Tangential Quadrilaterals and Cyclicity

Dr. Suzy Manuela Prajea

J.L.Chambers High School, Charlotte, NC, USA

Abstract

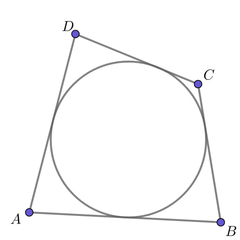
In the paper [2], N. Minculete proved some beautiful properties of tangential quadrilaterals using trigonometric computations. This paper will ease the role of trigonometry by providing new techniques based more on pure geometric considerations. These techniques will help further to deduce some characterizations for tangential cyclic quadrilaterals. From a didactic perspective, the content becomes in this way accessible to a larger variety of high school students who intend to improve their mathematical education in the field of geometry to reveal fascinating characterizations of some outstanding geometric configurations at a reasonable interference with basic algebra and trigonometry.

1 Notations

For the simplicity of the writing will denote by $\angle ABC$ both the angle and the measure of the angle $\angle ABC$. The difference will be deducted from the context of the usage of this notation.The distance from a point P to a line AB will be denoted by d_{AB} and the area of the triangle (ABC) will be denoted shortly by (ABC). To easiness the reading, the wording will prevail over the abstract formal mathematical notations when possible.

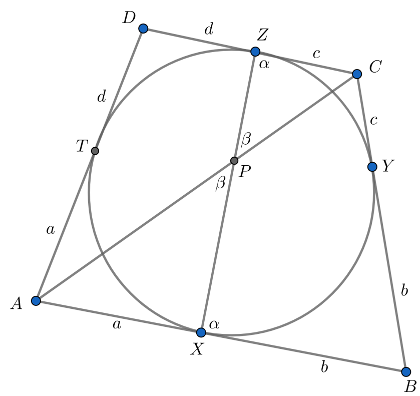
2 Tangential Quadrilaterals Characterizations

Definition 2.1. A convex quadrilateral is called tangential if there is a circle tangent to the sides of the quadrilateral (incircle).



Theorem 1 (Newton). *If $ABCD$ is a tangential quadrilateral and X, Y, Z, T are the tangency points of the incircle with the sides AB, BC, CD, DA then the lines AC, BD, XZ, YT are concurrent.*

Proof. Denote by a, b, c, d the lengths of the tangents from the vertices A, B, C, D to the incircle and by P, P' the intersection points of the diagonal AC with the lines XZ, YT respectively.



Notice that $\angle PZC = \angle PXB$ (subtend the same arc) and
$\angle ZPC = \angle APX$ as vertical angles.

Denote these angles with α and β respectively. Law of sines in $\triangle PCZ$ and $\triangle PXA$ provides

$\frac{PC}{\sin\alpha} = \frac{c}{\sin\beta}$ and $\frac{PA}{\sin(180^0 - \alpha)} = \frac{a}{\sin\beta}$.

Dividing these relationships it follows $\frac{PA}{PC} = \frac{a}{c}$.

Analogously we get $\frac{P'A}{P'C} = \frac{a}{c}$.

In conclusion we have

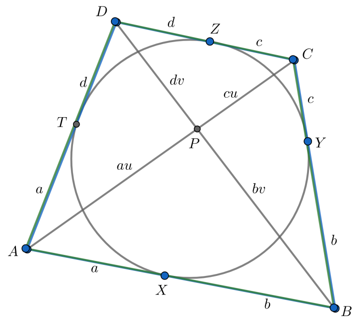
$$\frac{PA}{PC} = \frac{P'A}{P'C}$$

Finally, due to the fact that both points P and P' are on the segment AC and split it in the same ratio it follows that the points P and P' are identical. Therefore the lines AC, BD, XZ, and YT are concurrent. □

Corollary 1.1. *Let be $ABCD$ a tangential quadrilateral, X, Y, Z, T the tangency points of the incircle with the sides AB, BC, CD, DA, and P the intersection point of the lines AC, BD, XY, ZT. The length of the tangents from A, B, C, D to the incircle are denoted by a, b, c, d. Then there are u, v positive real numbers such that*

$$PA = au, \quad PC = cu, \quad PB = bv, \quad PD = dv$$

Proof. It follows immediately from $\frac{PA}{PC} = \frac{a}{c}$ and $\frac{PB}{PD} = \frac{b}{d}$. □



Theorem 2. *Let be $ABCD$ a convex quadrilateral. The following conditions are equivalent:*

(i) $ABCD$ *tangential*

(ii) $AB + CD = AD + BC$

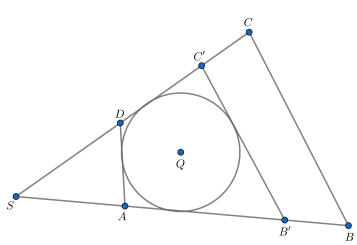
(iii) $\dfrac{1}{d_{AB}} + \dfrac{1}{d_{CD}} = \dfrac{1}{d_{BC}} + \dfrac{1}{d_{AD}}$

(iv) $a \sin A \sin B + c \sin C \sin D = b \sin B sin C + d \sin D \sin A$

Proof. (i) $\Rightarrow$ (ii) As in the proof of Theorem 1 it follows $AB = a + b$, $\quad CD = c + d$ and also $BC = b+c$, $\quad AD = a+d$. In consequence, $AB+CD = (a+b)+(c+d)$, $\quad BC+AD = (b+c)+(a+d)$ and therefore $AB + CD = BC + AD$. $\square$

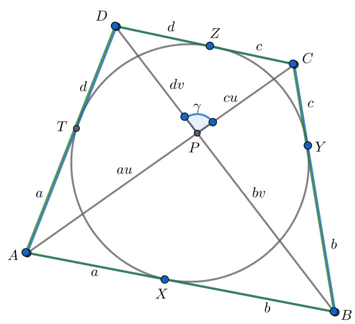
Proof. (ii)$\Rightarrow$(i) Two cases will be considered for this proof.
1. The pairs of opposite sides of $ABCD$ are parallel. It results $ABCD$ parallelogram so the opposite sides are equal. From (ii) it follows that the sums of the opposite sides are also equal therefore all the sides are equal so $ABCD$ is a rhombus. It is known that the center of the rhombus is equal distanced from its sides so there is a circle centered in the intersection of diagonals that is tangent to the sides of the rhombus (the radius is the common distance of the center to the rhombus sides) so $ABCD$ is tangential.

2.There is a pair of opposite sides of $ABCD$ that are not parallel, for instance AB and CD are not parallel. Denote by S their intersection point.



WLOG, assume the point A between S and B. Then D is between S ans C ($ABCD$ is convex). Assume by contradiction that $ABCD$ is not tangential. Define Q as the intersection points of angle bisectors from A and D so the point Q will be equal distanced from the sides AB, AD, CD and therefore the sides AB, AD, CD will be tangent to the circle centered in Q that has the radius the common distance from Q to the sides AB, AD, CD. There are two possible situations: BC to be external to the circle or BC to be secant to the circle. In the case that BC is external to the circle (the case BC secant can be proved in the same way), consider the parallel to the line BC that is tangent to the circle and denote by B' and C' the intersections with the sides AB and CD respectively. Due to the fact that BC is external to the circle, it follows that B' is on the side AB and C' on the side DC.The circle becomes incircle for the quadrilateral $AB'C'D$ so due to the fact that the tangents from A, B', C', D are equal it results that $AB' + C'D = AD + B'C'$ (the same argument as in (i)⇒ (ii)). From (ii) it holds also $AB + CD = AD + BC$. Subtracting the last two equations it follows $BB' + CC' = BC - B'C'$ or equivalently $BB' + CC' + B'C' = BC$. The last equation is a contradiction because $BC < BB' + CC' + B'C'$ (the length of the segment BC is less than the length of any polygonal line from B to C). In consequence, $ABCD$ is tangential. □

Proof. (ii)⇒ (iii) Denote $\angle CPD = \gamma$ and area of $\triangle UVW$ by (UVW). Then $(APB) = \dfrac{(a+b)d_{AB}}{2}$ and from Corollary 1.1 it follows $(APB) = \dfrac{(au)(bv)\sin\gamma}{2}$.



The last two equations help to conclude

$$\frac{1}{d_{AB}} = \frac{a+b}{abuv\sin\gamma}$$

A similar equation stands for CD

$$\frac{1}{d_{CD}} = \frac{c+d}{cduv\sin\gamma}$$

Adding the last two equations, it results

$$\frac{1}{d_{AB}} + \frac{1}{d_{CD}} = \frac{1}{uv\sin\gamma}\left(\frac{a+b}{ab} + \frac{c+d}{cd}\right) = \frac{1}{uv\sin\gamma}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

Similarly, the following holds

$$\frac{1}{d_{BC}} + \frac{1}{d_{AD}} = \frac{1}{uv\sin\gamma}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

and hence it results the conclusion

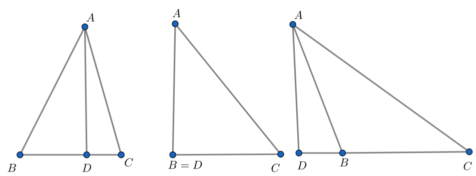
$$\frac{1}{d_{AB}} + \frac{1}{d_{CD}} = \frac{1}{d_{BC}} + \frac{1}{d_{AD}}$$

□

Proof. (iii)⇒(iv)

Lemma 2.1. *Consider $\triangle ABC$ and denote $BC = a, CA = b, AB = c$. Then $d_{BC} = c \sin B$.*

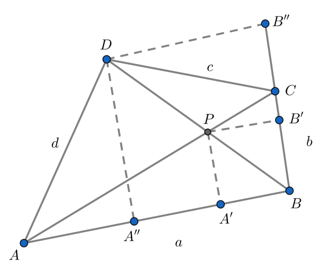
Proof. This can be easily justified (regarding the $\angle B$ is acute, right or obtuse) by applying the $\sin B$ ratio in $\triangle ABD$, where D is the foot of the perpendicular from A to BC.



□

If A', B', C', D' are the feet of the perpendiculars from P to A', B', C', D' then (iii) becomes

$$\frac{1}{PA'} + \frac{1}{PC'} = \frac{1}{PB'} + \frac{1}{PD'} \text{ or equivalently } 1 + \frac{PA'}{PC'} = \frac{PA'}{PB'} + \frac{PA'}{PD'} \quad (1)$$



Denote a, b, c, d the lengths of the sides AB, BC, CD, DA, by A', B', C', D' the feet of the perpendiculars from P to the sides AB, BC, CD, DA and by A'', B'' the feet of the perpendiculars from D to the sides AB, BC. According to Lemma 2.1, $DA'' = d \sin A, DB'' = c \sin C$. From $\triangle BPA' \sim \triangle BDA''$ and $\triangle BPB' \sim \triangle BDB''$ it follows $\dfrac{PA'}{DA''} = \dfrac{BP}{BD} = \dfrac{PB'}{DB''}$ and then $\dfrac{PA'}{PB'} = \dfrac{DA''}{DB''}$ or

$$\frac{PA'}{PB'} = \frac{d \sin A}{c \sin C} \quad (2)$$

Similarly,

$$\frac{PA'}{PD'} = \frac{b\sin B}{c\sin D} \quad (3)$$

Analogously holds $\dfrac{PD'}{PC'} = \dfrac{a\sin A}{b\sin C}$. Multiplying the last two equations it follows

$$\frac{PA'}{PC'} = \frac{a\sin A\sin B}{c\sin C\sin D} \quad (4)$$

Finally, replacing the ratios obtained in (2), (3) and (4) in the equation (1) it results

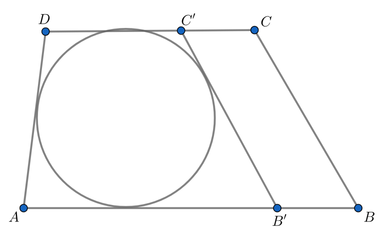
$$1 + \frac{a\sin A\sin B}{c\sin C\sin D} = \frac{d\sin A}{c\sin C} + \frac{b\sin B}{c\sin D}$$

Multiplying on both sides with $c\sin C\sin D$ it follows immediately that

$$c\sin C\sin D + a\sin A\sin B = d\sin D\sin A + b\sin B\sin C$$

so (iv) is proved. □

Proof. (iv)⇒ (i) Two cases will be considered for this proof.
1. $AB \parallel CD$. Assume by contradiction that $ABCD$ is not tangential (for proof will use a similar idea as in (ii)⇒(i)). Define Q as the intersection points of angle bisectors from A and D so the point Q will be equal distanced from the sides AB, AD, CD and therefore the sides AB, AD, CD will be tangent to the circle centered in Q that has the radius the common distance from Q to the sides AB, AD, CD.



There are two possible situations: BC to be external to the circle or BC to be secant to the circle. In the case that BC is external to the circle (the case BC secant can be proved in the same way), consider the parallel to the line BC that is tangent to the circle and denote by B' and C' the intersections with the sides AB and CD respectively. Due to the fact that BC is external to the circle, it follows that B' is on the side AB and C' on the side DC (5).The circle becomes incircle for the quadrilateral $AB'C'D$ so due to the equivalence (i) $\iff$ (ii) and the implications (ii) ⇒(iii)⇒ (iv) proved above, the following equation holds for $AB'C'D$

$$a'\sin A\sin B' + c'\sin C'\sin D = b'\sin B'\sin C' + d\sin D\sin A \quad (6)$$

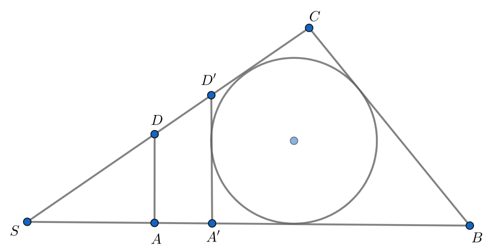
where a', b', c', d' are the sides' lenghts of the quadrilateral $AB'C'D$. For the quadrilateral $ABCD$, according to (iv) it holds also the equation

$$a\sin A\sin B + c\sin C\sin D = b\sin B\sin C + d\sin D\sin A \quad (7)$$

From $B'C' \parallel BC$ it follows $\angle B' = \angle B, \angle C' = \angle C$. Subtracting the equations (6) and (7) it results

$$(a - a')\sin A \sin B + (c - c')\sin C \sin D = (b - b')\sin D \sin A \quad (8)$$

From (5) it follows that $a > a', c > c'$ and from $AB \parallel CD$ and $B'C' \parallel BC$ it results $B'BCC'$ parallelogram so $b = b'$. Therefore the left hand side in (8) is positive while the right hand side is negative. This is a contradiction so $ABCD$ is tangential.
2. $AB \nparallel CD$. Denote S the intersection point of the lines AB and CD. WLOG, assume the point A lies between S and B. Then D is between S ans C ($ABCD$ is convex).



Assume by contradiction that $ABCD$ is not tangential (approximately the same idea as in(ii)⇒(i), strategically adapted). Define Q as the intersection point of angle bisectors from B and C so the point Q is equal distanced from the sides AB, BC, CD. Therefore the sides AB, BC, CD are tangent to the circle centered in Q that has the radius the common distance from Q to the sides AB, BC, CD. There are two possible situations: AD external to the circle or AD secant to the circle. In the case that AD is external to the circle (the case AD secant can be proved similarly), consider the parallel to the line AD that is tangent to the circle and denote by A' and D' the intersections with the sides AB and CD respectively. Due to the fact that AD is external to the circle, it follows that A' is on the side AB and D' on the side CD.The circle becomes incircle for the quadrilateral $A'BCD'$ so the following equation holds (due to the equivalence (i) $\iff$ (ii) and the implications (ii) ⇒(iii)⇒ (iv) proved above)

$$a' \sin A' \sin B + c' \sin C \sin D' = b \sin B \sin C + d' \sin D' \sin A \quad (9)$$

For $ABCD$ stands also a similar equation (from (iv))

$$a \sin A \sin B + c \sin C \sin D = b \sin B \sin C + d \sin D \sin A \quad (10)$$

From $A'D' \parallel AD$ it results $\angle A' = \angle A, \angle D' = \angle D$. Subtracting (10) and (9) it follows

$$(a - a')\sin A \sin B + (c - c')\sin C \sin D = (d - d')\sin D \sin A \quad (11)$$

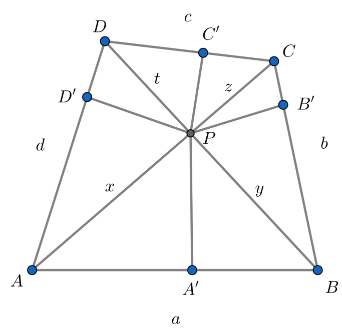
Because $a > a', c > c', d < d'$ (see the figure and the explanations provided) it follows that the left hand side of the equation (11) is positive while the right hand side is negative and hence contradiction. In conclusion, $ABCD$ is tangential. $\square$

3 Tangential Quadrilaterals Properties

Proposition 3.1. *If $ABCD$ is a tangential quadrilateral and P is the intersection of the diagonals then*

$$AB \cdot PC \cdot PD + CD \cdot PA \cdot PB = BC \cdot PD \cdot PA + DA \cdot PB \cdot PC$$

Proof. Denote $AB = a, BC = b, CD = c, DA = b,\ PA = x, PB = y, PC = z, PD = t,\ \angle APB = \alpha$. Consider A', B', C', D' the feet of the perpendiculars from P to the lines AB, BC, CD, DA.



From $(PAB) = \dfrac{xy\sin\alpha}{2}$ and $(PAB) = \dfrac{aPA'}{2}$ it follows $aPA' = xy\sin\alpha$ or equivalently $\dfrac{1}{PA'} = \dfrac{a}{xy\sin\alpha}$. Similarly, $\dfrac{1}{PC'} = \dfrac{c}{tz\sin\alpha}$, $\dfrac{1}{PB'} = \dfrac{b}{yz\sin\alpha}$, $\dfrac{1}{PD'} = \dfrac{d}{xt\sin\alpha}$.

Replacing in the equation from Theorem 2 (iii) it results

$$\frac{a}{xy\sin\alpha} + \frac{c}{tz\sin\alpha} = \frac{b}{yz\sin\alpha} + \frac{d}{xt\sin\alpha}$$

or equivalently

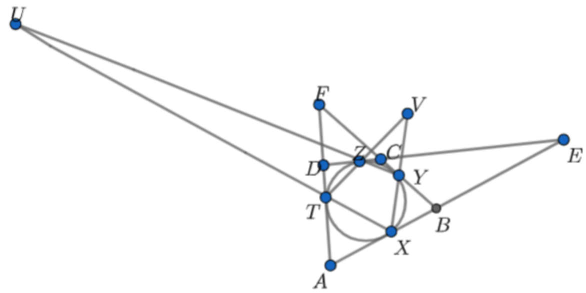
$$atz + cxy = bxt + dyz$$

□

Lemma 2.2 (Pascal)**.** *Let $ABCDEF$ be a hexagon (possible self-intersecting and possible degenerate) inscribed in a circle and P, Q, R the points of intersection of the opposite pairs of sides/diagonals in hexagon (AB, DE), (BC, EF), (CD, FA). Then P, Q, R are collinear.*

Proposition 3.2. *Let $ABCD$ be a tangential quadrilaterals and X, Y, Z, T the tangency points of the incircle with the sides AB, BC, CD, DA. Denote by E, F, U, V the intersection points of the pairs of lines $(AB, CD), (AD, BC), (XT, YZ), (XY, TZ)$. Then E, F, U, V are collinear.*

Proof. Consider the (degenerate) hexagon $TXXYZZ$ inscribed in the incircle and the opposite pairs of diagonals (TX, YZ) intersecting at U,then (XX, ZZ) intersecting at E, and (XY, ZT) intersecting at V.



From Pascal's Theorem it follows U, E, V collinear. (12)

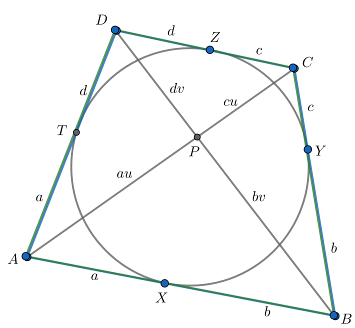
Similarly, the (degenerate) hexagon $TTXYYZ$ inscribed in the incircle has the opposite pairs of diagonals $(TT, YY), (TX, YZ), (XY, ZT)$ intersecting at F, U, V. From Pascal's Theorem it follows F, U, V collinear. (13) Finally, from (12) and (13) it follows that the points E, F belongs to the line UV hence the points E, F, U, V are collinear. □

4 Tangential Quadrilaterals and Cyclicity

Definition 4.1. A quadrilateral is called cyclic if there is a circle that passes through its vertices.

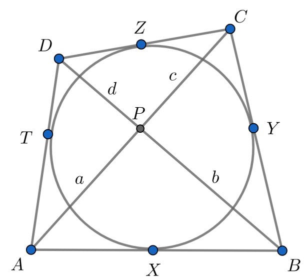
Proposition 4.1. *Let $ABCD$ be a tangential cyclic quadrilateral and P the intersection of the diagonals. Then the feet of the angle bisectors of $\angle PAB, \angle PBC, \angle PCD, \angle PDA$ to the sides AB, BC, CD, DA are exactly the tangency points of the incircle.*

Proof. Consider the notations and results from Theorems 1 and 2: $AB = AT = a, BX = BY = b, CY = CZ = c, DZ = DT = d$ and $PA = au, PC = cu, PB = bv, PD = dv$.



From $ABCD$ cyclic it follows $\angle PAB = \angle PDC$. Also $\angle AXP$ and $\angle DZP$ are half of the arc XTZ so $\angle AXP = \angle DZP$. In consequence, from $\triangle PAX$ and $\triangle PDZ$ (because the sum of the angles are 180^0 in each triangle) it results $\angle APX = \angle DPZ$ but $\angle DPZ = \angle BPX$ as vertical angles so finally $\angle APX = \angle BPX$. Hence the ray $(PX$ is the angle bisector of $\angle APB$. Analogously, the rays $(PY, (PZ, (PT$ are the angle bisectors of $\angle BPC, \angle CPD, \angle DPA$. □

Theorem 3. *Let $ABCD$ be a tangential quadrilateral and X, Y, Z, T the tangency points of the incircle to the sides AB, BC, CD, DA. Then $ABCD$ is cyclic iff $AX \cdot CZ = BY \cdot DT$.*



Proof. ($\Rightarrow$) Consider $ABCD$ cyclic. It is required to prove $AX \cdot CZ = BY \cdot DT$.

Denote $PA = a, PB = b, PC = c, PD = d$. From Proposition 4.1 it follows that $(PX$ is the angle bisector of $\angle PAB$. The angle bisector theorem in $\triangle PAB$ implies $\dfrac{AX}{BX} = \dfrac{AP}{BP}$ or $\dfrac{AX}{BX} = \dfrac{a}{b}$.

Hence there exists a positive number x s.t. $AX = ax, BX = bx$. Similarly there are y, z, t positive numbers s.t. $BY = by, CY = cy, CZ = cz, DZ = dz, AT = at, DT = dt$.

From the cyclicity of $ABCD$ it follows $\angle PAB = \angle PDC$ and from $(PX, (PZ$ angle bisectors of the congruent angles $\angle APB$ and $\angle CPD$ it results the similarity of the triangles $\triangle PBX$ and $\triangle PCZ$.

Hence $\dfrac{PB}{PC} = \dfrac{BX}{CZ}$ or $\dfrac{b}{c} = \dfrac{bx}{cz}$ that heads to $x = z$. Analogously it results $y = t$.

From $ABCD$ tangential it follows $AB + CD = BC + AD$ (Theorem 2) so it results

$$(ax + bx) + (cz + dz) = (by + cy) + (dt + at)$$

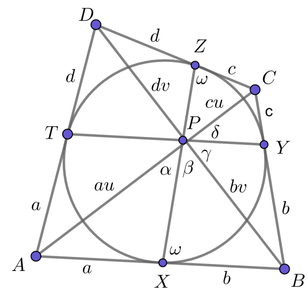
Also $x = z$ and $y = t$ heads to $(a+b+c+d)x = (b+c+d+a)y$ and then $x = y$. In consequence $x = y = z = t$.

Finally $AX \cdot CZ = (ax)(cx) = acx^2$ and $BY \cdot DT = (bx)(dx) = bdx^2$. The cyclicity of $ABCD$ involves $ac = bd$ (Ptolemy's Theorem) and therefore it results immediately $AX \cdot CZ = BY \cdot DT$.

($\Leftarrow$) Consider $AX \cdot CZ = BY \cdot DT$. It is required to prove that $ABCD$ is cyclic.

Denote a, b, c, d the length of tangents from A, B, C, D to the incircle, X, Y, Z, T the tangents points of the incircle to the sides AB, BC, CD, DA and P the intersection of the diagonals. From Corollary 1.1 it results $PA = au, PC = cu, PB = bv, PD = dv$ where u, v positive numbers.

Denote $\angle APX = \alpha, \angle BPX = \beta, \angle BPY = \gamma, \angle CPY = \delta$.



From $AX \cdot CZ = BY \cdot DT$ it follows $ac = bd$ or $\frac{a}{b} = \frac{d}{c}$. (14)

In the following, areas formulae for (APX) and (BPX) will be manipulated strategically several times. Starting with

$$(APX) = \frac{a \cdot d(P, AB)}{2}, \ (BPX) = \frac{b \cdot d(P, AB)}{2}$$

it follows

$$\frac{(APX)}{(BPX)} = \frac{a}{b} \quad (15)$$

then continue with

$$(APX) = \frac{au(\sin\alpha)PX}{2}, \ (BPX) = \frac{bv(\sin\beta)PX}{2}$$

it follows

$$\frac{au(\sin\alpha)PX}{bv(\sin\beta)PX} = \frac{a}{b} \quad (16)$$

or equivalently to

$$u \sin\alpha = v \sin\beta \quad (17)$$

Denote $\angle PXB = \angle PZC = \omega$ it follows $\angle PXA = \angle PZD = 180^0 - \omega$. From the $\triangle PAX$ it results $\angle PAX = \omega - \alpha$ and from the $\triangle PBX$ it results $\angle PBX = 180^0 - (\alpha + \beta)$.

Finally

$$(APX) = \frac{a(au)\sin(\omega - \alpha)}{2}, \ (BPX) = \frac{b(bv)\sin(\omega + \beta)}{2}$$

provides

$$\frac{a^2 u \sin(\omega - \alpha)}{b^2 v \sin(\omega + \beta)} = \frac{a}{b} \quad (18)$$

or equivalently

$$au\sin(\omega-\alpha)=bv\sin(\omega+\beta)\quad(19)$$

Analogously, using the same areas-based techniques for $\triangle CPZ$ and $\triangle DPZ$ it follows

$$dv\sin(\omega-\beta)=cu\sin(\omega+\alpha)\quad(20)$$

Multiplying (19) and (20 and using (14) it results after simplifications

$$u^2\sin(\omega+\alpha)\sin(\omega-\alpha)=v^2\sin(\omega+\beta)\sin(\omega-\beta)\quad(21)$$

Using on both sides of the equation the trig formula $\sin x\sin y=\dfrac{1}{2}(\cos(x-y)-\cos(x+y))$ it results

$$u^2(\cos 2\alpha-\cos 2\omega)=v^2(\cos 2\beta-\cos 2\omega)$$

Applying the double-angle formula $\cos 2x=1-2\sin^2 x$ it follows

$$u^2(1-2\sin^2\alpha-\cos 2\omega)=v^2(1-2\sin^2\beta-\cos 2\omega)$$

Using (17) and it follows

$$u^2(1-\cos 2\omega)=v^2(1-\cos 2\omega)$$

Due to the fact that $0^0<\omega<180^0$ it follows that $\cos 2\omega\neq 1$. The previous equation implies

$$u^2=v^2$$

and finally

$$u=v\quad(22)$$

Back to the length of the segments PA, PB, PC, PD used at the beginning of the sufficiency proof ($\Leftarrow$), from $u=v$ it results $PA=au, PB=bu, PC=cu, PD=du$ and because $ac=bd$ it results immediately $PA\cdot PC=PB\cdot PD$ i.e. $ABCD$ cyclic.

□

*Note for the sufficiency proof ($\Leftarrow$) : from (17) and (22) it follows also

$$\sin\alpha=\sin\beta$$

and due to the fact that α,β are in the interval $(0^0,180^0)$ and $0<\alpha+\beta<180^0$ it results $\alpha=\beta$. Analogously can be obtained $\gamma=\delta$.

In consequence, the rays $(PX, (PY, (PZ, (PT$ are the angle bisectors of $\angle APB, \angle BPC, \angle CPD$, $\angle DPA$ or equivalently $XZ\perp YT$. (23)

Theorem 4. *Let $ABCD$ be a tangential quadrilateral and X,Y,Z,T the tangency points of the incircle to the sides AB, BC, CD, DA. Then $ABCD$ is cyclic iff $XZ\perp YT$.*

Proof. It follows immediately from Theorem 3 and *Note (23). □

Proposition 4.2. *Let be $ABCD$ a cyclic quadrilateral, P the intersection of the diagonals and X,Y,Z,T the feet of the angle bisectors of $\angle PAB, \angle PBC, \angle PCD, \angle PDA$. If $XYZT$ is cyclic then $ABCD$ is tangential.*

Proof. Denote $PA = a, PB = b, PC = c, PD = d$.

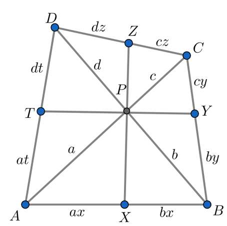
From the angle bisector theorem in $\triangle PAB$ it results $\dfrac{AX}{BX} = \dfrac{a}{b}$.

Hence there is a positive number x s.t. $AX = ax, BX = bx$. Analogously, there are y, z, t positive numbers s.t. $BY = by, CY = cy, CZ = cz, DZ = dz, DT = dt, AT = at$.

Denote $\angle APX = \angle BPX = \alpha, \angle BPY = \angle CPY = \beta$.

Notice that $\triangle PBX$ and $\triangle PCZ$ have two pairs of congruent angles so $\triangle PBX \sim \triangle PCZ$.

It results $\dfrac{PB}{PC} = \dfrac{BX}{CZ}$ or $\dfrac{b}{c} = \dfrac{bx}{cz}$ so $x = z$. Similarly, $y = t$. (24)



From $XYZT$ cyclic it follows $PX \cdot PZ = PY \cdot PT$ and hence $PX^2 \cdot PZ^2 = PY^2 \cdot PT^2$ (25)

From the angle bisector length theorem in $\triangle PAB$ it results

$$PX^2 = \frac{PA \cdot PB(PA + PB + AB)(PA + PB - AB)}{AB^2}$$

or equivalently

$$PX^2 = \frac{ab(a + b + (a + b)x)(a + b - (a + b)x)}{(a + b)^2}$$

$$= \frac{ab(a + b)^2(1 - x^2)}{(a + b)^2}$$

$$= ab(1 - x^2) \quad (26)$$

Analogously $PZ^2 = cd(1 - z^2), PY^2 = bc(1 - y^2), PT^2 = ad(1 - t^2)$. Using (24) it results

$PZ^2 = cd(1 - x^2), PY^2 = bc(1 - y^2), PT^2 = ad(1 - y^2)$ (27)

Using (26) and (27) in the equation (25) it follows

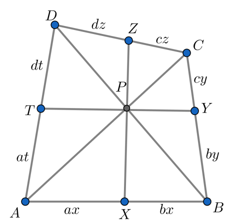
$abcd(1 - x^2)^2 = abcd(1 - y^2)^2$ or equivalently $|1 - x^2| = |1 - y^2|$ (28) newline

Notice that the triangle inequality $PA + PB > AB$ leads to $a + b > (a + b)x$ or $x < 1$. Similarly $y < 1$ and hence (28) becomes $1 - x^2 = 1 - y^2$ so $x = y$ and from (24) it results $x = y = z = t$. Finally $AB + CD = (a + b + c + d)x$ and $AD + BC = (a + b + c + d)x$ so $AB + CD = AD + BC$.

From Theorem 2 it results that the quadrilateral $ABCD$ is tangential. More than this, it results also $AX = AT, BX = BY, CY = CZ, DZ = DT$ so X, Y, Z, T are the contact points of the incircle with the quadrilateral$ABCD$. $\square$

Proposition 4.3. *Let be $ABCD$ a cyclic quadrilateral, P the intersection of the diagonals and X, Y, Z, T the feet of the angle bisectors of $\angle PAB, \angle PBC, \angle PCD, \angle PDA$. Then $XYZT$ is tangential if and only if $ABCD$ is trapezoid.*

Proof. Denote $PA = a, PB = b, PC = c, PD = d$. As in the proof of the Proposition 4.2 it results $AX = AT = ax, BX = BY = bz, CY = CZ = cx, DZ = DT = dx$.



Because the rays (PX, PY, PZ, PT are the angle bisectors of the $\angle APB, \angle BPC, \angle CPD, \angle DPA$ it results immediately $XZ \perp TY$.

From Proposition 4.2 it follows that

$$PX = \sqrt{ab(1-x^2)}, PZ = \sqrt{cd(1-x^2)}, PY = \sqrt{bc(1-x^2)}, PT = \sqrt{ad(1-x^2)}$$

The Pythagorean Theorem in $\triangle XPY$ delivers

$$XY = \sqrt{(1-x^2)b(a+c)}$$

Similarly

$$TZ = \sqrt{(1-x^2)d(a+c)}, TX = \sqrt{(1-x^2)a(b+d)}, YZ = \sqrt{(1-x^2)c(b+d)}$$

Hence $ABCD$ tangential $\iff XY + TZ = XT + ZY \iff$

$$\sqrt{1-x^2}(\sqrt{b(a+c)} + \sqrt{d(a+c)}) = \sqrt{1-x^2}(\sqrt{a(b+d)} + \sqrt{c(b+d)})$$

$\iff$ (because $0 < x < 1$ as in the proof of Proposition 4.2)

$$\sqrt{b(a+c)} + \sqrt{d(a+c)} = \sqrt{a(b+d)} + \sqrt{c(b+d)}$$

$\iff$

$$(a+c)(b+d+2\sqrt{bd}) = (b+d)(a+c+2\sqrt{ac})$$

$\iff$

$$(a+c)\sqrt{bd} = (b+d)\sqrt{ac}$$

From Theorem 3 it results $ac = bd$ so the previous equation is equivalent with

$$a + c = b + d$$

Denoting $a + c = b + d = S$ and $ac = bd = P$ it follows that a, c and b, d are the solutions of the same quadratic equation $x^2 - Sx + P = 0$ so there are two possible situations:

(i) $a = b, c = d$

(ii) $a = d, b = c$
Notice that (i) $\iff AB \parallel CD$ and (ii) $\iff AD \parallel BC$.

Therefore, $a + c = b + d \iff ABCD$ trapezoid.

As a remark, note that the $ABCD$ is an isosceles trapezoid due to the fact that it is cyclic.

□

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Dr. Suzy Manuela Prajea,
J.L. Chambers High School,
Charlotte, North Carolina, USA
prajeamanuela2012@gmail.com