Junior problems

J265. Let a, b, c be real numbers such that

$$5(a+b+c)-2(ab+bc+ca)=9.$$

Prove that any two of the equalities

$$|3a-4b|=|5c-6|, \quad |3b-4c|=|5a-6|, \quad |3c-4a|=|5b-6|$$

imply the third.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J266. Let $ABCD$ be a cyclic quadrilateral such that $AB > AD$ and $BC = CD$. The circle of center C and radius CD intersects again the line AD in E. The line BE intersects again the circumcircle of the quadrilateral in K. Prove that AK is perpendicular to CE.

Proposed by Mircea Becheanu, University of Bucharest, Romania

J267. Solve the system of equations

$$\begin{cases} x^5+x-1=(y^3+y^2-1)z \\ y^5+y-1=(z^3+z^2-1)x \\ z^5+z-1=(x^3+x^2-1)y, \end{cases}$$

where x, y, z are real numbers such that $x^3+y^3+z^3 \geq 3$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J268. Consider a convex m-gon $B_1 \ldots B_m$ lying inside a convex n-gon $A_1 \ldots A_n$. Their vertices define $m+n$ points in the plane. Prove that if $m+n \geq k^2-k+1$, then we can find a convex $(k+1)$-gon among these vertices that contains no other points inside it.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J269. Solve in positive integers the equation

$$(x^2-y^2)^2-6\min(x,y)=2013.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J270. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a+2b+5c}+\frac{1}{b+2c+5a}+\frac{1}{c+2a+5b} \leq \frac{9}{8}\frac{a+b+c}{(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})^2}.$$

Proposed by Tran Bach Hai, Bucharest, Romania

Senior problems

S265. Find all pairs (m, n) of positive integers such that $m^2 + 5n$ and $n^2 + 5m$ are both perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S266. Let $ABCD$ be a cyclic quadrilateral, $O = AC \cap BD$, M, N, P, Q be the midpoints of AB, BC, CD and DA, respectively, and X, Y, Z, T be the projections of O on AB, BC, CD and DA, respectively. Let $U = MP \cap YT$ and $V = NQ \cap XZ$. Prove that the U, O, V are collinear.

Proposed by Marius Stanean, Zalau, Romania

S267. Find all primes p, q, r such that $7p^3 - q^3 = r^6$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S268. Let C be a circle with center O and let W be a point in its interior. From W we draw $2k$ rays such that the angle between any two adjacent rays is equal to $\frac{\pi}{k}$. These rays intersect the circumference of the circle C in points $A_1, \dots, A_{2k}$. Prove that the centroid of $A_1 \dots A_{2k}$ is the midpoint of OW.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S269. Find all integers n for which the equation $(n^2 - 1)x^2 - y^2 = 2$ is solvable in integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S270. Complex numbers z_1, z_2, z_3 satisfy $|z_1| = |z_2| = |z_3| = 1$. If $z_1^k + z_2^k + z_3^k$ is an integer for $k \in \{1, 2, 3\}$, prove that $z_1^{12} = z_2^{12} = z_3^{12}$.

Proposed by Mihai Piticari and Sorin Radulescu, Romania

Undergraduate problems

U265. Let $a > 1$ be a real number and let $f : [1, a] \to \mathbb{R}$ be twice differentiable. Prove that if the map $x \mapsto xf(x)$ is increasing, then

$$f(\sqrt{a}) \le \frac{1}{\ln a} \int_1^a \frac{f(t)}{t} dt.$$

Proposed by Marcel Chirita, Bucharest, Romania

U266. Let $A, B \in M_n(\mathbb{R})$ be symmetric positive definite matrices. Prove that

$$\mathrm{tr}[(A^2 + AB^2A)^{-1}] \ge \mathrm{tr}[(A^2 + BA^2B)^{-1}].$$

Proposed by Cosmin Pohoata, Princeton University, USA

U267. A continuous map $f : [0, 1] \to [-\frac{1}{3}, \frac{2}{3}]$ is onto and satisfies $\int_0^1 f(x)dx = 0$. Prove that

$$\int_0^1 f(x)^3 dx \le \frac{1}{9}.$$

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

U268. Evaluate

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=2}^{n^2-1} \left\{ \frac{n}{\sqrt{k}} \right\},$$

where $\{x\}$ is the fractional part of x.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

U269. Let $a_1, \dots, a_k$ and $b_1, \dots, b_k$ be positive real numbers with $a_i > b_i$ for $i = 1, \dots, k$. If $\Delta_i = a_i - b_i$, prove that

$$\prod_{i=1}^k a_i - \prod_{i=1}^k b_i \ge k \sqrt[k]{\Delta_1 \cdots \Delta_k} \left(\prod_{i=1}^k a_i \right)^{\frac{k-1}{2k}} \left(\prod_{i=1}^k b_i \right)^{\frac{k-1}{2k}}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

U270. Let x_1 and x_2 be positive real numbers and define, for $n \ge 2$

$$x_{n+1} = \sqrt[n]{x_1} + \sqrt[n]{x_2} + \cdots + \sqrt[n]{x_n}.$$

Find $\displaystyle\lim_{n\to\infty} \frac{x_n - n}{\ln n}$.

Proposed by Gabriel Dospinescu, Ecole Normale Supérieure, Lyon, France

Olympiad problems

O265. Solve in nonnegative real numbers the system of equations

$$\begin{cases}(x+1)(y+1)(z+1)=5\\(\sqrt{x}+\sqrt{y}+\sqrt{z})^2-\min(x,y,z)=6.\end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O266. Let $a, b, c \geq 1$ be real numbers such that $a+b+c=6$. Prove that

$$(a^2+2)(b^2+2)(c^2+2) \leq 216.$$

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

O267. Find all primes p, q, r such that

$$\frac{p^{2q}+q^{2p}}{p^3-pq+q^3}=r.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O268. Let $a_1, \ldots, a_{2n+1}$ be real numbers that add up to 0. Consider function $f(x)=\sum_{i=1}^{2n+1}|a_i-x|$. Let y be the point at which $f(x)$ attains its minimum. For $n \geq 1$, prove that

$$y \leq \frac{1}{2(n+1)}\sum_{i=1}^{2n+1}|a_i|.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

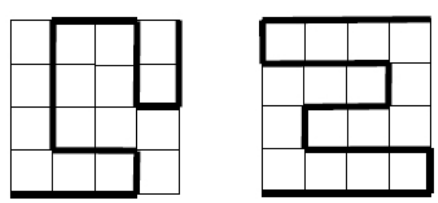
O269. Let ABC be a triangle with circumcenter Γ and nine-point center γ. Let X be a point on Γ and let Y, Z be on Γ so that the midpoints of segments XY and XZ are on γ.

a) Prove that the midpoint of YZ is on γ.

b) Find the locus of the symmedian point of triangle XYZ, as X moves along Γ.

Proposed by Cosmin Pohoata, Princeton University, USA

O270. The diagram shows a 4 by 4 grid made up of sixteen 1 by 1 squares.



A corner-to-corner path is a path that follows the edges of the 1 by 1 squares from the lower left corner of the grid to the upper right corner of the grid as shown in the two examples below. A path may not intersect itself by moving to a point where the path has already been. Find the number of corner-to-corner paths such as the second path shown below which are symmetric with respect to the center of the grid or, alternatively, are equal to themselves when the path is rotated 180 degrees.

Proposed by Jonathan Kane, University of Wisconsin, Whitewater, USA