Junior problems

J211. Let a, b, c be positive real numbers such that $a^3+b^3+c^3=1$. Prove that

$$\frac{1}{a^5(b^2+c^2)^2}+\frac{1}{b^5(c^2+a^2)^2}+\frac{1}{c^5(a^2+b^2)^2} \geq \frac{81}{4}.$$

Proposed by Titu Zvonaru, Comanesti, Romania

J212. Solve in real numbers the system of equations

$$\begin{aligned}(x-2y)(x-4z)&=6\\(y-2z)(y-4x)&=10\\(z-2x)(z-4y)&=-16.\end{aligned}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J213. For any positive integer n, let $S(n)$ denote the sum of digits in its decimal representation. Prove that the set of all positive integers n such that n is not divisible by 10 and $S(n) > S(n^2+2012)$ is infinite.

Proposed by Preudtanan Sriwongleang, Ramkamhaeng University, Bangkok, Thailand

J214. Let $a, b, c, d, e \in [1,2]$. Prove that

$$ab+bc+cd+de+ea \geq a^2+b^2+c^2+d^2-e^2$$

and find the values for which the equality occurs.

Proposed by Ion Dobrota, Romania and Adrian Zahariuc, Harvard University, USA

J215. Prove that for any prime $p>3$, $\frac{p^6-7}{3}+2p^2$ can be written as sum of two perfect cubes.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J216. Let ω be a circle and let M be a point outside it. Draw the lines l_1, l_2 and l_3 intersecting ω and consider the intersections $l_1 \cap \omega = \{A_1, A_2\}$, $l_2 \cap \omega = \{B_1, B_2\}$ and $l_3 \cap \omega = \{C_1, C_2\}$. Denote $P = A_1B_2 \cap A_2B_1$, $Q = B_1C_2 \cap B_2C_1$ and R is one of the points of intersection between PQ and ω. Prove that MR is tangent at ω.

Proposed by Catalin Turcas, Warwick University, United Kingdom

Senior problems

S211. Let (a, b, c, d, e, f) be a 6-tuple of positive real numbers satisfying simultaneously the equations:

$$\begin{aligned}2a^2-6b^2-7c^2+9d^2&=-1\\9a^2+7b^2+6c^2+2d^2&=e\\9a^2-7b^2-6c^2+2d^2&=f\\2a^2+6b^2+7c^2+9d^2&=ef.\end{aligned}$$

Prove that $a^2-b^2-c^2+d^2=0$ if and only if $7\frac{a}{b}=\frac{c}{d}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S212. Consider a circle $\omega(I, r)$ and let Γ be a point inside it such that $I\Gamma=\ell$. Using only the straightedge and the compass, construct a triangle such that ω is its incircle and Γ its Gergonne point.

//*Note*. The Gergonne point of a triangle is the intersection point of the lines determined by the vertices with the corresponding tangency points of the incircle with the opposite sides of the triangle.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S213. Let a, b, c be positive real numbers such that $a^2 \geq b^2+bc+c^2$. Prove that

$$a>\min(b, c)+\frac{\left|b^2-c^2\right|}{a}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S214. Let $x>y$ be positive rational numbers and R_0 a rectangle of dimensions $x \times y$. By a cut of R_0 we understand a dissection of the rectangle in two pieces: a square of dimensions $y \times y$ and a rectangle R_1 of dimensions $(x-y) \times y$. Similarly, R_2 is obtained from a cut of R_1, and so on. Prove that after finitely many cuts the sequence of rectangles $R_1, R_2, \ldots, R_k$ ends into the square R_k. Find k in terms of x, y and find the dimensions of R_k.

Proposed by Mircea Becheanu, University of Bucharest, Romania

S215. Let ABC be a given triangle and let ρ_A, ρ_B, ρ_C be the lines through the vertices A, B, C and parallel to the Euler line OH, where O and H are the circumcenter and orthocenter of ABC. Let X be the intersection of ρ_A with the sideline BC. The points Y, Z are defined analogously. If I_a, I_b, I_c are the corresponding excenters of triangle ABC, then the lines XI_a, YI_b, ZI_c are concurrent on the circumcircle of triangle $I_aI_bI_c$.

Proposed by Cosmin Pohoata, Princeton University, USA

S216. Let p be a prime number. Prove that for each positive integer n the polynomial $P(X) = (X^p + 1^2)(X^p + 2^2)\dots(X^p + n^2) + 1$ is irreducible in $\mathbb{Z}[X]$.

Proposed by Cezar Lupu, University of Pittsburgh, USA, and and Tudorel Lupu, Decebal High School Constanta, Romania

Undergraduate problems

U211. On the set $M = \mathbb{R} - \{3\}$ the following binary law is defined:

$$x * y = 3\left(xy - 3x - 3y\right) + m,$$

where $m \in \mathbb{R}$.

Find all possible values of m such that $(M, *)$ is a group.

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania

U212. Let G be a finite abelian group such that G contains a subgroup $K \neq \{e\}$ with the property that $K \subset H$ for each subgroup H of G such that $H \neq \{e\}$. Prove that G is a cyclic group.

Proposed by Daniel Lopez Aguayo, Institute of Mathematics, UNAM, Morelia, Mexico

U213. Let $x_0 \in (0, \pi)$ fixed. For $n \in \mathbb{N}$ we set $x_n = \sin x_{n-1}$. Show that

$$x_n = \frac{3^{1/2}}{n^{1/2}} - \frac{3^{3/2}\ln n}{10 n^{3/2}} + \mathcal{O}(n^{-3/2}).$$

Proposed by Anastasios Kotronis, Athens, Greece

U214. Prove that

$$\lim_{n\to+\infty} \prod_{k=1}^{n} \frac{\cosh\left(k^2 + k + \frac{1}{2}\right) + i\sinh\left(k + \frac{1}{2}\right)}{\cosh\left(k^2 + k + \frac{1}{2}\right) - i\sinh\left(k + \frac{1}{2}\right)} = \frac{e^2 - 1 + 2ie}{e^2 + 1}$$

Proposed by Moubinool Omarjee, Lycee Jean Murcat, Paris, France

U215. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_{-1}^{1} x^2 f(x)dx = 0$. Prove that

$$\int_{-1}^{1} f^2(x)dx \geq \frac{9}{8}\left(\int_{-1}^{1} f(x)dx\right)^2.$$

Proposed by Cezar Lupu, University of Pittsburgh, USA, and and Tudorel Lupu, Decebal High School Constanta, Romania

U216. Let N be a positive integer and let Δ_{f_N} be the discriminant of the polynomial $f_N(x) = x^N - x - 1$. Prove that for any prime p dividing Δ_{f_N}, the reduction modulo p of $f_N(x)$ has one double root and $N - 2$ simple roots.

Note. The discriminant Δ_f of a polynomial f of degree n is defined as

$$D_f = a_n^{2n-2} \prod_{i<j} \left(r_i - r_j\right)^2,$$

where a_n is the leading coefficient and $r_1, \ldots, r_n$ are the roots (counting multiplicity) of the polynomial in some splitting field.

Proposed by Cosmin Pohoata, Princeton University, USA

Olympiad problems

O211. Prove that for each positive integer n the number $4^n + 8^n + 16^n + 2(6^n + 9^n + 12^n)$ has at least three prime divisors.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O212. Let $f_0(i) = 1$ for all $i \in \mathbb{N}$ and let $f_k(n) = \sum_{i=1}^{n} f_{k-1}(i)$ for $k \geq 1$. Prove that

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} f_j(n-2j) = F_n,$$

where F_n is the n-th Fibonacci number.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O213. Let n be a positive integer and let z be a complex number such that $z^{2^n-1} - 1 = 0$. Evaluate

$$\prod_{k=1}^{n} \left(z^{2^k} + \frac{1}{z^{2^k}} - 1 \right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O214. The vertices $A_1, A_2, \ldots, A_n$ of a regular polygon lie on a circle $\mathcal{C}$ of center O. Is it true that the map $P \to \sum_{k=1}^{n} \frac{1}{PA_k^4}$, defined on the set of points of the plane outside $\mathcal{C}$, is a rational function in OP?

Proposed by Gabriel Dospinescu, Ecole Polytehnique, France

O215. Prove that there are no positive integers a, b, c, d that are consecutive terms of an arithmetic progression and also satisfy the condition that $ab+1$, $ac+1$, $ad+1$, $bc+1$, $bd+1$, $cd+1$ are all perfect squares.

Proposed by Cosmin Pohoata, Princeton University, USA

O216. Let $f \in \mathbb{Z}[X]$ be a monic polynomial of degree greater than 1. Suppose that $f(X^n)$ is a reducible polynomial in $\mathbb{Z}[X]$ for all $n \geq 2$. Does it follow that f is reducible in $\mathbb{Z}[X]$?

Proposed by Gabriel Dospinescu, Ecole Polytehnique, France