## Junior problems

J229. Adrian has a credit card from his dad. He cannot get cash without knowing the personal identification number (PIN). Adrian has sort of an emergency and asks dad to provide him with the PIN (a whole number from 0000 to 9999). Dad tells Adrian that the PIN is the largest prime that divides $3^{22} + 3$ and that he is not allowed to use a calculator. Adrian is able to to get the cash he needs by finding the PIN using his $7^{\text{th}}$ grade math knowledge. What is the PIN?

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

J230. Let $ABC$ be a triangle and let $M$ be the midpoint of the side $BC$. Suppose that there is some $0^\circ < x \le 30^\circ$ so that the measure of $\angle ACB$, $\angle ABC$, $\angle MAC$ are $x$, $60^\circ - x$, $2x$, respectively. Determine $x$.

*Proposed by Marius Stanean and Mircea Lascu, Zalau, Romania*

J231. Gigel and Costel have a collection $\mathcal{J}$ of empty jars of the same shape and a very large number of identical coins at their disposal. They decide to play the following game. Knowing that each jar has capacity of 100 coins, they take turns to pick a number of $k$ coins from the pile, with $1 \le k \le 10$, and then (in the same turn) choose a jar into which to put the selected coins. The winner is the one who fills the last jar. Assuming that Gigel goes first and that both players are smart, who wins the game?

*Proposed by Cosmin Pohoata, Princeton University, USA*

J232. Find with proof all integers that can be written as $a\{a\}\lfloor a\rfloor$ for some real number $a$. Here $\lfloor a\rfloor$ and $\{a\}$ denote the greatest integer less than or equal to $a$ and the fractional part of $a$, respectively.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

J233. Let $A_1A_2A_3A_4A_5$ be a regular pentagon and let $B_1B_2B_3B_4B_5$ be the pentagon formed by its diagonals. Prove that

$$\frac{K_{B_1B_2B_3B_4B_5}}{K_{A_1A_2A_3A_4A_5}} = \frac{7-3\sqrt{5}}{2}$$

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

J234. Let $ABC$ be a triangle with side-lengths $a, b, c$ that satisfy $a^{\frac{3}{2}} + b^{\frac{3}{2}} = c^{\frac{3}{2}}$. Prove that

$$\frac{\pi}{2} < \angle C < \frac{3\pi}{5}.$$

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

## Senior problems

S229. Let $a$, $b$, $c$ be the side-lengths of a triangle and let $R$ be its circumradius. Prove that

$$a^3 + b^3 + c^3 \leq 16R^3.$$

*Proposed by Arkady Alt, San Jose, USA and Ivan Borsenco, MIT, USA*

S230. Let $x$, $y$, $z$ be positive real numbers such that

$$xy + yz + zx \geq \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

Prove that $x + y + z \geq \sqrt{3}$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

S231. Let $ABC$ be a triangle with circumcenter $O$. Let $X$, $Y$, $Z$ be the circumcenters of triangles $BCO$, $CAO$, $ABO$ respectively. Furthermore, let $K$ be the circumcenter of triangle $XYZ$. Prove that $K$ lies on the Euler line of triangle $ABC$.

*Proposed by Andrew Kirk, Mearns Castle High School, UK*

S232. Let $x$, $y$, $z$ be real numbers such that $x + y + z = 0$ and $xy + yz + zx = -3$. Determine the extreme values of $x^4y + y^4z + z^4x$.

*Proposed by Marius Stanean, Zalau, Romania*

S233. In triangle $ABC$ with $\angle C = 60^\circ$, let $AA'$ and $BB'$ be the angle bisectors of $\angle A$ and $\angle B$. Prove that

$$\frac{a+b}{A'B'} \leq \left(1 + \frac{c}{a}\right)\left(1 + \frac{c}{b}\right).$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

S234. Let $ABC$ be a triangle. Denote by $D$, $E$, $F$ the feet of the internal angle bisectors such that $D \in (BC)$, $E \in (CA)$, $F \in (AB)$ and by $(I_a, r_a), (I_b, r_b), (I_c, r_c)$ its three excircles. If $\tau$ denotes the Feuerbach point of triangle $ABC$, prove that there is a choice of signs $+$ and $-$ such that the following equality holds

$$\pm D\tau \cdot \frac{I_aI}{I_aD} \cdot \sqrt{R + 2r_a} \pm E\tau \cdot \frac{I_bI}{I_bE} \cdot \sqrt{R + 2r_b} \pm F\tau \cdot \frac{I_cI}{I_cF} \cdot \sqrt{R + 2r_c} = 0.$$

*Proposed by Cosmin Pohoata, Princeton University, USA*

## Undergraduate problems

U229. Does the sequence $(x_n)_{n\geq 1}$ defined by $x_n = \{\log_n n!\}$ converge? Here $\{x\}$ denotes the fractional part of the real number $x$.

*Proposed by Cezar Lupu, University of Pittsburgh, USA*

U230. Let $(a_n)_n \geq 0$ be a sequence of positive real numbers such that $a_n^2 \geq a_{n-1}a_{n+1}$, for all $n \geq 1$. Prove that $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

U231. Define a sequence of maps on $[0,1]$ by $f_0(x) = 0$ and

$$f_{n+1}(x) = f_n(x) + \frac{x - f_n(x)^2}{2}.$$

It is well-known that $f_n$ converges uniformly to the function $x \to \sqrt{x}$ on $[0,1]$. Prove that there exists $c \in (0,\infty)$ such that

$$\lim_{n\to\infty} n \cdot \max_{x\in[0,1]} |f_n(x) - \sqrt{x}| = c.$$

*Proposed by Gabriel Dospinescu, Ecole Polytechnique, France*

U232. Let $\alpha > 0$ be any non-algebraic number. Prove that there is a funtion $f$ with period 1 and a countable set $A$ such that

$$x = f(\alpha x) - f(x) \qquad \text{for all } x \in \mathbb{R} \setminus A.$$

*Proposed by Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland*

U233. Let $X$ be a random variable with a mean $\mu$, variance $\sigma^2$, and median $m$. Denote by $MAD =$ median $|X - m|$ the median of the absolute deviations from the median of $X$. Prove that $MAD \leq 2\sigma$.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

U234. Consider the set of matrices $A \in M_n(\mathbf{R})$ whose coefficients are $-1$ or 1. What is the average value of $\det A^2$ when $A$ runs through the elements of this set?

*Proposed by Gabriel Dospinescu, Ecole Polytechnique, France*

## Olympiad problems

O229. Are there rational numbers $a$, $b$, $c$ such that $a^2b + b^2c + c^2a = ab^2 + bc^2 + ca^2 = 20.11$?

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

O230. Let $ABC$ be a triangle with incenter $I$ and let $X$, $Y$, $Z$ be points lying on the internal angle bisectors $AI$, $BI$, $CI$. Furthermore, let $M$, $N$, $P$ be the midpoints of the sides $BC$, $CA$, $AB$, and let $D$, $E$, $F$ be the tangency points of the incircle of $ABC$ with these sides. Prove that if $MX$, $NY$, $PZ$ are parallel, then $DX$, $EY$, $FZ$ are concurrent on the incircle of $ABC$.

This generalizes Problem S228 from the previous issue.

*Proposed by Luis Gonzalez, Maracaibo, Venezuela and Cosmin Pohoata, Princeton University, USA*

O231. Let $a$, $b$, $c$, $d$ be real numbers such that $a + b + c + d = 2$. Prove that

$$\frac{a}{a^2 - a + 1} + \frac{b}{b^2 - b + 1} + \frac{c}{c^2 - c + 1} + \frac{d}{d^2 - d + 1} \leq \frac{8}{3}.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

O232. Let $ABCDE$ be a convex pentagon with area $S$. The area of the pentagon formed by the intersections of its diagonals is equal to $S'$. Consider the statement $S' < cS$, where $c$ is constant. Prove that the statement holds when $c = \frac{1}{2}$.

Try to find the best constant you can achieve. For example, it can be proved that the statement still holds for $c = 2 - \sqrt{3}$.

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

O233. Let $ABC$ be a triangle with $\angle BAC = 90°$. Let $D$ be a point on hypotenuse $BC$ and let the parallel lines through $D$ to the legs $AC$ and $AB$ meet (for the second time) the circles with diameters $DC$ and $DB$ at $U$ and $V$, respectively. Furthermore, if $P$ is the second intersection of $AD$ with the circumcircle of triangle $ABC$, prove that $\angle UPV = 90°$.

*Proposed by Luis Gonzalez, Maracaibo, Venezuela*

O234. Let $f$ be a polynomial of degree 4 with integral coefficients. Prove that there are infinitely many positive integers $n$ such that $2^n - 1$ does not divide $f(n!)$.

Does the result still hold if we remove the condition $\deg(f) = 4$?

*Proposed by Gabriel Dospinescu, Ecole Polytechnique, France*