha)}{AF}=\frac{\sin\alpha}{AD}+\frac{\sin\theta}{AG'}.$$

By the subtended angle theorem, G', F, D are collinear. Therefore, $G = G'$.

Example 3. (*Butterfly Theorem*) Let A,C,E,B,D,F be points in cyclic order on a circle and $CD\cap EF=P$ is the midpoint of AB. Let $M = AB\cap DE$ and $N = AB\cap CF$. Prove that $MP = NP$.



Solution. By the intersecting chord theorem, $PC\cdot PD=PE\cdot PF$, call this x. Applying the subtended angle theorem to $\triangle PDE$ and $\triangle PCF$, we get

$$\frac{\sin(\alpha+\beta)}{PM}=\frac{\sin\alpha}{PD}+\frac{\sin\beta}{PE},$$
$$\frac{\sin(\alpha+\beta)}{PN}=\frac{\sin\alpha}{PC}+\frac{\sin\beta}{PF}.$$

Subtracting these equations, we get

$$\sin(\alpha+\beta)\left(\frac{1}{PM}-\frac{1}{PN}\right) \qquad (*)$$
$$=\sin\beta\frac{PF-PE}{x}-\sin\alpha\frac{PD-PC}{x}.$$

Let Q and R be the midpoints of EF and CD respectively. Since $OP\perp AB$, we have $PF-PE = 2PQ = 2OP\cos(90°-\alpha) = 2OP\sin\alpha$. Then similarly we have $PD-PC = 2OP\sin\beta$. Hence, the right side of (*) is zero. So the left side of (*) is also zero. Since $0 < \alpha+\beta < 180°$, we get $\sin(\alpha+\beta) \neq 0$. Then $PM = PN$.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***March 28, 2012.***

Problem 386. Observe that $7+1=2^3$ and $7^7+1=2^3\times113\times911$. Prove that for $n=2,3,4,\ldots$, in the prime factorization of $A_n=7^{7^n}+1$, the sum of the exponents is at least $2n+3$.

Problem 387. Determine (with proof) all functions $f:[0,+\infty)\to[0,+\infty)$ such that for every $x\ge0$, we have $4f(x)\ge3x$ and $f(4f(x)-3x)=x$.

Problem 388. In $\triangle ABC$, $\angle BAC=30°$ and $\angle ABC=70°$. There is a point M lying inside $\triangle ABC$ such that $\angle MAB=\angle MCA=20°$. Determine $\angle MBA$ (with proof).

Problem 389. There are 80 cities. An airline designed flights so that for each of these cities, there are flights going in both directions between that city and at least 7 other cities. Also, passengers from any city may fly to any other city by a sequence of these flights. Determine the least k such that no matter how the flights are designed subject to the conditions above, passengers from one city can fly to another city by a sequence of at most k flights.

Problem 390. Determine (with proof) all ordered triples (x, y, z) of positive integers satisfying the equation

$$x^2y^2=z^2(z^2-x^2-y^2).$$

****

Solutions

****

Problem 381. Let k be a positive integer. There are 2^k balls divided into a number of piles. For every two piles A and B with p and q balls respectively, if $p\ge q$, then we may transfer q balls from pile A to pile B. Prove that it is always possible to make finitely many such transfers so as to have all the balls end up in one pile.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Kevin LAU Chun Ting** (St. Paul's Co-educational College, S.3), **LO Shing Fung** (F3E, Carmel Alison Lam Foundation Secondary School) and **Andy LOO** (St. Paul's Co-educational College).

We induct on k. For $k=1$, we can merge the 2 balls in at most 1 transfer.

Suppose the case $k=n$ is true. For $k=n+1$, since 2^k is even, considering (odd-even) parity of the number of balls in each pile, we see the number of piles with odd numbers of balls is even. Pair up these piles. In each pair, after 1 transfer, both piles will result in even number of balls.

So we need to consider only the situation when all piles have even number of balls. Then in each pile, pair up the balls. This gives altogether 2^n pairs. Applying the case $k=n$ with the paired balls, we solve the case $k=n+1$.

Problem 382. Let $v_0=0$, $v_1=1$ and

$$v_{n+1}=8v_n-v_{n-1}\quad\text{for } n=1,2,3,\ldots.$$

Prove that v_n is divisible by 3 if and only if v_n is divisible by 7.

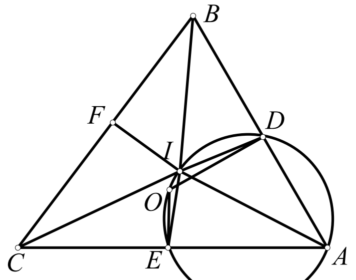
Solution. **Alumni 2011** (Carmel Alison Lam Foundation Secondary School) and **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia) and **Mihai STOENESCU** (Bischwiller, France).

For $n=1,2,3,\ldots$, $v_{n+2}=8(8v_n-v_{n-1})-v_n=63v_n-8v_{n-1}$. Then $v_{n+2}\equiv v_{n-1}$ (mod 3) and $v_{n+2}\equiv -v_{n-1}$ (mod 7). Since $v_0=0$, $v_1=1$, $v_2=8$, so $v_{3k+1}, v_{3k+2}\not\equiv 0$ (mod 3) and (mod 7) and $v_{3k}\equiv 0$ (mod 3) and (mod 7).

Other commended solvers: **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **CHAN Long Tin** (Diocesan Boys' School), **CHAN Yin Hong** (St. Paul's Co-educational College), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Kevin LAU Chun Ting** (St. Paul's Co-educational College, S.3), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Andy LOO** (St. Paul's Co-educational College), **NGUYEN van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (G.T.(Ellen Yeung) College), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Yan Yin WANG** (City University of Hong Kong, Computing Math, Year 2), **ZOLBAYAR Shagdar** (Orchlon School, Ulaanbaatar, Mongolia), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 383. Let O and I be the circumcenter and incenter of $\triangle ABC$ respectively. If $AB\ne AC$, points D, E are midpoints of AB, AC respectively and $BC=(AB+AC)/2$, then prove that the line OI and the bisector of $\angle CAB$ are perpendicular.

***Solution* 1. Kevin LAU Chun Ting** (St. Paul's Co-educational College, S.3).



From $BC=(AB+AC)/2=BD+CE$, we see there exists a point F be on side BC such that $BF=BD$ and $CF=CE$. Since BI bisects $\angle FBD$, by SAS, $\triangle IBD\cong\triangle IBF$. Then $\angle BDI=\angle BFI$. Similarly, $\angle CEI=\angle CFI$. Then

$$\begin{aligned}&\angle ADI+\angle AEI\\ &=(180°-\angle BDI)+(180°-\angle CEI)\\ &=360°-\angle BFI-\angle CFI=180°.\end{aligned}$$

So A,D,I,E are concyclic.

Since $OD\perp AD$ and $OE\perp AE$, so A,D,O,E are also conyclic. Then A,D,I,O are concyclic. So $\angle OIA=\angle ODA=90°$.

***Solution* 2. AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Ercole SUPPA** (Teramo, Italy), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Let $a=BC$, $b=CA$, $c=AB$ and let R, r, s be the circumradius, the inradius and the semiperimeter of $\triangle ABC$ respectively. By the famous formulas $OI^2=R^2-2Rr$, $s-a=AI\cos(A/2)$, $Rr=abc/(4s)$ and $\cos^2(A/2)=s(s-a)/(bc)$, we get

$$AI^2=\frac{(s-a)^2}{\cos^2(A/2)}=\frac{bc(s-a)}{s},$$

$$OI^2=R^2-2Rr=R^2-\frac{abc}{2s}.$$

If $a=(b+c)/2$, then we get $2s=3a$ and $bc(s-a)/s=abc/(2s)$. So $AI^2+OI^2=R^2=OA^2$. By the converse of Pythagoras' Theorem, we get $OI\perp AI$.

Comment: In the last paragraph, all steps may be reversed so that $OI\perp AI$ if and only if $a=(b+c)/2$.

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School), **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Andy LOO** (St. Paul's Co-educational College), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece), **NGUYEN van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **Mihai STOENESCU** (Bischwiller, France) and **ZOLBAYAR Shagdar** (Orchlon School, Ulaanbaatar, Mongolia).

Problem 384. For all positive real numbers a,b,c satisfying $a + b + c = 3$, prove that

$$\frac{a^2+3b^2}{ab^2(4-ab)}+\frac{b^2+3c^2}{bc^2(4-bc)}+\frac{c^2+3a^2}{ca^2(4-ca)}\ge 4.$$

Solution. **William PENG**.

Let

$$A=\frac{a}{b^2(4-ab)}+\frac{b}{c^2(4-bc)}+\frac{c}{a^2(4-ca)},$$

$$B=\frac{1}{a(4-ab)}+\frac{1}{b(4-bc)}+\frac{1}{c(4-ca)},$$

$$C=\frac{4-ab}{a}+\frac{4-bc}{b}+\frac{4-ca}{c}$$

and $D=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Then A+3B is the left side of the desired inequality. Now since $a + b + c = 3$, we have $C = 4D - 3$. By the Cauchy-Schwarz inequality, we have $(a+b+c)D \ge 3^2$, $AC \ge D^2$ and $BC \ge D^2$. The first of these gives us $D \ge 3$ so that $(D-3)(D-1)\ge 0$, which implies $D^2 \ge 4D-3$. The second and third imply

$$A+3B\ge\frac{4D^2}{C}=\frac{4D^2}{4D-3}\ge 4.$$

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School), **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Andy LOO** (St. Paul's Co-educational College), **NGUYEN van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Problem 385. To prepare for the IMO, in everyday of the next 11 weeks, Jack will solve at least one problem. If every week he can solve at most 12 problems, then prove that for some positive integer n, there are n consecutive days in which he can solve a total of 21 problems.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Andy LOO** (St. Paul's Co-educational College) and **Yan Yin WANG** (City University of Hong Kong, Computing Math, Year 2).

Let S_i be the total number of problems Jack solved from the first day to the end of the i-th day. Since he solves at least one problem everyday, we have $0 < S_1 < S_2 < S_3 < \cdots < S_{77}$. Since he can solve at most 12 problems every week, we have $S_{77} \le 12\times 11=132$.

Consider the two strictly increasing sequences $S_1, S_2, \cdots, S_{77}$ and $S_1+21, S_2+21, \cdots, S_{77}+21$. Now these 154 integers are at least 1 and at most 132+21=153. By the pigeonhole principle, since the two sequences are strictly increasing, there must be $m < k$ such that $S_k=S_m+21$. Therefore, Jack solved a total of 21 problems from the $(m+1)$-st day to the end of the k-th day.

Olympiad Corner

(continued from page 1)

Problem 4 (cont). Is it possible to empty both bags after a finite sequence of operations?

Operation b) is now replaced with

b') Triple the number of balls in one bag.

Is it now possible to empty both bags after a finite sequence of operations?

Problem 5. Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers.

Problem 6. Let ABC be an acute-angled triangle. The feet of all the altitudes from A, B and C are D, E and F respectively. Prove that $DE+DF \le BC$ and determine the triangles for which equality holds.

Zsigmondy's Theorem

(continued from page 2)

Example 8. Find all positive integral solutions to

$$(a+1)(a^2+a+1)\cdots(a^n+a^{n-1}+\cdots+1)$$
$$=a^m+a^{m-1}+\cdots+1.$$

Solution. Note that $n = m = 1$ is a trivial solution. Other than that, we must have $m>n$. Write the equation as

$$\frac{a^2-1}{a-1}\cdot\frac{a^3-1}{a-1}\cdot\ldots\cdot\frac{a^{n+1}-1}{a-1}=\frac{a^{m+1}-1}{a-1},$$

then rearranging we get

$$\left(a^2-1\right)\left(a^3-1\right)\ldots\left(a^{n+1}-1\right)$$
$$=\left(a^{m+1}-1\right)\left(a-1\right)^{n-1}.$$

By Zsigmondy's theorem, we must have $a = 2$ and $m + 1 = 6$, i.e. $m = 5$ (otherwise, a^m-1 has a prime factor that does not divide $a^2-1, a^3-1, \ldots, a^{n+1}-1$, a contradiction), which however does not yield a solution for n.

The above examples show that Zsigmondy's theorem can instantly reduce many number theoretic problems to a handful of small cases. We should bear in mind the exceptions stated in Zsigmondy's theorem in order not to miss out any solutions.

Below are some exercises for the readers.

Exercise 1 (1994 Romanian Team Selection Test). Prove that the sequence $a_n=3^n-2^n$ contains no three terms in geometric progression.

Exercise 2. Fermat's last theorem asserts that for a positive integer $n \ge 3$, the equation $x^n+y^n=z^n$ has no integral solution with $xyz\ne 0$. Prove this statement when z is a prime.

Exercise 3 (1996 British Math Olympiad Round 2). Determine all sets of non-negative integers x, y and z which satisfy the equation $2^x+3^y=z^2$.

Reference

[1] PISOLVE, *The Zsigmondy Theorem,*

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=721&t=422330

[2] Lola Thompson, *Zsigmondy's Theorem*,

http://www.math.dartmouth.edu/~thompson/Zsigmondy's%20Theorem.pdf

Mathematical Excalibur

Volume 16, Number 5 *March - April 2012*

Olympiad Corner

Below are the problems of the 2011 IMO Team Selection Contest from Estonia.

Problem 1. Two circles lie completely outside each other. Let A be the point of intersection of internal common tangents of the circles and let K be the projection of this point onto one of their external common tangents. The tangents, different from the common tangent, to the circles through point K meet the circles at M_1 and M_2. Prove that the line AK bisects angle M_1KM_2.

Problem 2. Let n be a positive integer. Prove that for each factor m of the number $1+2+\cdots+n$ such that $m \geq n$, the set $\{1,2,\cdots,n\}$ can be partitioned into disjoint subsets, the sum of the elements of each being equal to m.

Problem 3. Does there exist an operation $*$ on the set of all integers such that the following conditions hold simultaneously:

(1) for all integers x, y and z,
$$(x*y)*z = x*(y*z);$$

(2) for all integers x and y,
$$x*x*y = y*x*x = y\ ?$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 11, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

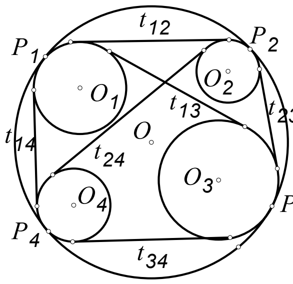
Casey's Theorem

Kin Y. Li

We recall *Ptolemy's theorem*, which asserts that for four noncollinear points A, B, C, D on a plane, we have

$$AB\cdot CD + AD\cdot BC = AC\cdot BD$$

if and only if $ABCD$ is a cyclic quadrilateral (cf *vol*. 2, *no*. 4 of Math Excalibur). In this article, we study a generalization of this theorem known as

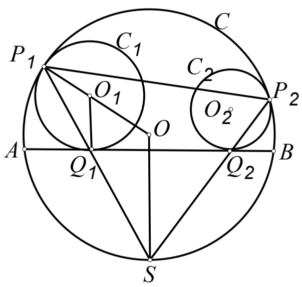


Casey's Theorem. If circles C_1, C_2, C_3, C_4 with centers O_1, O_2, O_3, O_4 are internally tangent to a circle C with center O at points P_1, P_2, P_3, P_4 in cyclic order respectively, then

$$t_{12}\cdot t_{34} + t_{14}\cdot t_{23} = t_{13}\cdot t_{24}, \qquad (*)$$

where t_{ik} denote the length of an external common tangent of circle C_i and C_k.

To prove this, consider the following figure.



Let line AB be an external common tangent to C_1, C_2 intersecting C_1 at Q_1, C_2 at Q_2. Let line P_1Q_1 intersect C at S. Let r_1, r be the respective radii of C_1, C. Then the isosceles triangles $P_1O_1Q_1$ and P_1OS are similar. So $O_1Q_1 \parallel OS$. Since $O_1Q_1 \perp AB$, so $OS \perp AB$, hence S is the midpoint of arc AB. Similarly, line P_2Q_2 passes through S. Now $\angle SQ_1Q_2 = \angle P_1Q_1A = ½\angle P_1O_1Q_1 = ½\angle P_1OS = \angle SP_2P_1$. Then $\triangle SQ_1Q_2$ and $\triangle SP_2P_1$ are similar. So

$$\frac{Q_1Q_2}{P_2P_1} = \frac{SQ_1}{SP_2} = \frac{SQ_2}{SP_1} = \sqrt{\frac{SQ_1\cdot SQ_2}{SP_1\cdot SP_2}} = \sqrt{\frac{OO_1\cdot OO_2}{OP_1\cdot OP_2}}$$

$$t_{12} = Q_1Q_2 = P_1P_2\frac{\sqrt{(r-r_1)(r-r_2)}}{r}. \quad (**)$$

The expressions for the other t_{ik}'s are similar. Since $P_1P_2P_3P_4$ is cyclic, by Ptolemy's theorem,

$$P_1P_2\cdot P_3P_4 + P_1P_4\cdot P_2P_3 = P_1P_3\cdot P_2P_4.$$

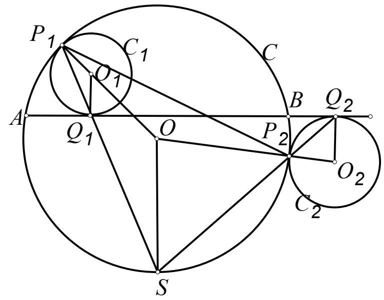
Multiplying all terms by

$$\frac{\sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}}{r^2}$$

and using (**), we get (*).

Casey's theorem can be *extended* to cover cases some C_k's are externally tangent to C. For this, define t_{ik} more generally to be the length of the external (resp. internal) common tangent of circles C_i and C_k when the circles are on the same (resp. opposite) side of C.

In case C_k is externally tangent to C, consider the following figure. The proof is the same as before except the factor $r-r_k$ should be replaced by $r+r_k$.



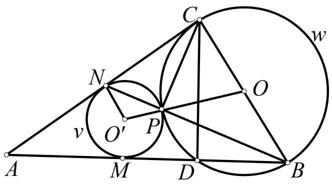
The converse of Casey's theorem and its extension are also true. However, the proofs are harder, longer and used inversion in some cases. For the curious readers, a proof of the converse can be found in *Roger A. Johnson*'s book Advanced Euclidean Geometry, published by Dover.

Next we will present some geometry problems that can be solved by Casey's theorem and its converse.

Example 1. (*2009 China Hong Kong Math Olympiad*) Let $\triangle ABC$ be a right-angled triangle with $\angle C=90^\circ$. CD is the altitude from C to AB, with D on AB. w is the circumcircle of $\triangle BCD$. v is a circle situated in $\triangle ACD$, it is tangent to the segments AD and AC at M and N respectively, and is also tangent to circle w.

(i) Show that $BD \cdot CN + BC \cdot DM = CD \cdot BM$.

(ii) Show that $BM = BC$.



Solution. (i) Think of B, C, D as circles with radius 0 externally tangent to w. Then $t_{BD} = BD$, $t_{Cv} = CN$, $t_{BC} = BC$, $t_{Dv} = DM$, $t_{CD} = CD$ and $t_{Bv} = BM$. By Casey's theorem, (*) yields

$$BD \cdot CN + BC \cdot DM = CD \cdot BM.$$

(ii) Let circles v and w meet at P. Then $\angle BPC=90^\circ$. Let O and O' be centers of circles w and v. Then O, P, O' are collinear. So

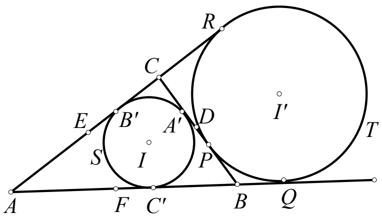
$$\angle PNC+\angle PCN = ½(\angle PO'N+\angle POC)$$
$$= ½(360°-\angle O'NC-\angle OCN) = 90°.$$

So $\angle NPC = 90^\circ$. Hence, B, P, N are collinear. By the power-of-a-point theorem, $BM^2=BP\cdot BN$. Also $\angle C=90^\circ$ and $CP\perp BN$ imply $BC^2 = BP\cdot BN$. Therefore, $BM=BC$.

Example 2. (*Feuerbach's Theorem*) Let D,E,F be the midpoints of sides AB, BC, CA of ΔABC respectively.

(i) Prove that the inscribed circle S of ΔABC is tangent to the (nine-point) circle N through D, E, F.

(ii) Prove that the described circle T on side BC is also tangent to N.



Solution. (1) We consider D, E, F as circles of radius 0. Let A', B', C' be the points of tangency of S to sides BC, CA, AB respectively.

First we recall that the two tangent segments from a point to a circle have the same length. Let $AB' = x = C'A$, $BC' = y = A'B$, $CA' = z = B'C$ and $s = (a+b+c)/2$, where $a=BC$, $b=CA$, $c=AB$. From $y+x = BA = c$, $z+y = CB = a$ and $x+z = AC = b$, we get $x = (c+b-a)/2 = s-a$, $y=s-b$, $z=s-c$. By the midpoint theorem, $t_{DE} = DE = ½BA = c/2$ and

$$t_{FS}= FC' = |FB-BC'| = |(c/2)-y|$$
$$= |c-2(s-b)|/2 = |b-a|/2.$$

Similarly, $t_{EF} = a/2$, $t_{DS}= |c-b|/2$, $t_{FD} = b/2$ and $t_{ES}=|a-c|/2$. Without loss of generality, we may assume $a\le b\le c$. Then

$$t_{DE} \cdot t_{FS} + t_{EF} \cdot t_{DS} = c(b-a)/4+a(c-b)/4$$
$$= b(c-a)/4$$
$$= t_{FD} \cdot t_{ES}.$$

By the converse of Casey's theorem, we get S is tangent to the circle N through D,E,F.

(2) Let I' be the center of T, let P,Q,R be the points of tangency of T to lines BC, AB, CA respectively. As in (1), $t_{DE}= c/2$.

To find t_{FT}, we need to know BQ. First note $AQ=AR$, $BP=BQ$ and $CR=CP$. So $2AQ=AQ+AR=AB+BP+CP+AC=2s$. So $AQ=s/2$. Next $BQ=AQ-AB=s-c$. Hence, $t_{FT}=FQ=FB+BQ = (c/2)+(s-c) = (b+a)/2$. Similarly, $t_{ET}= (a+c)/2$. Now $t_{DT}= DP = DB-BP = DB-BQ = (a/2) - (s-c) = (c-b)/2$. Then

$$t_{FD} \cdot t_{ET} + t_{EF} \cdot t_{DT} = b(a+c)/4+a(c-b)/4$$
$$= c(b+a)/4$$
$$= t_{DE} \cdot t_{FT}.$$

By the converse of Casey's theorem, we get T is tangent to the circle N through D,E,F.

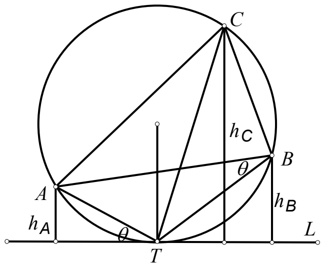
Example 3. (*2011 IMO*) Let ABC be an acute triangle with circumcircle Γ. Let L be a tangent line to Γ, and let L_a, L_b and L_c be the line obtained by reflecting L in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines L_a, L_b and L_c is tangent to the circle Γ.

Solution. (*Due to **CHOW Chi Hong**, 2011 Hong Kong IMO team member*)

Below for brevity, we will write $\angle A$, $\angle B$, $\angle C$ to denote $\angle CAB$, $\angle ABC$, $\angle BCA$ respectively.

Lemma. In the figure below, L is a tangent line to Γ, T is the point of tangency. Let h_A, h_B, h_C be the length of the altitudes from A, B, C to L respectively. Then

$$\sqrt{h_A}\sin\angle A+\sqrt{h_B}\sin\angle B=\sqrt{h_C}\sin\angle C.$$



Proof. By Ptolemy's theorem and sine law,

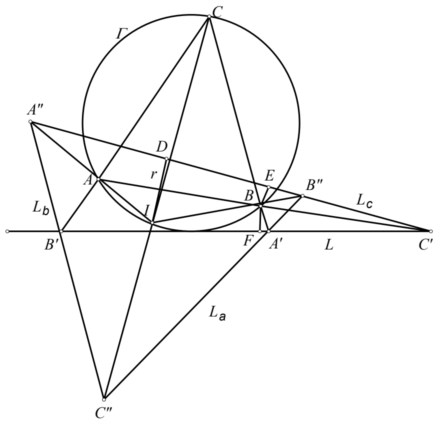
$$AT\cdot BC + BT\cdot CA = CT\cdot BC \quad \text{(or}$$

$$AT\sin\angle A + BT\sin\angle B = CT\sin\angle C\text{)}.$$

Let θ be the angle between lines AT and L as shown. Then $AT = h_A / \sin\theta = h_A(2k/AT)$, where k is the circumradius of ΔABC. Solving for AT (then using similar argument for BT and CT), we get

$$AT=\sqrt{2kh_A}, BT=\sqrt{2kh_B}, CT=\sqrt{2kh_C}.$$

Substituting these into (*), the result follows. This finishes the proof of the lemma.



For the problem, let $L_a\cap L=A'$, $L_b\cap L = B'$, $L_c\cap L=C'$, $L_a\cap L_b=C''$, $L_b\cap L_c=A''$, $L_c\cap L_a=B''$. Next

$$\angle A''C''B''=\angle A''B'A'-\angle C''A'B'$$
$$=2\angle CB'A'-(180°-2\angle CA'B')$$
$$=180°-2\angle C.$$

Similarly, $\angle A''B''C'' = 180° - 2\angle B$ and $\angle B''A''C'' = 180° - 2\angle A$. (***)

Consider $\Delta A'C'B''$. Now $A'B$ bisects $\angle B'A'B''$ and $C'B$ bisects $\angle A'C'B''$. So B is the excenter of $\Delta A'C'B''$ opposite C'. Hence $B''B$ bisects $\angle A''B''C''$. Similarly, $A''A$ bisects $\angle B''A''C''$ and $C''C$ bisects $\angle B''C''A''$. Therefore, they intersect at the incenter I of $\Delta A''B''C''$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 11, 2012.***

Problem 391. Let $S(x)$ denote the sum of the digits of the positive integer x in base 10. Determine whether there exist distinct positive integers a, b, c such that $S(a+b)<5$, $S(b+c)<5$, $S(c+a)<5$, but $S(a+b+c)>50$ or not.

Problem 392. Integers $a_0, a_1, \cdots, a_n$ are all greater than or equal to -1 and are not all zeros. If

$$a_0+2a_1+2^2a_2+\cdots+2^na_n = 0,$$

then prove that $a_0+a_1+a_2+\cdots+a_n>0$.

Problem 393. Let p be a prime number and $p \equiv 1 \pmod 4$. Prove that there exist integers x and y such that

$$x^2 - py^2 = -1.$$

Problem 394. Let O and H be the circumcenter and orthocenter of acute $\triangle ABC$. The bisector of $\angle BAC$ meets the circumcircle Γ of $\triangle ABC$ at D. Let E be the mirror image of D with respect to line BC. Let F be on Γ such that DF is a diameter. Let lines AE and FH meet at G. Let M be the midpoint of side BC. Prove that $GM \perp AF$.

Problem 395. One frog is placed on every vertex of a $2n$-sided regular polygon, where n is an integer at least 2. At a particular moment, each frog will jump to one of the two neighboring vertices (with more than one frog at a vertex allowed).

Find all n such that there exists a jumping of these frogs so that after the moment, all lines connecting two frogs at different vertices do not pass through the center of the polygon.

****

Solutions

****

Problem 386. Observe that $7+1=2^3$ and $7^7+1= 2^3\times 113\times 911$. Prove that for $n = 2, 3, 4,\ldots$, in the prime factorization of $A_n = 7^{7^n}+1$, the sum of the exponents is at least $2n+3$.

Solution. **Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **William PENG**.

The case $n = 0$ is given. Suppose the result is true for n. Let $x = A_n-1$. Then $A_{n+1} = x^7+1 = (x+1)P = A_nP$, where $P = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$. Comparing P with $(x+1)^6$, we find

$$\begin{aligned} P &= (x+1)^6 - 7x(x^4+2x^3+3x^2+2x+1) \\ &= (x+1)^6 - 7x(x^2+x+1)^2. \end{aligned}$$

Now $7x=7^{2m}$, where $m=(7^n+1)/2$. Then $P = [(x+1)^3+7^m(x^2+x+1)][(x+1)^3-7^m(x^2+x+1)]$. Next, $x > 7^m \geq 7$, $x^2+x+1 > (x+1)^2$ and

$$(x+1)^3-7^m(x^2+x+1) > (x+1)^2(x+1-7^m) > 1.$$

So P is the product of at least 2 more primes. Therefore, the result is true for $n+1$.

Problem 387. Determine (with proof) all functions $f : [0,+\infty) \to [0,+\infty)$ such that for every $x \geq 0$, we have $4f(x) \geq 3x$ and $f(4f(x) - 3x) = x$.

Solution. **Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **William PENG**.

We can check $f(x) = x$ is a solution. Assume there is another solution such that $f(c) \neq c$ for some $c \geq 0$. Let $x_0 = f(c)$, $x_1 = c$ and

$$x_{n+2} = 4x_n - 3x_{n+1} \quad \text{for } n = 0,1,2,\ldots.$$

From the given conditions, we can check by math induction that $x_n = f(x_{n+1}) \geq 0$ for $n = 0,1,2,\ldots$. Since $z^2+3z-4 = (z-1)(z+4)$, we see $x_n = \alpha+(-4)^n\beta$ for some real α and β. Taking $n = 0$ and 1, we get $f(c) = \alpha+\beta$ and $c = \alpha-4\beta$. Then $\beta = (f(c)-c)/5 \neq 0$.

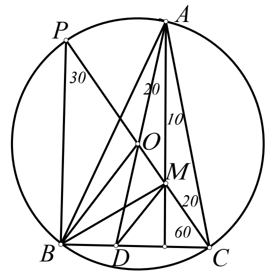
If $\beta > 0$, then $x_{2k+1} = \alpha+(-4)^{2k+1}\beta \to -\infty$ as $k \to \infty$, a contradiction. Similarly, if $\beta < 0$, then $x_{2k} = \alpha+(-4)^{2k}\beta \to -\infty$ as $k \to \infty$, yet another contradiction.

Other commended solvers: **CHAN Yin Hong** (St. Paul's Co-educational College) and **YEUNG Sai Wing** (Hong Kong Baptist University, Math, Year 1).

Problem 388. In $\triangle ABC$, $\angle BAC=30°$ and $\angle ABC=70°$. There is a point M lying inside $\triangle ABC$ such that $\angle MAB = \angle MCA = 20°$. Determine $\angle MBA$ (with proof).

Solution 1. **CHOW Chi Hong** (Bishop Hall Jubilee Schol) and **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia).

Extend CM to meet the circumcircle Γ of $\triangle ABC$ at P.



Then we have $\angle BPC = \angle BAC = 30°$ and $\angle PBC = 180° - \angle BPC - \angle BCM = 90°$. So line CM passes through center O of Γ.

Let lines AO and BC meet at D. Then $\angle AOB = 2\angle ACB = 160°$. Now $OA = OB$ implies $\angle OAB = 10°$. Then $\angle MAO = 10° = \angle MAC$ and $\angle ADC = 180° - 100° = 80° = \angle ACD$. These imply AM is the perpendicular bisector of CD. Then $MD = MC$. This along with $OB = OC$ and $\angle BOC = 60$ imply $\triangle OCB$ and $\triangle MCD$ are equilateral, hence $BOMD$ is cyclic. Then $\angle DBM = \angle DOM = 2\angle OAC = 40°$. So $\angle MBA = \angle ABC - \angle DBM = 30°$.

Solution 2. **CHAN Yin Hong** (St. Paul's Co-educational College), **Mathematics Group** (Carmel Alison Lam Foundation Secondary School), **O Kin Chit Alex** (G.T.(Ellen Yeung) College) and **Mihai STOENESCU** (Bischwiller, France).

Let $x = \angle MBA$. Applying the sine law to $\triangle ABC$, $\triangle ABM$, $\triangle AMC$ respectively, we get

$$\frac{AB}{AM} = \frac{\sin(20° + x)}{\sin x}, \frac{AB}{AC} = \frac{\sin 80°}{\sin 70°}, \frac{AC}{AM} = \frac{\sin 30°}{\sin 20°}.$$

Multiplying the last 2 equations, we get

$$\frac{\sin(20° + x)}{\sin x} = \frac{AB}{AM} = \frac{\sin 80°}{\sin 70°} \cdot \frac{\sin 30°}{\sin 20°}. \quad (\dagger)$$

Multiplying

$$\frac{\sin 80°}{\sin 40°} = 2\cos 40° = \frac{\sin 50°}{\sin 30°},$$

$$\frac{\sin 40°}{\sin 20°} = 2\cos 20° = \frac{\sin 70°}{\sin 30°},$$

we see (†) can be simplified to $\sin(20°+x)/\sin x = \sin 50°/\sin 30°$. Since the left side is equal to $\sin 20° \cot x + \cos 20°$, which is strictly decreasing (hence injective) for x between 0° to 70°, we must have $x=30°$.

Comments: One can get a similar equation as (†) directly by using the trigonometric form of Ceva's theorem.

Other commended solvers: **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **NG Ho Man** (La Salle

College, Form 5), **Bobby POON** (St. Paul's College), **St. Paul's College Mathematics Team**, **Aliaksei SEMCHANKAU** (Secondary School No.41, Minsk, Belarus) and **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia),

Problem 389. There are 80 cities. An airline designed flights so that for each of these cities, there are flights going in both directions between that city and at least 7 other cities. Also, passengers from any city may fly to any other city by a sequence of these flights. Determine the least k such that no matter how the flights are designed subject to the conditions above, passengers from one city can fly to another city by a sequence of at most k flights.

(*Source*: 2004 *Turkish MO*.)

***Solution*. William PENG**.

Below we denote the number of elements in a set S by $|S|$.

To show $k\geq 27$, take cities $A_1, A_2, \cdots, A_{28}$. For $i=1,2,\cdots,27$, design flights between A_i and A_{i+1}. For the remaining 52 cities, partition them into pairwise disjoint subsets $Y_0, Y_1, \cdots, Y_9$ so $|Y_0|=6=|Y_9|$ and the other $|Y_k|=5$. Let $Z_0=\{A_1,A_2\}\cup Y_0$, $Z_9=\{A_{27},A_{28}\}\cup Y_9$ and for $1\leq m\leq 8$, let $Z_m=\{A_{3m},A_{3m+1},A_{3m+2}\}\cup Y_m$. Then design flights between each pair of cities in Z_m for $1\leq m\leq 8$. In this design, from A_1 to A_{28} requires 27 flights.

Assume $k > 27$. Then there would exist two cities A_1 and A_{29} the shortest connection between them would involve a sequence of 28 flights from cities A_i to A_{i+1} for $i=1,2,\ldots,28$. Due to the shortest condition, each of A_1 and A_{29} has flights to 6 other cities not in $B=\{A_2,A_3,\ldots,A_{28}\}$. Each A_i in B has flights to 5 other cities not in $C = \{A_1,A_2,\ldots,A_{29}\}$.

Next for each A_i in $\{A_1,A_4,A_7,A_{10},A_{13},A_{16},A_{19},A_{22},A_{25},A_{29}\}$, let X_i be the set of cities not in C that have a flight to A_i. We have $|X_1|\geq 6$, $|X_{29}|\geq 6$ and the other $|X_i|\geq 5$. Now every pair of X_i's is disjoint, otherwise we can shorten the sequence of flights between A_1 and A_{29}. However, the union of C and all the X_i's would yield at least $29+6\times 2+5\times 8=81$ cities, contradiction. So $k = 27$.

Problem 390. Determine (with proof) all ordered triples (x, y, z) of positive integers satisfying the equation

$$x^2y^2 = z^2(z^2 - x^2 - y^2).$$

Solution. **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Ioan Viorel CODREANU** (Satulung Secondary School, Maramure, Romania) and **Aliaksei SEMCHANKAU** (Secondary School No.41, Minsk, Belarus).

Lemma. The system $a^2-b^2=c^2$ and $a^2+b^2=w^2$ has no solution in positive integers.

Proof. Assume there is a solution. Then consider a solution with minimal a^2+b^2. Due to minimality, $\gcd(a,b)=1$. Also $2a^2 = w^2+c^2$. Considering (mod 2), we see $w+c$ and $w-c$ are even. Then $a^2=r^2+s^2$, where $r=(w+c)/2$ and $s=(w-c)/2$.

Let $d=\gcd(a,r,s)$. Then d divides a and $r+s=w$. Since $a^2+b^2=w^2$, d divides b. As $\gcd(a,b)=1$, we get $d=1$. By the theorem on Pythagorean triples, there are relatively prime positive integers m,n with $m>n$ such that $\{r,s\}=\{m^2-n^2,2mn\}$ and $a=m^2+n^2$. Now $b^2 = (w^2-c^2)/2 = 2rs = 4mn(m^2-n^2)$ implies b is an even integer, say $b=2k$. Then $k^2=mn(m+n)(m-n)$. As $\gcd(m,n)=1$, we see $m, n, m+n, m-n$ are pairwise relatively prime integers. Hence, there exist positive integers d,e,f,g such that $m=d^2$, $n=e^2$, $m+n=f^2$ and $m-n=g^2$. Then $d^2-e^2=g^2$ and $d^2+e^2=f^2$, but

$$d^2+e^2=m+n < 4mn(m^2-n^2)=b^2 < a^2+b^2,$$

contradicting a^2+b^2 is minimal. The lemma is proved.

Now for the problem, the equation may be rearranged as $z^4-(x^2+y^2)z^2-x^2y^2=0$. If there is a solution (x,y,z) in positive integers, then considering discriminant, we see $x^4+6x^2y^2+y^4=w^2$ for some integer w. This can be written as $(x^2-y^2)^2+2(2xy)^2 = w^2$. Also, we have $(x^2-y^2)^2+(2xy)^2 = (x^2+y^2)^2$. Letting $c = |x^2-y^2|$, $b = 2xy$ and $a = x^2+y^2$. Then we have $c^2+b^2=a^2$ and $c^2+2b^2=w^2$ (or $a^2+b^2=w^2$). This contradicts the lemma above. So there is no solution.

Other commended solvers: **Mathematics Group** (Carmel Alison Lam Foundation Secondary School).

Olympiad Corner

(continued from page 1)

Problem 4. Let a, b, c be positive real numbers such that $2a^2+b^2=9c^2$. Prove that

$$\frac{2c}{a}+\frac{c}{b}\geq\sqrt{3}.$$

Problem 5. Prove that if n and k are positive integers such that $1 < k < n-1$, Then the binomial coefficient $\binom{n}{k}$ is divisible by at least two different primes.

Problem 6. On a square board with m rows and n columns, where $m \leq n$, some squares are colored black in such a way that no two rows are alike. Find the biggest integer k such that for every possible coloring to start with one can always color k columns entirely red in such a way that no two rows are still alike.

Casey's Theorem

(continued from page 2)

We have $\angle IAB = \angle AA''C' + \angle AC'A'' = \frac{1}{2}(\angle B'A''C' + \angle B'C'A'') = \frac{1}{2}\angle A'B'C''$ and similarly $\angle IBA = \frac{1}{2}\angle B'A'C''$. So

$$\begin{aligned}\angle AIB &= 180^\circ - \angle IA''B'' - \angle IB''A''\\ &= 180^\circ - \tfrac{1}{2}\angle C''A''B'' - \tfrac{1}{2}\angle C''B''A''\\ &= 90^\circ + \tfrac{1}{2}\angle A''C''B''\\ &= 90^\circ + \tfrac{1}{2}(180^\circ - 2\angle C) \text{ by } (***)\\ &= 180^\circ - \angle ACB.\end{aligned}$$

Hence, I lies on Γ.

Let D be the foot of the perpendicular from I to $A''B''$, then $ID=r$ is the inradius of $\Delta A''B''C''$. Let E, F be the feet of the perpendiculars from B to $A''B''$, $B'A'$ respectively. Then $BE = BF = h_B$.

Let $T(X)$ be the length of tangent from X to Γ, where X is outside of Γ. Since $\angle A''B''I = \frac{1}{2}\angle A''B''C'' = 90^\circ - \angle B$ by (***), we get

$$\begin{aligned}T(B'') &= \sqrt{B''B\cdot B''I}\\ &= \sqrt{\frac{BE}{\sin(90^\circ-\angle B)}\cdot\frac{ID}{\sin(90^\circ-\angle B)}}\\ &= \frac{\sqrt{h_B r}}{\cos B}.\end{aligned}$$

Let R be the circumradius of $\Delta A''B''C''$. Then

$$\begin{aligned}T(B'')\cdot C''A'' &= \frac{\sqrt{h_B r}}{\cos B}2R\sin(180^\circ-2\angle B)\\ &= 4R\sqrt{r}\sqrt{h_B}\sin B.\end{aligned}$$

Similarly, we can get expressions for $T(A'')\cdot B''C''$ and $T(C'')\cdot A''B''$. Using the lemma, we get

$$T(A'')\cdot B''C'' + T(B'')\cdot C''A'' = T(C'')\cdot A''B''.$$

By the converse of Casey's theorem, we have the result.