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On an Affine Variant of a Steinhaus Problem

Jean-Pierre Ehrmann

Abstract. Given a triangle ABC and three positive real numbers u, v, w, we prove that there exists a unique point P in the interior of the triangle, with cevian triangle $P_aP_bP_c$, such that the areas of the three quadrilaterals PP_bAP_c, PP_cBP_a , PP_aCP_b are in the ratio $u : v : w$. We locate P as an intersection of three hyperbolas.

In this note we study a variation of the theme of [2], a generalization of a problem initiated by H. Steinhaus on partition of a triangle (see [1]). Given a triangle ABC with interior $\mathscr{T}$, and a point $P \in \mathcal{T}$ with cevian triangle $P_aP_bP_c$, we denote by $\Delta_A(P)$, $\Delta_B(P)$, $\Delta_C(P)$ the areas of the oriented quadrilaterals PP_bAP_c, PP_cBP_a, PP_aCP_b. In this note we prove that given three arbitrary positive real numbers u, v, w, there exists a unique point $P \in \mathscr{T}$ such that

$$\Delta_A(P) : \Delta_B(P) : \Delta_C(P) = u : v : w.$$

To this end, we define

$$f(P) = \Delta_A(P) : \Delta_B(P) : \Delta_C(P).$$

This is the point of $\mathscr{T}$ such that

$$\Delta[BCf(P)] = \Delta_A(P), \quad \Delta[CAf(P)] = \Delta_B(P), \quad \Delta[ABf(P)] = \Delta_C(P).$$

Lemma 1. *If P has homogeneous barycentric coordinates $x : y : z$ with reference to triangle ABC, then*

$$f(P) = \frac{(y+z)(2x+y+z)}{x} : \frac{(z+x)(2y+z+x)}{y} : \frac{(x+y)(x+y+2z)}{z}.$$

Proof. If $P = x : y : z$, we have

$$\overrightarrow{AP_c} = \frac{y\overrightarrow{AB}}{x+y}, \quad \overrightarrow{AP} = \frac{y\,\overrightarrow{AB} + z\,\overrightarrow{AC}}{x+y+z}, \quad \overrightarrow{AP_b} = \frac{z\overrightarrow{AC}}{x+z},$$

so that

$$\Delta_a(P) = \Delta(AP_cP) + \Delta(APP_b) = \frac{y\,z}{x+y+z}(\frac{1}{x+y} + \frac{1}{x+z})\,\Delta(ABC).$$

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By cyclic permutations of x, y, z, we get the values of $\Delta_B(P)$ and $\Delta_C(P)$, and the result follows. $\square$

We shall prove that $f:\ \mathscr{T}\to\mathscr{T}$ is a bijection. We adopt the following notations.

(i) G_a, G_b, G_c are the vertices of the anticomplementary triangle. They are the images A, B, C under the homothety $\mathsf{h}(G,-2)$, G being the centroid of ABC.

(ii) P^* denotes the isotomic conjugate of P with respect to ABC. Its traces P_a^*, P_b^*, P_c^* on the sidelines of ABC are the reflections of P_a, P_b, P_c with respect to the midpoint of the corresponding side.

(iii) $[L]_\infty$ denotes the infinite point of a line L.

Proposition 2. *Let $P = x : y : z$ and $U = u : v : w$. The lines G_aP and P_a^*U are parallel if and only if P lies on the hyperbola $\mathscr{H}_{a,U}$ through A, G_a, U_a^*, the reflection of U_b^* in C and the reflection of U_c^* in B.*

Proof. As $P_a^* = 0 : z : y$ and $[G_aP]_\infty = -(2x+y+z) : z+x : x+y$, the lines G_aP and P_a^*U are parallel if and only if

$$\begin{aligned}
h_{a,U}(P) &:= \det([G_aP]_\infty,\ P_a^*,\ U)\\
&= \begin{vmatrix} -(2x+y+z) & z+x & x+y\\ 0 & z & y\\ u & v & w \end{vmatrix}\\
&= x((u+v)y-(w+u)z)+(x+y+z)(vy-wz)\\
&= 0.
\end{aligned}$$

It is clear that $h_{a,U}(P)=0$ defines a conic $\mathscr{H}_{a,U}$ through $A = 1:0:0$, and the infinite points of the lines $x=0$ and $(u+v)y-(w+u)z=0$. These are the lines BC and G_aU. It is also easy to check that it contains the points $G_a=-1:1:1$, $U_a^* = 0 : w : v$, and

$$\begin{aligned}
U_{bc}^* &:= -w : 0 : u+2w,\\
U_{cb}^* &:= -v : u+2v : 0.
\end{aligned}$$

These latter two are respectively the reflections of U_b^* in C and U_c^* in B. The conic $\mathscr{H}_{a,U}$ is a hyperbola since the four points A, G_a, U_{bc}^* and U_{cb}^* do not fall on two lines. $\square$

By cyclic permutations of coordinates, we obtain two hyperbolae $\mathscr{H}_{b,U}$ and $\mathscr{H}_{c,U}$ defined by

$$\begin{aligned}
h_{b,U}(P) &:= \det([G_bP]_\infty,\ P_b^*,\ U) = 0,\\
h_{c,U}(P) &:= \det([G_cP]_\infty,\ P_c^*,\ U) = 0.
\end{aligned}$$

It is easy to check that if $U = f(P)$, then

$$h_{a,U}(P) = h_{b,U}(P) = h_{c,U}(P) = 0.$$

From this we obtain a very easy construction of the point $f(P)$.

Corollary 3. *The point $f(P)$ is the intersection of the lines through P_a^*, P_b^* and P_c^* parallel to G_aP, G_bP, G_cP respectively.* See Figure 1.

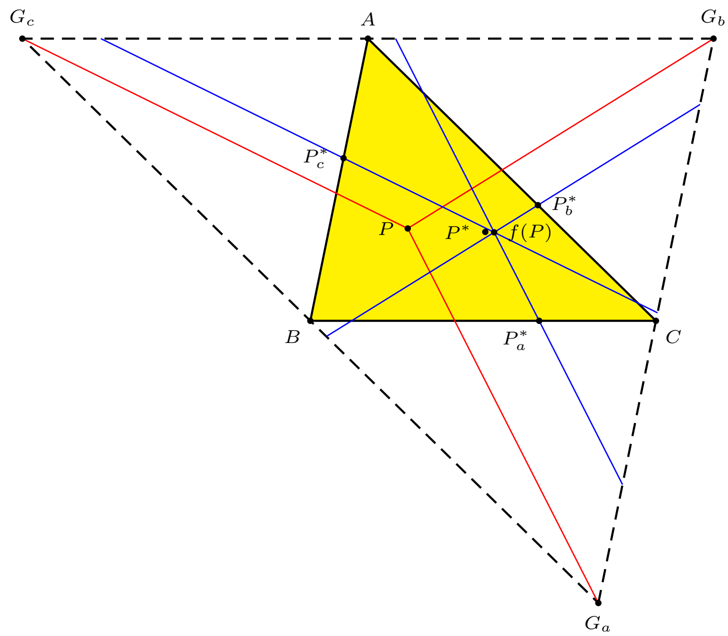

Figure 1.

Proof. The lines G_aP, G_bP, G_cP are parallel to $P_a^*f(P)$, $P_b^*f(P)$, $P_c^*f(P)$ respectively. □

Remarks. (1) $\mathscr{H}_{a,U}$ degenerates if and only if $v = w$, *i.e.*, when U lies on the median AG. In this case, $\mathscr{H}_{a,U}$ is the union of the median AG and of a line parallel to BC.

(2) P, P^*, $f(P)$ are collinear.

(3) As $h_{a,U}(P) + h_{b,U}(P) + h_{c,U}(P) = 0$, the three hyperbolae $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$, $\mathscr{H}_{c,U}$ are members of a pencil of conics. If $U \in \mathscr{T}$, the points P for which $f(P) = U$ are their common points lying in $\mathscr{T}$.

Lemma 4. *If $U \in \mathscr{T}$, $\mathscr{H}_{a,U}$ and $\mathscr{H}_{b,U}$ have a real common point in $\mathscr{T}$ and a real common point in $\mathscr{T}_A$, reflection in A of the open angular sector bounded by the half lines AB and AC.*

Proof. Using the fact that $\mathscr{H}_{a,U}$ passes through $[BC]_\infty$, we can cut $\mathscr{H}_{a,U}$ by lines parallel to BC to get a rational parametrization of $\mathscr{H}_{a,U}$. More precisely, let B_t and C_t be the images of B and C under the homothety $\mathsf{h}(A, 1-t)$. The point

$$(1-\mu)B_t + \mu C_t = t : (1-\mu)(1-t) : \mu(1-t)$$

lies on $\mathscr{H}_{a,U}$ if and only if

$$\mu = \mu_t = \frac{v + t(u+v)}{v + w + t(2u+v+w)}.$$

Let $P(t) = (1-\mu_t)B_t + \mu_t C_t$. It has homogeneous barycentric coordinates

$$t((v+w)+t(2u+v+w)) : (1-t)(w+t(w+u)) : (1-t)(v+t(u+v)).$$

with coordinate sum is $(v+w)+t(2u+v+w)$.

If $t \geq 0$, we have $0 < \mu_t < 1$. It follows that, for $0 < t < 1$, $P(t) \in \mathscr{T}$ and for $t > 1$, $P(t) \in \mathscr{T}_A$. Consider

$$\varphi(t) := \frac{h_{b,U}(P(t))}{(u+v+w)((v+w)+t(2u+v+w))^2}.$$

More explicitly,

$$\varphi(t) = \frac{2(u+v)(u+w)(u+v+w)t^4 + \text{lower degree terms of } t}{(u+v+w)(v+w+t(2u+v+w))^2}.$$

Clearly, $\varphi(0) = \frac{2vw}{(v+w)(u+v+w)} > 0$ and $\varphi(1) = -\frac{u}{u+v+w} < 0$. Note also that $\varphi(+\infty) = +\infty$. As φ is continuous for $t \geq 0$, the result follows. $\square$

Theorem 5. *If $U \in \mathscr{T}$, the three hyperbolas $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$, $\mathscr{H}_{c,U}$ have four distinct real common points, exactly one of which lies in $\mathscr{T}$. This point is the only point $P \in \mathscr{T}$ satisfying $f(P) = U$.*

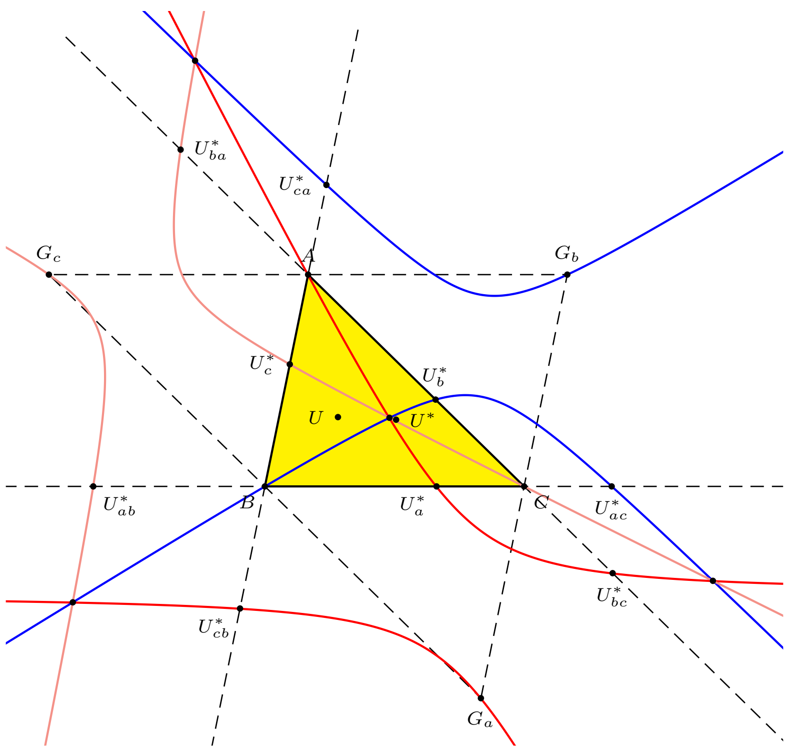

Figure 2.

Proof. In a similar way as in Lemma 4, we can see that $\mathscr{H}_{b,U}$ and $\mathscr{H}_{c,U}$ have a common point in $\mathscr{T}$ and a real common point in $\mathscr{T}_B$ and that $\mathscr{H}_{c,U}$ and $\mathscr{H}_{a,U}$ have a real common point in $\mathscr{T}$ and a real common point in $\mathscr{T}_B$. As the four sets $\mathscr{T}$, T_A, T_B, T_C pairwise have empty intersection, it follows that $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$, $\mathscr{H}_{c,U}$ have four real common points, one in each of $\mathscr{T}$, $\mathscr{T}_A$, $\mathscr{T}_B$ and $\mathscr{T}_C$. See Figure 2. $\square$

Remark. (4) If $U \in \mathscr{T}$, the points P such that

$$\Delta(AP_cP) + \Delta(APP_b) : \Delta(BP_aP) + \Delta(BPP_c) : \Delta(CP_bP) + \Delta(CPP_a) = u : v : w$$

are the four common points of $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$ and $\mathscr{H}_{c,U}$.

Remark (2) shows that $f^{-1}(U)$ lies on the isotomic cubic with pivot U. Clearly, $f(G) = f^{-1}(G) = G$.

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Two Triads of Congruent Circles from Reflections

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Abstract. Given a triangle, we construct two triads of congruent circles through the vertices, one associated with reflections in the altitudes, and the other reflections in the angle bisectors.

1. Reflections in the altitudes

Given triangle ABC with orthocenter H, let B_a and C_a be the reflections of B and C in the line AH. These are points on the sideline BC so that $\mathbf{BC_a} = \mathbf{CB_a}$. Similarly, consider the reflections C_b, A_b of C, A respectively in the line BH, and A_c, B_c of A, B in the line CH.

Theorem 1. *The circles AC_bB_c, BA_cC_a, and CB_aA_b are congruent.*

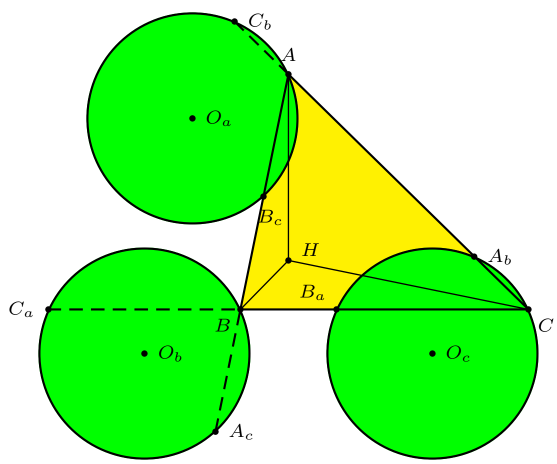

Figure 1.

Proof. Let O be the circumcenter of triangle ABC, and X its reflection in the A-altitude. This is the circumcenter of triangle AB_aC_a, the reflection of triangle ABC in its A-altitude. See Figure 2. It follows that H lies on the perpendicular bisector of OX, and $HX = OH$. Similarly, if Y and Z are the reflections of O in the lines BH and CH respectively, then $HY = HZ = OH$. It follows that O, X, Y, Z are concyclic, and H is the center of the circle containing them. See Figure 3.

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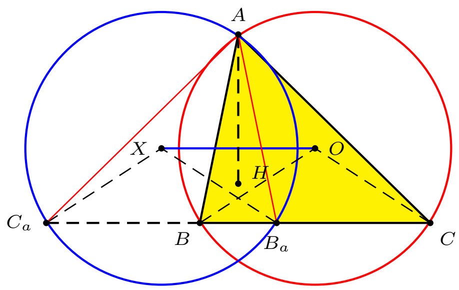

Figure 2

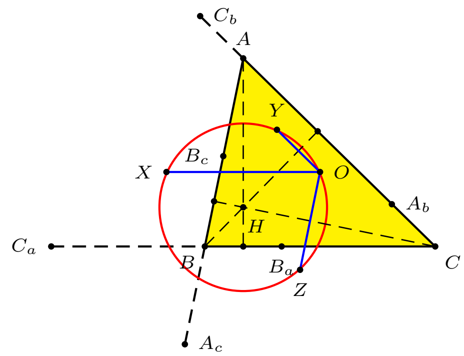

Figure 3

Let O be the circumcenter of triangle ABC. Note the equalities of vectors

$$\begin{aligned} \mathbf{OX} &= \mathbf{BC_a} = \mathbf{CB_a}, \\ \mathbf{OY} &= \mathbf{CA_b} = \mathbf{AC_b}, \\ \mathbf{OZ} &= \mathbf{AB_c} = \mathbf{BA_c}. \end{aligned}$$

The three triangles AC_bB_c, BA_cC_a, and CB_aA_b are the translations of OYZ by **OA**, OZX by **OB**, and OXY by **OC** respectively.

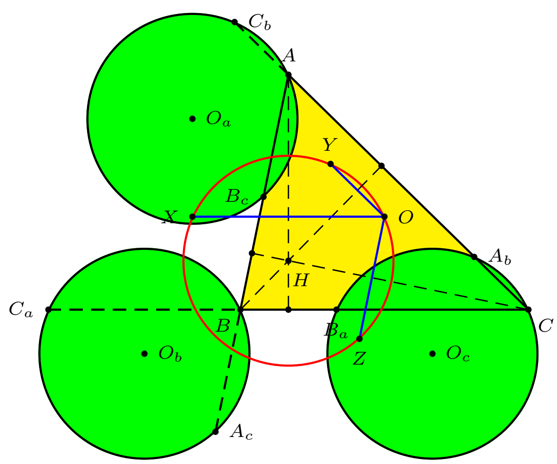

Figure 4.

Therefore, the circumcircles of the three triangles are all congruent and have radius OH. Their centers are the translations of H by the three vectors. $\square$

2. Reflections in the angle bisectors

Let I be the incenter of triangle ABC. Consider the reflections of the vertices in the angle bisectors: B'_a, C'_a of B, C in AI, C'_b, A'_b of C, A in BI, and A'_c, B'_c of A, B in CI. See Figure 5.

Theorem 2. *The circles $AC'_bB'_c$, $BA'_cC'_a$, and $CB'_aA'_b$ are congruent.*

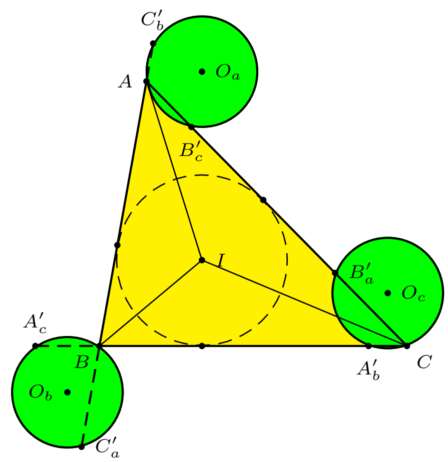

Figure 5.

Proof. Consider the reflections B''_c, C''_b of B'_c, C'_b in AI, C''_a, A''_c of C'_a, A'_c in BI, and A''_b, B''_a of A'_b, B'_a in CI. See Figure 6.

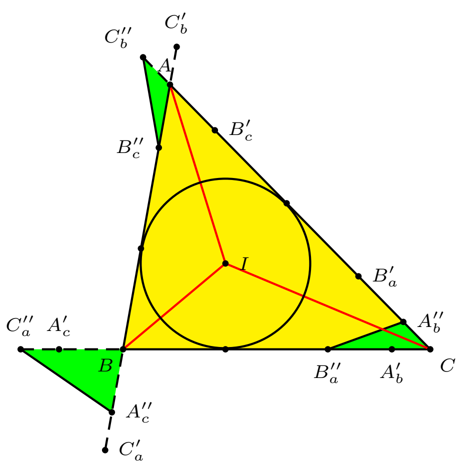

Figure 6

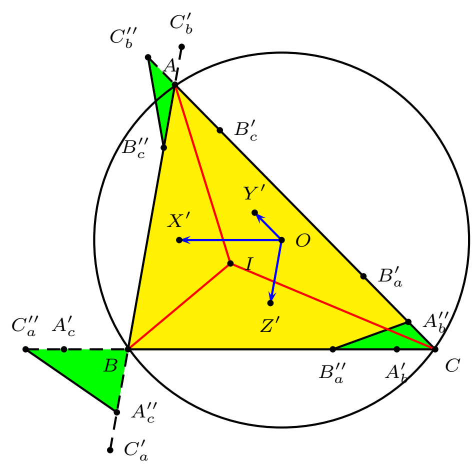

Figure 7

Note the equalities of vectors

$$\mathbf{BC''_a} = \mathbf{CB''_a}, \qquad \mathbf{CA''_b} = \mathbf{AC''_b}, \qquad \mathbf{AB''_c} = \mathbf{BA''_c}.$$

With the circumcenter O of triangle ABC, these define points X', Y', Z' such that

$$\begin{aligned} \mathbf{OX'} =& \mathbf{BC''_a} = \mathbf{CB''_a}, \\ \mathbf{OY'} =& \mathbf{CA''_b} = \mathbf{AC''_b}, \\ \mathbf{OZ'} =& \mathbf{AB''_c} = \mathbf{BA''_c}. \end{aligned}$$

The triangles $AC''_bB''_c$, $BA''_cC''_a$ and $CB''_aA''_b$ are the translations of $OY'Z'$, $OZ'X'$ and $OX'Y'$ by the vectors $\mathbf{OA}$, $\mathbf{OB}$ and $\mathbf{OC}$ respectively. See Figure 7.

Note, in Figure 8, that $OX'C''_aC$ is a symmetric trapezoid and $IC''_a = IC'_a = IC$. It follows that triangles IC''_aX' and ICO are congruent, and $IX' = IO$. Similarly, $IY' = IO$ and $IZ' = IO$. This means that the four points O, X', Y', Z' are on a circle center I. See Figure 9. The circumcenters O''_a, O''_b, O''_c of the triangles $AC''_bB''_c$, $BA''_cC''_a$ and $CB''_aA''_b$ are the translations of I by these vectors. These circumcircles are congruent to the circle $I(O)$.

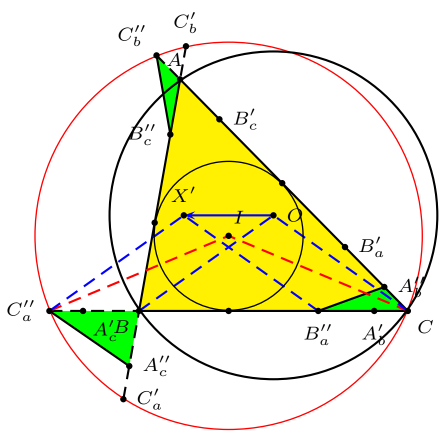

Figure 8

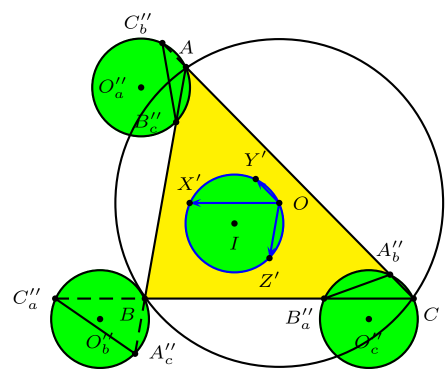

Figure 9

The segments AO''_a, BO''_b and CO''_c are parallel and equal in lengths. The triangles $AC'_bB'_c$, $BA'_cC'_a$ and $CB'_aA'_b$ are the reflections of $AC''_bB''_c$, $BA''_cC''_a$ and $CB''_aA''_b$ in the respective angle bisectors. See Figure 10. It follows that their circumcircles are all congruent to $I(O)$. □

Let O'_a, O'_b, O'_c be the circumcenters of triangles $AC'_bB'_c$, $BA'_cC'_a$ and $CB'_aA'_b$ respectively. The lines AO'_a and AO''_a are symmetric with respect to the bisector of angle A. Since AO''_a, BO''_b and CO''_c are parallel to the line OI, the reflections in the angle bisectors concur at the isogonal conjugate of the infinite point of OI. This is a point P on the circumcircle. It is the triangle center X_{104} in [1].

Finally, since $IO''_a = IO''_b = IO''_c$, we also have $IO'_a = IO'_b = IO'_c$. The 6 circumcenters all lie on the circle, center I, radius R.

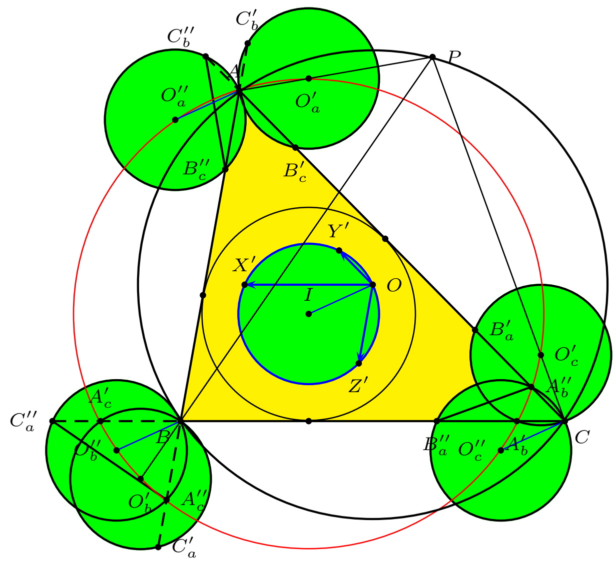

Figure 10.

To conclude this note, we establish an interesting property of the centers of the circles in Theorem 2.

Proposition 3. *The lines $O_a'I$, $O_b'I$ and $O_c'I$ are perpendicular to BC, CA and AB respectively.*

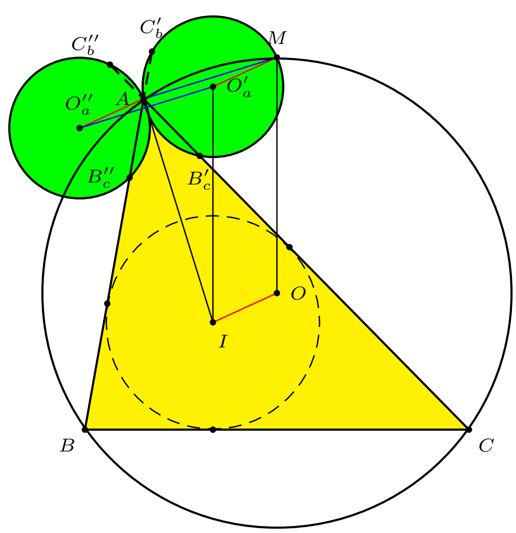

Figure 11.

Proof. It is enough to prove that for the line O'_aI. The other two cases are similar.

Let M be the intersection (other than A) of the circle (O'_a) with the circumcircle of triangle ABC. Since $IO'_a = OM$ (circumradius) and $O'_aM = IO$, O'_aMOI is a parallelogram. This means that $\mathbf{O'_aM} = \mathbf{IO} = \mathbf{O''_aA}$, and $AMO'_aO''_a$ is also a parallelogram. From this we conclude that AM, being parallel to $O''_aO'_a$, is perpendicular to the bisector AI. Thus, M is the midpoint of the arc BAC, and MO is perpendicular to BC. Since $\mathbf{O'_aI} = \mathbf{MO}$, the line O'_aI is also perpendicular to BC. □

Since the six circles (O'_a) and (O''_a) etc are congruent (with common radius OI) and their centers are all at a distance R from I, it is clear that there are two circles, center I, tangent to all these circles. These two circles are tangent to the circumcircle, the point of tangency being the intersection of the circumcircle with the line OI. These are the triangle centers X_{1381} and X_{1382} of [1].

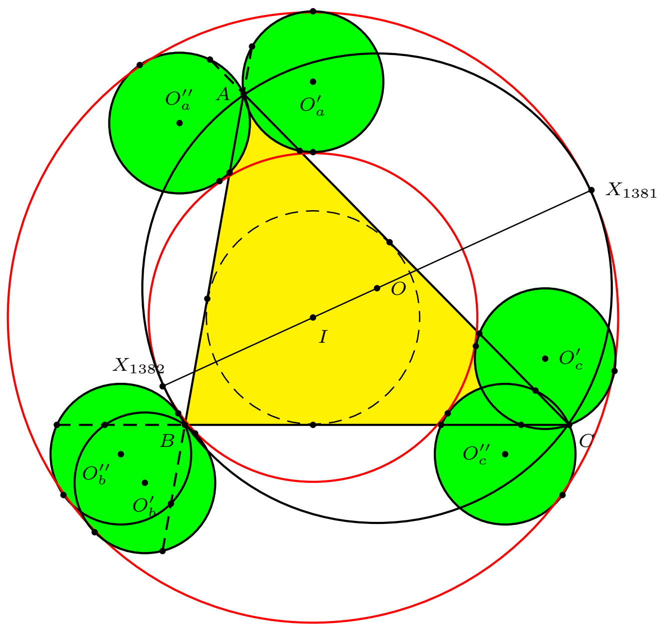

Figure 12.

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Angles, Area, and Perimeter Caught in a Cubic

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Abstract. The main goal of this paper is to establish sharp bounds for the angles and for the side ratios of any triangle of known area and perimeter. Our work is also related to the well known isoperimetric inequality.

1. Isosceles triangles sharing area and perimeter

Suppose we wish to determine all isosceles triangles, if any, of area 3 and perimeter 10 – a problem that is a bit harder than the corresponding well known problem for rectangles!

Let x be the length of the base and y the length of the two equal sides, $x < 2y$. Then the height of the isosceles triangles we wish to determine is equal to $\sqrt{y^2 - \frac{x^2}{4}}$. Thus $x+2y = 10$ while $\frac{x}{2}\sqrt{y^2 - \frac{x^2}{4}} = 3$. Hence $\frac{x}{2}\sqrt{\left(5 - \frac{x}{2}\right)^2 - \frac{x^2}{4}} = 3$, which leads to $5x^3 - 25x^2 + 36 = 0$. The positive roots of this cubic are $x_1 \approx 1.4177$ and $x_2 \approx 4.6698$, so that $y_1 \approx 4.2911$ and $y_2 \approx 2.6651$. Thus there are just two isosceles triangles of area 3 and perimeter 10 (see Figure 1).

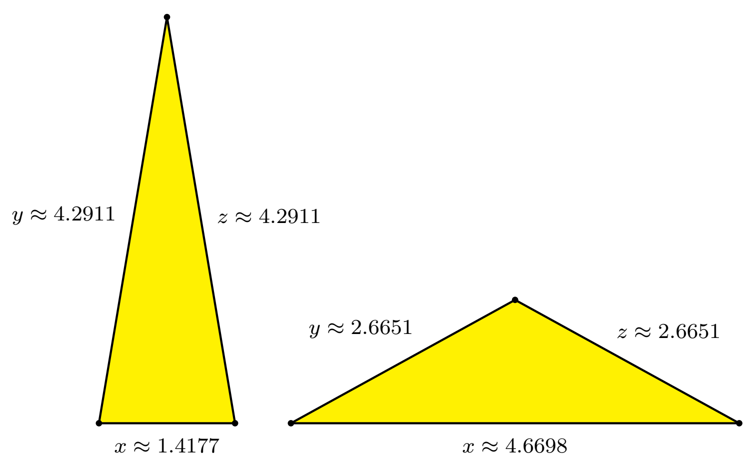

Figure 1. The two isosceles triangles of area 3 and perimeter 10

Are there always isosceles triangles of area A and perimeter P? A complete answer is provided by the following lemma and theorem.

Lemma 1. *Let x be the base of an isosceles triangle with given area A and perimeter P. Then*

$$2Px^3 - P^2x^2 + 16A^2 = 0. \tag{1}$$

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Proof. Working as in the above special case, we obtain $y = \frac{P-x}{2}$ and $\frac{x}{2}\sqrt{y^2 - \frac{x^2}{4}} = A$; substituting the former condition into the latter, we arrive at (1). □

Theorem 2. *There are exactly two distinct isosceles triangles of area A and perimeter P if and only if $P^2 > 12\sqrt{3}A$. There is exactly one if and only if $P^2 = 12\sqrt{3}A$ and the triangle is equilateral. The vertex angles $\phi_1 < \phi_2$ of these two isosceles triangles also satisfy $\phi_1 < \frac{\pi}{3} < \phi_2$.*

Proof. Let $f(x)$ be the cubic in (1). We first show that it has at most two distinct positive roots. Indeed the existence of three distinct positive roots would yield, by Rolle's theorem, two distinct positive roots for $f'(x) = 6Px^2 - 2P^2x$; but the roots of $f'(x)$ are $x = \frac{P}{3}$ and $x = 0$.

Notice now that $f''(x) = 12Px - 2P^2$, hence $f''(0) = -2P^2 < 0$ and $f''(\frac{P}{3}) = 2P^2 > 0$. So f has a positive local maximum of $16A^2$ at $x = 0$ and a local minimum at $x = \frac{P}{3}$ (Figure 2). It is clear that f has two distinct positive roots $x_1 < \frac{P}{3} < x_2$ if and only if $f(\frac{P}{3}) < 0$; but $f(\frac{P}{3}) = -\frac{P^4}{27} + 16A^2$, so $f(\frac{P}{3}) < 0$ is equivalent to $P^2 > 12\sqrt{3}A$.

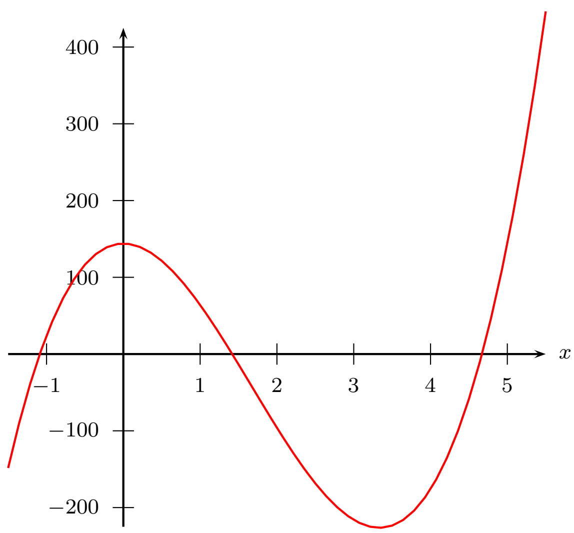

Figure 2. $2Px^3 - P^2x^2 + 16A^2$ for $A = 3$ and $P = 10$

Moreover, $f(\frac{P}{3}) = 0$ if and only if $P^2 = 12\sqrt{3}A$, implying that $f(x) = 0$ has precisely one ('tangential') positive solution if and only if $P^2 = 12\sqrt{3}A$. As it turns out, the cubic is then equivalent to $(3x - P)^2(6x + P) = 0$, and its unique positive solution corresponds to the equilateral triangle of side $\frac{P}{3}$.

As also noticed in [1], the vertex angles ϕ_1 and ϕ_2 of the two isosceles triangles of area A and perimeter P (that correspond to the positive roots x_1 and x_2 of (1)) do satisfy the inequalities $\phi_1 < \frac{\pi}{3} < \phi_2$. These inequalities follow from $x_1 < \frac{P}{3} < x_2$ since, in every triangle, the greater angle is opposite the greater side: indeed in every isosceles triangle of perimeter P, base x, vertex angle ϕ, and sides

$y = z$, the inequality $x < \frac{P}{3}$ implies $y = z > \frac{P}{3}$, so that $y = z > x$; therefore $\frac{\pi-\phi}{2} > \phi$, thus $\phi < \frac{\pi}{3}$. In a similar fashion one can prove that $x > \frac{P}{3}$ implies $\phi > \frac{\pi}{3}$. □

Remark. That the cubic in (1) can have at most two distinct positive roots may also be derived algebraically. Indeed, the existence of three distinct positive roots x_1, x_2, x_3 would imply that the cubic may be written as $c(x-x_1)(x-x_2)(x-x_3)$, with $c(x_1x_2 + x_2x_3 + x_3x_1)$ being the *positive* coefficient of the first power of x. That would contradict the fact that the cubic being analyzed has zero as the coefficient of the first power of x.

2. The isoperimetric inequality for arbitrary triangles

We have just seen that the inequality $P^2 \geq 12\sqrt{3}A$ holds for every isosceles triangle, with equality precisely when the triangle is equilateral. We will prove next that this *isoperimetric* inequality ([5, p.85], [3, p.42]) holds for every triangle.

First we notice that for every scalene triangle BCD, there exists an isosceles triangle ECD with BE parallel to CD (see Figure 3). Let ℓ be the line through B parallel to CD and F be the symmetric reflection of D with respect to ℓ. Let E and G be the points of ℓ on CF and DF, respectively. Clearly, $EG \| CD$ and $|FG| = |DG|$ imply $|FE| = |CE|$. Moreover, triangles FGE and DGE are congruent by symmetry, therefore $|FE| = |DE|$. We conclude that triangle ECD is isosceles with $|CE| = |DE|$.

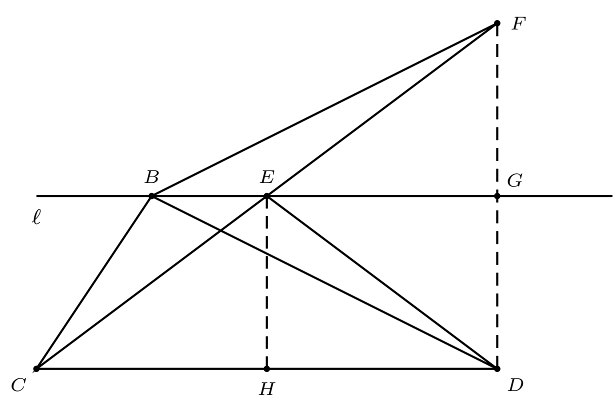

Figure 3. Reduction to the case of an isosceles triangle

It follows immediately from $BE \| CD$ that ΔECD and ΔBCD have equal areas. Less obviously, the perimeter of ΔECD is *smaller* than that of ΔBCD : $|CD| + |DE| + |EC| = |CD| + |FE| + |EC| = |CD| + |FC| < |CD| + |FB| + |BC| = |CD| + |DB| + |BC|$, with the last equality following from symmetry and the congruency of ΔFGB and ΔDGB.

So, given an arbitrary scalene triangle BCD of area A and perimeter P, there exists an isosceles triangle ECD of area A and perimeter $Q < P$. Since $Q^2 \geq$

$12\sqrt{3}A$, it follows that $P^2 > 12\sqrt{3}A$, so the isoperimetric inequality for triangles has been proven.

We invite the reader to use this geometrical technique to derive the isoperimetric inequality for quadrilaterals ($P^2 \geq 16A$ for every quadrilateral of area A and perimeter P), and possibly for other n-gons as well.

It should be mentioned here that the standard proof of the isoperimetric inequality for triangles (see for example [2, p.88]) relies on Heron's area formula (which we essentially derive later through a generalization of (1) for arbitrary triangles) and the arithmetic-geometric-mean inequalilty.

3. Newton's parametrization

Turning now to our main goal, namely the relations among a triangle's area, perimeter, and angles, we first find an expression for the sides of a triangle in terms of its area, perimeter, and *one* angle. To achieve this, we simply generalize Newton's derivation of the formula $x = \frac{P}{2} - \frac{2A}{P}$, expressing a right triangle's hypotenuse in terms of its area and perimeter; this work appeared in Newton's *Universal Arithmetick*, *Resolution of Geometrical Questions*, Problem III, p. 57 ([6, p.103]).

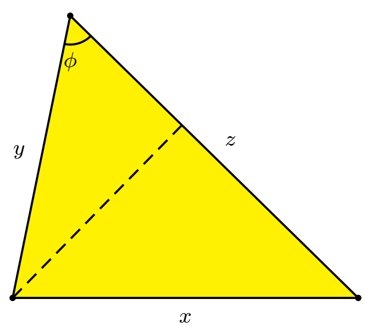

Figure 4. Toward 'Newton's parametrization'

Observe (as in Figure 4) that $A = \frac{1}{2}zy\sin\phi$, so $y^2 = Py - xy - \frac{2A}{\sin\phi}$; moreover, the law of cosines yields $y^2 = Px + Py - xy + \frac{2A\cos\phi}{\sin\phi} - \frac{P^2}{2}$. It follows that

$$x = x(\phi) = \frac{P}{2} - \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right), \tag{2}$$

extending Newton's formula for $0 < \phi < \pi$. Of course we need to have $\frac{P^2}{A} > 4\left(\frac{1+\cos\phi}{\sin\phi}\right)$ for x to be positive, so we need the condition $s(\phi) > 0$, where

$$s(\phi) = \frac{P^2\sin\phi}{4(1+\cos\phi)} - A. \tag{3}$$

Once x is determined, y and z are easily determined via $yz = \frac{2A}{\sin\phi}$ and $y + z = \frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)$: they are the roots of the quadratic $t^2 - \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)t + \frac{2A}{\sin\phi} = 0$, provided that $h(\phi) \geq 0$, where

$$h(\phi) = \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi} \tag{4}$$

is the discriminant; that is, $y = y(\phi)$ and $z = z(\phi)$ are given by

$$z,\ y = \left(\frac{P}{4} + \frac{A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right) \pm \frac{1}{2}\sqrt{\left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi}}. \tag{5}$$

Putting everything together, and observing that x, y, z as defined in (2) and (5) above do satisfy the triangle inequality and are the sides of a triangle of area A and perimeter P, we arrive at the following result.

Theorem 3. *The pair of conditions $s(\phi) > 0$ and $h(\phi) \geq 0$, where $s(\phi) = \frac{P^2\sin\phi}{4(1+\cos\phi)} - A$ and $h(\phi) = \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi}$, is equivalent to the existence of a triangle of area A, perimeter P, sides $x(\phi), y(\phi), z(\phi)$ as given in* (2), (5) *above, and angle ϕ between the sides y, z; that triangle is isosceles with vertex angle ϕ if and only if $h(\phi) = 0$.*

Figures 5 and 6 below offer visualizations of the three sides' parametrizations by the angle ϕ and of the two functions essential for the 'triangle conditions' of Theorem 3, respectively.

The 'vertical' intersections of $y(\phi)$ and $z(\phi)$ with each other in Figure 5 occur at $\phi \approx 0.33166 \approx 19.003°$ and $\phi \approx 2.13543 \approx 122.351°$: those are the positive roots of $h(\phi) = 0$, which are none other than the vertex angles of the two isosceles triangles in Figure 1. There are also intersections of $x(\phi)$ with $z(\phi)$ at $\phi \approx 1.40485 \approx 80.492°$ and of $x(\phi)$ with $y(\phi)$ at $\phi \approx 0.50305 \approx 28.822°$; which are again associated, via side renaming as needed and with ϕ being a *base* angle, with the isosceles triangles of Figure 1.

As we see in Figure 6, s and h cannot be simultaneously positive outside the interval defined by the two largest roots of h ($\phi \approx 0.33166$ and $\phi \approx 2.13543$): this fact remains true for arbitrary A and P and is going to be of central importance in what follows.

4. Angles 'bounded' by area and perimeter

We are ready to state and prove our first main result.

Theorem 4. *In every non-equilateral triangle of area A and perimeter P every angle ϕ must satisfy the inequality $\phi_1 \leq \phi \leq \phi_2$, where $\phi_1 < \frac{\pi}{3} < \phi_2$ are the vertex angles of the two isosceles triangles of area A and perimeter P; specifically,*

$$\arccos\left(\frac{P^2 - 2Px_1 - x_1^2}{P^2 - 2Px_1 + x_1^2}\right) \leq \phi \leq \arccos\left(\frac{P^2 - 2Px_2 - x_2^2}{P^2 - 2Px_2 + x_2^2}\right),$$

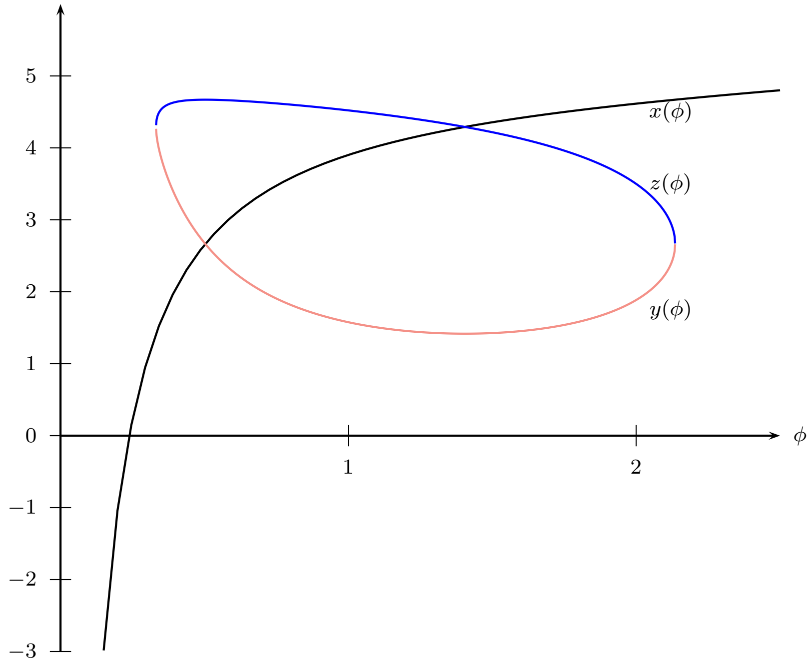

Figure 5. The triangle's three sides parametrized by ϕ for $19.003^\circ = 0.33166 \le \phi \le 2.13543 = 122.351^\circ$ at $A = 3$, $P = 10$

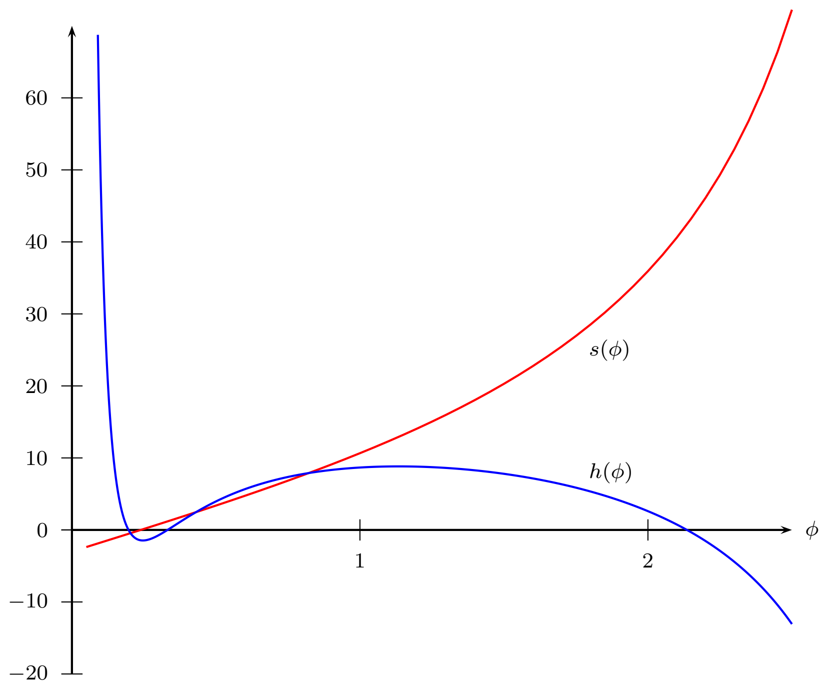

Figure 6. $s(\phi)$ and $h(\phi)$ for $0.1 \le \phi \le 2.3$ at $A = 3$, $P = 10$

where $x_1 < \frac{P}{3} < x_2$ *are the positive roots of* $2Px^3 - P^2x^2 + 16A^2 = 0$.

Proof. As we have seen in Lemma 1, the cubic (1) yields the base x of each of the two isosceles triangles of area A and perimeter P; and the formula above for the vertex angle ϕ of an isosceles triangle follows from $x^2 = 2y^2 - 2y^2\cos\phi$ (law of cosines) and $y = \frac{P-x}{2}$.

So it suffices to show that the inequality $\phi_1 \le \phi \le \phi_2$ is equivalent to the pair of conditions $s(\phi) > 0$ and $h(\phi) \ge 0$, where $s(\phi)$ and $h(\phi)$ are defined as in Theorem 3; for this, we need four lemmas.

Lemma 5. *For some* ψ *in* $(0, \phi_1)$, $s(\psi) = 0$.

Proof. Notice that $\lim_{\phi\to 0^+} s(\phi) = -A < 0$. On the other hand, the existence of an isosceles triangle with vertex angle ϕ_1 guarantees that $s(\phi_1) > 0$ (Theorem 3). By the continuity of s on $(0,\pi)$, there must exist ψ such that $0 < \psi < \phi_1$ and $s(\psi) = 0$. □

Lemma 6. *The function* s *is strictly increasing on* $(0,\pi)$ *and, for* $\phi \ge \phi_1$, $s(\phi) > 0$.

Proof. Since the derivative $s'(\phi) = \frac{P^2}{4(1+\cos\phi)}$ is positive on $(0,\pi)$, s is strictly increasing; it follows that $s(\phi) \ge s(\phi_1) > 0$ for $\phi \ge \phi_1$. □

Lemma 7. *For* $\phi > \phi_2$, $h(\phi) < 0$.

Proof. Recall that $h(\phi) = \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi}$. By L'Hospital's rule, we have $\lim_{\phi\to\pi} \frac{1+\cos\phi}{\sin\phi} = \lim_{\phi\to\pi} \frac{-\sin\phi}{\cos\phi} = 0$; it follows that $\lim_{\phi\to\pi^-} h(\phi) = \frac{P^2}{4} - \lim_{\phi\to\pi^-} \frac{8A}{\sin\phi} = -\infty$. Suppose $h(\phi) \ge 0$ for some $\phi > \phi_2$. Then $h(\phi_3) = 0$ for some $\phi_3 > \phi_2$ because h is continuous on $(0,\pi)$ and $\lim_{\phi\to\pi^-} h(\phi) = -\infty$. At the same time, $s(\phi_3) > 0$ (Lemma 6). Then by Theorem 3, there exists a third isosceles triangle of area A and perimeter P, which is impossible. □

Lemma 8. *There is no* ϕ *in* $(0,\pi)$ *for which* $h(\phi) = h'(\phi) = 0$.

Proof. Suppose $h(\phi) = h'(\phi) = 0$ for some ϕ in $(0,\pi)$. It follows that

$$\left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 = \frac{8A}{\sin\phi} \quad \text{and} \quad \frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right) = \frac{2P\cos\phi}{1+\cos\phi}.$$

Squaring the latter and dividing it by the former expression we get $P^2 = \frac{2A(1+\cos\phi)^2}{\sin\phi\cos^2\phi}$. Substituting this expression for P^2 into $\left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 = \frac{8A}{\sin\phi}$ we arrive at the equation $\frac{A(1+\cos\phi)^2}{2\sin\phi\cos^2\phi} + \frac{2A(1+\cos\phi)}{\sin\phi} + \frac{2A\cos^2\phi}{\sin\phi} = \frac{8A}{\sin\phi}$, which reduces to $(\cos\phi - 1)(2\cos\phi - 1)(2\cos^2\phi + 5\cos\phi + 1) = 0$. The only roots in $(0,\pi)$ are given by $\phi = \frac{\pi}{3}$ and $\phi = \arccos\left(\frac{-5+\sqrt{17}}{4}\right)$. It is easy to see that $h'(\phi) < 0$ for $\phi > \frac{\pi}{2}$, so $\arccos\left(\frac{-5+\sqrt{17}}{4}\right)$ is an extraneous solution. Moreover, $\phi = \frac{\pi}{3}$ turns

$P^2 = \frac{2A(1+\cos\phi)^2}{\sin\phi\cos^2\phi}$ into $P^2 = 12\sqrt{3}A$, contradicting the fact that the given triangle was assumed to be non-equilateral. We conclude that $h(\phi) = h'(\phi) = 0$ is impossible. $\square$

Completing the proof of Theorem 4.
Claim(a) *For $\phi_1 \le \phi \le \phi_2$, $s(\phi) > 0$ and $h(\phi) \ge 0$, with $h(\phi) > 0$ for $\phi_1 < \phi < \phi_2$.*

Recall from Lemma 6 that $s(\phi) > 0$ for $\phi \ge \phi_1$. So it remains to establish $h(\phi) \ge 0$ for $\phi_1 \le \phi \le \phi_2$. We will argue by contradiction.

Of course $h(\phi_1) = h(\phi_2) = 0$. Notice that $h(\phi) = 0$ for $\phi_1 < \phi < \phi_2$ is impossible for this would imply (by Theorem 3) the existence of a third isosceles triangle of area A and perimeter P. If $h(\phi_3) < 0$ for some ϕ_3 strictly between ϕ_1 and ϕ_2 then continuity of h, together with the impossibility of $h(\phi) = 0$ for $\phi_1 < \phi < \phi_2$, implies $h(\phi) < 0$ for *all* angles strictly between ϕ_1 and ϕ_2. But we already know from Lemma 7 that $h(\phi) < 0$ for all angles greater than ϕ_2. It follows that h has a local maximum at $\phi = \phi_2$, so that $h(\phi_2) = h'(\phi_2) = 0$, contradicting Lemma 8.

Recalling the statement immediately before Lemma 5, we see that the proof of Theorem 4 will be completed by establishing
Claim(b) *At least one of the conditions $s(\phi) > 0$ and $h(\phi) \ge 0$ fails when either $\phi < \phi_1$ or $\phi > \phi_2$.*

Of course the failure of $h(\phi) \ge 0$ for $\phi > \phi_2$ has been established in Lemma 7, so we only need to show either $s(\phi) \le 0$ or $h(\phi) < 0$ for $\phi < \phi_1$.

Lemma 5 asserts that there exists ψ in $(0, \pi)$ such that $\psi < \phi_1$ and $s(\psi) = 0$. Consider now an arbitrary $\phi < \phi_1$. If $\phi \le \psi$ then by Lemma 6 $s(\phi) \le s(\psi) = 0$, so we only need to pay attention to the possibility $\phi_1 > \phi > \psi$ and $s(\phi) > 0$. In that case we show below that $h(\phi) < 0$, arguing by contradiction.

The failure of $h(\phi) < 0$ implies, in the presence of $s(\phi) > 0$, that $h(\phi) > 0$: indeed $h(\phi) = 0$ and $s(\phi) > 0$ would yield a third isosceles triangle of area A and perimeter P, again by Theorem 3. The same argument applies in fact to all angles between ψ and ϕ_1. But we have already established through Claim(a) the strict positivity of h for all angles between ϕ_1 and ϕ_2. We conclude that h has a local minimum at $\phi = \phi_1$, so that $h(\phi_1) = h'(\phi_1) = 0$, contradicting Lemma 8. This completes the proof of Theorem 4.

Having completed the proof of Theorem 4, let us provide an example: the bases of the two isosceles triangles of area 3 and perimeter 10 (Figure 1) have already been computed as the positive roots of the cubic $5x^3 - 25x^2 + 36 = 0$; it follows then that *all* angles of *every* triangle of area 3 and perimeter 10 must be between about $19.003°$ and $122.351°$, the angles shown in Figure 5.

Remark. It can be shown that ϕ_1 and ϕ_2 are the two largest roots of

$$(P^2 \sin\phi + 4A + 4A\cos\phi)^2 - 32P^2 A \sin\phi = 0$$

in $(0, \pi)$, and that they also satisfy the equation

$$\sin\phi_2\left(1+\sin\frac{\phi_1}{2}\right)^2 = \sin\phi_1\left(1+\sin\frac{\phi_2}{2}\right)^2.$$

5. Heron's curve

Theorem 4 establishes bounds for the angles of every triangle of given area and perimeter; appealing to the law of sines, we see that it also yields bounds for the ratio of any two sides. Determining *sharp* bounds for side ratios relies on some machinery we develop next.

Instead of looking for isosceles triangles $(z = y)$ of area A and perimeter P, let us now look for triangles of area A and perimeter P where two sides have ratio r ($\frac{z}{y} = r$); without loss of generality, we may assume $r > 1$. (Observe here - as in fact noticed through Figure 5 and related discussion - that $r > 1$ does not rule out the possibilities $x = z$ (with $r \approx 3.0268$ at $A = 3$, $P = 10$) or $x = y$ (with $r \approx 1.7522$ at $A = 3$, $P = 10$).) Extending the procedure of Lemma 1 to arbitrary triangles, from $y^2 - x_1^2 = r^2y^2 - x_2^2$ and $x = x_2 \pm x_1$ (Figure 7) we find that

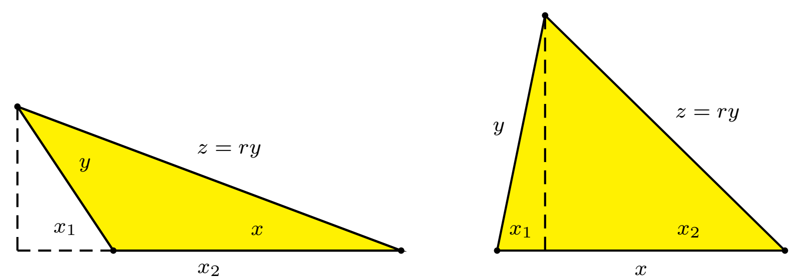

Figure 7. The case of an arbitrary triangle

$x_1 = \pm\frac{(1-r^2)y^2+x^2}{2x}$. In view of $\frac{x}{2}\sqrt{y^2 - x_1^2} = A$ and $y = \frac{P-x}{r+1}$, further algebraic manipulation leads to an equation that generalizes the isosceles triangle's cubic (1):

$$8rPx^3+4(r^2-3r+1)P^2x^2-4(1-r)^2P^3x+(1-r)^2P^4+16(1+r)^2A^2 = 0. \quad (6)$$

Appealing to Rolle's theorem as in the case of the isosceles triangle, we see that this cubic cannot have more than two positive roots. Indeed one of the derivative's roots, $\left(\frac{-(r^2-3r+1)-\sqrt{r^4-r^2+1}}{6r}\right)P$, is negative since $|r^2 - 3r + 1| < \sqrt{r^4 - r^2 + 1}$ for $r > 1$.

Unlike the case of the isosceles triangle, however, the isoperimetric inequality $P^2 > 12\sqrt{3}A$ does not guarantee the existence of two positive roots. So there can be *at most* two triangles of area A and perimeter P satisfying the condition $\frac{z}{y} = r > 1$.

Setting $x = P - y - z$ and $r = \frac{z}{y}$ in the cubic (6) leads to

$$P^4 - 4P^3(y + z) + 4P^2(y^2 + 3yz + z^2) - 8Pyz(y + z) + 16A^2 = 0, \quad (7)$$

which can be shown to be equivalent to Heron's area formula. The graph of this curve for $A = 3$ and $P = 10$ (Figure 8) illustrates the fact established above by (6): for every pair of A and P, there can be at most two triangles of area A and perimeter P satisfying $\frac{z}{y} = r > 1$. Indeed, the three unbounded regions shown in Figure 8 correspond to $x < 0$ (first quadrant), $y < 0$ (second quadrant), and $z < 0$ (fourth quadrant), hence it is only the boundary of the bounded region that corresponds to triangles of area 3 and perimeter 10; clearly, this boundary that we call *Heron's curve* (Figure 9) may be intersected by any line at most twice.

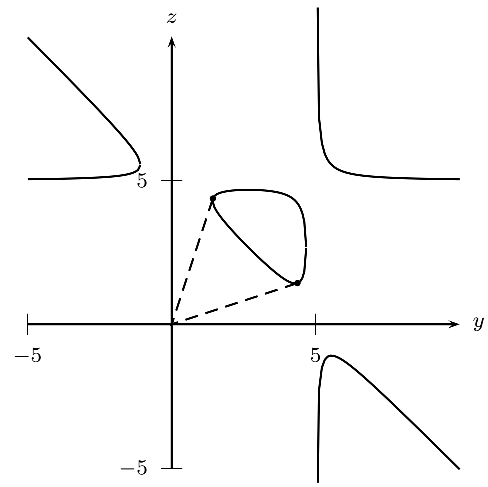

Figure 8. Graph of (7) for $A = 3$ and $P = 10$

Rather predictably, in view of its symmetry about $z = y$, the triangles corresponding to Heron's curve's intersections with (for example) $z = 2y$ and $z = \frac{y}{2}$ (see Figure 9) are mirror images of each other (about the third side x's perpendicular bisector); so it suffices to restrict our computations to $r > 1$, sticking to our initial assumption. These triangles are found by first solving the cubic (6) when $r = 2$ and are approximately $\{3.0077, 2.3307, 4.6615\}$ and $\{4.5977, 1.8007, 3.6015\}$; they are associated with parametrizing angles of about $33.529°$ and $112.315°$, respectively.

6. Side ratios 'bounded' by area and perimeter

We present now the following companion to Theorem 4.

Theorem 9. *In every non-equilateral triangle of area A and perimeter P, the ratio r of any two sides must satisfy the inequality $r_1 \leq r \leq r_2$, where $r_1 < 1 < r_2$, $r_1 r_2 = 1$ are the positive roots of the sextic*

$$32P^4A^2(2r^6 - 3r^4 - 3r^2 + 2) - P^8r^2(r-1)^2 + 6912A^4r^2(r+1)^2 = 0. \quad (8)$$

Proof. Figures 8 and 9 (and the discussion preceding them) make it clear that not all lines $z = ry$ intersect Heron's curve: such intersections (corresponding to triangles of area A and perimeter P satisfying $\frac{z}{y} = r$) occur only at $r = 1$ and a varying

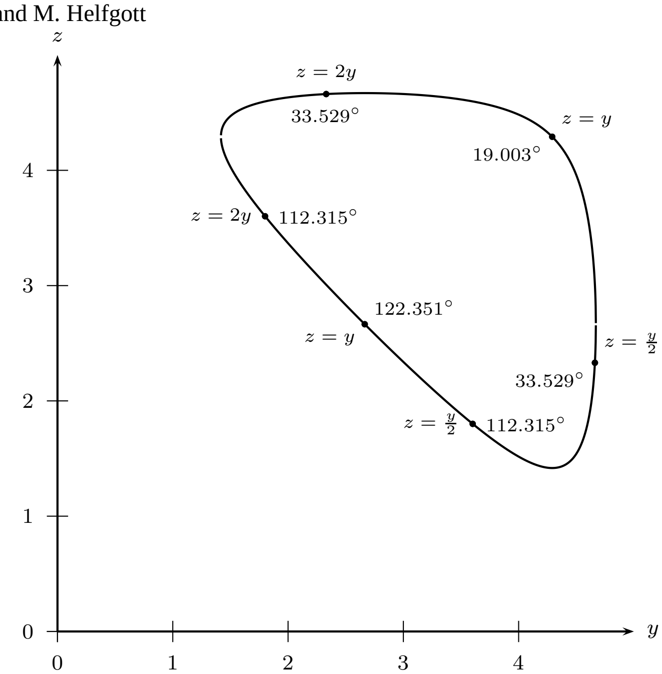

Figure 9. Heron's curve for $A = 3$ and $P = 10$

interval around it depending on A and P by way of (6). To establish sharp bounds for such 'intersecting' r, we observe that these bounds are none other than the slopes of the lines *tangent* to Heron's curve; in the familiar case $A = 3$, $P = 10$, these tangent lines are shown in Figure 8. But a line $z = ry$ is tangent to Heron's curve if and only if there is precisely one triangle of area A and perimeter P satisfying $\frac{z}{y} = r$; that is, if and only if the cubic (6) has a double root.

It is well known (see for example [4, p.91]) that the cubic $ax^3 + bx^2 + cx + d$ has a double root if and only if

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd = 0.$$

(The reader may arrive at this 'tangential' condition independently, arguing as in the proof of Theorem 2.) So we may conclude that the slopes of the two lines tangent to Heron's curve and passing through the origin are the positive roots of the polynomial $S(r) = -64P^2(r+1)^2Q(r)$, where $Q(r)$ is the sixth degree polynomial in (8).

It may not be obvious but Q, and therefore S as well, must have precisely two positive roots, as they ought to. This relies on the following facts (which imply a total of *four* real roots for Q): the leading coefficient of Q is positive and its highest power is even, so $\lim_{r\to\pm\infty} Q(r) = +\infty$; $Q(-1) = -4P^8 - 64P^4A^2 < 0$; $Q(0) = 64P^4A^2 > 0$; $Q(1) = -64A^2(P^4 - (12\sqrt{3})^2A^2) < 0$; $Q(\frac{1}{r}) = \frac{Q(r)}{r^6}$ for $r \neq 0$, so that r is a root of Q if and only if $\frac{1}{r}$ is. $\square$

In the familiar example of $A = 3$ and $P = 10$, the two positive roots of S are $r_1 \approx 0.3273$ and $r_2 \approx 3.0551$. As pointed out above, these two roots are inverses

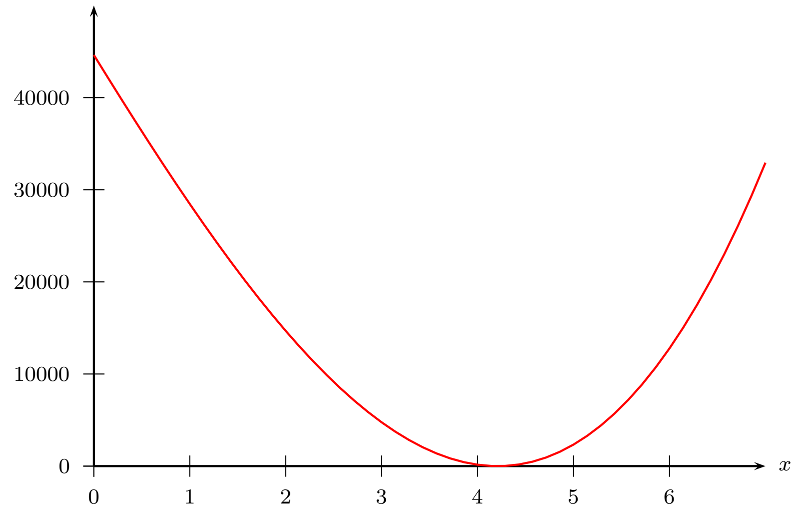

Figure 10. Graph of (6) for $A = 3$, $P = 10$, and $r \approx 3.0551$

of each other: this is geometrically justified by the fact that the two roots are the slopes of the two tangent lines in Figure 8, which are of course mirror images of each other about the diagonal $z = y$. Moreover, r_1 and r_2 lead to the *same* (modulo a factor) cubic in (6).

We conclude that the side ratios of every triangle of area 3 and perimeter 10 must be between approximately 0.3273 and 3.0551. To obtain the unique (modulo reflection) triangle of area 3 and perimeter 10 where these ratios are realized, we need to determine its third side x. It is the double root of the cubic (6) for r equal to approximately 3.0551 (Figure 10). It turns out that x equals approximately 4.2048.

The triangle is now fully determined through $y \approx \frac{10-4.2048}{3.0551+1} \approx 1.4291$ and $z \approx 3.0551 \times 1.4291 \approx 4.366$ (upper 'corner' in Figure 9). The angle-parameter (between sides y and z) at that 'corner' is now easy to find as $\arccos\left(\frac{y^2+z^2-x^2}{2yz}\right) \approx 74.079°$. The triangle obtained, approximately $\{4.2048, 4.3661, 1.4291\}$ (see Figure 11), is the furthest possible from being isosceles - or rather the furthest possible from being equilateral! - among all triangles of area 3 and perimeter 10.

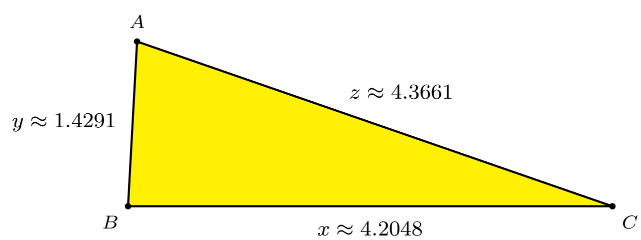

Figure 11. The unique extreme-side-ratio triangle of area 3 and perimeter 10

Our findings are confirmed in Figure 12 by a graph of $\frac{z(\phi)}{y(\phi)}$, where $z(\phi)$ and $y(\phi)$ are the Newton parametrizations of sides z and y in (5). That graph shows a maximum value of about 3.055 for $\frac{z(\phi)}{y(\phi)}$ with ϕ approximately equal to $1.293 \approx 74.08°$:

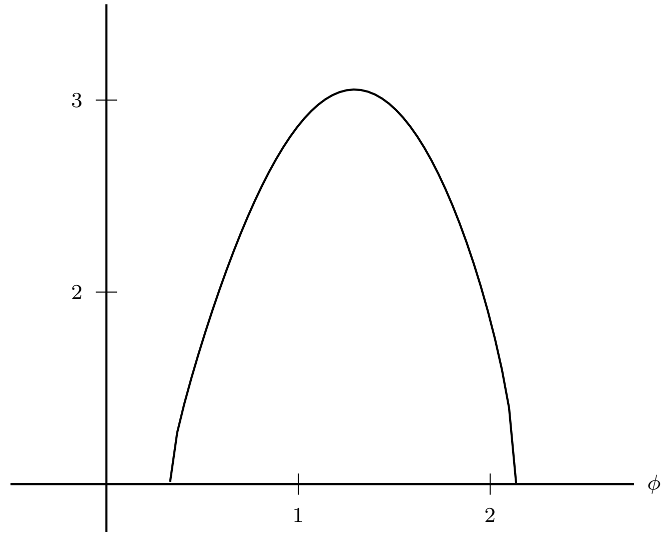

Figure 12. $\frac{z(\phi)}{y(\phi)}$ for $19.003° \approx 0.33166 \leq \phi \leq 2.13543 \approx 122.351°$, $A = 3$ and $P = 10$

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Kronecker's Approximation Theorem and a Sequence of Triangles

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Abstract. We investigate the dynamic behavior of the sequence of nested triangles with a fixed division ratio on their sides. We prove a result concerning a special case that was not examined in [1]. We also provide an answer to an open problem posed in [3].

1. Introduction

The dynamic behavior of a sequence of polygons is an intriguing research area and many articles have been devoted to it (see e.g. [1], [2], [3] and the references therein). The questions that arise about these sequences are mainly two. The first one is about the existence of a limiting point of the sequence. The second one is about the dynamic behavior of the shapes of the polygons that belong to the sequence. Thus, it is possible to find a limiting shape, periodical shapes or an even more complicated behavior. In this article we are interesting for the sequence of triangles with a fixed division ratio on their sides. Let $A_0B_0C_0$ be an initial triangle and let the points A_1 on B_0C_0, B_1 on A_0C_0 and C_1 on A_0B_0 such that:

$$\frac{B_0A_1}{A_1C_0} = \frac{C_0B_1}{B_1A_0} = \frac{A_0C_1}{C_1B_0} = \frac{t}{1-t},$$

where t is a fixed real number in $(0,1)$. Thus, the next triangle of the sequence is $A_1B_1C_1$. By using the fixed division ratio $t:(1-t)$ we produce the members of the sequence consecutively (see Figure 1 where $t=\frac{1}{3}$).

In [1] a more complicated sequence of triangles is investigated thoroughly. The author uses complex analysis and so the vertices of a triangle can be defined by three complex numbers A_n, B_n, C_n on the complex plane. The basic iterative process that is studied in [1] has the following matrix form:

$$V_n = TV_{n-1}, \tag{1}$$

where $V_n = \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1-t & t \\ t & 0 & 1-t \\ 1-t & t & 0 \end{pmatrix}$ is a circulant matrix and V_0 is a given initial triangle. Note that in [1] t is considered generally as a complex number. We stress also that throughout the article we ignore the scaling factor $1/r_n$ that appears at the above iteration in [1]. This factor does not affect the shape of the triangles. As an exceptional case in Section 5 in [1], it is studied the above sequence with t a real number in $(0,1)$. This is exactly the sequence that

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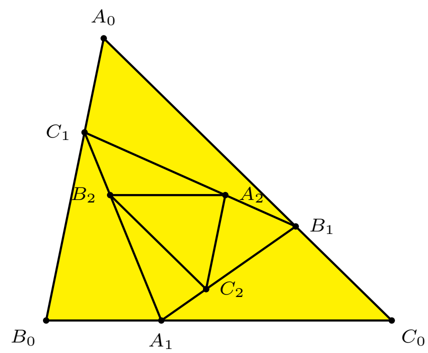

Figure 1.

we described previously and we study in this article. From now on we call this sequence the FDRS (*i.e.*, Fixed Division Ratio Sequence). Concerning the FDRS the author in [1] proved that if

$$t = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan(a\pi), \tag{2}$$

and a is a rational number, then the FDRS is periodic with respect to the shapes of the triangles. Apparently the same result is proved in [3] (although the proof is left as an exercise). At first sight the formula for the periodicity in [3] seems quite different from (2), but after some algebraic calculations it can be shown that is indeed the same. In [3] it is also proved, that the limiting point of the FDRS is the centroid of the initial triangle $A_0B_0C_0$. Obviously, this is a direct result from the recurrence (1) since it holds $A_{n+1} + B_{n+1} + C_{n+1} = A_n + B_n + C_n$, which means that all the triangles of the FDRS have the same centroid.

In this article we are interested in the behavior of the shapes of the triangles in the FDRS. Particularly, we examine the case when a in (2) is an irrational number. This case was not examined in [1] and [3]. Throughout the article we use the same nomenclature as in [1] and our results are an addendum to [1].

2. Preliminary results

In this Section we will repeat the formulation and the basic results from [1] and we will present some significant remarks. We use the recurrence (1) which is the FDRS as it represented on the complex plane. Without loss of generality as in [1], we can consider that the centroid of the initial triangle $A_0B_0C_0$ is at the origin (*i.e.*, $A_0 + B_0 + C_0 = 0$). This is legitimate since it is just a translation of the centroid to the origin and it does not affect the shapes of the triangles of the FDRS. By using results from circulant matrix theory in [1], it is proved that

$$V_n = T^n V_0 = s_1 \lambda_1^n F_{3,1} + s_2 \lambda_2^n F_{3,2} \tag{3}$$

where $F_{3,1} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \omega\\ \omega^2\end{pmatrix}$ and $F_{3,2} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \omega^2\\ \omega^4\end{pmatrix}$ are columns of the 3×3 Fourier matrix $F_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega^4\end{pmatrix}$. Moreover, $\lambda_j = (1-t)\omega^j + t\omega^{2j}$, $j = 0,1,2$ are the eigenvalues of T and $s = \begin{pmatrix}s_0\\ s_1\\ s_2\end{pmatrix}$ such that $F_3 s = V_0$. We also consider $\omega = e^{i2\pi/3}$, $\eta = e^{i\pi/3}$ and as $\overline{x}$ we denote the conjugate of x. The following function $z : \mathbb{C}^3 \to \mathbb{C}$ is also defined in [1]:

$$z(V_n) = \frac{C_n - A_n}{B_n - A_n}. \tag{4}$$

This is a very useful function. First, it signifies the orientation of the triangle on the complex plane. Thus, if $\arg(z(V_n)) > 0$ (< 0) the triangle is positively (negatively) oriented (see Figure 2). Note also the angle $\widehat{A}_n$ of the triangle $A_nB_nC_n$ is equal to $\arg(z(V_n))$, so $\widehat{A}_n$ can be regarded as positive or negative.

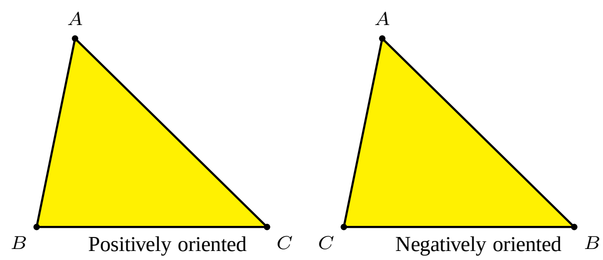

Figure 2.

Function $z(V_n)$ also signifies the ratio of the sides b_n, c_n since $|z(V_n)| = \frac{b_n}{c_n}$. If for instance we have that $|z(V_n)| = 1$, the triangle is isosceles ($b_n = c_n$). If additionally we have $\arg(z(V_n)) = \pi/3$ or $\arg(z(V_n)) = -\pi/3$ then the triangle is equilateral. From this observation we have the following Proposition:

Proposition 1. *A triangle $A_nB_nC_n$ on the complex plane is equilateral if and only if $z(V_n) = \eta$ (positively oriented) or $z(V_n) = \overline{\eta}$ (negatively oriented).*

All these facts stress the importance of function (4). It is apparent that the shape of a triangle on the complex plane is determined completely by function (4). Now, let us assume that the initial triangle $A_0B_0C_0$ of the FDRS is not degenerate (*i.e.*, two or three vertices do not coincide and the vertices are not collinear). Moreover, let us assume that $A_0B_0C_0$ is not equilateral (*i.e.*, $z(V_0) \neq \eta$ and $z(V_0) \neq \overline{\eta}$), because if it was equilateral then all members of the FDRS would be equilateral triangles. Let us next present two significant definitions and notations.

Firstly, after some algebraic calculations we define the following ratio:

$$\frac{s_2}{s_1} = \frac{B_0 - \omega A_0}{\omega^2 A_0 - B_0} = re^{i\rho}, \tag{5}$$

where $r = |\frac{s_2}{s_1}|$ and $\rho = \arg(\frac{s_2}{s_1})$. Note that (5) holds because we have considered $A_0 + B_0 + C_0 = 0$.

Secondly, from the eigenvalues λ_1 and λ_2 we can get the following definitions

$$\frac{\lambda_2}{\lambda_1} = e^{i\theta}, \quad and \quad \theta = 2\arctan(\sqrt{3}(2t-1)). \tag{6}$$

If we let $\theta = 2\pi a$ in the above equation we get directly equation (2). Now, we can consider the following cases:

(1) $\theta = 0$. In this case we have $t = 1/2$ and all the members of the FDRS are similar to $A_0B_0C_0$.
(2) $\theta = 2k\pi/m$. This case is studied in [1] where $a = k/m$ is rational. We have a periodical behavior and if $(k, m) = 1$ the period is equal to m (otherwise it is smaller than m).
(3) $\theta = 2a\pi$, where a is irrational. This is the case that we study in this article.

In what follows we prove a number of important facts about the FDRS.

Firstly, we note that it holds $s_1 \neq 0$ and $s_2 \neq 0$. This is a straightforward result from the equality $z(V_0) = \frac{s_1\eta + s_2}{s_1 + s_2\eta}$ (see [1]) and from the assumption that $z(V_n) \neq \eta$ and $z(V_n) \neq \overline{\eta}$.

Our next aim is to prove that $r \neq 1$. Let $A_0 = a_1 + ia_2$ and $B_0 = b_1 + ib_2$ and assume that $r = 1$ or equivalently $|B_0 - \omega A_0| = |\omega^2 A_0 - B_0|$. After some algebraic calculations we find $a_1b_2 = a_2b_1$, which means that the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$ and so the vectors $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ are linearly dependent. Thus, $A_0 = \lambda B_0$ where λ is real and $\lambda \neq 0$, $\lambda \neq 1$. Now from (4) we get

$$z(V_0) = \frac{C_0 - A_0}{B_0 - A_0} = \frac{-B_0 - 2A_0}{B_0 - A_0} = -\frac{1 + 2\lambda}{1 - \lambda} \in R.$$

Thus, $\arg(z(V_0)) = 0$ or $\arg(z(V_0)) = \pi$ which is impossible since the initial triangle is not degenerate. Consequently, it holds $r \neq 1$.

Next, we examine the case $r < 1$. From (3) and (6) we have

$$V_n = \lambda_1^n (s_1 F_{3,1} + s_2 e^{in\theta} F_{3,2}).$$

By using the above equation and (5), equation (4) becomes

$$z(V_n) = \frac{s_1\eta + s_2 e^{in\theta}}{s_1 + s_2\eta e^{in\theta}} = \eta \frac{1 + re^{i(\varphi_n - \pi/3)}}{1 + re^{i(\varphi_n + \pi/3)}},$$

where $\varphi_n = n\theta + \rho$. From the above equation we get directly that:

$$\begin{aligned} \arg(z(V_n)) = \ & \widehat{A}_n = \Phi(\varphi_n, r) = \\ = \ & \frac{\pi}{3} + \arctan\frac{r\sin(\varphi_n - \pi/3)}{1 + r\cos(\varphi_n - \pi/3)} - \arctan\frac{r\sin(\varphi_n + \pi/3)}{1 + r\cos(\varphi_n + \pi/3)}, \end{aligned} \tag{7}$$

and

$$|z(V_n)| = \frac{b_n}{c_n} = \mu(\varphi_n, r) = \tag{8}$$
$$= \sqrt{\frac{(1 + r\cos(\varphi_n - \pi/3))^2 + r^2\sin^2(\varphi_n - \pi/3)}{(1 + r\cos(\varphi_n + \pi/3))^2 + r^2\sin^2(\varphi_n + \pi/3)}}.$$

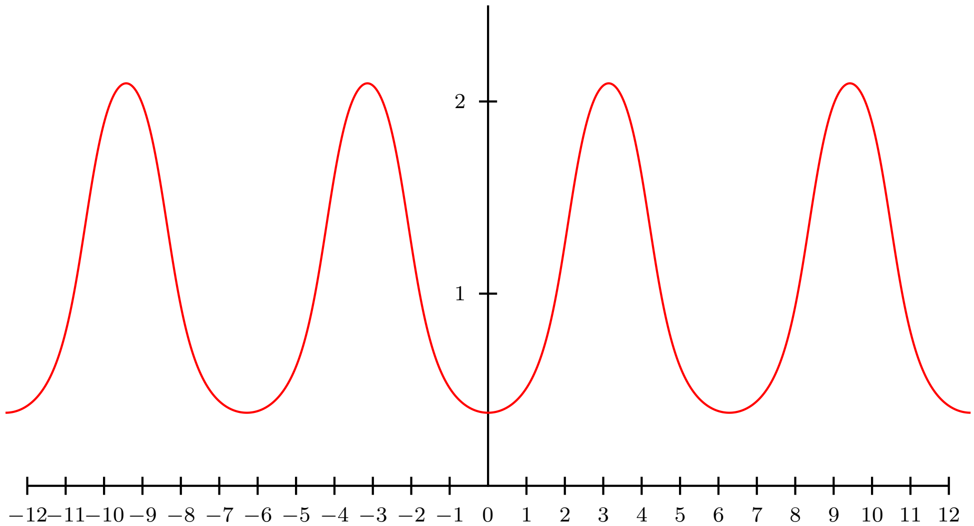

Figure 3(a)

Observe that in (7) and (8) functions $\Phi(\varphi, r)$ and $\mu(\varphi, r)$ are defined respectively. We also define $\Phi(\varphi) = \Phi(\varphi, r)$ and $\mu(\varphi) = \mu(\varphi, r)$. Function $\Phi(\varphi)$ is even (*i.e.*, $\Phi(\varphi) = \Phi(-\varphi)$) and periodic with period 2π (see Figure 3(a) where $r = 0.5$). The minima of $\Phi(\varphi)$ appear at $\varphi = 0, \pm 2\pi, \pm 4\pi, \ldots$ and the maxima at $\varphi = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$. Thus, $\arg(z(V_n)) = \Phi(\varphi_n, r) \in [m_1, m_2]$ where

$$m_1 = \Phi(0, r) = \frac{\pi}{3} - 2\arctan\frac{r\sqrt{3}}{2 + r}, \qquad m_2 = \Phi(\pi, r) = \frac{\pi}{3} + 2\arctan\frac{r\sqrt{3}}{2 - r}.$$

In Figure 3(b), where function Φ is depicted for different values of r, we can observe that the interval $[m_1, m_2]$ decreases as $r \to 0^+$ and increases as $r \to 1^-$. In every case since $r \in (0, 1)$ we find that $[m_1, m_2] \subset (0, \pi)$, which also means that the triangles of the FDRS are positively oriented.

Concerning function $\mu(\varphi)$ we have the following properties: $\mu(k\pi) = 1$ where k is integer, $\mu(-\varphi) = 1/\mu(\varphi)$ and $\mu(\varphi)$ is periodic with period 2π. Figure 3(c) depicts function $\mu(\varphi)$ in $[-4\pi, 4\pi]$ and $r = 0.5$.

Remark. Let us present a fact that we will need in Section 3. Let $r < 1$, since a similar argument applies for $r > 1$. Recall that function $\Phi(\varphi)$ is not injective (one-to-one) and so its inverse can not be determined uniquely. For an angle $\tilde{\theta} \in [m_1, m_2]$ (*i.e.*, $\tilde{\theta}$ belongs to the range of Φ), we want to find the elements $\tilde{\varphi}_m$

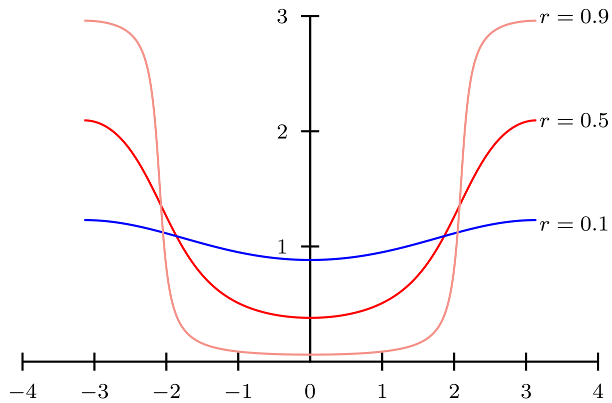

Figure 3(b)

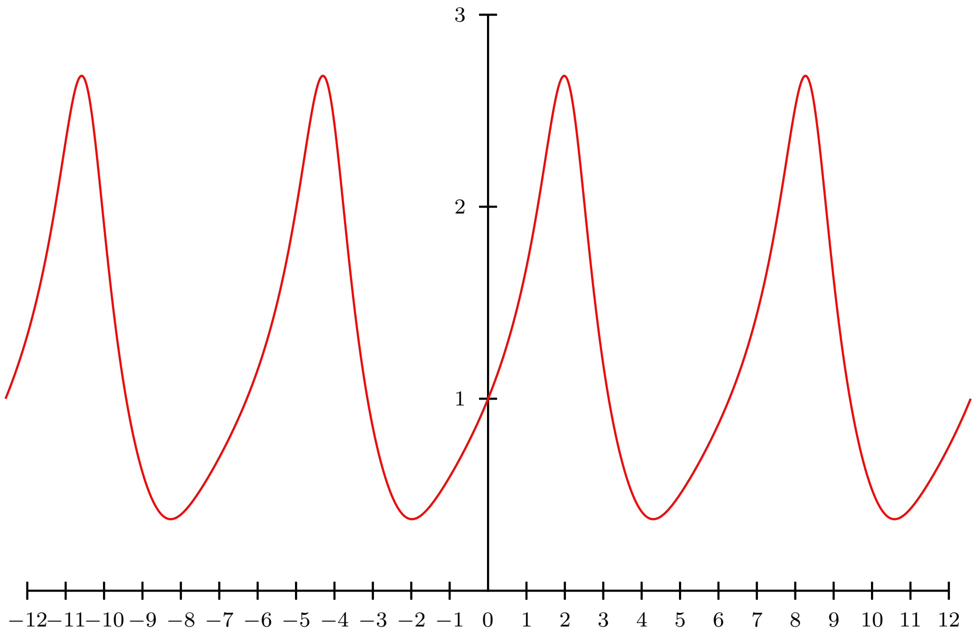

Figure 3(c)

which have the same image $\tilde{\theta}$ (*i.e.*, $\Phi(\tilde{\varphi}_m) = \tilde{\theta}$). Since $\Phi(\varphi)$ is periodic with period 2π, the elements $\tilde{\varphi}_m$ have the form: $2k\pi \pm \varphi_a(\tilde{\theta})$ (k is integer), where as $\varphi_a(\tilde{\theta})$ we define the minimum element $\tilde{\varphi}_m$ such that $\tilde{\varphi}_m \geq 0$ (see Figure 4). Apparently, $\varphi_a(\tilde{\theta}) \in [0, \pi]$ and it holds that $\Phi(2k\pi \pm \varphi_a(\tilde{\theta})) = \tilde{\theta}$ (*i.e.*, all the elements $2k\pi \pm \varphi_a(\tilde{\theta})$ have the same image $\tilde{\theta}$). Figure 4 depicts this characteristic of function $\Phi(\varphi)$.

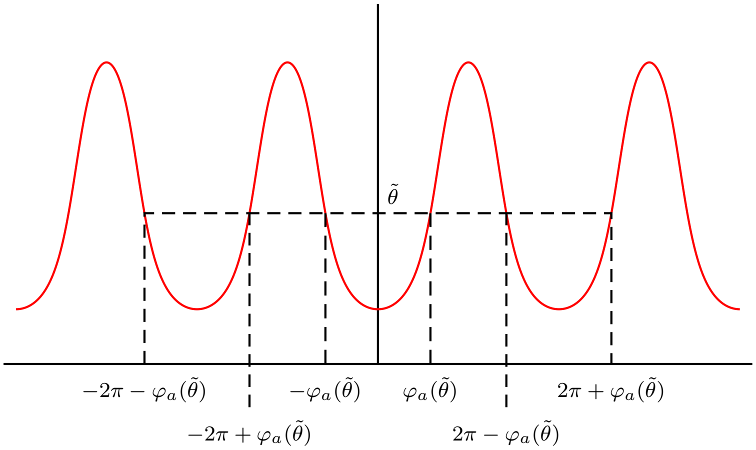

Figure 4.

Now, for the case $|\frac{s_2}{s_1}| = r > 1$ we can use the inverse ratios $\frac{s_1}{s_2} = \frac{1}{r}e^{-i\rho}$, $\frac{\lambda_1}{\lambda_2} = e^{-i\theta}$ and have that

$$z(V_n) = \frac{s_1\eta + s_2 e^{in\theta}}{s_1 + s_2\eta e^{in\theta}} = \overline{\eta}\frac{1+\frac{1}{r}e^{i(-\varphi_n+\pi/3)}}{1+\frac{1}{r}e^{i(-\varphi_n-\pi/3)}},$$

where again $\varphi_n = n\theta + \rho$. From the above we have as before:

$$\begin{aligned}\arg(z(V_n)) = \widehat{A}_n = -\Phi(-\varphi_n, 1/r) = \qquad\qquad (9)\\ = -\frac{\pi}{3} + \arctan\frac{\frac{1}{r}\sin(-\varphi_n+\pi/3)}{1+\frac{1}{r}\cos(-\varphi_n+\pi/3)} - \arctan\frac{\frac{1}{r}\sin(-\varphi_n-\pi/3)}{1+\frac{1}{r}\cos(-\varphi_n-\pi/3)},\end{aligned}$$

and

$$\begin{aligned}|z(V_n)| = \frac{b_n}{c_n} = \frac{1}{\mu(-\varphi_n, 1/r)} = \qquad\qquad (10)\\ = \sqrt{\frac{(1+\frac{1}{r}\cos(-\varphi_n+\pi/3))^2 + \frac{1}{r^2}\sin^2(-\varphi_n+\pi/3)}{(1+\frac{1}{r}\cos(-\varphi_n-\pi/3))^2 + \frac{1}{r^2}\sin^2(-\varphi_n-\pi/3)}}.\end{aligned}$$

It is now obvious that equations (7), (8) and equations (9), (10) signify similar triangles with different orientations provided of course that φ_n and r are common. When $r > 1$ the triangles of the FDRS are negatively oriented. Using similar arguments as before we can prove easily that $\arg(z(V_n)) = \widehat{A}_n \in [\overline{m}_1, \overline{m}_2]$ where

$$\overline{m}_1 = -\Phi(-\pi, 1/r) = -\frac{\pi}{3} - 2\arctan\frac{\frac{1}{r}\sqrt{3}}{2-\frac{1}{r}},$$

$$\overline{m}_2 = -\Phi(0, 1/r) = -\frac{\pi}{3} + 2\arctan\frac{\frac{1}{r}\sqrt{3}}{2+\frac{1}{r}}.$$

Thus, for any $r > 1$ we have $[\overline{m}_1, \overline{m}_2] \subset (-\pi, 0)$. The interval $[\overline{m}_1, \overline{m}_2]$ increases as $r \to 1^+$ and decreases as $r \to +\infty$. In the next Section we apply Kronecker's Approximation Theorem in order to get our main result for the FDRS when a in (2) is an irrational number.

3. Application of Kronecker's approximation theorem

First we present Kronecker's Approximation Theorem (see e.g. [4]).

Kronecker's approximation theorem *If ω is a given irrational number, then the sequence of numbers $\{n\omega\}$, where $\{x\} = x - \lfloor x \rfloor$, is dense in the unit interval. Explicitly, given any $\overline{p}$, $0 \leq \overline{p} \leq 1$, and given any $\epsilon > 0$, there exists a positive integer k such that $|\{k\omega\} - \overline{p}| < \epsilon$.*

We know that $\varphi_n = n\theta + \rho = 2\pi a n + \rho$ and recall that a is irrational and ρ is a function of A_0, B_0, so it is fixed. From Kronecker's Approximation Theorem we know that a member of the sequence $\{na\} = na - \lfloor na \rfloor$ will be arbitrarily close to any given $\overline{p} \in [0, 1]$. Similarly, a member of the sequence $2\pi\{na\} = \varphi_n - 2\pi \lfloor na \rfloor - \rho$ will be arbitrarily close to the angle $\overline{\theta} = 2\pi\overline{p} \in [0, 2\pi]$. Thus, a member of the sequence φ_n will be arbitrarily close to the angle $\overline{\theta} + 2\pi \lfloor na \rfloor + \rho$. Let us now define the sequence of angles φ_n on the unit circle. The quantity $2\pi \lfloor na \rfloor$ defines complete rotations on the unit circle and can be eliminated. This implies that a member of the sequence φ_n will be arbitrarily close to the angle $\overline{\theta} + \rho$ on the unit circle. If additionally, we imagine the unit circle to rotate by $-\rho$, we get that a member of the sequence φ_n will be arbitrarily close to the angle $\overline{\theta} = 2\pi\overline{p}$ on the unit circle. Since this holds for any given $\overline{p} \in [0, 1]$, we conclude that a member of the sequence φ_n will be arbitrarily close to any given angle $\overline{\theta} \in [0, 2\pi]$ on the unit circle. This important fact will be used in the proof of the next Theorem which is the main result of this article. Note that the Theorem uses the notation that has already been presented.

Theorem 2. *Let A_0, B_0, C_0 be complex numbers which define an initial non-degenerate and non-equilateral triangle on the complex plane such that its centroid is at the origin (*i.e.*, $A_0 + B_0 + C_0 = 0$). Suppose we apply the FDRS with $t = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan(a\pi)$ where a is an irrational number. Let $\epsilon_1 > 0$ and $\epsilon_2 > 0$. We have the following cases:*

(1) *If $r = |\frac{s_2}{s_1}| < 1$ (positively oriented triangles), choose a $\widetilde{\theta} \in [m_1, m_2] \subset (0, \pi)$. Then there is a member of the FDRS $A_k B_k C_k$ such that:*

$$|\widehat{A}_k - \widetilde{\theta}| < \epsilon_1,$$

and

$$either \quad \left| \frac{b_k}{c_k} - \mu(\varphi_a(\widetilde{\theta}), r) \right| < \epsilon_2 \quad or \quad \left| \frac{b_k}{c_k} - \mu^{-1}(\varphi_a(\widetilde{\theta}), r) \right| < \epsilon_2.$$

(2) *If $r = |\frac{s_2}{s_1}| > 1$ (negatively oriented triangles), choose a $\widetilde{\theta} \in [\overline{m}_1, \overline{m}_2] \subset (-\pi, 0)$. Then there is a member of the FDRS $A_k B_k C_k$ such that:*

$$|\widehat{A}_k - \widetilde{\theta}| < \epsilon_1,$$

and

$$either \quad \left|\frac{b_k}{c_k} - \mu(\varphi_a(\widetilde{\theta}), 1/r)\right| < \epsilon_2 \quad or \quad \left|\frac{b_k}{c_k} - \mu^{-1}(\varphi_a(\widetilde{\theta}), 1/r)\right| < \epsilon_2.$$

Proof: Let $r < 1$, we have seen that there is φ_k which is arbitrarily close to any given angle on the unit circle. Since function $\Phi(\varphi_n)$ is continuous with respect to φ_n, it is apparent that $\widehat{A}_k = \Phi(\varphi_k)$ can be arbitrarily close to a $\widetilde{\theta}$ chosen from the interval $[m_1, m_2]$ (the range of $\Phi(\varphi_n)$). This proves that $|\widehat{A}_k - \widetilde{\theta}| < \epsilon_1$. Since $\widehat{A}_k = \Phi(\varphi_k)$ can be arbitrarily close to $\widetilde{\theta}$, from Remark we conclude that φ_k will be arbitrarily close to an element of the form $2k\pi \pm \varphi_a(\tilde{\theta})$ (see Figure 4). Since we have considered that φ_k can be defined on the unit circle, we have that φ_k will be arbitrarily close either to $\varphi_a(\tilde{\theta})$ or to $2\pi - \varphi_a(\tilde{\theta})$ which are both defined in $[0, 2\pi]$. Observe that function $\mu(\varphi_n, r)$ is continuous with respect to φ_n and so from equation (8) we get that the ratio $\frac{b_k}{c_k} = \mu(\varphi_k, r)$ will be arbitrarily close either to $\mu(\varphi_a(\widetilde{\theta}), r)$ or to $\mu(2\pi - \varphi_a(\widetilde{\theta}), r) = \mu(-\varphi_a(\widetilde{\theta}), r) = \mu^{-1}(\varphi_a(\widetilde{\theta}), r)$ (recall the properties of function μ). This proves that either $\left|\frac{b_k}{c_k} - \mu(\varphi_a(\widetilde{\theta}), r)\right| < \epsilon_2$ or $\left|\frac{b_k}{c_k} - \mu^{-1}(\varphi_a(\widetilde{\theta}), r)\right| < \epsilon_2$. The case $r > 1$ can be treated analogously. This completes the proof. $\Box$

Concerning Theorem 2 we stress that ϵ_1 and ϵ_2 can be chosen independently. This is true since from the Kronecker's Approximation Theorem we can always find a φ_k as close as we want to a given $\widetilde{\theta}$. This implies that the angle $\widehat{A}_k$ can be as close as we want to $\widetilde{\theta}$, and so the ratio $\frac{b_k}{c_k}$ will be as close as we want either to $\mu(\varphi_a(\widetilde{\theta}), r)$ or to $\mu^{-1}(\varphi_a(\widetilde{\theta}), r)$. Ultimately, a φ_k will satisfy both inequalities no matter how small ϵ_1 and ϵ_2 are.

Although Theorem 2 and the analysis so far seem quite complicated, they have some interesting consequences. In what follows we consider that t is fixed and a is an irrational number as in Theorem 2.

We proved that there will be a member of the FDRS with an angle $\widehat{A}_k$ that will be arbitrarily close to any given $\widetilde{\theta} \in [m_1, m_2]$ or $\widetilde{\theta} \in [\overline{m}_1, \overline{m}_2]$. This means that the countable set of the angles $\widehat{A}_n$ (*i.e.*, $\{\widehat{A}_0, \widehat{A}_1, \dots\}$) is dense in $[m_1, m_2]$ or in $[\overline{m}_1, \overline{m}_2]$. Also by choosing ϵ_1, ϵ_2 as small as we want, we expect that some members $A_kB_kC_k$ of the FDRS will have their shapes as follows: $\widehat{A}_k \simeq \widetilde{\theta}$ and either $\frac{b_k}{c_k} \simeq \mu(\varphi_a(\widetilde{\theta}), r)$ or $\frac{b_k}{c_k} \simeq \mu^{-1}(\varphi_a(\widetilde{\theta}), r)$.

Let us now find if there is a member of the FDRS that is arbitrarily close to an equilateral triangle. If this was true then $\frac{b_k}{c_k}$ should be arbitrarily close to the unity. Thus from Theorem 2 (assume that $r < 1$ since for $r > 1$ the same argument applies), $\mu(\varphi_a(\widetilde{\theta}), r) = 1$ and from Section 2 we know that $\varphi_a(\widetilde{\theta}) = 0$ or $\varphi_a(\widetilde{\theta}) = \pi$. From these equalities we get $\widetilde{\theta} = m_1$ or $\widetilde{\theta} = m_2$. It should also hold that $\widetilde{\theta} = \pi/3$ (positively oriented equilateral triangle). So, it should be $m_1 = \pi/3 \Rightarrow r = 0$ or $m_2 = \pi/3 \Rightarrow r = 0$. Obviously, $r = 0$ is impossible. Consequently, for a specific $r > 0$ all the members of the FDRS will have at least a constant discrepancy from the shape of an equilateral triangle. This discrepancy can not be further decreased for a fixed $r > 0$, it can only be reduced if we chose another $r > 0$ closer to zero.

Let an isosceles triangle with $b = c$ and $\widehat{A} = \widetilde{\theta} < \pi/3$ be given. We want to find the value of $r < 1$ that will give a member of the FDRS arbitrarily close to the isosceles triangle. In the previous paragraph we show that for this case it holds $\widetilde{\theta} = m_1$ or $\widetilde{\theta} = m_2$. Let $\widetilde{\theta} = m_1$ and we have

$$\widetilde{\theta} = m_1 \Longleftrightarrow 2\arctan\frac{r\sqrt{3}}{2+r} = \frac{\pi}{3} - \widetilde{\theta} \Longleftrightarrow r = \frac{2\tan(\frac{\pi}{6} - \frac{\widetilde{\theta}}{2})}{\sqrt{3} - \tan(\frac{\pi}{6} - \frac{\widetilde{\theta}}{2})}.$$

The above formula gives the value of r for which a member of the FDRS would be arbitrarily close to the isosceles triangle with $\widehat{A} = \widetilde{\theta} < \pi/3$. The corresponding formula for an isosceles triangle with $b = c$ and a given $\widehat{A} = \widetilde{\theta} > \pi/3$ is

$$\widetilde{\theta} = m_2 \Longleftrightarrow 2\arctan\frac{r\sqrt{3}}{2-r} = \widetilde{\theta} - \frac{\pi}{3} \Longleftrightarrow r = \frac{2\tan(\frac{\widetilde{\theta}}{2} - \frac{\pi}{6})}{\sqrt{3} + \tan(\frac{\widetilde{\theta}}{2} - \frac{\pi}{6})}.$$

In the next Section we offer a simple geometric presentation of the FDRS, we examine closer the significance of the parameters r and φ_n and we answer a question posed in [3].

4. Geometric interpretations and final remarks

We have seen that equation (3) is the solution of the recurrence (1) provided that $A_0 + B_0 + C_0 = 0$. We can rewrite (3) as follows:

$$V_n = \frac{s_1\lambda_1^n}{\sqrt{3}}\left[\begin{pmatrix}1\\ \omega\\ \omega^2\end{pmatrix} + \frac{s_2}{s_1}\left(\frac{\lambda_2}{\lambda_1}\right)^n\begin{pmatrix}1\\ \omega^2\\ \omega\end{pmatrix}\right].$$

In this article we are interested in the shapes of the triangles. The complex number $\frac{s_1\lambda_1^n}{\sqrt{3}}$ at the above equation signifies a scaling factor and a rotation of the triangle V_n, and so it does not affect its shape. This means that we can define the shapes of the triangles of the FDRS simply as

$$S_n = P + re^{i\varphi_n}N, \tag{11}$$

where $P = \begin{pmatrix}1\\ \omega\\ \omega^2\end{pmatrix}$, $N = \begin{pmatrix}1\\ \omega^2\\ \omega\end{pmatrix}$ and r, φ_n as in Section 2. We stress that the triangles V_n and S_n have the same shape (*i.e.*, they are similar and they have the same orientation). Note also that P is a positively oriented equilateral triangle inscribed in the unit circle and N is a negatively oriented equilateral triangle inscribed in the unit circle (1, ω, ω^2 are the third roots of unity). It can be seen now that every member of the FDRS on the complex plane is represented as the sum of two equilateral triangles: P and $re^{i\varphi_n}N$. It is now obvious that the parameter r is the circumradius and the parameter φ_n is the angle of rotation of the equilateral triangle rN at the nth iteration. Thus, the parameters r and φ_n determine completely the contribution of the negatively oriented triangle in (11).

Let us next consider an open problem that is posed in [3]. The authors of [3] asked to find all values of the division ratio $t \in (0,1)$ for which the FDRS is divergent in shape. From the analysis so far, we have seen that the division ratio t can be given by equation (2). Equation (2) defines a function $t = t(a)$ which is one-to-one and for $a \in (-\frac{1}{3}, \frac{1}{3})$ its range is $(0,1)$. Thus, we can describe the behavior of the members of the FDRS with respect to t, by using equation (2). Similar to the analysis of Section 2 we have the following cases:

(1) $a = 0$. Equation (2) implies $t = \frac{1}{2}$. In this case all the members of the FDRS are similar to $A_0B_0C_0$ and the sequence is convergent in shape.
(2) $a \neq 0$ is a rational number in $(-\frac{1}{3}, \frac{1}{3})$ and t is given by (2). The FDRS is periodic in shape.
(3) a is an irrational number in $(-\frac{1}{3}, \frac{1}{3})$ and t is given by (2). From the analysis of Section 3 we conclude that the FDRS is neither convergent nor periodic in shape.

Thus, only when $t = \frac{1}{2}$ we have that the FDRS is convergent in shape. The second case above gives the values of t for which the FDRS is periodic in shape. The last case is described by Theorem 2 and the behavior of the FDRS is rather complex since it is neither convergent nor periodic in shape.

It is clear that only the change of an a rational to an a irrational in (2) is enough to produce a complicated dynamic behavior of the FDRS. We believe that only results of qualitative character like Theorem 2 can be used to describe this sequence of triangles. However, it would be interesting if one could prove another result (e.g. a statistical result), for the behavior of the FDRS when a is an irrational number.

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A Short Trigonometric Proof of the Steiner-Lehmus Theorem

Mowaffaq Hajja

Abstract. We give a short trigonometric proof of the Steiner-Lehmus theorem.

The well known Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. Unlike its trivial converse, this challenging statement has attracted a lot of attention since 1840, when Professor Lehmus of Berlin wrote to Sturm asking for a purely geometrical proof. Proofs by Rougevain, Steiner, and Lehmus himself appeared in the following few years. Since then, a great number of people, including several renowned mathematicians, took interest in the problem, resulting in as many as 80 different proofs. Extensive histories are given in [14], [15], [16], and [21], and biographies and lists of references can be found in [33], [37], and [19]. More references will be referred to later when we discuss generalizations and variations of the theorem.

In this note, we present a new trigonometric proof of the theorem. Compared with the existing proofs, such as the one given in [17, pp. 194–196], it is also short and simple. It runs as follows.

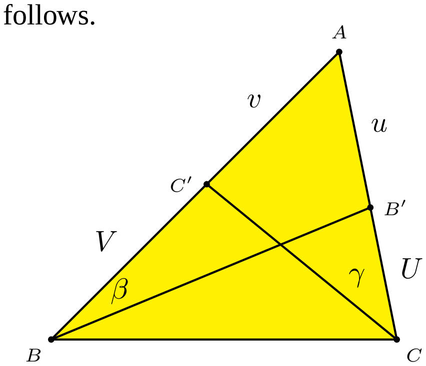

Figure 1

Let BB' and CC' be the respective internal angle bisectors of angles B and C in triangle ABC, and let a, b and c denote the sidelengths in the standard order. As shown in Figure 1, we set

$$B = 2\beta,\ C = 2\gamma, \quad u = AB',\ U = B'C, \quad v = AC',\ V = C'B.$$

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The author would like to thank the referee for suggestions that improved the exposition and for drawing his attention to references [4], [19], [33], and [37].

We shall see that the assumptions $BB' = CC'$ and $C > B$ (and hence $c > b$) lead to the contradiction that

$$\frac{b}{u} < \frac{c}{v}, \ \frac{b}{u} > \frac{c}{v}. \tag{1}$$

Geometrically, this means that the line $B'C'$ intersects both rays BC and CB.

To achieve (1), we use the law of sines, the angle bisector theorem, and the identity $\sin 2\theta = 2\sin\theta\cos\theta$ to obtain

$$\frac{b}{u} - \frac{c}{v} = \frac{u+U}{u} - \frac{v+V}{v} = \frac{U}{u} - \frac{V}{v} = \frac{a}{c} - \frac{a}{b} < 0, \tag{2}$$

$$\begin{aligned}\frac{b}{u} \div \frac{c}{v} &= \frac{b}{c}\,\frac{v}{u} = \frac{\sin B}{\sin C}\,\frac{v}{u} = \frac{2\cos\beta\sin\beta}{2\cos\gamma\sin\gamma}\,\frac{v}{u} = \frac{\cos\beta}{\cos\gamma}\,\frac{\sin\beta}{u}\,\frac{v}{\sin\gamma} \\ &= \frac{\cos\beta}{\cos\gamma}\,\frac{\sin A}{BB'}\,\frac{CC'}{\sin A} = \frac{\cos\beta}{\cos\gamma} > 1. \end{aligned}\tag{3}$$

Clearly (2) and (3) lead to the contradiction (1).

No new proofs of the Steiner-Lehmus theorem seem to have appeared in the past several decades, and attention has been focused on generalizations, variations, and certain foundational issues. Instead of taking angle bisectors, one may take r-sectors, i.e., cevians that divide the angles internally in the ratio $r : 1 - r$ for $r \in (0, 1)$. Then the result still holds; see [35], [15, X, p. 311], [36], and more recently, [5], [2], and [10]. In fact, the result still holds in absolute (or neutral) geometry; see [15, X, p. 311] and the references therein, and more recently [6, Exercise 7, p. 9; solution, p. 420] and [19, Exercise 15, p. 119]. One may also consider external angle bisectors. Then one sees that the equality of two external angle bisectors (and similarly the equality of one internal and one external angle bisectors) does not imply isoscelessness. This is considered in [16], [22], [23], and more recently in [11]; see also [30] and the references therein. The situation in spherical geometry was also considered by Steiner; see [16, IX, p. 310].

Variations on the Steiner-Lehmus theme have become popular in the past few decades with much of the contribution due to the late C. F. Parry. Here, one starts with a center P of triangle ABC, not necessarily the incenter, and lets the cevians AA', BB', CC' through P intersect the circumcircle of ABC at A^*, B^*, C^*, respectively. The classical Steiner-Lehmus theorem deals with the case when P is the incenter and considers the assumption $BB' = CC'$. One may start with any center and consider any of the assumptions $BB' = CC'$, $BB^* = CC^*$, $A'B' = A'C'$, $A^*B^* = A^*C^*$, etc. Such variations and others have appeared in [27], [28], [29], [34], [3], [12], [32], [31], [1], and [26, Problem 4, p. 31], and are surveyed in [13]. Some of these variations have been investigated in higher dimensions in [7] and interesting results were obtained. However, the generalization of the classical Steiner-Lehmus theorem to higher dimensions remains open: We still do not know what degree of regularity a d-simplex must enjoy so that two or even all the internal angle bisectors of the corner angles are equal. This problem is raised at the end of [7].

The existing proofs of the Steiner-Lehmus theorem are all indirect (many being proofs by contradiction or *reductio ad absurdum*) or use theorems that do not have direct proofs. The question, first posed by Sylvester in [36], whether there is a direct proof of the Steiner-Lehmus theorem is still open, and Sylvester's conjecture (and semi-proof) that no such proof exists seems to be commonly accepted; see the refutation made in [20] of the allegedly direct proof given in [24], and compare to [8], where we are asked on p. 58 (Problem 16) to *give a direct proof of the Steiner-Lehmus theorem*, and where such a *a proof* is given on p. 390 using Stewart's theorem. An interesting forum discussion can also be visited at [9]. We would like here to raise the question whether one can provide a direct proof of the following weaker version of the Steiner-Lehmus theorem: *If the three internal angle bisectors of the angles of a triangle are equal, then the triangle is equilateral.*

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On the Parry Reflection Point

Cosmin Pohoata

Abstract. We give a synthetic proof of C. F. Parry's theorem that the reflections in the sidelines of a triangle of three parallel lines through the vertices are concurrent if and only if they are parallel to the Euler line, the point of concurrency being the Parry reflection point. We also show that the Parry reflection point is common to a triad of circles associated with the tangential triangle and the triangle of reflections (of the vertices in their opposite sides). A dual result is also given.

1. The Parry reflection point

Theorem 1 (Parry). *Suppose triangle ABC has circumcenter O and orthocenter H. Parallel lines α, β, γ are drawn through the vertices A, B, C, respectively. Let α', β', γ' be the reflections of α, β, γ in the sides BC, CA, AB, respectively. These reflections are concurrent if and only if α, β, γ are parallel to the Euler line OH. In this case, their point of concurrency P is the reflection of O in E, the Euler reflection point.*

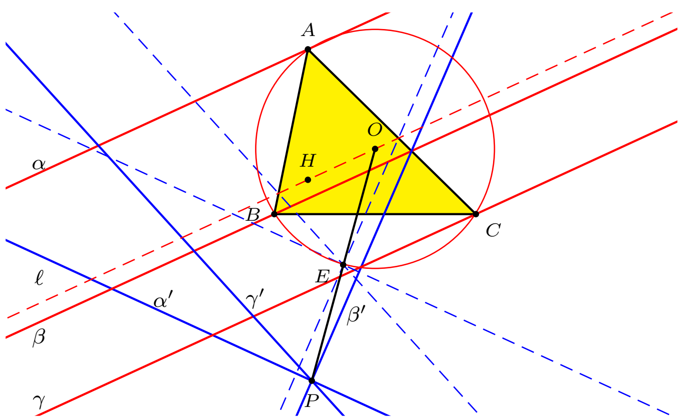

Figure 1.

We give a synthetic proof of this beautiful theorem below. C. F. Parry proposed this as a problem in the AMERICAN MATHEMATICAL MONTHLY, which was subsequently solved by R. L. Young using complex coordinates [6]. The point P in question is called the Parry reflection point. It appears as the triangle center X_{399}

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in [5]. The Euler reflection point E, on the other hand, is the point on the circumcircle which is the point of concurrency of the reflections of the Euler line in the sidelines. See Figure 1. It appears as X_{110} in [5]. The existence of E is justified by another elegant result on reflections of lines, which we use to deduce Theorem 1.

Theorem 2 (Collings)**.** *Let ℓ be a line in the plane of a triangle ABC. Its reflections in the sidelines BC, CA, AB are concurrent if and only if ℓ passes through the orthocenter H of ABC. In this case, their point of concurrency lies on the circumcircle.*

Synthetic proofs of Theorem 2 can be found in [1] and [3].

We denote by A', B', C' the reflections of A, B, C in their opposite sides, and by $A_tB_tC_t$ the tangential triangle of ABC.

Theorem 3. *The circumcircles of triangles $A_tB'C'$, $B_tC'A'$ and $C_tA'B'$ are concurrent at Parry's reflection point P.* See Figure 2.

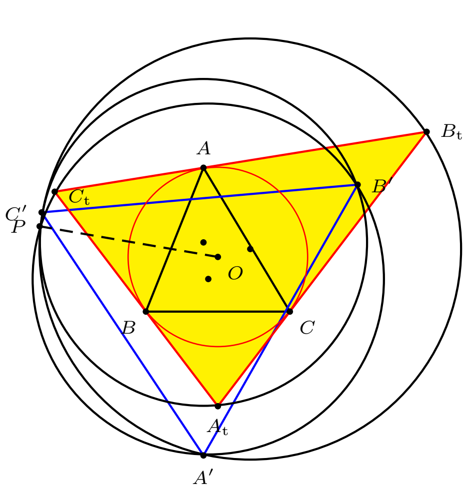

Figure 2

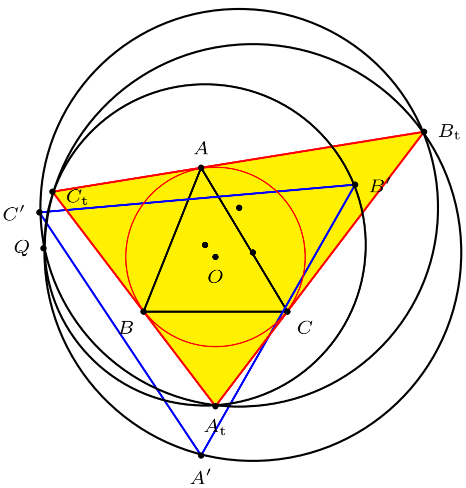

Figure 3

Theorem 4. *The circumcircles of triangles $A'B_tC_t$, $B'C_tA_t$ and $C'A_tB_t$ have a common point Q.* See Figure 3.

2. Proof of Theorem 1

Let $A_1B_1C_1$ be the image of ABC under the homothety $\mathsf{h}(O, 2)$. The orthocenter H_1 of $A_1B_1C_1$ is the reflection of O in H, and is on the Euler line of triangle ABC.

Consider the line ℓ through H parallel to the given lines α, β, γ. Let M be the midpoint of BC, and $M_1 = \mathsf{h}(O, 2)(M)$ on the line B_1C_1. The line AH intersects

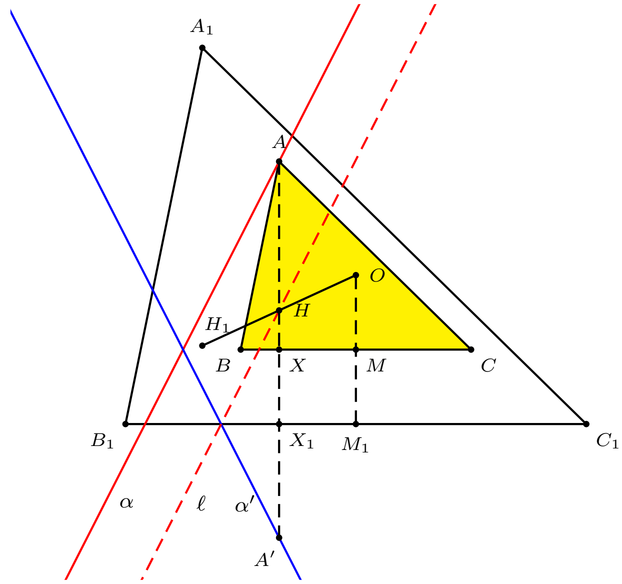

Figure 4.

BC and B_1C_1 at X and X_1 respectively. Note that the reflection of H in B_1C_1 is the reflection D of A in BC since $AH = 2 \cdot OM$ and

$$\begin{aligned} HA' =& AA' - AH = 2(AX - OM) = 2(AH + HX - OM) \\ =& 2(HX + OM) = 2(HX + XX_1) = 2HX_1. \end{aligned}$$

Therefore, α' coincides with the reflection of ℓ in the sides B_1C_1. Similarly, β' and γ' coincide with the reflections of ℓ in C_1A_1 and A_1B_1. By Theorem 2, the lines α', β', γ' are concurrent if and only if ℓ passes through the orthocenter H_1. Since H also lies on ℓ, this is the case when ℓ is the Euler line of triangle ABC, which is also the Euler line of triangle $A_1B_1C_1$. In this case, the point of concurrency is the Euler reflection point of $A_1B_1C_1$, which is the image of E under the homothety $\mathsf{h}(O, 2)$.

3. Proof of Theorem 3

We shall make use of the notion of directed angles $(\ell_1, \ \ell_2)$ between two lines ℓ_1 and ℓ_2 as the angle of rotation (defined modulo π) that will bring ℓ_1 to ℓ_2 in the same orientation as ABC. For the basic properties of directed angles, see [4, §§16–19].

Let α, β, γ be lines through the vertices A, B, C, respectively parallel to the Euler line. By Theorem 1, their reflections α', β', γ' in the sides BC, CA, AB pass through the Parry reflection point P.

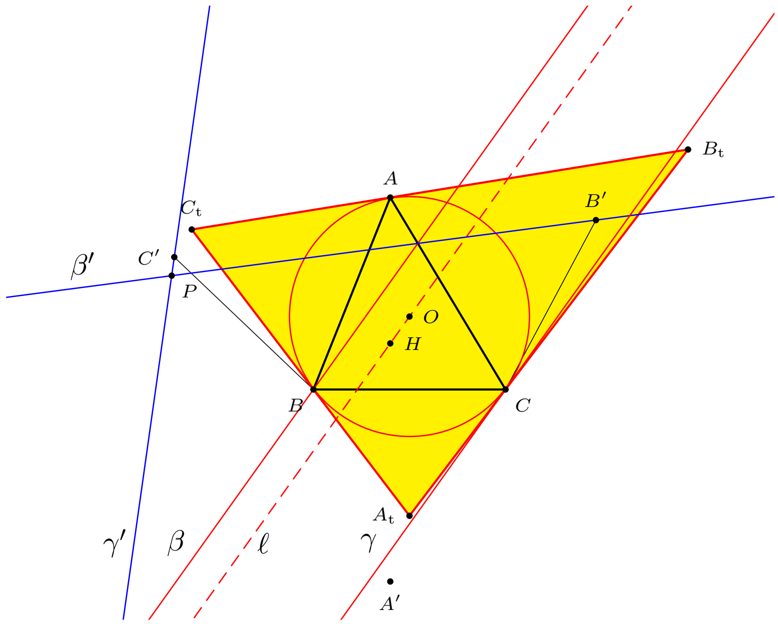

Figure 5

Now, since α, β, γ are parallel,

$$\begin{aligned}
(PB',\ PC') &= (\beta', \gamma') \\
&= (\beta', BC) + (BC, \gamma') \\
&= -(\beta', B'C) - (BC', \gamma') \quad \text{because of symmetry in } AC \\
&= (B'C,\ \beta) + (\gamma,\ BC') \\
&= (B'C,\ \beta) + (\beta,\ BC') \\
&= (B'C,\ BC') \\
&= (B'C,\ AC) + (AC,\ BC') \\
&= (AC,\ BC) + (AC,\ BC') \quad \text{because of symmetry in } AC \\
&= (AC,\ AB) + (AB,\ BC) + (AC,\ AB) + (AB,\ BC') \\
&= 2(AC,\ AB) \quad \text{because of symmetry in } AB \\
&= (OC,\ OB) \\
&= (A_tC,\ A_tB).
\end{aligned}$$

Since $A_tB = A_tC$ and $BC' = BC = B'C$, we conclude that the triangles A_tBC' and A_tCB' are directly congruent. Hence, $(A_tB',\ A_tC') = (A_tC,\ A_tB)$. This gives $(PB',\ PC') = (A_tB',\ A_tC')$, and the points P, A_t, B', C' are concyclic. The circle $A_tB'C'$ contains the Parry reflection point, so do the circles $B_tC'A'$ and $C_tA'B'$.

4. Proof of Theorem 4

Invert with respect to the Parry point P. By Theorem 3, the circles $A_tB'C'$, $B_tC'A'$, $C_tA'B'$ are inverted into the three lines bounding triangle $A'^*B'^*C'^*$. Here, A'^*, B'^*, C'^* are the inversive images of A', B', C' respectively. Since the points A_t^*, B_t^*, C_t^* lie on the lines $B'^*C'^*$, $C'^*A'^*$, $A'^*B'^*$, respectively, by Miquel's theorem, the circumcircles of triangles $A_t^*B'^*C'^*$, $B_t^*C'^*A'^*$, $C_t^*A'^*B'^*$ have a common point; so do their inversive images, the circles $A_tB'C'$, $B_tC'A'$, $C_tA'B'$. This completes the proof of Theorem 4.

The homogenous barycentric coordinates of their point of concurrency Q were computed by Javier Francisco Garcia Capitán [2] with the aid of Mathematica.

Added in proof. After the completion of this paper, we have found that the points P and Q are concyclic with the circumcenter O and the orthocenter H. See Figure 6. Paul Yiu has confirmed this by computing the coordinates of the center of the circle of these four points:

$$\begin{aligned}(&a^2(b^2-c^2)(a^8(b^2+c^2)-a^6(4b^4+3b^2c^2+4c^4)+2a^4(b^2+c^2)(3b^4-2b^2c^2+3c^4)\\&\quad -a^2(4b^8-b^6c^2+b^4c^4-b^2c^6+4c^8)+(b^2-c^2)^2(b^2+c^2)(b^4+c^4)))\\&:\cdots:\cdots),\end{aligned}$$

where the second and third coordinates are obtained by cyclic permutations of a, b, c.

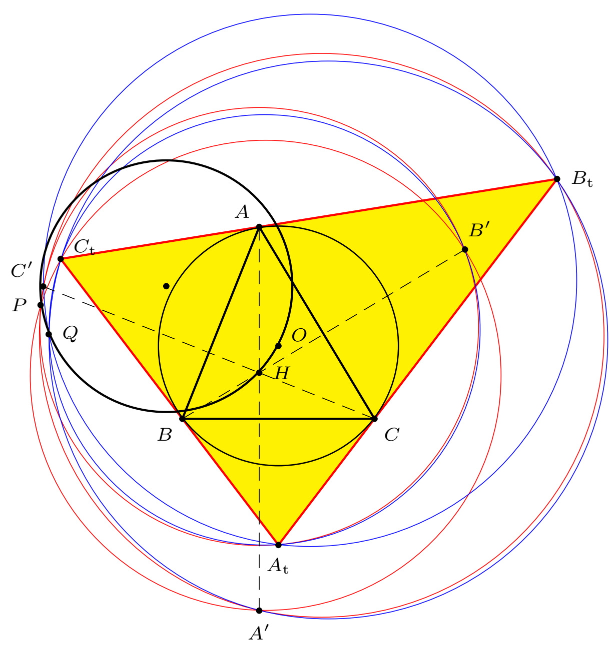

Figure 6.

For completeness, we record the coordinates of Q given by Garcia Capitán:

$$Q = (a^2 \sum_{k=0}^{10} a^{2(10-k)} f_{2k,a}(b,c) : b^2 \sum_{k=0}^{10} b^{2(10-k)} f_{2k,b}(c,a) : c^2 \sum_{k=0}^{10} c^{2(10-k)} f_{2k,c}(a,b)),$$

where

$$\begin{aligned}
f_{0,a}(b,c) &= 1,\\
f_{2,a}(b,c) &= -6(b^2+c^2)\\
f_{4,a}(b,c) &= 2(7b^4+12b^2c^2+7c^4),\\
f_{6,a}(b,c) &= -2(b^2+c^2)(7b^4+10b^2c^2+7c^4),\\
f_{8,a}(b,c) &= b^2c^2(18b^4+25b^2c^2+18c^4),\\
f_{10,a}(b,c) &= (b^2+c^2)(14b^8-15b^6c^2+8b^4c^4-15b^2c^6+14c^8),\\
f_{12,a}(b,c) &= -14b^{12}+b^{10}c^2+5b^8c^4-2b^6c^6+5b^4c^8+b^2c^{10}-14c^{12},\\
f_{14,a}(b,c) &= (b^2-c^2)^2(b^2+c^2)(6b^8+2b^6c^2+5b^4c^4+2b^2c^6+6c^8),\\
f_{16,a}(b,c) &= -(b^2-c^2)^2(b+c)^2(b^{12}-2b^{10}c^2-b^8c^4-6b^6c^6-b^4c^8-2b^2c^{10}+c^{12}),\\
f_{18,a}(b,c) &= -b^2c^2(b^2-c^2)^4(b^2+c^2)(3b^4+b^2c^2+3c^4),\\
f_{20,a}(b,c) &= b^2c^2(b^2-c^2)^6(b^2+c^2)^2.
\end{aligned}$$

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Construction of Malfatti Squares

Floor van Lamoen and Paul Yiu

Abstract. We give a very simple construction of the Malfatti squares of a triangle, and study the condition when all three Malfatti squares are inside the given triangle. We also give an extension to the case of rectangles.

1. Introduction

The Malfatti squares of a triangle are the three squares each with two adjacent vertices on two sides of the triangle and the two remaining adjacent vertices from those of a triangle in its interior. We borrow this terminology from [3] (see also [1, p.48]) where the lengths of the sides of the Malfatti squares are stated. In Figure 1, the Malfatti squares of triangle ABC are $B'C'Z_aY_a$, $C'A'X_bZ_b$ and $A'B'Y_cX_c$. We shall call $A'B'C'$ the Malfatti triangle of ABC, and present a simple construction of $A'B'C'$ from a few common triangle centers of ABC. Specifically, we shall make use of the isogonal conjugate of the Vecten point of ABC. [1] This is a point on the Brocard axis, the line joining the circumcenter O and the symmedian point K.

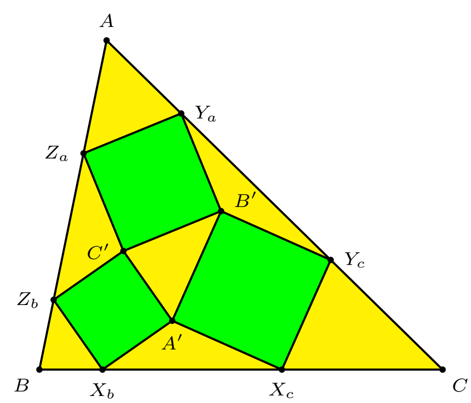

Figure 1.

Theorem 1. *Let P be the isogonal conjugate of the Vecten point of triangle ABC. The vertices of the Malfatti triangle are the intersections of the lines joining the centroid G to the pedals of the symmedian point K and the corresponding vertices to the pedals of P on the opposite sides of ABC.* See Figure 2.

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[1]The Vecten point of a triangle is the perspector of the (triangle whose vertices are) the centers of the squares erected externally on the sides. It appears as X_{485} in [6]. See Figure 6. Its isogonal conjugate appears as X_{371}, and is also called the Kenmotu point. It is associated with the construction of a triad of congruent squares in a triangle. In [5, p.268] the expression for the edge length of the squares should be reduced by a factor $\sqrt{2}$. A correct expression appears in [6] and [2, p.94].

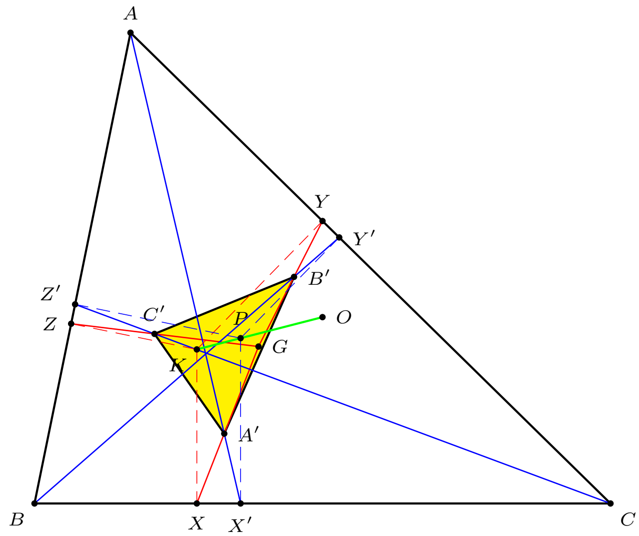

Figure 2.

2. Notations

We adopt the following notations. For a triangle of sidelengths a, b, c, let S denote *twice* the area of the triangle, and

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

These satisfy

$$S_BS_C + S_CS_A + S_AS_B = S^2.$$

More generally, for an arbitrary angle θ, $S_\theta = S \cdot \cot\theta$. In particular,

$$S_A + S_B + S_C = \frac{a^2+b^2+c^2}{2} = S_\omega,$$

where ω is the Brocard angle of triangle ABC.

3. The triangle of medians

Given a triangle ABC with sidelengths a, b, c, let m_a, m_b, m_c denote the lengths of the medians. By the Apollonius theorem, these are given by

$$\begin{aligned} m_a^2 =& \frac{1}{4}(2b^2+2c^2-a^2), \\ m_b^2 =& \frac{1}{4}(2c^2+2a^2-b^2), \\ m_c^2 =& \frac{1}{4}(2a^2+2b^2-c^2). \end{aligned} \tag{1}$$

There is a triangle whose sidelengths are m_a, m_b, m_c. See Figure 3A. We call this the triangle of medians of ABC. The following useful lemma can be easily established.

Lemma 2. *Two applications of the triangle of medians construction gives a similar triangle of similarity factor* $\frac{3}{4}$. See Figure 3B.

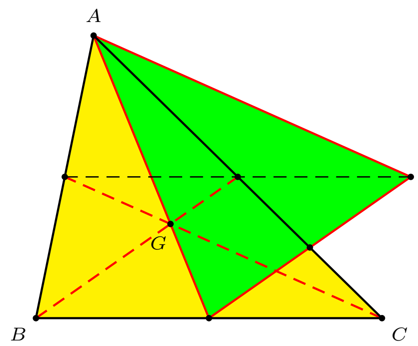

Figure 3A

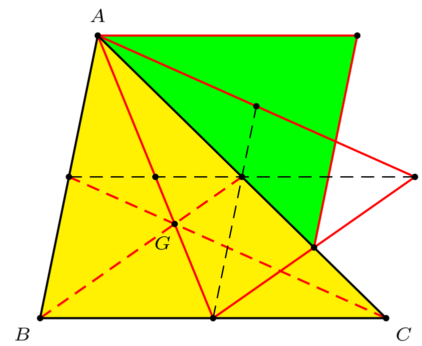

Figure 3B

We present an interesting example of a triangle similar to the triangle of medians which is useful for the construction of the Malfatti triangle.

Lemma 3. *The pedal triangle of the symmedian point is similar to the triangle of medians, the similarity factor being* $\tan\omega$.

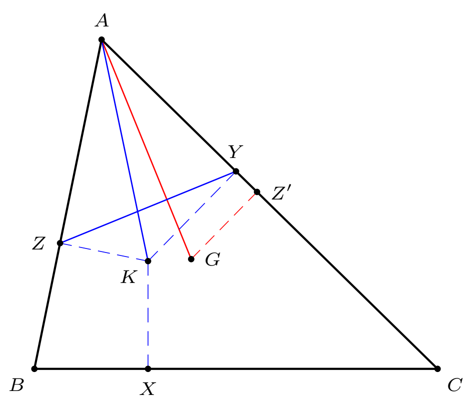

Figure 4

Proof. Since $S = bc\sin A$, the distance from the centroid G to AC is clearly $\frac{S}{3b}$. That from the symmedian point K to AB is

$$\frac{c^2}{a^2+b^2+c^2}\cdot\frac{S}{c}=\frac{S}{2S_\omega}\cdot c.$$

Since K and G are isogonal conjugates,

$$AK = AG\cdot\frac{\frac{S}{2S_\omega}\cdot c}{\frac{S}{3b}}=\frac{2}{3}m_a\cdot\frac{3bc}{2S_\omega}=\frac{bc}{S_\omega}\cdot m_a.$$

This is a diameter of the circle through A, K, and its pedals on AB and AC. It follows that the distance between the two pedals is

$$\frac{bc}{S_\omega} \cdot m_a \cdot \sin A = \frac{S}{S_\omega} \cdot m_a = \tan\omega \cdot m_a.$$

From this, it is clear that the pedal triangle is similar to the triangle of medians, the similarity factor being $\tan\omega$. □

Remark. The triangle of medians of ABC has the same Brocard angle as ABC.

Proposition 4. *Let P be a point with pedal triangle XYZ in ABC. The lines through A, B, C perpendicular to the sides YZ, ZX, XY concur at the isogonal conjugate of P.*

We shall also make use of the following characterization of the symmedian (Lemoine) point of a triangle.

Theorem 5 (Lemoine)**.** *The symmedian point is the unique point which is the centroid of its own pedal triangle.*

4. Proof of Theorem 1

Consider a triangle ABC with its Malfatti squares. Complete the parallelogram $A'B'A^*C'$. See Figure 5. Note that triangles $A'X_bX_c$ and $C'A'A^*$ are congruent. Therefore, $A'A^*$ is perpendicular to BC. Note that this line contains the centroid G' of triangle $A'B'C'$. Similarly, if we complete parallelograms $B'C'B^*A'$ and $C'A'C^*B'$, the lines $B'B^*$ and $C'C^*$ contain G' and are perpendicular to CA and AB respectively.

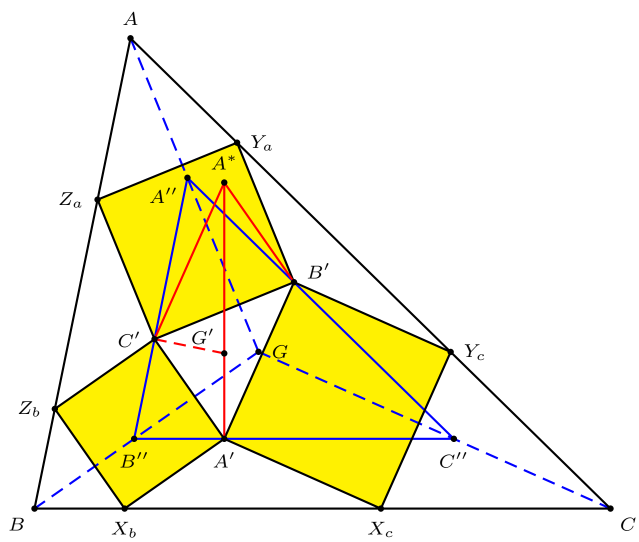

Figure 5.

Consider $A'B'C'$ as the pedal triangle of G' in a triangle $A''B''C''$ homothetic to ABC. By Lemoine's theorem, G' is the symmedian point of $A''B''C''$. Since

$A''B''C''$ is homothetic to ABC, $A'B'C'$ is homothetic to the pedal triangle of the symmedian point K of ABC.

In Figure 5, triangle $A''B'C'$ is the image of AY_aZ_a under the translation by the vector $\mathbf{Y_aB'} = \mathbf{Z_aC'} = \mathbf{AA''}$. This means that the line AA'' is perpendicular to $B'C'$, and to the A-side of the pedal triangle of K. Similarly, BB'' and CC'' are perpendicular to B- and C-sides of the same pedal triangle. By Proposition 4, the lines AA'', BB'', CC'' concur at the isogonal conjugate of K. This means that triangles $A''B''C''$ and ABC are homothetic at the centroid G of triangle ABC, and the sides of the Malfatti squares are parallel and perpendicular to the corresponding medians.

Denote by λ the homothetic ratio of $A''B''C''$ and ABC . This is also the homothetic ratio of the Malfatti triangle $A'B'C'$ and the pedal triangle of K. In Figure 5, $BX_b + X_cC = B''C'' = \lambda a$. Also, by Lemmas 2 and 3,

$$\begin{aligned} X_bX_c = A'A^* = {} & 2\lambda \cdot A\text{−median of pedal triangle of } K \\ & = 2\lambda \cdot \tan\omega \cdot A\text{−median of triangle of medians of } ABC \\ & = 2\lambda \cdot \tan\omega \cdot \frac{3}{4}a = \frac{3}{2}\lambda \cdot \tan\omega \cdot a. \end{aligned}$$

Since $BX_b + X_bX_c + X_cC = BX$, we have $\lambda(1 + \frac{3}{2}\tan\omega) = 1$ and

$$\lambda = \frac{2}{2 + 3\tan\omega} = \frac{2S_\omega}{3S + 2S_\omega}.$$

Let $\mathsf{h}(G, \lambda)$ be the homothety with center G and ratio λ. Since G' is the symmedian point of $A''B''C''$,

$$G' = \mathsf{h}(G, \lambda) = \lambda K + (1 - \lambda)G = \frac{1}{3S + 2S_\omega}(3S \cdot G + 2S_\omega \cdot K).$$

It has homogeneous barycentric coordinates $(a^2 + S : b^2 + S : c^2 + S)$. [2]

To compute the coordinates of the vertices of the Malfatti triangle, we make use of the pedals of the symmedian point K on the sidelines. The pedal on BC is the point

$$X = \frac{1}{2S_\omega}((S_A + 2S_C)B + (S_A + 2S_B)C).$$

A' is the point dividing the segment GX in the ratio $GA' : A'X = S_\omega : 3S$.

$$\begin{aligned} A' = & \frac{1}{3S + 2S_\omega}(3S \cdot G + 2S_\omega \cdot X) \\ = & \frac{1}{3S + 2S_\omega}\left(S \cdot A + (S + S_A + 2S_C)B + (S + S_A + 2S_B)C\right). \end{aligned}$$

[2]For a construction of G', see Proposition 6.

Similarly, we have B' and C'. In homogeneous barycentric coordinates, these are

$$\begin{aligned}A' =&(S : S + S_A + 2S_C : S + S_A + 2S_B),\\ B' =&(S + S_B + 2S_C : S : S + S_B + 2S_A),\\ C' =&(S + S_C + 2S_B : S + S_C + 2S_A : S).\end{aligned}$$

The lines AA', BB', CC' intersect the sidelines BC, CA, AB respectively at the points

$$\begin{aligned}X' =&(0 : S + S_A + 2S_C : S + S_A + 2S_B),\\ Y' =&(S + S_B + 2S_C : 0 : S + S_B + 2S_A),\\ Z' =&(S + S_C + 2S_B : S + S_C + 2S_A : 0).\end{aligned} \tag{2}$$

We show that these three intersections are the pedals of a specific point

$$P = (a^2(S_A + S) : b^2(S_B + S) : c^2(S_C + S)).$$

In absolute barycentric coordinates,

$$P = \frac{1}{2S(S + S_\omega)} \left((a^2(S_A + S)A + b^2(S_B + S)B + c^2(S_C + S)C\right).$$

The infinite point of the perpendiculars to BC being $-a^2 \cdot A + S_C \cdot B + S_B \cdot C$, the perpendicular from P to BC contains the point

$$\begin{aligned}& P + \frac{S_A + S}{2S(S + S_\omega)}(-a^2 \cdot A + S_C \cdot B + S_B \cdot C)\\ = & \frac{1}{2S(S + S_\omega)} \left((b^2(S_B + S) + S_C(S_A + S))B + (c^2(S_C + S) + S_B(S_A + S))C\right)\\ = & \frac{1}{2(S + S_\omega)} ((S + S_A + 2S_C)B + (S + S_A + 2S_B)C).\end{aligned}$$

This is the point X' whose homogeneous coordinates are given in (2) above. Similarly, the pedals of P on the other two lines CA and AB are the points Y' and Z' respectively.

These lead to a simple construction of the vertex A', as the intersection of the lines GX and the line joining A to the pedal of P on BC. This completes the proof of Theorem 1.

Remark. Apart from A', B', C', the vertices of the Malfatti squares on the sidelines are

$$\begin{aligned}X_b &= (0 : 3S + S_A + 2S_C : S_A + 2S_B), & X_c &= (0 : S_A + 2S_C : 3S + S_A + 2S_B),\\ Y_c &= (S_B + 2S_C : 0 : 3S + 2S_A + S_B), & Y_a &= (3S + S_B + 2S_C : 0 : 2S_A + S_B),\\ Z_a &= (3S + 2S_B + S_C : 2S_A + S_C : 0), & Z_b &= (2S_B + S_C : 3S + 2S_A + S_C : 0).\end{aligned}$$

5. An alternative construction

The vertices of the Malfatti triangle $A'B'C'$ are the intersections of the perpendiculars from G' to the sidelines of triangle ABC with the corresponding lines joining G to the pedals of K on the sidelines. A simple construction of G' would lead to the Malfatti triangle easily. Note that G' divides GK in the ratio

$$GG' : G'K = a^2 + b^2 + c^2 : 3S.$$

On the other hand, the point P is the isogonal conjugate of the Vecten point

$$V = \left(\frac{1}{S_A + S} : \frac{1}{S_B + S} : \frac{1}{S_C + S}\right).$$

As such, it can be easily constructed, as the intersection of the perpendiculars from A, B, C to the corresponding sides of the pedal triangles of V. See Figure 6. It is a point on the Brocard axis, dividing OK in the ratio

$$OP : PK = a^2 + b^2 + c^2 : 2S.$$

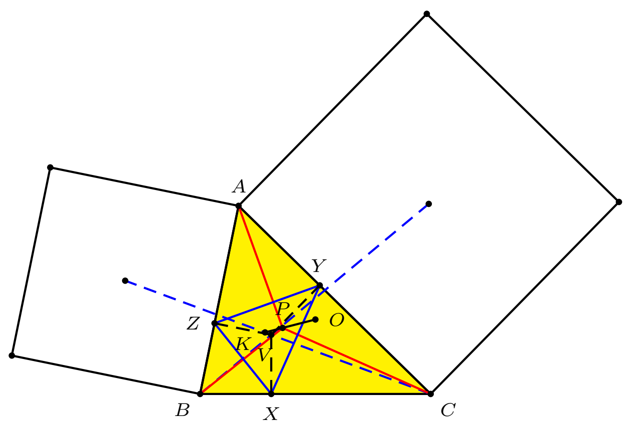

Figure 6.

This leads to a simple construction of the point G'.

Proposition 6. *G' is the intersection of GK with HP, where H is the orthocenter of triangle ABC.*

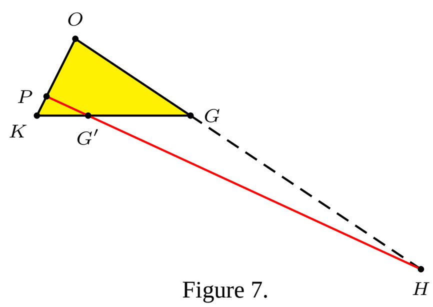

Figure 7.

Proof. Apply Menelaus' theorem to triangle OGK with transversal HP, noting that $OH : HG = 3 : -2$. See Figure 7. $\square$

6. Some observations

6.1. *Malfatti squares not in the interior of given triangle.* Sokolowsky [3] mentions the possibility that the Malfatti squares need not be contained in the triangle. Jean-Pierre Ehrmann pointed out that even the Malfatti triangle may have a vertex outside the triangle. Figure 8 shows an example in which both B' and Y_a are outside the triangle.

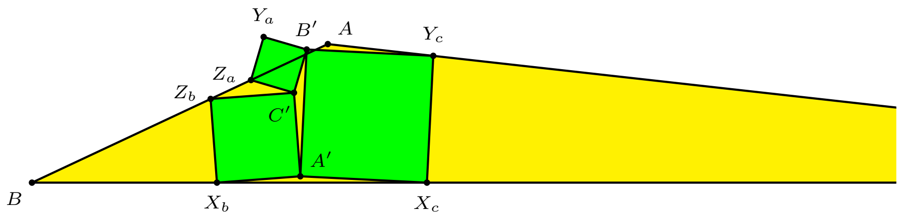

Figure 8.

Proposition 7. *At most one of the vertices the Malfatti triangle and at most one of the vertices of the Malfatti squares on the sidelines can be outside the triangle.*

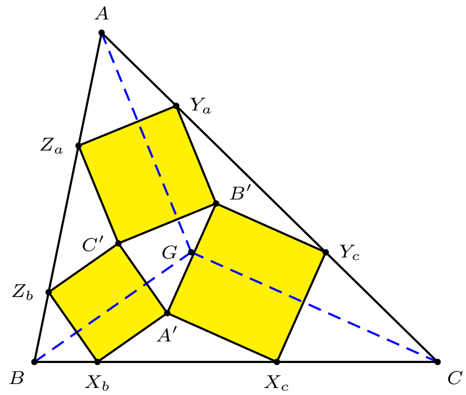

Figure 9.

Proof. If Y_a lies outside triangle ABC, then $\angle AZ_aC' < \frac{\pi}{2}$, and $\angle Z_bZ_aC' > \frac{\pi}{2}$. Since Z_aC' is parallel to AG, $\angle BAG = \angle Z_bZ_aC'$ is obtuse. Under the same hypothesis, if B' and C are on opposite sides of AB, then $\angle AZ_aC' < \frac{\pi}{4}$, and $\angle BAG > \frac{3\pi}{4}$.

Similarly, if any of Z_a, Z_b, X_b, X_c, Y_c lies outside the triangle, then correspondingly, $\angle CAG$, $\angle CBG$, $\angle ABG$, $\angle ACG$, $\angle BCG$ is obtuse. Since at most one of these angles can be obtuse, at most one of the six vertices on the sides and at most one of A', B', C' can be outside triangle ABC. $\square$

6.2. *A locus problem.* François Rideau [8] asked, given B and C, for the locus of A for which the Malfatti squares of triangle ABC are in the interior of the triangle. Here is a simple solution. Let M be the midpoint of BC, P the reflection of C in B, and Q that of B in C. Consider the circles with diameters PB, BM, MC, CQ, and the perpendiculars ℓ_P and ℓ_Q to BC at P and Q. See Figure 10.

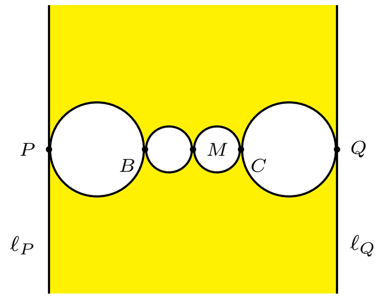

Figure 10.

For an arbitrary point A, consider ABC with centroid G.
(i) $\angle ABG$ is obtuse if A is inside the circle with diameter PB;
(ii) $\angle BAG$ is obtuse if A is inside the circle with diameter BM;
(iii) $\angle CAG$ is obtuse if A is inside the circle with diameter MC;
(iv) $\angle ACG$ is obtuse if A is inside the circle with diameter CQ;
(v) $\angle CBG$ is obtuse if A is on the side of ℓ_P opposite to the circles;
(vi) $\angle BCG$ is obtuse if A is on the side of ℓ_Q opposite to the circles.

Therefore, the locus of A for which the Malfatti squares of triangle ABC are in the interior of the triangle is the region between the lines ℓ_P and ℓ_Q with the four disks excised.

A similar reasoning shows that the locus of A for which the vertices A', B', C' of the Malfatti triangle of ABC are in the interior of triangle ABC is the shaded region in Figure 11.

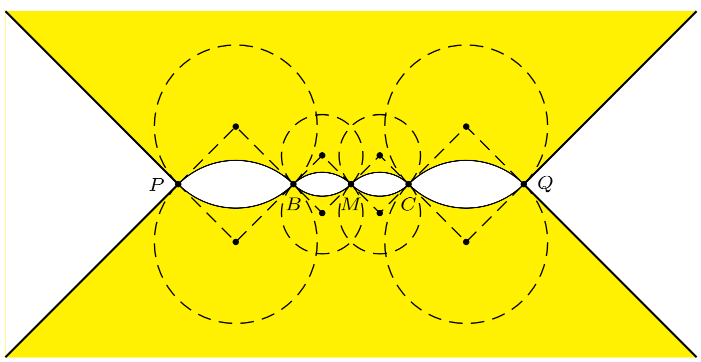

Figure 11.

7. Generalization

We present a generalization of Theorem 1 in which the Malfatti squares are replaced by rectangles of a specified shape. We say that a rectangle constructed on a side of triangle ABC has shape θ if its center is the apex of the isosceles triangle constructed on that side with base angle θ. We assume $0 < \theta < \frac{\pi}{2}$ so that the apex is on the opposite side of the corresponding vertex of the triangle. It is well known that for a given θ, the centers of the three rectangles of shape θ erected on the sides are perspective with ABC at the Kiepert perspector

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

The isogonal conjugate of $K(\theta)$ is the point

$$K^*(\theta) = (a^2(S_A + S_\theta) : b^2(S_B + S_\theta) : c^2(S_C + S_\theta))$$

on the Brocard axis dividing the segment OK in the ratio $\tan\omega \tan\theta : 1$.

Theorem 8. *For a given θ, let $A(\theta)$ be the intersection of the lines joining* (i) *the centroid G to the pedal of the symmedian point K on BC,* (ii) *the vertex A to the pedal of $K^*(\theta)$ on BC. Analogously construct points $B(\theta)$ and $C(\theta)$. Construct rectangles of shape θ on the sides of $A(\theta)B(\theta)C(\theta)$. The remaining vertices of these rectangles lie on the sidelines of triangle ABC.*

Figure 12 illustrates the case of the isodynamic point J. The Malfatti rectangles $B'C'Z_aY_a$, $C'A'X_bZ_b$ and $A'B'Y_cX_c$ have shape $\frac{\pi}{3}$, *i.e.*, lengths and widths in the ratio $\sqrt{3} : 1$.

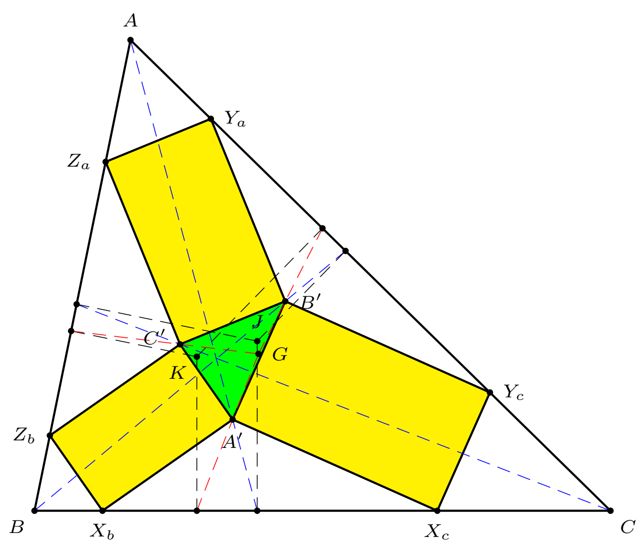

Figure 12.

The same reasoning in §6 shows that exactly one of the six vertices on the sidelines is outside the triangle if and only if a median makes an obtuse angle with an adjacent side. If this angle exceeds $\frac{\pi}{2} + \theta$, the corresponding vertex of Malfatti triangle is also outside ABC.

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A Simple Ruler and Rusty Compass Construction of the Regular Pentagon

Kurt Hofstetter

Abstract. We construct in 13 steps a regular pentagon with given sidelength using a ruler and rusty compass.

Suppose a line segment AB has been divided in the golden ratio at a point G. Figure 1 shows the construction of the vertices of a regular pentagon with four circles of radii equal to AB. Thus, let $\mathcal{C}_1 = A(AB)$, $\mathcal{C}_2 = B(AB)$, $\mathcal{C}_4 = G(AB)$, intersecting the half line AB at P_1, and $\mathcal{C}_5 = P_1(AB)$. Then, with $P_2 = \mathcal{C}_1 \cap \mathcal{C}_5$, $P_4 = \mathcal{C}_1 \cap \mathcal{C}_4$, and $P_5 = \mathcal{C}_2 \cap \mathcal{C}_5$. Since the radii of the circles involved are equal, this construction can be performed with a ruler and a rusty compass. We claim that the pentagon $P_1P_2AP_4P_5$ is regular.

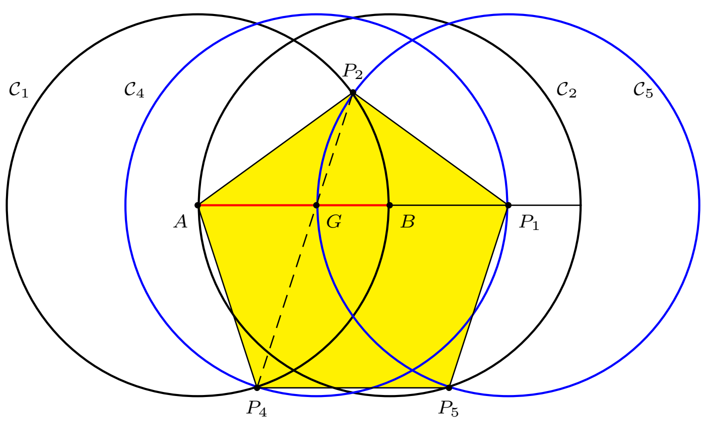

Figure 1

Here is a simple proof. Assume unit length for the segment AB. Let $\phi := \frac{\sqrt{5}+1}{2}$ be the golden ratio. It is well known that $AG = \frac{1}{\phi} = \phi - 1$. Now, $AP_1 = (\phi - 1) + 1 = \phi$. Therefore, the isosceles triangle AP_1P_2 consists of two sides and a diagonal of a regular pentagon. In particular, $\angle P_2AP_1 = 36^\circ$ and $\angle AP_2P_1 = 108^\circ$. On the other hand, triangle AGP_4 is also isosceles with sides in the proportions $1 : 1 : \frac{1}{\phi} = \phi : \phi : 1$. It consists of two diagonals and a side of a regular pentagon. In particular, $\angle GAP_4 = 72^\circ$ and $\angle AP_4G = 36^\circ$. From these, $\angle P_2AP_4 = 36^\circ + 72^\circ = 108^\circ$.

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Now, triangles AP_4P_2 and P_2AP_1 are congruent. It follows that $\angle AP_2P_4 = 36°$, and P_2, G, P_4 are collinear.

By symmetry, we also have $\angle P_2P_1P_5 = 108°$.

In the pentagon $P_1P_2AP_4P_5$, since the angles at P_1, P_2, A are all $108°$, those at P_4 and P_5 are also $108°$. On the other hand, since the circles $\mathcal{C}_2$ and $\mathcal{C}_5$ are the translations of $\mathcal{C}_1$ and $\mathcal{C}_4$ by the vector $\overrightarrow{AB}$, P_4P_5 has unit length. This shows that the pentagon $P_1P_2AP_4P_5$ is regular.

Now, using a rusty compass (set at a radius equal to AB) we have constructed in [1] the point G in 5 steps, which include the circles $\mathcal{C}_1$ and $\mathcal{C}_2$. (In Figure 2, M is the midpoint of AB, $\mathcal{C}_3 = M(AB)$ intersects $\mathcal{C}_2$ at E on the opposite side of C; $G = CE \cap AB$). It follows that the vertices of the regular pentagon $P_1P_2AP_4P_5$ can be constructed in $5 + 3 = 8$ steps. The pentagon can be completed in 5 more steps by filling in the sides.

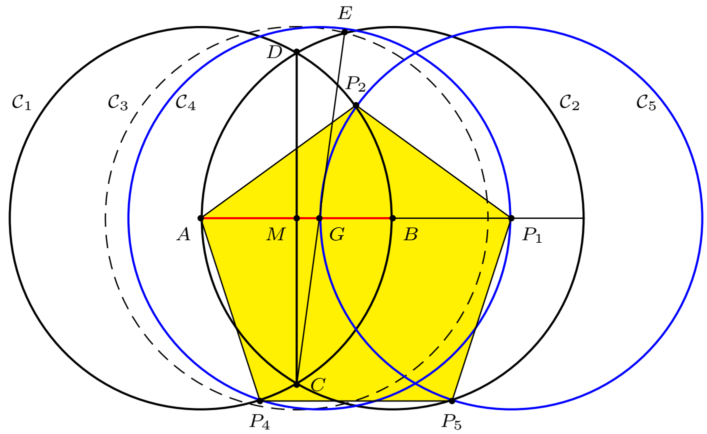

Figure 2

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Haruki's Lemma and a Related Locus Problem

Yaroslav Bezverkhnyev

Abstract. In this paper we investigate the nature of the constant in Haruki's Lemma and study a related locus problem.

1. Introduction

In his papers [2, 3], Ross Honsberger mentions a remarkably beautiful lemma that he accredits to Professor Hiroshi Haruki. The beauty and mystery of Haruki's lemma is in its apparent simplicity.

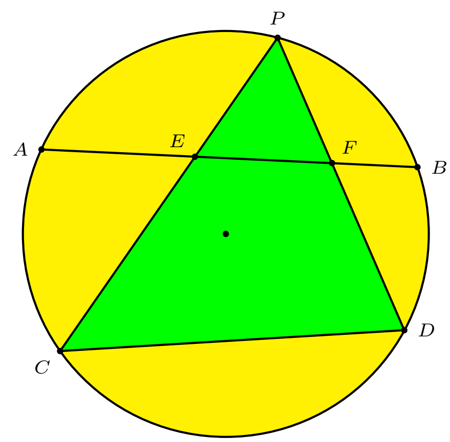

Figure 1. Haruki's lemma: $\frac{AE \cdot BF}{EF}$ = constant.

Lemma 1 (Haruki). *Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB respectively. The value of $\dfrac{AE \cdot BF}{EF}$ does not depend on the position of P.*

A very intriguing statement indeed. It should be duly noted that Haruki's Lemma leads to an easy proof of the Butterfly Theorem; see [2], [3, pp.135–140]. The nature of the constant, however, remains unclear. By looking at it in more detail we shall discover some interesting results.

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2. Proof of Haruki's lemma

A good interactive visualisation and proof of Haruki's lemma can be found in [1]. Here we present the proof essentially as it appeared in [3]. The proof is quite ingenious and relies on the fact that the angle $\angle CPD$ is constant.

We begin by constructing a circumcirlce of triangle PED and define point G to be the intersection of this cirumcircle with the line AB. Note that $\angle EGD = \angle EPD$ as they are subtended by the same chord ED of the circumcircle of $\triangle PED$ and so these angles remain constant as P varies on the arc AB. Hence, for all positions of P, $\angle EGD$ remains fixed and, therefore, point G remains fixed on the line AB (See Figure 2). So $BG = constant$.

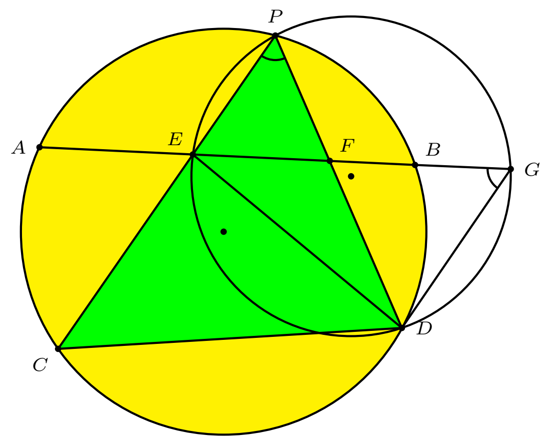

Figure 2. Point G is a fixed point on line AB.

Now, by applying the intersecting chords theorem to PD and AG in the two circles, we obtain the following:

$$AF \cdot FB = PF \cdot FD,$$
$$EF \cdot FG = PF \cdot FD.$$

From these, $(AE + EF) \cdot FB = EF \cdot (FB + BG)$, and $AE \cdot FB = EF \cdot BG$. Therefore, we have obtained $\dfrac{AE \cdot BF}{EF} = BG$, a constant. This completes the proof of Lemma 1.

Note that in the proof we could have used the circumcircle around $\triangle PFC$ instead of the one around $\triangle PED$.

3. An extension of Haruki's lemma

Haruki has apparently found the constant. However, finding it raises additional questions. Why is the ratio of distances that are bound to the circle (through points A, B, C, D, P) expressed by a constant that involves a point lying *outside* the circle? We explore the setup in Lemma 1. Consider an inversion with center P and

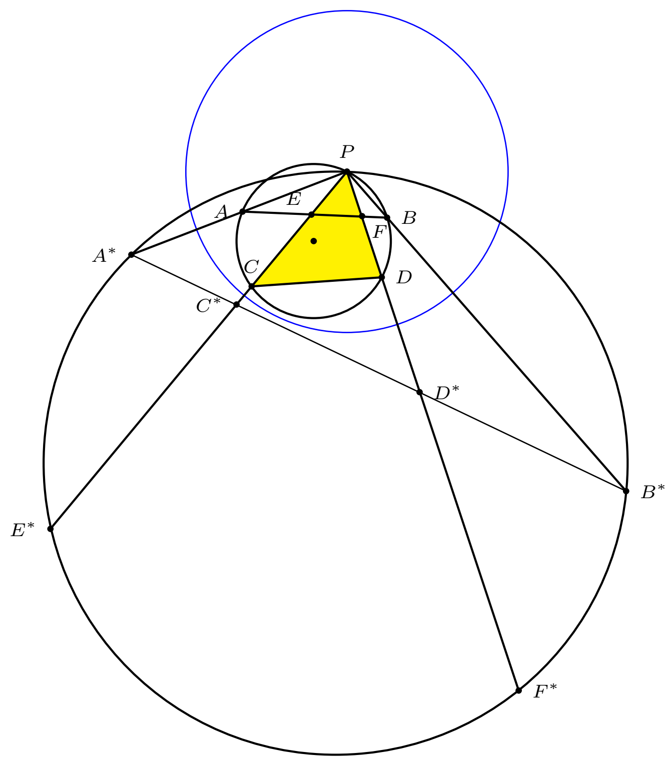

Figure 3. Applying inversion with center P

radius r that is bigger than the diameter of the circumcircle of $ABDC$ (See Figure 3).

Recall two basic facts about an inversion:

(a) It maps a line not through the center of inversion into a circle that goes through the center of inversion and vice versa.

(b) It maps the line that goes through the center of inversion into the same line.

Knowing these two facts, we can perform an inversion on the setup in Figure 1, the results of which are shown in Figure 3. We can see that the segments A^*E^*, B^*F^* and E^*F^* have taken the place of the segments AC, BD, CD. We shall use this hint to deduce the following extension of Haruki's Lemma.

Lemma 2. *Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB respectively. The following equalities hold:*

$$\frac{AE \cdot BF}{EF} = \frac{AC \cdot BD}{CD}, \tag{1}$$

$$\frac{AF \cdot BE}{EF} = \frac{AD \cdot BC}{CD}. \tag{2}$$

Proof. (1) Following the notation and proof of Lemma 1, we have $\frac{AE \cdot BF}{EF} = BG$. It remains to show that $BG = \frac{AC \cdot BD}{CD}$, or, equivalently,

$$\frac{BG}{BD} = \frac{AC}{CD}. \tag{3}$$

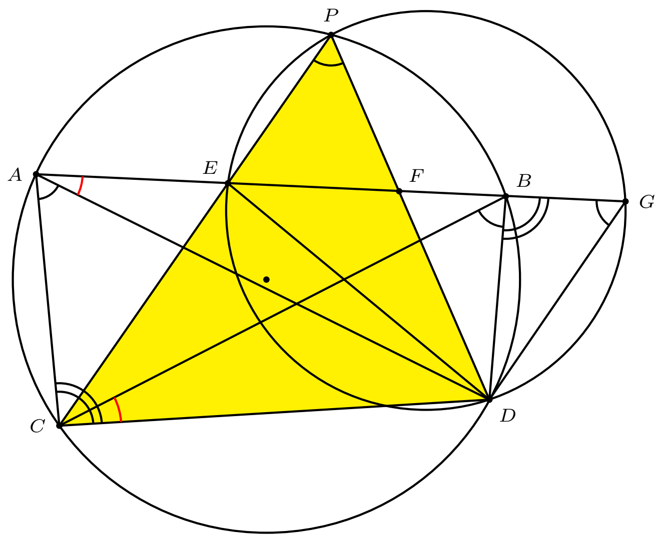

Figure 4. Triangles ACD and GBD are similar, as are AGD and CBD

Note that in Figure 4, $\angle CAD = \angle CPD = \angle EPD = \angle EGD$. Since $ABDC$ is a cyclic quadrilateral, we have $\angle ACD = \angle DBG$. This means that the triangles ACD and GBD are similar, thus yielding (3), and therefore (1).

For (2) we note that $\angle DCB = \angle DAB$. Also $\angle CBD = \angle CPD = \angle EPD = \angle EGD$, thus we get $\triangle AGD \sim \triangle CBD$ yielding:

$$\frac{AG}{AD} = \frac{BC}{CD} \Rightarrow AG = \frac{AD \cdot BC}{CD}.$$

However, $AF \cdot BE = (AE + EF) \cdot (EF + BF) = AE \cdot BF + AB \cdot EF$. We obtain, by using Lemma 1,

$$\frac{AF \cdot BE}{EF} = \frac{AE \cdot BF}{EF} + AB = BG + AB = AG = \frac{AD \cdot BC}{CD}.$$

□

Note that by switching the position of points C and D we effectively switch points E and F, thus equations (1) and (2) are equivalent. It may seem surprising; but the statement of Lemma 2 holds even for intersecting chords AB and CD and for any point P on the circle for which the points E and F are defined.

Theorem 3. *Given two distinct chords AB and CD in a circle and a point P on that circle distinct from A and B, let E and F be the intersections of the line AB with the lines PC and PD respectively. The equalities* (1) *and* (2) *hold.*

We leave the proof to the reader as an exercise. All that is necessary is to consider the different cases for the relative positions of A, B, C, D, P and to apply the ideas in the proofs of Lemmas 1 and 2, i.e. finding the point G as the intersection of the circumcircle of either $\triangle PED$ or $\triangle PFC$ with AB and then looking for similar triangles. Note that the point P may coincide with either C or D. In this case, by the line PC or PD we would mean the tangent to the circle at C or D.

4. A locus problem

Theorem 3 settles the case when points A, B, C, D lie on a circle. But what happens when points A, B, C, D do not belong to the same circle? Can we still find points P that will satisfy equation (1) or (2)? This gives rise to the following locus problem.

Problem. Given the points A, B, C, D find the locus $\mathcal{L}_1$ (respectively $\mathcal{L}_2$) of all points P that satisfy (1) (respectively (2)), where points E and F are the intersections of lines PC and AB, PD and AB respectively.

To investigate the loci $\mathcal{L}_1$ and $\mathcal{L}_2$, we begin with a result about the possibility of a point P belonging to both $\mathcal{L}_1$ and $\mathcal{L}_2$.

Lemma 4. *If there is a point P satisfying both* (1) *and* (2)*, then A, B, C, D are concyclic.*

Proof. First of all, points A, B, E, F are collinear, hence, they satisfy Euler's distribution theorem (See [4, p.3] and [5]), *i.e.*, if A, B, E, F are in this order, then, $AF \cdot BE + AB \cdot EF = AE \cdot BF$. Dividing through by EF, we obtain

$$\frac{AF \cdot BE}{EF} + AB = \frac{AE \cdot BF}{EF},$$

and so, by the fact that point P satisfies equations (1) and (2), we have:

$$\frac{AD \cdot BC}{CD} + AB = \frac{AC \cdot BD}{CD}.$$

Now multiplying by CD yields

$$AD \cdot BC + AB \cdot CD = AC \cdot BD,$$

which, by Ptolemy's inequality (See [6]), means that points A, B, C, D are concyclic with points A, C separating points B, D on the circle. The relative positions of A, B, E, F will influence the relative positions of points A, B, C, D on the circle. Similar argument can be applied to establish the validity of the statement of this lemma no matter the position of points A, B, E, F. $\square$

This lemma is interesting in the way it ties the "linear" Euler's equality, Ptolemy's inequality together with the extension of Haruki's lemma.

5. Barycentric coordinates

In order to find the loci $\mathcal{L}_1$ and $\mathcal{L}_2$ for the general position of points A, B, C, D we make use of the notion of homogeneous barycentric coordinates. Given a reference triangle ABC, any three numbers x, y, z proportional to the signed areas of oriented triangles PBC, PCA, PAB form a set of *homogeneous barycentric coordinates* of P, written as $(x : y : z)$.

With reference to triangle ABC, the absolute barycentric coordinates of the vertices are obviously $A(1:0:0)$, $B(0:1:0)$ and $C(0:0:1)$. We shall make use of the following basic property of barycentric coordinates.

Lemma 5. *Let P be point with homogeneous barycentric coordinates $(x : y : z)$ with reference to triangle ABC. The line AP intersects BC at a point X with coordinates $(0 : y : z)$, which divides BC in the ratio $BX : XC = z : y$. Similarly, BP intersects CA at $Y(x : 0 : z)$ such that $CY : YA = x : z$ and CP intersects AB at $Z(x : y : 0)$ such that $AZ : ZB = y : x$.*

Assume that D and P have barycentric coordinates $D(u : v : w)$ and $P(x : y : z)$. It is our aim to compute the coordinates of points E and F.

When there is no danger of confusion, we shall represent a line $p\alpha+q\beta+r\gamma = 0$ by $(p : q : r)$. The intersection of two lines $(p : q : r)$ and $(s : t : u)$ is the point

$$\left(\begin{vmatrix} q & r \\ t & u \end{vmatrix} : \begin{vmatrix} r & p \\ u & s \end{vmatrix} : \begin{vmatrix} p & q \\ s & t \end{vmatrix} \right).$$

This same expression also gives the line through the two points with homogeneous barycentric coordinates $(p : q : r)$ and $(s : t : u)$.

6. Solution of the locus problem

From the above formula we compute the coordinates of the lines AB, PC and PD:

Line	Coordinates
AB	$(0:0:1)$
PC	$(-y:x:0)$
PD	$\left(\begin{vmatrix} y & z \\ v & w \end{vmatrix} : \begin{vmatrix} z & x \\ w & u \end{vmatrix} : \begin{vmatrix} x & y \\ u & v \end{vmatrix}\right)$

From these we obtain the coordinates of E and F:

$$E\left(\begin{vmatrix} x & 0 \\ 0 & 1 \end{vmatrix} : \begin{vmatrix} 0 & -y \\ 1 & 0 \end{vmatrix} : \begin{vmatrix} -y & x \\ 0 & 0 \end{vmatrix}\right) = (x : y : 0),$$
$$F\left(\begin{vmatrix} z & x \\ w & u \end{vmatrix} : \begin{vmatrix} z & y \\ w & v \end{vmatrix} : 0\right) = (uz - wx : vz - wy : 0).$$

Assume $BC = a$, $CA = b$, and $AB = c$. Also, $AD = a'$, $BD = b'$, and $CD = c'$. These are also fixed quantities. From the coordinates of E and F, we obtain, by Lemma 5, the following *signed* lengths:

$$AE = \tfrac{y}{x+y} \cdot c, \qquad EB = \tfrac{x}{x+y} \cdot c;$$
$$AF = \tfrac{vz-wy}{z(u+v)-w(x+y)} \cdot c, \quad FB = \tfrac{uz-wx}{z(u+v)-w(x+y)} \cdot c.$$

Consequently,

$$EF = EB - FB = \frac{z(vx - uy)}{(x+y)(z(u+v) - w(x+y))} \cdot c.$$

Now we determine the loci $\mathcal{L}_1$ and $\mathcal{L}_2$.

Theorem 6. *Given the points A, B, C, D and a point P, define points E and F as the intersections of lines PC and AB, PD and AB respectively.*
(a) *The locus $\mathcal{L}_1$ of points P satisfying* (1) *is the union of two circumconics of $ABCD$ given by the equations*

$$(cc' + \varepsilon bb')uyz - \varepsilon bb'vzx - cc'wxy = 0, \quad \varepsilon = \pm 1. \tag{4}$$

(b) *The locus $\mathcal{L}_2$ of points P satisfying* (2) *is the union of two circumconics of $ABCD$ given by the equations*

$$\varepsilon aa'uyz + (cc' - \varepsilon aa')vzx - cc'wxy = 0, \quad \varepsilon = \pm 1. \tag{5}$$

Proof. In terms of signed lengths, (1) and (2) should be interpreted as $AE \cdot BF \cdot CD = \varepsilon \cdot AC \cdot BD \cdot EF$ and $AF \cdot BE \cdot CD = \varepsilon \cdot AD \cdot BC \cdot EF$ for $\varepsilon = \pm 1$. The results follow from direct substitutions. It is easy to see that the conics represented by (4) and (5) all contain the points A, B, C, D, with barycentric coordinates $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, $(u:v:w)$ respectively. □

7. Constructions

Theorem 6 tells us that the loci in question are each a union of two conics, each containing the four given points A, B, C, D. In order to construct these conics, we would need to find a fifth point on each of them. The following proposition helps with this problem.

Proposition 7. *The four intersections of the bisectors of angles ABD, ACD, and the four intersections of the bisectors of angles CAB and CDB are points on $\mathcal{L}_1$.*

Proof. First of all, it is routine to verify that for $P = (x : y : z)$, we have

$$[AEP] \cdot [BFP] \cdot [CDP] = [ACP] \cdot [BDP] \cdot [EFP], \tag{6}$$

where $[XYZ]$ denotes the signed area of the oriented triangle XYZ. Let d_{XY} be the distance from P to the line XY. In terms of distances, the relation (6) becomes

$$(AE \cdot d_{AE})(BF \cdot d_{BF})(CD \cdot d_{CD}) = (AC \cdot d_{AC})(BD \cdot d_{BD})(EF \cdot d_{EF}).$$

From this it is clear that (1) is equivalent to

$$d_{AE} \cdot d_{BF} \cdot d_{CD} = d_{AC} \cdot d_{BD} \cdot d_{EF}. \tag{7}$$

Since AE, BF, EF are the same line AB, this condition can be rewritten as

$$d_{AB} \cdot d_{CD} = d_{AC} \cdot d_{BD}. \tag{8}$$

If P is an intersection of the bisectors of angles ABD and ACD, then $d_{AB} = d_{BD}$ and $d_{AC} = d_{CD}$. On the other hand, if P is an intersection of the bisectors of angles CAB and CDB, then $d_{AC} = d_{AB}$ and $d_{CD} = d_{BD}$. In both cases, (7) is satisfied, showing that P is a point on the locus $\mathcal{L}_1$. □

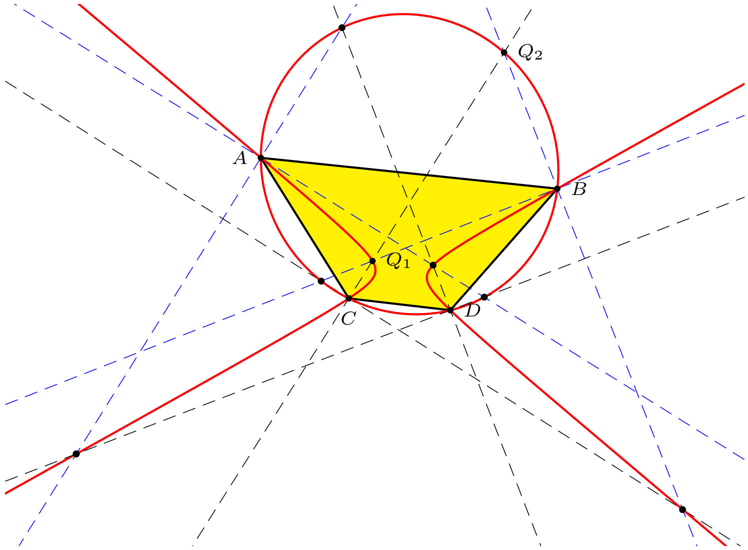

Figure 5. The locus $\mathcal{L}_1$

Let Q_1 be the intersection of the internal bisectors of angles ABD and ACD, and Q_2 as the intersection of the external bisector of angle ABD and the internal bisector of angle ACD. See Figure 5. Since Q_1, Q_2 and C are collinear, the points Q_1 and Q_2 must lie on distinct conics of $\mathcal{L}_1$.

Similarly, the locus $\mathcal{L}_2$ also contains the four intersections of the bisectors of angles BAD and BCD, and the four from angles ABC and ADC. Let Q_3 be the intersection of the internal bisectors and Q_4 the intersection of the internal bisector of BAD and the external bisector of angle BCD. See Figure 6. The points Q_3 and Q_4 are on different conics of $\mathcal{L}_2$.

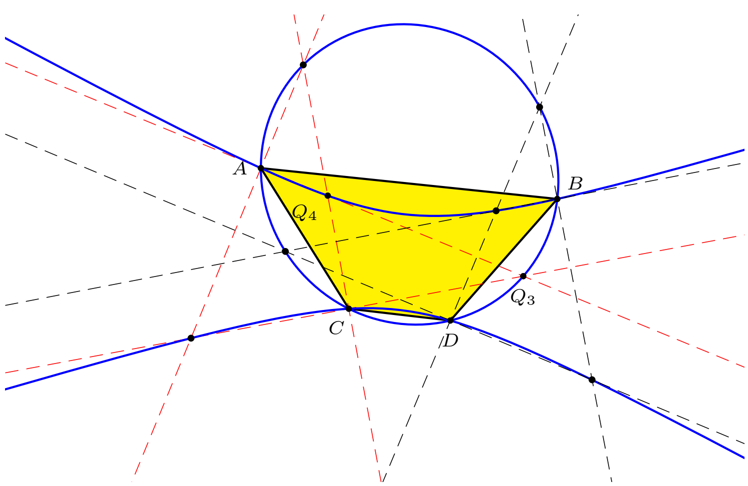

Figure 6. The locus $\mathcal{L}_2$

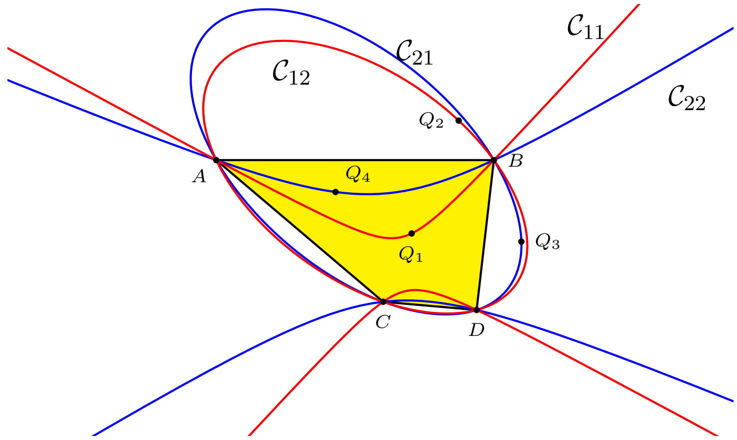

Figure 7.

Figure 7 shows the four conics, with $\mathcal{C}_{1,1}$, $\mathcal{C}_{1,2}$ forming $\mathcal{L}_1$ and $\mathcal{C}_{2,1}$, $\mathcal{C}_{2,2}$ forming $\mathcal{L}_2$.

Corollary 8. (a) *When points A, B, C, D all belong to the same circle $\mathcal{C}$, then one of the conics from $\mathcal{L}_1$ and one from $\mathcal{L}_2$ coincide with $\mathcal{C}$.*

(b) *If for some point P, (1) and (2) are both satisfied, then the points A, B, C, D, P are concyclic.*

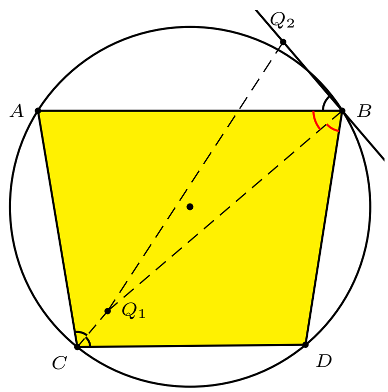

Figure 8.

Proof. (a) Assume Q_1 not on the circle $\mathcal{C}$. Suppose we have the situation as in Figure 8. In other cases the reasoning is similar. It is easy to see that $\angle ABQ_2 = \angle ACQ_2$ as Q_2 belongs to the external bisector of the angle ABD. This means that the points A, B, C and Q_2 are concyclic. But Q_2 lies on one of the conics from $\mathcal{L}_1$, therefore, this conic is actually a circle. Similarly, one can show that one of the conics from $\mathcal{L}_2$ coincides with $\mathcal{C}$. This proves (a).

(b) follows directly from (a) and Lemma 4. □

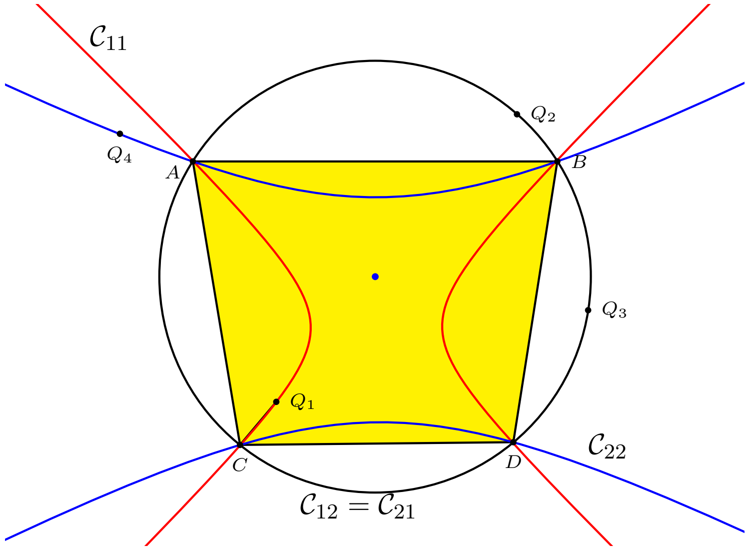

Figure 9. Loci $\mathcal{L}_1$ and $\mathcal{L}_2$ for cyclic quadrilateral $ABDC$

Theorem 3 together with Lemma 4 and part (b) of Corollary 8 provide us with the criteria for five points A, B, C, D and P to be concyclic. The case when $ABCD$ is a cyclic quadrilateral is depicted in Figure 9.

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An Inequality Involving the Angle Bisectors and an Interior Point of a Triangle

Wei-Dong Jiang

Abstract. We establish a new weighted geometric inequality involving the lengths of the angle bisectors and the radii of three circles through an interior point of a triangle. From this, several interesting geometric inequalities are derived.

1. Introduction

Throughout this paper we consider a triangle ABC with sidelengths a, b, c, circumradius R, and inradius r. Denote by w_a, w_b, w_c the lengths of the bisectors of angles A, B, C. Let P be an interior point. Denote by R_a, R_b, R_c the radii of the circles PBC, PCA, PAB respectively. Liu [2] has conjectured the inequality

$$\frac{w_a}{R_b+R_c}+\frac{w_b}{R_c+R_a}+\frac{w_c}{R_a+R_b}\leq\frac{9}{4}. \tag{1}$$

We prove a stronger inequality in Theorem 1 below, which include the

$$\frac{w_a}{\sqrt{R_bR_c}}+\frac{w_b}{\sqrt{R_cR_a}}+\frac{w_c}{\sqrt{R_aR_b}}\leq\frac{9}{2}. \tag{2}$$

Theorem 1. *For an interior point P and positive real numbers x, y, z, we have*

$$\frac{xw_a}{\sqrt{R_bR_c}}+\frac{yw_b}{\sqrt{R_cR_a}}+\frac{zw_c}{\sqrt{R_aR_b}}\leq\sqrt{2+\frac{r}{2R}}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right). \tag{3}$$

Equality holds if and only if the triangle ABC is equilateral, P its center, and $x=y=z$.

We shall make use of the following lemma.

Lemma 2. *For arbitrary nonzero real numbers $x,\ y,\ z$,*

$$x^2\sin^2A+y^2\sin^2B+z^2\sin^2C\leq\frac{1}{4}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right)^2. \tag{4}$$

Equality holds if and only if $x^2:y^2:z^2=\frac{1}{a^2(b^2+c^2-a^2)}:\frac{1}{b^2(c^2+a^2-b^2)}:\frac{1}{c^2(a^2+b^2-c^2)}$.

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The author is grateful to Professor Paul Yiu for his suggestions for the improvement of this paper.

Proof. We make use of Kooi's inequality [1, Inequality 14.1]: for real numbers λ_1, λ_2, λ_3 with $\lambda_1+\lambda_2+\lambda_3 \neq 0$,

$$(\lambda_1+\lambda_2+\lambda_3)^2R^2 \geq \lambda_2\lambda_3a^2+\lambda_3\lambda_1b^2+\lambda_1\lambda_2c^2;$$

equality holds if and only if the point with homogeneous barycentric coordinates $(\lambda_1 : \lambda_2 : \lambda_3)$ with reference to triangle ABC is the circumcenter of the triangle. Now, with $\lambda_1 = \frac{yz}{x}$, $\lambda_2 = \frac{zx}{y}$, $\lambda_3 = \frac{xy}{z}$, the result follows from the law of sines: $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$. □

2. Proof of Theorem 1

The length of the bisector of angle A is given by $w_a = \frac{2bc}{b+c}\cos\frac{A}{2}$. Clearly, $w_a \leq \sqrt{bc}\cos\frac{A}{2}$; equality holds if and only if $b = c$.

Let $\angle BPC = \alpha$, $\angle CPA = \beta$ and $\angle APB = \gamma$. Obviously, $0 < \alpha,\ \beta,\ \gamma < \pi$ and $\alpha+\beta+\gamma = 2\pi$. By the law of sines, $b = 2R_b\sin\beta$, $c = 2R_c\sin\gamma$. We have

$$\begin{aligned}
\frac{w_a}{\sqrt{R_bR_c}} &\leq \sqrt{\frac{bc}{R_bR_c}}\cdot\cos\frac{A}{2}\\
&= 2\sqrt{\sin\beta\sin\gamma}\cdot\cos\frac{A}{2}\\
&\leq (\sin\beta+\sin\gamma)\cos\frac{A}{2}\\
&= 2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{A}{2}\\
&\leq 2\sin\frac{\alpha}{2}\cos\frac{A}{2}.
\end{aligned}$$

Equality holds if and only if $b = c$ and $\beta = \gamma$. Similarly, $\frac{w_b}{\sqrt{R_cR_a}} \leq 2\sin\frac{\beta}{2}\cos\frac{B}{2}$ and $\frac{w_c}{\sqrt{R_aR_b}} \leq 2\sin\frac{\gamma}{2}\cos\frac{C}{2}$ with analogous conditions for equality. Therefore, for $x,\ y,\ z > 0$,

$$\begin{aligned}
&\frac{xw_a}{\sqrt{R_bR_c}}+\frac{yw_b}{\sqrt{R_cR_a}}+\frac{zw_c}{\sqrt{R_aR_b}}\\
&\leq 2x\sin\frac{\alpha}{2}\cos\frac{A}{2}+2y\sin\frac{\beta}{2}\cos\frac{B}{2}+2z\sin\frac{\gamma}{2}\cos\frac{C}{2} &(5)\\
&\leq 2\sqrt{\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\left(x^2\sin^2\frac{\alpha}{2}+y^2\sin^2\frac{\beta}{2}+z^2\sin^2\frac{\gamma}{2}\right)} &(6)\\
&\leq \sqrt{2+\frac{r}{2R}}\cdot\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right) &(7)
\end{aligned}$$

Here, the inequality in (6) follows from the Cauchy-Schwarz inequality. On the other hand, the inequality in (7) follows from the identity

$$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2} = 2+\frac{r}{2R},$$

and application of Lemma 2 to a triangle with angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$, $\frac{\gamma}{2}$. Equality holds in (5) holds if and only if $a = b = c$ and $\alpha = \beta = \gamma$. This means that ABC is equilateral and P is its center. Finally, by Lemma 2 again, equality holds in (7) if and only if $x^2 : y^2 : z^2 = 1 : 1 : 1$, *i.e.*, $x = y = z$. This completes the proof of Theorem 1.

3. Some applications

With $x = y = z$ in Theorem 1, we have

$$\frac{w_a}{\sqrt{R_bR_c}} + \frac{w_b}{\sqrt{R_cR_a}} + \frac{w_c}{\sqrt{R_aR_b}} \le 3\sqrt{2 + \frac{r}{2R}}.$$

By Euler's famous inequality $R \ge 2r$, we have (2).

Since $\sqrt{R_bR_c} \le \frac{1}{2}(R_b + R_c)$, $\sqrt{R_cR_a} \le \frac{1}{2}(R_c + R_a)$, $\sqrt{R_aR_b} \le \frac{1}{2}(R_a + R_b)$, we obtain from Theorem 1,

$$\frac{xw_a}{R_b + R_c} + \frac{yw_b}{R_c + R_a} + \frac{zw_c}{R_a + R_b} \le \frac{1}{2}\sqrt{2 + \frac{r}{2R}}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{8}$$

With $x = y = z$, we have

$$\frac{w_a}{R_b + R_c} + \frac{w_b}{R_c + R_a} + \frac{w_c}{R_a + R_b} \le \frac{3}{2}\sqrt{2 + \frac{r}{2R}}.$$

Liu's inequality (1) follows from $R \ge 2r$.

Again, from Euler's inequality, we immediately conclude from Theorem 1 that

$$\frac{xw_a}{\sqrt{R_bR_c}} + \frac{yw_b}{\sqrt{R_cR_a}} + \frac{zw_c}{\sqrt{R_aR_b}} \le \frac{3}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{9}$$

Corollary 3. *For an interior point P and positive real numbers x, y, z, we have*

$$x^2R_a + y^2R_b + z^2R_c \ge \frac{2}{3}(yzw_a + zxw_b + xyw_c).$$

Equality holds if and only if the triangle ABC is equilateral, P its center, and $x = y = z$.

Proof. Replace in (9) x, y, z respectively by $yz\sqrt{R_bR_c}$, $zx\sqrt{R_cR_a}$, $xy\sqrt{R_aR_b}$. □

In particular, with $x = y = z = 1$, we have

$$R_a + R_b + R_c \ge \frac{2}{3}(w_a + w_b + w_c);$$

equality holds if and only if the triangle is equilateral and P its center.

Corollary 4. *For an interior point P in a triangle ABC, $R_aR_bR_c \ge \frac{64}{27}w_aw_bw_c$. Equality holds if and only if ABC is equilateral and P its center.*

Proof. This follows from (9) by putting $x = y = z$ and applying the AM-GM inequality. □

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Cubics Related to Coaxial Circles

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Abstract. This note generalizes a result of Paul Yiu on a locus associated with a triad of coaxial circles. We present an interesting family of cubics with many properties similar to those of pivotal cubics. It is also an opportunity to show how different ways of writing the equation of a cubic lead to various geometric properties of the curve.

1. Introduction

In his Hyacinthos message [7], Paul Yiu encountered the cubic **K360** as the locus of point P (in the plane of a given triangle ABC) with cevian triangle XYZ such that the three circles $AA'X$, $BB'Y$, $CC'Z$ are coaxial. Here $A'B'C'$ is the circumcevian triangle of X_{56}, the external center of similitude of the circumcircle and incircle. See Figure 1. It is natural to study the coaxiality of the circles when $A'B'C'$ is the circumcevian triangle of a given point Q.

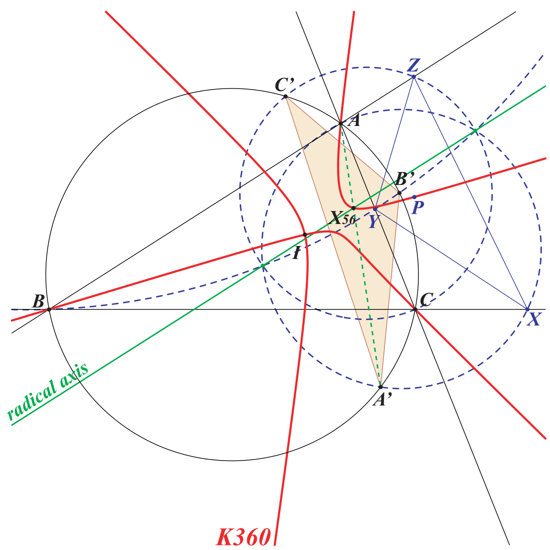

Figure 1. **K360** and coaxial circles

Throughout this note, we work with homogeneous barycentric coordinates with reference to triangle ABC, and adopt the following notations:

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gX the isogonal conjugate of X
tX the isotomic conjugate of X
cX the complement of X
aX the anticomplement of X
tg$X$ the isotomic conjugate of the isogonal conjugate of X

2. Preliminaries

Let $Q = p : q : r$ be a fixed point with circumcevian triangle $A'B'C'$ and P a variable point with cevian triangle $P_aP_bP_c$. Denote by $\mathcal{C}_A$ the circumcircle of triangle $AA'P_a$ and define $\mathcal{C}_B, \mathcal{C}_C$ in the same way.

Lemma 1. *The radical center of the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ is the point Q.*

Proof. The radical center of the circumcircle $\mathcal{C}$ of triangle ABC and $\mathcal{C}_B, \mathcal{C}_C$ must be Q. Indeed, it must be the intersection of BB' (the radical axis of $\mathcal{C}$ and $\mathcal{C}_B$) and CC' (the radical axis of $\mathcal{C}$ and $\mathcal{C}_C$). Hence the radical axis of $\mathcal{C}_B, \mathcal{C}_C$ contains Q. □

These three radical axes are in general distinct lines. For some choices of P, however, these circles are coaxial. For example, if $P = Q$, then the three circles degenerate into the cevian lines of Q and we regard these as infinite circles with radical axis the line at infinity. Another trivial case is when P is one of the vertices A, B, C, since two circles coincide with $\mathcal{C}$ and the third circle is not defined.

Lemma 2. *Let H be the orthocenter of triangle ABC. For any point $Q \neq H$ and $P = H$, the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ are coaxial with radical axis HQ.*

Proof. When $P = H$, the cevian triangle of P is the orthic triangle $H_aH_bH_c$. The inversion with respect to the polar circle swaps A, B, C and H_a, H_b, H_c respectively. Hence the products of signed distances $HA \cdot HH_a$, $HB \cdot HH_b$, $HC \cdot HH_c$ are equal but, since they represent the power of H with respect to the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$, H must be on their radical axes which turns out to be the line HQ. If $Q = H$, the property is a simple consequence of the lemma above. □

3. The cubic $\mathcal{K}(Q)$ and its construction

Theorem 3. *In general, the locus of P for which the circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ are coaxial is a circumcubic $\mathcal{K}(Q)$ passing through H, Q and several other remarkable points. This cubic is tangent at A, B, C to the symmedians of triangle ABC.*

This is obtained through direct and easy calculation. It is sufficient to write that the radical circle of $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ degenerates into the line at infinity and another line which is obviously the common radical axis of the circles. This calculation gives several equivalent forms of the barycentric equation of $\mathcal{K}(Q)$. In §§4 – 9 below, we explore these various forms, deriving essential geometric properties and identifying interesting points of the cubic. For now we examine the simplest of all these:

$$\sum_{\text{cyclic}} b^2c^2\, p\, x\,(y+z)(ry-qz) = 0 \iff \sum_{\text{cyclic}} \frac{x(y+z)}{a^2}\left(\frac{y}{q}-\frac{z}{r}\right) = 0. \quad (1)$$

It is clear that $\mathcal{K}(Q)$ contains A, B, C, Q and the vertices A_1, B_1, C_1 of the cevian triangle of $\mathbf{tg}Q = \frac{p}{a^2} : \frac{q}{b^2} : \frac{r}{c^2}$. Indeed, when we take $x = 0$ in equation (1) we obtain $(b^2ry - c^2qz)yz = 0$.

$\mathcal{K}(Q)$ also contains $\mathbf{ag}Q$. Indeed, if we write $\mathbf{ag}Q = u : v : w$ then $v+w = \dfrac{a^2}{p}$, etc, since this is the complement of $\mathbf{ag}Q$ i.e. $\mathbf{g}Q$. The second form of equation (1) obviously gives $\displaystyle\sum_{\text{cyclic}} \frac{u}{p}\left(\frac{v}{q} - \frac{w}{r}\right) = 0$.

Finally, it is easy to verify that $\mathcal{K}(Q)$ is tangent at A, B, C to the symmedians of triangle ABC. Indeed, when b^2z is replaced by c^2y in (1), the polynomial factorizes by y^2.

3.1. *Construction.* Given Q, denote by S be the second intersection of the Euler line with the rectangular circumhyperbola $\mathcal{H}_Q$ through Q.

Let $\mathcal{H}'_Q$ be the rectangular hyperbola passing through O, Q, S and with asymptotes parallel to those of $\mathcal{H}_Q$.

A variable line L_Q through Q meets $\mathcal{H}'_Q$ at a point Q'.

L_Q meets the rectangular circumhyperbola through $\mathbf{g}Q'$ (the isogonal transform of the line OQ') at two points M, M' of $\mathcal{K}(Q)$ collinear with Q.

Note that Q is the coresidual of A, B, C, H in $\mathcal{K}(Q)$ and that $\mathbf{ag}Q$ is the coresidual of A, B, C, Q in $\mathcal{K}(Q)$. Thus, the line through $\mathbf{ag}Q$ and M meets again the circumconic through Q and M at another point on $\mathcal{K}(Q)$.

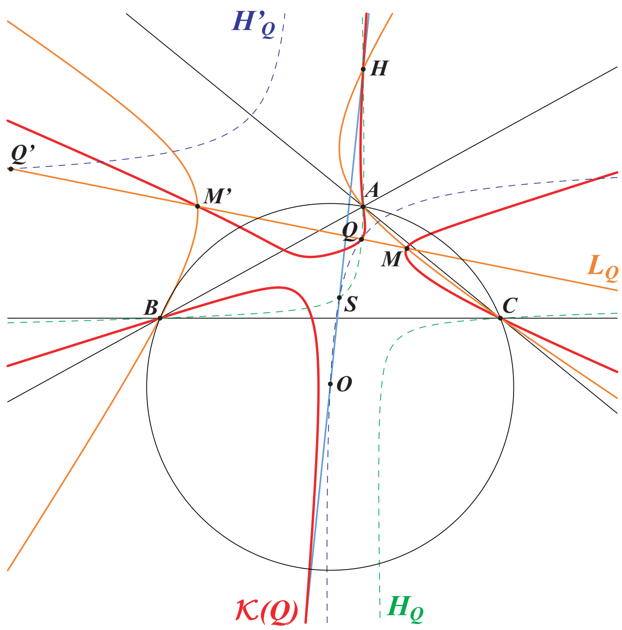

Figure 2. Construction of $\mathcal{K}(Q)$

4. Intersections with the circumcircle and the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$

Proposition 4. *$\mathcal{K}(Q)$ intersects the circumcircle at the same points as the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$ with pivot $\mathbf{ag}Q$.*

Proof. The equation of $\mathcal{K}(Q)$ can be written in the form

$$\sum_{\text{cyclic}} (-a^2qr + b^2rp + c^2pq)\, x\, (c^2y^2 - b^2z^2) \\ + (a^2yz + b^2zx + c^2xy) \sum_{\text{cyclic}} p\,(c^2q - b^2r)\, x = 0. \tag{2}$$

Any point common to $\mathcal{K}(Q)$ and the circumcircle also lies on the cubic

$$\sum_{\text{cyclic}} (-a^2qr + b^2pr + c^2pq)\, x\, (c^2y^2 - b^2z^2) = 0, \tag{3}$$

which is the pivotal isogonal circumcubic $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$. □

The two cubics $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$ must have three other common points on the line passing through G and $\mathbf{ag}Q$. One of them is $\mathbf{ag}Q$ and the two other points E_1, E_2 are not always real points. Indeed, the equation of this line is

$$\sum_{\text{cyclic}} p(c^2q - b^2r)x = 0.$$

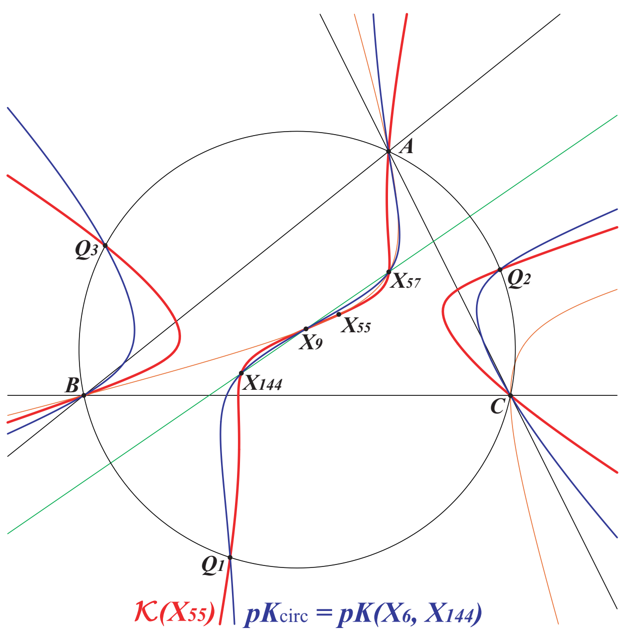

Figure 3. $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$ when $Q = X_{55}$

These points E_1, E_2 are the intersections of the line passing through G, $\mathbf{g}Q$, $\mathbf{ag}Q$ with the circumconic $ABCKQ$ which is its isogonal conjugate. It follows that these points are the last common points of $\mathcal{K}(Q)$ and the Thomson cubic **K002**.

Figure 3 shows these cubics when $Q = X_{55}$, the isogonal conjugate of the Gergonne point X_7. Here, the points E_1, E_2 are X_9, X_{57} and $\mathbf{ag}Q$ is X_{144}.

Thus, $\mathcal{K}(Q)$ meets the circumcircle at A, B, C with concurrent tangents at K and three other points Q_1, Q_2, Q_3 (one of them is always real). Following [4], **ag**Q must be the orthocenter of triangle $Q_1Q_2Q_3$.

4.1. *Construction of the points* Q_1, Q_2, Q_3. The construction of these points again follows a construction of [4] : the rectangular hyperbola having the same asymptotic directions as those of $ABCHQ$ and passing through Q, $\mathbf{ag}Q$, the antipode Z on the circumcircle of the isogonal conjugate Z' of the infinite point of the line $O\mathbf{g}Q$ meets the circumcircle at Z and Q_1, Q_2, Q_3. Note that Z' is the fourth point of $ABCHQ$ on the circumcircle. The sixth common point of the hyperbola and $\mathcal{K}(Q)$ is the second intersection Q' of the line $H\mathbf{ag}Q$ with both hyperbolas. It is the tangential of Q in $\mathcal{K}(Q)$. It is also the second intersection of the line ZZ' with both hyperbolas. See Figure 4.

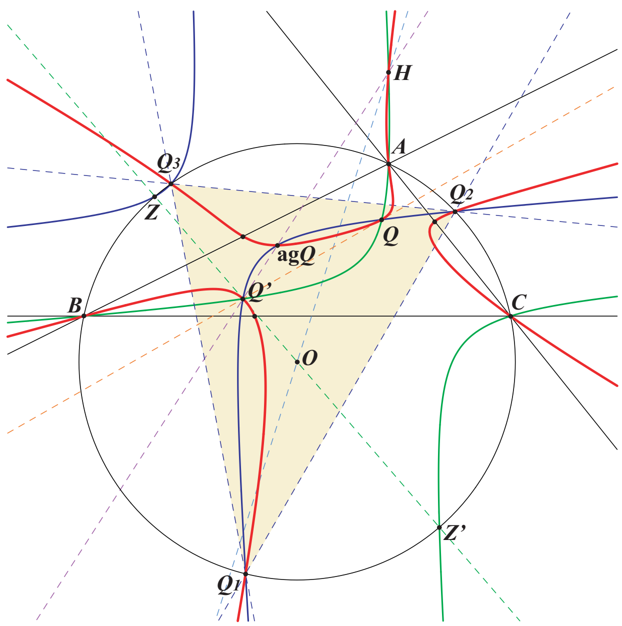

Figure 4. Construction of the points Q_1, Q_2, Q_3

These points Q_1, Q_2, Q_3 have several properties related with Simson lines obtained by manipulation of third degree polynomials. They derive from classical properties of triples of points on the circumcircle of ABC having concurring Simson lines.

Theorem 5. *The points Q_1, Q_2, Q_3 are the antipodes on the circumcircle of the three points Q'_1, Q'_2, Q'_3 whose Simson lines pass through* **g**Q.

It follows that Q_1, Q_2, Q_3 are three real distinct points if and only if **g**Q lies inside the Steiner deltoid $\mathcal{H}_3$.

Theorem 6. *The Simson lines of Q_1, Q_2, Q_3 are tangent to the inconic $\mathcal{I}(Q)$ with perspector* $\mathtt{tg}Q$ *and center* **cg**Q. *They form a triangle $S_1S_2S_3$ perspective at* **cg**Q *to $Q_1Q_2Q_3$.*

S_1 is the common point of the Simson lines of Q'_1, Q_2, Q_3. These points S_1, S_2, S_3 are the reflections of Q_1, Q_2, Q_3 in **cg**Q. See Figure 5.

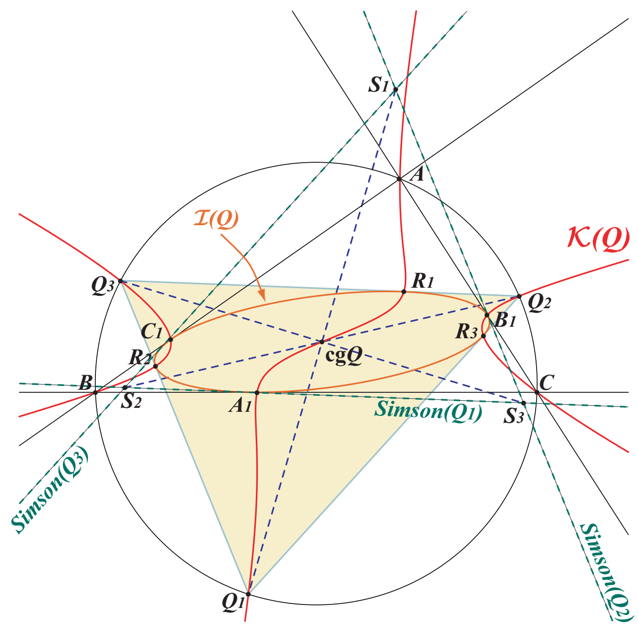

Figure 5. $\mathcal{K}(Q)$ and Simson lines

Another computation involving symmetric functions of the roots of a third degree polynomial gives

Theorem 7. *$\mathcal{K}(Q)$ meets the circumcircle at A, B, C with tangents concurring at the Lemoine point K of ABC and three other points Q_1, Q_2, Q_3 where the tangents are also concurrent at the Lemoine point of $Q_1Q_2Q_3$.*

This generalizes the property already encountered in a family of pivotal cubics seen in [4, §4]. Since the two triangles ABC and $Q_1Q_2Q_3$ are inscribed in the circumcircle, there must be a conic inscribed in both triangles. This gives

Theorem 8. *The inconic $\mathcal{I}(Q)$ with perspector* $\mathtt{tg}Q$ *is inscribed in the two triangles ABC and $Q_1Q_2Q_3$. It is also inscribed in the triangle formed by the Simson lines of Q_1, Q_2, Q_3.*

$\mathcal{K}(Q)$ meets $\mathcal{I}(Q)$ at six points which are the contacts of $\mathcal{I}(Q)$ with the sidelines of the two triangles. Three of them are the vertices A_1, B_1, C_1 of the cevian triangle of $\mathbf{tg}Q$ in ABC. The other points R_1, R_2, R_3 are the intersections of the sidelines of $Q_1Q_2Q_3$ with the cevian lines of H in $S_1S_2S_3$. In other words, $R_1 = HS_1 \cap Q_2Q_3$, etc. See Figure 5. Note that the reflections of R_1, R_2, R_3 in the center **cg**Q of $\mathcal{I}(Q)$ are the contacts T_1, T_2, T_3 of the Simson lines of Q_1, Q_2, Q_3 with $\mathcal{I}(Q)$.

5. Infinite points on $\mathcal{K}(Q)$ and intersection with $\mathrm{p}\mathcal{K}_{\inf}(Q)$

Proposition 9. *$\mathcal{K}(Q)$ meets the line at infinity at the same points as the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\inf}(Q)$ with pivot* **g**Q.

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} a^2\, qr\, x\, (c^2y^2 - b^2z^2) + (x+y+z) \sum_{\text{cyclic}} a^2\, p\, (c^2q - b^2r)\, yz = 0. \tag{4}$$

Any infinite point on $\mathcal{K}(Q)$ is also a point on the cubic

$$\sum_{\text{cyclic}} a^2\, qr\, x\, (c^2y^2 - b^2z^2) = 0 \iff \sum_{\text{cyclic}} \frac{x}{p}\left(\frac{y^2}{b^2} - \frac{z^2}{c^2}\right) = 0, \tag{5}$$

which is the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\inf}(Q)$ with pivot **g**Q. □

The six other common points of $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\inf}(Q)$ lie on the circumhyperbola through Q and K. They are A, B, C, Q and the two points E_1, E_2. Figure 6 shows these cubics when $Q = X_{55}$ thus **g**Q is the Gergonne point X_7. Recall that the points E_1, E_2 are X_9, X_{57}.

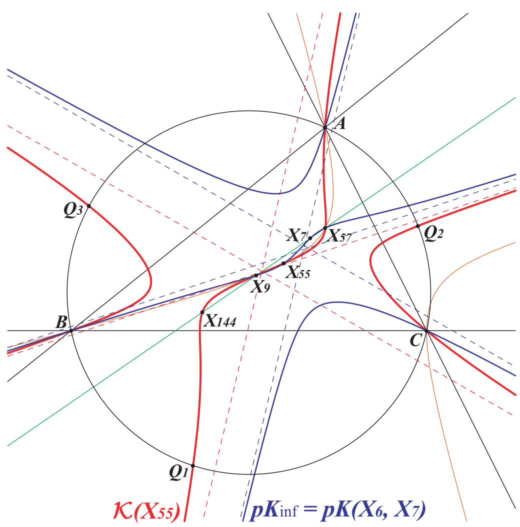

Figure 6. $\mathcal{K}(Q)$ and $\mathrm{p}\mathcal{K}_{\inf}(Q)$ when $Q = X_{55}$

6. $\mathcal{K}(Q)$ and the inconic with center cg$Q$

Proposition 10. *The cubic $\mathcal{K}(Q)$ contains the four foci of the inconic with center* **cg**Q *and perspector* **tg**Q.

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} px(c^2q-b^2r)(c^2y^2+b^2z^2)-2\left(\sum_{\text{cyclic}} a^2(b^2-c^2)qr\right)xyz \\ -\sum_{\text{cyclic}} px(c^2q+b^2r)(c^2y^2-b^2z^2)=0. \tag{6}$$

Indeed,

$$\sum_{\text{cyclic}} px(c^2q-b^2r)(c^2y^2+b^2z^2)-2\left(\sum_{\text{cyclic}} a^2(b^2-c^2)qr\right)xyz=0 \tag{7}$$

is the equation of the non-pivotal isogonal circular cubic $n\mathcal{K}_6(Q)$ which is the locus of foci of inconics with center on the line through G, **cg**Q and

$$\sum_{\text{cyclic}} px(c^2q+b^2r)(c^2y^2-b^2z^2)=0 \tag{8}$$

is the equation of the pivotal isogonal cubic p$\mathcal{K}_6(Q)$ with pivot **cg**Q. The two cubics $\mathcal{K}(Q)$ and p$\mathcal{K}_6(Q)$ obviously contain the above mentioned foci. □

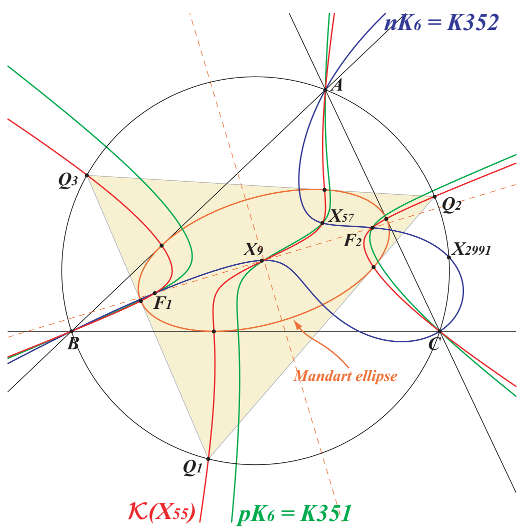

Figure 7. $\mathcal{K}(Q)$ and the related cubics $n\mathcal{K}_6(Q)$, p$\mathcal{K}_6(Q)$ when $Q=X_{55}$

These two cubics generate a pencil of cubics containing $\mathcal{K}(Q)$. Note that p$\mathcal{K}_6(Q)$ is a member of the pencil of isogonal pivotal cubics generated by p$\mathcal{K}_{\text{inf}}(Q)$ and

$\mathrm{p}\mathcal{K}_{\mathrm{circ}}(Q)$. The root of $n\mathcal{K}_6(Q)$ is the infinite point of the trilinear polar of $\mathbf{tg}Q$. Figure 7 shows these cubics when $Q = X_{55}$. The inscribed conic is the Mandart ellipse.

In the example above, $\mathcal{K}(Q)$ contains the center **cg**Q of the inconic $\mathcal{I}(Q)$ but this is not generally true. We have

Theorem 11. *$\mathcal{K}(Q)$ contains the center* **cg**Q *of $\mathcal{I}(Q)$ if and only if Q lies on the cubic* **K172** $= \mathrm{p}\mathcal{K}(X_{32}, X_3)$.

Since we know that $\mathcal{K}(Q)$ contains the perspector **tg**Q of this same inconic when it is a pivotal cubic, it follows that there are only two cubics $\mathcal{K}(Q)$ passing through the foci, the center, the perspector of $\mathcal{I}(Q)$ and its contacts with the sidelines of ABC. These cubics are obtained when

(i) $Q = X_6$: $\mathcal{K}(X_6)$ is the Thomson cubic **K002** and $\mathcal{I}(Q)$ is the Steiner inscribed ellipse,

(ii) $Q = X_{25}$: $\mathcal{K}(X_{25})$ is **K233** $= \mathrm{p}\mathcal{K}(X_{25}, X_4)$.

In the latter case, **cg**$Q = X_6$, **tg**$Q = X_4$, **ag**$Q = X_{193}$, $\mathcal{I}(Q)$ is the K-ellipse, [1] the infinite points are those of **K169** = $\mathrm{p}\mathcal{K}(X_6, X_{69})$, the points on the circumcircle are those of $\mathrm{p}\mathcal{K}(X_6, X_{193})$. See Figure 8.

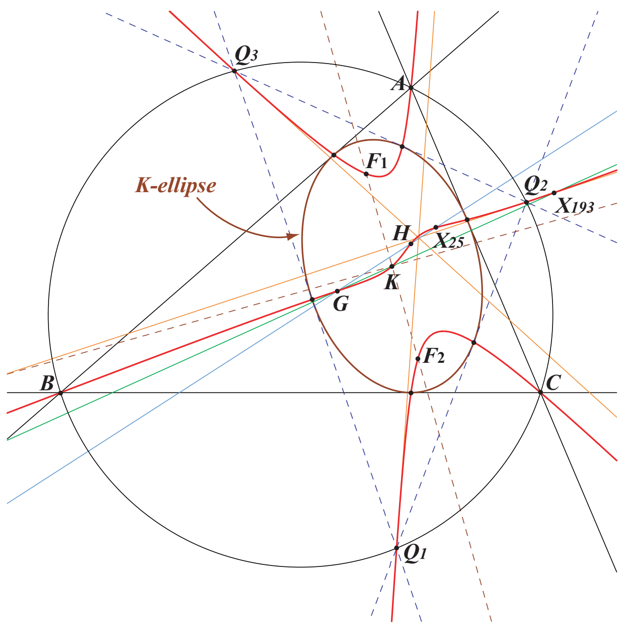

Figure 8. $\mathcal{K}(X_{25})$ and the related K-ellipse

[1]The $K-$ellipse is actually an ellipse only when triangle ABC is acute angled.

7. $\mathcal{K}(Q)$ and the Steiner ellipse

Proposition 12. *The cubic* $\mathcal{K}(Q)$ *meets the Steiner ellipse at the same points as* $\mathrm{p}\mathcal{K}(\mathbf{tg}Q,\ Q)$.

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} a^2\, p\, x\, (b^2 r y^2 - c^2 q z^2) + (xy + yz + zx) \sum_{\text{cyclic}} a^2\,(b^2 - c^2)\, qr\, x = 0. \tag{9}$$

Indeed,

$$\sum_{\text{cyclic}} a^2\, p\, x\, (b^2 r y^2 - c^2 q z^2) = 0 \iff \sum_{\text{cyclic}} x \left(\frac{y^2}{c^2 q} - \frac{z^2}{b^2 r}\right) = 0 \tag{10}$$

is the equation of the pivotal cubic $\mathrm{p}\mathcal{K}(\mathbf{tg}Q,\ Q)$. □

Note that $\sum_{\text{cyclic}} a^2\,(b^2 - c^2)\, qr\, x = 0$ is the equation of the line $Q\mathbf{tg}Q$. This will be construed in the next paragraph.

8. $\mathcal{K}(Q)$ and rectangular hyperbolas

Let $P = u : v : w$ be a given point and let $\mathcal{H}(P)$, $\mathcal{H}(\mathbf{g}P)$ be the two rectangular circum-hyperbolas passing through P, $\mathbf{g}P$ respectively. These have equations

$$\sum_{\text{cyclic}} u(S_B v - S_c w) yz = 0 \quad \text{and} \quad \sum_{\text{cyclic}} \left(\frac{S_B w}{c^2} - \frac{S_C v}{b^2}\right) yz = 0.$$

P must not lie on the McCay cubic in order to have two distinct hyperbolas. Indeed, $\mathbf{g}P$ lies on $\mathcal{H}(P)$ if and only if P lies on the line $O\mathbf{g}P$ i.e. P and $\mathbf{g}P$ are two isogonal conjugate points collinear with O.

Let $\mathcal{L}(Q)$ and $\mathcal{L}'(Q)$ be the two lines passing through Q with equations

$$\sum_{\text{cyclic}} a^2\,(vr(qx - py) - wq(rx - pz)) = 0$$

and

$$\sum_{\text{cyclic}} b^2 c^2 pu(v + w)(ry - qz) = 0.$$

These lines $\mathcal{L}(Q)$ and $\mathcal{L}'(Q)$ can be construed as the trilinear polars of the $Q-$isoconjugates of the infinite points of the polars of P and $\mathbf{g}P$ in the circumcircle.

The equation of $\mathcal{K}(Q)$ can be written in the form

$$\begin{aligned}\left(\sum_{\text{cyclic}} u(S_B v - S_c w) yz\right)\left(\sum_{\text{cyclic}} a^2\,(vr(qx - py) - wq(rx - pz))\right)\\ = \left(\sum_{\text{cyclic}} \left(\frac{S_B w}{c^2} - \frac{S_C v}{b^2}\right) yz\right)\left(\sum_{\text{cyclic}} b^2 c^2 pu(v + w)(ry - qz)\right)\end{aligned} \tag{11}$$

which will be loosely written under the form :

$$\mathcal{H}(P) \cdot \mathcal{L}(Q) = \mathcal{H}(\mathbf{g}P) \cdot \mathcal{L}'(Q).$$

If we recall that $\mathcal{K}(Q)$ and $\mathcal{H}(P)$ have already four common points namely A, B, C, H and that $\mathcal{K}(Q)$, $\mathcal{L}(Q)$ and $\mathcal{L}'(Q)$ all contain Q, then we have

Corollary 13. *$\mathcal{K}(Q)$ meets $\mathcal{H}(P)$ again at two points on the line $\mathcal{L}'(Q)$ and $\mathcal{H}(\mathbf{g}P)$ again at two points on the line $\mathcal{L}(Q)$.*

For example, with $P = G$, $\mathcal{H}(P)$ is the Kiepert hyperbola and $\mathcal{L}'(Q)$ is the line $Q\mathbf{gt}Q$, $\mathcal{H}(\mathbf{g}P)$ is the Jerabek hyperbola and $\mathcal{L}(Q)$ is the line $Q\mathbf{tg}Q$.

9. Further representations of $\mathcal{K}(Q)$

Proposition 14. *For varying Q, the cubics $\mathcal{K}(Q)$ form a net of cubics.*

Proof. This follows by writing the equation of $\mathcal{K}(Q)$ in the form

$$\begin{aligned} &\sum_{\text{cyclic}} a^2qr\,x\left(c^2y(x+z) - b^2z(x+y)\right) = 0 \\ \iff &\sum_{\text{cyclic}} a^2qr\,x\left(x(c^2y - b^2z) - (b^2-c^2)yz\right) = 0. \end{aligned} \tag{12}$$

The equation $c^2y(x+z) - b^2z(x+y) = 0$ is that of the rectangular circumhyperbola $\mathcal{H}_A$ tangent at A to the symmedian AK. Its center is the midpoint of BC. Its sixth common point with $\mathcal{K}(Q)$ is the intersection of the lines AQ and $A_1\mathbf{ag}Q$. Thus the net is generated by the three decomposed cubics which are the union of a sideline of ABC and the corresponding hyperbola such as $\mathcal{H}_A$. □

Proposition 15. *$\mathcal{K}(Q)$ is a pivotal cubic $\mathrm{p}\mathcal{K}(Q)$ if and only if Q lies on the circumhyperbola $\mathcal{H}$ passing through G and K.*

Proof. We write the equation of $\mathcal{K}(Q)$ in the form

$$\sum_{\text{cyclic}} b^2c^2\,p\,x\,(ry^2 - qz^2) + \left(\sum_{\text{cyclic}} a^2(b^2-c^2)qr\right)xyz = 0. \tag{13}$$

Recall that $\mathcal{K}(Q)$ meets the sidelines of triangle ABC again at the vertices of the cevian triangle of $\mathbf{tg}Q$. Thus, the cubic is a pivotal cubic is and only if the term in xyz vanishes. It is now sufficient to observe that the equation of the hyperbola $\mathcal{H}$ is $\sum_{\text{cyclic}} a^2(b^2-c^2)yz = 0$. □

See a more detailed study of these $\mathrm{p}\mathcal{K}(Q)$ in §10.1.

Proposition 16. *The cubic $\mathcal{K}(Q)$ belongs to another pencil of similar cubics generated by another pivotal cubic and another isogonal non-pivotal cubic.*

Proof.

$$\sum_{\text{cyclic}} px(c^2q - b^2r)(c^2y^2 + b^2z^2) - \left(\sum_{\text{cyclic}} a^2(b^2 - c^2)qr\right) xyz \\ + \sum_{\text{cyclic}} a^4qr(y - z)yz = 0. \tag{14}$$

Indeed,

$$\sum_{\text{cyclic}} px(c^2q - b^2r)(c^2y^2 + b^2z^2) - \left(\sum_{\text{cyclic}} a^2(b^2 - c^2)qr\right) xyz = 0 \tag{15}$$

is the equation of the non-pivotal isogonal cubic $n\mathcal{K}_7(Q)$ with root the infinite point of the trilinear polar of $\mathbf{tg}Q$ again and

$$\sum_{\text{cyclic}} a^4qr(y - z)yz = 0 \tag{16}$$

is the equation of the pivotal cubic $\mathrm{p}\mathcal{K}_7(Q)$ with pivot the centroid G and pole the $X_{32}-$isoconjugate of Q i.e. the point **gtg**Q. □

The cubics $n\mathcal{K}_6(Q)$ and $n\mathcal{K}_7(Q)$ obviously coincide when Q lies on the circumhyperbola $\mathcal{H}$ passing through G and K. Figure 9 shows these cubics when $Q = X_{55}$.

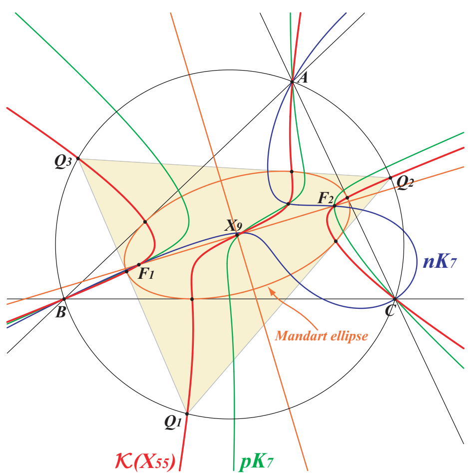

Figure 9. $\mathcal{K}(Q)$ and the related cubics $n\mathcal{K}_7(Q)$, $\mathrm{p}\mathcal{K}_7(Q)$ when $Q = X_{55}$

10. Special cubics $\mathcal{K}(Q)$

10.1. *Pivotal cubics* $\mathrm{p}\mathcal{K}(Q)$. Recall that for any point Q on the circumhyperbola $\mathcal{H}$ passing through G and K the cubic $\mathcal{K}(Q)$ becomes a pivotal cubic with pole Q and pivot $\mathbf{tg}Q$ on the Kiepert hyperbola. In this case, $\mathcal{K}(Q)$ has equation :

$$\sum_{\text{cyclic}} b^2c^2\, p\, x\,(ry^2 - qz^2) = 0 \iff \sum_{\text{cyclic}} \frac{x}{a^2}\left(\frac{y^2}{q} - \frac{z^2}{r}\right) = 0 \qquad (17)$$

The isopivot (secondary pivot) is clearly the Lemoine point K since the tangents at A, B, C are the symmedians. The points $\mathbf{g}Q$ and $\mathbf{ag}Q$ lie on the line GK namely the tangent at G to the Kiepert hyperbola.

These cubics form a pencil of pivotal cubics passing through A, B, C, G, H, K and tangent to the symmedians. Recall that they have the remarkable property to intersect the circumcircle at three other points Q_1, Q_2, Q_3 with concurrent tangents such that $\mathbf{ag}Q$ is the orthocenter of $Q_1Q_2Q_3$. See [4] for further informations.

This pencil is generated by the Thomson cubic **K002** (the only isogonal cubic) and by **K141** (the only isotomic cubic). See **CL043** in [2] for a selection of other cubics of the pencil among them **K273**, the only circular cubic, and **K233** seen above.

10.2. *Circular cubics* $\mathcal{K}(Q)$. We have seen that $\mathcal{K}(Q)$ meets the line at infinity at the same points as the pivotal isogonal cubic $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ with pivot $\mathbf{g}Q$. It easily follows that $\mathcal{K}(Q)$ is a circular cubic if and only if $\mathrm{p}\mathcal{K}_{\mathrm{inf}}(Q)$ is itself a circular cubic therefore if and only if $\mathbf{g}Q$ lies at infinity hence Q must lie on the circumcircle $\mathcal{C}$. Thus, we have :

Theorem 17. *For any point Q on the circumcircle, $\mathcal{K}(Q)$ is a circular cubic with singular focus on the circle with center O and radius $2R$. The tangent at Q always passes through O.*

The real asymptote envelopes a deltoid, the homothetic of the Steiner deltoid under $h(G, 4)$. See Figure 10.

For example, **K273** (obtained for $Q = X_{111}$, the Parry point) and **K306** (obtained for $Q = X_{759}$) are two cubics of this type in [2]. See also the bottom of the page **CL035** in [2].

10.3. *Lemoine generalized cubics* $\mathcal{K}(Q)$. A necessary (but not sufficient) condition to obtain a Lemoine generalized cubic $\mathcal{K}(Q)$ is that the cevian triangle of $\mathbf{tg}Q$ must be a pedal triangle. Hence, $\mathbf{tg}Q$ must be a point on the Lucas cubic **K007** therefore Q must be on its isogonal transform **K172**.

The only identified points that give a Lemoine generalized cubic are H and X_{56}.

$\mathcal{K}(H)$ is **K028**, the third Musselman cubic. It is also the only cubic with asymptotes making 60° angles with one another i.e. the only equilateral cubic of this type.

$\mathcal{K}(X_{56})$ is **K360**, at the origin of this note. See Figure 11.

The conic inscribed in the triangles ABC and $Q_1Q_2Q_3$ is the incircle of ABC since $\mathbf{tg}X_{56}$ is the Gergonne point X_7. $Q_1Q_2Q_3$ is a poristic triangle.

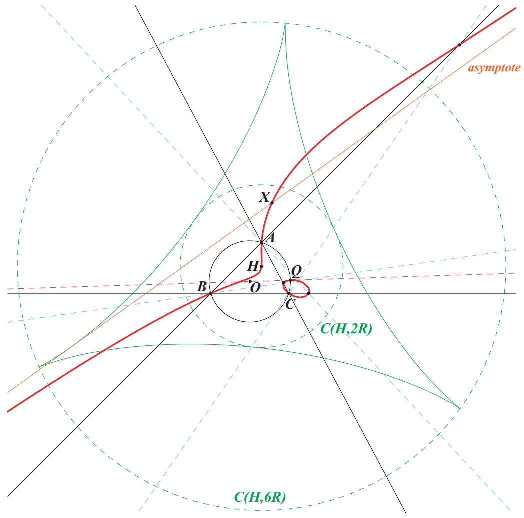

Figure 10. Circular cubics $\mathcal{K}(Q)$ and deltoid

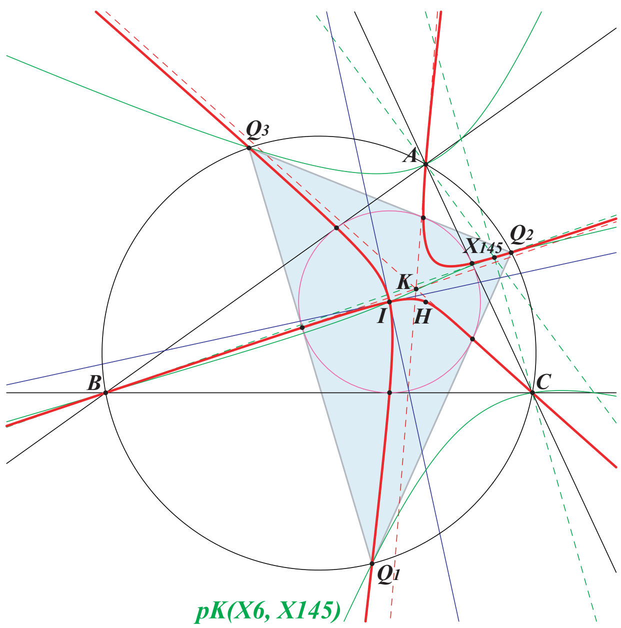

Figure 11. The Lemoine generalized cubic $\mathcal{K}(X_{56})$ = **K360**

10.4. $\mathcal{K}(X_{32})$. $\mathcal{K}(X_{32})$ has the remarkable property to have its six tangents at its common points with the circumcircle concurrent at the Lemoine point K. It follows that the triangles ABC and $Q_1Q_2Q_3$ have the same Lemoine point and the same Brocard axis. The polar conic of K is therefore the circumcircle.

The satellite conic of the circumcircle is the Brocard ellipse whose real foci Ω_1, Ω_2 (Brocard points) lie on the cubic. See Figure 12.

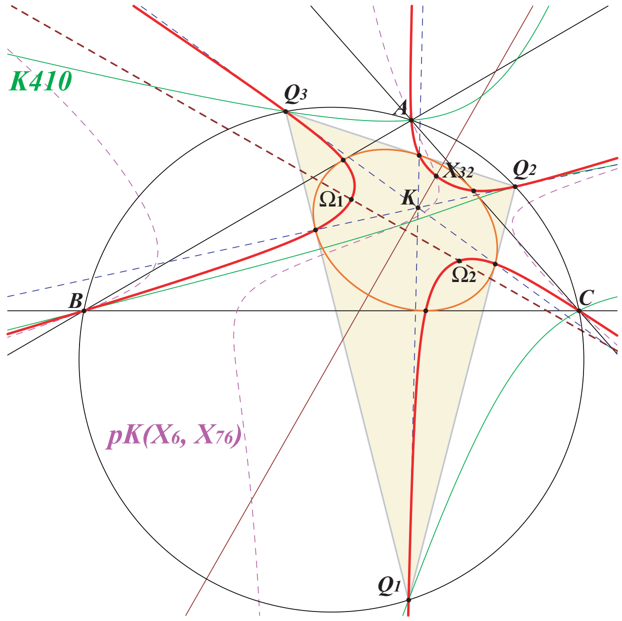

Figure 12. The cubic $\mathcal{K}(X_{32})$

Remark. $\mathcal{K}(X_{32})$ belongs to a pencil of circum-cubics having the same property to meet the circumcircle at six points A, B, C, Q_1, Q_2, Q_3 with tangents concurring at K hence the polar conic of K is always the circumcircle.

The cubic of the pencil passing through the given point $P = u : v : w$ has an equation of the form

$$\sum_{\text{cyclic}} a^4 vwyz \left((c^2v - b^2w)x - (u(c^2y - b^2z)\right) = 0,$$

which shows that the pencil is generated by three decomposed cubics, one of them being the union of the sidelines AB, AC and the line joining P to the feet K_a of the $A-$symmedian, the other two similarly. Each cubic meets the Brocard ellipse at six points which are the tangentials of the six points above. Three of them are K_a, K_b, K_c and the other points are the contacts of the Brocard ellipse with the sidelines of $Q_1Q_2Q_3$.

10.5. $\mathcal{K}(X_{54})$. $\mathcal{K}(X_{54}) =$ **K361** is the only cubic of the family meeting the circumcircle at the vertices of an equilateral triangle $Q_1Q_2Q_3$ namely the circumnormal triangle. The tangents at these points concur at O. **K361** is the isogonal transform of **K026**, the (first) Musselman cubic and the locus of pivots of pivotal cubics that pass through the vertices of the circumnormal triangle. See Figure 13 and further details in [2].

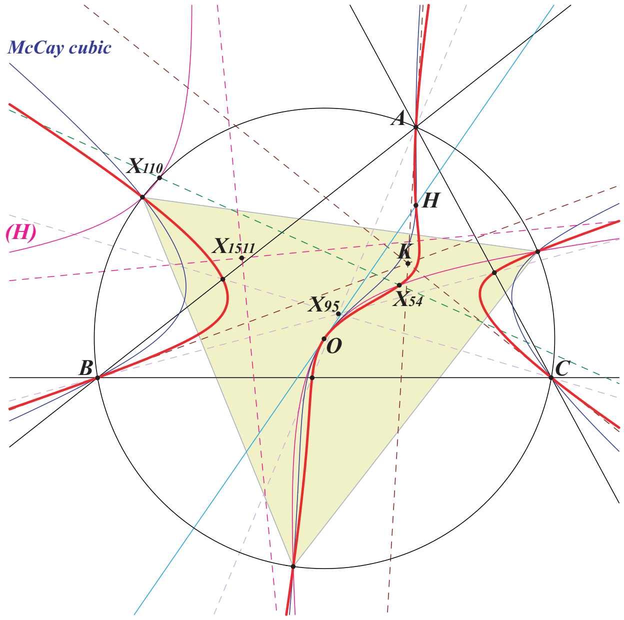

Figure 13. The cubic $\mathcal{K}(X_{54}) =$ **K361**

10.6. $\mathcal{K}(Q)$ *with concurring asymptotes.* $\mathcal{K}(Q)$ has three (not necessarily all real) concurring asymptotes if and only if Q lies on a circumcubic passing through O, H, X_{140}. This latter cubic is a $\mathcal{K}_{60}^{+}$ i.e. it has three real concurring asymptotes making $60°$ angles with one another. These are the parallels at X_{547} (the midpoint of X_2, X_5) to those of the McCay cubic **K003**. The cubic meets the cirumcircle at the same points as $\mathrm{p}\mathcal{K}(X_6, X_{140})$ where X_{140} is the midpoint of X_3, X_5. See Figure 14.

The two cubics $\mathcal{K}(H) =$ **K028** and $\mathcal{K}(X_{140})$ have concurring asymptotes but their common point is not on the curve. These are $\mathcal{K}^{+}$ cubics.

On the contrary, $\mathcal{K}(X_3)$ is a central cubic and the asymptotes meet at O on the curve. It is said to be a $\mathcal{K}^{++}$ cubic. See Figure 15.

11. Isogonal transform of $\mathcal{K}(Q)$

Under isogonal conjugation with respect to ABC, $\mathcal{K}(Q)$ is transformed into another circum-cubic **g**$\mathcal{K}(Q)$ meeting $\mathcal{K}(Q)$ again at the four foci of $\mathcal{I}(Q)$ and at the two points E_1, E_2 intersections of the line $G\mathbf{ag}Q$ with the conic $ABCKQ$.

Thus, $\mathcal{K}(Q)$ and **g**$\mathcal{K}(Q)$ have nine known common points. When they are distinct i.e. when Q is not K i.e. when $\mathcal{K}(Q)$ is not the Thomson cubic, they generate a pencil of cubics which contains $\mathrm{p}\mathcal{K}(X_6, \mathbf{cg}Q)$.

It is easy to verify that **g**$\mathcal{K}(Q)$

(i) contains the circumcenter O, **g**Q, the midpoints of ABC,

(ii) is tangent at A, B, C to the cevian lines of the $X_{32}-$isoconjugate of Q i.e. the point **gtg**Q,

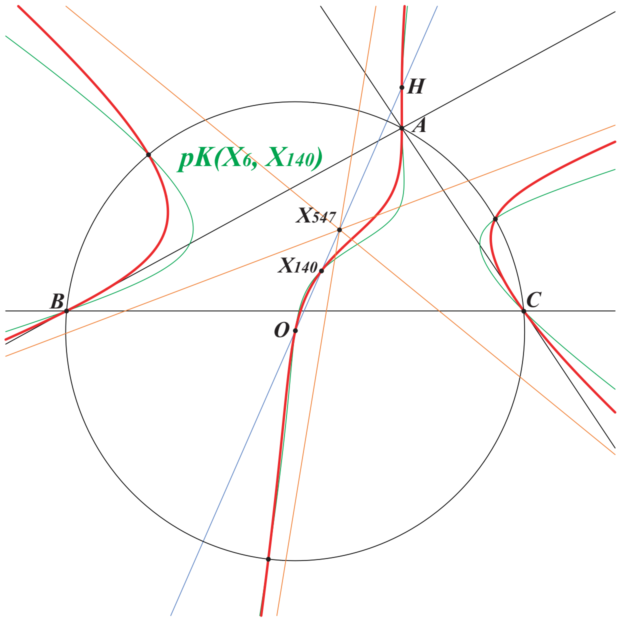

Figure 14. The cubic $\mathcal{K}_{60}^{+}$

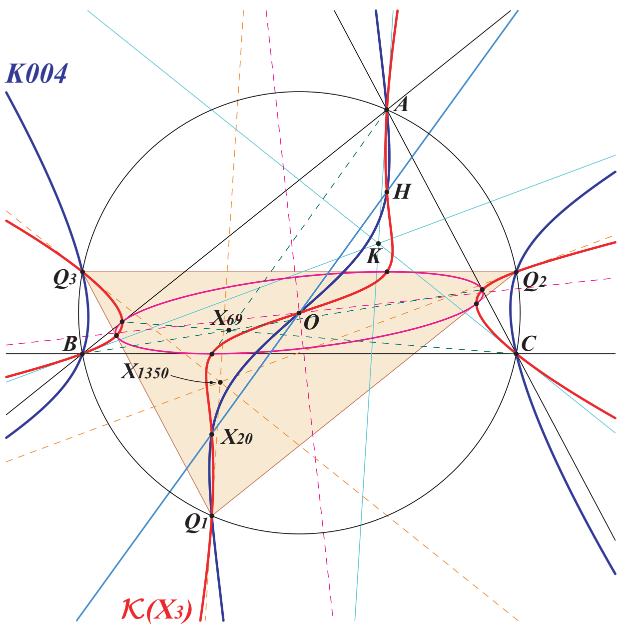

Figure 15. The cubic $\mathcal{K}(X_3)$

(iii) meets the circumcircle at the same points as $\mathrm{p}\mathcal{K}(X_6, \mathbf{g}Q)$ hence the orthocenter of the triangle $O_1O_2O_3$ formed by these points is $\mathbf{g}Q$; following a result of [4],

the inconic with perspector **tcg**Q is inscribed in ABC and $O_1O_2O_3$,
(iv) has the same asymptotic directions as $\mathrm{p}\mathcal{K}(X_6, \mathbf{ag}Q)$.

Except the case $Q = K$, **g**$\mathcal{K}(Q)$ cannot be a cubic of type $\mathcal{K}(Q)$.

The tangents to **g**$\mathcal{K}(Q)$ at A, B, C are still concurrent (at **gtg**Q) but in general, the tangents at the other intersections of **g**$\mathcal{K}(Q)$ with the circumcircle are not now concurrent unless Q lies on a circular circum-quartic which is the isogonal transform of **Q063**. This quartic contains X_1, X_3, X_6, X_{64}, X_{2574}, X_{2575}, the excenters.

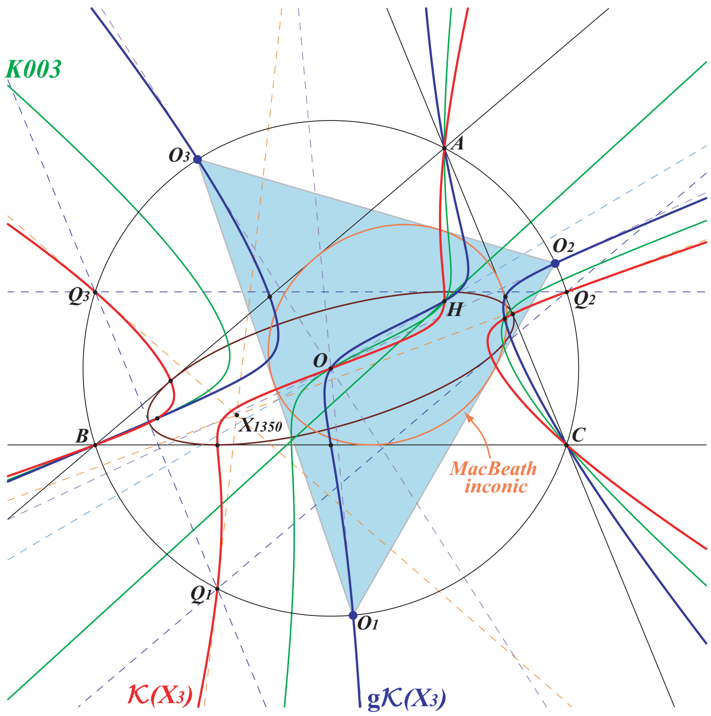

Figure 16. $\mathcal{K}(X_3)$, **g**$\mathcal{K}(X_3)$ and **K003**

Figure 16 presents $\mathcal{K}(X_3)$ and **g**$\mathcal{K}(X_3)$. These two cubics generate a pencil which contains the McCay cubic **K003** and the Euler isogonal focal cubic **K187**. The nine common points of these four cubics are A, B, C, O, H and the four foci of the inscribed conic with center O.

g$\mathcal{K}(X_3)$ meets the circumcircle at the same points O_1, O_2, O_3 as the Orthocubic **K006** and the triangles ABC, $O_1O_2O_3$ share the same orthocenter H therefore the same Euler line. The tangents at O_1, O_2, O_3 concur at O and those at A, B, C concur at X_{25}. The MacBeath inconic (with center X_5, foci O and H) is inscribed in ABC and $O_1O_2O_3$.

g$\mathcal{K}(X_3)$ meets the line at infinity at the same points as the Darboux cubic **K004**. Hence, its three asymptotes are parallel to the altitudes of ABC.

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A Short Proof of Lemoine's Theorem

Cosmin Pohoata

Abstract. We give a short proof of Lemoine's theorem that the Lemoine point of a triangle is the unique point which is the centroid of its own pedal triangle.

Lemoine's theorem states that the Lemoine (symmedian) point of a triangle is the unique point which is the centroid of its own pedal triangle. A proof of the fact that the Lemoine point has this property can be found in Honsberger [4, p.72]. The uniqueness part was conjectured by Clark Kimberling in the very first Hyacinthos message [6], and was subsequently confirmed by computations by Barry Wolk [7], Jean-Pierre Ehrmann [2], and Paul Yiu [8, §4.6.2]. Darij Grinberg [3] has given a synthetic proof. In this note we give a short proof by applying two elegant results on orthologic triangles.

Lemma 1. *If P is a point in plane of triangle ABC, with pedal triangle $A'B'C'$, then the perpendiculars from A to $B'C'$, from B to $C'A'$, from C to $A'B'$ are concurrent at Q, the isogonal conjugate of P.*

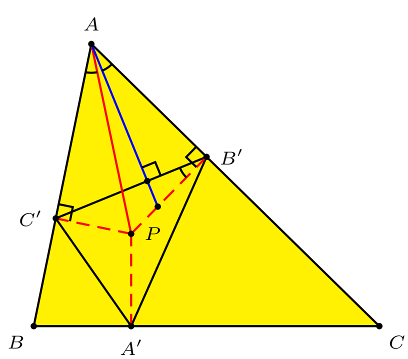

Figure 1

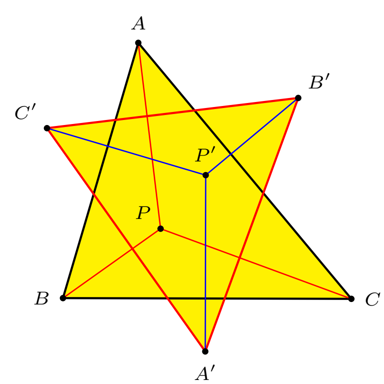

Figure 2

This is quite well-known. See, for example, [5, Theorem 237]. Figure 1 shows that AP and the perpendicular from A to $B'C'$ are isogonal with reference to A. From this Lemma 1 follows. The next beautiful result, illustrated in Figure 2, is the main subject of [1].

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Theorem 2 (Daneels and Dergiades)**.** *If triangles ABC and $A'B'C'$ are orthologic with centers P, P', with the perpendiculars from A, B, C to $B'C'$, $C'A'$, $A'B'$ intersecting at P and those from A', B', C' to BC, CA, AB intersecting at P', then the barycentric coordinates of P with reference to ABC are equal to the barycentric coordinates of P' with reference to $A'B'C'$.*

Now we prove Lemoine's theorem.

Let K be the Lemoine (symmedian) point of triangle ABC, and $A'B'C'$ its pedal triangle. According to Lemma 1, the perpendiculars from A to $B'C'$, from B to $C'A'$, from C to $A'B'$ are concurrent at the centroid G of ABC. Now since ABC and $A'B'C'$ are orthologic, with G as one of the orthology centers, by Theorem 2, the perpendiculars from A' to BC, from B' to CA, from C' to AB are concurrent at the centroid G' of $A'B'C'$. Hence, the symmedian point K coincides with the centroid of its pedal triangle.

Conversely, let P a point with pedal triangle $A'B'C'$, and suppose P is the centroid of $A'B'C'$; it has homogeneous barycentric coordinates $(1:1:1)$ with reference to $A'B'C'$. Since ABC and $A'B'C'$ are orthologic, by Theorem 2, we have that the perpendiculars from A to $B'C'$, from B to $C'A'$, from C to $A'B'$ are concurrent at a point Q with homogeneous barycentric coordinates $(1:1:1)$ with reference to ABC. This is the centroid G. By Lemma 1, this is also the isogonal conjugate of P. This shows that $P=K$, the Lemoine (symmedian) point.

This completes the proof of Lemoine's theorem.

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Means as Chords

Francisco Javier García Capitán

Abstract. On the circumcircle of a right triangle, we display chords whose lengths are the quadratic, arithmetic, geometric, and harmonic means of the two shorter sides.

Given two positive numbers a and b, the inequalities among their various means

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

are well known. In order, these are the harmonic, geometric, arithmetic, and quadratic means of a and b. Nelsen [1] has presented several few geometric proofs (without words). In the same spirit, we exhibit these various means as chords of a circle constructed from two segments of lengths a and b.

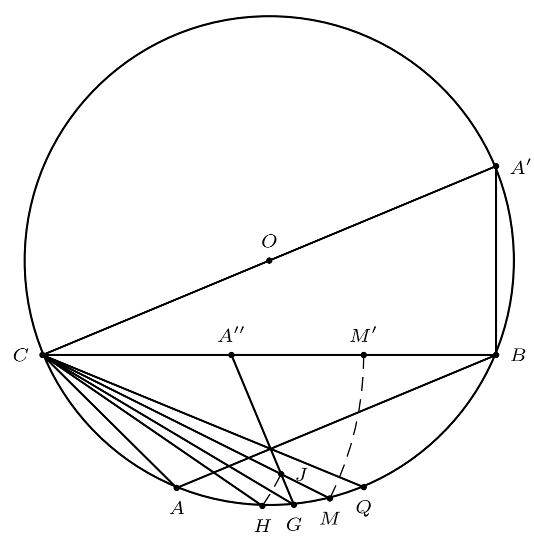

Figure 1

We shall assume $a \leq b$, and begin with a right triangle $A'BC$ with $A'B = a$, $CB = b$ and a right angle at B. Construct
(1) the circumcircle of the triangle (with center at the midpoint O of CA',

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(2) points A'' on the segment CB and and A the circle respectively such that $CA = CA'' = A'B$ and A, A' are on opposite sides of CB,
(3) the bisector AQ of angle ACB with Q on the circle (O),
(4) the midpoint M' of $A''B$ and the point M on the arc CAB with $CM = CM'$,
(5) the perpendicular from A'' to AB to intersect AM at J and the arc CAB at G,
(6) the point H on the arc CAB such that $CH = CJ$.

Proposition 1. *For the two segments CA and CB,*
(1) *CQ is the quadratic mean,*
(2) *CM is the arithmetic mean,*
(3) *CG is the geometric mean,*
(4) *CH is the harmonic mean.*

Proof. Note that the circle has radius $\frac{1}{2}\sqrt{a^2+b^2}$.

(1) Since $CA = A'B$, $CA'BA$ is an isosceles trapezoid, with AB parallel to CA'. Since CQ is the bisector of angle ACB, Q is the midpoint of the arc CAB, and OQ is perpendicular to AB. Hence, the radii OQ and OC are perpendicular to each other, and $CQ = \sqrt{2} \cdot OC = \frac{a^2+b^2}{2}$. This shows that CQ is the quadratic mean of a and b.

(2) $CM = CM' = \frac{1}{2}(a+b)$ is the arithmetic mean of a and b.

(3) Let $A''G$ intersect OA' at L. See Figure 2. Since CA' is parallel to AB, LG is perpendicular to CA'. From the similarity of the right triangles $CA''L$ and $CA'B$, we have $\frac{CL}{CA''} = \frac{CB}{CA'}$. In the right triangle $CA'G$, we have $CG^2 = CL \cdot CA' = CA'' \cdot CB = ab$. This shows that CG is the geometric mean of a and b.

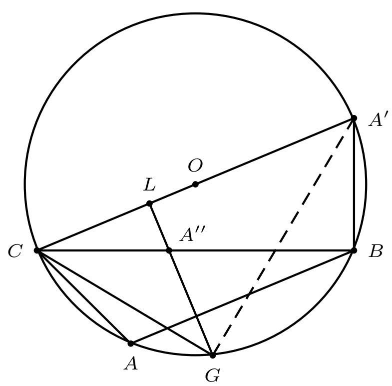

Figure 2

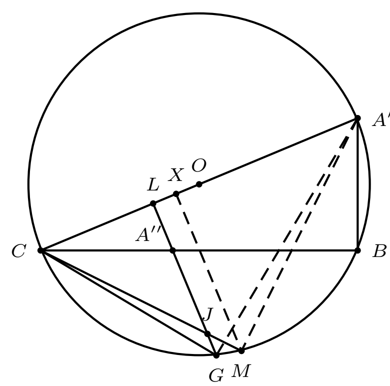

Figure 3

(4) Let the perpendicular from M to CA' intersect the latter at X. See Figure 3. From the similarity of triangles CLJ and CXM, we have

$$CJ = CM \cdot \frac{CL}{CX} = CM \cdot \frac{CL \cdot CA'}{CX \cdot CA'} = CM \cdot \frac{CG^2}{CM^2} = \frac{CG^2}{CM} = \frac{2ab}{a+b}.$$

This shows that $CH = CJ$ is the harmonic mean of a and b. □

We conclude with an interesting concurrency.

Proposition 2. *The lines AB, CQ, and $A''G$ are concurrent.*

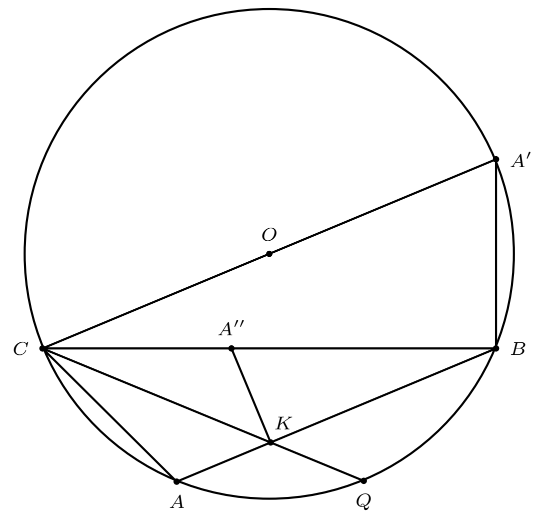

Figure 4

Proof. Let the bisector CQ of angle ACB intersect AB at K. See Figure 4. Clearly, the triangles ACK and $A''CK$ are congruent. Now,

$$\begin{aligned}\angle CA''K = \angle CAK = \angle CAB & \\ = 180^\circ - \angle CA'B & \qquad (C, A, B, A' \text{ concyclic}) \\ = \angle ABA' & \qquad (AB \text{ parallel to } CA') \\ = \angle ABC + 90^\circ. &\end{aligned}$$

It follows that $\angle A''KB = 90^\circ$, and $A''K$ is perpendicular to AB. This shows that K lies on $A''G$. □

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A Condition for a Circumscriptible Quadrilateral to be Cyclic

Mowaffaq Hajja

Abstract. We give a short proof of a characterization, given by M. Radić et al, of convex quadrilaterals that admit both an incircle and a circumcircle.

A convex quadrilateral is said to be *cyclic* if it admits a circumcircle (*i.e.*, a circle that passes through the vertices); it is said to be *circumscriptible* if it admits an incircle (*i.e.*, a circle that touches the sides internally). A quadrilateral is *bicentric* if it is both cyclic and circumscriptible. For basic properties of these quadrilaterals, see [7, Chapter 10, pp. 146–170]. One of the two main theorems in [5], namely Theorem 1 (p. 35), can be stated as follows:

Theorem. *Let $ABCD$ be a circumscriptible quadrilateral with diagonals AC and BD of lengths u and v respectively. Let a, b, c, and d be the lengths of the tangents from the vertices A, B, C, and D* (see Figure 1). *The quadrilateral $ABCD$ is cyclic if and only if $\frac{u}{v} = \frac{a+c}{b+d}$.*

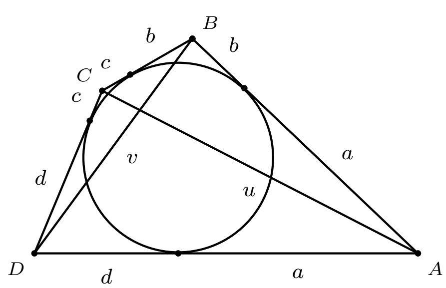

Figure 1

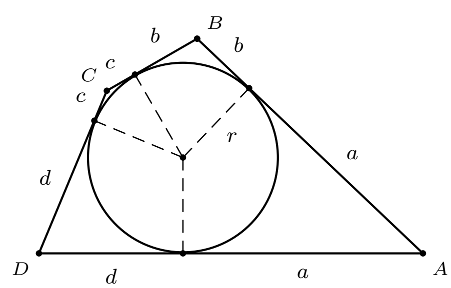

Figure 2

In this note, we give a proof that is much simpler than the one given in [5]. Our proof actually follows immediately from the three very simple lemmas below, all under the same hypothesis of the Theorem. Lemma 1 appeared as a problem in the MONTHLY [6] and Lemma 2 appeared in the solution of a quickie in the MAGAZINE [3], but we give proofs for the reader's convenience. Lemma 3 uses Lemma 2 and gives formulas for the lengths of the diagonals of a circumscriptible quadrilateral counterpart to those for cyclic quadrilaterals as given in [1], [7, § 10.2, p. 148], and other standard textbooks.

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Lemma 1. *$ABCD$ is cyclic if and only if $ac = bd$.*

Proof. Let $ABCD$ be any convex quadrilateral, not necessarily admitting an incircle, and let its vertex angles be $2A$, $2B$, $2C$, and $2D$. Then A, B, C, and D are acute, and $A + B + C + D = 180°$. We shall show that

$$ABCD \text{ is cyclic} \quad \Leftrightarrow \quad \tan A \tan C = \tan B \tan D. \tag{1}$$

If $ABCD$ is cyclic, then $A+C = B+D = 90°$, and $\tan A \tan C = \tan B \tan D$, each being equal to 1. Conversely, if $ABCD$ is not cyclic, then one may assume that $A + C > 90°$ and $B + D < 90°$. From

$$0 > \tan(A + C) = \frac{\tan A + \tan C}{1 - \tan A \tan C}$$

and the fact that A and C are acute, we conclude that $\tan A \tan C > 1$. Similarly $\tan B \tan D < 1$, and therefore $\tan A \tan C \neq \tan B \tan D$. This proves (1).

The result follows by applying (1) to the given quadrilateral, and using $\tan A = r/a$, etc., where r is the radius of the incircle (as shown in Figure 2). □

Lemma 2. *The radius r of the incircle is given by*

$$r^2 = \frac{bcd + acd + abd + abc}{a + b + c + d}. \tag{2}$$

Proof. Again, let the vertex angles of $ABCD$ be $2A$, $2B$, $2C$, and $2D$, and let

$$\alpha = \tan A,\ \beta = \tan B,\ \gamma = \tan C,\ \delta = \tan D.$$

Let $\varepsilon_1 = \sum \alpha$, $\varepsilon_2 = \sum \alpha\beta$, $\varepsilon_3 = \sum \alpha\beta\gamma$, and $\varepsilon_4 = \alpha\beta\gamma\delta$ be the elementary symmetric polynomials in α, β, γ, and δ. By [4, § 125, p. 132], we have

$$\tan(A + B + C + D) = \frac{\varepsilon_1 - \varepsilon_3}{1 - \varepsilon_2 + \varepsilon_4}.$$

Since $A+B+C+D = 180°$, it follows that $\tan(A+B+C+D) = 0$ and hence $\varepsilon_1 = \varepsilon_3$, *i.e.*,

$$\frac{r}{a} + \frac{r}{b} + \frac{r}{c} + \frac{r}{d} = \frac{r^3}{bcd} + \frac{r^3}{acd} + \frac{r^3}{abd} + \frac{r^3}{abc},$$

and (2) follows. □

Lemma 3.

$$u^2 = \frac{a + c}{b + d}\left((a + c)(b + d) + 4bd\right), \quad \textit{and} \quad v^2 = \frac{b + d}{a + c}\left((a + c)(b + d) + 4ac\right).$$

Proof. Again, let the vertex angles of $ABCD$ be $2A$, $2B$, $2C$, and $2D$. Then

$$\begin{aligned}
\cos 2A &= \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{a^2 - r^2}{a^2 + r^2} \\
&= \frac{a^2(a + b + c + d) - (bcd + acd + abd + abc)}{a^2(a + b + c + d) + (bcd + acd + abd + abc)}, \text{ by (2)} \\
&= \frac{a^2(a + b + c + d) - (bcd + acd + abd + abc)}{(a + b)(a + c)(a + d))}.
\end{aligned}$$

Therefore

$$\begin{aligned} v^2 &= (a+b)^2 + (a+d)^2 - 2(a+b)(a+d)\cos 2A \\ &= (a+b)^2 + (a+d)^2 - 2\frac{a^2(a+b+c+d) - (bcd+acd+abd+abc)}{a+c} \\ &= \frac{b+d}{c+a}\left((a+c)(b+d)+4ac\right). \end{aligned}$$

A similar formula holds for u. □

Proof of the main theorem. Using Lemmas 1 and 3 we see that

$$\begin{aligned} ABCD \text{ is cyclic} &\iff ac = bd, \text{ by Lemma 1} \\ &\iff (a+c)(b+d) + 4bd = (a+c)(b+d) + 4ac \\ &\iff \frac{u^2}{v^2} = \left(\frac{c+a}{b+d}\right)^2, \text{ by Lemma 3} \\ &\iff \frac{u}{v} = \frac{c+a}{b+d}, \end{aligned}$$

as desired. This completes the proof of the main theorem.

Remarks. (1) As mentioned earlier, Theorem 1 is one of the two main theorems in [5]. The other theorem is similar and deals with those quadrilaterals that admit an *excircle*. Note that the terms *chordal* and *tangential* are used in that paper to describe what we referred to as *cyclic* and *circumscriptible* quadrilaterals.

(2) Let $A_1 \dots A_n$ be circumscriptible n-gon and let $B_1, \dots, B_n$ be the points where the incircle touches the sides $A_1A_2, \dots, A_nA_1$. Let $|A_iB_i| = a_i$ for $i = 1, \dots, n$. Theorem 2 states that if $n = 4$, then the polygon is cyclic if and only if $a_1a_3 = a_2a_4$. One wonders whether a similar criterion holds for $n > 4$.

(3) It is proved in [2] that if $a_1, \dots, a_n$ are any positive numbers, then there exists a unique circumscriptible n-gon $A_1 \dots A_n$ such that the points $B_1, \dots, B_n$ where the incircle touches the sides $A_1A_2, \dots, A_nA_1$ have the property $|A_iB_i| = a_i$ for $i = 1, \dots, n$. Thus one can, in principle, express all the elements of the circumscriptible polygon in terms of the parameters $a_1, \dots, a_n$. Instances of this, when $n = 4$, are found in Lemms 2 and 3 where the inradius r and the lengths of the diagonals are so expressed. When $n > 4$, one can prove that r^2 is the unique positive zero of the polynomial

$$\sigma_{n-1} - r^2\sigma_{n-3} + r^4\sigma_{n-5} - \dots \dots = 0,$$

where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric polynomials in $a_1, \dots, a_n$, and where $a_1, \dots, a_n$ are as given in Remark 2. This is obtained in the same way we obtained (2) using the the formula

$$\tan(A_1 + \dots + A_n) = \frac{\varepsilon_1 - \varepsilon_3 + \varepsilon_5 - \dots}{1 - \varepsilon_2 + \varepsilon_4 - \dots},$$

where $\varepsilon_1, \dots, \varepsilon_n$ are the elementary symmetric polynomials in $\tan A_1, \dots, \tan A_n$, and where $A_1, \dots, A_n$ are half the vertex angles of the polygon.

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Periodic Billiard Trajectories in Polyhedra

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Abstract. We consider the billiard map inside a polyhedron. We give a condition for the stability of the periodic trajectories. We apply this result to the case of the tetrahedron. We deduce the existence of an open set of tetrahedra which have a periodic orbit of length four (generalization of Fagnano's orbit for triangles), moreover we can study completely the orbit of points along this coding.

1. Introduction

We consider the billiard problem inside polyhedron. We start with a point of the boundary of the polyhedron and we move along a straight line until we reach the boundary, where there is reflection according to the mirror law. A famous example of a periodic trajectory is Fagnano's orbit: we consider an acute triangle and the foot points of the altitudes. Those points form a billiard trajectory which is periodic [1].

For the polygons some results are known. For example we know that there exists a periodic orbit in all rational polygons (the angles are rational multiples of π), and recently Schwartz has proved in [8] the existence of a periodic billiard orbit in every obtuse triangle with angle less than 100 degrees . A good survey of what is known about periodic orbits can be found in the article [4] by Gal'perin, Stëpin and Vorobets or in the book of Masur, Tabachnikov [6]. In this article they define the notion of stability: They consider the trajectories which remain periodic if we perturb the polygon. They find a combinatorial rule which characterize the stable periodic words. Moreover they find some results about periodic orbits in obtuse triangles.

The study of the periodic orbits has also been done by famous physicists. Indeed Glashow and Mittag prove that the billiard inside a triangle is equivalent to the system of three balls on a ring, [5]. Some others results can be found in the article of Ruijgrok and Rabouw [7]. In the polyhedral case much less is known. The result on the existence of periodic orbit in a rational polygon can be generalized, but it is less important, because the rational polyhedra are not dense in the set of polyhedra. There is no other general result, the only result concerns the example of the tetrahedron. Stenman [10] shows that a periodic word of length four exists in a regular tetrahedron.

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The aim of this paper is to find Fagnano's orbit in a regular tetrahedron and to obtain a rule for the stability of periodic words in polyhedra. This allows us to obtain a periodic orbit in each tetrahedron in a neighborhood of the regular one. Moreover we give examples which prove that the trajectory is not periodic in all tetrahedra, and we find bounds for the size of the neighborhood. In the last section we answer a question of Gal'perin, Krüger, Troubetzkoy [3] by an example of periodic word v with non periodic points inside its beam.

2. Statement of results

The definitions are given in the following sections as appropriate. In Section 4 we prove the following result. Consider a periodic billiard orbit coded by the word v. In §4.2, we derive a certain isometry S_v from the combinatorics of the path.

Theorem 1. *Let P be a polyhedron and v the prefix of a periodic word of period $|v|$ in P. If the period is an even number, and S_v is different from the identity, then v is stable. If the period is odd, then the word is stable if and only if S_v is constant as a function of P.*

In Section 5 we prove

Theorem 2. *Assume the billiard map inside the tetrahedron is coded by a, b, c, d.*

(1) *The word $abcd$ is periodic for all the tetrahedra in a neighborhood of the regular one.* (This orbit will be referred to as Fagnano's orbit).

(2) *In any right tetrahedron Fagnano's orbit does not exist. There exists an open set of obtuse tetrahedron where Fagnano's orbit does not exist.*

The last section of this article is devoted to the study of the first return map of the billiard trajectory.

3. Background

3.1. *Isometries.* We recall some usual facts about affine isometries of $\mathbb{R}^3$. A general reference is [1].

To an affine isometry a, we can associate an affine map f and a vector u such that: f has a fixed point or is equal to the identity, and such that $a = t_u \circ f = f \circ t_u$ where t_u is the translation of vector u. Then f can be seen as an element of the orthogonal group $O_3(\mathbb{R})$.

Definition. First assume that f belongs to $O_3(+)$, and is not equal to the identity. If u is not an eigenvector of f, then a is called an affine rotation. The axis of a is the set of invariants points. If u is an eigenvector of f, a is called a screw motion. In this case the axis of a is the axis of the affine rotation.

If f, in $O_2(-)$ or $O_3(-)$, is a reflection and u is an eigenvector of f with eigenvalue 1, then a is called a glide reflection.

We recall Rodrigue's formula which gives the axis and the angle of the rotation product of two rotations. It can be done by the following method.

Lemma 3 ([2]). *We assume that the two rotations are not equal to the identity, or to a rotation of angle π. Let θ and u be the angle and axis of the first rotation, and denote by t the vector $\tan\frac{\theta}{2}\cdot u$ and t' the associated vector for the second rotation. Then the product of the two rotations is given by the vector t'' such that*

$$t'' = \frac{1}{1 - t\cdot t'}(t + t' + t \wedge t').$$

3.2. *Combinatorics.* Let $\mathcal{A}$ be a finite set called the alphabet. By a language L over $\mathcal{A}$ we mean always a factorial extendable language. A language is a collection of sets $(L_n)_{n\geq 0}$ where the only element of L_0 is the empty word, and each L_n consists of words of the form $a_1a_2\dots a_n$ where $a_i \in \mathcal{A}$ such that
(i) for each $v \in L_n$ there exist $a, b \in \mathcal{A}$ with $av, vb \in L_{n+1}$, and
(ii) for all $v \in L_{n+1}$, if $v = au = u'b$ with $a, b \in \mathcal{A}$, then $u, u' \in L_n$.

If $v = a_1a_2\dots a_n$ is a word, then for all $i \leq n$, the word $a_1\dots a_i$ is called a prefix of v.

4. Polyhedral billiard

4.1. *Definition.* We consider the billiard map T inside a polyhedron P. Let $X \subset \partial P \times \mathbb{PR}^3$ consist of (m,θ) for which $m + \mathbb{R}^*\theta$ does not intersect ∂P on an edge. The map T is defined by the rule

$$T(m,\theta) = (m', \theta')$$

if and only if mm' is collinear with θ, where $\theta' = S\theta$ and S is the linear reflection over the face which contains m'.

We identify $\mathbb{PR}^3$ with the unit vectors of $\mathbb{R}^3$ in the preceding definition.

4.2. *Coding.* We code the trajectory by the letters from a finite alphabet where we associate a letter to each face.

We call s_i the reflection in the face i, S_i the linear reflection in this face. If we start with a point of direction θ which has a trajectory of coding $v = v_0\cdots v_{n-1}$ the image of θ is: $S_{v_{n-1}}...S_{v_1}\theta$. Indeed the trajectory of the point first meets the face v_1, then the face v_2 etc.

If it is a periodic orbit, it meets the face v_0 after the face v_{n-1} and we have: $S_{v_0}S_{v_{n-1}}\dots S_{v_1}\theta = \theta = S_v\theta$, S_v is the product of the S_i, and s_v the product of the s_i.

We recall a result of [3]: the word v is the prefix of a periodic word of period $|v|$ if and only if there exists a point whose orbit is periodic and has v as coding.

Remark. If a point is periodic, the initial direction is an eigenvector of the map S_v with eigenvalue 1. It implies that in $\mathbb{R}^3$, for a periodic word of odd period, S is a reflection.

Definition. Let v be a finite word. The beam associated to v is the set of (m,θ) where m is in the face v_o (respectively edge), θ a vector of $\mathbb{R}^3$ (respectively $\mathbb{R}^2$), such that the orbit of (m,θ) has a coding which begins with v. We denote it σ_v.

A vector u of $\mathbb{R}^3$ (respectively $\mathbb{R}^2$) is admissible for v, with base point m, if there exists a point m in the face (edge) v_0 such that (m, u) belongs to the beam of v.

Lemma 4. *Let s be an isometry of $\mathbb{R}^3$ not equal to a translation. Let S be the associated linear map and u the vector of translation. Assume s is either a screw motion or a glide reflection. Then the points n which satisfy $\overrightarrow{ns(n)} \in \mathbb{R}u$, are either on the axis of s (if S is a rotation), or on the plane of reflection. In this case the vector $\overrightarrow{ns(n)}$ is the vector of the glide reflection.*

Proof. We call θ the eigenspace of S related to the eigenvalue one. We have $s(n) = s(o) + S\overrightarrow{on}$ where o, the origin of the base will be chosen later. Elementary geometry yields $\overrightarrow{ns(n)} = (S - Id)X + Y$ (where $X = \overrightarrow{on}$, $Y = \overrightarrow{os(o)}$) is inside the space θ.

The map s has no fixed point by assumption, thus $\overrightarrow{ns(n)}$ is nonzero. The condition gives that $(S - Id)X + Y$ is an eigenvector of S associated to the eigenvalue one. Thus,

$$\begin{aligned} S((S - Id)X + Y) &= (S - Id)X + Y, \\ (S - Id)^2 X &= -(S - I)Y. \end{aligned} \tag{1}$$

We consider first the case $\det S > 0$. We choose o on the axis of s. Then θ is a line, we call the direction of the line by the same name. Since $\det S > 0$ we have $S \in O_3(+)$ and thus in an appropriate basis S has the form $\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$, where R is a matrix of rotation of $\mathbb{R}^2$. The equation (1) is equivalent to

$$(R - Id)^2 X' = -(R - Id)Y',$$

where X' is the vector of $\mathbb{R}^2$ such that $X = \begin{pmatrix} X' \\ x \end{pmatrix}$ in this basis. Furthermore, since S is a screw motion with axis $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in these coordinates, Y has the following coordinates $\begin{pmatrix} Y' \\ y \end{pmatrix}$ where $Y' = 0$. Since $S \neq Id$, $R - Id$ is invertible and thus $X' = 0$. Thus the vectors X solutions of this equation are collinear with the axis.

Consider now the case $detS < 0$. By assumption S is a reflection, it implies that the eigenspace related to one is a plane. We will solve (1), keeping the notation $X = \begin{pmatrix} X' \\ x \end{pmatrix}$ and $Y = \begin{pmatrix} Y' \\ y \end{pmatrix}$.

We may assume that o is on the plane of reflection. Moreover we can choose the coordinates such that that this plane is orthogonal to the line $\mathbb{R}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. It implies that $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and $y = 0$. The equation (1) becomes $4x = 0$. It implies that X is on the plane of reflection. Since s is a glide reflection, the last point becomes obvious. $\square$

Proposition 5. *Let P a polyhedron, the following properties are equivalent.*
(1) *A word v is the prefix of a periodic word with period $|v|$.*
(2) *There exists $m \in v_0$ such that $\overrightarrow{s_v(m)m}$ is admissible with base point m for vv_0, and $\theta = \overrightarrow{s_v(m)m}$ is such that $S\theta = \theta$.*

Remark. Assume $|v|$ is even. In the polygonal case the matrix S_v can only be the identity, thus s_v is a translation. We see by unfolding that s_v can not have a fixed point, thus in the polyhedral case s_v is either a translation or a screw motion or a glide reflection. If we do not assume the admissibility in condition (2) it is not equivalent to condition (1) as can be seen in a obtuse triangle, or a right prism above the obtuse triangle and the word abc.

Proof of Proposition 5. First we claim the following fact. The vector connecting $T^{|v|}(m, \theta)$ to $s_v(m)$ is parallel to the direction of $T^{|v|}(m, \theta)$. For $|v| = 1$ if the billiard trajectory goes from (m, θ) to (m', θ') without reflection between, then the direction θ' is parallel to $\overrightarrow{s(m)m'}$, where s is the reflection over the face of m' (see Figure 1). Thus the claim follows combining this observation with an induction argument.

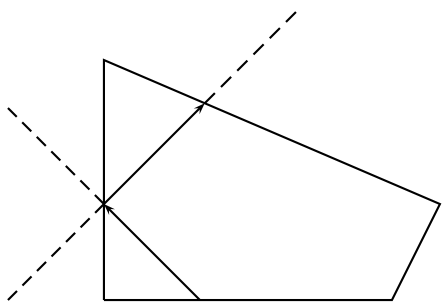

Figure 1. Billiard orbit and the associated map

Next assume (1). Then there exists (m, θ) periodic. We deduce that $S\theta = \theta$, moreover this direction is admissible. Then the claim implies that $\overrightarrow{s_v(m)m} = \theta$ and thus is admissible for vv_0.
Finally assume (2). First we consider the case where $S \neq Id$. Lemma 4 implies that m is on the axis of s if $|v|$ is even, otherwise on the plane of reflection. If $|v|$ is even then $\theta = \overrightarrow{s(m)m}$ is collinear to the axis of the screw motion. Since we have assumed $\overrightarrow{s_v(m)m}$ admissible we deduce that θ is admissible with base point m. If $|v|$ is odd then Lemma 4 implies that θ is the direction of the glide. The hypothesis implies that θ is admissible for v.

Now we prove that (m,θ) is a periodic trajectory. We consider the image $T^{|v|}(m,\theta)$. We denote this point (p,θ'). We have by hypothesis that p is in v_0. The above claim implies that $\overrightarrow{s_v(m)p}$ is parallel to the direction θ'. The equation $S\theta = \theta$ gives $\theta' = \theta$. Thus we have $\overrightarrow{s_v(m)m}$ is parallel to $\overrightarrow{s_v(m)p}$, since we do not consider direction included in a face of a polyhedron this implies $p = m$. Thus (m,θ) is a periodic point.

If $S = Id$, then s is a translation of vector $\overrightarrow{s_v(m)m} = u$. The vector u is admissible. Then we consider a point m on the face v_0 which is admissible. Then we show that (m,u) is a periodic point by the same argument related to the claim. □

Thus we have a new proof of the following result of [3].

Theorem 6. *Let v be a periodic word of even length. The set of periodic points in the face v_0 with code v and length $|v|$ can have two shapes. Either it is an open set or it is a point.*

If v is a periodic word of odd length, then the set of periodic points in the face v_0 with code v and period $|v|$ is a segment.

Proof. Let Π be a face of the polyhedron, and let $m \in \Pi$ be the starting point for a periodic billiard path. The first return map to m is an isometry of $\mathbb{R}^3$ that fixes both m and the direction u of the periodic billiard path.

Assume first $|v|$ is odd. Then the first return map is a reflection since it fixes a point. Then it fixes a plane Π'. Note that $u \in \Pi'$, and that the intersection $\Pi \cap \Pi'$ is a segment. Points in this segment sufficiently near m have a periodic orbit just as the one starting at v.

Assume now $|v|$ is even, we will use Proposition 5. If S_v is the identity, then the periodic points are the points such that the coding of the billiard orbit in the direction of the translation begins with v, otherwise there is a single point, at the intersection of the axis of s and v_0. However the set of points with code v is still an open set. □

Note that our proof gives an algorithm to locate this set in the face. We will use it in Section 6.

5. Stability

First of all we define the topology on the set of polyhedra with k vertices. As in the polygonal case we identify this set with $\mathbb{R}^{3(k-2)}$. But we remark the following fact. Consider a polyhedron P such that a face of P is not a triangle. Then we can find a perturbation of P, as small as we want, such that the new polyhedron has a different combinatorial type (*i.e.*, the numbers of vertices, edges and faces are different). In this case consider a triangulation of each face which does not add new vertices. Consider the set of all such triangulations of all faces. There are finitely many such triangulations. Each can be considered as a combinatorial type of the given polyhedron. Let $B(P,\varepsilon)$ be the ball of radius ε in $\mathbb{R}^{3(k-2)}$ of polyhedra Q. If P has a single combinatorial type, ε is chosen so small that all Q in the ball have the same combinatorial type. If P has several combinatorial types, then ε is taken so small that all Q have one of those combinatorial type. The definition of stability is now analogous to the definition in polygons. On the other hand, let v be a periodic word in P and g a piecewise similarity. Consider the polyhedron $g(P)$, and the same coding as in P. If v exists in $g(P)$ it is always a periodic word in $g(P)$. We note that the notion of periodicity only depends on the normal vectors to the planes of the faces.

Theorem 7. *Let P be a polyhedron and v the prefix of a periodic word of period $|v|$ in P.*
(1) *If the period is even, and S_v is different from the identity, then v is stable.*
(2) *If the period is odd, then the word is stable if and only if S_v is constant as a function of P.*

Remark. The second point has no equivalence in dimension two, since each element of $O(2,-)$ is a reflection. It is not the case for $O(3,-)$.

Proof of Theorem 7. First consider the case of period even. The matrix $S = S_v$ is not the identity, and $\theta = \theta_v$ is the eigenvector associated to the eigenvalue one. First note that by continuity v persists for sufficiently small perturbations of the polyhedron. Fix a perturbation and let $B = S_v^Q$ be the resulting rotation for the new polyhedron Q. We will prove that the eigenvalue of S is a continuous function of P. We take the reflections which appears in v two by two. The product of two of those reflections is a rotation. We only consider the rotations different of the identity. The axes of the rotations are continuous map as function of P since they are at the intersection of two faces. Then Rodrigue's formula implies that the axes of the rotation, product of two of those rotations, are continuous maps of the polyhedron, under the assumption that the rotation is not the identity (because t must be of non-zero norm). Since S^P is not equal to Id, there exists a neighborhood of P where $S^Q \neq Id$. It implies that the axis of S^P is a continuous function of P. Thus the two eigenvectors of B, S are near if B is sufficiently close to S. The direction θ was admissible for v, we know that the beam of v is an open set of the phase space [3], so we have for Q sufficiently close to P that α (the real eigenvector of B) is admissible for the same word. Moreover the foot points are not far from the initial points because they are on the axis of the isometries. Thus the perturbated word is periodic by Proposition 5.

If the length of v is odd, then Remark 4.2 implies that S is a reflection. We have two cases: either S_v is constant, or not. If it is not a constant function, then in any neighborhood there exists a polyhedron Q such that S_v^Q is different from a reflection. Then the periodic trajectory can not exist in Q. If S_v is constant, then it is always a reflection, and a similar argument to the even case shows that the plane of reflection of S is a continuous map of P. It completes the proof of Theorem 7. $\square$

Corollary 8. (1) *All the words of odd length are stable in a polygon.*
(2) *Consider a periodic billiard path in a right prism. Then its projection inside the polygonal basis is a billiard path. We denote the coding of the projected trajectory*

as the projected word. Assume that the projected word is not stable in the polygonal basis. Then the word is unstable.
(3) *All the words in the cube are unstable.*

Proof. (1) was already mentioned in [4]. The proof is the same as that of Theorem 7. Indeed is $|v|$ is odd then s has a real eigenvector, and we can apply the proof.

For (2) we begin with the period two trajectory which hits the top and the bottom of the prism. It is clearly unstable, for example we can change one face and keep the other. Let v be any other periodic word, and w the word corresponding to the projection of v to the base of the prism assumed to be unstable. We perturb a vertical face of the prism such that this face contains an edge which appears in the coding of w. The word v can not be periodic in this polyhedron by unstability of w.

For (3), let v be a periodic word, by preceding point its projection on each coordinate plane must be stable. But an easy computation shows that no word is stable in the square. □

We remark that the two and three dimensional cases are different for the periodic trajectories of odd length. They are all stable in one case, and all unstable in the second. Recently Vorobets has shown that if $S_v = Id$ then the word is not stable [11].

6. Tetrahedron

In the following two Sections we prove the following result.

Theorem 9. *Assume the billiard map inside the tetrahedron is coded by a, b, c, d.*
(1) *The word $abcd$ is periodic for all the tetrahedra in a neighborhood of the regular one.*
(2) *In any right tetrahedron Fagnano's orbit does not exist. There exists an open set of obtuse tetrahedron where Fagnano's orbit does not exist.*

Remark. Steinhaus in his book [9], cites Conway for a proof that $abcd$ is periodic in all tetrahedra, but our theorem gives a counter example. Moreover our proof gives an **algorithm** which find the coordinates of the periodic point, when it exists.

For the definition of obtuse tetrahedron, see Section 7.

We consider a regular tetrahedron. We can construct a periodic trajectory of length four, which is the generalization of Fagnano's orbit. To do this we introduce the appropriate coding (see Figure 2 in which the letter a is opposite to the vertex A, etc).

Lemma 10. *Let $ABCD$ be a regular tetrahedron, with the natural coding. If v is the word $adcb$, there exists a direction θ, there exists an unique point m such that (m, θ) is periodic and has v as prefix of its coding. Moreover m is on the altitude of the triangle BCD which starts at C.*

Remark. If we consider the word v^n, the preceding point m is the unique periodic point for v^n. Indeed the map s_{v^n} has the same axis as s_v, and we use Proposition 5.

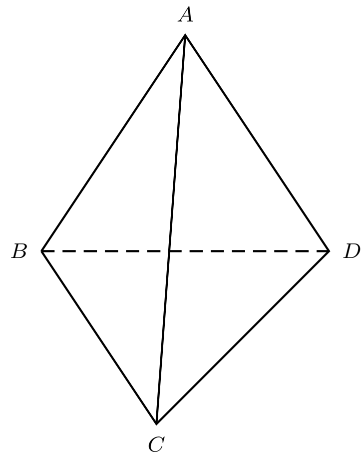

Figure 2. Coding of a tetrahedron

We use the following coordinates for two reasons. First these coordinates were used by Ruijgrok and Rabouw [7]. Secondly with these coordinates the matrix S_v has rational entries, and the computations seems more simples.

Proof. The lemma has already been proved in [10], but we rewrite it in a different form with the help of Proposition 5.

We have $S_v = S_a.S_b.S_c.S_d = R_{DC}.R_{AB}$ where R_{DC} is the linear rotation of axis DC, it is a product of the two reflections. We compute the real eigenvector of S_v, and we obtain the point m at the intersection of the axis of s and the face BCD. We consider an orthonormal base of $\mathbb{R}^3$ such that the points have the following coordinates (see [10]):

$$A = \frac{\sqrt{2}}{4}\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \quad D = \frac{\sqrt{2}}{4}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \quad C = \frac{\sqrt{2}}{4}\begin{pmatrix}1\\1\\-1\end{pmatrix}, \quad B = \frac{\sqrt{2}}{4}\begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

The matrices of S_a, S_d, S_c, S_b are:

$$\frac{1}{3}\begin{pmatrix}1 & -2 & -2\\-2 & 1 & -2\\-2 & -2 & 1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1 & 2 & 2\\2 & 1 & -2\\2 & -2 & 1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1 & -2 & 2\\-2 & 1 & 2\\2 & 2 & 1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1 & 2 & -2\\2 & 1 & 2\\-2 & 2 & 1\end{pmatrix}.$$

From these we obtain

$$S = S_aS_bS_cS_d = \frac{1}{81}\begin{pmatrix}-79 & -8 & 16\\8 & 49 & 64\\-16 & 64 & -47\end{pmatrix}.$$

This has a real eigenvector $u = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\2\\1\end{pmatrix}$. Now we compute the vector N such that $s(X) = SX + N$. To do this we use the relation $s(A) = s_a(A)$. s_a is the product of S_a and a translation of vector v. We obtain

$$v = \frac{\sqrt{2}}{6}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad s(A) = \frac{5\sqrt{2}}{12}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad N = \frac{\sqrt{2}}{81}\begin{pmatrix}16\\64\\34\end{pmatrix}.$$

We see that s is a screw motion. Finally we find the point at the intersection of the axis and the face a. The points of the axis verify the equation

$$SX + N = X + \lambda u.$$

where X are the coordinates of the point of the axis, and λ is a real number. The point m is on the face a if we have the dot product

$$\overrightarrow{Cm} \cdot (\overrightarrow{CB} \wedge \overrightarrow{CD}) = 0.$$

So X is the root of the system made by those two equations. The last equation gives $x + y + z = \frac{\sqrt{2}}{4}$. We obtain

$$m = \frac{\sqrt{2}}{20}\begin{pmatrix}2\\2\\1\end{pmatrix}.$$

We remark that $\overrightarrow{Cm} \cdot \overrightarrow{DB} = 0$ which proves that m is on the altitude of the triangle BCD. □

In fact there are six periodic trajectories of length four, one for each of the word

$$abcd,\ abdc,\ acbd,\ acdb,\ adbc,\ adcb.$$

The six orbits come in pairs which are related by the natural involution of direction reversal. Now we can ask the same question in a non regular tetrahedron. Applying Theorem 7 yield the first part of Theorem 9.

Now the natural question is to characterize the tetrahedron which contains this periodic word.

7. Stability for the tetrahedron

A tetrahedron is acute if and only if in each face the orthogonal projection of the other vertex is inside the triangle. It is a right tetrahedron if and only if there exists a vertex, where the three triangles are right triangles.

Proof of second part of Theorem 9. We consider a tetrahedron $ABCD$ with vertices

$$A = (0,0,0) \quad B = (a,0,0) \quad C = (0,b,0) \quad D = (0,0,1).$$

We study the word $v = abcd$. We have $S = S_a * S_d * S_c * S_b$. Since

$$S_b = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad S_c = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad S_d = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},$$

we obtain $S = -S_a$. Thus S has 1 for eigenvalue, and the associated eigenvector is the normal vector to the plane a. We remark that $s(A) = s_a(A)$. The fact that

$S = -S_a$ implies that S is a rotation of angle π, thus s is the product of a rotation of angle π and a translation.

Consider the plane which contains A and orthogonal to the axis of S, let O the point of intersection. Then S is a rotation of angle π, thus O is the middle of $[AE]$, where E is given by $S(\overrightarrow{OE}) = \overrightarrow{OA}$. It implies that the middle M of the edge $[As(A)]$ is on the axis of s, see Figure 3.

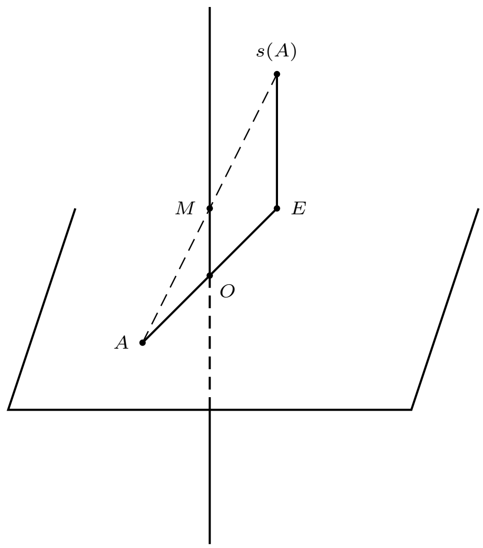

Figure 3. Screw motion associated to the word abcd

Clearly m is a point in the side ABC. If v is periodic then applying Proposition 5 yields that M is the base point of the periodic trajectory. Moreover, since the direction of the periodic trajectory is the normal vector to the plane a, we deduce that A is on the trajectory. So the periodic trajectory cannot exist.

Now we prove the second part of the theorem. We give an example of obtuse tetrahedron where Fagnano's orbit does not exist.

In this example the point on the initial face, which must be periodic see Proposition 5, is not in the interior of the triangle.

We consider the tetrahedron $ABCD$ with vertices

$$A(0,0,0), \quad B(2,0,0), \quad C(1,1,0), \quad D(3,2,1).$$

We study the word $v = abcd$. We obtain the matrix of S_v

$$\begin{pmatrix} \frac{1}{33} & \frac{8}{33} & \frac{32}{33} \\ \frac{104}{165} & \frac{-25}{33} & \frac{28}{165} \\ \frac{128}{165} & \frac{20}{33} & \frac{-29}{165} \end{pmatrix}.$$

Now s is the map $SX+N$ where $N=\begin{pmatrix}\frac{4}{11}\\ \frac{4}{11}\\ -\frac{12}{11}\end{pmatrix}$. S has eigenvector $u=\begin{pmatrix}\frac{9}{8}\\ \frac{1}{2}\\ 1\end{pmatrix}$.

Now s is a screw motion and we find the point at the intersection of the axis of s and the face a by solving the system

$$Sm+N=m+\lambda u \tag{2}$$

$$\overrightarrow{Bm}\cdot n=0. \tag{3}$$

This is equivalent to the system

$$\begin{pmatrix}S-Id & -u\\ n^t & 0\end{pmatrix}\begin{pmatrix}m\\ \lambda\end{pmatrix}=\begin{pmatrix}-N\\ 2\end{pmatrix}.$$

where $n=\begin{pmatrix}1\\ 1\\ -3\end{pmatrix}$ is the normal vector to the face BCD.

We obtain the matrix

$$\begin{pmatrix}-\frac{32}{33} & \frac{8}{33} & \frac{32}{33} & -\frac{9}{8}\\ \frac{104}{165} & -\frac{58}{33} & \frac{28}{165} & -\frac{1}{2}\\ \frac{128}{165} & \frac{20}{33} & -\frac{194}{165} & -1\\ 1 & 1 & -3 & 0\end{pmatrix}$$

and

$$m=\begin{pmatrix}\frac{22}{161}\\ \frac{6}{23}\\ -\frac{86}{161}\end{pmatrix}.$$

But this point is not inside BCD. Moreover we see that this point is not on the altitude at BD which passes through C.

The tetrahedron is obtuse, due to the triangle ABD. The triangle BCD is acute, and the axis of s does not cut this face in the interior of the triangle.

Moreover we obtain that there exists a neighborhood of this tetrahedron, where Fagnano's word is not periodic. Indeed in a neighborhood the point m can not be in the interior of $ABCD$.

Remark. We can remark that our proof gives a criterion for the existence of a periodic billiard path of this type. One computes the axis of the screw motion, and finds if it intersects the relevant faces.

For a generic tetrahedron we can use it to know if there exists a Fagnano's orbit. But we have not find a good system of coordinates where the computations are easy. Thus we are not able to caracterize the tetrahedra with a Fagnano's orbit.

8. First return map

In this section we use the preceding example to study a related problem for periodic billiard paths. We answer to a question of Gal'perin, Krüger and Troubetzkoy [3] by an example of periodic word v with non periodic points inside its beam.

We consider the word $v = (abcd)^\infty$ and the set σ_v. The projection of this set on the face a is an open set. Each point in this open set return to the face a after three reflections. We study this return map and the set $\pi_a(\sigma_v)$. We consider the same basis as in Section 6. Moreover, in the face a we consider the following basis

$$\begin{pmatrix} \frac{\sqrt{2}}{4} \\ 0 \\ 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Theorem 11. *In the regular tetrahedron, consider the word $v = (abcd)^\infty$. Then the set $\pi_a(\sigma_v)$ is an open set. There exists only one point in this set with a periodic billiard orbit.*

The theorem of [3] explains that some such cases could appear, but there were no example before this result.

Theorem 11 means that for all point in $\pi_a(\sigma_v)$, except one, the billiard orbit is coded by a periodic word, but it is never a periodic trajectory. For the proof we make use of the following lemma.

Lemma 12. *In the regular tetrahedron, consider the word $v = (abcd)^\infty$. The first return map r on $\pi_a(\sigma_v)$ is given by*

$$r\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + B,$$

where

$$A = \frac{1}{81}\begin{pmatrix} -83 & 28 \\ -12 & -75 \end{pmatrix}, \qquad B = \frac{1}{81}\begin{pmatrix} -15 \\ 9 \end{pmatrix}.$$

The set $\pi_a(\sigma_v)$ is the interior of the biggest ellipse of center m related to the matrix A.

Proof. If m is a point of the face a, the calculation in Section 6 shows that

$$r(m) = \frac{1}{81}\begin{pmatrix} -79x - 8y + 16\sqrt{2} \\ 66x - 21y + 42z + \frac{3\sqrt{2}}{2} \\ 13x + 29y - 58z\frac{33\sqrt{2}}{12} \end{pmatrix}$$

Now we compute m and rm in the basis of the face a. We obtain the matrices A, B. $\square$

Proof of Theorem 11. We can verify that the periodic point $\frac{\sqrt{2}}{20}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ is fixed by r. Indeed in this basis, it becomes $\frac{\sqrt{2}}{20}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Now the orbit of a point under r

is contained on an ellipse related to the matrix A. This shows that the set $\pi_a(\sigma_v)$ is the biggest ellipse included in the triangle. And an obvious computation shows that only one point is fixed by r.

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On the Centroids of Polygons and Polyhedra

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Abstract. In this paper we introduce the centroid of any finite set of points of the space and we find some general properties of centroids. These properties are then applied to different types of polygons and polyhedra.

1. Introduction

In elementary geometry the centroid of a figure in the plane or space (triangle, quadrilateral, tetrahedron, $\dots$) is introduced as the common point of some elements of the figure (medians or bimedians), once it has been proved that these elements are indeed concurrent. The proofs are appealing and have their own beauty in the spirit of Euclidean geometry. But they are different from figure to figure, and often use auxiliary elements. For example, the centroid of a triangle is defined as the common point of its three medians, after proving that they are concurrent. It is usually proved considering, as an auxiliary figure, the Varignon parallelogram of the quadrilateral whose vertices are the vertices of the triangle and the common point to two medians ([3, p. 10]). We can also define the centroid of a tetrahedron after proving that the four medians of the tetrahedron are concurrent (Commandino's Theorem, [1, p.57]). A natural question is: is it possible to characterize the properties of centroids of geometric figures with one unique and systematic method? In this paper we introduce the centroid of a finite set of points of the space, called a system, and find some of its general properties. These properties are then applied to different types of polygons and polyhedra. Then it is possible to obtain, in a simple and immediate way, old and new results of elementary geometry. At the end of the paper we introduce the notion of an extended system. This allows us to find some unexpected and charming properties of some figures, highlighting the great potential of the method that is used.

2. Systems and centroids

Throughout this paper, the ambient space is either a plane or a 3-dimensional space. Let $\mathcal{S}$ be a set of n points of the space. We call this an n-system or a system of order n. Let $\mathcal{S}'$ be a nonempty subset of S of k points, that we call a k-subsystem of $\mathcal{S}$ or a subsystem of order k of $\mathcal{S}$. There are $\binom{n}{k}$ different subsystems of order k. We say that two subsystems $\mathcal{S}'$ and $\mathcal{S}''$ of an n-system $\mathcal{S}$ are *complementary* if

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$\mathcal{S}' \cup S'' = S$ and $\mathcal{S}' \cap S'' = \emptyset$. We also say that $\mathcal{S}'$ is complementary to $\mathcal{S}''$ and $\mathcal{S}''$ is complementary to S'. If $\mathcal{S}'$ is a k-subsystem, $\mathcal{S}''$ is an $(n-k)$-subsystem. Let A_i, $i = 1, 2, \ldots, n$, be the points of an n-system S and $\mathbf{x}_i$ be the position vector of A_i with respect to a fixed point P. We call the *centroid* of $\mathcal{S}$ the point C whose position vector with respect to P is

$$\mathbf{x} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i.$$

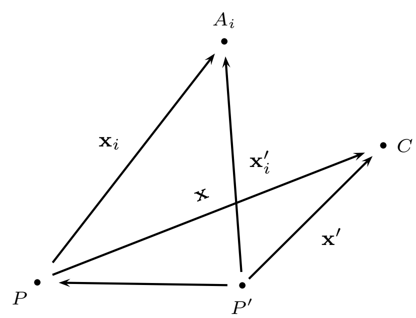

Figure 1

The point C does not depend on P. In fact, let P' be another point of the space and $\mathbf{x}'_i$ be the position vector of A_i with respect to P'. Since $\mathbf{x}'_i = \mathbf{x}_i + \overrightarrow{P'P}$, we have

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}'_i = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i + \overrightarrow{P'P}.$$

Every subsystem of $\mathcal{S}$ has its own centroid. The centroid of a 1-subsystem $\{A_i\}$ is A_i. The centroid of a 2-subsystem $\{A_i,\ A_j\}$ is the midpoint of the segment A_iA_j.

Let $\mathcal{S}'$ be a k-subsystem of $\mathcal{S}$ and C' its centroid. Let $\mathcal{S}''$ be the subsystem of $\mathcal{S}$ complementary to $\mathcal{S}'$ and C'' its centroid. We call the segment $C'C''$ the *median* of $\mathcal{S}$ relative to S'. The median relative to $\mathcal{S}''$ coincides with the one relative to $\mathcal{S}'$.

Let $\mathcal{S}$ be an n-system and C its centroid.

Theorem 1. *The medians of $\mathcal{S}$ are concurrent in C. Moreover, C divides the median $C'C''$ relative to a k-subsystem $\mathcal{S}'$ of $\mathcal{S}$ into two parts such that:*

$$\frac{C'C}{CC''} = \frac{n-k}{k}. \qquad (*)$$

Proof. In fact, let $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{v}''$ the position vectors of C, C', C'' respectively. It is easy to prove that

$$\mathbf{v} - \mathbf{v}' = \frac{n-k}{k}(\mathbf{v}'' - \mathbf{v}).$$

This relation means that $\overrightarrow{C'C} = \frac{n-k}{k}\overrightarrow{CC''}$. Hence, C, C', C'' are collinear and (*) holds. $\square$

Here are some interesting consequences of Theorem 1.

Corollary 2. *The system of centroids of the* $k-$*subsystems of* $\mathcal{S}$ *is the image of the system of centroids of the* $(n-k)$*-subsystems of* $\mathcal{S}$ *in the dilatation with ratio* $-\frac{n-k}{k}$ *and center* C*. In this dilatation the centroid of a* k*-subsystem is the image of the centroid of its complementary.*

Corollary 3. *The segment* $C_1'C_2'$ *that joins the centroids of two* k*-subsystems* $\mathcal{S}_1'$*,* $\mathcal{S}_2'$ *of* $\mathcal{S}$ *is parallel to the segment* $C_1''C_2''$ *that joins the centroids of the* $(n-k)$*-subsystems complementary to* $\mathcal{S}_1'$*,* $\mathcal{S}_2'$*. Moreover,*

$$\frac{C_1'C_2'}{C_1''C_2''} = \frac{n-k}{k}.$$

Corollary 4. *If* $n = 2k$*,* C *is the center of symmetry of the system of centroids of the* k*-subsystems of* $\mathcal{S}$*. Moreover, the segment* $C_1'C_2'$ *that joins the centroids of two* k*-subsystems* S_1'*,* $\mathcal{S}_2'$ *of* $\mathcal{S}$ *is parallel and equal to the segment* $C_1''C_2''$ *that joins the centroids of the* k*-subsystems complementary to* $\mathcal{S}_1'$*,* S_2'*.*

We conclude this section by the following theorem which is easily verified.

Theorem 5. *The centroid* C *of* $\mathcal{S}$ *is also the centroid of the system of centroids of the* k*-subsystems of* $\mathcal{S}$*.*

3. Applications

We propose here some applications to polygons and polyhedra. Let $\mathcal{P}$ be a polygon or a polyhedron. We associate with it the system $\mathcal{S}$ whose points are the vertices of $\mathcal{P}$.

3.1. *Triangles.* Let $\mathcal{T}$ be a triangle, with associated system $\mathcal{S}$ and centroid C. The 1-subsystems of $\mathcal{S}$ detect the vertices of $\mathcal{T}$, the 2-subsystems detect the sides. The centroids of the 2-subsystems of $\mathcal{S}$ are the midpoints of the sides of $\mathcal{T}$ and detect the medial triangle of $\mathcal{T}$. The medians of $\mathcal{S}$ are the medians of $\mathcal{T}$.

As a consequence of Theorem 1, we have

Proposition 6 ([3, p.10], [4, p.8])**.** *The three medians of a triangle all pass through one point which divides each median into two segments in the ratio* $2:1$.

It follows that the centroid of $\mathcal{T}$ coincides with the centroid C of $\mathcal{S}$.

From Theorem 5 and Corollary 2, we deduce

Proposition 7 ([4, p.18], [5, p.11])**.** *A triangle* $\mathcal{T}$ *and its medial triangle have the same centroid* C*. Moreover, the medial triangle is the image of* $\mathcal{T}$ *in the dilatation with ratio* $-\frac{1}{2}$ *and center* C*.* See Figure 2.

Corollary 3 yields

Proposition 8 ([4, p.53])**.** *The segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long as that third side.*

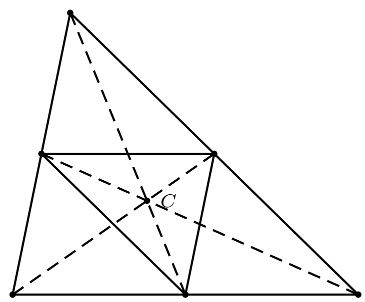

Figure 2.

3.2. *Quadrilaterals.* Let $A_1A_2A_3A_4$ be a quadrilateral which we denote by $\mathcal{Q}$. Let $\mathcal{S}$ be the system associated with $\mathcal{Q}$ and C its centroid. The 1-subsystems of $\mathcal{S}$ detect the vertices of $\mathcal{Q}$, the 2-subsystems detect the sides and the diagonals, the 3-subsystems detect the sub-triangles of $\mathcal{Q}$. The centroids of the 2-subsystems of $\mathcal{S}$ are the midpoints of the sides and of the diagonals of $\mathcal{Q}$. The centroids of the 3-subsystems are the centroids C_1, C_2, C_3, C_4 of the triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$ respectively. We call $C_1C_2C_3C_4$ the quadrilateral of centroids and denote it by $\mathcal{Q}_c$ ([6]). The medians of $\mathcal{S}$ relative to the 2-subsystems are the *bimedians* of $\mathcal{Q}$ and the segment that joins the midpoints of the diagonals of $\mathcal{Q}$. The medians of $\mathcal{S}$ relative to the 1-subsystems are the segments A_iC_i, $i = 1, 2, 3, 4$.

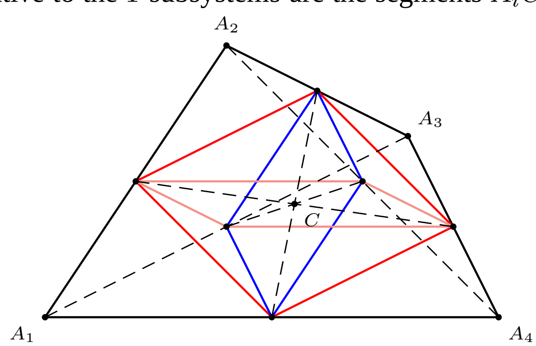

Figure 3

From Theorem 1 it follows that

Proposition 9 ([4, p.54])**.** *The bimedians of a quadrilateral and the segment joining the midpoints of the diagonals are concurrent and bisect one another.* See Figure 3.

Thus, the centroid of the quadrilateral $\mathcal{Q}$, *i.e.*, the intersection point of the bimedians, coincides with the centroid C of $\mathcal{S}$. From Corollary 4, we obtain

Proposition 10 ([4, p.53])**.** *The quadrilateral whose vertices are the midpoints of the sides of a quadrilateral is a parallelogram* (Varignon's Theorem). *Moreover, the quadrilateral whose vertices are the midpoints of the diagonals and of two opposite sides of a quadrilateral is a parallelogram.*

Thus, three parallelograms are naturally associated with a quadrilateral. These have the same centroid, which, by Theorem 1, coincides with the centroid of the quadrilateral.

Theorem 5 and Corollary 2 then imply

Proposition 11 ([6])**.** *The quadrilaterals $\mathcal{Q}$ and $\mathcal{Q}_c$ have the same centroid C. Moreover, $\mathcal{Q}_c$ is the image of $\mathcal{Q}$ in the dilatation with ratio $-\frac{1}{3}$ and center C.* See Figure 4.

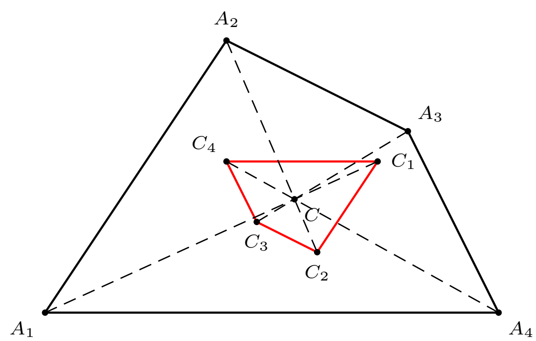

Figure 4

Some of these properties, with appropriate changes, hold also for polygons with more than four edges. For example, from Theorem 1 it follows that

Proposition 12. *The five segments that join the midpoint of a side of a pentagon with the centroid of the triangle whose vertices are the remaining vertices and the five segments that join a vertex of a pentagon with the centroid of the quadrilateral whose vertices are the remaining vertices are all concurrent in a point C that divides the first five segments in the ratio 3:2 and the other five in the ratio 4:1.*

The point C is the centroid of the system $\mathcal{S}$ associated with the pentagon. C will also be called the centroid of the pentagon.

3.3. *Tetrahedra.* Let $\mathcal{T}$ be a tetrahedron. Let $\mathcal{S}$ be the system associated with $\mathcal{T}$ and C its centroid. The subsystem of $\mathcal{S}$ of order 1, 2, and 3 detect the vertices, the edges and the faces of $\mathcal{T}$, respectively. The centroids of the 2-subsystems are the midpoints of the edges. Those of the 3-subsystems are the centroids of the faces of $\mathcal{T}$, which detect the medial tetrahedron of $\mathcal{T}$. The medians of $\mathcal{S}$ relative to the 2-subsystems are the bimedians of $\mathcal{T}$, *i.e.*, the segments that join the midpoints of two opposite sides. The medians of $\mathcal{S}$ relative to the 1-subsystems are the medians of $\mathcal{T}$, *i.e.*, the segments that join one vertex of $\mathcal{T}$ with the centroid of the opposite face.

From Theorem 1 follows Commandino's Theorem:

Proposition 13 ([1, p.57])**.** *The four medians of a tetrahedron meet in a point which divides each median in the ratio* $1:3$*.* See Figure 5.

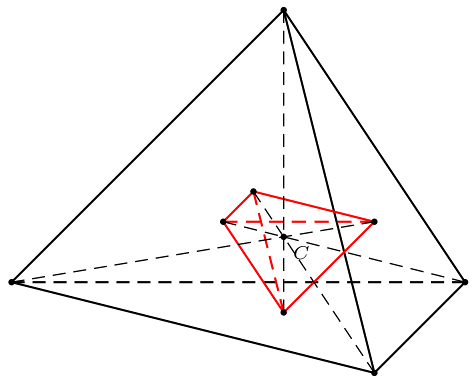

Figure 5

It follows that the centroid of the tetrahedron $\mathcal{T}$, intersection point of the medians, coincides with the centroid C of $\mathcal{S}$. From Theorem 5 and from Corollary 2 it follows that

Proposition 14 ([1, p.59])**.** *A tetrahedron $\mathcal{T}$ and its medial tetrahedron have the same centroid C. Moreover the medial tetrahedron is the image of $\mathcal{T}$ in the dilatation with ratio $-\frac{1}{3}$ and center C. The faces and the edges of the medial tetrahedron of a tetrahedron $\mathcal{T}$ are parallel to the faces and the edges of $\mathcal{T}$.*

Finally, Theorem 1 and Corollary 2 yield

Proposition 15 ([1, pp.54,58])**.** *The three bimedians of a tetrahedron are concurrent in the centroid of the tetrahedron and are bisected by it. Moreover, the midpoints of two pairs of opposite edges of tetrahedron are the vertices of a parallelogram.* See Figure 6.

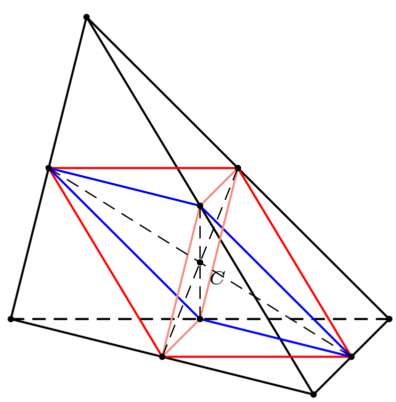

Figure 6

By using the theorems of the theory it is possible to find lots of interesting properties on polyhedra. For example, Corollary 4 gives

Proposition 16. *The centroids of the faces of an octahedron with triangular faces are the vertices of a parallelepiped. The centroids of the faces of a hexahedron with quadrangular faces are the vertices of an octahedron with triangular faces having a symmetry center C.* See Figures 7A and 7B.

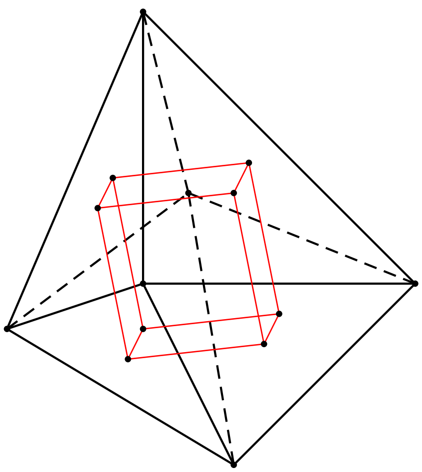

Figure 7A

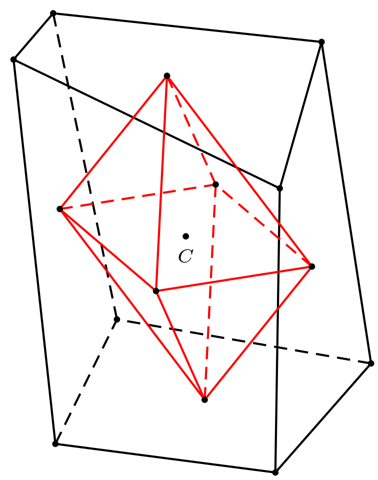

Figure 7B

The point C is the centroid of the system $\mathcal{S}$ associated with the hexahedron. This point is also called the centroid of the hexahedron.

4. Extended systems and applications

Let $\mathcal{S}$ be an n-system and h a fixed positive integer. Let H be a set of h points such that $\mathcal{S} \cap H = \emptyset$. We call h-extension of $\mathcal{S}$ the system $\mathcal{S}_H = \mathcal{S} \cup H$.

Let t be a fixed integer such that $1 \leq t < n$. Consider the system $\mathcal{C}_{H,t}$ of centroids of the subsystems of $\mathcal{S}_H$, of order $h+t$, that contain H. The complementary subsystems of these subsystems are the subsystems of $\mathcal{S}$ of order $n-t$ and we denote the system of their centroids by $\mathcal{C}'_{n-t}$.

Let us consider now two h-extensions of $\mathcal{S}$, $\mathcal{S}_{H_1}$ and $\mathcal{S}_{H_2}$, and let C_1 and C_2 be their centroids. Consider the systems $\mathcal{C}_{H_{1,t}}$ and $\mathcal{C}_{H_{2,t}}$, and the system $\mathcal{C}'_{n-t}$.

From Corollary 2 applied to the system $\mathcal{S}_{H_1}$ (respectively $\mathcal{S}_{H_2}$) it follows that $\mathcal{C}_{H_{1,t}}$ (respectively $\mathcal{C}_{H_{2,t}}$) is the image of $\mathcal{C}'_{n-t}$ in the dilatation with ratio $-\frac{n-t}{h+t}$ and center C_1 (respectively C_2).

Thus, we have

Theorem 17. *If $\mathcal{S}_{H_1}$ and $\mathcal{S}_{H_2}$ are two h-extension of $\mathcal{S}$, then the systems $\mathcal{C}_{H_{1,t}}$ and $\mathcal{C}_{H_{2,t}}$ are correspondent in a translation.*

It is easy to see that the vector of the translation transforming $\mathcal{C}_{H_{1,t}}$ into $\mathcal{C}_{H_{2,t}}$ is $\frac{n+h}{h+t}\overrightarrow{C_1C_2}$.

The following theorem is also of interest.

Theorem 18. *If $\mathcal{S}$ is an n-system, $\mathcal{S}_H$ is a 1-extension of $\mathcal{S}$, $\mathcal{S}_K$ is a $(n-1)$-extension of $\mathcal{S}$, then the systems $\mathcal{C}_{H,n-1}$ and $\mathcal{C}_{K,1}$ are correspondent is a half-turn.*

Proof. Let C and C_K be the centroids of $\mathcal{S}_H$ and K respectively. From Corollary 2 the system $\mathcal{C}_{H,n-1}$ is the image of the system $\mathcal{C}'_1 = \mathcal{S}$ in the dilatation with ratio $\frac{1}{n}$ and center C that is, $\mathcal{S}$ is the image of $\mathcal{C}_{H,n-1}$ in the dilatation with ratio $-n$ and center C.

Let $C' \in \mathcal{C}_{K,1}$ and suppose that C' is the centroid of the n-subsystem $\mathcal{S}' = K \cup \{A\}$ of $\mathcal{S}_K$, with $A \in \mathcal{S}$. From Theorem 1, C' lies on the median C_KA of $\mathcal{S}'$ and is such that $\frac{C_KC'}{C'A} = \frac{1}{n-1}$. It follows that $\frac{C_KC'}{C_KA} = \frac{1}{n}$, and $\mathcal{C}_{K,1}$ is the image of $\mathcal{S}$ in the dilatation with ratio $\frac{1}{n}$ and center C_K.

Since $\mathcal{S}$ is the image of $\mathcal{C}_{H,n-1}$ in the dilatation with ratio $-n$ and center C and $\mathcal{C}_{K,1}$ is the image of $\mathcal{S}$ in the dilatation with ratio $\frac{1}{n}$ and center C_K, then $\mathcal{C}_{H,n-1}$ and $\mathcal{C}_{K,1}$ are correspondent in a dilatation with ratio -1, *i.e.*, in a half-turn. $\square$

It is easy to see that the center $\overline{C}$ of the half-turn is the point of the segment CC_K such that $\frac{C\overline{C}}{\overline{C}C_K} = \frac{n-1}{n+1}$.

Now, we offer some applications of Theorems 17 and 18.

4.1. *Triangles.* Let $\mathcal{T}$ be a triangle and $\mathcal{S}$ its associated system. Let S_H be a 1-extension of $\mathcal{S}$, with $H = \{P\}$, and $\mathcal{S}_K$ be a 2-extension of $\mathcal{S}$, with $K = \{P_1, P_2\}$. The points of the system $C_{H,2}$ are vertices of a triangle $\mathcal{T}_H$ and the points of the system $C_{K,1}$ are vertices of a triangle $\mathcal{T}_K$. Theorem 18 gives

Proposition 19. *The triangles $\mathcal{T}_H$ and $\mathcal{T}_K$ are correspondent in a half-turn.* See Figure 8.

Let $\{\mathcal{T}_H\}$ be the family of triangles $\mathcal{T}_H$ obtained by varying the point P and $\{\mathcal{T}_K\}$ be the family of triangles $\mathcal{T}_K$ obtained by varying the points P_1 and P_2.

From Theorem 17 the triangles of the family $\{\mathcal{T}_H\}$ are all congruent and have corresponding sides that are parallel. The same property also holds for the triangles of the family $\{\mathcal{T}_K\}$. On the other hand, each triangle $\mathcal{T}_H$ and each triangle $\mathcal{T}_K$ are correspondent in a half-turn, then:

Proposition 20. *The triangles of the family $\{\mathcal{T}_H\} \cup \{\mathcal{T}_K\}$ are all congruent and have corresponding sides that are parallel.*

4.2. *Quadrilaterals.* Let $\mathcal{Q}$ be a quadrilateral $A_1A_2A_3A_4$ and $\mathcal{S}$ its associated system. Let $\mathcal{S}_H$ be a 1-extension of $\mathcal{S}$, with $H = \{P\}$, and let C be its centroid.

Let us consider the subsystems $\{P,\ A_1,\ A_2\}$, $\{P,\ A_2,\ A_3\}$, $\{P,\ A_3,\ A_4\}$, $\{P,\ A_4,\ A_1\}$ of $\mathcal{S}_H$ and their centroids C_1, C_2, C_3, C_4 respectively, that are points of $C_{H,2}$. From Corollary 3 applied to the system $\mathcal{S}_H$, the segments C_1C_2, C_2C_3, C_3C_4, C_4C_1 are parallel to the sides of the Varignon parallelogram of $\mathcal{Q}$ respectively. Thus, $C_1C_2C_3C_4$ is a parallelogram, that we denote by $\mathcal{Q}_H$. Moreover,

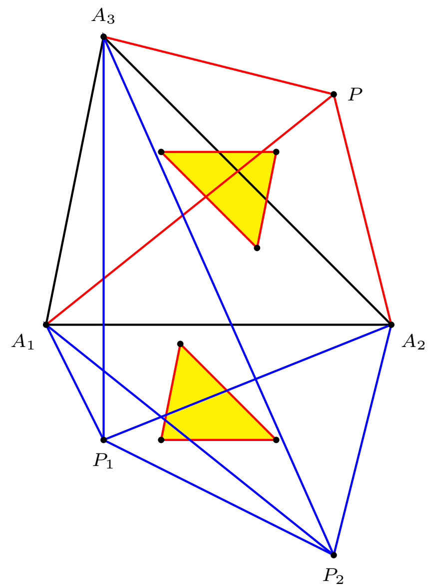

Figure 8.

from Corollary 2, $\mathcal{Q}_H$ is the image of the Varignon parallelogram of $\mathcal{Q}$ in the dilatation with ratio $-\frac{2}{3}$ and center C. In the case when P is the intersection point of the diagonals of Q, the existence of a dilatation between $\mathcal{Q}_H$ and the Varignon parallelogram of $\mathcal{Q}$ has already been proved ([2, p.424], [7, p.23]).

If we consider two 1-extensions of $\mathcal{S}$, the systems $\mathcal{C}_{H,2}$, for Theorem 17, are correspondent in a translation. Thus, if $\{\mathcal{Q}_H\}$ is the family of the parallelograms obtained as P varies, we obtain

Proposition 21. *The parallelograms of the family $\{\mathcal{Q}_H\}$ are all congruent and their corresponding sides are parallel.*

Moreover, taking P as the vertex of a pyramid with base $\mathcal{Q}$, we are led to

Proposition 22. *The centroids of the faces of a pyramid with a quadrangular base are vertices of the parallelogram that is the image to Varignon parallelogram of $\mathcal{Q}$ in the dilatation with ratio $-\frac{2}{3}$ and center C. Moreover, as P varies, the parallelograms whose vertices are the centroids of the faces are all congruent.* See Figure 9.

The point C is called the centroid of the pyramid.

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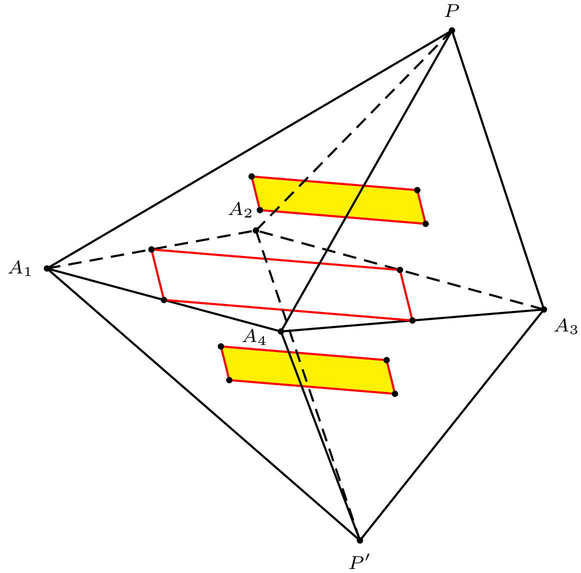

Figure 9

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Another Variation on the Steiner-Lehmus Theme

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Abstract. Let the internal angle bisectors BB' and CC' of angles B and C of triangle ABC be extended to meet the circumcircle at B^* and C^*. The Steiner-Lehmus theorem states that if $BB' = CC'$, then $AB = AC$. In this article, we investigate those triangles for which $BB^* = CC^*$ and we address several issues that arise within this investigation.

1. Introduction

The celebrated Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. In terms of triangle centers and cevians, it states that if two cevians through the *incenter* of a triangle are equal, then the triangle is isosceles. Variations on the theme can be obtained by replacing the incenter by any of the hundreds of centers known in the literature; see [6] and the website [7]. Other variations on this theme are obtained by letting the cevians of ABC through a center P meet the circumcircle of ABC at A^*, B^*, and C^* and asking whether the equality $BB^* = CC^*$ implies that $AB = AC$, where XY denotes the length of the line segment XY. This variation, together with several others, is investigated in [5] where it is proved that if P is the incenter, the orthocenter, or the Fermat-Torricelli point, then $BB^* = CC^*$ if and only if $AB = AC$ or $A = \frac{\pi}{3}$. When P is the centroid, the triangles for which $BB^* = CC^*$ are proved, in Theorem 9 below, to be the ones whose side lengths satisfy the relation $a^4 = b^4 - b^2c^2 + c^4$, a relation that has no geometric interpretation and cannot be fitted into a traditional geometry context such as Euclid's *Elements*.

Using geometric arguments, we show that if the centroid P of a scalene triangle ABC is such that $BB^* = CC^*$, then $\angle BAC$ must lie in the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$ and that to every θ in $[\frac{\pi}{3}, \frac{\pi}{2}]$ there is essentially a unique scalene triangle with $\angle BAC = \theta$ and with $BB^* = CC^*$. The proof uses a generalization of Proposition 7 of Book III of Euclid's *Elements*, in brief Euclid III.7 [1], and deserves recording on its own.

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[1]Throughout, the symbol Euclid $*.**$ designates Proposition $**$ of Book $*$ in Euclid's *Elements*.

2. Euclid III.7 and a generalization

Euclid III.7, not that well known, states that if Ω is a circle centered at O, if $M \neq O$ is a point inside Ω, and if the intersection of a ray MX with Ω is denoted by X', then
(i) the maximum value of MX' is attained when the ray MX passes through O and the minimum is attained when the ray MX is the opposite ray $-MO$,
(ii) as the ray MX rotates from the position MO to the opposite position $-MO$, the quantity MX' changes monotonically. We restate this proposition in Theorem 1 as a preparation for the generalization that is made in Theorem 5.

Theorem 1 (Euclid III.7)**.** *Let BC be a chord in a circle Ω, let M be the mid-point of BC, and let the line perpendicular to BC through M meet Ω at E and F. As a point P moves from E to F along the arc ECF of Ω, the length MP changes monotonically. It increases or decreases according as E is closer or farther than F from M.*

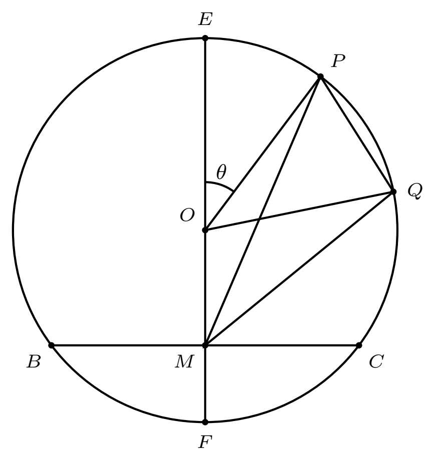

Figure 1.

Proof. Referring to Figure 1, we shall show that if $EM > MF$, *i.e.*, if the center O of Ω is between E and M, and if P and Q are any points on the arc ECF such that P is closer to E than Q, then $MP > MQ$. Under these assumptions,

$$\angle MQP > \angle OQP = \angle OPQ > \angle MPQ.$$

Thus $\angle MQP > \angle MPQ$, and therefore $MP > MQ$, as desired. □

Remark. The proof above uses the fairly simple-minded fact that in a triangle, the greater angle is subtended by the greater side. This is Euclid I.19. It is interesting that Euclid's proof uses the more sophisticated Euclid I.24. This theorem, referred to in [8, Theorem 6.3.9, page 140] as the *Open Mouth Theorem*, states that if triangles ABC and $A'B'C'$ are such that $AB = A'B'$, $AC = A'C'$, $\angle BAC > \angle B'A'C'$, then $BC > B'C'$. Quoting [8], this says that *the wider you open your mouth, the farther apart your lips are.* Although this follows immediately from the law of cosines, the intricate proofs given by Euclid and in [8] have the advantage of showing that the theorem is a theorem in neutral geometry.

Theorem 5 below generalizes Theorem 1. In fact Theorem 1 follows from Theorem 5 by taking BC to be a diameter of one of the circles Ω and Ω'. For the proof of Theorem 5, we need the following simple lemmas.

Lemma 2. *Let ABC be a triangle and let D and E be points on the sides AB and AC respectively* (see Figure 2)*. Then $\frac{AD}{AB}$ is greater than, less than, or equal to $\frac{AE}{AC}$ according as $\angle ABC$ is greater than, less than, or equal to $\angle ADE$, respectively.*

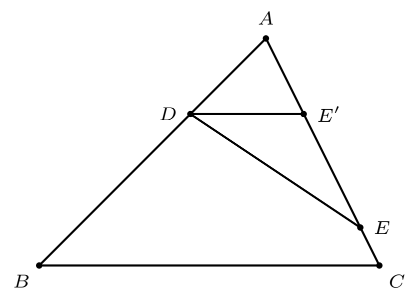

Figure 2

Proof. Let E' be the point on AC such that $\frac{AE'}{AC} = \frac{AD}{AB}$; *i.e.*, DE' is parallel to BC. If $\frac{AE}{AC} = \frac{AD}{AB}$, then $E' = E$ and $\angle ABC = \angle ADE$. If $\frac{AE}{AC} > \frac{AD}{AB}$, then E lies between E' and C, and $\angle ABC = \angle ADE' < \angle ADE$. Similarly for the case $\frac{AE}{AC} < \frac{AD}{AB}$. $\square$

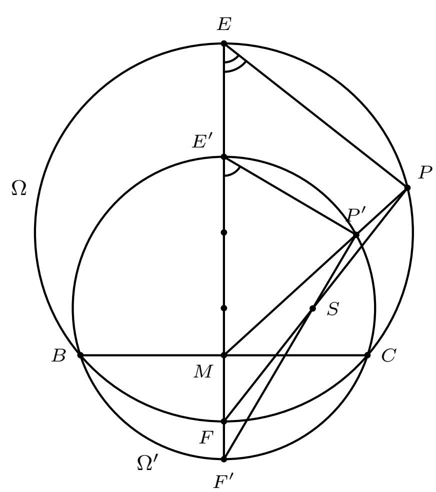

Figure 3

Lemma 3. *Two circles Ω and Ω' intersect at B and C, and the line perpendicular to BC through the midpoint M of BC meets Ω and Ω' at E and E', respectively, such that E' is inside Ω* (see Figure 3)*. If P is any point on the arc ECF of Ω and if the ray MP meets Ω' at P', then $\frac{MP'}{MP} > \frac{ME'}{ME}$.*

Proof. Let S be the point of intersection of FP and $F'P'$. Since $\angle EPF = \frac{\pi}{2} = \angle E'P'F'$, it follows that $\angle ME'P' + \angle MF'P' = \frac{\pi}{2} = \angle MEP + \angle MFP$. But $\angle MFP > \angle MF'P'$, by the exterior angle theorem. Hence $\angle ME'P' > \angle MEP$. By Lemma 2, we have $\frac{MP'}{MP} > \frac{ME'}{ME}$, as desired. □

Lemma 4. *Let EBC be an isosceles triangle having $EB = EC$. Let M be the midpoint of BC and let E' be the circumcenter of EBC* (see Figure 4). *Then $\frac{ME'}{ME}$ is greater than, equal to, or less than $\frac{1}{3}$ according as $\angle BEC$ is less than, equal to, or greater than $\frac{\pi}{3}$, respectively.*

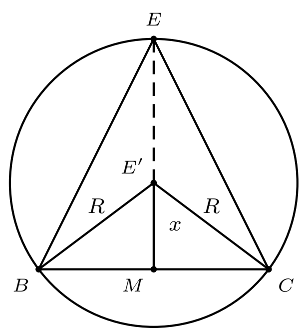

Figure 4.

Proof. Let $\theta = \angle BEC$, $x = ME'$, and let R be the circumradius of EBC. Then $\angle ME'C = \theta$ and

$$\frac{ME'}{ME} - \frac{1}{3} = \frac{x}{x+R} - \frac{1}{3} = \frac{R\cos\theta}{R\cos\theta + R} - \frac{1}{3} = \frac{2\cos\theta - 1}{3(\cos\theta + 1)}.$$

$\cos\theta$ is greater than, equal to, or less than $\frac{1}{2}$. □

Theorem 5. *Two circles Ω and Ω' intersect at B and C and the line perpendicular to BC through the midpoint M of BC meets Ω at E and F and meets Ω' at E' and F'. For every point P on Ω, let P' be the point where the ray MP meets Ω'. As a point P moves from E to F along the arc ECF, the ratio $\frac{MP'}{MP}$ changes monotonically. It decreases or increases according as E' is inside or outside Ω.*

Proof. Referring to Figure 5, suppose that E' lies inside Ω and let P and Q be two points on the arc ECF of Ω such that P is closer to E than Q. we are to show that $\frac{MP'}{MP} < \frac{MQ'}{MQ}$.

Extend QM to meet Ω at U and Ω' at U'. Let T be the point of intersection of EU and $E'U'$. Since the quadrilaterals $EPQU$ and $E'P'Q'U'$ are cyclic, it follows that

$$\angle UQP + \angle UEP = \pi = \angle U'Q'P' + \angle U'E'P'. \tag{1}$$

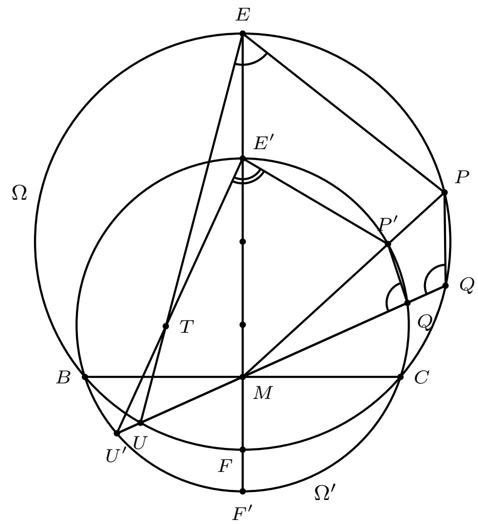

Figure 5

But

$$\begin{aligned}\angle U'E'P' &= \angle U'E'M + \angle ME'P'\\ &> \angle UEM + \angle ME'P' \text{ (by the exterior angle theorem)}\\ &> \angle UEM + \angle MEP \text{ (by Lemmas 3 and 2)}\\ &= \angle UEP.\end{aligned}$$

From this and (1) it follows that $\angle U'Q'P' > \angle UQP$. By Lemma 2, we conclude that $\frac{MP'}{MP} < \frac{MQ'}{MQ}$, as desired.

Note that if P is on the arc EC and Q is on the arc CF, then $\frac{MP'}{MP} < 1 < \frac{MQ'}{MQ}$. □

3. Conditions of equality of two chords through a given point

The next simple lemma exhibits the relation between two geometric properties of a point P inside a triangle ABC. It will be used in the proof of Theorem 9.

Lemma 6. *Let P be a point inside triangle ABC and let the rays BP and CP meet the circumcircle of ABC at B^* and C^* respectively* (see Figure 6)*. Then*
(a) $BB^* = CC^*$ *if and only if* $PB = PC$ *or* $\angle BPC = 2\angle BAC$*;*
(b) $\angle BPC = 2\angle BAC \iff PB^* = PC \iff B^*C \parallel C^*B$.
Moreover, if P is the centroid, then
(c) $PB = PC \iff AB = AC \iff B^*C^* \parallel BC$.

Proof. (a) It is clear that

$$\begin{aligned} BB^* = CC^* &\iff \angle BAB^* = \angle CAC^* \text{ or } \angle BAB^* + \angle CAC^* = \pi \\ &\iff \angle CAB^* = \angle BAC^* \text{ or } \angle CAB^* + \angle BAC^* + 2\angle BAC = \pi \\ &\iff \angle CBB^* = \angle BCC^* \text{ or } \angle CBB^* + \angle BCC^* + 2\angle BAC = \pi \\ &\iff PB = PC \text{ or } \angle BPC = 2\angle BAC. \end{aligned}$$

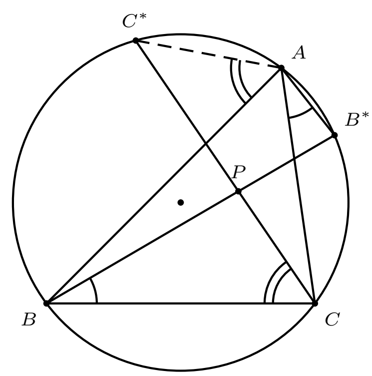

Figure 6.

(b) Also,

$$\begin{aligned} \angle BPC = 2\angle BAC &\iff \angle PB^*C + \angle PCB^* = 2\angle PB^*C \\ &\iff \angle PCB^* = \angle PB^*C \\ &\iff \angle PB^* = \angle PC. \end{aligned}$$

This proves the first part of (b). The implication $PB^* = PC \iff B^*C \parallel C^*B$ is easy.

(c) Let the lengths of the medians from B and C be β and γ, respectively. By Apollonius theorem, we have

$$\frac{b^2}{2} + 2\beta^2 = a^2 + c^2 \,, \; \frac{c^2}{2} + 2\gamma^2 = a^2 + b^2.$$

The rest follows from the facts that $PB = \frac{2\beta}{3}$ and $PC = \frac{2\gamma}{3}$. □

4. Chords of circumcircle through the centroid

In Theorem 7, we focus on triangles ABC whose centroid G has the property that $\angle BGC = 2\angle BAC$. Interest in this property stems from Lemma 6. Note that Part (i) provides a solution of the problem in [4].

Theorem 7. (i) *If ABC is a triangle whose centroid G has the property that $\angle BGC = 2\,\angle BAC$, then $\frac{\pi}{3} \le \angle BAC < \frac{\pi}{2}$ with $\angle BAC = \frac{\pi}{3}$ if and only if ABC is equilateral.*

(ii) *If θ is any angle in the interval $(\frac{\pi}{3}, \frac{\pi}{2})$ and if BC is any line segment, then there is a triangle ABC, unique up to reflection about BC and about the perpendicular bisector of BC, having $\angle BAC = \theta$ and whose centroid G has the property $\angle BGC = 2\,\angle BAC$.*

Proof. (i) Let Ω be the circumcircle of ABC and let E' be its circumcenter. Let Ω' be the circumcircle of $E'BC$. Let M be the midpoint of BC and let the perpendicular bisector of BC meet Ω at E and F and meet Ω' at (E' and) F', where E is on the arc BAC of Ω (see Figure 7). Let $\angle BAC = \theta$, and let G be the centroid of ABC. Also, for every P on Ω, let P' be the point where the ray MP meets Ω'.

Suppose that $\angle BGC = 2\angle BAC$. Since $\angle BE'C = 2\angle BAC$, it follows that G lies on the arc $BE'C$ of Ω'. Also, G lies on the median AM of ABC. Therefore, G is the point A' where the ray MA meets Ω'. In particular, $\frac{MA'}{MA} = \frac{1}{3}$. As P moves from E to F along the arc ECF, the ratio $\frac{MP'}{MP}$ increases by Theorem 5. Therefore

$$\frac{ME'}{ME} \leq \frac{MA'}{MA} = \frac{1}{3}.$$

By Lemma 4, $\theta \geq \frac{\pi}{3}$, with equality if and only if $A = E$, or equivalently if and only if ABC is equilateral. The possibility that $\angle BAC \geq \frac{\pi}{2}$ is ruled out since it would lead to the contradiction $\angle BGC \geq \pi$.

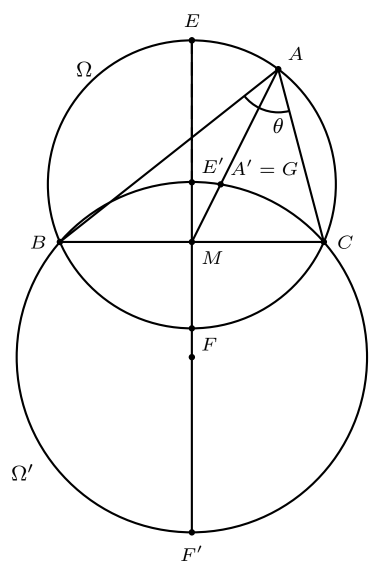

Figure 7

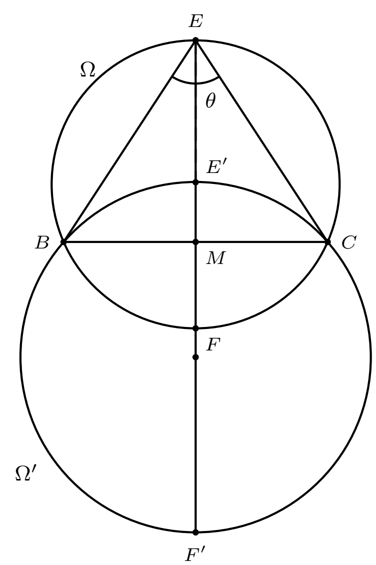

Figure 8

(ii) Suppose that θ is a given angle such that $\frac{\pi}{3} \leq \theta < \frac{\pi}{2}$ and that BC is a given segment. Let EBC be an isosceles triangle with $EB = EC$ and with $\angle BEC = \theta$. Let Ω be the circumcircle of EBC and let E' be its circumcenter. Let Ω' be the circumcircle of $E'BC$. Let M be the midpoint of BC and let the perpendicular bisector of BC meet Ω at (E and) F and meet Ω' at (E' and) F'; see Figure 8. For every P on Ω, we let P' be the point where the ray MP meets Ω'. Let $t = \frac{ME'}{ME}$. Since $\theta \geq \frac{\pi}{3}$, it follows from Lemma 4 that $t \leq \frac{1}{3}$. Also, $C' = C$ and $\frac{MC'}{MC} = 1$. Thus as P moves from E to C along one of the arcs EC of Ω, the ratio $\frac{MP'}{MP}$ increases from $t \leq \frac{1}{3}$ to 1. By continuity and the intermediate value theorem, there is a unique point A on that arc EC for which $\frac{MA'}{MA} = \frac{1}{3}$. If we think of MC as the x-axis and of ME as the y-axis, then the point A is the only point in the first quadrant for which ABC has the desired property. Points in the other quadrants are obtained by reflection about the x- and y-axes.

This is precisely the point A on the arc ECF for which A' is the centroid of ABC. This triangle ABC is the unique triangle (up to reflection about BC and about the perpendicular bisector of BC) whose vertex angle at A is θ and whose centroid G has the property that $\angle BGC = 2\angle BAC$. □

Theorem 9 characterizes those triangles whose centroid has the property $BB^* = CC^*$. For the proof, we need the following simple lemma.

Lemma 8. *Let ABC be a triangle with side-lengths a, b, and c* (in the standard order) *and with centroid G. Let the rays BG and CG meet the circumcircle of ABC at B^* and C^* respectively.*

$$BB^{*2} = \frac{(a^2 + c^2)^2}{2a^2 + 2c^2 - b^2}.$$

Proof. Let $m = BB'$, $x = BB^*$. By Apollonius' theorem, $m^2 = \frac{2(a^2+c^2)-b^2}{4}$. Since $BB'B^*$ and $AB'C$ are diagonals of a cyclic quadrilateral, $m(x - m) = \frac{b^2}{4}$. It follows that $mx = \frac{a^2+c^2}{2}$ and $x^2 = \frac{(a^2+c^2)^2}{4m^2} = \frac{(a^2+c^2)^2}{2a^2+2c^2-b^2}$. □

Theorem 9. *Let ABC be a triangle with side-lengths a, b, and c* (in the standard order) *and with centroid G. Let the rays BG and CG meet the circumcircle of ABC at B^* and C^*, respectively. If $b \neq c$, then the following are equivalent:*
(i) $BB^* = CC^*$,
(ii) $\angle BGC = 2\angle BAC$,
(iii) $a^4 = b^4 + c^4 - b^2c^2$.

Proof. Since $b \neq c$, it follows that $GB \neq GC$. By Lemma 6, (i) is equivalent to (ii). To see that (i) is equivalent to (iii), let $x = BB^*$, $y = CC^*$, and let $s = a^2 + b^2 + c^2$. By Lemma 8,

$$x^2 = \frac{(s - b^2)^2}{2s - 3b^2}, \qquad y^2 = \frac{(s - c^2)^2}{2s - 3c^2}.$$

Therefore

$$\begin{aligned}
x = y \iff & \frac{(s-b^2)^2}{2s-3b^2} = \frac{(s-c^2)^2}{2s-3c^2} \\
\iff & (s^2 - 2b^2 s + b^4)(2s - 3c^2) = (s^2 - 2c^2 s + c^4)(2s - 3b^2) \\
\iff & s^2(c^2 - b^2) - 2s(c^2 - b^2)(c^2 + b^2) + 3c^2 b^2 (c^2 - b^2) = 0 \\
\iff & s^2 - 2s(c^2 + b^2) + 3c^2 b^2 = 0 \qquad \text{(because } b \neq c\text{)} \\
\iff & (s - (c^2 + b^2))^2 = (c^2 + b^2)^2 - 3c^2 b^2 \\
\iff & a^4 = c^4 + b^4 - c^2 b^2,
\end{aligned}$$

as claimed. □

It follows from [1, Theorem 2.3.3., page 83] (or [9, page 20]) that the only positive solutions of the diophantine equation

$$a^4 + b^4 - a^2 b^2 = c^4 \tag{2}$$

are given by $a = b = c$. Thus there are no non-isosceles triangles ABC with integer side-lengths whose centroid G has the property $BB^* = CC^*$.

We end this note by a Euclidean construction, provided by a referee, of triangles ABC whose centroid has the property $BB^* = CC^*$. We start with any segment BC.

(i) Take any point A' on the major arc BA_0C of an equilateral triangle A_0BC.

(ii) Extend $A'C$ and $A'B$ to Y and Z respectively such that $CY = BZ = BC$.

(iii) Construct a circle with diameter $A'Z$ and the perpendicular at B to $A'Z$, intersecting the circle at B'

(iii′) Construct a circle with diameter $A'Y$ and the perpendicular at C to $A'Y$, intersecting the circle at C'

(iv) Construct the circles centered at B and C and passing through B' and C', respectively.

Letting A be a point of intersection of the two circles in (iv), one can verify that triangle ABC satisfies $BB^* = CC^*$.

In this regard, one may ask whether one can construct a triangle ABC having the property $BB^* = CC^*$ and having preassigned side BC and angle A (in $[\frac{\pi}{3}, \frac{\pi}{2}]$). The answer is affirmative as seen below.

Without loss of generality, assume $BC = 1$. Let $b = AC$, $c = AB$, and $t = \cos A$. We are to show that b and c are constructible. These are defined by

$$b^4 + c^4 - b^2 c^2 = 1\ ,\ b^2 + c^2 = 2bct + 1.$$

Subtracting the square of the second from the first and simplifying, we obtain $bc = \frac{4t}{3-4t^2}$. Thus bc is constructible. Since $b^2 + c^2 = 2bct + 1$, it follows that $b^2 + c^2$ is constructible. Thus both b^2c^2 and $b^2 + c^2$ are constructible, and hence b^2 and c^2, being the zeros of $f(T) := T^2 - (b^2 + c^2)T + b^2c^2$, are constructible. This shows that b and c are constructible, as desired. The restriction $A \in [60^\circ, 90^\circ]$, i.e., $t \in \left[0, \frac{1}{2}\right]$, guarantees that the zeros of $f(T)$ are real (and positive).

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Haruki's Lemma for Conics

Yaroslav Bezverkhnyev

Abstract. We extend Haruki's lemma to conics.

1. Main results

In this paper we continue to explore Haruki's lemma introduced by Ross Honsberger in [2, 3]. In [1], we gave an extension of Haruki's lemma (Theorem 1 below) and studied a related locus problem, leading to certain interesting conics. [1]

Theorem 1 ([1, Lemma 2])**.** *Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB respectively. The following equalities hold:*

$$\frac{AE \cdot BF}{EF} = \frac{AC \cdot BD}{CD}, \tag{1}$$

$$\frac{AF \cdot BE}{EF} = \frac{AD \cdot BC}{CD}. \tag{2}$$

In this paper we generalize this result to conics.

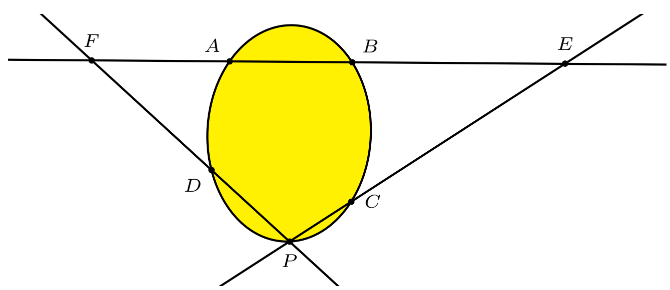

Figure 1.

Theorem 2. *Given a nondegenerate conic $\mathcal{C}$ with fixed points A, B, C, D on it, let P be a variable point distinct from A and B. Let E and F be the intersections of the lines PC, AB, and of PD, AB respectively. Then the ratios $\dfrac{AE \cdot BF}{EF}$ and $\dfrac{AF \cdot BE}{FE}$ are independent of the choice of P.*

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The author wishes to thank Paul Yiu for his invaluable additions and help with the preparation of the article.

[1]See Remark following the proof of Theorem 2 below.

It turns out that this result still holds when the points A and B coincide. In this case, we replace the line AB by the tangent to the conic at A. With a minor change of notations, we have the following result.

Theorem 3. *Given a nondegenerate conic $\mathcal{C}$ with fixed points A, B, C on it, let P be a variable point distinct from A. Let E and F be the intersections of the lines PB, PC with the tangent to the conic at A. Then the ratio $\dfrac{AE \cdot AF}{EF}$ is independent of the choice of P.*

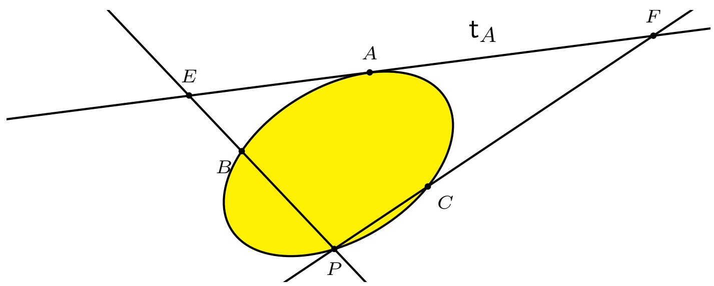

Figure 2

2. Proof of Theorem 2

We choose ABC as reference triangle. The nondegenerate conic $\mathcal{C}$ has equation of the form

$$fyz + gzx + hxy = 0 \tag{3}$$

for nonzero constants f, g, h. See Figure 1. Suppose D has homogeneous barycentric coordinates $(u : v : w)$, *i.e.*,

$$fvw + gwu + huv = 0. \tag{4}$$

Clearly, u, v, w are all nonzero. For an arbitrary point P with barycentric coordinates $(x : y : z)$, the coordinates of the intersections $E = AB \cap DC$ and $F = AB \cap PD$ can be easily determined:

$$E = (x : y : 0), \qquad F = (uz - wx : vz - wy : 0).$$

See [1, §6]. From these, we have the signed lengths of the various relevant segments:

$$\begin{aligned} AE &= \tfrac{y}{x+y} \cdot c, & EB &= \tfrac{x}{x+y} \cdot c, \\ AF &= \tfrac{vz-wy}{z(u+v)-w(x+y)} \cdot c, & FB &= \tfrac{uz-wx}{z(u+v)-w(x+y)} \cdot c, \\ EF &= \tfrac{z(vx-uy)}{(x+y)(z(u+v)-w(x+y))} \cdot c, \end{aligned}$$

where $c = AB$. It follows that $\dfrac{AE \cdot BF}{EF} = \dfrac{y(wx - uz)}{z(vx - uy)} \cdot c$. To calculate this fraction, note that from (4), we have $\frac{fw}{h} = -u(1 + k)$ for $k = \frac{gw}{hv}$. Now, from (3), we have

$$
\begin{aligned}
&\frac{fw}{h} \cdot yz + \frac{gw}{h} \cdot zx + w \cdot xy = 0, \\
&-u(1+k)yz + kvzx + wxy = 0, \\
&y(wx - uz) + kz(vx - uy) = 0.
\end{aligned}
$$

Hence, $\dfrac{AE \cdot BF}{EF} = \dfrac{y(wx-uz)}{z(vx-uy)} \cdot c = -kc$, a constant.

A similar calculation gives $\dfrac{AF \cdot BE}{FE} = (1+k)c$, a constant. This completes the proof of the theorem.

Remark. Note that we have actually proved that

$$\frac{AE \cdot BF}{EF} = -\frac{gw}{hv} \cdot c \quad \text{and} \quad \frac{AF \cdot BE}{FE} = -\frac{fw}{hu} \cdot c.$$

In [1, Theorem 6], we have solved two loci problems in connection with Haruki's lemma. Denote, in Figure 1, $BC = a$, $CA = b$, $AB = c$, and $AD = a'$, $BD = b'$, $CD = c'$. The locus of points P satisfying (1) is the union of the two circumconics of $ABCD$

$$(cc' + \varepsilon bb')uyz - \varepsilon bb'vzx - cc'wxy = 0, \quad \varepsilon = \pm 1.$$

Now, with

$$f = (cc' + \varepsilon bb')u, \qquad g = -\varepsilon bb'v, \qquad h = -cc'w,$$

we have

$$\frac{AE \cdot BF}{EF} = -\frac{-\varepsilon bb'vw}{-cc'wv} \cdot c = -\varepsilon \cdot \frac{bb'}{c'} = \varepsilon \cdot \frac{AC \cdot BD}{CD}.$$

Similarly, the locus of points P satisfying (2) is the union of the two circumconics of $ABCD$

$$\varepsilon aa'uyz + (cc' - \varepsilon aa')vzx - cc'wxy = 0, \quad \varepsilon = \pm 1.$$

Now, with

$$f = \varepsilon aa'u, \qquad g = (cc' - \varepsilon aa')v, \qquad h = -cc'w,$$

we have

$$\frac{AF \cdot BE}{FE} = -\frac{fw}{hu} \cdot c = -\frac{\varepsilon aa'uw}{-cc'wu} \cdot c = \varepsilon \cdot \frac{aa'}{c'} = -\varepsilon \cdot \frac{AD \cdot BC}{DC}.$$

These confirm that Theorem 2 is consistent with Theorem 6 of [1].

3. Proof of Theorem 3

Again, we choose ABC as the reference triangle, and write the equation of the nondegenerate conic $\mathcal{C}$ in the form (3) with $fgh \neq 0$. The tangent at A is the line

$$\mathtt{t}_A: \qquad hy + gz = 0.$$

For an arbitrary point P with homogeneous barycentric coordinates $(x : y : z)$, the lines PB and PC intersect $\mathtt{t}_A$ respectively at

$$\begin{aligned} E =& (hx : -gz : hz), \\ F =& (gx : gy : -hy). \end{aligned}$$

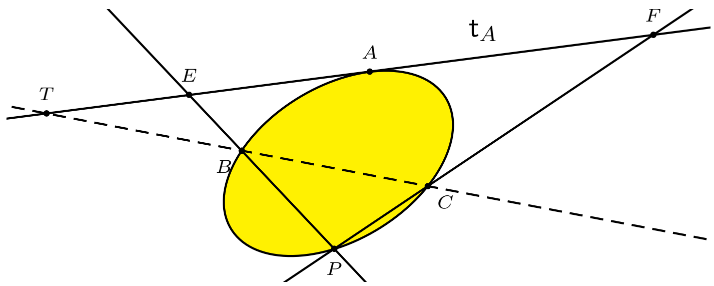

Figure 3

On the tangent line there is the point $T = (0 : -g : h)$, the intersection with the line BC. It is clearly possible to express the points E and F in terms of A and T. In fact, from

$$\begin{aligned} (hx, -gz, hz) =& hx(1, 0, 0) - z(0, g, -h), \\ (gx, gy, -hy) =& gx(1, 0, 0) + y(0, g, -h), \end{aligned}$$

we have, in absolute barycentric coordinates,

$$\begin{aligned} E =& \frac{hx}{hx - (g-h)z} \cdot A + \frac{-(g-h)z}{hx - (g-h)z} \cdot T, \\ F =& \frac{gx}{gx + (g-h)y} \cdot A + \frac{(g-h)y}{gx + (g-h)y} \cdot T. \end{aligned}$$

From these,

$$\frac{AE}{AT} = \frac{-(g-h)z}{hx - (g-h)z}, \qquad \frac{AF}{AT} = \frac{(g-h)y}{gx + (g-h)y}.$$

It follows that

$$\begin{aligned} \frac{EF}{AT} = \frac{AF - AE}{AT} &= \frac{(g-h)y}{gx + (g-h)y} + \frac{(g-h)z}{hx - (g-h)z} \\ &= \frac{(g-h)x(hy + gz)}{(gx + (g-h)y)(hx - (g-h)z)}. \end{aligned}$$

Therefore,

$$\frac{AE \cdot AF}{EF} = \frac{-(g-h)z \cdot (g-h)y}{(g-h)x(hy+gz)} \cdot AT = \frac{-(g-h)yz}{gzx+hxy} \cdot AT$$
$$= \frac{-(g-h)yz}{-fyz} \cdot AT = \frac{g-h}{f} \cdot AT.$$

This is independent of the choice of the point $P(x : y : z)$ on the conic. This completes the proof of Theorem 3.

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A Simple Compass-Only Construction of the Regular Pentagon

Kurt Hofstetter

Abstract. In 7 steps we give a simple compass-only (Mascheroni) construction of the vertices of a regular pentagon .

In [1] we have given a simple 5-step compass-only (Mascheroni) construction of the golden section. Here we note that with two additional circles, it is possible to construct the vertices of a regular pentagon. As usual, we denote by $P(Q)$ the circle with center P and passing through Q.

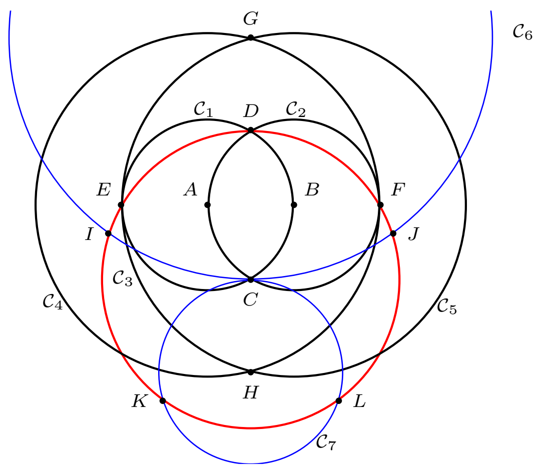

Figure 1

Construction 1. *Given two points A and B,*
(1) $\mathcal{C}_1 = A(B)$,
(2) $\mathcal{C}_2 = B(A)$ *to intersect* $\mathcal{C}_1$ *at* C *and* D,
(3) $\mathcal{C}_3 = C(D)$ *to intersect* $\mathcal{C}_1$ *at* E *and* $\mathcal{C}_2$ *at* F,
(4) $\mathcal{C}_4 = A(F)$,
(5) $\mathcal{C}_5 = B(E)$ *to intersect* $\mathcal{C}_4$ *at* G *and* H.
(6) $\mathcal{C}_6 = G(C)$ *to intersect* $\mathcal{C}_3$ *at* I *and* J,
(7) $\mathcal{C}_7 = H(C)$ *to intersect* $\mathcal{C}_3$ *at* K *and* L.

Then $DIKLJ$ is a regular pentagon.

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Proof. In [1] we have shown that the first five steps above lead to four collinear points C, D, G, H such that D divides CG, and C divides DH, in the golden section.

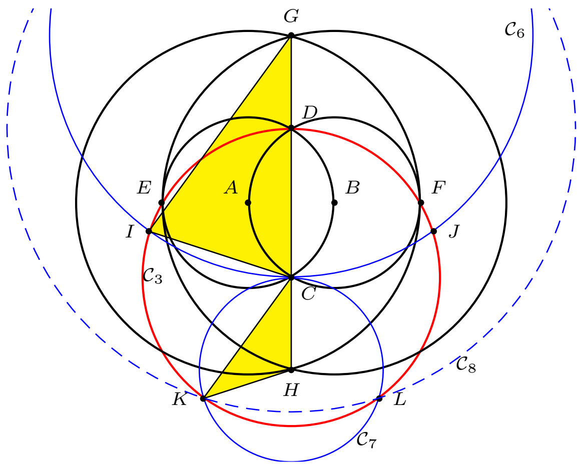

Figure 2

(i) This means that in the isosceles triangle GCI, $\frac{GC}{IC} = \frac{GC}{DC} = \phi$. The base angles are 72°. Therefore, $\angle DCI = 72^\circ$. By symmetry, $\angle DCJ = 72^\circ$.

(ii) Also, in the isosceles triangle HCK, $\frac{KC}{CH} = \frac{DC}{CH} = \phi$. The base angles are 36°. It follows that $\angle KCH = 36^\circ$. By symmetry, $LCH = 36^\circ$, and $KCL = 72^\circ$.

(iii) Since C is on the line GH, $\angle ICK = 180^\circ - \angle GCI - \angle KCH = 72^\circ$. By symmetry, $\angle JCL = 72^\circ$.

Therefore, the five points D, I, K, L, J are equally spaced on the circle $\mathcal{C}_3$. They form the vertices of a regular pentagon. □

Remark. The circle $\mathcal{C}_7$ can be replaced by $\mathcal{C}_8$ with center D and radius IJ. This intersects $\mathcal{C}_3$ at the same points K and L.

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Two More Powerian Pairs in the Arbelos

Quang Tuan Bui

Abstract. We construct two more pairs of Archimedes circles analogous to those of Frank Power, in addition to those by Floor van Lamoen and the author.

Consider an arbelos with semicircles (O), (O_1), (O_2) with diameters AB, AC, BC as diameters respectively. Denote by r_1 and r_2 respectively the radii of (O_1) and (O_2), and D the intersection of (AB) with the perpendicular to AB at C. If P is a point such that $OP^2 = r_1^2 + r_2^2$, then the circles tangent to (O) and to OP at P are Archimedean. Examples were first given in Power [3], subsequently also in [1, 2].

We construct two more Powerian pairs.

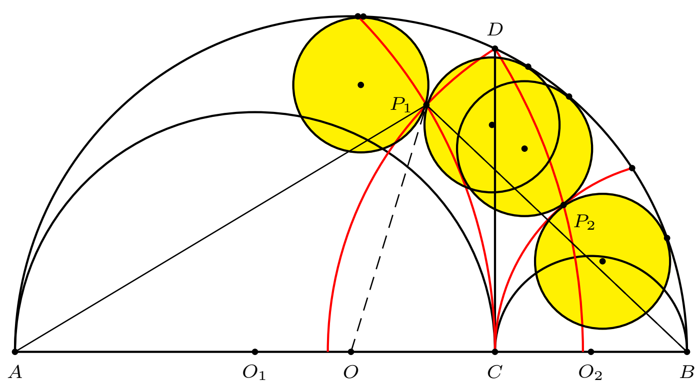

Figure 1

Let P_1 be the intersection of the circles $A(C)$ and $B(D)$. Consider OP_1 as a median of triangle P_1AB, we have, by Apollonius' theorem (see, for example, [4]),

$$\begin{aligned} OP_1^2 &= \frac{1}{2}\left(AP_1^2 + BP_1^2\right) - OA^2 \\ &= \frac{1}{2}\left((2r_1)^2 + 2r_2 \cdot 2(r_1 + r_2)\right) - (r_1 + r_2)^2 \\ &= r_1^2 + r_2^2. \end{aligned}$$

Similarly, for P_2 the intersection of $B(C)$ and $A(D)$, $OP_2^2 = r_1^2 + r_2^2$. Therefore, we have two Powerian pairs at P_1, P_2.

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On the Generalized Gergonne Point and Beyond

Miklós Hoffmann and Sonja Gorjanc

Abstract. In this paper we further extend the generalization of the concept of Gergonne point for circles concentric to the inscribed circle. Given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_1, q_2, q_3 from I, we find a distance $x = IQ_1 = IQ_2 = IQ_3$ along these directions, for which the three cevians V_iQ_i are concurrent. Types and number of solutions, which can be obtained by the common intersection points of three conics, are also discussed in detail.

1. Introduction

The Gergonne point is a well-known center of the triangle. It is the intersection of the three cevians defined by the touch points of the inscribed circle [3]. Konečný [1] has generalized this to circles concentric with the inscribed circle. Let $\mathcal{C}(I)$ be a circle with center I, the incenter of triangle $V_1V_2V_3$. Let Q_1, Q_2, Q_3 be the points of intersection of $\mathcal{C}(I)$ with the lines from I that are perpendicular to the sides V_2V_3, V_3V_1, V_1V_2 respectively. Then the lines V_iQ_i, $i = 1, 2, 3$, are concurrent (see Figure 1).

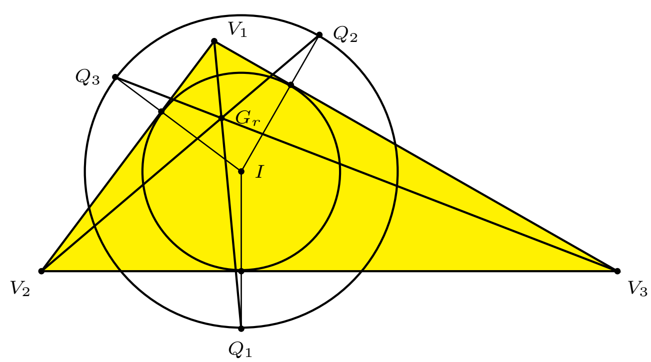

Figure 1. Lines V_iQ_i are also concurrent for circles concentric to the inscribed circle

The first question naturally arises: if the radius of the circle is altered, what will be the locus of the point G_r? Boyd and Raychowdhury [4] computed the convex coordinates of G_r, from which it is clear that the locus is a hyperbola.

Now instead of the inscribed circle consider an inscribed conic (see Figure 2). The lines V_iQ_i, $i = 1, 2, 3$, are still concurrent, at a point called the Brianchon point of the conic (c.f. [5]). There are infinitely many inscribed conics, thus the center

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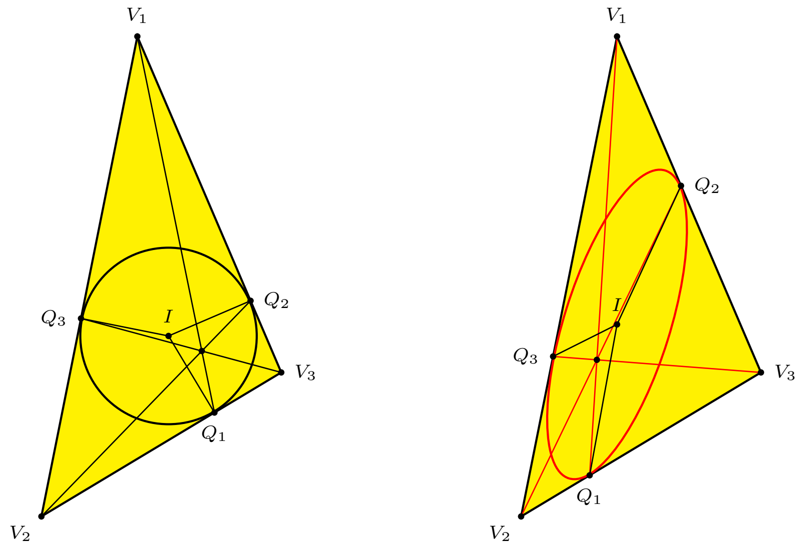

Figure 2. Inscribed ellipse and its directions IQ_i

I and directions q_i (corresponding to the line connecting I and Q_i) can be chosen in many ways, but not arbitrarily. Note that the center completely determines the inscribed conic and the points of tangency Q_1, Q_2, Q_3.

Using these directions we generalize the concept of concentric circles: given a triangle $V_1V_2V_3$ and an inscribed conic with center I and touch points Q_1, Q_2, Q_3, consider the three lines q_i connecting I and Q_i respectively. A circle with center I has to be found which meets the lines q_i at $\bar{Q}_i$ such that the lines $V_i\bar{Q}_i$, $i = 1, 2, 3$, are concurrent. In fact, as we will see in the next section, we do not have to restrict ourselves in terms of the position of the center and the given directions.

2. The general problem and its solution

The general problem can be formulated as follows: given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_i, find a distance $x = IQ_1 = IQ_2 = IQ_3$ along these directions, for which the three cevians V_iQ_i are concurrent. In general these lines will not meet in one point (see Figure 3): instead of one single center G we have three different intersection points G_{12}, G_{13} and G_{23}.

In the following theorem we will prove that altering the value x, the points G_{12},G_{13} and G_{23} will separately move on three conics. If there is a solution to our generalized problem, it would mean that these conics have to meet in one common point. It is easy to observe that each pair of conics have two common points at I and a vertex of the triangle. Here we prove that the other two intersection points can be common for all the three conics. Previously mentioned special cases are excluded from this point.

Theorem 1. *Let V_1, V_2, V_3 and I be four points in the plane in general positions. Let q_1, q_2, q_3 be three different oriented lines through I ($V_i \notin q_i$). There exist*

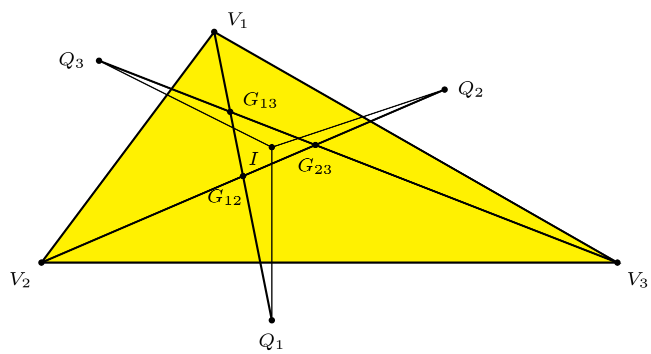

Figure 3. For arbitrary directions and distance, cevians V_iQ_i are generally not concurrent, but meet at three different points

at most two values $x \in \mathbb{R}\backslash\{0\}$ such that for points Q_i along the lines q_i with $IQ_1 = IQ_2 = IQ_3 = x$, the lines V_iQ_i are concurrent.

Proof. For a real number x and $i = 1, 2, 3$, let $Q_i(x)$ be a point on q_i for which $IQ_i(x) = x$. The correspondences $Q_i(x) \leftrightarrow Q_j(x)$ define perspectivities $(q_i) \doublebarwedge (q_j)$, $(i \neq j)$.

Now let $l_i(x)$ be the line connecting V_i and $Q_i(x)$. The correspondences $l_i(x) \leftrightarrow l_j(x)$ define projectivities $(V_i) \barwedge (V_j)$, $(i \neq j)$. The intersection points of corresponding lines of these projectivities lie on three conics:

$$\begin{aligned}
(V_1) \barwedge (V_2) &\Rightarrow c_3 \\
(V_1) \barwedge (V_3) &\Rightarrow c_2 \\
(V_2) \barwedge (V_3) &\Rightarrow c_1.
\end{aligned}$$

We find the intersection points of these conics. Since $Q_i(0) = I$, then $I \in c_i, (i = 1, 2, 3)$. $V_3 \in c_1 \cap c_2$, $V_2 \in c_1 \cap c_3$ and $V_1 \in c_2 \cap c_3$ also hold. Denote the other two intersection points of c_1 and c_2 by S_1 and S_2, *i.e.*,

$$c_2 \cap c_3 = \{I, V_1, S_1, S_2\}.$$

The points S_1 and S_2 can be real and distinct, real and identical, or imaginary in pair.

(i) If they are real and distinct, then for some x_1 and x_2,

$$\begin{aligned}
S_1 = l_1(x_1) \cap l_2(x_1) = l_1(x_1) \cap l_3(x_1) \quad &\Rightarrow \quad S_1 = l_2(x_1) \cap l_3(x_1) \\
S_2 = l_1(x_2) \cap l_2(x_2) = l_1(x_2) \cap l_3(x_2) \quad &\Rightarrow \quad S_2 = l_2(x_2) \cap l_3(x_2)
\end{aligned}$$

which immediately yields $S_1, S_2 \in c_1$ as well.

(ii) If they are identical, then for the unique x,

$$S = l_1(x) \cap l_2(x) = l_1(x) \cap l_3(x) \quad \Rightarrow \quad S = l_2(x) \cap l_3(x)$$

which yields $S \in c_1$.

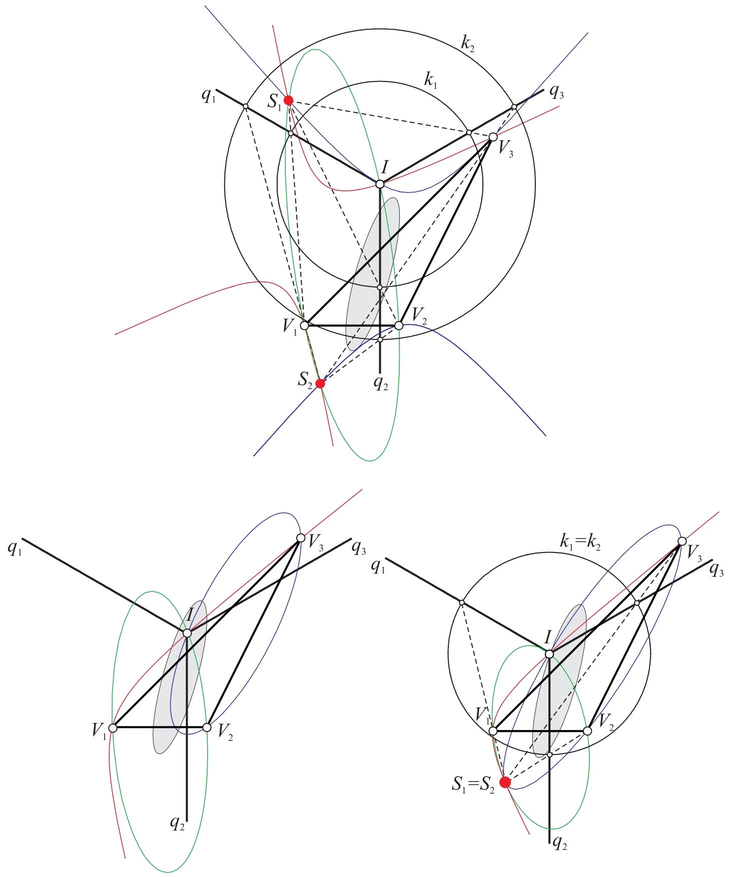

Figure 4. Given a triangle $V_1V_2V_2$ and directions $q_i(i = 1, 2, 3)$ there can be two different real solutions (upper figure), two coinciding solutions (bottom right) and two imaginary solutions (bottom left). Cevians are plotted by dashed lines. The type of solutions depends on the relative position of I to the shaded conic. The three conic paths of G_{12} (green), G_{13} (red) and G_{23} (blue) are also shown. (This figure is computed and plotted by the software *Mathematica*)

(iii) If the points S_1 and S_2 are the pair of imaginary points there are no real number x for which the lines V_iQ_i are concurrent. □

Figure 4 shows the three different possibilities mentioned in the proof. If the triangle and the directions q_i, $i = 1, 2, 3$, are fixed, then the radius of the circle can be obtained by the solutions of a quadratic equation in which the only unknown is the point I. The type of the solutions depends on the discriminant, which is a quadratic function of I. This means that for every triangle and triple of directions there exists a conic which separates the possible positions of I in the following way: if I is outside the conic (discriminant > 0) then there are two different real solutions, if I is on the conic (discriminant $= 0$) then there are two coinciding real solutions, while if I is inside the conic (discriminant < 0) then there are two imaginary solutions. This conic is also shown in Figure 4.

Remarks. (1) Note that there are no further restrictions for the positions of the center and the directed lines. The center can even be outside the reference triangle.

(2) According to the projective principles in the proof, the statement remains valid if we replace the condition $IQ_1 = IQ_2 = IQ_3 = x$ with the more condition that the ratios of these lengths be fixed.

3. Further research

The conics c_i, $i = 1, 2, 3$, play a central role in the proof. The affine types of these conics however, can only be determined by analytical approach or by closer study the type of involutive pencils determined by cevians. It is also a topic of further research how the types of solutions depend on the ratios mentioned in Remark 2. The exact representation of the length of the radius by the given data can also be discussed analytically in a further study.

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Stronger Forms of the Steiner-Lehmus Theorem

Mowaffaq Hajja

Abstract. We give a short proof based on Breusch's lemma of a stronger form of the Steiner-Lehmus theorem, and we discuss other possible stronger forms.

1. A stronger form of Steiner-Lehmus Theorem

Let a, b, c, A, B, C denote, in the standard manner, the side lengths and angles of a triangle ABC. An elegant lemma that was designed by Robert Breusch for solving an interesting 1961 MONTHLY problem [4] states that

$$\frac{p(ABC)}{a} = \frac{2}{1 - \tan(B/2)\tan(C/2)}, \tag{1}$$

where $p(\dots)$ denotes the perimeter. Its simple proof is reproduced in [1], where it is used to give a very short proof of a theorem of Urquhart.

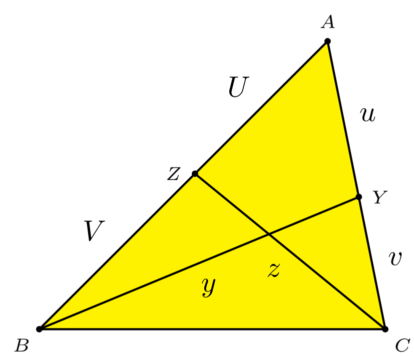

Figure 1

We now consider the Steiner-Lehmus configuration shown in Figure 1, where BY and CZ are the internal angle bisectors of angles B and C. Applying Breusch's lemma to triangles YBC and ZBC, we obtain

$$\frac{p(YBC)}{p(ZBC)} = \frac{1 - \tan(B/2)\tan(C/4)}{1 - \tan(B/4)\tan(C/2)}.$$

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The author would like to thank Yarmouk University for supporting this work, and Nikolaos Dergiades for his valuable comments and additions. Nikolaos is responsible for much of §2. Specifically, he is responsible for proving (8), which appeared as a conjecture in the first draft, and for strengthening the conjecture in (13) by testing more triangles and by introducing and evaluating the limit in (12).

If $c > b$, then

$$\tan\frac{C}{2}\ \tan\frac{B}{4} = \frac{2\tan(B/4)\tan(C/4)}{1-\tan^2(C/4)} > \frac{2\tan(B/4)\tan(C/4)}{1-\tan^2(B/4)} = \tan\frac{B}{2}\ \tan\frac{C}{4},$$

and therefore $p(YBC) > p(ZBC)$. Letting

$$|BY| = y\ ,\ |CZ| = z\ , |AZ| = U\ ,\ |ZB| = V\ ,\ |AY| = u\ ,\ |YC| = v,$$

we have proved the stronger form

$$c > b \Longleftrightarrow y + v > z + V \tag{2}$$

of the traditional Steiner-Lehmus theorem

$$c > b \Longleftrightarrow y > z. \tag{3}$$

To see that (2) is indeed stronger than (3), we need to show that $V > v$. By the angle bisector theorem, we have $\frac{V}{U} = \frac{a}{b}$. Therefore, $\frac{V}{V+U} = \frac{a}{a+b}$, and $V = \frac{ac}{a+b}$. A similar formula holds for v. Thus we have

$$V = \frac{ac}{a+b}\ ,\ v = \frac{ab}{a+c}, \tag{4}$$

and

$$V - v \quad = \quad \frac{ac}{a+b} - \frac{ab}{a+c} \;=\; \frac{a(c(a+c) - b(b+c))}{(a+b)(a+c)} \;=\; \frac{a(a+b+c)(c-b)}{(a+b)(a+c)}$$

Thus

$$c > b \Longrightarrow V > v, \tag{5}$$

and (2) is stronger than (3).

It follows from (4) that

$$U = \frac{bc}{a+b}\ ,\ u = \frac{bc}{a+c}, \tag{6}$$

and therefore

$$c > b \Longrightarrow U > u.$$

Thus the statement

$$c > b \Longrightarrow y + b > z + c \tag{7}$$

would be stronger, and more pleasant, than (2). Unfortunately, (7) is not true. In fact, a recent MONTHLY problem [3] states that if $a \geq c > b$, then the reverse inequality $y + b < z + c$ holds.

2. Additive stronger forms

2.1. One then wonders about the statement

$$c > b \Longrightarrow y + u > z + U. \tag{8}$$

This is also stronger than the classical form (3). In order to prove (8), since $c > b \Longrightarrow y > z$ and $U > u$, it is sufficient to prove that

$$\frac{y^2 - z^2}{U - u} > y + z,\ \text{or}\ \left(\frac{y^2 - z^2}{U - u}\right)^2 > (y+z)^2,\ \text{or}\ \left(\frac{y^2 - z^2}{U - u}\right)^2 > 2(y^2 + z^2),$$

or

$$\frac{a(a+b+c)(a^8+s_7a^7+\cdots+s_1a+s_0)}{b^2c^2(a+c)^2(a+b)^2}>0,$$

which is true because

$$\begin{array}{ll}
s_7=3(b+c), & s_6=3(b^2+4bc+c^2),\\
s_5=b^2+16bc+16c^2, & s_4=bc(2b+5c)(5b+2c),\\
s_3=bc(b+c)(2b^2+17bc+2c^2), & s_2=5b^2c^2(b+c)^2,\\
s_1=b^2c^2(b+c)(b^2+c^2), & s_0=2b^3c^3(b-c)^2.
\end{array}$$

Combining (8) with (2) would yield the form

$$c>b\Longrightarrow y+\frac{b}{2}>z+\frac{c}{2}$$

or

$$c>b\Longrightarrow y-z>\frac{1}{2}(c-b). \tag{9}$$

2.2. In all cases, it is interesting to investigate the best constant λ for which

$$c>b\Longrightarrow y-z\geq\lambda(c-b). \tag{10}$$

Similar questions can be asked about the best constants in

$$c>b\Longrightarrow y-z\geq\lambda(U-u),\quad c>b\Longrightarrow y-z\geq\lambda(V-v). \tag{11}$$

These may turn out to be quite easy given available computer packages such as BOTTEMA. For example, the stronger form $c>b\Longrightarrow y-z>0.8568(c-b)$ of (9) was verified for all triangles whose side lengths are integers less than 51. The minimum value 0.8568 of the fraction $\frac{y-z}{c-b}$, verified for all triangles whose side lengths are integers less than 51, is attained for $(a,b,c)=(48,37,38)$, and the minimum value 0.856762, verified for all triangles whose side lengths are integers less than 501, is attained for $(a,b,c)=(499,388,389)$. Hence one may cnjecture that the minimum value of $\frac{y-z}{c-b}$ is attained when c tends to b. Note that

$$\lim_{c\to b}\frac{y-z}{c-b}=\frac{\sqrt{(a+2b)b}(a^2+ab+2b^2)}{2b(a+b)^2}. \tag{12}$$

Let $f(x):=\frac{\sqrt{x+2}(x^2+x+2)}{2(x+1)^2}$, so that the above limit is $f(\frac{a}{b})$. Since

$$f'(x)=\frac{x^3+4x^2+x-10}{4(x+1)^3\sqrt{x+2}},$$

we conclude that the minimum is $f(q)=0.856762$, where $q=1.284277$ is the unique real zero of x^3+4x^2+x-10. Hence, we may conjecture that

$$c>b\Longleftrightarrow y-z>f(q)(c-b). \tag{13}$$

3. Multiplicative stronger forms

3.1. One may also wonder about possibilities such as

$$c > b \Longrightarrow yb > zc.$$

This is again false. In fact, it is proved in [5, Exercise 4, p. 15] that

$$y^2b^2 - z^2c^2 = \frac{abc(c-b)(a+b+c)^2(b^2-bc+c^2-a^2)}{(a+b)^2(a+c)^2}, \tag{14}$$

and therefore

$$c > b \Longrightarrow (yb > zc \Longleftrightarrow A < 60^\circ)\,. \tag{15}$$

However, it is direct to check that

$$y^2b - z^2c = \frac{abc(c-b)(a+b+c)(b^2+c^2+ab+ac)}{(a+b)^2(a+c)^2}, \tag{16}$$

showing that

$$c > b \Longleftrightarrow y^2b > z^2c, \tag{17}$$

yet another stronger form of the Steiner-Lehmus theorem (3).

3.2. Formulas (14) and (16) are derived from the formulas

$$y^2 = ac\left(1 - \left(\frac{b}{a+c}\right)^2\right), \quad z^2 = ab\left(1 - \left(\frac{c}{a+b}\right)^2\right). \tag{18}$$

These follow from Stewart's theorem using (6) and (4); see [5, Exercise 1, p. 15], where these are used to give a proof of the Steiner-Lehmus theorem via

$$y^2 - z^2 \quad = \quad \frac{(c-b)a(a+b+c)(a^2(a+b+c)+bc(b+c+3a))}{(a+b)^2(a+c)^2}.$$

Similarly one can prove the stronger forms

$$c > b \quad \Longrightarrow \quad y^2u - z^2U > a(c-b), \tag{19}$$

$$c > b \quad \Longrightarrow \quad y^2 - z^2 > V^2 - v^2 \tag{20}$$

using

$$\begin{aligned} y^2u - z^2U \quad &= \quad \frac{(c-b)abc(a+b+c)Q_1}{(a+b)^3(a+c)^3}, \\ (y^2+v^2) - (z^2+V^2) \quad &= \quad \frac{(c-b)a(a+b+c)Q_2}{(a+b)^2(a+c)^2}. \end{aligned}$$

where

$$\begin{aligned} Q_1 \quad &= \quad (a^3+2abc)(a+b+c) + bc(a^2+b^2+c^2), \\ Q_2 \quad &= \quad a^3 + b^2c + c^2b + 3abc - b^2a - c^2a. \end{aligned}$$

Here Q_2 can be seen to be positive by substituting $a = \beta+\gamma$, $b = \alpha+\gamma$, $c = \alpha+\beta$.

Remark. It is well known that the Steiner-Lehmus theorem (3) is valid in neutral (or absolute) geometry; see [2, p. 119]. One wonders whether the same is true of the stronger forms (2), (8), and the other possible forms discussed in §2.

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A New Proof of a Weighted Erdős-Mordell Type Inequality

Yu-Dong Wu

Dedicated to Miss Xiao-Ping Lü
on the occasion of the 24-th Teachers' Day

Abstract. In this short note, by making use of one of Liu's theorems and Cauchy-Schwarz Inequality, we solve a conjecture posed by Liu [3] and give a new proof of a weighted Erdős–Mordell type inequality. Some interesting corollaries are also given at the end.

1. Introduction and Main Results

Let P be an arbitrary point in the plane of triangle ABC. Denote by R_1, R_2, and R_3 the distances from P to the vertices A, B, and C, and r_1, r_2, and r_3 the signed distances from P to the sidelines BC, CA, and AB, respectively. The neat and famous inequality

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3), \tag{1}$$

conjectured by Paul Erdős in 1935, was first proved by L. J. Mordell and D. F. Barrow (see [2]). In 2005, Jian Liu [3] obtained a weighted Erdős-Mordell type inequality as follows.

Theorem 1. *For* $x, y, z \in \mathbb{R}$,

$$\begin{aligned} &x^2\sqrt{R_2 + R_3} + y^2\sqrt{R_3 + R_1} + z^2\sqrt{R_1 + R_2} \\ \geq &\sqrt{2}(yz\sqrt{r_2 + r_3} + zx\sqrt{r_3 + r_1} + xy\sqrt{r_1 + r_2}). \end{aligned} \tag{2}$$

Liu's proof, however, is quite complicated. We give a simple proof of Theorem 1 as a corollary of a more general result, also conjectured by Liu in [3].

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The author would like to thank Professor Li Zhou for valuable comments which helped improve the presentation, and Professor Zhi-Hua Zhang for his careful reading of this paper.

Theorem 2. *For $x, y, z \in \mathbb{R}$ and arbitrary positive real numbers u, v, w, we have*

$$\begin{aligned} & x^2\sqrt{v+w} + y^2\sqrt{w+u} + z^2\sqrt{u+v} \\ \geq & 2\left(yz\sqrt{u\sin\frac{A}{2}} + zx\sqrt{v\sin\frac{B}{2}} + xy\sqrt{w\sin\frac{C}{2}} \right). \end{aligned} \tag{3}$$

2. Preliminary Results

In order to prove our main results, we shall require the following two lemmas.

Lemma 3 ([4, 5])**.** *For $x, y, z \in \mathbb{R}$, $p_i \in (-\infty, 0)\bigcup(0, +\infty)$, and $q_i \in \mathbb{R}$ for $i = 1, 2, 3$, the quadratic inequality of three variables*

$$p_1x^2 + p_2y^2 + p_3z^2 \geq q_1yz + q_2zx + q_3xy$$

holds if and only if

$$\begin{cases} p_i > 0,\ i = 1,\ 2,\ 3; \\ 4p_2p_3 > q_1^2,\ 4p_3p_1 > q_2^2,\ 4p_1p_2 > q_3^2, \\ 4p_1p_2p_3 \geq p_1q_1^2 + p_2q_2^2 + p_3q_3^2 + q_1q_2q_3. \end{cases}$$

Lemma 4. *In $\triangle ABC$, we have*

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1.$$

Proof. This follows from the formula $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$ and the known identity

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

□

3. Proof of Theorem 2

(1) For $u, v, w > 0$,

$$\begin{cases} \sqrt{v+w} > 0, \\ \sqrt{w+u} > 0, \\ \sqrt{u+v} > 0. \end{cases} \tag{4}$$

(2) From $\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2} \in (0, 1)$, we easily get

$$\begin{cases} 4\sqrt{(w+u)(u+v)} > 4u > 4u\sin\frac{A}{2}, \\ 4\sqrt{(u+v)(v+w)} > 4v > 4v\sin\frac{B}{2}, \\ 4\sqrt{(v+w)(w+u)} > 4w > 4w\sin\frac{C}{2}. \end{cases} \tag{5}$$

By the Cauchy-Schwarz inequality and Lemma 4, we have

$$\begin{aligned}
&\left(u\sqrt{v+w}\sin\frac{A}{2}+v\sqrt{w+u}\sin\frac{B}{2}+w\sqrt{u+v}\sin\frac{C}{2}+\sqrt{2uvw}\sqrt{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\right)^2\\
\leq&\left(u^2(v+w)+v^2(w+u)+w^2(u+v)+2uvw\right)\\
&\cdot\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\\
=&(u+v)(v+w)(w+u). \qquad (6)
\end{aligned}$$

From Lemma 3 and (4)–(6), we conclude that inequality (3) holds. The proof of Theorem 2 is complete.

4. Applications of Theorem 2

Proof of Theorem 1. If we take $u = R_1$, $v = R_2$, $w = R_3$ and with known inequalities (see [1])

$$2R_1\sin\frac{A}{2}\geq r_2+r_3,\quad 2R_2\sin\frac{B}{2}\geq r_3+r_1,\quad 2R_3\sin\frac{C}{2}\geq r_1+r_2,$$

we obtain Theorem 1 immediately. This completes the proof of Theorem 1.

Many further inequalities can be obtained from various substitutions for (u, v, w). Here are two examples.

Corollary 5. *For $\triangle ABC$ and real numbers x, y, z, we have*

$$\begin{aligned}
&x^2\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+y^2\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}+z^2\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}\\
\geq&2\left(yz\sin\frac{A}{2}+zx\sin\frac{B}{2}+xy\sin\frac{C}{2}\right).
\end{aligned}$$

Corollary 6. *For $\triangle ABC$ and real numbers x, y, z, we have*

$$\begin{aligned}
&x^2\sqrt{\csc\frac{B}{2}+\csc\frac{C}{2}}+y^2\sqrt{\csc\frac{C}{2}+\csc\frac{A}{2}}+z^2\sqrt{\csc\frac{A}{2}+\csc\frac{B}{2}}\\
\geq&2(yz+zx+xy).
\end{aligned}$$

Further inequalities can also be obtained from substitutions of (x, y, z) by geometric elements of $\triangle ABC$. The reader is invited to experiment with the possibilities.

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Another Compass-Only Construction of the Golden Section and of the Regular Pentagon

Michel Bataille

Abstract. We present a compass-only construction of the point dividing a *given* segment in the golden ratio. As a corollary, we obtain a very simple construction of a regular pentagon inscribed in a *given* circle.

Various constructions of the golden section and of the regular pentagon have already appeared in this journal. In particular, in $[1, 2]$, Kurt Hofstetter offers very interesting compass-only constructions that require only a small number of circles. However, the constructed divided segment and pentagon come into sight as fortunate outcomes of the completed figures and are not subject to any prior constraint. As a result, these constructions do not adjust easily to the usual cases when the segment to be divided or the circumcircle of the pentagon are given at the start. The purpose of this note is to propose direct, simple compass-only constructions adapted to such situations.

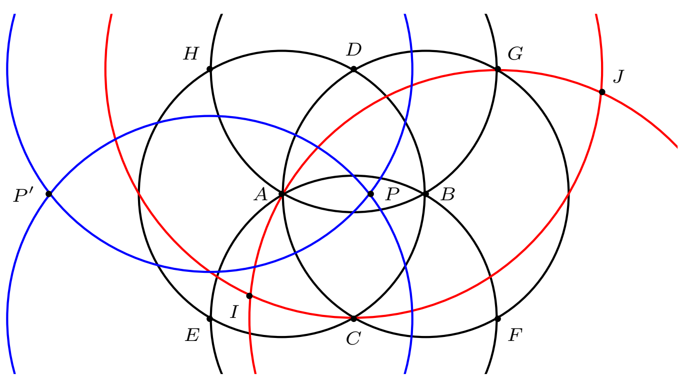

Figure 1

Construction 1. *Given two distinct points A, B, to obtain the point P of the line segment AB such that $\frac{AP}{AB} = \frac{\sqrt{5}-1}{2}$, construct*
(1) *with the same radius AB, the circles with centers A and B, to intersect at C and D,*
(2) *with the same radius AB, the circles with centers C and D, to intersect the two circles in* (1) *at E, F, G, H* (see Figure 1),

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(3) *with the same radius DC, the circles with centers D and F, to intersect at I and J,*
(4) *with the same radius BI, the circles with centers E and H.*
The points of intersection of these two circles are on the line AB, and P is the one between A and B.

Note that eight circles are needed, but if the line segment AB has been drawn, the number of circles drops to six, as it is easily checked. Note also that only three different radii are used.

Construction 2. *Given a point B on a circle Γ with center A, to obtain a regular pentagon inscribed in Γ with vertex B, construct*
(1) *the point P which divides AB in the golden section,*
(2) *the circle with center P and radius AB, to intersect Γ at B_1 and B_4,*
(3) *the circles $B_1(B)$ and $B_4(B)$ to intersect Γ, apart from B, at B_2 and B_3 respectively.*
The pentagon $BB_1B_2B_3B_4$ is the desired one (see Figure 2).

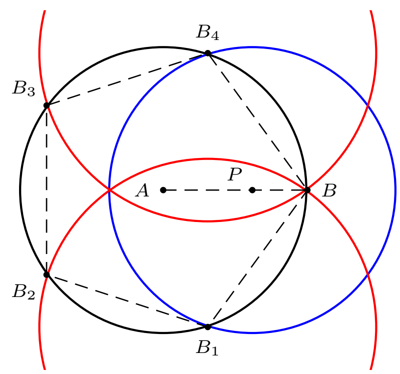

Figure 2

Proof of Construction 1. Let $a = AB$. Clearly, E, F (respectively F, D) are diametrically opposite on the circle with center C (respectively B) and radius a. It follows that EB is the perpendicular bisector of DF and since $IF = ID$, I is on the line EB. Therefore ΔIBF is right-angled at B, and $IB = a\sqrt{2}$ (since $IF = CD = a\sqrt{3}$ and $BF = a$). Now, the circles in (4) do intersect (since $HE = CD < 2BI$) and are symmetrical in the line AB, hence their intersections P, P' are certainly on this line. As for the relation $AP = \frac{\sqrt{5}-1}{2}AB$, it directly results from the following key property:

Let triangle BAE satisfy $AE = AB = a$ and $\angle BAE = 120^\circ$ and let P be on the side AB such that $EP = a\sqrt{2}$. Then $AP = \frac{\sqrt{5}-1}{2}\,a$ (see Figure 3).

Indeed, the law of cosines yields $PE^2 = AE^2 + AP^2 - 2AE \cdot AP \cdot \cos 120^\circ$ and this shows that AP is the positive solution to the quadratic $x^2 + ax - a^2 = 0$. Thus, $AP = \frac{\sqrt{5}-1}{2}\,a$. $\square$

Note that $AP' = \frac{\sqrt{5}+1}{2}\,a$ is readily obtained in a similar manner.

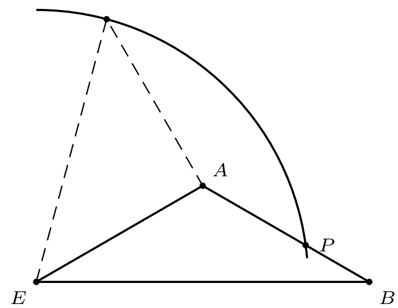

Figure 3

Proof of Construction 2. Since ΔAB_4P is isosceles with $B_4A = B_4P = a$, we have $\cos BAB_4 = \frac{1}{2}\frac{AP}{a} = \frac{\sqrt{5}-1}{4}$. Hence $\angle BAB_4 = 72^\circ$ and the result immediately follows. $\square$

As a final remark, Figure 3 and the property above lead to a quick construction of the golden section with ruler and compass.

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Some Identities Arising From Inversion of Pappus Chains in an Arbelos

Giovanni Lucca

Abstract. We consider the inversive images, with respect to the incircle of an arbelos, of the three Pappus chains associated with the arbelos, and establish some identities connecting the radii of the circles involved.

In a previous work [1], we considered the three Pappus chains that can be drawn inside the arbelos and demonstrated some identities relating the radii of the circles in these chains. In Figure 1, the diameter AC of the left semicircle $\mathcal{C}_{\rm a}$ is $2a$, the diameter CB of the right semicircle $\mathcal{C}_{\rm b}$ is $2b$, and the diameter AB of the outer semicircle $\mathcal{C}_{\rm r}$ is $2r$, $r = a + b$. The first circle Γ_1 is common to all three chains and is the incircle of the arbelos.

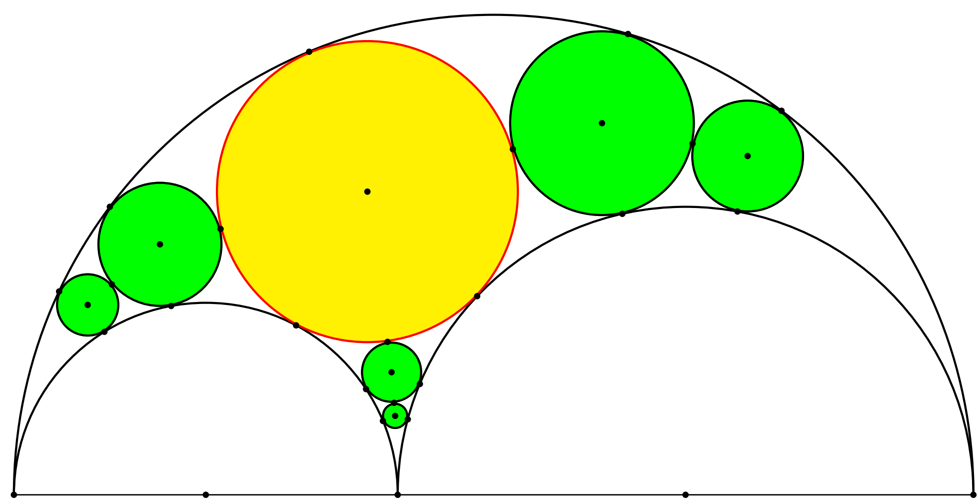

Figure 1. The Pappus chains in an arbelos

With reference to Figure 1, we denote by $\Gamma_{\rm r}$, $\Gamma_{\rm a}$ and $\Gamma_{\rm b}$ the chains converging to C, A, B respectively. Table 1 gives the coordinates of the centers and the radii of the circles in the chains, referring to a Cartesian reference system with origin at C and x-axis along AB.

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Table 1: Center coordinates and radii of the circles in the Pappus chains

Chain	Γ_{r}	Γ_{a}	Γ_{b}
Abscissa of n-th circle	$x_{\mathrm{r}n} = \frac{ab(a-b)}{n^2r^2-ab}$	$x_{\mathrm{a}n} = 2b - \frac{rb(r+b)}{n^2a^2+rb}$	$x_{\mathrm{b}n} = -2a + \frac{ra(r+a)}{n^2b^2+ra}$
Ordinate of n-th circle	$y_{\mathrm{r}n} = \frac{2nrab}{n^2r^2-ab}$	$y_{\mathrm{a}n} = \frac{2nrab}{n^2a^2+rb}$	$y_{\mathrm{b}n} = \frac{2nrab}{n^2b^2+ra}$
Radius of n-th circle	$\rho_{\mathrm{r}n} = \frac{rab}{n^2r^2-ab}$	$\rho_{\mathrm{a}n} = \frac{rab}{n^2a^2+rb}$	$\rho_{\mathrm{b}n} = \frac{rab}{n^2b^2+ra}$

The following proposition was established in [1].

Proposition 1. *Given a generic arbelos with its three Pappus chains, the following identities hold for each integer n:*

$$\rho_{\mathrm{inc}}\left(\frac{1}{\rho_{\mathrm{r}n}}+\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{b}n}}\right)=2n^2+1, \tag{1}$$

$$\rho_{\mathrm{inc}}^2\left(\frac{1}{\rho_{\mathrm{r}n}^2}+\frac{1}{\rho_{\mathrm{a}n}^2}+\frac{1}{\rho_{\mathrm{b}n}^2}\right)=2n^4+1, \tag{2}$$

$$\rho_{\mathrm{inc}}^2\left(\frac{1}{\rho_{\mathrm{r}n}}\cdot\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{a}n}}\cdot\frac{1}{\rho_{\mathrm{b}n}}+\frac{1}{\rho_{\mathrm{b}n}}\cdot\frac{1}{\rho_{\mathrm{r}n}}\right)=n^4+2n^2. \tag{3}$$

In particular, the center of the incircle of the arbelos is the point

$$(x_{\mathrm{inc}},\ y_{\mathrm{inc}})=\left(\frac{ab(a-b)}{a^2+ab+b^2},\ \frac{2ab(a+b)}{a^2+ab+b^2}\right).$$

Its radius is

$$\rho_{\mathrm{inc}}=\frac{ab(a+b)}{a^2+ab+b^2}.$$

We now consider the inversion of these three Pappus chains with respect to the incircle of arbelos. See Figure 2. For convenience, we record a useful formula, which can be found in [2], we use for the computation of the centers and radii of the inversive images of the circles in the Pappus chains.

Lemma 2. *With respect the circle of center (x_0, y_0) and radius R_0, the inversive image of the circle with center $(x_C,\ y_C)$ and radius R is the circle with center $(x_C^{\mathrm{i}},\ y_C^{\mathrm{i}})$ and radius R^{i} given by*

$$x_C^{\mathrm{i}}=x_0+\frac{R_0^2}{(x_C-x_0)^2+(y_C-y_0)^2-R^2}(x_C-x_0),$$

$$y_C^{\mathrm{i}}=y_0+\frac{R_0^2}{(x_C-x_0)^2+(y_C-y_0)^2-R^2}(y_C-y_0),$$

$$R^{\mathrm{i}}=\left|\frac{R_0^2}{(x_C-x_0)^2+(y_C-y_0)^2-R^2}\right|R.$$

We give in Table 2 the coordinates of the centers of the inversive images of the circles in the Pappus chains, and their radii.

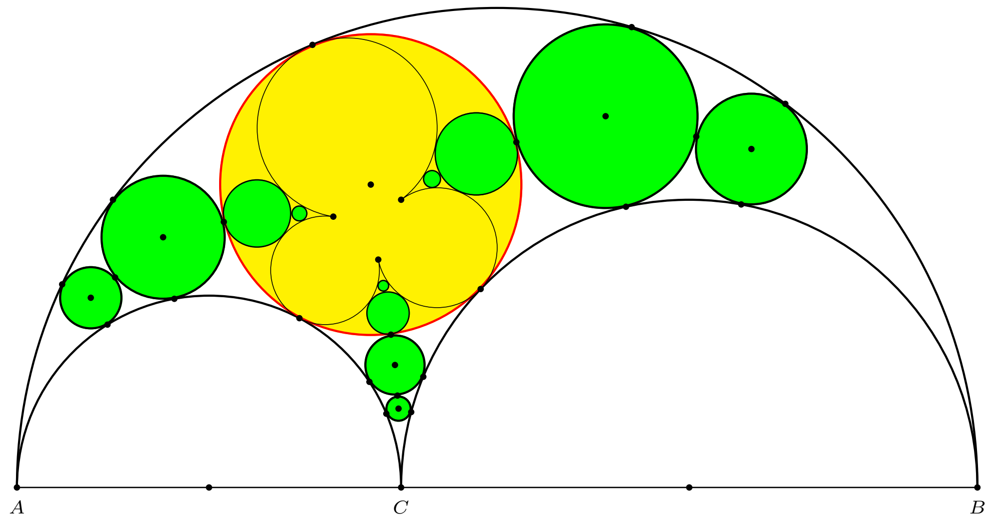

Figure 2. Inversive images of the Pappus chains

Table 2: Center coordinates and radii of inversive images of circles in the Pappus chains

	Inverted chain $\Gamma_{\mathrm{r}}^{\mathrm{i}}$
Abscissa of n-th circle	$x_{\mathrm{r}n}^{\mathrm{i}}=x_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(x_{\mathrm{r}n}-x_{\mathrm{inc}})}{(x_{\mathrm{r}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{r}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Ordinate of n-th circle	$y_{\mathrm{r}n}^{\mathrm{i}}=y_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(y_{\mathrm{r}n}-y_{\mathrm{inc}})}{(x_{\mathrm{r}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{r}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Radius of n-th circle	$\rho_{\mathrm{r}n}^{\mathrm{i}}=\left\lvert\frac{\rho_{\mathrm{inc}}^2}{(x_{\mathrm{r}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{r}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}\right\rvert\rho_{\mathrm{r}n}$
	Inverted chain $\Gamma_{\mathrm{a}}^{\mathrm{i}}$
Abscissa of n-th circle	$x_{\mathrm{a}n}^{\mathrm{i}}=x_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(x_{\mathrm{a}n}-x_{\mathrm{inc}})}{(x_{\mathrm{a}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{a}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Ordinate of n-th circle	$y_{\mathrm{a}n}^{\mathrm{i}}=y_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(y_{\mathrm{a}n}-y_{\mathrm{inc}})}{(x_{\mathrm{a}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{a}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Radius of n-th circle	$\rho_{\mathrm{a}n}^{\mathrm{i}}=\left\lvert\frac{\rho_{\mathrm{inc}}^2}{(x_{\mathrm{a}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{a}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}\right\rvert\rho_{\mathrm{a}n}$
	Inverted chain $\Gamma_{\mathrm{b}}^{\mathrm{i}}$
Abscissa of n-th circle	$x_{\mathrm{b}n}^{\mathrm{i}}=x_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(x_{\mathrm{b}n}-x_{\mathrm{inc}})}{(x_{\mathrm{b}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{b}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Ordinate of n-th circle	$y_{\mathrm{b}n}^{\mathrm{i}}=y_{\mathrm{inc}}+\frac{\rho_{\mathrm{inc}}^2(y_{\mathrm{b}n}-y_{\mathrm{inc}})}{(x_{\mathrm{b}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{b}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Radius of n-th circle	$\rho_{\mathrm{b}n}^{\mathrm{i}}=\left\lvert\frac{\rho_{\mathrm{inc}}^2}{(x_{\mathrm{b}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{b}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}\right\rvert\rho_{\mathrm{b}n}$

From these data, we can deduce some identities connecting the radii of these circles.

Theorem 3. *For the circles in the Pappus chains and their inversive images in the incircle, the following identities hold. For $n \geq 2$,*

$$\frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm r}n}} - \frac{\rho_{\rm inc}}{\rho_{{\rm r}n}} = \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}} - \frac{\rho_{\rm inc}}{\rho_{{\rm a}n}} = \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm b}n}} - \frac{\rho_{\rm inc}}{\rho_{{\rm b}n}} = 4n^2 - 8n + 2, \tag{4}$$

$$\frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm r}n}} + \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}} + \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm b}n}} = 14n^2 - 24n + 7, \tag{5}$$

$$\frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm r}n}} + \frac{\rho_{\rm inc}}{\rho_{{\rm a}n}} + \frac{\rho_{\rm inc}}{\rho_{{\rm b}n}} = \frac{\rho_{\rm inc}}{\rho_{{\rm r}n}} + \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}} + \frac{\rho_{\rm inc}}{\rho_{{\rm b}n}} = \frac{\rho_{\rm inc}}{\rho_{{\rm r}n}} + \frac{\rho_{\rm inc}}{\rho_{{\rm a}n}} + \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm b}n}} = 6n^2 - 8n + 3, \tag{6}$$

$$\frac{\rho_{\rm inc}}{\rho_{{\rm r}n}} + \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}} + \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm b}n}} = \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm r}n}} + \frac{\rho_{\rm inc}}{\rho_{{\rm a}n}} + \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm b}n}} = \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm r}n}} + \frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}} + \frac{\rho_{\rm inc}}{\rho_{{\rm b}n}} = 10n^2 - 16n + 5, \tag{7}$$

$$\frac{\rho^2_{\rm inc}}{\rho_{{\rm r}n}\rho^{\rm i}_{{\rm r}n}} + \frac{\rho^2_{\rm inc}}{\rho_{{\rm a}n}\rho^{\rm i}_{{\rm a}n}} + \frac{\rho^2_{\rm inc}}{\rho_{{\rm b}n}\rho^{\rm i}_{{\rm b}n}} = 10n^4 - 16n^3 + 8n^2 - 8n + 3, \tag{8}$$

$$\frac{\rho^2_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}\rho^{\rm i}_{{\rm b}n}} + \frac{\rho^2_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}\rho^{\rm i}_{{\rm r}n}} + \frac{\rho^2_{\rm inc}}{\rho^{\rm i}_{{\rm b}n}\rho^{\rm i}_{{\rm r}n}} = 65n^4 - 224n^3 + 258n^2 - 112n + 16, \tag{9}$$

$$\left(\frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm r}n}}\right)^2 + \left(\frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}}\right)^2 + \left(\frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm b}n}}\right)^2 = 66n^4 - 224n^3 + 256n^2 - 112n + 17. \tag{10}$$

From (9), (10) above, and also (2), (3) in Proposition 1, we have

$$\begin{aligned}
&\left(\frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm r}n}}\right)^2 + \left(\frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}}\right)^2 + \left(\frac{\rho_{\rm inc}}{\rho^{\rm i}_{{\rm b}n}}\right)^2 - \left(\frac{\rho^2_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}\rho^{\rm i}_{{\rm b}n}} + \frac{\rho^2_{\rm inc}}{\rho^{\rm i}_{{\rm a}n}\rho^{\rm i}_{{\rm r}n}} + \frac{\rho^2_{\rm inc}}{\rho^{\rm i}_{{\rm b}n}\rho^{\rm i}_{{\rm r}n}}\right) \\
=\ &\left(\frac{\rho_{\rm inc}}{\rho_{{\rm r}n}}\right)^2 + \left(\frac{\rho_{\rm inc}}{\rho_{{\rm a}n}}\right)^2 + \left(\frac{\rho_{\rm inc}}{\rho_{{\rm b}n}}\right)^2 - \left(\frac{\rho^2_{\rm inc}}{\rho_{{\rm a}n}\rho_{{\rm b}n}} + \frac{\rho^2_{\rm inc}}{\rho_{{\rm a}n}\rho_{{\rm r}n}} + \frac{\rho^2_{\rm inc}}{\rho_{{\rm b}n}\rho_{{\rm r}n}}\right) \\
=\ &(n^2 - 1)^2.
\end{aligned}$$

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Second-Degree Involutory Symbolic Substitutions

Clark Kimberling

Abstract. Suppose a, b, c are algebraic indeterminates. The mapping $(a, b, c) \to (bc, ca, ab)$ is an example of a second-degree involutory symbolic substitution (SISS) which maps the transfigured plane of a triangle to itself. The main result is a classification of SISSs as four individual mappings and two families of mappings. The SISS $(a, b, c) \to (bc, ca, ab)$ maps the circumcircle onto the Steiner ellipse. This and other examples are considered.

1. Introduction

This article is a sequel to [2], in which symbolic substitutions are introduced. A brief summary follows. The symbols a, b, c are algebraic indeterminates over the field of complex numbers. Suppose α, β, γ are nonzero homogeneous algebraic functions of (a, b, c) :

$$\alpha(a, b, c), \beta(a, b, c), \gamma(a, b, c), \tag{1}$$

all of the same degree of homogeneity. Throughout this work, triples with notations such as $U = (u, v, w)$ and $X = (x, y, z)$ are understood to be as in (1), except that one or two (but not all three) of the coordinates can be 0. Triples (x, y, z) and (x', y', z') are *equivalent* if $xy' = yx'$ and $yz' = zy'$. The equivalence class containing any particular (x, y, z) is denoted by $x : y : z$ and is a *point*. The set of points is the *transfigured plane*, denoted by $\mathcal{P}$. A well known model of $\mathcal{P}$ is obtained by taking a, b, c to be sidelengths of a euclidean triangle ABC and taking $x : y : z$ to be the point whose directed distances [1] from the sidelines BC, CA, AB are respectively proportional to x, y, z.

A simple example of a symbolic substitution is indicated by

$$(a, b, c) \to (bc, ca, ab).$$

This means that a point

$$x : y : z = x(a, b, c) : y(a, b, c) : z(a, b, c) \tag{2}$$

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[1] The coordinates $x : y : z$ are *homogeneous trilinear coordinates*, or simply *trilinears*. The notation (x, y, z), in this paper, represents an ordinary ordered triple, as when x, y, z are actual directed distances or when (x, y, z) is the argument of a function. Unfortunately, the notation (x, y, z) has sometimes been used for homogeneous coordinates, so that, for example $(2x, 2y, 2z) = (x, y, z)$, which departs from ordinary ordered triple notation. On the other hand, using colons, we have $2x : 2y : 2z = x : y : z$.

maps to the point

$$x : y : z = x(bc, ca, ab) : y(bc, ca, ab) : z(bc, ca, ab), \tag{3}$$

so that $\mathcal{P}$ is mapped to itself. We are interested in the effects of such substitutions on various points and curves. Consider, for example the Thompson cubic, $\mathcal{Z}(X_2, X_1)$, given by the equation[2]

$$bc\alpha(\beta^2 - \gamma^2) + ca\beta(\gamma^2 - \alpha^2) + ab\gamma(\alpha^2 - \beta^2) = 0. \tag{4}$$

For each point (2) on (4), the point (3) is on the cubic $\mathcal{Z}(X_6, X_1)$, given by the equation

$$a\alpha(\beta^2 - \gamma^2) + b\beta(\gamma^2 - \alpha^2) + cc(\alpha^2 - \beta^2) = 0. \tag{5}$$

Letting $\mathcal{S}(X_i)$ denote the image of X_i under the substitution $(a, b, c) \to (bc, ca, ab)$, specific points on $\mathcal{Z}(X_2, X_1)$ map to points on $\mathcal{Z}(X_6, X_1)$ as shown in Table 1:

Table 1. From $\mathcal{Z}(X_2, X_1)$to $\mathcal{Z}(X_6, X_1)$

X_i on $\mathcal{Z}(X_2, X_1)$	X_1	X_2	X_3	X_4	X_6	X_9	X_{57}
$\mathcal{S}(X_i)$ on $\mathcal{Z}(X_6, X_1)$	X_1	X_6	X_{194}	X_{3224}	X_2	X_{43}	X_{87}

As suggested by Table 1, $\mathcal{S}(\mathcal{S}(X)) = X$ for every X, which is to say that $\mathcal{S}$ is involutory. The main purpose of this article is to find explicitly all second-degree involutory symbolic substitutions.

2. Terminology and Examples

A *polynomial triangle center* is a point U which has a representation

$$u(a, b, c) : v(a, b, c) : w(a, b, c),$$

where $u(a, b, c)$ is a polynomial in a, b, c and these conditions hold:

$$v(a, b, c) = \quad u(b, c, a); \tag{6}$$
$$w(a, b, c) = \quad u(c, a, b); \tag{7}$$
$$|u(a, c, b)| = \quad |u(a, b, c)| . \tag{8}$$

If $u(a, b, c)$ has degree 2, then U is a *second-degree triangle center*. A *second-degree symbolic substitution* is a transformation of $\mathcal{P}$ or some subsert thereof, with images in $\mathcal{P}$, given by a symbolic substitution of the form

$$(a, b, c) \longrightarrow (u(a, b, c), v(a, b, c), w(a, b, c))$$

for some second-degree triangle center U. The mapping (whether of polynomial form or not) is *involutory* if its compositional square is the identity; that is, if

$$u(u, v, w) : v(u, v, w) : w(u, v, w) = a : b : c,$$

[2]Triangle centers are indexed as in [1]: $X_1 =$ incenter, $X_2 =$ centroid, etc. The cubic $\mathcal{Z}(U, P)$ is defined as the set of points $\alpha : \beta : \gamma$ satisfying

$$up\alpha(q\beta^2 - r\gamma^2) + vq\beta(r\gamma^2 - p\alpha^2) + wr\gamma(p\alpha^2 - q\beta^2) = 0$$

where $U = u : v : w$ and $P = p : q : r$. Geometrically, $\mathcal{Z}(U, P)$ is the locus of $X = x : y : z$ such that the P-isoconjugate of X is on the line UX. The P-isoconjugate of X (and the X-isoconjugate of P) is the point $qryz : rpzx : pqxy$.

where

$$u = u(a, b, c),\ v = v(a, b, c),\ w = w(a, b, c).$$

Equivalently, a symbolic substitution $(a, b, c) \longrightarrow (u, v, w)$ is involutory if

$$u(u, v, w) = ta$$

for some function t of (a, b, c) that is symmetric in a, b, c. Henceforth we shall abbreviate “second-degree involutory symbolic substitution” as SISS. Following are four examples.

Example 1. The SISS

$$(a, b, c) \longrightarrow (bc, ca, ab) \tag{9}$$

gives

$$\begin{aligned} u(u, v, w) &= u(bc, ca, ab) \\ &= (bc)(ca) \\ &= ta, \end{aligned}$$

where $t = abc$.

Example 2. The SISS

$$(a, b, c) \longrightarrow (a^2 - bc, b^2 - ca, c^2 - ab) \tag{10}$$

gives

$$\begin{aligned} u(u, v, w) &= u(a^2 - bc, b^2 - ca, c^2 - ab) \\ &= (a^2 - bc)^2 - (b^2 - ca)(c^2 - ab) \\ &= ta, \end{aligned}$$

where

$$t = (a + b + c)\,(a^2 + b^2 + c^2 - bc - ca - ab).$$

Note that (10) is meaningless for $a = b = c$. As a, b, c, are indeterminates, however, such cases do not require additional writing, just as, when one writes “$\tan\theta$” where θ is a variable, it is understood that $\theta \neq \frac{\pi}{2}$.

Example 3. The SISS

$$(a, b, c) \longrightarrow (b^2 + c^2 - ab - ac, c^2 + a^2 - bc - ba, a^2 + b^2 - ca - cb) \tag{11}$$

gives

$$u(u, v, w) = ta,$$

where

$$t = 2\,(a + b + c)\,(a^2 + b^2 + c^2 - bc - ca - ab).$$

Example 4. The SISS

$$(a, b, c) \longrightarrow (a(a - b - c), b(b - c - a), c(c - a - b)) \tag{12}$$

gives

$$u(u, v, w) = ta,$$

where

$$t = (a - b - c)\,(b - c - a)(c - a - b).$$

3. Main result

Theorem. *In addition to the four SISSs (9)-(12), there are two families of SISSs given below by (17) and (18), and there is no other SISS.*

Proof. Equations (6) –(8) and the requirement that u be a polynomial of degree 2 imply that u is expressible in one of these two forms:

$$u = ja^2 + k(b^2 + c^2) + lbc + ma(b + c) \tag{13A}$$

$$u = (b - c)(ja + k(b + c)) \tag{14}$$

for some complex numbers j, k, l, m. The proof will be given in two parts, depending on (13A) and (14).

Part 1: u given by (13A). In this case,

$$v = jb^2 + k(c^2 + a^2) + lca + mb(c + a), \tag{13B}$$

$$w = jc^2 + k(a^2 + b^2) + lab + mc(a + b). \tag{13C}$$

Let $P = u(u, v, w)$. We wish to find all j, k, l, m for which P factors as ta, where t is symmetric in a, b, c. The polynomial P can be written as $aQ + R$, where Q and R are polynomials and the R is invariant of a. In order to have $P = ta$, the coefficients j, k, l, m must make $R(a, b, c)$ identically 0. We have

$$R = (b^4 + c^4)S_1 + 2bc(b^2 + c^2)S_2 + b^2c^2S_3,$$

where

$$S_1 = jkl + jkm + k^3 + jk^2 + j^2k + k^2m,$$

$$S_2 = jkl + jkm + jlm + klm + km^2 + k^2m,$$

$$S_3 = 2jkm + 6jk^2 + jl^2 + j^2l + k^2l + 2km^2 + 2k^2m + 3lm^2.$$

Thus, we seek j, k, l, m for which $S_1 = S_2 = S_3 = 0$.

Case 1: $j = 0$. Here,
$S_1 = (k + m)\,k^2$, so that $k = 0$ or $k = -m$.
$S_2 = mk\,(k + l + m)$, so that $m = 0$ or $k = 0$ or $k + l + m = 0$.
$S_3 = k^2l + 2km^2 + 2k^2m + 3lm^2$.

Subcase 1.1: $j = 0$ and $k = 0$. Here, $S_2 = 0$, $S_3 = 3lm^2$, so that $l = 0$ or $m = 0$ but not both. If $l = 0$ and $m \neq 0$, then by (13A-C),

$$P = mu(v + w) = -m^3a\,(ab + ac + 2bc)\,(b + c)\,,$$

not of the required form aQ where Q is symmetric in a, b, c. On the other hand, if $m = 0$ and $l \neq 0$, then $P = lvw = l^3a^2bc$, so that, on putting $l = 1$, we have $(u, v, w) = (bc, ab, ca)$, as in (9).

Subcase 1.2: $j = 0$ and $k = -m \neq 0$. Here, with $S_2 = 0$, $k \neq 0$, $m \neq 0$, and $k + l + m = 0$, we have $l = 0$, and (13A-C) give

$$P = k(v^2 + w^2) - ku(v + w) = 2a\,(a + b + c)\left(a^2 + b^2 + c^2 - bc - ca - ab\right)k^3,$$

so that taking $(j, k, l, m) = (0, 1, 0, -1)$ gives the SISS (11).

Case 2: $k = 0$. Here, $S_1 = 0$, $S_2 = jlm$, and $S_3 = l\left(jl + j^2 + 3m^2\right)$.

Subcase 2.1: $k = 0$ and $j = 0$. Here, since $S_3 = 0$, we have $3lm^2 = 0$. If $l = 0$, then

$$\begin{aligned} u &= ma(b + c), v = mb(c + a), w = mc(a + b), \\ P &= mu(v + w) = -\,(ab + ac + 2bc)\,(b + c)\,am^3, \end{aligned}$$

not of the required form aQ. On the other hand, if $m = 0$, then

$$u = lbc, \quad v = lca, \quad w = lab,$$

so that taking $(j, k, l, m) = (0, 0, 1, 0)$ gives the SISS (9).

Subcase 2.2: $k = 0$ and $l = 0$. Here, $S_2 = S_3 = 0$, and (13A-C) give

$$\begin{aligned} P &= ju^2 + mu(v + w) \\ &= a\,(aj + bm + cm) \\ &\quad \cdot \left(abjm + acjm + abm^2 + acm^2 + 2bcm^2 + b^2jm + c^2jm + a^2j^2\right). \end{aligned}$$

In order for P to have the form aQ with Q symmetric in a, b, c, the factor

$$\left(abjm + acjm + abm^2 + acm^2 + 2bcm^2 + b^2jm + c^2jm + a^2j^2\right)$$

must factor as

$$(bj + cm + am)(cj + am + bm).$$

The identity

$$\begin{aligned} &\left(abjm + acjm + abm^2 + acm^2 + 2bcm^2 + b^2jm + c^2jm + a^2j^2\right) \\ &- (bj + cm + am)(cj + am + bm) \\ =\ &(m - j)\,(j + m)\left(bc - a^2\right) \end{aligned}$$

shows that this factorization occurs if and only if $j = m$ or $j = -m$. If $j = m$, then

$$u = a^2 + a(b + c), \quad v = b^2 + b(c + a), \quad w = c^2 + c(a + b),$$

leading to $(j, k, l, m) = (1, 0, 0, 1)$, but this is simply the identity substitution $(a, b, c) \to (a, b, c)$, not an SISS.

On the other hand, if $j = -m$, then

$$P = a\,(aj + bm + cm)\,(bj + cm + am)(cj + am + bm),$$

so that for $(j, k, l, m) = (1, 0, 0, -1)$, we have the SISS (12).

Subcase 2.3: $k = 0$ and $m = 0$. Here,

$$\begin{aligned} P &= ju^2 + lvw \\ &= a^4j^3 + a^2bcl^3 + ab^3jl^2 + ac^3jl^2 + 2a^2bcj^2l + b^2c^2jl^2 + b^2c^2j^2l, \end{aligned}$$

which has the form aQ only if $b^2c^2jl^2 + b^2c^2j^2l = 0$, which means that $jl(j+l) = 0$. If $j = 0$ or $l = 0$, we have solutions already found. If $j = -l$, then

$$\begin{aligned} P &= ju^2 - jvw \\ &= l^3a\,(a+b+c)\left(ab+ac+bc-a^2-b^2-c^2\right), \end{aligned}$$

giving $(j,k,l,m) = (1,0,-1,0)$, the SISS (10).

Case 3: $l = 0$. Here,

$$\begin{aligned} S_1 &= \left(jk+jm+km+j^2+k^2\right)k, \\ S_2 &= 2mk\,(j+k+m)\,, \\ S_3 &= 2k\left(3jk+jm+km+m^2\right). \end{aligned}$$

Subcase 3.1: $l = 0$, $m = 0$, $S_1 = \left(jk+j^2+k^2\right)k$, $S_2 = 0$, and $S_3 = 6jk^2$. Since $S_3 = 0$, we have $j = 0$ or $k = 0$, already covered.

Subcase 3.2: $l = 0$, and either $j = 0$ or $k = 0$, already covered.

Case 4: $m = 0$. Here, $S_1 = k\left(jk+jl+j^2+k^2\right)$, $S_2 = 2jkl$, and $S_3 = \left(6jk^2+jl^2+j^2l+k^2l\right)$. Since $S_2 = 0$, we must have $j = 0$ or $k = 0$ or $l = 0$. All of these possibilities are already covered.

Case 5: none of j,k,l,m is 0. Here,

$$\begin{aligned} S_1 &= k\left(jk+jl+jm+km+j^2+k^2\right), \\ S_2 &= jkl+jkm+jlm+klm+km^2+k^2m, \\ S_3 &= 2jkm+6jk^2+jl^2+j^2l+k^2l+2km^2+2k^2m+3lm^2. \end{aligned}$$

As $j \neq 0$ and $k \neq 0$, the requirement that $S_1 = 0$ gives

$$l = -\frac{jk+jm+km+j^2+k^2}{j}. \tag{15}$$

Substitute l into the expression for S_2 and factor, getting

$$S_2 = -\frac{(k+m)(j+m)(jk+j^2+k^2)}{j} = 0. \tag{16}$$

Subcase 5.1: $m = -j$. Here, $l = -\frac{k^2}{j}$. This implies $S_1 = S_2 = S_3 = 0$ and

$$P = \frac{a(ak+(a-b-c)j)(bk+(b-c-a)j)(ck+(c-a-b)j)(k-j)^3}{j^3},$$

which is of the form aQ with Q symmetric in a, b, c. Because of homogeneity, we can without loss of generality take $(j, k, l, m) = (1, k, -k^2, -1)$, where $k \notin \{0, 1, -2\}$. This leaves a family of SISSs:

$$(a, b, c) \to (u, v, w), \tag{17}$$

where

$$\begin{aligned} u &= a^2 + k(b^2 + c^2) - k^2 bc - a(b + c), \\ v &= b^2 + k(c^2 + a^2) - k^2 ca - b(c + a), \\ w &= c^2 + k(a^2 + b^2) - k^2 ab - c(a + b), \\ P &= a(k-1)^3(a - b - c + ak)(b - a - c + bk)(c - b - a + ck). \end{aligned}$$

Note that for $k = -2$, we have $u = (a + b + c)(a - 2b - 2c)$, so that the involutory substitution

$$(a, b, c) \to (a - 2b - 2c, b - 2c - 2a, c - 2a - 2b)$$

is actually of first-degree, not second. (It is easy to check that for

$$u = a + mb + mc,$$

the only values of m for which the substitution $(a, b, c) \to (u, v, w)$ is involutory are 0 and -2.)

Subcase 5.2: $m = -k$. Here, $l = -j$. This implies $S_1 = S_2 = S_3 = 0$ and

$$P = a(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)(j + 2k)(k - j)^2,$$

which is of the form aQ with Q symmetric in a, b, c. Thus, if $j \neq k$ and $j \neq -2k$, we take $(j, k, l, m) = (j, k, -j, -k)$ and have a family of SISSs:

$$(a, b, c) \to (u, v, w) \tag{18}$$

where

$$\begin{aligned} u &= a^2 j + b^2 k + c^2 k - bcj - abk - ack, \\ v &= b^2 j + c^2 k + a^2 k - caj - bck - bak, \\ w &= c^2 j + a^2 k + b^2 k - abj - cak - cbk. \end{aligned}$$

Note that $u = (a^2 - bc)j + (b^2 + c^2 - ab - ac)k$, a linear combination of second-degree polynomials appearing in (10) and (11).

Subcase 5.3: Equation (16) leaves one more subcase: $jk + j^2 + k^2 = 0$. This and (15) give $l = \frac{(j+k)k}{j}$, implying $S_1 = (k + m)(j + k)k$. Since $k \neq 0$ and $j + k \neq 0$ (because $l \neq 0$), we have $S_1 = 0$ only if $m = -k$, already covered in subcase 5.2.

Part 2: u given by (14). In this case,

$$P = (aj + bk + ck)(b - c)(j - k)\left(2bcj - acj - abj - 2a^2 k + b^2 k + c^2 k\right),$$

which is not, for any (j, k, l, m), of the form aQ where Q is symmetric in a, b, c. $\square$

4. Mappings by symbolic substitutions

To summarize from [2], a symbolic substitution $\mathcal{S}$ maps points to points, lines to lines, conics to conics, cubics to cubics, circumconics to circumconics, and inconics to inconics. Regarding cubics, $\mathcal{S}$ maps each cubic $\mathcal{Z}(U,P)$ to the cubic $\mathcal{Z}(\mathcal{S}(U),\mathcal{S}(P))$ and each cubic $\mathcal{ZP}(U,P)$ to the cubic $\mathcal{ZP}(\mathcal{S}(U),\mathcal{S}(P))$. Symbolic substitutions thus have in common with projections and collineations various incidence properties and degree-preserving properties. On the other hand, symbolic substitutions are fundamentally different from strictly geometric transformations: given an ordinary 2-dimensional triangle ABC and a point $X = x(a,b,c) : y(a,b,c) : z(a,b,c)$ there seems no opportunity to apply geometric methods for describing the image-point

$$\mathcal{S}(X) = x' : y' : z' = x(bc,ca,ab) : y(bc,ca,ab) : z(bc,ca,ab).$$

Algebraically, however, it is clear if X lies on the circumcircle, which is to say that X is on the locus $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$, and if $\mathcal{S}$ is the symbolic substitution in (9), then $\mathcal{S}(X)$ satisfies $bcy'z' + caz'x' + abx'y' = 0$, which is to say that $\mathcal{S}(X)$ lies on the Steiner ellipse, $bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta = 0$.

Table 2. From circumcircle Γ to Steiner ellipse $\mathbb{E}$

X_i on Γ	X_{98}	X_{99}	X_{100}	X_{101}	X_{105}	X_{106}	X_{110}	X_{111}
$\mathcal{S}(X_i)$ on $\mathbb{E}$	X_{3225}	X_{99}	X_{190}	X_{668}	X_{3226}	X_{3227}	X_{670}	X_{3228}

As a final example, note that the point $X_{101} = b-c : c-a : a-b$ is a fixed point of the SISS (10), as verified here:

$$b - c \to b^2 - ca - (c^2 - ab) = (a+b+c)(b-c).$$

Consequently, the line X_1X_6, given by the equation

$$(b-c)\alpha + (c-a)\beta + (a-b)\gamma = 0,$$

is left fixed by the SISS $\mathcal{S}$ in (10), as typified by Table 3.

Table 3. From X_1X_6 to X_1X_6

X_i on X_1X_6	X_1	X_6	X_9	X_{37}	X_{44}	X_{238}
$\mathcal{S}(X_i)$ on X_1X_6	X_1	X_{238}	X_{1757}	X_{518}	X_{44}	X_6

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On the Nagel Line and a Prolific Polar Triangle

Jan Vonk

Abstract. For a given triangle ABC, the polar triangle of the medial triangle with respect to the incircle is shown to have as its vertices the orthocenters of triangles AIB, BIC and AIC. We prove results which relate this polar triangle to the Nagel line and, eventually, to the Feuerbach point.

1. A prolific triangle

In a triangle ABC we construct a triad of circles $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$ that are orthogonal to the incircle Γ of the triangle, with their centers at the midpoints D, E, F of the sides BC, AC, AB. These circles pass through the points of tangency X, Y, Z of the incircle with the respective sides. We denote by ℓ_a (respectively ℓ_b, ℓ_c) the radical axis of Γ and $\mathcal{C}_a$ (respectively $\mathcal{C}_b$, $\mathcal{C}_c$), and examine the triangle $A^*B^*C^*$ bounded by these lines (see Figure 1). J.-P. Ehrmann [1] has shown that this triangle has the same area as triangle ABC.

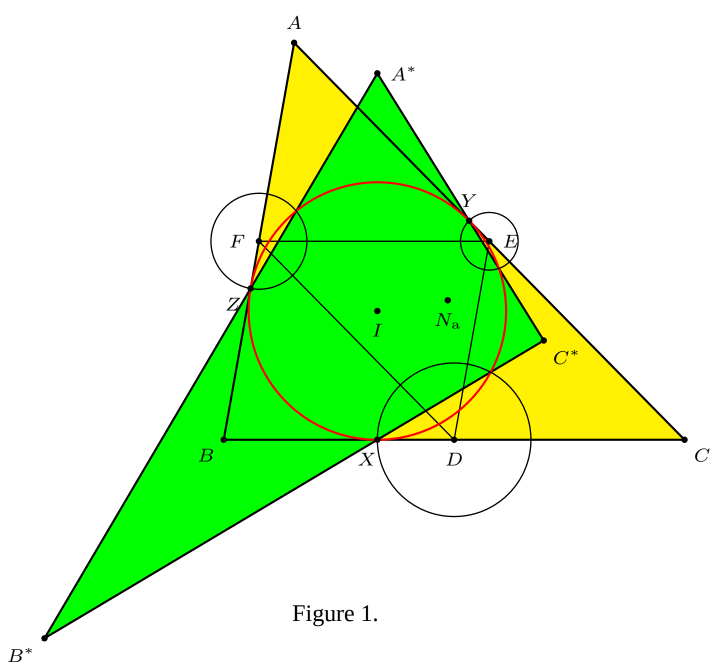

Figure 1.

Lemma 1. *The triangle $A^*B^*C^*$ is the polar triangle of the medial triangle DEF of triangle ABC with respect to Γ.*

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Proof. Because $\mathcal{C}_a$ is orthogonal to Γ, the line ℓ_a is the polar of D with respect to Γ. Similarly, ℓ_b and ℓ_c are the polars of E and F with respect to the same circle. $\square$

Note that Lemma 1 implies that triangles $A^*B^*C^*$ and XYZ are perspective with center I: $A^*I \perp EF$ because EF is the polar line of A^* with respect to Γ. Because $EF \parallel BC$ and $BC \perp XI$, the assertion follows.

Lemma 2. *The lines XY, BI, EF, and AC^* are concurrent at a point of $\mathcal{C}_b$, as are the lines YZ, BI, DE, and AB^** (see Figure 2).

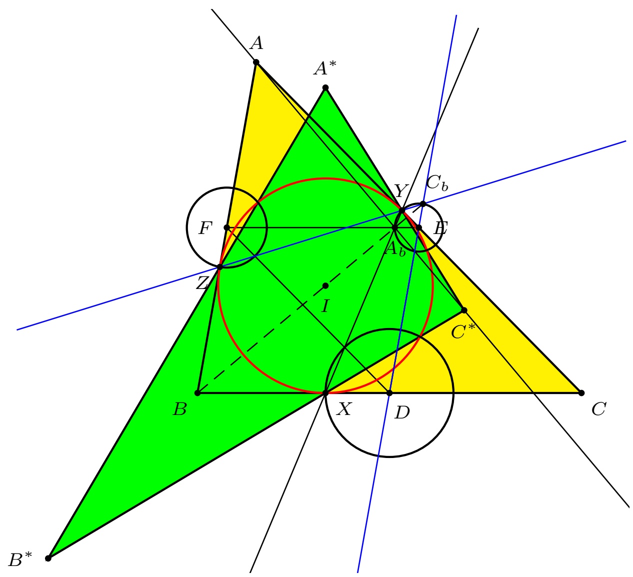

Figure 2.

Proof. Let A_b as the point on EF, on the same side of F as E, so that $FA_b = FA$.

(i) Because $FA = FA_b = FB$, the points A, A_b and B all lie on a circle with center F. This implies that $\angle ABC = \angle AFA_b = 2\angle ABA_b$, yielding $\angle ABI = \angle ABA_b$. This shows that A_b lies on BI.

(ii) Because $YC = \frac{1}{2}(AC + CB - BA) = EC + EF - FA$, we have

$$EY = YC - EC = EF - FA = FE - FA_b = EA_b,$$

showing that A_b lies on $\mathcal{C}_b$. Also, noting that $CX = CY$, we have $\dfrac{EY}{CY} = \dfrac{EA_b}{CX}$. This implies that triangles EYA_b and CYX are isosceles and similar. From this we deduce that A_b lies on XY.

A similar argument shows that DE, BI, YZ are concurrent at a point C_b on the circle $\mathcal{C}_b$. We will use this to prove the last part of this lemma.

(iii) Because YZ and DE are the polar lines of A and C^* with respect to Γ, AC^* is the polar line of C_b, which also lies on BI. This implies that $AC^* \perp BI$, so the intersection of AC^* and BI lies on the circle with diameter AB. We have shown that A_b lies on this circle, and on BI, so A_b also lies on AC^*.

Similarly, C_b also lies on the line AB^*. □

Note that the points A_b and C_b are the orthogonal projections of A and C on BI. Analogous statements can be made of quadruples of lines intersecting on the circles $\mathcal{C}_a$ and $\mathcal{C}_c$. Reference to this configuration can be found, for example, in a problem on the $2002 - 2003$ Hungarian Mathematical Olympiad. A solution and further references can be found in *Crux Mathematicorum with Mathematical Mayhem*, 33 (2007) 415–416.

We are now ready for our first theorem, conjectured in 2002 by D. Grinberg [2].

Theorem 3. *The points A^*, B^*, and C^* are the respective orthocenters of triangles BIC, CIA, and AIB.*

Proof. Because the point A_b lies on the polar lines of A^* and C with respect to Γ, we know that $A^*C \perp BI$. Combining this with the fact that $A^*I \perp BC$ we conclude that A^* is indeed the orthocenter of triangle BIC. □

Theorem 4. *The medial triangle DEF is perspective with triangle $A^*B^*C^*$, at the Mittenpunkt M_{t}* [1] *of triangle ABC* (see Figure 3).

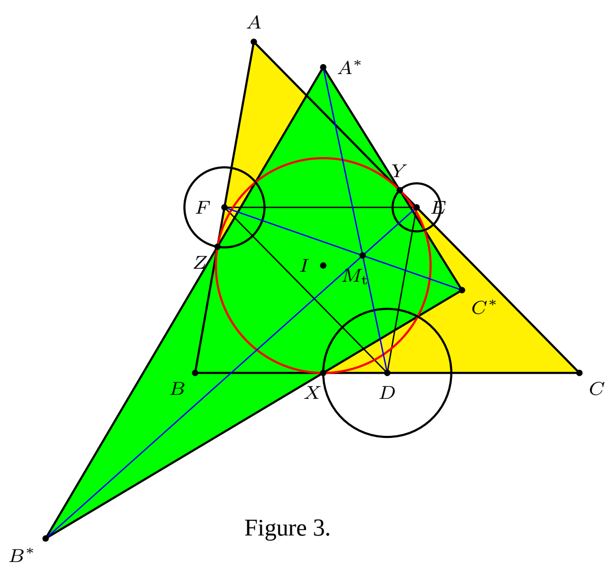

Figure 3.

Proof. Because A^*C is perpendicular to BI, it is parallel to the external bisector of angle B. A similar argument holds for BA^*, so we conclude that A^*BI_aC is a parallelogram. It follows that A^*, D, and I_a are collinear. This shows that M_{t} lies on I_aD, and similar arguments show that M_{t} lies on the lines I_bE and I_cF. □

We already know that triangle $A^*B^*C^*$ and triangle XYZ are perspective at the incenter I. By proving Theorem 4, we have in fact found two additional triangles that are perspective with triangle $A^*B^*C^*$: the medial triangle DEF and the

[1]The Mittenpunkt (called $X(9)$ in [4]) is the point of concurrency of the lines joining D to the excenter I_a, E to the excenter I_b, and C to the excenter I_c. It is also the symmedian point of the excentral triangle $I_aI_bI_c$.

excentral triangle $I_aI_bI_c$, both with center M_{t}. This is however just a taste of the many special properties of triangle $A^*B^*C^*$, which will be treated throughout the rest of this paper.

Theorem 3 shows that B, C, A^*, I are four points that form an orthocentric system. A consequence of this is that I is the orthocenter of triangles A^*BC, AB^*C, ABC^*. In the following theorem we prove a similar result that will produce an unexpected point.

Theorem 5. *The Nagel point N_{a} of triangle ABC is the common orthocenter of triangles AB^*C^*, A^*BC^*, A^*B^*C.*

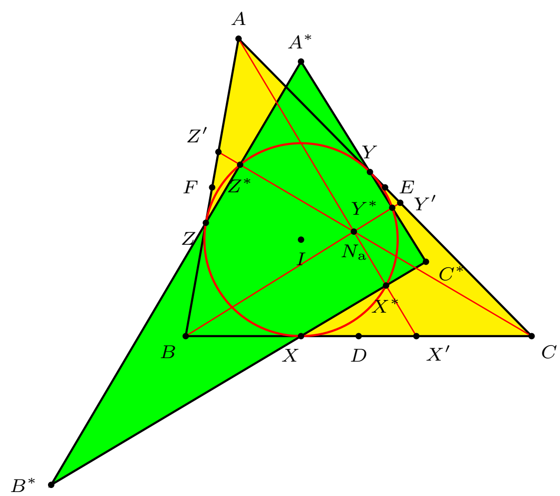

Figure 4.

Proof. Consider the homothety $\zeta := \mathsf{h}(D, -1)$. [2] This carries A into the vertex A' of the anticomplementary triangle $A'B'C'$ of ABC. It follows from Theorem 4 that $\zeta(A^*) = I_a$. This implies that $A'A^*$ is the bisector of $\angle BA'C$.

The Nagel line is the line IG joining the incenter and the centroid. It is so named because it also contains the Nagel point N_{a}. Since $2IG = GN_{\mathrm{a}}$, the Nagel point N_{a} is the incenter of the anticomplementary triangle. This implies that $A'N_{\mathrm{a}}$ is the bisector of $\angle BA'C$. Hence, ζ carries A^*N_{a} into AI, so A^*N_{a} and AI are parallel. From this, $A^*N_{\mathrm{a}} \perp CB^*$.

Similarly, we deduce that $B^*N_{\mathrm{a}} \perp CA^*$, so N_{a} is the orthocenter of triangle A^*B^*C. $\square$

The next theorem was proved by J.-P. Ehrmann in [1] using barycentric coordinates. We present a synthetic proof here.

Theorem 6 (Ehrmann)**.** *The centroid G^* of triangle $A^*B^*C^*$ is the point dividing IH in the ratio $IG^* : G^*H = 2 : 1$.*

[2] A homothety with center P and factor k is denoted by $\mathsf{h}(P, k)$.

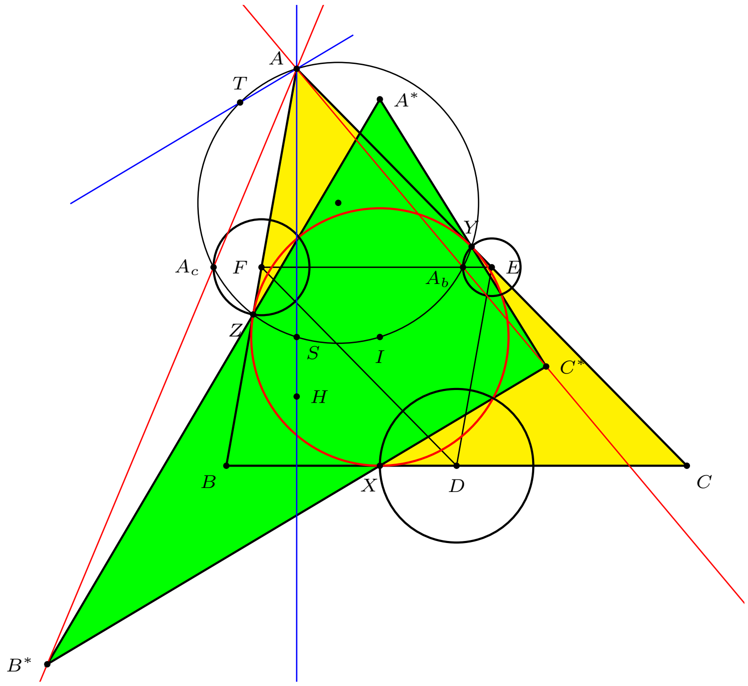

Figure 5.

Proof. The four points A, A_b, I, A_c all lie on a circle with diameter IA, which we will call $\mathcal{C}'_a$. Let H be the orthocenter of triangle ABC, and S the (second) intersection of $\mathcal{C}'_a$ with the altitude AH. Construct also the parallel AT to B^*C^* through A to intersect the circle at T (see Figure 5).

Denote by R_b and R_c the circumradii of triangles AIC and AIB respectively. Because C^* is the orthocenter of triangle AIB, we can write $AC^* = R_c \cdot \cos\frac{A}{2}$, and similarly for AB^*. Using this and the property $B^*C^* \parallel AT$, we have

$$\frac{\sin TAA_b}{\sin TAA_c} = \frac{\sin AC^*B^*}{\sin AB^*C^*} = \frac{AB^*}{AC^*} = \frac{R_b}{R_c} = \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}} = \frac{IC}{IB}.$$

The points A_b, A_c are on EF according to Lemma 2, so triangle IA_bA_c and triangle IBC are similar. This implies $\dfrac{IC}{IB} = \dfrac{IA_c}{IA_b}$.

In any triangle, the orthocenter and circumcenter are known to be each other's isogonal conjugates. Applying this to triangle AA_bA_c, we find that $\angle SAA_b = \angle A_cAI$. We can now see that $\dfrac{SA_b}{SA_c} = \dfrac{IA_c}{IA_b}$.

Combining these results, we obtain

$$\frac{SA_b}{SA_c} = \frac{IA_c}{IA_b} = \frac{IC}{IB} = \frac{\sin TAA_b}{\sin TAA_c} = \frac{TA_b}{TA_c}.$$

This proves that $TA_c \cdot SA_b = SA_c \cdot TA_b$, so TA_cSA_b is a harmonic quadrilateral. It follows that AC^*, AB^* divide AH, AT harmonically. Because $AT \parallel B^*C^*$, we know that AH must pass through the midpoint of B^*C^*.

Let us call D^* the midpoint of B^*C^*, and consider the homothety $\xi = \mathsf{h}(G^*, -2)$. Because ξ takes D^* to B^* while $AH \parallel A^*X$, we know that ξ takes AH to A^*X. Similar arguments applied to B and B^* establish that ξ takes H to I. □

2. Two more triads of circles

Consider again the orthogonal projections A_b, A_c of A on the bisectors BI and CI. It is clear that the circle $\mathcal{C}'_a$ with diameter IA in Theorem 6 contains the points Y and Z as well. Similarly, we consider the circles $\mathcal{C}'_b$ and $\mathcal{C}'_c$ with diameters IB and IC (see Figure 6). It is easy to determine the intersections of the circles from the two triads $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$, and $\mathcal{C}'_a$, $\mathcal{C}'_b$, $\mathcal{C}'_c$, which we summarize in the following table.

Table 1. Intersections of circles

	$\mathcal{C}'_a$	$\mathcal{C}'_b$	$\mathcal{C}'_c$
$\mathcal{C}_a$		$X,\ B_a$	$X,\ C_a$
$\mathcal{C}_b$	$Y,\ A_b$		$Y,\ X_b$
$\mathcal{C}_c$	$Z,\ A_c$	$Z,\ B_c$	

Now we introduce another triad of circles.

Let X^* (respectively Y^*, Z^*) be the intersection of Γ with $\mathcal{C}_a$ (respectively $\mathcal{C}_b$, $\mathcal{C}_c$) different from X (respectively Y, Z). Consider also the orthogonal projections A^*_b and A^*_c of A^* onto B^*N_a and C^*N_a, and similarly defined B^*_a, B^*_c, C^*_a, C^*_b.

Lemma 7. *The six points A^*, A^*_b, A^*_c, Y^*, Z^*, and N_a all lie on the circle with diameter A^*N_a* (see Figure 6).

Proof. The points A^*_b and A^*_c lie on the circle with diameter A^*N_a by definition.

We know that the Nagel point and the Gergonne point are isotomic conjugates, so if we call Y' the intersection of BN_a and AC, it follows that $YE = Y'E$. Therefore, Y' lies on $\mathcal{C}_b$.

Clearly YY' is a diameter of $\mathcal{C}_b$. It follows from Theorem 5 that BN_a is perpendicular to A^*C^*, so their intersection point must lie on $\mathcal{C}_b$. Since Y^* is the intersection point of A^*C^* and $\mathcal{C}_b$ different from Y, it follows that Y^* lies on BN_a.

Combining the above results, we obtain that $N_\mathrm{a}Y^* \perp A^*Y^*$, so Y^* lies on the circle with diameter A^*N_a. A similar proof holds for Z^*. □

We will call this circle $\mathcal{C}^*_a$. Likewise, $\mathcal{C}^*_b$ and $\mathcal{C}^*_c$ are the ones with diameters B^*N_a and C^*N_a. Here are the intersections of the circles in the two triads $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$, and $\mathcal{C}^*_a$, $\mathcal{C}^*_b$, $\mathcal{C}^*_c$.

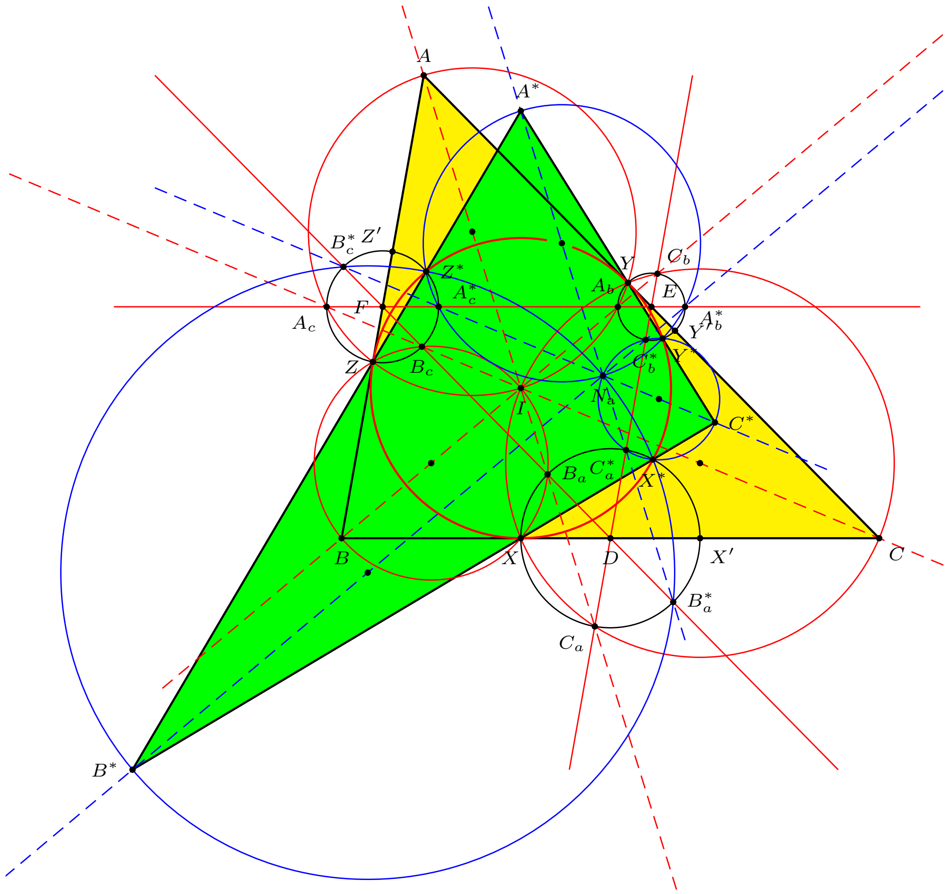

Figure 6.

Table 2. Intersections of circles

	$\mathcal{C}_a^*$	$\mathcal{C}_b^*$	$\mathcal{C}_c^*$
$\mathcal{C}_a$		$X^*,\ B_a^*$	$X^*,\ C_a^*$
$\mathcal{C}_b$	$Y^*,\ A_b^*$		$Y^*,\ X_b^*$
$\mathcal{C}_c$	$Z^*,\ A_c^*$	$Z^*,\ B_c^*$	

Lemma 8. *The circle $\mathcal{C}_a^*$ intersects $\mathcal{C}_b$ in the points Y^* and A_b^*. The point A_b^* lies on EF* (see Figure 7).

Proof. The point Y^* lies on $\mathcal{C}_b$ by definition, and on $\mathcal{C}_a^*$ by Lemma 7.

Consider the homothety $\phi := \mathsf{h}(E, -1)$. We already know that $\phi(AC^*) = CA^*$ and $\phi(BI) = B^*N_\mathrm{a}$. This shows that the intersection points are mapped onto each other, or $\phi(A_b) = A_b^*$. It follows that A_b^* lies on $\mathcal{C}_b$ and EF. $\square$

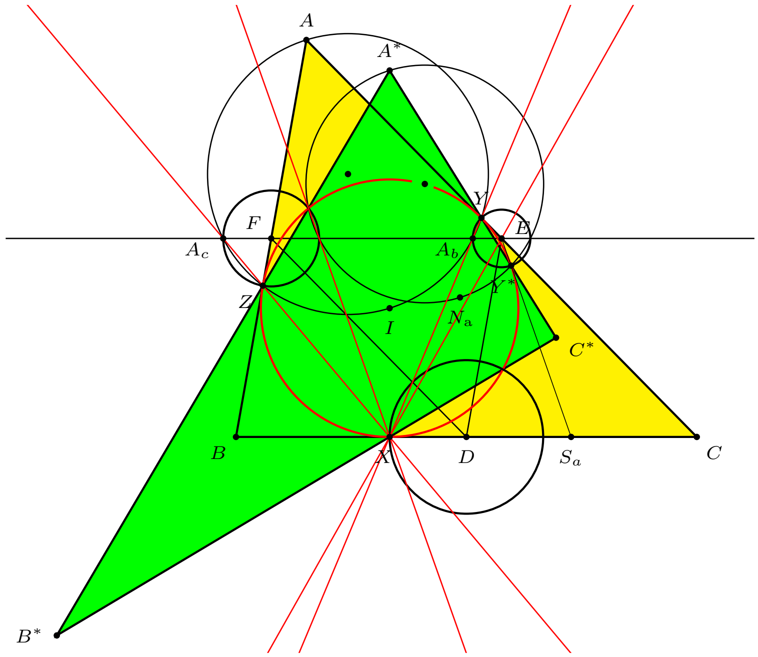

Figure 7.

The two triads of circles have some remarkable properties, strongly related to the Nagel line and eventually to the Feuerbach point. We will start with a property that may be helpful later on.

Theorem 9. *The point X has equal powers with respect to the circles $\mathcal{C}_b$, $\mathcal{C}_c$, $\mathcal{C}_a^*$, and $\mathcal{C}_a'$* (see Figure 7).

Proof. Let us call S_a the intersection of EY^* and BC, and S_b the intersection of XY^* and EF. Because EY^* is tangent to Γ, we have $S_aY^* = S_aX$. Because triangles XS_aY^* and S_bEY^* are similar, it follows that $EY^* = ES_b$. This implies that S_b lies on $\mathcal{C}_b$ so in fact S_b and A_b^* coincide. This shows that X lies on $Y^*A_b^*$. Similar arguments can be used to prove that X lies on $Z^*A_c^*$.

From Table 1, it follows that A_bY (respectively A_cZ) is the radical axis of the circles $\mathcal{C}_a'$ and $\mathcal{C}_b$ (respectively $\mathcal{C}_c$). Lemma 2 implies that X lies on both A_bY and A_cZ, so it is the radical center of $\mathcal{C}_a'$, $\mathcal{C}_b$ and $\mathcal{C}_c$.

From Lemma 8, it follows that $Y^*A_b^*$ (respectively $Z^*A_c^*$) is the radical axis of the circles $\mathcal{C}_b$ and $\mathcal{C}_a^*$ (respectively $\mathcal{C}_c$ and $\mathcal{C}_b^*$). We have just proved that X lies on both $Y^*A_b^*$ and $Z^*A_c^*$, so it is the radical center of $\mathcal{C}_a^*$, $\mathcal{C}_b$, and $\mathcal{C}_c$. The conclusion follows. □

3. Some similitude centers and the Nagel line

Denote by U, V, W the intersections of the Nagel line IG with the lines EF, DF and DE respectively (see Figure 8).

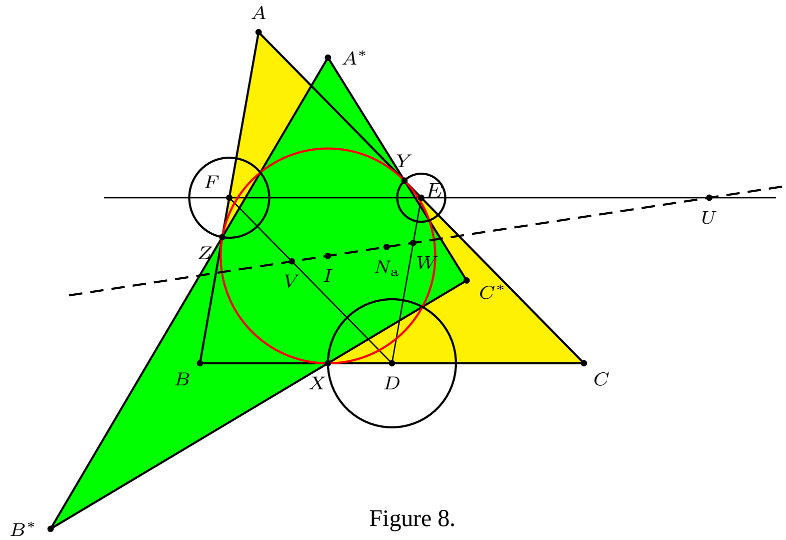

Figure 8.

Theorem 10. *The point U is a center of similitude of circles $\mathcal{C}'_a$ and $\mathcal{C}^*_a$. Likewise, V is a center of similitude of circles $\mathcal{C}'_b$ and $\mathcal{C}^*_b$, and W of $\mathcal{C}'_c$ and $\mathcal{C}^*_c$.*

Proof. We know from Lemma 2 and Theorem 5 that $A^*A^*_b \parallel AA_b$, and $AI \parallel A^*N_\mathrm{a}$, as well as $A^*_bN_\mathrm{a} \parallel A_bI$. Hence triangles triangle $A^*N_\mathrm{a}A^*_b$ and triangle AIA_b have parallel sides. It follows from Desargues' theorem that AA^*, $A_bA^*_b$, IN_a are concurrent. Clearly, the point of concurrency is a center of similitude of both triangles, and therefore also of their circumcircles, $\mathcal{C}^*_a$ and $\mathcal{C}_a$. This point of concurrency is the intersection point of EF and the Nagel line as shown above, so the theorem is proved. □

Theorem 11. *The point U is a center of similitude of circles $\mathcal{C}_b$ and $\mathcal{C}_c$. Likewise, V is a center of similitude of circles $\mathcal{C}_c$ and $\mathcal{C}_a$, and W of $\mathcal{C}_a$ and $\mathcal{C}_b$.*

Proof. By Theorem 10, we know that

$$\frac{A_bU}{A_cU} = \frac{A^*_bU}{A^*_cU}. \tag{1}$$

By Table 1 and Theorem 8, we know that A_b, A^*_c lie on $\mathcal{C}_c$ and A_b, A^*_b lie on $\mathcal{C}_b$. Knowing that U lies on EF, the line connecting the centers of $\mathcal{C}_b$ and $\mathcal{C}_c$, relation (1) now directly expresses that U is a center of similitude of $\mathcal{C}_b$ and $\mathcal{C}_c$. □

Depending on the shape of triangle ABC, the center of similitude of $\mathcal{C}_b$ and $\mathcal{C}_c$ which occurs in the above theorem could be either external or internal. Whichever it is, we will meet the other in the next theorem.

Theorem 12. *The lines BV and CW intersect at a point on EF. This point is the center of similitude different from U of $\mathcal{C}_b$ and $\mathcal{C}_c$* (see Figure 9).

Proof. Let us call U' the point of intersection of BV and EF. We have that $G = BE \cap CF$ and $V = DF \cap BU'$. By the theorem of Pappus-Pascal applied to the collinear triples E, U', F and C, D, B, the intersection of $U'C$ and DE must lie

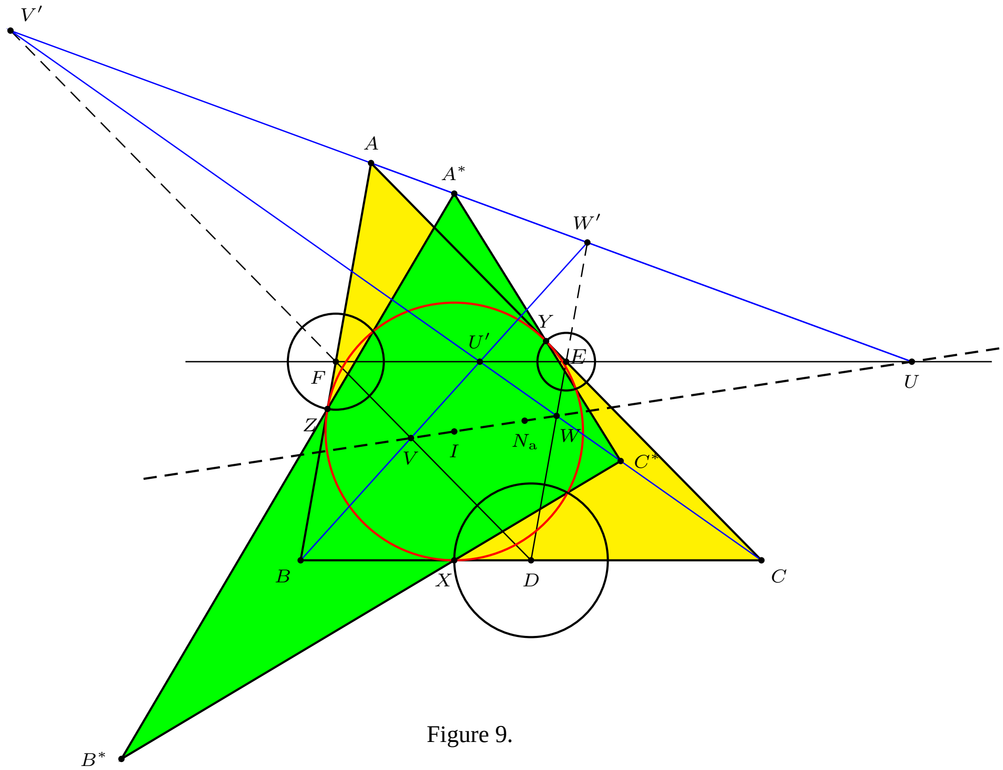

Figure 9.

on GV, and therefore, it must be W. It follows that BV and CW are concurrent in the point U' on EF.

By similarity of triangles, we have $\dfrac{DB}{DV} = \dfrac{FU'}{FV}$ and $\dfrac{DC}{DW} = \dfrac{EU'}{EW}$.

This gives us:

$$\frac{WE}{WD} \cdot \frac{VD}{VF} \cdot \frac{U'F}{U'E} = \frac{EU'}{DC} \cdot \frac{DB}{FU'} \cdot \frac{U'F}{U'E} = \frac{DB}{DC} = -1.$$

Hence DU', EV, FW are concurrent by Ceva's theorem applied to triangle DEF. By Menelaus's theorem applied to the transversal WVU we obtain that U' is the harmonic conjugate of U with respect to E and F. Therefore, it is a center of similitude of $\mathcal{C}_b$ and $\mathcal{C}_c$. □

Let us call X'', Y'', Z'' the antipodes of X, Y, Z respectively on the incircle Γ.

Theorem 13. *The point X'' is the center of similitude different from U of circles $\mathcal{C}'_a$ and $\mathcal{C}^*_a$. Likewise, Y'' is a center of similitude of $\mathcal{C}'_b$ and $\mathcal{C}^*_b$, and Z'' one of $\mathcal{C}'_c$ and $\mathcal{C}^*_c$.*

Proof. We construct the line $l_{X''}$ which passes through X'' and is parallel to BC. The triangle bounded by $AC, AB, l_{X''}$ has Γ as its excircle opposite A. This implies that its Nagel point lies on AX'', and because it is homothetic to triangle ABC from center A, we have that X'' lies on AN_{a}. We have also proved that A^*,

I, X are collinear, so it follows that X'' lies on A^*I. Hence the intersection point of AN_{a} and A^*I is X'', a center of similitude of $\mathcal{C}_a$ and $\mathcal{C}_a^*$, different from U. □

Having classified all similitude centers of the pairs of circles $\mathcal{C}_a', \mathcal{C}_a^*$ and $\mathcal{C}_b, \mathcal{C}_c$ (and we obtain similar results for the other pairs of circles), we now establish a surprising concurrency. Not only does this involve hitherto inconspicuous points introduced at the beginning of §2, it also strongly relates the triangle $A^*B^*C^*$ to the Nagel line of ABC.

Theorem 14. *The triangles $A^*B^*C^*$ and $X^*Y^*Z^*$ are perspective at a point on the Nagel line* (see Figure 10).

Proof. Considering the powers of A^*, B^*, C^* with respect to the incircle Γ of triangle ABC, we have

$$A^*Z{\cdot}A^*Z^* = A^*Y^*{\cdot}A^*Y, \quad B^*X^*{\cdot}B^*X = B^*Z^*{\cdot}B^*Z, \quad C^*X{\cdot}C^*X^* = C^*Y{\cdot}C^*Y^*.$$

From these,

$$\begin{aligned}\frac{B^*X^*}{X^*C^*}\cdot\frac{C^*Y^*}{Y^*A^*}\cdot\frac{A^*Z^*}{Z^*B^*} &= \frac{B^*X^*}{Z^*B^*}\cdot\frac{C^*Y^*}{X^*C^*}\cdot\frac{A^*Z^*}{Y^*A^*}\\ &= \frac{B^*Z}{XB^*}\cdot\frac{C^*X}{YC^*}\cdot\frac{A^*Y}{ZA^*} = \frac{B^*Z}{ZA^*}\cdot\frac{C^*X}{XB^*}\cdot\frac{A^*Y}{YC^*} = 1\end{aligned}$$

since $A^*B^*C^*$ and XYZ are perspective. By Ceva's theorem, we conclude that $A^*B^*C^*$ and $X^*Y^*Z^*$ are perspective, *i.e.*, A^*X^*, B^*Y^*, C^*Z^* intersect at a point Q.

To prove that Q lies on the Nagel line, however, we have to go a considerable step further. First, note that $A_b^*Y^*ZA_c$ is a cyclic quadrilateral, because $XA_b^*\cdot XY^* = XA_c\cdot XZ$ using Theorem 9. We call N_c the point where DE meets ZY^* and working with directed angles we deduce that

$$\measuredangle ZY^*A_b^* = \measuredangle ZA_cU = \measuredangle N_cA_bU = \measuredangle N_cA_bA_b^* = \measuredangle N_cY^*A_b^*$$

We conclude that N_c, Y^*, Z and therefore also Z, Y^*, U are collinear. Similar proofs show that

$$U\in YZ^*,\ V\in XZ^*,\ V\in ZX^*, W\in XY^*,\ W\in YX^*.$$

If we construct the intersection points

$$J = FZ^*\cap BC \quad \text{and} \quad K = DX^*\cap AB,$$

we know that the pole of JK with respect to Γ is the intersection of XZ^* with X^*Z, which is V. The fact that JK is the polar line of V shows that B^* lies on JK, and that JK is perpendicular to the Nagel line.

Now we construct the points

$$O = EF\cap DX^*, \quad P = DE\cap FZ^*, \quad R = OD\cap FZ^*.$$

Recalling Lemma 1 and the definitions of X^* and Z^* following Lemma 3, we see that OP is the polar line of Q with respect to Γ. We also know by similarity of the triangles ORF and DRJ that $OR\cdot RJ = DR\cdot RF$. Likewise, we find by similarity of the triangles KFR and DPR that $RF\cdot DR = KR\cdot RP$. Combining these identities we get $OR\cdot RJ = KR\cdot RP$, and this proves that OP and JK are

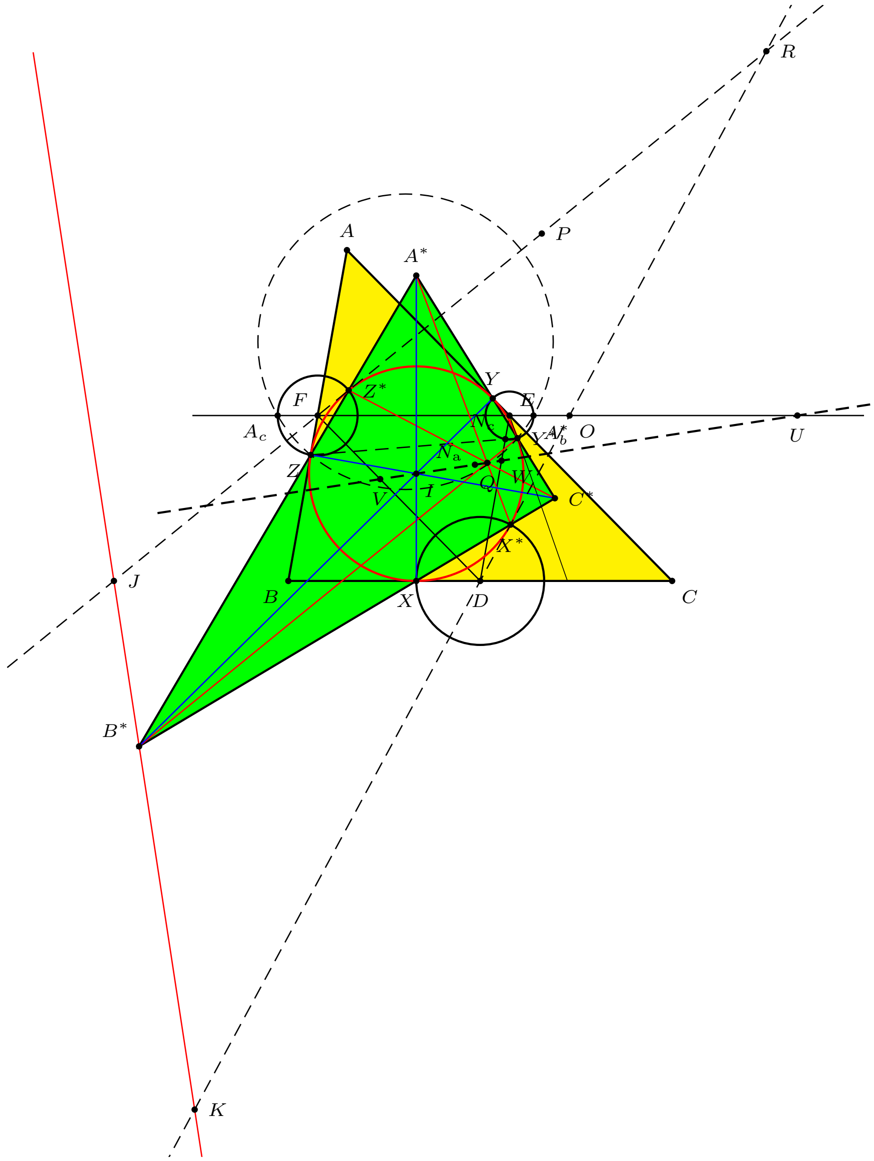

Figure 10.

parallel. Thus, OP is perpendicular to the Nagel line, whence its pole Q lies on the Nagel line. $\square$

4. The Feuerbach point

Theorem 15. *The line connecting the centers of* $\mathcal{C}'_a$ *and* $\mathcal{C}^*_a$ *passes through the Feuerbach point of triangle* ABC*; so do the lines joining the centers of* $\mathcal{C}'_b$, $\mathcal{C}^*_b$ *and those of* $\mathcal{C}'_c$, $\mathcal{C}^*_c$ (see Figure 11).

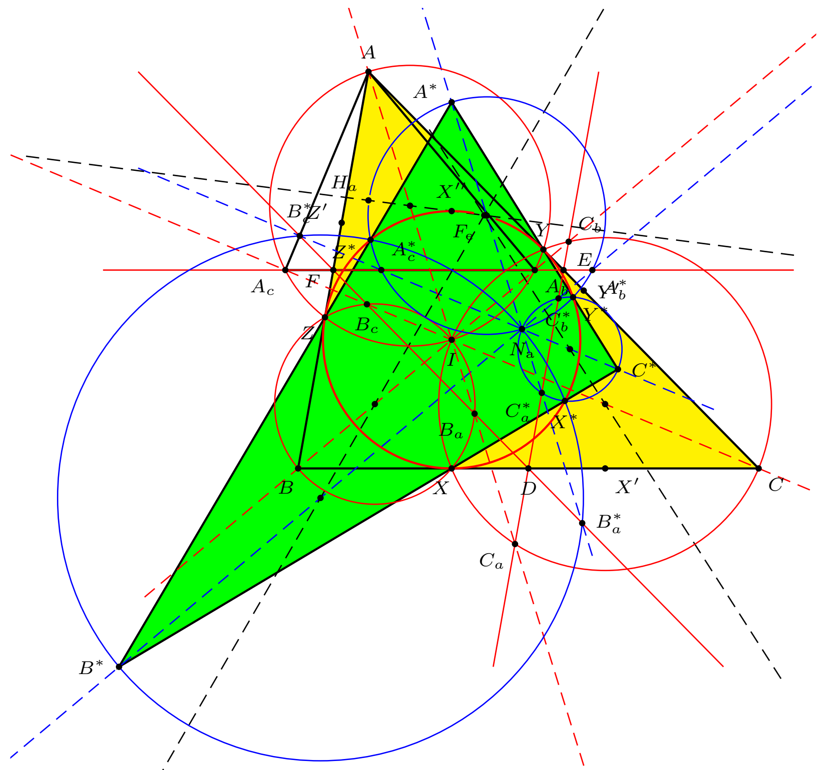

Figure 11.

Proof. Let us call H_a the orthocenter of triangle AA_bA_c. Since AI is the diameter of $\mathcal{C}'_a$ (as in the proof of Theorem 6), we have $AH_a = AI \cdot \cos A_bAA_c = AI \cdot \sin \frac{A}{2}$, where the last equality follows from $\frac{\pi}{2} - \frac{A}{2} = \angle BIC = \angle A_bIA_c = \pi - \angle A_bAA_c$. By observing triangle AIZ, for instance, and writing r for the inradius of triangle ABC we find that

$$AH_a = AI \cdot \sin \frac{A}{2} = r.$$

Now consider the homothety χ with factor -1 centered at the midpoint of AI (which is also the center of $\mathcal{C}'_a$). We have that $\chi(A) = I$ and $\chi(AH_a) = A^*I$. But we just proved that $AH_a = r = IX''$, so it follows that $\chi(H_a) = X''$. This shows that X'' lies on the Euler line of triangle AA_bA_c, so the line joining the centers of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ is exactly the Euler line of triangle AA_aA_b.

According to A. Hatzipolakis ([3]; see also [5]), the Euler line of triangle AA_bA_c passes through the Feuerbach point of triangle ABC. From this our conclusion follows immediately. □

In summary, the Euler line of triangle AA_bA_c and the Nagel line of triangle ABC intersect on EF. We will show that the circles $\mathcal{C}_a$, $\mathcal{C}^*_a$ have another amazing connection to the Feuerbach point.

Theorem 16. *The radical axis of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ passes through the Feuerbach point of triangle ABC; so do the radical axes of $\mathcal{C}'_b$, $\mathcal{C}^*_b$, and of $\mathcal{C}'_c$, $\mathcal{C}^*_c$* (see Figure 12).

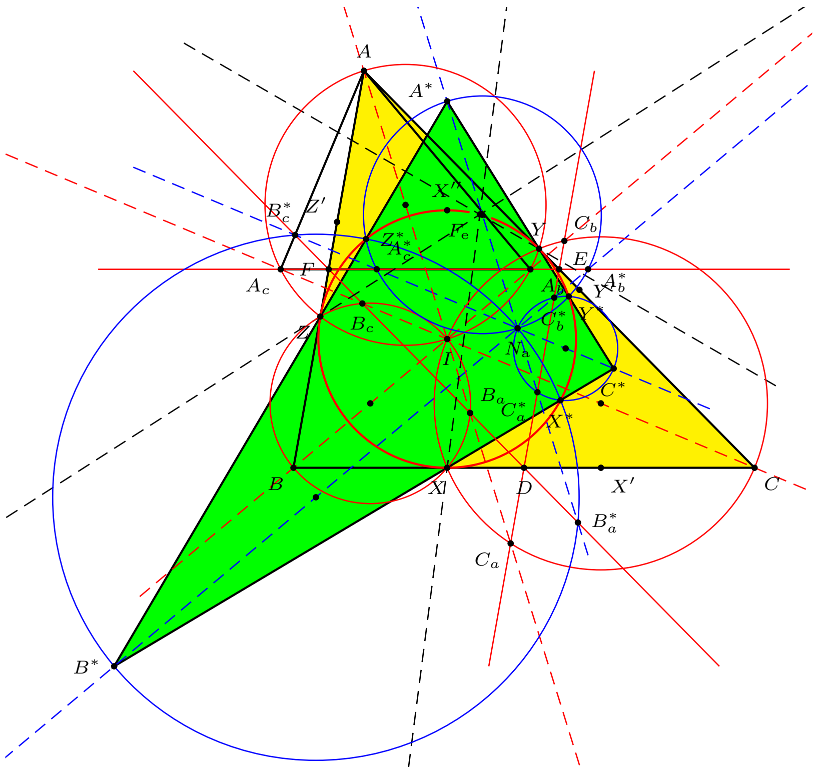

Figure 12.

Proof. Because the radical axis of two circles is perpendicular to the line joining the centers of the circles, the radical axis $\mathcal{R}_a$ of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ is perpendicular to the Euler line of triangle AA_bA_c. Since this Euler line contains X'', and $\mathcal{R}_a$ contains X (see Theorem 9), their intersection lies on Γ. This point is also the intersection point of the Euler line with Γ, different from X''. It is the Feuerbach point of ABC. □

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A Purely Geometric Proof of the Uniqueness of a Triangle With Prescribed Angle Bisectors

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Abstract. We give a purely geometric proof of triangle congruence on three angle bisectors without using trigonometry, analysis and the formulas for triangle angle bisector length.

It is known that three given positive numbers determine a unique triangle with the angle bisectors lengths equal to these numbers [1]. Therefore two triangles are congruent on three angle bisectors. In this note we give a pure geometric proof of this fact. We emphasize that the proof does not use trigonometry, analysis and the formulas for triangle angle bisector length, but only synthetic reasoning.

Lemma 1. *Suppose triangles ABC and $AB'C'$ have a common angle at A, and that the incircle of $AB'C'$ is not greater than the incircle of ABC. If $C' > C$, then the bisector of C' is less than the bisector of C.*

Proof. Let CF and $C'F'$ be the bisectors of angles C, C' of triangles ABC, $AB'C'$. Assuming $C' > C$, we shall prove that $C'F' < CF$.

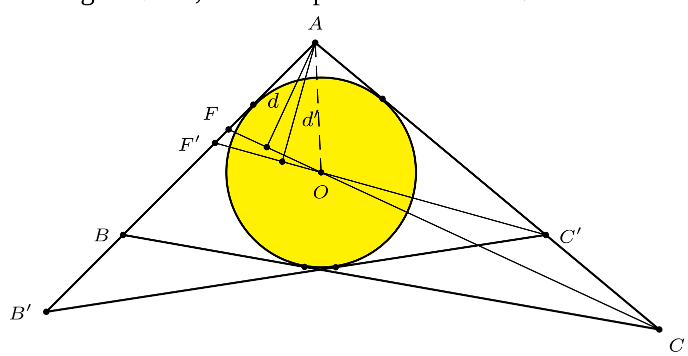

Figure 1.

Case 1. The triangles have equal incircles (see Figure 1). Without loss of generality assume $B > B'$ and the point C' between A and C. Let O be the center of the common incircle of the triangles. It is known that $OF < OC$ and $OF' < OC'$. Hence, in areas,

$$\triangle OFF' < \triangle OCC'. \tag{1}$$

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Let d, d' be the distances of A from the bisectors CF, $C'F'$ respectively. Since $\angle AOF' = \angle OAC' + \angle AC'O = \frac{A+C'}{2} < 90°$, we have $\angle AOF < \angle AOF' < 90°$, and $d < d'$. Now, from (1), we have

$$\triangle OFF' + \triangle OC'AF < \triangle OCC' + OC'AF.$$

This gives $\triangle AF'C' < \triangle AFC$, or $\frac{1}{2}d' \cdot C'F' < \frac{1}{2}d \cdot CF$. Since $d < d'$, we have $C'F' < CF$.

Case 2. The incircle of $AB'C'$ is smaller than the incircle of ABC (see Figure 2). Since the incircle of $AB'C'$ is inside triangle ABC, we construct a tangent $B''C''$ parallel to BC that is closer to A than BC. Let $C''F''$ be the bisector of triangle $AB''C''$. We have $C''F''||CF$ and

$$C''F'' < CF. \tag{2}$$

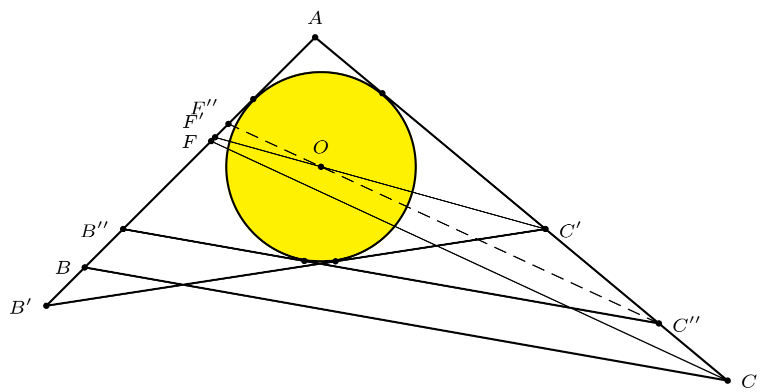

Figure 2.

Since $\angle AC''B'' = \angle ACB < \angle AC'B'$, from Case 1 we have

$$C'F' < C''F'' \tag{3}$$

From (2) and (3) we have $C'F' < CF$. $\square$

Lemma 2. *Suppose triangles ABC and $AB'C'$ have a common angle at A, and a common angle bisector AD, the common angle not greater than any other angle of $AB'C'$. If $C' > C$, then the bisector of C' is less than the bisector of C.*

Proof. If the incirle of triangle $AB'C'$ is not greater than that of ABC, then the result follows from Lemma 1.

Assume the incircle of $AB'C'$ greater than the incircle of ABC (see Figure 3). The line BC cuts the incircle of $AB'C'$ incircle. Hence, the tangent from C to this incircle meets AB' at a point B'' between B and B'. Let CF, $C'F'$ be the bisectors of angles C, C' in triangles ABC and $AB'C'$ respectively. We shall prove that $C'F' < CF$.

Consider also the bisector CF'' in triangle $AB''C$. Since B is between A and B'', F is between A and F''. From lemma 1 we have

$$C'F' < CF'' \tag{4}$$

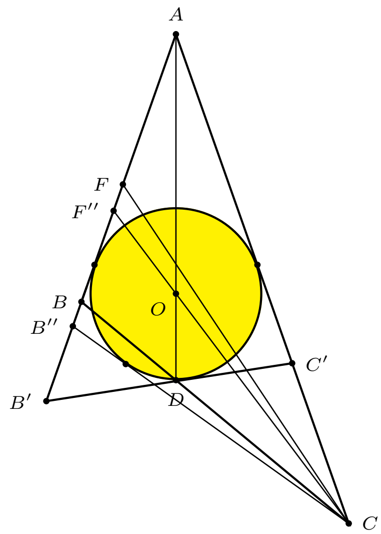

Figure 3.

Since $\angle CB''A > \angle C'B'A \geq \angle B'AC'$, we have $\angle CF''A > 90°$, and from triangle CFF''

$$CF'' < CF. \tag{5}$$

From (4) and (5) we conclude that $C'F' < CF$. $\square$

Now we prove the main theorem of this note.

Theorem 3. *If three internal angle bisectors of triangle ABC are respectively equal to three internal angle bisectors of triangle $A'B'C'$, then the triangles are congruent.*

Proof. Denote the angle bisectors of ABC by AD, BE, CF and let $AD = A'D'$, $BE = B'E'$, $CF = C'F'$.

If for the angles of the triangles we have $A = A'$, $B = B'$, $C = C'$, then from the similarity of ABC with $A'B'C'$ and of ABD with $A'B'D'$ we conclude the congruence of ABC with $A'B'C'$.

Let A' be an angle that is not greater than any other angle of triangles $A'B'C'$ and ABC. We construct a triangle AB_1C_1 congruent to $A'B'C'$ that has AD as bisector of angle B_1AC_1.

If $A' = A$ and $C' > C$, then the triangles ABC and AB_1C_1 satisfy the conditions of Lemma 2. It follows that $C'F' < CF$, a contradiction.

If $A' < A$ and the lines AB_1, AC_1 meet BC at the points B_2, C_2 respectively, without loss of generality we assume C_1 between A and C_2, possibly coinciding with C_2 (see Figure 4). Suppose the bisector of angle AC_2B_2 meets AB_2 at F_2 and AB at F_3. Since triangles AB_1C_1 and AB_2C_2 satisfy the conditions of Lemma 2, we have

$$C'F' \leq C_2F_2 < C_2F_3. \tag{6}$$

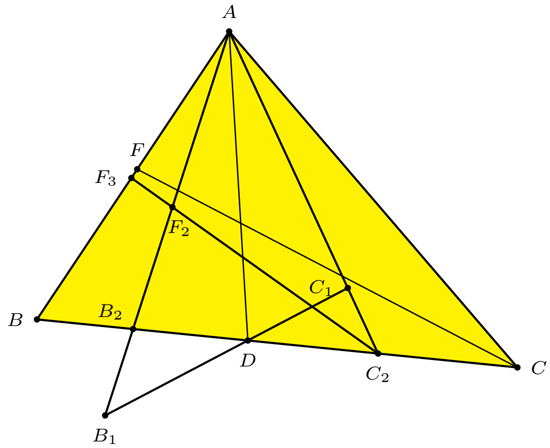

Figure 4.

The incircle of triangle ABC_2 is smaller than that of triangle ABC. Since $\angle AC_2B > \angle ACB$, by Lemma 1, $C_2F_3 < CF$ and from (6) we conclude $C'F' < CF$. This again is a contradiction. Hence, triangles ABC and $A'B'C'$ are congruent. $\square$

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An Elementary Proof of a Theorem by Emelyanov

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Abstract. In this note, we provide an alternative proof of a theorem by Lev Emelyanov stating that the Miquel point of any complete quadrilateral (in general position) lies on the nine-point circle of the triangle formed by the diagonals of that same complete quadrilateral.

1. Introduction and terminology

In their recent book on the geometry of conics, Akopyan and Zaslavsky prove a curious theorem by Lev Emelyanov on complete quadrilaterals. Their proof is very concise, but it does rely on the theory of conic sections, as presumably does Emelyanov's original proof. Indeed, it is the authors' contention that the theorem does not seem to allow for a "short and simple" proof without using the so-called inscribed parabola of the complete quadrilateral.[1] In this note, we will show that actually it is possible to avoid the use of conic sections and to give a proof that uses elementary means only. It is left to the reader to decide whether our proof is reasonably short and simple.

Recall that a *complete* quadrilateral is usually defined as the configuration of four given lines, no three of which are concurrent, and the six points at which they intersect each other. For this paper, we will also assume that no two of the lines are parallel. Without loss of generality, we can think of a complete quadrilateral as the configuration associated with a quadrilateral $ABCD$ in the traditional sense with no two sides parallel and no two vertices coinciding, together with the points $F = AD \cap BC$ and $G = AB \cap CD$. By abuse of notation, we will refer to a generic complete quadrilateral as a *complete* quadrilateral $\square ABCD$, where we will assume that none of the sides of $ABCD$ are parallel and no three are concurrent. [2] The lines AC, BD and FG are known as the *diagonals* of $\square ABCD$. Let $AC \cap BD$ be denoted by E_{FG} and so on. Then, the triangle $\triangle E_{AC}E_{BD}E_{FG}$ formed by the diagonals of $\square ABCD$ is usually referred to as the *diagonal triangle* of $\square ABCD$ (see Figure 2). With these notations, we are now ready to prove Emelyanov's Theorem.

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[1] See [1, pp.110–111] for both the proof (which relies on two propositions proved earlier) and the authors' contention.

[2] Thus, for any quadrilateral $ABCD$ with F and G as above, $\square ABCD$, $\square AFCG$, and $\square BGDF$ and so on, all denote the same configuration.

2. Emelyanov's Theorem

We will prove Emelyanov's Theorem as a corollary to a slightly more general result. For this we first need the following lemma (see Figure 1).

Lemma 1. *For any complete quadrilateral* $\square ABCD$ (as defined above)*, let* F_{BC} *be the unique point on* AD *such that* $F_{BC}E_{FG}$ *is parallel to* BC *and let* F_{DA}*,* G_{AB} *and* G_{CD} *be defined similarly. Finally, let* F_G *and* G_F *be the midpoints of* FE_{FG} *and* GE_{FG}*, respectively. Then* F_{BC}*,* F_{DA}*,* G_{AB}*,* G_{CD} *all four lie on the line* F_GG_F*.*

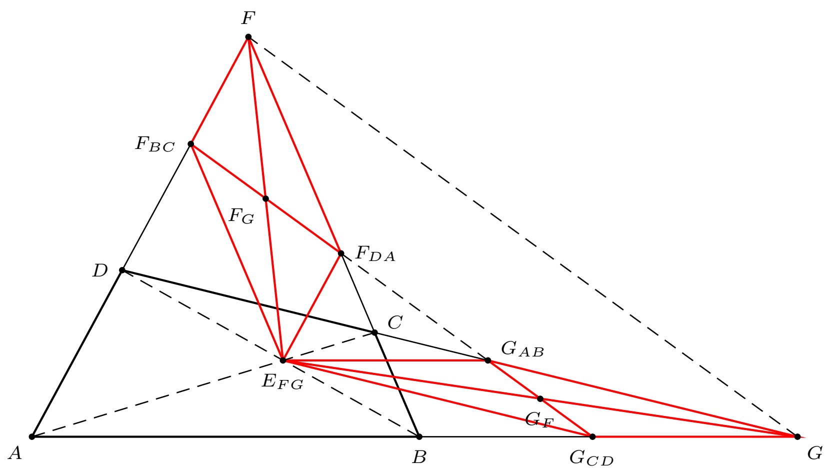

Figure 1. Collinearity of F_{BC}, F_{DA}, G_{AB}, G_{CD} and of F_G, G_F

Proof. Note that by the harmonic property of quadrilaterals, the sides DA and BC are harmonically separated by FE_{FG} and FG. Therefore, the points F_{DA} and F_{BC} are harmonically separated by the points of intersection $FE_{FG} \cap F_{DA}F_{BC}$ and $FG \cap F_{DA}F_{BC}$. By the construction of F_{DA} and F_{BC}, $E_{FG}F_{DA}FF_{BC}$ is a parallelogram and therefore $FE_{FG} \cap F_{DA}F_{BC}$ coincides with F_G. As F_G is also the midpoint of $F_{DA}F_{BC}$, it follows that $FG \cap F_{DA}F_{BC}$ has to be the point at infinity of $F_{DA}F_{BC}$. In other words, FG and $F_{DA}F_{BC}$ are parallel. As F_GG_F is parallel to FG as well and F_G also lies on $F_{DA}F_{BC}$, it follows that $F_{DA}F_{BC}$ and F_GG_F coincide. By the same argument, $G_{AB}G_{CD}$ coincides with F_GG_F as well. It follows that the six points are collinear. $\square$

Corollary 2. *With the notation introduced above, the directed ratios* $\dfrac{F_{BC}D}{F_{BC}A}$ *and* $\dfrac{F_{DA}C}{F_{DA}B}$ *are equal, as are the ratios* $\dfrac{G_{CD}A}{F_{CD}B}$ *and* $\dfrac{G_{AB}D}{F_{AB}C}$.

Proof. It suffices to prove the first part of the statement. Note that by construction the ratio $\dfrac{F_{BC}D}{F_{BC}A}$ is equal to the cross ratio $[E_{FG}D, E_{FG}A; E_{FG}F_{BC}, E_{FG}F_{DA}]$ of the lines $E_{FG}D$, $E_{FG}A$, $E_{FG}F_{BC}$, and $E_{FG}F_{DA}$. Similarly, the ratio $\dfrac{F_{DA}C}{F_{DA}B}$

equals the cross ratio $[E_{FG}C, E_{FG}B; E_{FG}F_{DA}, E_{FG}F_{BC}]$. As ED is parallel to EB, while EA is parallel to EC, the two cross ratios are equal. Therefore, the two ratios are equal as well. $\square$

We are now ready to derive our main result. We start with a lemma about Miquel points, which we prefer to associate to a complete quadrilateral $\square ABCD$, rather than to $ABCD$.

Lemma 3. *For any quadrilateral $ABCD$* (with its sides in general position), *the Miquel points of $\square ABF_{DA}F_{BC}$ and $\square CDF_{BC}F_{DA}$ both coincide with the Miquel point M of $\square ABCD$.*

Proof. Let M be constructed as the second point of intersection (other than F) of the circumcircles of $\triangle FAB$ and $\triangle FCD$. By Corollary 2, the ratio of the power of F_{BC} with respect to the circumcircle of $\triangle FCD$ and the power of F_{BC} with respect to the circumcircle of $\triangle FAB$ equals the ratio of the power of F_{DA} with respect to the same two circles. This means that F_{BC} and F_{DA} lie on the same circle of the coaxal system generated by the circumcircles of $\triangle FCD$ and $\triangle FAB$. In other words, F, F_{BC}, F_{DA} and M are co-cyclic. Since M lies on both the circumcircle of $\triangle F_{BC}F_{DA}F$ and the circumcircle of $\triangle FAB$, it follows that M is also the Miquel point of $\square ABF_{DA}F_{BC}$. By a similar argument, M is the Miquel point of $\square CDF_{BC}F_{DA}$ as well. $\square$

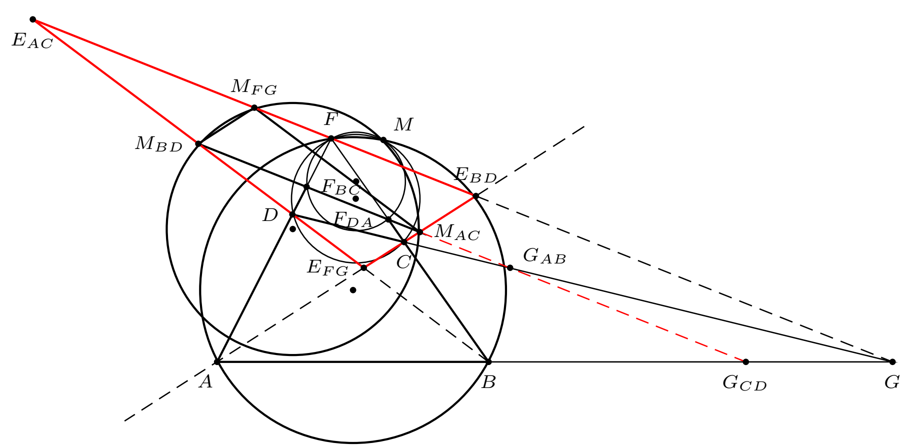

Figure 2. Coincidence of Miquel points

Corollary 4. *For any quadrilateral $ABCD$* (with sides in general position), *the (orthogonal) projection of M on $F_{BC}F_{DA}$ lies on the pedal line of $\square ABCD$.*

Proof. By Lemma 3 and the properties of Miquel points, the (orthogonal) projection of M on $F_{BC}F_{DA}$ is collinear with the (orthogonal) projections of M on AB, BF_{DA} and $F_{BC}A$, i.e. its projections on AB, BC, and DA. But for $ABCD$ in general position, the latter points do not all three coincide. As they also lie on the pedal line of $\square ABCD$, they therefore define the pedal line and the (orthogonal) projection of M on $F_{BC}F_{DA}$ has to lie on it. $\square$

Now, let M_{AC} be the midpoint of $E_{BD}E_{FG}$ and so on. Clearly, $M_{AC}M_{BD}$ coincides with $F_{BC}F_{DA}$. Furthermore, Corollary 4 applies to the quadrilaterals $AFCG$ and $BFDG$ as well. Since $\Box AFCG$ and $\Box BFDG$ coincide with $\Box ABCD$, their Miquel points also coincide. These observations immediately lead to our main result.

Theorem 5. *For any quadrilateral $ABCD$* (with sides in general position), *the (orthogonal) projections of the Miquel point M of $\Box ABCD$ on the sides of the triangle $\triangle M_{AC}M_{BD}M_{FG}$ all three lie on the pedal line of $\Box ABCD$.*

Emelyanov's Theorem follows from Theorem 5 as a corollary.

Corollary 6 (Emelyanov)**.** *For any quadrilateral $ABCD$* (with sides in general position), *the Miquel point M of $\Box ABCD$ lies on the nine-point circle of the diagonal triangle $\triangle E_{AC}E_{BD}E_{FG}$ of $\Box ABCD$.*

Proof. Since the (orthogonal) projections of M on the sides of $\triangle M_{AC}M_{BD}M_{FG}$ are collinear, M has to lie on the circumcircle of $\triangle M_{AC}M_{BD}M_{FG}$. But this is the same as saying that M lies on the nine-point circle of $\triangle E_{AC}E_{BD}E_{FG}$. $\Box$

3. Conclusion

In this note we derived an elementary proof of Emelyanov's Theorem as stated in [**?**] from a more general result. At this point, it is unclear to us whether this Theorem 5 may have any other implications than Emelyanov's Theorem, but it was not our goal to look for such implications. Similarly, we could have shortened our proof a little bit by noting that Corollary 2 implies that $F_{BC}F_{DA}$ is a tangent line to the unique inscribed parabola of $\Box ABCD$. The same parabola therefore is also the inscribed parabola to $\Box ABF_{DA}F_{BC}$ and $\Box CDF_{BC}F_{DA}$. Since the focal point of the parabola inscribing a complete quadrilateral is the Miquel point of the same, Lemma 3 immediately follows. As stated in the introduction, however, our goal was to provide a proof of the theorem without using the theory of conic sections.

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A Generalization of Thébault's Theorem on the Concurrency of Three Euler Lines

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Abstract. We prove a generalization of Victor Thébault's theorem that if $H_aH_bH_c$ is the orthic triangle of ABC, then the Euler lines of triangles AH_cH_b, BH_aH_c, and CH_bH_a are concurrent at the center of the Jerabek hyperbola which is the isogonal tranform of the Euler line.

In this note we generalize a theorem of Victor Thébault's as given in [1, Theorem 1]. Given a triangle ABC with orthic triangle $H_aH_bH_c$, the Euler lines of the triangles AH_bH_c, BH_cH_a, and CH_aH_b are concurrent at a point on the nine-point circle, which is the center of the Jerabek hyperbola, the isogonal transform of the Euler line of triangle ABC.

Since triangle AH_cH_b is similar to ABC, it is the reflection in the bisector of angle A of a triangle AB_aC_a, which is a homothetic image of ABC. Let P be a triangle center of triangle ABC. Its counterpart in AH_cH_b is the point P_a constructed as the reflection in the bisector of angle A of the point on AP which is the intersection of the parallels to BP, CP through C_a, B_a respectively (see Figure 1).

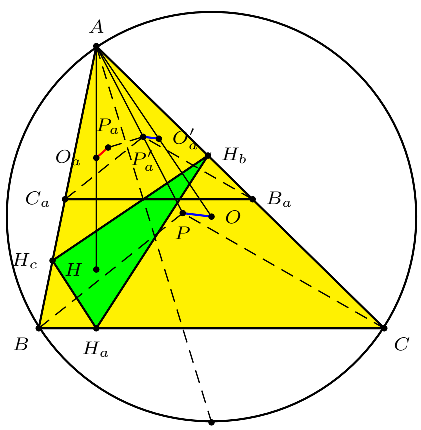

Figure 1.

Note that the circumcenter O_a of triangle AH_cH_b is the midpoint of AH. It is also the reflection (in the bisector of angle A) of the circumcenter O'_a of triangle AB_aC_a. The line O_aP_a is the reflection of $O'_aP'_a$ in the bisector of angle A.

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Here is an alternative description of the line O_aP_a that leads to an interesting result. Consider the line ℓ'_a through A parallel to OP, and its reflection ℓ_a in the bisector of angle A. It is well known that ℓ_a intersects the circumcircle at a point Q' which is the isogonal conjugate of the infinite point of OP. Now, the line O_aP_a is clearly the image of ℓ_a under the homothety $\mathsf{h}(H, \frac{1}{2})$. As such, it contains the midpoint Q of the segment HQ'.

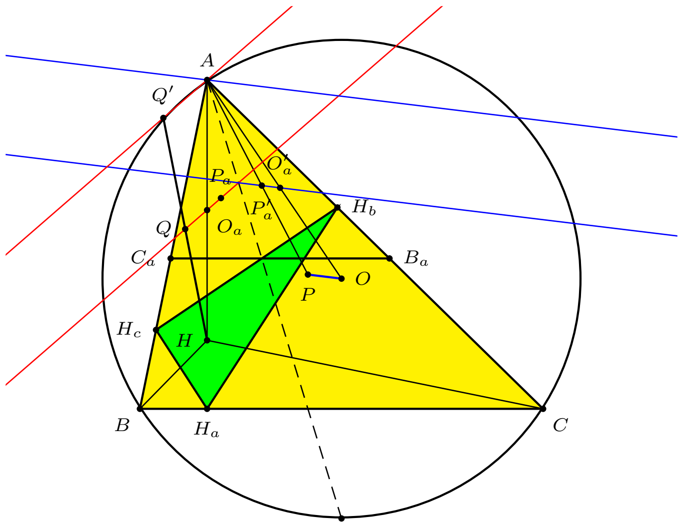

Figure 2.

The above reasoning applies to the lines O_bP_b and O_cP_c as well. The reflections of the parallels to OP through B and C in the respective angle bisectors intersect the circumcircle of ABC at the same point Q', which is the isogonal conjugate of the infinite point of OP (see Figure 3). Therefore, the lines O_bP_b and O_cP_c also contain the same point Q, which is the image of the Q' under the homothety $\mathsf{h}(H, \frac{1}{2})$. As such, it lies on the nine-point circle of triangle BAC. It is well known (see [3]) that Q is the center of the rectangular circum-hyperbola which is the isogonal transform of the line OP.

We summarize this in the following theorem.

Theorem. *Let P be a triangle center of triangle ABC. If P_a, P_b, P_c are the corresponding triangle centers in triangles AH_cH_b, BH_aH_c, CH_bH_a respectively, the lines O_aP_a, O_bP_b, O_cP_c intersect at a point Q on the nine-point circle of ABC, which is the center of the rectangular circumhyperbola which is the isogonal transform of the line OP.*

Thébault's theorem is the case when P is the orthocenter.

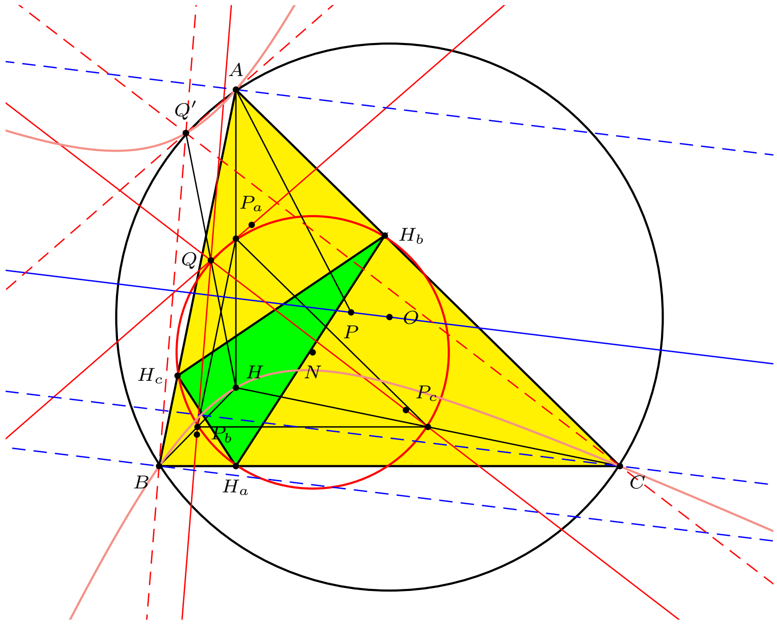

Figure 3.

We conclude with a record of coordinates. Suppose P has homogeneous barycentric coordinates $(u : v : w)$ in reference to triangle ABC. The line O_aP_a, O_bP_b, O_cP_c intersect at the point

$$\begin{aligned} Q = \big(&(b^2-c^2)u + a^2(v-w))(c^2(a^2+b^2-c^2)v - b^2(c^2+a^2-b^2)w) \\ :&\,(c^2-a^2)v + b^2(w-u))(a^2(b^2+c^2-a^2)w - c^2(a^2+b^2-c^2)u) \\ :&\,(a^2-b^2)w + c^2(u-v))(b^2(c^2+a^2-b^2)u - a^2(b^2+c^2-a^2)v)\big) \end{aligned}$$

on the nine-point circle, which is the center of the rectangular hyperbola through A, B, C, H and

$$\begin{aligned} Q' = \Bigg(&\frac{a^2}{((b^2-c^2)^2 - a^2(b^2+c^2))u + a^2(b^2+c^2-a^2)(v+w)} \\ :&\,\frac{b^2}{((c^2-a^2)^2 - b^2(c^2+a^2))v + b^2(c^2+a^2-b^2)(w+u)} \\ :&\,\frac{c^2}{((a^2-b^2)^2 - c^2(a^2+b^2))w + c^2(a^2+b^2-c^2)(u+v)}\Bigg). \end{aligned}$$

on the circumcircle. Here are some examples. The labeling of triangle centers follows [2].

P	Q on nine-point circle	Q' on circumcircle
Orthocenter X_4	Jerabek center X_{125}	X_{74}
Symmedian point X_6	Kiepert center X_{115}	X_{98}
Incenter X_1	Feuerbach point X_{11}	X_{104}
Nagel point X_8	X_{3259}	X_{953}
Spieker center X_{10}	X_{124}	X_{102}
X_{66}	X_{127}	X_{1297}
Steiner point X_{99}	X_{2679}	X_{2698}

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[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

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