Junior problems

J217. If a, b, c are integers such that $a^2 + 2bc = 1$ and $b^2 + 2ca = 2012$, find all possible values of $c^2 + 2ab$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J218. Prove that in any triangle with sides of lenghts a, b, c, circumradius R, and inradius r, the following inequality holds

$$\frac{\sqrt{ab}}{a+b-c} + \frac{\sqrt{bc}}{b+c-a} + \frac{\sqrt{ca}}{c+a-b} \leq 1 + \frac{R}{r}.$$

Proposed by Cezar Lupu, University of Pittsburgh, USA, and Virgil Nicula, Bucharest, Romania

J219. Trying to solve a problem, Jimmy used the following "formula": $\log_{ab} x = \log_a x \log_b x$, where a, b, x are positive real numbers different from 1. Prove that this is correct only if x is a solution to the equation $\log_a x + \log_b x = 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J220. Find the least prime p for which $p = a_k^2 + kb_k^2$, $k = 1, \ldots, 5$, for some (a_k, b_k) in $\mathbb{Z} \times \mathbb{Z}$.

Proposed by Cosmin Pohoata, Princeton University, USA

J221. Solve in integers the system of equations

$$\begin{aligned} xy - \frac{z}{3} &= xyz + 1 \\ yz - \frac{x}{3} &= xyz - 1 \\ zx - \frac{y}{3} &= xyz - 9. \end{aligned}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J222. Give a ruler and straightedge construction of a triangle ABC given its orthocenter and the intersection points of the internal and external angle bisectors of one of its angles with the corresponding opposite side.

Proposed by Cosmin Pohoata, Princeton University, USA

Senior problems

S217. Find all integer solutions of the equation $2x^2 - y^{14} = 1$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

S218. Let ABC be a triangle with incircle $\mathcal{C}$ and incenter I. Let D, E, F be the tangency points of $\mathcal{C}$ with the sides BC, CA, and AB, respectively, and furthermore, let S be the intersection of BC and EF. Let P, Q be the intersection points of SI with $\mathcal{C}$ such that P, Q lie on the small arcs DE and FD respectively. Prove that the lines AD, BP, CQ are concurrent.

Proposed by Marius Stanean, Zalau, Romania

S219. Let $ABCD$ be a quadrilateral and let $\{P\} = AC \cap BD$, $\{E\} = AD \cap BC$, and $\{F\} = AB \cap CD$. Denote by $\mathrm{isog}_{XYZ}(P)$ the isogonal conjugate of P with respect to triangle XYZ. Prove that $\mathrm{isog}_{ABE}(P) = \mathrm{isog}_{CDE}(P) = \mathrm{isog}_{ADF}(P) = \mathrm{isog}_{BCF}(P)$ if and only if AC and BD are perpendicular.

Proposed by Cosmin Pohoata, Princeton University, USA

S220. Let a, b, c be nonnegative real numbers. Prove that

$$\sqrt[3]{a^3 + b^3 + c^3 - \frac{1}{2}\left(ab(a+b) + bc(b+c) + ca(c+a)\right)} \geq \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}.$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

S221. Let ABC be a triangle with centroid G and let F be a point that minimizes the quantity $PA + PB + PC$ over all points P lying in the plane of ABC. Prove that

$$FG \leq \min(AG, BG, CG).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S222. Solve the equation $3\phi(n) = 4\tau(n)$ where $\phi(n)$ is the Euler totient function and $\tau(n)$ is the number of divisors of n.

Proposed by Roberto Bosch Cabrera, Florida, USA

Undergraduate problems

U217. Define an increasing sequence $(a_k)_{k\in\mathbb{Z}_+}$ to be *attractive* if $\sum_{k=1}^{\infty}\frac{1}{a_k}$ diverges and $\sum_{k=1}^{\infty}\frac{1}{a_k^2}$ converges. Prove that there is an attractive sequence a_k such that $a_k\sqrt{k}$ is also an attractive sequence.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U218. Let $*$ be an associative and "totally non-commutative" ($x \neq y$ implies $x * y \neq y * x$) binary operation on a set S. Prove that $x * y * z = x * z$ for all x, y, z in S.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania

U219. a) Let $f : [0, \infty) \to \mathbb{R}$ be a convex differentiable function with $f(0) = 0$. Prove that

$$\int_0^x f(t)dt \le \frac{x^2}{2} f'(x) \quad \text{for all} \quad x \in [0, \infty).$$

b) Find all differentiable functions $f : [0, \infty) \to \mathbb{R}$ for which we have equality in the above inequality.

Proposed by Dorin Andrica, Babes-Bolyai University of Cluj-Napoca, and Mihai Piticari National College "Dragos Voda" Campulung Moldovenesc, Romania

U220. Evaluate

$$\lim_{n\to\infty}\left((n+1)\sqrt[n+1]{\Gamma\left(\frac{1}{n+1}\right)} - n\sqrt[n]{\Gamma\left(\frac{1}{n}\right)}\right),$$

where Γ denotes the classical Gamma function.

Proposed by Cezar Lupu, University of Pittsburgh, USA and Moubinool Omarjee, Lycee Jean Murcat, Paris, France

U221. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function such that $f(2012) \neq 0$. Prove that there exists $c > 0$ such that for all $n \ge 2012$ we have

$$\sum_{k=1}^{n} \frac{|f(k)|}{k} > c \cdot \ln n.$$

Proposed by Gabriel Dospinescu, Ecole Polytechnique, Paris, France

U222. Let p and q be distinct odd primes and let d be a divisor of $q - 1$. Prove that

a) $\mathbb{Q}[\zeta_q]$ has a unique subfield K_d that has degree d over $\mathbb{Q}$, where ζ_q denotes a primitive q-th root of unity.

b) p splits completely in K_d if and only if q splits completely in $\mathbb{Q}\left[\sqrt[d]{p}\right]$.

Proposed by Cosmin Pohoata, Princeton University, USA

Olympiad problems

O217. Equilateral triangles ACB' and BDC' are drawn on the diagonals of a convex quadrilateral $ABCD$ so that B and B' are on the same side of AC, and C and C' are on the same side of BD. Find $\angle BAD + \angle CAD$ if $B'C' = AB + CD$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O218. Find all integers n such that $2^n + 3^n + 13^3 - 14^n$ is the cube of an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O219. Let a, b, c, d be positive real numbers that satisfy

$$\frac{1-c}{a} + \frac{1-d}{b} + \frac{1-a}{c} + \frac{1-b}{d} \geq 0.$$

Prove that $a(1-b) + b(1-c) + c(1-d) + d(1-a) \geq 0$.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, Paris, France

O220. Let $A_1, \ldots, A_n$ be distinct points in the plane and let G be their center of gravity. Consider a point F in plane for which $A_1F + \cdots + A_nF$ is minimal. Prove that

$$\sum_{i=1}^{n} A_iG \leq 2\left(1 - \frac{1}{n}\right) \sum_{i=1}^{n} A_iF.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O221. Suppose that an ant starts at a vertex of complete graph K_4 and moves on edges with probability $\frac{1}{3}$. Determine the probability that the ant returns to the original vertex within n moves.

Proposed by Antonio Blanca Pimentel, UC Berkeley and Roberto Bosch Cabrera, Florida, USA

O222. Let $a_1,\ a_2, \ldots, a_n$ and $b_1,\ b_2, \ldots b_n$ be nonnegative reals, where n is a positive integer. Let σ be a permutation of $\{1, 2, \ldots, n\}$. For every $k \in \{1, 2, \ldots, n\}$, let

$$c_k = \max\left(\{a_1b_k, a_2b_k, \ldots, a_kb_k\} \cup \{a_kb_1, a_kb_2, \ldots, a_kb_k\}\right).$$

Prove that

$$a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + \cdots + a_nb_{\sigma(n)} \leq c_1 + c_2 + \cdots + c_n.$$

Proposed by Darij Grinberg, Massachusetts Institute of Technology, USA