Mathematical Excalibur

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Olympiad Corner

Below were the problems of the Balkan Mathematical Olympiad which took place in Belgrade, Serbia on May 9, 2018.

Time allowed was 270 minutes. Each problem was worth 10 points

Problem 1. A quadrilateral $ABCD$ is inscribed in a circle k, where $AB > CD$ and AB is not parallel to CD. Point M is the intersection of the diagonals AC and BD and the perpendicular from M to AB intersects the segment AB at the point E. If EM bisects the angle CED, prove that AB is a diameter of the circle k. (*Bulgaria*)

Problem 2. Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the n-th minute (n=1,2,…) each of them chooses whether to walk due north, east, south or west and then walks the distance of q^n metres. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q. (*United Kingdom*)

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 1, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Miscellaneous Inequalities

Kin Y. Li

There are many kinds of inequality problems in mathematical Olympiad competitions. Some of these can be solved by applying certain powerful inequalities such as rearrangement or majorization or Muirhead's inequalities. Some can be solved by techniques like tangent line methods using a bit of differential calculus.

In this article, we will be looking at some inequality problems that are not solved by these kinds of powerful tools and techniques.

Example 1. (*1983 IMO Shortlisted Problem proposed by Finland*) Let p and q be integers with q>0. Show that there exists an interval I of length $1/q$ and a polynomial P with integral coefficients such that

$$\left|P(x)-\frac{p}{q}\right|<\frac{1}{q^2}$$

for all $x\in I$.

Solution. Pick $P(x) = p((qx-1)^{2n+1}+1)/q$ and $I = [1/(2q), 3/(2q)]$. Then all the coefficients of P are integers and

$$\left|P(x)-\frac{p}{q}\right|=\left|\frac{p}{q}(qx-1)^{2n+1}\right|\le\left|\frac{p}{q}\right|\frac{1}{2^{2n+1}}$$

for all $x\in I$. Choose n large so that $2^{2n+1} > |pq|$. Then we are done.

Example 2 (*1994 IMO*) Let m and n be positive integers. The set $A=\{a_1, a_2, \ldots, a_m\}$ is a subset of 1,2,…,n. Whenever $a_i+a_j\le n$, $1\le i\le j\le m$, a_i+a_j also belong to A. Prove that

$$\frac{a_1+a_2+\cdots+a_m}{m}\ge\frac{n+1}{2}.$$

Solution. We may assume that $a_1>a_2>\cdots>a_m$. We claim that for i=1,2,…,m,

$$a_i+a_{m+1-i}\ge n+1. \qquad (*)$$

If not, then $a_i+a_{m+1-i}, \ldots, a_i+a_{m-1}, a_i+a_m$ are i different elements of A greater than a_i, which is impossible. By adding the cases i=1,2,…,m of (*), we get

$$2(a_1+\cdots+a_m)\ge m(n+1).$$

The result follows.

Example 3 (2001 IMO Shortlisted Problem proposed by Bulgaria). Find all positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{99}{100}=\frac{a_0}{a_1}+\frac{a_1}{a_2}+\cdots+\frac{a_{n-1}}{a_n},$$

where a_0=1 and $(a_{k+1}-1)a_{k-1}\ge a_k^2(a_k-1)$ for k=1,2,…,$n-1$.

Solution. Let $a_1, a_2, \ldots, a_n$ satisfy the conditions of the problem. Then $a_k>a_{k-1}$ and hence $a_k\ge 2$ for k=1,2,…,n. The inequality $(a_{k+1}-1)a_{k-1}\ge a_k^2(a_k-1)$ can be rewritten as

$$\frac{a_{k-1}}{a_k}+\frac{a_k}{a_{k+1}-1}\le\frac{a_{k-1}}{a_k-1}.$$

Adding these inequalities for $k = i$+1,…, $n-1$ and using $a_{n-1}/a_n < a_{n-1}/(a_n-1)$, we obtain

$$\frac{a_i}{a_{i+1}}+\cdots+\frac{a_{n-1}}{a_n}<\frac{a_i}{a_{i+1}-1}.$$

Then

$$\frac{a_i}{a_{i+1}}\le\frac{99}{100}-\frac{a_0}{a_1}-\cdots-\frac{a_{i-1}}{a_i}<\frac{a_i}{a_{i+1}-1} \qquad (*)$$

for i=1,2,⋯,$n-1$. Now given $a_0,a_1,\ldots, a_i$, there is at most one possibility for a_{i+1}. By (*), this yields a_1=2, a_2=5, a_3=56, a_4=78400. These values satisfy the condition of the problem. So this is a unique solution.

Example 4 (1999 Polish Math Olympiad). Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be integers. Prove that

$$\sum_{1\le i<j\le n}(|a_i-a_j|+|b_i-b_j|)\le\sum_{1\le i<j\le n}|a_i-b_j|.$$

(continued on page 2)

Solution. For integer x, let $f_{\{a,b\}}(x)=1$ if either $a\le x<b$ or $b\le x<a$ and $f_{\{a,b\}}(x)=0$ otherwise. Observe that when a,b are integers, $|a-b|$ equals the sum of $f_{\{a,b\}}(x)$ over all integers x. Now fix an integer x and suppose $a_\le$ is the number of values of i for which $a_i\le x$.

Define $a_>$, $b_\le$, $b_<$ analogously. We have

$$\begin{aligned}&(a_\le-b_\le)+(a_>-b_>)\\&=(a_\le+a_<)-(b_\le+b_>)\\&=n-n=0,\end{aligned}$$

which implies $(a_\le-b_\le)(a_>-b_>)\le 0$. Thus

$$a_\le a_> + b_\le b_> \le a_\le b_> + a_> b_\le.$$

Now

$$a_\le a_> = \sum_{1\le i<j\le n} f_{\{a_i,b_j\}}(x).$$

because both sides count the same set of pairs and the other terms reduce similarly, yielding

$$\sum_{1\le i<j\le n} f_{\{a_i,a_j\}}(x)+f_{\{b_i,b_j\}}(x)\le \sum_{1\le i<j\le n} f_{\{a_i,b_j\}}(x).$$

Because x was an arbitrary integer, this last inequality holds for all integers x. Summing over all integers x and using our first observation, we get the desired inequality. Equality holds if and only if the above inequality is an inequality for all x, which is true precisely when the a_i equal the b_i in some order.

Example 5 (2007 Chinese Math Olympiad). Let a,b,c be complex numbers. Let $|a+b|=m$, $|a-b|=n$ and $mn\ne 0$. Prove that

$$\max\{|ac+b|,|a+bc|\}\ge \frac{mn}{\sqrt{m^2+n^2}}.$$

Solution. Since

$$\begin{aligned}&\max\{|ac+b|,|a+bc|\}\\ \ge\ &\frac{|b|\cdot|ac+b|+|a|\cdot|a+bc|}{|b|+|a|}\\ \ge\ &\frac{|b(ac+b)-a(a+bc)|}{|a|+|b|}=\frac{|b^2-a^2|}{|a|+|b|}\\ \ge\ &\frac{|b+a|\cdot|b-a|}{\sqrt{2(|a|^2+|b|^2)}}\end{aligned}$$

and $m^2+n^2=|a-b|^2+|a+b|^2=2(|a|^2+|b|^2)$, so

$$\max\{|ac+b|,|a+bc|\}\ge \frac{mn}{\sqrt{m^2+n^2}}.$$

Example 6 (1999 Balkan Math Olympiad). Let $x_0,x_1,x_2,\ldots$ be a non-decreasing sequence of nonnegative integers such that for every $k\ge 0$, the number of terms of the sequence which are less than or equal to k is finite; let this number be y_k. Prove that for all positive integers m and n,

$$\sum_{i=0}^n x_i+\sum_{j=0}^m y_j\ge (n+1)(m+1).$$

Solution. Under the given construction, $y_s\le t$ if and only if $x_t>s$. Thus the sequences $x_0,x_1,x_2,\ldots$ and $y_0,y_1,y_2,\ldots$ are dual, meaning that applying the given algorithm to $y_0,y_1,y_2,\ldots$ will restore the original $x_0,x_1,x_2,\ldots$.

To find $x_0+x_1+\cdots+x_n$, observe that among the numbers $x_0,x_1,\cdots,x_n$, there are exactly y_0 terms equal to 0, y_1-y_0 terms equal to 1, … and $y_{x_{n-1}}-y_{x_{n-2}}$ terms equal to x_{n-1}, while the remaining $n+1-x_{n-1}$ terms equal to x_n. Hence, $x_0+x_1+\cdots+x_n$ equals

$$\begin{aligned}&\sum_{i=1}^{x_n-1} i(y_i-y_{i-1})+x_n(n+1-y_{x_n-1})\\&=-y_0-y_1-\cdots-y_{x_{n-1}}+(n+1)x_n.\end{aligned}$$

First suppose that $x_n-1\ge m$. Write $x_n-1=m+k$ for $k\ge 0$. Because $x_n>m+k$, from our initial observations we have $y_{m+k}\le n$. Then

$$n+1\ge y_{m+k}\ge y_{m+k-1}\ge\cdots\ge y_m.$$

So

$$\begin{aligned}\sum_{i=0}^n x_i+\sum_{j=0}^m y_j &= (n+1)x_n-\left(\sum_{j=0}^{x_{n-1}} y_j-\sum_{i=0}^m y_i\right)\\ &= (n+1)x_n-\sum_{i=m+1}^{m+k} y_i\\ &\ge (n+1)(m+k+1)-k(n+1)\\ &= (n+1)(m+1).\end{aligned}$$

Next suppose that $x_n-1<m$. Then $x_n\le m$ implies $y_m>n$, which implies $y_m-1\ge n$. Because $x_0,x_1,x_2,\ldots$ and $y_0,y_1,y_2,\ldots$ are dual, we may apply the same argument with the roles of the two sequences reversed. This completes the proof.

Example 7 (2007 Chinese Girls' Math Olympiad). Let m,n be integers, $m>n\ge 2$, $S=\{1,2,\ldots,m\}$ and $T=\{a_1,a_2,\ldots,a_n\}$ be a subset of S. Suppose every two elements of T are not both the divisors of any element of S. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{m}.$$

Solution. For $i=1,2,\ldots,n$, let k_i be the integer such that $k_i\le m/a_i<k_i+1$. Let $T_i=\{ka_i : k=1,\ldots,k_i\}$. Then $|T_i|=k_i$ Since every two elements of T are not both the divisors of any element of S, so if $i\ne i'$, then $T_i\cap T_{i'}$ is empty. Hence,

$$\sum_{i=1}^n k_i=\sum_{i=1}^n |T_i|=|T|\le|S|=m.$$

Since $m/a_i<k_i+1$, we have

$$m\sum_{i=1}^n 1/a_i\le\sum_{i=1}^n (k_i+1)\le m+n.$$

Dividing by m, we get the desired conclusion.

Example 8 (1987 IMO Shortlisted Problem proposed by Netherland). Given five real numbers u_0, u_1, u_2, u_3, u_4, prove that it is always possible to find five real numbers v_0, v_1, v_2, v_3, v_4 that satisfy the following conditions:

(i) $u_i-v_i\in\mathbb{N}$.
(ii) $\Sigma_{0\le i<j\le 4}(v_i-v_j)^2<4$.

Solution. Observe that

$$\begin{aligned}\sum_{0\le i<j\le 4}(v_i-v_j)^2 &= \sum_{0\le i<j\le 4}[(v_i-v)-(v_j-v)]^2\\ &= 5\sum_{i=0}^4(v_i-v)^2-\left(\sum_{i=0}^4(v_i-v)\right)^2\\ &\le 5\sum_{i=0}^4(v_i-v)^2.\end{aligned}$$

Let us take v_i's satisfying the last line with $v_0\le v_1\le v_2\le v_3\le v_4\le 1+v_0$. Define $v_5=1+v_0$. We see that one of the differences $v_{i+1}-v_i$, $i=0,\ldots,4$, is at most 1/5. Let $v=(v_{i+1}+v_i)/2$. Then place the other three v_i's in $[v-1/2,v+1/2]$. Now we have $|v-v_i|\le 1/10$, $|v-v_{i+1}|\le 1/10$ and $|v-v_k|\le 1/2$ for any k other than i and $i+1$. Finally, we have

$$\sum_{0\le i<j\le 4}(v_i-v_j)^2\le 5(2(1/10)^2+3(1/2)^2)<4.$$

Example 9 (2000 Romanian Math Olympiad). Let $n\ge 1$ be an odd positive integer and $x_1, x_2, \ldots, x_n$ be real numbers such that $|x_{k+1}-x_k|\le 1$ for $k=1,2,\ldots,n-1$. Show that

$$\sum_{k=1}^n|x_k|-\left|\sum_{k=1}^n x_k\right|\le\frac{n^2-1}{4}.$$

Solution. Let P, N be the sets of positive, negative numbers among $x_1, x_2, \ldots, x_n$ respectively. Without loss of generality, assume that there are more k such that x_k is negative than there are k such that x_k is positive. Let $(a_1,\ldots,a_n)$ be a permutation of $(x_1,\ldots,x_n)$ such that $a_1,\ldots,a_n$ is a nondecreasing sequence. By construction, $|P|\le(n-1)/2$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***December 1, 2018.***

Problem 521. Given 20 points in space so that no three of them are collinear, prove that the number of planes determined by these points is not equal to 1111.

Problem 522. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x and y,

$$(x-2)\,f(y)+f\,(y+2\,f(x))=f\,(x+y\,f(x)).$$

Problem 523. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that $P(d)=(n/d)^2$ for each positive divisor d of n.

Problem 524. (proposed by *Andrew WU*, St. Albans School, Mc Lean, VA, USA) In ΔABC with centroid G, M and N are the midpoints of AB and AC, and the tangents from M and N to the circumcircle of ΔAMN meet BC at R and S, respectively. Point X lies on side BC satisfying $\angle CAG=\angle BAX$. Show that GX is the radical axis of the circumcircles of ΔBMS and ΔCNR.

Problem 525. Find all positive integer n such that $n(n+2)(n+4)$ has at most 15 positive divisors.

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Solutions

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Problem 516. Determine all triples (p,m,n) of positive integers such that p is prime and $2^mp^2+1=n^5$ holds.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School) and **ZHANG Yupei** (HKUST).

Let $q=n^4+n^3+n^2+n+1$. Then $2^mp^2=(n-1)q$ and $\gcd(n-1,q)=\gcd(n-1,5)=1$ or 5. Now $q>1$ is odd and so p is an odd prime. Let $p=2k+1$. Then $\gcd(2^m,p^2)=1$. So $n-1=2^m$, $q=p^2$. Then $n=2^m+1$. So $n^4+n^3+n^2+n=p^2-1$ can be expressed as

$$(2^{2m}+2^{m+1}+2)(2^{2m}+3\cdot 2^m+2)=4k(k+1).$$

If $m\geq 2$, then the left side is 4 (mod 8) and the right side is 0 (mod 8). Hence, $m=1$. Then $p=11$ and $n=3$. So $(p,m,n)=(11,1,3)$ only.

Other commended solvers: **Ioan Viorel CODREANU** (Satulung, Maramures, Romania), **Akash Singha ROY** (West Bengal, India), **Ioannis D. SFIKAS** (Athens, Greece), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Problem 517. For all positive x and y, prove that

$$x^2y^2(x^2+y^2-2)\geq(xy-1)(x+y).$$

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School).

Let $k=xy$. We have

$$2\sqrt{k}-\frac{2k+2}{2\sqrt{k}}-\frac{k-1}{k^2}$$
$$=\frac{(\sqrt{k}-1)^2(k\sqrt{k}+2k+2\sqrt{k}+1)}{k^2}\geq 0$$

Since $2\sqrt{k}=2\sqrt{xy}\leq x+y,$ so

$$\frac{x^2+y^2-2}{x+y}=x+y-\frac{2xy+2}{x+y}$$

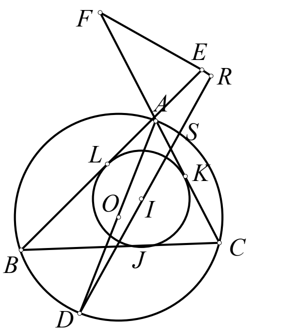
$$\geq 2\sqrt{k}-\frac{2k+2}{2\sqrt{k}}\geq\frac{k-1}{k^2}=\frac{xy-1}{x^2y^2}.$$

Then $x^2y^2(x^2+y^2-2)\geq(xy-1)(x+y)$.

Other commended solvers: **LEUNG Hei Chun**, **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Ioannis D. SFIKAS** (Athens, Greece), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Problem 518. Let I be the incenter and AD be a diameter of the circumcircle of ΔABC. Let point E be on the ray BA and point F be on the ray CA. If the lengths of BE and CF are both equal to the semiperimeter of ΔABC, then prove that lines EF and DI are perpendicular.

Solution. **ZHANG Yupei** (HKUST).



Let circle ABC intersect line DI at S. Let K, J, L be the feet of the perpendiculars from I to sides AC, CB, BA of ΔABC respectively. Since AD is a diameter of the circumcircle of ΔABC, we get $\angle ASD=\angle AKI=\angle ALI=90°$. So A,S,K,I,L are concyclic.

Next, $\angle BLS=180°-\angle ALS=180°-\angle AKS=\angle CKS$ and $\angle LBS=\angle KCS$. So ΔBLS, ΔCKS are similar. Since $BE=CF$, $AF/AE=BL/CK=SB/SC$. We get $\angle EAF=\angle CAB=\angle CSB$. So $\Delta EAF\cong\Delta CSB$. Then $\angle SBC=\angle SAC=\angle EFA$. We get $EF\parallel AS$. Then $DI\perp EF$.

Other commended solvers: **Andrea FANCHINI** (Cantù, Italy), **William KAHN** (Sidney, Australia), **Akash Singha ROY** (West Bengal, India), **Ioannis D. SFIKAS** (Athens, Greece), and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 519. Let A and B be subsets of the positive integers with 10 and 9 elements respectively. Suppose for every $x,y,u,v\in A$ satisfying $x+y=u+v$, we have $\{x,y\}=\{u,v\}$. Prove that the set $A+B=\{a+b\colon a\in A,\, b\in B\}$ has at least 50 elements.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School).

If a_1, $a_2\in A$ and b_1, $b_2\in B$ such that $a_1+b_1=a_2+b_2$, then $a_1-a_2=b_2-b_1$ (with $a_1\neq a_2$ and $b_1\neq b_2$). Assume the equation $x+b_1=y+b_2$ has two distinct solutions $(x,y)=(a_3,\,a_4)$ and $(a_5,\,a_6)$ such that a_3, a_4, a_5, $a_6\in A$. Then we have $a_3-a_4=b_2-b_1=a_5-a_6$, which implies $a_3+a_6=a_4+a_5$. By the condition of A, we have $\{a_3,\,a_6\}=\{a_4,\,a_5\}$. Then we have 2 cases.

Case 1: $a_3=a_4$ and $a_5=a_6$. From $a_3+b_1=a_4+b_2$, we get $b_1=b_2$. Then $|a_3-a_4|+|b_1-b_2|=0$, contradiction.

Case 2: $a_3=a_5$ and $a_4=a_6$. Then $(a_3,\,a_4)=(a_5,\,a_6)$, contradiction.

So $x+b_1=y+b_2$ has at most one solution. Since there are 36 choices of $b_1\neq b_2\in B$, so there must be 36 solutions of $(a_1,\,a_2,\,b_1,\,b_2)$ such that $a_1\neq a_2\in A$, $b_1\neq b_2\in B$ and $a_1+b_1=a_2+b_2$.

However, we have a_1+b_1, $a_2+b_2\in A+B$. Since $A+B$ has 90 not necessary distinct elements, so $A+B$ has at least 54 distinct elements. In particular, $A+B$ has at least 50 distinct elements.

Other commended solvers: **William KAHN** (Sidney, Australia), **Akash Singha ROY** (West Bengal, India), **George SHEN, Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Problem 520. Let P be the set of all polynomials $f(x)=ax^2+bx$, where a,b are nonnegative integers less than 2010^{18}. Find the number of polynomials f in P for which there is a polynomial g in P such that $g(f(k))\equiv k \pmod{2010^{18}}$ for all integers k.

Solution. **William KAHN** (Sidney, Australia) and **George SHEN**.

We will show that there exists $Q(x) = cx^2+dx$ for $P(x) = ax^2+bx$ if and only if $2^8 1005^9|a$ and $\gcd(2010,b)=1$. Then it follows that the answer is $2\cdot2010^9\cdot 2010^{18}(1-1/2)(1-1/3)(1-1/5)(1-1/67) = 2^5 3\cdot 11\cdot 2010^{26}$.

Assume that $Q(P(n))\equiv n \pmod{2010^{18}}$ for all n. Then $n\rightarrow P(n)$ is one-to-one $\pmod{2010^{18}}$ and using the Chinese remainder theorem we deduce that $n\rightarrow P(n)$ is one-to-one $\pmod{p^{18}}$ for p in $\{2,3,5,67\}$.

Let $p\in\{2,3,5,67\}$. If $p|b$, then $P(p^{17})\equiv P(0) \pmod{p^{18}}$ gives a contradiction. Hence, $p\nmid b$. If $p\nmid a$, then $P(-a^{-1}b)\equiv P(0) \pmod{p^{18}}$ gives a contradiction. So $p\mid a$. Hence $2010\mid a$ and $\gcd(2010,b) = 1$. In particular, $(b(a^2-b^2))^{-1} \pmod{2010^{18}}$ exists. Since

$$\begin{aligned}
&Q(P(1))\equiv 1 \pmod{2010^{18}}\\
\Rightarrow\ &c(a+b)^2+d(a+b)\equiv 1 \pmod{2010^{18}}\\
\Rightarrow\ &2b(a^2-b^2)c\equiv 2a \pmod{2010^{18}}
\end{aligned}$$

and

$$\begin{aligned}
&Q(P(-1))\equiv -1 \pmod{2010^{18}}\\
\Rightarrow\ &c(a-b)^2+d(a-b)\equiv -1 \pmod{2010^{18}}\\
\Rightarrow\ &2b(a^2-b^2)d\equiv -2(a^2+b^2) \pmod{2010^{18}}
\end{aligned}$$

we have

$c\equiv(b(a^2-b^2))^{-1}a+2010^{18}e \pmod{2010^{18}}$ and $d\equiv -(b(a^2-b^2))^{-1}(a^2+b^2) + 2010^{18}e \pmod{2010^{18}}$, where $e = 0$ or ½.

Therefore,

$$\begin{aligned}
&Q(P(x))-x\\
\equiv\ &-(b(a^2-b^2))^{-1}a^2x(x-1)(x+1)(ax+b)\\
&+2010^{18}ex(x-1)\\
\equiv\ &-(b(a^2-b^2))^{-1}a^2x(x-1)(x+1)(ax+2b)\\
&\pmod{2010^{18}}.
\end{aligned}$$

Now if $x=2$, we get $2010^{18}\mid 2^2 3a^2$, hence $2^8 1005^9\mid a$.

Conversely, if $2^8 1005^9\mid a$ and $\gcd(2010,b)=1$, then we can define c and d as above. Since $2\mid n(n-1)$ and $2\mid an+2b$ for all n, $Q(P(n))\equiv n \pmod{2010^{18}}$ follows.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Olympiad Corner

(*Continued from page 1*)

Problem 3. Alice and Bob play the following game: They start with two non-empty piles of coins. Taking turns, with Alice playing first, each player choose a pile with an even number of coins and moves half of the coins of this pile to the other piles. The game ends if a player cannot move, in which case the other player wins. (*Cyprus*)

Problem 4. Find all primes p and q such that $3p^{q-1}+1$ divides 11^p+17^p. (*Bulgaria*)

Miscellaneous Inequalities

(*Continued from page 2*)

Suppose that $1\le i\le n-1$. In the sequence $x_1,\ldots,x_n$, there must be two adjacent terms x_k and x_{k+1} which are separated by the interval (a_i, a_{i+1}), i.e. such that either $x_k\le a_i\le a_{i+1}\le x_{k+1}$ or $x_{k+1}\le a_i\le a_{i+1}\le x_k$. So $a_{i+1}-a_i\le|x_k-x_{k+1}|\le 1$. That is $a_1,\ldots,a_n$ is a nondecreasing sequence of terms, such that any two adjacent terms differ by at most 1.

Let σ_P denote the sum of the numbers in P. We claim that $\sigma_P\le(n^2-1)/8$. This is certainly true if P is empty.

If P is nonempty, then the elements of P are $a_i\le a_{i+1}\le\cdots\le a_n$ for some $2\le i\le n$. Because $a_{i-1}\le 0$ by assumption and $a_i\le a_{i-1}+1$ from the previous paragraph, we have $a_i\le 1$.

Similarly, $a_{i+1}\le a_i+1\le 2$ and so on up to $a_n\le|P|$. Hence, $\sigma_P\le 1+2+\cdots+|P|$. From $|P|\le(n-1)/2$, we get $\sigma_P\le(n^2-1)/8$, as claimed.

Let σ_N denote the sum of the numbers in N. The left-hand side of the required inequality then equals

$$\begin{aligned}
&|\sigma_P-\sigma_N|-|-\sigma_P-\sigma_N|\\
\le\ &|2\sigma_P|\\
\le\ &2\left(\frac{n^2-1}{8}\right)=\frac{n^2-1}{4}
\end{aligned}$$

as needed.

<u>*Example 10 (2000 Asia Pacific Math Olympiad).*</u> Let n, k be positive integers with $n > k$. Prove that

$$\frac{1}{n+1}\cdot\frac{n^n}{k^k(n-k)^{n-k}}<\frac{n!}{k!(n-k)!}<\frac{n^n}{k^k(n-k)^{n-k}}.$$

<u>*Solution.*</u> By the binomial theorem, we have $n^n=(k+(n-k))^n=a_0+\cdots+a_n$, where for $i=0,1,\ldots,n$,

$$a_i=\binom{n}{i}k^i(n-k)^{n-i}>0.$$

We claim that

$$\frac{n^n}{n+1}<a_i<n^n.$$

The right inequality holds because $n^n=a_0+\cdots+a_n>a_i$. To prove the left inequality, it suffices to prove that a_i is larger than $a_0,\ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ because then

$$n^n=\sum_{m=0}^{n}a_m<\sum_{m=0}^{n}a_i=(n+1)a_i.$$

Next, we will show a_i is increasing for $i\le k$ and decreasing for $i\ge k$. Observe that

$$\binom{n}{i}=\frac{i+1}{n-i}\binom{n}{i+1}.$$

Hence

$$\frac{a_i}{a_{i+1}}=\frac{\binom{n}{i}k^i(n-k)^{n-i}}{\binom{n}{i+1}k^{i+1}(n-k)^{n-i-1}}=\frac{n-k}{n-i}\cdot\frac{i+1}{k}.$$

This expression is less than 1 when $i<k$ and it is greater than 1 when $i\ge k$. In other words, $a_0<\cdots<a_k$ and $a_k>\cdots>a_n$ as desired.

Mathematical Excalibur

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Olympiad Corner

Below were the Day 1 problems of the Croatian Mathematical Olympiad which took place on May 5, 2018.

Problem A1. Let a, b and c be positive real numbers such that $a+b+c=2$. Prove that

$$\frac{(a-1)^2}{b}+\frac{(b-1)^2}{c}+\frac{(c-1)^2}{a} \geq \frac{1}{4}\left(\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\right).$$

Problem C1. Let n be a positive integer. A *good word* is a sequence of $3n$ letters, in which each of the letters A, B and C appears exactly n times. Prove that for every good word X there exists a good word Y such that Y cannot be obtained from X by swapping neighbouring letters fewer than $3n^2/2$ times.

Problem G1. Let k be a circle centered at O. Let $\overline{AB}$ be a chord of that circle and M its midpoint. Tangent on k at points A and B intersect at T. The line ℓ goes through T, intersects the shorter arc AB at the point C and the longer arc AB at the point D, so that $|BC|=|BM|$.

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 15, 2019***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Austrian Math Problems

Kin Y. Li

In this article, we would like to look at some of the Austrian Math Olympiad problems. This competition is going into its 50th year. For the young math students, the Austrian math problems are treasures that are everlasting, especially the problems appeared in the recent decades. Below are some examples that we hope you will enjoy.

<u>*Example 1.*</u> (*Beginners Competition: June 7th, 2001*) Prove that the number n^n-1 is divisible by 24 for all odd positive integer values of n.

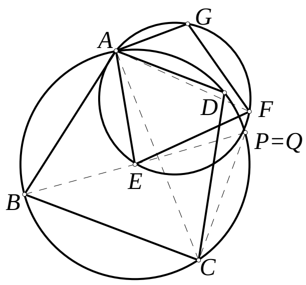
<u>*Solution.*</u> Since n is an odd positive integer, we can write $n=2k+1$ with $k=0,1,2,\ldots$. Substituting yields

$$n^n-1=n(n^{n-1}-1)=n(n^{2k}-1).$$

Since $1^2\equiv3^2\equiv5^2\equiv7^2\equiv1 \pmod 8$, we see that $n^{2k}\equiv1 \pmod 8$ certainly holds, and $n^{2k}-1$ is therefore divisible by 8.

If n is divisible by 3, we see that $n(n^{2k}-1)$ is certainly divisible by $3\cdot8=24$ as required. If n is not divisible by 3, we note that $1^2\equiv2^2\equiv1 \pmod 3$, and $n^{2k}\equiv1 \pmod 3$ holds, so that $n^{2k}-1$ is not only divisible by 8, but also by 3. It follows that $n^{2k}-1$ is therefore divisble by $3\cdot8=24$, and therefore so is $n(n^{2k}-1)$ as required.

<u>*Example 2*</u> (*National Competition: June 6th, 2002*) Let $ABCD$ and $AEFG$ be similar inscribed quadrilaterals, whose vertices are labeled counter-clockwise. Let P be the second common point of the circumcircles of the quadrilaterals beside A. Show that P must lie on the line connecting B and E.



<u>*Solution.*</u> Rotation and stretching with center A, $\angle BAC$ and factor $AB{:}AC$ maps B onto C and E onto F. This mapping therefore transforms the line $BE=BQ$ onto the line $FC=FQ$, whereby we let Q denote the point of intersection of lines BE and FC. Since this mapping rotates by $\angle BAC$, this is also the angle between the lines BQ and FQ, and since this is equal to $\angle BAC$ (or its supplement), Q must lie on the circumcircle of ΔABC, which is also the circumcircle of $ABCD$. By analogous reasoning, it must also lie on the circumcircle of $AEFG$, and we see that $P=Q$ must hold, which proves that P must lie on the line BE, as required.

<u>*Example 3*</u> (*National Competition: May 26th, 2004*). Prove without the use of calculus:

a) If a, b, c and d are real numbers, then

$$a^6+b^6+c^6+d^6-6abcd \geq -2$$

holds. When does equality hold?

b) For which positive integers k does there exist an inequality of the form

$$a^k+b^k+c^k+d^k-kabcd \geq M_k$$

that holds for all real values of a, b, c and d? Determine the largest possible values of M_k and determine when equality holds.

<u>*Solution.*</u> a) The given inequality can be proved by applying the AM-GM inequality as

$$\frac{a^6+b^6+c^6+d^6+1^6+1^6}{6} \geq |abcd| \geq abcd.$$

Equality holds for $|a|=|b|=|c|=|d|=1$, more precisely when (a,b,c,d) equals one of

$(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,1,-1)$, $(-1,1,1,-1)$, $(1,-1,-1,1)$, $(-1,1,-1,1)$, $(-1,-1,1,1)$ or $(-1,-1,-1,-1)$.

(continued on page 2)

b) First of all, we note that no such number M_k can possibly exist if k is odd, since a choice of negative values for a, b, c and d with sufficiently large absolute value yields negative values with arbitrary large absolute value for the expression $a^k+b^k+c^k+d^k-kabcd$.

Similarly, no such number exists for k=2, since a choice of $a=b=c=d=r$ yields $a^2+b^2+c^2+d^2-2abcd = 4r^2-2r^4$, for which a choice of sufficiently large values of r again yields negative values with arbitrarily large absolute value.

This leaves even values of k with $k\geq 4$ to consider. In this case, choosing $a=b=c=d=1$ yield $a^k+b^k+c^k+d^k-kabcd = 4-k$, and as in a), we can apply AM-GM inequality to get

$$\frac{a^k+b^k+c^k+d^k+(k-4)1^k}{k} \geq |abcd| \geq abcd$$

with equality for the same values of (a,b,c,d) as in a).

<u>*Example 4*</u> (*National Competition: June 6th, 2007*) We are given a convex n-gon with a triangulation, i.e. a division into triangles by non-intersecting diagonals. Prove that the n corners of the n-gon can each be labeled by the digits of 2007 such that any quadrilateral composed of two triangles in the triangulation with a common side has corners labeled by digits with the sum 9.

<u>*Solution.*</u> We shall prove this by induction on n. If n=4, we label the vertices 2, 0, 0, 7 and the claim holds. (Note that this is the only possible combination of digits summing to 9, since $4\cdot 2<9$ and $2\cdot 7>9$ hold. Also note that the three corners of any triangle must be labeled with three of the digits 2, 0, 0, 7.)

We now assume that the claim holds as stated for any convex n-gon, and consider a convex $(n+1)$-gon. Any triangulation of such an $(n+1)$-gon certainly contains at least one triangle (in fact, at least two), two of whose sides are consecutive sides of the $(n+1)$-gon with common vertex V. The n-gon obtained by removing this one triangle from the triangulation with the implied triangulation in the remaining n-gon as given can certainly be labeled as required.

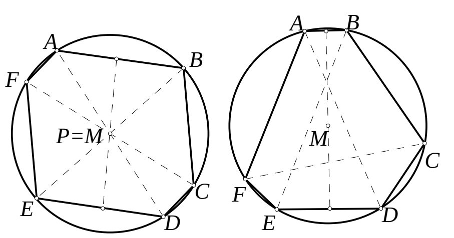
We now note that the triangle with vertex V only has a side in common with one other triangle of the triangulation, the corners of which are already labeled with three of the four required digits. Labelling V with the fourth digit results in a labeling of the $(n+1)$-gon with the required property.

<u>*Example 5*</u> (*National Competition: June 3rd, 2010*) A diagonal in a hexagon is considered a <u>*long*</u> diagonal if it divides the hexagon into two quadrilaterals. Any two long diagonals divide the hexagon into two triangles and two quadrilaterals.

We are given a convex hexagon with the property that the division into pieces by any two long diagonals always yields two isosceles triangles with sides of the hexagon as bases. Show that such a hexagon must have a circumcircle.

<u>*Solution.*</u> Since any two opposing isosceles triangles (such as ABP and DEP) have a common angle at their vertices, they must be similar, and their bases therefore parallel. The angle bisector in their common vertex is therefore also the common altitude.

If all three diagonals of the hexagon intersect at M, this point is also a common point of all angle bisectors. It must therefore be the same distance from A to B, as it lies on the bisector of AB, but the same holds for B and C, C and D, and so on. This point is therefore equidistant from all corners of the hexagon, and is therefore the mid-point of the circumcircle of the hexagon.



If the diagonals of the hexagon do not have a common point, they form a triangle. The angle bisectors have a common point, namely the incenter of this triangle, which we again call M. The same holds for this point M as in the previous situation, and we once again have established the existence of a circumcircle of the hexagon, as claimed.

<u>*Example 6*</u> (*National Competition: May 1st, 2015*) A <u>*police emergency number*</u> is a positive integer that ends with the digits 133 in decimal representation. Prove that every police emergency number has a prime factor larger than 7.

(*In Austria, 133 is the emergency number of the police*.)

<u>*Solution.*</u> Let $n=1000k+133$ be a police emergency number and assume that all its prime divisors are at most 7. It is clear from the last digit that n is odd and that n is not divisible by 5, so $1000k+133 = 3^a7^b$ for suitable integers $a,b\geq 0$. Thus, $3^a7^b\equiv 133 \pmod{1000}$.

This also implies $3^a7^b\equiv 133\equiv 5 \pmod 8$. We know that 3^a is congruent to 1 or 3 modulo 8 and 7^b is congruent to 1 or 7 modulo 8. In order for the product 3^a7^b to be congruent to 5 modulo 8, 3^a must therefore be congruent to 3 and 7^b must be congruent to 7. Therefore, we can conclude that a and b are both odd.

We also have $3^a7^b\equiv 133\equiv 3 \pmod 5$. As a and b are odd, 3^a and 7^b are each congruent to 3 or 2 modulo 5. Neither 3^2, nor $3\cdot 2$ is congruent to 3 modulo 5, a contradiction.

<u>*Example 7*</u> (*National Competition: April 30th, 2016*) Consider 2016 points arranged on a circle. We are allowed to jump ahead by 2 or 3 points in clockwise direction. What is the minimum number of jumps required to visit all points and return to the starting point?

<u>*Solution.*</u> Clearly it takes at least 2016 jumps to visit all points. It is impossible to use only jumps of length 2 or only jumps of length 3 because this would confine us to a single residue class modulo 2 or 3 respectively.

If the problem could be solved with 2016 jumps, the total distance covered by these jumps would be strictly between $2\cdot 2016$ and $3\cdot 2016$ which makes a return to the original point impossible. Therefore, at least 2017 jumped are required.

This is indeed possible, for example with the following sequence of points on the circle

0,3,6,…,2013,2015, 2,5,…,2012,2014, 1,4,…, 2011, 2013,0.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***February 15, 2019.***

Problem 526. Let $a_1=b_1=c_1=1$, $a_2=b_2=c_2=3$ and for $n\geq 3$, $a_n=4a_{n-1}-a_{n-2}$,

$$b_n=\frac{b_{n-1}^2+2}{b_{n-2}},\, c_n=2c_{n-1}+\sqrt{3c_{n-1}^2-2}.$$

Prove that $a_n=b_n=c_n$ for all n=1,2,3,….

Problem 527. Let points O and H be the circumcenter and orthocenter of acute ΔABC. Let D be the midpoint of side BC. Let E be the point on the angle bisector of $\angle BAC$ such that $AE\perp HE$. Let F be the point such that $AEHF$ is a rectangle. Prove that points D, E, F are collinear.

Problem 528. Determine all positive integers m satisfying the condition that there exists a unique positive integer n such that there exists a rectangle which can be decomposed into n congruent squares and can also be decomposed into $n+m$ congruent squares.

Problem 529. Determine all ordered triples (x,y,n) of positive integers satisfying the equation $x^n+2^{n+1}=y^{n+1}$ with x is odd and the greatest common divisor of x and n+1 is 1.

Problem 530. A square can be decomposed into 4 rectangles with 12 edges. If square $ABCD$ is decomposed into 2005 convex polygons with degrees of A, B, C, D at least 2 and degrees of all other vertices at least 3, then determine the maximum number of edges in the decomposition.

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Solutions

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Problem 521. Given 20 points in space so that no three of them are collinear, prove that the number of planes determined by these points is not equal to 1111.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Eren KIZILDAG** (MIT), **LEUNG Hei Chun** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Assume the number of planes is 1111. The 20 points would define $(20\cdot 19\cdot 18)/3!$ = 1140 planes so that 1140−1111=29 triplets of points lie in the planes already determined by other triplets. If one of the planes contain 7 or more points, then there are $(7\cdot 6\cdot 5)/3!$ = 35 triplets of points in this plane and the number of triplets is greater than the number of planes by at least 35−1=34. So the greatest possible number of planes is 1140−34=1105. Clearly, this cannot happen if there are 1111 planes.

So each plane can contain at most 6 of the points. Let a, b, c be the number of planes containing 4, 5, 6 points respectively. When counting triplets, in cases k=4,5,6, we consider each plane containing k points $k(k-1)(k-2)/3!$ = 4, 10, 20 times, which are 3, 9, 19 times too many, respectively. So the number of planes satisfies $1140-3a-9b-19c = 1111$. Hence $3a+9b+19c=29$. However, there are no nonnegative integers a,b,c satisfying $3a+9b+19c=29$. So we arrive at a contradiction.

Other commended solvers: **ZHANG Yupei** (HKUST).

Problem 522. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x and y,

$$(x-2)\, f(y) + f\,(y + 2f(x)) = f\,(x + y\, f(x)).$$

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Eren KIZILDAG** (MIT), **Akash Singha ROY** (West Bengal, India), **Ioannis D. SFIKAS** (Athens, Greece), **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

We will refer to the given equation as (*). In case $f(0)=0$, setting x=0 in (*), we get $f(y)=0$ for all y. In case $f(0)\neq 0$, setting y=0, (*) becomes $(x-2)f(0)+f(2f(x)) = f(x)$ for all real x. If $f(x)=f(x')$, then $x=x'$ and so f is injective.

Next, putting x=2 into (*), we get $f(y+2f(2)) = f(2+yf(2))$ for all real y. Since f is injective, we get $y+2f(2) = 2+yf(2)$ for all real y. Setting y=0, we get $f(2)$=1. Since f is injective, $f(3)\neq 1$. Setting x=3 and $y=3/(1-f(3))$ (which is $y=3+yf(3)$) into (*), we get $f(y+2f(3))=0$. So f has a root at $r=y+f(3)$. Next, setting $y=r$ in (*), we get $f(r+2f(x))=f(x+rf(x))$ for all real x. Since f is injective, we get $r+2f(x) = x+rf(x)$ for all real x.

Now due to $f(2)=1\neq 0$, $r\neq 2$. So $f(x)=(x-r)/(2-r)$. Finally, substituting $f(x)$ by $(x-r)/(2-r)$ we get r=1 so that $f(x)=x-1$. As a result, it is easy to check (*) has the two solutions $f(x)$=0 and $f(x)=x-1$.

Other commended solvers: **Alex Kin Chit O** (G.T. (Ellen Yeung) College).

Problem 523. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that $P(d) = (n/d)^2$ for each positive divisor d of n.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Eren KIZILDAG** (MIT), **LEUNG Hei Chun**, **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

For n=1, let $P(x)=x$, then $P(1)$=1 satisfies the condition. If n is a prime, then its only positive divisors are 1 and n and the conditions on P is $P(1)=n^2$ and $P(n)$=1. We can satisfy this with $P(x)=n^2+(n+1)(1-x)$.

Next we consider $n=km$ is not prime with k,m>1. We have conditions $P(1)=n^2$, $P(k)=m^2$, $P(m)=k^2$ and $P(n)$=1. For arbitrary integers a, b, by factoring, we see $P(a)-P(b)$ is divisible by $a-b$. So $n-k=k(m-1)$ divides $P(n)-P(k)$ = $1-m^2 = (1-m)(1+m)$. This leads to k divides m+1. Similarly, $n-m$ divides $P(n)-P(m)$ and so $m(k-1)$ divides $(1-k)(1+k)$ and m divides k+1. Hence, km divides $(k+1)(m+1)$ and it also divides $(k+1)(m+1)-km = k+m+1$. We must have $km\leq k+m+1$, which implies that $km-k-m+1\leq 2$ or $(k-1)(m-1)\leq 2$. We may assume $k\leq m$. Then the only possible case is k=2 and m=3 so that n=6.

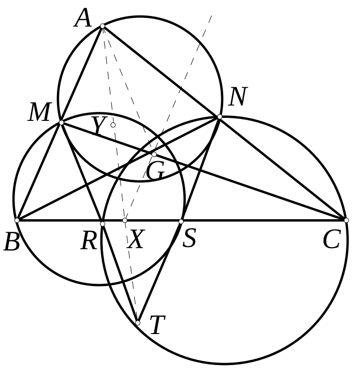
For n=6, we will find a polynomial P such that $P(1)$=36, $P(2)$=9, $P(3)$=4 and $P(6)$=1. We can apply the Lagrange interpolation formula to get $P(x)$ = $1-(x-6)(1+(x-3)(2x-5))$, which can be easily checked to satisfy $P(1)$=36, $P(2)$=9, $P(3)$=4 and $P(6)$=1.

Other commended solvers: **Akash Singha ROY** (West Bengal, India).

Problem 524. (proposed by *Andrew WU*, St. Albans School, Mc Lean, VA, USA) In ΔABC with centroid G, M

and N are the midpoints of AB and AC, and the tangents from M and N to the circumcircle of ΔAMN meet BC at R and S, respectively. Point X lies on side BC satisfying $\angle CAG = \angle BAX$. Show that GX is the radical axis of the circumcircles of ΔBMS and ΔCNR.

Solution. By Proposer.



Observe that BN is the radical axis of the circumcircles of ΔANM and ΔCNR. To prove this, we will show $BM{\cdot}BA=BR{\cdot}BC$ or equivalently that $AMRC$ is a cyclic quadrilateral. By the tangency condition, we have $\angle AMR = 180°-\angle ANM=180°-\angle ACR$, so $AMRC$ is cyclic, as desired. Similarly, we have CM is the radical axis of the circumcircles of ΔANM and ΔBMS. Thus, by the radical center theorem, BN, CM and the radical axis of the circumcircles of ΔBMS and ΔCNR concur. This implies the centroid G lies on the radical axis.

Next, by properties of symmedians, we get lines MR, AX, NS concur at some point T. Suppose lines AX and MN meet at Y. Then by similar triangles, we have $RX/XS=MY/YN=BX/XC$ due to the facts that $\Delta TRS\sim\Delta TMN$ and $\Delta AMN\sim\Delta ABC$.

Thus, it follows that $XR{\cdot}XC=XS{\cdot}XB$. So X has equal power with respect to the circumcircles of ΔBMS and ΔCNR. Then line GX is the radical axis of ΔBMS and ΔCNR.

Other commended solvers: **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Andrea FANCHINI** (Cantù, Italy), **LEUNG Hei Chun** and **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Problem 525. Find all positive integer n such that $n(n+2)(n+4)$ has at most 15 positive divisors.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Ioan Viorel CODREANU** (Satulung, Maramures, Romania), **Eren KIZILDAG** (MIT), **LEUNG Hei Chun**, **Ioannis D. SFIKAS** (Athens, Greece), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Let $a_n=n(n+2)(n+4)$ and let b_n be the number of positive divisors of a_n. The values of b_1 to b_{10} are 4, 10, 8, 14, 12, 24, 12, 28, 12, 40. Next, we recall if a positive integer m has prime factorization $p_1^{e_1}\cdots p_j^{e_j}$, then m has $(e_1+1)\cdots(e_j+1)$ positive divisors. If m divides a positive integer M, then M has at least as many divisors as m.

Let $n\geq 11$. If n is even, say $n=2k$, then $a_n=2^3k(k+1)(k+2)$. At least one of the numbers k, $k+1$, $k+2$ is divisible by 2 and exactly one of them is divisible by 3. Since $k\geq 6$, the numbers k, $k+1$, $k+2$ cannot all be powers of 2 or 3. So $k(k+1)(k+2)$ has a prime divisor p not equal to 2 or 3. Hence, $2^4 3p$ divides a_n and this implies that a_n has at least $5{\cdot}2{\cdot}2 =20$ positive divisors.

Let $n\geq 11$ be odd. Then the numbers n and $n+2$ are relativity prime, as are $n+2$ and $n+4$ and also n and $n+4$. One of these three numbers is divisible by 3. This number has at least one other prime divisor p or else is a power of 3. In the latter case it is divisible by 3^3 since $n\geq 11$. Let q and r be prime divisors of the other two numbers. In the first case the number a_n is divisible by $3pqr$. The number n, $n+2$, $n+4$ are relatively prime, so 3, p, q, r are relatively prime. This implies that a_n has at least $2{\cdot}2{\cdot}2{\cdot}2 =16$ divisors. In the second case a_n is divisible by 3^3qr. The primes 3, q, r are again distinct. So a_n has at least $4{\cdot}2{\cdot}2=16$ divisors.

The number a_n has at most 15 positive divisors only for n=1, 2, 3, 4, 5, 7, 9.

Other commended solvers: **Christos ALVANOS** (Mandoulides, Thessaloniki, Greece), **Alex Kin Chit O** (G.T. (Ellen Yeung) College) and **Akash Singha ROY** (West Bengal, India).

Olympiad Corner

(Continued from page 1)

Problem G1. (*cont.*) Prove that the circumcenter of the triangle ADM is the reflection of O across the line AD.

Problem N1. Determine all pairs (m,n) of positive integers such that

$$2^m = 7n^2+1.$$

Austrian Math Problems

(Continued from page 2)

Example 8 (*National Competition: April 30th, 2017*) Anna and Berta play a game in which they take turns in removing marbles from a table. Anna takes the first turn. When at the beginning of a turn there are $n\geq 1$ marbles on the table, then the player whose turn it is removes k marbles, where $k\geq 1$ either is an even number with $k\leq n/2$ or an odd number with $n/2\leq k\leq n$. A player wins the game if she removes the last marble from the table. Find the smallest $N\geq 100{,}000$ such that Berta can enforce a victory if there are exactly N marbles on the table in the beginning.

Solution. We claim that the losing situations are those with exactly $n=2^a-2$ marbles left on the table for all integers $a\geq 2$. All other situation are winning situations.

For n=1, the player wins by taking the single remaining marble. For n=2, the only possible move is to take k=1 marbles and the opponent wins in the next move. For $n\geq 3$, (1) if n is odd, the player takes all n marbles and wins; (2) if n is even, but not of the form 2^a-2, then n lies between two other numbers of that form, so there is a unique b with $2^b-2<n<2^{b+1}-2$. From $n\geq 3$, we get $b\geq 2$. So all 3 parts of the inequalities are even and so $2^b\leq n\leq 2^{b+1}-4$. By the induction hypothesis, we know 2^b-2 is a losing situation. Taking $k = n-(2^b-2) \leq n/2$ marbles, we leave it to the opponent; (3) if n is even of the form 2^a-2, the player cannot leave a losing situation with 2^b-2 marbles to the opponent (where $b<a$ holds due to at least 1 marble must be removed and $b\geq 2$ holds as after a legal move starting from an even n, at least 1 marble remains). The player would then remove $k=2^a-2^b$ marbles. As $b\geq 2$, k is even and greater than $n/2$ due to $k\geq 2^{a-1}>2^{a-1}-1= n/2$, which is impossible. This means Berta can enforce a victory if and only if N is of the form 2^a-2. The smallest number $N\geq 100{,}000$ of this form is $N = 2^{17}-2 = 131{,}070$.

Mathematical Excalibur

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Olympiad Corner

Below were the Day 2 problems of the Croatian Mathematical Olympiad which took place on May 6, 2018.

Problem A2. determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xf(y))=(1-y)f(xy)+x^2y^2f(y)$$

holds for all real numbers x and y.

Problem C2. Let n be a positive integer. Points $A_1, A_2, \ldots, A_n$ are located on the inside of a circle, and points $B_1, B_2, \ldots, B_n$ are on the circle, so that the lines $A_1B_1, A_2B_2, \ldots, A_nB_n$ are mutually disjoint. A grasshopper can jump from point A_i to point A_j (for $i,j\in\{1,\ldots,n\}$, $i\neq j$) if and only if the lines A_iA_j does not go through any of the inner points of the lines $A_1B_1, A_2B_2, \ldots, A_nB_n$.

Problem G2. Let ABC be an acute-angled triangle such that $|AB| < |AC|$. Point D is the midpoint of the shorter arc BC of the circumcircle of the triangle ABC. Point I is the incenter of the triangle ABC, and point J is the reflection of I across the line BC.

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 25, 2019***.

For individual subscription for the next five issues for the 18-19 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Sum of Digits of Positive Integers

Pedro Pantoja, Natal/RN, Brazil

In this short article we will explore some types of problems in number theory about the sum of digits of a positive integer.

Throughout this article, $S(a)$ will denote the sum of the digits of a positive integer a. For example $S(12)=1+2=3$, $S(349)=3+4+9=16$. Let $c(n,m)$ denote the total number of carries, which arises when adding a and b, for example $c(100,4)=0$, $c(23,17)=1$, $c(88,99)=2$.

Proposition 1. For positive integer a, we have

i) $S(a)\le a$;
ii) $S(a)\equiv a \pmod 9$;
iii) if a is even, then $S(a+1)-S(a)=1$;
iv) $S(a+b)=S(a)+S(b)-9c(a,b)$, in particular, $S(a+b)\le S(a)+S(b)$;
v) $S(ab)\le\min\{aS(b),bS(a)\}$;
vi) $S(ab)\le S(a)S(b)$;
vii) $S(a)\le 9([\log a]+1)$.

Proof. i) and ii) are obvious.

iii) If a is even, then $S(a+1)-S(a)=1$. In fact, a and $a+1$ differ only in the unit digit, which for a will be 0, 2, 4, 6 or 8 and for $a+1$ will be, respectively, 1, 3, 5, 7 or 9.

iv) We proceed by induction on the maximal number of digits k of b and a. If both b and a are single digit numbers, then we have just two cases. If $b+a<10$, then we have nocarries and clearly $S(b+a)=b+a=S(b)+S(a)$. If on the other hand, $b+a=10+k\ge 10$, then

$$S(b+a) = 1+k = 1+(b+a-10) = S(b)+S(a)-9.$$

Assume that the claim holds for all pairs with at most k digits each. Let

$$b = b_1+n\cdot 10^{k+1} \text{ and } a = a_1+n\cdot 10^{k+1},$$

where b_1 and a_1 are at most k digit numbers. If there is no carry at the $k+1^{\text{st}}$ digit, then $c(b,a)=c(b_1,a_1)$ and thus

$$\begin{aligned} S(b+a) &= S(b_1+a_1) + m + n \\ &= S(b_1)+m+S(a_1)+n-9c(n_1,m_1) \\ &= S(b)+S(a)-9c(b,a). \end{aligned}$$

If there is a carry, then $c(n,a) = 1 + c(n_1,ma_1)$ and thus

$$\begin{aligned} S(b+a) &= S(b_1+a_1)+m+n-9 \\ &= S(b_1)+m+S(a_1)+n-9(c(b_1,a_1)+1) \\ &= S(b)+S(a)-9c(b,a). \end{aligned}$$

This finishes the induction and we are done.

v) Because of symmetry, in order to prove v), it suffices to prove that $S(ab) \le aS(b)$. The last inequality follows by applying the subadditivity (iv) property repeatedly. Indeed, $S(2b)=S(b+b)\le S(b)+S(b) = 2S(b)$. After a steps we obtain

$$S(ab) = S(b+\cdots+b) \le S(b)+\cdots+S(b) = aS(b).$$

vi) and vii) Left as exercises for the reader.

For applications, we provide

Example 1: Find all positive integers with $n \le 1000$ such that $n = (S(n))^3$.

Solution: The perfect cube numbers smaller than 1000 are 1, 8, 27, 64, 125, 216, 343, 512, 729. From these numbers the only one that satifies the conditions of the problem is $n = 512$.

Example 2: (*MAIO-2012*) Evaluate

$$S(1) - S(2) + S(3) - S(4) + \cdots + S(2011) - S(2012).$$

Solution: The problem becomes trivial using Proposition 1, item iii). We have $S(3)-S(2)=1$, $S(5)-S(4) = 1$, …, $S(2011) - S(2010) = 1$ and $S(1) = 1$, $S(2012) = 5$. Therefore, $S(1) - S(2) + S(3) - S(4) + \cdots + S(2011) - S(2012) =1+1005-5 =1001$.

(continued on page 2)

Example 3: (*Nordic Contest 1996*) Show that there exists an integer divisible by 1996 such that the sum of its decimal digits is 1996.

Solution. We affirm that the number m = 199619961996…199639923992 satisfies the conditions of the statement. Note that $S(m)=25\cdot 78+2\cdot 23=1996$. On the other hand, m is divisible by 1996, since m equals

$$1996\cdot 100010001000\ldots 1000200002.$$

Example 4: Find $S(S(S(S(2018^{2018}))))$.

Solution: Using proposition 1, item vii) several times we have

$$S(2018^{2018})\le 9([2018\log 2018]+1) < 60030,$$

$$S(S(2018^{2018}))\le 9([\log 60030]+1) < 45,$$

$$S(S(S(2018^{2018})))\le 9([\log 45]+1) < 18.$$

On the other hand, $2018^{2018}\equiv 2^{2018}=(2^3)^{672}\cdot 2^2\equiv 4 \pmod 9$. Hence,

$$S(S(S(2018^{2018}))) = 4 \text{ or } 13.$$

So $S(S(S(S(2018^{2018})))) = 4$.

Example 5: Prove that $S(n)+S(n^2)+S(n^3)$ is a perfect square for infinitely many positive integers n that are not divisible by 10.

Solution: Let us prove that the numbers of the form $n=10^{m^2}-1$ satisfy the problem. The result follows immediately because there are infinitely many number of this form. Firstly, $S(n)=9m^2$ and

$$n^2=10^{2m^2}-2\cdot 10^{m^2}+1=99..9800..01$$

where there are m^2-1 9's and 0's. Then $S(n^2)=9m^2$. Similarly,

$$S(n^3)=99\ldots 9700\ldots 0299\ldots 9,$$

where there are m^2-1 9's and 0's and m^2 9's at the end. Then $S(n^3)=18m^2$. Finally, $S(n)+S(n^2)+S(n^3)=36m^2$.

Remark 1: The numbers of the previous problem are registered in On-Line Encyclopedia of Integer Sequences (OEIS) A153185. Some examples of such numbers: 9, 18, 45, 90, 171, 180, 207, 279, 297, 396, 414, 450, 459, ….

Remark 2: Notice that sometimes mathematical intuition deceives us. That is, the nine numbers 1, 11, 111, …, 111…1 satisfy $S(n^2)=(S(n))^2$. Unfortunately, the next number in this family is

$$1111111111^2=1234567900987654321.$$

So $S(1111111111)=10$, but $S(1111111111^2)=82$. The smallest positive integer such that $S(n)=10$ and $S(n^2)=100$ is $n=1101111211$.

Example 6: We say that a superstitious number is equal to 13 times a sum of its digits. Find all superstitious numbers.

Solution: Obviously there is no superstitious number with one digit. If a two digit number ab is superstitious, then $10a+b=13(a+b)$, that is $3a+12b=0$, which is impossible.

If a three-digit number abc is superstitious, we would have $100a+10b+c=13(a+b+c)$. that is $29a=b+4c$. The maximum possible value for $b+4c$ is 45 (for $b=c=9$). So a must be 1 and the equation $29=b+4c$ has solutions $(b,c)=(1,7),(5,6)$, and $(9,5)$. The numbers 117, 156 and 195 are the only superstitious numbers with three digits.

If a four-digit number abcd is superstitious, it would result in $1000a+100b+10c+d=13(a+b+c+d)$. As the number on the left is at least 1000 and the number on the right is at most $13\cdot 36=468$, there is no superstitious numbers of four digits. Finally, there is no superstitious number with more than four digits, since each added digit contributes at least 1,000 to the number on the left, while the one on the right contributes at most $13\cdot 9=117$. So the only superstitious numbers are 117, 156 and 195.

Example 7: (*Romanian Team Selection Test 2002*) Let $a, b>0$. Prove that the sequence $S([an+b])$ contains a constant subsequence.

Solution. For any positive integer k, let n_k equals $[(10^k+a-b)/b]$. Then

$$\begin{aligned}10^k &= a\left(\frac{10^k+a-b}{a}-1\right)+b\\ &< an_k+b=a\left[\frac{10^k+a-b}{a}\right]+b\\ &\le 10^k+b.\end{aligned}$$

It follows that $10^k=[an_k+b]\le 10^k+b$.

If k is sufficiently large, that is $10^{k-1}>b$, it follows from above that S_{n_k} is one plus the sum of the digits of one of the numbers t in the set $\{0,1,\ldots,[b]\}$. Since k takes infinitely many values and the set of the numbers t is finite, it follows that for infinitely many k, the sum of digits of numbers $[an_k+b]$ is the same.

Example 8: (*2016 IMO Shortlisted Problem*) Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n\ge 2016$, the integer $P(n)$ is positive and

$$S(P(n))=P(S(n)). \qquad (*)$$

Solution: Let

$$P(x)=a_dx^d+a_{d-1}x^{d-1}+\cdots+a_0.$$

Clearly $a_d>0$. There exists an integer $m>1$ such that $|a_i|<10^m$ for all $0\le i\le d$. Consider $n=9\cdot 10^k$ for a sufficiently large integer k in (*). If there exists an index $0\le i\le d-1$ such that $a_i<0$, then all digits of $P(n)$ in positions from 10^{ik+m+1} to $10^{(i+1)k-1}$ are all 9's Hence, we have $S(P(n))>9(k-m-1)$. On the other hand, $P(S(n))=P(9)$ is a fixed constant. Therefore, (*) cannot hold for large k. This shows $a_i>0$ and for all $0\le i\le d-1$. Hence, $P(n)$ is an integer formed by the nonnegative integers $a_d9^d, a_{d-1}9^{d-1},\ldots, a_0$ by inserting some zeros in between.

This yields

$$S(P(n))=S(a_d9^d)+S(a_{d-1}9^{d-1})+\cdots+S(a_0).$$

Combining with (*), we have

$$\begin{aligned}S(a_d9^d)+S(a_{d-1}9^{d-1})+\cdots+S(a_0) &= P(9)\\ &= a_d9^d+a_{d-1}9^{d-1}+\cdots+a_0.\end{aligned}$$

As $S(m)\le m$ for any positive integer m, with equality when $1\le m\le 9$, this forces each a_i9^i to be a positive integer between 1 and 9. In particular, this shows $a_i=0$ for $i>2$ and hence $d\le 1$. Also, we have $a_1\le 1$ and $a_0\le 9$. If $a_1=1$ and $1\le a_0\le 9$, we take $n=10^k+(10-a_0)$ for sufficiently large k in (*). This yields a contradiction. Since

$$\begin{aligned}S(P(n)) &= S(10^k+10)=2\\ &\ne 11=P(11-a_0)=P(S(n)).\end{aligned}$$

The zero polynomial is also rejected since $P(n)$ is positive for large n. The remaining candidates are $P(x)=x$ or $P(x)=a_0$ where $1\le a_0\le 9$, all of which satisfy (*), and hence are the only solutions.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 25, 2019.***

Problem 531. $BCED$ is a convex quadrilateral such that $\angle BDC = \angle CEB = 90^\circ$ and BE intersects CD at A. Let F, G be the midpoints of sides DE, BC respectively. Let O be the circumcenter of ΔBAC. Prove that lines AO and FG are parallel.

Problem 532. Prove that there does not exist a function f:(0,+∞)→(0,+ ∞) such that for all $x,y>0$,

$$f^2(x) \ge f(x+y)(f(x)+y).$$

Problem 533. Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of all integers and all positive integers respectively. Let $r,\ s \in \mathbb{N}$. Prove that there are exactly $(r+1)^{s+1}-r^{s+1}$ functions $g:[1,s]\cap\mathbb{N}\to[-r,r]\cap\mathbb{Z}$ such that for all $x,y\in[1,s]\cap\mathbb{N}$, we have $|g(x)-g(y)|\le r$.

Problem 534. Prove that for any two positive integers m and n, there exists a positive integer k such that $2^k - m$ has at least n distinct prime divisors.

Problem 535. Determine all integers $n>4$ such that it is possible to color the vertices of a regular n-sided polygon using at most 6 colors such that any 5 consecutive vertices have distinct colors.

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Solutions

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Problem 526. Let $a_1=b_1=c_1=1$, $a_2=b_2=c_2=3$ and for $n\ge3$, $a_n=4a_{n-1}-a_{n-2}$,

$$b_n=\frac{b_{n-1}^2+2}{b_{n-2}},\ c_n=2c_{n-1}+\sqrt{3c_{n-1}^2-2}.$$

Prove that $a_n=b_n=c_n$ for all n =1,2,3,….

Solution. **Angel Gerardo Napa BERNUY** (PUCP University, Lima, Peru), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **DBS Maths Solving Team** (Diocesan Boy's School), **Prithwijit DE** (HBCSE, Mumbai, India), **O Long Kin Oscar** (St. Joseph's College), **TAM Choi Nang Julian** (Yan Chai Hospital Law Chan Chor Si College), **Duy Quan TRAN** (University of Medicine and Pharmacy at Ho Chi Minh City, Vietnam) and **Bruce XU** (West Island School).

The cases n = 1,2 can easily be checked. For $n\ge3$, $b_nb_{n-2} = b_{n-1}^2+2$ implies $b_{n+1}b_{n-1} = b_n^2+2$. Subtracting these and factoring, we get $(b_{n+1}-b_{n-1})/b_n=(b_n-b_{n-2})/b_{n-1}$. Then

$$(b_n-b_{n-2})/b_{n-1}= (b_{n-1}-b_{n-3})/b_{n-2}$$
$$= \cdots = (b_3-b_1)/b_2 = 4.$$

Hence, $b_n = 4b_{n-1}- b_{n-2}$ for $n \ge 3$. Since $a_1=b_1$ and $a_2=b_2$, $a_n=b_n$ for all n = 1,2,3,…. Next, from

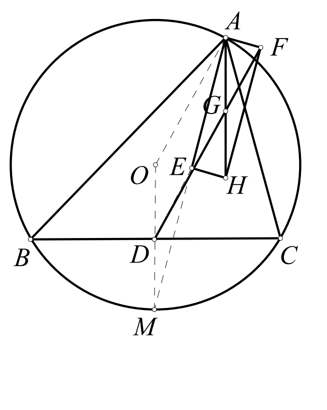
$$c_n=2c_{n-1}+\sqrt{3c_{n-1}^2-2},$$

we can see c_n is strictly increasing and for $n\ge2$, $(c_n-2c_{n-1})^2 = 3c_{n-1}^2-2$. Then $c_n^2 - 4c_nc_{n-1} + c_{n-1}^2 = -2$ and $c_{n+1}^2-4c_{n+1}c_n + c_n^2 = -2$. Subtracting these and factoring, we get $(c_{n+1}-c_{n-1})(c_{n+1}-4c_n+c_{n-1}) = 0$. As $c_{n+1} > c_{n-1}$, we get $c_{n+1}=4c_n-c_{n-1}$ for $n\ge2$. So $a_n=b_n=c_n$ for all n = 1,2,3,….

Other commended solvers: **AISINGIUR To To**, **Alvin LUKE** (Portland, Oregon, USA), **Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania), **Ioannis D. SFIKAS** (Athens, Greece), **Toshihiro SHIMIZU** (Kawasaki, Japan), **SO Tsz To** (S.K.H. Lam Woo Memorial Secondary School), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** (Buzău, Romania).

Problem 527. Let points O and H be the circumcenter and orthocenter of acute ΔABC. Let D be the midpoint of side BC. Let E be the point on the angle bisector of $\angle BAC$ such that $AE\perp HE$. Let F be the point such that $AEHF$ is a rectangle. Prove that points D, E, F are collinear.

Solution. **Alvin LUKE** (Portland, Oregon, USA).



Connect AO, OD and extend OD to meet the circumcircle of ΔABC at M. Then $OD\perp BC$ and M bisects arc BC. Also, A, E, M are collinear. Observe AE, AF are internal and external bisectors of $\angle BAC$. So $AE\perp AF$.

Since $HE\perp AE$ and $HF\perp AF$, so $AEHF$ is a rectangle. Hence, segments AH and EF bisect each other. Let AH and EF meet at G. Then AG=½AH= ½EF=EG.

Also, $OA=OM$ and $OD \parallel AH$. So

$$\angle OAE=\angle OME=\angle EAG=\angle GEA.$$

So (*) $EG \parallel OA$.

Next, observe O and H are the circumcenter and the orthocenter of of ΔABC respectively. Since $OD\perp BC$, so OD=½AH=AG. Finally, connect DG. We see $AODG$ is a parallelogram. So (**) $DG \parallel OA$. Therefore, by (*) and (**), D, E, G, F are collinear.

Other commended solvers: **Angel Gerardo Napa BERNUY** (PUCP University, Lima, Peru), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **DBS Maths Solving Team** (Diocesan Boy's School), **Prithwijit DE** (HBCSE, Mumbai, India), **Andrea FANCHINI** (Cantú, Italy), **Jon GLIMMS, Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania), **Apostolos MANOLOUDIS**, **George SHEN, Toshihiro SHIMIZU** (Kawasaki, Japan), **Mihai STOENESCU** (Bischwiller, France), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** (Buzău, Romania).

Problem 528. Determine all positive integers m satisfying the condition that there exists a unique positive integer n such that there exists a rectangle which can be decomposed into n congruent squares and can also be decomposed into $n+m$ congruent squares.

Solution. **Angel Gerardo Napa BERNUY** (PUCP University, Lima, Peru), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School) , and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Suppose rectangle $ABCD$ can be decomposed into $n+m$ unit squares and also into n squares with sides equal x. Let $x = a/b$ with gcd(a,b) = 1. Then the area of rectangle $ABCD$ is $n+m$ as well as $n(a/b)^2$. Then from $n+m = n(a/b)^2$, we can solve for n to get

$$n = \frac{mb^2}{a^2 - b^2} = \frac{mb^2}{(a-b)(a+b)}.$$

Since $\gcd(b,a+b) = \gcd(b,a-b) = \gcd(a,b) = 1$, so $(a-b)(a+b) \mid m$. Now $a+b$, $a-b$ are of the same parity. If m is the product of positive integers i, j, k with j, k odd and greater than 1, then $(a+b,a-b) = (j,k)$ or $(jk,1)$ leading to $n=i(j-k)^2/4$ or $i(jk-1)^2/4$, contradicting the uniqueness of n. So m can have at most one odd factor greater than 1, i.e. $m=2^c$ or $2^c p$ with p an odd prime.

In case $m=2^c$, for c=1,2, there is no n; for c=3, m=8 and (a,b)=(2,4), n=1; for $c\geq 4$, $(a+b,a-b)$=(4,2) or (8,2) resulting in $n = 2^{c-3}$ or 2^{c-4} contradicting the uniqueness of n.

In case $m=2^c p$, for c=0, m=p and $(a+b,a-b) = (p,1)$, $n = (p-1)^2/4$; for c = 1, $(a+b,a-b) = (p,1)$, $n = (p-1)^2/2$; for c = 2, $(a+b,a-b) = (p,1)$, $n = (p-1)^2$; for $c\geq 3$, $(a+b,a-b)$ = (4,2) or $(p,1)$ contradict the uniqueness of n.

So the only solutions are m = 8, p, $2p$, $4p$, where p is an odd prime.

Other commended solvers: **Victor LEUNG Chi Shing** and **Charles POON Tsz Chung**.

Problem 529. Determine all ordered triples (x,y,n) of positive integers satisfying the equation $x^n+2^{n+1} = y^{n+1}$ with x is odd and the greatest common divisor of x and $n+1$ is 1.

Solution. **Alvin LUKE** (Portland, Oregon, USA) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

When n=1, let $y=t$ be an integer at least 3 and $x=t^2-4$ are solutions. When $n\geq 2$,

$$x^n = y^{n+1} - 2^{n+1} = (y-2)\sum_{k=0}^{n} 2^k y^{n-k}.$$

For any prime factor p of y−2, from above, we see x must be a multiple of p. As x is odd, p is also odd. As $\gcd(x,n+1) = 1$, we see $\gcd(x,(n+1)2^n) = 1$. Then p is not a factor of $(n+1)2^n$. Now

$$S = \sum_{k=0}^{n} 2^k y^{n-k} \equiv \sum_{k=0}^{n} 2^n = (n+1)2^n \pmod{y-2}.$$

Hence, p is not a factor of S. So we have $\gcd(y-2,S) = 1$. So $S=T^n$ for some positive integer T. Since y is positive, y is at least 3.

When $n\geq 2$, we have

$$y^n < S = T^n < (y+2)^n. \qquad (*)$$

So $T = y+1$. However, when y is even, $S \equiv y^n \pmod 2$ is even, but then $S = (y+1)^n$ is odd by (*). Similarly, when y is odd, $S \equiv y^n \pmod 2$ is odd, but then $S=(y+1)^n$ is even by (*). Again this leads to a contradiction.

In conclusion, when integer n is at least 2, there are no solutions. So the only solution are $x=t^2-4$, $y=t$, n=1, where integer $t\geq 3$.

Other commended solvers: **Ioannis D. SFIKAS** (Athens, Greece).

Problem 530. A square can be decomposed into 4 rectangles with 12 edges. If square $ABCD$ is decomposed into 2005 convex polygons with degrees of A, B, C, D at least 2 and degrees of all other vertices at least 3, then determine the maximum number of edges in the decomposition.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **DBS Maths Solving Team** (Diocesan Boy's School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let v, e, f be the number of vertices, edges and faces used in decomposing the square respectively. By Euler's formula, we have $v-e+f = 1$ (omitting the exterior of the square).

Let $d(V)$ be the number of edges connected to V. Let V be a vertex on the square other than A,B,C,D. Then $d(V)\geq 3$, which is the same as $d(V)\leq 3d(V)-6$.

Now there are $v-4$ vertices not equal to A, B, C, D. The sum of the degrees of the $v-4$ vertices other than A, B, C, D is $2e-[d(A)+d(B)+d(C)+d(D)]$, which is at least $3(v-4)$. Since $d(A)$, $d(B)$, $d(C)$, $d(D) \geq 2$, we get

$$2e-8\geq 2e-[d(A)+d(B)+d(C)+d(D)] \geq 3(v-4) = 3v-12.$$

Since $v-e+f = 1$, $3e = 3v+3f-3 \leq 2e+1+3f$, which simplies to $e \leq 3f+1$.

For equality case, we can decompose the unit square into rectangles of size 1 by 1/2005, which has 3×2005+1=6016 edges.

Olympiad Corner

(*Continued from page 1*)

Problem G2. (*cont.*) Line DJ intersects the circumcircle of the triangle ABC at the point E which lies on the shorter arc AB. Prove that $|AI|=|IE|$ holds.

Problem N2. Let n be a positive integer. Prove that there exists a positive integer k such that

$$51^k - 17$$

is divisible by 2^n.

Sums of Digits …

(*Continued from page 2*)

Next, we will provide some exercises for the readers.

Problem 1: (*Mexico 2018*) Find all pairs of positive integers (a,b) with $a>b$ which simultaneously satisfy the following two conditions

$$a \mid b+S(a) \quad \text{and} \quad b \mid a+S(b).$$

Problem 2: (*Lusophon 2018*) Determine the smallest positive integer a such that there are infinitely many positive integer n for which you have $S(n)-S(n+a) = 2018$.

Problem 3: (*Cono Sur 2016*) Find all n such that $S(n)(S(n)-1) = n-1$.

Problem 4: (*Iberoamerican 2014*) Find the smallest positive integer k such that

$$S(k) = S(2k) = S(3k) = \cdots = S(2013k) = S(2014k).$$

Problem 5: (*OMCC 2010*) Find all solutions of the equation $n(S(n)-1) = 2010$.

Problem 6: (*Iberoamerican 2012*) Show that for all positive integers n there are n consecutive positive integers such that none is divisible by the sum of their respective digits.

Mathematical Excalibur

Volume 22, Number 4 *May 2019 – October 2019*

Olympiad Corner

Below were the Hong Kong (China) Mathematical Olympiad on December 1, 2018.

Problem 1. Given that a, b and c are positive real numbers such that $ab+bc+ca\geq 1$, prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{\sqrt{3}}{abc}.$$

Problem 2. Find the number of nonnegative integers k, $0\leq k\leq 2188$, and such that $2188!/(k!(2188-k)!)$ is divisible by 2188.

Problem 3. The incircle of ΔABC, with incenter I, meets BC, CA and AB at D, E, F respectively. The line EF cuts the lines BI, CI, BC and DI at points K, L, M and Q respectively. The line through the midpoint of CL and M meets CK at P.

(a) Determine $\angle BKC$.
(b) Show that the lines PQ and CL are parallel.

Problem 4. Find all integers $n\geq 3$ with the following property: there exist n distinct points on the plane such that each point is the circumcenter of a triangle formed by 3 of the points.

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 2, 2019***.

For individual subscription for the next five issues for the 18-19 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Notes on IMO 2019

Tat Wing LEUNG

Despite all its sham, drudgery and broken dreams, the Gifted Section of the Education Department (EDB), the Hong Kong Academy of Gifted Education (HKAGE), and our Committee (International Mathematical Olympiad Hong Kong Committee, IMOHKC) managed to send a team to the 60th International Mathematical Olympiad (IMO 2019). The competition was held from July 11 to July 22, 2019, in Bath, United Kingdom.

The team was composed as follows: Leader: Leung Tat Wing, Deputy Leader: Cesar Jose C. Jr. Alaban (CJ), Members: Bruce Changlong Xu, Daniel Weili Sheremeta, Harris Leung, Wan Lee, Nok To Omega Tong, Sui Kei Ho. A lady from EDB (Miriam Cheung) also went with us as an observer.

Let me briefly discuss the problems of the two contests.

Problem 1 was very interesting. It was initially selected as the easy algebra problem and later selected as the easy pair. Although it was most liked, it was also most hated. I supposed it was because some leaders thought the problem was simply too easy. By substituting suitable values (say a by 0 and b by n+1 one quickly comes to the conclusion that the function is linear (or by Cauchy), and hence by using some initial values to get the answers. Some leaders first tried to replace the easy algebra by another easy problem (which was actually classified as a combinatorial problem), and later tried to add alternate option pairs to the option pairs that contained the easy algebra problem. I myself could not say if it was right or wrong, I just found it funny. Indeed the problem was selected using the approach as agreed, why tried to change it in the middle of the process? At the end of the day, totally 73 students did not get anything in this problem, and only slightly more than half (382 out of 621) scored full mark.

Problem 4 was an easy Diophantine equation. By putting small values of n, one quickly comes up with the solutions (1,1) and (3,2), the hard part is to show that there are no more. Many students lost partial marks while trying to compare values (or 2-adic valuations) of the two sides of the equation. As learned from leaders of stronger teams, I found they considered Legendre's formula and/or the lifting exponent lemma rather common tools, although the lemma was not really necessary. So yes, do we need to ask our students to further enhance their toolkit?

Problem 5 was an *ouroboros*-type problem, namely part of the problem is relating to other part of itself. In this case we are given a sequence of heads and tails of n coins, the k^{th} coin is flipped if there are exactly k heads in the sequence. The problem is not too hard, and given its "natural" condition, it is probably known. Indeed if the first coin is head, then basically we need to deal with the remaining sequence of length $n-1$, and the final step is to flip the first coin. If the last coin is a tail, then it will never be flipped, and we are basically dealing with the first $n-1$ coins.

If the first coin is a tail, and the last coin is a head, then we first deal with the middle $n-2$ coins. After that only one head remaining (at the end), then the first $n-1$ coins are flipped successively and all become heads, then starting from the end, each coin is flipped, until the first one and every coin becomes tail. Using these, we can make up recursive relations and get the answer relatively easy. Our team members, using their own ingenuity and persistence, managed to do the problem well.

(continued on page 4)

Wilson's Theorem

Kin Y. Li

In solving number theory problems, Fermat's or Euler's theorems as well as the Chinese remainder theorem are often applied. In this article, we will look at examples of number theory problems involving factorials. For this type of problems, Wilson's theorem asserts that for every prime number p, we have $(p-1)! \equiv -1 \pmod p$. Below are problems using Wilson's theorem.

Problem 1. Let p be an odd integer greater than 1. Prove that

$$1^2\cdot 3^2\cdot 5^2\cdots(p-2)^2 \equiv (-1)^{(p+1)/2} \pmod p.$$

Solution. By Wilson's theorem, $(p-1)! \equiv -1 \pmod p$ when p is an odd prime. Also, we have $i \equiv -(p-i) \pmod p$. Multiplying the cases $i = 1, 3, \cdots, p-2$, we get

$$1\cdot 3\cdots(p-2)\equiv(-1)^{(p-1)/2}(p-1)(p-3)\cdots 2 \pmod p.$$

Multiplying both sides by $1\cdot 3\cdots(p-2)$, we get

$$1^2\cdot 3^2\cdot 5^2\cdots(p-2)^2 \equiv (-1)^{(p-1)/2}(p-1)! \equiv (-1)^{(p+1)/2} \pmod p.$$

Problem 2. Let p be a prime number and $N = 1+2+3+\cdots+(p-1) = (p-1)p/2$. Prove that $(p-1)! \equiv p-1 \pmod N$.

Solution. Since p is prime, by Wilson's theorem, $(p-1)! \equiv -1 \pmod p$. Then there exists an integer m such that

(*) $(p-1)!=mp-1=(m-1)p+(p-1)$.

So $(m-1)p = (p-1)!-(p-1) = (p-1)k$, where $k=(p-2)!-1$ and $p|(p-1)k$. Since $\gcd(p,p-1)=1$, so $p|k$. Let $k=np$, then

(**) $(m-1)p=(p-1)pn$,

so $m-1=n(p-1)$. Putting (**) into (*), we get

$$(p-1)!=[n(p-1)+1]\,p-1=n(p-1)p+p-1 =2n[(p-1)p/2]+p-1=2nN+p-1.$$

So $(p-1)! \equiv p-1 \pmod N$.

Problem 3. Determine all positive integers n having the property that there exists a permutation $a_1, a_2, \ldots, a_n$ of $0,1,2,\ldots,n-1$ such that when divided by n, the remainders of a_1, a_1a_2, … , $a_1a_2\cdots a_n$ are distinct.

Solution. When n is a prime number p, let $a_1=1$ and other integers a_i satisfy $0\le a_i\le p-1$ and $ia_{i+1} \equiv i+1 \pmod p$ for $i = 2,\cdots, p$.

Then a_1, a_1a_2, … , $a_1a_2\cdots a_n$ when divided by n have remainders $1,2,\cdots, p$. Also, from $ia_{i+1} \equiv i + 1 \pmod p$, we see $a_{i+1}-1$ is the inverse of i. So $a_1, a_2, \ldots, a_n$ are distinct.

When $n = 1$ or 4, the permutations (0), (1,3,2,0) satisfy the condition. When $n>4$ is composite, if $n = p^2$, let $q = 2p < n$. Otherwise $n=pq$ with $1<p<q<n$ so that $pq|(n-1)!$.

If the required permutation exists, then $a_n=0$ and $a_1a_2\cdots a_{n-1}=(n-1)!\equiv 0 \pmod n$, which is a contradiction. (In fact, when $n>4$ is composite, $n \mid (n-1)!$ and $3! \equiv -2 \pmod 4$ so that the converse of Wilson's theorem also hold.

Problem 4. For integers n, q satisfying $n\ge 5$ and $n\ge q\ge 2$, prove that $[(n-1)!/q]$ is divisible by $q-1$.

Solution. (1) If $n>q$, then $(q-1)q \mid (n-1)!$. Hence, $(q-1) \mid [(n-1)!/q]$.

(2) If $q=n$ and q is composite, then $[(n-1)!/q]=(n-1)!/n$. Since $\gcd(n-1,n)=1$ and $q-1=(n-1) \mid (n-1)!$. So $q-1$ divides $[(n-1)!/q]$.

(3) If $q=n$ is prime, then by Wilson's theorem, $(n-1)! \equiv -1 \pmod n$ so that $(n-1)!+1=kn$ for some integer k. Then $[(n-1)!/q]=k-1$ and $(k-1)n=(n-1)!+1-n$ so that $k-1=((n-2)!-1)(n-1)/n$ is an integer. Since $\gcd(n-1,n)=1$, so n divides $(n-2)!-1$. Therefore, $[(n-1)!/q]=k-1$ is a multiple of $n-1$.

Problem 5. Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $a_0, a_1, \cdots, a_n$ are integers, $a_n> 0$ and $n\ge 2$. Then prove that there exists a positive integer m such that $P(m!)$ is a composite number.

Solution. If $a_0=0$, then $m! \mid P(m!)$ and the conclusion follows.

Next let $S(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. Suppose $a_0\ne 0$. By Wilson's theorem, for every prime p and positive even integer $k < p$, we have

$$\begin{aligned}&(k-1)!(p-k)!\\ &\equiv (-1)^{k-1}(p-k)!(p-k+1)(p-k+2)(p-1)\\ &= -(p-1)! \equiv 1 \pmod p.\end{aligned}$$

So $(p-1)!\equiv-1 \pmod p$ and

$$((k-1)!)^n\, P((p-k)!)\equiv S((k-1)!) \pmod p.$$

So $p \mid P((p-k)!)$ if and only if $p \mid S((k-1)!)$. Take $k > 2a_n+1$. Then $u = (k-1)!/a_n$ is an integer divisible by all primes not greater than k.

Problem 6. If p and $p+2$ are both prime numbers, then we say they are twin primes. Show that if p and $p+2$ are twin primes, then $4(p-1)!+4+p$ is divisible by $p(p+2)$.

Solution. If p and $p+2$ are prime, then $p>2$ so that p and $p+2$ are odd. By Wilson's theorem, $(p-1)! \equiv -1 \pmod p$ and also $(p+1)! \equiv -1 \pmod{p+2}$. Then we have

$$4(p-1)!+4+p \equiv 0 \pmod p.$$

Also

$$4(p-1)!+4 \equiv -p(p+1)p[(p-1)!+1] \equiv -p[(p+1)!+2] \equiv -p \pmod{p+2},$$

which is $4(p-1)!+4+p \equiv 0 \pmod{p+2}$. As $\gcd(p,p+2)=1$, we get $4(p-1)!+4+p \equiv 0 \pmod{p(p+2)}$.

Problem 6. (Wolstenholme's Theorem) Let p be a prime greater than or equal to 5. For positive integers m and n that are relatively prime and

$$\frac{m}{n}=\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(p-1)^2}.$$

Prove that p is a divisor of m and p^2 is a divisor of

$$(p-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{p-1}\right).$$

Solution. If integer k is not divisible by p, then there are integers a, b such that $ak+bp = \gcd(k,p) = 1$. We say a is the inverse of k in mod p and denote a as k^{-1}. We have

$$\begin{aligned}((p-1)!)^2\frac{m}{n}&=\sum_{k=1}^{p-1}\frac{((p-1)!)^2}{k^2}\\ &\equiv(-1)^2(1^2+2^2+\cdots+(p-1)^2)\\ &\equiv\frac{(p-1)p(2p-3)}{6}\equiv 0 \pmod p.\end{aligned}$$

Since $\gcd((p-1)!, p) = 1$, so $p \mid m$. Next, let $S=(p-1)!(1+1/2+\cdots+1/(p-1))$. Then

$$\begin{aligned}2S&=(p-1)!\sum_{i=1}^{p-1}\left(\frac{1}{i}+\frac{1}{p-i}\right)\\ &=p\sum_{i=1}^{p-1}\frac{(p-1)!}{i(p-i)}=pT,\end{aligned}$$

where $2S$, p and T are integers. Since $\gcd(p,2)=1$, so p divides S. Due to $p|m$,

$$T=\sum_{i=1}^{p-1}\frac{(p-1)!}{i(p-i)}\equiv(p-1)!\frac{m}{n}\equiv 0 \pmod p.$$

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***November 2, 2019.***

Problem 536. Determine whether there exists a function $f:\mathbb{R}\to\mathbb{R}$ such that for all real x, we have $f(x^3+x) \le x \le (f(x))^3+f(x)$.

Problem 537. Distinct points A, B, C are on the unit circle Γ with center O inside ΔABC. Suppose the feet of the perpendiculars from O to sides BC, CA, AB are D, E, F. Determine the largest value of $OD+OE+OF$.

Problem 538. Determine all prime numbers p such that there exist integers a and b satisfying $p=a^2+b^2$ and a^3+b^3-4 is divisible by p.

Problem 539. In an exam, there are 5 multiple choice problems, each with 4 distinct choices. For every problem, every one of the 2000 students is required to choose exactly 1 of the 4 choices. Among the 2000 exam papers received, it is discovered that there exists a positive integer n such that among any n exam papers, there exist 4 such that for every 2 of the exam papers, there are at most 3 problems having the same choices. Determine the least such n.

Problem 540. Do there exist a positive integer k and a non-constant sequence $a_1, a_2, a_3, \ldots$ of positive integers such that $a_n=\gcd(a_{n+k}, a_{n+k+1})$ for all positive integer n?

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Solutions

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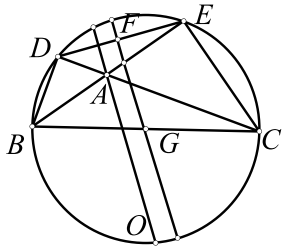
Problem 531. $BCED$ is a convex quadrilateral such that $\angle BDC=\angle CEB=90°$ and BE intersects CD at A. Let F, G be the midpoints of sides DE, BC respectively. Let O be the circumcenter of ΔBAC. Prove that lines AO and FG are parallel.

Solution 1. **Jon GLIMMS, Hei Chun LEUNG** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Since $\angle CAO = (180°-\angle COA)/2 = 90°-\angle COA/2 = 90°-\angle CBA = 90°-\angle CBE = 90°-\angle CDE = 90°-\angle ADE$, we have OA and DE are perpendicular. Also, since FG passes through the center G of the circle $CEDG$ and midpoint F of chord DE, FG is perpendicular to DE. Thus, both AO, FG are perpendicular to DE. So lines AO and FG are parallel.

Solution 2. **Prithwijit DE** (HBCSE, Mumbai, India).

Let R be the radius of the circumcircle of triangle BAC. As $\angle BAC>90°$, BC is not the diameter of the circle ABC and therefore D and E are outside the circle ABC. Observe that $EA\cdot EB=EO^2-R^2$ and $DA\cdot DC=DO^2-R^2$. Thus

$$\begin{aligned}EO^2-DO^2&=EA\cdot EB-DA\cdot DC\\&=EA^2-DA^2+EA\cdot AB-DA\cdot DC\\&=EA^2-DA^2.\end{aligned}$$

This implies $OA\perp DE$. Now $FG\perp DE$ because G is the centre of the circle passing through B, C, E and D, and F is the midpoint of chord DE of this circle. Therefore, lines AO and FG are parallel.

Other commended solvers: **CHUI Tsz Fung, Andrea FANCHINI** (Cantù, Italy), **Panagiotis N. KOUMANTOS** (Athens, Greece), **LAU Chung Man** (Lee Kau Yan Memorial School), **LW Maths Solving Team** (SKH Lam Woo Memorial Secondary School), **Jim MAN**, **Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania) and **Apostolis MANOLOUDIS**.

Problem 532. Prove that there does not exist a function $f:(0,+\infty)\to(0,+\infty)$ such that for all $x,y>0$,

$$f^2(x) \ge f(x+y)(f(x)+y).$$

Solution. **Jon GLIMMS, Alvin LUKE** (Portland, Oregon, USA) and **Toshihiro SHIMIZU** (Kawasaki, Japan),

Assume such function exists. The given inequality can be rewritten as

$$f(x)-f(x+y)\ge\frac{f(x)y}{f(x)+y}>0.$$

Then f is strictly decreasing. Fix $x>0$ and choose positive integer n such that $nf(x+1)\ge 1$. For $k=0,1,\ldots,n-1$, we have

$$\begin{aligned}f\left(x+\frac{k}{n}\right)-f\left(x+\frac{k+1}{n}\right)&\ge\frac{f(x+k/n)}{nf(x+k/n)+1}\\&\ge\frac{1}{2n}.\end{aligned}$$

Summing the first and third parts above from $k=0$ to $n-1$, we get

$$f(x)-f(x+1)\ge\frac{1}{2}.$$

From this it follows that for all positive integers m, we have

$$f(x+2m)\le f(x)-m.$$

Choosing $m \ge f(x)$, we easily get a contradiction with the condition $f(x)>0$.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania), **Apostolos MANOLOUDIS**, **George SHEN** and **Thomas WOO.**

Problem 533. Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of all integers and all positive integers respectively. Let $r, s \in \mathbb{N}$. Prove that there are exactly $(r+1)^{s+1}-r^{s+1}$ functions $g:[1,s]\cap\mathbb{N}\to[-r,r]\cap\mathbb{Z}$ such that for all $x,y\in[1,s]\cap\mathbb{N}$, we have $|g(x)-g(y)|\le r$.

Solution. **LAU Chung Man** (Lee Kau Yan Memorial School), **George SHEN** and **Thomas WOO.**

If integer k is in $[-r,r]\cap\mathbb{Z}$, then there are $(\min\{r+1,r-k+1\})^s$ functions satisfying the given conditions which attain values only in $\{k,\ldots,k+r\}$. Of these, $(\min\{r,r-k\})^s$ functions attain values only in $\{k+1, \ldots, k+r\}$. Hence, exactly

$$(\min\{r+1,r+1-k\})^s-(\min\{r,r-k\})^s$$

functions satisfying the given conditions have minimum value k.

This expression equals $(r+1)^s-r^s$ for each of the $r+1$ values $k\le 0$, and it equals $(r+1-k)^s-(r-k)^s$ when $k>0$. Thus, the sum of the expression over all $k\le 0$ is $(r+1)((r+1)^s-r^s)$, while the sumof the expression over all $k>0$ is the telescoping sum

$$\sum_{k=1}^{r}((r+1-k)^s-(r-k)^s)=r^s.$$

Adding these two sums, we find that the total number of functions satisfying the given conditions is $(r+1)^{s+1}-r^{s+1}$.

Other commended solvers: **Jon GLIMMS, Michael HUI** and **Jeffrey HUI**, **Hei Chun LEUNG**, **Alvin LUKE** (Portland, Oregon, USA) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 534. Prove that for any two positive integers m and n, there exists a positive integer k such that 2^k-m has at least n distinct prime divisors.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

We show by induction that there is $k\in\mathbb{N}$ such that 2^k-m has at least n odd prime divisors. If m is even, we can write $n=2^es$ (with odd integer s) and take $k>e$ so we have $2^k-m=2^e(2^{k-e}-s)$. Then it is sufficient to show for $m=s$ (odd). Thus, we assume m is odd.

Taking $k\in\mathbb{N}$ such that $2^k-m>1$, we can take an odd prime divisor p of 2^k-m (which is odd). Assume we have $k\in\mathbb{N}$ such that 2^k-m has n odd prime divisors p_1, p_2, …, p_n. For any i the pattern of 2^j (mod p_i) is periodic for j, which implies there are e_i, $f_i\in\mathbb{N}$ such that $2^j\equiv m$ (mod p_i) if and only if $j=e_it+f_i$ for some $t\in\mathbb{N}$. Since $p_i>2$, each e_i is greater than 1. Thus, we can take f_i' such that $f_i\not\equiv f_i'$ (mod e_i). By the Chinese remainder theorem, we can take k' such that $k'\equiv f_i'$ (mod e_i) and we have $p_i\nmid 2^{k'}-m$ for $1\le i\le n$. We can also select k' such that $2^{k'}-m>1$. Then we can take odd prime divisor p_{n+1} of $2^{k'}-m$, where p_{n+1} is different from any one of p_1, p_2, …, p_n. Then we can choose j such that $2^j\equiv m$ (mod p_{n+1}), where $j=e_{n+1}t+f_{n+1}$ for some e_{n+1}, f_{n+1}. By the Chinese remainder theorem again, we can take K such that $K\equiv f_i'$ (mod e_i) and we have $p_i\nmid 2^K-m$ for $1\le i\le n+1$. Then 2^K-m has at least $n+1$ prime factors $p_1, p_2, \ldots, p_{n+1}$, completing the induction.

Problem 535. Determine all integers $n>4$ such that it is possible to color the vertices of a regular n-sided polygon using at most 6 colors such that any 5 consecutive vertices have distinct colors.

Solution. **CHUI Tsz Fung**, **Hei Chun LEUNG**, **LAU Chung Man** (Lee Kau Yan Memorial School), **LW Maths Solving Team** (SKH Lam Woo Memorial Secondary School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let the colors be a, b, c, d, e, f. Denote by S_1 the sequence a, b, c, d, e and by S_2 the sequence a, b, c, d, e, f. If $n>0$ is representable in the form $5x+6y$ for $x,y\ge0$, then n satisfies the conditions of the problem: we may place x consecutive S_1 sequences, followed by y consecutive S_2 sequences, around the polygon. Setting y equal to 0, 1, 2, 3 or 4, we find that n may equal any number of the form $5x$, $5x+6$, $5x+12$, $5x+18$ or $5x+24$. The only numbers greater than 4 not of this form are 7, 8, 9, 13, 14 and 19. Below we will show that none of these numbers has the required property.

Assume for a contradiction that a coloring exists for n equal to one of 7, 8, 9, 13, 14 and 19. There exists a number k such that $6k < n < 6(k+1)$. By the pigeonhole principle, at least $k+1$ vertices of the n-gon have the same color. Between any two of these vertices are at least 4 others, because any 5 consecutive vertices have different colors. Hence, there are at least $5k+5$ vertices, and $n\ge 5k+5$. However, this inequality fails for $n = 7, 8, 9, 13, 14, 19$, a contradiction. Hence, a coloring is possible for all $n\ge5$ except 7, 8, 9, 13, 14 and 19.

Notes on IMO 2019

(Continued from page 1)

Problem 3 is a graph algorithmic problem. The problem is not real hard, but the essential difficulty is hidden by the numbers, students also might find it hard because they do not have the language of graph theory. Namely the graph is connected, with at least three vertices and is not complete, and there is a vertex of odd degree. Then it is possible to find a vertex and apply the operation, and reduce the number of edges by 1, yet maintaining the essential initial conditions. There is no worry of the existence of a cycle, for instance, during the operations. Otherwise the cycle can only be shrunk to a triangle and get stuck. At least a solution is conceivable.

I do not know what to say about problem 2 and 6 (medium and hard geometry problem). Our team did not do too well. It suffices to say, problem 2 may be done by careful angle chasing, while problem 6 is more complicated, but there is a nice and not too complicated complex number solution.

In short, leaders generally agreed that those problems are do-able. If one understands what is going on, one should be able to do those problems, and there is no need of deep and/or obscure theorems. I recalled one of my teachers told us, there really is "no mystery", if you get the point. Also it came to my mind Hilbert's motto: *wir mussen wissen*, *wir werden wussen* (we must know, we will know). Indeed at the end, the cut-off scores were relatively high, 17 for bronze, 24 for silver, and 31 for gold, and in total 6 contestants obtained full mark.

After coordination and the final Jury meeting, we managed to get 1 silver medal (Harris) and 3 bronze (Wan, Daniel and Omega). Surely it was not too good, but not too bad either. Indeed they could do better. For instance, Sui Kei was only 1 point below bronze, and Bruce 3 points (he got a honorable mention by scoring full mark in a problem), should they not making several trivial mistakes (also made by members of several strong teams), they should get medals. Both Daniel and Wan solved three problems, and in my opinion potential silver medalists. On the whole, I notice they have been working hard during the last two months, so I don't think I should blame them too much. One thing however I think our team members should watch out is, in case they will come back next time, they should know how much further effort they need to devote and know what they expect.

I have given my opinions and suggestions. Accordingly 2020 IMO will be held in Russia, 2021 in USA, 2022 in Norway, 2023 in Japan, 2024 in Shanghai China (probably) and 2025 in Australia. Some people have been working hard to make future IMOs possible. I hope Hong Kong will continue to join. However I cannot be too sure. For one thing, not sure if Hong Kong will be as relatively free/peaceful/prosperous to sustain events of this kind. Even so, I am not quite sure if our students may maintain their interest. Life is hard (as usual). Let's hope for the best. Good Luck.