Junior problems

J205. Find the greatest n-digit number $a_1a_2...a_n$ with the following properties:

i) all its digits are different from zero and distinct;

ii) for each $k = 2, ..., n-1$, $\frac{1}{a_{k-1}}, \frac{1}{a_k}, \frac{1}{a_{k+1}}$ is either an arithmetic sequence or a geometric sequence.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J206. Let A, B, C, X, Y, Z be points in the plane. Prove that the circumcircles of triangles AYZ, BZX, CXY are concurrent if and only if the circumcircles of triangles XBC, YCA, ZAB are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

J207. Find the greatest number of the form $2^a5^b + 1$, with a and b nonnegative integers, that divides a number all whose digits are distinct.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J208. Let K be the symmedian point of triangle ABC and let R be its circumradius. Prove that

$$AK + BK + CK \leq 3R.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J209. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{(b+c)^5}{a} + \frac{(c+a)^5}{b} + \frac{(a+b)^5}{c} \geq \frac{32}{9}(ab + bc + ca).$$

Proposed by Marius Stanean and Mircea Lascu, Zalau, Romania

J210. Let P and Q be points in the plane of triangle ABC such that $\{AP, BP, CP\} = \{AQ, BQ, CQ\}$. Prove that

$$OP^2 + \frac{1}{3}PG^2 = OQ^2 + \frac{1}{3}QG^2,$$

where O and G are the circumcenter and centroid of triangle ABC, respectively.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Senior problems

S205. Let $C_0(O, R)$ be a circle and let I be a point at distance $d < R$ from O. Consider circles $C_1(I, r_1)$ and $C_2(I, r_2)$ such that there is a triangle inscirbed in C_0 and circumscribed about C_1 and there is a quadrilateral inscribed in C_0 and circumscribed about C_2. Prove that $1 < \frac{r_2}{r_1} \leq \sqrt{2}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S206. Find all integers $n \geq 2$ having a prime divisor p such that $n-1$ is divisible by the exponent of p in $n!$.

Proposed by Tigran Hakobyan, Yerevan, Armenia

S207. Let a, b, c be distinct nonzero real numbers such that $ab + bc + ca = 3$ and $a + b + c \neq abc + \frac{2}{abc}$. Prove that

$$\left(\sum_{cyc} \frac{a(b-c)}{bc-1}\right) \cdot \left(\sum_{cyc} \frac{bc-1}{a(b-c)}\right)$$

is the square of an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S208. Let $f \in \mathbb{Z}[X]$ be such that $f(1) + f(2) + \ldots + f(n)$ is a perfect square for all positive integers n. Prove that there exist a positive integer k and a polynomial $g \in \mathbb{Z}[X]$ with $g(0) = 0$ and $k^2 f(X) = g^2(X) - g^2(X-1)$.

Proposed by Vlad Matei, University of Cambridge, United Kingdom

S209. Let a, b, c be the sidelengths, s the semiperimeter, r the inradius, and R the circumradius of a triangle ABC. Prove that

$$\frac{sr}{R}\left(1 + \frac{R-2r}{4R+r}\right) \leq \frac{(s-b)(s-c)}{a} + \frac{(s-c)(s-a)}{b} + \frac{(s-a)(s-b)}{c}.$$

Proposed by Darij Grinberg, Massachusetts Institute of Technology, and Cosmin Pohoata, Princeton University, USA

S210. Let p be an odd prime and let $F(X) = \sum_{k=0}^{p-1} \binom{2k}{k}^2 \cdot X^k$. Prove that for all $x \in \mathbb{Z}$,

$$(-1)^{\frac{p-1}{2}} F(x) \equiv F\left(\frac{1}{16} - x\right).$$

Proposed by Gabriel Dospinescu, Ecole Polytehnique, France

Undergraduate problems

U205. Let E be a vector space with norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Decide if $\min(\|\cdot\|_1, \|\cdot\|_2)$ is a norm.

Proposed by Roberto Bosch Cabrera, Florida, USA

U206. Prove that there is precisely one group with 30 elements and 8 automorphisms.

Proposed by Gabriel Dospinescu, Ecole Polytehnique, France

U207. Let $n \geq 3$ be an odd integer. Evaluate

$$\sum_{k=1}^{\frac{n-1}{2}} \sec \frac{2k\pi}{n}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U208. Let X and Y be standard Cauchy random variables $C(0,1)$. Prove that the probability density function of random variable $Z = X^2 + Y^2$ is given by

$$f_Z(t) = \frac{2}{\pi} \cdot \frac{1}{(t+2)\sqrt{t+1}}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U209. Let $k \geq 2$ be a positive integer and G be an $(k-1)$-edge-connected k-regular graph with an even number of vertices. Prove that for every edge e of the graph there is a perfect matching of G containing e.

Proposed by Cosmin Pohoata, Princeton University, USA

U210. A graph G arises from G_1 and G_2 by pasting them along S if G has induced subgraphs G_1, G_2 with $G = G_1 \cup G_2$ and S is such that $S = G_1 \cap G_2$. A graph is called chordal if it can be constructed recursively by pasting along complete subgraphs, starting from complete subgraphs. For a graph $G(V, E)$ define its Hilbert polynomial $H_G(x)$ to be

$$H_G(x) = 1 + Vx + Ex^2 + c(K_3)x^3 + c(K_4)x^4 + \ldots + c(K_{\omega(G)})x^{\omega(G)},$$

where $c(K_i)$ is the number of $i-$*cliques* in G and $\omega(G)$ is the clique number of G. Prove that $H_G(-1) = 0$ if and only if G is chordal.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Olympiad problems

O205. Find all n such that each number containing n 1's and one 3 is prime.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O206. Let $D \in BC$ be the foot of the A-symmedian of triangle ABC with centroid G. The circle passing through A and tangent to BC at D intersects sides AB and AC at E and F, respectively. If $3AD^2 = AB^2 + AC^2$, prove that G lies on EF.

Proposed by Marius Stanean, Zalau, Romania

O207. Define a sequence $(x_n)_{n\geq 1}$ of rational numbers by $x_1 = x_2 = x_3 = 1$ and $x_n x_{n-3} = x_{n-1}^2 + x_{n-1}x_{n-2} + x_{n-2}^2$ for all $n \geq 4$. Prove that x_n is an integer for every positive integer n.

Proposed by Darij Grinberg, Massachusetts Institute of Technology, USA

O208. Let $z_1, z_2, ..., z_n$ be complex numbers such that $z_1^k + z_2^k + ... + z_n^k$ is the k-th power of a rational number for all $k > 2011$. Prove that at most one of the numbers z_i is nonzero.

Proposed by Gabriel Dospinescu, Ecole Polytehnique, France and Octav Dragoi, ICHB, Bucharest, Romania

O209. Let P be a point on the side BC of triangle ABC with circumcircle Γ, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the circles internally tangent to Γ and also to AP, BP, and AP, CP, respectively. If I is the incenter of triangle ABC and M is the midpoint of the arc BC of Γ not containing the vertex A, prove that the radical axis of $\mathcal{T}_1$ and $\mathcal{T}_2$ is the line determined by M and the midpoint of the segment IP.

Proposed by Cosmin Pohoata, Princeton University, USA

O210. Suppose that the set of positive integers is partitioned into a set of sequences $(L_{n,i})_{i\geq 1}$ such that $L_{n,i}$ divides $L_{n,i+1}$ for all positive integers n and i. Prove that for all positive integers t, there are infinitely many n such that $\omega(L_{n,1}) = t$, where for a positive integer a, decomposed into primes as $a = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, $\omega(a) = \alpha_1 + \alpha_2 + ... + \alpha_r$.

Proposed by Radu Bumbacea, Bucharest, Romania