

## On a Triad of Similar Triangles Associated With the Perpendicular Bisectors of the Sides of a Triangle

Quang Tuan Bui

**Abstract.** We prove some perspectivity and orthology results related to a triad of similar triangles associated with the perpendicular bisectors of the sides of a given triangle.

### 1. A triad of triangles from the perpendicular bisectors

Given a triangle  $ABC$ , let the perpendicular bisectors of the sides  $AB$  and  $AC$  intersect the sideline  $BC$  at  $A_b$  and  $A_c$  respectively. The angles of triangle  $AA_bA_c$  have measures  $\pi - 2A$ ,  $\pi - 2B$ ,  $\pi - 2C$  if  $ABC$  is acute angled, and  $2A - \pi$ ,  $2B$ ,  $2C$  if angle  $A$  is obtuse (see Figure 1).

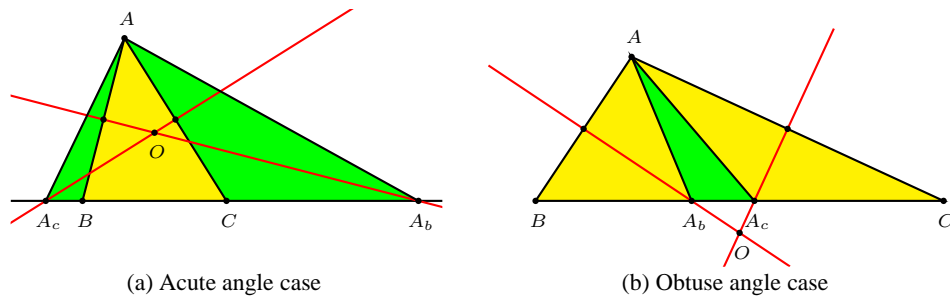


Figure 1.

Similarly, we consider the intersections  $B_a$  and  $B_c$  of  $CA$  with the perpendicular bisectors of  $BA$  and  $BC$ , yielding triangle  $B_aBB_c$ , and the intersections  $C_a$  and  $C_b$  of  $AB$  with the perpendicular bisectors of  $CA$ ,  $CB$  yielding triangle  $C_aC_bC$ . It is clear that the triangles  $AA_bA_c$ ,  $B_aBB_c$  and  $C_aC_bC$  are all similar (see Figure 2). We record the homogeneous barycentric coordinates of these points.

$$\begin{aligned} A_b &= (0 : -S_A + S_B : S_A + S_B), & A_c &= (0 : S_C + S_A : S_C - S_A); \\ B_c &= (S_B + S_C : 0 : -S_B + S_C), & B_a &= (S_A - S_B : 0 : S_A + S_B); \\ C_a &= (-S_C + S_A : S_C + S_A : 0), & C_b &= (S_B + S_C : S_B - S_C : 0). \end{aligned}$$

Given a finite point  $P$  with homogeneous barycentric coordinates  $(x : y : z)$  relative to  $ABC$ , we consider the points with the same coordinates in the triangles  $AA_bA_c$ ,  $B_aBB_c$  and  $C_aC_bC$ . These are the points

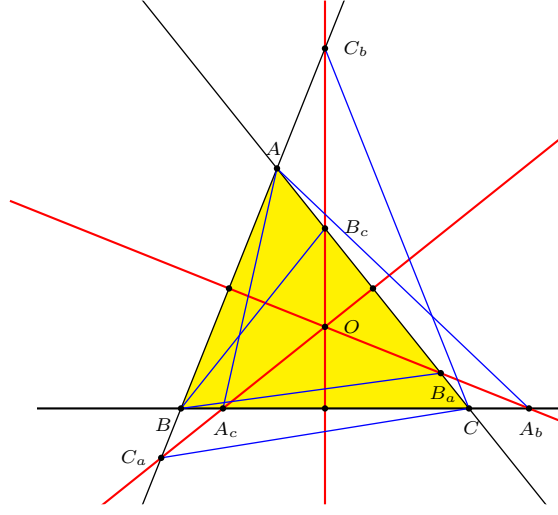


Figure 2.

$$\begin{aligned}
 A_P &= (2S_{BC}x : -S_C(S_A - S_B)y + S_B(S_C + S_A)z : S_C(S_A + S_B)y - S_B(S_C - S_A)z), \\
 B_P &= (S_A(S_B + S_C)z - S_C(S_A - S_B)x : 2S_{CA}y : -S_A(S_B - S_C)z + S_C(S_A + S_B)x), \\
 C_P &= (-S_B(S_C - S_A)x + S_A(S_B + S_C)y : S_B(S_C + S_A)x - S_A(S_B - S_C)y : 2S_{AB}z).
 \end{aligned}$$

with coordinates with respect to triangle  $ABC$  (see Figure 3).

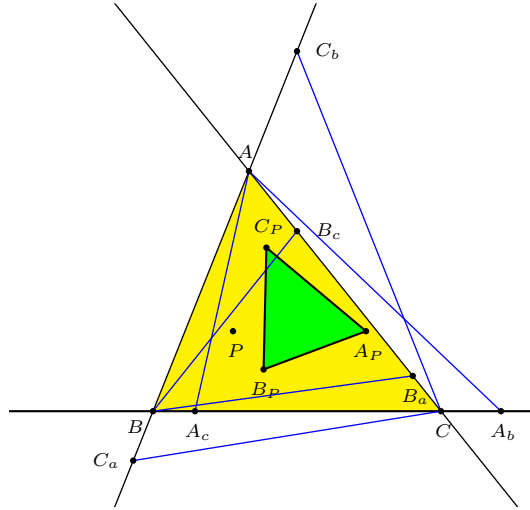


Figure 3.

**Proposition 1.** *Triangle  $A_P B_P C_P$  is oppositely similar to  $ABC$ .*

*Proof.* The square lengths of the sides are

$$(B_PC_P)^2 = k \cdot a^2, \quad (C_PA_P)^2 = k \cdot b^2, \quad (A_PB_P)^2 = k \cdot c^2,$$

where

$$k = \frac{S^2 \left( \sum_{\text{cyclic}} a^4 S_{AAyz} \right) - (x+y+z) \left( \sum_{\text{cyclic}} b^2 c^2 S_{BB} S_{CC} x \right)}{(2S_{ABC}(x+y+z))^2}.$$

From this the similarity is clear. Note that the multiplier  $k$  is nonnegative. Now, the areas are related by

$$\Delta A_PB_PC_P = -k \cdot \Delta ABC.$$

It follows that the two triangles are oppositely oriented.  $\square$

## 2. Perspectivity

**Proposition 2.** *Triangle  $A_PB_PC_P$  is perspective with the medial triangle at*

$$\left( \frac{x}{S_A} \left( -\frac{x}{S_A} + \frac{y}{S_B} + \frac{z}{S_C} \right) : \cdots : \cdots \right). \quad (1)$$

*Proof.* Let  $DEF$  be the medial triangle. The line joining  $DA_P$  has equation

$$S_A(S_C y - S_B z)\mathbb{X} + S_{BC}x\mathbb{Y} - S_{BC}x\mathbb{Z} = 0,$$

or

$$\left( \frac{y}{S_B} - \frac{z}{S_C} \right) \mathbb{X} + \frac{x}{S_A} (\mathbb{Y} - \mathbb{Z}) = 0.$$

This clearly contains the point  $\left( -\frac{x}{S_A} : \frac{y}{S_B} : \frac{z}{S_C} \right)$ , which is a vertex of the anticevian triangle of  $Q = \left( \frac{x}{S_A} : \frac{y}{S_B} : \frac{z}{S_C} \right)$ . Similarly, the lines  $EB_P$  and  $FC_P$  contain the corresponding vertices of the same anticevian triangle. It follows that  $A_PB_PC_P$  and  $DEF$  are perspective at the cevian quotient  $G/Q$  whose coordinates are given by (1) above.  $\square$

**Proposition 3.** *Triangle  $A_PB_PC_P$  is perspective with  $ABC$  if and only if  $P$  lies on the Jerabek hyperbola of  $ABC$ . In this case, the perspector traverses the Euler line.*

*Proof.* The lines  $AA_P$ ,  $BB_P$ ,  $CC_P$  have equations

$$\begin{aligned} (S_C(S_A + S_B)y + S_B(S_C - S_A)z)\mathbb{Y} + (S_C(S_A - S_B)y - S_B(S_C + S_A)z)\mathbb{Z} &= 0, \\ (S_A(S_B - S_C)z - S_C(S_A + S_B)x)\mathbb{X} + (S_A(S_B + S_C)z + S_C(S_A - S_B)x)\mathbb{Z} &= 0, \\ (S_B(S_C + S_A)x + S_A(S_B - S_C)y)\mathbb{X} + (S_B(S_C - S_A)x - S_A(S_B + S_C)y)\mathbb{Y} &= 0. \end{aligned}$$

They are concurrent if and only if

$$2S_{ABC}(x+y+z) \left( \sum_{\text{cyclic}} S_A(S_{BB} - S_{CC})yz \right) = 0.$$

Since  $P$  is a finite point, this is the case when  $P$  lies on the conic

$$\sum_{\text{cyclic}} S_A(S_{BB} - S_{CC})yz = 0,$$

which is the Jerabek hyperbola, the isogonal conjugate of the Euler line.

Now, if  $P$  is the isogonal conjugate of the point  $(S_{BC} + t : S_{CA} + t : S_{AB} + t)$ , then the perspector is the point

$$(a^2((2S_{ABC} + t(S_A + S_B + S_C)) \cdot S_A - t \cdot S_{BC}) : \dots : \dots).$$

Since  $(a^2S_{BC} : b^2S_{CA} : c^2S_{AB})$  is the point  $X_{25}$  (the homothetic center of the tangential and orthic triangles) on the Euler line, the locus of the perspector is the Euler line.  $\square$

### 3. Orthology

**Proposition 4.** *The triangles  $ABC$  and  $A_P B_P C_P$  are orthologic.*

(a) *The perpendiculars from  $A$  to  $B_P C_P$ ,  $B$  to  $C_P A_P$ , and  $C$  to  $A_P B_P$  intersect at the point*

$$Q = \left( \frac{a^2 S_A}{-\frac{x}{S_{ABC}} + \frac{y}{b^2 S_{BB}} + \frac{z}{c^2 S_{CC}}} : \frac{b^2 S_B}{\frac{x}{a^2 S_{AA}} - \frac{y}{S_{ABC}} + \frac{z}{c^2 S_{CC}}} : \frac{c^2 S_C}{\frac{x}{a^2 S_{AA}} + \frac{y}{b^2 S_{BB}} - \frac{z}{S_{ABC}}} \right).$$

(b) *The perpendiculars from  $A_P$  to  $BC$ ,  $B_P$  to  $CA$ , and  $C_P$  to  $AB$  intersect at the point*

$$Q' = \left( \left( \sum_{\text{cyclic}} \frac{x}{a^2 S_{AA}} \right) \frac{1}{S_A} - \frac{S^2}{a^2 b^2 c^2 S_{ABC}} \cdot \frac{(S^2 - S_{AA})x}{S_{AA}} : \dots : \dots \right).$$

*Remarks.* (1) The orthology center  $Q$  lies on the circumcircle of triangle  $ABC$ . The orthology centers  $Q(P)$  and  $Q(P')$  are the same if and only if the line  $PP'$  contains the circumcenter  $O$ . Here are some examples.

$P$	$G$	$I$	$K$	$N_a$	$X_{110}$	$X_{69}$	$X_{184}$
$Q$	$X_{1294}$	$X_{1295}$	$X_{1297}$	$X_{2734}$	$X_{2693}$	$X_{98}$	$X_{74}$

(2) The coordinates of  $Q'$  with respect to  $A_P B_P C_P$  are the same as those of  $Q$  with respect to  $ABC$ .

(3) Therefore,  $Q'$  lies on the circumcircle of triangle  $A_P B_P C_P$ .

### 4. An example: Gergonne points of the triad

We assume  $ABC$  to be acute angled, so that the triangles  $AA_b A_c$ ,  $B_a B B_c$  and  $C_a C_b C$  have angles  $\pi - 2A$ ,  $\pi - 2B$  and  $\pi - 2C$  and sidelengths in the proportion of  $a^2 S_A : b^2 S_B : c^2 S_C$ . In this case, the Gergonne point  $A_g$  of  $AA_b A_c$  has homogeneous barycentric coordinates  $(S_A : S_B : S_C)$  with respect to the triangle. Therefore, with  $P = (S_A : S_B : S_C)$  (the triangle center  $X_{69}$ , we have  $A_g = A_P = (2S_A : S_B + S_C : S_B + S_C)$ , and likewise  $B_g = (S_C + S_A : 2S_B : S_C + S_A)$  and  $C_g = (S_A + S_B : S_A + S_B : 2S_C)$ . From the table above, the orthology center

$Q$  is  $X_{98} = \left( \frac{1}{S_{AA}-S_{BC}} : \frac{1}{S_{BB}-S_{CA}} : \frac{1}{S_{CC}-S_{AB}} \right)$ , the Tarry point (see Figure 4). The other orthology center  $Q'$ , being the Tarry point of  $A_g B_g C_g$ , is

$$\begin{aligned} & (S_{BB} - S_{CA})(S_{CC} - S_{AB})(2S_A, S_B + S_C, S_B + S_C) \\ & + (S_{CC} - S_{AB})(S_{AA} - S_{BC})(S_C + S_A, 2S_B, S_C + S_A) \\ & + (S_{AA} - S_{AB})(S_{BB} - S_{CA})(S_A + S_B, S_A + S_B, 2S_C) \\ & = \left( \sum_{\text{cyclic}} S_{BC} - S_{AA} \right) ((S_{BC} + S_{CA} + S_{AB})(S_A, S_B, S_C) \\ & \quad + (S_A + S_B + S_C)(S_{BC}, S_{CA}, S_{AB})). \end{aligned}$$

This is the triangle center  $X_{1352}$ , which is the midpoint of  $H$  and  $X_{69}$ .

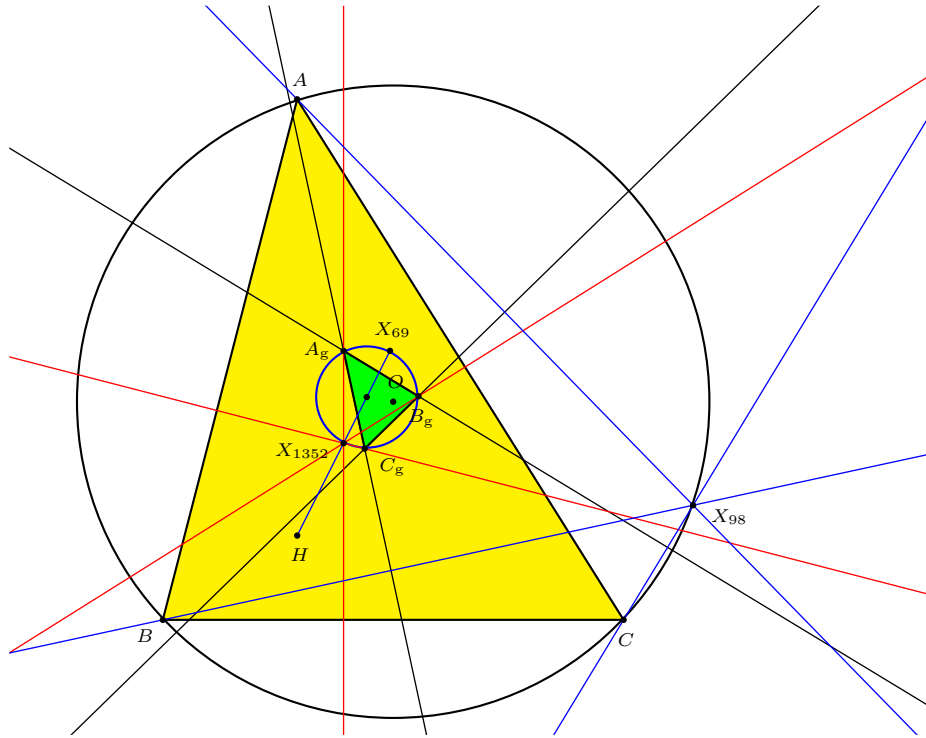


Figure 4.

*Remarks.* (1) The circumcenter of  $A_g B_g C_g$  is the point

$$\begin{aligned} & S_A(S_B + S_C)(2S_A, S_B + S_C, S_B + S_C) \\ & + S_B(S_C + S_A)(S_C + S_A, 2S_B, S_C + S_A) \\ & + S_C(S_A + S_B)(S_A + S_B, S_A + S_B, 2S_C) \\ & = 3(S_{BC} + S_{CA} + S_{AB})(S_A, S_B, S_C) + (S_A + S_B + S_C)(S_{BC}, S_{CA}, S_{AB}). \end{aligned}$$

This is the midpoint of  $X_{69}$  and  $X_{1352}$ .

(2) Since the Tarry point and the Steiner point are antipodal on the circumcircle, we conclude that  $X_{69}$  is the Steiner point of triangle  $A_g B_g C_g$  (see [1]).

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## A New Proof of a “Hard But Important” Sangaku Problem

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**Abstract.** We present a solution to a well-known *sangaku* problem, previously solved using the Sawayama Lemma, that does not require it.

### 1. Introduction

The following problem has been described as “hard but important”.

**Problem 1** ([5, Problem 2.2.8]). *M is the midpoint of chord AB of circle (O). Given triangle ABC with sides a, b, c, and semiperimeter s, let N be the midpoint of the arc AB opposite the incircle I(r), and MN = v. Given circle Q(q) tangent to AC, BC, and (O) (see Figure 1), prove that*

$$q - r = \frac{2v(s - a)(s - b)}{cs}. \quad (1)$$

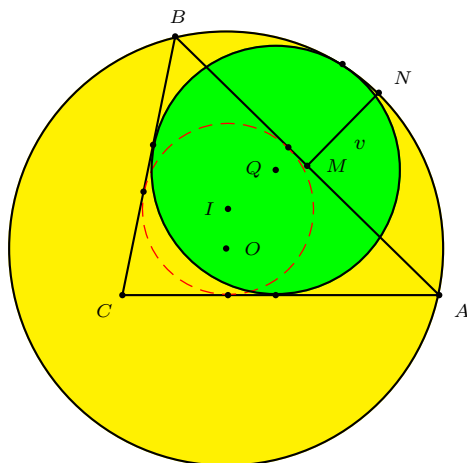


Figure 1.

Problem 1 is certainly hard. One solution [6], summarized in [8, pp. 25–26], entails a series of algebraic manipulations so lengthy that it is hard to believe anyone would have pursued it without knowing in advance that (1) holds. Another solution [3, 9] involves somewhat less algebra, but requires the Sawayama Lemma, the proof of which [1] is itself no simple matter. Neither proof gives much insight into the motivation for (1).

Problem 1 is important in that solutions of other *sangaku* problems depend on (1). For instance, in the special case in which  $AC$  and  $BC$  are the sides of squares  $ACDE$  and  $BCFG$  with  $E$  and  $G$  on  $(O)$ , it can be shown using (1) that  $q = 2r$  [5, Example 3.2], [8, pp. 20–21]. But the solution we offer here follows from that of another *sangaku* problem, which it cannot be used to solve in a straight-forward way.

## 2. The source problem

**Problem 2.** Triangle  $ABC$  has incircle  $I(r)$ , to which  $(O)$ , through  $B$  and  $C$ , is internally tangent. Circle  $P(p)$  is tangent to  $AB$  and  $AC$  and externally tangent to  $(O)$ . The sagitta from  $M$ , the midpoint of  $BC$ , to the arc opposite the incircle has length  $v$ . Prove that  $r^2 = 2pv$ .

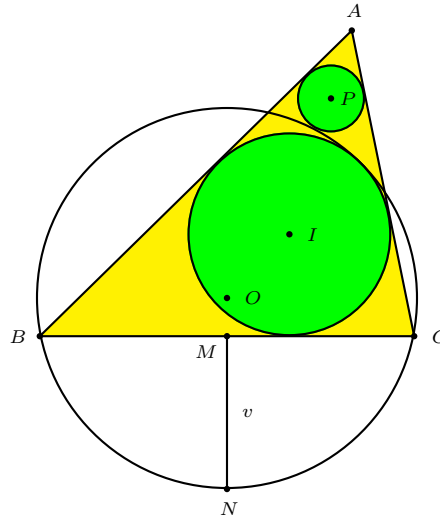


Figure 2.

In the original problem [5, Problem 2.4.2],  $v$  in Figure 2 is replaced by a circle of diameter  $v$ , which gives the figure a pleasing harmony, but is something of a red herring. The solution relies on the following theorem, which we state as it pertains to the foregoing figure.

**Theorem 1.** A circle  $(O)$  passing through  $B$ ,  $C$ , tangent to the circles  $(I)$  (internally) and  $(P)$  (externally), exists if and only if one of the intangents of  $(P)$  and  $(I)$  is parallel to  $BC$ .

*Proof.* Consider  $P$  in general position (see Figure 3). Since the two right triangles formed by  $AI$ , the two intangents, and radii of  $(I)$  are congruent, we see that

- (1)  $DE$ ,  $FG$ , and  $AI$  are concurrent in  $X$ ,
- (2) the triangles  $AEX$  and  $AGX$  are congruent,
- (3) the triangles  $ADE$  and  $AFG$  are congruent.

The equal vertical angles  $EXF$  and  $GXD$  are both equal to  $\angle AGF - \angle ADE = \angle AED - \angle AFG$ . Hence, if  $\angle AFG = \angle ABC$  and  $\angle AGF = \angle ACB$ , then  $FG$



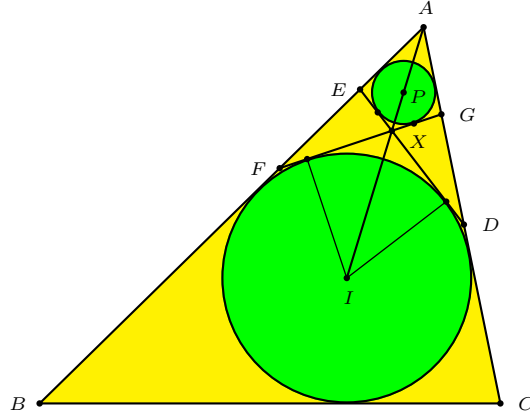


Figure 3.

and  $BC$  are parallel, and  $\angle ACB - \angle ADE = \angle AED - \angle ABC$ . But since  $\angle ADE < \angle ACB$  and  $\angle AED > \angle ABC$ , this can be true only if  $\angle ADE = \angle ABC$  and  $\angle AED = \angle ACB$ . Thus,

(4)  $FG$  (respectively  $DE$ ) is parallel to  $BC$  if and only if  $DE$  (respectively  $FG$ ) is its antiparallel in triangle  $ABC$ .

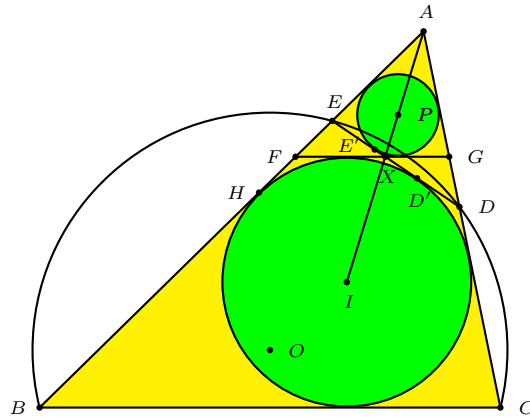


Figure 4.

Now consider the coaxial system  $\Gamma$  of circles with centers on the perpendicular bisector of  $BC$ . If a circle in  $\Gamma$  cuts  $AB$  and  $AC$  in two distinct points, then, together with  $B$  and  $C$ , they form a cyclic quadrilateral. On the other hand, if  $FG \parallel BC$ ,  $BCDE$  is a cyclic quadrilateral because, as just shown,  $\angle ADE = \angle ABC$  and  $\angle AED = \angle ACB$ . Say  $DE$  touches  $(I)$  and  $(P)$  in  $D'$  and  $E'$  respectively. Because chords of all circles in  $\Gamma$  cut off by  $AB$  and  $AC$  are parallel to  $DE$ , a circle in  $\Gamma$  that passes through either  $D'$  or  $E'$  passes through the other. In light of statement (4), this is the circle  $(O)$  in Problem 2 (compare Figure 2 with Figure 4).  $\square$

Let  $H$  be the point on  $AB$  touched by  $(I)$ .  $AH = s - a$ , where  $s$  is the semiperimeter of triangle  $ABC$ . Since  $(I)$  is the excircle of  $AFG$ ,  $AH$  is also the semiperimeter of  $AFG$ . Hence  $\frac{p}{r} = \frac{s-a}{s}$ . Therefore,  $2pv = \frac{2vr(s-a)}{s}$ , and for this to be equal to  $r^2$ , we must have  $2v(s-a) = rs = (\text{area of } ABC)$ . But that is the case if and only if  $2v$  is the radius of the excircle  $(S)$  touching  $BC$  at  $T$  (see Figure 5).

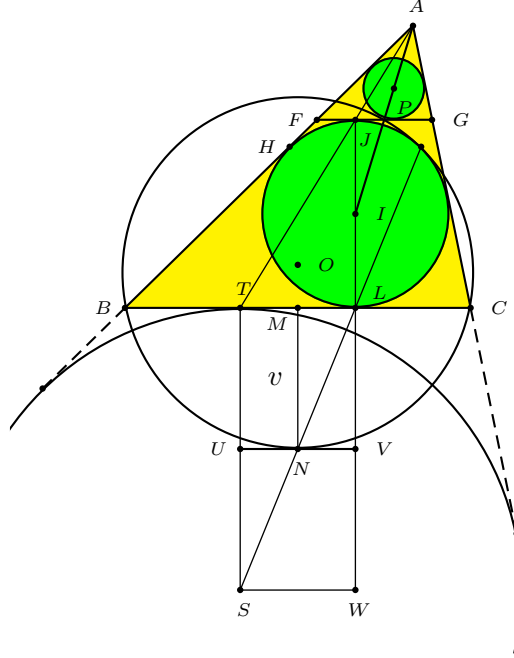


Figure 5.

$AI$  is concurrent in  $S$  with the bisectors of the exterior angles at  $B$  and  $C$ , and  $ST \perp BC$  just as  $JL \perp BC$ . Therefore, drawing parallels to  $BC$  through  $N$  and  $S$ , we obtain rectangles  $LTUV$ ,  $LTSW$ , and  $SUVW$ . By construction,  $MN = v = TU = LV$ . Since  $BT = CL = s - c$ ,  $MT = ML$ . Hence triangles  $LMN$  and  $NUS$  are congruent:  $LNS$  is a straight angle, and  $US = VW = v$ . Therefore  $2v$  is indeed the radius of  $(S)$ .

### 3. Extension to Problem 1

We have just proved  $\frac{2v(s-a)}{s} = r$  in case  $(O)$  is tangent to  $(I)$ . To extrapolate to Problem 1, we add circle  $(Q)$  tangent to  $AB$ ,  $AC$ , and  $(O)$  (see Figure 6).

Say  $(Q)$  touches  $AB$  in  $H'$ , and  $(I)$  touches  $AC$  in  $Y$ . Since triangles  $AIH$  and  $AQH'$  are similar,  $\frac{q-r}{r} = \frac{AH'-AH}{AH} = \frac{HH'}{AH}$ . But  $AH = s - a$ . Hence

$$q - r = \frac{r \cdot HH'}{s - a} = \frac{2v(s-a)}{s} \cdot \frac{HH'}{s-a} = \frac{2v \cdot HH'}{s}. \quad (2)$$

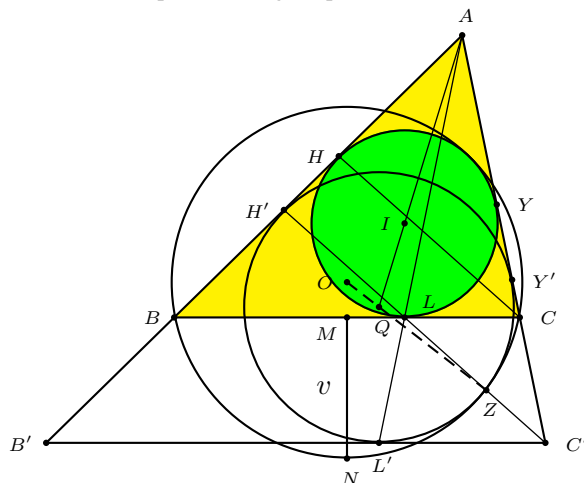


Figure 6.

This gives  $q - r = \frac{2v(s-b)(s-c)}{as}$  if  $\frac{HH'}{s-b} = \frac{s-c}{a}$ . Since  $BH = s - b$  and  $CL = s - c$ , this proportion is equivalent to the similarity of triangles  $BLH'$  and  $BCH$ , i.e.,  $LH' \parallel CH$ . Extend  $AB$  and  $AC$  to intersect the tangent to  $(Q)$  at  $L'$  parallel to  $BC$  on the side opposite  $(I)$ . This makes triangles  $ABC$  and  $AB'C'$  similar and  $(Q)$  the incircle of the latter. Since  $LY$  is a side of the intouch triangle of  $ABC$ ,  $CH \perp LY$ . Likewise  $C'H' \perp L'Y'$ . Since  $L'Y' \parallel LY$ ,  $C'H' \parallel CH$ . Thus  $C'LH'$  is a straight angle and  $LH' \parallel CH$ . Indeed, for this choice of  $O$ ,  $L$  is the Gergonne point of  $AB'C'$ . Now as  $O$  varies on the perpendicular bisector of  $BC$ , points  $B'$ ,  $C'$ ,  $H'$ ,  $L'$ ,  $N$ ,  $Y'$  and  $Z$  vary relative to  $A$ ,  $B$ ,  $C$ ,  $I$ ,  $L$ ,  $M$ ,  $H$ , and  $Y$ , which are unaffected, but the relations  $B'C' \parallel BC$ ,  $H'C' \parallel HC$ , and  $L'Y' \parallel LY$  remain the same. We still have the similarity of triangles  $AIH$  and  $AQH'$ ; hence (2) is valid, and, although the Gergonne point of  $AB'C'$  no longer coincides with  $L$ , we continue to have  $\frac{HH'}{s-b} = \frac{s-c}{a}$  as before. Therefore,  $q - r = \frac{2v(s-b)(s-c)}{as}$ . With the change of notation  $a, b, c \rightarrow c, a, b$ , we obtain (1).

## 4. Historical remarks

The problem presented in Figure 7 is found on p. 26 fasc. 3 of *Seiyō sanpō* (精要算法) [4] (1781) of Fujita Sadasuke (藤田定資), sometimes called Fujita Teishi, who lived from 1734 to 1807. Like most *sangaku* problems, this one is presented as a concrete example: one is told that the lengths of the sagitta, chord, and left and right diagonal lines are  $v = 5$ ,  $c = 30$ ,  $a = 8$ , and  $b = 26$  inches (*sun* 寸), respectively, and asked for the diameter of the inner circle. The answer 9 inches is then given, following which the method of calculation is presented. It uses the labels East, West, South, and North for intermediate results; freely translated into Western notation, if

$$E := a + b + c, \quad W := E(a + b - c), \quad S := 4ab - W, \quad N := \sqrt{WS},$$



then the diameter sought is  $\frac{N+2vS}{E^c}$ . This reduces to equation (1) because  $N = 4rs$ . There is, however, no explicit mention of  $(I)$  or  $r$  in the text. Indeed, the sides  $a$  and  $b$  of the triangle are treated as arbitrary line segments emanating from the endpoints of  $c$ , and the fact that  $N$  is four times the area of the triangle goes unmentioned. Looking at just this excerpt, we get no idea of how Fujita arrived at this method, but it is doubtful that he had the Sawayama Lemma in mind. Another special choice of  $O$  to consider is the circumcenter of  $ABC$ . In that case,  $Q$  is particularly easy to construct, but standard proofs of the construction [2], [10, pp. 56–57] involve trigonometric equations not seen in Edo period Japanese mathematics [7]. One runs into a snag if one assumes (1) and tries to solve Problem 2 by adding circle  $(Q)$ . If, in Figure 2,  $(P)$  is tangent to  $AB$  at  $K$ , this leads in short order to  $HH' \cdot HK = 2pv$ , but proving  $HH' \cdot HK = r^2$  is equivalent to showing that  $H'IK$  is a right triangle. If there is a way to do this other than by solving Problem

2 independently, it must be quite difficult. At any rate, because  $FG$  is not a cevian of  $ABC$ , the Sawayama Lemma does not help when taking this approach.

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# The Poncelet Pencil of Rectangular Hyperbolas

Roger C. Alperin

**Abstract.** We describe a pencil of rectangular hyperbolas depending on a triangle and the relations of the hyperbolas with several triangle centers.

## 1. Isogonal Transforms

In this section we collect some background on the isogonal transform and its connections to our study. The results are somewhat scattered in the literature so we review some of the details necessary for our investigation.

The isogonal transformation determined by a triangle  $ABC$  is a birational quadratic transformation taking a general line to a conic passing through  $A, B, C$ . The fixed points of this transformation are the incenter and excenters of the triangle, see [2]; the vertices of the triangle are singular points. The isogonal transform of the point  $P$  is denoted  $P'$ . The quadratic nature of the transform means that a pencil of lines through  $P$  is transformed to a pencil of conics passing through  $A, B, C$  and  $P'$ .

**Theorem 1.** *The pencil of lines through the circumcenter  $O$  is transformed to a pencil of rectangular hyperbolas passing through the vertices and orthocenter  $H$  of triangle  $ABC$ . The centers of these rectangular hyperbolas lie on the Euler circle.*

*Proof.* The isogonal transform of  $O$  is  $O' = H$ . Thus all the conics in the pencil pass through  $A, B, C, H$ , an orthocentric set, i.e., any one of the points is the orthocenter of the other three. Hence the pencil of conics is a pencil of rectangular hyperbolas.

It is a well-known theorem of Feuerbach [2] that a pencil of rectangular hyperbolas has a circle as its locus of centers.

The pencil has three degenerate rectangular hyperbolas. These are determined by the lines through  $O$  and the vertices; the transform of, say  $OA$  is the reducible conic consisting of the opposite sides  $BC$  together with  $HA$ , the altitude. These reducible conics have centers at the feet of the altitudes (this also makes sense when the triangle is equilateral); thus since the locus of centers is a circle it is the Euler nine-point circle.  $\square$

We refer to this pencil of hyperbolas as the *Poncelet pencil*. The referee has kindly pointed out that Poncelet discovered this pencil in 1822.

The points on the circumcircle of triangle  $ABC$  are sent by the isogonal transform to points on the line at infinity in the projective plane. The isogonal transformation from the circumcircle to the line at infinity halves the angle of arc on the circle, measuring the angle between the lines from  $O$  to the corresponding points at infinity. The orientation of these angles is reversed by the isogonal transformation.

For every line  $\ell$  at  $O$  the isogonal transform is a conic of the Poncelet pencil. The intersections of  $\ell$  with the circumcircle are transformed to the line at infinity; thus antipodal points on the circumcircle are taken to asymptotic directions which are consequently perpendicular. Hence again we see that the conics of the Poncelet pencil are rectangular hyperbolas.

## 2. Parametrizations

Every point of the plane (other than  $A, B, C, H$ ) is on a unique conic of the Poncelet pencil. This gives a curious way of distinguishing the notable ‘centers’ of a triangle. We give four ways of parameterizing this pencil.

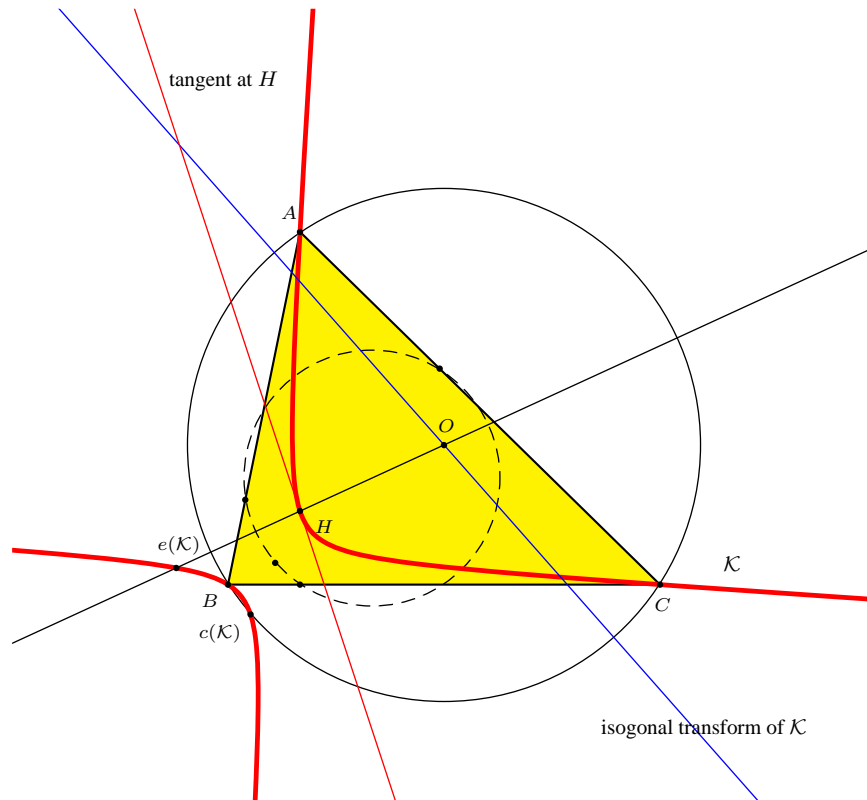


Figure 1. Parametrizations of hyperbolas in the Poncelet pencil

2.1. *Lines at  $O$ .* Our first parametrization (via isogonal transform) uses the pencil of lines passing through  $O$ ; the isogonal transform of this pencil yields the Poncelet pencil of rectangular hyperbolas.

2.2. *Euler Point.* A conic  $\mathcal{K}$  of the Poncelet pencil meets the Euler line at  $H$  and a second time at  $e(K)$ , the *Euler point*. Intersections of a conic in a pencil with a line affords an involution in general (by Desargues Involution Theorem), except when the line passes through a common point of the pencil; in this case the Euler line passes through  $H$  so  $e(K)$  is linear in  $\mathcal{K}$ .

2.3. *Tangents.* Our third parametrization of the Poncelet pencil uses the tangents at  $H$ , a pencil of lines at  $H$ .

2.4. *Circumcircle Point.* The intersection of the circumcircle with any hyperbola of the Poncelet pencil consists of the vertices of triangle  $ABC$  and a fourth intersection  $c(K)$  called the circumcircle point of the hyperbola  $K$ . Our fourth parametrization uses the circumcircle point.

For a point  $S$  on the circumcircle we determine  $S'$  on the line at infinity. Then the isogonal transform of  $OS'$  is the hyperbola in the Poncelet pencil passing through  $S$ .

### 3. Notable Hyperbolas

We next give some notable members in the Poncelet pencil of rectangular hyperbolas.

3.1. *Kiepert's hyperbola  $\mathcal{K}_t$ .* The first Euler point is the centroid  $G$ . Let  $L$  denote the (Lemoine) symmedian point,  $L = G'$ . The isogonal transform of the line  $OL$  is the Kiepert hyperbola. Let  $S$  be the center of Spieker's circle;  $OLS'$  lie on a line [3] so  $S$  lies on this hyperbola.

3.2. *Jerabek's hyperbola  $\mathcal{J}_k$ .* This is the isogonal transform of the Euler line  $OH$ . Hence  $O = H'$  and the symmedian point  $L = G'$  lie on this hyperbola. Thus  $c(\mathcal{J}_k)'$  is the end (at infinity) of the Euler line.

Given a line  $m$  through  $O$  meeting Jerabek's hyperbola at  $X$ , the isogonal conjugate of  $X$  lies on the Euler line and on the isogonal of  $m = OX$ , hence  $X' = e(m')$ .

3.3. *Feuerbach's hyperbola  $\mathcal{F}_h$ .* Transform line  $OI$  where  $I$  denotes the incenter to obtain this hyperbola. Since  $OIM'K'$  lie on a line where  $M$  is the Nagel point and  $K$  is the Gergonne point [3], the Nagel and Gergonne points lie on Feuerbach's hyperbola. Feuerbach's hyperbola is tangent to  $OI$  at  $I$  since  $I$  is a fixed point of the isogonal transformation

The three *excentral Feuerbach* hyperbolas pass through the excenters, and hence are tangent to the lines through  $O$  there, since the excenters are also fixed points of the isogonal transformation (see Figure 2).



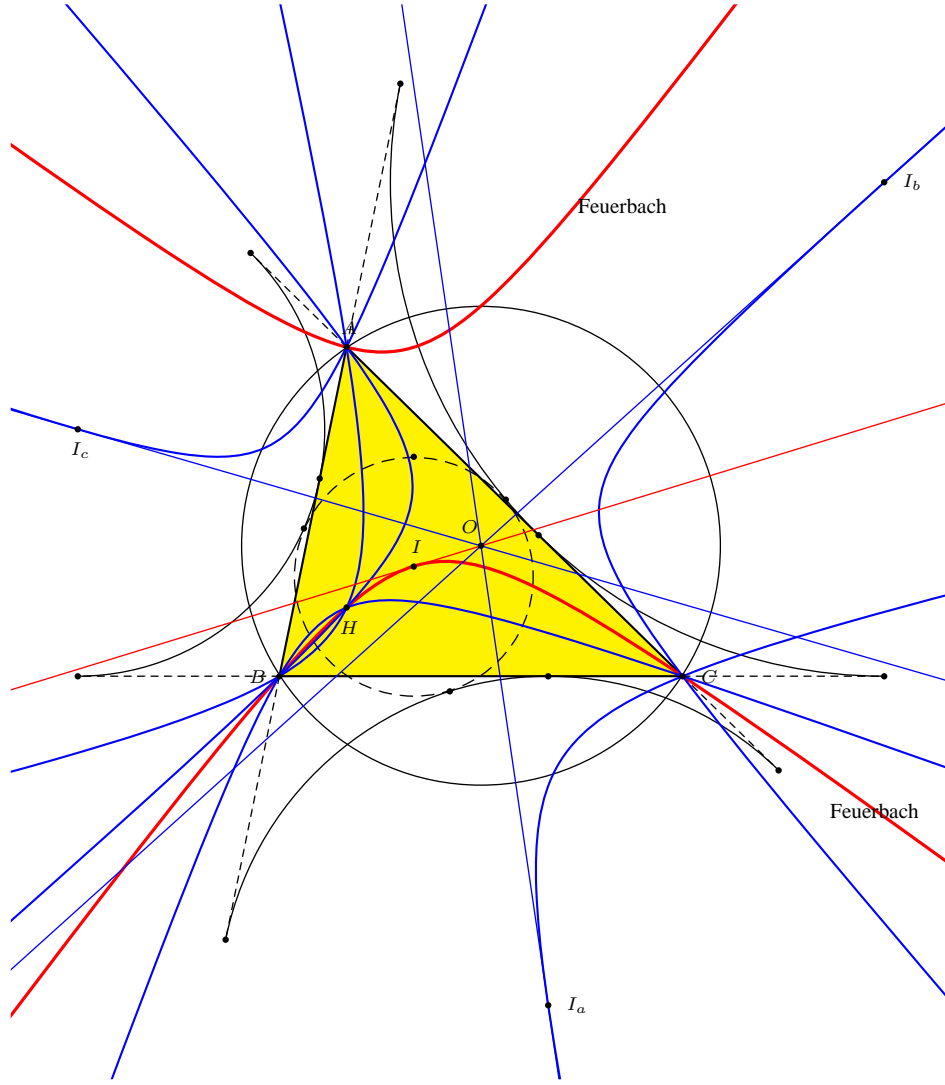


Figure 2. The Feuerbach hyperbolas in the Poncelet pencil

The names for the next two conics are chosen for convenience. Huygens did use a hyperbola however in his solution of Alhazen's problem.

3.4. *Huygens' Hyperbola  $\mathcal{H}_n$* . We define the isogonal transform of the tangent line to Jerabek at  $O$  as Huygens' hyperbola. Hence Huygens' rectangular hyperbola is tangent to the Euler line at  $H$  since  $O' = H$ .

3.5. *Euler's Hyperbola  $\mathcal{E}_r$* . This hyperbola corresponds to the end of the Euler line using the parametrization by its second intersection with the Euler line. It has the property that the isogonal transform is line  $OW$  for  $W = c(\mathcal{J}_k)$  on the circumcircle. The tangent at  $H$  meets circumcircle at  $W$ .

*Remark.* Here are the Euler points for the different hyperbolas mentioned:

$$e(\mathcal{H}_n) = H, \quad e(\mathcal{K}_t) = G, \quad e(\mathcal{J}_k) = O, \quad e(\mathcal{E}_r) = \infty.$$

The Euler point of Feuerbach's hyperbola is the Schiffler point, which is the common point of the Euler lines of the four triangles  $IBC$ ,  $ICA$ ,  $IAB$ , and  $ABC$  (see [4]).

#### 4. Orthic Poncelet pencil

Consider the locus  $\mathcal{K}$  of duals of the Euler line of triangle  $ABC$  in each conic of the Poncelet pencil. We show next that this locus is a rectangular hyperbola in the Poncelet pencil of the orthic triangle.

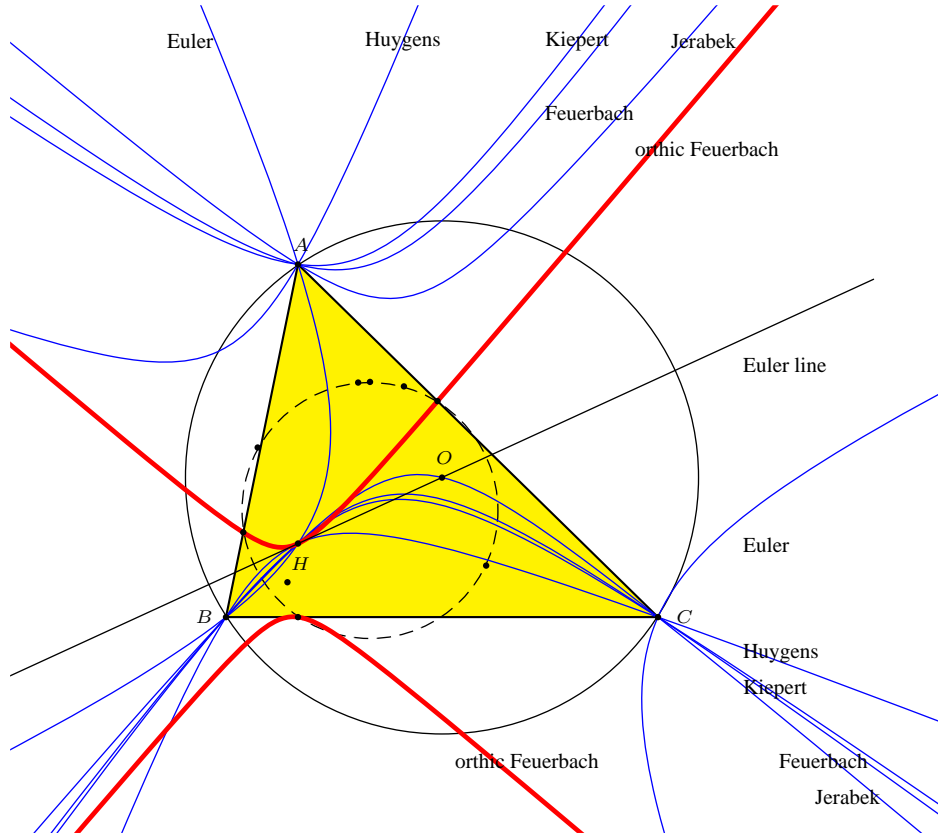


Figure 3. Poncelet pencil with orthic Feuerbach

To see this we recall the close connection between the duality relation and conjugacy: every point  $P$  of a line  $\ell$  is conjugate to the dual  $Q = \ell^*$  of the line, since the dual of  $P$  is a line through  $Q$ . But the conjugacy relation is the same as isogonal conjugacy, [1, §4]; thus the dual of the line  $\ell$  in every conic of the pencil is the same as the isogonal conjugate of the line with respect to the triangle of diagonal points. In our situation here the quadrangle is  $A, B, C, H$  and its diagonal triangle

is the orthic triangle of triangle  $ABC$ . Thus the locus  $K$  is a conic passing through the vertices of the orthic triangle and it is the isogonal conjugate of the Euler line of triangle  $ABC$  with respect to the orthic triangle.

Notice that the isogonal conjugate (with respect to the orthic triangle) of the center of nine-point circle  $N$  is the orthocenter  $h$  of the orthic triangle, since the circumcircle of the orthic triangle is the nine-point circle. Since  $N$  lies on the Euler line then  $h$  lies on  $K$  since  $K$  is the isogonal conjugate of the Euler line. Thus the conic  $K$  is a rectangular hyperbola since it contains an orthocentric set. Since  $H, A, B, C$  are the incenter and excenters of the orthic triangle,  $K$  is tangent to the Euler line at  $H$  (see Figure 3). Thus this conic is a Feuerbach hyperbola of the orthic triangle. If the triangle is acute, then the incenter is  $H$ .

**Theorem 2.** *The locus of duals of the Euler line in the conics in Poncelet pencil is the Feuerbach hyperbola of the orthic triangle.*

As a consequence if we consider the excentral triangle, the original triangle is its orthic triangle. We obtain the following result.

**Corollary 3.** *Feuerbach's hyperbola is the locus of duals of the Euler line of the excentral triangle in the Poncelet pencil of the excentral triangle.*

*Remark.* The Euler line of the excentral triangle is the line  $OI$ .

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## A Maximal Parallelogram Characterization of Ovals Having Circles as Orthoptic Curves

Nicolae Anghel

**Abstract.** It is shown that ovals admitting circles as orthoptic curves are precisely characterized by the property that every one of their points is the vertex of exactly one maximal-perimeter inscribed parallelogram. This generalizes an old property of ellipses, most recently revived by Connes and Zagier in the paper *A Property of Parallelograms Inscribed in Ellipses*.

Let  $\mathcal{C}$  be a centrally symmetric smooth strictly convex closed plane curve (oval with a center). It has been known for a long time, see for instance [1], that if  $\mathcal{C}$  is an ellipse then among all the inscribed parallelograms those of maximal perimeter have the property that any point of  $\mathcal{C}$  is the vertex of exactly one. The proof given in [1] makes it clear that this property is related to the fact that the Monge orthoptic curve, *i.e.*, the locus of all the points from where a given closed curve can be seen at a right angle, of an ellipse is a circle.

The purpose of this note is to show that the maximal-perimeter property of parallelograms inscribed in centrally symmetric ovals described above for ellipses is characteristic precisely to the class of ovals admitting circles as orthoptic curves. For a parallel result, proved by analytic methods, see [2].

**Theorem 1.** *Let  $\mathcal{C}$  be a centrally symmetric oval. Then every point of  $\mathcal{C}$  is the vertex of an unique parallelogram of maximal perimeter among those inscribed in  $\mathcal{C}$  if and only if the orthoptic curve of  $\mathcal{C}$  is a circle.*

The proof of Theorem 1 will be an immediate consequence of the following lemma.

**Lemma 2.** *Let  $ABCD$  be a parallelogram of maximal perimeter among those inscribed in a given centrally symmetric oval  $\mathcal{C}$ . Then the tangent lines to  $\mathcal{C}$  at  $A$ ,  $B$ ,  $C$ , and  $D$ , form with the sides of the parallelogram equal angles, respectively, (for the ‘table’  $\mathcal{C}$ , the parallelogram is a billiard of period 4), and intersect at the vertices of a rectangle  $PQRS$ , concentric to  $\mathcal{C}$  (see Figure 1). Moreover, the perimeter of the parallelogram  $ABCD$  equals four times the radius of the circle circumscribed about the rectangle  $PQRS$ .*

*Proof.* The strict convexity and central symmetry of  $\mathcal{C}$  imply that the center of any parallelogram inscribed in  $\mathcal{C}$  is the same as the center  $O$  of  $\mathcal{C}$ . Therefore, any parallelogram inscribed in  $\mathcal{C}$  is completely determined by two consecutive vertices. The function from  $\mathcal{C} \times \mathcal{C}$  to  $[0, \infty)$  which gives the perimeter of the associated

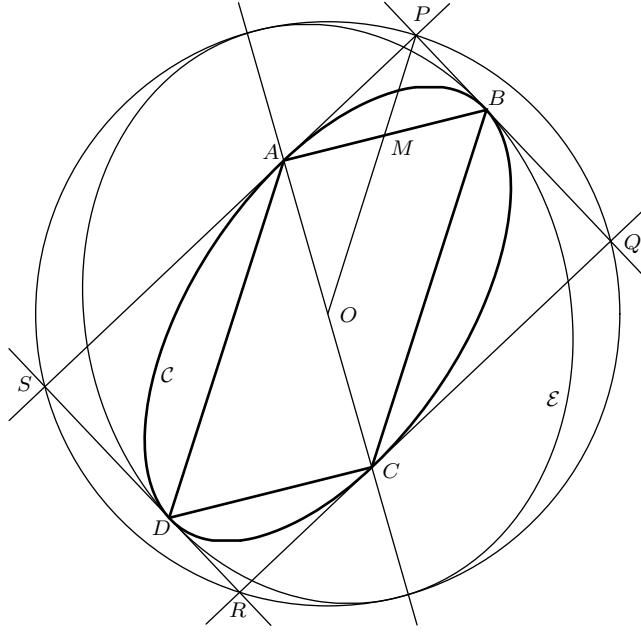


Figure 1. A maximal-perimeter inscribed parallelogram  $ABCD$  and its associated circumscribed rectangle  $PQRS$

parallelogram being continuous, an inscribed parallelogram of maximal perimeter always exists.

Let now  $ABCD$  be such a parallelogram. In the family of ellipses with foci  $A$  and  $C$  there is a unique ellipse  $\mathcal{E}$  such that  $\mathcal{C} \cap \mathcal{E} \neq \emptyset$ , but  $\mathcal{C} \cap \mathcal{E}' = \emptyset$  for any ellipse  $\mathcal{E}'$  in the family whose major axis is greater than the major axis of  $\mathcal{E}$ . In other words,  $\mathcal{E}$  is the largest ellipse with foci  $A$  and  $C$  intersecting  $\mathcal{C}$ . Consequently,  $\mathcal{C}$  and  $\mathcal{E}$  have the same tangent lines at any point in  $\mathcal{C} \cap \mathcal{E}$ . Notice that  $\mathcal{C} \cap \mathcal{E}$  contains at least two points, symmetric with respect to the center  $O$  of  $\mathcal{C}$ . The maximality of  $ABCD$  implies that  $B$  and  $D$  belong to  $\mathcal{C} \cap \mathcal{E}$  and the familiar reflective property of ellipses guarantees the reflective property with respect to the ‘mirror’  $\mathcal{C}$  of the parallelogram  $ABCD$  at the vertices  $B$  and  $D$ . A similar argument applied to the family of ellipses with foci  $B$  and  $D$  yields the reflective property of the parallelogram  $ABCD$  at the other two vertices,  $A$  and  $C$ .

Assume now that the tangent lines to  $\mathcal{C}$  at  $A$ ,  $B$ ,  $C$ , and  $D$ , meet at  $P$ ,  $Q$ ,  $R$ , and  $S$  (see Figure 1). By symmetry,  $PQRS$  is a parallelogram with the same center as  $\mathcal{C}$ . The reflective property of  $ABCD$  and symmetry imply that, say  $\triangle APB$  and  $\triangle CQB$  are similar triangles. Consequently, the parallelogram  $PQRS$  has two adjacent angles congruent, so it must be a rectangle.

Referring to Figure 1, let  $M$  be the midpoint of  $\overline{AB}$ . Similarity inside  $\triangle ABC$  shows that the mid-segment  $\overline{MO}$  is parallel to  $\overline{BC}$  and  $MO = \frac{BC}{2}$ . Now, in the right triangle  $APB$ ,  $PM = BM = \frac{AB}{2}$ , since the segment  $\overline{PM}$  is the median

relative to the hypotenuse  $\overline{AB}$ . It follows that  $\angle MBP$  is congruent to  $\angle MPB$ , as base angles in the isosceles triangle  $MBP$ . Since by the reflective property of the parallelogram  $ABCD$ ,  $\angle MBP$  is congruent to  $\angle CBQ$ , the transversal  $\overleftrightarrow{BM}$  cuts on  $\overleftrightarrow{BC}$  and  $\overleftrightarrow{PM}$  congruent corresponding angles, and so  $\overline{BC}$  is parallel to  $\overline{PM}$ . In conclusion, the points  $P$ ,  $M$ , and  $O$  are collinear and  $PO = \frac{AB + BC}{2}$ , or equivalently the perimeter of  $ABCD = 4PO =$  four times the radius of the circle circumscribed about the rectangle  $PQRS$ . Being inscribed in the rectangle  $PQRS$ , clearly the oval  $\mathcal{C}$  is completely contained inside the circle circumscribed about  $PQRS$ .

As a byproduct of Lemma 2, in a centrally symmetric oval a vertex can be prescribed to at most one parallelogram of maximal perimeter, in which case the other vertices belong to the unique rectangle circumscribed about  $\mathcal{C}$  and sharing a side with the tangent line to  $\mathcal{C}$  at the prescribed point. Such a construction can be performed for any point of  $\mathcal{C}$  but it does not always yield maximal-perimeter parallelograms.  $\square$

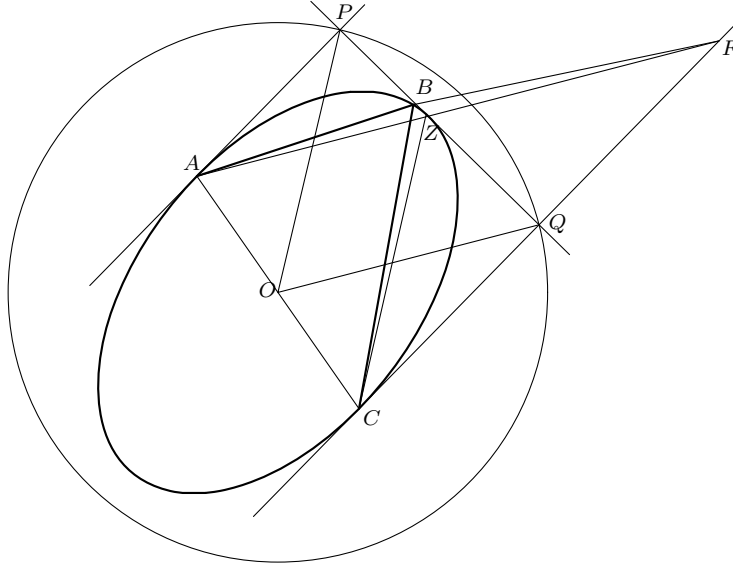


Figure 2. Ovals whose orthoptic curves are circles satisfy the maximal parallelogram property

*Proof of Theorem 1.* Let  $p$  be the common perimeter of all the maximal parallelograms inscribed in the oval  $\mathcal{C}$  of center  $O$ . By Lemma 2, the vertices of the rectangle  $PQRS$  associated to some maximal parallelogram  $ABCD$  inscribed in  $\mathcal{C}$  all belong to the orthoptic curve of  $\mathcal{C}$  and, at the same time, to the circle centered at  $O$  and of radius  $\frac{p}{4}$ . So, when the orthoptic curve of  $\mathcal{C}$  is a circle it must be the circle of center  $O$  and radius  $\frac{p}{4}$ .

Assume now the existence of maximal-perimeter inscribed parallelograms for all the points of  $\mathcal{C}$ . If  $X$  is a point on the orthoptic curve of  $\mathcal{C}$  and  $\overleftrightarrow{XA}$  and  $\overleftrightarrow{XB}$ ,

$A, B \in \mathcal{C}$ , are the two tangent lines from  $X$  to  $\mathcal{C}$ , then  $\overline{AB}$  must be a side of the maximal parallelogram inscribed in  $\mathcal{C}$  and based at  $A$ , by Lemma 2. Again by Lemma 2,  $XO = \frac{p}{4}$ , or  $X$  belongs to the circle centered at  $O$  and of radius  $\frac{p}{4}$ . Conversely, let  $X$  be a point on the circle. Then  $X$  is located outside  $\mathcal{C}$  and if  $\overrightarrow{XA}$  is one of the two tangent lines from  $X$  to  $\mathcal{C}$ ,  $A \in \mathcal{C}$ , and  $ABCD$  is the maximal-perimeter inscribed parallelogram with associated circumscribed rectangle  $PQRS$ ,  $\overrightarrow{PS} = \overrightarrow{XA}$ , then  $P$  belongs to  $\overrightarrow{AX}$ ,  $PO = SO = XO = \frac{p}{4}$ , and so  $X = P$ , since  $S \notin \overrightarrow{AX}$ . Thus,  $X$  belongs to the orthoptic curve of  $\mathcal{C}$ .

Assume now the orthoptic curve of  $\mathcal{C}$  is the circle with center  $O$  and radius  $\frac{p}{4}$ . For a given point  $A \in \mathcal{C}$ , consider the inscribed parallelogram  $ABCD$  such that the tangent lines to  $\mathcal{C}$  at  $A, B, C$ , and  $D$ , intersect at the vertices of circumscribed rectangle  $PQRS$ . By hypothesis,  $PO = QO = \frac{p}{4}$ . We claim that the perimeter of  $ABCD$  equals  $p$ , so  $ABCD$  will be the maximal-perimeter inscribed parallelogram based at  $A$ . On the one hand,  $AB + BC \leq \frac{p}{2}$ . On the other hand,

$$AB + BC \geq \min_{Y \in \overline{PQ}} (AY + YC)$$

This minimum occurs exactly at the point  $Z \in \overline{PQ}$  where  $\overline{AF}$ ,  $F$  the symmetric point of  $C$  with respect to  $\overline{PQ}$ , intersects  $\overline{PQ}$  (see Figure 2). By construction,  $\angle AZP$  is congruent to  $\angle CZQ$ , and then an argument similar to that given in Lemma 2 shows that  $\overline{CZ}$  is parallel to  $\overline{PO}$  and  $AZ + CZ = 2PO = \frac{p}{2}$ . As a result,  $AB + BC \geq \frac{p}{2}$ , which concludes the proof of the Theorem.  $\square$

We end this note with an application to the Theorem. It gives a ‘quarter-oval’ geometric description of the centrally symmetric ovals whose orthoptic curves are circles.

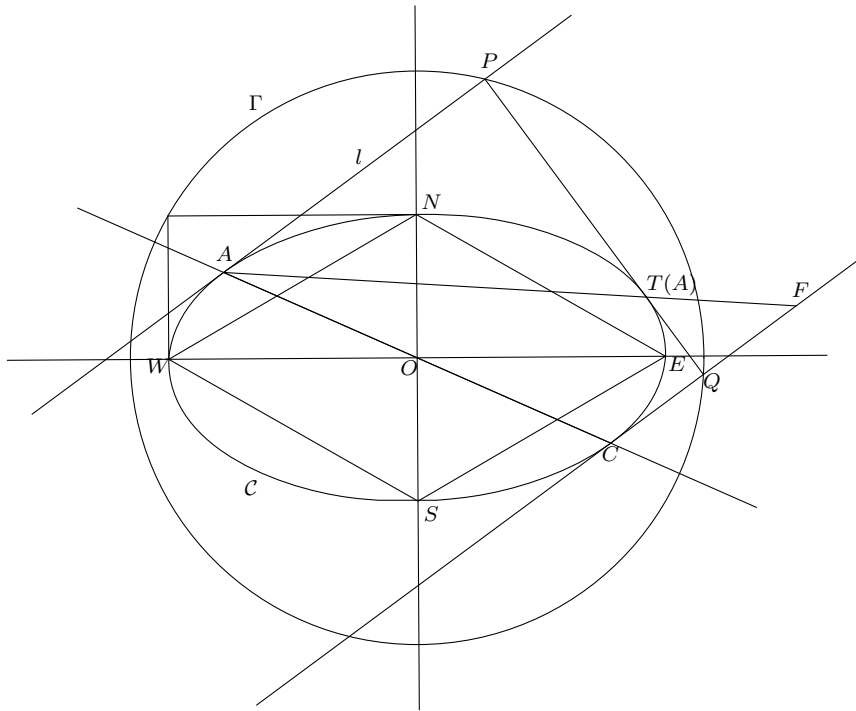
**Corollary 3.** *Let  $\mathcal{C}$  be an oval with center  $O$ , having a circle  $\Gamma$  centered at  $O$  as orthoptic curve. By continuity, there is a point  $W \in \mathcal{C}$  such that the associated maximal-perimeter inscribed parallelogram is a rhombus,  $\widehat{WNE}$  (see Figure 3). Then  $\mathcal{C}$  is completely determined by the quarter-oval  $WN$  according to the following recipe:*

(a) *Consider the coordinate system centered at  $O$ , with axes  $\overrightarrow{WO}$  and  $\overrightarrow{ON}$ , and such that the quarter-oval  $WN$  is situated in the third quadrant and the circle  $\Gamma$  has radius  $\sqrt{OW^2 + ON^2}$ .*

(b) *For an arbitrary point  $A \in \widehat{WN}$ , the tangent line  $l$  to  $\widehat{WN}$  at  $A$  and the parallel line to  $l$  through  $C$ , the symmetric point of  $A$  with respect to  $O$ , intersect the circle  $\Gamma$  in two points,  $P$ , respectively  $Q$ , situated on the same side of the line  $\overrightarrow{AC}$  as  $N$ .*

(c) *The line  $\overrightarrow{AF}$ ,  $F$  being the symmetric point of  $C$  with respect to  $\overline{PQ}$ , intersects  $\overline{PQ}$  in a point  $T(A)$ .*

*Then the transformation  $\widehat{WN} \ni A \mapsto T(A)$  sends bijectively and clockwise increasingly the quarter-oval  $WN$  onto the portion of  $\mathcal{C}$  situated in the first quadrant of the coordinate system, in fact another quarter-oval,  $\widehat{NE}$ . Moreover, symmetry with respect to  $O$  of the half-oval  $WNE$  completes the oval  $\mathcal{C}$ .*



The proof of the Corollary is a simple consequence of all the facts considered in the proof of the Theorem.

(a) The quarter-oval must globally satisfy a certain curvature growth-condition, which amounts to the fact that the transformation ‘abscissa of  $A \mapsto$  abscissa of  $T(A)$ ’ must be strictly increasing.

These matters are better addressed by analytic methods and in keeping with the strictly geometric character of this note they will not be considered here.

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# On the Inradius of a Tangential Quadrilateral

Martin Josefsson

**Abstract.** We give a survey of known formulas for the inradius  $r$  of a tangential quadrilateral, and derive the possibly new formula

$$r = 2\sqrt{\frac{(M - uvx)(M - vxy)(M - xyu)(M - yuv)}{uvxy(uv + xy)(ux + vy)(uy + vx)}}$$

where  $u, v, x$  and  $y$  are the distances from the incenter to the vertices, and  $M = \frac{1}{2}(uvx + vxy + xyu + yuv)$ .

## 1. Introduction

A *tangential quadrilateral*<sup>1</sup> is a convex quadrilateral with an incircle, that is, a circle which is tangent to all four sides. Not all quadrilaterals are tangential. The most useful characterization is that its two pairs of opposite sides have equal sums,  $a + c = b + d$ , where  $a, b, c$  and  $d$  are the sides in that order [1, pp. 64–67].<sup>2</sup>

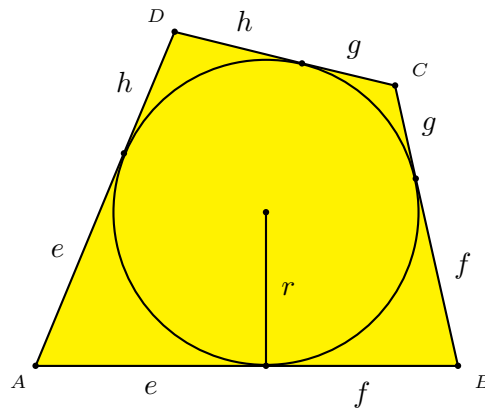


Figure 1. Klamkin's problem

It is well known that the inradius  $r$  of the incircle is given by

$$r = \frac{K}{s}$$

where  $K$  is the area of the quadrilateral and  $s$  is the semiperimeter<sup>3</sup>. The area of a tangential quadrilateral  $ABCD$  with sides  $a, b, c$  and  $d$  is according to P. Yiu [10]

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<sup>1</sup>Other names for these quadrilaterals are circumscribable quadrilateral [10], circumscribable quadrilateral [9] and circumscribed quadrilateral [4].

<sup>2</sup>There exists a lot of other interesting characterizations, see [9] and [7].

<sup>3</sup>This formula holds for all polygons with an incircle, where  $K$  is the area of the polygon.

given by<sup>4</sup>

$$K = \sqrt{abcd} \sin \frac{A+C}{2}.$$

From the formulas for the radius and area we conclude that the inradius of a tangential quadrilateral is not determined by the sides alone; there must be at least one angle given, then the opposite angle can be calculated by trigonometry.

Another formula for the inradius is

$$r = \sqrt{\frac{efg + fgh + ghe + hef}{e + f + g + h}}$$

where  $e$ ,  $f$ ,  $g$  and  $h$  are the distances from the four vertices to the points where the incircle is tangent to the sides (see Figure 1). This is interesting, since here the radius is only a function of four distances and no angles! The problem of deriving this formula was a quickie by M. S. Klamkin, with a solution given in [5].

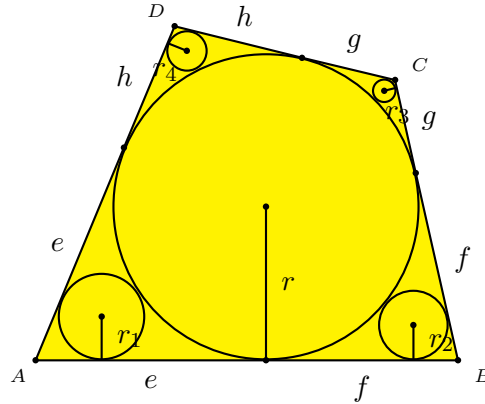


Figure 2. Minkus' 5 circles

If there are four circles with radii  $r_1$ ,  $r_2$ ,  $r_3$  and  $r_4$  inscribed in a tangential quadrilateral in such a way, that each of them is tangent to two of the sides and the incircle (see Figure 2), then the radius  $r$  of the incircle is a root of the quadratic equation

$$r^2 - (\sqrt{r_1 r_2} + \sqrt{r_1 r_3} + \sqrt{r_1 r_4} + \sqrt{r_2 r_3} + \sqrt{r_2 r_4} + \sqrt{r_3 r_4}) r + \sqrt{r_1 r_2 r_3 r_4} = 0$$

according to J. Minkus in [8, editorial comment].<sup>5</sup>

In [2, p.83] there are other formulas for the inradius, whose derivation was only a part of the solution of a contest problem from China. If the incircle in a tangential quadrilateral  $ABCD$  is tangent to the sides at points  $W$ ,  $X$ ,  $Y$  and  $Z$ , and if  $E$ ,

<sup>4</sup>A long synthetic proof can be found in [4]. Another way of deriving the formula is to use the formula  $K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \frac{A+C}{2}}$  for the area of a general quadrilateral, derived in [10, pp.146–147], and the characterization  $a + c = b + d$ .

<sup>5</sup>The corresponding problem for the triangle is an old Sangaku problem, solved in [8], [3, pp. 30, 107–108.].

$F$ ,  $G$  and  $H$  are the midpoints of  $ZW$ ,  $WX$ ,  $XY$  and  $YZ$  respectively, then the inradius is given by the formulas

$$r = \sqrt{AI \cdot IE} = \sqrt{BI \cdot IF} = \sqrt{CI \cdot IG} = \sqrt{DI \cdot IH}$$

where  $I$  is the incenter (see Figure 3). The derivation is easy. Triangles  $IWA$  and  $IEW$  are similar, so  $\frac{r}{AI} = \frac{IE}{r}$  which gives the first formula and the others follow by symmetry.

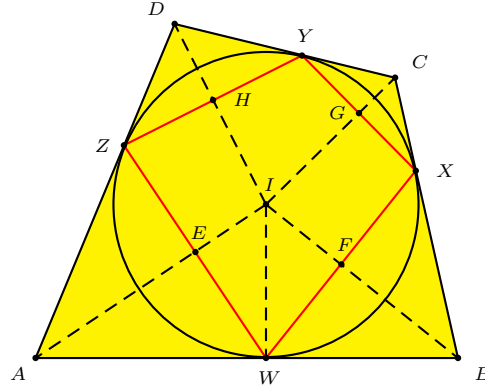


Figure 3. The problem from China

The main purpose of this paper is to derive yet another (perhaps new) formula for the inradius of a tangential quadrilateral. This formula is also a function of only four distances, which are from the incenter  $I$  to the four vertices  $A$ ,  $B$ ,  $C$  and  $D$  (see Figure 4).

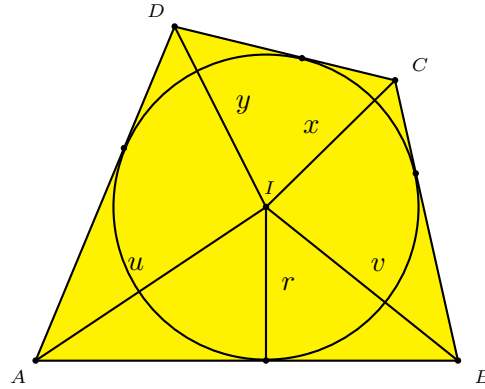


Figure 4. The main problem

**Theorem 1.** *If  $u$ ,  $v$ ,  $x$  and  $y$  are the distances from the incenter to the vertices of a tangential quadrilateral, then the inradius is given by the formula*

$$r = 2\sqrt{\frac{(M - uvx)(M - vxy)(M - xyu)(M - yuv)}{uvxy(uv + xy)(ux + vy)(uy + vx)}} \quad (1)$$

where

$$M = \frac{uvx + vxy + xyu + yuv}{2}.$$

*Remark.* It is noteworthy that formula (1) is somewhat similar to Parameshvaras' formula for the circumradius  $R$  of a *cyclic* quadrilateral,<sup>6</sup>

$$R = \frac{1}{4} \sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}}$$

where  $s$  is the semiperimeter, which is derived in [6, p.17].

## 2. Preliminary results about triangles

The proof of formula (1) uses two equations that holds for all triangles. These are two cubic equations, and one of them is a sort of correspondence to formula (1). The fact is, that while it is possible to give  $r$  as a function of the distances  $AI$ ,  $BI$ ,  $CI$  and  $DI$  in a tangential quadrilateral, the same problem of giving  $r$  as a function of  $AI$ ,  $BI$  and  $CI$  in a triangle  $ABC$  is not so easy to solve, since it gives a cubic equation. The second cubic equation is found when solving the problem, in a triangle, of finding an exradius as a function of the distances from the corresponding excenter to the vertices.

**Lemma 2.** *If  $x$ ,  $y$  and  $z$  are the distances from the incenter to the vertices of a triangle, then the inradius  $r$  is a root of the cubic equation*

$$2xyzr^3 + (x^2y^2 + y^2z^2 + z^2x^2)r^2 - x^2y^2z^2 = 0. \quad (2)$$

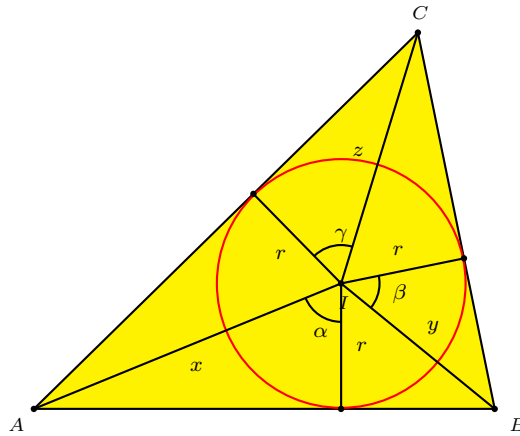


Figure 5. The incircle

<sup>6</sup>A quadrilateral with a circumcircle.

*Proof.* If  $\alpha$ ,  $\beta$  and  $\gamma$  are the angles between these distances and the inradius (see Figure 5), we have  $\alpha + \beta + \gamma = \pi$ , so  $\cos(\alpha + \beta) = \cos(\pi - \gamma)$  and it follows that  $\cos \alpha \cos \beta - \sin \alpha \sin \beta = -\cos \gamma$ . Using the formulas  $\cos \alpha = \frac{r}{x}$ ,  $\cos \beta = \frac{r}{y}$  and  $\sin^2 \alpha + \cos^2 \alpha = 1$ , we get

$$\frac{r}{x} \cdot \frac{r}{y} - \sqrt{1 - \frac{r^2}{x^2}} \sqrt{1 - \frac{r^2}{y^2}} = -\frac{r}{z}$$

or

$$\frac{r^2}{xy} + \frac{r}{z} = \frac{\sqrt{(x^2 - r^2)(y^2 - r^2)}}{xy}.$$

Multiplying both sides with  $xyz$ , reducing common factors and squaring, we get

$$(zr^2 + xyr)^2 = z^2(x^2 - r^2)(y^2 - r^2)$$

which after expansion and simplification reduces to (2).  $\square$

**Lemma 3.** *If  $u$ ,  $v$  and  $z$  are the distances from an excenter to the vertices of a triangle, then the corresponding exradius  $r_c$  is a root of the cubic equation*

$$2uvzr_c^3 - (u^2v^2 + v^2z^2 + z^2u^2)r_c^2 + u^2v^2z^2 = 0. \quad (3)$$

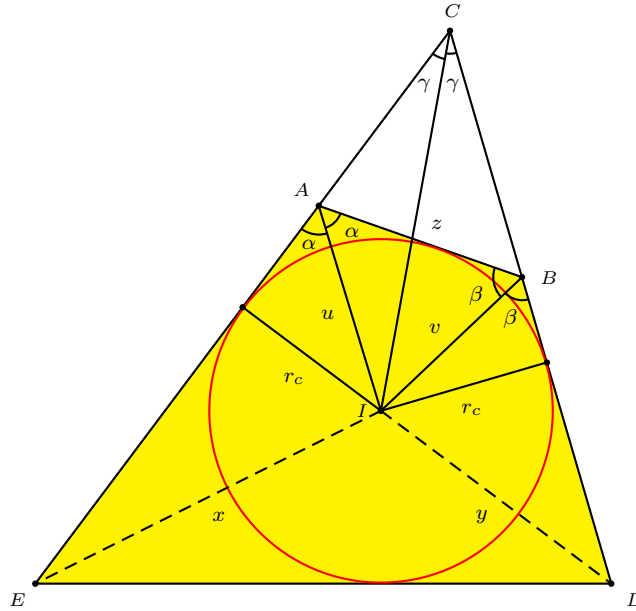


Figure 6. Excircle to triangle  $ABC$  and incircle to  $ABDE$

*Proof.* Define angles  $\alpha$ ,  $\beta$  and  $\gamma$  to be between  $u$ ,  $v$ ,  $z$  and the sides of the triangle  $ABC$  or their extensions (see Figure 6). Then  $2\alpha + A = \pi$ ,  $2\beta + B = \pi$  and  $2\gamma = C$ . From the sum of angles in a triangle,  $A + B + C = \pi$ , this simplifies to  $\alpha + \beta = \frac{\pi}{2} + \gamma$ . Hence  $\cos(\alpha + \beta) = \cos(\frac{\pi}{2} + \gamma)$  and it follows that  $\cos \alpha \cos \beta -$

$\sin \alpha \sin \beta = -\sin \gamma$ . For the exradius  $r_c$ , we have  $\sin \alpha = \frac{r_c}{u}$ ,  $\sin \beta = \frac{r_c}{v}$ ,  $\sin \gamma = \frac{r_c}{z}$ , and so

$$\sqrt{1 - \frac{r_c^2}{u^2}} \sqrt{1 - \frac{r_c^2}{v^2}} - \frac{r_c}{u} \cdot \frac{r_c}{v} = -\frac{r_c}{z}.$$

This can, in the same way as in the proof of Lemma 2, be rewritten as

$$z^2(u^2 - r_c^2)(v^2 - r_c^2) = (zr_c^2 - uvr_c)^2$$

which after expansion and simplification reduces to (3).  $\square$

### 3. Proof of the theorem

Given a tangential quadrilateral  $ABDE$  where the distances from the incenter to the vertices are  $u, v, x$  and  $y$ , we see that if we extend the two sides  $DB$  and  $EA$  to meet at  $C$ , then the incircle in  $ABDE$  is both an incircle in triangle  $CDE$  and an excircle to triangle  $ABC$  (see Figure 6). The incircle and the excircle therefore have the same radius  $r$ , and from (2) and (3) we get that

$$2xyzr^3 + (x^2y^2 + y^2z^2 + z^2x^2)r^2 - x^2y^2z^2 = 0, \quad (4)$$

$$2uvzr^3 - (u^2v^2 + v^2z^2 + z^2u^2)r^2 + u^2v^2z^2 = 0. \quad (5)$$

We shall use these two equations to eliminate the common variable  $z$ . To do so, equation (4) is multiplied by  $uv$  and equation (5) by  $xy$ , giving

$$2uvxyzr^3 + uv(x^2y^2 + y^2z^2 + z^2x^2)r^2 - uvx^2y^2z^2 = 0,$$

$$2uvxyzr^3 - xy(u^2v^2 + v^2z^2 + z^2u^2)r^2 + xyu^2v^2z^2 = 0.$$

Subtracting the second of these from the first gives

$$(uv(x^2y^2 + y^2z^2 + z^2x^2) + xy(u^2v^2 + v^2z^2 + z^2u^2))r^2 - uvx^2y^2z^2 - xyu^2v^2z^2 = 0$$

from which it follows

$$uvxyr^2(xy + uv) + z^2((uvy^2 + uvx^2 + xyv^2 + xyu^2)r^2 - uvxy(xy + uv)) = 0.$$

Solving for  $z^2$ ,

$$z^2 = \frac{uvxy(uv + xy)r^2}{uvxy(uv + xy) - r^2(uvy^2 + uvx^2 + xyv^2 + xyu^2)}. \quad (6)$$

Now we multiply (4) by  $u^2v^2$  and (5) by  $x^2y^2$ , which gives

$$2u^2v^2xyzr^3 + u^2v^2(x^2y^2 + y^2z^2 + z^2x^2)r^2 - u^2v^2x^2y^2z^2 = 0,$$

$$2x^2y^2uvzr^3 - x^2y^2(u^2v^2 + v^2z^2 + z^2u^2)r^2 + u^2v^2x^2y^2z^2 = 0.$$

Adding these we get

$$2uvxyz(uv + xy)r^3 + (u^2v^2y^2 + u^2v^2x^2 - x^2y^2v^2 - x^2y^2u^2)z^2r^2 = 0$$

and since  $zr^2 \neq 0$  this reduces to

$$2uvxy(uv + xy)r + (u^2v^2y^2 + u^2v^2x^2 - x^2y^2v^2 - x^2y^2u^2)z = 0.$$

Solving for  $z$ , we get

$$z = -\frac{2uvxy(uv + xy)r}{u^2v^2y^2 + u^2v^2x^2 - x^2y^2v^2 - x^2y^2u^2}.$$

Squaring and substituting  $z^2$  from (6), we get the equality

$$\begin{aligned} & \frac{uvxy(uv + xy)r^2}{uvxy(uv + xy) - r^2(uvy^2 + uvx^2 + xyv^2 + xyu^2)} \\ &= \frac{4(uvxy(uv + xy))^2 r^2}{(u^2v^2y^2 + u^2v^2x^2 - x^2y^2v^2 - x^2y^2u^2)^2}, \end{aligned}$$

which, since  $uvxy(uv + xy)r^2 \neq 0$ , rewrites as

$$\begin{aligned} & 4uvxy(uv + xy)(ux(vx + uy) + vy(uy + vx))r^2 \\ &= (2uvxy(uv + xy))^2 - (u^2v^2y^2 + u^2v^2x^2 - x^2y^2v^2 - x^2y^2u^2)^2. \end{aligned}$$

What is left is to factor this equation. Using the basic algebraic identities  $a^2 - b^2 = (a + b)(a - b)$ ,  $a^2 + 2ab + b^2 = (a + b)^2$  and  $a^2 - 2ab + b^2 = (a - b)^2$  we get

$$\begin{aligned} & 4uvxy(uv + xy)(uy + vx)(ux + vy)r^2 \\ &= (2uvxy(uv + xy) + (uvy)^2 + (uvx)^2 - (xyv)^2 - (xyu)^2) \\ & \quad \cdot (2uvxy(uv + xy) - (uvy)^2 - (uvx)^2 + (xyv)^2 + (xyu)^2) \\ &= ((uvy + uvx)^2 - (xyv - xyu)^2)((xyv + xyu)^2 - (uvy - uvx)^2) \\ &= ((uvy + uvx + xyv - xyu)(uvy + uvx - xyv + xyu)) \\ & \quad \cdot ((xyv + xyu + uvy - uvx)(xyv + xyu - uvy + uvx)). \end{aligned} \tag{7}$$

Now using  $M = \frac{1}{2}(uvx + vxy + xyu + yuv)$  we get

$$uvy + uvx + xyv - xyu = (uvx + vxy + xyu + yuv) - 2xyu = 2(M - xyu)$$

and in the same way

$$\begin{aligned} uvy + uvx - xyv + xyu &= 2(M - vxy), \\ xyv + xyu + uvy - uvx &= 2(M - uvx), \\ xyv + xyu - uvy + uvx &= 2(M - uvy). \end{aligned}$$

Thus, (7) is equivalent to

$$\begin{aligned} & 4uvxy(uv + xy)(uy + vx)(ux + vy)r^2 \\ &= 2(M - uxy) \cdot 2(M - vxy) \cdot 2(M - uvx) \cdot 2(M - uvy). \end{aligned}$$

Hence

$$r^2 = \frac{4(M - uxy)(M - vxy)(M - uvx)(M - uvy)}{uvxy(uv + xy)(uy + vx)(ux + vy)}.$$

Extracting the square root of both sides finishes the derivation of formula (1).

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# Some Triangle Centers Associated with the Circles Tangent to the Excircles

Boris Odehnal

**Abstract.** We study those tritangent circles of the excircles of a triangle which enclose exactly one excircle and touch the two others from the outside. It turns out that these three circles share exactly the Spieker point. Moreover we show that these circles give rise to some triangles which are in perspective with the base triangle. The respective perspectors turn out to be new polynomial triangle centers.

## 1. Introduction

Let  $\mathbf{T} := ABC$  be a triangle in the Euclidean plane, and  $\Gamma_a, \Gamma_b, \Gamma_c$  its excircles, lying opposite to  $A, B, C$  respectively, with centers  $I_a, I_b, I_c$  and radii  $r_a, r_b, r_c$ . There are eight circles tangent to all three excircles: the side lines of  $\mathbf{T}$  (considered as circles with infinite radius), the Feuerbach circle (see [2, 4]), the so-called Apollonius circle (enclosing all the three excircles (see for example [3, 6, 9])), and three remaining circles which will in the following be denoted by  $\mathcal{K}_a, \mathcal{K}_b, \mathcal{K}_c$ . The circle  $\mathcal{K}_a$  is tangent to  $\Gamma_a$  and externally to  $\Gamma_b$  and  $\Gamma_c$ ; similarly for  $\mathcal{K}_b$  and  $\mathcal{K}_c$ . The radii of these circles are computed in [1]. These circles have the Spieker center  $X_{10}$  as a common point. In this note we study these circles in more details, and show that the triangle of contact points  $K_{a,a}K_{b,b}K_{c,c}$  is perspective with  $\mathbf{T}$ . Surprisingly, the triangle  $M_aM_bM_c$  of the centers these circles is also perspective with  $\mathbf{T}$ .

## 2. Main results

The problem of constructing the circles tangent to three given circles is well studied. Applying the ideas of J. D. Gergonne [5] to the three excircles we see that the construction of the circles  $\mathcal{K}_a$  etc can be accomplished simply by a ruler. Let  $K_{a,b}$  be the contact point of circle  $\Gamma_a$  with  $\mathcal{K}_b$ , and analogously define the remaining eight contact points. The contact points  $K_{a,a}, K_{b,a}, K_{c,a}$  are the intersections of the excircles  $\Gamma_a, \Gamma_b, \Gamma_c$  with the lines joining their contact points with the sideline  $BC$  to the radical center radical center of the three excircles, namely, the Spieker point

$$X_{10} = (b + c : c + a : a + b)$$

Let  $s := \frac{1}{2}(a + b + c)$  be the semiperimeter. The contact points of the excircles with the sidelines are the points

A conic is be represented by an equation in the form  $x^T M x = 0$ , where  $x^T = (x_0 \ x_1 \ x_2)$  is the vector collecting the homogeneous barycentric coordinates of a

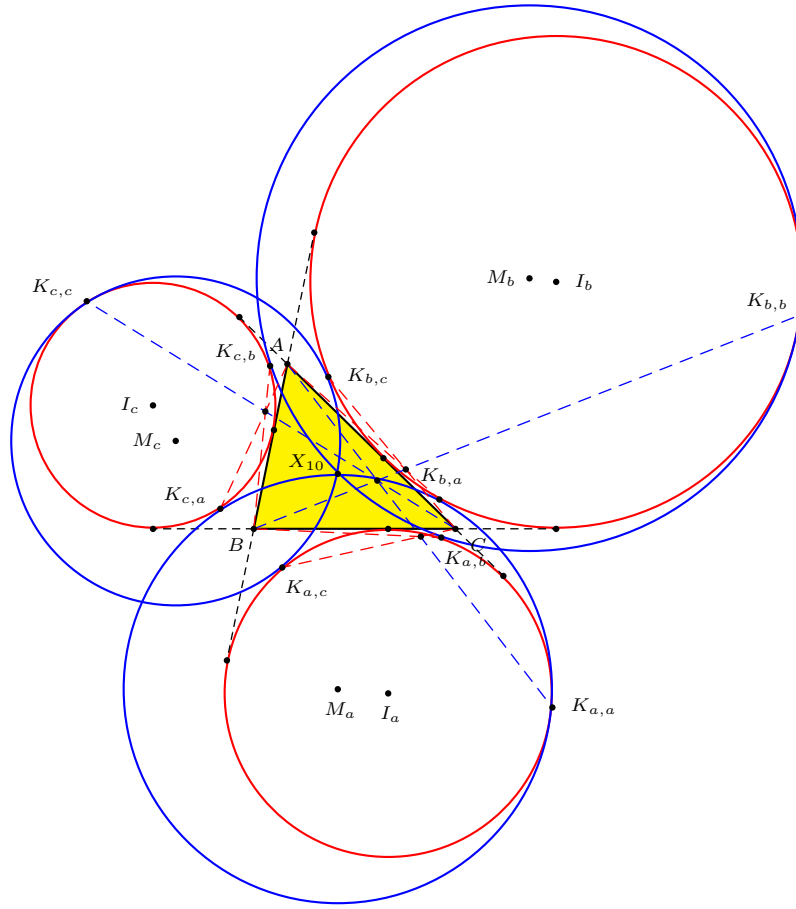


Figure 2.

point  $X$ , and  $M$  is a symmetric  $3 \times 3$ -matrix. For the excircles, these matrices are

$$\begin{aligned}
 M_a &= \begin{pmatrix} s^2 & s(s-c) & s(s-b) \\ s(s-c) & (s-c)^2 & -(s-b)(s-c) \\ s(s-b) & -(s-b)(s-c) & (s-b)^2 \end{pmatrix}, \\
 M_b &= \begin{pmatrix} (s-c)^2 & s(s-c) & -(s-a)(s-c) \\ s(s-c) & s^2 & s(s-a) \\ -(s-a)(s-c) & s(s-a) & (s-a)^2 \end{pmatrix}, \\
 M_c &= \begin{pmatrix} (s-b)^2 & -(s-a)(s-b) & s(s-b) \\ -(s-a)(s-b) & (s-a)^2 & s(s-a) \\ s(s-b) & s(s-a) & s^2 \end{pmatrix}.
 \end{aligned}$$

It is elementary to verify that the homogeneous barycentrics of the contact points are given by:

$$\begin{aligned}
K_{a,a} &= (-(b+c)^2(s-b)(s-c) : c^2s(s-b) : b^2s(s-c)), \\
K_{a,b} &= (-c^2s(s-b) : (c+a)^2(s-b)(s-c) : (as+bc)^2), \\
K_{a,c} &= (-b^2s(s-c) : (as+bc)^2 : (a+b)^2(s-b)(s-c)); \\
K_{b,a} &= ((b+c)^2(s-a)(s-c) : -c^2s(s-a) : (bs+ac)^2), \\
K_{b,b} &= (c^2s(s-a) : -(c+a)^2(s-a)(s-c) : a^2s(s-c)), \\
K_{b,c} &= ((bs+ac)^2 : -a^2s(s-c) : (a+b)^2(s-a)(s-c)); \\
K_{c,a} &= ((b+c)^2(s-a)(s-b) : (cs+ab)^2 : -b^2s(s-a)), \\
K_{c,b} &= ((cs+ab)^2 : (c+a)^2(s-a)(s-c) : -a^2s(s-b)), \\
K_{c,c} &= (b^2s(s-a) : a^2s(s-b) : -(a+b)^2(s-a)(s-b)).
\end{aligned} \tag{1}$$

**Theorem 1.**

The triangle  $K_{a,a}K_{b,b}K_{c,c}$  of contact points is perspective with  $\mathbf{T}$  at a point with homogeneous barycentric coordinates

$$\left( \frac{s-a}{a^2} : \frac{s-b}{b^2} : \frac{s-c}{c^2} \right). \tag{2}$$

*Proof.* The coordinates of  $K_{a,a}$ ,  $K_{b,b}$ ,  $K_{c,c}$  can be rewritten as

$$\begin{aligned}
K_{a,a} &= \left( -\frac{(b+c)^2(s-b)(s-c)}{b^2c^2s} : \frac{s-b}{b^2} : \frac{s-c}{c^2} \right), \\
K_{b,b} &= \left( \frac{s-a}{a^2} : -\frac{(c+a)^2(s-a)(s-c)}{c^2a^2s} : \frac{s-c}{c^2} \right), \\
K_{c,c} &= \left( \frac{s-a}{a^2} : \frac{s-b}{b^2} : -\frac{(a+b)^2(s-a)(s-b)}{a^2b^2s} \right).
\end{aligned} \tag{3}$$

From these, it is clear that the lines  $AK_{a,a}$ ,  $BK_{b,b}$ ,  $CK_{c,c}$  meet in the point given in (2).  $\square$

*Remark.* The triangle center  $P_K$  is not listed in [7].

**Theorem 2.**

The lines  $AK_{a,a}$ ,  $BK_{a,b}$ , and  $CK_{a,c}$  are concurrent.

*Proof.* The coordinates of the points  $K_{a,a}$ ,  $K_{a,b}$ ,  $K_{a,c}$  can be rewritten in the form

$$\begin{aligned}
K_{a,a} &= \left( -\frac{(b+c)^2(s-b)(s-c)}{b^2c^2s} : \frac{s-b}{b^2} : \frac{s-c}{c^2} \right), \\
K_{a,b} &= \left( -\frac{s(s-b)(s-c)}{(as+bc)^2} : \frac{(c+a)^2(s-b)(s-c)^2}{c^2(as+b^2)^2} : \frac{s-c}{c^2} \right), \\
K_{a,c} &= \left( -\frac{s(s-b)(s-c)}{(as+bc)^2} : \frac{s-b}{b^2} : \frac{(a+b)^2(s-b)^2(s-c)}{b^2(as+bc)^2} \right).
\end{aligned} \tag{4}$$

From these, the lines  $AK_{a,a}$ ,  $BK_{a,b}$ , and  $CK_{a,c}$  intersect at the point

$$\left( -\frac{s(s-b)(s-c)}{(as+bc)^2} : \frac{s-b}{b^2} : \frac{s-c}{c^2} \right).$$

$\square$

Let  $M_i$  be the center of the circle  $\mathcal{K}_i$ .

**Theorem 3.**

The triangle  $M_1M_2M_3$  is perspective with  $\mathbf{T}$  at the point

$$\left( \frac{1}{-a^5 - a^4(b+c) + a^3(b-c)^2 + a^2(b+c)(b^2+c^2) + 2abc(b^2+bc+c^2) + 2(b+c)b^2c^2} \right. \\ \left. : \dots : \dots \right).$$

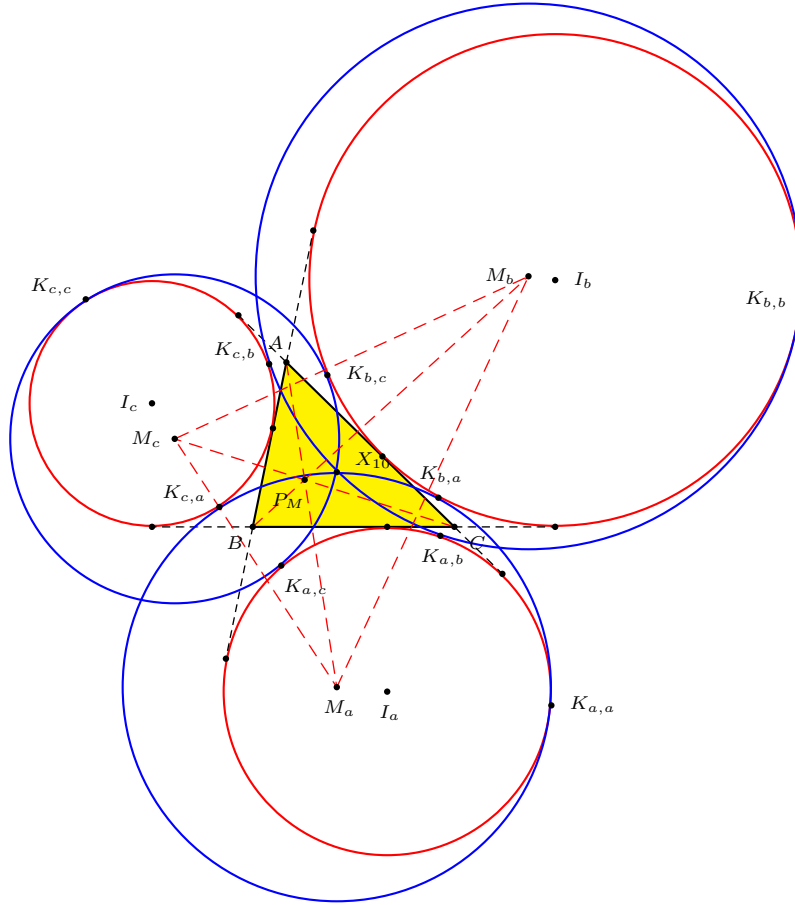


Figure 3.

*Proof.* The center of the circle  $K_a$  is the point

$$\begin{aligned} M_a &= -2a^4(b+c) - a^3(4b^2 + 4bc + 3c^2) + a^2(b+c)(b^2+c^2) - (b+c-a)(b^2-c^2)^2 \\ &: -c^5 - c^4(a+b) + c^3(a-b)^2 + c^2(a+b)(a^2+b^2) + 2abc(a^2+ab+b^2) + 2a^2b^2(a+b) \\ &: -b^5 - b^4(c+a) + b^3(c-a)^2 + b^2(c+a)(c^2+a^2) + 2abc(c^2+ca+a^2) + 2c^2a^2(c+a). \end{aligned}$$

Similarly, the coordinates of  $M_b$  and  $M_c$  can be written down. From these, the perspectivity of  $\mathbf{T}$  and  $M_a M_b M_c$  follows, with the perspector given above.  $\square$

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# Conics Tangent at the Vertices to Two Sides of a Triangle

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**Abstract.** We study conics tangent to two sides of a given triangle at two vertices, and construct two interesting triads of such conics, one consisting of parabolas, the other rectangular hyperbolas. We also construct a new triangle center, the barycentric cube root of  $X_{25}$ , which is the homothetic center of the orthic and tangential triangles.

## 1. Introduction

In the plane of a given triangle  $ABC$ , a conic is the locus of points with homogeneous barycentric coordinates  $x : y : z$  satisfying a second degree equation of the form

$$fx^2 + gy^2 + hz^2 + 2pyz + 2qzx + 2hxy = 0, \quad (1)$$

where  $f, g, h, p, q, r$  are real coefficients (see [4]). In matrix form, (1) can be rewritten as

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} f & r & q \\ r & g & p \\ q & p & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

If we have a circumconic, *i.e.*, a conic that passes through the vertices  $A, B, C$ , then from (1),  $f = g = h = 0$ . Since the equation becomes  $pyz + qzx + rxy = 0$ , the circumconic is the isogonal (or isotomic) conjugate of a line.

If the conic passes through  $B$  and  $C$ , we have  $g = h = 0$ . The conic has equation

$$fx^2 + 2pyz + 2qzx + 2rxy = 0. \quad (2)$$

## 2. Conic tangent to two sides of $ABC$

Consider a conic given by (2) which is tangent at  $B, C$  to the sides  $AB, AC$ . We call this an  $A$ -conic. Since the line  $AB$  has equation  $z = 0$ , the conic (2) intersects  $AB$  at  $(x : y : 0)$  where  $x(fx + 2ry) = 0$ . The conic and the line are tangent at  $B$  only if  $r = 0$ . Similarly the conic is tangent to  $AC$  at  $C$  only if  $q = 0$ . The  $A$ -conic has equation

$$fx^2 + 2pyz = 0. \quad (3)$$

We shall also consider the analogous notions of  $B$ - and  $C$ -conics. These are respectively conics with equations

$$gy^2 + 2qzx = 0, \quad hz^2 + 2rxy = 0. \quad (4)$$

If triangle  $ABC$  is scalene, none of these conics is a circle.

**Lemma 1.** *Let  $BC$  be a chord of a conic with center  $O$ , and  $M$  the midpoint of  $BC$ . If the tangents to the conic at  $B$  and  $C$  intersect at  $A$ , then the points  $A$ ,  $M$ ,  $O$  are collinear.*

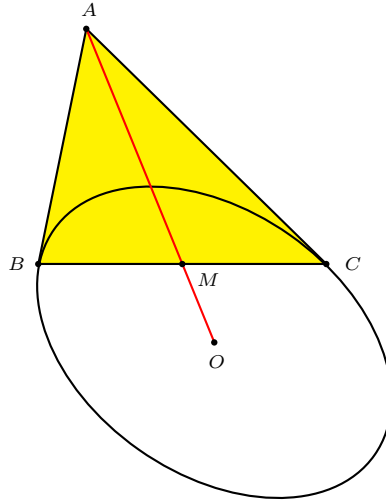


Figure 1.

*Proof.* The harmonic conjugate  $S$  of the point  $M$  relative to  $B$ ,  $C$  is the point at infinity of the line  $BC$ . The center  $O$  is the midpoint of every chord passing through  $O$ . Hence, the polar of  $O$  is the line at infinity. The polar of  $A$  is the line  $BC$  (see Figure 1). Hence, the polar of  $S$  is a line that must pass through  $O$ ,  $M$  and  $A$ .  $\square$

From Lemma 1 we conclude that the center of an  $A$ -conic lies on the  $A$ -median. In the case of a parabola, the axis is parallel to the median of  $ABC$  since the center is a point at infinity. In general, the center of the conic (1) is the point

$$\left( \begin{vmatrix} 1 & r & q \\ 1 & g & p \\ 1 & p & h \end{vmatrix} : \begin{vmatrix} f & 1 & q \\ r & 1 & p \\ q & 1 & h \end{vmatrix} : \begin{vmatrix} f & r & 1 \\ r & g & 1 \\ q & p & 1 \end{vmatrix} \right). \quad (5)$$

Hence the centers of the  $A$ -,  $B$ -,  $C$ -conics given in (3) and (4) are the points  $(p : f : f)$ ,  $(g : q : g)$ , and  $(h : h : r)$  respectively.

We investigate two interesting triads when the three conics (i) are parabolas, (ii) rectangular hyperbolas, and (iii) all pass through a given point  $P = (u : v : w)$ , and shall close with a generalization.



### 3. A triad of parabolas

The  $A$ -conic (3) is a parabola if the center  $(p : f : f)$  is on the line at infinity. Hence,  $p + f + f = 0$ , and the  $A$ -parabola has equation  $x^2 - 4yz = 0$ . We label this parabola  $\mathcal{P}_a$ . Similarly, the  $B$ - and  $C$ -parabolas are  $\mathcal{P}_b : y^2 - 4zx = 0$  and  $\mathcal{P}_c : z^2 - 4xy = 0$  respectively. These are also known as the Artzt parabolas (see Figure 2).

**3.1. Construction.** A conic can be constructed with a dynamic software by locating 5 points on it. The  $A$ -parabola  $\mathcal{P}_a$  clearly contains the vertices  $B, C$ , and the points  $M_a = (2 : 1 : 1)$ ,  $P_b = (4 : 1 : 4)$  and  $P_c = (4 : 4 : 1)$ . Clearly,  $M_a$  is the midpoint of the median  $AA_1$ , and  $P_b, P_c$  are points dividing the medians  $BB_1$  and  $CC_1$  in the ratio  $BP_b : P_bB_1 = CP_c : P_cC_1 = 8 : 1$ .

Similarly, if  $M_b, M_c$  are the midpoints of the medians  $BB_1$  and  $CC_1$ , and  $P_a$  divides  $AA_1$  in the ratio  $AP_a : P_aA_1 = 8 : 1$ , then the  $B$ -parabola  $\mathcal{P}_b$  is the conic containing  $C, A, M_b, P_c, P_a$ , and the  $C$ -parabola  $\mathcal{P}_c$  contains  $A, B, M_c, P_a, P_b$ .

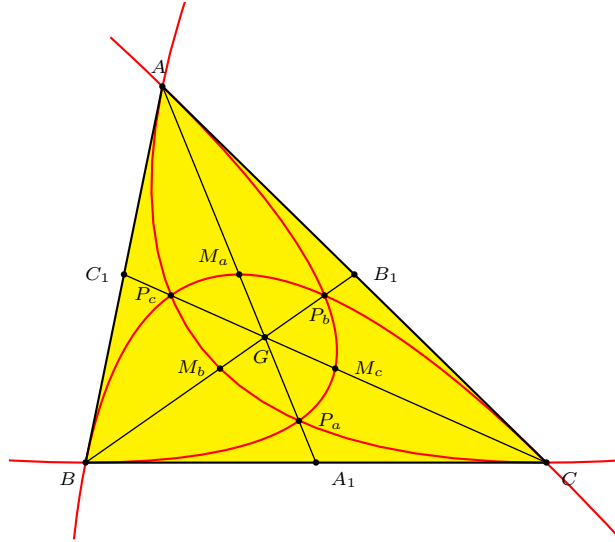


Figure 2.

*Remarks.* (1)  $P_a, P_b, P_c$  are respectively the centroids of triangles  $GBC, GCA, GAB$ .

(2) Since  $A_1$  is the midpoint of  $BC$  and  $M_aA_1$  is parallel to the axis of  $\mathcal{P}_a$ , by Archimedes' celebrated quadrature, the area of the parabola triangle  $M_aBC$  is  $\frac{4}{3} \cdot \triangle M_aBC = \frac{2}{3} \cdot \triangle ABC$ .

(3) The region bounded by the three Artzt parabolas has area  $\frac{5}{27}$  of triangle  $ABC$ .

**3.2. Foci and directrices.** We identify the focus and directrix of the  $A$ -parabola  $\mathcal{P}_a$ . Note that the axis of the parabola is parallel to the median  $AA_1$ . Therefore,

the parallel through  $B$  to the median  $AA_1$  is parallel to the axis, and its reflection in the tangent  $AB$  passes through the focus  $F_a$ . Similarly the reflection in  $AC$  of the parallel through  $C$  to the median  $AA_1$  also passes through  $F_a$ . Hence the focus  $F_a$  is constructible (in the Euclidean sense).

Let  $D, E$  be the orthogonal projections of the focus  $F_a$  on the sides  $AB, AC$ , and  $D', E'$  the reflections of  $F_a$  with respect to  $AB, AC$ . It is known that the line  $DE$  is the tangent to the  $A$ -parabola  $\mathcal{P}_a$  at its vertex and the line  $D'E'$  is the directrix of the parabola (see Figure 3).

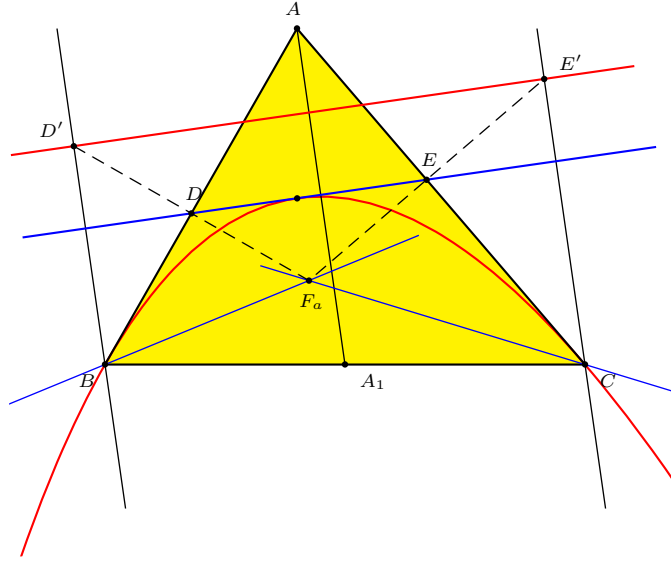


Figure 3.

**Theorem 2.** *The foci  $F_a, F_b, F_c$  of the three parabolas are the vertices of the second Brocard triangle of  $ABC$  and hence are lying on the Brocard circle. The triangles  $ABC$  and  $F_a F_b F_c$  are perspective at the Lemoine point  $K$ .*

*Proof.* With reference to Figure 3, in triangle  $ADE$ , the median  $AA_1$  is parallel to the axis of the  $A$ -parabola  $\mathcal{P}_a$ . Hence,  $AA_1$  is an altitude of triangle  $ADE$ . The line  $AF_a$  is a diameter of the circumcircle of  $ADE$  and is the isogonal conjugate to  $AA_1$ . Hence the line  $AF_a$  is a symmedian of  $ABC$ , and it passes through the Lemoine point  $K$  of triangle  $ABC$ .

Similarly, if  $F_b$  and  $F_c$  are the foci of the  $B$ - and  $C$ -parabolas respectively, then the lines  $BF_b$  and  $CF_c$  also pass through  $K$ , and the triangles  $ABC$  and  $F_a F_b F_c$  are perspective at the Lemoine point  $K$ .

Since the reflection of  $BF_a$  in  $AB$  is parallel to the median  $AA_1$ , we have

$$\angle F_a B A = \angle D' B A = \angle B A A_1 = \angle F_a A C, \quad (6)$$

and the circle  $F_a A B$  is tangent to  $AC$  at  $A$ . Similarly,

$$\angle A C F_a = \angle A C E' = \angle A_1 A C = \angle B A F_a, \quad (7)$$

and the circle  $F_aCA$  is tangent to  $AB$  at  $A$ . From (6) and (7), we conclude that the triangles  $F_aAB$  and  $F_aCA$  are similar, so that

$$\angle AF_aB = \angle CF_aA = \pi - A \quad \text{and} \quad \angle BF_aC = 2A. \quad (8)$$

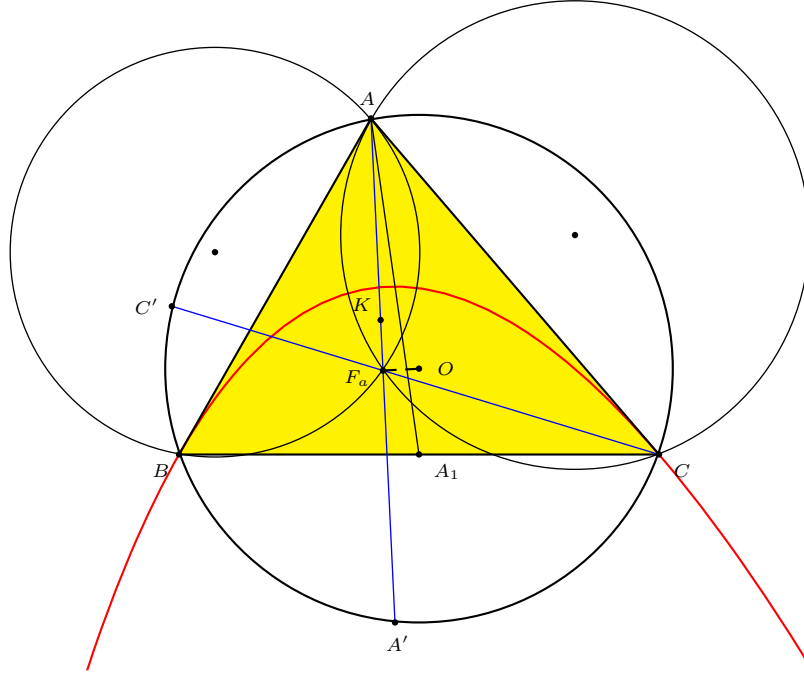


Figure 4.

If  $O$  is the circumcenter of  $ABC$  and the lines  $AF_a$ ,  $CF_a$  meet the circumcircle again at the points  $A'$ ,  $C'$  respectively then from the equality of the arcs  $AC'$  and  $BA'$ , the chords  $AB = A'C'$ . Since the triangles  $ABF_a$  and  $A'C'F_a$  are similar to triangle  $CAF_a$ , they are congruent. Hence,  $F_a$  is the midpoint of  $AA'$ , and is the orthogonal projection of the circumcenter  $O$  on the  $A$ -symmedian. As such, it is on the Brocard circle with diameter  $OK$ . Likewise, the foci of the  $B$ - and  $C$ -parabolas are the orthogonal projections of the point  $O$  on the  $B$ - and  $C$ -symmedians respectively. The three foci form the second Brocard triangle of  $ABC$ .  $\square$

*Remarks.* (1) Here is an alternative, analytic proof. In homogeneous barycentric coordinates, the equation of a circle is of the form

$$a^2yz + b^2zx + c^2xy - (x + y + z)(Px + Qy + Rz) = 0,$$

where  $a, b, c$  are the lengths of the sides of  $ABC$ , and  $P, Q, R$  are the powers of  $A, B, C$  relative to the circle. The equation of the circle  $F_aAB$  tangent to  $AC$  at  $A$  is

$$a^2yz + b^2zx + c^2xy - b^2(x + y + z)z = 0, \quad (9)$$

and that of the circle  $F_aCA$  tangent to  $AB$  at  $A$  is

$$a^2yz + b^2zx + c^2xy - c^2(x + y + z)y = 0. \quad (10)$$

Solving these equations we find

$$F_a = (b^2 + c^2 - a^2 : b^2 : c^2).$$

It is easy to verify that this lies on the Brocard circle

$$a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{a^2 + b^2 + c^2}(x + y + z) \left( \frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} \right) = 0.$$

(2) This computation would be more difficult if the above circles were not tangent at  $A$  to the sides  $AC$  and  $AB$ . So it is interesting to show another method. We can get the same result, as we know directed angles (defined modulo  $\pi$ ) from

$$\theta = (F_aB, F_aC) = 2A, \quad \varphi = (F_aC, F_aA) = -A, \quad \psi = (F_aA, F_aB) = -A.$$

Making use of the formula given in [1], we obtain

$$\begin{aligned} F_a &= \left( \frac{1}{\cot A - \cot \theta} : \frac{1}{\cot B - \cot \varphi} : \frac{1}{\cot C - \cot \psi} \right) \\ &= \left( \frac{1}{\cot A - \cot 2A} : \frac{1}{\cot B + \cot A} : \frac{1}{\cot C + \cot A} \right) \\ &= \left( \frac{\sin A \sin 2A}{\sin A} : \frac{\sin A \sin B}{\sin(A+B)} : \frac{\sin A \sin C}{\sin(A+C)} \right) \\ &= (2 \cos A \sin B \sin C : \sin^2 B : \sin^2 C) \\ &= (b^2 + c^2 - a^2 : b^2 : c^2). \end{aligned}$$

#### 4. Rectangular $A$ -, $B$ -, $C$ -hyperbolas

The  $A$ -conic  $fx^2 + 2pyz = 0$  is a rectangular hyperbola if it contains two orthogonal points at infinity  $(x_1 : y_1 : z_1)$  and  $(x_2 : y_2 : z_2)$  where

$$x_1 + y_1 + z_1 = 0 \quad \text{and} \quad x_2 + y_2 + z_2 = 0.$$

and . Putting  $x = sy$  and  $z = -(x + y)$  we have  $fs^2 - 2ps - 2p = 0$  with roots  $s_1 + s_2 = \frac{2p}{f}$  and  $s_1s_2 = -\frac{2p}{f}$ . The two points are orthogonal if

$$S_Ax_1x_2 + S_By_1y_2 + S_Cz_1z_2 = 0.$$

From this,  $S_Ax_1x_2 + S_By_1y_2 + S_C(x_1 + y_1)(x_2 + y_2) = 0$ ,  $S_As_1s_2 + S_B + S_C(s_1 + 1)(s_2 + 1) = 0$ , and

$$\frac{p}{f} = \frac{S_B + S_C}{2S_A} = \frac{a^2}{b^2 + c^2 - a^2}.$$

This gives the rectangular  $A$ -hyperbola

$$\mathcal{H}_a : (b^2 + c^2 - a^2)x^2 + 2a^2yz = 0,$$

with center

$$O_a = (a^2 : b^2 + c^2 - a^2 : b^2 + c^2 - a^2).$$

This point is the orthogonal projection of the orthocenter  $H$  of  $ABC$  on the  $A$ -median and lies on the orthocentroidal circle, *i.e.*, the circle with diameter  $GH$ . Similarly the centers  $O_b, O_c$  lie on the orthocentroidal circle as projections of  $H$  on the  $B$ -,  $C$ -medians. Triangle  $O_aO_bO_c$  is similar to the triangle of the medians of  $ABC$  (see Figure 5).

The construction of the  $A$ -hyperbola can be done since we know five points of it: the points  $B, C$ , their reflections  $B', C'$  in  $O_a$ , and the orthocenter of triangle  $B'BC$ .

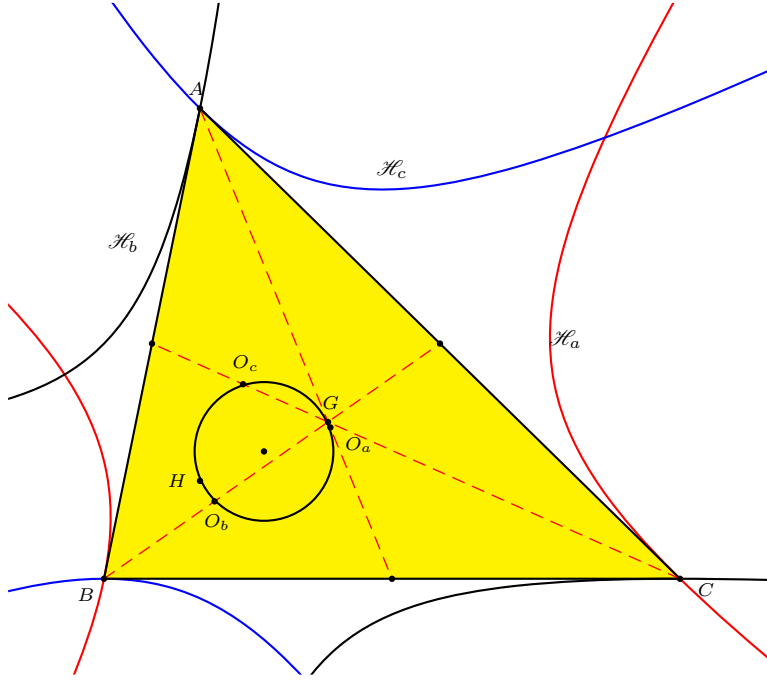


Figure 5.

### 5. Triad of conics passing through a given point

Let  $P = (u : v : w)$  be a given point. We denote the  $A$ -conic through  $P$  by  $(\mathcal{A}_P)$ ; similarly for  $(\mathcal{B}_P)$  and  $(\mathcal{C}_P)$ . These conics have equations

$$vwx^2 - u^2yz = 0, \quad wuy^2 - v^2zx = 0, \quad uvz^2 - w^2xy = 0.$$

Let  $XYZ$  be the cevian triangle of  $P$  and  $X'Y'Z'$  be the trilinear polar of  $P$  (see Figure 6). It is known that the point  $X'$  is the intersection of  $YZ$  with  $BC$ , and is the harmonic conjugate of  $X$  relative to the points  $B, C$ ; similarly for  $Y'$  and  $Z'$  are the harmonic conjugates of  $Y$  and  $Z$  relative to  $C, A$  and  $A, B$ . These points have coordinates

$$X' = (0 : -v : w), \quad Y' = (u : 0 : -w), \quad Z' = (-u : v : 0),$$

and they lie on the trilinear polar

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

Now, the polar of  $A$  relative to the conic  $(\mathcal{A}_P)$  is the line  $BC$ . Hence, the polar of  $X'$  passes through  $A$ . Since  $X$  is harmonic conjugate of  $X'$  relative to  $B, C$ , the line  $AX$  is the polar of  $X'$  and the line  $X'P$  is tangent to the conic  $(\mathcal{A}_P)$  at  $P$ .

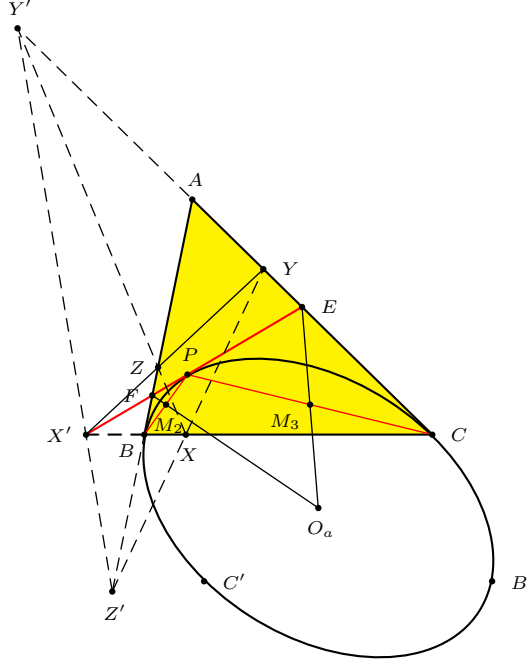


Figure 6.

*Remark.* The polar of an arbitrary point or the tangent of a conic at the point  $P = (u : v : w)$  is the line given by

$$(x \ y \ z) \begin{pmatrix} f & r & q \\ r & g & p \\ q & p & h \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0.$$

Hence the tangent of  $(\mathcal{A}_P)$  at  $P$  is the line

$$(x \ y \ z) \begin{pmatrix} 2vw & 0 & 0 \\ 0 & 0 & -u^2 \\ 0 & -u^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0,$$

or  $2vwx - wuy - uvz = 0$ . This meets  $BC$  at the point  $X' = (0 : -v : w)$  as shown before.

5.1. *Construction of the conic  $(\mathcal{A}_P)$ .* We draw the line  $YZ$  to meet  $BC$  at  $X'$ . Then  $X'P$  is tangent to the conic at  $P$ . Let this meet  $AC$  and  $AB$  at  $E$  and  $F$  respectively. Let  $M_2$  and  $M_3$  be the midpoints of  $PB$  and  $PC$ . Since  $AB$ ,  $AC$  and  $EF$  are tangents to  $(\mathcal{A}_P)$  at the points  $B$ ,  $C$ ,  $P$  respectively, the lines  $EM_3$  and  $FM_2$  meet at the center  $O_a$  of the conic  $(\mathcal{A}_P)$ . If  $B'$ ,  $C'$  are the symmetric points of  $B$ ,  $C$  relative to  $O_a$ , then  $(\mathcal{A}_P)$  is the conic passing through the five points  $B$ ,  $P$ ,  $C$ ,  $B'$ ,  $C'$ .

The conics  $(\mathcal{B}_P)$  and  $(\mathcal{C}_P)$  can be constructed in a similar way.

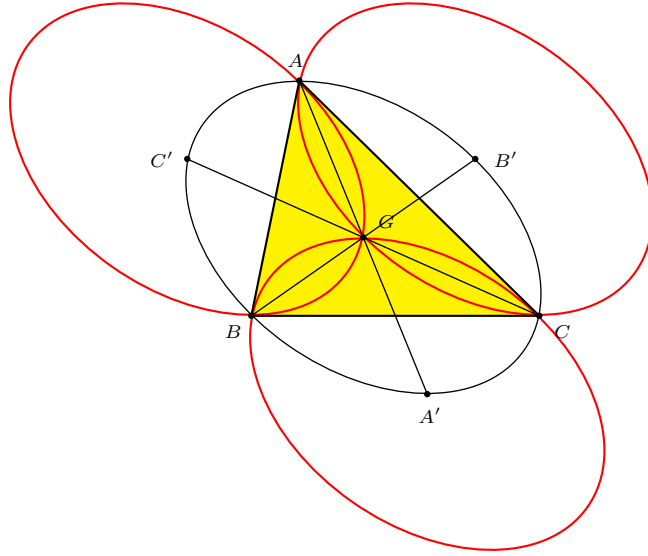


Figure 7.

In the special case when  $P$  is the centroid  $G$  of triangle  $ABC$ ,  $X'$  is a point at infinity and the line  $YZ$  is parallel to  $BC$ . The center  $O_a$  is the symmetric  $A'$  of  $A$  relative to  $G$ . It is obvious that  $(\mathcal{A}_P)$  is the translation of the Steiner circumellipse (with center  $G$ ), by the vector  $AG$ ; similarly for  $(\mathcal{B}_P)$ ,  $(\mathcal{C}_P)$  (see Figure 7).

**Theorem 3.** *For an arbitrary point  $Q$  the line  $X'Q$  intersects the line  $AP$  at the point  $R$ , and we define the mapping  $h(Q) = S$ , where  $S$  is the harmonic conjugate of  $Q$  with respect to  $X'$  and  $R$ . The mapping  $h$  swaps the conics  $(\mathcal{B}_P)$  and  $(\mathcal{C}_P)$ .*

*Proof.* The mapping  $h$  is involutive because  $h(Q) = S$  if and only if  $h(S) = Q$ . Since the line  $AP$  is the polar of  $X'$  relative to the pair of lines  $AB$ ,  $AC$  we have  $h(A) = A$ ,  $h(P) = P$ ,  $h(C) = B$ , so that  $h(AB) = AC$ , and  $h(BC) = CB$ . Hence, the conic  $h((\mathcal{B}_P))$  is the one passing through  $A$ ,  $P$ ,  $B$  and tangent to the sides  $AC$ ,  $BC$  at  $A$ ,  $B$  respectively. This is clearly the conic  $(\mathcal{C}_P)$ .  $\square$

Therefore, a line passing through  $X'$  tangent to  $(\mathcal{B}_P)$  is also tangent to  $(\mathcal{C}_P)$ . This means that the common tangents of the conics  $(\mathcal{B}_P)$  and  $(\mathcal{C}_P)$  intersect at  $X'$ .

Similarly we can define mappings with pivot points  $Y'$ ,  $Z'$  swapping  $(\mathcal{C}_P)$ ,  $(\mathcal{A}_P)$  and  $(\mathcal{A}_P)$ ,  $(\mathcal{B}_P)$ .

Consider the line  $X'P$  tangent to the conic  $(\mathcal{A}_P)$  at  $P$ . It has equation

$$-2vwx + wuy + uvz = 0.$$

This line meets the conic  $(\mathcal{B}_P)$  at the point  $X_b = (u : -2v : 4w)$ , and the conic  $(\mathcal{C}_P)$  at the point  $X_c = (u : 4v : -2w)$ .

Similarly, the tangent  $Y'P$  of  $(\mathcal{B}_P)$  intersects  $(\mathcal{C}_P)$  again at  $Y_c = (4u : v : -2w)$  and  $(\mathcal{A}_P)$  again at  $Y_a = (-2u : v : 4w)$ . The tangent  $Z'P$  of  $(\mathcal{C}_P)$  intersects  $(\mathcal{A}_P)$  again at  $Z_a = (-2u : 4v : w)$  and  $(\mathcal{B}_P)$  again at  $Z_b = (4u : -2v : w)$  respectively.

**Theorem 4.** *The line  $Z_bY_c$  is a common tangent of  $(\mathcal{B}_P)$  and  $(\mathcal{C}_P)$ , so is  $X_cZ_a$  of  $(\mathcal{C}_P)$  and  $(\mathcal{A}_P)$ , and  $Y_aX_b$  of  $(\mathcal{A}_P)$  and  $(\mathcal{B}_P)$ .*

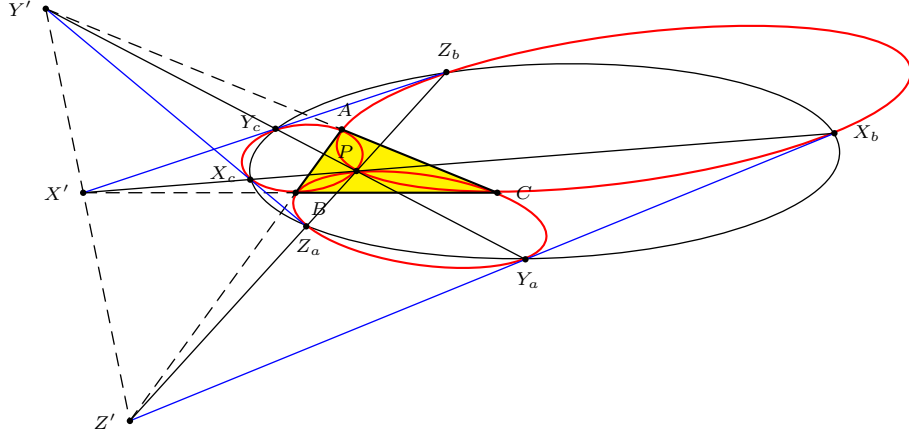


Figure 8.

*Proof.* We need only prove the case  $Y_cZ_b$ . The line has equation

$$vwx + 4wuy + 4uvz = 0.$$

This is tangent to  $(\mathcal{B}_P)$  at  $Z_b$  and to  $(\mathcal{C}_P)$  at  $Y_c$  as the following calculation confirms.

$$\begin{pmatrix} 0 & 0 & v^2 \\ 0 & -2wu & 0 \\ v^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4u \\ -2v \\ w \end{pmatrix} = v \begin{pmatrix} vw \\ 4wu \\ 4uv \end{pmatrix},$$

$$\begin{pmatrix} 0 & w^2 & 0 \\ w^2 & 0 & 0 \\ 0 & 0 & -2uv \end{pmatrix} \begin{pmatrix} 4u \\ v \\ -2w \end{pmatrix} = w \begin{pmatrix} vw \\ 4wu \\ 4uv \end{pmatrix}.$$

□

**Theorem 5.** *The six points  $X_b, X_c, Y_c, Y_a, Z_a, Z_b$  lie on a conic (see Figure 7).*

*Proof.* It is easy to verify that the six points satisfy the equation of the conic

$$2v^2w^2x^2 + 2w^2u^2y^2 + 2u^2v^2z^2 + 7u^2vwy + 7uv^2wzx + 7uvw^2xy = 0, \quad (11)$$



which has center

$$(u(-11u + 7v + 7w) : v(7u - 11v + 7w) : w(7u + 7v - 11w)).$$

□

*Remark.* The equation of the conic (11) can be rewritten as

$$u^2(2v^2 - 7vw + 2w^2)yz + v^2(2w^2 - 7wu + 2u^2)zx + w^2(2u^2 - 7uv + 2v^2)xy - 2(x + y + z)(v^2w^2x + w^2u^2y + u^2v^2z) = 0.$$

This is a circle if and only if

$$\frac{u^2(2v^2 - 7vw + 2w^2)}{a^2} = \frac{v^2(2w^2 - 7wu + 2u^2)}{b^2} = \frac{w^2(2u^2 - 7uv + 2v^2)}{c^2}.$$

Equivalently, the isotomic conjugate of  $P$ , namely,  $(\frac{1}{u} : \frac{1}{v} : \frac{1}{w})$  is the intersection of the three conics defined by

$$\frac{2y^2 - 7yz + 2z^2}{a^2} = \frac{2x^2 - 7zx + 2x^2}{b^2} = \frac{2x^2 - 7xy + 2y^2}{c^2}.$$

5.2. *The type of the three conics*  $(\mathcal{A}_P)$ ,  $(\mathcal{B}_P)$ ,  $(\mathcal{C}_P)$ . The type of a conic with given equation (1) can be determined by the quantity

$$d = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & f & r & q \\ 1 & r & g & p \\ 1 & q & p & h \end{vmatrix}$$

For the conic  $\mathcal{A}_P : vwx^2 - u^2yz = 0$ , this is

- (i) a parabola if  $u^2 - 4vw = 0$  (and  $\mathcal{A}_P = \mathcal{P}_a$ ),
- (ii) an ellipse if  $u^2 - vw < 0$ , i.e.,  $P$  lying inside  $\mathcal{P}_a$ ,
- (iii) a hyperbola if  $u^2 - vw > 0$ , i.e.,  $P$  lying outside  $\mathcal{P}_a$ .

Thus the conics in the triad  $(\mathcal{A}_P, \mathcal{B}_P, \mathcal{C}_P)$  are all ellipses if and only if  $P$  lies in the interior of the curvilinear triangle  $P_aP_bP_c$  bounded by arcs of the Artzt parabolas (see Figure 2).

*Remarks.* (1) It is impossible for all three conics  $(\mathcal{A}_P, \mathcal{B}_P, \mathcal{C}_P)$  to be parabolas.

(2) The various possibilities of conics of different types are given in the table below.

Ellipses	3	2	2	1	1	1		
Parabolas		1		2	1		1	
Hyperbolas			1		1	2	2	3

Note that the conics in the triad  $(\mathcal{A}_P, \mathcal{B}_P, \mathcal{C}_P)$  cannot be all rectangular hyperbolas. This is because, as we have seen in §4 that the only rectangular  $A$ -hyperbola is  $\mathcal{H}_a$ . The conic  $\mathcal{A}_P$  is a rectangular hyperbola if and only if  $P$  lies on  $\mathcal{H}_a$ , and in this case,  $\mathcal{A}_P = \mathcal{H}_a$ .

## 6. Generalization

Slightly modifying (3) and (4), we rewrite the equations of a triad of  $A$ -,  $B$ -,  $C$ -conics as

$$x^2 + 2Pyz = 0, \quad y^2 + 2Qzx = 0, \quad z^2 + 2Rxy = 0.$$

Apart from the vertices, these conics intersect at

$$\begin{aligned} Q_a &= \left( -\left(\frac{1}{QR}\right)^{\frac{1}{3}} : 2Q^{\frac{1}{3}} : 2R^{\frac{1}{3}} \right), \\ Q_b &= \left( 2P^{\frac{1}{3}} : -\left(\frac{1}{RP}\right)^{\frac{1}{3}} : 2R^{\frac{1}{3}} \right), \\ Q_c &= \left( 2P^{\frac{1}{3}} : 2Q^{\frac{1}{3}} : -\left(\frac{1}{PQ}\right)^{\frac{1}{3}} \right), \end{aligned}$$

where, for a real number  $x$ ,  $x^{\frac{1}{3}}$  stands for the real cube root of  $x$ . Clearly, the triangles  $Q_aQ_bQ_c$  and  $ABC$  are perspective at

$$Q = \left( P^{\frac{1}{3}} : Q^{\frac{1}{3}} : R^{\frac{1}{3}} \right).$$

Let  $XYZ$  be the cevian triangle of  $Q$  relative to  $ABC$  and  $X'$ ,  $Y'$ ,  $Z'$  the intersection of the sidelines with the trilinear polar of  $Q$ .

**Proposition 6.** *For an arbitrary point  $S$ , the line  $X'S$  meets the line  $AQ$  at the point  $T$ , and we define the mapping  $h(S) = U$ , where  $U$  is the harmonic conjugate of  $S$  with respect to  $X'$  and  $T$ . The mapping  $h$  swaps the conics  $B$ - and  $C$ -conics.*

Similarly we can define mappings with pivot points  $Y'$ ,  $Z'$  swapping the  $C$ - and  $A$ -conics, and the  $A$ - and  $B$ -conics.

The tangent to	at	intersects	at the point
$\mathcal{C}_a$	$Q_b$	$\mathcal{C}_c$	$X_c = \left( -(RP^2)^{\frac{1}{3}} : 2 : 2(R^2P)^{\frac{1}{3}} \right)$
$\mathcal{C}_a$	$Q_c$	$\mathcal{C}_b$	$X_b = \left( -(P^2Q)^{\frac{1}{3}} : 2(PQ^2)^{\frac{1}{3}} : 2 \right)$
$\mathcal{C}_b$	$Q_c$	$\mathcal{C}_a$	$Y_a = \left( 2(P^2Q)^{\frac{1}{3}} : -(PQ^2)^{\frac{1}{3}} : 2 \right)$
$\mathcal{C}_b$	$Q_a$	$\mathcal{C}_c$	$Y_c = \left( 2 : -(Q^2R)^{\frac{1}{3}} : 2(QR^2)^{\frac{1}{3}} \right)$
$\mathcal{C}_c$	$Q_a$	$\mathcal{C}_b$	$Z_b = \left( 2 : 2(Q^2R)^{\frac{1}{3}} : -(QR^2)^{\frac{1}{3}} \right)$
$\mathcal{C}_c$	$Q_b$	$\mathcal{C}_a$	$Z_a = \left( 2(RP^2)^{\frac{1}{3}} : 2 : -(R^2P)^{\frac{1}{3}} \right)$

From these data we deduce the following theorem.

**Theorem 7.** *The line  $Z_bY_c$  is a common tangent of the  $B$ - and  $C$ -conics; so is  $X_cZ_a$  of the  $C$ - and  $A$ -conics, and  $Y_aX_b$  of the  $A$ - and  $B$ -conics.*

Figure 9 shows the case of the triad of rectangular hyperbolas ( $\mathcal{H}_a$ ,  $\mathcal{H}_b$ ,  $\mathcal{H}_c$ ). The perspector  $Q$  is the barycentric cube root of  $X_{25}$  (the homothetic center of the orthic and tangential triangles).  $Q$  does not appear in [3].

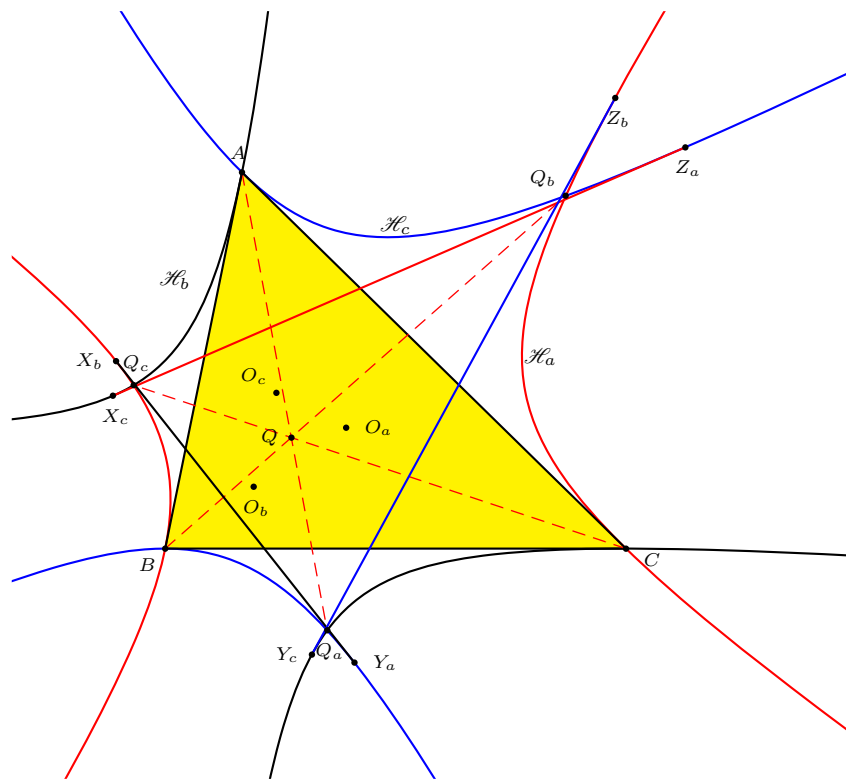


Figure 9.

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## Simple Relations Regarding the Steiner Inellipse of a Triangle

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**Abstract.** By applying analytic geometry within a special Cartesian reference, based on the Kiepert hyperbola, we prove a great number of relations (collinearities, similarities, inversions etc.) regarding central points, central lines and central conics of a triangle. Most - not all - of these statements are well-known, but somehow dispersed throughout the literature. Some relations turn out to be easy consequences of the action of a conjugation - an involutory Möbius transformation - whose fixed points are the foci of the Steiner inellipse.

### 1. Introduction

With the aim of making proofs simpler and more uniform, we applied analytic geometry to revisit a number of theorems regarding the triangle centers. The choice of an intrinsic Cartesian frame, which we call the *Kiepert reference*, turned out to be very effective in dealing with a good part of the standard results on central points and related conics: along with several well-known statements, a number of simple relations which seem to be new have emerged. Here is, perhaps, the most surprising example:

**Theorem 1.** *Let  $G, F_+, F_-$  denote, respectively, the centroid, the first and second Fermat points of a triangle. The major axis of its Steiner inellipse is the inner bisector of the angle  $\angle F_+GF_-$ . The lengths of the axes are  $|GF_-| \pm |GF_+|$ , the sum and difference of the distances of the Fermat points from the centroid.*

As a consequence, if  $2c$  denotes the focal distance,  $c$  is the geometric mean between  $|GF_-|$  and  $|GF_+|$ . This means that  $F_+$  and  $F_-$  are interchanged under the action of an involutory Möbius transformation  $\mu$ , the product of the reflection in the major axis by the inversion in the circle whose diameter is defined by the foci. This conjugation plays an interesting role in the geometry of the triangle. In fact, one easily discovers the existence of many other  $\mu$ -coupled objects: the isodynamic points; the circumcenter and the focus of the Kiepert parabola; the orthocenter and the center of the Jerabek hyperbola; the Lemoine point and the Parry point; the circumcircle and the Brocard circle; the Brocard axis and the Parry circle, etc. By applying standard properties of homographies, one can then recognize various sets

of concyclic central points, parallel central lines, orthogonal central circles, similar central triangles etc.

*Notation and terminology.* If  $A, B$  are points,  $AB$  will indicate both the segment and the line through  $A, B$ ,  $|AB|$  is the length of the segment  $AB$ ,  $\overrightarrow{AB}$  is a vector; sometimes we also write  $\overrightarrow{AB} = B - A$ . The angle between  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$  is  $\angle ABC$ .  $\overrightarrow{AB} \cdot \overrightarrow{CD}$  is the scalar product.  $A^B = C$  means that a half-turn about  $B$  maps  $A$  onto  $C$ ; equivalently, we write  $B = \frac{A+C}{2}$ .

For the Cartesian coordinates of a point  $A$  we write  $A = [x_A, y_A]$ ; for a vector,  $\overrightarrow{AB} = [x_B - x_A, y_B - y_A]$ .

In order to identify central points of a triangle  $\mathbf{T} = A_1A_2A_3$  we shall use both capital letters and numbers, as listed by Clark Kimberling in [3, 2]; for example,  $G = X_2$ ,  $O = X_3$ ,  $H = X_4$ , etc.

## 2. The Kiepert reference

The Kiepert hyperbola  $\mathcal{K}$  of a (non equilateral) triangle  $\mathbf{T} = A_1A_2A_3$  is the (unique) rectangular hyperbola which is circumscribed to  $\mathbf{T}$  and passes through its centroid  $G$ . We shall adopt an orthogonal Cartesian reference such that the equation for  $\mathcal{K}$  is  $xy = 1$ . This is always possible unless  $\mathcal{K}$  reduces to a pair of perpendicular lines; and this only happens if  $\mathbf{T}$  is isosceles, an easy case that we shall treat separately in §14. How to choose between  $x$  and  $y$ , as well as orientations, will be soon treated. Within this *Kiepert reference*, for the vertices of  $\mathbf{T}$  we write  $A_1 = [x_1, \frac{1}{x_1}]$ ,  $A_2 = [x_2, \frac{1}{x_2}]$ ,  $A_3 = [x_3, \frac{1}{x_3}]$ .

Since central points are symmetric functions of  $A_1, A_2, A_3$ , for their coordinates we expect to find symmetric functions of  $x_1, x_2, x_3$  and hopefully algebraic functions of the elementary symmetric polynomials

$$s_1 := x_1 + x_2 + x_3, \quad s_2 := x_1x_2 + x_2x_3 + x_3x_1, \quad s_3 := x_1x_2x_3.$$

This is true for many, but not all of the classical central points. For example, for the centroid  $G$  we obviously have

$$G = \left[ \frac{1}{3}(x_1 + x_2 + x_3), \frac{1}{3} \left( \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \right) \right] = \left[ \frac{s_1}{3}, \frac{s_2}{3s_3} \right].$$

However, for points like the incenter or the Feuerbach point we cannot avoid encountering functions like  $\sqrt{1 + x_i^2 x_j^2}$  which cannot be expressed explicitly in terms of  $s_1, s_2, s_3$ . Therefore this paper will only deal with a part of the standard geometry of central points, which nevertheless is of importance. Going back to the centroid, since  $G$ , by definition, lies on  $\mathcal{K}$ , we must have  $\frac{s_2}{3s_3} = \frac{3}{s_1}$ , so that  $s_2 = \frac{9s_3}{s_1}$  can be eliminated and we are only left with functions of  $s_1, s_3$ . (Note that, under our assumptions, we always have  $s_1$  and  $s_3$  nonzero).

From now on, it will be understood that this reduction has been made, and we shall write, for short,

$$s_1 = x_1 + x_2 + x_3 = s \quad \text{and} \quad s_3 = x_1x_2x_3 = p.$$

The location of  $G = \left[\frac{s}{3}, \frac{3}{s}\right]$  will determine what was still ambiguous about the reference: without loss of generality, we shall assume that its coordinates are positive:  $s > 0$ . We claim that this implies  $p < 0$ . In fact the square of the Vandermonde product  $V := (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$  is symmetric:

$$V^2 = \frac{-4p(s^3 - 27p)^2}{s^3}.$$

Since we want to deal with proper triangles only, we assume  $s^3 - 27p \neq 0$  and therefore  $p < 0$ , as we wanted. These inequalities will be essential when dealing with square roots as  $\sqrt{-sp}$ ,  $\sqrt{\frac{-s}{p}}$  etc., that we want to be (positive) real numbers. In fact, our calculations will take place within the field  $F = \mathbb{Q}(s, p)$  and its real quadratic extension  $F(u)$ , where  $u = \sqrt{\frac{-sp}{3}}$ .

As we shall see, the advantage of operating within the Kiepert reference can be summarized as follows: once coordinates and equations have been derived, which may require a moderate amount of accurate geometric and algebraic work, many statements will look evident at a glance, without any computing effort.

### 3. Central points

The center of the Kiepert hyperbola  $\mathcal{K}$  is the origin of our reference: the Kiepert center

$$K = X_{115} = [0, 0],$$

By reflecting the centroid

$$G = X_2 = \left[\frac{s}{3}, \frac{3}{s}\right] = \frac{1}{3s}[s^2, 9]$$

upon  $K$ , we obviously find another point of  $\mathcal{K}$ :

$$G^K = X_{671} = \left[\frac{-s}{3}, \frac{-3}{s}\right].$$

As the hyperbola is rectangular, we also find on  $\mathcal{K}$  the orthocenter

$$H = X_4 = \left[\frac{-1}{p}, -p\right] = \frac{-1}{p}[1, p^2]$$

and the Tarry point:

$$T = H^K = X_{98} = \left[\frac{1}{p}, p\right].$$

For the circumcenter we calculate

$$O = X_3 = \left[\frac{1+sp}{2p}, \frac{9+sp}{2s}\right] = \frac{1}{2sp}[s(sp+1), p(sp+9)],$$

and check the collinearity of  $O, G, H$  and Euler's equation:  $\overline{GH} = 2\overline{OG}$ . By comparing coordinates, we notice that  $sp = -3$  would imply  $G = H = O$ . Since this only holds for equilateral triangles (a case we have excluded), we may assume  $t = sp + 3 \neq 0$ . We shall soon find an important interpretation for the sign of  $t$ .

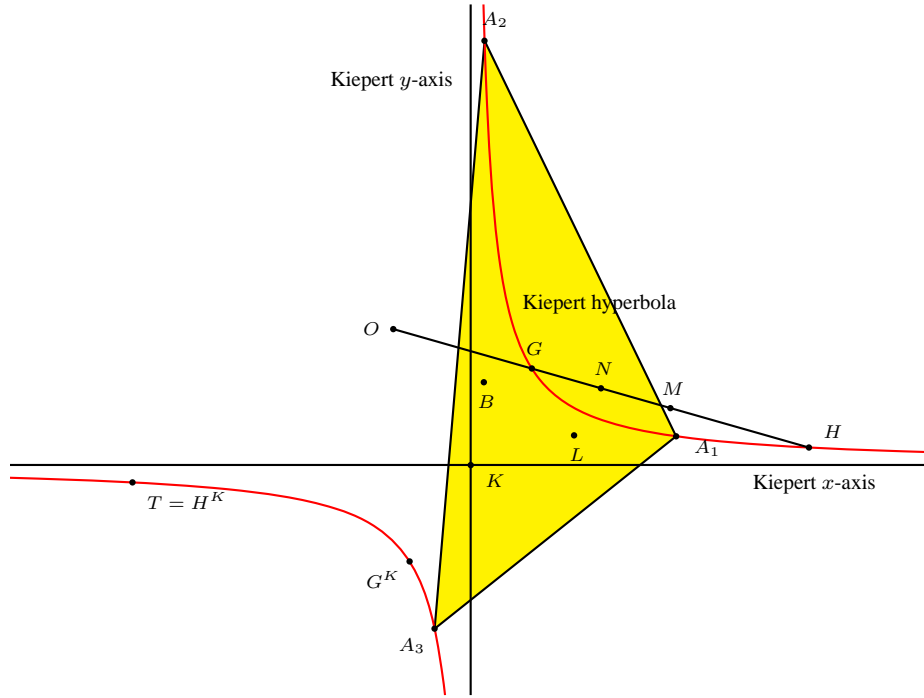


Figure 1.

We also want to calculate the nine-point center  $N$  and the center  $M$  of the orthocentroidal circle (see Figure 1):

$$N = \frac{O + H}{2} = X_5 = \left[ \frac{sp - 1}{4p}, \frac{9 - sp}{4s} \right] = \frac{1}{-4sp} [s(1 - sp), -p(9 - sp)],$$

$$M = \frac{G + H}{2} = O^G = X_{381} = \frac{3 - sp}{-6sp} [s, -3p].$$

We now want the symmedian or Lemoine point  $L$ , the isogonal conjugate of  $G$ . To find its coordinates, we can use a definition of isogonal conjugation which is based on reflections: if we reflect  $G$  in the three sides of  $\mathbf{T}$  and take the circumcenter of the resulting triangle, we find

$$L = X_6 = \frac{2}{3 - sp} [s, -3p].$$

It appears, at a glance, that  $K, M, L$  are collinear. Another central point we want is the Brocard point

$$B = \frac{O + L}{2} = X_{182} = \frac{1}{-4sp(3 - sp)} [s(s^2p^2 - 6sp - 3), p(s^2p^2 + 18sp - 27)].$$

Notice that  $G, N, M, H, L$  (unlike  $O$ ) always have positive coordinates, and this gives interesting information about their location.

#### 4. Central lines

The central points  $O, G, N, M, H$  are collinear on the

**Euler line:**  $-3px + sy + sp - 3 = 0.$

The line through  $M, L, K$  is known as the

**Fermat axis:**  $3px + sy = 0.$

We shall often apply reflections in the  $xy$ -axes or parallel lines and map a vector  $[X, Y]$  onto  $[X, -Y]$  or  $[-X, Y]$ . For example, looking at coefficients in the equations above, we notice that such a reflection maps  $[3p, -s]$  onto  $\pm[3p, s]$ . This proves that the asymptotic directions of  $\mathcal{K}$  bisect the angles between the Euler line and the Fermat axis.

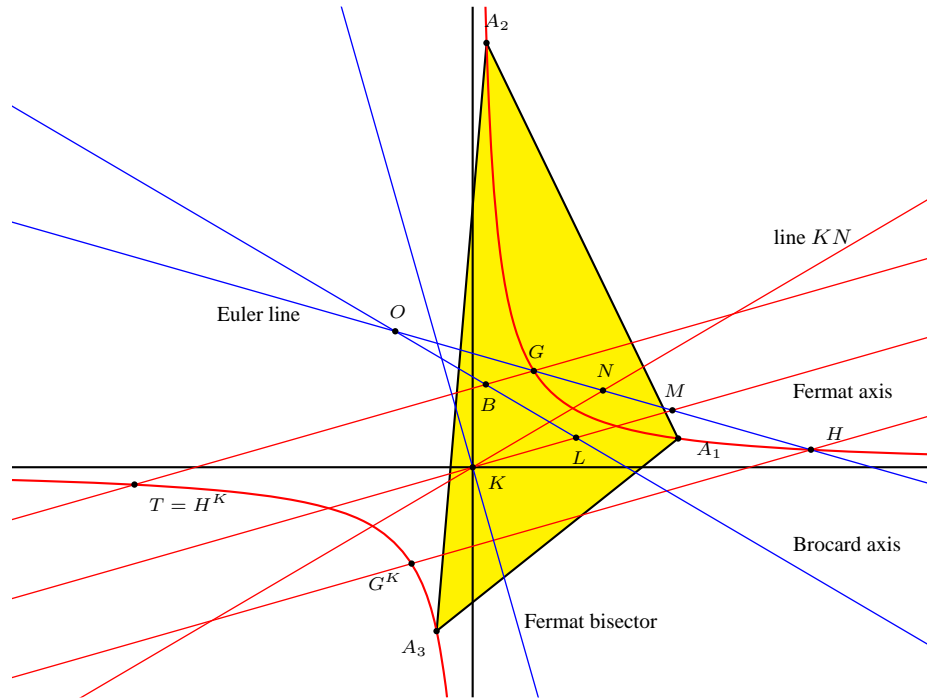


Figure 2.

The line through  $K$ , perpendicular to the Fermat axis, will be called the

**Fermat bisector:**  $sx - 3py = 0.$

These names clearly anticipate the location of the Fermat points. The line through  $O, B, L$  is known as the

**Brocard axis:**  $-p(9 - sp)x + s(1 - sp)y + 8sp = 0.$

A comparison with the line

**KN :**  $p(9 - sp)x + s(1 - sp)y = 0$



shows that, by the same argument, the asymptotic directions of  $\mathcal{K}$  bisect also the angles formed by the Brocard axis and the line  $KN$ .

## 5. Central circles

The circumcircle has equation

$$\mathcal{O} : \quad x^2 + y^2 - \frac{sp+1}{p}x - \frac{sp+9}{s}y + \frac{s^2+9p^2}{sp} = 0$$

and for the circumradius  $r_O$  we find

$$r_O^2 = \frac{s^2 - 2s^3p + 81p^2 + s^4p^2 - 18sp^3 + s^2p^4}{4s^2p^2}.$$

By direct substitution, it is easy to check that the Tarry point  $T$  lies on  $\mathcal{O}$ . Indeed,  $T$  is the fourth intersection (after the three vertices) of  $\mathcal{K}$  and  $\mathcal{O}$ . The antipode of  $T$  on  $\mathcal{O}$  is the

**Steiner point:** 
$$S = T^O = X_{99} = \left[ s, \frac{9}{s} \right] = \frac{1}{s}[s^2, 9].$$

$S$  is the Kiepert center of the complementary triangle of  $\mathbf{T}$ , and therefore can be equivalently defined by the relation  $\overline{GS} = 2\overline{KG}$ .

The second intersection (after  $S$ ) of  $\mathcal{O}$  with the line  $GK$  is the

**Parry point:** 
$$P = X_{111} = \frac{s^2+9p^2}{p(s^4+81)}[s^2, 9].$$

The nine-point circle, with center in  $N$  and radius  $r_N = \frac{r_O}{2}$ , has equation

$$\mathcal{N} : \quad x^2 + y^2 + \frac{1-sp}{2p}x - \frac{9-sp}{2s}y = 0,$$

This shows that  $K$  lies on  $\mathcal{N}$ , as expected for the center of a circumscribed rectangular hyperbola. This is also equivalent to stating that  $S$  lies on  $\mathcal{O}$ . Since  $KN$  and  $SO$  are parallel, the last remark of §4 reads: the angle  $\angle SOL$  is bisected by the asymptotic directions of  $\mathcal{K}$ .

The orthocentroidal circle, centered in  $M$ , is defined by its diameter  $GH$  and has equation:

$$\mathcal{M} : \quad x^2 + y^2 + \frac{3-sp}{3p}x - \frac{3-sp}{s}y - \frac{s^2+9p^2}{3sp} = 0.$$

We now introduce a central circle  $\mathcal{D}$  whose role in the geometry of the triangle has been perhaps underestimated (although it is mentioned in [2, p.230]). First define a central point  $D$  as the intersection of the Fermat bisector with the line through  $G$ , normal to the Euler line:

$$D := \frac{s^3+27p}{18s^2p}[3p, s].$$

Then consider the circle centered in  $D$ , passing through  $G$ :

$$\mathcal{D} : \quad x^2 + y^2 - \frac{s^3+27p}{3s^2}x - \frac{s^3+27p}{9sp}y + \frac{s^2+9p^2}{3sp} = 0.$$

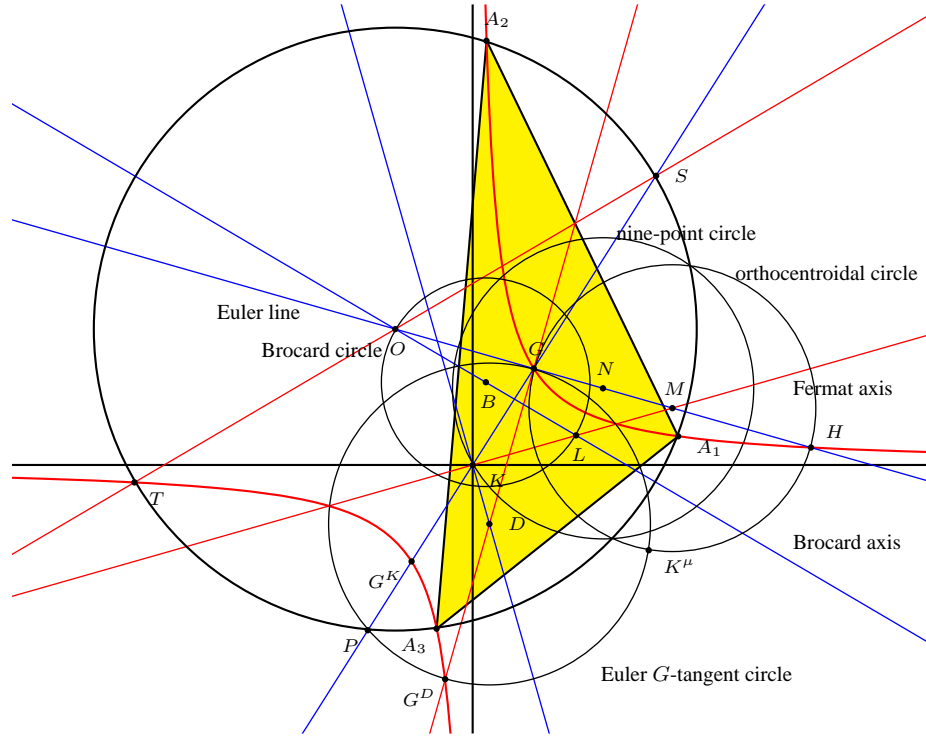


Figure 3.

We shall call  $\mathcal{D}$  the Euler  $G$ -tangent circle because, by construction, it is tangent to the Euler line in  $G$ . This circle will turn out to contain several interesting central points besides  $G$ . For example: the Parry point  $P$  is the second intersection (after  $G$ ) of  $\mathcal{D}$  with the line  $GK$ ; the antipode of  $G$  on  $\mathcal{D}$  is

$$G^D = \left[ \frac{9p}{s^2}, \frac{s^2}{9p} \right]$$

which is clearly a point of  $\mathcal{K}$ . Another point on  $\mathcal{D}$  that we shall meet later is the reflection of  $P$  in the Fermat bisector:

$$K^\mu = \frac{1}{p(s^4 + 81)} [-s(s^3 - 54p - 9sp^2), 3(3s^2 + 2s^3p - 27p^2)].$$

This point (the symbol  $K^\mu$  will be clear later) is collinear with  $G$  and  $L$ . In fact  $K^\mu$  is the second intersection (after  $G$ ) of the circles  $\mathcal{M}$  and  $\mathcal{D}$ , whose radical axis is therefore the line  $GL$ .

Most importantly, the Fermat points will also be shown to lie on  $\mathcal{D}$ .

Lastly, let us consider the *Brocard circle*  $\mathcal{B}$ , centered at  $B$  and defined by its diameter  $OL$ ; rather than writing down its equation, we shall just calculate its radius  $r_B$ . The resulting formula looks rather complicated:

$$r_B^2 = \frac{1}{4} \overline{OL} \cdot \overline{OL} = \frac{(sp + 3)^2 (s^2 - 2s^3p + 81p^2 + s^4p^2 - 18sp^3 + s^2p^4)}{16(3 - sp)^2 s^2 p^2}.$$

But we notice the appearance of the same polynomial which we have found for the circumradius. In fact, we have a very simple ratio of the radii of the Brocard and the nine-point circles:

$$\frac{r_B}{r_N} = \left| \frac{3 + sp}{3 - sp} \right|.$$

We shall find a surprisingly simple geometrical meaning of this ratio in the next section.

## 6. The Steiner inellipse

The Steiner inellipse  $\mathcal{S}$  of a triangle  $\mathbf{T}$  is the unique conic section which is centered at  $G$  and tangent to the sides of  $\mathbf{T}$ . From this definition one calculates the equation:

$$\mathcal{S} : 3x^2 - spy^2 - 2sx + 6py = 0.$$

Since the term in  $xy$  is missing, the axes of  $\mathcal{S}$  (the Steiner axes) are parallel to the asymptotes of the Kiepert hyperbola  $\mathcal{K}$ . Just looking at the equation, we also notice that  $K$  lies on  $\mathcal{S}$  and the line tangent to  $\mathcal{S}$  at  $K$  is parallel to the Fermat bisector (see Figure 4).

By introducing the traditional parameters  $a, b$  for the lengths of the semi-axes, the equation for  $\mathcal{S}$  can be rewritten as

$$\frac{(x - \frac{s}{3})^2}{a^2} + \frac{(y - \frac{3}{s})^2}{b^2} = 1,$$

where  $a^2 = \frac{s^3 - 27p}{9s}$  and  $b^2 = \frac{s^3 - 27p}{-3s^2p}$ .

We cannot distinguish between the major and the minor axis unless we take into account the sign of  $a^2 - b^2 = \frac{(s^3 - 27p)(sp + 3)}{9s^2p}$ . This gives a meaning to the sign of  $t = sp + 3$ , with respect to our reference. In fact we must distinguish two cases:

Case 1:  $t < 0$ ,  $a > b$ : the major Steiner axis is parallel to the  $x$ -axis.

Case 2:  $t > 0$ ,  $a < b$ : the major Steiner axis is parallel to the  $y$ -axis.

Notice that the possibility that  $\mathcal{S}$  is a circle ( $t = 0$ ,  $a = b$ ) has been excluded, as the triangle  $\mathbf{T}$  would be equilateral.

This reduction to cases will appear frequently. For example, we can use a single formula  $2c = 2\sqrt{|a^2 - b^2|}$  for the focal distance, but for the foci  $U_+$ ,  $U_-$  we must write, respectively,

$$U_{\pm} = \begin{cases} \frac{1}{3sp}[s^2p \pm \sqrt{p(3 + sp)(s^3 - 27p)}, 9p], & \text{if } sp + 3 < 0, \\ \frac{1}{3sp}[s^2p, 9p \pm \sqrt{-p(3 + sp)(s^3 - 27p)}], & \text{if } sp + 3 > 0. \end{cases}$$

The number  $u = \sqrt{\frac{-sp}{3}} = \frac{a}{b}$  is the tangent of an angle  $\frac{\alpha}{2}$  which measures the eccentricity  $e$  of  $\mathcal{S}$ . Notice, however, that either  $e^2 = 1 - u^{-2}$  or  $e^2 = 1 - u^2$  according as  $sp + 3 < 0$  or  $> 0$ . What we do not expect is for the number

$$|\cos \alpha| = \frac{|1 - u^2|}{1 + u^2} = \frac{|a^2 - b^2|}{a^2 + b^2} = \left| \frac{3 + sp}{3 - sp} \right|$$

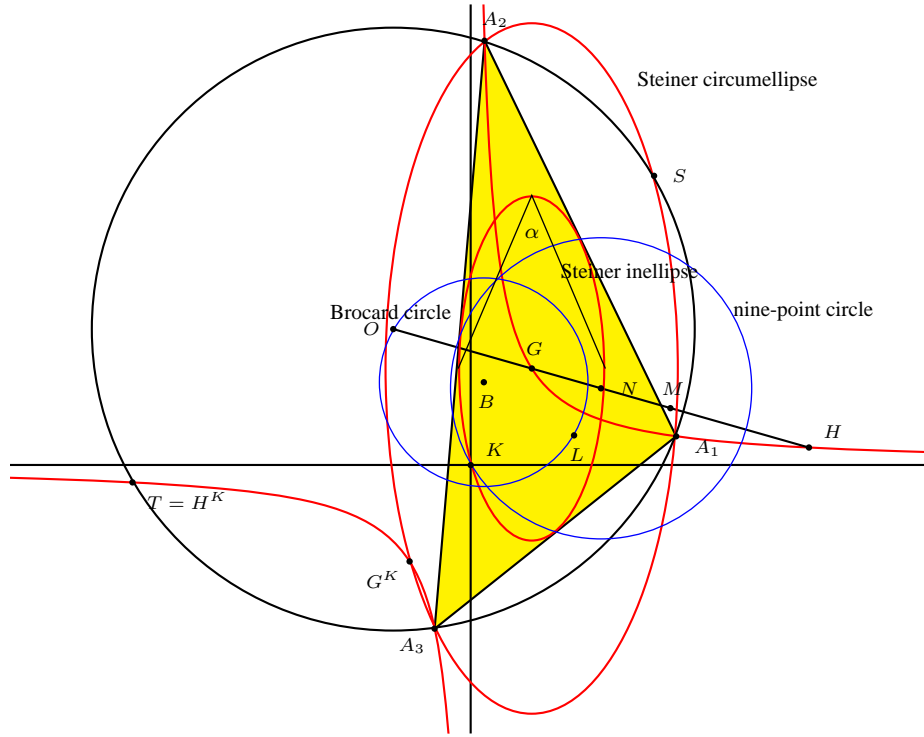


Figure 4.

to be precisely the ratio of the radii that we have found at the end of §5. Taking into account the meaning of  $t = sp + 3$  we conclude that, in any case,

**Theorem 2.** *The ratio between the radii of the Brocard circle and the nine-point circle equals the cosine of the angle under which the minor axis of the Steiner ellipse is viewed from an extreme of the major axis:  $\frac{r_B}{r_N} = |\cos \alpha|$ .*

By applying a homothety of coefficient 2 and fixed point  $G$ , the Steiner inellipse  $S$  is transformed into the Steiner circumellipse. This conic is in fact circumscribed to  $T$  and passes through the Steiner point  $S$ , which is therefore the fourth intersection (after the triangle vertices) of the Steiner circumellipse with the circumcircle  $\mathcal{O}$ . The fourth intersection with  $\mathcal{K}$  is  $S^G = G^K = X_{671}$ .

## 7. The Kiepert parabola and its focus

The Kiepert parabola of a triangle  $T$  is the (unique) parabola  $\mathcal{P}$  which is tangent to the sides of the triangle, and has the Euler line as directrix. By applying this definition one finds for  $\mathcal{P}$  a rather complicated equation:

$$\mathcal{P} : s^2 \left( s \left( x + \frac{s}{3} \right) + 3p \left( y + \frac{3}{s} \right) \right)^2 + \frac{8}{9} ((s^3 - 27p)^2 - 3s(s^4x + 81p^2y)) = 0$$

or

$$s^4x^2 + 9s^2p^2y^2 + 6s^3pxy - 2s^2x(s^3 - 9p) + 2spy((s^3 - 81p) + s^6 - 42s^3p + 729p^3) = 0.$$

From this formula one can check that the tangency points  $E_i$  (on the sides  $A_j A_h$  of  $\mathbf{T}$ ) and the vertices  $A_i$  are perspective; the center of perspective (sometimes called the Brianchon point) is the Steiner point  $S = X_{99}$ . Less well-known, but not difficult to prove, is the fact that the orthocenter of the Steiner triangle  $E_1 E_2 E_3$  is  $O$ , the circumcenter of  $\mathbf{T}$ .

Direct calculations show that the focus of  $\mathcal{P}$  is

$$E = X_{110} = \frac{1}{s(s^2 + 9p^2)}[s(s^3 - 18p + 3sp^2), 3s^2 - 2s^3p + 81p^2].$$

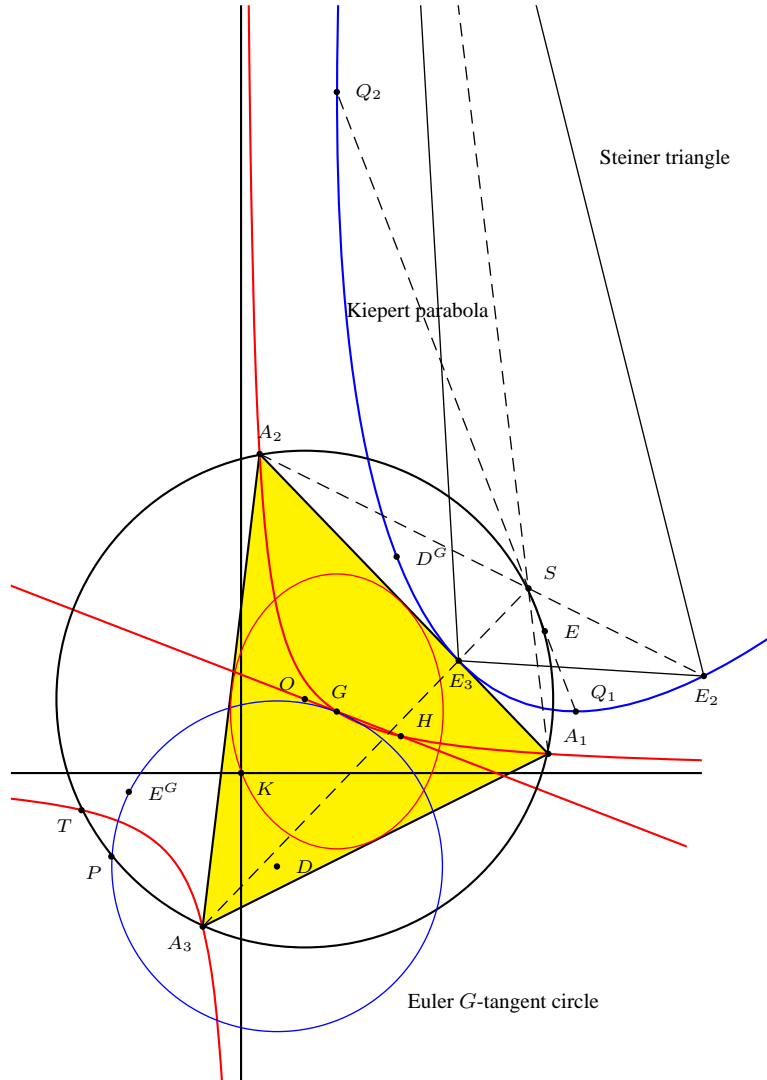


Figure 5.

One can verify that  $E$  is a point of the circumcircle  $\mathcal{O}$ ; this also follows from the well-known fact that, when reflecting the Euler line in the sides of the triangle,

these three lines intersect at  $E$ . Thus  $E$  is the isogonal conjugate of the point at infinity, normal to the Euler line. Further calculations show that  $G, E, T$  are collinear on the line

$$\text{GE :} \quad 3px + sy - 3sp - 3 = 0$$

which is clearly parallel to the Fermat axis. Another well-known collinearity regards the points  $E, L$ , and  $P$ . The proof requires less easy calculations and the equation for this line will not be reported. On the other hand, the line

$$\text{ES :} \quad sx - 3py - \frac{s^3 - 27p}{s} = 0$$

is parallel to the Fermat bisector. This line meets  $\mathcal{P}$  at the points

$$Q_1 = \left[ \frac{s^3 - 18p}{s^2}, \frac{3}{s} \right] \quad \text{and} \quad Q_2 = \left[ \frac{s}{3}, \frac{81p - 2s^3}{9sp} \right],$$

each of which lies on a Steiner axis. When substituting the values  $y = \frac{3}{s}$  or  $x = \frac{s}{3}$  in the equation of  $\mathcal{P}$  one discovers a property that we have not found in literature:

**Theorem 3.** *The axes of the Steiner ellipse of a triangle are tangent to its Kiepert parabola. The tangency points are collinear with the focus and the Steiner point (see Figure 5).*

As a consequence, the images of  $E$  under reflections in the Steiner axes both lie on the Euler line. The relatively poor list of central points which are known to lie on  $\mathcal{P}$  may be enriched, besides by  $Q_1$  and  $Q_2$ , by the addition of

$$D^G = \frac{1}{-18s^2p}[-9p(s^3 - 9p), s(s^3 - 81p)].$$

The tangent to  $\mathcal{P}$  at  $D^G$  is the perpendicular bisector of  $GE$ . We recall that  $D$  was defined in §5 as the center of the  $G$ -tangent circle  $\mathcal{D}$ . The close relation between  $\mathcal{P}$  and  $\mathcal{D}$  is confirmed by the fact that, somehow symmetrically, the point

$$E^G = \frac{1}{3s(s^2 + 9p^2)}[-s(s^3 - 54p - 9sp^2), 3(3s^2 + 2s^3p - 27p^2)]$$

lies on  $\mathcal{D}$ .

## 8. Reflections and angle bisectors

In what follows we shall make frequent use of reflections of vectors in lines parallel to the  $xy$ -axes and write the new coordinates by just changing signs, as explained in §4. Consider, for example,

$$\overline{MG} = G - M = \left[ \frac{s}{3}, \frac{3}{s} \right] - \left[ \frac{sp - 3}{6p}, -\frac{sp - 3}{2s} \right] = \frac{sp + 3}{6sp} [s, 3p]$$

and its reflection in the  $x$ -axis:  $\frac{sp+3}{6sp} [s, -3p]$ . By comparing coordinates, we see that the latter is parallel to the vector

$$\overline{ML} = L - M = \left[ \frac{2s}{3 - sp}, -\frac{6p}{3 - sp} \right] - \left[ \frac{sp - 3}{6p}, -\frac{sp - 3}{2s} \right] = \frac{(3 + sp)^2}{6sp(3 - sp)} [s, -3p].$$

If we apply the same argument to other pairs of vectors, as

we can conclude similarly:

**Theorem 4.** *The inner bisectors of the angles  $\angle GML$  and  $\angle SOL$  are parallel to the minor Steiner axis. The inner bisector of  $\angle EGO$  is the major Steiner axis (see Figure 6).*

These are refinements of well-known statements. Similar results regarding other angles will appear later.

## 9. The Fermat points

We now turn our attention to the Fermat points. For their coordinates we cannot expect to find symmetric polynomials in  $x_1, x_2, x_3$ , as these points are interchanged by an odd permutation of the triangle vertices. In fact, by applying the traditional constructions (through equilateral triangles constructed on the sides of  $\mathbf{T}$ ) one ends up with the twin points

$$\frac{s \cdot V}{2\sqrt{3}p(s^3 - 27p)} \left[ \frac{s}{3}, \frac{-1}{p} \right] \quad \text{and} \quad \frac{s \cdot V}{2\sqrt{3}p(s^3 - 27p)} \left[ -\frac{s}{3}, \frac{1}{p} \right],$$

where  $V$  is the Vandermonde determinant (see §2). We cannot yet tell which is which, but we already see that they both lie on the Fermat axis and their midpoint is the Kiepert center  $K$ . Less obvious, but easy to check analytically, is the fact that both the Fermat points lie on the  $G$ -tangent circle  $\mathcal{D}$ . (Incidentally, this permits a non traditional construction of the Fermat points from the central points  $G, O, K$  via  $M$ ). By squaring and substituting for  $V^2$ , the expressions above can be rewritten as

$$F_+ = \left[ \sqrt{\frac{-s}{3p}}, \sqrt{\frac{-3p}{s}} \right], \quad F_- = \left[ -\sqrt{\frac{-s}{3p}}, -\sqrt{\frac{-3p}{s}} \right].$$

This shows that  $F_+$  and  $F_-$  belong to the Kiepert hyperbola  $\mathcal{K}$  (see Figure 7).

In the next formulas we want to avoid the symbol  $\sqrt{\phantom{x}}$  and use instead the (positive) parameter  $u = \sqrt{\frac{-sp}{3}} = \frac{a}{b}$ , which was introduced in connection with the Steiner inellipse in §6. We know that  $u \neq 1$ . Moreover,  $u > 1$  or  $< 1$  according as  $sp + 3 < 0$  or  $> 0$ . We now want to distinguish between the two Fermat points and claim that

$$F_+ = -\frac{u}{sp}[s, -3p], \quad \text{and} \quad F_- = -\frac{u}{sp}[-s, 3p].$$

Note that  $F_+$  and  $F_-$  are always in the first and third quadrants respectively. This follows by applying the distance inequality  $|GX_{13}| < |GX_{14}|$ , a consequence of their traditional definitions, and only checking the inequality:

$$|GF_-|^2 - |GF_+|^2 = \frac{4u(s^3 - 27p)}{3s^2p^2} > 0.$$

Let us now apply the reflection argument, as described in section 8, to the vectors  $\overline{GF_+}$  and  $\overline{GF_-}$ . We claim that the major Steiner axis is the inner bisector of  $\angle F_+GF_-$ . We shall show, equivalently, that the reflection  $\tau$  in the major axis maps  $\overline{GF_+}$  onto a vector  $\overline{GF_+^\tau}$  that has the same direction and orientation as  $\overline{GF_-}$ . Again, we must treat two cases separately.



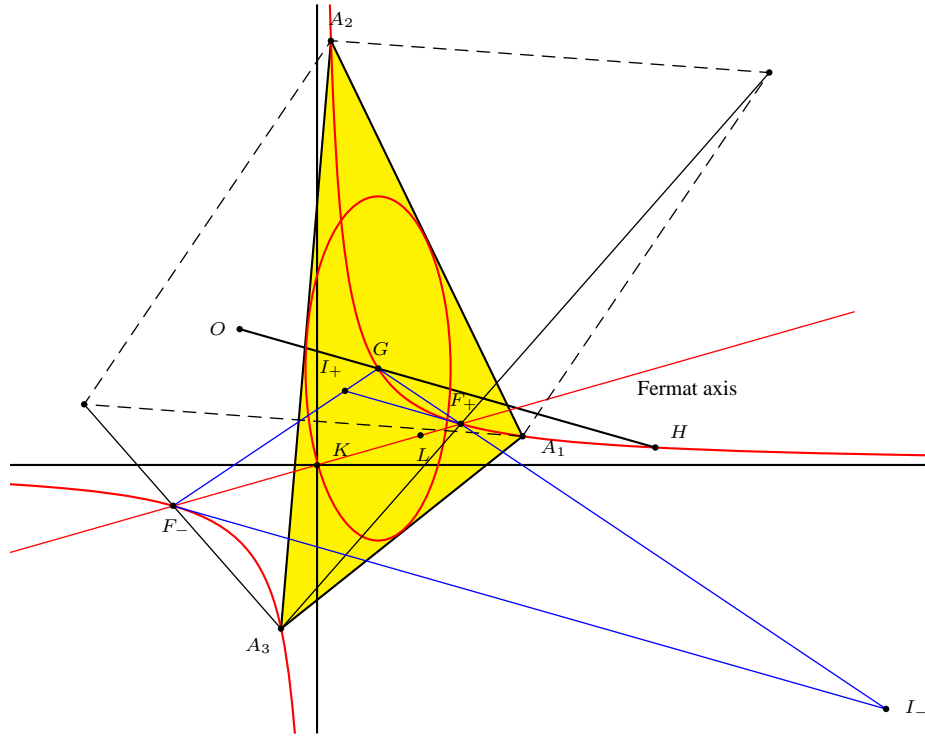


Figure 7.

Case 1:  $a < b$ . In this case  $u < 1$ . The major Steiner axis is parallel to the  $y$ -axis.

$$\begin{aligned}\overline{GF_+} &= F_+ - G = \left[ \frac{-u}{p} - \frac{s}{3}, \frac{3u}{s} - \frac{3}{s} \right], \\ \overline{GF_+^\tau} &= \left[ \frac{u}{p} + \frac{s}{3}, \frac{3u}{s} - \frac{3}{s} \right]; \\ \overline{GF_-} &= \left[ \frac{u}{p} - \frac{s}{3}, \frac{-3u}{s} - \frac{3}{s} \right].\end{aligned}$$

We now calculate both the vector and the scalar products of the last two vectors:

$$\begin{aligned}-\left(\frac{u}{p} + \frac{s}{3}\right)\left(\frac{-3u}{s} - \frac{3}{s}\right) + \left(\frac{3u}{s} - \frac{3}{s}\right)\left(\frac{u}{p} - \frac{s}{3}\right) &= \frac{3u^2}{sp} + \frac{3u}{sp} + u + 1 + \frac{3u^2}{sp} - u - \frac{3u}{sp} + 1 = 0, \\ \left(\frac{u}{p} + \frac{s}{3}\right)\left(\frac{u}{p} - \frac{s}{3}\right) + \left(\frac{3u}{s} - \frac{3}{s}\right)\left(\frac{-3u}{s} - \frac{3}{s}\right) &= \frac{u^2}{p^2} - \frac{s^2}{9} + \frac{9}{s^2} - \frac{9u^2}{s^2} = \frac{(s^3 - 27p)(sp + 3)}{9s^2p} > 0.\end{aligned}$$

Since the last fraction equals  $b^2 - a^2 > 0$ , this is what we wanted. Moreover, since the vectors  $\overline{GF_+^\tau}$  and  $\overline{GF_-}$  share both directions and orientations, we can easily calculate absolute values as follows

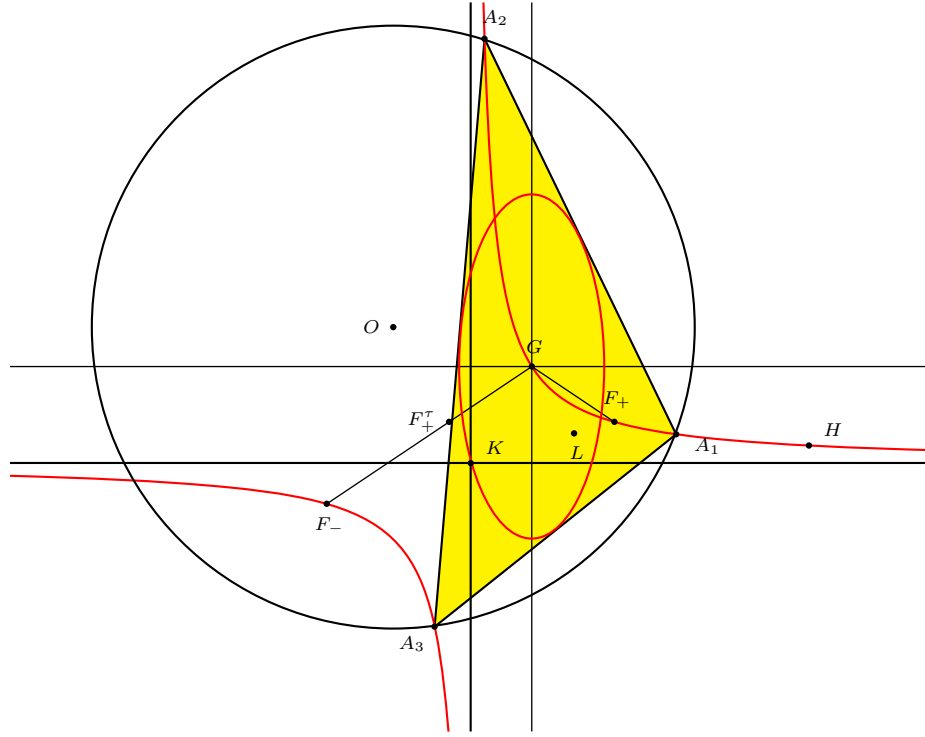


Figure 8.

$$\begin{aligned}
 |GF_+| + |GF_-| &= |\overline{GF_+^{\tau}}| + |\overline{GF_-}| = |\overline{GF_+^{\tau}} + \overline{GF_-}| \\
 &= \left| \left[ \frac{u}{p} + \frac{s}{3}, \frac{3u}{s} - \frac{3}{s} \right] + \left[ \frac{u}{p} - \frac{s}{3}, \frac{-3u}{s} - \frac{3}{s} \right] \right| = \left| \left[ \frac{2u}{p}, \frac{-6}{s} \right] \right| \\
 &= \sqrt{\frac{4u^2}{p^2} + \frac{36}{s^2}} = 2\sqrt{\frac{s^3 - 27p}{3s^2p}} = 2b; \\
 |GF_+| - |GF_-| &= |\overline{GF_+^{\tau}}| - |\overline{GF_-}| = |\overline{GF_+^{\tau}} - \overline{GF_-}| \\
 &= \left| \left[ \frac{u}{p} + \frac{s}{3}, \frac{3u}{s} - \frac{3}{s} \right] - \left[ \frac{u}{p} - \frac{s}{3}, \frac{-3u}{s} - \frac{3}{s} \right] \right| = \left| \left[ \frac{2s}{3}, \frac{6u}{s} \right] \right| \\
 &= \sqrt{\frac{4s^2}{9} + \frac{36u^2}{s^2}} = 2\sqrt{\frac{s^3 - 27p}{9s}} = 2a.
 \end{aligned}$$

The last computation could be spared by deriving the difference from the sum and the product

$$|GF_+||GF_-| = \overline{GF_+^{\tau}} \cdot \overline{GF_-} = \frac{(s^3 - 27p)(3 + sp)}{-9s^2p} = b^2 - a^2.$$

*Case 2:*  $a > b$ . In this case  $u > 1$ . The major Steiner axis is parallel to the  $x$ -axis and  $\overline{GF_+^\tau} = [\frac{-u}{p} - \frac{s}{3}, \frac{-3u}{s} + \frac{3}{s}]$ . Taking products of  $\overline{GF_+^\tau}$  and  $\overline{GF_-}$  leads to similar expressions: the vector product vanishes, while the scalar product equals  $a^2 - b^2 > 0$ . As for absolute values, the results are  $|GF_+| + |GF_-| = 2a$  and  $|GF_-| - |GF_+| = 2b$ , and  $|GF_+||GF_-| = a^2 - b^2$ .

All these results together prove Theorem 1.

## 10. Relations regarding areas

There are some well-known relations regarding areas, which could somehow anticipate the close relation between the Steiner ellipse and the Fermat point, as described in Theorem 1.

The area  $\Delta$  of the triangle  $\mathbf{T} = A_1A_2A_3$  can be calculated from the coordinates of the vertices  $A_i = [x_i, \frac{1}{x_i}]$  as a determinant which reduces to Vandermonde (see §2):

$$\Delta = \left| \frac{V}{2p} \right| = \frac{1}{2} \sqrt{\frac{(s^3 - 27p)^2}{-s^3p}}.$$

If we compare this area with that of the Steiner inellipse  $\mathcal{S}$ , we find that the ratio is invariant:

$$\Delta(\mathcal{S}) = \pi ab = \pi \sqrt{\frac{(s^3 - 27p)}{9s}} \cdot \sqrt{\frac{(s^3 - 27p)}{-s^2p}} = \frac{\pi}{3\sqrt{3}} \sqrt{\frac{V^2}{4p^2}} = \frac{\pi}{3\sqrt{3}} \Delta.$$

There actually exists a more elegant argument to prove this result, based on invariance under affine transformations.

Another famous area relation has to do with the Napoleon triangles  $\mathbf{N}_{\text{ap}+}$  and  $\mathbf{N}_{\text{ap}-}$ . It is well-known that these equilateral triangles are both centered in  $G$  and their circumcircles pass through  $F_-$  and  $F_+$  respectively. Their areas are easily calculated in terms of their radius:

$$\Delta(\mathbf{N}_{\text{ap}+}) = \frac{3\sqrt{3}}{4} |GF_-|^2, \quad \Delta(\mathbf{N}_{\text{ap}-}) = \frac{3\sqrt{3}}{4} |GF_+|^2.$$

The difference turns out to be precisely the area of  $\mathbf{T}$ :

$$\Delta(\mathbf{N}_{\text{ap}+}) - \Delta(\mathbf{N}_{\text{ap}-}) = \frac{3\sqrt{3}}{4} (|GF_-|^2 - |GF_+|^2) = 3\sqrt{3}ab = \sqrt{\frac{V^2}{4p^2}} = \Delta.$$

## 11. An involutory Möbius transformation

Let  $U_+$  and  $U_-$  be the foci of the Steiner inellipse. The focal distance is

$$|U_+U_-| = 2c = 2\sqrt{|a^2 - b^2|} = 2\sqrt{|GF_+||GF_-|}.$$

If we introduce the circle  $\mathcal{U}$ , centered at  $G$ , with  $U_+U_-$  as diameter, we know from §9 that the reflection  $\tau$  in the major axis maps  $\overline{GF_+}$  onto  $\overline{GF_+^\tau}$ , a vector which has the same direction and orientation as  $\overline{GF_-}$ . Furthermore, we know from Theorem 1 that  $c$  is the geometric mean between  $|GF_+|$  and  $|GF_-|$ . This means that the

inversion in the circle  $\mathcal{U}$  maps  $\tau(F_+)$  onto  $F_-$ . Thus  $F_-$  is the inverse in  $\mathcal{U}$  of  $F_+^T$ , the reflection of  $F_+$  in the major axis. We shall denote by  $\mu$  the composite of the reflection  $\tau$  in the major axis of  $\mathcal{S}$  and the inversion in the circle  $\mathcal{U}$  whose diameter is given by the foci of  $\mathcal{S}$ . Note that this composite is independent of the order of the reflection and the inversion.

The mapping  $\mu$  is clearly an involutory Möbius transformation. Its fixed points are the foci, its fixed lines are the Steiner axes. The properties of the mapping  $\mu$  become evident after introducing in the plane a complex coordinate  $z$  such that the foci are the points  $z = 1, z = -1$ . Then  $\mu$  is the complex inversion:  $\mu(z) = \frac{1}{z}$ . What we have proved so far is that  $\mu$  interchanges the Fermat points. But  $\mu$  acts similarly on other pairs of central points. For example, if we go back to the end of §8, we realize that we have partially proved that  $\mu$  interchanges the circumcenter  $O$  with the focus  $E$  of the Kiepert parabola; what we still miss is the equality  $|GE||GO| = c^2$ , which only requires a routine check. In order to describe more examples, let us consider the isodynamic points  $I_+$  and  $I_-$ , namely, the isogonal conjugates of  $F_+$  and  $F_-$  respectively. A straightforward calculation gives for these points:

$$I_+ = \frac{1}{sp(sp+3)}[4s^2p + 3us(1-sp), 12sp^2 + 3up(9-sp)],$$

$$I_- = \frac{1}{sp(sp+3)}[4s^2p - 3us(1-sp), 12sp^2 - 3up(9-sp)].$$

We claim that  $\mu$  interchanges  $I_+$  and  $I_-$ . One can proceed as before: discuss the cases  $u > 1$  and  $u < 1$  separately, reflect  $\overline{GI_+}$  to get  $\overline{GI_+^T}$ , then calculate the vanishing of the vector product of  $\overline{GI_+^T}$  and  $\overline{GI_-}$ , and finally check that the scalar product is  $|GI_+||GI_-| = c^2$ . In the present case, however, one may use an alternative argument. In fact, from the above formulas it is possible to derive several well-known properties such as:

- (i)  $I_+$  and  $I_-$  both lie on the Brocard axis;
- (ii) the lines  $F_+I_+$  and  $F_-I_-$  are both parallel to the Euler line;
- (iii)  $G, F_+, I_-$  are collinear;
- (iv)  $G, F_-, I_+$  are collinear.

These statements imply, in particular, that there is a homothety which has  $G$  as a fixed point and maps  $F_+$  onto  $I_-$ ,  $F_-$  onto  $I_+$  (see Figure 7). Combined with what we know about the Fermat points, this proves that  $\mu$  interchanges  $I_+$  and  $I_-$ , as we wanted.

The next theorem gives a list of  $\mu$ -conjugated objects.

**Theorem 5.** *The mapping  $\mu$  interchanges the following pairs of central points*

circumcenter $O = X_3$	Focus of Kiepert parabola $E = X_{110}$
orthocenter $H = X_4$	Jerabek center $J = X_{125}$
Lemoine point $L = X_6$	Parry point $P = X_{111}$
Fermat point $F_+ = X_{13}$	Fermat point— $F_- = X_{14}$
isodynamic point $I_+ = X_{15}$	isodynamic point $I_- = X_{16}$
center of Brocard circle $X_{182}$	inverse of centroid in circumcircle $U = X_{23}$

and the following pairs of central lines and circles

Euler line: $G, O, H, U, T^\mu$	line: $G, E, J, B, T$
Fermat axis: $M, L, K, F_+, F_-$	Euler G-tangent circle: $G, P, E^G = G^J, K^\mu, F_-, F_+$
circumcircle: $E, P, S, T$	Brocard circle: $O, L, S^\mu, T^\mu$
Fermat bisector: $K, D, J$	orthocentroidal circle: $G, K^\mu, D^\mu, H$
Brocard axis: $O, B, L, I_+, I_-$	Parry circle: $G, E, U, P, I_-, I_+$
line: $D^\mu, K^\mu, M^\mu = E^G, K$	circle: $G, D, K, M, K^\mu$

All the  $\mu$ -coupling of points can be proved through the argument of §8. The  $\mu$ -coupling of lines and circles follows from properties of Möbius transformations. Some central points mentioned in Theorem 5 are not listed in [3] but can be rather naturally characterized:

- (i)  $K^\mu$  (whose coordinates have been calculated in §5) is the reflection of  $P$  in the Fermat bisector and also the second intersection (after  $G$ ) of  $\mathcal{M}$  and  $\mathcal{D}$ ;
- (ii)  $T^\mu$  is the second intersection (after  $O$ ) of the Euler line with the Brocard circle;
- (iii)  $S^\mu$  is the second intersection (after  $L$ ) of the line  $GL$  with the Brocard circle;
- (iv)  $D^\mu$  is the reflection of  $G$  in the Fermat axis;
- (v)  $M^\mu = E^G = G^J$ .

Further well-known properties of homographies can be usefully applied, such as the conservation of orthogonality between lines or circles and the *reflection principle*: if a point  $P$  is reflected (inverted) onto  $Q$  in a line (circle)  $\mathcal{L}$ , then  $P^\mu$  is reflected (inverted) onto  $Q^\mu$  by the line (circle)  $\mathcal{L}^\mu$ . A great number of statements are therefore automatically proved. Here are some examples:

- (i) inversion in the orthocentroidal circle interchanges the Fermat points; it also interchanges  $L$  and  $K$ ;
- (ii) inversion in the Brocard circle interchanges the isodynamic points;
- (iii)  $\mathcal{M}$  and  $\mathcal{D}$  are orthogonal;
- (iv) the Parry circle is orthogonal to both the circumcircle and the Brocard circle, etc.

These statements are surely present in literature but not so easily found.

Among relations which have probably passed unnoticed, we mention equalities of angles, deriving from similarities which also follow from general properties of homographies. Consider any two pairs of  $\mu$ -coupled points, say  $Z_+ \leftrightarrow Z_-$ ,  $W^+ \leftrightarrow W^-$ . Then there exists a direct similarity which fixes  $G$  and simultaneously transforms  $Z_+$  onto  $W^-$  and  $W^+$  onto  $Z_-$ . As a consequence, we are able to recognize a great number of direct similarities (dilative rotations around  $G$ ) between triangles, such as the following pairs:

$$\begin{aligned} (GOF_-, GF_+E), & (GEF_-, GF_+O), (GOI_-, GI_+E), \\ (GOI_+, GI_-E), & (GEL, GPO). \end{aligned}$$

A special case regards the Steiner foci, which are fixed under the action of  $\mu$ . In fact, any pair of  $\mu$ -corresponding points, say  $Z_+, Z_-$ , belong to a circle passing through the foci  $U_+, U_-$ . This cyclic quadrangle  $U_+Z_+U_-Z_-$  is therefore split into two pairs of directly similar triangles having  $G$  as a common vertex:

$GZ_+U_- \leftrightarrow GU_-Z_+$ ,  $GZ_+U_+ \leftrightarrow GU_-Z_-$ . All these similarities can be read in terms of geometric means.

## 12. Construction of the Steiner foci and a proof of Marden's theorem

Conversely, having at disposal the centroid  $G$  and any pair of  $\mu$ -corresponding points, say  $Z_+$ ,  $Z_-$ , the Steiner foci  $U_+$ ,  $U_-$  can be easily constructed (by ruler and compass) through the following simple steps:

- (1) Construct the major and minor Steiner axes, as inner and outer bisectors of  $\angle Z_+GZ_-$ .
- (2) Construct the perpendicular bisector of  $Z_+Z_-$ .
- (3) Find the intersection  $W$  of the line in (2) with the minor axis.
- (4) Construct the circle centered in  $W$ , passing through  $Z_+$ ,  $Z_-$ .

The foci  $U_+$  and  $U_-$  are the intersections of the circle in (4) with the major axis (see Figure 9).

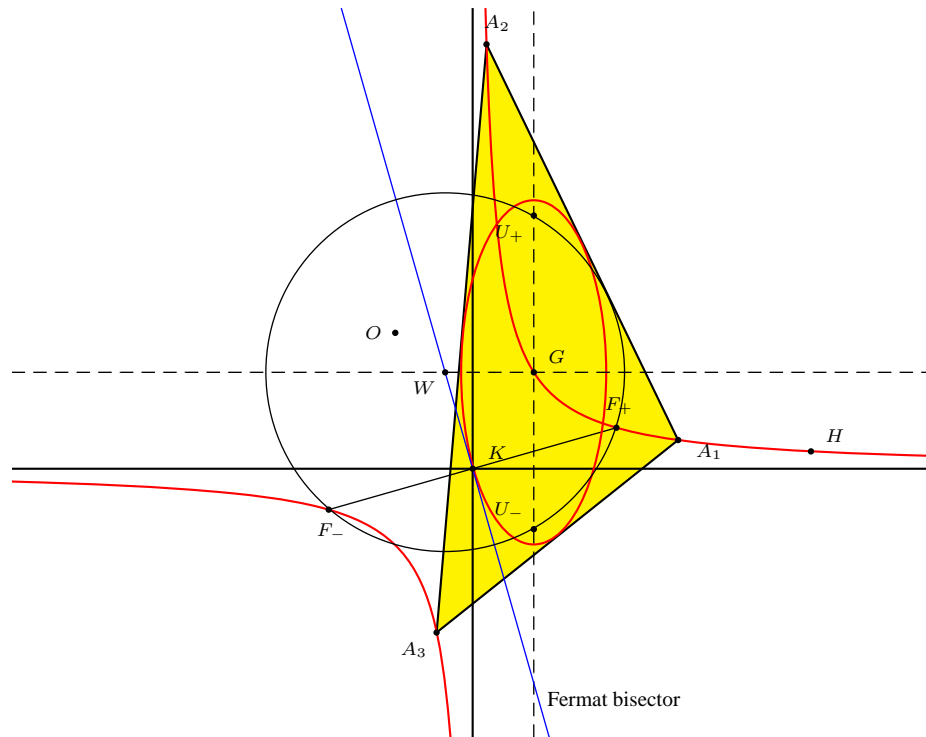


Figure 9.

Avoiding all sorts of calculations, a short synthetic proof of this construction relies on considering the reflection of  $Z_+$  in the major axis and the power of  $G$  with respect to the circle in (4). In particular, by choosing the Fermat points for  $Z_+$  and  $Z_-$ , then we obtain

**Theorem 6.** *The foci of the Steiner inellipse of a triangle are the intersections of the major axis and the circle through the Fermat points and with center on the minor axis.*

A direct analytic proof of this statement is achieved by considering, as usual, separate cases as shown below:

*Case 1:  $a > b$ .*  $W = [\frac{s}{3}, \frac{s^2}{9p}]$ . The circle in (4) has equation

$$x^2 + y^2 - x\frac{2s}{3} - y\frac{2s^2}{9p} + \frac{s^2 + 9p^2}{3sp} = 0,$$

and intersects the line  $y = \frac{3}{s}$ . As expected, we find the foci

$$U_{\pm} = \left[ \frac{s}{3} \pm c, \frac{3}{s} \right],$$

where  $c = \sqrt{a^2 - b^2}$ .

*Case 2:  $a < b$ .*  $W = [\frac{9p}{s^2}, \frac{3}{p}]$ . The circle in (4) has equation

$$x^2 + y^2 - x\frac{18p}{s^2} - y\frac{6}{s} + \frac{s^2 + 9p^2}{3sp} = 0$$

and intersects the line  $x = \frac{s}{3}$ . The foci are the points

$$U_{\pm} = \left[ \frac{s}{3}, \frac{3}{s} \pm c \right],$$

where  $c = \sqrt{b^2 - a^2}$ .

Regarding the Steiner foci, we mention a beautiful result often referred to as Marden's Theorem. If one adopts complex coordinates  $z = x + iy$ , the following curious property was proved by J. Sieberg in 1864 (for this reference and a different proof, see [1]; also [4]). Assume the triangle vertices are  $z_1 = x_1 + iy_1$ ,  $z_2 = x_2 + iy_2$ ,  $z_3 = x_3 + iy_3$ , and let  $F(z) = (z - z_1)(z - z_2)(z - z_3)$ . Then the foci of the Steiner inellipse are the zeros of the derivative  $F'(z)$ .

We shall give a short proof of this statement by applying our Kiepert coordinates. Write

$$\begin{aligned} F(z) &= z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_3, \\ F'(z) &= 3z^2 - 2\sigma_1 z + \sigma_2, \end{aligned}$$

where

$$\begin{aligned} \sigma_1 &= z_1 + z_2 + z_3 = s_1 + i\frac{s_2}{s_3} = s + i\frac{9}{s}, \\ \sigma_2 &= z_1 z_2 + z_2 z_3 + z_3 z_1 = s_2 - \frac{s_1}{s_3} + 6i = \frac{9p^2 - s^2}{sp} + 6i. \end{aligned}$$

Again we have two cases. Assuming, for example,  $a > b$ , we have found for the Steiner foci  $u^\pm = \pm c + \frac{s}{3} + i \cdot \frac{3}{s}$ , where  $c = \sqrt{a^2 - b^2}$ . Therefore what we have to check is just

$$\begin{aligned} u_+ + u_- &= 2\frac{s}{3} + i\frac{6}{s} = \frac{2\sigma_1}{3}, \\ u_+ u_- &= (c + \frac{s}{3})(-c + \frac{s}{3}) - \frac{9}{s^2} + i\frac{3}{s} \cdot \frac{2s}{3} \\ &= \frac{s^2}{9} + 2i - \frac{9}{s^2} - \frac{(s^3 - 27p)(3 + sp)}{9s^2 p} = \frac{\sigma_2}{3}. \end{aligned}$$

When  $a < b$ , the foci are  $u_\pm = \frac{s}{3} + i(\pm c + \frac{3}{s})$ , where  $c = \sqrt{b^2 - a^2}$  and the values of  $u_+ + u_-$  and  $u_+ u_-$  turn out again to be what we wanted.

### 13. Further developments and possible obstacles

There are many other central points, lines and conics that can be conveniently treated analytically within the Kiepert reference, leading to coordinates and coefficients which still belong to the field  $F = Q(s, p)$  or its quadratic extensions. This is the case, for example, for the Napoleon points  $X_{17}$  and  $X_{18}$  which are proved to lie on  $\mathcal{K}$  and be collinear with  $L$ . Incidentally, this line

$$p(27 + sp)x - s(1 + 3sp)y - 16sp = 0$$

turns out to be the radical axis of the orthocentroidal circle and the Lester circle. The Jarabek hyperbola, centered at  $J = X_{125}$ , can also be treated within the Kiepert reference. These points, lines and conics, however, produce relatively complicated formulas. On the other hand, they do not seem to be strictly connected with the involution  $\mu$ , whose action is the main subject of the present paper.

As we said in the Introduction, the Kiepert reference may be inconvenient in dealing with many other problems regarding central points: serious difficulties arise if one tries to treat the incenter, the excenters, the Gergonne and Nagel points, the Feuerbach points and, more generally, any central point whose definition involves the angle bisectors. The main obstacle is the fact that the corresponding coordinates are no longer elements of the fields  $Q(s, q)$  nor of a quadratic extension. Typically, for this family of points one encounters rational functions of  $\sqrt{1 + x_1^2 x_2^2}$ ,  $\sqrt{1 + x_2^2 x_3^2}$ ,  $\sqrt{1 + x_3^2 x_1^2}$ , which can hardly be reduced to the basic parameters  $s = s_1, p = s_3$ .

### 14. Isosceles triangles

In all of the foregoing sections we have left out the possibility that the triangle  $T$  is isosceles, in which case the Kiepert hyperbola  $\mathcal{K}$  degenerates into a pair of orthogonal lines and cannot be represented by the equation  $xy = 1$ . However, unless the triangle is equilateral - an irrelevant case - all results remain true, although most become trivial. To prove such results, instead of the Kiepert reference, one makes use of a Cartesian reference in which the vertices have coordinates

$$A_1 = [-1, 0], \quad A_2 = [1, 0], \quad A_3 = [0, h]$$



and assume  $0 < h \neq \sqrt{3}$ . Thanks to symmetry, all central points turn out to lie on the  $y$ -axis. Here are some examples.

$$\begin{aligned} G &= [0, \frac{h}{3}], & O &= [0, \frac{h^2-1}{2h}], \\ H &= [0, \frac{1}{h}], & M &= [0, \frac{3+h^2}{6h}], \\ N &= [0, \frac{h^2+1}{4h}], & L &= [0, \frac{2h}{3+h^2}], \\ E &= [0, h] = S, & P &= [0, \frac{-1}{h}] = T, \\ F_+ &= [0, \frac{\sqrt{3}}{3}], & F_- &= [0, \frac{-\sqrt{3}}{3}], \\ I_+ &= [0, \frac{\sqrt{3}(h^2+1)-4h}{h^2-3}], & I_- &= [0, \frac{-\sqrt{3}(h^2+1)-4h}{h^2-3}]. \end{aligned}$$

Central lines are the reference axes: either  $x = 0$  (Euler, Fermat, Brocard) or  $y = 0$  (Fermat bisector). Here are the familiar central conics:

$$\begin{aligned} \text{circumcircle:} & \quad x^2 + y^2 - y \frac{h^2-1}{h} - 1 = 0 \\ \text{nine-point circle:} & \quad x^2 + y^2 - y \frac{h^2+1}{2h} = 0 \\ \text{Kiepert hyperbola:} & \quad xy = 0 \\ \text{Kiepert parabola:} & \quad y = h \\ \text{Steiner inellipse:} & \quad \frac{x^2}{a} + \frac{(y-\frac{h}{3})^2}{b^2} = 1, \quad a = \frac{\sqrt{3}}{3}, \quad b = \frac{h}{3} \end{aligned}$$

All the relations between the Fermat points and the ellipse  $\mathcal{S}$  remain true, and proofs still require us to consider separate cases :

*Case 1:*  $h < \sqrt{3}$ .  $a > b$ ;  $c^2 = \frac{h^2-3}{9}$ . The major axis is parallel to the  $x$ - axis.

$$|GF_-| = a + b = \frac{\sqrt{3} + h}{3}, \quad |GF_+| = a - b = \frac{\sqrt{3} - h}{3}.$$

The foci are cut on the line  $y = \frac{h}{3}$  by the circle  $x^2 + y^2 - \frac{1}{3} = 0$ :

$$U_{\pm} = \frac{1}{3}[\pm\sqrt{3-h^2}, h].$$

The mapping  $\mu$  is the product of the reflection in the  $y$ -axis by the inversion in the circle

$$x^2 + (y - \frac{h}{3})^2 - \frac{3-h^2}{9} = 0.$$

*Case 2:*  $h > \sqrt{3}$ .  $a < b$ ;  $c^2 = \frac{3-h^2}{9}$ . The major axis is parallel to the  $y$ - axis;

$$|GF_-| = a + b = \frac{\sqrt{3} + h}{3}, \quad |GF_+| = b - a = \frac{h - \sqrt{3}}{3}.$$

The foci are cut by the same circle on the line  $x = 0$ :

$$U_{\pm} = \frac{1}{3}[0, h \pm \sqrt{h^2-3}].$$

The mapping  $\mu$  is the product of the reflection in the line  $y = \frac{h}{3}$  by the inversion in the circle

$$x^2 + (y - \frac{h}{3})^2 - \frac{h^2-3}{9} = 0.$$

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# Orthic Quadrilaterals of a Convex Quadrilateral

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**Abstract.** We introduce the orthic quadrilaterals of a convex quadrilateral, based on the notion of valtitudes. These orthic quadrilaterals have properties analogous to those of the orthic triangle of a triangle.

## 1. Orthic quadrilaterals

The orthic triangle of a triangle  $T$  is the triangle determined by the feet of the altitudes of  $T$ . The orthic triangle has several and interesting properties (see [2, 4]). In particular, it is the triangle of minimal perimeter inscribed in a given acute-angled triangle (Fagnano’s problem). It is possible to define an analogous notion for quadrilaterals, that is based on the valtitudes of quadrilaterals [6, p.20]. In this case, though, given any quadrilateral we obtain a family of “orthic quadrilaterals”. Precisely, let  $A_1A_2A_3A_4$  be a convex quadrilateral, which from now on we will denote by  $Q$ . We call v-parallelogram of  $Q$  any parallelogram inscribed in  $Q$  and having the sides parallel to the diagonals of  $Q$ . We denote by  $V$  a v-parallelogram of  $Q$  with vertices  $V_i, i = 1, 2, 3, 4$ , on the side  $A_iA_{i+1}$  (with indices taken modulo 4).

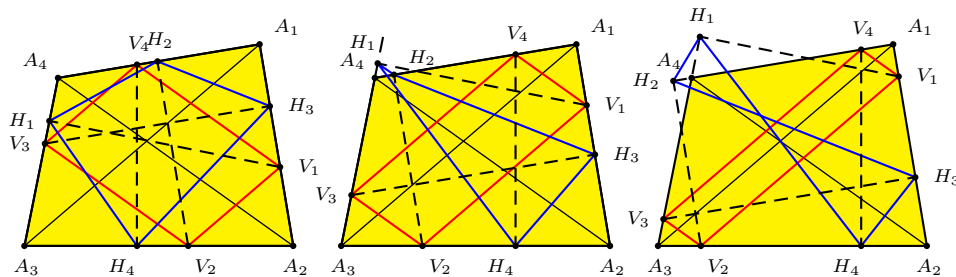


Figure 1.

The v-parallelograms of  $Q$  can be constructed as follows. Fix an arbitrary point  $V_1$  on the segment  $A_1A_2$ . Draw from  $V_1$  the parallel to the diagonal  $A_1A_3$  and let  $V_2$  be the intersection point of this line with the side  $A_2A_3$ . Draw from  $V_2$  the parallel to the diagonal  $A_2A_4$  and let  $V_3$  be the intersection point of this line with the side  $A_3A_4$ . Finally, draw from  $V_3$  the parallel to the diagonal  $A_1A_3$  and let  $V_4$  be the intersection point of this line with the side  $A_4A_1$ . The quadrilateral  $V_1V_2V_3V_4$  is a v-parallelogram ([6, p.19]). By moving  $V_1$  on the segment  $A_1A_2$ , we obtain all possible v-parallelograms of  $Q$ . The v-parallelogram  $M_1M_2M_3M_4$ , with  $M_i$  the midpoint of the segment  $A_iA_{i+1}$ , is the Varignon’s parallelogram of  $Q$ .

Given a  $v$ -parallelogram  $V$  of  $Q$ , let  $H_i$  be the foot of the perpendicular from  $V_i$  to the line  $A_{i+2}A_{i+3}$ . We say that  $H_1H_2H_3H_4$  is an orthic quadrilateral of  $Q$ , and denote it by  $Q_o$ . Note that  $Q_o$  may be convex, concave or self-crossing (see Figure 1). The lines  $V_iH_i$  are called the valtitudes of  $Q$  with respect to  $V$ .

The orthic quadrilateral relative to the Varignon's parallelogram ( $V_i = M_i$ ) will be called principal orthic quadrilateral of  $Q$  and will be denoted by  $Q_{po}$ . The line  $M_iH_i$  is the maltitude of  $Q$  on the side  $A_{i+2}A_{i+3}$  (see Figure 2).

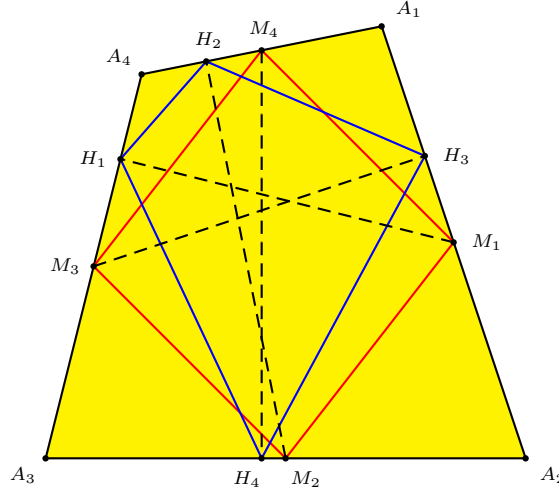


Figure 2.

The study of the orthic quadrilaterals, and in particular of the principal one, allows us to find some properties that are analogous to those of the orthic triangle. In §2 we study the orthic quadrilaterals of an orthogonal quadrilateral, in §3 we consider the case of cyclic and orthodiagonal quadrilaterals. In §4 we find some particular properties of the principal orthic quadrilateral of a cyclic and orthodiagonal quadrilateral. Finally, in §5 we introduce the notion of orthic axis of an orthodiagonal quadrilateral.

## 2. Orthic quadrilaterals of an orthodiagonal quadrilateral

We recall that the maltitudes of  $Q$  are concurrent if and only if  $Q$  is cyclic ([6]). If  $Q$  is cyclic, the point  $H$  of concurrence of the maltitudes is called anticenter of  $Q$  (see Figure 3). Moreover, if  $Q$  is cyclic and orthodiagonal, the anticenter is the common point to the diagonals of  $Q$  (Brahmagupta's theorem, [2, p.44]). In general, if  $Q$  is cyclic, with circumcenter  $O$  and centroid  $G$ , then  $H$  is the symmetric of  $O$  with respect to  $G$ , and the line containing the three points  $H$ ,  $O$  and  $G$  is called Euler line of  $Q$ .

The valtitudes of  $Q$  relative to a  $v$ -parallelogram may concur only if  $Q$  is cyclic or orthodiagonal [6]. Precisely, when  $Q$  is cyclic they concur if and only if they are the maltitudes of  $Q$ . When  $Q$  is orthodiagonal there exists one and only one  $v$ -parallelogram of  $Q$  with concurrent valtitudes. In this case they concur in the

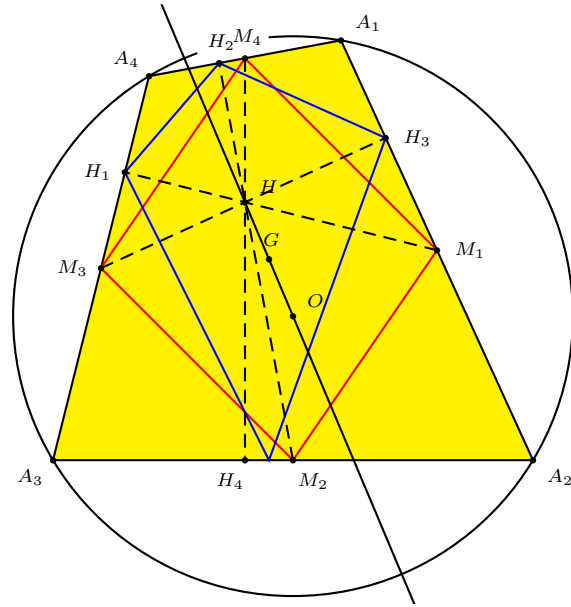


Figure 3.

point  $D$  common to the diagonals of  $\mathbf{Q}$ , and are perpendicular to the sides of  $\mathbf{Q}$  through  $D$ .

**Lemma 1.** *If  $\mathbf{Q}$  is orthodiagonal, the valtitudes  $V_iH_i$  and  $V_{i+1}H_{i+1}$  ( $i = 1, 2, 3, 4$ ) with respect to a  $v$ -parallelogram  $\mathbf{V}$  of  $\mathbf{Q}$  meet on the diagonal  $A_{i+1}A_{i+3}$  of  $\mathbf{Q}$ .*

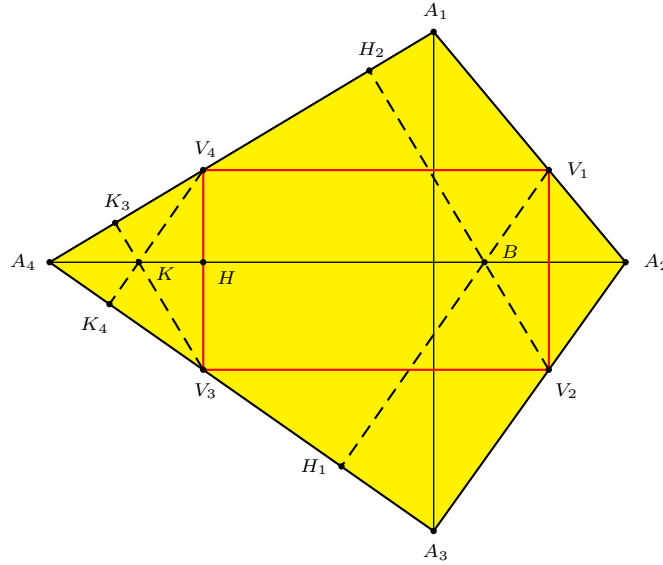


Figure 4.

*Proof.* Let  $\mathbf{Q}$  be orthodiagonal and  $\mathbf{V}$  a  $v$ -parallelogram of  $\mathbf{Q}$ . Let us prove that the valtitudes  $V_1H_1$  and  $V_2H_2$  meet on the line  $A_2A_4$  (see Figure 4). The altitudes  $V_3K_3$ ,  $V_4K_4$ ,  $A_4H$  of triangle  $V_3V_4A_4$  concur at a point  $K$  on the line  $A_2A_4$ . Let  $B$  be the common point to  $V_1H_1$  and  $A_2A_4$ . We prove that  $B$  is on  $V_2H_2$  as well. The quadrilateral  $V_1BKV_4$  is a parallelogram, because its opposite sides are parallel. Thus,  $BK$  is equal and parallel to  $V_1V_4$  and to  $V_2V_3$ , and the quadrilateral  $V_2V_3KB$  is a parallelogram because it has two opposite sides equal and parallel. It follows that  $V_2B$  is parallel to  $V_3K$ , and  $B$  lies on  $V_2H_2$ .

Analogously we can proceed for the other pairs of valtitudes.  $\square$

**Theorem 2.** *Let  $\mathbf{Q}$  be orthodiagonal. Let  $\mathbf{V}$  be a  $v$ -parallelogram of  $\mathbf{Q}$  and  $\mathbf{Q}_0$  be the orthic quadrilateral of  $\mathbf{Q}$  relative to  $\mathbf{V}$ . The vertices of  $\mathbf{V}$  and those of  $\mathbf{Q}_0$  lie on the same circle.*

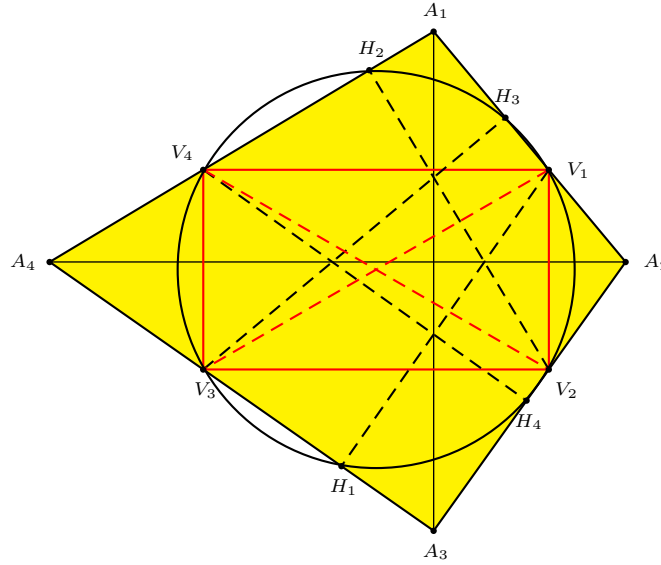


Figure 5.

*Proof.* In fact, since  $\mathbf{Q}$  is orthodiagonal,  $\mathbf{V}$  is a rectangle and it is inscribed in the circle  $\mathcal{C}$  of diameter  $V_1V_3 = V_2V_4$ . The vertices of  $\mathbf{Q}_0$  lie on  $\mathcal{C}$ , because, for example,  $\angle V_1H_1V_3$  is a right angle, and  $H_1$  lie on  $\mathcal{C}$  (see Figure 5).  $\square$

Note that if  $\mathbf{V}$  is the Varignon's parallelogram, the center of the circle  $\mathcal{C}$  is the centroid  $G$  of  $\mathbf{Q}$ . In this case  $\mathcal{C}$  is known as the eight-point circle of  $\mathbf{Q}$  (see [1, 3]).

**Corollary 3.** *If  $\mathbf{Q}$  is orthodiagonal, then each orthic quadrilateral of  $\mathbf{Q}$ , in particular  $\mathbf{Q}_{po}$ , is cyclic.*

### 3. Orthic quadrilaterals of a cyclic and orthodiagonal quadrilateral

The orthic quadrilaterals of  $\mathbf{Q}$  may not be inscribed in  $\mathbf{Q}$ . In particular,  $\mathbf{Q}_{po}$  is inscribed in  $\mathbf{Q}$  if and only if the angles formed by each side of  $\mathbf{Q}$  with the lines joining its endpoints with the midpoint of the opposite side are acute. It follows that if  $\mathbf{Q}$  is cyclic and orthodiagonal, then  $\mathbf{Q}_{po}$  is inscribed in  $\mathbf{Q}$ .

**Theorem 4.** *If  $\mathbf{Q}$  is cyclic and orthodiagonal and  $\mathbf{Q}_o$  is an orthic quadrilateral of  $\mathbf{Q}$  that is inscribed in  $\mathbf{Q}$ , the valtitudes that detect  $\mathbf{Q}_o$  are the internal angle bisectors of  $\mathbf{Q}_o$ .*

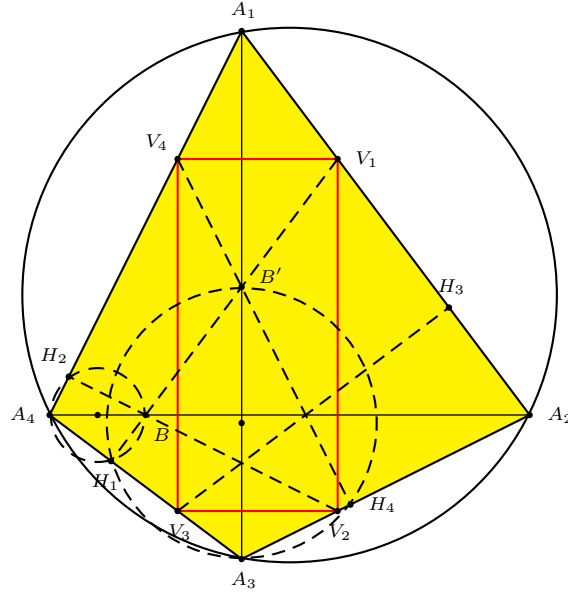


Figure 6.

*Proof.* We prove that the valtitude  $V_1H_1$  is the bisector of  $\angle H_2H_1H_4$  (see Figure 6).

Since  $\mathbf{Q}$  is cyclic, we have

$$\angle A_1A_4A_2 = \angle A_1A_3A_2, \quad (1)$$

because they are subtended by the same arc  $A_1A_2$ . Let  $B$  be the common point to the valtitudes  $V_1H_1$  and  $V_2H_2$  and  $B'$  the common point to the valtitudes  $V_1H_1$  and  $V_4H_4$ . The quadrilateral  $BH_1A_4H_2$  is cyclic because the angles in  $H_1$  and in  $H_2$  are right angles; it follows that

$$\angle H_2H_1B = \angle H_2A_4B, \quad (2)$$

because they are subtended by the same arc  $H_2B$ . Analogously the quadrilateral  $B'H_1A_3H_4$  is cyclic and

$$\angle B'H_1H_4 = \angle B'A_3H_4. \quad (3)$$

But, for Lemma 1,  $\angle H_2A_4B = \angle A_1A_4A_2$  and  $\angle B'A_3H_4 = \angle A_1A_3A_2$ , then, from (1), (2), (3), it follows that  $\angle H_2H_1V_1 = \angle V_1H_1H_4$ .  $\square$

From Corollary 3 and Theorem 4 applied to the case of maltitudes, we obtain

**Corollary 5.** *If  $Q$  is cyclic and orthodiagonal, then  $Q_{po}$  is bicentric and its centers are the centroid and the anticenter of  $Q$  (see Figure 7).*

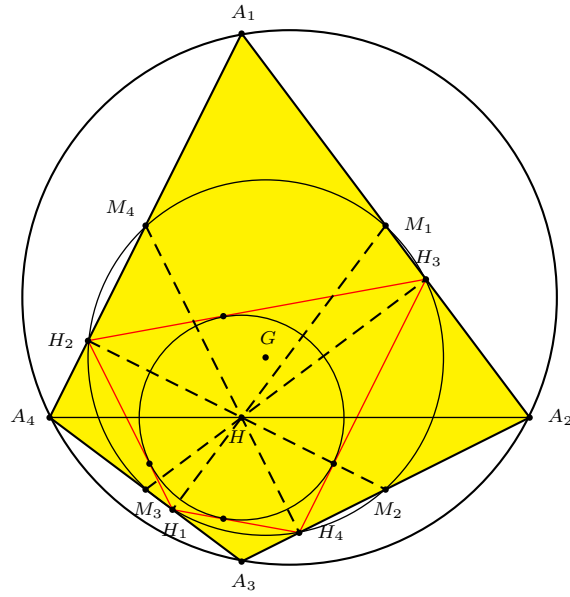


Figure 7.

**Theorem 6.** *If  $Q$  is cyclic and orthodiagonal, the bimedians of  $Q$  are the axes of the diagonals of  $Q_{po}$ .*

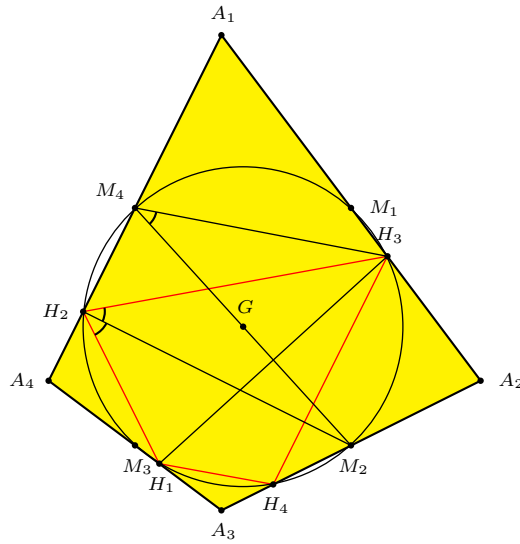


Figure 8.



*Proof.* It is enough to consider the eight-point circle of  $\mathbf{Q}$  and prove that the bi-median  $M_2M_4$  is the axis of the diagonal  $H_1H_3$  of  $\mathbf{Q}_{po}$  (see Figure 8). Note that  $\angle H_3M_4M_2 = \angle H_3H_2M_2$ , because they are subtended by the same arc  $H_3M_2$ . Moreover,  $\angle H_3H_2M_2 = \angle H_1H_2M_2$ , because  $H_2M_2$  bisects  $\angle H_1H_2H_3$  (Theorem 4). It follows that  $\angle H_3M_4M_2 = \angle H_1H_2M_2$ . Then  $M_2$  is the midpoint of the arc  $H_1H_3$  and  $M_2M_4$  is the axis of  $H_1H_3$ .  $\square$

Note that  $M_2$  and  $M_4$  are the midpoints of the two arcs with endpoints  $H_1, H_3$ , and  $M_1, M_3$  are the midpoints of the two arcs with endpoints  $H_2, H_4$ .

**Theorem 7.** *If  $\mathbf{Q}$  is cyclic and orthodiagonal, the orthic quadrilaterals of  $\mathbf{Q}$  inscribed in  $\mathbf{Q}$  have the same perimeter. Moreover, they have the minimum perimeter of any quadrilateral inscribed in  $\mathbf{Q}$ .*

*Proof.* Let  $\mathbf{Q}$  be cyclic and orthodiagonal and let  $\mathbf{Q}_o$  be any orthic quadrilateral of  $\mathbf{Q}$  inscribed in  $\mathbf{Q}$  (see Figure 9). Let  $\bar{\mathbf{Q}}$  be any quadrilateral inscribed in  $\mathbf{Q}$ , different from  $\mathbf{Q}_o$ . In the figure,  $\mathbf{Q}_o$  is the red quadrilateral and  $\bar{\mathbf{Q}}$  is the blue quadrilateral.

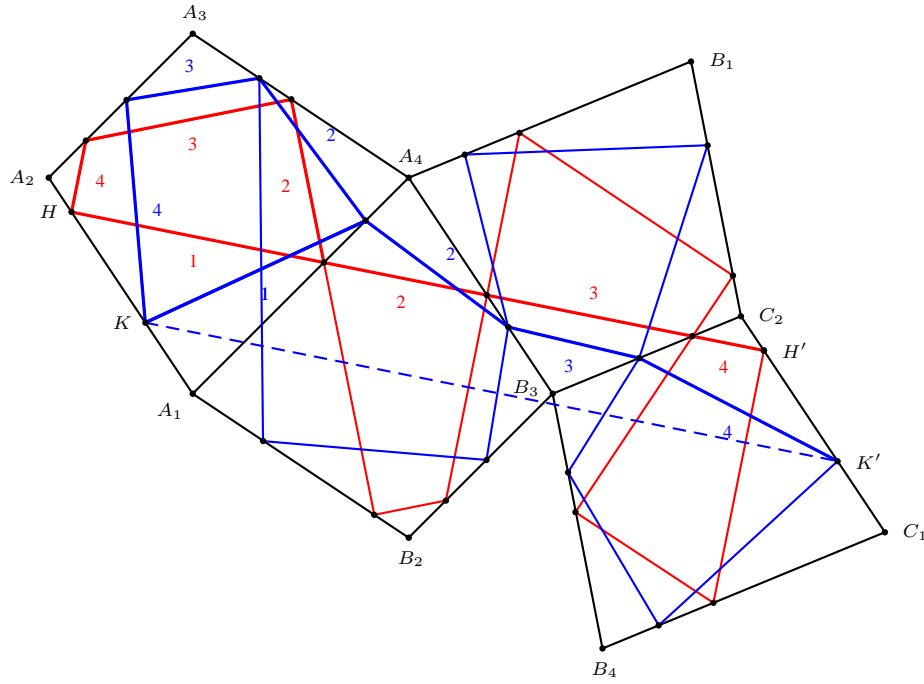


Figure 9.

Let us consider the reflection in the line  $A_1A_4$ , that transforms  $A_1A_2A_3A_4$  in  $A_1B_2B_3A_4$ , the reflection in the line  $B_3A_4$ , that transforms  $A_1B_2B_3A_4$  in  $B_1C_2B_3A_4$ , and the reflection in the line  $C_2B_3$ , that transforms  $B_1C_2B_3A_4$  in

$C_1C_2B_3B_4$ . Let  $H$  and  $K$  be the vertices of  $\mathbf{Q}_o$  and  $\overline{\mathbf{Q}}$  on the segment  $A_1A_2$  respectively, and  $H'$  and  $K'$  the correspondent points of  $H$  and  $K$  in the product of the three reflections.

Let us consider the broken line  $A_2A_1A_4B_3C_2C_1$ . The angles formed by its sides, measured counterclockwise, are  $\angle A_1$ ,  $-\angle A_4$ ,  $\angle A_3$ ,  $-\angle A_2$ . The sum of these angles is equal to zero, because  $\mathbf{Q}$  is cyclic, then the final side  $C_1C_2$  is parallel to  $A_1A_2$ . It follows that the segments  $HH'$  and  $KK'$  are congruent by translation.

For Theorem 4 the valtitudes of  $\mathbf{Q}$  relative to  $\mathbf{Q}_o$  are the internal angles bisectors of  $\mathbf{Q}_o$ , then with the three reflections in the lines  $A_1A_4$ ,  $B_3A_4$  and  $C_2B_3$ , the sides of  $\mathbf{Q}_o$  will lie on the segment  $HH'$ , whose length is then equal to the perimeter of  $\mathbf{Q}_o$ . But, the segment  $HH'$  is equal to the segment  $KK'$ , that has the same endpoints of the broken line formed by the sides of  $\overline{\mathbf{Q}}$ . It follows that the perimeter of  $\overline{\mathbf{Q}}$  is greater than or equal to the one of  $\mathbf{Q}_o$ , then the theorem is proved.  $\square$

#### 4. Properties of the principal orthic quadrilateral of a cyclic and orthodiagonal quadrilateral

Let  $\mathbf{Q}_o$  be an orthic quadrilateral of  $\mathbf{Q}$  inscribed in  $\mathbf{Q}$ . Subtracting from  $\mathbf{Q}$  the quadrilateral  $\mathbf{Q}_o$  we produce the corner triangles  $A_iH_{i+1}H_{i+2}$ , ( $i = 1, 2, 3, 4$ ).

**Lemma 8.** *Let  $\mathbf{Q}$  be cyclic and orthodiagonal and let  $\mathbf{Q}_o$  be an orthic quadrilateral of  $\mathbf{Q}$  inscribed in  $\mathbf{Q}$ . The triangle  $A_iH_{i+1}H_{i+2}$  ( $i = 1, 2, 3, 4$ ) is similar to the triangle  $A_iA_{i+1}A_{i+3}$ .*

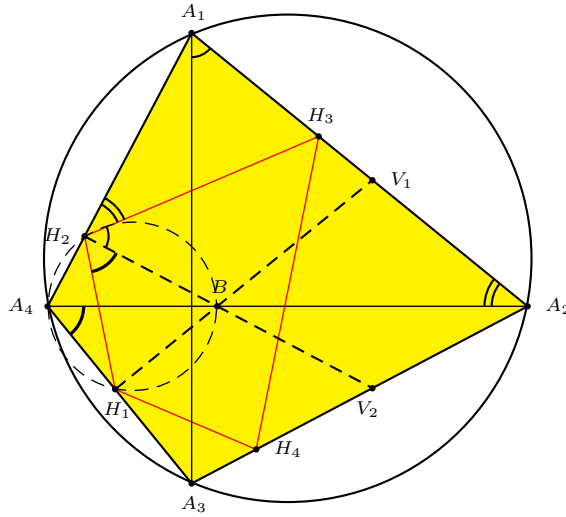


Figure 10.

*Proof.* Let us prove that the triangles  $A_1H_2H_3$  and  $A_1A_2A_4$  are similar. Then all we need to prove is that  $\angle A_1H_2H_3 = \angle A_1A_2A_4$  (see Figure 10).

Let  $B$  be the common point to the altitudes  $V_1H_1$  and  $V_2H_2$ . Since the quadrilateral  $A_4H_1BH_2$  is cyclic, it is  $\angle BH_2H_1 = \angle BA_4H_1$ . Moreover,  $\angle BH_2H_3 = \angle BH_2H_1$ , because the altitude  $V_2H_2$  bisects  $\angle H_1H_2H_3$ . We have  $\angle A_3A_1A_2 = \angle A_2A_4A_3$ , because  $\mathbf{Q}$  is cyclic. Then  $\angle A_3A_1A_2 = \angle BH_2H_3$ . Since  $\angle A_1H_2H_3 = 90^\circ - \angle BH_2H_3$  and  $\angle A_1A_2A_4 = 90^\circ - \angle A_3A_1A_2$ , because  $\mathbf{Q}$  is orthodiagonal, it is  $\angle A_1H_2H_3 = \angle A_1A_2A_4$ .  $\square$

Suppose now that  $\mathbf{Q}$  is cyclic and orthodiagonal. Let us find some properties that hold for the principal orthic quadrilateral  $\mathbf{Q}_{po}$ , but not for any orthic quadrilateral of  $\mathbf{Q}$ .

Consider the quadrilateral  $\mathbf{Q}'$  whose vertices are the points  $A'_i$  in which  $\mathbf{Q}_{po}$  is tangent to its incircle (Corollary 5) and the quadrilateral  $\mathbf{Q}_t$  whose sides are tangent to the circumcircle of  $\mathbf{Q}$  at its vertices. We say that  $\mathbf{Q}_{po}$  is the tangential quadrilateral of  $\mathbf{Q}'$  and  $\mathbf{Q}_t$  is the tangential quadrilateral of  $\mathbf{Q}$ .

**Theorem 9.** *If  $\mathbf{Q}$  is cyclic and orthodiagonal, the quadrilaterals  $\mathbf{Q}'$  and  $\mathbf{Q}$  and the quadrilaterals  $\mathbf{Q}_{po}$  and  $\mathbf{Q}_t$  are correspondent in a homothetic transformation whose center lies on the Euler line of  $\mathbf{Q}$ .*

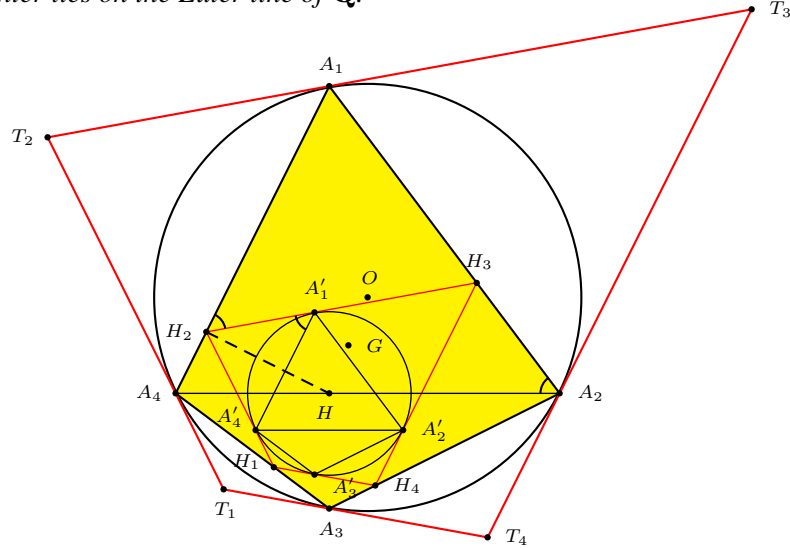


Figure 11.

*Proof.* It suffices to prove that the quadrilaterals  $\mathbf{Q}'$  and  $\mathbf{Q}$  are homothetic (see Figure 11).

Let us start proving that the sides of  $\mathbf{Q}$  are parallel to the sides of  $\mathbf{Q}'$ , for example that  $A_1A_4$  is parallel to  $A'_1A'_4$ . In fact, the altitude  $HH_2$  is perpendicular to  $A_1A_4$ ; moreover, it bisects  $\angle A'_1H_2H_4$ , then it is perpendicular to  $A'_1A'_4$  also, thus  $A_1A_4$  and  $A'_1A'_4$  are parallel. It follows, in particular, that the angles of  $\mathbf{Q}$  are equal to those of  $\mathbf{Q}'$ , precisely  $\angle A_i = \angle A'_i$ .

Let us prove now that the sides of  $Q$  are proportional to the sides of  $Q'$ . It is  $\angle A_1 H_2 H_3 = \angle H_2 A'_1 A'_4$ , because  $A_1 A_4$  and  $A'_1 A'_4$  are parallel, and  $\angle H_2 A' D' = \angle A' B' D'$ , because they are subtended by the same arc  $A' D'$ , then  $\angle A H_2 H_3 = \angle A'_1 A'_2 A'_4$ . It follows that the triangles  $A_1 H_2 H_3$  and  $A'_1 A'_2 A'_4$  are similar. But, for Lemma 8,  $A_1 H_2 H_3$  is similar to  $A_1 A_2 A_4$ , then the triangles  $A_1 A_2 A_4$  and  $A'_1 A'_2 A'_4$  are similar. Analogously it is possible to prove that the triangles  $A_3 A_2 A_4$  and  $A'_3 A'_2 A'_4$  are similar. It follows that the sides of  $Q$  are proportional to the sides of  $Q'$ . Then it is proved that the quadrilaterals  $Q'$  and  $Q$  are homothetic. Finally, the homothetic transformation that transforms  $Q'$  in  $Q$  transforms the circumcenter  $H$  of  $Q'$  in the circumcenter  $O$  of  $Q$ , then the center  $P$  of the homothetic transformation lies on the Euler line of  $Q$ .  $\square$

It is known that given a circumscribable quadrilateral and considered the quadrilateral whose vertices are the points of contact of the incircle with the sides, the diagonals of the two quadrilaterals intersect at the same point (see [5] and [7, p.156]). By applying this result to  $Q_t$  and  $Q$  it follows that the diagonals of  $Q_t$  are concurrent in  $H$ . Thus the common point to the diagonals of  $Q_{po}$ ,  $N$ , lies on the Euler line. Moreover,  $Q_t$  is cyclic, because  $Q_{po}$  is cyclic, and its circumcenter  $T$  lies on the Euler line (see Figure 12).

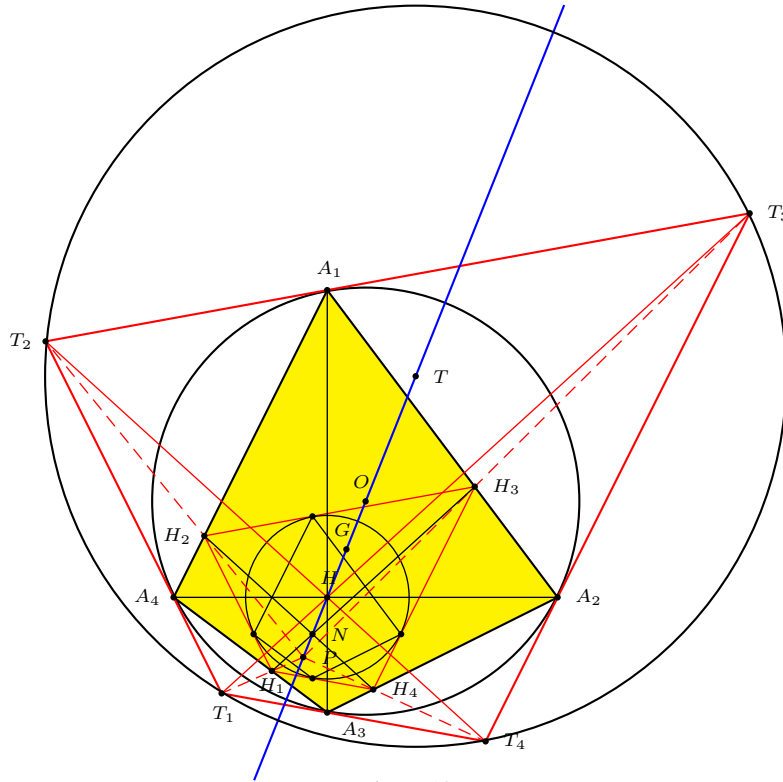


Figure 12.

**Theorem 10.** *If  $Q$  is cyclic and orthodiagonal and  $Q_o$  is an orthic quadrilateral of  $Q$  inscribed in  $Q$ , the perimeter of  $Q_o$  is twice the ratio between the area of  $Q$  and the radius of the circumcircle of  $Q$ .*

*Proof.* In fact, from Theorem 7 all orthic quadrilaterals inscribed in  $Q$  have the same perimeter, then it suffices to prove the property for  $Q_{po}$ . The segments  $H_1H_2$  and  $T_1T_2$  are parallel, because  $Q$  and  $Q_t$  are homothetic, then they both are perpendicular to  $OA_4$ , radius of the circumcircle of  $Q$  (see Figure 13). It follows that the area of the quadrilateral  $OH_1A_4H_2$  is equal to  $\frac{1}{2} \cdot OA_4 \cdot H_1H_2$ .  $\square$

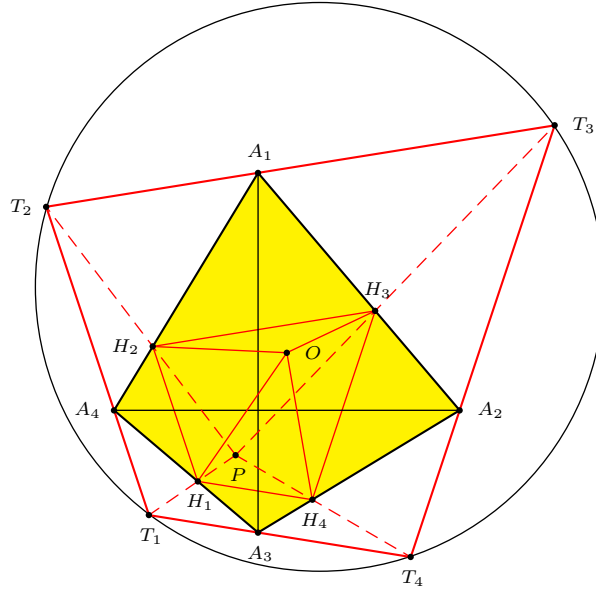


Figure 13.

**Conjecture.** If  $Q$  is cyclic and orthodiagonal, among all orthic quadrilaterals of  $Q$  inscribed in  $Q$  the one of maximum area is  $Q_{po}$ .

The conjecture, which we have been unable to prove, arises from several proofs that we made by using Cabri Géomètre.

## 5. Orthic axis of an orthodiagonal quadrilateral

Suppose that  $Q$  is not a parallelogram. If  $Q$  does not have parallel sides, let  $\mathcal{R}$  be the straight line joining the common points of the lines containing opposite sides of  $Q$ ; if  $Q$  is a trapezium, let  $\mathcal{R}$  be the line parallel to the basis of  $Q$  and passing through the common point of the lines containing the oblique sides of  $Q$ . Let  $Q_o$  be any orthic quadrilateral of  $Q$  and let  $S_i$  ( $i = 1, 2, 3, 4$ ) be the common point of the lines  $H_iH_{i+1}$  and  $V_iV_{i+1}$ , when these lines intersect (see Figure 14).

**Theorem 11.** *If  $Q$  is orthodiagonal and is not a square, for any orthic quadrilateral  $Q_o$  of  $Q$  the points  $S_1, S_2, S_3, S_4$  lie on a line  $\mathcal{R}$ .*

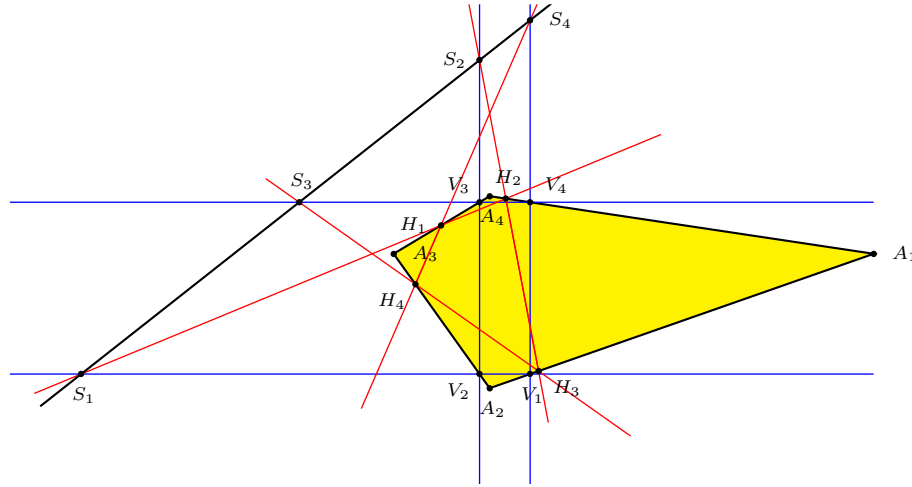


Figure 14.

*Proof.* Set up an orthogonal coordinate system whose axes are the diagonals of  $\mathbf{Q}$ ; then the vertices of  $\mathbf{Q}$  have coordinates  $A_1 = (a_1, 0)$ ,  $A_2 = (0, a_2)$ ,  $A_3 = (a_3, 0)$ ,  $A_4 = (0, a_4)$ . The equation of line  $\mathcal{R}$  is

$$a_2a_4(a_1 + a_3)x + a_1a_3(a_2 + a_4)y - 2a_1a_2a_3a_4 = 0. \quad (4)$$

If  $\mathbf{V}$  is a v-parallelogram of  $\mathbf{Q}$ , with  $x$ -coordinate  $\alpha$  for the vertex  $V_1$ , then

$$\begin{aligned} S_1 &= \left( \frac{a_3(a_2(\alpha - a_1) + a_4(\alpha + a_1))}{a_4(a_1 + a_3)}, \frac{a_2(a_1 - \alpha)}{a_1} \right), \\ S_2 &= \left( \frac{\alpha a_3}{a_1}, \frac{a_2a_4(2a_1^2 - \alpha(a_1 + a_3))}{a_1^2(a_2 + a_4)} \right), \\ S_3 &= \left( \frac{a_3(a_2(\alpha + a_1) + a_4(\alpha - a_1))}{a_2(a_1 + a_3)}, \frac{a_4(a_1 - \alpha)}{a_1} \right), \\ S_4 &= \left( \alpha, \frac{a_2a_4(2a_1a_3 - \alpha(a_1 + a_3))}{a_1a_3(a_2 + a_4)} \right). \end{aligned}$$

It is not hard to verify that the coordinates of the points  $S_i$  satisfy (4).  $\square$

We call the line  $\mathcal{R}$  the orthic axis of  $\mathbf{Q}$ . It is possible to verify that if  $\mathbf{Q}$  is cyclic and orthodiagonal, i.e.,  $a_1a_3 = a_2a_4$ , the orthic axis of  $\mathbf{Q}$  is perpendicular to the Euler line of  $\mathbf{Q}$  (see Figure 15). Moreover, it is known that in a cyclic quadrilateral  $\mathbf{Q}$  without parallel sides the tangent lines to the circumcircle of  $\mathbf{Q}$  in two opposite vertices meet on the line joining the common points of the lines containing the opposite sides of  $\mathbf{Q}$  (see [2, p. 76]). It follows that if  $\mathbf{Q}$  is cyclic and orthodiagonal and it has not parallel sides, the common points to the lines tangent to the circumcircle of  $\mathbf{Q}$  in the opposite vertices of  $\mathbf{Q}$  lie on the orthic axis of  $\mathbf{Q}$ .

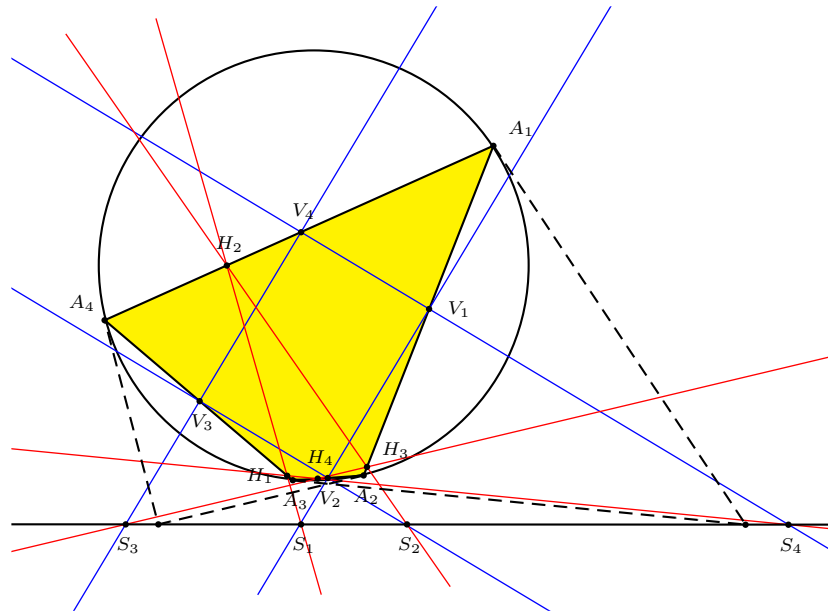


Figure 15.

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# Diophantine Steiner Triples and Pythagorean-Type Triangles

Bojan Hvala

**Abstract.** We present a connection between Diophantine Steiner triples (integer triples related to configurations of two circles, the larger containing the smaller, in which the Steiner chain closes) and integer-sided triangles with an angle of  $60^\circ$ ,  $90^\circ$  or  $120^\circ$ . We introduce an explicit formula and provide a geometrical interpretation.

## 1. Introduction

In [3] we described all integer triples  $(R, r, d)$ ,  $R > r + d$ , for which a configuration of two circles of radii  $R$  and  $r$  with the centers  $d$  apart possesses a closed Steiner chain. This means that there exists a cyclic sequence of  $n$  circles  $L_1, \dots, L_n$  each tangent to the two circles of radii  $R$  and  $r$ , and to its two neighbors in the sequence. Such triples are called *Diophantine Steiner (DS) triples*. For obvious reasons the consideration can be limited to *primitive* DS triples, *i.e.*, DS triples with  $\gcd(R, r, d) = 1$ . We also proved in [3] that the only possible length of a Steiner chain in a DS triple is 3, 4 or 6. Therefore, the set of primitive DS triples can be divided into three disjoint sets  $DS_n$  for  $n = 3, 4, 6$ . The elements of these sets are solutions of the following Diophantine equations:

$n$	relation
3	$R^2 - 14Rr + r^2 - d^2 = 0,$
4	$R^2 - 6Rr + r^2 - d^2 = 0,$
6	$3R^2 - 10Rr + 3r^2 - 3d^2 = 0.$

The sequence of  $R$  in  $DS_4$  is

$$6, 15, 20, 28, \dots$$

In the *ENCYCLOPEDIA OF INTEGER SEQUENCES (EIS)* [6], this is the sequence A020886 of semi-perimeters of Pythagorean triangles. This suggested to us that  $DS_4$  might be closely connected with the Pythagorean triangles. It turns out that in the same manner the sets  $DS_3$  and  $DS_6$  are connected with integer sided triangles having an angle of  $120^\circ$  or of  $60^\circ$ , respectively. Such triangles were considered in



papers [1, 4, 5]. Together with Pythagorean triangles, these form a set of triangles that we will call *Pythagorean-type triangles*.

It is surprising that bijective correspondences between three pairs of triples sets are given by the same formula (Theorem 1 below). It is the purpose of this paper to present this formula, provide a geometrical interpretation and derive some further curiosities.

## 2. Bijective correspondence between the sets $Q_\varphi$ and $DS_n$

The sides of Pythagorean-type triangles form three sets of triples, which we denote by  $Q_{60}$ ,  $Q_{90}$  and  $Q_{120}$  respectively. The set  $Q_\varphi$  contains all primitive integer triples  $(a, b, c)$  such that a triangle with the sides  $a, b, c$  contains the angle  $\varphi$  degrees opposite to side  $c$ . We also require  $b > a$ . (This excludes the triple  $(1, 1, 1)$  from  $Q_{60}$  and avoids duplication of triples with the roles of  $a$  and  $b$  interchanged.)

It is also convenient to slightly modify the sets  $Q_{60}$  and  $Q_{120}$  to sets  $Q'_{60}$  and  $Q'_{120}$  as follows: triple  $(a, b, c)$  with three odd numbers  $a, b, c$  is replaced with a triple  $(2a, 2b, 2c)$ . Other triples remain unchanged. Modification in  $Q_{90}$  is not necessary, since primitive Pythagorean triples always include exactly one even number.

**Theorem 1.** *The correspondences  $DS_4 \longleftrightarrow Q_{90}$ ,  $DS_3 \longleftrightarrow Q'_{120}$  and  $DS_6 \longleftrightarrow Q'_{60}$  given by*

$$(R, r, d) \mapsto \left(\frac{1}{2}(R + r - d), \frac{1}{2}(R + r + d), R - r\right) \quad (1)$$

$$(a, b, c) \mapsto \left(\frac{1}{2}(a + b + c), \frac{1}{2}(a + b - c), b - a\right) \quad (2)$$

*are bijective and inverse to each other.*

*Proof.* It is straightforward that the above maps are mutually inverse and that they map the solution  $(R, r, d)$  of the equation  $R^2 - 6Rr + r^2 - d^2 = 0$  into the solution  $(a, b, c)$  of the equation  $a^2 + b^2 - c^2 = 0$ , and vice versa. The same could be proved for the pair  $R^2 - 14Rr + r^2 - d^2 = 0$  and  $a^2 + b^2 + ab - c^2 = 0$ , as well as for the pair  $3R^2 - 10Rr + 3r^2 - 3d^2 = 0$  and  $a^2 + b^2 - ab - c^2 = 0$ .

Using standard arguments, we also prove that the given primitive triple of  $DS_4$  corresponds to the primitive triple of  $Q_{90}$ , and vice versa. In the other two cases, consideration is similar but with a slight difference: the triples from  $Q_{60}$  and  $Q_{120}$  can have three odd components; therefore, the multiplication by 2 was needed. Now we prove that triples  $(R, r, d)$  from  $DS_3$  and  $DS_6$  with an even  $d$  correspond to the modified triples of  $Q'_{60}$  and  $Q'_{120}$  of the form  $(2a, 2b, 2c)$ ,  $a, b, c$  being odd; and triples with odd  $d$  correspond to the untouched triples of  $Q'_{60}$  and  $Q'_{120}$ . In each case, the primitiveness of the triples from  $Q_{60}$  and  $Q_{120}$  implies the primitiveness of those from  $DS_3$  and  $DS_6$ , and vice versa.  $\square$

*Remark.* Without restriction to integer values, these correspondences extend to the configurations  $(R, r, d)$  with Steiner chains of length  $n = 3, 4, 6$  and triangles containing an angle  $\frac{180^\circ}{n}$ .

We present a geometrical interpretation of the relations (1) and (2). Let  $(R, r, d)$  be a DS triple from  $DS_n$ ,  $n \in \{3, 4, 6\}$ . Beginning with two points  $S_1, S_2$  at a distance  $d$  apart, we construct two circles  $S_1(R)$  and  $S_2(r)$ . Let the line  $S_1S_2$  intersect the circle  $S_1(R)$  at the points  $U, V$  and  $S_2(r)$  at  $W$  and  $Z$ . On opposite sides of  $S_1S_2$ , construct two similar isosceles triangles  $VUI_c$  and  $WZI$  on the segments  $UV$  and  $WZ$ , with angle  $\frac{180^\circ}{n}$  between the legs. Complete the triangle  $ABC$  with  $I$  as incenter  $I$ . Then  $I_c$  is the excenter on the side  $c$  along the line  $S_1S_2$ . This is the corresponding Pythagorean-type triangle (see Figure 1 for the case of  $n = 4$ ). To prove this, it is enough to show that the sides of triangle  $ABC$  are  $a = \frac{1}{2}(R + r - d)$ ,  $b = \frac{1}{2}(R + r + d)$ ,  $c = R - r$ , i.e. the sides given in (1).

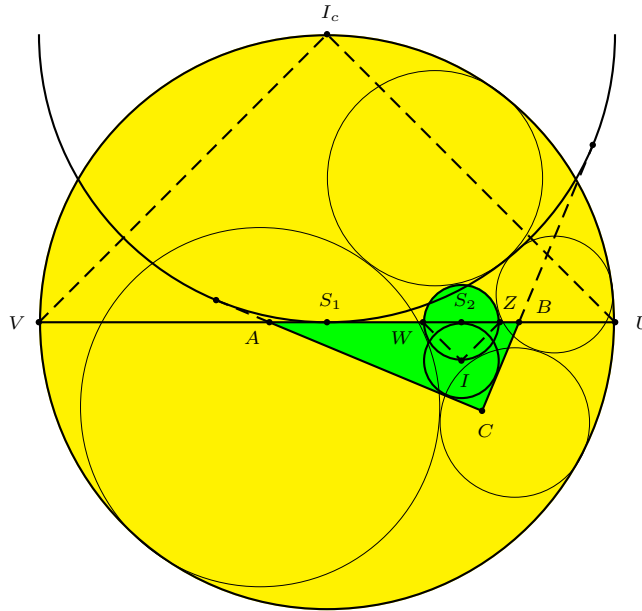


Figure 1.

$$\begin{aligned} a &= \frac{1}{2} \left( \sqrt{4r_i r_c + d^2} \cdot \frac{r_c + r_i}{r_c - r_i} - d \right), \\ b &= \frac{1}{2} \left( \sqrt{4r_i r_c + d^2} \cdot \frac{r_c + r_i}{r_c - r_i} + d \right), \\ c &= \sqrt{4r_i r_c + d^2}. \end{aligned}$$

Now let us consider different  $n \in \{3, 4, 6\}$ . In the case  $n = 3$ , according to the construction,  $r_i = r\sqrt{3}$ ,  $r_c = R\sqrt{3}$  and  $4r_i r_c + d^2 = 12Rr + d^2$ . Since triples from  $DS_3$  satisfy  $R^2 - 14Rr + r^2 - d^2 = 0$ , we have  $12Rr + d^2 = (R - r)^2$ . Hence,  $\sqrt{4r_i r_c + d^2} = R - r$ .

For  $n = 4$  and  $n = 6$ , we have different  $r_i$  and  $r_c$  and apply different Diophantine equations, but end up with the same value of the square root. Applying all these to the formulas above, we get the desired sides  $a, b, c$ .

#### 4. The relation between sets $DS_3$ and $DS_6$

In [3] we found an injective (but not surjective) map from  $DS_3$  to  $DS_6$ . In this section, we will explain the background and provide a geometrical interpretation of this relation. In §2, we have the bijections  $DS_3 \leftrightarrow Q'_{120} \leftrightarrow Q_{120}$  and  $DS_6 \leftrightarrow Q'_{60} \leftrightarrow Q_{60}$ . Besides, it is clear from Figure 2 that the map  $(a, b, c) \mapsto (a, a + b, c)$  represent an injective map from  $Q_{120}$  to  $Q_{60}$ .

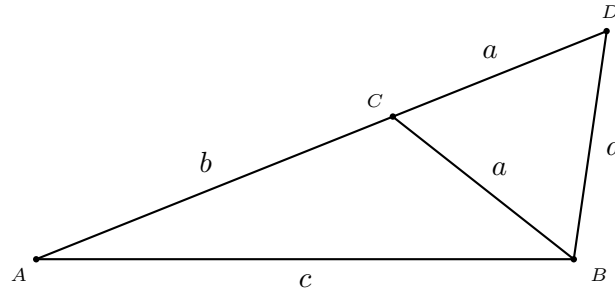


Figure 2.

The same is true for the map  $(a, b, c) \mapsto (b, a + b, c)$ . We therefore have two maps  $Q_{120} \rightarrow Q_{60}$ , the union of their disjoint images being the whole  $Q_{60}$ . Therefore, the sequence of maps  $DS_3 \rightarrow Q'_{120} \rightarrow Q_{120} \rightarrow Q_{60} \rightarrow Q'_{60} \rightarrow DS_6$  defines two maps  $DS_3 \rightarrow DS_6$ . Following step by step, we can easily find both explicit formulas:

$$\begin{aligned} g_1(R, r, d) &= k \cdot \left( \frac{1}{4}(5R + r - d), \frac{1}{4}(R + 5r - d), \frac{1}{2}(R + r + d) \right) \\ g_2(R, r, d) &= k \cdot \left( \frac{1}{4}(5R + r + d), \frac{1}{4}(R + 5r + d), \frac{1}{2}(R + r - d) \right) \end{aligned}$$

with the appropriate factor  $k \in \{2, 1, \frac{1}{2}\}$ .

The correspondence noticed in [3] is, in fact, just  $g_2$  with the chosen maximal possible factor  $k = 2$  (in multiplying by a larger factor, we only lose primitiveness). The existence of two maps  $g_1$  and  $g_2$  whose images cover  $DS_6$  explains why the image of  $g_2$  alone covered only “one half” of  $DS_6$ .

Now we give a geometric interpretation of these maps. Let us start with a triple  $(R, r, d) \in DS_3$  and construct the associated triangle  $ABC$  from  $Q'_{120}$ . According to (1), the sides  $a$  and  $b$  of this triangle are  $a = \frac{R+r-d}{2}$  and  $b = \frac{R+r+d}{2}$ . Hence,  $g_1(R, r, d) = k \cdot (R + \frac{a}{2}, r + \frac{a}{2}, b)$ . To get the first possible configuration of two circles with the closed Steiner triple of the length  $n = 6$ , we draw circles with centers  $A$  and  $C$  with the radii  $R' = R + \frac{a}{2}$  and  $r' = r + \frac{a}{2}$  (see Figure 3). Similarly, drawing the circles with centers  $B$  and  $C$  with the radii  $R' = R + \frac{b}{2}$  and  $r' = r + \frac{b}{2}$ ,

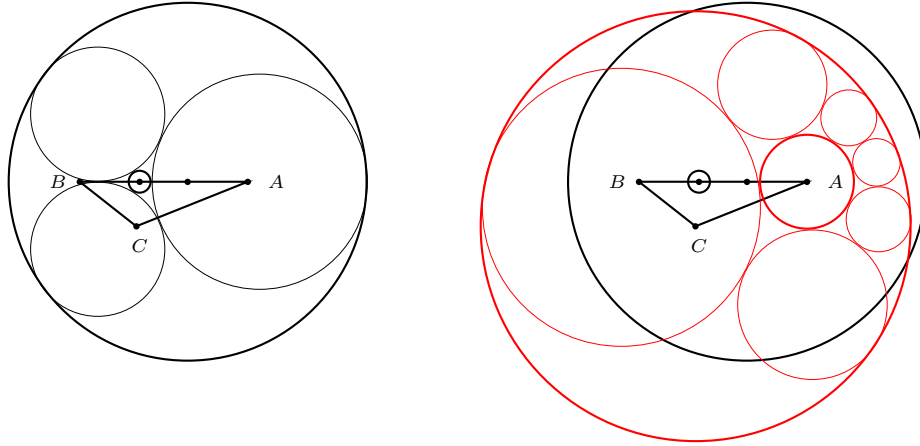


Figure 3.

we get the second possibility, arising from  $g_2$ . To obtain triples in  $DS_6$ , we must consider the effect of  $k$ : *i.e.*, it is possible that the elements  $(R', r', d')$  need to be multiplied or divided by 2.

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# Three Maximal Cyclic Quadrangles

Paris Pamfilos

**Abstract.** We study three problems concerning the determination of the maximal cyclic quadrangle under certain restrictions on its diagonals: (a) assuming that one of the diagonals as well as the intersection point of the two diagonals are fixed, (b) assuming that the location of the intersection point of the two diagonals is fixed, and (c) assuming that the intersection point of the diagonals is fixed and the diagonals intersect under a constant angle.

## 1. Introduction

Consider a convex cyclic quadrangle  $ABCD$  and assume that one of its diagonals  $AC$  is a diameter of its circumcircle. Assume also that the intersection point  $E$  of the two diagonals is a fixed point and that the second diagonal  $BD$  turns about that point. For which position of the turning diagonal does the resulting quadrangle have maximum area? Figure 1 displays such a case of maximal quadrangle indicating that the solution is asymmetric, whereas changing the position of  $E$  on  $AC$  towards the center we do obtain symmetric maximal quadrangles (kites).

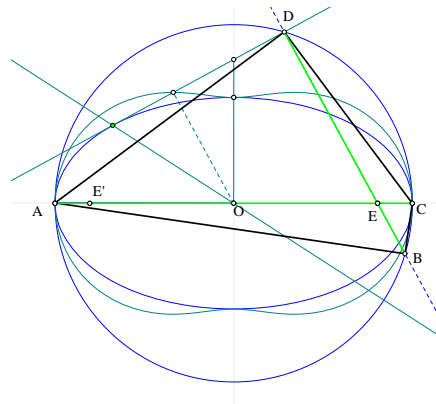


Figure 1. Maximal for fixed  $AC$  and  $E$

Figure 1 also displays a nice quartic curve which controls this behavior. §2 below explores the geometric structure of the problem and establishes its connection to the aforementioned quartic. §3 solves the problem of finding the maximal cyclic quadrangles in the two cases:

- (a) one diagonal  $AC$  and the intersection  $E$  of the diagonals are held fixed, and
- (b) only the intersection  $E$  is held fixed.

§4 handles the problem of finding the maximal cyclic quadrangle under the condition that its diagonals pass through a given point and intersect there at a fixed angle.

## 2. The Hippopede of Proclus

The basic configuration in this section is suggested by Figure 2. In this a chord  $AC$  turns about a fixed point  $E$  lying inside the circle  $c$ . The segment  $OG$  drawn from the center of the circle parallel and equal to this chord has its other end-point  $G$  moving on a curve called the *Hippopede of Proclus* [7, p. 88], [8, p. 35], [3] or the *Lemniscate of Booth* [2, vol.II, p. 163], [5, p.127].

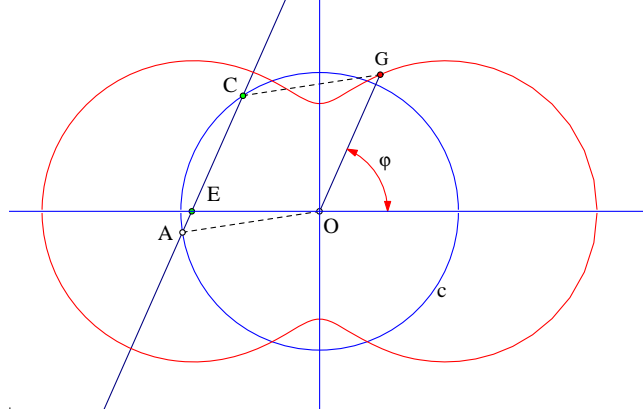


Figure 2. Hippopede of Proclus

This curve is a *quartic* described by the equation

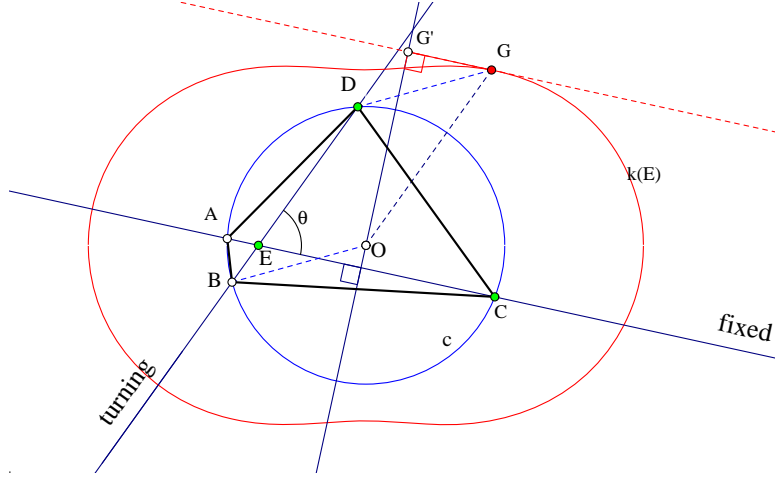
$$(x^2 + y^2)^2 - 4R^2(x^2 + (1 - s^2)y^2) = 0.$$

Here  $s \in (0, 1)$  defines the distance  $OE = sR$  in terms of the radius  $R$  of the circle. A compendium of the various aspects of this interesting curve can be seen in [3]. Regarding the maximum quadrangle the following fact results immediately.

**Proposition 1.** *Let  $ABCD$  be a cyclic quadrangle,  $E$  be the intersection point of its diagonals and  $k(E)$  the corresponding hippopede. Let the positions of the diagonal  $AC$  and  $E$  be fixed and consider the other diagonal  $BD$  turning about  $E$ . Further, let  $G$  be the corresponding point on the hippopede, such that  $OG$  is parallel and equal and equally oriented to  $BD$ . Then the quadrangle attains its maximum area when the tangent  $t_G$  at  $G$  is parallel to the fixed diagonal  $AC$  and at maximum distance from the center  $O$  (see Figure 3).*

The proof follows trivially from the formula  $|AC| \cdot |BD| \sin \theta$  expressing the area of the quadrangle in terms of the diagonals and their angle. Since  $AC$  is fixed the problem reduces to maximizing the product  $|BD| \sin \theta$ , which is the length of the projection of  $OG$  on the axis orthogonal to  $AC$ . The claim follows at once.

In general the construction of the point  $G$  cannot be effected through elementary operations by ruler and compasses. This is because the position of  $G$ , which

Figure 3. Maximal quadrangle for fixed  $AC$  and  $E$ 

determines the *width* of the hippopede in a certain direction, amounts to a cubic equation. In fact, it can be easily seen that the parameterization of the curve in polar coordinates is given by:

$$(x, y) = 2R\sqrt{1 - s^2 \sin^2 \phi} \cdot (\cos \phi, \sin \phi).$$

Then, the inner product  $\langle \nabla f, \bar{e}_\alpha \rangle$  of the gradient of the function defining the curve with the direction  $\bar{e}_\alpha = (\cos \alpha, \sin \alpha)$  of the fixed diagonal  $AC$  defining the width of the curve in the direction  $\alpha + \frac{\pi}{2}$  is:

$$\begin{aligned} \frac{1}{4} \langle \nabla f, \bar{e}_\alpha \rangle &= ((x^2 + y^2) - 2R^2)x \cos \alpha + ((x^2 + y^2) - 2R^2(1 - s^2))y \sin \alpha \\ &= 4R^3 \sqrt{1 - s^2 \sin^2 \phi} (\cos^2 \phi + (1 - 2s^2) \sin^2 \phi) \cos \phi \cos \alpha \\ &\quad + 4R^3 \sqrt{1 - s^2 \sin^2 \phi} ((1 + s^2) \cos^2 \phi + (1 - s^2) \sin^2 \phi) \sin \phi \sin \alpha \\ &= 0. \end{aligned}$$

Assuming  $x \neq 0$ , i.e.,  $\phi \neq \frac{\pi}{2}$ , and also  $\alpha \neq \frac{\pi}{2}$ , simplifying, dividing by  $\cos^3 \phi \cos \alpha$ , and setting  $z = \tan \phi$ ,  $a = \tan \alpha$ , we get the cubic

$$a(1 - s^2)z^3 + (1 - 2s^2)z^2 + a(1 + s^2)z + 1 = 0. \quad (1)$$

This, as expected, controls the position of the variable diagonal  $BD$  in dependence of the position ( $s$ ) of  $E$  and the direction  $\alpha$  of the fixed diagonal  $AC$ . In dependence of these parameters the previous cubic can have one, two or three real solutions. In fact, the cubic depends on the form of the hippopede which in turn depends only on  $s$ . It is easily seen that for  $s \leq \frac{1}{\sqrt{2}}$  the hippopede is convex, whereas for  $s > \frac{1}{\sqrt{2}}$  the curve is non-convex and has two real horizontal bitangents. In the latter case, depending on the orientation of the line  $AC$  (i.e., on  $\alpha$ ), there may be





### 3. The set of maximal quadrangles

Proposition 2 summarizes some well known facts about the various ways to generate the hippopede, as pedal of the ellipse with respect to its center (as does point I in Figure 6) [1, p. 148], but mainly, for us, as envelope of circles passing through the center of an ellipse with eccentricity  $s$  (with axes  $(R, \sqrt{1-s^2}R)$ ) and having their centers at points of this ellipse, called the *deferent* of the hippopede [3].

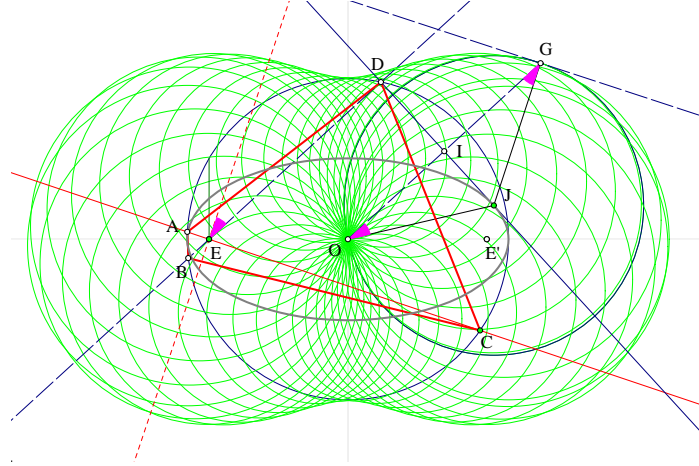


Figure 6. Hippopede as envelope

**Proposition 2.** Let  $c(O, R)$  be a circle of radius  $R$  centered at the origin and  $E : |OE| = sR, s \in (0, 1)$  be a fixed point inside the circle.

- (1) The ellipse  $e(s)$  centered at  $O$  having one focus at  $E$  and eccentricity equal to  $s$  has  $c$  as its auxiliary circle and its pedal with respect to the center  $O$  is the homothetic with ratio  $\frac{1}{2}$  of the hippopede.
- (2) The hippopede is the envelope of all circles passing through  $O$  and having their centers on the ellipse  $e(s)$ .
- (3) For each point  $J$  on the ellipse the circle  $c(J, |OJ|)$  is tangent to the hippopede at a point  $G$  which is the reflection of  $O$  on the tangent  $t_J$  to the ellipse at  $J$ . The tangent  $t_G$  of the hippopede at  $G$  is the reflection of the tangent  $t_O$  of  $c(O, |OJ|)$  with respect to  $t_J$ .

Using these facts and the symmetry of the figure we can eliminate the difficulties of selecting the furthest to  $O$  tangent of the hippopede and show that, by restricting the locations of the diagonal  $AC$  to the absolutely necessary needed to cover all possible quadrangles, the other diagonal  $BD$  of the corresponding maximal quadrangle depends continuously and in an invertible correspondence from  $AC$ . This among other things guarantees also the *existence* of maximal quadrangles under the restrictions we are considering. Denote by  $P_\alpha = ABCD$  the quadrangle with maximum area among all quadrangles inscribed in  $c$  with diagonals intersecting

at  $E$  and having the diagonal  $AC$  fixed in the direction  $\bar{e}_\alpha = (\cos \alpha, \sin \alpha)$ . To obtain all such quadrangles it suffices to consider directions  $\bar{e}_\alpha$  of the fixed diagonal  $AC$  which are restricted to one quadrant of the circle, for instance by selecting  $\alpha \in [-\frac{\pi}{2}, 0]$ . Then the polar angle  $\phi$  of the corresponding point  $G$  on the hippopede, defining the other diagonal  $BD$  can be considered to lie in an interval  $[0, \Phi]$  with  $\Phi \leq \frac{\pi}{2}$  (see Figure 7). With these conventions it is trivial to prove the following.

**Proposition 3.** *The correspondence  $p : \alpha \mapsto \phi$  mapping each  $\alpha \in [-\frac{\pi}{2}, 0]$  to the polar angle  $\phi \in [0, \Phi]$  such that the tangent  $t_G$  of the hippopede at its point  $G$  is parallel to  $\bar{e}_\phi$  and at maximum distance from  $O$  is a differentiable and invertible one.*

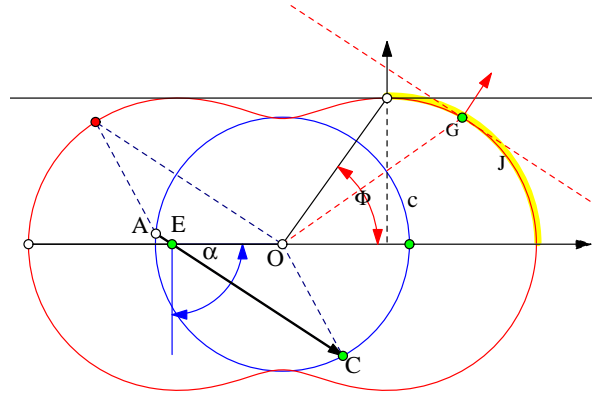
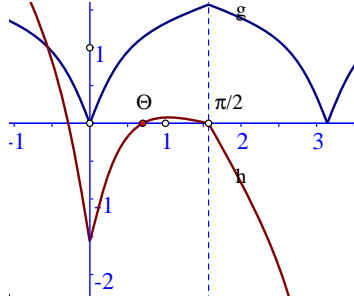


Figure 7. Domain  $\Phi$

This is obvious from the geometry of the figure. The analytic proof follows from the angle relations of the triangle  $OJG$  in Figure 6. In fact, if  $(x, y) = (a \cos \theta, b \sin \theta)$  is a parametrization of the ellipse, then the direction of  $BD$  is that of the normal of the ellipse at  $(x, y)$ , which is  $\phi = \arccos \left( \frac{x/a^2}{\sqrt{x^2/a^4 + y^2/b^4}} \right)$ . Setting  $x = a \cos \theta = R \cos \theta$  and  $y = b \sin \theta = R\sqrt{1-s^2} \sin \theta$ , we obtain the function  $\phi = g(\theta) = \arccos \left( \frac{1}{\sqrt{1 + \frac{1}{1-s^2} \tan^2 \theta}} \right)$ , increasing for  $\theta \in [0, \frac{\pi}{2}]$  (see Figure 8).

On the other hand, by the aforementioned relations, we obtain  $\alpha = h(\theta) = 2\phi - \theta + \frac{\pi}{2}$  with its first root at  $\Theta$ , such that  $\alpha = h(\Theta) = 2\Phi - \Theta + \frac{\pi}{2} = 0$ . It follows easily that in  $[0, \Theta]$  the function  $\alpha = h(\theta)$  is also increasing. The relation between the two angles is given by  $\phi = p(\alpha) = g(h^{-1}(\alpha))$ , which proves the claim.

In the following we restrict ourselves to quadrangles  $P\alpha$  with  $\alpha \in [0, \frac{\pi}{2}]$  and denote the set of all these maximal quadrangles by  $\mathcal{P}$ .

Figure 8. Composing the function  $\phi(\alpha)$ 

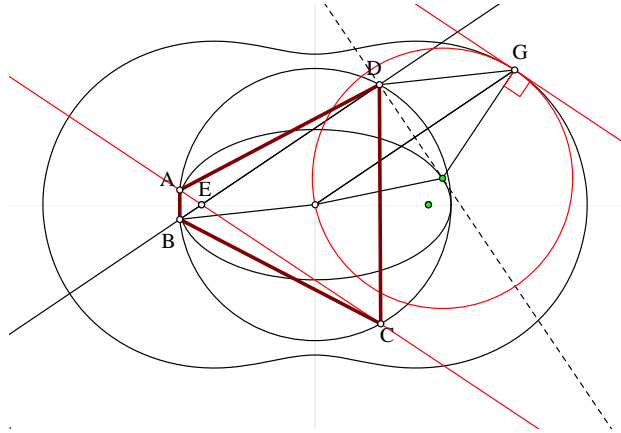
**Proposition 4.** *For each point  $E$  there is a unique quadrangle  $P\alpha \in \mathcal{P}$  having the diagonals symmetric with respect to the diameter  $OE$  hence equal, so that  $P\alpha$  in this case is an isosceles trapezium. This quadrangle is also the maximal one in area among all quadrangles whose diagonals pass through  $E$ .*

In fact, by the previous discussion we see that the diametral position of  $AC$  does not deliver equal diagonals, except in the trivial case for  $s = 0$  defining the square inscribed in the circle. Thus, we can assume  $\phi \neq \frac{\pi}{2}$  and the proof follows by observing the role of  $a = \tan \alpha$  in the cubic equation. The symmetry condition implies  $z = \tan \phi = -a$ , hence the biquadratic equation in  $a$ :

$$(1 - s^2)a^4 + 4s^2a^2 - 1 = 0.$$

The only acceptable solution for  $a^2$  producing two symmetric solutions, hence a unique quadrangle (see Figure 9), is

$$a^2 = \frac{1}{1 - s^2}(\sqrt{4s^4 - s^2 + 1} - 2s^2).$$

Figure 9. Maximal for diagonals through  $E$ 

The last claim results again from the necessary cubic equation holding for  $AC$  as well as for  $BD$  in the case of the maximal quadrangle among all quadrangles

with both diagonals varying but also passing through  $E$ . Taking the symmetric quadrangle with respect to the diameter  $EO$  we may assume that the cubic equation holds true two times with the roles of  $\alpha$  and  $\phi$  reversed. Subtracting the resulting cubic equations we obtain

$$\begin{aligned} & (a(1-s^2)z^3 + (1-2s^2)z^2) - (z(1-s^2)a^3 + (1-2s^2)a^2) \\ &= (1-s^2)az(z^2-a^2) + (1-2s^2)(z^2-a^2) \\ &= ((1-s^2)az + (1-2s^2))(z-a)(z+a) = 0. \end{aligned}$$

It is readily seen that only  $z+a=0$  is possible, and this proves the last claim.

#### 4. Diagonals making a fixed angle

Here we turn to the problem of finding the quadrangle with maximum area among all quadrangles  $ABCD$  inscribed in a circle  $c(O, R)$  whose diagonals pass through the fixed point  $E$  and intersect there at a fixed angle  $\alpha$ . By the discussion in §2 we know that the area of the quadrilateral in this case can be expressed by the product:

$$A = 4R^2 \sqrt{1-s^2 \sin^2 \phi} \sqrt{1-s^2 \sin^2(\phi+\alpha)} \cdot \sin \alpha.$$

Essentially this amounts to maximizing the product

$$(1-s^2 \sin^2 \phi)(1-s^2 \sin^2(\phi+\alpha)).$$

Using the identity  $\sin^2 \phi = \frac{1-\cos 2\phi}{2}$ , we rewrite this product as

$$\left( \left(1 - \frac{s^2}{2}\right) + \frac{s^2}{2}x \cos \alpha \right)^2 - \left( \frac{s^4}{4}(1-x^2) \sin^2 \alpha \right),$$

with  $x = \cos(2\phi+\alpha)$ . A further short calculation leads to the quadratic polynomial

$$\begin{aligned} & \frac{s^4}{4} \cdot x^2 + \left( s^2 \left(1 - \frac{s^2}{2}\right) \cos \alpha \right) x + \left( \frac{s^4}{4} \cos^2 \alpha + 1 - s^2 \right) \\ &= (1-s^2) + \frac{s^2}{2} \left( \frac{s^2}{2}x^2 + \left(1 - \frac{s^2}{2}\right)x \cos \alpha + \frac{s^2}{2} \cos^2 \alpha \right) \end{aligned}$$

The quadratic in  $x$  is easily seen to maximize among all values  $|x| \leq 1$  for  $x = 1$ . Since  $x = \cos(2\phi+\alpha)$ , this corresponds to  $2\phi+\alpha = 2k\pi$  i.e.,  $\phi = -\frac{\alpha}{2} + k\pi$ . This establishes the following result.

**Proposition 5.** *The maximal in area quadrangle inscribed in a circle  $c(O, R)$  and having its diagonals passing through a fixed point  $E$  and intersecting there under a constant angle  $\alpha$  is the equilateral trapezium whose diagonals are inclined to the diameter  $OE$  by an angle of  $\pm \frac{\alpha}{2}$ .*

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# Iterates of Brocardian Points and Lines

Gotthard Weise

**Abstract.** We establish some interesting results on Brocardians in relation to the Steiner ellipses of a triangle.

## 1. Notations

Let  $ABC$  be a triangle with vertices  $A, B, C$  and sidelines  $a, b, c$ . For the representation of points and lines we use barycentric coordinates, and write

$$P = (u : v : w) \text{ for a point,}$$

$$p = [u : v : w] := ux + vy + wz = 0 \text{ for a line.}$$

The point  $P = (u : v : w)$  and the line  $p = [u : v : w]$  are said to be dual, and we write  $P = \star p$  and  $p = \star P$ . The trilinear polar (or simply tripolar) of a point  $(u : v : w)$  is the line  $[\frac{1}{u} : \frac{1}{v} : \frac{1}{w}]$ . The trilinear pole (or simply tripole) of a line  $[u : v : w]$  is the point  $(\frac{1}{u} : \frac{1}{v} : \frac{1}{w})$ . The conjugate<sup>1</sup> of a point  $P = (u : v : w)$  is the point  $P^\bullet = (\frac{1}{u} : \frac{1}{v} : \frac{1}{w})$ , and the conjugate of a line  $p = [u : v : w]$  is the line  $p^\bullet = [\frac{1}{u} : \frac{1}{v} : \frac{1}{w}]$ .

## 2. Brocardians of a point

Let  $P$  be a point (not on the sidelines of  $ABC$  and different from the centroid  $G$ ) with cevian traces  $P_a, P_b, P_c$ . The parallels of  $b$  through  $P_a$ ,  $c$  through  $P_b$ , and  $a$  through  $P_c$  intersect the sidelines  $c, a, b$  respectively in the point  $P_{ab}, P_{bc}, P_{ca}$ . These points are the traces of a point  $P_{\rightarrow}$ , called the forward (or right) Brocardian of  $P$ . Similarly, the parallels of  $c$  through  $P_a$ ,  $a$  through  $P_b$ , and  $b$  through  $P_c$  intersect the sidelines  $b, c, a$  respectively in the point  $P_{ac}, P_{ba}, P_{cb}$ , the traces of a point  $P_{\leftarrow}$ , the backward (or left) Brocardian of  $P$  (see Figure 1). In barycentric coordinates,

$$P_{\rightarrow} = \left( \frac{1}{w} : \frac{1}{u} : \frac{1}{v} \right), \quad P_{\leftarrow} = \left( \frac{1}{v} : \frac{1}{w} : \frac{1}{u} \right).$$

We say that a point  $P = (u_1 u_2 : v_1 v_2 : w_1 w_2)$  is the barycentric product of the points  $P_1 = (u_1 : v_1 : w_1)$  and  $P_2 = (u_2 : v_2 : w_2)$ . A point  $P$  is obviously the barycentric product of its two Brocardians.

For  $P = K$ , the symmedian point, the Brocardian points are the Brocard points of the reference triangle  $ABC$ .

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<sup>1</sup>In this paper, the term “conjugate” always means isotomic conjugate.

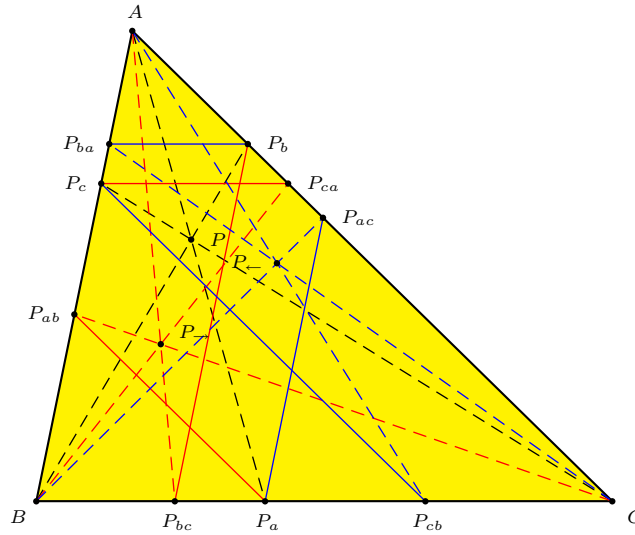


Figure 1.

### 3. Brocardians of a line

Let  $p$  be a line not parallel to the sidelines, intersecting  $a, b, c$  respectively at  $X_a, X_b, X_c$ . The parallels of  $c$  through  $X_a$ ,  $a$  through  $X_b$ , and  $b$  through  $X_c$  intersect  $b, c, a$  respectively at  $Y_b, Y_c, Y_a$ . These three points are collinear in a line  $p_{\rightarrow}$  which we call the right Brocardian line of  $p$ . Likewise, the parallels of  $b$  through  $X_a$ ,  $c$  through  $X_b$ , and  $a$  through  $X_c$  intersect  $c, a, b$  respectively at  $Z_c, Z_a, Z_b$ . These points are on a line  $p_{\leftarrow}$ , the left Brocardian line of  $p$  (see Figure 2).

If  $p = [u : v : w]$ , then the representation of its Brocardian lines ( $p$ -Brocardians)

$$p_{\rightarrow} = \left[ \frac{1}{w} : \frac{1}{u} : \frac{1}{v} \right] \quad \text{and} \quad p_{\leftarrow} = \left[ \frac{1}{v} : \frac{1}{w} : \frac{1}{u} \right]$$

can be derived by an easy calculation.

Here is an interesting connection between Brocardian points, Brocardian lines, and their trilinear elements.

**Proposition 1.** (a) *The tripolar of the right (left) Brocardian point of  $P$  is the right (left) Brocardian line of the tripolar of  $P$ .*

(b) *The tripole of the right (left) Brocardian line of  $p$  is the right (left) Brocardian point of the tripole of  $p$ .*

*Proof.*  $\star P_{\rightarrow} = p_{\rightarrow}$  and  $\star p_{\leftarrow} = P_{\leftarrow}$ . □

We say that the line  $p = [u_1 u_2 : v_1 v_2 : w_1 w_2]$  is the barycentric product of the lines  $p_1 = [u_1 : v_1 : w_1]$  and  $p_2 = [u_2 : v_2 : w_2]$ . Hence, a line is the barycentric product of its Brocardian lines.

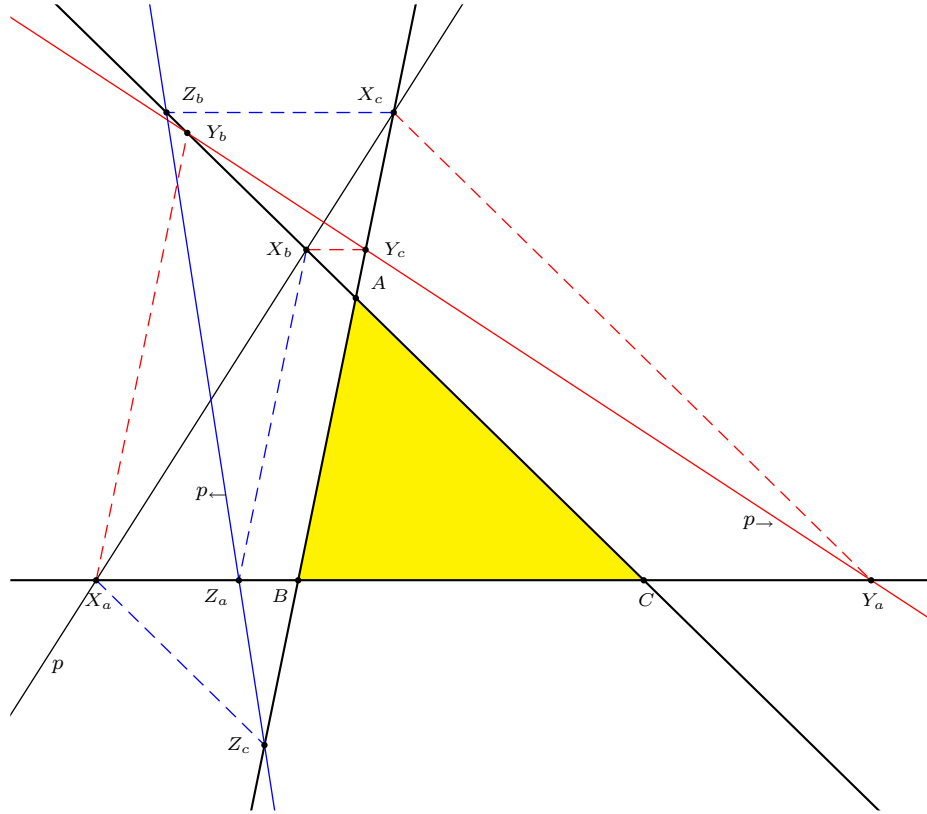


Figure 2.

#### 4. Iterates of Brocardian points

Now we examine on what happens when we repeat the Brocardian operations. First of all,

$$P_{\rightarrow\rightarrow} = (v : w : u), \quad P_{\rightarrow\leftarrow} = P_{\leftarrow\rightarrow} = P, \quad P_{\leftarrow\leftarrow} = (w : u : v).$$

With respect to (isotomic) conjugation,

$$(P^\bullet)_{\rightarrow} = (P_{\rightarrow})^\bullet = P_{\leftarrow\leftarrow}, \quad (P^\bullet)_{\leftarrow} = (P_{\leftarrow})^\bullet = P_{\rightarrow\rightarrow}.$$

More generally, for a positive integer, let  $P_{n\rightarrow}$  denote the  $n$ -th iterate of  $P_{\rightarrow}$  and  $P_{n\leftarrow}$  the  $n$ -th iterate of  $P_{\leftarrow}$ . These operations form a cycle of period 6 (see Figure 3). The “neighbors” of each point are its Brocardians, and the “antipode” its conjugate, *i.e.*,

$$P_{3\rightarrow} = P_{3\leftarrow} = P^\bullet.$$

For example, consider the case of the symmedian point  $P = K = X_6$  (in Kimberling’s notation [1, 2]). The conjugate of  $X_6$  is the third Brocard point  $X_{76}$ , the Brocardians of  $X_6$  are the bicentric pair  $P(1)$ ,  $U(1)$ , and the Brocardians of  $X_{76}$  are the bicentric pair  $P(11)$ ,  $U(11)$ , which are also the conjugates of the Brocard points.



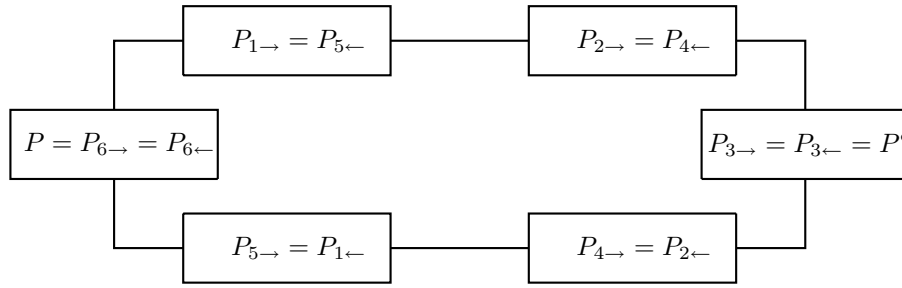


Figure 3.

The 6-cycle of Brocardians can be divided into two 3-cycles by selecting alternate points. We call  $(P, P_{2\rightarrow}, P_{2\leftarrow})$  the  $P$ -Brocardian triple, and  $(P^\bullet, P_{\leftarrow}, P_{\rightarrow})$  the conjugate  $P$ -Brocardian triple (or simply the  $P^\bullet$ -Brocardian triple). For each point in such a triple, the remaining two points are the Brocardians of its conjugate.

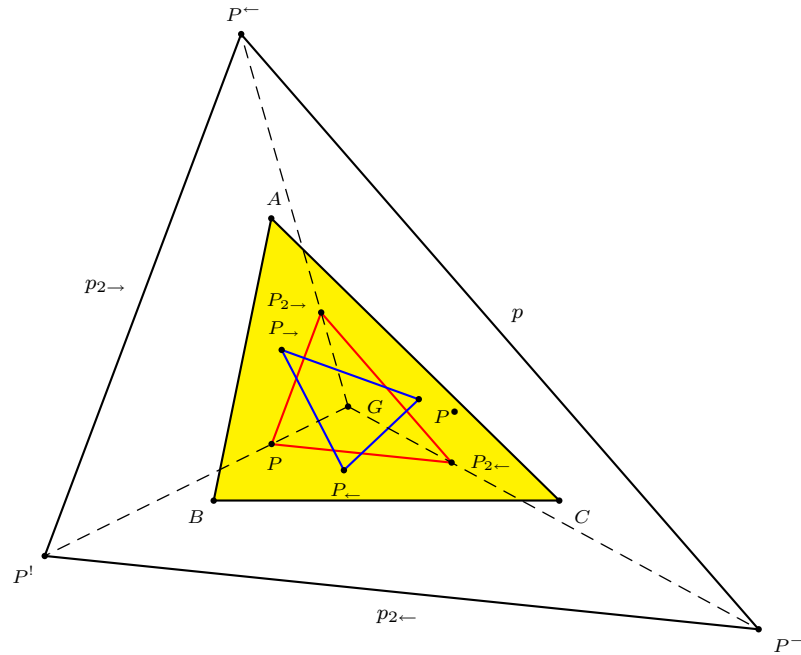


Figure 4.

## 5. Iterates of Brocardian lines

Analogous to the Brocardian operation for points one can iterate this process for lines. The results of two and three operations are the following:

$$p_{2\rightarrow} = [v : w : u], \quad p_{\rightarrow\leftarrow} = p_{\leftarrow\rightarrow} = p, \quad p_{2\leftarrow} = [w : u : v],$$

$$p_{3\rightarrow} = p_{3\leftarrow} = \left[ \frac{1}{u} : \frac{1}{v} : \frac{1}{w} \right] = p^\bullet.$$

The lines  $p_{2\rightarrow}$  and  $p_{2\leftarrow}$  are the tripolars of  $P_{\leftarrow}$  and  $P_{\rightarrow}$  (and the duals of  $P_{2\rightarrow}$  and  $P_{2\leftarrow}$ ) respectively. The line  $p_{3\rightarrow}$  is the tripolar of  $P$  and the barycentric product of  $p_{2\rightarrow}$  and  $p_{2\leftarrow}$ . It is obvious that the iterates of a Brocardian operation for lines have the same structure as those of a Brocardian operation for points. This is not surprising because the iterates of Brocardian lines are the duals of the iterates of Brocardian points.

We call the line triple  $\{p, p_{2\rightarrow}, p_{2\leftarrow}\}$  a  $p$ -Brocardian triple. It is the dual of the  $P$ -Brocardian triple and has the same centroid as the reference triangle. The triangles formed by the  $P$ -Brocardian triple and  $p$ -triple are homothetic at  $G$ , *i.e.*, they are similar and their corresponding sides are parallel. (The vertices of the  $p$ -triple in Figure 4 are defined in the next section.

## 6. Brocardian points on a line

There are geometric causes to complete the above structure of six points and their duals. For instance, it is desirable to solve following problem: Given a line  $p = [u : v : w]$  and its dual  $P$ , does there exist a point  $X$  with Brocardians  $X_{\rightarrow}$  and  $X_{\leftarrow}$  lying on  $p$ ? How can such a point be constructed?

Consider the lines  $p_{2\rightarrow}$  and  $p_{2\leftarrow}$ . They generate three new points:

$$P^! := p_{2\rightarrow} \cap p_{2\leftarrow} = (u^2 - vw : v^2 - wu : w^2 - uv),$$

$$P^{\rightarrow} := p \cap p_{2\rightarrow} = (w^2 - uv : u^2 - vw : v^2 - wu),$$

$$P^{\leftarrow} := p \cap p_{2\leftarrow} = (v^2 - wu : w^2 - uv : u^2 - vw).$$

The point  $P^!$  is also called the Steiner inverse of  $P$  (see [3]). It is interesting that  $P^{\rightarrow}$  and  $P^{\leftarrow}$  are the Steiner inverses of  $P_{2\leftarrow}$  and  $P_{2\rightarrow}$  respectively. The point with its Brocardians  $P^{\rightarrow}$  and  $P^{\leftarrow}$  is the conjugate of  $P^!$ :

$$P^{!^\bullet} = \left( \frac{1}{u^2 - vw} : \frac{1}{v^2 - wu} : \frac{1}{w^2 - uv} \right).$$

The line containing  $P_{2\rightarrow}$  and  $P_{2\leftarrow}$  has tripole  $P^{!^\bullet}$ . For  $P = K$ , the symmedian point, we have  $P^! = X_{385}$  and  $P^{!^\bullet} = X_{1916}$ .

## 7. Brocardian lines through a point

Given a point  $P$  not on the sidelines and different from the centroid  $G$ , are there two lines through  $P$  which are the Brocardian lines of a third line? This is easy to answer by making use of duality. The Brocardian lines are the duals of the points  $P^{\rightarrow}$  and  $P^{\leftarrow}$ , and the third line is the dual of the point  $P^{!^\bullet}$ .

## 8. Generation of further Brocardian triples

There are many possibilities to create new Brocardian triples from a given one. We consider a few of these.

8.1. The midpoints of each pair in a  $P$ -Brocardian triple form a new Brocardian triple with coordinates

$$(v + w : w + u : u + v), \quad (w + u : u + v : v + w), \quad (u + v : v + w : w + u).$$

8.2. The  $P$ -Brocardian and  $P^\bullet$ -Brocardian triples have an interesting property: they are triply perspective:

$PP^\bullet, P_{2\rightarrow}P_{\rightarrow}, P_{2\leftarrow}P_{\leftarrow}$  are concurrent in  $P_1 := \left( \frac{u^2-vw}{u} : \frac{v^2-wu}{v} : \frac{w^2-uv}{w} \right)$ ,

$PP_{\rightarrow}, P_{2\rightarrow}P_{\leftarrow}, P_{2\leftarrow}P^\bullet$  are concurrent in  $P_2 := \left( \frac{v^2-wu}{v} : \frac{w^2-uv}{w} : \frac{u^2-vw}{u} \right)$ ,

$PP_{\leftarrow}, P_{2\rightarrow}P^\bullet, P_{2\leftarrow}P_{\rightarrow}$  are concurrent in  $P_3 := \left( \frac{w^2-uv}{w} : \frac{u^2-vw}{u} : \frac{v^2-wu}{v} \right)$ .

These intersections form a Brocardian triple.

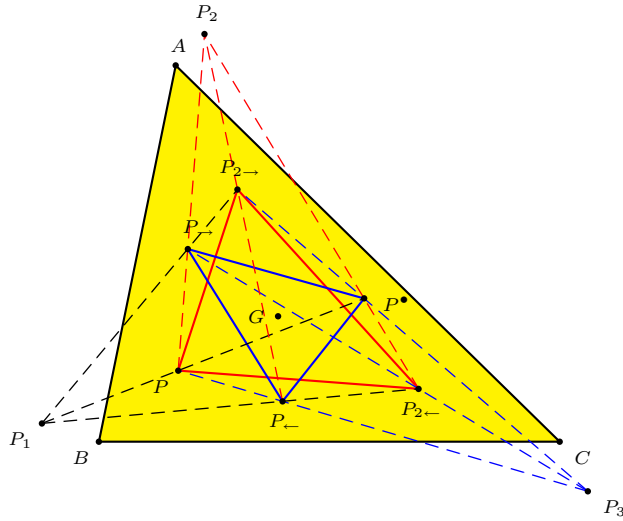


Figure 5.

8.3. Given a  $P$ -Brocardian triple and its dual, the lines  $p, p_{2\rightarrow}, p_{2\leftarrow}$  of that dual intersect the infinite line at the points of another Brocardian triple:

$$\begin{aligned} Q^\bullet &:= (v - w : w - u : u - v), \\ Q_{2\rightarrow}^\bullet &= (w - u : u - v : v - w), \\ Q_{2\leftarrow}^\bullet &= (u - v : v - w : w - u). \end{aligned}$$

Likewise, their conjugates

$$Q = \left( \frac{1}{v-w} : \frac{1}{w-u} : \frac{1}{u-v} \right),$$

$$Q_{2\rightarrow} = \left( \frac{1}{w-u} : \frac{1}{u-v} : \frac{1}{v-w} \right),$$

$$Q_{2\leftarrow} = \left( \frac{1}{u-v} : \frac{1}{v-w} : \frac{1}{w-u} \right)$$

form a Brocardian triple with points lying on the Steiner circumellipse  $yz + zx + xy = 0$ . The connecting line of  $Q_{2\rightarrow}$  and  $Q_{2\leftarrow}$  is the dual of  $Q$ :

$$\star Q = \left[ \frac{1}{v-w} : \frac{1}{w-u} : \frac{1}{u-v} \right]$$

and is tangent to the Steiner inellipse  $x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0$  at

$$R := ((v-w)^2 : (w-u)^2 : (u-v)^2).$$

The point  $Q$  is the fourth intersection of the Steiner circumellipse with the  $P$ -circumconic. The midpoints of  $QQ_{2\rightarrow}$  and  $QQ_{2\leftarrow}$  are the points

$$R_{2\rightarrow} := ((u-v)^2 : (v-w)^2 : (w-u)^2),$$

$$R_{2\leftarrow} := ((w-u)^2 : (u-v)^2 : (v-w)^2)$$

lying on the Steiner inellipse, which, together with  $R$ , form a Brocardian triple.

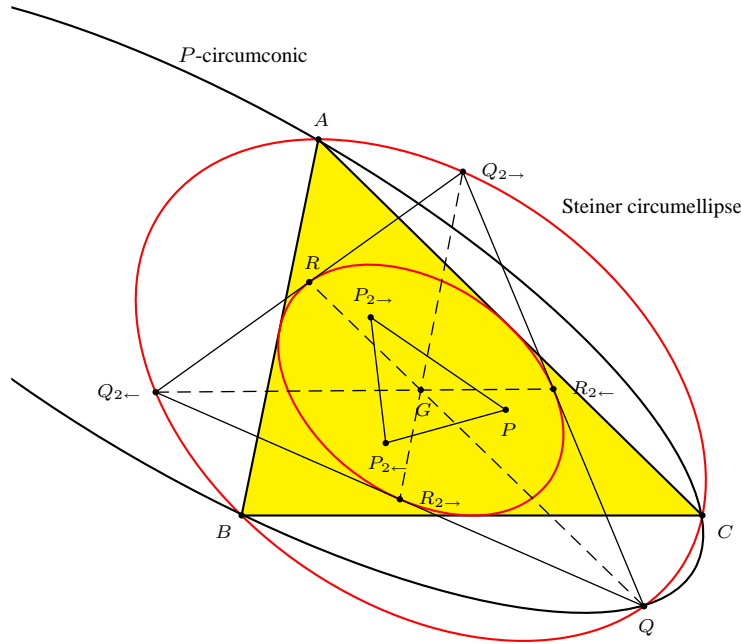


Figure 6.

### 9. Brocardians on a circumconic

Let  $\mathcal{Q}_P$  be the circumconic  $uyz + vzx + wxy = 0$ . The triangle formed by the tangents of  $\mathcal{Q}_P$  at the vertices is perspective with  $ABC$  at  $P = (u : v : w)$ . We call  $\mathcal{Q}_P$  the  $P$ -circumconic of triangle  $ABC$ .

A natural question is the following: Are there a pair of points  $X_{\rightarrow}$  and  $X_{\leftarrow}$  on a given conic  $\mathcal{Q}$  which are the Brocardians of a point  $X$ ?

Let us begin with the special case of the Steiner circumellipse  $\mathcal{Q} = \mathcal{Q}_G : xy + yz + zx = 0$  with perspector  $G = (1 : 1 : 1)$ . If  $X = (x : y : z)$  is a point on  $\mathcal{Q}_G$ , its conjugate  $X^\bullet$  is an infinite point and has Brocardians

$$X_{\rightarrow}^\bullet = (z : x : y), \quad X_{\leftarrow}^\bullet = (y : z : x),$$

which also lie on  $\mathcal{Q}_G$ . These three points on the Steiner ellipse obviously form a Brocardian triple. The Brocardians of a point  $Y$  lie on the Steiner ellipse if and only if  $Y$  is an infinite point.

We can answer the above question for the Steiner circumellipse as follows: the set of points  $X$  constitutes the infinite line.

Now, for the case of  $\mathcal{Q} = \mathcal{Q}_P$  with  $P \neq G$ , consider a variable point  $X = (x : y : z)$  on  $\mathcal{Q}_P$ . It is easy to see that the Brocardians  $X_{\rightarrow}$  and  $X_{\leftarrow}$  lie on the tripolars of  $P_{\rightarrow}$  and  $P_{\leftarrow}$  respectively. The intersection of these lines is the Steiner inverse  $P^!$  of  $P$ . Hence there must be a pair of points  $X_1$  and  $X_2$  on  $\mathcal{Q}$  such that  $(X_2)_{\rightarrow} = (X_1)_{\leftarrow} = P^!$ . Then we have

$$P_{\rightarrow}^! = (X_1)_{\leftarrow\rightarrow} = X_1 \quad P_{\leftarrow}^! = (X_2)_{\rightarrow\leftarrow} = X_2$$

with coordinates

$$P_{\rightarrow}^! = \left( \frac{1}{w^2 - uv} : \frac{1}{u^2 - vw} : \frac{1}{v^2 - wu} \right),$$

$$P_{\leftarrow}^! = \left( \frac{1}{v^2 - wu} : \frac{1}{w^2 - uv} : \frac{1}{u^2 - vw} \right).$$

Since the  $P$ -circumconic is the point-by-point conjugate of  $p$ , it is clear that these points are the conjugates of  $P^{\leftarrow}$  and  $P^{\rightarrow}$ .

Now we want to list some possibilities to construct the points  $P_{\rightarrow}^!$  and  $P_{\leftarrow}^!$ :

- (1) Construct the Brocardians of  $P^!$ .
- (2) Construct the conjugates of  $P^{\rightarrow}$  and  $P^{\leftarrow}$ .
- (3) Construct the tripoles of the lines  $PP_{2\rightarrow}$  and  $PP_{2\leftarrow}$ .
- (4) Reflect the Brocardians  $P_{\rightarrow}$  and  $P_{\leftarrow}$  in  $R_{2\leftarrow}$  and  $R_{2\rightarrow}$  respectively.
- (5) The tripoles of the lines  $PP_{\rightarrow}$  and  $PP_{\leftarrow}$  are the points

$$Z_1 = \left( \frac{u}{u - w} : \frac{v}{v - u} : \frac{w}{w - v} \right) \quad \text{and} \quad Z_2 = \left( \frac{u}{u - v} : \frac{v}{v - w} : \frac{w}{w - u} \right).$$

These are the barycentric products of  $P$  and  $Q_{2\rightarrow}$  and  $Q_{2\leftarrow}$  respectively. They lie on the  $P$ -circumconic and are the intersections of the lines  $QQ_{2\leftarrow}$  with  $GP_{\rightarrow}^!$  and  $QQ_{2\rightarrow}$  with  $GP_{\leftarrow}^!$  respectively. The line  $Z_1Z_2$  is tangent to the Steiner inellipse. Construct the intersections of the  $P$ -circumconic with the lines  $GZ_1$  and  $GZ_2$  to obtain the points  $P_{\rightarrow}^!$  and  $P_{\leftarrow}^!$ .

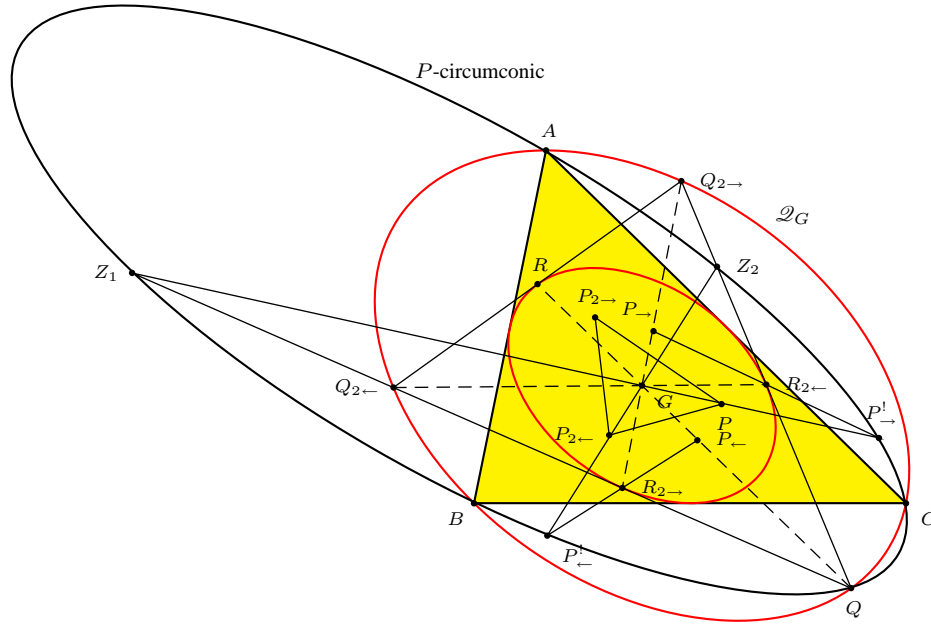


Figure 7.

## 10. Brocardians of a curve

Are there simple types of curves with the property that a point of one such curve has Brocardians on some simple curves?

Here is one special case for lines.

Given a line  $p = [u : v : w]$  with dual  $P$ , the circumconics  $\mathcal{Q} = \mathcal{Q}_P$ ,  $\mathcal{Q}_\rightarrow = \mathcal{Q}_{P_2\rightarrow}$  and  $\mathcal{Q}_\leftarrow = \mathcal{Q}_{P_2\leftarrow}$ , the following statements hold for  $X$  is a point on  $p$  (see Figure 8).

- (1) The Brocardians  $X_\rightarrow$  and  $X_\leftarrow$  lie on the circumconics  $\mathcal{Q}_\leftarrow : wyz + uzx + vxy = 0$  and  $\mathcal{Q}_\rightarrow : vyz + wxz + uxy = 0$ .
- (2) The fourth intersection of  $\mathcal{Q}_\rightarrow$  and  $\mathcal{Q}_\leftarrow$  is the point  $P^!$ .
- (3) The fourth intersections of the Steiner circumellipse with  $\mathcal{Q}_\rightarrow$  and  $\mathcal{Q}_\leftarrow$  are the points  $Q_{2\rightarrow}$  and  $Q_{2\leftarrow}$  respectively.
- (4) The fourth intersections of  $\mathcal{Q}$  with  $\mathcal{Q}_\rightarrow$  and  $\mathcal{Q}_\leftarrow$  are  $P_\rightarrow^!$  and  $P_\leftarrow^!$  respectively.

It is easy to show that the Brocardians  $X_\rightarrow$  and  $X_\leftarrow$  of all points  $X$  on a circumconic lie on the Brocardians  $\ell_\rightarrow$  and  $\ell_\leftarrow$  of a line  $\ell$ .

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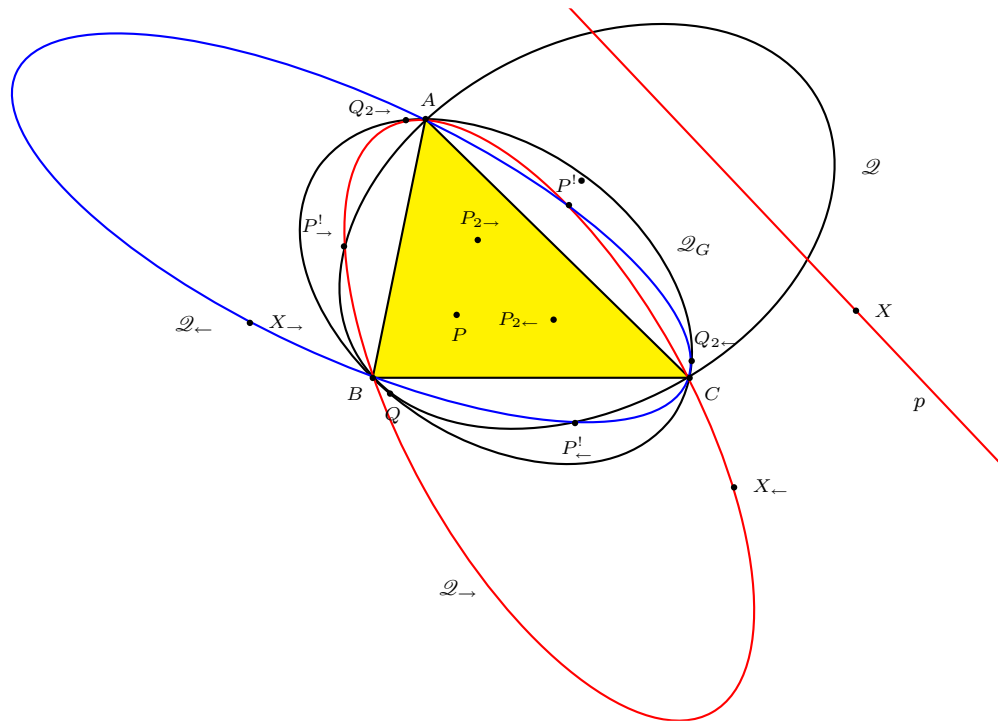


Figure 8.

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# Calculations Concerning the Tangent Lengths and Tangency Chords of a Tangential Quadrilateral

Martin Josefsson

**Abstract.** We derive formulas for the length of the tangency chords and some other quantities in a tangential quadrilateral in terms of the tangent lengths. Three formulas for the area of a bicentric quadrilateral are also proved.

## 1. Introduction

A *tangential quadrilateral* is a quadrilateral with an incircle, *i.e.*, a circle tangent to its four sides. We will call the distances from the four vertices to the points of tangency the *tangent lengths*, and denote these by  $e$ ,  $f$ ,  $g$  and  $h$ , as indicated in Figure 1.

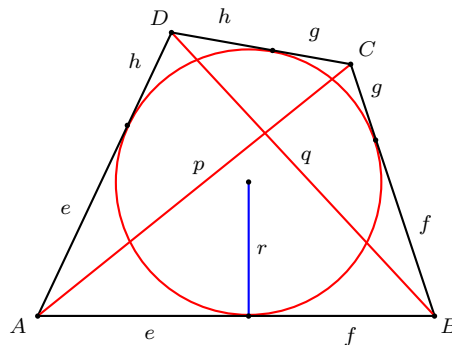


Figure 1. The tangent lengths

What is so interesting about the tangent lengths is that they alone can be used to calculate for instance the inradius  $r$ , the area of the quadrilateral  $K$  and the length of the diagonals  $p$  and  $q$ . The formula for  $r$  is

$$r = \sqrt{\frac{efg + fgh + ghe + hef}{e + f + g + h}} \quad (1)$$

and its derivation can be found in [5, p.26], [6, pp.187-188] and [13, 15]. Using the well known formula  $K = rs = r(e + f + g + h)$ , where  $s$  is the semiperimeter, we get the area of the tangential quadrilateral [6, p.188]

$$K = \sqrt{(e + f + g + h)(efg + fgh + ghe + hef)}. \quad (2)$$



Hajja [13] has also derived formulas for the length of the diagonals  $p = AC$  and  $q = BD$ . They are given by

$$\begin{aligned} p &= \sqrt{\frac{e+g}{f+h}((e+g)(f+h) + 4fh)}, \\ q &= \sqrt{\frac{f+h}{e+g}((e+g)(f+h) + 4eg)}. \end{aligned} \quad (3)$$

In this paper we prove some formulas that express a few other quantities in a tangential quadrilateral in terms of the tangent lengths.

## 2. The length of the tangency chords

If the incircle in a tangential quadrilateral  $ABCD$  is tangent to the sides  $AB$ ,  $BC$ ,  $CD$  and  $DA$  at  $W$ ,  $X$ ,  $Y$  and  $Z$  respectively, then the segments  $WY$  and  $XZ$  are called the *tangency chords* according to Dörrie [10, pp.188-189]. One interesting property of the tangency chords is that their intersection is also the intersection of the diagonals  $AC$  and  $BD$  (see [12, 20] and [24, pp.156-157]; the paper by Tan contains nine different proofs).

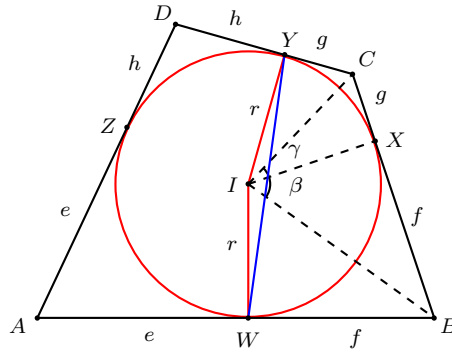


Figure 2. The tangency chord  $k = WY$

**Theorem 1.** *The lengths of the tangency chords  $WY$  and  $XZ$  in a tangential quadrilateral are respectively*

$$\begin{aligned} k &= \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+f)(g+h)(e+g)(f+h)}}, \\ l &= \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+h)(f+g)(e+g)(f+h)}}. \end{aligned}$$

*Proof.* If  $I$  is the incenter and angles  $\beta$  and  $\gamma$  are defined as in Figure 2, by the law of cosines in triangle  $WYI$  we get

$$k^2 = 2r^2 - 2r^2 \cos(2\beta + 2\gamma) = 2r^2(1 - \cos(2\beta + 2\gamma)).$$

Hence, using the addition formula

$$\frac{k^2}{2r^2} = 1 - \cos 2\beta \cos 2\gamma + \sin 2\beta \sin 2\gamma.$$

From the double angle formulas, we have

$$\cos 2\beta = \frac{1 - \tan^2 \beta}{1 + \tan^2 \beta} = \frac{r^2 - r^2 \tan^2 \beta}{r^2 + r^2 \tan^2 \beta} = \frac{r^2 - f^2}{r^2 + f^2} \quad (4)$$

and

$$\sin 2\beta = \frac{2 \tan \beta}{1 + \tan^2 \beta} = \frac{2rf}{r^2 + f^2}.$$

Similar formulas holds for  $\gamma$ , with  $g$  instead of  $f$ . Thus, we have

$$\frac{k^2}{2r^2} = 1 - \frac{r^2 - f^2}{r^2 + f^2} \cdot \frac{r^2 - g^2}{r^2 + g^2} + \frac{2rf}{r^2 + f^2} \cdot \frac{2rg}{r^2 + g^2} = 2r^2 \cdot \frac{(f + g)^2}{(r^2 + f^2)(r^2 + g^2)}$$

so

$$k^2 = (2r^2)^2 \cdot \frac{(f + g)^2}{(r^2 + f^2)(r^2 + g^2)}.$$

Now we factor  $r^2 + f^2$ , where  $r$  is given by (1). We get

$$\begin{aligned} r^2 + f^2 &= \frac{efg + fgh + ghe + hef + f^2(e + f + g + h)}{e + f + g + h} \\ &= \frac{e(fg + fh + gh + f^2) + f(gh + f^2 + fg + fh)}{e + f + g + h} \\ &= \frac{(e + f)(g(f + h) + f(h + f))}{e + f + g + h} \\ &= \frac{(e + f)(f + g)(f + h)}{e + f + g + h}. \end{aligned}$$

In the same way

$$r^2 + g^2 = \frac{(e + g)(f + g)(g + h)}{e + f + g + h} \quad (5)$$

so

$$k^2 = (2r^2)^2 \cdot \frac{(f + g)^2(e + f + g + h)^2}{(e + f)(f + g)(f + h)(e + g)(f + g)(g + h)}.$$

After simplification

$$k = 2r^2 \cdot \frac{e + f + g + h}{\sqrt{(e + f)(f + h)(h + g)(g + e)}}$$

and using (1) we finally get

$$k = \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e + f)(f + h)(h + g)(g + e)}}.$$

The formula for  $l$  can either be derived the same way, or we can use the symmetry in the tangential quadrilateral and need only to make the change  $f \leftrightarrow h$  in the formula for  $k$ .  $\square$

From Theorem 1 we get the following result, which was Problem 1298 in the MATHEMATICS MAGAZINE [8].

**Corollary 2.** *In a tangential quadrilateral with sides  $a$ ,  $b$ ,  $c$  and  $d$ , the quotient of the tangency chords satisfy*

$$\left(\frac{k}{l}\right)^2 = \frac{bd}{ac}.$$

*Proof.* Taking the quotient of  $k$  and  $l$  from Theorem 1, after simplification we get

$$\frac{k}{l} = \sqrt{\frac{(e+h)(f+g)}{(e+f)(h+g)}} = \sqrt{\frac{db}{ac}},$$

and the result follows.  $\square$

**Corollary 3.** *The tangency chords in a tangential quadrilateral are of equal length if and only if it is a kite.*

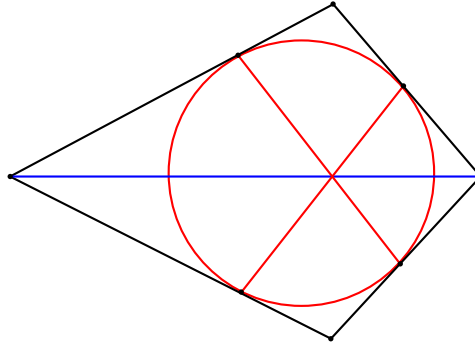


Figure 3. The tangency chords in a kite

*Proof.* ( $\Rightarrow$ ) If the quadrilateral is a kite it directly follows that the tangency chords are of equal length because of the mirror symmetry in the longest diagonal (see Figure 3).

( $\Leftarrow$ ) Conversely, if the tangency chords are of equal length in a tangential quadrilateral, from Corollary 2 we get  $ac = bd$ . In all tangential quadrilaterals the consecutive sides  $a$ ,  $b$ ,  $c$  and  $d$  satisfy  $a + c = b + d (= e + f + g + h$ ; see also [1, p.135], [2, pp.65-67] and [23]). Squaring, this implies  $a^2 + 2ac + c^2 = b^2 + 2bd + d^2$  and using  $ac = bd$  it follows that  $a^2 + c^2 = b^2 + d^2$ . This is the characterization for orthodiagonal quadrilaterals<sup>1</sup> [24, p.158]. The only tangential quadrilateral with perpendicular diagonals is the kite. We give an algebraic proof of this claim. Rewriting two of the equations above, we have

$$a - b = d - c, \tag{6}$$

$$a^2 - b^2 = d^2 - c^2 \tag{7}$$

<sup>1</sup>A quadrilateral with perpendicular diagonals.

Factorizing the second, we get

$$(a - b)(a + b) = (d - c)(d + c). \quad (8)$$

*Case 1.* If  $a = b$  we also have  $d = c$  using (6).

*Case 2.* If  $a \neq b$ , then we get  $a + b = d + c$  after division in (8) by  $a - b$  and  $d - c$  on respective sides (which by (6) are equal). Now adding  $a + b = d + c$  and  $a - b = d - c$ , we get  $2a = 2d$ . Hence  $a = d$  and also  $b = c$  using (6).

In both cases two pairs of adjacent sides are equal, so the quadrilateral is a kite.  $\square$

### 3. The angle between the tangency chords

In the proof of the next theorem we will use the following simple lemma.

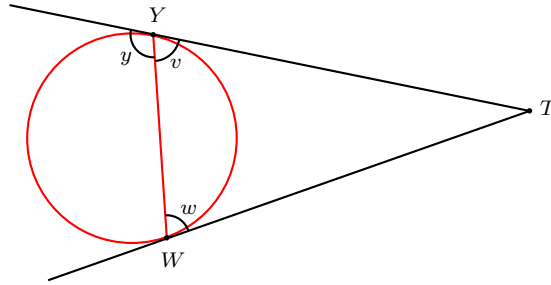


Figure 4. Alternate angles  $w$  and  $y$

**Lemma 4.** *The alternate angles between a chord and two tangents to a circle are supplementary angles, i.e.,  $w + y = \pi$  in Figure 4.*

*Proof.* Extend the tangents at  $W$  and  $Y$  to intersect at  $T$ , see Figure 4. Triangle  $TWY$  is isosceles according to the two tangent theorem, so the angles at the base are equal,  $w = v$ . Also,  $v + y = \pi$  since they are angles on a straight line. Hence  $w + y = \pi$ .  $\square$

Now we derive a formula for the angle between the two tangency chords.

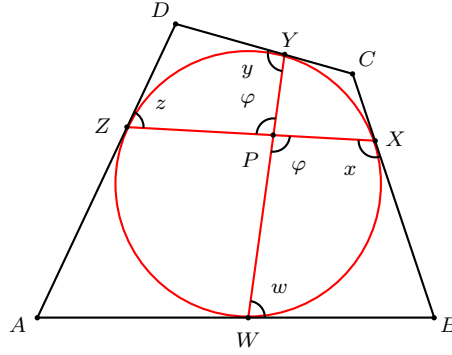
**Theorem 5.** *If  $e, f, g$  and  $h$  are the tangent lengths in a tangential quadrilateral, the angle  $\varphi$  between the tangency chords is given by*

$$\sin \varphi = \sqrt{\frac{(e + f + g + h)(efg + fgh + ghe + hef)}{(e + f)(f + g)(g + h)(h + e)}}.$$

*Proof.* We start by relating the angle  $\varphi$  to two opposite angles in the tangential quadrilateral (see Figure 5).

From the sum of angles in quadrilaterals  $BWPX$  and  $DYPZ$  we have  $w + x + \varphi + B = 2\pi$  and  $y + z + \varphi + D = 2\pi$ . Adding these,

$$w + x + y + z + 2\varphi + B + D = 4\pi. \quad (9)$$

Figure 5. The angle  $\varphi$  between the tangency chords

Using the lemma,  $w + y = \pi$  and  $x + z = \pi$ . Inserting these into (9), we get

$$2\pi + 2\varphi + B + D = 4\pi \Leftrightarrow B + D = 2\pi - 2\varphi. \quad (10)$$

For the area  $K$  of a tangential quadrilateral we have the formula

$$K = \sqrt{abcd} \sin \frac{B + D}{2} \quad (11)$$

where  $a, b, c$  and  $d$  are the sides of the tangential quadrilateral [9, p.28]. Inserting (10), we get

$$K = \sqrt{abcd} \sin (\pi - \varphi) = \sqrt{abcd} \sin \varphi,$$

hence

$$\sin \varphi = \frac{K}{\sqrt{abcd}} = \frac{\sqrt{(e + f + g + h)(efg + fgh + ghe + hef)}}{\sqrt{(e + f)(f + g)(g + h)(h + e)}}$$

where we used (2). □

From equation (10) we also get the following well known characterization for a quadrilateral to be *bicentric*, i.e., both tangential and cyclic. We will however formulate it as a characterization for the tangency chords to be perpendicular. Our proof is similar to that given in [10, pp.188-189] (if we include the derivation of (10) from the last theorem). Other proofs are given in [4, 11].

**Corollary 6.** *The tangency chords in a tangential quadrilateral are perpendicular if and only if it is a bicentric quadrilateral.*

*Proof.* In any tangential quadrilateral,  $B + D = 2\pi - 2\varphi$  by (10). The tangency chords are perpendicular if and only if

$$\varphi = \frac{\pi}{2} \Leftrightarrow B + D = \pi$$

which is a well known characterization for a quadrilateral to be cyclic. Hence this is a characterization for the quadrilateral to be bicentric. □

#### 4. The area of the contact quadrilateral

If the incircle in a tangential quadrilateral  $ABCD$  is tangent to the sides  $AB$ ,  $BC$ ,  $CD$  and  $DA$  at  $W$ ,  $X$ ,  $Y$  and  $Z$  respectively, then in [11] Yetti<sup>2</sup> calls the quadrilateral  $WXYZ$  the *contact quadrilateral* (see Figure 6). Here we shall derive a formula for its area in terms of the tangent lengths.

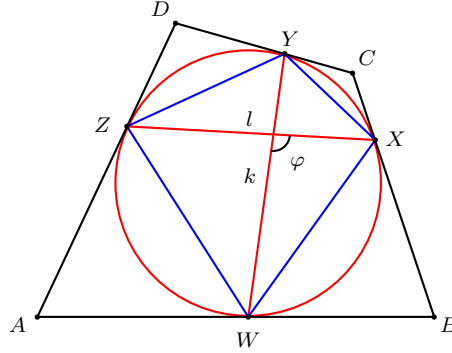


Figure 6. The contact quadrilateral  $WXYZ$

**Theorem 7.** *If  $e$ ,  $f$ ,  $g$  and  $h$  are the tangent lengths in a tangential quadrilateral, then the contact quadrilateral has area*

$$K_c = \frac{2\sqrt{(e+f+g+h)(efg+fgh+ghe+hfe)^5}}{(e+f)(e+g)(e+h)(f+g)(f+h)(g+h)}.$$

*Proof.* The area of any convex quadrilateral is

$$K = \frac{1}{2}pq \sin \theta \quad (12)$$

where  $p$  and  $q$  are the length of the diagonals and  $\theta$  is the angle between them (see [21, p.213] and [22]). Hence for the area of the contact quadrilateral we have

$$K_c = \frac{1}{2}kl \sin \varphi$$

where  $k$  and  $l$  are the length of the tangency chords and  $\varphi$  is the angle between them. Using Theorems 1 and 5, the formula for  $K_c$  follows at once after simplification.  $\square$

#### 5. The angles of the tangential quadrilateral

The next theorem gives formulas for the sines of the half angles of a tangential quadrilateral in terms of the tangent lengths.

<sup>2</sup>Yetti is the username of an American physicist at the website *Art of Problem Solving* [3].

**Theorem 8.** If  $e, f, g$  and  $h$  are the tangent lengths in a tangential quadrilateral  $ABCD$ , then its angles satisfy

$$\sin \frac{A}{2} = \sqrt{\frac{efg + fgh + ghe + hef}{(e+f)(e+g)(e+h)}},$$

$$\sin \frac{B}{2} = \sqrt{\frac{efg + fgh + ghe + hef}{(f+e)(f+g)(f+h)}},$$

$$\sin \frac{C}{2} = \sqrt{\frac{efg + fgh + ghe + hef}{(g+e)(g+f)(g+h)}},$$

$$\sin \frac{D}{2} = \sqrt{\frac{efg + fgh + ghe + hef}{(h+e)(h+f)(h+g)}}.$$

*Proof.* If the incircle has center  $I$  and is tangent to sides  $AB$  and  $AD$  at  $W$  and  $Z$  (see Figure 7), then by the law of cosines in triangle  $WZI$

$$WZ^2 = 2r^2(1 - \cos 2\alpha) = \frac{4e^2r^2}{r^2 + e^2}$$

where we used

$$\cos 2\alpha = \frac{r^2 - e^2}{r^2 + e^2}$$

which we get from (4) when making the change  $f \leftrightarrow e$ .

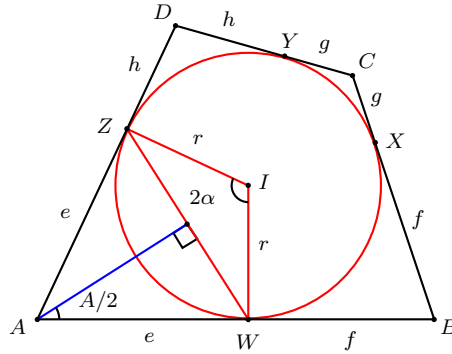


Figure 7. Half the angle of  $A$

Now using (1) and

$$r^2 + e^2 = \frac{(e+f)(e+g)(e+h)}{e+f+g+h}$$

which by symmetry follows from (5) when making the change  $g \leftrightarrow e$ , we have

$$WZ^2 = 4e^2 \cdot \frac{efg + fgh + ghe + hef}{e+f+g+h} \cdot \frac{e+f+g+h}{(e+f)(e+g)(e+h)}$$

hence

$$WZ = 2e \sqrt{\frac{efg + fgh + ghe + hef}{(e+f)(e+g)(e+h)}}.$$

Finally, from the definition of sine, we get (see Figure 7)

$$\sin \frac{A}{2} = \frac{\frac{1}{2}WZ}{e} = \sqrt{\frac{efg + fgh + ghe + hef}{(e+f)(e+g)(e+h)}}.$$

The other formulas can be derived in the same way, or we get them at once using symmetry.  $\square$

## 6. The area of a bicentric quadrilateral

The formula for the area of a *bicentric quadrilateral* (see Figure 8) is almost always derived in one of two ways.<sup>3</sup> Either by inserting  $B + D = \pi$  into formula (11) or by using  $a + c = b + d$  in Brahmagupta's formula<sup>4</sup>

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

for the area of a cyclic quadrilateral, where  $s$  is the semiperimeter. A third derivation was given by Stapp as a solution to a problem<sup>5</sup> by Rosenbaum in an old number of the MONTHLY [18]. Another possibility is to use the formula<sup>6</sup>

$$K = \frac{1}{2} \sqrt{(pq)^2 - (ac - bd)^2}$$

for the area of a tangential quadrilateral [9, p.29], inserting Ptolemy's theorem  $pq = ac + bd$  (derived in [1, pp.128-129], [9, p.25] and [24, pp.148-150]) and factorize the radicand.

Here we shall give a fifth proof, using the tangent lengths in a way different from what Stapp did in [18].

**Theorem 9.** *A bicentric quadrilateral with sides  $a, b, c$  and  $d$  has area*

$$K = \sqrt{abcd}.$$

*Proof.* From formula (2) we get

$$\begin{aligned} K^2 &= (efg + fgh + ghe + hef)(e + f + g + h) \\ &= ef(g+h)(e+f) + ef(g+h)^2 + gh(e+f)^2 + gh(e+f)(g+h) \\ &= (e+f)(g+h)(ef + gh + eg + hf - eg - hf) + ef(g+h)^2 + gh(e+f)^2 \\ &= (e+f)(g+h)(f+g)(e+h) - (eg - fh)^2 \end{aligned}$$

where we used the factorizations  $ef + gh + eg + hf = (f+g)(e+h)$  and

$$(e+f)(g+h)(-eg - hf) + ef(g+h)^2 + gh(e+f)^2 = -(eg - fh)^2,$$

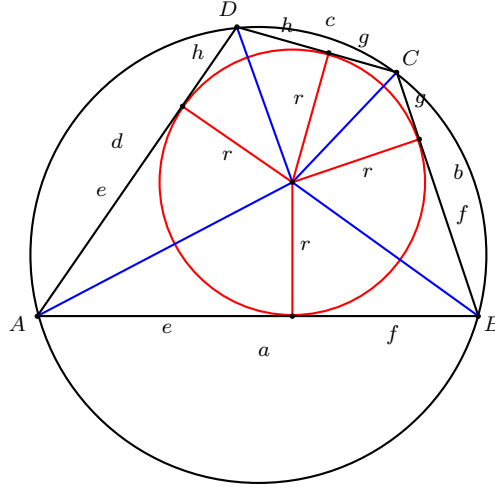
<sup>3</sup>Or intended to be derived so; in many books [1, 7, 16, 24] this is an exercise rather than a theorem.

<sup>4</sup>For a derivation, see [7, pp.57-58] or [9, p.24].

<sup>5</sup>The problem was to prove our Theorem 9. Stapp used the tangent lengths in his calculation.

<sup>6</sup>This formula can be derived independently from (11) and Brahmagupta's formula.



Figure 8. A bicentric quadrilateral  $ABCD$ 

which are easy to check. Hence

$$K^2 = acbd - (eg - fh)^2$$

and we have  $K = \sqrt{abcd}$  if and only if  $eg = fh$ , which according to Hajja<sup>7</sup> [13] is a characterization for a tangential quadrilateral to be cyclic, *i.e.*, bicentric.  $\square$

In a bicentric quadrilateral there is a simpler formula for the area in terms of the tangent lengths than (2), according to the next theorem.

**Theorem 10.** *A bicentric quadrilateral with tangent lengths  $e, f, g$  and  $h$  has area*

$$K = \sqrt[4]{efgh}(e + f + g + h).$$

*Proof.* The quadrilateral has an incircle, so (see Figure 8)

$$r = e \tan \frac{A}{2} = f \tan \frac{B}{2} = g \tan \frac{C}{2} = h \tan \frac{D}{2},$$

hence

$$r^4 = efgh \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \tan \frac{D}{2}. \quad (13)$$

It also has a circumcircle, so  $A + C = \pi$ . Hence  $\frac{A}{2} = \frac{\pi}{2} - \frac{C}{2}$  and it follows that

$$\tan \frac{A}{2} = \cot \frac{C}{2} \Leftrightarrow \tan \frac{A}{2} \tan \frac{C}{2} = 1.$$

In the same way

$$\tan \frac{B}{2} \tan \frac{D}{2} = 1.$$

<sup>7</sup>Note that Hajja uses  $a, b, c$  and  $d$  for the tangent lengths.

Thus, in a bicentric quadrilateral we get<sup>8</sup>

$$r^4 = efgh.$$

Finally, the area<sup>9</sup> is given by

$$K = rs = \sqrt[4]{efgh}(e + f + g + h)$$

where  $s$  is the semiperimeter.  $\square$

We conclude with another interesting and possibly new formula for the area of a bicentric quadrilateral in terms of the lengths of the tangency chords and the diagonals.

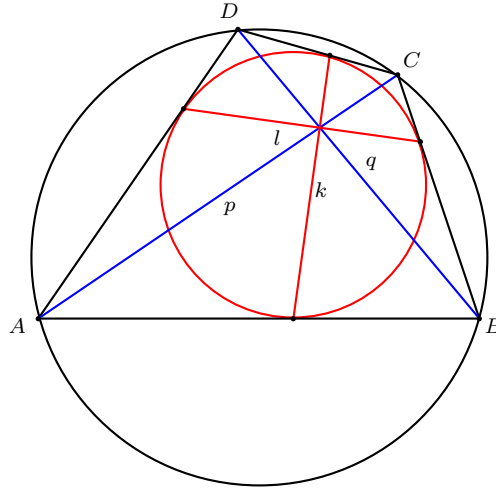


Figure 9. The tangency chords and diagonals

**Theorem 11.** *A bicentric quadrilateral with tangency chords  $k$  and  $l$ , and diagonals  $p$  and  $q$  has area*

$$K = \frac{klpq}{k^2 + l^2}.$$

*Proof.* Using (12), Theorem 9 and Ptolemy's theorem, we have

$$K = \frac{1}{2}pq \sin \theta \Leftrightarrow \sqrt{abcd} = \frac{1}{2}(ac + bd) \sin \theta.$$

Hence

$$\frac{2}{\sin \theta} = \frac{ac + bd}{\sqrt{abcd}} = \sqrt{\frac{ac}{bd}} + \sqrt{\frac{bd}{ac}} = \frac{l}{k} + \frac{k}{l} = \frac{k^2 + l^2}{kl}$$

<sup>8</sup>This derivation was done by Yetti in [19], where there are also some proofs of formula (1).

<sup>9</sup>This formula also gives the area of a tangential trapezoid. Since it has two adjacent supplementary angles,  $\tan \frac{A}{2} \tan \frac{D}{2} = \tan \frac{B}{2} \tan \frac{C}{2} = 1$  or  $\tan \frac{A}{2} \tan \frac{B}{2} = \tan \frac{C}{2} \tan \frac{D}{2} = 1$ ; thus the formula for  $r$  is still valid.

where we used Corollary 2. Then we get the area of the bicentric quadrilateral using (12) again

$$K = \frac{\sin \theta}{2} pq = \frac{klpq}{k^2 + l^2}$$

completing the proof. □

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# Generalized Fibonacci Circle Chains

Giovanni Lucca

**Abstract.** We consider a particular circle chain where, each circle belonging to it, is tangent to the two previous ones and to a common straight line. We give a simple formula for calculating the limit point of the chain in terms of the radii of the first two circles and of the golden ratio.

## 1. Introduction

By taking a cue from [4] we define a generalized Fibonacci circle chain as follows. Let  $\mathcal{L}$  be a line and  $\mathcal{C}_1$  and  $\mathcal{C}_2$  two circles of radii  $a$  and  $b$  respectively, tangent to each other and both tangent to  $\mathcal{L}$ . The generalized Fibonacci circles chain is the sequence of circles  $\mathcal{C}_3, \mathcal{C}_4, \dots$ , where  $\mathcal{C}_n$  is tangent to  $\mathcal{C}_{n-1}, \mathcal{C}_{n-2}$  and  $\mathcal{L}$  (see Figure 1). Denote by  $r_n$  the radius of  $\mathcal{C}_n$ , with  $r_1 = a$  and  $r_2 = b$ .

Let  $X_n$  be the point of tangency of the circle  $\mathcal{C}_n$  with the line  $\mathcal{L}$ , with coordinates  $(x_n, 0)$ . Assuming  $x_1 < x_2$ , we have  $x_3 < x_2$ ,  $x_4 > x_3$ , and more generally,  $x_{n+1} - x_n > 0$  or  $< 0$  according as  $n$  is odd or even.

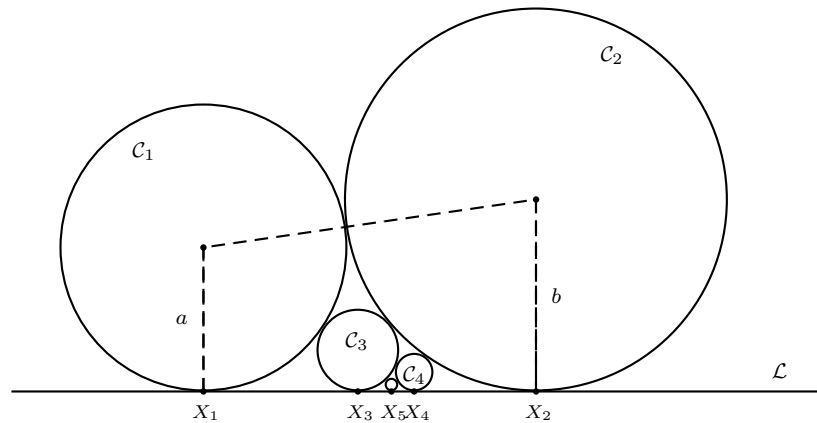


Figure 1.

In Figure 1, a simple application of the Pythagorean theorem shows that

$$(x_2 - x_1)^2 = (a + b)^2 - (a - b)^2 = 4ab,$$

so that  $x_2 - x_1 = 2\sqrt{ab}$ . Applying the same relation to three consecutive circles with points of tangency  $X_n, X_{n+1}$  and  $X_{n+2}$  with the line  $\mathcal{L}$ , we have

$$\begin{aligned}
x_{n+1} - x_n &= (-1)^{n-1} \cdot 2\sqrt{r_{n+1}r_n}, \\
x_{n+1} - x_{n+2} &= (-1)^{n-1} \cdot 2\sqrt{r_{n+2}r_{n+1}}, \\
x_{n+2} - x_n &= (-1)^{n-1} \cdot 2\sqrt{r_{n+2}r_n}.
\end{aligned} \tag{1}$$

Since  $x_{n+1} - x_n = (x_{n+2} - x_n) + (x_{n+1} - x_{n+2})$ , we have

$$2\sqrt{r_{n+1}r_n} = 2\sqrt{r_{n+2}r_n} + 2\sqrt{r_{n+2}r_{n+1}},$$

and

$$\frac{1}{\sqrt{r_{n+2}}} = \frac{1}{\sqrt{r_{n+1}}} + \frac{1}{\sqrt{r_n}}. \tag{2}$$

If we put  $G_n = \frac{1}{\sqrt{r_n}}$ , then (2) becomes a Fibonacci-like recursive relation

$$G_{n+2} = G_{n+1} + G_n, \quad G_1 = \frac{1}{\sqrt{a}}, \quad G_2 = \frac{1}{\sqrt{b}}. \tag{3}$$

The solution of this recurrence relation can be expressed in terms of the Fibonacci numbers  $F_n$  according to a formula in [3]:

$$G_{n+2} = G_2 F_{n+1} + G_1 F_n = \frac{F_{n+1}}{\sqrt{b}} + \frac{F_n}{\sqrt{a}}. \tag{4}$$

Here,  $(F_n)$  is the Fibonacci sequence defined by

$$F_{n+2} = F_{n+1} + F_n, \quad F_1 = F_2 = 1. \tag{5}$$

Relations (4) can be easily verified by induction. It is well known that

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \varphi := \frac{\sqrt{5} + 1}{2},$$

the golden ratio. In particular, when  $a = b = 1$ , (3) becomes the classical Fibonacci recurrence (5). Thus, the circles in the chain shown in Figure 1 can be regarded as generalized Fibonacci circles.

## 2. Limit point of the circle chain

We are interested in locating the limit point of the generalized Fibonacci circle chain. From (1), we have

$$x_{n+1} = x_n + (-1)^{n-1} \frac{2}{G_n G_{n+1}} = x_1 + 2 \sum_{k=1}^n \frac{(-1)^{k-1}}{G_k G_{k+1}}. \tag{6}$$

The sum  $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{G_k G_{k+1}} = \frac{\varphi}{G_1(G_1 + \varphi G_2)}$  can be found as a particular case of formulas given in [1, 2]. We give a direct proof here. First of all, by induction, it is easy to establish

$$G_k G_{k+2} - G_{k+1}^2 = -(G_{k-1} G_{k+1} - G_k^2).$$

From this it follows that

$$\begin{aligned}
 G_k G_{k+2} - G_{k+1}^2 &= (-1)^{k-1} (G_1 G_3 - G_2^2) \\
 &= (-1)^{k-1} \frac{b + \sqrt{ab} - a}{ab} \\
 &= (-1)^{k-1} \frac{(\sqrt{b} + \varphi\sqrt{a})(\varphi\sqrt{b} - \sqrt{a})}{\varphi ab}.
 \end{aligned}$$

Using this, we rewrite (6) as

$$\begin{aligned}
 x_{n+1} &= x_1 + \frac{2\varphi ab}{(\sqrt{b} + \varphi\sqrt{a})(\varphi\sqrt{b} - \sqrt{a})} \sum_{k=1}^n \frac{G_k G_{k+2} - G_{k+1}^2}{G_k G_{k+1}} \\
 &= x_1 + \frac{2\varphi ab}{(\sqrt{b} + \varphi\sqrt{a})(\varphi\sqrt{b} - \sqrt{a})} \sum_{k=1}^n \left( \frac{G_{k+2}}{G_{k+1}} - \frac{G_{k+1}}{G_k} \right) \\
 &= x_1 + \frac{2\varphi ab}{(\sqrt{b} + \varphi\sqrt{a})(\varphi\sqrt{b} - \sqrt{a})} \left( \frac{G_{n+2}}{G_{n+1}} - \frac{G_2}{G_1} \right).
 \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \frac{G_{n+2}}{G_{n+1}} = \varphi$ , which is easy to see from (4), we obtain the limit point

$$\begin{aligned}
 x_\infty &= x_1 + \frac{2\varphi ab}{(\sqrt{b} + \varphi\sqrt{a})(\varphi\sqrt{b} - \sqrt{a})} \left( \varphi - \frac{\sqrt{a}}{\sqrt{b}} \right) \\
 &= x_1 + 2\sqrt{ab} \cdot \frac{\sqrt{a}}{(\varphi - 1)\sqrt{b} + \sqrt{a}}.
 \end{aligned}$$

Since  $x_2 = x_1 + 2\sqrt{ab}$ , this limit point divides  $X_1 X_2$  in the ratio

$$x_\infty - x_1 : x_2 - x_\infty = \sqrt{a} : (\varphi - 1)\sqrt{b} = \varphi\sqrt{a} : \sqrt{b}.$$

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## Trilinear Distance Inequalities for the Symmedian Point, the Centroid, and Other Triangle Centers

Clark Kimberling

**Abstract.** Seven inequalities which appear to be new are derived using Hölder’s inequality and the arithmetic-mean–geometric-mean inequality. In particular, bounds are found for power sums  $x^q + y^q + z^q$ , where  $x, y, z$  are the directed distances of a point to the sidelines of a triangle  $ABC$ , and the centroid maximizes the product  $xyz$ .

We begin with a *very* special point inside a triangle  $ABC$  and prove its well known extremal property; however, the proof is not the one often cited (e.g., [2, p.75]). Instead, the proof given here depends on Hölder’s inequality and extends to extremal properties of other special points. Then, a second classical inequality, namely the arithmetic-mean–geometric-mean inequality, is used to prove that yet another special point attains an extreme. These results motivate an open question stated at the end of the note.

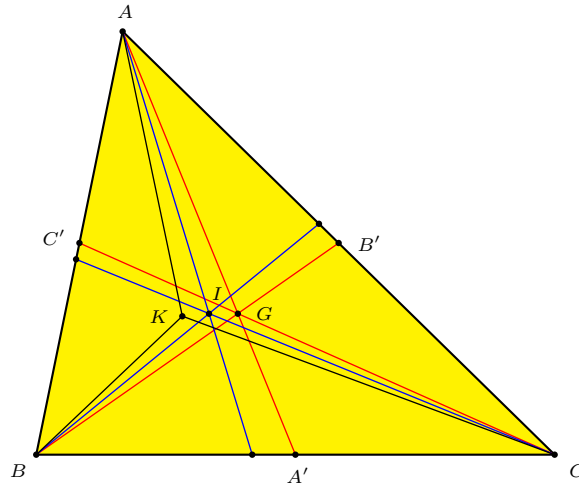


Figure 1.

Regarding the *very* special point, Honsberger [2] writes, “The symmedian point is one of the crown jewels of modern geometry.” A construction of the symmedian point,  $K$ , of a triangle  $ABC$ , goes like this: let  $M_A$  be the  $A$ -median; i.e., the line of the vertex  $A$  and the midpoint  $A'$  of segment  $BC$ . Let  $m_A$  be the  $A$ -symmedian; i.e., the reflection of  $M_A$  in the bisector of angle  $A$ . Let  $m_B$  be the  $B$ -symmedian

and  $m_C$  the  $C$ -symmedian. Then the three symmedians  $m_A, m_B, m_C$  concur in  $K$ , as shown in Figure 1, where you can also see the medians concurring in the centroid,  $G$ , and the bisectors concurring in the incenter,  $I$ .

In 1873, Emile Lemoine proved that if a point  $X$  inside a triangle  $ABC$  has distances  $x, y, z$  from the lines  $BC, CA, AB$ , then

$$\alpha^2 + \beta^2 + \gamma^2 \leq x^2 + y^2 + z^2, \quad (1)$$

where  $\alpha, \beta, \gamma$  are the distances from  $K$  to those lines. That is,  $K$  minimizes  $x^2 + y^2 + z^2$ . Lemoine's proof [4] can be found by googling "minimum aura donc lieu pour le pied". Expositions in English can be found by googling the titles of John MacKay's historical articles ([6], [7]); these describe the proposal of "Yanto" about 1803, a proof in 1809 by Simon Lhuillier [5, pages 296-8], and rediscoveries. Hölder's inequality [1, pages 19 and 51] of 1889, generalizes Cauchy's inequality [1, pages 2 and 50] of 1821. (Perhaps rediscoverers of the minimal property of  $K$  recognized that it follows easily from Cauchy's inequality.)

We shall now generalize (1), starting with Hölder's inequality [1]: that if  $a, b, c, x, y, z$  are positive real numbers, then

$$ax + by + cz \leq (a^p + b^p + c^p)^{1/p} (x^q + y^q + z^q)^{1/q} \quad (2)$$

for all real  $p, q$  satisfying

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ and } q > 1,$$

with the inequality (2) reversed if  $q < 1$  and  $q \neq 0$ . It is easy to check that Hölder's inequality is equivalent to

$$\frac{(ax + by + cz)^q}{[a^{q/(q-1)} + b^{q/(q-1)} + c^{q/(q-1)}]^{q-1}} \leq x^q + y^q + z^q \quad (3)$$

if  $q > 1$  or  $q < 0$ , with (3) reversed if  $0 < q < 1$ .

Let  $ABC$  be a triangle with sidelengths  $a = |BC|$ ,  $b = |CA|$ ,  $c = |AB|$  and area  $\Delta$ . If  $U$  is any point inside  $ABC$ , then homogeneous trilinear coordinates (or simply trilinears) for  $U$ , written as  $u : v : w$ , are any triple of numbers proportional to the respective perpendicular distances  $x, y, z$  from  $U$  to the sidelines  $BC, CA, AB$ . Thus, there is a constant  $h$  such that  $(x, y, z) = (hu, hv, hw)$ . In fact,

$$h = \frac{2\Delta}{au + bv + cw},$$

since  $\Delta = \frac{1}{2}(ax + by + cz)$  is the sum of areas  $\frac{1}{2}ahu$ ,  $\frac{1}{2}bhv$ ,  $\frac{1}{2}chw$  of the triangles  $BCU, CAU, ABU$ . Writing  $S(q)$  for  $x^q + y^q + z^q$ , we now recast (2), via (3), as

$$\frac{(2\Delta)^q}{[aa^{1/(q-1)} + bb^{1/(q-1)} + cc^{1/(q-1)}]^{q-1}} \begin{cases} \leq S(q) & \text{if } q < 0 \text{ or } q > 1, \\ \geq S(q) & \text{if } 0 < q < 1. \end{cases} \quad (4)$$

These two inequalities show that the point  $a^{1/(q-1)} : b^{1/(q-1)} : c^{1/(q-1)}$  is an extreme point of  $S(q)$  for  $q \neq 1$ . Taking  $q = 2$  (which reduces Hölder's inequality to Cauchy's) gives (1). Other choices of  $q$  give inequalities not mentioned in [8]. In order to list several, we use the indexing of special points in the *Encyclopedia*



of *Triangle Centers*[3], where  $K$  is indexed as  $X_6$ , and many of its properties, and properties of other special points to be mentioned here, are recorded.

The point  $X_6 = a : b : c$  minimizes  $S(2)$  :

$$\frac{4\Delta^2}{a^2 + b^2 + c^2} \leq x^2 + y^2 + z^2$$

The point  $X_{365} = a^{1/2} : b^{1/2} : c^{1/2}$  minimizes  $S(3)$  :

$$\frac{8\Delta^3}{(a^{3/2} + b^{3/2} + c^{3/2})^2} \leq x^3 + y^3 + z^3$$

The point  $X_{31} = a^2 : b^2 : c^2$  minimizes  $S(3/2)$  :

$$\left( \frac{8\Delta^3}{a^3 + b^3 + c^3} \right)^{1/2} \leq x^{3/2} + y^{3/2} + z^{3/2}$$

The point  $X_{32} = a^3 : b^3 : c^3$  minimizes  $S(4/3)$  :

$$\left( \frac{16\Delta^4}{a^4 + b^4 + c^4} \right)^{1/3} \leq x^{4/3} + y^{4/3} + z^{4/3}$$

The point  $X_{75} = a^{-2} : b^{-2} : c^{-2}$  maximizes  $S(1/2)$  :

$$[2\Delta(a^{-1} + b^{-1} + c^{-1})]^{1/2} \geq x^{1/2} + y^{1/2} + z^{1/2}$$

The point  $X_{76} = a^{-3} : b^{-3} : c^{-3}$  maximizes  $S(2/3)$  :

$$[4\Delta^2(a^{-2} + b^{-2} + c^{-2})]^{1/3} \geq x^{2/3} + y^{2/3} + z^{2/3}$$

The point  $X_{366} = a^{-1/2} : b^{-1/2} : c^{-1/2}$  minimizes  $S(-1)$  :

$$\frac{(a^{1/2} + b^{1/2} + c^{1/2})^2}{2\Delta} \leq x^{-1} + y^{-1} + z^{-1}$$

In these examples, the extreme point is of the form  $P(t) = a^t : b^t : c^t$ . If  $ABC$  has distinct sidelengths then  $P(t)$  traces the *power curve*, as in Figure 2. Taking limits as  $t \rightarrow \infty$  and  $t \rightarrow -\infty$  shows that this curve is tangent to the shortest and longest sides of  $ABC$  at the vertices opposite those sides. The incenter  $I = P(0)$  lies on the power curve, and for each  $t \neq 0$ , the points  $P(t)$  and  $P(-t)$  are a pair of isogonal conjugates; *e.g.*, for  $t = 1$ , they are the symmedian point and the centroid.

Here are two more choices of  $q$  : the solutions of  $1/(q-1) = q$ , namely the golden ratio  $\frac{1}{2}(1 + \sqrt{5})$  and  $\frac{1}{2}(1 - \sqrt{5})$ . These are the only values of  $q$  for which the point  $a^q : b^q : c^q$  minimizes  $x^q + y^q + z^q$ .

We turn now to a different sort of inequality for points inside  $ABC$ . As a special case of the arithmetic-mean–geometric-mean inequality [1],

$$27uvw \leq (u + v + w)^3$$

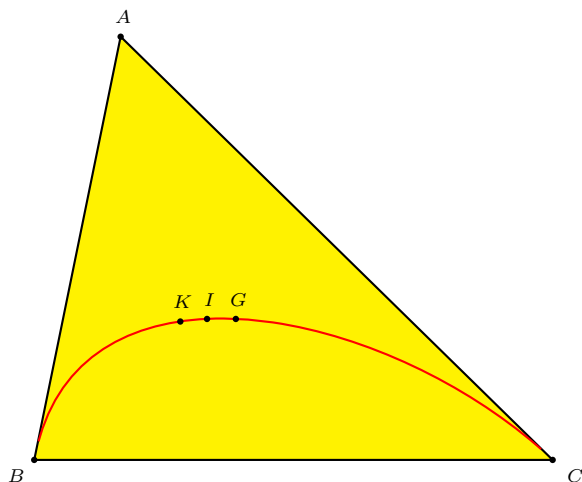


Figure 2.

for all positive  $u, v, w$ , so that if  $a, b, c, x, y, z$  are arbitrary positive real numbers, then

$$xyz \leq \frac{(ax + by + cz)^3}{27abc}. \quad (5)$$

The distances from the centroid ( $X_2 = a^{-1} : b^{-1} : c^{-1}$ , labeled  $G$  in Figures 1 and 2) to the sidelines  $BC, CA, AB$  are given by

$$\left( \frac{2\Delta}{3a}, \frac{2\Delta}{3b}, \frac{2\Delta}{3c} \right),$$

so that their product is  $8\Delta^3/(27abc)$ . Since  $ax + by + cz = 2\Delta$ , we may interpret (5) thus: *the centroid maximizes the product  $xyz$ .*

What other extrema are attained by special points inside a triangle? To seek others, it is easy and entertaining to devise computer-based searches – either visual searches with dynamic distance measurements using *Cabri* or *The Geometer's Sketchpad*, or algebraic using trilinears or other coordinates. Such searches prompt an intriguing problem: to determine conditions on a function in  $x, y, z$  for which an extreme value is attained inside  $ABC$ . If the function is homogeneous and symmetric in  $a, b, c$ , as in the examples considered above, then must such an extreme point be a triangle center? Is there a simple example in which the extreme point is not on the power curve?

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# Constructions with Inscribed Ellipses in a Triangle

Nikolaos Dergiades

**Abstract.** We give a simple construction of the axes and foci of an inscribed ellipse with prescribed center, and as an application, a simple solution of the problem of construction of a triangle with prescribed circumcevian triangle of the centroid.

## 1. Construction of the axes and foci of an inscribed ellipse

Given a triangle  $ABC$ , we give a simple construction of the axes and foci of an inscribed ellipse with given center or perspector. If a conic touches the sides  $BC$ ,  $CA$ ,  $AB$  at  $X$ ,  $Y$ ,  $Z$  respectively, then the lines  $AX$ ,  $BY$ ,  $CZ$  are concurrent at a point  $P$  called the perspector of the conic. The center  $M$  of the conic is the complement of the isotomic conjugate of  $P$ . In homogeneous barycentric coordinates, if  $P = (p : q : r)$ , then

$$M = \left( \frac{1}{q} + \frac{1}{r} : \frac{1}{r} + \frac{1}{p} : \frac{1}{p} + \frac{1}{q} \right) = (p(q+r) : q(r+p) : r(p+q)).$$

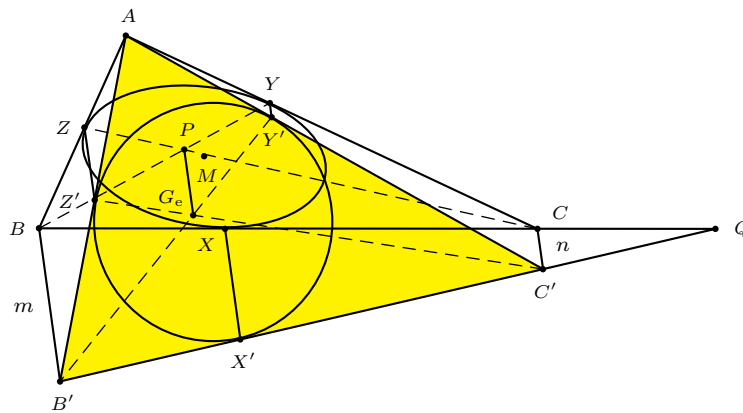


Figure 1

Consider triangle  $ABC$  and its inscribed ellipse with center  $M = (u : v : w)$  as the orthogonal projections of a triangle  $AB'C'$  and its incircle. We may take  $A' = A$  and assume  $BB' = m$ ,  $CC' = n$ . If the incircle of  $AB'C'$  touches  $B'C'$ ,  $C'A$  and  $AB'$  at  $X'$ ,  $Y'$ ,  $Z'$  respectively, then  $X$ ,  $Y$ ,  $Z$  are the orthogonal projections of  $X'$ ,  $Y'$ ,  $Z'$ . It follows that the perspector  $P$  is the projection of

the Gergonne point  $G_e$  of  $AB'C'$ . Suppose triangle  $AB'C'$  has sides  $B'C' = a'$ ,  $C'A = b'$  and  $AB' = c'$ , then

$$b' + c' - a' : c' + a' - b' : a' + b' - c' = \frac{1}{p} : \frac{1}{q} : \frac{1}{r}.$$

It follows that

$$a' : b' : c' = \frac{1}{q} + \frac{1}{r} : \frac{1}{r} + \frac{1}{p} : \frac{1}{p} + \frac{1}{q} = u : v : w.$$

From this we draw a remarkable conclusion.

**Proposition 1.** *If a triangle and an inscribed ellipse are the orthogonal projections of a triangle and its incircle, the sidelengths of the latter triangle are proportional to the barycentric coordinates of the center of the ellipse.*

Since  $a', b', c'$  satisfy the triangle inequality, the center  $M = (u : v : w)$  of the ellipse is an interior point of the medial triangle of  $BAC$ .

We determine the relative positions of  $ABC$  and  $AB'C'$  leading to a simple construction of the axes and foci of the inscribed ellipse of given center  $M$ . We first construct two triangles  $A_+BC$  and  $A_-BC$  with sidelengths proportional to the barycentric coordinates of  $M$ .

**Construction 2.** *Let  $M = (u : v : w)$  be a point in the interior of the medial triangle of  $ABC$ , with cevian triangle  $A_0B_0C_0$ . Construct*

(i) *the parallels of  $BB_0$  and  $CC_0$  through  $A$  to intersect  $BC$  at  $D$  and  $E$  respectively,*

(ii) *the circles  $B(D)$  and  $C(E)$  to intersect at two points  $A_+$  and  $A_-$  symmetric in the line  $BC$ . Label  $A_+$  the one on the opposite side of  $BC$  as  $A$ .*

*Each of the triangles  $A_+BC$  and  $A_-BC$  have sidelengths  $u : v : w$ .*

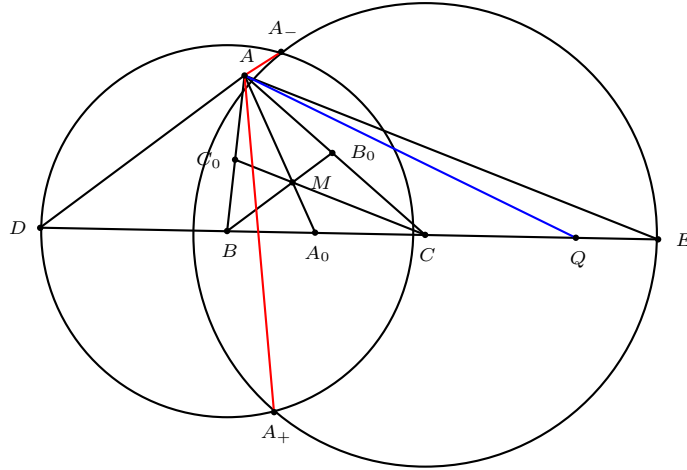


Figure 2

*Proof.* Note that

$$\begin{aligned}\frac{BC}{CA_-} &= \frac{BC}{CE} = \frac{BC_0}{C_0A} = \frac{u}{v}, \\ \frac{BC}{A_-B} &= \frac{BC}{DB} = \frac{B_0C}{AB_0} = \frac{u}{w}.\end{aligned}$$

From these,  $BC : CA_- : A_-B = u : v : w$ ; similarly for  $BC : CA_+ : A_+B$ .  $\square$

**Lemma 3.** *The lengths of  $AA_+$  and  $AA_-$  are given by*

$$AA_+ = \frac{\sqrt{Q + 16\Delta\Delta'}}{\sqrt{2}u} \quad \text{and} \quad AA_- = \frac{\sqrt{Q - 16\Delta\Delta'}}{\sqrt{2}u},$$

where

$$Q = (b^2 + c^2 - a^2)u^2 + (c^2 + a^2 - b^2)v^2 + (a^2 + b^2 - c^2)w^2, \quad (1)$$

and  $\Delta'$  is the area of the triangle with sidelengths  $u, v, w$ .

*Proof.* Applying the law of cosines to triangle  $ABA_-$ , we have

$$\begin{aligned}AA_-^2 &= AB^2 + BA_-^2 - 2AB \cdot BA_- \cos(\angle ABC - \angle A_-BC) \\ &= c^2 + \left(\frac{aw}{u}\right)^2 - 2c \cdot \frac{aw}{u} (\cos B \cos \angle A_-BC + \sin B \sin \angle A_-BC) \\ &= \frac{c^2u^2 + a^2w^2 - 2ca \cos B \cdot wu \cos \angle A_-BC - 2ca \sin B \cdot wu \sin \angle A_-BC}{u^2} \\ &= \frac{2c^2u^2 + 2a^2w^2 - (c^2 + a^2 - b^2)(w^2 + u^2 - v^2) - 16\Delta\Delta'}{2u^2} \\ &= \frac{Q - 16\Delta\Delta'}{2u^2}.\end{aligned}$$

The case of  $AA_+$  is similar.  $\square$

Let  $Q$  be the intersection of the lines  $BC$  and  $B'C'$ . The line  $AQ$  is the intersection of the planes  $ABC$  and  $AB'C'$ . The diameters of the incircle of  $AB'C'$  parallel and orthogonal to  $AQ$  project onto the major and minor axes of the inscribed ellipse.

**Proposition 4.** *The line  $AQ$  is the internal bisector of angle  $A_+AA_-$ .*

*Proof.* The sidelengths of triangle  $A_-BC$  are  $BC = a$ ,  $CA_- = \frac{av}{u} = \frac{ab'}{a'}$  and  $A_-B = \frac{aw}{u} = \frac{ac'}{a'}$ . Set up a Cartesian coordinate system so that  $A = (x_1, y_1)$ ,  $B = (-a, 0)$ ,  $C = (0, 0)$ ,  $A_+ = (x_0, -y_0)$  and  $A_- = (x_0, y_0)$ , where

$$x_0 = -CA_- \cos \angle A_-CB = -\frac{a'^2 + b'^2 - c'^2}{2a'^2} \cdot a, \quad (2)$$

$$x_1 = -b \cos C = -\frac{a^2 + b^2 - c^2}{2a}. \quad (3)$$

Since the lines  $AA_+$  and  $AA_-$  have equations

$$\begin{aligned}(y_1 + y_0)x - (x_1 - x_0)y - (x_1y_0 + x_0y_1) &= 0, \\ (y_1 - y_0)x - (x_1 - x_0)y + (x_1y_0 - x_0y_1) &= 0,\end{aligned}$$

a bisector of angle  $A_+AA_-$  meets  $BC$  at a point with coordinates  $(x, 0)$  satisfying

$$\frac{((y_1 - y_0)x + (x_1y_0 - x_0y_1))^2}{(y_1 - y_0)^2 + (x_1 - x_0)^2} = \frac{((y_1 + y_0)x - (x_1y_0 + x_0y_1))^2}{(y_1 + y_0)^2 + (x_1 - x_0)^2}.$$

Simplifying this into

$$(x_1 - x_0)x^2 + (x_0^2 + y_0^2 - x_1^2 - y_1^2)x + x_0(x_1^2 + y_1^2) - x_1(x_0^2 + y_0^2) = 0,$$

and making use of (2) and (3), we obtain

$$\begin{aligned} & \left( \frac{a'^2 + b'^2 - c'^2}{2a'^2} - \frac{a^2 + b^2 - c^2}{2a^2} \right) x^2 + a \left( \frac{b'^2}{a'^2} - \frac{b^2}{a^2} \right) x \\ & - \left( b^2 \cdot \frac{a'^2 + b'^2 - c'^2}{2a'^2} - b'^2 \cdot \frac{a^2 + b^2 - c^2}{2a^2} \right) = 0. \end{aligned} \quad (4)$$

Now, since

$$a'^2 = (m - n)^2 + a^2, \quad b'^2 = n^2 + b^2, \quad c'^2 = m^2 + c^2, \quad (5)$$

we reorganize (4) into the form

$$((m - n)x - na)((m(a^2 + b^2 - c^2) + n(c^2 + a^2 - b^2))x - (n(b^2 + c^2 - a^2) - 2mb^2)a) = 0.$$

Note that the two roots

$$X_1 = \frac{na}{m - n} \quad \text{and} \quad X_2 = \frac{(n(b^2 + c^2 - a^2) - 2mb^2)a}{m(a^2 + b^2 - c^2) + n(c^2 + a^2 - b^2)}$$

satisfy

$$(x_0 - x_1)(X_1 - X_2) = \frac{m^2b^2 - mn(b^2 + c^2 - a^2) + n^2c^2}{a^2 + (m - n)^2} > 0,$$

since the discriminant of  $m^2b^2 - mn(b^2 + c^2 - a^2) + n^2c^2$  (as a quadratic form in  $m, n$ ) is  $(b^2 + c^2 - a^2)^2 - 4b^2c^2 = -4b^2c^2 \sin^2 A < 0$ . We conclude that if  $x_1 < x_0$ , then  $X_1 > X_2$  and  $X_1$  is the root that corresponds to the internal bisector. Since  $\frac{X_1}{X_1 + a} = \frac{n}{m}$ , this intercept is the intersection  $Q$  of the lines  $BC$  and  $BC'$ .  $\square$

**Theorem 5.** *The inscribed ellipse of triangle  $ABC$  with center  $M = (u : v : w)$  (inside the medial triangle) has semiaxes given by*

$$a_{\max} = \frac{u(AA_+ + AA_-)}{2(u + v + w)} \quad \text{and} \quad a_{\min} = \frac{u(AA_+ - AA_-)}{2(u + v + w)}.$$

*Proof.* Making use of (5), we have

$$2mn = m^2 + n^2 - (m - n)^2 = (b'^2 + c'^2 - a'^2) - (b^2 + c^2 - a^2).$$

It follows that

$$4(b'^2 - b^2)(c'^2 - c^2) = ((b'^2 + c'^2 - a'^2) - (b^2 + c^2 - a^2))^2,$$

and

$$\begin{aligned}
 & 4b'^2c'^2 - (b'^2 + c'^2 - a'^2)^2 + 4b^2c^2 - (b^2 + c^2 - a^2)^2 \\
 &= 4b^2c'^2 + 4c^2b'^2 - 2(b^2 + c^2 - a^2)(b'^2 + c'^2 - a'^2) \\
 &= 2((b^2 + c^2 - a^2)a'^2 + (c^2 + a^2 - b^2)b'^2 + (a^2 + b^2 - c^2)c'^2). \quad (6)
 \end{aligned}$$

Note that  $4b^2c^2 - (b^2 + c^2 - a^2)^2 = 4b^2c^2(1 - \cos^2 A) = 4b^2c^2 \sin^2 A = 16\Delta^2$  and similarly,  $4b'^2c'^2 - (b'^2 + c'^2 - a'^2)^2 = 16\Delta(AB'C')^2$ . Let  $\rho$  and  $\rho'$  be the inradii of triangle  $AB'C'$  and the one with sidelengths  $u, v, w$ . These two triangles have ratio of similarity  $\frac{\rho}{\rho'} = \frac{\rho(u+v+w)}{2\Delta'}$ , we have

$$\Delta(AB'C') = \Delta' \left( \frac{\rho(u+v+w)}{2\Delta} \right)^2 = \frac{\rho^2(u+v+w)^2}{4\Delta'}.$$

With this, (6) can be rewritten as

$$2(u+v+w)^4\rho^4 - (u+v+w)^2\mathcal{Q}\rho^2 + 32\Delta^2\Delta'^2 = 0.$$

This has roots  $\pm\rho_1$  and  $\pm\rho_2$ , where

$$\rho_1 = \frac{\sqrt{\mathcal{Q} + 16\Delta\Delta'} + \sqrt{\mathcal{Q} - 16\Delta\Delta'}}{2\sqrt{2}(u+v+w)}, \quad \rho_2 = \frac{\sqrt{\mathcal{Q} + 16\Delta\Delta'} - \sqrt{\mathcal{Q} - 16\Delta\Delta'}}{2\sqrt{2}(u+v+w)}.$$

Note that  $\rho_1\rho_2 = \frac{4\Delta\Delta'}{(u+v+w)^2}$ , and

$$\frac{a_{\min}}{a_{\max}} = \frac{\Delta}{\Delta(AB'C')} = \frac{4\Delta\Delta'}{\rho^2(u+v+w)^2} = \frac{\rho_1\rho_2}{\rho^2}.$$

Since  $\rho_1 > \rho_2$ , it follows that  $a_{\max} = \rho = \rho_1$  and  $a_{\min} = \rho_2$ . □

Now we construct the axes and foci of an inscribed ellipse.

**Construction 6.** Let  $M$  be a point in the interior of the medial triangle of  $ABC$ , and  $A_+BC$ ,  $A_-BC$  triangles with sidelengths proportional to the barycentric coordinates of  $M$  (see Construction 2). Construct

- (1) the internal bisector of angle  $A_+AA_-$  to intersect the line  $BC$  at  $Q$ ,
- (2) the parallel of  $AQ$  through  $M$ . This is the major axis of the ellipse.

Further construct

- (3) the orthogonal projection  $S$  of  $Q$  on  $AA_+$ ,
- (4) the parallel through  $M$  to  $AA_+$  to intersect  $A_1S$  at  $T$ ,
- (5) the circle  $M(T)$ . This is the auxiliary circle of the ellipse.

Finally, construct

- (6) the perpendiculars to the sides at the intersections with the circle  $M(T)$  to intersect the major axis at  $F$  and  $F'$ . These are the foci of the ellipse. See Figure 3.



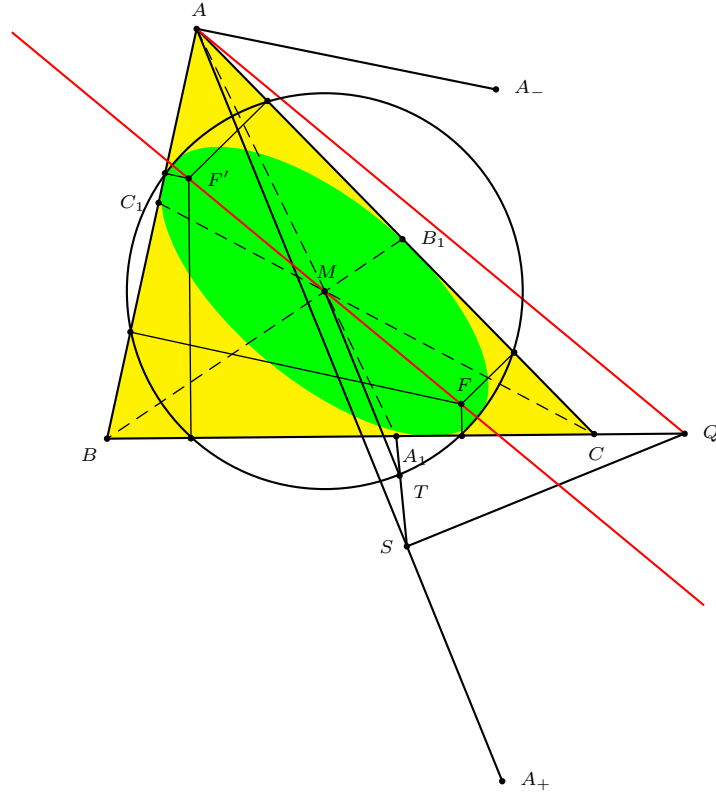


Figure 3.

## 2. The circumcevian triangle of the centroid

For the Steiner inellipse with center  $G = (1 : 1 : 1)$ , the triangles  $A_+BC$  and  $A_-BC$  are equilateral triangles, and

$$AA_+ = \sqrt{\frac{a^2 + b^2 + c^2 + 4\sqrt{3}\Delta}{2}} \quad \text{and} \quad AA_- = \sqrt{\frac{a^2 + b^2 + c^2 - 4\sqrt{3}\Delta}{2}}.$$

By Theorem 5,

$$a_{\max} = \frac{AA_+ + AA_-}{6} \quad \text{and} \quad a_{\min} = \frac{AA_+ - AA_-}{6}.$$

**Theorem 7.** Let  $G$  be the centroid of  $ABC$ , with circumcevian triangle  $A_1B_1C_1$ . The pedal circle of  $G$  relative to  $A_1B_1C_1$  has center the centroid  $G_1$  of  $A_1B_1C_1$ . Hence,  $G$  is a focus of the Steiner inellipse of triangle  $A_1B_1C_1$ .

*Proof.* Let  $a, b, c$  and  $a_1, b_1, c_1$  be the sidelengths of triangles  $ABC$  and  $A_1B_1C_1$  respectively. Let  $x = AG, y = BG$  and  $z = CG$ . By Apollonius' theorem,

$$x^2 = \frac{2b^2 + 2c^2 - a^2}{9}, \quad y^2 = \frac{2c^2 + 2a^2 - b^2}{9}, \quad z^2 = \frac{2a^2 + 2b^2 - c^2}{9}.$$

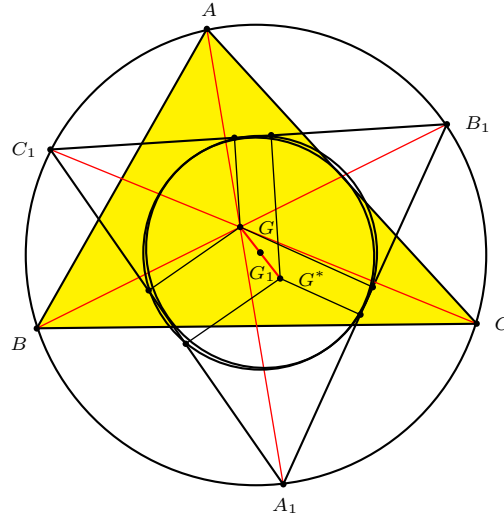


Figure 4

If  $\mathcal{P}$  is the power of  $G$  relative to circumcircle of  $ABC$ , then

$$GA_1 = \frac{\mathcal{P}}{x}, \quad GB_1 = \frac{\mathcal{P}}{y}, \quad GC_1 = \frac{\mathcal{P}}{z}.$$

From the similarity of triangles  $GBC$  and  $GC_1B_1$ , we have

$$\frac{a_1}{a} = \frac{GC_1}{BG} = \frac{CG \cdot GC_1}{BG \cdot GC} = \frac{\mathcal{P}}{yz}.$$

From this we obtain  $a_1$ , and similarly  $b_1$  and  $c_1$ :

$$a_1 = \frac{a\mathcal{P}}{yz}, \quad b_1 = \frac{b\mathcal{P}}{zx}, \quad c_1 = \frac{c\mathcal{P}}{xy}. \quad (7)$$

The homogeneous barycentric coordinates of  $G$  relative to  $A_1B_1C_1$  are

$$\begin{aligned} & \Delta GB_1C_1 : \Delta GC_1A_1 : \Delta GA_1B_1 \\ &= \frac{\Delta GB_1C_1}{\Delta GBC} : \frac{\Delta GC_1A_1}{\Delta GCA} : \frac{\Delta GA_1B_1}{\Delta GAB} \\ &= \frac{GB_1 \cdot GC_1}{GB \cdot GC} : \frac{GC_1 \cdot GA_1}{GC \cdot GA} : \frac{GA_1 \cdot GB_1}{GA \cdot GB} \\ &= \frac{(BG \cdot GB_1)(CG \cdot GC_1)}{BG^2 \cdot CG^2} : \frac{(CG \cdot GC_1)(AG \cdot GA_1)}{CG^2 \cdot AG^2} : \frac{(AG \cdot GA_1)(BG \cdot GB_1)}{AG^2 \cdot BG^2} \\ &= \frac{\mathcal{P}^2}{y^2 z^2} : \frac{\mathcal{P}^2}{z^2 x^2} : \frac{\mathcal{P}^2}{x^2 y^2} \\ &= x^2 : y^2 : z^2. \end{aligned}$$

The isogonal conjugate of  $G$  (relative to triangle  $A_1B_1C_1$ ) is the point

$$G^* = \frac{a_1^2}{x^2} : \frac{b_1^2}{y^2} : \frac{c_1^2}{z^2} = a^2 : b^2 : c^2.$$

We find the midpoint of  $GG^*$  by working with absolute barycentric coordinates. Since  $x^2 + y^2 + z^2 = \frac{1}{3}(a^2 + b^2 + c^2)$ , and

$$3x^2 + a^2 = 3y^2 + b^2 = 3z^2 + c^2 = \frac{2(a^2 + b^2 + c^2)}{3},$$

we have for the first component,

$$\frac{1}{2} \left( \frac{x^2}{x^2 + y^2 + z^2} + \frac{a^2}{a^2 + b^2 + c^2} \right) = \frac{3x^2 + a^2}{2(a^2 + b^2 + c^2)} = \frac{1}{3}.$$

Similarly, the other two components are also equal to  $\frac{1}{3}$ . It follows that the midpoint of  $GG^*$  is the centroid  $G_1$  of  $A_1B_1C_1$ . As such, it is the center of the pedal circle of the points  $G$  and  $G^*$ , which are the foci of an inconic that has center and perspector  $G_1$ . This inconic is the Steiner inellipse of triangle  $A_1B_1C_1$ .  $\square$

### 3. A construction problem

Theorem 7 gives an elementary solution to a challenging construction problem in Altshiller-Court [1, p.292, Exercise 11]. The interest on this problem was rejuvenated by a recent message on the Hyacinthos group [2].

**Problem 8.** *Construct a triangle given, in position, the traces of its medians on the circumcircle.*

*Solution.* Let a given triangle  $A_1B_1C_1$  be the circumcevian triangle of the (unknown) centroid  $G$  of the required triangle  $ABC$ . We construct the equilateral triangles  $A_+B_1C_1$  and  $A_-B_1C_1$  on the segment  $B_1C_1$ . Let  $G_1$  be the centroid of  $A_1B_1C_1$  and  $r = \frac{1}{6}(A_1A_+ + A_1A_-)$ . The circle  $G_1(r)$  is the auxiliary circle of the Steiner inellipse, hence the pedal circle of  $G$  (one of the foci) relative to  $A_1B_1C_1$ . From the intersections of this circle with the sides of  $A_1B_1C_1$  and the parallel from  $G_1$  to the internal bisector of angle  $A_+A_1A_-$  we determine the point  $G$  (two solutions). The second intersections of the circle with the lines  $A_1G$ ,  $B_1G$ ,  $C_1G$  give the points  $A$ ,  $B$ ,  $C$ .

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## Integer Triangles with $R/r = N$

Allan J. MacLeod

**Abstract.** We consider the problem of finding integer-sided triangles with  $R/r$  an integer, where  $R$  and  $r$  are the radii of the circumcircle and incircle respectively. We find that such triangles are relatively rare.

### 1. Introduction

Let  $ABC$  be a triangle with sides of integer length  $a, b, c$ . Possibly the two most fundamental circles associated with the triangle are the circumcircle which passes through the three vertices, and the incircle which has the three sides as tangents. The radii of these circles are denoted  $R$  and  $r$  respectively. If we look at the basic equilateral triangle with sides of length 1, it is clear that both circles will have their centers at the centroid of the triangle. Simple trigonometry gives  $R = \frac{1}{\sqrt{3}}$  and  $r = \frac{1}{\sqrt{12}}$  so that  $\frac{R}{r} = 2$ . It is interesting to speculate whether  $\frac{R}{r}$  (a positive dimensionless quantity) can be an integer value for larger integers.

If we denote the distance between the centers of these two circles as  $d$ , then it is a standard result in triangle geometry that  $d^2 = R(R - 2r)$ , so that  $\frac{R}{r} \geq 2$ . For the equilateral triangle,  $d = 0$ , and it is easy to prove the converse – if  $d = 0$  then the triangle must be equilateral.

Basic trigonometry gives the formulae

$$R = \frac{abc}{4\Delta}, \quad r = \frac{\Delta}{s} \quad (1)$$

with  $s$  being the semi-perimeter  $\frac{1}{2}(a + b + c)$  and  $\Delta$  the area. Thus,

$$\frac{R}{r} = \frac{abcs}{4\Delta^2}. \quad (2)$$

Now,  $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ , so that

$$\frac{R}{r} = \frac{2abc}{(a+b-c)(a+c-b)(b+c-a)}. \quad (3)$$

If  $\frac{R}{r} = N$ , with  $N$  a strictly positive integer, we must find integers  $a, b, c$  such that

$$\frac{2abc}{(a+b-c)(a+c-b)(b+c-a)} = N, \quad (4)$$

which bears a very strong resemblance to the integer representation problems in [1, 2]. In all cases, we look to express  $N$  as a ratio of two homogeneous cubics in 3 variables.

Expressing equation (4) as a single fraction, we derive the cubic

$$Na^3 - N(b+c)a^2 - (b^2N - 2bc(N+1) + c^2N)a + N(b+c)(b-c)^2 = 0. \quad (5)$$

We cannot get very far with this form, but we can proceed quickly if we replace  $c$  by  $2s - a - b$ .

$$2(4Ns - b(4N+1))a^2 - 2(b^2(4N+1) - 2bs(6N+1) + 8Ns^2)a + 8Ns(b-s)^2 = 0, \quad (6)$$

and it should be noted that we only need to consider rational  $a, b, c$  and scale to integer values, since this would not affect the value of  $\frac{R}{r}$ .

Since we want rational values for  $a$ , this quadratic must have a discriminant which is a rational square, so that there must be a rational  $d$  with

$$d^2 = (4N+1)^2b^4 - 4(2N+1)(4N+1)b^3s + 4(4N^2+8N+1)b^2s^2 - 16Nbs^3. \quad (7)$$

Define  $d = \frac{s^2y}{4N+1}$  and  $g = \frac{sx}{4N+1}$ , giving the quartic

$$y^2 = x^4 - 4(2N+1)x^3 + 4(4N^2+8N+1)x^2 - 16N(4N+1)x, \quad (8)$$

which can be transformed to an equivalent elliptic curve by a birational transformation.

We find the curve to be

$$E_N : v^2 = u^3 + 2(2N^2 - 2N - 1)u^2 + (4N+1)u \quad (9)$$

with the transformation

$$\frac{b}{s} = \frac{v - (4N+1 - (2N+1)u)}{(u-1)(4N+1)}. \quad (10)$$

As an example, consider the case of  $N = 7$ , so that  $E_7$  is the curve  $v^2 = u^3 + 166u^2 + 29u$ . It is moderately easy to find the rational point  $(u, v) = (\frac{29}{169}, \frac{6902}{2197})$  which lies on the curve. This gives  $\frac{b}{s} = \frac{63}{65}$ , and the equation for  $a$  is  $-14a^2 + 938a + 14560 = 0$ , from which we find the representation  $(a, b, c) = (-13, 63, 80)$ , clearly not giving a real triangle.

## 2. Elliptic Curve Properties

We have shown that integer solutions to equation (4) are related to rational points on the curves  $E_N$  defined in equation (9). The problem is that equation (4) can be satisfied by integers which could be negative as in the representation problems of [1, 2].

To find triangles for the original form of the problem, we must enforce extra constraints on  $a, b, c$ . To investigate the effect of this, we must examine the properties of the curves.

We first note that the discriminant of the curve is given by

$$D = 256N^3(N-2)(4N+1)^2$$

so that the curve is singular for  $N = 2$ , which we now exclude from possible values, having seen before that the equilateral triangle gives  $N = 2$ .

Given the standard method of addition of rational points on an elliptic curve, see [3], the set of rational points forms a finitely-generated group. The points of finite order are called torsion points, and we look for these first. The point at infinity is considered the identity of the group. The form of  $E_N$  implies, by the Nagell-Lutz theorem, that the coordinates of the torsion points are integers.

The points of order 2 are integer solutions of

$$u^3 + 2(2N^2 - 2N - 1)u^2 + (4N + 1)u = 0$$

and it is easy to see that the roots are  $u = 0$  and  $u = 2N + 1 - 2N^2 \pm \sqrt{N(N - 2)}$ . The latter two are clearly irrational and negative for  $N$  a positive integer, so there is only one point of order 2.

The fact that there are 3 real roots implies that the curve consists of two components - an infinite component for  $u \geq 0$  and a closed finite component (usually called the “egg”).

Points of order 2 allow one to look for points of order 4, since if  $P = (j, k)$  has order 4,  $2P$  must have order 2. For a curve of the form given by (9), the  $u$ -coordinate of  $2P$  must be of the form  $\frac{(j^2 - 4N - 1)^2}{4k^2}$ . Thus, we must have  $j^2 = 4N + 1 = (2t + 1)^2$ , so  $N = t^2 + t$ . Substituting these values into (9), we see that we get a rational point only if  $t(t + 2)$  is an integer square, which never happens. There are thus no points of order 4.

Points of order 3 are points of inflexion of the curve. Simple analysis shows that there are points of inflexion at  $(1, \pm 2N)$ . We can also use the doubling formula to show that there are 2 points of order 6, namely  $(4N + 1, \pm 2N(4N + 1))$ .

The presence of points of orders 2, 3, 6, together with Mazur’s theorem on possible torsion structures, shows that the torsion subgroup must be isomorphic to  $\mathbb{Z}_6$  or  $\mathbb{Z}_{12}$ .

The latter possibility would need a point  $P$  of order 12, with  $2P$  of order 6, and thus an integer solution of

$$\frac{(j^2 - (4N + 1))^2}{4k^2} = 4N + 1$$

implying that  $N = t^2 + t$ . Substituting into this equation, we get an integer solution if either  $t^2 - 1$  or  $t(t + 2)$  are integer squares - which they are not, unless  $N = 2$  which has been excluded previously.

Thus the torsion subgroup is isomorphic to  $\mathbb{Z}_6$ , with finite points  $(0, 0)$ ,  $(1, \pm 2N)$ ,  $(4N + 1, \pm 2N(4N + 1))$ . Substituting these points into the  $\frac{b}{s}$  transformation formula leads to  $\frac{b}{s}$  being 0, 1 or undefined, none of which lead to a practical solution of the problem.

Thus, we must look at the second type of rational point - those of infinite order.

### 3. Practical Solutions

If there are rational points of infinite order, Mordell's theorem implies that there are  $r$  generator points  $G_1, \dots, G_r$ , such that any rational point  $P$  can be written

$$P = T + n_1 G_1 + \dots + n_r G_r \quad (11)$$

with  $T$  one of the torsion points, and  $n_1, \dots, n_r$  integers. The value  $r$  is called the *rank* of the curve.

Unfortunately, there is no simple method of determining firstly the rank, and then the generators. We used a computational approach to estimate the rank using the Birch and Swinnerton-Dyer conjecture. A useful summary of the computations involved can be found in the paper of Silverman [4].

Applying the calculations to a range of values of  $N$ , we find several examples of curves with rank greater than zero, mostly with rank 1. A useful byproduct of the calculations in the rank 1 case is an estimate of the height of the generator, where the height gives an indication of number of digits in the rational coordinates. For curves of rank greater than 1 and rank 1 curves with small height, we can reasonably easily find generators. However, when we backtrack the calculations to solutions of the original problem, we hit a significant problem.

The elliptic curve generators all give solutions to equation (4), but for the vast majority of  $N$  values, these include at least one negative value of  $a, b, c$ . Thus we find extreme difficulty in finding real triangles with strictly positive sides. In fact, for  $3 \leq N \leq 99$ , there are only 2 values of  $N$  where this occurs, at  $N = 26$  and  $N = 74$ .

To investigate this problem, consider the quadratic equation (6), but written as

$$a^2 + (b - 2s)a + \frac{4Ns(b - s)^2}{4Ns - b(4N + 1)}. \quad (12)$$

The sum of the roots of this is clearly  $2s - b$ , but since  $a + b + c = 2s$ , this means that the roots of this quadratic are in fact  $a$  and  $c$ . Positive triangles thus require  $s > 0$ ,  $b > 0$ ,  $2s - b > 0$  and  $4Ns - b(4N + 1) > 0$ , all of which reduce to

$$0 < \frac{b}{s} < \frac{4N}{4N + 1}. \quad (13)$$

Looking at equation (10), we see that the analysis splits first according as  $u > 1$  or  $u < 1$ . Consider first  $u > 1$ , so that, for  $\frac{b}{s} > 0$  we need  $v > 4N + 1 - (2N + 1)u$ . The line  $v = 4N + 1 - (2N + 1)u$  meets  $E_N$  in only two points,  $(1, 2N)$  and  $(4N + 1, -2N(4N + 1))$  with the line actually being a tangent to the curve at the latter point. Thus, if  $u > 1$  we need  $v > 0$  to give points on the curve with  $\frac{b}{s} > 0$ .

For the second half of the inequality with  $u > 1$ , we need  $v < 1 + (2N - 1)u$ . The line  $v = 1 + (2N - 1)u$  meets  $E_N$  only at  $(1, 2N)$ , so none of the points with  $u > 1, v > 0$  are satisfactory. Thus to satisfy (13) we must look in the range  $u < 1$ . Firstly, in the interval  $[0, 1)$ , we have  $\frac{b}{s} > 0$ , since the numerator and denominator of (10) are negative. For the second half, however, we need  $v > 1 + (2N - 1)u$ , but the previous analysis shows this cannot happen.

Thus, the only possible way of achieving real-world triangles is to have points on the egg component. From the previous analysis it is clear that any point on the egg leads to  $\frac{b}{s} > 0$ . For the other part of (10), we must consider where the egg lies in relation to the line  $v = 1 + (2N - 1)u$ . Since the line only meets  $E_N$  at  $u = 1$ , the entire egg either lies above or below the line. The line meets the  $u$ -axis at  $u = \frac{-1}{2N-1}$ , and the extreme left-hand end of the egg is at  $u = 2N + 1 - 2N^2 - \sqrt{N(N-2)}$  which is less than  $-1$  for  $N \geq 3$ . Thus the entire egg lies above the line so  $v > 1 + (2N - 1)u$  and so (10) is satisfied.

Thus, we get a real triangle if and only if  $(u, v)$  is a rational point on  $E_N$  with  $u < 0$ .

Consider now the effect of the addition  $P + T = Q$  where  $P$  lies on the egg and  $T$  is one of the torsion points. All of the five finite torsion points lie on the infinite component. Since  $E_N$  is symmetrical about the  $u$ -axis,  $P, T, -Q$  all lie on a straight line, and since the egg is a closed convex region, simple geometry implies that  $-Q$  and hence  $Q$  must lie on the egg. Similarly, if  $P$  lies on the infinite component, then  $Q$  must also lie on the infinite component.

Geometry also shows that  $2P$  must lie on the infinite component irrespective of where  $P$  lies.

TABLE 1  
Integer sided triangles with  $\frac{R}{r} = N$

$N$	$a$	$b$	$c$
2	1	1	1
26	11	39	49
74	259	475	729
218	115	5239	5341
250	97	10051	10125
314	177487799	55017780825	55036428301
386	1449346321141	2477091825117	3921344505997
394	12017	2356695	2365193
458	395	100989	101251
586	3809	18411	22201
602	833	14703	15523
634	10553413	1234267713	1243789375
674	535	170471	170859
746	47867463	6738962807	6782043733
778	1224233861981	91266858701995	92430153628659
866	3025	5629	8649

This shows that if the generators of  $E_N$  all lie on the infinite component then there is NO rational point on the egg, and hence no real triangle.



We have, for  $1 \leq N \leq 999$ , found 16 examples of integer triangles, which are given in Table 1. There are probably more to be found, but these almost certainly come from rank 1 curves with generators having a large height and therefore difficult to find. I am not sure that the effort to find more examples is worthwhile.

A close look at the values of  $N$  shows that they all satisfy  $N \equiv 2 \pmod{8}$ . Is this always true? If so, why?

#### 4. Nearly-equilateral Triangles

As mentioned in the introduction, if we have an equilateral triangle with side 1 then  $N = 2$ . This suggests investigating how close we can get to  $N = 2$  with non-equilateral integer triangles. We thus investigate

$$\frac{R}{r} = 2 + \frac{1}{M}$$

with  $M$  a positive integer.

We can proceed in an almost identical manner to before, so the precise details are left out, but we use the same names for the lengths and semi-perimeter. The problem is equivalent to finding rational points on the elliptic curve

$$F_M : v^2 = u^3 + (6M^2 + 12M + 4)u^2 + (9M^4 + 4M^3)u \quad (14)$$

with

$$\frac{b}{s} = \frac{v - (9M^3 + 4M^2 - (5M + 2)u)}{(u - M^2)(9M + 4)}.$$

The curves  $F_M$  have an obvious point of order 2 at  $(0, 0)$ , and can be shown to have points of order 3 at  $(M^2, \pm 2M^2(2M + 1))$  and order 6 at  $(9M^2 + 4M, \pm 2M(2M + 1)(9M + 4))$ . In general these are the only torsion points, none of which lead to a practical solution.

For  $M = 2k^2 + 2k$ , however, with  $k > 0$ , the elliptic curve has 3 points of order 2, which lead to the isosceles triangles with  $a = 2k$ ,  $b = c = 2k + 1$ . This shows that we can get as close to  $N = 2$  as we like with an isosceles triangle. If we reverse the process and start with an isosceles triangle, we can show that  $M$  must be of the form  $2k^2 + 2k$ .

The curves  $F_M$  have two components, the infinite one and the egg, and, as before, we can show that real triangles can only come from rational points on the egg. Numerical experiments show that these are much more common than for  $E_N$ .

As an example, the analysis for  $M = 2009$  leads to an elliptic curve of rank 1, with generator of height 33.94, and finally to sides

$$\begin{aligned} a &= 893780436979684590267493037241340104559255616 \\ b &= 877646641306278516279522874129152375921514449 \\ c &= 885805950860882235231118974654122876065715369 \end{aligned}$$

with angles  $A = 60.9045^\circ$ ,  $B = 59.0969^\circ$  and  $C = 59.9987^\circ$ .

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# On the Euler Reflection Point

Cosmin Pohoata

**Abstract.** The Euler reflection point  $E$  of a triangle is known in literature as the common point of the reflections of its Euler line  $OH$  in each of its sidelines, where  $O$  and  $H$  are the circumcenter and the orthocenter of the triangle, respectively. In this note we prove that  $E$  lies on six circles associated with the triangles of Napoleon.

## 1. Introduction

The Euler reflection point  $E$  of a triangle  $ABC$  is the concurrency point of the reflections of the Euler line in the sidelines of the triangle. The existence of  $E$  is justified by the following more general result.

**Theorem 1** (S. N. Collings). *Let  $\rho$  be a line in the plane of a triangle  $ABC$ . Its reflections in the sidelines  $BC$ ,  $CA$ ,  $AB$  are concurrent if and only if  $\rho$  passes through the orthocenter  $H$  of  $ABC$ . In this case, their point of concurrency lies on the circumcircle.*

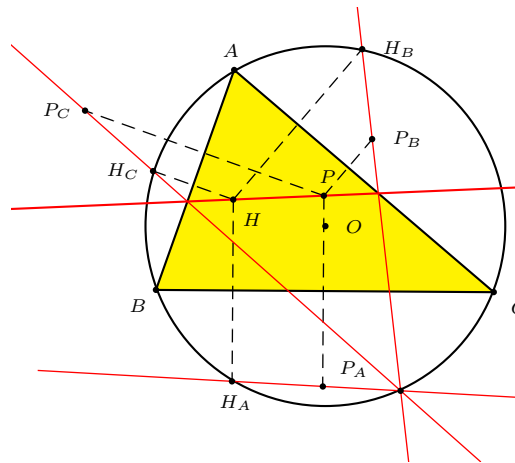


Figure 1

Synthetic proofs of Theorem 1 can be found in [1] and [2]. Known as  $X_{110}$  in Kimberling's list of triangle centers, the Euler reflection point is also the focus of the Kiepert parabola (see [8]) whose directrix is the line containing the reflections of  $E$  in the three sidelines.

Before proceeding to our main theorem, we give two preliminary results.

**Lemma 2** (J. Rigby). *The three lines joining the vertices of a given triangle  $ABC$  with the circumcenter of the triangle formed by the other two vertices of  $ABC$  and the circumcenter  $O$  are concurrent at the isogonal conjugate of the nine-point center.*

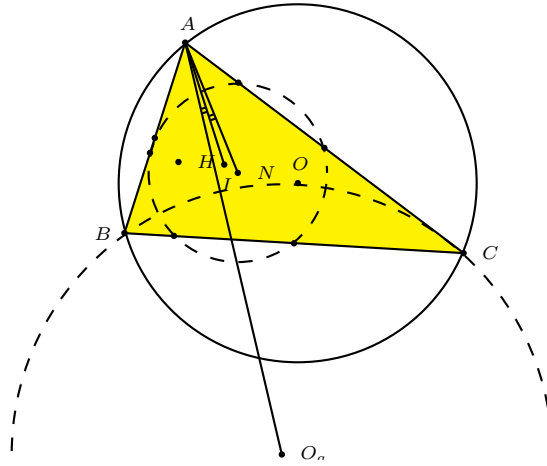


Figure 2.

The common point of these lines is also known as the Kosnita point of triangle  $ABC$ . For a synthetic proof of this result, see [7]. For further references, see [3] and [5].

**Lemma 3.** *The three lines joining the vertices of a given triangle  $ABC$  with the reflections of the circumcenter  $O$  into the opposite sidelines are concurrent at the nine-point center of triangle  $ABC$ .*

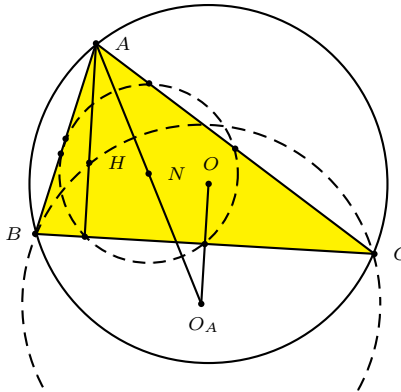


Figure 3.

This is a simple consequence of the fact that the reflection  $O_A$  of  $O$  into the sideline  $BC$  is the circumcenter of triangle  $BHC$ , where  $H$  is the orthocenter of

$ABC$ . In this case, according to the definition of the nine-point circle, the circum-circle of  $BHC$  is the homothetic image of the nine-point circle under  $h(A, 2)$ . See also [4].

## 2. The Euler reflection point and the triangles of Napoleon

Let  $A_+, B_+, C_+, A_-, B_-, C_-$  be the apices of the outer and inner equilateral triangles erected on the sides  $BC, CA$  and  $AB$  of triangle  $ABC$ , respectively. Denote by  $N_A, N_B, N_C, N'_A, N'_B, N'_C$  the circumcenters of triangles  $BCA_+, CAB_+, BCA_-, CAB_-, ABC_+, ABC_-$ , respectively. The triangles  $N_A N_B N_C$  and  $N'_A N'_B N'_C$  are known as the two triangles of Napoleon (the outer and the inner).

**Theorem 4.** *The circumcircles of triangles  $AN_B N_C, BN_C N_A, CN_A N_B, AN'_B N'_C, BN'_C N'_A, CN'_A N'_B$  are concurrent at the Euler reflection point  $E$  of triangle  $ABC$ .*

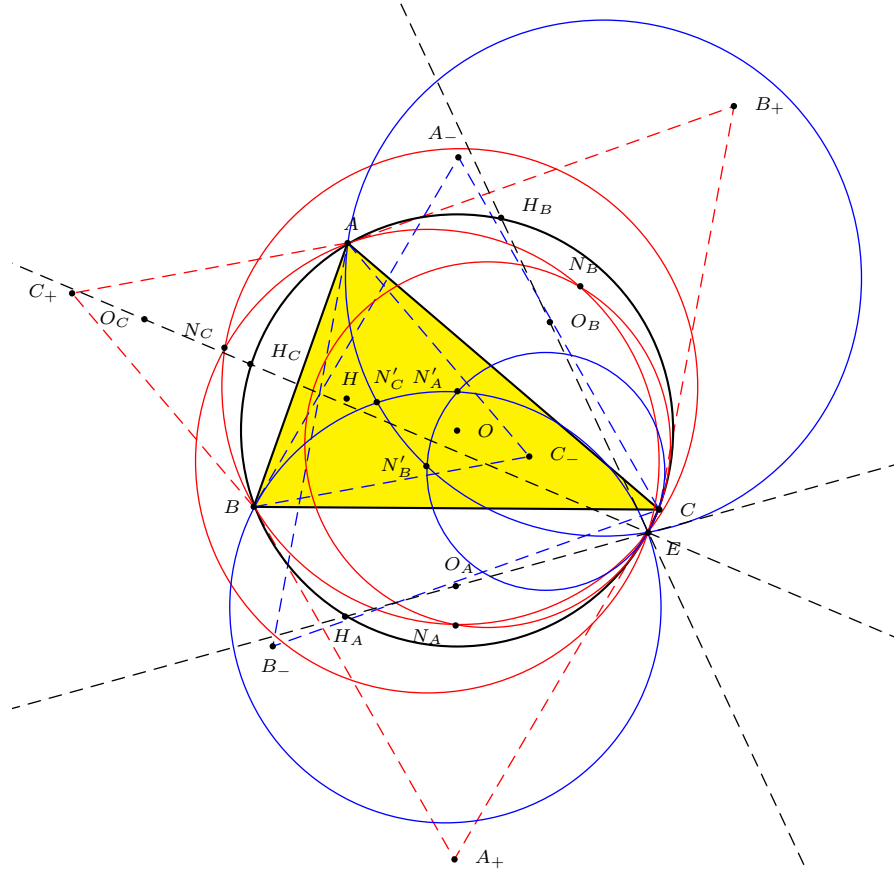


Figure 4

*Proof.* We shall show that each of these circles contains  $E$ . It is enough to consider the circle  $AB_BN_C$ .

Denote by  $O_B, O_C$  the reflections of the circumcenter  $O$  into the sidelines  $CA$  and  $AB$ , respectively. The lines  $EO_B, EO_C$  are the reflections of the Euler line  $OH$  in the sidelines  $CA$  and  $AB$ , respectively. Computing directed angles, we have

$$\begin{aligned} (EO_C, EO_B) &= (EO_C, OH) + (OH, EO_B) \\ &= 2(AB, OH) + 2(OH, AC) \\ &= 2(AB, AC) \pmod{\pi}. \end{aligned}$$

On the other hand,

$$(AO_C, AO_B) = (AO_C, AO) + (AO, AO_B) = 2(AB, AO) + 2(AO, AC) = 2(AB, AC).$$

Therefore, the quadrilateral  $O_CAO_BE$  is cyclic. We show that the centers of the three circles  $O_BAO_C$ ,  $ABC$  and  $AN_BN_C$  are collinear. Since they all contain  $A$ , it follows that they are coaxial with two common points. Since  $E$  lies on the first two circles, it must also lie on the the third circle  $AN_BN_C$ .  $\square$

**Proposition 5.** *Let  $ABC$  be a triangle with circumcenter  $O$  and orthocenter  $H$ . Consider the points  $Y$  and  $Z$  on the sides  $CA$  and  $AB$  respectively such that the directed angles  $(AC, HY) = -\frac{\pi}{3}$  and  $(AB, HZ) = \frac{\pi}{3}$ . Let  $U$  be the circumcenter of triangle  $HYZ$ .*

- (a)  $A_-, U, H$  are collinear.
- (b)  $A, U, O_A$  are collinear.

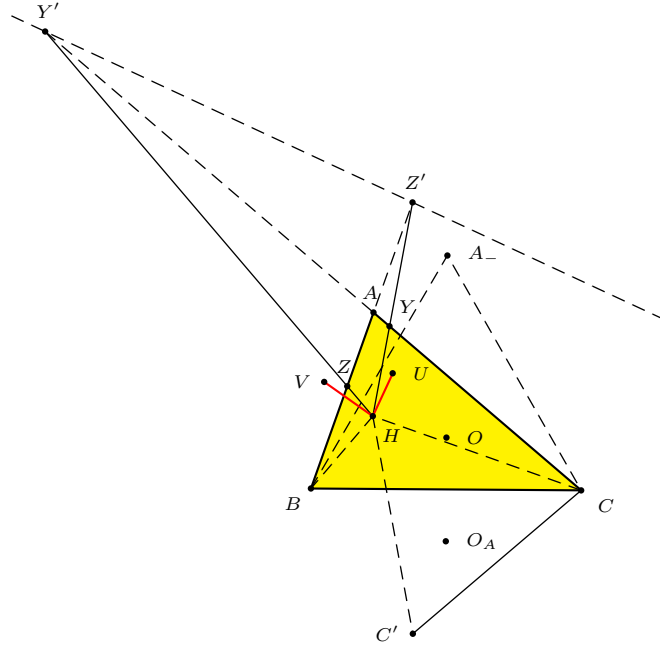


Figure 5

*Proof.* (a) Let  $V$  be the orthocenter of triangle  $HYZ$ , and denote by  $Y'$ ,  $Z'$  the intersections of the lines  $HZ$  with  $CA$  and  $HY$  with  $AB$ . Since the quadrilateral  $YZY'Z'$  is cyclic, the lines  $Y'Z'$ ,  $YZ$  are antiparallel. Since the lines  $HU$ ,  $HV$  are isogonal conjugate with respect to the angle  $YHZ$ , it follows that the lines  $HU$  and  $Y'Z'$  are perpendicular.

Let  $C'$  be the reflections of  $C$  in the line  $HY'$ . Triangle  $HC'C$  is equilateral since

$$\begin{aligned} (HY', HC) &= (HY', CA) + (CA, HC) \\ &= (AB, AC) - \frac{\pi}{3} + \frac{\pi}{2} - (AB, AC) \\ &= \frac{\pi}{6}. \end{aligned}$$

Now, triangles  $Y'HC$  and  $Z'HB$  are similar since  $\angle HY'C = \angle HZ'B$  and  $\angle HCY' = \angle HBZ'$ . Since  $Y'HC'$  is the reflection of  $Y'HC$  in  $HY'$ , we conclude that triangles  $Y'HC'$  and  $Z'HB$  are similar. This means

$$\frac{HY'}{HZ'} = \frac{HC'}{HB} \quad \text{and} \quad \angle Y'HC' = \angle Z'HB,$$

and

$$\frac{HY'}{HC'} = \frac{HZ'}{HB} \quad \text{and} \quad \angle Z'HY' = \angle BHC'.$$

Hence,  $Z'HY'$  and  $BHC'$  are directly similar. This implies that  $A_-H$  and  $Y'Z'$  are perpendicular:

$$\begin{aligned} (Y'Z', A_-H) &= (Y'Z', BC') + (BC', A_-H) \\ &= (Z'H, BH) + (BC, A_-C) \\ &= \frac{\pi}{2}. \end{aligned}$$

Together with the perpendicularity of  $HU$  and  $Y'Z'$ , this yields the collinearity of  $A_-$ ,  $U$ , and  $H$ .

(b) Note that the triangles  $BC'C$  and  $A_-HC$  are congruent since  $BC = A_-C$ ,  $C'C = HC$ , and  $\angle BCC' = \angle A_-CH$ . Applying the law of sines to triangle  $HYZ$ , we have

$$UH = \frac{YZ}{2 \sin YHZ} = \frac{YZ}{2 \sin \left( \frac{2\pi}{3} - A \right)}.$$

From the similarity of triangles  $Z'HY'$  and  $BHC'$  and of  $HYZ$  and  $HY'Z'$ , we have

$$A_-H = BC' = Y'Z' \cdot \frac{BH}{Y'H} = YZ \cdot \frac{Y'H}{ZH} \cdot \frac{BH}{Y'H} = YZ \cdot \frac{\cos \frac{\pi}{6}}{|\cos A|} = YZ \cdot \frac{\sqrt{3}}{2|\cos A|}.$$

Therefore,

$$\frac{UH}{A_-H} = \frac{|\cos A|}{\sqrt{3} \sin \left( \frac{2\pi}{3} - A \right)}.$$

Since  $AH = 2R \cos A$ , we have

$$AH + A_-O_A = 3R \cos A + a \sin \frac{\pi}{3} = R \cos A + \sqrt{3}R \sin A = 2\sqrt{3}R \sin \left( \frac{2\pi}{3} - A \right),$$

and

$$\frac{|AH|}{AH + A_-O_A} = \frac{|\cos A|}{\sqrt{3} \sin \left( \frac{2\pi}{3} - A \right)} = \frac{UH}{A_-H} = \frac{UH}{A_-U + UH}.$$

Since  $U$ ,  $H$  and  $A_-$  are collinear by (a), we have  $\frac{AH}{A_-O_A} = \frac{UH}{A_-U}$ . Combining this with the parallelism of the lines  $AH$  and  $A_-O_A$ , we have the direct similarity of triangles  $AHU$  and  $O_A A_-U$ . We now conclude that the angles  $HUA$  and  $A_-UO'$  are equal. This, together with (a) above, implies the collinearity of the points  $A$ ,  $U$ ,  $O_A$ .

On the other hand, according to Lemma 3, the points  $A$ ,  $N$ ,  $O_A$  are collinear. Hence, the points  $A$ ,  $U$ ,  $N$  are collinear as well.  $\square$

According to Lemma 1, the lines  $AO_a$  and  $AN$  are isogonal conjugate with respect to angle  $BAC$ . Thus, by reflecting the figure in the internal bisector of angle  $BAC$ , and following Lemma 6, we obtain the following result.

**Corollary 6.** *Given a triangle  $ABC$  with circumcenter  $O$ , let  $Y$ ,  $Z$  be points on the sides  $AC$ ,  $AB$  satisfying  $(AC, OY) = -\frac{\pi}{3}$  and  $(AB, OZ) = \frac{\pi}{3}$ . The circumcenters of triangles  $OYZ$  and  $BOC$ , and the vertex  $A$  are collinear.*

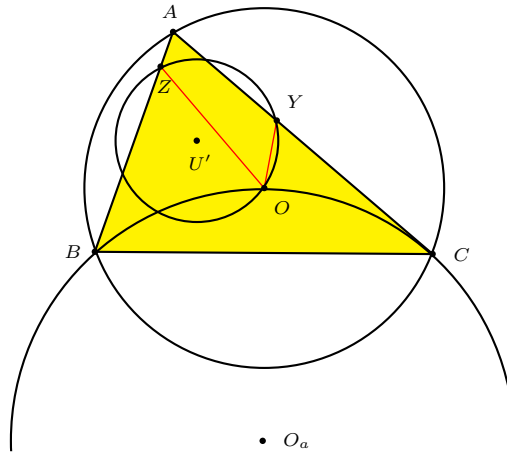


Figure 6

Now we complete the proof of Theorem 4. By applying Corollary 6 to triangle  $OO_BO_C$  with the points  $N_B$ ,  $N_C$  lying on the sidelines  $OO_B$  and  $OO_C$  such that  $(OO_B, AN_B) = -\frac{\pi}{3}$  and  $(OO_C, AN_C) = \frac{\pi}{3}$ , we conclude that the circumcenters of triangles  $AN_BN_C$ ,  $O_BAO_C$  and  $ABC$  are collinear.



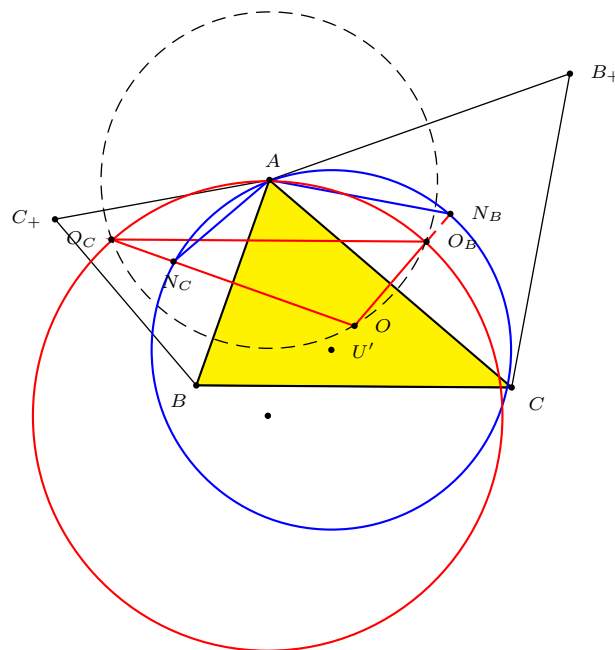


Figure 7

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# Characterizations of Bicentric Quadrilaterals

Martin Josefsson

**Abstract.** We will prove two conditions for a tangential quadrilateral to be cyclic. According to one of these, a tangential quadrilateral is cyclic if and only if its Newton line is perpendicular to the Newton line of its contact quadrilateral.

## 1. Introduction

A *bicentric quadrilateral* is a convex quadrilateral with both an incircle and a circumcircle. One characterization of these quadrilaterals is obtained by combining the most useful characterizations of tangential and cyclic quadrilaterals, that the consecutive sides  $a, b, c$  and  $d$ , and angles  $A, B, C$  and  $D$  satisfy

$$\begin{aligned}a + c &= b + d, \\A + C &= B + D = \pi.\end{aligned}$$

We review a few other characterizations of bicentric quadrilaterals before proving two possibly new ones.

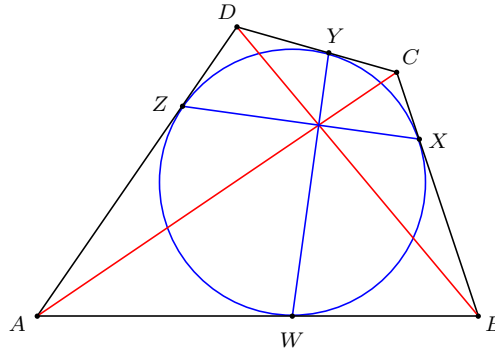


Figure 1. The tangency chords and diagonals

If the incircle in a tangential quadrilateral  $ABCD$  is tangent to the sides  $AB, BC, CD$  and  $DA$  at  $W, X, Y$  and  $Z$  respectively, then the segments  $WY$  and  $XZ$  are called the *tangency chords* in [8, pp.188-189]. See Figure 1. In [4, 9, 13] it is proved that *a tangential quadrilateral is cyclic if and only if the tangency chords are perpendicular*.

Problem 10804 in the MONTHLY [14] states that a tangential quadrilateral is cyclic if and only if

$$\frac{AW}{WB} = \frac{DY}{YC}.$$

Using the same notations, Hajja proved in [11] that a tangential quadrilateral is cyclic if and only if

$$\frac{AC}{BD} = \frac{AW + CY}{BX + DZ}.$$

If  $E, F, G$  and  $H$  are the midpoints of  $WX, XY, YZ$  and  $ZW$  respectively (see Figure 2), then *the tangential quadrilateral  $ABCD$  is cyclic if and only if the quadrilateral  $EFGH$  is a rectangle*. This characterization was Problem 6 on China Western Mathematical Olympiad 2003 [5, pp.182-183].

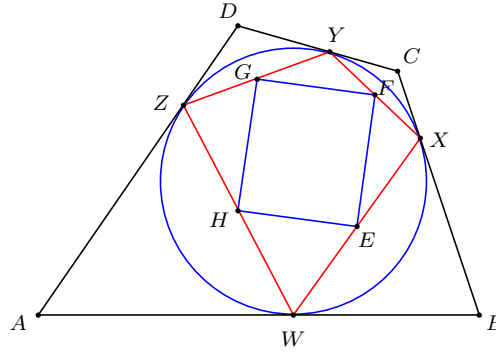


Figure 2.  $ABCD$  is cyclic iff  $EFGH$  is a rectangle

## 2. Two characterizations of right triangles

To prove one of the characterizations of bicentric quadrilaterals we will need the following characterization of right triangles. The direct part of the theorem is an easy exercise<sup>1</sup>, but we have found no reference of the converse result.

**Theorem 1.** *In a non-isosceles triangle the median and altitude to one of the sides divide the opposite angle into three parts. This angle is a right one if and only if the angle between the median and the longer of the sides at the considered vertex is equal to the angle between the altitude and the shorter side at that vertex.*

*Proof.* We use notations as in Figure 3. If  $C = \frac{\pi}{2}$ , we shall prove that  $\alpha = \beta$ . Triangle  $AMC$  is isosceles<sup>2</sup> with  $AM = CM$ , so  $A = \alpha$ . Triangles  $ACB$  and  $CHB$  are similar, so  $A = \beta$ . Hence  $\alpha = \beta$ .

Conversely, if  $\alpha = \beta$ , we shall prove that  $C = \frac{\pi}{2}$ . By the exterior angle theorem, angle  $CMB = A + \alpha$ , so in triangle  $MCH$  we have

$$A + \alpha + \gamma = \frac{\pi}{2}. \quad (1)$$

<sup>1</sup>A similar problem also including the angle bisector can be found in [1, pp.46-49] and [12, p.32].

<sup>2</sup>The midpoint of the hypotenuse is the circumcenter.

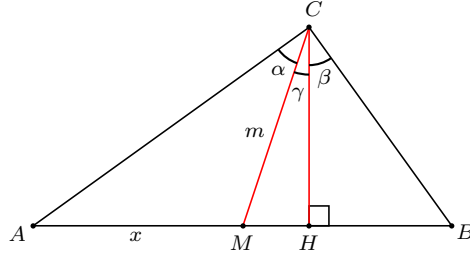


Figure 3. Median and altitude in a triangle

Let  $x = AM = BM$  and  $m = CM$ . Using the law of sines in triangles  $CAM$  and  $CMB$ ,

$$\frac{\sin \alpha}{x} = \frac{\sin A}{m} \Leftrightarrow \frac{x}{m} = \frac{\sin \alpha}{\sin A}$$

and with  $\alpha = \beta$ ,

$$\frac{\sin(\alpha + \gamma)}{x} = \frac{\sin B}{m} \Rightarrow \frac{x}{m} = \frac{\sin(\frac{\pi}{2} - A)}{\sin(\frac{\pi}{2} - \alpha)} = \frac{\cos A}{\cos \alpha}$$

since  $B + \alpha = \frac{\pi}{2}$  in triangle  $BCH$ . Combining the last two equations, we get

$$\frac{\sin \alpha}{\sin A} = \frac{\cos A}{\cos \alpha} \Leftrightarrow \sin 2\alpha = \sin 2A.$$

This equation has the two solutions  $2\alpha = 2A$  and  $2\alpha = \pi - 2A$ , hence  $\alpha = A$  or  $\alpha = \frac{\pi}{2} - A$ . The second solution combined with  $B + \alpha = \frac{\pi}{2}$  gives  $A = B$ , which is impossible since the triangle is not isosceles by the assumption in the theorem. Thus  $\alpha = A$  is the only valid solution. Hence

$$C = \alpha + \gamma + \beta = A + \gamma + \alpha = \frac{\pi}{2}$$

according to (1), completing the proof.  $\square$

**Corollary 2.** *Let  $CM$ ,  $CD$  and  $CH$  be a median, an angle bisector and an altitude respectively in triangle  $ABC$ . The angle  $C$  is a right angle if and only if  $CD$  bisects angle  $HCM$ .*

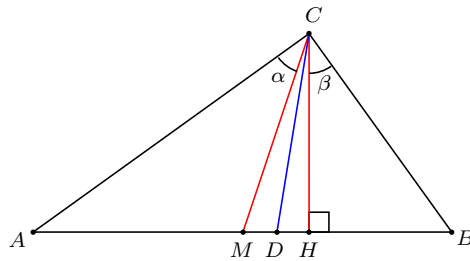


Figure 4. Median, angle bisector and altitude in a triangle

*Proof.* Since  $CD$  is an angle bisector in triangle  $ABC$ , we have (see Figure 4)

$$\alpha + \angle MCD = \angle HCD + \beta. \quad (2)$$

Using Theorem 1 and (2), we get

$$C = \frac{\pi}{2} \Leftrightarrow \alpha = \beta \Leftrightarrow \angle MCD = \angle HCD.$$

□

### 3. Corollaries of Pascal's theorem and Brocard's theorem

Pascal's theorem states that *if a hexagon is inscribed in a circle and the three pairs of opposite sides are extended until they meet, then the three points of intersection are collinear*. A proof is given in [6, pp.74-75]. Pascal's theorem is also true in degenerate cases.

In [7, p.15], the following theorem is called Brocard's theorem: *if the extensions of opposite sides in a cyclic quadrilateral intersect at  $J$  and  $K$ , and the diagonals intersect at  $P$ , then the circumcenter  $O$  of the quadrilateral is also the orthocenter in triangle  $JKP$*  (see Figure 5). An elementary proof of this theorem can be found at [16].

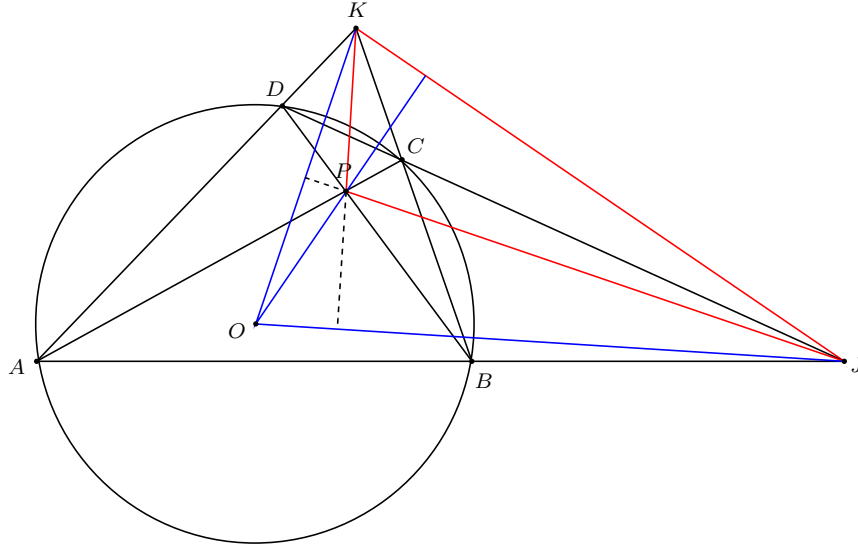


Figure 5. Brocard's theorem

To prove our second characterization of bicentric quadrilaterals we will need two corollaries of these theorems that are quite well known. The first is a special case of Pascal's theorem in a quadrilateral. If the incircle in a tangential quadrilateral  $ABCD$  is tangent to the sides  $AB$ ,  $BC$ ,  $CD$  and  $DA$  at  $W$ ,  $X$ ,  $Y$  and  $Z$  respectively, then in [9] Yetti<sup>3</sup> calls the quadrilateral  $WXYZ$  the *contact quadrilateral*.

<sup>3</sup>Yetti is the username of an American physicist at the website *Art of Problem Solving* [3].

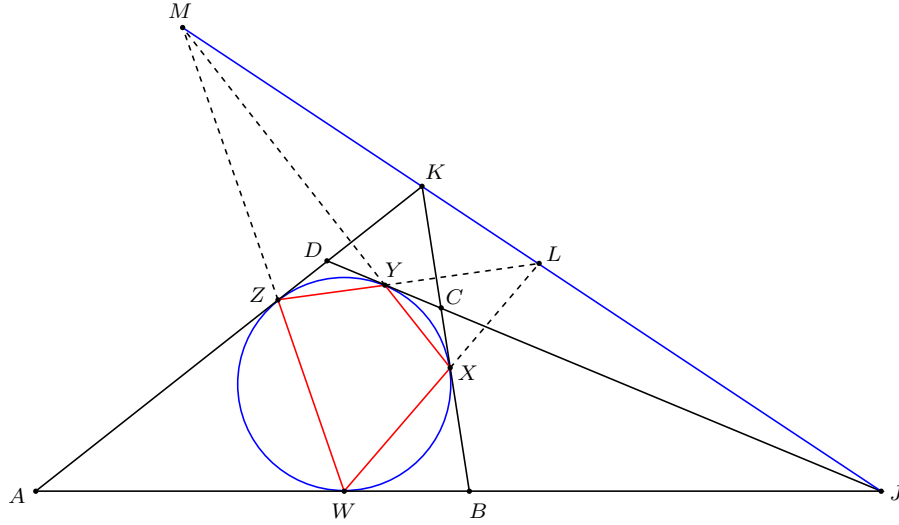


Figure 6. Pascal's theorem in a tangential quadrilateral

**Corollary 3.** *If the extensions of opposite sides in a tangential quadrilateral intersect at  $J$  and  $K$ , and the extensions of opposite sides in its contact quadrilateral intersect at  $L$  and  $M$ , then the four points  $J$ ,  $L$ ,  $K$  and  $M$  are collinear.*

*Proof.* Consider the degenerate cyclic hexagon  $WWXYYZ$ , where  $W$  and  $Y$  are double vertices. The extensions of the sides at these vertices are the tangents at  $W$  and  $Y$ , see Figure 6. According to Pascal's theorem, the points  $J$ ,  $L$  and  $M$  are collinear.

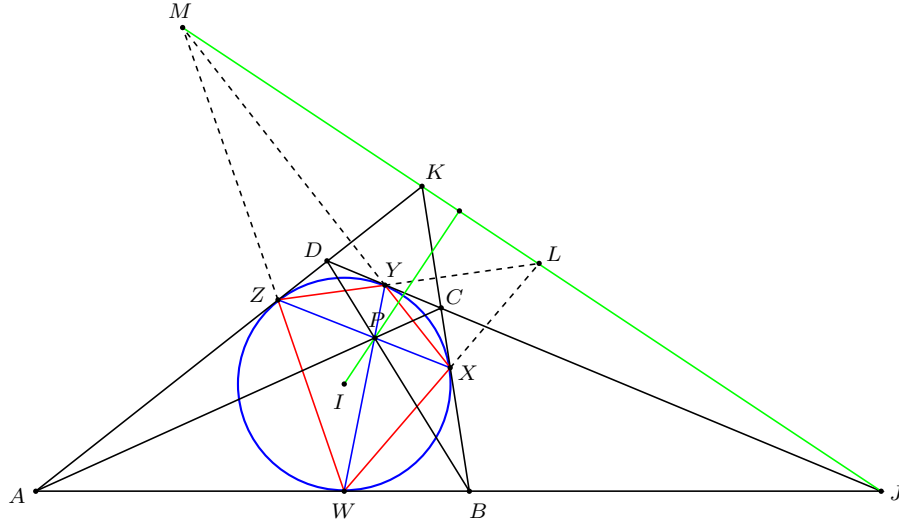
Next consider the degenerate cyclic hexagon  $WXXYZZ$ . In the same way the points  $M$ ,  $K$  and  $L$  are collinear. This proves that the four points  $J$ ,  $L$ ,  $K$  and  $M$  are collinear, since  $M$  and  $L$  are on both lines, so these lines coincide.  $\square$

**Corollary 4.** *If the extensions of opposite sides in a tangential quadrilateral intersect at  $J$  and  $K$ , and the diagonals intersect at  $P$ , then  $JK$  is perpendicular to the extension of  $IP$  where  $I$  is the incenter.*

*Proof.* The contact quadrilateral  $WXYZ$  is a cyclic quadrilateral with circumcenter  $I$ , see Figure 7. It is well known that the point of intersection of  $WY$  and  $XZ$  is also the point of intersection of the diagonals in the tangential quadrilateral  $ABCD$ , see [10, 15, 17]. If the extensions of opposite sides in the contact quadrilateral  $WXYZ$  intersect at  $L$  and  $M$ , then by Brocard's theorem  $ML \perp IP$ . According to Corollary 3,  $ML$  and  $JK$  are the same line. Hence  $JK \perp IP$ .  $\square$

#### 4. Two characterizations of bicentric quadrilaterals

Many problems on quadrilaterals in text books and on problem solving web sites are formulated as implications of the form: if the quadrilateral is a special type (like a bicentric quadrilateral), then you should prove it has some property. How

Figure 7. Perpendicular lines  $JK$  and  $IP$ 

about the converse statement? Sometimes it is considered, but far from always. The two characterizations we will prove here was found when considering if the converse statement of two such problems are also true. The first is a rather easy one and it would surprise us if it hasn't been published before; however we have been unable to find a reference for it. Besides, it will be used in the proof of the second characterization.

**Theorem 5.** *Let the extensions of opposite sides in a tangential quadrilateral intersect at  $J$  and  $K$ . If  $I$  is the incenter, then the quadrilateral is also cyclic if and only if  $JIK$  is a right angle.*

*Proof.* We use notations as in Figure 8, where  $G$  and  $H$  are the midpoints of the tangency chords  $WY$  and  $XZ$  respectively and  $P$  is the point of intersection of  $WY$  and  $XZ$ . In isosceles triangles  $WJY$  and  $XKZ$ ,  $IJ \perp WY$  and  $IK \perp XZ$ . Hence opposite angles  $IGP$  and  $IHP$  in quadrilateral  $GIHP$  are right angles, so by the sum of angles in quadrilateral  $GIHP$ ,

$$\angle JIK = \angle GIH = 2\pi - 2 \cdot \frac{\pi}{2} - \angle WPZ.$$

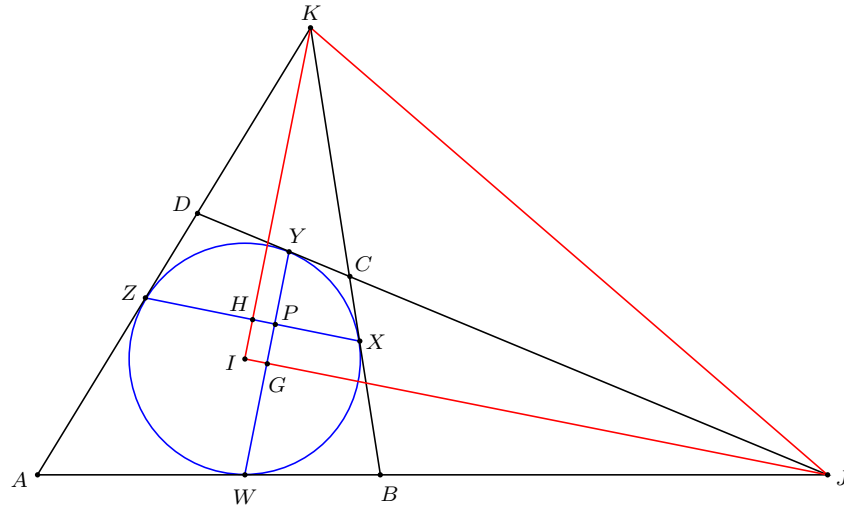
Hence we have

$$\angle JIK = \frac{\pi}{2} \Leftrightarrow \angle WPZ = \frac{\pi}{2} \Leftrightarrow WY \perp XZ$$

and according to [4, 9, 13] the tangency chords in a tangential quadrilateral are perpendicular if and only if it is cyclic<sup>4</sup>.  $\square$

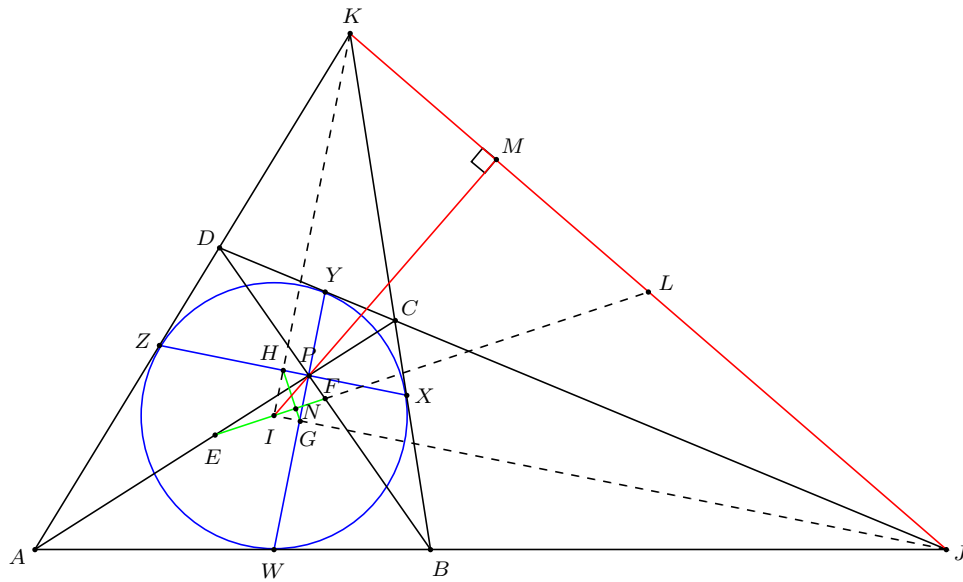
Now we are ready for the main theorem in this paper, our second characterization of bicentric quadrilaterals. The direct part of the theorem was a problem

<sup>4</sup>This was also mentioned in the introduction to this paper.



studied at [2]. The *Newton line*<sup>5</sup> of a quadrilateral is the line defined by the mid-points of the two diagonals.

**Theorem 6.** *A tangential quadrilateral is cyclic if and only if its Newton line is perpendicular to the Newton line of its contact quadrilateral.*



<sup>5</sup>It is sometimes known as the Newton-Gauss line.



*Proof.* We use notations as in Figure 9, where  $P$  is the point where both the diagonals and the tangency chords intersect (see [10, 15, 17]) and  $L$  is the midpoint of  $JK$ . If  $I$  is the incenter, then the points  $E, I, F$  and  $L$  are collinear on the Newton line, see Newton's theorem in [7, p.15] (this is proved in two different theorems in [1, p.42]<sup>6</sup> and [17, p.169]). Let  $M$  be the intersection of  $JK$  and the extension of  $IP$ . By Corollary 4  $IM \perp JK$ . In isosceles triangles  $ZKX$  and  $WJY$ ,  $IK \perp ZX$  and  $IJ \perp WY$ .

Since it has two opposite right angles ( $\angle IHP$  and  $\angle IGP$ ), the quadrilateral  $GIHP$  is cyclic, so  $\angle HGI = \angle HPI$ . From the sum of angles in a triangle, we have

$$\angle ING = \pi - (\angle GIF + \angle HGI) = \pi - (\angle JIL + \angle HPI)$$

where  $N$  is the intersection of  $EF$  and  $GH$ . Thus

$$\angle ING = \pi - \angle JIL - \left(\frac{\pi}{2} - \angle HIP\right) = \frac{\pi}{2} - \angle JIL + \angle KIM.$$

So far we have only used properties of tangential quadrilaterals, so

$$\angle ING = \frac{\pi}{2} - \angle JIL + \angle KIM$$

is valid in all tangential quadrilaterals where no pair of opposite sides are parallel<sup>7</sup>.

Hence we have

$$EF \perp GH \Leftrightarrow \angle ING = \frac{\pi}{2} \Leftrightarrow \angle JIL = \angle KIM \Leftrightarrow \angle JIK = \frac{\pi}{2}$$

where the last equivalence is due to Theorem 1 and the fact that  $IM \perp JK$  (Corollary 4). According to Theorem 5,  $\angle JIK = \frac{\pi}{2}$  if and only if the tangential quadrilateral is also cyclic.

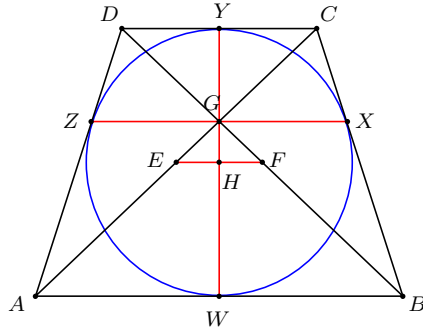


Figure 10. An isosceles tangential trapezoid

It remains to consider the case when at least one pair of opposite sides are parallel. Then the tangential quadrilateral is a trapezoid, so<sup>8</sup>

$$A + D = B + C \Leftrightarrow A - B = C - D.$$

<sup>6</sup>That the incenter  $I$  lies on the Newton line  $EF$  is actually a solved problem in this book.

<sup>7</sup>Otherwise at least one of the points  $J$  and  $K$  do not exist.

<sup>8</sup>We suppose without loss of generality that  $AB \parallel CD$ .

The trapezoid has a circumcircle if and only if

$$A + C = B + D \Leftrightarrow A - B = D - C.$$

Hence the quadrilateral is bicentric if and only if

$$C - D = D - C \Leftrightarrow C = D \Leftrightarrow A = B,$$

that is, the quadrilateral is bicentric if and only if it is an isosceles tangential trapezoid. In these  $EF \perp GH$  (see Figure 10, where  $EF \parallel AB$  and  $GH \perp AB$ ) completing the proof.  $\square$

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# The Circles of Lester, Evans, Parry, and Their Generalizations

Paul Yiu

**Abstract.** Beginning with the famous Lester circle containing the circumcenter, nine-point center and the two Fermat points of a triangle, we survey a number of interesting circles in triangle geometry.

## 1. Introduction

This paper treats a number of interesting circles discovered by June Lester, Lawrence Evans, and Cyril Parry. We prove their existence and establish their equations. Lester [12] has discovered that the Fermat points, the circumcenter, and the nine-point center are concyclic. We call this the first Lester circle, and study it in §§3 – 6. Lester also conjectured in [12] the existence of a circle through the symmedian point, the Feuerbach point, the Clawson point, and the homothetic center of the orthic and the intangent triangles. This conjecture is validated in §15. Evans, during the preparation of his papers in Forum Geometricorum, has communicated two conjectures on circles through two perspectors  $V_{\pm}$  which has since borne his name. In §9 we study in detail the first Evans circle in relation to the excentral circle. The second one is established in §14. In [9], a great number of circles have been reported relating to the Parry point, a point on the circumcircle. These circles are studied in §§10 – 12.

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## 2. Preliminaries

We refer to [15] for the standard notations of triangle geometry. Given a triangle  $ABC$ , with sidelengths  $a, b, c$ , the circumcircle is represented in homogeneous barycentric coordinates by the equation

$$a^2yz + b^2zx + c^2xy = 0.$$

The equation of a general circle  $\mathcal{C}$  is of the form

$$a^2yz + b^2zx + c^2xy + (x + y + z) \cdot L(x, y, z) = 0 \quad (1)$$

where  $L(x, y, z)$  is a linear form, and the line  $L(x, y, z) = 0$  is the radical axis of the circle  $\mathcal{C}$  and the circumcircle.

**2.1. Intersection of a circle with the circumcircle .** The intersections of the circle  $\mathcal{C}$  with the circumcircle can certainly be determined by solving the equations

$$\begin{aligned} a^2yz + b^2zx + c^2xy &= 0, \\ L(x, y, z) &= 0 \end{aligned}$$

simultaneously. Here is an interesting special case where these intersections can be easily identified. We say that a triangle center function  $f(a, b, c)$  represents an infinite point if

$$f(a, b, c) + f(b, c, a) + f(c, a, b) = 0. \quad (2)$$

**Proposition 1.** *If a circle  $\mathcal{C}$  is represented by an equation (1) in which*

$$L(x, y, z) = F(a, b, c) \cdot \sum_{\text{cyclic}} b^2 c^2 \cdot f(a, b, c) \cdot g(a, b, c)x, \quad (3)$$

where  $F(a, b, c)$  is symmetric in  $a, b, c$ , and  $f(a, b, c), g(a, b, c)$  are triangle center functions representing infinite points, then the circle intersects the circumcircle at the points

$$Q_f := \left( \frac{a^2}{f(a, b, c)} : \frac{b^2}{f(b, c, a)} : \frac{c^2}{f(c, a, b)} \right)$$

and

$$Q_g := \left( \frac{a^2}{g(a, b, c)} : \frac{b^2}{g(b, c, a)} : \frac{c^2}{g(c, a, b)} \right).$$

*Proof.* The line  $L(x, y, z) = 0$  clearly contains the point  $Q_f$ , which by (2) is the isogonal conjugate of an infinite point, and so lies on the circumcircle. It is therefore a common point of the circumcircle and  $\mathcal{C}$ . The same reasoning applies to the point  $Q_g$ .  $\square$

For an application, see Remark after Proposition 11.

**2.2. Construction of circle equation.** Suppose we know the equation of a circle through two points  $Q_1$  and  $Q_2$ , in the form of (1), and the equation of the line  $Q_1 Q_2$ , in the form  $L'(x, y, z) = 0$ . To determine the equation of the circle through  $Q_1, Q_2$  and a third point  $Q = (x_0, y_0, z_0)$  not on the line  $Q_1 Q_2$ , it is enough to find  $t$  such that

$$a^2 y_0 z_0 + b^2 z_0 x_0 + c^2 x_0 y_0 + (x_0 + y_0 + z_0)(L(x_0, y_0, z_0) + t \cdot L'(x_0, y_0, z_0)) = 0.$$

With this value of  $t$ , the equation

$$a^2 yz + b^2 zx + c^2 xy + (x + y + z)(L(x, y, z) + t \cdot L'(x, y, z)) = 0$$

represents the circle  $Q_1 Q_2 Q$ . For an application of this method, see §6.3 (11) and Proposition 11.

**2.3. Some common triangle center functions.** We list some frequently occurring homogeneous functions associated with the coordinates of triangle centers or coefficients in equations of lines and circles. An asterisk indicates that the function represents an infinite point.

### Quartic forms

	$f_{4,1} := a^2(b^2 + c^2) - (b^2 - c^2)^2$
	$f_{4,2} := a^2(b^2 + c^2) - (b^4 + c^4)$
	$f_{4,3} := a^4 - (b^4 - b^2c^2 + c^4)$
	$f_{4,4} := (b^2 + c^2 - a^2)^2 - b^2c^2$
*	$f_{4,5} := 2a^4 - a^2(b^2 + c^2) - (b^2 - c^2)^2$
	$f_{4,6} := 2a^4 - 3a^2(b^2 + c^2) + (b^2 - c^2)^2$
*	$f_{4,7} := 2a^4 - 2a^2(b^2 + c^2) - (b^4 - 4b^2c^2 + c^4)$

### Sextic forms

	$f_{6,1} := a^6 - 3a^4(b^2 + c^2) + a^2(3b^4 - b^2c^2 + 3c^4) - (b^2 + c^2)(b^2 - c^2)^2$
*	$f_{6,2} := 2a^6 - 2a^4(b^2 + c^2) + a^2(b^4 + c^4) - (b^2 + c^2)(b^2 - c^2)^2$
*	$f_{6,3} := 2a^6 - 6a^4(b^2 + c^2) + 9a^2(b^4 + c^4) - (b^2 + c^2)^3$

### Octic forms

	$f_{8,1} := a^8 - 2a^6(b^2 + c^2) + a^4b^2c^2 + a^2(b^2 + c^2)(2b^4 - b^2c^2 + 2c^4) - (b^8 - 2b^6c^2 + 6b^4c^4 - 2b^2c^6 + c^8)$
*	$f_{8,2} := 2a^8 - 2a^6(b^2 + c^2) - a^4(3b^4 - 8b^2c^2 + 3c^4) + 4a^2(b^2 + c^2)(b^2 - c^2)^2 - (b^2 - c^2)^2(b^4 + 4b^2c^2 + c^4)$
*	$f_{8,3} := 2a^8 - 5a^6(b^2 + c^2) + a^4(3b^4 + 8b^2c^2 + 3c^4) + a^2(b^2 + c^2)(b^4 - 5b^2c^2 + c^4) - (b^2 - c^2)^4$
	$f_{8,4} := 3a^8 - 8a^6(b^2 + c^2) + a^4(8b^4 + 7b^2c^2 + 8c^4) - a^2(b^2 + c^2)(4b^4 - 3b^2c^2 + 4c^4) + (b^4 - c^4)^2$

### 3. The first Lester circle

**Theorem 2 (Lester).** *The Fermat points, the circumcenter, and the nine-point center of a triangle are concyclic.*

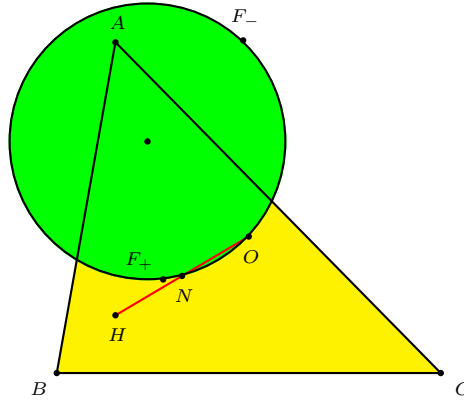


Figure 1. The first Lester circle through  $O$ ,  $N$  and the Fermat points

Our starting point is a simple observation that the line joining the Fermat points intersects the Euler line at the midpoint of the orthocenter  $H$  and the centroid  $G$ .

Clearing denominators in the homogeneous barycentric coordinates of the Fermat point

$$F_+ = \left( \frac{1}{\sqrt{3}S_A + S}, \frac{1}{\sqrt{3}S_B + S}, \frac{1}{\sqrt{3}S_C + S} \right),$$

we rewrite it in the form

$$F_+ = (3S_{BC} + S^2, 3S_{CA} + S^2, 3S_{AB} + S^2) + \sqrt{3}S(S_B + S_C, S_C + S_A, S_A + S_B).$$

This expression shows that  $F_+$  is a point of the line joining the symmedian point

$$K = (S_B + S_C, S_C + S_A, S_A + S_B)$$

to the point

$$\begin{aligned} M &= (3S_{BC} + S^2, 3S_{CA} + S^2, 3S_{AB} + S^2) \\ &= 3(S_{BC}, S_{CA}, S_{AB}) + S^2(1, 1, 1). \end{aligned}$$

Note that  $M$  is the midpoint of the segment  $HG$ , where  $H = (S_{BC}, S_{CA}, S_{AB})$  is the orthocenter and  $G = (1, 1, 1)$  is the centroid. It is the center of the orthocentroidal circle with  $HG$  as diameter. Indeed,  $F_+$  divides  $MK$  in the ratio

$$MF_+ : F_+K = 2\sqrt{3}S(S_A + S_B + S_C) : 6S^2 = (S_A + S_B + S_C) : \sqrt{3}S.$$

With an obvious change in sign, we also have the negative Fermat point  $F_-$  dividing  $MK$  in the ratio

$$MF_- : F_-K = (S_A + S_B + S_C) : -\sqrt{3}S.$$

We have therefore established

**Proposition 3.** *The Fermat points divide  $MK$  harmonically.*

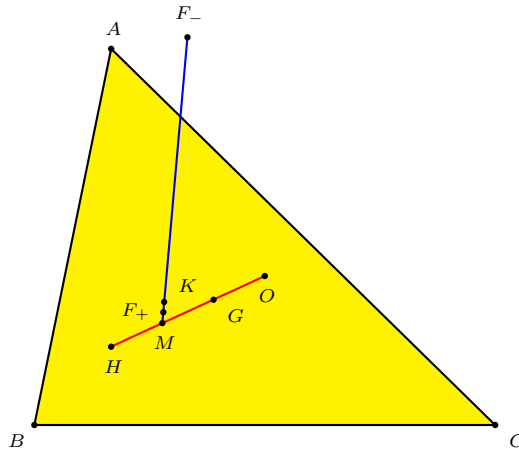


Figure 2. Intersection of Fermat line and Euler line

This simple observation suggests a proof of Lester's circle theorem by the intersection chords theorem.

**Proposition 4.** *The following statements are equivalent.*

(A)  $MF_+ \cdot MF_- = MO \cdot MN$ .

(B) *The circle  $F_+F_-G$  is tangent to the Euler line at  $G$ , i.e.,  $MF_+ \cdot MF_- = MG^2$ .*

(C) *The circle  $F_+F_-H$  is tangent to the Euler line at  $H$ , i.e.,  $MF_+ \cdot MF_- = MH^2$ .*

(D) *The Fermat points are inverse in the orthocentroidal circle.*

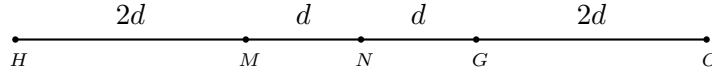


Figure 3. The Euler line

*Proof.* Since  $M$  is the midpoint of  $HG$ , the statements (B), (C), (D) are clearly equivalent. On the other hand, putting  $OH = 6d$ , we have

$$MO \cdot MN = (MH)^2 = (MG)^2 = 4d^2,$$

see Figure 3. This shows that (A), (B), (C) are equivalent.  $\square$

Note that (A) is Lester's circle theorem (Theorem 2). To complete its proof, it is enough to prove (D). We do this by a routine calculation.

**Theorem 5.** *The Fermat points are inverse in the orthocentroidal circle.*

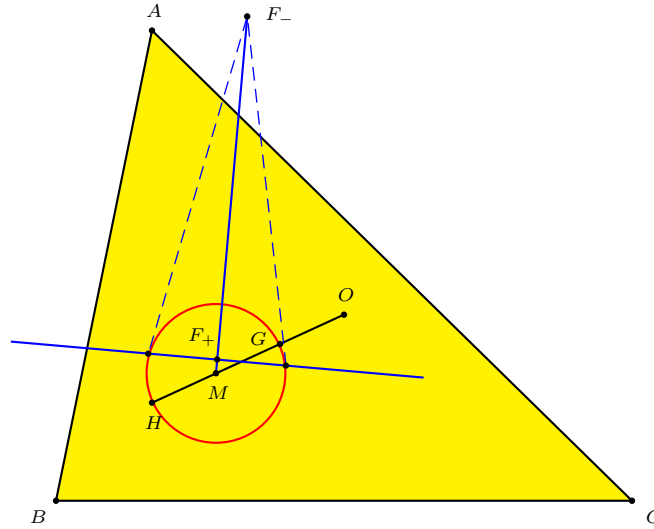


Figure 4.  $F_+$  on the polar of  $F_-$  in the orthocentroidal circle

*Proof.* The equation of the orthocentroidal circle is

$$3(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(S_Ax + S_By + S_Cz) = 0,$$



equivalently,<sup>1</sup>

$$-2(S_A x^2 + S_B y^2 + S_C z^2) + ((S_B + S_C)yz + (S_C + S_A)zx + (S_A + S_B)xy) = 0.$$

This is represented by the matrix

$$M = \begin{pmatrix} -4S_A & S_A + S_B & S_A + S_C \\ S_A + S_B & -4S_B & S_B + S_C \\ S_A + S_C & S_B + S_C & -4S_C \end{pmatrix}.$$

The coordinates of the Fermat points can be written as

$$F_+ = X + Y \quad \text{and} \quad F_- = X - Y,$$

with

$$X = (3S_{BC} + S^2 \quad 3S_{CA} + S^2 \quad 3S_{AB} + S^2),$$

$$Y = \sqrt{3}S (S_B + S_C \quad S_C + S_A \quad S_A + S_B).$$

With these, we have

$$XMX^t = YMY^t = 6S^2(S_A(S_B - S_C)^2 + S_B(S_C - S_A)^2 + S_C(S_A - S_B)^2),$$

and

$$F_+MF_-^t = (X + Y)M(X - Y)^t = XMX^t - YMY^t = 0.$$

This shows that the Fermat points are inverse in the orthocentroidal circle.  $\square$

The proof of Theorem 2 is now complete, along with tangency of the Euler line with the two circles  $F_+F_-G$  and  $F_+F_-H$  (see Figure 5). We call the circle through  $O$ ,  $N$ , and  $F_\pm$  the first Lester circle.

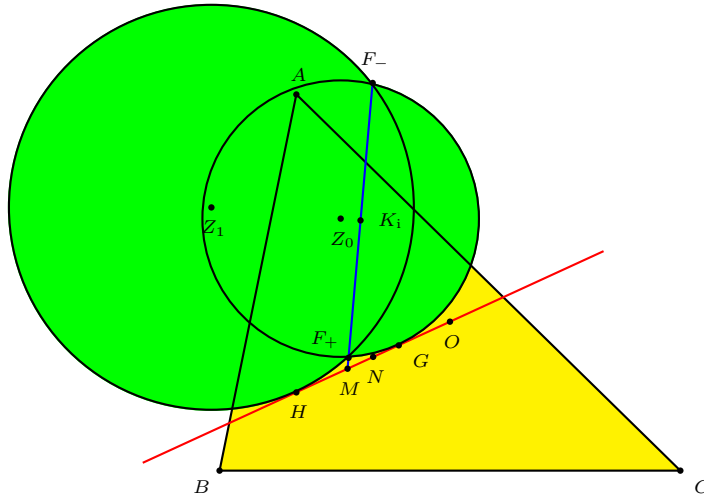


Figure 5. The circles  $F_+F_-G$  and  $F_+F_-H$

<sup>1</sup>It is easy to see that this circle contains  $H$  and  $G$ . The center of the circle (see [15, §10.7.2]) is the point  $M = (S_A(S_B + S_C) + 4S_{BC} : S_B(S_C + S_A) + 4S_{CA} : S_C(S_A + S_B) + 4S_{AB})$  on the Euler line, which is necessarily the midpoint of  $HG$ .

*Remarks.* (1) The tangency of the circle  $F_+F_-G$  and the Euler line was noted in [9, pp.229–230].

(2) The symmedian point  $K$  and the Kiepert center  $K_i$  (which is the midpoint of  $F_+F_-$ ) are inverse in the orthocentroidal circle.

#### 4. Gibert's generalization of the first Lester circle

Bernard Gibert [7] has found an interesting generalization of the first Lester circle, which we explain as a natural outgrowth of an attempt to compute the equations of the circles  $F_+F_-G$  and  $F_+F_-H$ .

**Theorem 6** (Gibert). *Every circle whose diameter is a chord of the Kiepert hyperbola perpendicular to the Euler line passes through the Fermat points.*

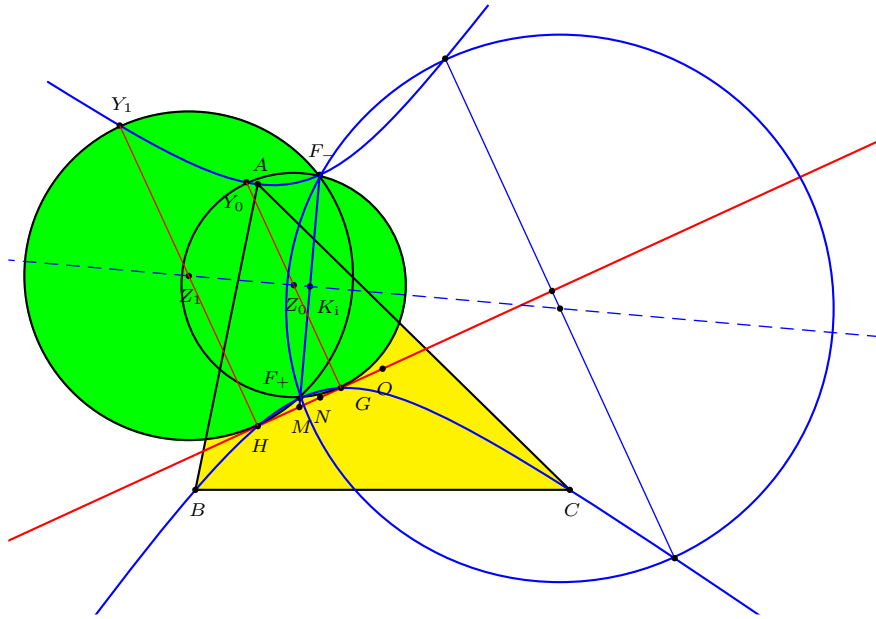


Figure 6. Gibert's generalization of the first Lester circle

*Proof.* Since  $F_{\pm}$  and  $G$  are on the Kiepert hyperbola, and the center of the circle  $F_+F_-G$  is on the perpendicular to the Euler line at  $G$ , this line intersects the Kiepert hyperbola at a fourth point  $Y_0$  (see Figure 6), and the circle is a member of the pencil of conics through  $F_+$ ,  $F_-$ ,  $G$  and  $Y_0$ . Let  $L(x, y, z) = 0$  and  $L_0(x, y, z) = 0$  represent the lines  $F_+F_-$  and  $GY_0$  respectively. We may assume the circle given by

$$k_0((b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy) - L(x, y, z) \cdot L_0(x, y, z) = 0$$

for an appropriately chosen constant  $k_0$ .

Replacing  $G$  by  $H$  and  $Y_0$  by another point  $Y_1$ , the intersection of the Kiepert hyperbola with the perpendicular to the Euler line at  $H$ , we write the equation of the circle  $F_+F_-H$  in the form

$$k_1((b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy) - L(x, y, z) \cdot L_1(x, y, z) = 0,$$

where  $L_1(x, y, z) = 0$  is the equation of the line  $HY_1$ .

The midpoints of the two chords  $GY_0$  and  $HY_1$  are the centers of the two circles  $F_+F_-G$  and  $F_+F_-H$ . The line joining them is therefore the perpendicular bisector of  $F_+F_-$ .

Every line perpendicular to the Euler line is represented by an equation

$$L_t(x, y, z) := tL_0(x, y, z) + (1 - t)L_1(x, y, z) = 0$$

for some real number  $t$ . Let  $k_t := tk_0 + (1 - t)k_1$  correspondingly. Then the equation

$$k_t((b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy) - L(x, y, z) \cdot L_t(x, y, z) = 0$$

represents a circle  $C_t$  through the Fermat points and the intersections of the line  $L_t(x, y, z) = 0$  and the Kiepert hyperbola. The perpendicular bisector of  $F_+F_-$  is the diameter of the family of parallel lines  $L_t(x, y, z) = 0$ . Therefore the center of the circle is the midpoint of the chord cut out by  $L_t(x, y, z) = 0$ .  $\square$

*Remark.* If the perpendicular to the Euler line intersects it outside the segment  $HG$ , then the circle intersects the Euler line at two points dividing the segment  $HG$  harmonically, say in the ratio  $\tau : 1 - \tau$  for  $\tau < 0$  or  $\tau > 1$ . In this case, the line divides  $HG$  in the ratio  $-\tau^2 : (1 - \tau)^2$ .

## 5. Center of the first Lester circle

Since the circumcenter  $O$  and the nine-point center  $N$  divides the segment  $HG$  in the ratio  $3 : \mp 1$ , the diameter of the first Lester circle perpendicular to the Euler line intersects the latter at the point  $L$  dividing  $HG$  in the ratio  $9 : -1$ . This is the midpoint of  $ON$  (see Figure 7), and has coordinates

$$(f_{4,6}(a, b, c) : f_{4,6}(b, c, a) : f_{4,6}(c, a, b)).$$

As such it is the nine-point center of the medial triangle, and appears as  $X_{140}$  in [10].

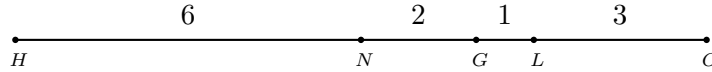


Figure 7. The Euler line

**Proposition 7.** (a) *Lines perpendicular to the Euler line have infinite point*

$$X_{523} = (b^2 - c^2, c^2 - a^2, a^2 - b^2).$$

(b) *The diameter of the first Lester circle perpendicular to the Euler line is along the line*

$$\sum_{\text{cyclic}} f_{6,1}(a, b, c)x = 0. \quad (4)$$

**Proposition 8.** (a) *The equation of the line  $F_+F_-$  is*

$$\sum_{\text{cyclic}} (b^2 - c^2)f_{4,4}(a, b, c)x = 0. \quad (5)$$

(b) *The perpendicular bisector of  $F_+F_-$  is the line*

$$\frac{x}{b^2 - c^2} + \frac{y}{c^2 - a^2} + \frac{z}{a^2 - b^2} = 0. \quad (6)$$

*Proof.* (a) The line  $F_+F_-$  contains the symmedian point  $K$  and the Kiepert center

$$K_i = ((b^2 - c^2)^2, (c^2 - a^2)^2, (a^2 - b^2)^2).$$

(b) The perpendicular bisector of  $F_+F_-$  is the perpendicular at  $K_i$  to the line  $KK_i$ , which has infinite point

$$X_{690} = ((b^2 - c^2)(b^2 + c^2 - 2a^2), (c^2 - a^2)(c^2 + a^2 - 2b^2), (a^2 - b^2)(a^2 + b^2 - 2c^2)).$$

□

**Proposition 9.** *The center of the first Lester circle has homogeneous barycentric coordinates*

$$((b^2 - c^2)f_{8,3}(a, b, c) : (c^2 - a^2)f_{8,3}(b, c, a) : (a^2 - b^2)f_{8,3}(c, a, b)).$$

*Proof.* This is the intersection of the lines (4) and (6). □

*Remarks.* (1) The center of the first Lester circle appears as  $X_{1116}$  in [10].

(2) The perpendicular bisector of  $F_+F_-$  also contains the Jerabek center

$$J_e = ((b^2 - c^2)^2(b^2 + c^2 - a^2), (c^2 - a^2)^2(c^2 + a^2 - b^2), (a^2 - b^2)^2(a^2 + b^2 - c^2)),$$

which is the center of the Jerabek hyperbola, the isogonal conjugate of the Euler line. It follows that  $J_e$  is equidistant from the Fermat points. The points  $K_i$  and  $J_e$  are the common points of the nine-point circle and the pedal circle of the centroid.

## 6. Equations of circles

6.1. *The circle  $F_+F_-G$ .* In the proof of Theorem 6, we take

$$L(x, y, z) = \sum_{\text{cyclic}} (b^2 - c^2)f_{4,4}(a, b, c)x, \quad (7)$$

$$L_0(x, y, z) = \sum_{\text{cyclic}} (b^2 + c^2 - 2a^2)x \quad (8)$$

for the equation of the line  $F_+F_-$  (Proposition 8(a)) and the perpendicular to the Euler line at  $G$ . Now, we seek a quantity  $k_0$  such that the member

$$k_0((b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy) - L(x, y, z) \cdot L_0(x, y, z) = 0$$

of the pencil of conic through the four points  $F_{\pm}$ ,  $G$ ,  $Y_0$  is a circle. For this,

$$k_0 = -3(a^2(c^2 - a^2)(a^2 - b^2) + b^2(a^2 - b^2)(b^2 - c^2) + c^2(b^2 - c^2)(c^2 - a^2)),$$

and the equation can be reorganized as

$$9(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2yz + b^2zx + c^2xy) + (x + y + z) \left( \sum_{\text{cyclic}} (b^2 - c^2)(b^2 + c^2 - 2a^2)f_{4,4}(a, b, c)x \right) = 0. \quad (9)$$

The center of the circle  $F_+F_-G$  is the point

$$Z_0 := ((b^2 - c^2)f_{4,7}(a, b, c) : (c^2 - a^2)f_{4,7}(b, c, a) : (a^2 - b^2)f_{4,7}(c, a, b)).$$

The point  $Y_0$  has coordinates  $\left( \frac{b^2 - c^2}{b^2 + c^2 - 2a^2} : \dots : \dots \right)$ .

6.2. *The circle  $F_+F_-H$ . With the line*

$$L_1(x, y, z) = \sum_{\text{cyclic}} (b^2 + c^2 - a^2)(2a^4 - a^2(b^2 + c^2) - (b^2 - c^2)^2)x = 0$$

perpendicular to the Euler line at  $H$ , we seek a number  $k_1$  such that

$$k_1((b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy) - L(x, y, z) \cdot L_1(x, y, z) = 0$$

of the pencil of conic through the four points  $F_{\pm}$ ,  $H$ ,  $Y_1$  is a circle. For this,

$$k_1 = 16\Delta^2(a^4(b^2 + c^2 - a^2) + b^4(c^2 + a^2 - b^2) + c^4(a^2 + b^2 - c^2) - 3a^2b^2c^2),$$

and the equation can be reorganized as

$$48(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)\Delta^2(a^2yz + b^2zx + c^2xy) - (x + y + z) \left( \sum_{\text{cyclic}} (b^2 - c^2)(b^2 + c^2 - a^2)f_{4,4}(a, b, c)f_{4,5}(a, b, c)x \right) = 0. \quad (10)$$

This is the equation of the circle  $F_+F_-H$ . The center is the point

$$Z_1 := ((b^2 - c^2)f_{8,2}(a, b, c) : (c^2 - a^2)f_{8,2}(b, c, a) : (a^2 - b^2)f_{8,2}(c, a, b)).$$

The triangle center

$$Y_1 = \left( \frac{b^2 - c^2}{f_{4,5}(a, b, c)} : \frac{c^2 - a^2}{f_{4,5}(b, c, a)} : \frac{a^2 - b^2}{f_{4,5}(c, a, b)} \right)$$

is  $X_{2394}$ .

6.3. *The first Lester circle.* Since the line joining the Fermat points has equation  $L(x, y, z) = 0$  with  $L$  given by (7), every circle through the Fermat points is represented by

$$9(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2yz + b^2zx + c^2xy) + (x + y + z) \left( \sum_{\text{cyclic}} (b^2 - c^2)(b^2 + c^2 - 2a^2 + t)f_{4,4}(a, b, c)x \right) = 0 \quad (11)$$

for an appropriate choice of  $t$ . The value of  $t$  for which this circle passes through the circumcenter is

$$t = \frac{a^2(c^2 - a^2)(a^2 - b^2) + b^2(a^2 - b^2)(b^2 - c^2) + c^2(b^2 - c^2)(c^2 - a^2)}{32\Delta^2}.$$

The equation of the circle is

$$96\Delta^2(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2yz + b^2zx + c^2xy) + (x + y + z) \left( \sum_{\text{cyclic}} (b^2 - c^2)f_{4,4}(a, b, c)f_{6,1}(a, b, c)x \right) = 0.$$

## 7. The Brocard axis and the Brocard circle

7.1. *The Brocard axis.* The isogonal conjugate of the Kiepert perspector  $K(\theta)$  is the point

$$K^*(\theta) = (a^2(S_A + S_\theta), b^2(S_B + S_\theta), c^2(S_C + S_\theta)),$$

which lies on the line joining the circumcenter  $O$  and the symmedian point  $K$ . The line  $OK$  is called the Brocard axis. It is represented by the equation

$$\sum_{\text{cyclic}} b^2c^2(b^2 - c^2)x = 0. \quad (12)$$

7.2. *The Brocard circle.* The Brocard circle is the circle with  $OK$  as diameter. It is represented by the equation

$$(a^2 + b^2 + c^2)(a^2yz + b^2zx + c^2xy) - (x + y + z)(b^2c^2x + c^2a^2y + a^2b^2z) = 0. \quad (13)$$

It is clear from

$$K^*(\theta) = (a^2S_A, b^2S_B, c^2S_C) + S_\theta(a^2, b^2, c^2), \\ K^*(-\theta) = (a^2S_A, b^2S_B, c^2S_C) - S_\theta(a^2, b^2, c^2)$$

that  $K^*(\theta)$  and  $K^*(-\theta)$  divide  $O$  and  $K$  harmonically, and so are inverse in the Brocard circle. The points  $K^*(\pm\frac{\pi}{3})$  are called the isodynamic points, and are more simply denoted by  $J_\pm$ .

**Proposition 10.**  $K^*(\pm\theta)$  are inverse in the circumcircle if and only if they are the isodynamic points.

7.3. *The isodynamic points.* The isodynamic points  $J_{\pm}$  are also the common points of the three Apollonian circles, each orthogonal to the circumcircle at a vertex (see Figure 8). Thus, the  $A$ -Apollonian circle has diameter the endpoints of the bisectors of angle  $A$  on the sidelines  $BC$ . These are the points  $(b, \pm c)$ . The center of the circle is the midpoint of these, namely,  $(b^2, -c^2)$ . The circle has equation

$$(b^2 - c^2)(a^2yz + b^2zx + c^2xy) + a^2(x + y + z)(c^2y - b^2z) = 0.$$

Similarly, the  $B$ - and  $C$ -Apollonian circles have equations

$$(c^2 - a^2)(a^2yz + b^2zx + c^2xy) + b^2(x + y + z)(a^2z - c^2x) = 0,$$

$$(a^2 - b^2)(a^2yz + b^2zx + c^2xy) + c^2(x + y + z)(b^2x - a^2y) = 0.$$

These three circles are coaxial. Their centers lie on the Lemoine axis

$$\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = 0, \quad (14)$$

which is the perpendicular bisector of the segment  $J_+J_-$ .

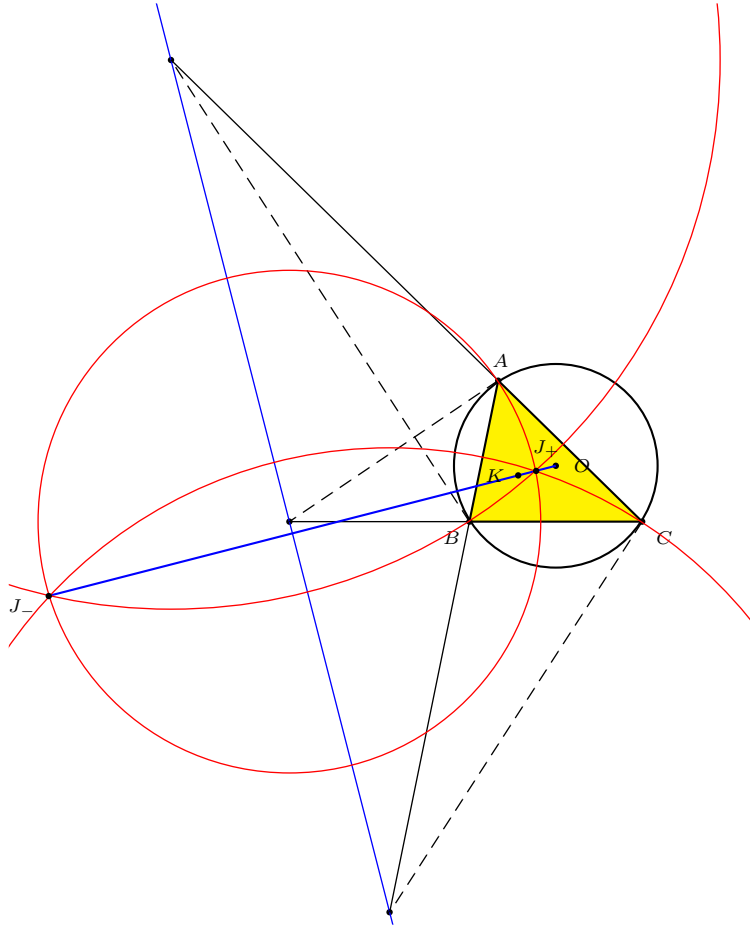


Figure 8. The Apollonian circles and the isodynamic points

**Proposition 11.** *Every circle through the isodynamic points can be represented by an equation*

$$3(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2yz + b^2zx + c^2xy) + (x + y + z) \left( \sum_{\text{cyclic}} b^2c^2(b^2 - c^2)(b^2 + c^2 - 2a^2 + t)x \right) = 0 \quad (15)$$

for some choice of  $t$ .

*Proof.* Combining the above equations for the three Apollonian circles, we obtain

$$3(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2yz + b^2zx + c^2xy) + (x + y + z) \sum_{\text{cyclic}} a^2(c^2 - a^2)(a^2 - b^2)(c^2y - b^2z) = 0.$$

A simple rearrangement of the terms brings the radical axis into the form

$$3(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2yz + b^2zx + c^2xy) + (x + y + z) \left( \sum_{\text{cyclic}} b^2c^2(b^2 - c^2)(b^2 + c^2 - 2a^2)x \right) = 0. \quad (16)$$

Now, the line containing the isodynamic points is the Brocard axis given by (12). It follows that every circle through  $J_{\pm}$  is represented by (15) above for some choice of  $t$  (see §2.2).  $\square$

*Remark.* As is easily seen, equation (16) is satisfied by  $x = y = z = 1$ , and so represents the circle through  $J_{\pm}$  and  $G$ . Since the factors  $b^2 - c^2$  and  $b^2 + c^2 - 2a^2$  yield infinite points, applying Proposition 1, we conclude that this circle intersects the circumcircle at the Euler reflection point  $E = \left( \frac{a^2}{b^2 - c^2} : \cdots : \cdots \right)$  and the Parry point  $\left( \frac{a^2}{b^2 + c^2 - 2a^2} : \cdots : \cdots \right)$ .

This is the Parry circle we consider in §10 below.

**Proposition 12.** *The circle through the isodynamic points and the orthocenter has equation*

$$16\Delta^2 \cdot (b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2yz + b^2zx + c^2xy) + (x + y + z) \left( \sum_{\text{cyclic}} b^2c^2(b^2 - c^2)(b^2 + c^2 - a^2)f_{4,1}(a, b, c)x \right) = 0.$$

*Its center is the point*

$$Z_3 := (a^2(b^2 - c^2)(b^2 + c^2 - a^2)f_{4,1}(a, b, c) : \cdots : \cdots).$$



## 8. The excentral triangle

The excentral triangle  $I_a I_b I_c$  has as vertices the excenters of triangle  $ABC$ . It has circumradius  $2R$  and circumcenter  $I'$ , the reflection of  $I$  in  $O$  (see Figure 9). Since the angles of the excentral triangle are  $\frac{1}{2}(B + C)$ ,  $\frac{1}{2}(C + A)$ , and  $\frac{1}{2}(A + B)$ , its sidelengths  $a' = I_b I_c$ ,  $b' = I_c I_a$ ,  $c' = I_a I_b$  satisfy

$$\begin{aligned} a'^2 : b'^2 : c'^2 &= \cos^2 \frac{A}{2} : \cos^2 \frac{B}{2} : \cos^2 \frac{C}{2} \\ &= a(b + c - a) : b(c + a - b) : c(a + b - c). \end{aligned}$$

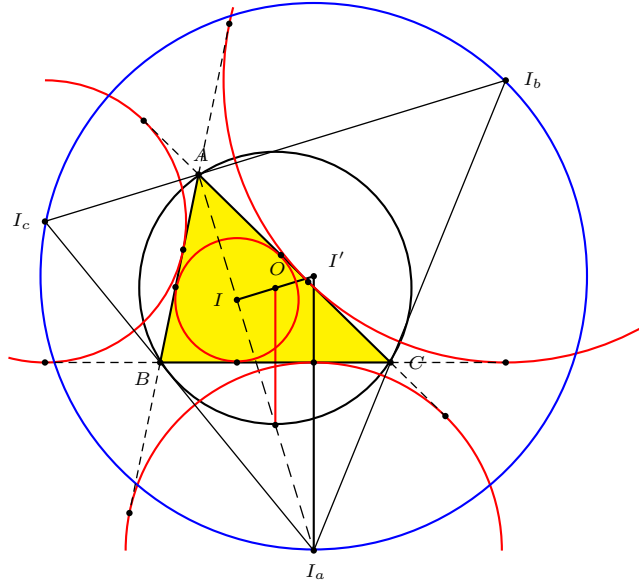


Figure 9. The excentral triangle and its circumcircle

**8.1. Change of coordinates.** A point with homogeneous barycentric coordinates  $(x, y, z)$  with reference to  $ABC$  has coordinates

$$(x', y', z') = (a(b + c - a)(cy + bz), b(c + a - b)(az + cx), c(a + b - c)(bx + ay))$$

with reference to the excentral triangle.

Consider, for example, the Lemoine axis of the excentral triangle, with equation

$$\frac{x'}{a'^2} + \frac{y'}{b'^2} + \frac{z'}{c'^2} = 0.$$

With reference to triangle  $ABC$ , the same line is represented by the equation

$$\frac{a(b + c - a)(cy + bz)}{a(b + c - a)} + \frac{b(c + a - b)(az + cx)}{b(c + a - b)} + \frac{c(a + b - c)(bx + ay)}{c(a + b - c)} = 0,$$

which simplifies into

$$(b + c)x + (c + a)y + (a + b)z = 0. \quad (17)$$

On the other hand, the circumcircle of the excentral triangle, with equation

$$\frac{a'^2}{x'} + \frac{b'^2}{y'} + \frac{c'^2}{z'} = 0,$$

is represented by

$$\frac{1}{cy + bz} + \frac{1}{az + cx} + \frac{1}{bx + ay} = 0$$

with reference to triangle  $ABC$ . This can be rearranged as

$$a^2yz + b^2zx + c^2ay + (x + y + z)(bcx + cay + abz) = 0. \quad (18)$$

## 9. The first Evans circle

9.1. *The Evans perspector  $W$ .* Let  $A^*$ ,  $B^*$ ,  $C^*$  be respectively the reflections of  $A$  in  $BC$ ,  $B$  in  $CA$ ,  $C$  in  $AB$ . The triangle  $A^*B^*C^*$  is called the triangle of reflections of  $ABC$ . Larry Evans has discovered the perspectivity of the excentral triangle and  $A^*B^*C^*$ .

**Theorem 13.** *The excentral triangle and the triangle of reflections are perspective at a point which is the inverse image of the incenter in the circumcircle of the excentral triangle.*

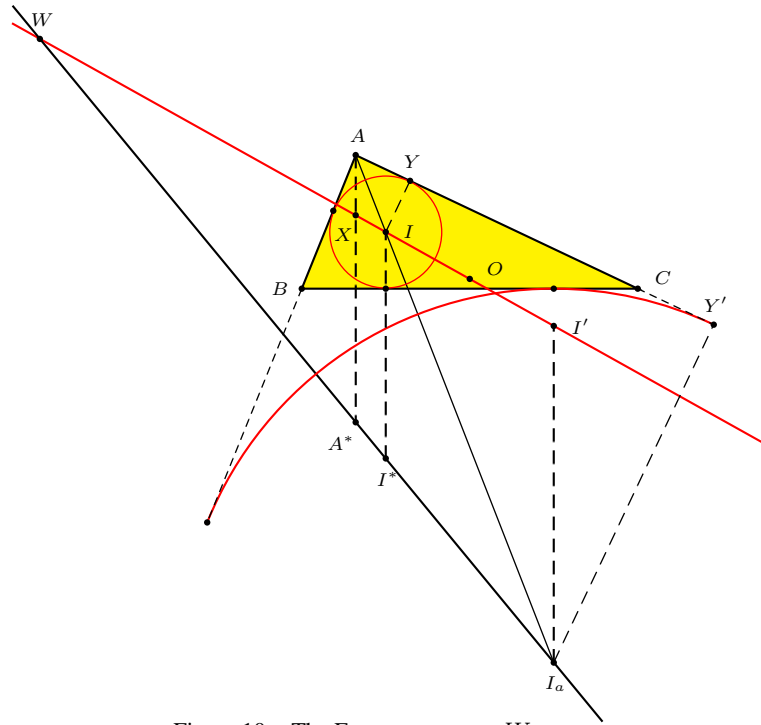


Figure 10. The Evans perspector  $W$

*Proof.* We show that the lines  $I_aA^*$ ,  $I_bB^*$ , and  $I_cC^*$  intersect the line  $OI$  at the same point. Let  $X$  be the intersection of the lines  $AA^*$  and  $OI$  (see Figure 10). If  $h_a$  is the  $A$ -altitude of triangle  $ABC$ , and the parallel from  $I$  to  $AA^*$  meets  $I_aA^*$  at  $I^*$ , then since

$$\frac{II^*}{2h_a} = \frac{II^*}{AA^*} = \frac{I_aI}{I_aA} = \frac{YY'}{AY'} = \frac{a}{s},$$

we have  $II^* = \frac{2ah_a}{s} = 4r$  and  $\frac{WI'}{WI} = \frac{I'I_a}{II^*} = \frac{2R}{4r}$ . Therefore,  $W$  divides  $II'$  in the ratio

$$I'W : WI = R : -2r.$$

Since this ratio is a symmetric function of the sidelengths, we conclude that the same point  $W$  lies on the lines  $I_bB^*$  and  $I_cC^*$ . Moreover, since  $I'W = \frac{R}{R-2r} \cdot I'I$ , by the famous Euler formula  $OI^2 = R(R-2r)$ , we have

$$I'W \cdot I'I = \frac{R}{R-2r} \cdot I'I^2 = \frac{4R}{R-2r} \cdot OI^2 = \frac{4R}{R-2r} \cdot R(R-2r) = (2R)^2.$$

This shows that  $W$  and  $I$  are inverse in the circumcircle of the excentral triangle.  $\square$

The point  $W$  is called the Evans perspector; it has homogeneous barycentric coordinates,

$$\begin{aligned} W &= (a(a^3 + a^2(b+c) - a(b^2 + bc + c^2) - (b+c)(b-c)^2) : \dots : \dots) \\ &= (a((a+b+c)(c+a-b)(a+b-c) - 3abc) : \dots : \dots). \end{aligned}$$

It appears as  $X_{484}$  in [10].

## 9.2. Perspectivity of the excentral triangle and Kiepert triangles.

**Lemma 14.** *Let  $XBC$  and  $X'I_bI_c$  be oppositely oriented similar isosceles triangles with bases  $BC$  and  $I_bI_c$  respectively. The lines  $I_aX$  and  $I_aX'$  are isogonal with respect to angle  $I_a$  the excentral triangle (see Figure 11).*

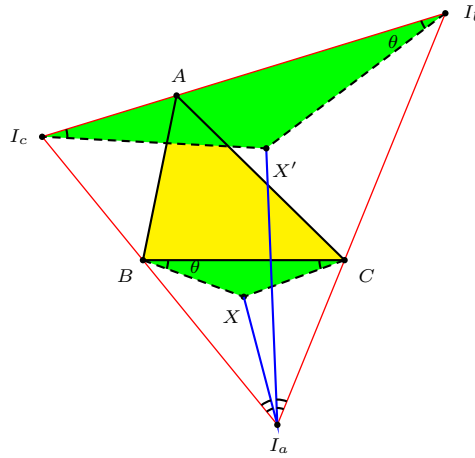


Figure 11. Isogonal lines joining  $I_a$  to apices of similar isosceles on  $BC$  and  $I_bI_c$

*Proof.* The triangles  $I_aBC$  and  $I_aI_bI_c$  are oppositely similar since  $BC$  and  $I_bI_c$  are antiparallel. In this similarity  $X$  and  $X'$  are homologous points. Hence, the lines  $I_aX$  and  $I_aX'$  are isogonal in the excentral triangle.  $\square$

We shall denote Kiepert perspectors with reference to the excentral triangle by  $K_e(-)$ .

**Theorem 15.** *The excentral triangle and the Kiepert triangle  $\mathcal{K}(\theta)$  are perspective at the isogonal conjugate of  $K_e(-\theta)$  in the excentral triangle.*

*Proof.* Let  $XYZ$  be a Kiepert triangle  $\mathcal{K}(\theta)$ . Construct  $X', Y', Z'$  as in Lemma 14 (see Figure 11).

(i)  $I_aX', I_bY', I_cZ'$  concur at the Kiepert perspector  $K_e(-\theta)$  of the excentral triangle.

(ii) Since  $I_aX$  and  $I_aX'$  are isogonal with respect to  $I_a$ , and similarly for the pairs  $I_bY, I_bY'$  and  $I_cZ, I_cZ'$ , the lines  $I_aX, I_bY, I_cZ$  concur at the isogonal conjugate of  $K_e(-\theta)$  in the excentral triangle.  $\square$

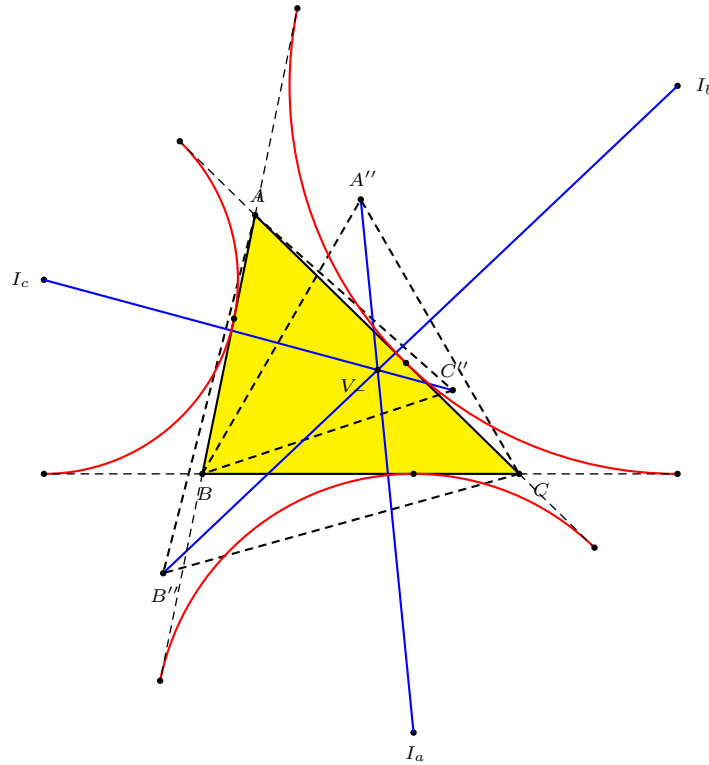


Figure 12. Evans' perspector  $V_-$  of  $\mathcal{K}(-\frac{\pi}{3})$  and excentral triangle

We denote the perspector in Theorem 15 by  $V(\theta)$ , and call this a generalized Evans perspector. In particular,  $V(\frac{\pi}{3})$  and  $V(-\frac{\pi}{3})$  are the isodynamic points of the excentral triangle, and are simply denoted by  $V_+$  and  $V_-$  respectively (see

Figure 12 for  $V_-$ ). These are called the second and third Evans perspectors respectively. They are  $X_{1276}$  and  $X_{1277}$  of [10].

**Proposition 16.** *The line  $V_+V_-$  has equation*

$$\sum_{\text{cyclic}} (b-c)(b^2+c^2-a^2)x = 0. \quad (19)$$

*Proof.* The line  $V_+V_-$  is the Brocard axis of the excentral triangle, with equation

$$\sum_{\text{cyclic}} \frac{b'^2 - c'^2}{a'^2} \cdot x' = 0$$

with reference to the excentral triangle (see §7.1). Replacing these by parameter with reference to triangle  $ABC$ , we have  $\sum_{\text{cyclic}} (b-c)(b+c-a)(cy+bz) = 0$ . Rearranging terms, we have the form (19) above.  $\square$

**Proposition 17.** *The Kiepert triangle  $\mathcal{K}(\theta)$  is perspective with the triangle of reflections  $A^*B^*C^*$  if and only if  $\theta = \pm\frac{\pi}{3}$ . The perspector is  $K^*(-\theta)$ , the isogonal conjugate of  $K(-\theta)$ .*

This means that for  $\varepsilon = \pm 1$ , the Fermat triangle  $\mathcal{K}(\varepsilon \cdot \frac{\pi}{3})$  and the triangle of reflections are perspective at the isodynamic point  $J_{-\varepsilon}$  (see Figure 13 for the case  $\varepsilon = -1$ ).

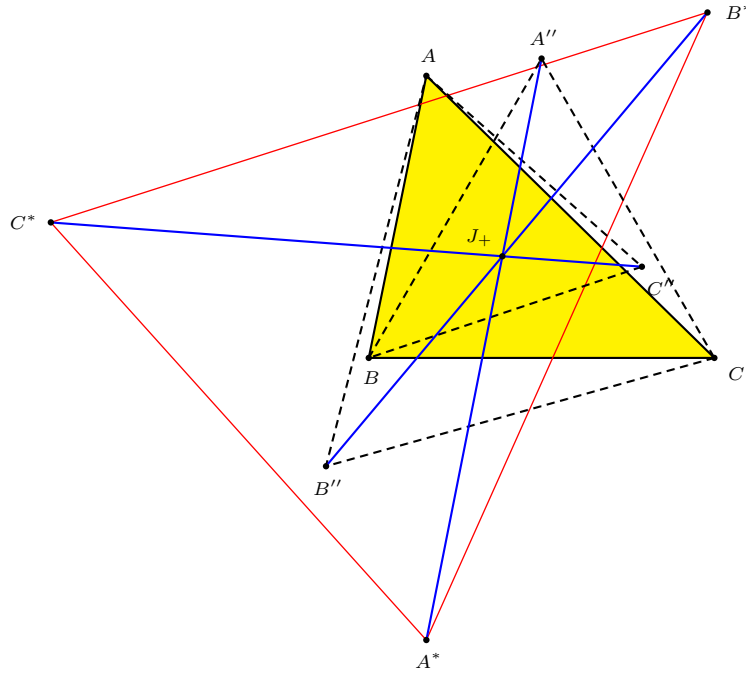


Figure 13.  $\mathcal{K}(-\frac{\pi}{3})$  and  $A^*B^*C^*$  perspective at  $J_+$

9.3. *The first Evans circle.* Since  $V_{\pm}$  are the isodynamic points of the excentral triangle, they are inverse in the circumcircle of the excentral triangle. Since  $W$  and  $I$  are also inverse in the same circle, we conclude that  $V_+$ ,  $V_-$ ,  $I$ , and  $W$  are concyclic (see [1, Theorem 519]). We call this the first Evans circle. A stronger result holds in view of Proposition 10.

**Theorem 18.** *The four points  $V(\theta)$ ,  $V(-\theta)$ ,  $I$ ,  $W$  are concyclic if and only if  $\theta = \pm \frac{\pi}{3}$ .*

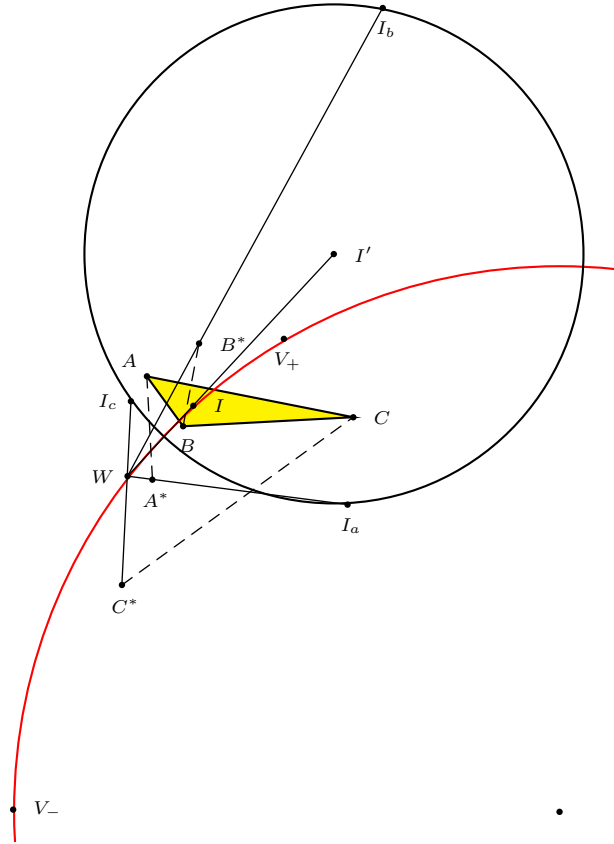


Figure 14. The first Evans circle

We determine the center of the first Evans circle as the intersection of the perpendicular bisectors of the segments  $IW$  and  $V_+V_-$ .

**Lemma 19.** *The perpendicular bisector of the segment  $IW$  is the line*

$$bc(b+c)x + ca(c+a)y + ab(a+b)z = 0. \quad (20)$$

*Proof.* If  $M'$  is the midpoint of  $IW$ , then since  $O$  is the midpoint of  $II'$ , from the degenerate triangle  $II'W$  we have  $OM' = \frac{I'W}{2} = \frac{R^2}{OI}$ . This shows that the

midpoint of  $IW$  is the inverse of  $I$  in the circumcircle. Therefore, the perpendicular bisector of  $IW$  is the polar of  $I$  in the circumcircle. This is the line

$$(a \ b \ c) \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

which is the same as (20).  $\square$

*Remark.*  $M' = (a^2(a^2 - b^2 + bc - c^2) : b^2(b^2 - c^2 + ca - a^2) : c^2(c^2 - a^2 + ab - b^2))$  is the triangle center  $X_{36}$  in [10].

**Lemma 20.** *The perpendicular bisector of the segment  $V_+V_-$  is the line*

$$(b + c)x + (c + a)y + (a + b)z = 0. \quad (21)$$

*Proof.* Since  $V_+$  and  $V_-$  are the isodynamic points of the excentral triangle, the perpendicular bisector of  $V_+V_-$  is the polar of the symmedian point of the excentral triangle with respect to its own circumcircle. With reference to the excentral triangle, its Lemoine axis has equation

$$\frac{x'}{a'^2} + \frac{y'}{b'^2} + \frac{z'}{c'^2} = 0.$$

Changing coordinates, we have, with reference to  $ABC$ , the same line represented by the equation

$$\frac{a(b + c - a)(cy + bz)}{a(b + c - a)} + \frac{b(c + a - b)(az + cx)}{b(c + a - b)} + \frac{c(a + b - c)(bx + ay)}{c(a + b - c)} = 0,$$

which simplifies into (21).  $\square$

**Proposition 21.** *The center of the first Evans circle is the point*

$$\left( \frac{a(b - c)}{b + c} : \frac{b(c - a)}{c + a} : \frac{c(a - b)}{a + b} \right).$$

*Proof.* This is the intersection of the lines (20) and (21).  $\square$

*Remark.* The center of the first Evans circle is the point  $X_{1019}$  in [10]. It is also the perspector of excentral triangle and the cevian triangle of the Steiner point.

**Proposition 22.** *The equation of the first Evans circle is*

$$(a - b)(b - c)(c - a)(a + b + c)(a^2yz + b^2zx + c^2xy) - (x + y + z) \left( \sum_{\text{cyclic}} bc(b - c)(c + a)(a + b)(b + c - a)x \right) = 0. \quad (22)$$

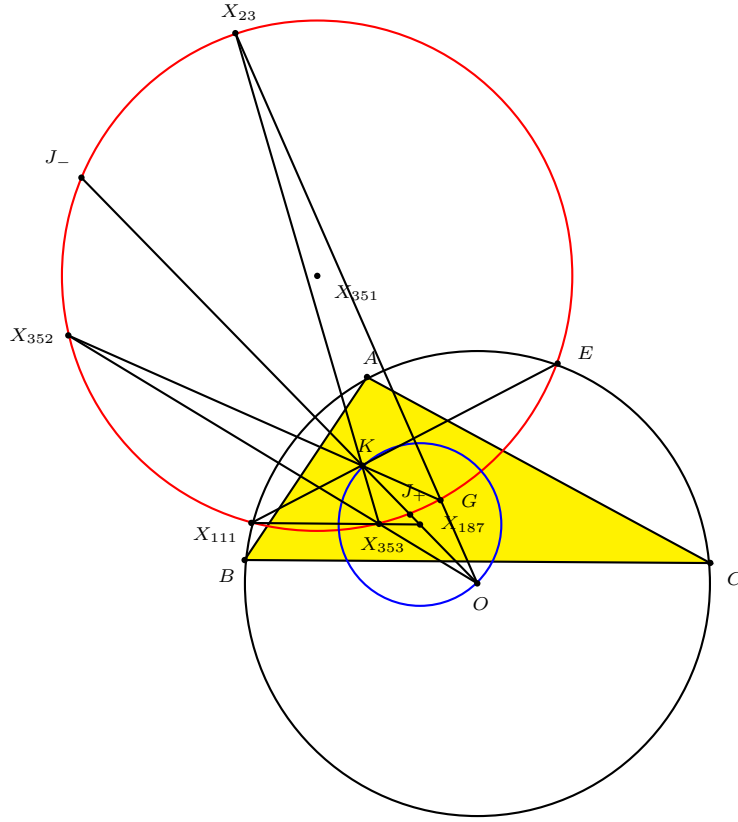


Figure 15. The Parry circle

### 10. The Parry circle and the Parry point

The Parry circle  $\mathcal{C}_P$ , according to [9, p.227], is the circle through a number of interesting triangle centers, including the isodynamic points and the centroid. We shall define the Parry circle as the circle through these three points, and seek to explain the incidence of the other points.

First of all, since the isodynamic points are inverse in each of the circumcircle and the Brocard circle, the Parry circle is orthogonal to each of these circles. In particular, it contains the inverse of  $G$  in the circumcircle. This is the triangle center

$$X_{23} = (a^2(a^4 - b^4 + b^2c^2 - c^4), b^2(b^4 - c^4 + c^2a^2 - a^4), c^2(c^4 - a^4 + a^2b^2 - b^4)). \quad (23)$$

The equation of the Parry circle has been computed in §7.3, and is given by (16). Applying Proposition 1, we see that this circle contains the Euler reflection point

$$E = \left( \frac{a^2}{b^2 - c^2}, \frac{b^2}{c^2 - a^2}, \frac{c^2}{a^2 - b^2} \right) \quad (24)$$



and the point

$$P = \left( \frac{a^2}{b^2 + c^2 - 2a^2}, \frac{b^2}{c^2 + a^2 - 2b^2}, \frac{c^2}{a^2 + b^2 - 2c^2} \right) \quad (25)$$

which we call the Parry point.

*Remark.* The line  $EP$  also contains the symmedian point  $K$ .

**Lemma 23.** *The line  $EG$  is parallel to the Fermat line  $F_+F_-$ .*

*Proof.* The line  $F_+F_-$  is the same as the line  $KK_i$ , with equation given by (5). The line  $EG$  has equation

$$\sum_{\text{cyclic}} (b^2 - c^2) f_{4,2}(a, b, c) x = 0. \quad (26)$$

Both of these lines have the same infinite point

$$X_{542} = (f_{6,2}(a, b, c) : f_{6,2}(b, c, a) : f_{6,2}(c, a, b)).$$

□

**Proposition 24.** *The Euler reflection point  $E$  and the centroid are inverse in the Brocard circle.*

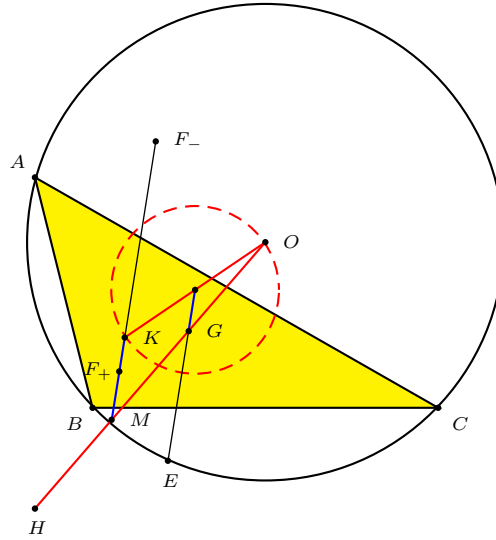


Figure 16.  $E$  and  $G$  are inverse in the Brocard circle

*Proof.* Note that the line  $F_+F_-$  intersects the Euler line at the midpoint  $M$  of  $HG$ , and  $G$  is the midpoint of  $OM$ . Since  $EG$  is parallel to  $MK$ , it intersects  $OK$  at its midpoint, the center of the Brocard circle. Since the circle through  $E, G, J_{\pm}$  is orthogonal to the Brocard circle,  $E$  and  $G$  are inverse to each other with respect to this circle. □

The following two triangle centers on the Parry circle are also listed in [9]:

(i) the second intersection with the line joining  $G$  to  $K$ , namely,

$$X_{352} = (a^2(a^4 - 4a^2(b^2 + c^2) + (b^4 + 5b^2c^2 + c^4)), \dots, \dots),$$

(ii) the second intersection with the line joining  $X_{23}$  to  $K$ , namely,

$$X_{353} = (a^2(4a^4 - 4a^2(b^2 + c^2) - (2b^4 + b^2c^2 + 2c^4)), \dots, \dots),$$

which is the inverse of the Parry point  $P$  in the Brocard circle, and also the inverse of  $X_{352}$  in the circumcircle.

#### 10.1. The center of the Parry circle.

**Proposition 25.** *The perpendicular bisector of the segment  $GE$  is the line*

$$\frac{x}{b^2 + c^2 - 2a^2} + \frac{y}{c^2 + a^2 - 2b^2} + \frac{z}{a^2 + b^2 - 2c^2} = 0. \quad (27)$$

*Proof.* The midpoint of  $EG$  is the point

$$Z_4 := ((b^2 + c^2 - 2a^2)f_{4,5}(a, b, c) : (c^2 + a^2 - 2b^2)f_{4,5}(b, c, a) : (a^2 + b^2 - 2c^2)f_{4,5}(c, a, b)).$$

By Lemma 23 and Proposition 8(b), the perpendicular bisector of  $EG$  has infinite point  $X_{690}$ . The line through  $Z_4$  with this infinite point is the perpendicular bisector of  $EG$ .  $\square$

**Proposition 26.** *The center of the Parry circle  $\mathcal{C}_P$  is the point*

$$(a^2(b^2 - c^2)(b^2 + c^2 - 2a^2), \dots, \dots).$$

*Proof.* This is the intersection of the line (27) above and the the Lemoine axis (14).  $\square$

*Remark.* The center of the Parry circle appears in [10] as  $X_{351}$ .

### 11. The generalized Parry circles

We consider the generalized Parry circle  $\mathcal{C}_P(\theta)$  passing through the centroid and the points  $K^*(\pm\theta)$  on the Brocard axis. Since  $K^*(\theta)$  and  $K^*(-\theta)$  are inverse in the Brocard circle (see §7.2), the generalized Parry circle  $\mathcal{C}_P(\theta)$  is orthogonal to the Brocard circle, and must also contain the Euler reflection point  $E$ . Its equation is

$$3(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(16\Delta^2 \sin^2 \theta - (a^2 + b^2 + c^2)^2 \cos^2 \theta)(a^2 yz + b^2 zx + c^2 xy) \\ + (x + y + z) \sum_{\text{cyclic}} b^2 c^2 (b^2 - c^2)(f_{6,2}(a, b, c) \sin^2 \theta + f_{6,3}(a, b, c) \cos^2 \theta) x = 0.$$

The second intersection with the circumcircle is the point

$$Q(\theta) = \left( \frac{a^2}{f_{6,2}(a, b, c) \sin^2 \theta + f_{6,3}(a, b, c) \cos^2 \theta}, \dots, \dots \right).$$

The Parry point  $P$  is  $Q(\theta)$  for  $\theta = \frac{\pi}{3}$ . Here are two more examples.

(i) With  $\theta = \frac{\pi}{2}$ , we have the circle  $GEO$  tangent to the Brocard axis and with center

$$Z_5 := (a^2(b^2 - c^2)(b^2 + c^2 - 2a^2)f_{4,4}(a, b, c) : \dots : \dots).$$

It intersects the circumcircle again at the point

$$\left( \frac{a^2}{f_{6,2}(a, b, c)}, \frac{b^2}{f_{6,2}(b, c, a)}, \frac{c^2}{f_{6,2}(c, a, b)} \right). \quad (28)$$

This is the triangle center  $X_{842}$ .

(ii) With  $\theta = 0$ , we have the circle  $GEK$  tangent to the Brocard axis and with center

$$Z_6 := (a^2(b^2 - c^2)(b^2 + c^2 - 2a^2)((a^2 + b^2 + c^2)^2 - 9b^2c^2) : \dots : \dots).$$

It intersects the circumcircle again at the point

$$Z_7 := \left( \frac{a^2}{f_{6,3}(a, b, c)} : \frac{b^2}{f_{6,3}(b, c, a)} : \frac{c^2}{f_{6,3}(c, a, b)} \right). \quad (29)$$

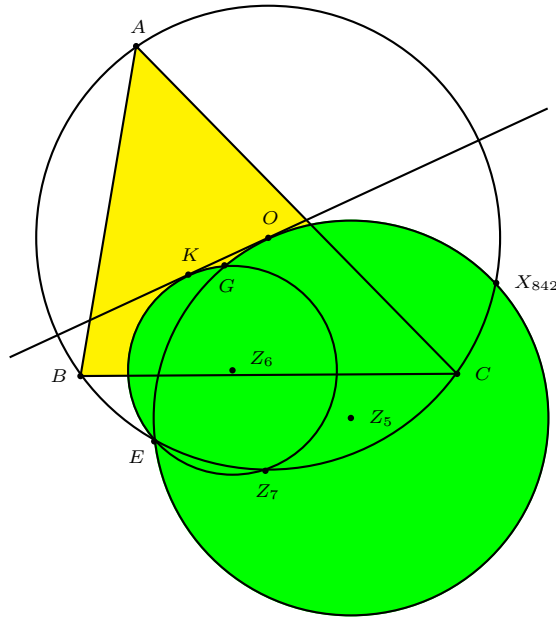


Figure 17. The circles  $GEO$  and  $GEK$

## 12. Circles containing the Parry point

12.1. *The circle  $F_+F_-G$ .* The equation of the circle  $F_+F_-G$  has been computed in §6.1, and is given by (9). Applying Proposition 1, we see that the circle  $F_+F_-G$  contains the Parry point and the point

$$Q = \left( \frac{1}{(b^2 - c^2)f_{4,4}(a, b, c)} : \frac{1}{(c^2 - a^2)f_{4,4}(b, c, a)} : \frac{1}{(a^2 - b^2)f_{4,4}(c, a, b)} \right).$$

This is the triangle center  $X_{476}$  in [10]. It is the reflection of the Euler reflection point in the Euler line.<sup>2</sup>

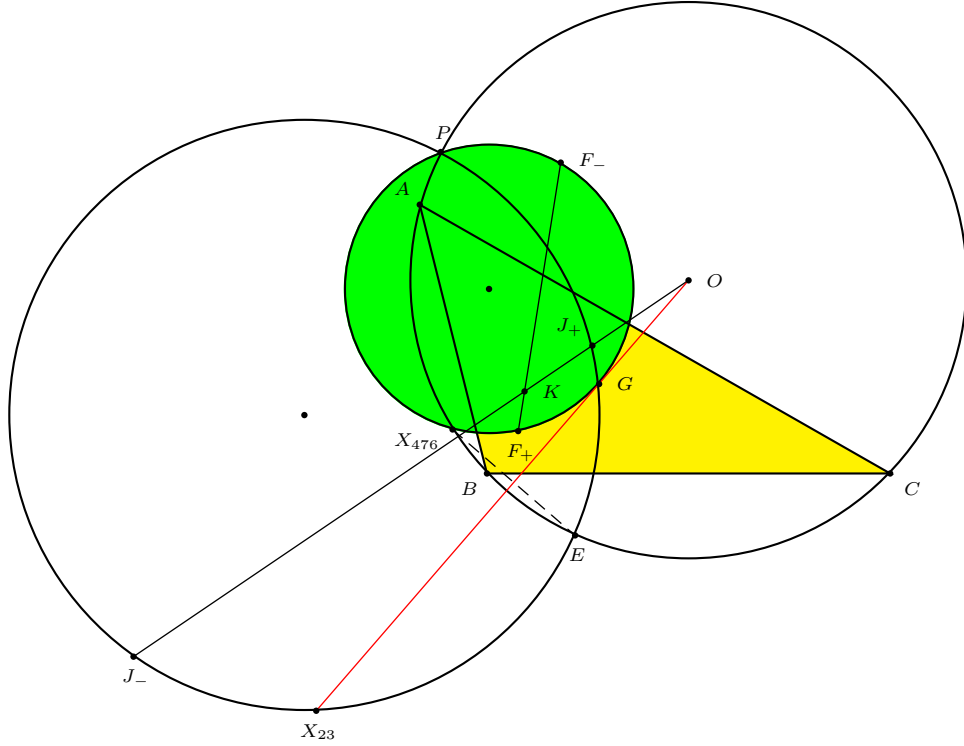


Figure 18. Intersections of  $F_+F_-G$  and the circumcircle

12.2. *The circle GOK.* Making use of the equations (13) of the Brocard circle and (12) of the Brocard axis, we find equations of circles through  $O$  and  $K$  in the form

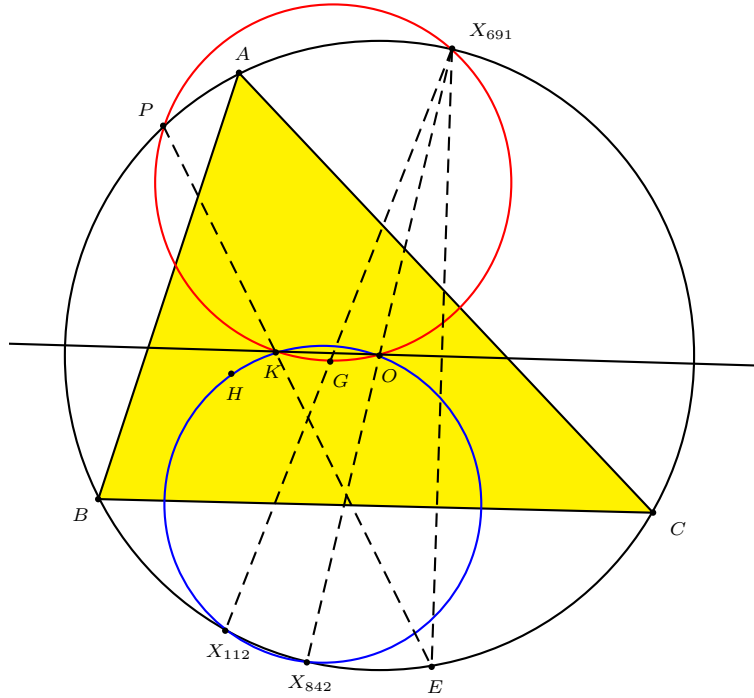
$$(a^2 + b^2 + c^2)(a^2 yz + b^2 zx + c^2 xy) - (x + y + z) \left( \sum_{\text{cyclic}} b^2 c^2 ((b^2 - c^2)t + 1)x \right) = 0 \quad (30)$$

for suitably chosen  $t$ .

With  $t = -\frac{a^4 + b^4 + c^4 - b^2 c^2 - c^2 a^2 - a^2 b^2}{3(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)}$ , and clearing denominators, we obtain the equation of the circle  $GOK$ .

$$3(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2 + b^2 + c^2)(a^2 yz + b^2 zx + c^2 xy) + (x + y + z) \left( \sum_{\text{cyclic}} b^2 c^2 (b^2 - c^2)(b^2 + c^2 - 2a^2)^2 x \right) = 0.$$

<sup>2</sup>To justify this, one may compute the infinite point of the line  $EQ$  and see that it is  $X_{523} = (b^2 - c^2 : c^2 - a^2 : a^2 - b^2)$ . This shows that  $EQ$  is perpendicular to the Euler line.

Figure 19. The circles  $GOK$  and  $HOK$ 

This circle  $GOK$  contains the Parry point  $P$  and the point

$$Q' = \left( \frac{a^2}{(b^2 - c^2)(b^2 + c^2 - 2a^2)} : \dots : \dots \right),$$

which is the triangle center  $X_{691}$ . It is the reflection of  $E$  in the Brocard axis.<sup>3</sup>

*Remark.* The line joining  $P$  to  $X_{691}$  intersects

- (i) the Brocard axis at  $X_{187}$ , the inversive image of  $K$  in the circumcircle,
- (ii) the Euler line at  $X_{23}$ , the inversive image of the centroid in the circumcircle.

### 13. Some special circles

13.1. *The circle  $HOK$ .* By the same method, with

$$t = -\frac{a^4(c^2 - a^2)(a^2 - b^2) + b^4(a^2 - b^2)(b^2 - c^2) + c^4(b^2 - c^2)(c^2 - a^2)}{16\Delta^2(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)}$$

<sup>3</sup>This may be checked by computing the infinite point of the line  $EQ'$  as  $X_{512} = (a^2(b^2 - c^2), b^2(c^2 - a^2), c^2(a^2 - b^2))$ , the one of lines perpendicular to  $OK$ .

in (30), we find the equation of the circle  $HOK$  as

$$16\Delta^2(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2yz + b^2zx + c^2xy) \\ + (x + y + z) \left( \sum_{\text{cyclic}} b^2c^2(b^2 - c^2)(b^2 + c^2 - a^2)f_{6,2}(a, b, c)x \right) = 0.$$

Therefore, the circle  $HOK$  intersects the circumcircle at

$$\left( \frac{a^2}{(b^2 - c^2)(b^2 + c^2 - a^2)}, \frac{b^2}{(c^2 - a^2)(c^2 + a^2 - b^2)}, \frac{c^2}{(a^2 - b^2)(a^2 + b^2 - c^2)} \right),$$

which is the triangle center  $X_{112}$ , and  $X_{842}$  given by (28).

*Remarks.* (1) The circle  $HOK$  has center

$$Z_8 := (a^2(b^2 - c^2)f_{4,2}(a, b, c)f_{4,3}(a, b, c) : \cdots : \cdots).$$

(2)  $X_{112}$  is the second intersection of the circumcircle with the line joining  $X_{74}$  with the symmedian point.

(3)  $X_{842}$  is the second intersection of the circumcircle with the parallel to  $OK$  through  $E$ . It is also the antipode of  $X_{691}$ , which is the reflection of  $E$  in the Brocard axis.

(4) The radical axis with the circumcircle intersects the Euler line at  $X_{186}$  and the Brocard axis at  $X_{187}$ . These are the inverse images of  $H$  and  $K$  in the circumcircle.

13.2. *The circle  $NOK$ .* The circle  $NOK$  contains the Kiepert center because both  $O$ ,  $N$  and  $K$ ,  $K_1$  are inverse in the orthocentroidal circle.

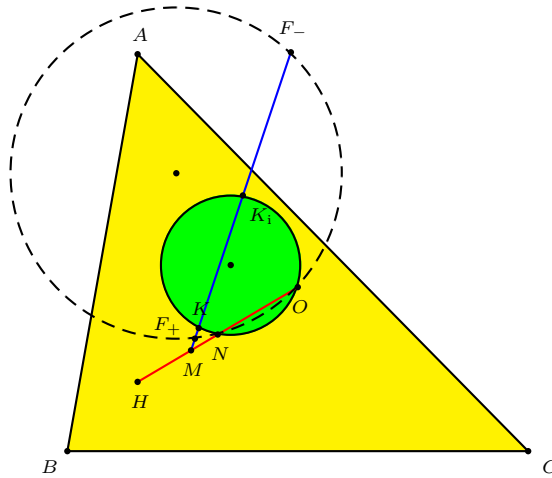


Figure 20. The circle  $NOK$

This circle has equation

$$32(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)(a^2 + b^2 + c^2)\Delta^2(a^2yz + b^2zx + c^2xy) \\ + \sum_{\text{cyclic}} b^2c^2(b^2 - c^2)f_{8,4}(a, b, c)x = 0.$$

Its center is the point

$$Z_9 := (a^2(b^2 - c^2)f_{8,1}(a, b, c) : \cdots : \cdots).$$

#### 14. The second Evans circle

Evans also conjectured that the perspectors  $V_{\pm} = V(\pm\frac{\pi}{3})$  and  $X_{74}$ ,  $X_{399}$  are conyclic. Recall that

$$X_{74} = \left( \frac{a^2}{f_{4,5}(a, b, c)} : \frac{b^2}{f_{4,5}(b, c, a)} : \frac{c^2}{f_{4,5}(c, a, b)} \right)$$

is the antipode on the circumcircle of the Euler reflection point  $E$  and  $X_{399}$ , the Parry reflection point, is the reflection of  $O$  in  $E$ .

We confirm Evans' conjecture indirectly, by first finding the circle through  $V_{\pm}$  and the point

$$X_{101} = (a^2(c - a)(a - b) : b^2(a - b)(b - c) : c^2(b - c)(c - a))$$

on the circumcircle. Making use of the equation (22) of the first Evans circle, and the equation (19) of the line  $V_+V_-$ , we seek a quantity  $t$  such that

$$(a - b)(b - c)(c - a)(a + b + c)(a^2yz + b^2zx + c^2xy) \\ - (x + y + z) \left( \sum_{\text{cyclic}} (bc(b - c)(c + a)(a + b)(b + c - a) + t(b - c)(b^2 + c^2 - a^2))x \right) = 0,$$

represents a circle through the point  $X_{101}$ . For this, we require

$$t = - \frac{abc(a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) + abc)}{(b + c - a)(c + a - b)(a + b - c)},$$

and the equation of the circle through  $V_{\pm}$  and  $X_{101}$  is

$$16\Delta^2(a - b)(b - c)(c - a)(a^2yz + b^2zx + c^2xy) \\ - (x + y + z) \left( \sum_{\text{cyclic}} b^2c^2(b - c)f_{4,5}(a, b, c)x \right) = 0.$$

It is clear that this circle does also contain the point  $X_{74}$ .

The center of the circle is the point

$$Z_{10} = (a^2(b - c)((b^2 + c^2 - a^2)^2 - b^2c^2), \dots, \dots).$$

Now, the perpendicular bisector of the segment  $OE$  is the line

$$\sum \frac{x}{a^2f_{4,4}(a, b, c)} = 0,$$

which clearly contains the center of the circle. Therefore, the circle also contains the point which is the reflection of  $X_{74}$  in the midpoint of  $OE$ . This is the same as the reflection of  $O$  in  $E$ , the Parry reflection point  $X_{399}$ .

**Theorem 27** (Evans). *The four points  $V_{\pm}$ , the antipode of the Euler reflection point  $E$  on the circumcircle, and the reflection of  $O$  in  $E$  are concyclic (see Figure 21).*

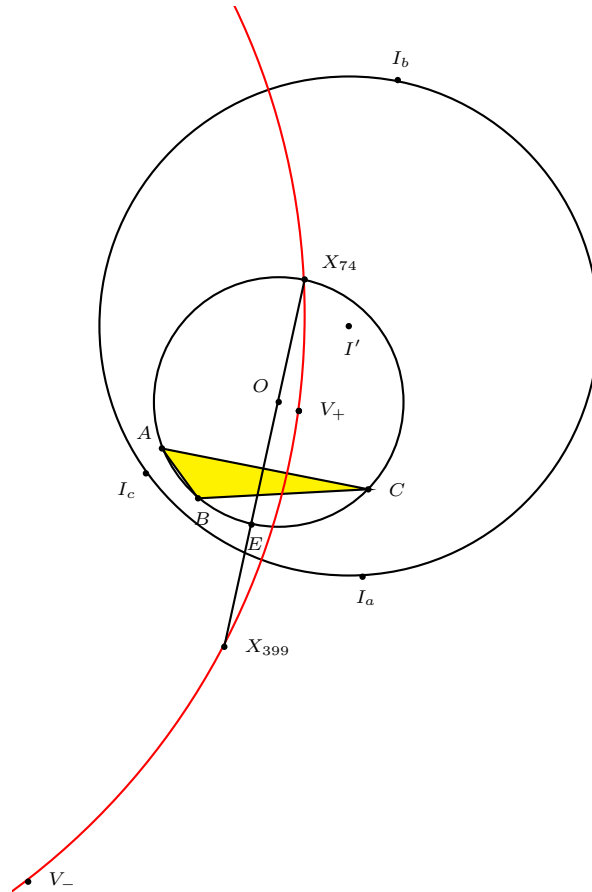


Figure 21. The second Evans circle

## 15. The second Lester circle

In Kimberling's first list of triangle centers [8], the point  $X_{19}$ , the homothetic center of the orthic triangle and the triangular hull of the three excircles, was called the crucial point. Kimberling explained that this name "derives from the name of the publication [13] in which the point first appeared". In the expanded list in [9], this point was renamed after J.W. Clawson. Kimberling gave the reference [2], and



commented that this is “possibly the earliest record of this point”.<sup>4</sup>

$$C_w = \left( \frac{a}{b^2 + c^2 - a^2}, \frac{b}{c^2 + a^2 - b^2}, \frac{c}{a^2 + b^2 - c^2} \right). \quad (31)$$

**Proposition 28.** *The Clawson point  $C_w$  is the perspector of the triangle bounded by the radical axes of the circumcircle with the three excircles (see Figure 24).*

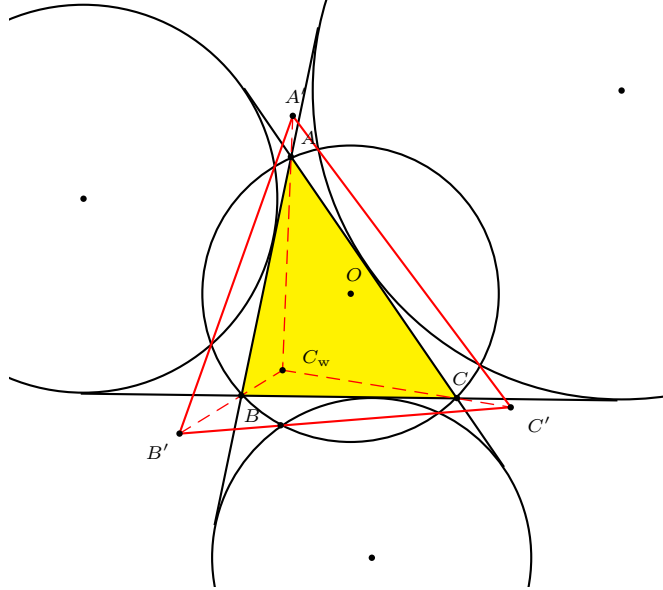


Figure 22. The Clawson point

*Proof.* The equations of the excircles are given in [15, §6.1.1]. The radical axes with circumcircle are the lines

$$L_a := s^2x + (s - c)^2y + (s - b)^2z = 0,$$

$$L_b := (s - c)^2x + s^2y + (s - a)^2z = 0,$$

$$L_c := (s - b)^2x + (s - a)^2y + s^2z = 0.$$

These lines intersect at

$$A' = (0 : b(a^2 + b^2 - c^2) : c(c^2 + a^2 - b^2)),$$

$$B' = (a(a^2 + b^2 - c^2) : 0 : c(b^2 + c^2 - a^2)),$$

$$C' = (a(c^2 + a^2 - b^2) : b(b^2 + c^2 - a^2) : 0).$$

It is clear that the triangles  $ABC$  and  $A'B'C'$  are perspective at a point whose coordinates are given by (31).  $\square$

<sup>4</sup>According to the current edition of [10], this point was studied earlier by E. Lemoine [11].

Apart from the circle through the circumcenter, the nine-point center and the Fermat points, Lester has discovered another circle through the symmedian point, the Clawson point, the Feuerbach point and the homothetic center of the orthic and the intangents triangle. The intangents are the common separating tangents of the incircle and the excircles apart from the sidelines. These are the lines

$$\begin{aligned} L'_a &:= bcx + (b-c)cy - (b-c)bz = 0, \\ L'_b &:= -(c-a)cx + cay + (c-a)az = 0, \\ L'_c &:= (a-b)bx - (a-b)ay + abz = 0. \end{aligned}$$

These lines are parallel to the sides of the orthic triangles, namely,

$$\begin{aligned} -(b^2 + c^2 - a^2)x + (c^2 + a^2 - b^2)y + (a^2 + b^2 - c^2)z &= 0, \\ (b^2 + c^2 - a^2)x - (c^2 + a^2 - b^2)y + (a^2 + b^2 - c^2)z &= 0, \\ (b^2 + c^2 - a^2)x + (c^2 + a^2 - b^2)y - (a^2 + b^2 - c^2)z &= 0. \end{aligned}$$

The two triangles are therefore homothetic. The homothetic center is

$$T_o = \left( \frac{a(b+c-a)}{b^2+c^2-a^2}, \frac{b(c+a-b)}{c^2+a^2-b^2}, \frac{c(a+b-c)}{a^2+b^2-c^2} \right). \quad (32)$$

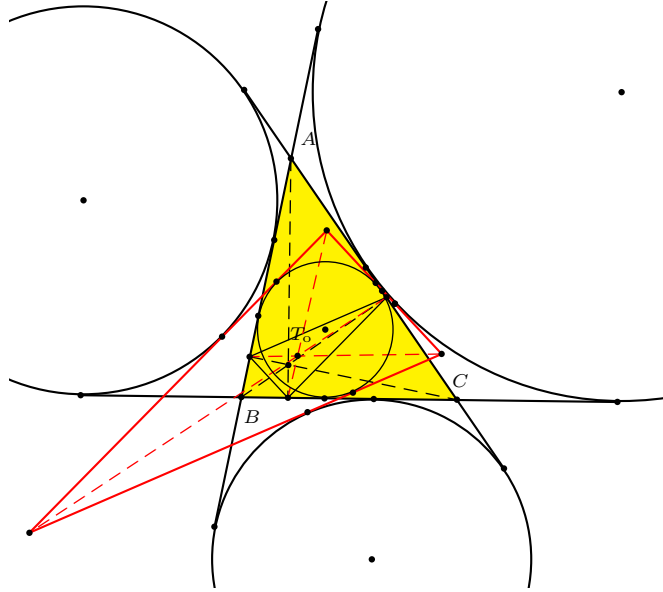


Figure 23. The intangents triangle

**Theorem 29 (Lester).** *The symmedian point, the Feuerbach point, the Clawson point, and the homothetic center of the orthic and the intangent triangles are concyclic.*

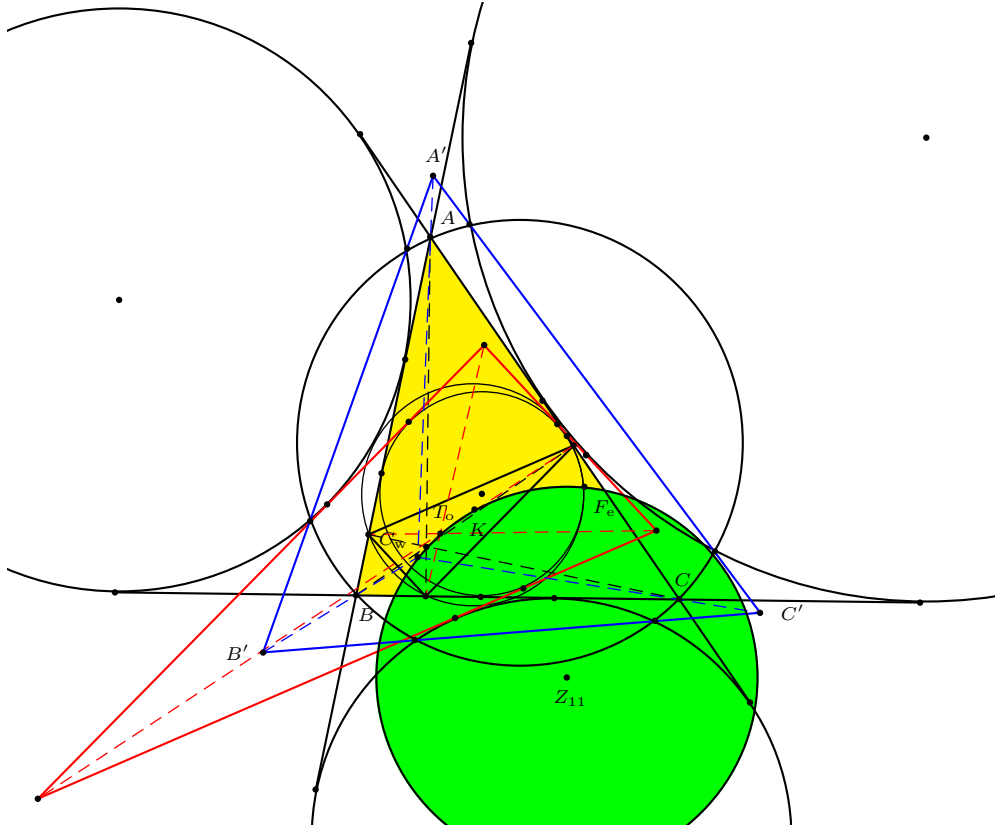


Figure 24. The second Lester circle

There are a number of ways of proving this theorem, all very tedious. For example, it is possible to work out explicitly the equation of the circle containing these four points. Alternatively, one may compute distances and invoke the intersecting chords theorem. These proofs all involve polynomials of large degrees. We present here a proof given by Nikolaos Dergiades which invokes only polynomials of relatively small degrees.

**Lemma 30.** *The equation of the circle passing through three given points  $R_1 = (u_1 : v_1 : w_1)$ ,  $P_2(u_2 : v_2 : w_2)$  and  $P_3 = (u_3 : v_3 : w_3)$  is*

$$a^2yz + b^2zx + c^2xy - (x + y + z)(px + qy + rz) = 0$$

where

$$p = \frac{D(u_1, u_2, u_3)}{s_1 s_2 s_3 D(1, 2, 3)}, \quad q = \frac{D(v_1, v_2, v_3)}{s_1 s_2 s_3 D(1, 2, 3)}, \quad r = \frac{D(w_1, w_2, w_3)}{s_1 s_2 s_3 D(1, 2, 3)},$$

with

$$s_1 = u_1 + v_1 + w_1, \quad s_2 = u_2 + v_2 + w_2, \quad s_3 = u_3 + v_3 + w_3, \quad D(1, 2, 3) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix},$$

and

$$\begin{aligned} D(u_1, u_2, u_3) &= \begin{vmatrix} a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1 & s_1v_1 & s_1w_1 \\ a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2 & s_2v_2 & s_2w_2 \\ a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3 & s_3v_3 & s_3w_3 \end{vmatrix}, \\ D(v_1, v_2, v_3) &= \begin{vmatrix} s_1u_1 & a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1 & s_1w_1 \\ s_2u_2 & a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2 & s_2w_2 \\ s_3u_3 & a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3 & s_3w_3 \end{vmatrix}, \\ D(w_1, w_2, w_3) &= \begin{vmatrix} s_1u_1 & s_1v_1 & a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1 \\ s_2u_2 & s_2v_2 & a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2 \\ s_3u_3 & s_3v_3 & a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3 \end{vmatrix}. \end{aligned}$$

*Proof.* This follows from applying Cramer's rule to the system of linear equations

$$\begin{aligned} a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1 - s_1(pu_1 + qv_1 + rw_1) &= 0, \\ a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2 - s_2(pu_2 + qv_2 + rw_2) &= 0, \\ a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3 - s_3(pu_3 + qv_3 + rw_3) &= 0. \end{aligned}$$

□

**Lemma 31.** Four points  $P_i = (u_i : v_i : w_i)$ ,  $i = 1, 2, 3, 4$ , are concyclic if and only if

$$\frac{D(u_1, u_2, u_4)}{D(u_1, u_2, u_3)} = \frac{D(v_1, v_2, v_4)}{D(v_1, v_2, v_3)} = \frac{D(w_1, w_2, w_4)}{D(w_1, w_2, w_3)} = \frac{s_4 D(1, 2, 4)}{s_3 D(1, 2, 3)}.$$

*Proof.* The circumcircles of triangles  $P_1P_2P_3$  and  $P_1P_2P_4$  have equations

$$\begin{aligned} a^2yz + b^2zx + c^2xy - (x + y + z)(px + qy + rz) &= 0, \\ a^2yz + b^2zx + c^2xy - (x + y + z)(p'x + q'y + r'z) &= 0 \end{aligned}$$

where  $p, q, r$  are given in Lemma 30 above and  $p', q', r'$  are calculated with  $u_3, v_3, w_3$  replaced by  $u_4, v_4, w_4$  respectively. These two circles are the same if and only if  $p = p', q = q', r = r'$ . The condition  $p = p'$  is equivalent to  $\frac{D(u_1, u_2, u_4)}{D(u_1, u_2, u_3)} = \frac{s_4 D(1, 2, 4)}{s_3 D(1, 2, 3)}$ ; similarly for the remaining two conditions. □

Finally we complete the proof of the second Lester circle theorem. For

$$P_1 = K = (a^2 : b^2 : c^2),$$

$$P_2 = F_e = ((b - c)^2(b + c - a) : (c - a)^2(c + a - b) : (a - b)^2(a + b - c)),$$

$$P_3 = C_w = (aS_{BC} : bS_{CA} : cS_{AB}),$$

$$P_4 = T_o = (a(b + c - a)S_{BC} : b(c + a - b)S_{CA} : c(a + b - c)S_{AB}),$$

we have

$$\frac{D(u_1, u_2, u_4)}{D(u_1, u_2, u_3)} = \frac{(b + c - a)(c + a - b)(a + b - c)}{a + b + c} = \frac{s_4 D(1, 2, 4)}{s_3 D(1, 2, 3)}.$$

The cyclic symmetry also shows that

$$\frac{D(v_1, v_2, v_4)}{D(v_1, v_2, v_3)} = \frac{D(w_1, w_2, w_4)}{D(w_1, w_2, w_3)} = \frac{(b + c - a)(c + a - b)(a + b - c)}{a + b + c}.$$

It follows from Lemma 31 that the four points  $K$ ,  $F_e$ ,  $C_w$ , and  $T_o$  are concyclic. This completes the proof of Theorem 29.

For completeness, we record the coordinates of the center of the second Lester circle, namely,

$$Z_{11} := (a(b-c)f_5(a, b, c)f_{12}(a, b, c) : \cdots : \cdots),$$

where

$$\begin{aligned} f_5(a, b, c) &= a^5 - a^4(b+c) + 2a^3bc - a(b^4 + 2b^3c - 2b^2c^2 + 2bc^3 + c^4) + (b-c)^2(b+c)^3, \\ f_{12}(a, b, c) &= a^{12} - 2a^{11}(b+c) + 9a^{10}bc + a^9(b+c)(2b^2 - 13bc + 2c^2) \\ &\quad - a^8(3b^4 - 2b^3c - 22b^2c^2 - 2bc^3 + 3c^4) + 4a^7(b+c)((b^2 - c^2)^2 - b^2c^2) \\ &\quad - 10a^6bc(b^2 - c^2)^2 - 2a^5(b+c)(b-c)^2(b^2 - 4bc + c^2)(2b^2 + 3bc + 2c^2) \\ &\quad + a^4(b-c)^2(3b^6 + 2b^5c - 19b^4c^2 - 32b^3c^3 - 19b^2c^4 + 2bc^5 + 3c^6) \\ &\quad - 2a^3(b+c)(b-c)^2(b^6 + 2b^5c - 3b^4c^2 - 2b^3c^3 - 3b^2c^4 + 2bc^5 + c^6) \\ &\quad + a^2bc(b^4 - c^4)^2 + a(b+c)(b-c)^4(b^2 + c^2)^2(2b^2 + 3bc + 2c^2) \\ &\quad - (b-c)^4(b+c)^2(b^2 + c^2)^3. \end{aligned}$$

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