Mathematical Excalibur

Volume 9, Number 1 *January 2004 – April 2004*

Olympiad Corner

The Sixth Hong Kong (China) Mathematical Olympiad took place on December 20, 2003. Here are the problems. Time allowed: 3 hours

Problem 1. Find the greatest real K such that for every positive u, v and w with $u^2 > 4vw$, the inequality

$$(u^2 - 4vw)^2 > K(2v^2 - uw)(2w^2 - uv)$$

holds. Justify your claim.

Problem 2. Let $ABCDEF$ be a regular hexagon of side length 1, and O be the center of the hexagon. In addition to the sides of the hexagon, line segments are drawn from O to each vertex, making a total of twelve unit line segments. Find the number of paths of length 2003 along these line segments that start at O and terminate at O.

Problem 3. Let $ABCD$ be a cyclic quadrilateral. K, L, M, N are the midpoints of sides AB, BC, CD and DA respectively. Prove that the orthocentres of triangles AKN, BKL, CLM, DMN are the vertices of a parallelogram.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance and to Lee Man Fui and Poon Ming Fung for typesetting.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 25, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Geometry via Complex Numbers

Kin Y. Li

Complex numbers are wonderful. In this article we will look at some applications of complex numbers to solving geometry problems. If a problem involves points and chords on a circle, often we can without loss of generality assume it is the unit circle. In the following discussion, we will use the same letter for a point to denote the same complex number in the complex plane. To begin, we will study the equation of lines through points. Suppose Z is an arbitrary point on the line through W_1 and W_2. Since the vector from W_1 to Z is a multiple of the vector from W_1 to W_2, so in terms of complex numbers, we get $Z - W_1 = t(W_2 - W_1)$ for some real t. Now $t = \bar{t}$ and so

$$\frac{Z - W_1}{W_2 - W_1} = \frac{\overline{Z} - \overline{W_1}}{\overline{W_2} - \overline{W_1}}$$

Reversing the steps, we can see that every Z satisfying the equation corresponds to a point on the line through W_1 and W_2. So this is the equation of a line through two points in the complex variable Z.

Next consider the line passing through a point C and perpendicular to the line through W_1 and W_2. Let Z be on this line. Then the vector from C to Z is perpendicular to the vector from W_1 to W_2. In terms of complex numbers, we get $Z - C = it(W_2 - W_1)$ for some real t. So

$$\frac{Z - C}{i(W_2 - W_1)} = \frac{\overline{Z} - \overline{C}}{\overline{i}(\overline{W_2} - \overline{W_1})}.$$

Again reversing steps, we can conclude this is the equation of the line through C perpendicular to the line through W_1 and W_2.

In case the points W_1 and W_2 are on the unit circle, we have $W_1\overline{W_1} = 1 = W_2\overline{W_2}$. Multiplying the numerators and denominators of the right sides of the two displayed equations above by W_1W_2, we can simplify them to

$$Z + W_1W_2\overline{Z} = W_1 + W_2$$

and $$Z - W_1W_2\overline{Z} = C - W_1W_2\overline{C}$$

respectively.

By moving W_2 toward W_1 along the unit circle, in the limit, we will get the equation of the tangent line at W_1 to the unit circle. It is $Z + W_1{}^2\overline{Z} = 2W_1$.

Similarly, the equation of the line through C perpendicular to this tangent line is $Z - W_1{}^2\overline{Z} = C - W_1{}^2\overline{C}$.

For a given triangle $A_1A_2A_3$ with the unit circle as its circumcircle, in terms of complex numbers, its circumcenter is the origin O, its centroid is $G = (A_1 + A_2 + A_3)/3$, its orthocenter is $H = A_1 + A_2 + A_3$ (because $OH = 3OG$) and the center of its nine point circle is $N = (A_1 + A_2 + A_3)/2$ (because N is the midpoint of OH).

Let us proceed to some examples.

Example 1. *(2000 St. Petersburg City Math Olympiad, Problem Corner 188)* The line S is tangent to the circumcircle of acute triangle ABC at B. Let K be the projection of the orthocenter of triangle ABC onto line S (i.e. K is the foot of perpendicular from the orthocenter of triangle ABC to S). Let L be the midpoint of side AC. Show that triangle BKL is isosceles.

Solution. (*Due to POON Ming Fung, STFA Leung Kau Kui College, Form 6*) Without loss of generality, let the circumcircle of triangle ABC be the unit circle on the plane. Let $A = a + bi$, $B = -i$, $C = c + di$. Then the orthocenter is $H = A + B + C$ and $K = (a + c) - i$, $L = (a + c)/2 + (b + d)i/2$. Since

$$LB = \frac{1}{2}\sqrt{(a+c)^2 + (b+d+2)^2} = KL,$$

triangle BKL is isosceles.

Example 2. Consider triangle ABC and its circumcircle S. Reflect the circle with respect to AB, AC and BC to get three new circles S_{AB}, S_{AC} and S_{BC} (with the same radius as S). Show that these three new circles intersect at a common point. Identify this point.

Solution. Without loss of generality, we may assume S is the unit circle. Let the center of S_{AB} be O', then O' is the mirror image of O with respect to the segment AB. So $O' = A + B$ (because segments OO' and AB bisect each other). Similarly, the centers of S_{AC} and S_{BC} are $A + C$ and $B + C$ respectively. We need to show there is a point Z such that Z is on all three new circles, i.e.

$$|Z - (A + B)| = |Z - (A + C)| = |Z - (B + C)| = 1.$$

We easily see that the orthocenter of triangle ABC, namely $Z = H = A + B + C$, satisfies these equations. Therefore, the three new circles intersect at the orthocenter of triangle ABC.

Example 3. A point A is taken inside a circle. For every chord of the circle passing through A, consider the intersection point of the two tangents at the endpoints of the chord. Find the locus of these intersection points.

Solution. Without loss of generality we may assume the circle is the unit circle and A is on the real axis. Let WX be a chord passing through A with W and X on the circle. The intersection point Z of the tangents at W and X satisfies $Z + W^2\overline{Z} = 2W$ and $Z + X^2\overline{Z} = 2X$. Solving these equations together for Z, we find $Z = 2/(\overline{W} + \overline{X})$.

Since A is on the chord WX, the real number A satisfies the equation for line WX, i.e. $A + WXA = W + X$. Using $W\overline{W} = 1 = X\overline{X}$, we see that

$$\operatorname{Re} Z = \frac{1}{\overline{W} + \overline{X}} + \frac{1}{W + X} = \frac{WX + 1}{W + X} = \frac{1}{A}.$$

So the locus lies on the vertical line through $1/A$.
Conversely, for any point Z on this line, draw the two tangents from Z to the unit circle and let them touch the unit circle at the point W and X. Then the above equations are satisfied by reversing the argument. In particular, $A + WXA = W + X$ and so A is on the chord WX. Therefore, the locus is the line perpendicular to OA at a distance $1/OA$ from O.

Example 4. Let A_1, A_2, A_3 be the midpoints of W_2W_3, W_3W_1, W_1W_2 respectively. From A_i drop a perpendicular to the tangent line to the circumcircle of triangle $W_1W_2W_3$ at W_i. Prove that these perpendicular lines are concurrent. Identify this point of concurrency.

Solution. Without loss of generality, let the circumcircle of triangle $W_1W_2W_3$ be the unit circle. The line perpendicular to the tangent at W_1 through $A_1 = (W_2 + W_3)/2$ has equation

$$Z - W_1^2\overline{Z} = \frac{W_2 + W_3}{2} - W_1^2\,\frac{\overline{W_2} + \overline{W_3}}{2}.$$

Using $W_1\overline{W_1} = 1$, we may see that the right side is the same as

$$\frac{W_1 + W_2 + W_3}{2} - W_1^2\,\frac{\overline{W_1} + \overline{W_2} + \overline{W_3}}{2}.$$

From this we see that $N = (W_1 + W_2 + W_3)/2$ satisfies the equation of the line and so N is on the line. Since the expression for N is symmetric with respect to W_1, W_2, W_3, we can conclude that N will also lie on the other two lines. Therefore, the lines concur at N, the center of the nine point circle of triangle $W_1W_2W_3$.

Example 5. *(Simson Line Theorem)* Let W be on the circumcircle of triangle $Z_1Z_2Z_3$ and P, Q, R be the feet of the perpendiculars from W to Z_3Z_1, Z_1Z_2, Z_2Z_3 respectively. Prove that P, Q, R are collinear. (This line is called the *Simson line* of triangle $Z_1Z_2Z_3$ from W.)

Solution. Without loss of generality, we may assume the circumcircle of triangle $Z_1Z_2Z_3$ is the unit circle.
Then $|Z_1| = |Z_2| = |Z_3| = |W| = 1$. Now P is on the line Z_3Z_1 and the line through W perpendicular to Z_3Z_1. So P satisfies the equations $P + Z_1Z_3\overline{P} = Z_1 + Z_3$ and $P - Z_1Z_3\overline{P} = W - Z_1Z_3\overline{W}$. Solving these together for P, we get

$$P = \frac{Z_1 + Z_3 + W - Z_1Z_3\overline{W}}{2}.$$

Similarly,

$$Q = \frac{Z_1 + Z_2 + W - Z_1Z_2\overline{W}}{2}$$

and

$$R = \frac{Z_2 + Z_3 + W - Z_2Z_3\overline{W}}{2}.$$

To show P, Q, R are collinear, it suffices to check that

$$\frac{P - R}{Q - R} = \frac{\overline{P} - \overline{R}}{\overline{Q} - \overline{R}}.$$

Now the right side is

$$\frac{\overline{Z_1} - \overline{Z_2} - \overline{Z_1Z_3}W + \overline{Z_2Z_3}W}{\overline{Z_1} - \overline{Z_3} - \overline{Z_1Z_2}W + \overline{Z_2Z_3}W}.$$

Multiplying the numerator and denominator by $Z_1Z_2Z_3\overline{W}$ and using $Z_i\overline{Z_i} = 1 = W\overline{W}$, we get

$$\frac{Z_2Z_3\overline{W} - Z_1Z_3\overline{W} - Z_2 + Z_1}{Z_2Z_3\overline{W} - Z_1Z_2\overline{W} - Z_3 + Z_1}.$$

This equals the left side $(P - R)/(Q - R)$ and we complete the checking.

Example 6. (*2003 IMO, Problem 4*) Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ meet on AC.

Solution. (*Due to SIU Tsz Hang, 2003 Hong Kong IMO team member*) Without loss of generality, assume A, B, C, D lies on the unit circle and the perpendicular bisector of AC is the real axis. Let $A = \cos\theta + i\sin\theta$, then $C = \overline{A} = \cos\theta - i\sin\theta$ so that $AC = 1$ and $A + C = 2\cos\theta$. Since the bisectors of $\angle ABC$ and $\angle ADC$ pass through the midpoints of the major and minor arc AC, we may assume the bisectors of $\angle ABC$ and $\angle ADC$ pass through 1 and −1 respectively. Let AC intersect the bisector of $\angle ABC$ at Z, then Z satisfies $Z + AC\overline{Z} = A + C$, (which is $Z + \overline{Z} = 2\cos\theta$), and $Z + B\overline{Z} = B + 1$. Solving for Z, we get

$$Z = \frac{2B\cos\theta - B - 1}{B - 1}.$$

Similarly, the intersection point Z' of AC with the bisector of $\angle ADC$ is

$$Z' = \frac{2D\cos\theta + D - 1}{D + 1}.$$

Next, R is on the line AB and the line through D perpendicular to AB. So $R + AB\overline{R} = A + B$ and $R - AB\overline{R} = D - AB\overline{D}$. Solving for R, we find

$$R = \frac{A + B + D - AB\overline{D}}{2}.$$

Similarly,

$$P = \frac{B + C + D - BC\overline{D}}{2}$$

and

$$Q = \frac{C + A + D - CA\overline{D}}{2}.$$

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 25, 2004.***

Problem 196. (*Due to John PANAGEAS, High School "Kaisari", Athens, Greece*) Let $x_1, x_2, ..., x_n$ be positive real numbers with sum equal to 1. Prove that for every positive integer m,

$$n \le n^m (x_1^m + x_2^m + ... + x_n^m).$$

Problem 197. In a rectangular box, the length of the three edges starting at the same vertex are prime numbers. It is also given that the surface area of the box is a power of a prime. Prove that exactly one of the edge lengths is a prime number of the form $2^k - 1$.

Problem 198. In a triangle ABC, $AC = BC$. Given is a point P on side AB such that $\angle ACP = 30^\circ$. In addition, point Q outside the triangle satisfies $\angle CPQ = \angle CPA + \angle APQ = 78^\circ$. Given that all angles of triangles ABC and QPB, measured in degrees, are integers, determine the angles of these two triangles.

Problem 199. Let R^+ denote the positive real numbers. Suppose $f: R^+ \to R^+$ is a strictly decreasing function such that for all $x, y \in R^+$,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that $f(f(x)) = x$ for every $x > 0$. (*Source: 1997 Iranian Math Olympiad*)

Problem 200. Aladdin walked all over the equator in such a way that each moment he either was moving to the west or was moving to the east or applied some magic trick to get to the opposite point of the Earth. We know that he travelled a total distance less than half of the length of the equator altogether during his westward moves. Prove that there was a moment when the difference between the distances he had covered moving to the east and moving to the west was at least half of the length of the equator.

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Solutions

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Due to an editorial mistake in the last issue, solutions to problems 186, 187, 188 by **POON Ming Fung** (*STFA Leung Kau Kui College, Form 6*) were overlooked and his name was not listed among the solvers. We express our apology to him and point out that his clever solution to problem 188 is printed in example 1 of the article "Geometry via Complex Numbers" in this issue.

Problem 191. Solve the equation

$$x^3 - 3x = \sqrt{x+2}.$$

Solution. **Helder Oliveira de CASTRO (**ITA-Aeronautic Institue of Technology, Sao Paulo, Brazil) and **Yufei ZHAO** (Don Mills Collegeiate Institute, Toronoto, Canada, Grade 10).

If $x < -2$, then the right side of the equation is not defined. If $x > 2$, then

$$x^3 - 3x = \frac{x^3 + 3x(x+2)(x-2)}{4} > \frac{x^3}{4} > \sqrt{x+2}.$$

So the solution(s), if any, must be in [-2, 2]. Write $x = 2\cos a$, where $0 \le a \le \pi$. The equation becomes

$$8\cos^3 a - 6\cos a = \sqrt{2\cos a + 2}.$$

Using the triple angle formula on the left side and the half angle formula on the right side, we get

$$2\cos 3a = 2\cos\frac{a}{2} (\ge 0).$$

Then $3a \pm (a/2) = 2n\pi$ for some integer n. Since $3a \pm (a/2) \in [-\pi/2, 7\pi/2]$, we get $n = 0$ or 1. We easily checked that $a = 0$, $4\pi/5$, $4\pi/7$ yield the only solutions $x = 2$, $2\cos(4\pi/5)$, $2\cos(4\pi/7)$.

Other commended solvers: **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 7), **LEE Man Fui** (CUHK, Year 1), **LING Shu Dung**, **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SINN Ming Chun** (STFA Leung Kau Kui College, Form 4), **SIU Ho Chung** (Queen's College, Form 5), **TONG Yiu Wai** (Queen Elizabeth School), **YAU Chi Keung** (CNC Memorial College, Form 7) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Problem 192. Inside a triangle ABC, there is a point P satisfies $\angle PAB = \angle PBC = \angle PCA = \varphi$. If the angles of the triangle are denoted by α, β and γ, prove that

$$\frac{1}{\sin^2\varphi} = \frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta} + \frac{1}{\sin^2\gamma}.$$

Solution. **LEE Tsun Man Clement** (St. Paul's College), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SIU Ho Chung** (Queen's College, Form 5) and **Yufei ZHAO** (Don Mills Collegiate Institute, Tornoto, Canada, Grade 10).

Let AP meet BC at X. Since $\angle XBP = \angle BAX$ and $\angle BXP = \angle AXB$, triangles XPB and XBA are similar. Then $XB/XP = XA/XB$. Using the sine law and the last equation, we have

$$\frac{\sin^2\varphi}{\sin^2\beta} = \frac{\sin^2\angle XAB}{\sin^2\angle XBA} = \frac{XB^2}{XA^2}$$

$$= \frac{XP \cdot XA}{XA^2} = \frac{XP}{XA}$$

Using [ ] to denote area, we have

$$\frac{XP}{XA} = \frac{[XBP]}{[XBA]} = \frac{[XCP]}{[XCA]} = \frac{[BPC]}{[ABC]}$$

Combining the last two equations, we have $\sin^2\varphi/\sin^2\beta = [BPC]/[ABC]$. By similar arguments, we have

$$\frac{\sin^2\varphi}{\sin^2\alpha} + \frac{\sin^2\varphi}{\sin^2\phi} + \frac{\sin^2\varphi}{\sin^2\gamma}$$
$$= \frac{[APB]}{[ABC]} + \frac{[BPC]}{[ABC]} + \frac{[CPA]}{[ABC]}$$
$$= \frac{[ABC]}{[ABC]} = 1$$

The result follows.

Other commended solvers: **CHENG Tsz Chung** (La Salle College, Form 5), **LEE Man Fui** (CUHK, Year 1) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Comments: Professor Murray KLAMKIN (University of Alberta, Edmonton, Canada) informed us that the result $\csc^2\varphi = \csc^2\alpha + \csc^2\beta + \csc^2\gamma$ in the problem is a known relation for the Brocard angle φ of a triangle. Also known is $\cot\varphi = \cot\alpha + \cot\beta + \cot\gamma$. He mentioned these relations and others are given in R.A. Johnson, *Advanced Euclidean Geometry*, Dover, N.Y., 1960, pp. 266-267. (For the convenience of interested readers, the Chinese translation of this book can be found in many bookstore.–Ed) LEE Man Fui and Achilleas PORFYRIADIS gave a proof of the cotangent relation and use it to derive the cosecant relation, which is the equation in the problem, by trigonometric manipulations.

Problem 193. Is there any perfect square, which has the same number of positive divisors of the form $3k + 1$ as of the form $3k + 2$? Give a proof of your answer.

Solution 1. **K.C. CHOW** (Kiangsu-Chekiang College Shatin, Teacher), **LEE Tsun Man Clement** (St. Paul's College), **SIU Ho Chung** (Queen's College, Form 5) and **Yufei ZHAO** (Don Mills Collegiate Institute, Toronto, Canada, Grade 10).

No. For a perfect square m^2, let $m = 3^a b$ with b not divisible by 3. Then $m^2 = 3^{2a}b^2$. Observe that divisors of the form $3k + 1$ or $3k + 2$ for m^2 and for b^2 consist of the same numbers because they cannot have any factor of 3. Since b^2 has an odd number of divisors and they can only be of the form $3k + 1$ or $3k + 2$, so the number of divisors of the form $3k + 1$ cannot be the same as the number of divisors of the form $3k + 2$. Therefore, the same is true for m^2.

Solution 2. **Helder Oliveira de CASTRO** (ITA-Aeronautic Institute of Technology, Sao Paulo, Brazil), **LEE Man Fui** (CUHK, Year 1), **LING Shu Dung, POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Alan T.W. WONG** (Markham, Ontario, Canada) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

No. For a perfect square, its prime factorization is of the form $2^{2e_1}3^{2e_2}5^{2e_3}\cdots$. Let x, y, z be the number of divisors of the form $3k$, $3k + 1$, $3k + 2$ for this perfect square respectively. Then $x + y + z = (2e_1 + 1)(2e_2 + 1)(2e_3 + 1)\cdots$ is odd. Now divisor of the form $3k$ has at least one factor 3, so $x = (2e_1 + 1)(2e_2)(2e_3 + 1)\cdots$ is even. Then $y + z$ is odd. Therefore y cannot equal z.

Other commended solvers: **CHENG Tsz Chung** (La Salle College, Form 5) and **YEUNG Wai Kit** (STFA Leung Kau Kui College).

Problem 194. (*Due to Achilleas Pavlos PORFYRIADIS, American College of Thessaloniki "Anatolia", Thessaloniki, Greece*) A circle with center O is internally tangent to two circles inside it, with centers O_1 and O_2, at points S and T respectively. Suppose the two circles inside intersect at points M, N with N closer to ST. Show that S, N, T are collinear if and only if $SO_1/OO_1 = OO_2/TO_2$.

Solution. **CHENG Tsz Chung** (La Salle College, Form 5), **K. C. CHOW** (Kiangsu-Chekiang College Shatin, Teacher), **Helder Oliveira de CASTRO** (ITA-Aeronautic Institute of Technology, Sao Paulo, Brazil), **LEE Tsun Man** Clement (St. Paul's College), **LING Shu Dung, POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SIU Ho Chung** (Queen's College, Form 5), **YEUNG Wai Kit** (STFA Leung Kau Kui College), **Yufei ZHAO** (Don Mills Collegiate Institute, Toronto, Canada, Grade 10) and the proposer.

If S, N, T are collinear, then triangles SO_1N and SOT are isosceles and share the common angle OST, which imply they are similar. Thus $\angle SO_1N = \angle SOT$ and so lines O_1N and OT are parallel. Similarly, lines O_2N and OS are parallel. Hence, OO_1NO_2 is a parallelogram and $OO_2 = O_1N = O_1S$, $OO_1 = O_2N = O_2T$. Therefore, $SO_1/OO_1 = OO_2/TO_2$. Conversely, if $SO_1/OO_1 = OO_2/TO_2$, then using $OO_1 = OS - O_1S$ and $OO_2 = OT - O_2T$, we get

$$\frac{O_1S}{OS - O_1S} = \frac{OT - O_2T}{O_2T},$$

which reduces to $O_1S + O_2T = OS$. Then $OO_1 = OS - O_1S = O_2T = O_2N$ and $OO_2 = OT - O_2T = O_1S = O_1N$. Hence OO_1NO_2 is again a parallelogram. Then

$$\begin{aligned}&\angle O_1NS + \angle O_1NO_2 + \angle O_2NT \\ &= \angle O_1SN + \angle O_1NO_2 + \angle O_2TN \\ &= \frac{1}{2}\angle OO_1N + \angle O_1NO_2 + \frac{1}{2}\angle OO_2N \\ &= 180^\circ.\end{aligned}$$

Therefore, S, N, T are collinear.

Other commended solver: **TONG Yiu Wai** (Queen Elizabeth School).

Problem 195. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Given n ($n > 3$) points on a plane, no three of them are collinear, x pairs of these points are connected by line segments. Prove that if

$$x \geq \frac{n(n-1)(n-2)+3}{3(n-2)},$$

then there is at least one triangle having these line segments as edges. Find all possible values of integers $n > 3$ such that $\frac{n(n-1)(n-2)+3}{3(n-2)}$ is an integer and the minimum number of line segments guaranteeing a triangle in the above situation is this integer.

Solution. **SIU Ho Chung** (Queen's College, Form 5), **Yufei ZHAO** (Don Mills Collegiate Institute, Toronto, Canada, Grade 10) and the proposer.

For every three distinct points A, B, C, form a pigeonhole containing the three segments AB, BC, CA. (Each segment may be in more than one pigeonholes.) There are C_3^n pigeonholes. For each segment joining a pair of endpoints, that segment will be in $n - 2$ pigeonholes. So if $x(n-2) \geq 2C_3^n + 1$, that is

$$x \geq \frac{2C_3^n + 1}{n-2} = \frac{n(n-1)(n-2)+3}{3(n-2)},$$

then by the pigeonhole principle, there is at least one triangle having these line segments as edges.

If $f(n) = (n(n-1)(n-2) + 3) / (3(n-2))$ is an integer, then $3(n-2)f(n) = n(n-1)(n-2) + 3$ implies 3 is divisible by $n - 2$. Since $n > 3$, we must have $n = 5$. Then $f(5) = 7$. For the five vertices A, B, C, D, E of a regular pentagon, if we connected the six segments BC, CD, DE, EA, AC, BE, then there is no triangle. So a minimum of $f(5) = 7$ segments is needed to get a triangle.

Other commended solvers: **K. C. CHOW** (Kiangsu-Chekiang College Shatin, Teacher) and **POON Ming Fung** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 4. Find, with reasons, all integers a, b, and c such that

$$\frac{1}{2}(a+b)(b+c)(c+a) + (a+b+c)^3 = 1 - abc.$$

Geometry via Complex Numbers

(continued from page 2)

By Simson's theorem, P, Q, R are collinear. So $PQ = QR$ if and only if $Q = (P + R)/2$. In terms of A, B, C, D, this may be simplified to

$$C + A - 2B = (2CA - AB - BC)\overline{D}.$$

In terms of B, D, θ, this is equivalent to $(2\cos\theta - 2B)D = 2 - 2B\cos\theta$. This is easily checked to be the same as

$$\frac{2\cos\theta - B - 1}{B - 1} = \frac{2D\cos\theta + D - 1}{D + 1},$$

i.e. $Z = Z'$.

Comments: The official solution by pure geometry is shorter, but it takes a fair amount of time and cleverness to discover. Using complex numbers as above reduces the problem to straight computations.

Mathematical Excalibur

Volume 9, Number 2 *May 2004 – July 2004*

Olympiad Corner

The XVI Asian Pacific Mathematical Olympiad took place on March 2004. Here are the problems. Time allowed: 4 hours.

Problem 1. Determine all finite nonempty sets S of positive integers satisfying

$\frac{i+j}{(i,j)}$ is an element of S for all i, j in S,

where (i, j) is the greatest common divisor of i and j.

Problem 2. Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, COH is equal to the sum of the areas of the other two.

Problem 3. Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\mathcal{L}$ denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to color the points of S with at most two colors, such that for any points p, q of S, the number

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 9, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Inversion

Kin Y. Li

In algebra, the method of logarithm *transforms* tough problems involving multiplications and divisions into simpler problems involving additions and subtractions. For every positive number x, there is a unique real number $\log x$ in base 10. This is a one-to-one correspondence between the positive numbers and the real numbers.

In geometry, there are also transformation methods for solving problems. In this article, we will discuss one such method called *inversion*. To present this, we will introduce the *extended plane*, which is the plane together with a point that we would like to think of as infinity. Also, we would like to think of *all* lines on the plane will go through *this point at infinity*! To understand this, we will introduce the *stereographic projection*, which can be described as follow.

Consider a sphere sitting on a point O of a plane. If we remove the north pole N of the sphere, we get a *punctured* sphere. For every point P on the plane, the line NP will intersect the punctured sphere at a *unique* point S_P. So this gives a one-to-one correspondence between the plane and the punctured sphere. If we consider the points P on a circle in the plane, then the S_P points will form a circle on the punctured sphere. However, if we consider the points P on *any* line in the plane, then the S_P points will form a punctured circle on the sphere with N as the point removed from the circle. *If we move a point P on any line on the plane toward infinity, then S_P will go toward the same point N!* Thus, in this model, all lines can be thought of as going to the same infinity.

Now for the method of inversion, let O be a point on the plane and r be a positive number. The *inversion* with center O and radius r is the function on the extended plane that sends a point $X \neq O$ to the *image* point X' on the ray $\overrightarrow{OX}$ such that

$$OX \cdot OX' = r^2.$$

When $X = O$, X' is taken to be the point at infinity. When X is infinity, X' is taken to be O. The circle with center O and radius r is called the *circle of inversion*.

The method of inversion is based on the following facts.

(1) The function sending X to X' described above is a one-to-one correspondence between the extended plane with itself. (This follows from checking $(X')' = X$.)

(2) If X is on the circle of inversion, then $X' = X$. If X is outside the circle of inversion, then X' is the midpoint of the chord formed by the tangent points T_1, T_2 of the tangent lines from X to the circle of inversion. (This follows from

$$OX \cdot OX' = (r \sec \angle T_1OX)(r \cos \angle T_1OX) = r^2.\ )$$

(3) A circle not passing through O is sent to a circle not passing through O. In this case, the images of concyclic points are concyclic. The point O, the centers of the circle and the image circle are collinear. *However, the center of the circle is not sent to the center of the image circle!*

(4) A circle passing through O is sent to a line which is not passing through O and is parallel to the tangent line to the circle at O. Conversely, a line not passing through O is sent to a circle passing through O with the tangent line at O parallel to the line.

(5) A line passing through O is sent to itself.

(6) If two curves intersect at a certain angle at a point $P \neq O$, then the image curves will also intersect at the same angle at P'. If the angle is a right angle, the curves are said to be orthogonal. So in particular, orthogonal curves at P are sent to orthogonal curves at P'. A circle orthogonal to the circle of inversion is sent to itself. Tangent curves at P are sent to tangent curves at P'.

(7) If points A, B are different from O and points O, A, B are not collinear, then the equation $OA \cdot OA' = r^2 = OB \cdot OB'$ implies $OA/OB = OB'/OA'$. Along with $\angle AOB = \angle B'OA'$, they imply ΔOAB, $\Delta OB'A'$ are similar. Then

$$\frac{A'B'}{AB} = \frac{OA'}{OB} = \frac{r^2}{OA \cdot OB}$$

so that

$$A'B' = \frac{r^2}{OA \cdot OB} AB .$$

The following are some examples that illustrate the powerful method of inversion. In each example, when we do inversion, it is often that we take the point that plays the most significant role and where many circles and lines intersect.

***Example* 1**. (*Ptolemy's Theorem*) For coplanar points A, B, C, D, if they are concyclic, then

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

Solution. Consider the inversion with center D and any radius r. By fact (4), the circumcircle of ΔABC is sent to the line through A', B', C'. Since $A'B' + B'C' = A'C'$, we have by fact (7) that

$$\frac{r^2}{AD \cdot BD} AB + \frac{r^2}{BD \cdot CD} BC = \frac{r^2}{AD \cdot CD} AC.$$

Multiplying by $(AD \cdot BD \cdot CD)/r^2$, we get the desired equation.

Remarks. The steps can be reversed to get the converse statement that if

$$AB \cdot CD + AD \cdot BC = AC \cdot BD,$$

then A,B,C,D are concyclic.

***Example* 2**. (*1993 USAMO*) Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let O be their intersection point. Prove that the reflections of O across AB, BC, CD, DA are concyclic.

Solution. Let P, Q, R, S be the feet of perpendiculars from O to AB, BC, CD, DA, respectively. The problem is equivalent to showing P, Q, R, S are concyclic (since they are the midpoints of O to its reflections). Note $OSAP$, $OPBQ$, $OQCR$, $ORDS$ are cyclic quadrilaterals. Let their circumcircles be called C_A, C_B, C_C, C_D, respectively.

Consider the inversion with center O and any radius r. By fact (5), lines AC and BD are sent to themselves. By fact (4), circle C_A is sent to a line L_A parallel to BD, circle C_B is sent to a line L_B parallel to AC, circle C_C is sent to a line L_C parallel to BD, circle C_D is sent to a line L_D parallel to AC.

Next C_A intersects C_B at O and P. This implies L_A intersects L_B at P'. Similarly, L_B intersects L_C at Q', L_C intersects L_D at R' and L_D intersects L_A at S'.

Since $AC \perp BD$, $P'Q'R'S'$ is a rectangle, hence cyclic. Therefore, by fact (3), P, Q, R, S are concyclic.

***Example* 3.** (*1996 IMO*) Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

Solution. Let lines AP, BD intersect at X, lines AP, CE intersect at Y. We have to show $X = Y$. By the angle bisector theorem, $BA/BP = XA/XP$. Similarly, $CA/CP = YA/YP$. As X, Y are on AP, we get $X = Y$ if and only if $BA/BP = CA/CP$.

Consider the inversion with center A and any radius r. By fact (7), ΔABC, $\Delta AC'B'$ are similar, ΔAPB, $\Delta AB'P'$ are similar and ΔAPC, $\Delta AC'P'$ are similar. Now

$$\begin{aligned}\angle B'C'P' &= \angle AC'P' - \angle AC'B' \\ &= \angle APC - \angle ABC \\ &= \angle APB - \angle ACB \\ &= \angle AB'P - \angle AB'C' \\ &= \angle C'B'P'.\end{aligned}$$

So $\Delta B'C'P'$ is isosceles and $P'B' = P'C'$. From ΔAPB, $\Delta AB'P'$ similar and ΔAPC, $\Delta AC'P'$ similar, we get

$$\frac{BA}{BP} = \frac{P'A}{P'B'} = \frac{P'A}{P'C'} = \frac{CA}{CP}.$$

Therefore, $X = Y$.

***Example* 4.** (*1995 Israeli Math Olympiad*) Let PQ be the diameter of semicircle H. Circle O is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that $AB \perp PQ$ and is tangent to O. Prove that AC bisects $\angle PAB$.

Solution. Consider the inversion with center C and any radius r. By fact (7), ΔCAP, $\Delta CP'A'$ similar and ΔCAB, $\Delta CB'A'$ similar. So AC bisects PAB if and only if $\angle CAP = \angle CAB$ if and only if $\angle CP'A' = \angle CB'A'$.

By fact (5), line PQ is sent to itself. Since circle O passes through C, circle O is sent to a line O' parallel to PQ. By fact (6), since H is tangent to circle O and is orthogonal to line PQ, H is sent to the semicircle H' tangent to line O' and has diameter $P'Q'$. Since segment AB is tangent to circle O and is orthogonal to PQ, segment AB is sent to arc $A'B'$ on the semicircle tangent to line O' and has diameter CB'. Now observe that arc $A'Q'$ and arc $A'C$ are symmetrical with respect to the perpendicular bisector of CQ' so we get $\angle CP'A' = \angle CB'A'$.

In the solutions of the next two examples, we will consider the nine-point circle and the Euler line of a triangle. Please consult Vol. 3, No. 1 of Mathematical Excalibur for discussion if necessary.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for submitting solutions is ***August 9, 2004.***

Problem 201. (*Due to Abderrahim OUARDINI, Talence, France*) Find which nonright triangles ABC satisfy

$$\tan A \tan B \tan C > [\tan A] + [\tan B] + [\tan C],$$

where $[t]$ denotes the greatest integer less than or equal to t. Give a proof.

Problem 202. (*Due to LUK Mee Lin, La Salle College*) For triangle ABC, let D, E, F be the midpoints of sides AB, BC, CA, respectively. Determine which triangles ABC have the property that triangles ADF, BED, CFE can be folded above the plane of triangle DEF to form a tetrahedron with AD coincides with BD; BE coincides with CE; CF coincides with AF.

Problem 203. (*Due to José Luis DÍAZ-BARRERO, Universitat Politecnica de Catalunya, Barcelona, Spain*) Let a, b and c be real numbers such that $a + b + c \neq 0$. Prove that the equation

$$(a+b+c)x^2 + 2(ab+bc+ca)x + 3abc = 0$$

has only real roots.

Problem 204. Let n be an integer with $n > 4$. Prove that for every n distinct integers taken from 1, 2, …, $2n$, there always exist two numbers whose least common multiple is at most $3n + 6$.

Problem 205. *(Due to HA Duy Hung, Hanoi University of Education, Vietnam)* Let a, n be integers, both greater than 1, such that $a^n - 1$ is divisible by n. Prove that the greatest common divisor (or highest common factor) of $a - 1$ and n is greater than 1.

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Solutions

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Problem 196. (*Due to John PANAGEAS, High School "Kaisari", Athens, Greece*) Let $x_1, x_2, ..., x_n$ be positive real numbers with sum equal to 1. Prove that for every positive integer m,

$$n \leq n^m (x_1^m + x_2^m + ... + x_n^m).$$

Solution. **CHENG Tsz Chung** (La Salle College, Form 5), **Johann Peter Gustav Lejeune DIRICHLET** (Universidade de Sao Paulo – Campus Sao Carlos), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 6), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Ho Chung** (Queen's College, Form 5) and **YU Hok Kan** (STFA Leung Kau Kui College, Form 6).

Applying Jensen's inequality to $f(x) = x^m$ on [0, 1] or the power mean inequality, we have

$$\left(\frac{x_1 + \cdots + x_n}{n}\right)^m \leq \frac{x_1^m + \cdots + x_n^m}{n}.$$

Using $x_1 + \cdots + x_n = 1$ and multiplying both sides by n^{m+1}, we get the desired inequality.

Other commended solvers: **TONG Yiu Wai** (Queen Elizabeth School, Form 6), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 3) and **YEUNG Yuen Chuen** (La Salle College, Form 4).

Problem 197. In a rectangular box, the lengths of the three edges starting at the same vertex are prime numbers. It is also given that the surface area of the box is a power of a prime. Prove that exactly one of the edge lengths is a prime number of the form $2^k - 1$. (*Source: KöMaL Gy.3281*)

Solution. **CHAN Ka Lok** (STFA Leung Kau Kui College, Form 4), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 6), **John PANAGEAS** (Kaisari High School, Athens, Greece), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Ho Chung** (Queen's College, Form 5), **TO Ping Leung** (St. Peter's Secondary School), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 3), **YEUNG Yuen Chuen** (La Salle College, Form 4) and **YU Hok Kan** (STFA Leung Kau Kui College, Form 6).

Let the prime numbers x, y, z be the lengths of the three edges starting at the same vertex. Then $2(xy + yz + zx) = p^n$ for some prime p and positive integer n. Since the left side is even, we get $p = 2$. So $xy + yz + zx = 2^{n-1}$. Since x, y, z are at least 2, the left side is at least 12, so n is at least 5. If none or exactly one of x, y, z is even, then $xy + yz + zx$ would be odd, a contradiction. So at least two of x, y, z are even and prime, say $x = y = 2$. Then $z = 2^{n-3} - 1$. The result follows.

Other commended solvers: **NGOO Hung Wing** (Valtorta College).

Problem 198. In a triangle ABC, $AC = BC$. Given is a point P on side AB such that $\angle ACP = 30°$. In addition, point Q outside the triangle satisfies $\angle CPQ = \angle CPA + \angle APQ = 78°$. Given that all angles of triangles ABC and QPB, measured in degrees, are integers, determine the angles of these two triangles. (*Source: KöMaL C. 524*)

Solution. **CHAN On Ting Ellen** (True Light Girls' College, Form 4), **CHENG Tsz Chung** (La Salle College, Form 5), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **TONG Yiu Wai** (Queen Elizabeth School, Form 6), **YEUNG Yuen Chuen** (La Salle College, Form 4) and **YU Hok Kan** (STFA Leung Kau Kui College, Form 6).

As $\angle ACB > \angle ACP = 30°$, we get

$$\angle CAB = \angle CBA < (180° - 30°)/2 = 75°.$$

Hence $\angle CAB \leq 74°$. Then

$$\angle CPB = \angle CAB + \angle ACP \leq 74° + 30° = 104°.$$

Now

$$\angle QPB = 360° - \angle QPC - \angle CPB \geq 360° - 78° - 104° = 178°.$$

Since the angles of triangle QPB are positive integers, we must have

$$\angle QPB = 178°, \angle PBQ = 1° = \angle PQB$$

and all less-than-or-equal signs must be equalities so that

$$\angle CAB = \angle CBA = 74° \text{ and } \angle ACB = 32°.$$

Other commended solvers: **CHAN Ka Lok** (STFA Leung Kau Kui College, Form 4), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Ho Chung** (Queen's College, Form 5), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 3), **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 6) and **YIP Kai Shing** (STFA Leung Kau Kui College, Form 4).

Problem 199. Let R^+ denote the positive real numbers. Suppose $f: R^+ \to R^+$ is a strictly decreasing function such that for all $x, y \in R^+$,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that $f(f(x)) = x$ for every $x > 0$. (*Source: 1997 Iranian Math Olympiad*)

Solution. **Johann Peter Gustav Lejeune DIRICHLET** (Universidade de Sao Paulo – Campus Sao Carlos) and **Achilleas P. PORFYRIADIS** (American College of

Thessaloniki "Anatolia", Thessaloniki, Greece).

Setting $y = x$ gives

$$f(2x) + f(2f(x)) = f(2f(x + f(x))).$$

Setting both x and y to $f(x)$ in the given equation gives

$$f(2f(x)) + f(2f(f(x))) \\ = f(2f(f(x) + f(f(x)))).$$

Subtracting this equation from the one above gives

$$f(2f(f(x))) - f(2x) = f(2f(f(x) + f(f(x)))) \\ - f(2f(x + f(x))).$$

Assume $f(f(x)) > x$. Then $2f(f(x)) > 2x$. Since f is strictly decreasing , we have $f(2f(f(x))) < f(2x)$. This implies the left side of the last displayed equation is negative. Hence,

$$f(2f(f(x) + f(f(x)))) < f(2f(x + f(x))).$$

Again using f strictly decreasing, this inequality implies

$$2f(f(x) + f(f(x))) > 2f(x + f(x)),$$

which further implies

$$f(x) + f(f(x)) < x + f(x).$$

Canceling $f(x)$ from both sides leads to the contradiction that $f(f(x)) < x$.

Similarly, $f(f(x)) < x$ would also lead to a contradiction as can be seen by reversing all inequality signs above. Therefore, we must have $f(f(x)) = x$.

Problem 200. Aladdin walked all over the equator in such a way that each moment he either was moving to the west or was moving to the east or applied some magic trick to get to the opposite point of the Earth. We know that he travelled a total distance less than half of the length of the equator altogether during his westward moves. Prove that there was a moment when the difference between the distances he had covered moving to the east and moving to the west was at least half of the length of the equator. (*Source: KöMaL F. 3214*)

Solution.

Let us abbreviate Aladdin by A. At every moment let us consider a twin, say $\forall$, of A located at the opposite point of the position of A. *Now draw the equator circle*. Observe that at every moment either both are moving east or both are moving west. Combining the movement swept out by A and $\forall$, we get two continuous paths on the equator. At the same moment, each point in one path will have its opposite point in the other path.

Let N be the initial point of A in his travel and let $P(N)$ denote the path beginning with N. Let W be the westernmost point on $P(N)$. Let N' and W' be the opposite points of N and W respectively. By the westward travel condition on A, W cannot be as far as N'.

Assume the conclusion of the problem is false. Then the easternmost point reached by $P(N)$ cannot be as far as N'. So $P(N)$ will not cover the inside of minor arc WN' and the other path will not cover the inside of minor arc $W'N$. Since A have walked over all points of the equator (and hence A and $\forall$ together walked every point at least twice), $P(N)$ must have covered every point of the minor arc $W'N$ at least twice. Since $P(N)$ cannot cover the entire equator, every point of minor arc $W'N$ must be traveled westward at least once by A or $\forall$. Then A travelled westward at least a distance equal to the sum of lengths of minor arcs $W'N$ and NW, i.e. half of the equator. We got a contradiction.

Other commended solvers: **POON Ming Fung** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

of lines in $\mathscr{L}$ which separate p from q is odd if and only if p and q have the same color.

Note: A line ℓ separates two points p and q if p and q lie on opposite sides of ℓ with neither point on ℓ.

Problem 4. For a real number x, let $\lfloor x \rfloor$ stand for the largest integer that is less than or equal to x. Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n.

Problem 5. Prove that

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$$

for all real numbers a, b, $c > 0$.

Inversion

(continued from page 2)

***Example* 5.** (*1995 Russian Math Olympiad*) Given a semicircle with diameter AB and center O and a line, which intersects the semicircle at C and D and line AB at M ($MB < MA$, $MD < MC$). Let K be the second point of intersection of the circumcircles of triangles AOC and DOB. Prove that $\angle MKO = 90^\circ$.

Solution. Consider the inversion with center O and radius $r = OA$. By fact (2), A, B, C, D are sent to themselves. By fact (4), the circle through A, O, C is sent to line AC and the circle through D, O, B is sent to line DB. Hence, the point K is sent to the intersection K' of lines AC with DB and the point M is sent to the intersection M' of line AB with the circumcircle of ΔOCD. Then the line MK is sent to the circumcircle of $OM'K'$.

To solve the problem, note by fact (7), $\angle MKO=90^\circ$ if and only if $\angle K'M'O= 90^\circ$.

Since $BC\perp AK'$, $AD\perp BK'$ and O is the midpoint of AB, so the circumcircle of ΔOCD is the nine-point circle of $\Delta ABK'$, which intersects side AB again at the foot of perpendicular from K' to AB. This point is M'. So $\angle K'M'O = 90^\circ$ and we are done.

***Example* 6.** (*1995 Iranian Math Olympiad*) Let M, N and P be points of intersection of the incircle of triangle ABC with sides AB, BC and CA respectively. Prove that the orthocenter of ΔMNP, the incenter of ΔABC and the circumcenter of ΔABC are collinear.

Solution. Note the incircle of ΔABC is the circumcircle of ΔMNP. So the first two points are on the Euler line of ΔMNP.

Consider inversion with respect to the incircle of ΔABC with center I. By fact (2), A, B, C are sent to the midpoints A', B', C' of PM, MN, NP, respectively. The circumcenter of $\Delta A'B'C'$ is the center of the nine point circle of ΔMNP, which is on the Euler line of ΔMNP. By fact (3), the circumcircle of ΔABC is also on the Euler line of ΔMNP.

Mathematical Excalibur

Volume 9, Number 3 *August 2004 – September 2004*

Olympiad Corner

The 45th International Mathematical Olympiad took place on July 2004. Here are the problems.

Day 1 *Time allowed: 4 hours 30 minutes.*

Problem 1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of the side BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

Problem 2. Find all polynomials $P(x)$ with real coefficients which satisfy the equality

$$P(a-b)+P(b-c)+P(c-a) = 2P(a+b+c)$$

for all real numbers a, b, c such that $ab + bc + ca = 0$.

Problem 3. Define a *hook* to be a figure made up of six unit squares as shown in the diagram

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 20, 2004***.

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IMO 2004

T. W. Leung

The 45th International Mathematical Olympiad (IMO) was held in Greece from July 4 to July 18. Since 1988, we have been participating in the Olympiads. This year our team was composed as follows.

<u>*Members*</u>

Cheung Yun Kuen (Hong Kong Chinese Women's Club College)

Chung Tat Chi (Queen Elizabeth School)

Kwok Tsz Chiu (Yuen Long Merchant Association Secondary School)

Poon Ming Fung (STFA Leung Kau Kui College)

Tang Chiu Fai (HKTA Tang Hin Memorial Secondary School)

Wong Hon Yin (Queen's College)

Cesar Jose Alaban (Deputy Leader)

Leung Tat Wing (Leader)

I arrived at Athens on July 6. After waiting for a couple of hours, leaders were then delivered to Delphi, a hilly town 170 km from the airport, corresponding to 3 more hours of journey. In these days the Greeks were still ecstatic about what they had achieved in the Euro 2004, and were busy preparing for the coming Olympic Games in August. Of course Greece is a small country full of legend and mythology. Throughout the trip, I also heard many times that they were the originators of democracy, their contribution in the development of human body and mind and their emphasis on fair play.

After receiving the short-listed problems leaders were busy studying them on the night of July 8. However obviously some leaders had strong opinions on the beauty and degree of difficulty of the problems, so selections of all six problems were done in one day. Several problems were not even discussed in details of their own merits.

The following days were spent on refining the wordings of the questions and translating the problems into different languages.

The opening ceremony was held on July 11. In the early afternoon we were delivered to Athens. After three hours of ceremony we were sent back to Delphi. By the time we were in Delphi it was already midnight. Leaders were not allowed to talk to students in the ceremony.

Contests were held in the next two days. The days following the contests were spent on coordination, i.e. leaders and coordinators discussed how many points should be awarded to the answers of the students. This year the coordinators were in general very careful. I heard several teams spent more than three hours to go over six questions. Luckily coordination was completed on the afternoon of July 15. The final Jury meeting was held that night. In the meeting the cut-off scores were decided, namely 32 points for gold, 24 for silver and 16 for bronze. Our team was therefore able to obtain two silver medals (Kwok and Chung) and two bronze medals (Tang and Cheung). Other members (Poon and Wong) both solved at least one problem completely, thus received honorable mention. Unofficially our team ranked 30 out of 85. The top five teams in order were respectively China, USA, Russia, Vietnam and Bulgaria.

In retrospect I felt that our team was good and balanced, none of the members was particularly weak. In one problem we were as good as any strong team. Every team members solved problem 4 completely. Should we did better in the geometry problems our rank would be much higher. Curiously geometry is in our formal school curriculum while number theory and combinatorics are not. In this Olympiad we had two geometry problems, but fittingly so, after all, it was Greece.

Extending an IMO Problem

Hà Duy Hung

Dept. of Math and Informatics
Hanoi Univ. of Education

In this brief note we give a generalization of a problem in the 41st International Mathematical Olympiad held in Taejon, South Korea in 2000.

IMO 2000/5. Determine whether or not there exists a positive integer n such that n is divisible by exactly 2000 different prime divisors, and 2^n+1 is divisible by n.

The answer to the question is positive. This intriguing problem made me recall a well-known theorem due to O. Reutter in [1] as follows.

Theorem 1. If a is a positive integer such that $a+1$ is not a power of 2, then a^n+1 is divisible by n for infinitely many positive integers n.

We frequently encounter the theorem in the case $a=2$. The theorem and the IMO problem prompted me to think of more general problem. Can we replace the number 2 in the IMO problem by other positive integers? The difficulty partly lies in the fact that the two original problems are solved independently. After a long time, I finally managed to prove a generalization as follows.

Theorem 2. Let s, a, b be given positive integers, such that a, b are relatively prime and $a+b$ is not a power of 2. Then there exist infinitely many positive integers n such that

- n has exactly s different prime divisors; and
- a^n+b^n is divisible by n.

We give a proof of Theorem 2 below. We shall make use of two familiar lemmas.

Lemma 1. Let n be an odd positive integer, and a, b be relative prime positive integers. Then

$$\frac{a^n+b^n}{a+b}$$

is an odd integer ≥ 1, equality if and only if $n=1$ or $a=b=1$.

The proof of Lemma 1 is simple and is left for the reader.

Also, we remind readers the usual notations $r \mid s$ means s is divisible by r and $u \equiv v \pmod m$ means $u-v$ is divisible by m.

Lemma 2. Let a, b be distinct and relatively prime positive integers, and p an odd prime number which divides $a+b$. Then for any non-negative integer k,

$$p^{k+1} \mid a^m+b^m,$$

where $m=p^k$.

Proof. We prove the lemma by induction. It is clear that the lemma holds for $k=0$. Suppose the lemma holds for some non-negative integer k, and we proceed to the case $k+1$.

Let $x=a^{p^k}$ and $y=b^{p^k}$. Since

$$x^p+y^p=(x+y)\sum_{i=0}^{p-1}(-1)^i x^{p-1-i}y^i,$$

it suffices to show that the whole summation is divisible by p. Since $x \equiv -y \pmod{p^{k+1}}$, we have

$$\begin{aligned}&\sum_{i=0}^{p-1}(-1)^i x^{p-1-i}y^i\\ &\equiv \sum_{i=0}^{p-1}(-1)^{2i}x^{p-1}\\ &\equiv px^{p-1} \pmod{p^{k+1}}\end{aligned}$$

completing the proof.

In the rest of this note we shall complete the proof of Theorem 2.

Proof of Theorem 2. Without loss of generality, let $a>b$. Since $a+b$ is not a power of 2, it has an odd prime factor p. For natural number k, set

$$x_k=a^{p^k}+b^{p^k},\ y_k=\frac{x_{k+1}}{x_k}.$$

Then y_k is a positive integer and

$$\begin{aligned}y_k&=\sum_{i=0}^{p-1}(-1)^i(a^{p^k})^{p-1-i}(b^{p^k})^i\\ &\equiv \sum_{i=0}^{p-1}(-1)^{2i}(a^{p^k})^{p-1}\\ &\equiv px^{p-1} \pmod{p^{k+1}}\end{aligned}$$

which implies that $\frac{y_k}{p}$ is a positive integer. Also, we have

$$\frac{y_k}{p} \equiv b^{p^k(p-1)} \quad \left(\operatorname{mod} \frac{x_k}{p}\right),$$

so that

$$\gcd\left(\frac{x_k}{p},\frac{y_k}{p}\right)=1$$

for $k=1, 2, \ldots$ By Lemma 2, we also have

$$\gcd\left(\frac{y_k}{p},p^k\right)=1$$

for $k=1, 2, \ldots$. Moreover, we have $x_k \ge p^k$. This leads us to

$$\begin{aligned}y_k&=b^{p^k(p-1)}+\sum_{i=1}^{\frac{p-1}{2}}[(a^{p^k})^{2i}(b^{p^k})^{p-1-2i}\\ &\qquad -(a^{p^k})^{2i-1}(b^{p^k})^{p-2i}]\\ &>b^{p^k}+a^{p^k}\\ &=x_k\\ &\ge p^{k+1}\end{aligned}$$

It follows that

$$\frac{y_k}{p} \ge p^k >1.$$

By Lemma 1, $\frac{y_k}{p}$ is an odd positive integer, so we can choose an odd prime divisor q_k of $\frac{y_k}{p}$.

We now have a sequence of odd prime numbers $\{q_k\}_{k=1}^{+\infty}$ satisfying the following properties

(i) $\gcd(x_k, q_k)=1$

(ii) $\gcd(p, q_k)=1$

(iii) $q_k \mid x_{k+1}$

(iv) $x_k \mid x_{k+1}$.

We shall now show that the sequence $\{q_k\}_{k=1}^{+\infty}$ consists of distinct prime numbers and is thus infinite. Indeed, if $k_0<k_1$ are positive integers and $q_{k_0}=q_{k_1}$, then

$$q_{k_1}=q_{k_0} \mid x_{k_0+1} \mid \cdots \mid x_{k_1}$$

by properties (iii) and (iv). But this contradicts property (i).

Next, set $n_0=p^s q_1 \ldots q_{s-1}$ and $n_{k+1}=pn_k$ for $k=0, 1, 2, \ldots$. It is evident that $\{n_k\}_{k=0}^{+\infty}$ is a strictly increasing sequence

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***October 20, 2004.***

Problem 206. (*Due to Zdravko F. Starc, Vršac, Serbia and Montenegro*) Prove that if a, b are the legs and c is the hypotenuse of a right triangle, then

$$(a+b)\sqrt{a}+(a-b)\sqrt{b}<\sqrt{2\sqrt{2}}\,c\sqrt{c}.$$

Problem 207. Let $A = \{0, 1, 2, ..., 9\}$ and $B_1, B_2, ..., B_k$ be nonempty subsets of A such that B_i and B_j have at most 2 common elements whenever $i \neq j$. Find the maximum possible value of k.

Problem 208. In ΔABC, $AB > AC > BC$. Let D be a point on the minor arc BC of the circumcircle of ΔABC. Let O be the circumcenter of ΔABC. Let E, F be the intersection points of line AD with the perpendiculars from O to AB, AC, respectively. Let P be the intersection of lines BE and CF. If $PB = PC + PO$, then find $\angle BAC$ with proof.

Problem 209. Prove that there are infinitely many positive integers n such that $2^n + 2$ is divisible by n and $2^n + 1$ is divisible by $n - 1$.

Problem 210. Let $a_1 = 1$ and

$$a_{n+1}=\frac{a_n}{2}+\frac{1}{a_n}$$

for $n = 1, 2, 3, ...$. Prove that for every integer $n > 1$,

$$\frac{2}{\sqrt{a_n^2-2}}$$

is an integer.

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Solutions

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Problem 201. (*Due to Abderrahim OUARDINI, Talence, France*) Find which nonright triangles ABC satisfy

$$\tan A\tan B\tan C > [\tan A]+[\tan B]+[\tan C],$$

where $[t]$ denotes the greatest integer less than or equal to t. Give a proof.

Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4). , **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece),

From

$$\tan C = \tan(180^\circ - A - B) = -\tan(A+B) = -(\tan A + \tan B)/(1-\tan A\tan B),$$

we get

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Let $x = \tan A$, $y = \tan B$ and $z = \tan C$. If $xyz \le [x]+[y]+[z]$, then $x+y+z \le [x] + [y] + [z]$. As $[t] \le t$, x, y, z must be integers.

If triangle ABC is obtuse, say $A > 90^\circ$, then $x < 0 < 1 \le y \le z$. This implies $1 \le yz = (x + y + z)/x = 1 + (y + z)/x < 1$, a contradiction. If triangle ABC is acute, then we may assume $1 \le x \le y \le z$. Now $xy = (x+y+z)/z \le (3z)/z = 3$. Checking the cases $xy = 1, 2, 3$, we see $x+y+z = xyz$ can only happen when x=1, y=2 and z=3. This corresponds to $A = \tan^{-1} 1$, $B = \tan^{-1} 2$ and $C = \tan^{-1} 3$. Reversing the steps, we see among nonright triangles, the inequality in the problem holds except only for triangles with angles equal $45^\circ = \tan^{-1} 1$, $\tan^{-1} 2$ and $\tan^{-1} 3$.

Problem 202. (*Due to LUK Mee Lin, La Salle College*) For triangle ABC, let D, E, F be the midpoints of sides AB, BC, CA, respectively. Determine which triangles ABC have the property that triangles ADF, BED, CFE can be folded above the plane of triangle DEF to form a tetrahedron with AD coincides with BD; BE coincides with CE; CF coincides with AF.

Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Observe that $ADEF$, $BEFD$ and $CFDE$ are parallelograms. Hence $\angle BDE = \angle BAC$, $\angle ADF = \angle ABC$ and $\angle EDF = \angle BCA$. In order for AD to coincide with BD in folding, we need to have $\angle BDE + \angle ADF > \angle EDF$. So we need $\angle BAC + \angle ABC > \angle BCA$. Similarly, for BE to coincide with CE and for CF to coincide with AF, we need $\angle ABC + \angle BCA > \angle BAC$ and $\angle BCA + \angle BAC > \angle ABC$. So no angle of ΔABC is 90 or more. Therefore, ΔABC is acute.

Conversely, if ΔABC is acute, then reversing the steps, we can see that the required tetrahedron can be obtained.

Problem 203. (*Due to José Luis DÍAZ-BARRERO, Universitat Politecnica de Catalunya, Barcelona, Spain*) Let a, b and c be real numbers such that $a + b + c \neq 0$. Prove that the equation

$$(a+b+c)x^2 + 2(ab+bc+ca)x + 3abc = 0$$

has only real roots.

Solution. **CHAN Pak Woon (**Wah Yan College, Kowloon, Form 6), **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Hoi Kit** (SKH Lam Kau Mow Secondary School, Form 7), **CHEUNG Yun Kuen** (HKUST, Math, Year 1), **Murray KLAMKIN** (University of Alberta, Edmonton, Canada), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

The quadratic has real roots if and only if its discriminant

$$\begin{aligned}D &= 4(ab+bc+ca)^2 - 12(a+b+c)abc\\ &= 4[(ab)^2+(bc)^2+(ca)^2-(a+b+c)abc]\\ &= 4[(ab-bc)^2+(bc-ca)^2+(ca-ab)^2]\end{aligned}$$

is nonnegative, which is clear.

Other commended solvers: **Jason CHENG Hoi Sing** (SKH Lam Kau Mow Secondary School, Form 7), **POON Ho Yin** (Munsang College (Hong Kong Island), Form 4) and **Anderson TORRES** (Universidade de Sao Paulo – Campus Sao Carlos).

Problem 204. Let n be an integer with $n > 4$. Prove that for every n distinct integers taken from 1, 2, …, $2n$, there always exist two numbers whose least common multiple is at most $3n + 6$.

Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Let S be the set of n integers taken and k be the minimum of these integers. If $k \le n$, then either $2k$ is also in S or $2k$ is not in S. In the former case, $\text{lcm}(k,2k) = 2k \le 2n < 3n+6$. In the latter case, we replace k in S by $2k$. Note this will not decrease the least common multiple of any pair of numbers. So if the new S satisfies the problem, then the original S will also satisfy the problem. As we repeat this, the new minimum will increase strictly so that we eventually reach either k and $2k$ both in S, in which case we are done, or the new S will consist of $n+1$, $n+2$, …, $2n$. So we need to consider the latter case only.

If $n > 4$ is even, then $3(n+2)/2$ is an integer at most $2n$ and $\operatorname{lcm}(n+2, 3(n+2)/2) = 3n+6$. If $n > 4$ is odd, then $3(n+1)/2$ is an integer at most $2n$ and $\operatorname{lcm}(n+1,3(n+1)/2) = 3n+3$.

Problem 205. *(Due to HA Duy Hung, Hanoi University of Education, Vietnam)* Let a, n be integers, both greater than 1, such that $a^n - 1$ is divisible by n. Prove that the greatest common divisor (or highest common factor) of $a - 1$ and n is greater than 1.

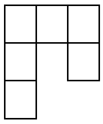
Solution. **CHENG Hao** (The Second High School Attached to Beijing Normal University), **CHEUNG Yun Kuen** (HKUST, Math, Year 1) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Let p be the smallest prime divisor of n. Then $a^n - 1$ is divisibly by p so that $a^n \equiv 1 \pmod p$. In particular, a is not divisible by p. Then, by Fermat's little theorem, $a^{p-1} \equiv 1 \pmod p$.

Let d be the smallest positive integer such that $a^d \equiv 1 \pmod p$. Dividing n by d, we get $n = dq + r$ for some integers q, r with $0 \le r < d$. Then $a^r \equiv (a^d)^q a^r = a^n \equiv 1 \pmod p$. By the definition of d, we get $r = 0$. Then n is divisible by d. Similarly, dividing $p - 1$ by d, we see $a^{p-1} \equiv 1 \pmod p$ implies $p - 1$ is divisible by d. Hence, $\gcd(n, p - 1)$ is divisible by d. Since p is the smallest prime dividing n, we must have $\gcd(n, p - 1) = 1$. So $d = 1$. By the definition of d, we get $a - 1$ is divisible by p. Therefore, $\gcd(a - 1, n) \ge p > 1$.

Olympiad Corner

(continued from page 1)



or any of the figures obtained by applying rotations and reflections to this figure.

Determine all $m \times n$ rectangles that can be covered with hooks so that

● the rectangle is covered without gaps and without overlaps;

● no part of a hook covers area outside the rectangle.

Day 2 Time allowed: 4 hours 30 minutes.

Problem 4. Let $n \ge 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \le i < j < k \le n$.

Problem 5. In a convex quadrilateral $ABCD$ the diagonal BD bisects neither the angle ABC not the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Problem 6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers n such that n has a multiple which is alternating.

Extending an IMO Problem

(continued from page 2)

of positive integers and each term of the sequence has exactly s distinct prime divisors.

It remains to show that

$$n_k \mid a^{n_k} + b^{n_k}$$

for $k = 0, 1, 2, \ldots$ Note that for odd positive integers m, n with $m \mid n$, we have $a^m + b^m \mid a^n + b^n$. By property (iii), we have, for $0 \le k < s$,

$$q_k \mid x_{k+1} \mid x_s \mid a^{n_0} + b^{n_0} \mid a^{n_j} + b^{n_j}$$

for $j = 0, 1, 2, \ldots$. Now it suffices to show that

$$p^{k+s} \mid a^{n_k} + b^{n_k}$$

for $k = 0, 1, 2, \ldots$. But this follows easily from Lemma 2 since

$$p^{s+k} \mid x_{k+s} \mid a^{n_k} + b^{n_k}.$$

This completes the proof of Theorem 2.

References:

[1] O. Reutter, *Elemente der Math.*, 18 (1963), 89.

[2] W. Sierpinski, *Elementary Theory of Numbers*, English translation, Warsaw, 1964.

2004 Hong Kong team to IMO: From left to right, Cheung Yun Kuen, Poon Ming Fung, Tang Chiu Fai, Cesar Jose Alaban (Deputy Leader), Leung Tat Wing (Leader), Chung Tat Chi, Kwok Tsz Chiu & Wong Hon Yin.

Mathematical Excalibur

Volume 9, Number 4 October 2004 – December 2004

Olympiad Corner

The Czech-Slovak-Polish Match this year took place in Bilovec on June 21-22, 2004.Here are the problems.

Problem 1. Show that real numbers p, q, r satisfy the condition

$$p^4(q-r)^2 + 2p^2(q+r) + 1 = p^4$$

if and only if the quadratic equations

$$x^2 + px + q = 0 \text{ and } y^2 - py + r = 0$$

have real roots (not necessarily distinct) which can be labeled by x_1, x_2 and y_1, y_2, respectively, in such way that the equality $x_1y_1 - x_2y_2 = 1$ holds.

Problem 2. Show that for each natural number k there exist at most finitely many triples of mutually distinct primes p, q, r for which the number $qr - k$ is a multiple of p, the number $pr - k$ is a multiple of q, and the number $pq - k$ is a multiple of r.

Problem 3. In the interior of a cyclic quadrilateral $ABCD$, a point P is given such that $|\angle BPC|=|\angle BAP|+|\angle PDC|$. Denote by E, F and G the feet of the perpendiculars from the point P to the lines AB, AD and DC, respectively. Show that the triangles FEG and PBC are similar.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 20, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Homothety

Kin Y. Li

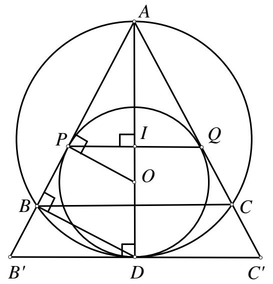
A <u>*geometric transformation*</u> of the plane is a function that sends every point on the plane to a point in the same plane. Here we will like to discuss one type of geometric transformations, called *homothety*, which can be used to solve quite a few geometry problems in some international math competitions.

A <u>*homothety with center O and ratio k*</u> is a function that sends every point X on the plane to the point X' such that

$$\overrightarrow{OX'} = k\overrightarrow{OX}.$$

So if $|k| > 1$, then the homothety is a magnification with center O. If $|k| < 1$, it is a reduction with center O. *A homothety sends a figure to a similar figure.* For instance, let D, E, F be the midpoints of sides BC, CA, AB respectively of ΔABC. The homothety with center A and ratio 2 sends ΔAFE to ΔABC. The homothety with center at the centroid G and ratio $-1/2$ sends ΔABC to ΔDEF.

<u>Example 1</u>. (*1978 IMO*) In ΔABC, $AB = AC$. A circle is tangent internally to the circumcircle of ABC and also to the sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of ΔABC.

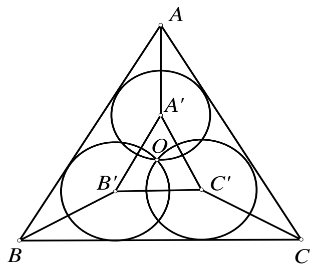


Solution. Let O be the center of the circle. Let the circle be tangent to the circumcircle of ΔABC at D. Let I be the midpoint of PQ. Then A, I, O, D are collinear by symmetry. Consider the homothety with center A that sends ΔABC to $\Delta AB'C'$ such that D is on $B'C'$. Thus, $k=AB'/AB$. As right triangles AIP, ADB', ABD, APO are similar, we have

$$AI/AO = (AI/AP)(AP/AO) = (AD/AB')(AB/AD) = AB/AB' = 1/k.$$

Hence the homothety sends I to O. Then O being the incenter of $\Delta AB'C'$ implies I is the incenter of ΔABC.

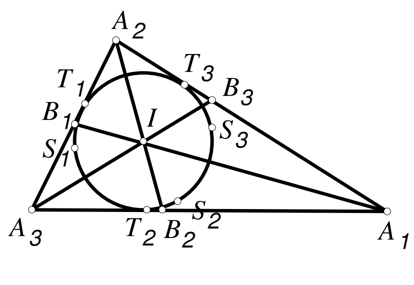
<u>Example 2</u>. (*1981 IMO*) Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear.



Solution. Consider the figure shown. Let A', B', C' be the centers of the circles. Since the radii are the same, so $A'B'$ is parallel to AB, $B'C'$ is parallel to BC, $C'A'$ is parallel to CA. Since AA', BB' CC' bisect $\angle A$, $\angle B$, $\angle C$ respectively, they concur at the incenter I of ΔABC. Note O is the circumcenter of $\Delta A'B'C'$ as it is equidistant from A', B', C'. Then the homothety with center I sending $\Delta A'B'C'$ to ΔABC will send O to the circumcenter P of ΔABC. Therefore, I, O, P are collinear.

<u>Example 3</u>. (*1982 IMO*) A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all i=1, 2, 3, M_i is the midpoint of side a_i, and T_i is the point where the incircle touchs side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i.

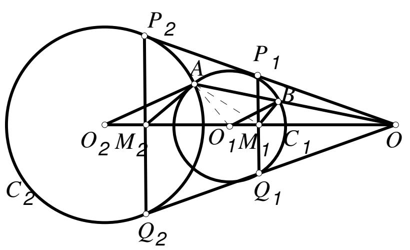
Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.



Solution. Let I be the incenter of $\Delta A_1A_2A_3$. Let B_1, B_2, B_3 be the points where the internal angle bisectors of $\angle A_1$, $\angle A_2$, $\angle A_3$ meet a_1, a_2, a_3 respectively. We will show S_iS_j is parallel to M_iM_j. With respect to A_1B_1, the reflection of T_1 is S_1 and the reflection of T_2 is T_3. So $\angle T_3IS_1 = \angle T_2IT_1$. With respect to A_2B_2, the reflection of T_2 is S_2 and the reflection of T_1 is S_3. So $\angle T_3IS_2 = \angle T_1IT_2$. Then $\angle T_3IS_1 = \angle T_3IS_2$. Since IT_3 is perpendicular to A_1A_2, we get S_2S_1 is parallel to A_1A_2. Since A_1A_2 is parallel to M_2M_1, we get S_2S_1 is parallel to M_2M_1. Similarly, S_3S_2 is parallel to M_3M_2 and S_1S_3 is parallel to M_1M_3.

Now the circumcircle of $\Delta S_1S_2S_3$ is the incircle of $\Delta A_1A_2A_3$ and the circumcircle of $\Delta M_1M_2M_3$ is the nine point circle of $\Delta A_1A_2A_3$. Since $\Delta A_1A_2A_3$ is not equilateral, these circles have different radii. Hence $\Delta S_1S_2S_3$ is not congruent to $\Delta M_1M_2M_3$ and there is a homothety sending $\Delta S_1S_2S_3$ to $\Delta M_1M_2M_3$. Then M_1S_1, M_2S_2 and M_3S_3 concur at the center of the homothety.

Example 4. (*1983 IMO*) Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.



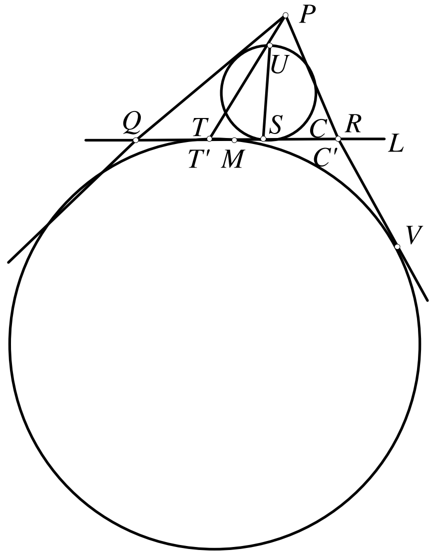
Solution. By symmetry, lines O_2O_1, P_2P_1, Q_2Q_1 concur at a point O. Consider the homothety with center O which sends C_1 to C_2. Let OA meet C_1 at B, then A is the image of B under the homothety. Since ΔBM_1O_1 is sent to ΔAM_2O_2, so $\angle M_1BO_1 = \angle M_2AO_2$.

Now ΔOP_1O_1 similar to ΔOM_1P_1 implies $OO_1/OP_1 = OP_1/OM_1$. Then

$$OO_1 \cdot OM_1 = OP_1^2 = OA \cdot OB,$$

which implies points A, B, M_1, O_1 are concyclic. Then $\angle M_1BO_1 = \angle M_1AO_1$. Hence $\angle M_1AO_1 = \angle M_2AO_2$. Adding $\angle O_1AM_2$ to both sides, we have $\angle O_1AO_2 = \angle M_1AM_2$.

Example 5. (*1992 IMO*) In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exist two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of ΔPQR.



Solution. Let L be the tangent to C at S. Let T be the reflection of S with respect to M. Let U be the point on C diametrically opposite S. Take a point P on the locus. The homothety with center P that sends C to the excircle C' will send U to T', the point where QR touches C'. Let line PR touch C' at V. Let s be the semiperimeter of ΔPQR, then

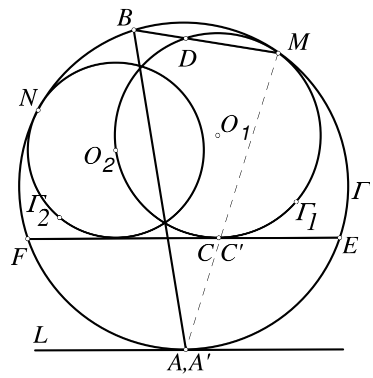
$$TR = QS = s - PR = PV - PR = VR = T'R$$

so that P, U, T are collinear. Then the locus is on the part of line UT, opposite the ray $\overrightarrow{UT}$.

Conversely, for any point P on the part of line UT, opposite the ray $\overrightarrow{UT}$, the homothety sends U to T and T', so $T = T'$. Then $QS = s - PR = PV - PR = VR = T'R = TR$ and $QM = QS - MS = TR - MT = RM$. Therefore, P is on the locus.

For the next example, the solution involves the concepts of power of a point with respect to a circle and the radical axis. We will refer the reader to the article "*Power of Points Respect to Circles*," in *Math Excalibur*, *vol.* 4, *no.* 3, *pp.* 2, 4.

Example 6. (*1999 IMO*) Two circles Γ_1 and Γ_2 are inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the center of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D, respectively. Prove that CD is tangent to Γ_2.



Solution. (*Official Solution*) Let EF be the chord of Γ which is the common tangent to Γ_1 and Γ_2 on the same side of line O_1O_2 as A. Let EF touch Γ_1 at C'. The homothety with center M that sends Γ_1 to Γ will send C' to some point A' and line EF to the tangent line L of Γ at A'. Since lines EF and L are parallel, A' must be the midpoint of arc $FA'E$. Then $\angle A'EC' = \angle A'FC' = \angle A'ME$. So $\Delta A'EC$ is similar to $\Delta A'ME$. Then the power of A' with respect to Γ_1 is $A'C' \cdot A'M = A'E^2$. Similar, the power of A' with respect to Γ_2 is $A'F^2$. Since $A'E = A'F$, A' has the same power with respect to Γ_1 and Γ_2. So A' is on the radical axis AB. Hence, $A' = A$. Then $C' = C$ and C is on EF.

Similarly, the other common tangent to Γ_1 and Γ_2 passes through D. Let O_i be the center of Γ_i. By symmetry with respect to O_1O_2, we see that O_2 is the midpoint of arc CO_2D. Then

$$\angle DCO_2 = \angle CDO_2 = \angle FCO_2.$$

This implies O_2 is on the angle bisector of $\angle FCD$. Since CF is tangent to Γ_2, therefore CD is tangent to Γ_2.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for submitting solutions is ***January 20, 2005.***

Problem 211. For every a, b, c, d in [1,2], prove that

$$\frac{a+b}{b+c}+\frac{c+d}{d+a}\le 4\,\frac{a+c}{b+d}.$$

(*Source: 32nd Ukranian Math Olympiad*)

Problem 212. Find the largest positive integer N such that if S is any set of 21 points on a circle C, then there exist N arcs of C whose endpoints lie in S and each of the arcs has measure not exceeding 120°.

Problem 213. Prove that the set of all positive integers can be partitioned into 100 nonempty subsets such that if three positive integers a, b, c satisfy $a + 99\,b = c$, then at least two of them belong to the same subset.

Problem 214. Let the inscribed circle of triangle ABC be tangent to sides AB, BC at E and F respectively. Let the angle bisector of $\angle CAB$ intersect segment EF at K. Prove that $\angle CKA$ is a right angle.

Problem 215. Given a 8×8 board. Determine all squares such that if each one is removed, then the remaining 63 squares can be covered by 21 3×1 rectangles.

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Solutions

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Problem 206. (*Due to Zdravko F. Starc, Vršac, Serbia and Montenegro*) Prove that if a, b are the legs and c is the hypotenuse of a right triangle, then

$$(a+b)\sqrt{a}+(a-b)\sqrt{b}<\sqrt{2\sqrt{2}c}\sqrt{c}.$$

Solution. **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Problem Group Discussion Euler-Teorema**(Fortaleza, Brazil), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **TO Ping Leung** (St. Peter's Secondary School) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

By Pythagoras' theorem,

$$a+b\le\sqrt{(a+b)^2+(a-b)^2}=\sqrt{2}c.$$

Equality if and only if $a = b$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}&(a+b)\sqrt{a}+(a-b)\sqrt{b}\\ &\le\sqrt{(a+b)^2+(a-b)^2}\sqrt{a+b}\\ &\le\sqrt{2}c\sqrt{\sqrt{2}c}.\end{aligned}$$

For equality to hold throughout, we need $a+b : a-b=\sqrt{a}:\sqrt{b}=1:1$, which is not possible for legs of a triangle. So we must have strict inequality.

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania) and **TONG Yiu Wai** (Queen Elizabeth School, Form 7).

Problem 207. Let $A = \{ 0, 1, 2, ..., 9\}$ and $B_1, B_2, ..., B_k$ be nonempty subsets of A such that B_i and B_j have at most 2 common elements whenever $i \ne j$. Find the maximum possible value of k.

Solution. **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

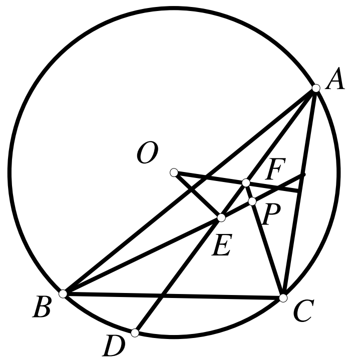
If we take all subsets of A with 1, 2 or 3 elements, then these 10 + 45 + 120 = 175 subsets satisfy the condition. So $k \ge 175$.

Let $B_1, B_2, ..., B_k$ satisfying the condition with k maximum. If there exists a B_i with at least 4 elements, then every 3 element subset of B_i cannot be one of the B_j, $j \ne i$, since B_i and B_j can have at most 2 common elements. So adding these 3 element subsets to $B_1, B_2, ..., B_k$ will still satisfy the conditions. Since B_i has at least four 3 element subsets, this will increase k, which contradicts maximality of k. Then every B_i has at most 3 elements. Hence, $k \le 175$. Therefore, the maximum k is 175.

Other commended solvers: **CHAN Wai Hung** (Carmel Divine Grace Foundation Secondary School, Form 6), **LI Sai Ki** (Carmel Divine Grace Foundation Secondary School, Form 6), **LING Shu Dung**, **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Problem 208. In ΔABC, $AB > AC > BC$. Let D be a point on the minor arc BC of the circumcircle of ΔABC. Let O be the circumcenter of ΔABC. Let E, F be the intersection points of line AD with the perpendiculars from O to AB, AC, respectively. Let P be the intersection of lines BE and CF. If $PB = PC + PO$, then find $\angle BAC$ with proof.

Solution. **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Problem Group Discussion Euler - Teorema** (Fortaleza, Brazil) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).



Since E is on the perpendicular bisector of chord AB and F is on the perpendicular bisector of chord AC, $AE = BE$ and $AF = CF$. Applying exterior angle theorem,

$$\begin{aligned}\angle BPC &= \angle AEP + \angle CFD\\ &= 2\,(\angle BAD+\angle CAD)\\ &= 2\angle BAC = \angle BOC.\end{aligned}$$

Hence, B, C, P, O are concyclic. By Ptolemy's theorem,

$$PB\cdot OC = PC\cdot OB + PO\cdot BC.$$

Then $(PB - PC)\cdot OC = PO\cdot BC$. Since $PB - PC = PO$, we get $OC = BC$ and so ΔOBC is equilateral. Then

$$\angle BAC=\tfrac{1}{2}\angle BOC = 30°$$

Other commended solvers: **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7), **TONG Yiu Wai**

(Queen Elizabeth School, Form 7) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Problem 209. Prove that there are infinitely many positive integers n such that $2^n + 2$ is divisible by n and $2^n + 1$ is divisible by $n - 1$.

Solution. **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7) and **Problem Group Discussion Euler-Teorema**(Fortaleza, Brazil).

As $2^2 + 2 = 6$ is divisible by 2 and $2^2 + 1 = 5$ is divisible by 1, $n = 2$ is one such number.

Next, suppose $2^n + 2$ is divisible by n and $2^n + 1$ is divisible by $n - 1$. We will prove $N = 2^n + 2$ is another such number. Since $N - 1 = 2^n + 1 = (n - 1)k$ is odd, so k is odd and n is even. Since $N = 2^n + 2 = 2(2^{n-1} + 1) = nm$ and n is even, so m must be odd. Recall the factorization

$$x^i + 1 = (x + 1)(x^{i-1} - x^{i-3} + \cdots + 1)$$

for odd positive integer i. Since k is odd, $2^N + 2 = 2(2^{N-1} + 1) = 2(2^{(n-1)k} + 1)$ is divisible by $2(2^{n-1} + 1) = 2^n + 2 = N$ using the factorization above. Since m is odd, $2^N + 1 = 2^{nm} + 1$ is divisible by $2^n + 1 = N - 1$. Hence, N is also such a number. As $N > n$, there will be infinitely many such numbers.

Problem 210. Let $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$$

for $n = 1, 2, 3, \ldots$. Prove that for every integer $n > 1$,

$$\frac{2}{\sqrt{a_n^2 - 2}}$$

is an integer.

Solution. **G.R.A. 20 Problem Group** (Roma, Italy), **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **Problem Group Discussion Euler – Teorema** (Fortaleza, Brazil), **TO Ping Leung** (St. Peter's Secondary School) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Note $a_n = p_n / q_n$, where $p_1 = q_1 = 1$, $p_{n+1} = p_n^2 + 2q_n^2$, $q_{n+1} = 2p_nq_n$ for $n = 1, 2, 3, \ldots$. Then

$$\frac{2}{\sqrt{a_n^2 - 2}} = \frac{2q_n}{\sqrt{p_n^2 - 2q_n^2}}.$$

It suffices to show by mathematical induction that $p_n^2 - 2q_n^2 = 1$ for $n > 1$. We have $p_2^2 - 2q_2^2 = 3^2 - 2\cdot 2^2 = 1$. Assuming case n is true, we get

$$p_{n+1}^2 - 2q_{n+1}^2 = (p_n^2 + 2q_n^2)^2 - 2(2p_nq_n)$$
$$= (p_n^2 - 2q_n^2)^2 = 1.$$

Other commended solvers: **Ellen CHAN On Ting** (True Light Girls' College, Form 5), **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **LAW Yau Pui** (Carmel Divine Grace Foundation Secondary School, Form 6), **Asger OLESEN** (Toender Gymnasium (grammar school), Denmark), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **Steve ROFFE**, **TONG Yiu Wai** (Queen Elizabeth School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 4).

Olympiad Corner

(continued from page 1)

Problem 4. Solve the system of equations

$$\frac{1}{xy} = \frac{x}{z} + 1, \quad \frac{1}{yz} = \frac{y}{x} + 1, \quad \frac{1}{zx} = \frac{z}{y} + 1$$

in the domain of real numbers.

Problem 5. In the interiors of the sides AB, BC and CA of a given triangle ABC, points K, L and M, respectively, are given such that

$$\frac{|AK|}{|KB|} = \frac{|BL|}{|LC|} = \frac{|CM|}{|MA|}.$$

Show that the triangles ABC and KLM have a common orthocenter if and only if the triangle ABC is equilateral.

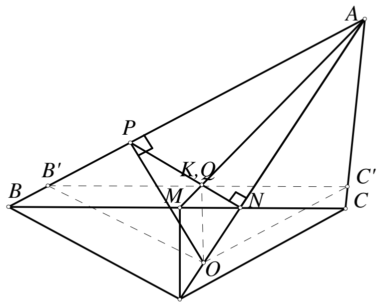
Problem 6. On the table there are k heaps of 1, 2, …, k stones, where $k \geq 3$. In the first step, we choose any three of the heaps on the table, merge them into a single new heap, and remove 1 stone (throw it away from the table) from this new heap. In the second step, we again merge some three of the heaps together into a single new heap, and then remove 2 stones from this new heap. In general, in the i-th step we choose any three of the heaps, which contain more than i stones when combined, we merge them into a single new heap, then remove i stones from this new heap. Assume that after a number of steps, there is a single heap left on the table, containing p stones. Show that the number p is a perfect square if and only if the numbers $2k+2$ and $3k+1$ are perfect squares. Further, find the least number k for which p is a perfect square.

Homothety

(continued from page 2)

Example 7. (*2000APMO*) Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA respectively and O the point in which the perpendicular at P to BA meets AN produced.

Prove that QO is perpendicular to BC.



Solution (*due to Bobby Poon*). The case $AB = AC$ is clear.

Without loss of generality, we may assume $AB > AC$. Let AN intersect the circumcircle of ΔABC at D. Then

$$\angle DBC = \angle DAC = \tfrac{1}{2}\angle BAC$$
$$= \angle DAB = \angle DCB.$$

So $DB = DC$ and MD is perpendicular to BC.

Consider the homothety with center A that sends ΔDBC to $\Delta OB'C'$. Then $OB' = OC'$ and BC is parallel to $B'C'$. Let $B'C'$ intersect PN at K. Then

$$\angle OB'K = \angle DBC = \angle DAB$$
$$= 90° - \angle AOP = \angle OPK.$$

So points P, B', O, K are concyclic. Hence $\angle B'KO = \angle B'PO = 90°$ and $B'K = C'K$. Since $BC \parallel B'C'$, this implies K is on MA. Hence, $K = Q$. Now $\angle B'KO = 90°$ implies $QO = KO \perp B'C'$. Finally, $BC \parallel B'C'$ implies QO is perpendicular to BC.

Mathematical Excalibur

Volume 9, Number 5 *January 2005 – February 2005*

Olympiad Corner

The 7th China Hong Kong Math Olympiad took place on December 4, 2004. Here are the problems.

Problem 1. For $n \ge 2$, let $a_1, a_2, \ldots, a_n, a_{n+1}$ be positive and $a_2 - a_1 = a_3 - a_2 = \cdots = a_{n+1} - a_n \ge 0$. Prove that

$$\frac{1}{a_2^2}+\frac{1}{a_3^2}+\cdots+\frac{1}{a_n^2} \le \frac{n-1}{2}\cdot\frac{a_1a_n+a_2a_{n+1}}{a_1a_2a_na_{n+1}}.$$

Determine when equality holds.

Problem 2. In a school there are b teachers and c students. Suppose that

(i) each teacher teaches exactly k students; and

(ii) for each pair of distinct students, exactly h teachers teach both of them.

Show that $\dfrac{b}{h}=\dfrac{c(c-1)}{k(k-1)}$.

Problem 3. On the sides AB and AC of triangle ABC, there are points P and Q respectively such that $\angle APC = \angle AQB = 45°$. Let the perpendicular line to side AB through P intersects line BQ at S. Let the perpendicular line to side AC through Q intersects line CP at R. Let D be on side BC such that $AD \perp BC$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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例析數學競賽中的計數問題 (一)

費振鵬 （江蘇省鹽城市城區永豐中學 224054）

組合數學中的計數問題，數學競賽題中的熟面孔，看似司空見慣，不足爲奇．很多同學認爲只要憑藉單純的課內知識就可左右逢源，迎刃而解．其實具體解題時，卻會使你挖空心思，也無所適從．對於這類問題往往首先要通過構造法描繪出對象的簡單數學模型，繼而借助在計數問題中常用的一些數學原理方可得出所求對象的總數或其範圍．

1 運用分類計數原理與分步計數原理

分類計數原理與分步計數原理（即加法原理與乘法原理）是關於計數的兩個基本原理，是解決競賽中計數問題的基礎．下面提出的三個問題，注意結合排列與組合的相關知識，構造出相應的模型再去分析求解．

<u>例 1</u> 已知兩個實數集合 $A=\{a_1,a_2,\cdots,a_{100}\}$ 與 $B=\{b_1,b_2,\cdots,b_{50}\}$，若從$A$到$B$的映射$f$使得$B$中每個元素都有原象，且 $f(a_1) \le f(a_2) \le \ldots \le f(a_{100})$，則這樣的映射共有（ ）個．

（A）C_{100}^{50}（B）C_{99}^{48}（C）C_{100}^{49}（D）C_{99}^{49}

<u>解答</u> 設 $b_1,b_2,\cdots,b_{50}$ 按從小到大排列爲 $c_1<c_2<\cdots<c_{50}$（因集合元素具有互異性，故其中不含相等情形）．

將A中元素 $a_1,a_2,\cdots,a_{100}$ 分成 50 組，每組依次與B中元素 $c_1,c_2,\cdots,c_{50}$ 對應．這裏，我們用 $a_1a_2a_3c_1a_4a_5c_2\cdots$，表示 $f(a_1)=f(a_2)=f(a_3)=c_1$，$f(a_4)=f(a_5)=c_2$，…

考慮 $f(a_1) \le f(a_2) \le \cdots \le f(a_{100})$，容易得到 $f(a_{100})=c_{50}$，這就是說 c_{50} 只能寫在 a_{100} 的右邊，故只需在 $a_1□a_2□a_3□\cdots□a_{98}□a_{99}□a_{100}c_{50}$ 之間的99個空位“□”中選擇49個位置並按從左到右的順序依次填上 $c_1,c_2,\cdots,c_{49}$．從而構成滿足題設要求的映射共有 C_{99}^{49} 個．因此選 D．

<u>例 2</u> 有人要上樓，此人每步能向上走1階或2階，如果一層樓有18階，他上一層樓有多少種不同的走法？

<u>解答 1</u> 此人上樓最多走18步，最少走9步．這裏用 $a_{18},a_{17},a_{16},\cdots,a_9$ 分別表示此人上樓走18步，17步，16步，…，9步時走法（對於任意前後兩者的步數，因後者少走2步1階，而多走1步2階，計後者少走1步）的計數結果．考慮步子中的每步2階情形，易得 $a_{18}=C_{18}^0$，$a_{17}=C_{17}^1$，$a_{16}=C_{16}^2$，…，$a_9=C_9^9$．

綜上，他上一層樓共有 $C_{18}^0+C_{17}^1+C_{16}^2+\cdots+C_9^9=1+17+120+\cdots+1=4181$ 種不同的走法．

<u>解答 2</u> 設 F_n 表示上n階的走法的計數結果．

顯然，$F_1=1$，$F_2=2$（2步1階；1步2階）．對於 $F_3,F_4,\cdots$，起步只有兩種不同走法：上1階或上2階．

因此對於 F_3，第1步上1階的情形，還剩 $3-1=2$ 階，計 F_2 種不同的走法；對於第1步上2階的情形，還剩 $3-2=1$ 階，計 F_1 種不同的走法．總計 $F_3=F_2+F_1=2+1=3$．

同理，$F_4=F_3+F_2=3+2=5$，$F_5=F_4+F_3=5+3=8$，…，$F_{18}=F_{17}+F_{16}=2584+1597=4181$．

<u>例 3</u> 在世界盃足球賽前，F國教練爲了考察 $A_1,A_2,\ldots,A_7$ 這七名隊員，準備讓他們在三場訓練比賽（每場90分鐘）都上場．假設在比賽的任何時刻，這

些隊員中有且僅有一人在場上，並且 A_1,A_2,A_3,A_4 每人上場的總時間（以分鐘爲單位）均被 7 整除，A_5,A_6,A_7 每人上場的總時間（以分鐘爲單位）均被 13 整除．如果每場換人次數不限，那麼按每名隊員上場的總時間計算，共有多少種不同的情況．

解答　設 $A_i\left(i=1,2,\cdots,7\right)$ 上場的總時間分別爲 $a_i\left(i=1,2,\cdots,7\right)$ 分鐘．

根據題意，可設 $a_i=7k_i\left(i=1,2,3,4\right)$，$a_i=13k_i\left(i=5,6,7\right)$，其中 $k_i\left(i=1,2,\cdots,7\right)\in Z^+$．

令 $\sum_{i=1}^{4}k_i=m$，$\sum_{i=5}^{7}k_i=n$，其中 $m\geq 4$，$n\geq 3$，且 $m,n\in Z^+$．則 $7m+13n=270$．易得其一個整數特解爲 $\begin{cases}m=33\\n=3\end{cases}$，又因 $(7,13)=1$，故其整數通解爲 $\begin{cases}m=33+13t\\n=3-7t\end{cases}$．再由 $\begin{cases}33+13t\geq 4\\3-7t\geq 3\end{cases}$，得 $-\frac{29}{13}\leq t\leq 0$，故整數 $t=0,-1,-2$．

從而其滿足條件的所有整數解爲 $\begin{cases}m=33,\\n=3;\end{cases}\begin{cases}m=20,\\n=10;\end{cases}\begin{cases}m=7,\\n=17.\end{cases}$

對於 $\sum_{i=1}^{4}k_i=33$ 的正整數解，可以寫一橫排共計 33 個 1，在每相鄰兩個 1 之間共 32 個空位中任選 3 個填入“+”號，再把 3 個“+”號分隔開的 4 個部分裏的 1 分別統計，就可得到其一個正整數解，故 $\sum_{i=1}^{4}k_i=33$ 有 C_{32}^3 個正整數解 $\left(k_1,k_2,k_3,k_4\right)$；同理 $\sum_{i=5}^{7}k_i=3$ 有 C_2^2 個正整數解 $\left(k_5,k_6,k_7\right)$；從而此時滿足條件的正整數解 $\left(k_1,k_2,k_3,k_4,k_5,k_6,k_7\right)$ 有 $C_{32}^3\cdot C_2^2$ 個．…

因此滿足條件的所有正整數解 $\left(k_1,k_2,k_3,k_4,k_5,k_6,k_7\right)$ 有

$C_{32}^3\cdot C_2^2+C_{19}^3\cdot C_9^2+C_6^3\cdot C_{16}^2=42244$

個，即按每名隊員上場的總時間計算，共有 42244 種不同的情況．

2 運用容斥原理

容斥原理，又稱包含排斥原理或逐步淘汰原理．顧名思義，即先計算一個較大集合的元素的個數，再把多計算的那一部分去掉．它由英國數學家 J.J.西爾維斯特首先創立．這個原理有多種表達形式，其中最基本的形式爲：

設 $A_1,A_2,\cdots,A_n$ 是任意 n 個有限集合，以 *card* (*S*) 代表 *S* 的元素的個數，則

$$\begin{aligned}&card\left(A_1\cup A_2\cup\cdots\cup A_n\right)\\&=\sum_{1\leq i\leq n}card(A_i)-\sum_{1\leq i<j\leq n}card(A_i\cap A_j)\\&+\sum_{1\leq i<j<k\leq n}card(A_i\cap A_j\cap A_k)-\cdots\\&+(-1)^{n-1}card(A_1\cap A_2\cap\cdots\cap A_n).\end{aligned}$$

例 4 由數字 1，2，3 組成 n 位數，且在這個 n 位數中，1，2，3 的每一個至少出現一次，問這樣的 n 位數有多少個？

解答 設 U 是由 1，2，3 組成的 n 位元數的集合，A_1 是 U 中不含數字 1 的 n 位元數的集合，A_2 是 U 中不含數字 2 的 n 位元數的集合，A_3 是 U 中不含數字 3 的 n 位元數的集合，則

$card\left(U\right)=3^n$,
$card\left(A_1\right)=card\left(A_2\right)=card\left(A_3\right)=2^n$,
$card\left(A_1\cap A_2\right)=card\left(A_2\cap A_3\right)=card\left(A_3\cap A_1\right)=1$,
$card\left(A_1\cap A_2\cap A_3\right)=0.$ ．

因此

$$\begin{aligned}&card(U)-card(A_1\cup A_2\cup A_3)\\&=3^n-3\cdot 2^n+3\cdot 1-0=3^n-3\cdot 2^n+3\end{aligned}$$ ．

即符合題意的 n 位數的個數爲 $3^n-3\cdot 2^n+3$ ．

下面，我們再來看一個關於容斥原理應用的變異問題．

例 5 參加大型團體表演的學生共 240 名，他們面對教練站成一行，自左至右按 1，2，3，4，5，…依次報數．教練要求全體學生牢記各自所報的數，並做下列動作：先讓報的數是 3 的倍數的全體同學向後轉；接著讓報的數是 5 的倍數的全體同學向後轉；最後讓報的數是 7 的倍數的全體同學向後轉．問：

⑴此時還有多少名同學面對教練？

⑵面對教練的同學中，自左至右第 66 位同學所報的數是幾？

解答 ⑴設 $U=\{1,2,3,\cdots,240\}$，A_i 表示由 U 中所有 i 的倍數組成的集合．則

$$card\left(U\right)=240,\quad card\left(A_3\right)=\left[\frac{240}{3}\right]=80,$$

$$card\left(A_5\right)=\left[\frac{240}{5}\right]=48,\ card\left(A_7\right)=\left[\frac{240}{7}\right]=34$$

$$card\left(A_{15}\right)=\left[\frac{240}{15}\right]=16,\ card\left(A_{21}\right)=\left[\frac{240}{21}\right]=11,$$

$$card\left(A_{35}\right)=\left[\frac{240}{35}\right]=6,\ card\left(A_{105}\right)=\left[\frac{240}{105}\right]=2.$$

從而此時有

$$\begin{aligned}&card(U)-[card(A_3)+card(A_5)+card(A_7)]\\&+2[card(A_{15})+card(A_{21})+card(A_{35})]\\&-4card(A_{105})=136\end{aligned}$$

名同學面對教練．

如果我們借助威恩圖進行分析，利用上面所得數據分別填入圖 1，注意按從內到外的順序填．

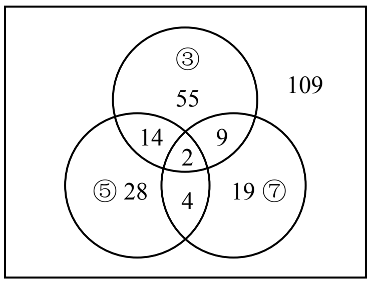

圖 1

如圖 1，此時面對教練的同學一目了然，應有 109+14+4+9=136 名．

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***March 31, 2005.***

Problem 216. (*Due to Alfred Eckstein, Arad, Romania*) Solve the equation

$$4x^6 - 6x^2 + 2\sqrt{2} = 0.$$

Problem 217. Prove that there exist infinitely many positive integers which cannot be represented in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11},$$

where x_1, x_2, x_3, x_4, x_5 are positive integers. (*Source:* 2002 *Belarussian Mathematical Olympiad, Final Round*)

Problem 218. Let O and P be distinct points on a plane. Let $ABCD$ be a parallelogram on the same plane such that its diagonals intersect at O. Suppose P is not on the reflection of line AB with respect to line CD. Let M and N be the midpoints of segments AP and BP respectively. Let Q be the intersection of lines MC and ND. Prove that P, Q, O are collinear and the point Q does not depend on the choice of parallelogram $ABCD$. (*Source:* 2004 *National Math Olympiad in Slovenia, First Round*)

Problem 219. (*Due to Dorin Mărghidanu, Coleg. Nat. "A.I. Cuza", Corabia, Romania)* The sequences $a_0,a_1,a_2,\ldots$ and $b_0,b_1,b_2,\ldots$ are defined as follows: $a_0,b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n}, \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for $n = 1,2,3,\ldots$. Prove that

$$\max\{a_{2004}, b_{2004}\} > \sqrt{2005}.$$

Problem 220. (*Due to Cheng HAO**,** The Second High School Attached to Beijing Normal University*) For $i = 1,2,\ldots,n$, and $k \geq 4$, let $A_i = (a_{i1}, a_{i2}, \ldots, a_{ik})$ with $a_{ij} = 0$ or 1 and every A_i has at least 3 of the k coordinates equal 1. Define the distance between A_i and A_j to be

$$\sum_{m=1}^{k} |a_{im} - a_{jm}|.$$

If the distance between any A_i and A_j ($i \neq j$) is greater than 2, then prove that

$$n \leq 2^{k-3} - 1.$$

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Solutions

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Problem 211. For every a, b, c, d in [1,2], prove that

$$\frac{a+b}{b+c} + \frac{c+d}{d+a} \leq 4\,\frac{a+c}{b+d}.$$

(*Source: 32*[nd] *Ukranian Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (HKUST, Math Major, Year 1), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania).

Since $0 < b + d \leq 4$, it suffices to show

$$\frac{a+b}{b+c} + \frac{c+d}{d+a} \leq a + c.$$

Without loss of generality, we may assume $1 \leq a \leq c$, say $c = a + e$ with $e \geq 0$. Then

$$\frac{a+b}{b+c} + \frac{c+d}{d+a} \leq 1 + \left(1 + \frac{e}{d+a}\right)$$

$$\leq 2 + e$$

$$\leq 2a + e = a + c.$$

In passing, we observe that equality holds if and only if $e = 0$, $a = c = 1$, $b = d = 2$.

Other commended solvers: **CHENG Hei** (Tsuen Wan Government Secondary School, Form 5), **LAW Yau Pui** (Carmel Divine Grace Foundation Secondary School, Form 6) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 5).

Problem 212. Find the largest positive integer N such that if S is any set of 21 points on a circle C, then there exist N arcs of C whose endpoints lie in S and each of the arcs has measure not exceeding 120°.

Solution.

We will N = 100. To see that $N \leq 100$, consider a diameter AB of C. Place 11 points close to A and 10 points close to B. The number of desired arcs is then

$$\binom{11}{2} + \binom{10}{2} = 100.$$

To see that $N \geq 100$, we need to observe that for every set T of $k = 21$ points on C, there exists a point X in T such that there are at least $[(k-1)/2]$ arcs XY (with Y in T, $Y \neq X$) each having measure not exceeding 120°. This is because we can divide the circle C into three arcs C_1, C_2, C_3 of 120° (only overlapping at endpoints) such that the common endpoint of C_1 and C_2 is a point X of T. If X does not have the required property, then there are $1 + [(k-1)/2]$ points of T lies on C_3 and any of them can serve as X.

Next we remove X and apply the same argument to $k = 20$, then remove that point, and repeat with $k = 19, 18, \ldots, 3$. We get a total of $10 + 9 + 9 + 8 + 8 + \ldots + 1 + 1 = 100$ arcs.

Problem 213. Prove that the set of all positive integers can be partitioned into 100 nonempty subsets such that if three positive integers a, b, c satisfy $a + 99b = c$, then at least two of them belong to the same subset.

Solution. **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Let $f(n)$ be the <u>largest</u> nonnegative integer k for which n is divisible by 2^k. Then given three positive integers a, b, c satisfying $a + 99\,b = c$ at least two of $f(a)$, $f(b)$, $f(c)$ are equal. To prove this, if $f(a) = f(b)$, then we are done. If $f(a) < f(b)$, then $f(c) = f(a)$. If $f(a) > f(b)$, then $f(c) = f(b)$.

Therefore, the following partition suffices:

$$S_i = \{n \mid f(n) \equiv i \pmod{100}\}$$

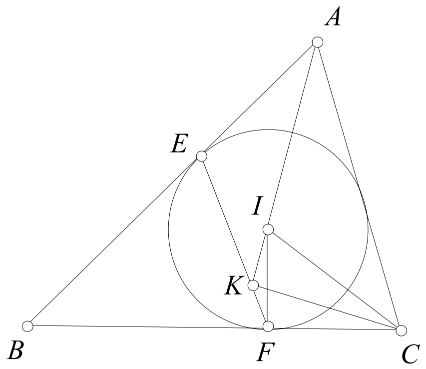
for $1 \leq i \leq 100$.

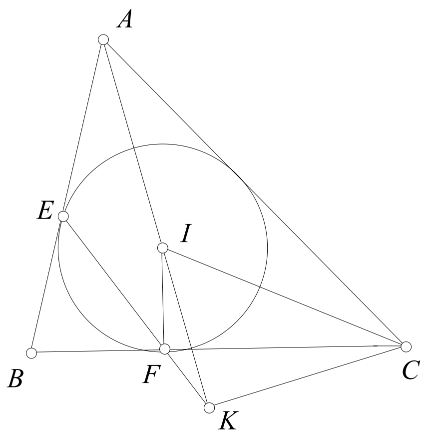
Problem 214. Let the inscribed circle of triangle ABC be tangent to sides AB, BC at E and F respectively. Let the angle bisector of $\angle CAB$ intersect segment EF at K. Prove that $\angle CKA$ is a right angle.

Solution. **CHENG Hei** (Tsuen Wan Government Secondary School, Form 5), **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **YIM Wing Yin** (South Tuen Mun

Government Secondary School, Form 5) and **YUNG Ka Chun** (Carmel Divine Grace Foundation Secondary School, Form 6).

Most of the solvers pointed out that the problem is still true if the angle bisector of $\angle CAB$ intersect line EF at K *outside the segment EF*. So we have two figures.





Let I be the center of the inscribed circle. Then A, I, K are collinear. Now $\angle CIK$ = ½($\angle BAC$+$\angle ACB$). Next, $BE = BF$ implies that $\angle BFE = 90° - ½ \angle CBA$ = ½ ($\angle BAC$+$\angle ACB$) = $\angle CIK$. (In the second figure, we have $\angle CFK = \angle BFE$ = $\angle CIK$.) Hence C, I, K, F are concyclic. Therefore, $\angle CKI = \angle CFI = 90°$.

Other commended solvers: **CHEUNG Yun Kuen** (HKUST, Math Major, Year 1).

Problem 215. Given a 8×8 board. Determine all squares such that if each one is removed, then the remaining 63 squares can be covered by 21 3×1 rectangles.

Solution. **CHEUNG Yun Kuen** (HKUST, Math Major, Year 1).

Let us number the squares of the board from 1 to 64, with 1 to 8 on the first row, 9 to 16 on the second row and so on.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Using this numbering, a 3×1 rectangle will cover three numbers with a sum divisible by 3. Since $64 \equiv 1 \pmod 3$, only squares with numbers congruent to 1 (mod 3) need to be considered for our problem.

If there is a desired square for the problem, then considering the left-right symmetry of the board and the up-down symmetry of the board, the images of a desired square under these symmetries are also desired squares. Hence they must also have numbers congruent to 1 (mod 3) in them.

However, the only such square and its image squares having this property are the squares with numbers 19, 22, 43 and 46.

Finally square 19 has the required property (and hence also squares 22, 43, 46 by symmetry) by putting 3×1 rectangles as shown in the following figure (those squares having the same letter are covered by the same 3×1 rectangle).

A	A	A	B	B	B	F	G
C	C	C	D	D	D	F	G
H	I		E	E	E	F	G
H	I	J	J	J	K	K	K
H	I	L	L	L	M	M	M
N	O	P	Q	R	S	T	U
N	O	P	Q	R	S	T	U
N	O	P	Q	R	S	T	U

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **NG Siu Hong** (Carmel Divine Grace Foundation Secondary School, Form 6) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Olympiad Corner

(continued from page 1)

Problem 3. *(cont.)* Prove that the lines PS, AD, QR meet at a common point and lines SR and BC are parallel.

Problem 4. Let $S = \{1, 2, \ldots, 100\}$. Determine the number of functions $f : S \to S$ satisfying the following conditions.

(i) $f(1) = 1$;
(ii) f is bijective (i.e. for every y in S, the equation $f(x) = y$ has exactly one solution);
(iii) $f(n) = f(g(n))\, f(h(n))$ for every n in S.

Here $g(n)$ and $h(n)$ denote the uniquely determined positive integers such that $g(n) \le h(n)$, $g(n)\, h(n) = n$ and $h(n) - g(n)$ is as small as possible. (For instance, $g(80) = 8$, $h(80) = 10$ and $g(81) = h(81) = 9$.)

例析數學競賽中的計數問題 (一)

(continued from page 2)

⑵用上面類似的方法可算得自左至右第 1 號至第 105 號同學中面對教練的有 60 名．

考慮所報的數不是 3，5，7 的倍數的同學沒有轉動，他們面對教練；所報的數是 3，5，7 中的兩個數的倍數的同學經過兩次轉動，他們仍面對教練；其餘同學轉動了一次或三次，都背對教練．

作如下分析：106，107，~~108~~（3 的倍數），109，~~110~~（5 的倍數），~~111~~（3 的倍數），~~112~~（7 的倍數），113，~~114~~（3 的倍數），~~115~~（5 的倍數），116，~~117~~（3 的倍數），118，~~119~~（7 的倍數），120（3、5 的倍數），……，可知面對教練的第 66 位同學所報的數是 118.

(*to be continued*)