Junior problems

J277. Is there an integer n such that $4^{5^n} + 5^{4^n}$ is a prime?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J278. Find all positive integers n for which

$$\left\{\sqrt[3]{n}\right\} \leq \frac{1}{n},$$

where $\{x\}$ denotes the fractional part of x.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J279. Find all triples (p, q, r) of primes such that $pqr = p + q + r + 2000$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J280. Let a, b, c, d be positive real numbers. Prove that

$$2(ab + cd)(ac + bd)(ad + bc) \geq (abc + bcd + cda + dab)^2.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J281. Solve the equation

$$x + \sqrt{(x+1)(x+2)} + \sqrt{(x+2)(x+3)} + \sqrt{(x+3)(x+1)} = 4.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J282. Given an $m \times n$ board, k cells are painted such that if the centers of four cells are the vertices of a quadrilateral with parallel sides to the borders of the board then at most two must be painted. Find the greatest value of k.

Proposed by Roberto Bosch Cabrera, Texas, USA

Senior problems

S277. Let a, b, c be positive real numbers such that

$$\frac{1}{a^3+b^3}+\frac{1}{b^3+c^3}+\frac{1}{c^3+a^3}\leq\frac{3}{a+b+c}.$$

Prove that

$$2(a^2+b^2+c^2)+(a-b)^2+(b-c)^2+(c-a)^2\geq 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S278. Let a, b, c be complex numbers such that $|a| = |b| = |c| = 1$. If there is a positive integer n such that $|a+b|^{2^n}+|b+c|^{2^n}+|c+a|^{2^n}\leq 3$, prove that a, b, c are the affixes of the vertices of an equilateral triangle.

Proposed by Marcel Chirita, Bucharest, Romania

S279. Solve in integers the equation

$$(2x+y)(2y+x)=9\min(x,y).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S280. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$x^4y^4z^4(x^3+y^3+z^3)\leq 3.$$

Proposed by Sayan Das, ISI Kolkata, India

S281. Let n be an integer greater than 1. For $a\in\mathbb{C}\setminus\mathbb{R}$ with $|a|=1$, consider the equation

$$\sum_{k=0}^{n}\binom{n}{k}(a^k+1)x^k=0.$$

Prove that

(a) All roots of the equation lie on a line d_a.

(b) Lines d_a and d_b are perpendicular if and only if $a+b=0$.

Proposed by Dorin Andrica, Babes Bolyai University, Cluj-Napoca, Romania

S282. Let ABC be a triangle, G its centroid, and O its circumcenter. Lines AG, BG, CG intersect the circumcircle of triangle ABC at A', B', C'. Denote by G' the centroid of triangle $A'B'C'$. Prove that $OG\geq OG'$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Undergraduate problems

U277. For $n \in \mathbb{N}$, $n \geq 2$, find the greatest integer less than $2\left(e^{\frac{1}{n+1}} + \ldots + e^{\frac{1}{n+n}}\right)$.

Proposed by Marius Cavachi, Constanta, Romania

U278. Evaluate

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{1}{(kn+1)k!}$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U279. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with lateral limits at every point. Prove that there is some real number x_0 such that

$$\lim_{x\to x_0, x>x_0} f(x) \leq x_0 \leq \lim_{x\to x_0, x<x_0} f(x).$$

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

U280. Let S be an uncountable set of circles in the plane. Prove that there is an uncountable subset S' of S such that all the circles in S' have a common interior point.

Proposed by Marius Cavachi, Constanta, Romania

U281. Let G be a graph on n vertices so that for every connected subgraph H of G, the graph $G - H$ is connected. Prove that G is either a cycle or a complete graph.

Proposed by Cosmin Pohoata, Princeton University, USA

U282. Let $P = \{2, 3, 5, 7, 11, \ldots,\}$ denote the set of all primes less than 2^{100}. Prove that $\sum\limits_{p\in P} \frac{1}{p} < 8$.

Proposed by Marius Cavachi, Constanta, Romania

Olympiad problems

O277. Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 1$. Prove that

$$\frac{a+b}{\sqrt{ab+c}}+\frac{b+c}{\sqrt{bc+a}}+\frac{c+a}{\sqrt{ca+b}}\geq 3\sqrt[6]{abc}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O278. Find all primes p,q,r such that $p \mid 2qr+r$, $q \mid 2rp+p$, and $r \mid 2pq+q$.

Proposed by Roberto Bosch Cabrera, Texas, USA

O279. One hundred boys and one hundred girls go to prom. Knowing that each boy dances with a girl at most once and that there are 1050 couples dancing, prove that there are two boys and two girls who dance with both of the boys.

Proposed by Marius Cavachi, Constanta, Romania

O280. Find all positive integers that can be written as

$$\frac{(3a_1^2+2a_1-4)(3a_2^2+2a_2-4)\dots(3a_k^2+2a_k-4)}{(3b_1^2+2b_1-4)(3b_2^2+2b_2-4)\dots(3b_k^2+2b_k-4)}$$

for some positive integers a_k, b_k and some $k\in\mathbb{N}^*$.

Proposed by Vlad Matei, University of Wisconsin, Madison, USA

O281. Let $a_1,a_2,\dots,a_n$ be a decreasing sequence of positive real numbers. Prove that

$$\sqrt{a_1^2+\cdots+a_n^2}\leq a_1+\frac{a_2}{\sqrt{2}+1}+\cdots+\frac{a_n}{\sqrt{n}+\sqrt{n-1}}\leq\sqrt{\left(1+\frac{1}{4}\ln n\right)(a_1^2+\cdots+a_n^2)}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O282. Let ABC be a triangle and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle of triangle ABC. Lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C', respectively, and X, Y, Z are the intersections $BB'\cap CC'$, $CC'\cap AA'$, and $AA'\cap BB'$. Parallels through A, B, C to BC, CA, AB determine a triangle MNQ and α, β, γ are the reflections of NQ, QM, MN into AX, BY, CZ, respectively. Prove that:

a) α, β, γ concur on the circumcircle of triangle ABC;

b) $AX\|BY\|CZ\|PP'$, where P' is the concurrence point from part a).

Proposed by Cosmin Pohoata, Princeton University, USA