Mathematical Excalibur

Volume 4, Number 1 December, 1997 - February, 1998

Olympiad Corner

International Mathematics Tournament of the Towns, Spring 1997:

Junior A-Level Paper

Problem 1. One side of a triangle is equal to one third of the sum of the other two. Prove that the angle opposite the first side is the smallest angle of the triangle. (3 points)

Problem 2. You are given 25 pieces of cheese of different weights. Is it always possible to cut one of the pieces in two parts and put the 26 pieces in two packets so that

(i) each packet contains 13 pieces;

(ii) the total weights of the two packets are equal;

(iii) the two parts of the piece which has been cut are in different packets?

(5 points)

Problem 3. In a chess tournament, each of $2n$ players plays every other player once in each of two rounds. A win is worth 1 point and a draw is worth ½point. Prove that if for every player, the total score in the first round differs from that in the second round by at least n points, then the difference is exactly n points for every player. (5 points)

(*continued on page 4*)

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On-line: http: //www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is April 15, 1998.

For individual subscription for the three remaining issues for the 97-98 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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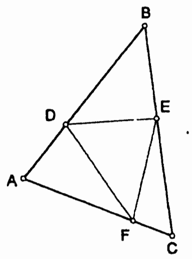
老師不教的幾何（五）

張百康

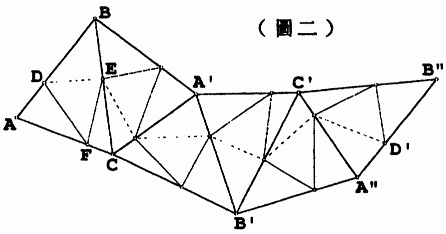
圖一顯示了一個銳角三角形ABC和它的一個內切三角形DEF。



（圖一）

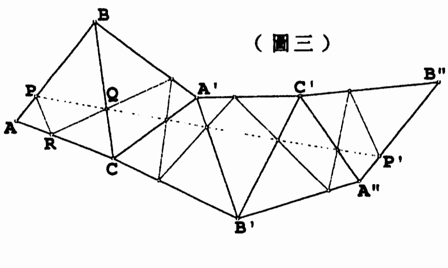
大家想一想：隨意在ΔABC上作內切三角形，哪一個的周界最短？這問題早在十八世紀時，由數學家Fagnano最先提出，並且用微分方法，經過繁複的運算和簡化，求得一個最短周長的內切三角形。因此這問題又名Fagnano問題。

經過整整一個世紀，才有另一位數學家Schwarz找到一個漂亮的初等幾何解法：如圖二所示，Schwarz將ΔABC以它的邊輪流作鏡面反射，得到六個相連的全等三角形。



（圖二）

圖二的虛折線$DE\cdots D'$全長剛好是ΔDEF周長的兩倍。將D、E、F沿ΔABC的三邊移動，是否可以找到另一個內切三角形PQR，使虛折線成一直線PP'（圖三）？



（圖三）

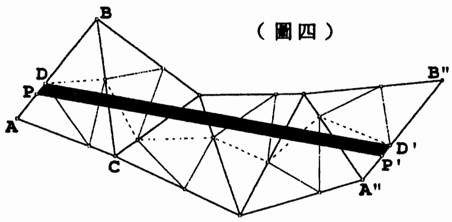
如果我們用一線段連D和D'，這線段肯定較折線$DE\cdots D'$短。但PP'和DD'的長度又如何比較呢？破綻正正在於爲甚麼Schwarz要作不多不少的五次鏡面反射。大家細心看一看圖二的AB和$A''B''$，它們不但等長，而且好像互相平行呢。Schwarz巧妙地用旋轉的觀念來證明AB平行$A''B''$：

AB繞點B旋轉$2\angle B$得$A'B$，再繞點A'旋轉$2\angle A$得$A'B'$；$A'B'$繞點B'旋轉$-2\angle B$得$A''B$，再繞點A''旋轉$-2\angle A$得$A''B''$。因此AB和$A''B''$的夾角是

$$2\angle B+2\angle A-2\angle B-2\angle A=0，$$

也就是說，AB平行$A''B''$。

由於$AD=A''D'$和$AP=A''P'$，因此$PDD'P'$是平行四邊形（圖四）。換言之，ΔPQR的周長$(=\frac{1}{2}PP')$是所有ΔABC的內切三角形中最短的。

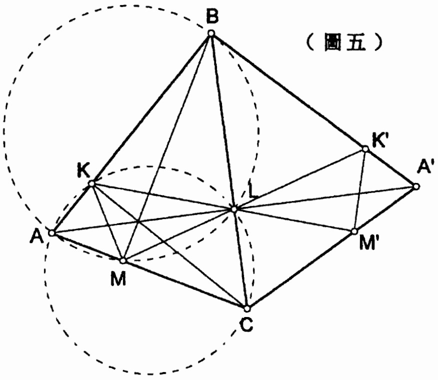


（圖四）

這ΔPQR究竟有甚麼特性？大家不妨再看一遍圖三，不難發現ΔPQR好像是ΔABC的垂足三角形(orthic triangle)。各同學可利用圖五證明圖中的垂足三角形KLM的角$\angle KLM$被高AL平分，關鍵在於圖中的一些四點共圓特性，留待各同學自行理解。

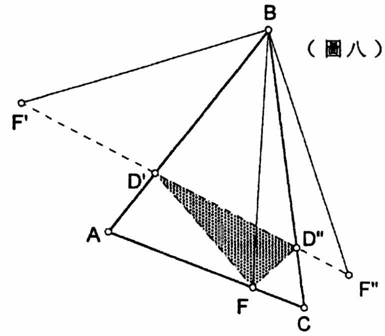
由於$\Delta A'BC$是ΔABC以BC爲鏡面的反射影象，所以高AL和$A'L$成一直線，並且$A'L$也平分角$K'LM$。由此推知，KL和LM也成一直線。餘此類推，如按圖二

連續進行鏡面反射，則ΔKLM就是我們要找尋的最短周長內切三角形。



（圖五）

接著，我們改變F的位置，找尋上述這種$\Delta FD'D''$中周長最短者，便是我們要找的最短周長內切三角形。



（圖八）

利用鏡面反射的對稱性質可知：圖八中的$BF' = BF = BF''$，$\angle F'BD' = \angle FBD'$及$\angle FBD'' = \angle F''BD''$，因此

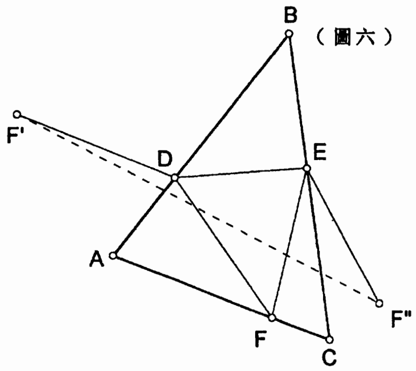
$$F'F'' = 2BF \sin B。$$

其中只有BF可改變，而BF長度的最小值是當它是ΔABC的高，即F是垂足。同理可知另外兩頂點D和E也必定是垂足方可使ΔDEF成爲周長最短的內切三角形。

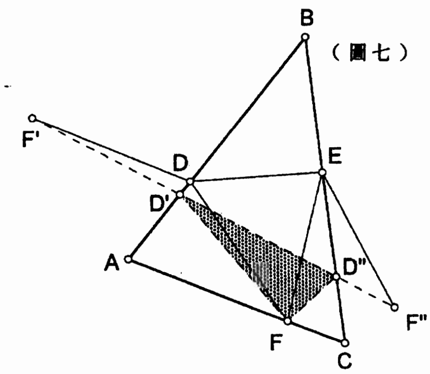
青出於藍勝於藍，Schwarz看過學生Fejér的證明後，也讚賞不已。

Schwarz的證明固然巧妙，但好戲還在後頭。在1900年，當時還在柏林唸書的匈牙利數學家Fejér找到一個比Schwarz的證明還精簡的證法：

分別以ΔABC的邊BA和BC作鏡面，找到F的影像F'和F''（圖六）。由鏡面反射的性質可知：$FD = F'D$及$FE = F''E$，因此折線$F'DEF''$全長等於內切三角形DEF的周長。明顯地，祇要F點不變，不管其餘兩點D和E在AB和BC上如何移動，所得的內切三角形周長肯定大於直線$F'F''$的長度。設$F'F''$與AB及BC分別交於點D'和D''（圖七），則$\Delta FD'D''$的周長是所有一頂點在F的內切三角形中最短的。



（圖六）



（圖七）

A Proof for The Lambek and Moser Theorem

Two sequences $f(n)$ and $f^*(n)$ are called inverse sequences if

$$f^*(n) = k, \text{ where } f(k) < n \le f(k+1).$$

Two sequences $F(n)$ and $G(n)$ are called complementary sequences if $F(n)$ and $G(n)$ together contain each natural number exactly once. (c.f. vol. 3 no. 4)

Theorem: $f(n)$ and $f^*(n)$ are inverse sequences if and only if $F(n) = f(n) + n$ and $G(n) = f^*(n) + n$ are complementary sequences (with the minor conditions that (i) $f(n)$ and $f^*(n)$ are non-decreasing sequences of non-negative integers; (ii) $F(n)$ and $G(n)$ are strictly increasing sequences of positive integers.)

Proof: We will first prove the converse. Let $F(n)$ and $G(n)$ be strictly increasing sequences of positive integers such that F and G are complementary. For example,

$$F(n) = \overbrace{1,2,3,\quad 6,\ 8,}^{r}\quad 10,\ 11,\ \cdots$$
$$G(n) = \qquad \underbrace{4,5,\ 7,\ 9,}_{s}\qquad 12,\ \cdots$$

(Note the inserted spaces in the above illustration so that the natural numbers are in increasing order from left to right in relative position.) Let N be a natural number. Let r and s be the number of terms in $F(n)$ and $G(n)$ that are $\le N$ respectively. (In the above illustration, $N = 9$, $r = 5$ and $s = 4$.) Note that $r + s = N$.

Now consider $f(n) = F(n) - n$ and $f^*(n) = G(n) - n$.

$$f(n) = \overbrace{0,0,0,\quad 2,\ 3,}^{r}\quad 4,\ 4,\ \cdots$$
$$f^*(n) = \qquad \underbrace{3,3,\ 4,\ 5,}_{s}\qquad 7,\ \cdots$$

We observe that

$$f^*(s) = G(s) - s = N - s = r.$$

That is,

$f^*(s) =$ the number of terms in f appear on the left hand side (in position) of the term $f^*(s)$.

Likewise,

$f(r) =$ the number of terms in f^* appear on the left hand side (in position) of the term $f(r)$.

Since the term $f(r)$ appear on the left hand side of $f^*(s)$, $f(r) < s$. We may similarly show that $f(r+1) \ge s$ and thus

$$f(r) < s \le f(r+1).$$

That is, $f^*(n)$ is the frequency distribution of $f(n)$ and thus $f(n)$ and $f^*(n)$ are inverse sequences. The fact that $f(r) < s \le f(r+1)$ can also be proved formally as follows.

$$f(r) = F(r) - r < N - r = s;$$
$$f(r+1) = F(r+1) - (r+1) > N - (r+1) = s - 1,$$
$$f(r+1) \ge s.$$

We will now show that if $f(n)$ is a non-decreasing sequence of non-negative integers and $f^*(n)$ is the frequency distribution function of $f(n)$, then $F(n) = f(n) + n$ and $G(n) = g(n) + n$ are complementary. Given the sequence $f(n)$, we can first construct the sequence $F(n) = f(n) + n$. Let $H(n)$ be the complementary sequence of $F(n)$ and let $h(n) = H(n) - n$. From the converse proof, $h(n)$ must be the frequency distribution of $f(n)$. Since the frequency distribution of a given function is unique, $h(n) = f^*(n)$ and thus

$$G(n) = f^*(n) + n = h(n) + n = H(n)$$

is the complementary sequence of $F(n)$.

Q.E.D.

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is April 15, 1998.

Problem 71. Find all real solutions of the system

$$x+\log\left(x+\sqrt{x^2+1}\right)=y,$$
$$y+\log\left(y+\sqrt{y^2+1}\right)=z,$$
$$z+\log\left(z+\sqrt{z^2+1}\right)=x.$$

(*Source:* 1995 Israel Math Olympiad.)

Problem 72. Is it possible to write the numbers 1, 2, ... , 121 in an 11×11 table so that any two consecutive numbers be written in cells with a common side and all perfect squares lie in a single column? (*Source:* 1995 Russian Math Olympiad.)

Problem 73. Prove that if a and b are rational numbers satisfying the equation $a^5+b^5=2a^2b^2$, then $1-ab$ is the square of a rational number. (*Source:* 26th British Math Olympiad.)

Problem 74. Points A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1 of acute triangle ABC, respectively. Find the sum of $\angle B_2A_1C_2$, $\angle C_2B_1A_2$, $\angle A_2C_1B_2$. (*Source:* 1995 Russian Math Olympiad.)

Problem 75. Let $P(x)$ be any polynomial with integer coefficients such that $P(21) = 17$, $P(32) = -247$, $P(37) = 33$. Prove that if $P(N) = N + 51$, for some integer N, then $N = 26$. (*Source:* 23rd British Math Olympiad.)

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Solutions

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Problem 66.

(a) Find the first positive integer whose square ends in three 4′s.

(b) Find all positive integers whose squares end in three 4′s.

(c) Show that no perfect square ends with four 4′s.

(*Source:* 1995 British Mathematical Olympiad.)

Solution: Andy CHAN Kin Hang (Bishop Hall Jubilee School, Form 4) and **SHUM Ho Keung** (PLK No. 1 W. H. Cheung College, Form 5).

(a) Since $21^2 < 444 < 22^2$ and $1444 = 38^2$, the first such positive integer is 38.

(b) Assume n is such an integer. Then

$$n^2 - 1444 = (n-38)(n+38)$$

is divisible by $1000 = 2^3 5^3$. This implies at least one of $n-38$, $n+38$ is divisible by 4. Since their difference is 76, hence both must be divisible by 4. Since 76 is not divisible by 5, hence one of $n-38$, $n+38$ is divisible by $4\cdot 5^3 = 500$. Then $n = 500k \pm 38$ for some nonnegative integer k. Conversely, for such n,

$$n^2 = 1000(250k^2 \pm 38k) + 1444$$

always ends in three 4's.

(c) Since $250k^2 \pm 38k$ is even, no perfect square ends with four 4's.

Other commended solvers: **KWOK Chi Hang** (Valtorta College, Form 6), **LAI Chi Fung, Brian** (Queen Elizabeth School, Form 5), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **LI Fung** (HK Taoist Association Ching Chung Secondary School, Form 7), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5) and **WONG Shu Fai** (Valtorta College, Form 6).

Problem 67. Let Z and R denote the integers and real numbers, respectively. Find all functions $f: Z \to R$ such that

$$f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{2}$$

for all integers x, y such that $x + y$ is divisible by 3. (*Source:* a modified problem from the 1995 Iranian Mathematical Olympiad.)

Solution: CHAN Wing Sum (City U) and **TSANG Sai Wing** (Valtorta College, Form 7).

For all integer n,

$$f(0)+f(3n)=2f(n)=f(n)+f(2n).$$

This implies

$$f(n)=f(2n)=\frac{f(3n)+f(3n)}{2}=f(3n).$$

So $f(n)=f(0)$ for all integer n. It is also clear that all constant functions are solutions.

Other commended solvers: **Andy CHAN Kin Hang** (Bishop Hall Jubilee School, Form 4), **CHING Wai Hung** (STFA Leung Kau Kui College, Form 6), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **LI Fung** (HK Taoist Association Ching Chung Secondary School, Form 7), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5) and **WONG Hau Lun** (STFA Leung Kau Kui College, Form 6).

Problem 68. If the equation

$$ax^2+(c-b)x+(e-d)=0$$

has real roots greater than 1, show that the equation

$$ax^4+bx^3+cx^2+dx+e=0$$

has at least one real root. (*Source*: 1995 Greek Mathematical Olympiad.)

Solution: CHAN Wing Chiu (La Salle College, Form 5).

Suppose

$$p(x)=ax^4+bx^3+cx^2+dx+e$$

has no real root. Let $y > 1$ be a root of $ay^2 + (c-b)y + (e-d) = 0$ and $z=\sqrt{y}$. Since

$$p(x) = ax^4+(c-b)x^2+(e-d)+(x-1)(bx^2+d),$$

we get

$$p(z)=(z-1)(bz^2+d)$$

and

$$p(-z)=(-z-1)(bz^2+d).$$

Now $z > 1$ implies one of $p(z)$, $p(-z)$ is positive, while the other is negative. Therefore, $p(x)$ has a root between z and $-z$, a contradiction.

Problem 69. *ABCD* is a quadrilateral such that $AB = AD$ and $\angle B = \angle D = 90°$. Points F and E are chosen on BC and CD, respectively, so that $DF \perp AE$. Prove that $AF \perp BE$. (*Source:* 1995 Russian Mathematical Olympiad.)

Solution 1: WONG Hau Lun (STFA Leung Kau Kui College, Form 6).

Let E' be the mirror image of E with

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

respect to AC. Let X be the intersection of DF and AE. Let Y be the intersection of AF and BE. Since $\angle ADE = 90° = \angle AXD$, we have $\angle ADF = \angle DEA = \angle BE'A = 180° - \angle AE'F$. So A, D, F, E' are concyclic. Then $\angle AFD = \angle AE'D = \angle AEB$. So X, E, F, Y are concyclic. Therefore $\angle EYF = \angle EXF = 90°$.

Solution 2: CHING Wai Hung (STFA Leung Kau Kui College, Form 6).

Since $DF \perp AE$ and $DA \perp DE$, so

$$\begin{aligned} 0 &= \overrightarrow{DF} \cdot \overrightarrow{AE} \\ &= (\overrightarrow{DA} + \overrightarrow{AF}) \cdot \overrightarrow{AE} \\ &= \overrightarrow{DA} \cdot (\overrightarrow{AD} + \overrightarrow{DE}) + \overrightarrow{AF} \cdot \overrightarrow{AE} \end{aligned}$$

which simplifies to

$$\overrightarrow{AF} \cdot \overrightarrow{AE} = |\overrightarrow{AD}|^2 .$$

Since $BF \perp BA$, so

$$\begin{aligned} \overrightarrow{AF} \cdot \overrightarrow{BE} &= \overrightarrow{AF} \cdot (\overrightarrow{BA} + \overrightarrow{AE}) \\ &= (\overrightarrow{AB} + \overrightarrow{BF}) \cdot \overrightarrow{BA} + \overrightarrow{AF} \cdot \overrightarrow{AE} \\ &= -\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AD}\right|^2 \\ &= 0 \end{aligned}$$

which implies $AF \perp BE$.

Other commended solver: **TSANG Kam Wing** (Valtorta College, Form 5).

Problem 70. Lines $l_1, l_2, \cdots, l_k$ are on a plane such that no two are parallel and no three are concurrent. Show that we can label the C_2^k intersection points of these lines by the numbers $1, 2, \cdots, k-1$ so that in each of the lines $l_1, l_2, \cdots, l_k$ the numbers $1, 2, \cdots, k-1$ appear exactly once if and only if k is even. (*Source:* a modified problem from the 1995 Greek Mathematical Olympiad.)

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 5).

If such labeling exists for an integer k, then the label 1 must occur once on each line and each point labeled 1 lies on exactly 2 lines. Hence there are $k/2$ 1's, i.e. k is even.

Conversely, if k is even, then the following labeling works: for $1 \le i < j \le k - 1$, give the intersection of lines l_i and l_j the label $i + j - 1$ when $i + j \le k$, the label $i + j - k$ when $i + j > k$. For the intersection of lines l_k and l_i ($i = 1, 2, \ldots, k - 1$), give the label $2i - 1$ when $2i \le k$ the label $2i - k$ when $2i > k$.

Comments: The official solution made use of the special symmetry of an odd number sided regular polygon to construct the labeling as follow: for k even, consider the $k - 1$ sided regular polygon with the vertices labeled $1, 2, \ldots, k - 1$. For $1 \le i < j \le k - 1$, the perpendicular bisector of the segment joining vertices i and j passes through a unique vertex, give the intersection of lines l_i and l_j the label of that vertex. For the intersection of lines l_k and l_i ($i = 1, 2, \ldots, k - 1$), give the label i.

Other commended solver: **LAW Ka Ho** (Queen Elizabeth School, Form 5).

Olympiad Corner

(*continued from page 1*)

Problem 4. $AC'BA'CB'$ is a convex hexagon such that $AB' = AC'$, $BC' = BA'$ and $CA' = CB'$. Moreover, $\angle A + \angle B + \angle C = \angle A' + \angle B' + \angle C'$. Prove that the area of triangle ABC is half of the area of the hexagon. (6 points)

Problem 5. Prove that the number

(a) 97^{97}; (4 points)

(b) 1997^{17} (4 points)

is not representable as a sum of cubes of several consecutive integers.

Problem 6. Let P be a point inside the triangle ABC with $AB = BC$, $\angle ABC = 80°$, $\angle PAC = 40°$, and $\angle ACP = 30°$. Find $\angle BPC$. (7 points)

Problem 7. You are given a balance and one copy of each ten weights of 1, 2, 4, 8, 16, 32, 64, 128, 256 and 512 grams. An object weighing M grams, where M is a positive integer, may be balanced in different ways by placing various combinations of the given weights on either pans of the balance.

(a) Prove that no object may be balanced in more than 89 ways. (5 points)

(b) Find a value of M such that an object weighing M grams can be balanced in 89 ways. (4 points)

Senior A-Level Paper

Problem 1. same as Junior A-Level Paper Problem 2. (4 points)

Problem 2. D is the point on BC and E is the point on CA such that AD and BE are the bisectors of $\angle A$ and $\angle B$ of triangle ABC. If DE is the bisector of $\angle ADC$, find $\angle A$. (5 points)

Problem 3. You are given 20 positive weights such that any object of integer weight m, $1 \le m \le 1997$, can be balanced by placing in it one pan of a balance and a subset of the weights on the other pan. What is the minimal value of the largest of the 20 weights if the weights are

(a) all integers; (3 points)

(b) not necessarily integers? (3 points)

Problem 4. A convex polygon G is placed inside a convex polygon F so that their boundaries have no common points. A segment s containing two points on the boundary of F is called a support chord for G if s contains a side or only a vertex of G. Prove that

(a) there exists a support chord for G whose midpoint lies on the boundary of G; (6 points)

(b) there exist at least two such chords. (2 points)

Problem 5. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \le 1,$$

where a, b and c are positive numbers such that $abc = 1$. (8 points)

Problem 6. Prove that if $F(x)$ and $G(x)$ are polynomials with coefficients 0 and 1 such that

$$F(x)G(x) = 1 + x + x^2 + \ldots + x^{n-1}$$

holds for some $n > 1$, then one of them is representable in the form

$$(1 + x + x^2 + \ldots + x^{k-1})T(x)$$

for some $k > 1$ and some polynomial $T(x)$ with coefficients 0 and 1. (8 points)

Problem 7. Several strips and a circle of radius 1 are drawn on the plane. The sum of the widths of the strips is 100. Prove that one can translate each strip parallel to itself so that together they cover the circle. (8 points)

Mathematical Excalibur

Volume 4, Number 2 March, 1998 - December, 1998

Olympiad Corner

Tenth Asian Pacific Mathematics Olympiad, March, 1998:

Each question is worth 7 points.

Problem 1. Let F be the set of all n-tuples $(A_1, A_2,\ldots, A_n)$ where each A_i, $i = 1, 2, \ldots, n$ is a subset of $\{1,2,\ldots,1998\}$. Let $|A|$ denote the number of elements of the set A. Find the number

$$\sum_{(A_1,A_2,\ldots,A_n)} \left|A_1 \cup A_2 \cup \ldots A_n\right|.$$

Problem 2. Show that for any positive integers a and b, $(36a+b)(a+36b)$ cannot be a power of 2.

Problem 3. Let a, b, c be positive real numbers. Prove that

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \ge 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Problem 4. Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math Dept, Catherine NG, EEE Dept, HKUST and Tam Siu Lung for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is December 31, 1998.

For individual subscription for the three remaining issues for the 98-99 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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A Taste of Topology

Wing-Sum Chan

Beauty is the first test: there is no permanent place in the world for ugly mathematics.
(G. H. Hardy)

In topology, there are many abstractions of geometrical ideas, such as continuity and closeness. ‘Topology’ is derived from the Greek words $\tau o\pi o\sigma$, a place and $\lambda o\gamma o\sigma$, a discourse. It was introduced in 1847 by Johann Benedict Listing (1808-1882), who was a student of Carl Friedrich Gauss (1777-1855). In the early days, people called it *analysis situs*, that is, analysis of position. Rubber-sheet geometry is a rather descritpive term to say what it is. (Just think of properties of objects drawn on a sheet of rubber which are not changed when the sheet is being distorted.) Hence, topologists could not distinguish a triangle from a rectangle and they may even consider a basketball as a ping-pong ball.

Topologists consider two objects to be the same (homeomorphic) if one can be continuously deformed to look like the other. Continuous deformations include bending, stretching and squashing without gluing or tearing points.

Example 1. The following are homeomorphic: (See Figure 1.)

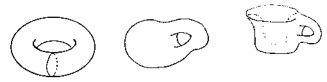


Fig 1.

Example 2. The following are non-homeomorphic: (See Figure 2.)

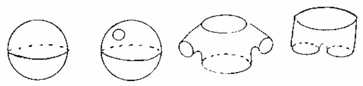


Fig 2.

In practise, continuous deformations may not be easy to carry out. In fact, there is a simple method to see two objects are non-homeomorphic, by seeking their Poincaré-Euler characteristics, (in short, Euler numbers). In order to see what the Euler number is, we need to introduce the concept of subdivision on an n-manifold (here $n \le 2$ throughout). (An n-manifold is roughly an n dimensional object in which each point has a neighborhood homeomorphic to an open interval (if $n = 1$) or an open disk (if $n = 2$). For example, a circle is a 1-manifold and a sphere is a 2-manifold.)

Basically, we start with an n manifold and subdividing it into a finite number of vertices, edges and faces. A vertex is a point. An edge is a curve with endpoints that are vertices. A face is a region with boundary that are edges.

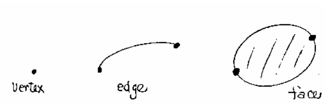


Fig 3

Here are typical pictures of vertex, edge and face, (see Figure 3.)

The Euler number (χ) of a compact (loosely speaking, bounded) 1-manifold is defined to be the number of vertices(v) minus the number of edges(e), and for a compact 2-manifold (surface), it is defined to be the number of vertices(v) minus the number of edges (e) plus the number of

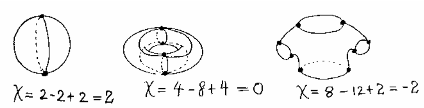


Fig.4

faces (*f*) (see Figure 4.) The following theorem is a test to distinguish non-homeomorphic objects.

Theorem 1. *If two n-manifolds are homeomorphic, then they have the same Euler number.*

So figure 4 and theorem 1 imply the sphere and the torus are not homeomorphic, i.e. the sphere cannot be continuously deformed to look like the torus and vice versa.

Here are two terms we need before we can state the next theorem. A connected manifold is one where any two points on the manifold can be connected by a curve on the manifold. The manifold is orientable if it has 2 sides, an inside and an outside.

Theorem 2. *Two connected orientable n-manifolds* ($n \le 2$) *with the same number of boundary components are homeomorphic if and only if they have the same Euler number.*

Here are some important results that tell us the general pictures of one and two manifolds.

Classification I. *Any connected compact one-manifold is either homeomorphic to an open interval or a circle.*

Classification II. *Any connected, orientable and compact two-manifolds is homeomorphic to one of the followings:* (see Figure 5.)

Finally, we mention a famous open problem (the Poincaré conjecture), which is to show that every compact, simply connected three-manifold is homeomorphic to a three-sphere, where simply connected means any circle on the manifold can be

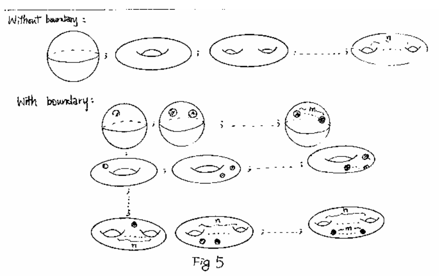


Fig 5

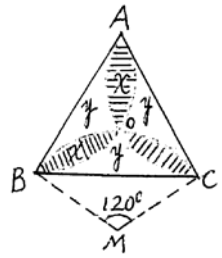
shrunk to a point on the manifold.

巧列一次方程組　妙解陰影面積題

于志洪

江蘇泰州橡膠總廠中學

有一類關於求陰影部分面積的問題，我們可根據題意適當設元，通過一次方程組求得結果。這種數形結合，將幾何面積問題轉化爲解一次方程組代數問題的方法，由於方法新穎、思路清晰，因而頗受師生重視。現舉三例分析說明如下：



一．列二元一次方程組求陰影面積

例一：如上圖，O 爲正三角形 ABC 的中心， $AB = 8\sqrt{3}\text{cm}$ ，則 $\overset{\frown}{AOB}$ 、 $\overset{\frown}{BOC}$ 、 $\overset{\frown}{COA}$ 所圍成的陰影部分的面積是____ cm^2 。(1996 年陝西省中考題)

分析：上圖中含有形狀不同的兩類圖形，分別爲 x 和 y，由圓形特徵得知，2 個 x 和 1 個 y 組成一個圓心角爲 $120°$ 的弓形，而 3 個 x 和 3 個 y 組成一個正三角形 ABC。由於正三角形 ABC 的高 $= \frac{\sqrt{3}}{2} \times 8\sqrt{3} = 12$ ，又 O 爲正三角形 ABC 的中心，故 $BO = \frac{2}{3} \times 12 = 9 = MB$ 。

$$\therefore 2x + y = S_{扇形MBC} - S_{\Delta MBC}$$

$$= \frac{120\pi \cdot 8^2}{360} - \frac{1}{2} \times 8\sqrt{3} \times 4 = \frac{64\pi}{3} - 16\sqrt{3}$$

$$= \frac{1}{2} \times 8\sqrt{3} \times 12 = 48\sqrt{3}$$

解下列方程組

$$\begin{cases} 2x + y = \dfrac{64\pi}{3} - 16\sqrt{3} & \cdots\cdots\cdots(1) \\ 3x + 3y = 48\sqrt{3} & \cdots\cdots\cdots(2) \end{cases}$$

，得 $3x = 64\pi - 93\sqrt{3}$ 。這就是所求陰影部分的面積。

二．列三元一次方程組求陰影面積

例 2：如下圖，在正方形 $ABCD$ 中，有一個以正方形的中心爲圓心，以邊長一半爲半徑的圓。另分別以 A、B、C、D 爲圓心，以邊長一半爲半徑畫四條弧。若正方形的邊長爲 $2a$，求所圍成的陰影部分的面積。（1997 年泰州市中考模擬題）

分析：圖中含有形狀不同三類圖形，分別爲 x、y、z。由圖形特徵得知：4 個 x 和 1 個 y 組成一個圓；1 個 x 和 1 個 z 組成一個以 a 爲半徑、圓心角爲直角的的扇形；4 個 x、4 個 z 和 1 個 y 組成一個正方形。

故此，可列出方程組

$$\begin{cases} 4x + y = \pi a^2 & \cdots\cdots\cdots(1) \\ x + z = \frac{1}{4}\pi a^2 & \cdots\cdots\cdots(2) \\ 4x + y + 4z = 4a^2 & \cdots\cdots\cdots(3) \end{cases}$$

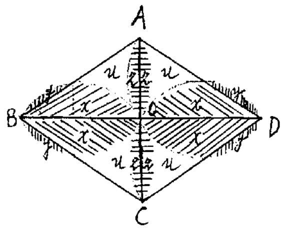
$(3) - (1)$ 得 $z = \frac{1}{4}(4 - \pi)a^2$

再代入 (2) 得 $x = (\frac{\pi}{2} - 1)a^2$ 。

$\therefore S_{陰影} = 4x = (2\pi - 4)a^2$ 。

三．列四元一次方程組求陰影面積

例 3：如左圖，菱形 $ABCD$ 的兩條對角線長分別爲 a、b，分別以每邊爲直徑向形內作半圓。求 4 條半圓弧圍成的花瓣形面積(陰影部分的面積）。（人教版九年義務教材初中《幾何》第三冊 P. 212）



分析：圖中含有形狀不同的四類圖形，分別爲 x、y、z、u，則由圖形特徵得知：$2x$、$2y$、z、u 組成一個以邊長爲直徑的半圓；x、z、u 組成直角三角形 BOC。

解：設 x、y、z、u 如圖所示，則依題意得

$$\begin{cases} x + z + u = \dfrac{1}{2} \cdot \dfrac{a}{2} \cdot \dfrac{b}{2} & \cdots(1) \\ 2x + 2z + y + u = \dfrac{1}{2}\pi\left(\dfrac{1}{2}\sqrt{\dfrac{a^2+b^2}{4}}\right)^2 & \cdots(2) \end{cases}$$

$(2) - (1)$ 再乘 4 得

（續於第四頁）

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *Dec 31, 1998.*

Problem 76. Find all positive integers N such that in base 10, the digits of $9N$ is the reverse of the digits of N and N has at most one digit equal 0. (*Source:* 1977 unused IMO problem proposed by Romania)

Problem 77. Show that if ΔABC satisfies

$$\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\cos^2 A+\cos^2 B+\cos^2 C}=2,$$

then it must be a right triangle. (*Source:* 1967 unused IMO problem proposed by Poland)

Problem 78. If $c_1, c_2,\ldots, c_n$ ($n \geq 2$) are real numbers such that

$$(n-1)(c_1^2+c_2^2+\ldots+c_n^2) =(c_1+c_2+\ldots+c_n)^2,$$

show that either all of them are nonnegative or all of them are nonpositive. (*Source:* 1977 unused IMO problem proposed by Czechoslovakia)

Problem 79. Which regular polygons can be obtained (and how) by cutting a cube with a plane? (*Source:* 1967 unused IMO problem proposed by Italy)

Problem 80. Is it possible to cover a plane with (infinitely many) circles in such a way that exactly 1998 circles pass through each point? (*Source:* Spring 1988 Tournament of the Towns Problem)

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Solutions

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Problem 71. Find all real solutions of the system

$$x+\log(x+\sqrt{x^2+1})=y,$$
$$y+\log(y+\sqrt{y^2+1})=z,$$
$$z+\log(z+\sqrt{z^2+1})=x.$$

(*Source:* 1995 Israel Math Olympiad)

Solution: CHOI Fun Ieng (Pooi To Middle School (Macau), Form 5).

If $x < 0$, then $0<x+\sqrt{x^2+1}<1$. So $\log(x+\sqrt{x^2+1})<0$, which implies $y < x < 0$. Similarly, we get $z < y < 0$ and $x < z < 0$, yielding the contradiction $x < z < y < x$. If $x > 0$, then $x+\sqrt{x^2+1}>1$. So $\log(x+\sqrt{x^2+1})>0$, which implies $y > x > 0$. Similarly, we get $z > y > 0$ and $x > z > 0$, yielding the contradiction $x > z > y > x$. If $x = 0$, then $x = y = z = 0$ is the only solution.

Other commended solvers: **AU Cheuk Yin** (Ming Kei College, Form 5). **CHEUNG Kwok Koon** (HKUST), **CHING Wai Hung** (STFA Leung Kau Kui College, Form 6), **HO Chung Yu** (Ming Kei College, Form 6), **KEE Wing Tao Wilton** (PLK Centenary Li Shiu Chung Memorial College, Form 6), **KU Wah Kwan** (Heep Woh College, Form 7), **KWOK Chi Hang** (Valtorta College, Form 6), **LAM Yee** (Valtorta College, Form 6), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5), **TAM Siu Lung** (Queen Elizabeth School, Form 5), **WONG Chi Man** (Valtoria College, Form 3) and **WONG Hau Lun** (STFA Leung Kau Kui College, Form 6).

Problem 72. Is it possible to write the numbers 1,2,…,121 in an 11x11 table so that any two consecutive numbers be written in cells with a common side and all perfect squares lie in a single column? (*Source:* 1995 Russian Math Olympiad)

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 5).

Suppose such a table exists. The table would be divided into 2 parts by the single column of perfect squares, with one side $11n$ $(0 \le n \le 5)$ cells and the other side $110 - 11n$ cells. Note that numbers between 2 successive perfect squares, say a^2, $(a+1)^2$, lie on one side since they cannot cross over the perfect square column, and those between $(a+1)^2$, $(a+2)^2$ lie on opposite side. Now the number of integers (strictly) between 1, 4, 9, 16, …, 100, 121 is 2, 4, 6, 8, …, 20, respectively. So one side has 2 + 6 + 10 + 14 + 18 = 50 numbers while the other side has 4 + 8 + 12 + 16 + 20 = 60 numbers. Both 50 and 60 are not multiple of 11, a contradiction.

Other commended solvers: **CHEUNG Kwok Koon** (HKUST), **HO Chung Yu** (Ming Kei College, Form 6), **LAI Chi Fung Brian** (Queen Elizabeth School, Form 4), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **TAM Siu Lung** (Queen Elizabeth School, Form 5), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 6) and **WONG Shu Fai** (Valtorta College, Form 6).

Problem 73. Prove that if a and b are rational numbers satisfying the equation $a^5+b^5=2a^2b^2$, then $1-ab$ is the square of a rational number. (*Source:* 26th British Math Olympiad)

Solution: CHAN Wing Sum (City U).

If $b=0$, then $1-ab=1^2$. If $b\neq 0$, then $a^6+ab^5=2a^3b^2$. So $a^6-2a^3b^2+b^4$ $=b^4-ab^5=b^4(1-ab)$. Therefore, $1-ab$ $=(a^6-2a^3b^2+b^4)/b^4$ is the square of the rational number $(a^3-b^2)/b^2$.

Other recommended solvers: **CHING Wai Hung** (STFA Leung Kau Kui College, Form 6), **CHOI Fun Ieng** (Pooi To Middle School (Macau), Form 5), **KU Wah Kwan** (Heep Woh College, Form 7) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5).

Problem 74. Points A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1 of acute triangle ABC, respectively. Find the sum of $\angle B_2A_1C_2$, $\angle C_2B_1A_2$ and $\angle A_2C_1B_2$. (*Source:* 1995 Russian Math Olympiad)

Solution: LAM Po Leung (Ming Kei College, Form 5)

Let A_3, B_3, C_3 be the midpoints of BC, CA, AB, respectively, and H be the orthocenter of ΔABC. Since C_3A_3 is parallel to AC, so $\angle HB_2A_3=90°$ $=\angle HA_1A_3$, which implies H, B_2, A_3, A_1 are concyclic. So $\angle B_2A_1H=\angle B_2A_3H$. Since B_3A_3 is parallel to AB, so

$\angle HC_2A_3 = 90° = \angle HA_1A_3$, which implies H, C_2, A_3, A_1 are concyclic. So $\angle C_2A_1H = \angle C_2A_3H$. Then $\angle B_2A_1C_2 = \angle B_2A_1H + \angle C_2A_1H = \angle B_2A_3H + \angle C_2A_3H = \angle C_3A_3B_3 = \angle BAC$ (because $\Delta A_3B_3C_3$ is similar to ΔABC). Similarly, $\angle B_2C_1A_2 = \angle BCA$ and $\angle A_2B_1C_2 = \angle ABC$. Therefore, the sum of $\angle B_2A_1C_2$, $\angle C_2B_1A_2$, $\angle A_2B_1C_2$ is 180^o.

Other commended solvers: **HO Chung Yu** (Ming Kei College, Form 6).

Problem 75. Let $P(x)$ be any polynomial with integer coefficients such that $P(21) = 17$, $P(32) = -247$, $P(37) = 33$. Prove that if $P(N) = N + 51$ for some integer N, then $N = 26$. (*Source:* 23rd British Math Olympiad)

Solutions: HO Chung Yu (Ming Kei College, Form 6).

If $P(N) = N + 51$ for some integer N, then $P(x) – x – 51 = (x - N)Q(x)$ for some polynomial $Q(x)$ by the factor theorem. Note $Q(x)$ has integer coefficients because $P(x) – x – 51 = P(x) – P(N) – (x – N)$ is a sum of $a_i(x^i - N^i)$ terms (with a_i's integer). Since $Q(21)$ and $Q(37)$ are integers, $P(21) – 21 – 51 = -55$ is divisible by $21 – N$ and $P(3) – 37 – 51 = -55$ is divisible by $37 – N$ is 16, we must have $N = 26$ or 32. However, if $N = 32$, then we get $-247 = P(32) = 32 + 51$, a contradiction. Therefore $N = 26$.

Other commended solvers: **CHEUNG Kwok Koon** (HKUST), **KU Wah Kwan** (Heep Who College, Form 7), **TAM Siu Lung** (Queen Elizabeth School, Form 5) and **WONG Shu Fai** (Valtorta College, Form 6).

Olympiad Corner
(continued from page 1)

Problem 5. Determine the largest of all integers n with the property that n is divisible by all positive integers that are less than $\sqrt[3]{n}$.

（續第二頁）

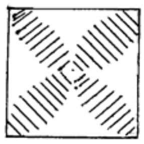
$$4(x+y+z)=\frac{\pi a^2+\pi b^2-4ab}{8}\ 。$$

這就是所求陰影部分的面積。

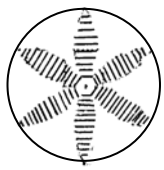
綜上所述可知：一般的陰影圖形大多是由多種規則圖形組成的，所以利用方程式組解決這類問題時，首先要根據圖形的特徵（尤其是對稱性）把圖形分成幾類，用字母表示各類圖形的面積；其次要仔細觀察圖形的組成，分析圖形中各部分之間及各部分與整體圖形的關係，通過規則圖形面積公式列出方程組；最後解方程求出陰影面積。

附練習題

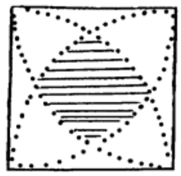
１．如右圖，已知一塊正方形的地瓷磚邊長爲 a，瓷磚上的圖案是以各邊爲直徑在正方形內畫圓所圍成的（陰影部分），那麼陰影部分的面積是多少？（1997 年寮夏回族自治區中考題）



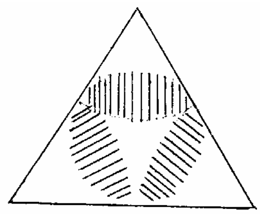
２．如右圖，已知圓形 O 的半徑爲 R，求圖中陰影部分的面積。（1998 年泰州市中考模擬題）



３．如右圖，正方形的邊長爲 a，分別以正方形的四個頂點爲圓心，邊長爲半徑，在正方形內畫弧，那麼這四條弧所圍成的陰影部分的面積是多少？（1994 年安徽省中考題）



４．如右圖，圓 O 內切於邊長爲 a 的正三角形，分別以三角形的三頂點爲圓心，$\frac{a}{2}$ 爲半徑畫弧相交成圓中所示的陰影，求陰影部分的面積。（1996 年泰州市中考模擬題）



參考資料

1．《陰影部分面積的幾種解法》
《初中生數學園地》
安義人(華南師大主辦) 1997 年 3 月

2．《列一次方程組解陰影面積題》
《中小學數學》
于志洪(中國教育學會主辦)1997年11月

練習題答案:

1. $(\frac{\pi}{2}-1)a^2$ 2. $2\pi R^2-3\sqrt{3}R^2$

3. $(1-\sqrt{3}+\frac{\pi}{3})a^2$ 4. $\frac{5\pi-6\sqrt{3}}{24}a^2$

(Hong Kong team to IMO 98: (from left to right) Lau Wai Tong (Deputy Leader), Law Ka Ho, Chan Kin Hang, Choi Ming Cheung, Lau Lap Ming, Cheung Pok Man, Leung Wing Chung, Liu Kam Moon (Leader).)

Mathematical Excalibur

Volume 4, Number 3 January, 1999 - March, 1999

Rearrangement Inequality

Kin-Yin Li

Olympiad Corner

39th International Mathematical Olympiad, July 1998:

Each problem is worth 7 points.

Problem 1. In the convex quadrilateral *ABCD*, the diagonals *AC* and *BD* are perpendicular and the opposite sides *AB* and *DC* are not parallel. Suppose that the point *P*, where the perpendicular bisectors of *AB* and *DC* meet, is inside *ABCD*. Prove that *ABCD* is a cyclic quadrilateral if and only if the triangles *ABP* and *CDP* have equal areas.

Problem 2. In a competition, there are *a* contestants and *b* judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose *k* is a number such that, for any two judges, their ratings coincide for at most *k* contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

Problem 3. For any positive integer *n*, let *d(n)* denote the number of positive divisions of *n* (including 1 and *n* itself).

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math Dept, Catherine NG, EEE Dept, HKUST and Tam Siu Lung for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is April 30, 1999.

For individual subscription for the two remaining issues for the 98-99 academic year, send us two stamped self-addressed envelopes. Send all correspondence to:

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The rearrangement inequality (or the permutation inequality) is an elementary inequality and at the same time a powerful inequality. Its statement is as follow. Suppose $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Let us call

$$A = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

the *ordered sum* of the numbers and

$$B = a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$$

the *reverse sum* of the numbers. If $x_1, x_2, ..., x_n$ is a rearrangement (or permutation) of the numbers $b_1, b_2, ..., b_n$ and if we form the *mixed sum*

$$X = a_1x_1 + a_2x_2 + \cdots + a_nx_n,$$

then the rearrangement inequality asserts that $A \geq X \geq B$. In the case the a_i's are strictly increasing, then equality holds if and only if the b_i's are all equal.

We will look at $A \geq X$ first. The proof is by mathematical induction. The case $n = 1$ is clear. Suppose the case $n = k$ is true. Then for the case $n = k + 1$, let $b_{k+1} = x_i$ and $x_{k+1} = b_j$. Observe that $(a_{k+1} - a_i)(b_{k+1} - b_j) \geq 0$. We get

$$a_ib_j + a_{k+1}b_{k+1} \geq a_ib_{k+1} + a_{k+1}b_j.$$

So in *X*, we may switch x_i and x_{k+1} to get a possibly larger sum. After switching, we can apply the case $n = k$ to the first *k* terms to conclude that $A \geq X$. The inequality $X \geq B$ follows from $A \geq X$ using $-b_n \leq -b_{n-1} \leq \cdots \leq -b_1$ in place of $b_1 \leq b_2 \leq \cdots \leq b_n$.

Now we will give some examples.

Example 1. (*Chebysev's Inequality*) Let *A* and *B* be as in the rearrangement inequality, then

$$A \geq \frac{(a_1 + \cdots + a_n)(b_1 + \cdots + b_n)}{n} \geq B.$$

Proof. Cyclically rotating the b_i's, we get *n* mixed sums

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n,$$
$$a_1b_2 + a_2b_3 + \cdots + a_nb_1,$$
$$...,$$
$$a_1b_n + a_2b_1 + \cdots + a_nb_{n-1}.$$

By the re-arrangement inequality, each of these is between *A* and *B*, so their average is also between *A* and *B*. This average is just the expression given in the middle of Chebysev's inequality.

Example 2. (*RMS-AM-GM-HM In-equality*) Let $c_1, c_2, ..., c_n \geq 0$. The *root mean square* (RMS) of these numbers is $[(c_1^2 + \cdots + c_n^2)/n]^{1/2}$, the *arithmetic mean* (AM) is $(c_1 + c_2 + \cdots + c_n)/n$ and the *geometric mean* (GM) is $(c_1c_2 \cdots c_n)^{1/n}$. We have $RMS \geq AM \geq GM$. If the numbers are positive, then the *harmonic mean* (HM) is $n/[(1/c_1) + \cdots + (1/c_n)]$. We have $GM \geq HM$.

Proof. Setting $a_i = b_i = c_i$ in the left half of Chebysev's inequality, we easily get $RMS \geq AM$. Next we will show $AM \geq GM$. The case $GM = 0$ is clear. So suppose $GM > 0$. Let $a_1 = c_1/GM$, $a_2 = c_1c_2/GM^2$, ..., $a_n = c_1c_2 \cdots c_n/GM^n = 1$ and $b_i = 1/a_{n-i+1}$ for $i = 1, 2, ... n$. (Note the a_i's may not be increasing, but the b_i's will be in the reverse order as the a_i's). So the mixed sum

$$a_1b_1 + a_2b_n + \cdots + a_nb_2 =$$
$$c_1/GM + c_2/GM + \cdots + c_n/GM$$

is greater than or equal to the reverse sum $a_1b_n + \cdots + a_nb_1 = n$. The AM-GM inequality follows easily. Finally $GM \geq HM$ follows by applying $AM \geq GM$ to the numbers $1/c_1, ..., 1/c_n$.

Example 3. (1974 *USA Math Olympiad*) If $a, b, c > 0$, then prove that

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

Solution. By symmetry, we may assume $a \leq b \leq c$, then $\ln a \leq \ln b \leq \ln c$. By Chebysev's inequality,

$$a \ln a + b \ln b + c \ln c$$
$$\geq \frac{(a+b+c)(\ln a + \ln b + \ln c)}{3}.$$

The desired inequality follows from exponentiation.

Example 4. *(1978 IMO)* Let $c_1, c_2, \ldots, c_n$ be distinct positive integers. Prove that

$$c_1 + \frac{c_2}{4} + \cdots + \frac{c_n}{n^2} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Solution. Let $a_1, a_2, \ldots, a_n$ be the c_i's arranged in increasing order. Since a_i's are distinct positive integers, $a_i \geq i$. Since $1 > 1/4 > \ldots > 1/n^2$, by the re-arrangement inequality,

$$c_1 + \frac{c_2}{4} + \cdots + \frac{c_n}{n^2}$$
$$\geq a_1 + \frac{a_2}{4} + \cdots + \frac{a_n}{n^2}$$
$$\geq 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Example 5. *(1995 IMO)* Let $a, b, c > 0$ and $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution. (HO Wing Yip, Hong Kong Team Member) Let $x = bc = 1/a$, $y = ca = 1/b$, $z = ab = 1/c$. The required inequality is equivalent to

$$\frac{x^2}{z+y} + \frac{y^2}{x+z} + \frac{z^2}{y+x} \geq \frac{3}{2}.$$

By symmetry, we may assume $x \leq y \leq z$, then $x^2 \leq y^2 \leq z^2$ and $1/(z+y) \leq 1/(x+z) \leq 1/(y+x)$. The left side of the required inequality is just the ordered sum A of the numbers. By the rearrangement inequality,

$$A \geq \frac{x^2}{y+x} + \frac{y^2}{z+y} + \frac{z^2}{x+z},$$
$$A \geq \frac{x^2}{x+z} + \frac{y^2}{y+x} + \frac{z^2}{z+y}.$$

(continued on page 4)

Power of Points Respect to Circles

Kin-Yin Li

Intersecting Chords Theorem. *Let two lines through a point P not on a circle intersect the inside of the circle at chords AA' and BB', then PA × PA' = PB × PB'. (When P is outside the circle, the limiting case A = A' refers to PA tangent to the circle.)*

This theorem follows from the observation that triangles *ABP* and *A'B'P* are similar and the corresponding sides are in the same ratio. In the case *P* is inside the circle, the product *PA* × *PA*' can be determined by taking the case the chord *AA*' passes through *P* and the center *O*. This gives *PA* × *PA*' = $r^2 - d^2$, where r is the radius of the circle and $d = OP$. In the case *P* is outside the circle, the product *PA* × *PA*' can be determined by taking the limiting case *PA* is tangent to the circle. Then *PA* × *PA*' = $d^2 - r^2$.

The *power* of a point *P* with respect to a circle is the number $d^2 - r^2$ as mentioned above. (In case *P* is on the circle, we may define the power to be 0 for convenience.) For two circles C_1 and C_2 with different centers O_1 and O_2, the points whose power with respect to C_1 and C_2 are equal form a line perpendicular to line $O_1 O_2$. (This can be shown by setting coordinates with line $O_1 O_2$ as the *x*-axis.) This line is called the *radical axis* of the two circles. In the case of the three circles C_1, C_2, C_3 with noncollinear centers O_1, O_2, O_3, the three radical axes of the three pairs of circles intersect at a point called the *radical center* of the three circles. (This is because the intersection point of any two of these radical axes has equal power with respect to all three circles, hence it is on the third radical axis too.)

If two circles C_1 and C_2 intersect, their radical axis is the line through the intersection point(s) perpendicular to the line of the centers. (This is because the intersection point(s) have 0 power with respect to both circles, hence they are on the radical axis.) If the two circles do not intersect, their radical axis can be found by taking a third circle C_3 intersecting both C_1 and C_2. Let the radical axis of C_1, C_3 intersect the radical axis of C_2, C_3 at *P*. Then the radical axis of C_1, C_2 is the line through *P* perpendicular to the line of centers of C_1, C_2.

We will illustrate the usefulness of the intersecting chords theorem, the concepts of power of a point, radical axis and radical center in the following examples.

Example 1. (*1996 St. Petersburg City Math Olympiad*) Let *BD* be the angle bisector of angle *B* in triangle *ABC* with *D* on side *AC*. The circumcircle of triangle *BDC* meets *AB* at *E*, while the circumcircle of triangle *ABD* meets *BC* at *F*. Prove that *AE* = *CF*.

Solution. By the intersecting chords theorem, *AE* × *AB* = *AD* × *AC* and *CF* × *CB* = *CD* × *CA*, so *AE/CF* = (*AD/CD*)(*BC/AB*). However, *AB/CB* = *AD/CD* by the angle bisector theorem. So *AE* = *CF*.

Example 2. (*1997 USA Math Olympiad*) Let *ABC* be a triangle, and draw isosceles triangles *BCD, CAE, ABF* externally to *ABC*, with *BC, CA, AB* as their respective bases. Prove the lines through *A, B, C,* perpendicular to the lines *EF, FD, DE,* respectively, are concurrent.

Solution. Let C_1 be the circle with center *D* and radius *BD*, C_2 be the circle with center *E* and radius *CE*, and C_3 be the circle with center *F* and radius *AF*. The line through *A* perpendicular to *EF* is the radical axis of C_2, C_3, the line through *B* perpendicular to *FD* is the radical axis of C_3, C_1 and the line through *C* perpendicular to *DE* is the radical axis of C_1, C_2. These three lines concur at the radical center of the three circles.

Example 3. (*1985 IMO*) A circle with center *O* passes through vertices *A* and *C* of triangle *ABC* and intersects side *AB* at *K* and side *BC* at *N*. Let the circumcircles of triangles *ABC* and *KBN* intersect at *B* and *M*. Prove that *OM* is perpendicular to *BM*.

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Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *April 30, 1999.*

Problem 81. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area. (*Source*: 1996 Irish Mathematical Olympiad)

Problem 82. Show that if n is an integer greater than 1, then n^4+4^n cannot be a prime number. (*Source:* 1977 Jozsef Kurschak Competition in Hungary)

Problem 83. Given an alphabet with three letters *a, b, c*, find the number of words of *n* letters which contain an even number of *a*'s. (*Source*: 1996 Italian Mathematical Olympiad)

Problem 84. Let M and N be the midpoints of sides AB and AC of ΔABC, respectively. Draw an arbitrary line through A. Let Q and R be the feet of the perpendiculars from B and C to this line, respectively. Find the locus of the intersection P of the lines QM and RN as the line rotates about A.

Problem 85. Starting at (1, 1), a stone is moved in the coordinate plane according to the following rules:

(a) From any point (a, b), the stone can be moved to $(2a, b)$ or $(a, 2b)$.

(b) From any point (a, b), the stone can be moved to $(a - b, b)$ if $a > b$, or to $(a, b - a)$ if $a < b$.

For which positive integers *x, y*, can the stone be moved to (x, y)? (*Source:* 1996 German Mathematical Olympiad)

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Solutions

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Problem 76. Find all positive integers N such that in base 10, the digits of $9N$ is the reverse of the digits of N and N has at most one digit equal 0. (*Source:* 1977 unused IMO problem proposed by Romania)

Solution. LAW Ka Ho (Queen Elizabeth School, Form 6) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 6).

Let $[a_1a_2 \ldots a_n]$ denote N in base 10 with $a_1 \neq 0$. Since $9N$ has the same number of digits as N, we get $a_1 = 1$ and $a_n = 9$. Since $9 \times 19 \neq 91$, $n > 2$. Now $9[a_2 \ldots a_{n-1}] + 8 = [a_{n-1} \ldots a_2]$. Again from the number of digits of both sides, we get $a_2 \leq 1$. The case $a_2 = 1$ implies $9a_{n-1} + 8$ ends in a_2 and so $a_{n-1} = 7$, which is not possible because $9[1 \ldots 7] + 8 > [7 \ldots 1]$. So $a_2 = 0$ and $a_{n-1} = 8$. Indeed, 1089 is a solution by direct checking. For $n > 4$, we now get $9[a_3 \ldots a_{n-2}] + 8 = [8\, a_{n-2} \ldots a_3]$. Then $a_3 \geq 8$. Since $9a_{n-2} + 8$ ends in a_3, $a_3 = 8$ will imply $a_{n-2} = 0$, causing another 0 digit. So $a_3 = 9$ and $a_{n-2} = 9$. Indeed, 10989 and 109989 are solutions by direct checking. For $n > 6$, we again get $9[a_4 \ldots a_{n-3}] + 8 = [8\, a_{n-3} \ldots a_4]$. So $a_4 = \ldots = a_{n-3} = 9$. Finally direct checking shows these numbers are solutions.

Other recommended solvers: **CHAN Siu Man** (Ming Kei College, Form 6), **CHING Wai Hung** (STFA Leung Kau Kui College, Form 7), **FANG Wai Tong Louis** (St. Mark's School, Form 6), **KEE Wing Tao Wilton** (PLK Centenary Li Shiu Chung Memorial College, Form 7), **KWOK Chi Hang** (Valtorta College, Form 7), **TAM Siu Lung** (Queen Elizabeth School, Form 6)**, WONG Chi Man** (Valtorta College, Form 4), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 7) and **WONG Shu Fai** (Valtorta College, Form 7).

Problem 77. Show that if $\Delta\, ABC$ satisfies

$$\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\cos^2 A+\cos^2 B+\cos^2 C}=2,$$

then it must be a right triangle. (*Source*: 1967 unused IMO problem proposed by Poland)

Solution. (All solutions received are essentially the same.)

Using $\sin^2 x = (1-\cos 2x)/2$ and $\cos^2 x = (1 + \cos 2x)/2$, the equation is equivalent to

$$\cos 2A + \cos 2B + \cos 2C + 1 = 0.$$

This yields $\cos(A + B)\cos(A - B) + \cos^2 C = 0$. Since $\cos(A + B) = -\cos C$, we get $\cos C\,(\cos(A - B) + \cos(A + B)) = 0$. This simplifies to $\cos C \cos A \cos B = 0$. So one of the angles *A, B, C* is 90^0.

Solvers: **CHAN Lai Yin, CHAN Man Wai**, **CHAN Siu Man**, **CHAN Suen On**, **CHEUNG Kin Ho**, **CHING Wai Hung**, **CHOI Ching Yu**, **CHOI Fun Ieng**, **CHOI Yuet Kei**, **FANG Wai Tong Louis**, **FUNG Siu Piu, HUNG Kit**, **KEE Wing Tao Wilton**, **KO Tsz Wan**, **KWOK Chi Hang, LAM Tung Man**, **LAM Wai Hung**, **LAM Yee**, **LAW Ka Ho**, **LI Ka Ho**, **LING Hoi Sheung**, **LOK Chan Fai, LUNG Chun Yan**, **MAK Wing Hang**, **MARK Kai Pan**, **Gary NG Ka Wing**, **OR Kin**, **TAM Kwok Cheong**, **TAM Siu Lung**, **TSANG Kam Wing**, **TSANG Pui Man**, **TSANG Wing Kei**, **WONG Chi Man**, **WONG Hau Lun**, **YIM Ka Wing** and **YU Tin Wai**.

Problem 78. If $c_1, c_2, \ldots, c_n\ (n \geq 2)$ are real numbers such that

$$(n-1)(c_1^2+c_2^2+\cdots+c_n^2) = (c_1+c_2+\cdots+c_n)^2,$$

show that either all of them are non-negative or all of them are non-positive. (*Source*: 1977 unused IMO problem proposed by Czechoslovakia)

Solution. CHOY Ting Pong (Ming Kei College, Form 6).

Assume the conclusion is false. Then there are at lease one negative and one positive numbers, say $c_1 \leq c_2 \leq \cdots \leq c_k \leq 0 < c_{k+1} \leq \cdots \leq c_n$ with $1 \leq k < n$, satisfying the condition. Let $w = c_1 + \ldots + c_k$, $x = c_{k+1}+\cdots+c_n$, $y = c_1^2+\cdots+c_k^2$ and $z = c_{k+1}^2+\cdots+c_n^2$. Expanding w^2 and x^2 and applying the inequality $a^2+b^2 \geq 2ab$, we get $ky \geq w^2$ and $(n-k)z \geq x^2$. So

$$(w+x)^2 = (n-1)(y+z) \geq ky + (n-k)z \geq w^2+x^2.$$

Simiplifying, we get $wx \geq 0$, contradicting $w < 0 < x$.

Other commended solvers: **CHAN Siu Man** (Ming Kei College, Form 6), **FANG Wai Tong Louis** (St. Mark's School, Form 6), **KEE Wing Tao Wilton** (PLK Centenary Li Shiu Chung Memorial College, Form 7), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 6), **TAM Siu Lung** (Queen Elizabeth School, Form 6), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 7) and **YEUNG Kam Wah** (Valtorta College, Form 7).

Problem 79. Which regular polygons can be obtained (and how) by cutting a cube with a plane? (*Source*: 1967 unused IMO problem proposed by Italy)

Solution. FANG Wai Tong Louis (St. Mark's school, Form 6), **KEE Wing Tao** (PLK Centenary Li Shiu Chung Memorial School, Form 7), **TAM Siu Lung** (Queen Elizabeth School, Form 6) and **YEUNG Kam Wah** (Valtorta College, Form 7).

Observe that if two sides of a polygon is on a face of the cube, then the whole polygon lies on the face. Since a cube has 6 faces, only regular polygon with 3, 4, 5 or 6 sides are possible. Let the vertices of the bottom face of the cube be *A, B, C, D* and the vertices on the top face be *A', B', C', D'* with *A'* on top of *A*, *B'* on top of *B* and so on. Then the plane through *A, B', D'* cuts an equilateral triangle. The perpendicular bisecting plane to edge *AA'* cuts a square. The plane through the mid-points of edges *AB, BC, CC', C'D', D'A', A'A* cuts a regular hexagon. Finally, a regular pentagon is impossible, otherwise the five sides will be on five faces of the cube implying two of the sides are on parallel planes, but no two sides of a regular pentagon are parallel.

Problem 80. Is it possible to cover a plane with (infinitely many) circles in such a way that exactly 1998 circles pass through each point? (*Source*: Spring 1988 Tournament of the Towns Problem)

Solution. Since no solution is received, we will present the modified solution of Professor Andy Liu (University of Alberta, Canada) to the problem.

First we solve the simpler problem where 1998 is replaced by 2. Consider the lines $y = k$, where k is an integer, on the coordinate plane. Consider every circle of diameter 1 tangent to a pair of these lines. Every point (x, y) lies on exactly two of these circles. (If y is an integer, then (x, y) lies on one circle on top of it and one below it. If y is not an integer, then (x, y) lies on the right half of one circle and on the left half of another.) Now for the case 1998, repeat the argument above 998 times (using lines of the form $y = k + (j/999)$ in the j-th time, $j = 1, 2, ..., 998$.)

Olympiad Corner

(continued from page 1)

Determine all positive integers k such that

$$\frac{d(n^2)}{d(n)} = k$$

for some n.

Problem 4. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

Problem 5. Let I be the incentre of triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at K, L and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that $\angle RIS$ is acute.

Problem 6. Consider all functions f from the set $\mathbb{N}$ of all positive integers into itself satisfying

$$f(t^2 f(s)) = s(f(t))^2,$$

for all s and t in $\mathbb{N}$. Determine the least possible value of $f(1998)$.

Rearrangement Inequality

(continued from page 2)

So

$$A \geq \frac{1}{2}\left(\frac{y^2+x^2}{y+x}+\frac{z^2+y^2}{z+y}+\frac{x^2+z^2}{x+z}\right).$$

Applying the RMS-AM inequality $r^2 + s^2 \geq (r+s)^2/2$, the right side is at least $(x+y+z)/2$, which is at least $3(xyz)^{1/3}/2 = 3/2$ by the AM-GM inequality.

Power of Points Respect to Circles

(continued from page 2)

Solution. For the three circles mentioned, the radical axes of the three pairs are lines AC, KN and BM. (The centers are noncollinear because two of them are on the perpendicular bisector of AC, but not the third.) So the axes will concur at the radical center P. Since $\angle PMN = \angle BKN = \angle NCA$, it follows that P, M, N, C are concyclic. By power of a point, $BM \times BP = BN \times BC = BO^2 - r^2$ and $PM \times PB = PN \times PK = PO^2 - r^2$, where r is the radius of the circle through A, C, N, K. Then $PO^2 - BO^2 = BP(PM - BM) = PM^2 - BM^2$. This implies OM is perpendicular to BM. (See remarks below.)

Remarks. By coordinate geometry, it can be shown that the locus of points X such that $PO^2 - BO^2 = PX^2 - BX^2$ is the line through O perpendicular to line BP. This is a useful fact.

Example 4. (*1997 Chinese Math Olympiad*) Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the tangents QE and QF to the circle, where E and F are the points of tangency. Prove that P, E, F are collinear.

Solution. Let M be a point on PQ such that $\angle CMP = \angle ADC$. Then D, C, M, Q are concyclic and also, B, C, M, P are concyclic. Let r_1 be the radius of the circumcircle C_1 of $ABCD$ and O_1 be the center of C_1. By power of a point, $PO_1^2 - r_1^2 = PC \times PD = PM \times PQ$ and $QO_1^2 - r_1^2 = QC \times QB = QM \times PQ$. Then $PO_1^2 - QO_1^2 = (PM - QM)PQ = PM^2 - QM^2$, which implies $O_1M \perp PQ$. The circle C_2 with QO_1 as diameter passes through M, E, F and intersects C_1 at E, F. If r_2 is the radius of C_2 and O_2 is the center of C_2, then $PO_1^2 - r_1^2 = PM \times PQ = PO_2^2 - r_2^2$. So P lies on the radical axis of C_1, C_2, which is the line EF.

Mathematical Excalibur

Volume 4, Number 4 April 1999 - September 1999

Olympiad Corner

11th Asian Pacific Mathematical Olympiad, March 1999:

Time allowed: 4 Hours
Each problem is worth 7 points.

Problem 1. Find the smallest positive integer n with the following property: There does not exist an arithmetic progression of 1999 terms of real numbers containing exactly n integers.

Problem 2. Let a_1, a_2, ... be a sequence of real numbers satisfying $a_{i+j} \le a_i + a_j$ for all i, j = 1, 2, ... Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \ge a_n$$

for each positive integer n.

Problem 3. Let Γ_1 and Γ_2 be two circles interecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, Catherine Ng, EEE Dept, HKUST and Tam Siu Lung for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is September 30, 1999.

For individual subscription for the two remaining issues for the 98-99 academic year, send us two stamped self-addressed envelopes. Send all correspondence to:

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費馬最後定理 (一)

梁子傑
香港道教聯合會青松中學

大約在 1637 年，當法國業餘數學家費馬 (Pierre de Fermat, 1601-1665) 閱讀古希臘名著《算術》時，在書邊的空白地方，他寫下了以下的一段說話：「將個立方數分成兩個立方數，一個四次冪分成兩個四次冪，或者一般地將一個高於二次冪的數分成兩個相同次冪，這是不可能的。 我對這個命題有一個美妙的證明，這裏空白太小，寫不下。」換成現代的數學術語，費馬的意思就即是：「當整數 $n > 2$ 時，方程 $x^n + y^n = z^n$ 沒有正整數解。」

費馬當時相信自己已發現了對以上命題的一個數學證明。 可惜的是，當費馬死後，他的兒子為他收拾書房時，並沒有發現費馬的「美妙證明」。到底，費馬有沒有證實這個命題呢？又或者，費馬這個命題是否正確呢？

費馬這個命題並不難理解，如果大家用計算機輸入一些數字研究一下，（注意：費馬的時代並未發明任何電子計算工具，）那麼就會「相信」費馬這個命題是正確的。由於 費馬在生時提出的其他數學命題，都逐步被證實或否定，就祇剩下這一個看似正確，但無法證明的命題未能獲證，所以數學家就稱它為「費馬最後定理」。

說也奇怪，最先對「費馬最後定理」的證明行出第一步的人，就是費馬本人！有人發現，在費馬的書信中，曾經提及方程 $x^4 + y^4 = z^4$ 無正整數解的證明。費馬首先假設方程 $x^4 + y^4 = z^2$ 是有解的，即是存在 三個正整數 a、b 和 c，並且 $a^4 + b^4$ 剛好等於 c^2。 然後他通過「勾股數組」的通解，構作出另外三個正整數 e、f 和 g，使得 $e^4 + f^4 = g^2$ 並且 $c > g$。 費馬指出這是不可能的，因為如果這是正確的，那麼重覆他的構作方法，就可以構造出一連串遞降的數字，它們全都滿足方程 $x^4 + y^4 = z^2$。 但是 c 是一個有限數， 不可能如此無窮地遞降下去！ 所以前文中假設方程 $x^4 + y^4 = z^2$ 有解這個想法不成立，亦即是說方程 $x^4 + y^4 = z^2$ 無整數解。

又由於方程 $x^4 + y^4 = z^2$ 是無解的，方程 $x^4 + y^4 = z^4$ 亦必定無解。 否則 將後者的解寫成 $x^4 + y^4 = (z^2)^2$ 就會 變成前一個方程的解，從而導出矛盾。 由此可知，當 $n = 4$ 時，「費馬最後定理」成立。

為「費馬最後定理」踏出另一步的人，是瑞士大數學家歐拉（Leonhard Euler, 1707-1783）。他利用了複數 $a + b\sqrt{-3}$ 的性質，證實了方程 $x^3 + y^3 = z^3$ 無解。但由於歐拉在他的證明中，在沒有足夠論據的支持下，認為複數 $a + b\sqrt{-3}$ 的立方根必定可以再次寫成 $a + b\sqrt{-3}$ 的形式， 因此他的證明 未算圓滿。 歐拉證明的缺憾，又過了近半個世紀，才由德國數學家高斯（Carl Friedrich Gauss, 1777-1855）成 功地補充。 同時，高斯更為此而引進了「複整數」的概念，即形如 $a + b\sqrt{-k}$ 的複數，其中 k 為正整數，a 和 b 為整數。

1823 年，七十一歲高齡的法國數學家勒讓德（Adrien Marie Legendre, 1752 - 1833）提出了「費馬最後定理」當 $n = 5$ 時的證明。 1828 年，年青的德國數學家狄利克雷（Peter Gustav Lejeune Dirichlet, 1805 -1859）亦獨立地

Pierre Fermat

Leonhard Euler

Carl Friedrich Gauss

Lejeune Dirichlet

證得同樣的結果。 其後，在 1832 年，狄利克雷更證明當 $n = 14$ 時，「費馬最後定理」成立。

1839 年，另一位法國人拉梅（Gabriel Lamé, 1795 -1870）就證到 $n = 7$。1847 年，拉梅更宣稱他已完成了「費馬最後定理」的證明。

拉梅將 $x^n + y^n$ 分解成 $(x + y)(x + \zeta y)(x + \zeta^2 y)\ldots(x + \zeta^{n-1} y)$，其中 $\zeta = \cos(2\pi/n) + i\sin(2\pi/n)$ ，即方程 $r^n = 1$ 的複數根。 如果 $x^n + y^n = z^n$，那麼拉梅認爲每一個 $(x + \zeta^k y)$ 都會是 n 次冪乘以一個複數單位，從而可導出矛盾，並能證明「費馬最後定理」成立。 不過，拉梅的證明很快便證實爲無效，這是因爲拉梅所構作的複數，並不一定滿足「唯一分解定理」。

甚麼是「唯一分解定理」呢？ 在一般的整數中，每一個合成數都衹可能被分解成一種「質因數連乘式」。 但在某些「複整數」中，情況就未必相同。例如：$6 = 2 \times 3 = (1 + \sqrt{-5}) \times (1 - \sqrt{-5})$ ，而在 $a + b\sqrt{-5}$ 的複整數中，2、3、$(1 + \sqrt{-5})$ 和 $(1 - \sqrt{-5})$ 都是互不相同的質數。換句話說，形如 $a + b\sqrt{-5}$ 的複整數，並不符合「唯一分解定理」。

如果能夠滿足「唯一分解定理」，那麼當 $z^n = ab$ 時，我們就確信可以找到兩個互質的整數 u 和 v ，使得 $a = u^n$ 和 $b = v^n$ 了。 但如果未能滿足「唯一分解定理」，以上的推論就不成立了。例如：$6^2 = 2 \times 3 \times (1 + \sqrt{-5}) \times (1 - \sqrt{-5})$ ，但右方的四個數，都並非是一個平方數，故此，當 $6^2 = ab$ 時，我們就不能肯定 a 和 b 是不是平方數了！ 這一點，亦正好是拉梅證明的一大漏洞！

爲了解決未能滿足「唯一分解定理」所帶來的問題，德國數學家庫默爾（Ernst Edward Kummer, 1810 -1893）就提出了「理想數」的想法。

已知 n 爲一個質數。 假設 $\zeta = \cos(2\pi/n) + i\sin(2\pi/n)$，即方程 $r^n = 1$ 的複數根，則稱

$$a_0 + a_1\zeta + a_2\zeta + \cdots + a_{n-1}\zeta^{n-1}$$

爲「分圓整數」，其中 a_i 爲整數。 並非每一個分圓整數集合都滿足「唯一分解定理」，但如果能夠加入一個額外的「數」，使到該分圓整數集合滿足「唯一分解定理」，則稱該數爲「理想數」。庫默爾發現，當 n 爲一些特殊的質數時，（他稱之爲「正規質數」，）就可以利用「理想數」來證明「費馬最後定理」在這情況下成立。

由此，庫默爾證明了當 $n < 100$ 時，「費馬最後定理」成立。

德國商人沃爾夫斯凱爾（Paul Friedrich Wolfskehl, 1856 - 1908）在他的遺囑上訂明，如果有人能夠在他死後一百年內證實「費馬最後定理」，則可以獲得十萬馬克的獎金。自此，「費馬最後定理」就吸引到世上不同人仕的注意，不論是數學家或者是業餘學者，都紛紛作出他們的「證明」。在 1909 至 1934 年間，「沃爾夫斯凱爾獎金」的評審委員會，就收到了成千上萬個「證明」，可惜的是當中並沒有一個能夠成立。自從經過了兩次世界大戰之後，該筆獎金的已大幅貶值，「費馬最後定理」的吸引力和熱潮，亦慢慢地降低了。

其實，研究「費馬最後定理」有甚麼好處呢？首先，就是可以滿足人類的求知慾。「費馬最後定理」是一道簡單易明的命題，但是它的證明卻並非一般人所能理解，這已經是一個非常之有趣的事情。其次，在證明該定理的過程之中，我們發現了不少新的數學現象，產生了不少新的數學工具，同時亦豐富了我們對數學，特別是數論的知識。有數學家更認爲，「費馬最後定理」就好像一隻會生金蛋的母雞，由它所衍生出來的數學理論，例如：「唯一分解定理」、「分圓整數」、「理想數」……等等，都是人類思想中最珍貴的產物。

(to be continued next issue)

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *October 1, 1999.*

Problem 86. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$
$$\sqrt{7y}\left(1-\frac{1}{x+y}\right)=4\sqrt{2}.$$

(Source: 1996 Vietnamese Math Olympiad)

Problem 87. Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with an *O*. Then, player II chooses another square and marks it with *X*. They play until one of the players marks a row or a column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning. (*Source: 1995 Israeli Math Olympiad*)

Problem 88. Find all positive integers n such that $3^{n-1}+5^{n-1}$ divides 3^n+5^n. (*Source: 1996 St. Petersburg City Math Olympiad*)

Problem 89. Let *O* and *G* be the circumcenter and centroid of triangle *ABC*, respectively. If *R* is the circumradius and *r* is the inradius of *ABC*, then show that $OG\le\sqrt{R(R-2r)}$. (*Source: 1996 Balkan Math Olympiad*)

Problem 90. There are n parking spaces (numbered 1 to n) along a one-way road down which n drivers $d_1,d_2,\ldots,d_n$ in that order are traveling. Each driver has a favorite parking space and parks there if it is free; otherwise, he parks at the nearest free place down the road. (Two drivers may have the same favorite space.) If there is no free space after his favorite, he drives away. How many lists $a_1,a_2,\ldots,a_n$ of favorite parking spaces are there which permit all of the drivers to park? Here a_i is the favorite parking space number of d_i. (*Source: 1996 St. Petersburg City Math Olympiad*)

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Solutions

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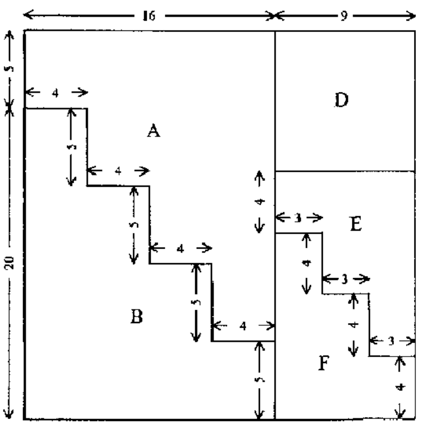
Problem 81. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area. (*Source: 1996 Irish Math Olympiad*)

Solution. SHAM Wang Kei (St. Paul's College, Form 4).

In the following diagram, *A* and *B* can be reassembled to form a 20×20 square and *E* and *F* can be reassembled to form a 12×12 square.



25×25

Other recommended solvers: **CHAN Man Wai** (St. Stephen's Girls' College, Form 4).

Problem 82. Show that if n is an integer greater than 1, then n^4+4^n cannot be a prime number. (*Source: 1977 Jozsef Kurschak Competition in Hungary*).

Solution. Gary NG Ka Wing (STFA Leung Kau Kui College, Form 6) and **NG Lai Ting** (True Light Girls' College, Form 6).

For even n, n^4+4^n is an even integer greater than 2, so it is not a prime. For odd $n>1$, write $n=2k-1$ for a positive integer $k>1$. Then n^4+4^n = $(n^2+2^n)^2-2^{n+1}n^2=(n^2+2^n-2^k n)$ $(n^2+2^n+2^k n)$. Since the smaller factor $n^2+2^n-2^k n=(n-2^{k-1})^2+2^{2k-2}>1, n^4+4^n$ cannot be prime.

Other recommended solvers: **FAN Wai Tong** (St. Mark's School, Form 6), **LAW Ka Ho** (Queen Elizabeth School, Form 6), **SHAM Wang Kei** (St. Paul's College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4) and **TAM Siu Lung** (Queen Elizabeth School, Form 6).

Problem 83. Given an alphabet with three letters *a, b, c*, find the number of words of *n* letters which contain an even number of *a*'s. (*Source: 1996 Italian Math Olympiad*).

Solution I. CHAO Khek Lun Harold (St. Paul's College, Form 4) **and Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 6).

For a nonnegative even integer $2k\le n$, the number of *n* letter words with $2k$ *a*'s is $C^n_{2k}2^{n-2k}$. The answer is the sum of these numbers, which can be simplified to $((2+1)^n+(2-1)^n)/2$ using binomial expansion.

Solution II. TAM Siu Lung (Queen Elizabeth School, Form 6).

Let S_n be the number of *n* letter words with even number of *a*'s and T_n be the number of *n* letter words with odd number of *a*'s. Then $S_n+T_n=3^n$. Among the S_n words, there are T_{n-1} words ended in *a* and $2S_{n-1}$ words ended in *b* or *c*. So we get $S_n=T_{n-1}+2S_{n-1}$. Similarly $T_n=S_{n-1}+2T_{n-1}$. Subtracting these, we get $S_n-T_n=S_{n-1}-T_{n-1}$. So $S_n-T_n=S_1-T_1=2-1=1$. Therefore, $S_n=(3^n+1)/2$.

Problem 84. Let *M* and *N* be the midpoints of sides *AB* and *AC* of ΔABC, respectively. Draw an arbitrary line through *A*. Let *Q* and *R* be the feet of the perpendiculars from *B* and *C* to this line, respectively. Find the locus of the intersection *P* of the lines *QM* and *RN* as the line rotates about *A*.

Solution. CHAO Khek Lun Harold (St. Paul's College, Form 4).

Let *S* be the midpoint of side *BC*. From midpoint theorem, it follows $\angle MSN$ = $\angle BAC$. Since *M* is the midpoint of the hypotenuse of right triangle *AQM*, we get $\angle BAQ=\angle AQM$. Similarly, $\angle CAR$ = $\angle ARN$.

If the line intersects side *BC*, then either $\angle MPN=\angle QPR$ or $\angle MPN+\angle QPR$ = 180°. In the former case, $\angle MPN=180^\circ$ $-\angle PQR-\angle PRQ$ = $180^\circ-\angle AQM$ -

$\angle ARN = 180^\circ - \angle BAC$. So $\angle MPN$ + $\angle MSN = 180^\circ$. Then, *M, N, S, P* are concyclic. In the later case, $\angle MPN$ = $\angle PQR + \angle PRQ$ = $\angle AQM + \angle ARN$ = $\angle BAC$ = $\angle MSN$. So again *M, N, S, P* are concyclic. Similarly, if the line does not intersect side *BC*, there are 2 cases both lead to *M, N, S, P* concyclic. So the locus is on the circumcircle of *M, N, S*.
Conversely, for every point *P* on this circle, draw line *MP* and locate *Q* on line *MP* so that *QM* = *AM*. The line *AQ* is the desired line and *QM, RN* will intersect at *P*.
Comments: The circle through *M, N, S* is the nine point circle of ΔABC. As there are 4 cases to deal with, it may be better to use coordinate geometry.

Other commended solvers: **FAN Wai Tong** (St. Mark's School, Form 6) and **TAM Siu Lung** (Queen Elizabeth School, Form 6).

Problem 85. Starting at $(1,1)$, a stone is moved in the coordinate plane according to the following rules:

(a) Form any point (*a, b*), the stone can be moved to (2*a, b*) or (*a,* 2*b*).

(b) From any point (*a, b*), the stone can be moved to $(a-b, b)$ if $a > b$, or to $(a, b-a)$ if $a < b$.

For which positive integers *x, y*, can the stone be moved to (*x, y*)? (*Source: 1996 German Math Olympiad*)

Solution. Let gcd(*x, y*) be the greatest common divisor (or highest common factor) of *x* and *y*. After rule (a), the gcd either remained the same or doubled. After rule (b), the gcd remain the same. So if (*x, y*) can be reached from (*a, b*), then gcd (*x, y*) = 2^n gcd(*a, b*) for a nonnegative integer *n*. If *a* = *b* = 1, then gcd(*x, y*) = 2^n.

Conversely, suppose gcd(*x, y*) = 2^n. Of those points (*a, b*) from which (*x, y*) can be reached, choose one that minimizes the sum *a* + *b*. If *a* or *b* is even, then (*x, y*) can be reached from (*a*/2, *b*) or (*a, b*/2) with a smaller sum. So *a* and *b* are odd. If *a* > *b* (or *a* < *b*), then (*x, y*) can be reached from ((*a* + *b*)/2, *b*) (or (*a*, (*a* + *b*)/2)) with a smaller sum. So *a* = *b*. Since 2^n = gcd(*x, y*) is divisible by *a* = gcd(*a, b*) and *a* is odd, so *a* = *b* = 1. Then (*x, y*) can be reached from (1, 1).

Olympiad Corner

(continued from page 1)

Problem 4. Determine all pairs (*a, b*) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Problem 5. Let *S* be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of *S* on its circumference, $n-1$ points in its interior and $n-1$ in its exterior. Prove that the number of good circles has the same parity as *n*.

Equation $x^4 + y^4 = z^4$

Recall the following theorem, see *Mathematical Excalibur*, Vol. 1, No. 2, pp. 2, 4 available at the web site

www.math.ust.hk/mathematical_excalibur/

Theorem. If *u, v* are *relatively prime* positive (i.e. *u, v* have no common prime divisor), $u > v$ and one is odd, the other even, then $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$ give a *primitive* solution of $a^2 + b^2 = c^2$ (i.e. a solution where *a, b, c* are relatively prime). Conversely, every primitive solution is of this form, with a possible permutation of *a* and *b*.

Using this theorem, Fermat was able to show $x^4 + y^4 = z^4$ has no positive integral solutions. We will give the details below.

It is enough to show the equation $x^4 + y^4 = w^2$ has no positive integral solutions. Suppose $x^4 + y^4 = w^2$ has positive integral solutions. Let $x = a$, $y = b$, $w = c$ be a positive integral solution with *c* taken to be the least among all such solution. Now *a, b, c* are relatively prime for otherwise we can factor a common prime divisor and reduce *c* to get contradiction. Since $(a^2)^2 + (b^2)^2 = c^2$, by the theorem, there are relatively prime positive integers *u, v* (one is odd, the other even) such that $a^2 = u^2 - v^2$, $b^2 = 2uv$, $c = u^2 + v^2$. Here *u* is odd and *v* is even for otherwise $a^2 \equiv -1 \pmod 4$, which is impossible. Now $a^2 + v^2 = u^2$ and *a, u, v* are relatively prime. By the theorem again, there are relatively prime positive integers *s, t* such that $a = s^2 - t^2$, $v = 2st$, $u = s^2 + t^2$. Now $b^2 = 2uv$ = $4st(s^2 + t^2)$. Since $s^2, t^2, s^2 + t^2$ are relatively prime, we must have $s = e^2$, $t = f^2, s^2 + t^2 = g^2$ for some positive integers *e, f, g*. Then $e^4 + f^4 = g^2$ with $g \le g^2 = s^2 + t^2 = u \le u^2 < c$. This contradicts the choice *c* being least. Therefore, $x^4 + y^4 = w^2$ has no positive integral solutions.

IMO1999

This year the International Mathematical Olympiad will be held in Romania. Based on their performances in qualifying examinations, the following students are selected to be Hong Kong team members:

Chan Ho Leung (Diocesan Boys' School, Form 7)
Chan Kin Hang (Bishop Hall Jubilee School, Form 5)
Chan Tsz Hong (Diocesan Boys' School, Form 7)
Law Ka Ho (Queen Elizabeth School, Form 6)
Ng Ka Wing (STFA Leung Kau Kui College, Form 6)
Wong Chun Wai (Choi Hung Estate Catholic Secondary School, Form 6)

Both Chan Kin Hang and Law Ka Ho were Hong Kong team members last year. This year the team leader is Dr. Tam Ping Kwan (Chinese University of Hong Kong) and the deputy leader will be Miss Luk Mee Lin (La Salle College).

Corrections

In the last issue of the *Mathematical Excalibur*, the definition of *power* given in the article *Power of Points Respect to Circles* should state "The *power* of a point *P* with respect to a circle is the number $d^2 - r^2$ as mentioned above." In particular, the power is positive when the point is outside the circle. The power is 0 when the point is on the circle. The power is negative when the point is inside the circle.

Mathematical Excalibur

Volume 4, Number 5 October 1999 - December 1999

Olympiad Corner

40th International Mathematical Olympiad, July 1999:

Time allowed: 4.5 Hours
Each problem is worth 7 points.

Problem 1. Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

Problem 2. Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \le i < j \le n} x_i x_j \left(x_i^2 + x_j^2\right) \le C\left(\sum_{1 \le i \le n} x_i\right)^4$$

holds for all real numbers $x_1, x_2, \ldots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

Problem 3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are *adjacent* if they have a common side.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is December 15, 1999.

For individual subscription for the next five issues for the 99-00 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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費馬最後定理（二）

梁子傑
香港道教聯合會青松中學

在「數論」的研究之中，有一門分枝不可不提，它就是「橢圓曲線」(Elliptic Curve) $y^2 = x^3 + ax^2 + bx + c$（見 page 2 附錄）。

「橢圓曲線」並非橢圓形，它是計算橢圓周長時的一件「副產品」。但「橢圓曲線」本身卻有著一些非常有趣的數學性質，吸引著數學家的注視。

提到「橢圓曲線」，又不可不提「谷山 － 志村猜想」了。

1954 年，志村五郎 (Goro Shimura) 在東京大學結識了比他太一歲的谷山豊 （Yutaka Taniyama, 1927 - 1958），之後，就開始了二人對「模形式」(modular form) 的研究。「模形式」，起源於法國數學家龐加萊（Henry Poincaré, 1854 - 1912）對「自守函數」的研究。所謂「自守函數」，可以說是「週期函數」的推廣，而「模形式」則可以理解為在複平面上的「週期函數」。

1955 年，谷山開始提出他的驚人猜想。三年後，谷山突然自殺身亡。其後，志村繼續谷山的研究，總結出以下的一個想法：「每條橢圓曲線，都可以對應一個模形式。」之後，人們就稱這猜想為「谷山 – 志村猜想」。

起初，大多數數學家都不相信這個猜想，但經過十多年的反覆檢算後，又沒有理據可以將它推翻。到了 70 年代，相信「谷山 － 志村猜想」的人越來越多，甚至以假定「谷山 － 志村猜想」成立的前提下進行他們的論證。

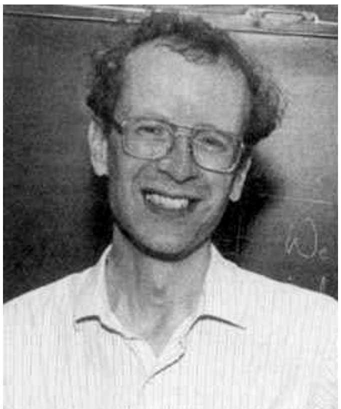
1984 年秋，德國數學家弗賴（Gerhand Frey），在一次數學會議上，提出了以下的觀點：

首先，假設「費馬最後定理」不成立。即能夠發現正整數 A、B、C 和 N，使得 $A^N + B^N = C^N$。於是利用這些數字構作橢圓曲線：$y^2 = x(x - A^N)(x + B^N)$。弗賴發現這條曲線有很多非常特別的性質，特別到不可能對應於任何一個「模形式」！換句話說，弗賴認為：如果「費馬最後定理」不成立，那麼「谷山 – 志村猜想」也是錯的！但倒轉來說，如果「谷山 – 志村猜想」成立，那麼「費馬最後定理」就必定成立！因此，弗賴其實是指出了一條證明「費馬最後定理」的新路徑：這就是去證明「谷山 － 志村猜想」！

可惜的是，弗賴在 1984 年的研究，並未能成功地證實他的觀點。不過，美國數學家里貝特（Kenneth Ribet ），經過多次嘗試後，終於在 1986 年證實了有關的問題。

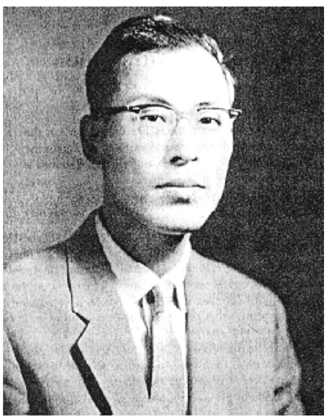
似乎，要證明「費馬最後定理」，現在祇需要證明「谷山 –志村猜想」就可以了。不過自從該猜想

Henri Poincaré



Andrew Wiles

Yutaka Taniyama



Goro Shimura

被提出以來，已經歷過差不多三十年的時間，數學家對這個證明，亦沒有多大的進展。不過，在這時候，英國數學家懷爾斯就開始他偉大而艱巨的工作。

懷爾斯（Andrew Wiles），出生於 1953 年。10 歲已立志要證明「費馬最後定理」。1975 年，開始在劍橋大學進行研究，專攻「橢圓曲線」和「岩澤理論」。在取得博士學位之後，就轉到美國的普林斯頓大學繼續工作。當他知道里貝特證實了弗賴的猜想後，就決定放棄當時手上的所有研究，專心於「谷山－志村猜想」的證明。由於他不想被人騷擾，他更決定要秘密地進行此項工作。

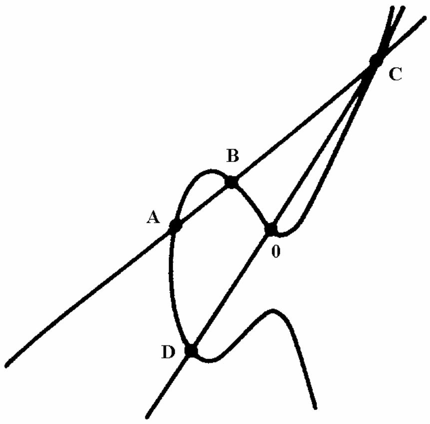
經過了七年的秘密工作後，懷爾斯認為他已證實了「谷山－志村猜想」，並且在 1993 年 6 月 23 日，在劍橋大學的牛頓研究所中，以「模形式、橢圓曲線、伽羅瓦表示論」為題，發表了他對「谷山－志村猜想」重要部份（即「費馬最後定理」）的證明。當日的演講非常成功，「費馬最後定理」經已被證實的消息，很快就傳遍世界。

不過，當懷爾斯將他長達二百頁的證明送給數論專家審閱時，卻發現當中出現漏洞。起初，懷爾斯以為很容易便可以將這個漏洞修補，但事與願遺，到了 1993 年的年底，他承認他的證明出現問題，而且要一段時間才可解決。

到了 1994 年的 9 月，懷爾斯終於突破了證明中的障礙，成功地完成了一項人類史上的創舉，證明了「費馬最後定理」。1995 年 5 月，懷爾斯的證明，發表在雜誌《數學年鑑》之中。到了 1997 年 6 月 27 日，懷爾斯更獲得價值五萬美元的「沃爾夫斯凱爾獎金」，實現了他的童年夢想，正式地結束了這個長達 358 年的數學證明故事。

附錄：橢圓曲線

「橢圓曲線」是滿足方程 $y^2 = x^3 + ax^2 + bx + c$ 的點所組成的曲線，其中 a, b, c 為有理數使 $x^3 + ax^2 + bx + c$ 有不同的根。在曲線上定一個有理點 O。不難證明，當直線穿過兩個曲線上的有理點 A, B 後，該直線必定與曲線再相交於第三個有理點 C。由 C 和 O 再得一點 D 如下圖。我們可以將曲線上的有理點以 $A + B = D$ 為定義看成一個「群」(group)。由於以上性質可以用來解答很多相關的問題，故此「橢圓曲線」就成為數學研究的一個焦點。現時，「橢圓曲線」的理論，主要應用於現代編寫通訊密碼的技術方面。



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Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *December 4, 1999.*

Problem 91. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1-\frac{1}{x+y}\right)=4\sqrt{2}\,.$$

(This is the corrected version of problem 86.)

Problem 92. Let $a_1, a_2, \ldots, a_n$ $(n>3)$ be real numbers such that $a_1 + a_2 + \ldots + a_n \geq n$ and $a_1^2 + a_2^2 + \ldots + a_n^2 \geq n^2$. Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$. (*Source: 1999 USA Math Olympiad*)

Problem 93. Two circles of radii R and r are tangent to line L at points A and B respectively and intersect each other at C and D. Prove that the radius of the circumcircle of triangle ABC does not depend on the length of segment AB. (*Source: 1995 Russian Math Olympiad*)

Problem 94. Determine all pairs (m, n) of positive integers for which $2^m + 3^n$ is a square.

Problem 95. Pieces are placed on an $n\times n$ board. Each piece "attacks" all squares that belong to its row, column, and the northwest-southeast diagonal which contains it. Determine the least number of pieces which are necessary to attack all the squares of the board. (*Source: 1995 Iberoamerican Math Olympiad*)

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Solutions

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Problem 86. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1+\frac{1}{x+y}\right)=4\sqrt{2}\,.$$

(*Source: 1996 Vietnamese Math Olympiad*)

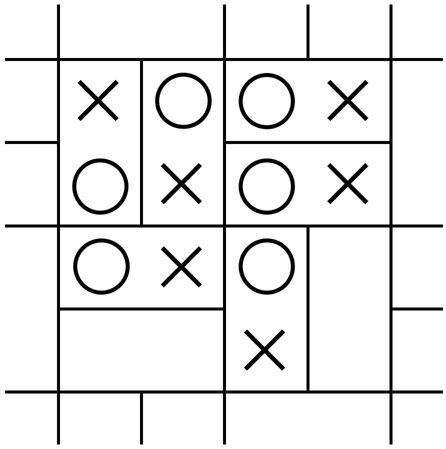
Solution. CHAO Khek Lun Harold (St. Paul's College, Form 5), **FAN Wai Tong Louis** (St. Marks' School, Form 7), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7) and **NG Lai Ting** (True Light Girls' College, Form 7).

Clearly, x and y are nonzero. Dividing the second equation by the first equation, we then simplify to get $y = 24x/7$. So $x + y = 31x/7$. Substituting this into the first equation, we then simplifying, we get $x - (2/\sqrt{3})\sqrt{x} + 7/31 = 0$. Applying the quadratic formula to find $\sqrt{x}$, then squaring, we get $x = (41 \pm 2\sqrt{310})/93$. Then $y = 24x/7 = (328 \pm 16\sqrt{310})/217$, respectively. By direct checking, we see that both pairs (x, y) are solutions.

Other recommended solvers: **CHAN Hiu Fai Philip** (STFA Leung Kau Kui College, Form 6), **CHAN Kwan Chuen** (HKSYC & IA Wong Tai Shan Memorial School, Form 4), **CHUI Man Kei** (STFA Leung Kau Kui College, Form 5), **HO Chung Yu** (HKU), **LAW Siu Lun Jack** (Ming Kei College, Form 5), **LEUNG Yiu Ka** (STFA Leung Kau Kui College, Form 4), **KU Hong Tung** (Carmel Divine Grace Foundation Secondary School, Form 6), **SUEN Yat Chung** (Carmel Divine Grace Foundation Secondary School, Form 6), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 5), **WONG Chi Man** (Valtorta College, Form 5), **WONG Chun Ho Terry** (STFA Leung Kau Kui College, Form 5), **WONG Chung Yin** (STFA Leung Kau Kui College), **WONG Tak Wai Alan** (University of Waterloo, Canada), **WU Man Kin Kenny** (STFA Leung Kau Kui College) and **YUEN Pak Ho** (Queen Elizabeth School, Form 6).

Problem 87. Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with an O. Then, player II chooses another square and marks it with X. They play until one of the players marks a row or a column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning. (*Source: 1995 Israeli Math Olympiad*).

Solution. CHAO Khek Lun Harold (St. Paul's College, Form 5).



Divide the board into 2×2 blocks. Then bisect each 2×2 block into two 1×2 tiles so that for every pair of blocks sharing a common edge, the bisecting segment in one will be horizontal and the other vertical. Since every five consecutive squares on the board contain a tile, after player I choose a square, player II could prevent player I from winning by choosing the other square in the tile.

Problem 88. Find all positive integers n such that $3^{n-1}+5^{n-1}$ divides 3^n+5^n. (*Source: 1996 St. Petersburg City Math Olympiad*).

Solution. CHAO Khek Lun Harold (St. Paul's College, Form 5), **HO Chung Yu** (HKU), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NG Lai Ting** (True Light Girls' College, Form 7), **SHUM Ho Keung** (PLK No.1 W.H. Cheung College, Form 6) and **TSE Ho Pak** (SKH Bishop Mok Sau Tseng Secondary School, Form 5).

For such an n, since

$$3(3^{n-1}+5^{n-1}) < 3^n+5^n < 5(3^{n-1}+5^{n-1}),$$

so $3^n+5^n = 4(3^{n-1}+5^{n-1})$. Cancelling, we get $5^{n-1} = 3^{n-1}$. This forces $n = 1$. Since 2 divides 8, $n = 1$ is the only solution.

Other recommended solvers: **CHAN Hiu Fai Philip** (STFA Leung Kau Kui College, Form 6), **CHAN Kwan Chuen** (HKSYC & IA Wong Tai Shan Memorial School, Form 4), **CHAN Man Wai** (St. Stephen's Girls' College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **HON Chin Wing** (Pui Ching Middle School, Form 5), **LAW**

Siu Lun Jack (Ming Kei College, Form 5), **LEUNG Yiu Ka** (STFA Leung Kau Kui College, Form 4), **NG Ka Chun** (Queen Elizabeth School), **NG Tin Chi** (TWGH Chang Ming Thien College, Form 7), **TAI Kwok Fung** (Carmel Divine Grace Foundation Secondary School, Form 6), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 5), **TSUI Ka Ho Willie** (Hoi Ping Chamber of Commerce Secondary School, Form 6), **WONG Chi Man** (Valtorta College, Form 5), **WONG Chun Ho Terry** (STFA Leung Kau Kui College, Form 5), **WONG Tak Wai Alan** (University of Waterloo, Canada), **YU Ka Lok** (Carmel Divine Grace Foundation Secondary School, Form 6) and **YUEN Pak Ho** (Queen Elizabeth School, Form 6).

Problem 84. Let O and G be the circumcenter and centroid of triangle ABC, respectively. If R is the circumradius and r is the inradius of ABC, then show that $OG \le \sqrt{R(R-2r)}$. *(Source: 1996 Balkan Math Olympiad)*

Solution I. CHAO Khek Lun Harold (St. Paul's College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **NG Lai Ting** (True Light Girls' College, Form 7) and **YUEN Pak Ho** (Queen Elizabeth School, Form 6)

Let line AG intersect side BC at A' and the circumcircle again at A''. Since $\cos BA'A + \cos CA'A = 0,$ we can use the cosine law to get

$$A'A^2 = (2b^2 + 2c^2 - a^2)/4,$$

where *a, b, c* are the usual side lengths of the triangle. By the inter-secting chord theorem,

$$A'A \times A'A'' = A'B \times A'C = a^2/4.$$

Consider the chord through O and G interecting AA'' at G. By the intersecting chord theorem,

$$(R+OG)(R-OG) = GA \times GA''$$
$$= (2A'A/3)(A'A/3 + A'A'')$$
$$= (a^2+b^2+c^2)/9.$$

Then

$$OG = \sqrt{R^2 - (a^2+b^2+c^2)/9}\,.$$

By the AM-GM inequality,

$$(a+b+c)(a^2+b^2+c^2) \ge$$
$$(3\sqrt[3]{abc}\,)(3\sqrt[3]{a^2b^2c^2}\,) = 9abc.$$

Now the area of the triangle is $(ab \sin C)/2 = abc/(4R)$ (by the extended sine law) on one hand and $(a + b + c)r/2$ on the other hand. So, $a + b + c = abc/(2rR)$. Using this, we simplify the inequality to get $(a^2+b^2+c^2)/9 \ge 2rR$. Then

$$\sqrt{R^2-2rR} \ge \sqrt{R^2-(a^2+b^2+c^2)/9}$$
$$= OG.$$

Solution II. NG Lai Ting (True Light Girls' College, Form 7)

Put the origin at the circumcenter. Let z_1, z_2, z_3 be the complex numbers corresponding to *A, B, C*, respectively on the complex plane. Then $OG^2 = \left|(z_1+z_2+z_3)/3\right|^2$. Using $|\omega|^2 = \omega\overline{\omega}$, we can check the right side equals $(3|z_1|^2 + 3|z_2|^2 + 3|z_3|^2 - |z_1-z_2|^2 - |z_2-z_3|^2 - |z_3-z_1|^2)/9$. Since $|z_1| = |z_2| = |z_3| = R$ and $|z_1-z_2| = c$, $|z_2-z_3| = a$, $|z_3-z_1| = b$, we get

$$OG^2 = (9R^2 - a^2 - b^2 - c^2)/9.$$

The rest is as in solution 1.

Problem 90. There are n parking spaces (numbered 1 to n) along a one-way road down which n drivers $d_1, d_2, \ldots, d_n$ in that order are traveling. Each driver has a favorite parking space and parks there if it is free; otherwise, he parks at the nearest free place down the road. (Two drivers may have the same favorite space.) If there is no free space after his favorite, he drives away. How many lists $a_1, a_2, \ldots, a_n$ of favorite parking spaces are there which permit all of the drivers to park? Here a_i is the favorite parking space number of d_i. (*Source: 1996 St. Petersburg City Math Olympiad*).

Solution: Call a list of favorite parking spaces $a_1, a_2, \ldots, a_n$ which permits all drivers to park a *good* list. To each good list, associate the list $b_2, \ldots, b_n$, where b_i is the difference (mod n + 1) between the number a_i and the number of the space driver d_{i-1} took. Note from a_1 and $b_2, \ldots, b_n$, we can reconstruct $a_2, \ldots, a_n$. It follows that different good lists give rise to different lists of b_i's. Since there are n + 1 possible choices for each b_i, there are $(n+1)^{n-1}$ possible lists of $b_2, \ldots, b_n$. For each of these lists of the b_i's, imagine the n parking spaces are arranged in a circle with an extra parking space put at the end. Let d_1 park anywhere temporarily and put $d_i (i > 1)$ in the first available space after the space b_i away from the space taken by d_{i-1}. By shifting the position of d_1, we can ensure the extra parking space is not taken. This implies the corresponding list of $a_1, a_2, \ldots, a_n$ is good. So the number of good lists is $(n+1)^{n-1}$.

Comments: To begin the problem, one could first count the number of good lists in the cases n = 2 and n = 3. This will lead to the answer $(n+1)^{n-1}$. From the $n+1$ factor, it becomes natural to consider an extra parking space. The difficulty is to come up with the *one-to-one correspondence* between the good lists and the b_i lists. For this problem, only one incomplete solution with correct answer and right ideas was sent in by **CHAO Khek Lun Harold** (St. Paul's College, Form 5)

Olympiad Corner

(continued from page 1)

Problem 3. (cont'd) N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.
Determine the smallest possible value of N.

Problem 4. Determine all pairs (n, p) of positive integers such that p is a prime, $n \le 2p$, and $(p-1)^n + 1$ is divisible by n^{p-1}.

Problem 5. Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the centre of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B, respectively. The lines MA and MB meet Γ_1 at C and D, respectively.
Prove that CD is tangent to Γ_2.

Problem 6. Determine all functions $f: \boldsymbol{R} \to \boldsymbol{R}$ such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$ for all $x, y \in \boldsymbol{R}$.