Junior problems

J229. Adrian has a credit card from his dad. He cannot get cash without knowing the personal identification number (PIN). Adrian has sort of an emergency and asks dad to provide him with the PIN (a whole number from 0000 to 9999). Dad tells Adrian that the PIN is the largest prime that divides $3^{22}+3$ and that he is not allowed to use a calculator. Adrian is able to get the cash he needs by finding the PIN using his 7th grade math knowledge. What is the PIN?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Adrian can easily find that $3, 3^7+1$ divide $3^{22}+3$, and that $3^{14}-3^7+1 = 3^{14}+2\cdot 3^7+1-3^8$, or

$$3^{22}+3 = 3\left(3^7+1\right)\left(3^7+3^4+1\right)\left(3^7-3^4+1\right).$$

Clearly no prime factor of $3^{22}+3$ can be larger than $3^7+3^4+1 = 2187+81+1 = 2269$, which is the largest of the four factors that Adrian can easily find this way. But 2269 itself is prime, as Adrian can find by dividing it by all primes less than $\sqrt{2269} < \sqrt{2304} = 48$. Adrian can then find that the PIN is 2269.

Also solved by Arkady Alt , San Jose ,California, USA, Daniel Vacaru, Pitesti, Romania

J230. Let ABC be a triangle and let M be the midpoint of the side BC. Suppose that there is some $0^\circ < x \leq 30^\circ$ so that the measure of $\angle ACB, \angle ABC, \angle MAC$ are $x, 60^\circ - x, 2x$, respectively. Determine x.

Proposed by Marius Stanean and Mircea Lascu, Zalau, Romania

Solution by Christopher Wiriawan, Jakarta, Indonesia

First, we see immediately that $\angle BAM = 120^\circ - 2x$. Then, by the Law of Sines, applied in triangle ABM, we can write

$$\frac{BM}{AM} = \frac{\sin \angle BAM}{\sin \angle ABM} = \frac{\sin(120^\circ - 2x)}{\sin(60^\circ - x)} = 2\cos(60^\circ - x).$$

Similarly, the Law of Sines in triangle AMC tells us

$$\frac{MC}{AM} = \frac{\sin \angle CAM}{\sin \angle ACM} = \frac{\sin 2x}{\sin x} = 2\cos x,$$

and thus, since M is the midpoint of side BC, it follows that

$$2\cos(60^\circ - x) = 2\cos x, \text{ i.e. } \cos(60^\circ - x) = \cos x.$$

Since $0^\circ < x \leq 30^\circ$, we conclude that $60^\circ - x = x$, which yields $x = 30^\circ$. This completes the proof.

Also solved by Francesco Bonesi and Giulia Giovannotti, Università di Roma "Tor Vergata", Roma, Italy; Zolbayar Shagdar, Ulaanbaatar, Mongolia; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Universidad Pública de Navarra, Spain, Nicusor Zlota, Focsani, Romania.

J231. Gigel and Costel have a collection $\mathcal{J}$ of empty jars of the same shape and a very large number of identical coins at their disposal. They decide to play the following game. Knowing that each jar has capacity of 100 coins, they take turns to pick a number of k coins from the pile, with $1 \le k \le 10$, and then (in the same turn) choose a jar into which to put the selected coins. The winner is the one who fills the last jar. Assuming that Gigel goes first and that both players are smart, who wins the game?

Proposed by Cosmin Pohoata, Princeton University, USA

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

At any given moment in the game, we will name the players A (the player who plays next) and B (the player who plays in the following to the next turn).

We say that a (possibly full) jar is a *loosing jar* if it contains a number of coins which leaves a remainder of 1 when divided by 11. If a player puts $1 \le k \le 10$ coins in a loosing jar, the other player may counter by putting $1 \le 11-k \le 10$ coins in the jar, thus maintaining the situation, until reaching $100 = 1 + 9 \cdot 11$ coins. In other words, if at a given moment not all jars are full but there are only loosing jars (including possibly some full jars), B can make sure to win, by placing at each turn exactly $1 \le 11-k \le 10$ coins in the jar where A has just placed $1 \le k \le 10$ coins, ensuring that it will be him who will place the last coin(s) in each one of the jars not yet full.

We say that a (possibly empty) jar is a *winning jar* if it contains a number of coins which is a multiple of 11. At any given moment in the game, if there are at least two winning jars, B can ensure to keep the parity of the number of winning jars unchanged; indeed, if A places k coins in a loosing jar, B can counter by keeping that jar a loosing jar as shown earlier. If A places k coins in a winning jar, there are two possibilities:

- The winning jar chosen by A had initially 99 coins, or A can place exactly one coin, making the jar a loosing jar. Then, B can choose any other winning jar and put one coin in it, making it a loosing jar (and possibly filling it completely), thus reducing the number of winning jars by 2, and increasing the number of loosing jars by 2.

- The winning jar chosen by A had initially ≤ 88 coins, or B counters by placing $1 \le 11-k \le 10$ coins in it after A has placed $1 \le k \le 10$ coins in it, or all initially winning jars are still winning jars.

Assume that the total number j of jars involved in the game is odd. Then Gigel wins by placing one coin in any jar. Note that, after Gigel's first move, there is exactly one loosing jar and even number $j-1$ of winning jars. From this point on, Gigel can always counter Costel's moving by either keeping all loosing jars loosing and all winning jars winning, or by reducing exactly by 2 the number of winning jars while increasing exactly by 2 the number of loosing jars. After exactly $9j + \frac{j+1}{2}$ moves from Gigel and $9j + \frac{j-1}{2}$ moves by Costel, all jars will be full and the last move will have been Gigel's.

Assume that the total number j of jars involved in the game is even. Then Costel moves by countering each of Gigel's moves as in the previous game, except that now the roles are reversed. After exactly $9j + \frac{j}{2}$ moves from each player all jars will be full and the last move will have been Costel's.

Second solution by Francesco Bonesi and Antonello Cirulli, Università di Roma "Tor Vergata", Roma, Italy

This is an finite impartial game. A position is given by $(p_j)_{j\in\mathcal{J}}$ where p_j is equal to the number of coins in the jth jar. It is easy to see that for one jar the Sprague-Grundy function is $g(p) = 100 - p \equiv 1 - p \pmod{11}$. For a finite collection of jars, the Sprague-Grundy function is

$$G((p_j)_{j\in\mathcal{J}}) = \bigoplus_{j\in\mathcal{J}} g(j)$$

where $\bigoplus$ is the nim-addition (binary sum without carry). Now the initial position is favourable to the second player, Costel, iff

$$G((0)_{j\in\mathcal{J}}) = \bigoplus_{j\in\mathcal{J}} 1 = 0$$

that is iff $|\mathcal{J}|$, the cardinality of $\mathcal{J}$, is even.

Also solved by Zolbayar Shagdar, Ulaanbaatar, Mongolia.

J232. Find with proof all integers that can be written as $a\{a\}\lfloor a\rfloor$ for some real number a. Here $\lfloor a\rfloor$ and $\{a\}$ denote the greatest integer less than or equal to a and the fractional part of a, respectively.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Francesco Bonesi, Università di Roma "Tor Vergata", Roma, Italy

Since $\lfloor a\rfloor \geq 0$ iff $a \geq 0$ and $\{a\} \in [0,1)$, it follows that $a\{a\}\lfloor a\rfloor \geq 0$ and no negative integer m can be written in that way. If $m = 0$ then we can take a equal to any integer. Assume that $m > 0$ and let us find a real number $a > 0$ such that $m = \lfloor a\rfloor$ and $\{a\} = a - m = 1/a \in (0,1)$. Hence $a^2 - ma - 1 = 0$ and by solving this quadratic equation we find that for $m > 0$ we can take

$$a = \frac{m + \sqrt{m^2+4}}{2}.$$

Note that this formula works also when $m = 0$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain, Zolbayar Shagdar, Ulaanbaatar, Mongolia, Daniel Vacaru, Pitesti, Romania

J233. Let $A_1A_2A_3A_4A_5$ be a regular pentagon and let $B_1B_2B_3B_4B_5$ be the pentagon formed by its diagonals. Prove that

$$\frac{K_{B_1B_2B_3B_4B_5}}{K_{A_1A_2A_3A_4A_5}} = \frac{7-3\sqrt{5}}{2}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Arkady Alt, San Jose, California, USA

First, recall that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ and $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$. Let a and b be the common value for the sidelengths of pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$. We then have that

$$\frac{K_{B_1B_2B_3B_4B_5}}{K_{A_1A_2A_3A_4A_5}} = \frac{b^2}{a^2}.$$

Now, we know that $A_2A_5 = 2a\cos 36^\circ$ and $x = \dfrac{a}{2\cos 36^\circ}$, thus

$$\begin{aligned} b &= A_2A_5 - 2x = 2a\cos 36^\circ - \frac{a}{\cos 36^\circ} = a\left(2\cos 36^\circ - \frac{1}{\cos 36^\circ}\right) \\ &= \frac{a\left(2\cos^2 36^\circ - 1\right)}{\cos 36^\circ} = \frac{a\cos 72^\circ}{\cos 36^\circ} = \frac{a\sin 18^\circ}{\cos 36^\circ} = a\cdot\frac{\sqrt{5}-1}{\sqrt{5}+1} \\ &= a\cdot\frac{3-\sqrt{5}}{2}, \end{aligned}$$

and, therefore,

$$\frac{b^2}{a^2} = \left(\frac{3-\sqrt{5}}{2}\right)^2 = \frac{7-3\sqrt{5}}{2}.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Zolbayar Shagdar, Ulaanbaatar, Mongolia; G.R.A.20 Problem Solving Group, Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

J234. Let ABC be a triangle with side-lengths a, b, c that satisfy $a^{\frac{3}{2}} + b^{\frac{3}{2}} = c^{\frac{3}{2}}$. Prove that

$$\frac{\pi}{2} < \angle C < \frac{3\pi}{5}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

First, we prove that $\frac{\pi}{2} < \angle C$. Obviously, it suffices to show that sufficient to show that $c^2 > a^2 + b^2$. To do this, we write

$$c^{\frac{3}{2}} = a^{\frac{3}{2}} + b^{\frac{3}{2}} \Rightarrow c^2 = \left(a^{\frac{3}{2}} + b^{\frac{3}{2}}\right)^{\frac{4}{3}} = a^2\left(1 + \left(\frac{b}{a}\right)^{\frac{3}{2}}\right)^{\frac{4}{3}},$$

and so, if we prove the inequality

$$\left(1 + \left(\frac{b}{a}\right)^{\frac{3}{2}}\right)^{\frac{4}{3}} > 1 + \left(\frac{b}{a}\right)^2,$$

this will yield precisely what we want. Now, to prove the inequality, we assume, WLOG, that $b \geq a$, and let $\frac{b}{a} = x$; we then have $x \geq 1$. Next, consider the following function

$$f(x) = (1 + x^{\frac{3}{2}})^{\frac{4}{3}} - (1 + x^2), \quad \text{defined for } x \geq 1.$$

$$\begin{aligned} We note that f'(x) &= \frac{4}{3}(1 + x^{\frac{3}{2}})^{\frac{1}{3}} \cdot \frac{3}{2} \cdot x^{\frac{1}{2}} - 2x = 2(1 + x^{\frac{3}{2}})^{\frac{1}{3}} \cdot x^{\frac{1}{2}} - 2x \\ &> 2x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} - 2x = 0, \end{aligned}$$

whence the function $f(x)$ is increasing on $[1, \infty)$. Thus, it follows that $f(x) \geq f(1)$, which yields

$$(1 + x^{\frac{3}{2}})^{\frac{4}{3}} - (1 + x^2) \geq 2\sqrt[3]{2} - 2.$$

However, $2\sqrt[3]{2} - 2 > 0$, so we have that $(1 + x^{\frac{3}{2}})^{\frac{4}{3}} > (1 + x^2)$, from which we deduce that

$$\left(1 + \left(\frac{b}{a}\right)^{\frac{3}{2}}\right)^{\frac{4}{3}} > 1 + \left(\frac{b}{a}\right)^2,$$

which proves the first inequality.

We now prove that $\angle C < \frac{3\pi}{5}$. To do this, it is sufficient to show that

$$\cos(\pi - \angle C) < \frac{\sqrt{5} - 1}{4};$$

We write

$$\cos(\pi - \angle C) = \frac{c^2 - a^2 - b^2}{2ab} = \frac{(a^{\frac{3}{2}} + b^{\frac{3}{2}})^{\frac{4}{3}} - (a^2 + b^2)}{2ab},$$

and so we need to prove that

$$(a^{\frac{3}{2}} + b^{\frac{3}{2}})^{\frac{4}{3}} < \frac{\sqrt{5} - 1}{2} \cdot ab + a^2 + b^2.$$

Again, we take $\frac{b}{a} = x$, and assume WLOG that $a \leq b$, so that $x \geq 1$; consider the function

$$g(x) = x^2 + 1 + \frac{\sqrt{5} - 1}{2}x - (1 + x^{\frac{3}{2}})^{\frac{4}{3}}, \text{ defined for } x \geq 1.$$

We have that

$$g'(x) = \frac{1}{2}\left(\sqrt{5} - 1 + 4x - 4\sqrt{x}\sqrt[3]{1 + x^{\frac{3}{2}}}\right),$$

and

$$g''(x) = 2 - \frac{x}{(1 + x^{\frac{3}{2}})^{\frac{2}{3}}} - \frac{\sqrt[3]{1 + x^{\frac{3}{2}}}}{\sqrt{x}},$$

so we note that $g''(x) > 0$ for all $x \geq 1$. Hence the function $g'(x)$ is increasing on $[1, \infty)$. Therefore $g'(x) \geq g'(1)$ for all $x \geq 1$. But

$$g'(1) = 2 + \frac{\sqrt{5} - 1}{2} = \frac{\sqrt{5} + 3}{2} - 2\sqrt[3]{2} > 0;$$

whence $g'(x) > 0$ for all $x \geq 1$; thus the function $g(x)$ is increasing on $[1, \infty)$. Again, this means that $g(x) \geq g(1)$ for all $x \geq 1$, but

$$g(1) = 2 + \frac{\sqrt{5} - 1}{2} - 2\sqrt[3]{2} > 0,$$

so $g(x) > 0$ for all $x \geq 1$, which yields

$$g(x) = x^2 + 1 + \frac{\sqrt{5} - 1}{2}x - (1 + x^{\frac{3}{2}})^{\frac{4}{3}} > 0.$$

From this, we deduce that

$$x^2 + 1 + \frac{\sqrt{5} - 1}{2}x > (1 + x^{\frac{3}{2}})^{\frac{4}{3}},$$

and this is precisely what we needed. This completes the proof.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

Senior problems

S229. Let a, b, c be the side-lengths of a triangle and let R be its circumradius. Prove that

$$a^3+b^3+c^3 \le 16R^3.$$

Proposed by Arkady Alt, San Jose, USA and Ivan Borsenco, MIT, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain
Denote by s, r the semiperimeter and inradius of ABC. Note first that

$$\frac{s-2R-r}{2R} = \frac{\sin A+\sin B+\sin C}{2} - 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} =$$

$$= 2\sin\frac{A}{2}\sin\frac{B}{2}\left(\cos\frac{C}{2}-\sin\frac{C}{2}\right)+\sin C-1=$$

$$= \left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)\left(\cos\frac{B}{2}-\sin\frac{B}{2}\right)\left(\cos\frac{C}{2}-\sin\frac{C}{2}\right).$$

If ABC is not acute, then one and only one of the factors in the RHS is non-positive, yielding $s \le 2R+r$. Otherwise, denote $u=\cos\frac{A}{2}-\sin\frac{A}{2}$, $v=\cos\frac{B}{2}-\sin\frac{B}{2}$, $w=\cos\frac{C}{2}-\sin\frac{C}{2}$, or

$$u+vw = \sin\frac{B+C}{2} - \cos\frac{B+C}{2} + \left(\cos\frac{B}{2}-\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}-\sin\frac{C}{2}\right) = 2\sin\frac{B}{2}\sin\frac{C}{2},$$

for

$$\frac{r^2}{2R^2} = (u+vw)(v+wu)(w+uv) \ge uvw(1+u+v+w+u^2+v^2+w^2+uvw).$$

Now, $\cos(x)-\sin(x)$ has first derivative $-\sin(x)-\cos(x)$, clearly negative, or assuming wlog $A\le B\le C$, it follows that $u \ge \cos(30^\circ)-\sin(30^\circ) = \frac{\sqrt{3}-1}{2}$, and $1+u+u^2 \ge 1+\frac{2\sqrt{3}+1}{2} > \frac{3}{2}$. We conclude that $uvw \le \frac{r^2}{3R^2}$, and $s \le 2R+r+\frac{r^2}{3R}$.

Note now that

$$a^3+b^3+c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc=$$

$$= 2s\left(4s^2-3(ab+bc+ca)+6Rr\right).$$

It is relatively well known (or easily proved) that $4Rr+r^2 = bc+ca+ab-s^2$. It therefore suffices to show that

$$s\left(s^2-6Rr-3r^2\right) \le 8R^3.$$

But

$$s^2-6Rr-3r^2 \le 4R^2-2Rr-\frac{2r^2}{3}+\frac{2r^3}{3R}+\frac{r^4}{9R^2},$$

$$s\left(s^2-6Rr-3r^2\right) \le 8R^3-2Rr^2+\frac{2r^4}{3R}+\frac{r^5}{3R^2}+\frac{r^6}{27R^3} \le 8R^3-\frac{773Rr^2}{432} \le 8R^3,$$

where we have used that $R \ge 2r$. The conclusion follows, equality holds iff ABC is a non-obtuse triangle with zero inradius, ie iff $a=b=2R$ and $c=0$. For non-degenerate triangles, equality cannot hold, but we may get arbitrarily close to equality by taking $a=b$ and letting $c\to 0$, and consequently $a=b\to 2R$.

Also solved by Paolo Perfetti, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Ioan Viorel Codreanu, Satulung, Maramure, Romania; Marin Sandu and Mihai Sandu, Bucharest, Romania; Isha Maini, Delhi, India, Nicusor Zlota, Focsani, Romania.

S230. Let x, y, z be positive real numbers such that

$$xy + yz + zx \geq \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

Prove that $x + y + z \geq \sqrt{3}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA
Note that

$$xy + yz + zx \geq \frac{1}{\sqrt{x^2 + y^2 + z^2}} \text{ if and only if } x^2 + y^2 + z^2 \geq \frac{1}{(xy + yz + zx)^2},$$

which rewrites as

$$(x + y + z)^2 \geq 2(xy + yz + zx) + \frac{1}{(xy + yz + zx)^2}.$$

But the AM-GM Inequality yields

$$2(xy + yz + zx) + \frac{1}{(xy + yz + zx)^2} \geq 3\sqrt[3]{(xy + yz + zx)^2 \cdot \frac{1}{(xy + yz + zx)^2}} = 3,$$

thus, it follows that $(x + y + z)^2 \geq 3$, which yields $x + y + z \geq \sqrt{3}$. This completes the proof.

Also solved by Shamil Asgarli, University of British Columbia, Canada; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramure, Romania; Christopher Wiriawan, Jakarta, Indonesia; Marin Sandu and Mihai Sandu, Bucharest, Romania; Sayan Das, Kolkata, India; Paolo Perfetti, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Vazgen Mikayelyan.

S231. Let ABC be a triangle with circumcenter O. Let X, Y, Z be the circumcenters of triangles BCO, CAO, ABO respectively. Furthermore, let K be the circumcenter of triangle XYZ. Prove that K lies on the Euler line of triangle ABC.

Proposed by Andrew Kirk, Mearns Castle High School, UK, Nicusor Zlota, Focsani, Romania.

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Lemma: Let ABC be a triangle, and I_a, I_b, I_c its excenters. Then, the symmetric of the incenter I of ABC, with respect to the circumcircle O of ABC, is the circumcenter of $I_aI_bI_c$.

Proof: Denote by P the symmetric of I with respect to O. It is well known (or easily provable using basic results) that the second point A' where AI intersects the circumcircle of ABC, is the circumcenter of $BICI_a$, or A' is the midpoint of II_a. The median theorem then yields

$$R^2 = OA'^2 = \frac{OI^2 + OI_a{}^2}{2} - IA'^2, \qquad OI_a{}^2 = 2R^2 + 2IA'^2 - OI^2.$$

Again by the median theorem,

$$OI_a{}^2 = \frac{II_a{}^2 + PI_a{}^2}{2} - OI^2, \qquad PI_a{}^2 = 2OI_a{}^2 + 2OI^2 - II_a{}^2.$$

Moreover, by similarity we have $\frac{II_a}{AI_a} = \frac{AI_a - AI}{AI_a} = \frac{a}{s}$, where s is the semiperimeter of ABC, while $s = AI_a \cos\frac{A}{2}$, or $II_a = \frac{a}{\cos\frac{A}{2}} = \frac{4Rr}{AI}$, where using that the power of I with respect to the circumcircle of ABC is $2Rr$, we conclude that $II_a = 2IA'$. Combining these results yields $PI_a = 2R$. Similarly, $PI_b = PI_c = 2R$. The conclusion follows. Incidentally, point P thus defined is called the Bevan point of ABC.

Note that OX is clearly the perpendicular bisector of BC, or OB, OC are symmetric with respect to OX, and so are their perpendicular bisectors ZX, XY. Therefore, OX is the internal bisector of $\angle ZXY$, and similarly for OY, OZ. We conclude that O is the incenter of XYZ.

Let perpendicular lines to OX, OY, OZ at X, Y, Z determine a triangle $A'B'C'$ homothetic to ABC, since $OX \perp BC$. Clearly A', B', C' are the excenters of XYZ, while X, Y, Z are the feet of the perpendicular from A', B', C' onto $B'C', C'A', A'B'$, respectively, and O is also the orthocenter of triangle $A'B'C'$. The homothety that transforms ABC into $A'B'C'$ has center T, such that the circumcircle O' of $A'B'C'$ (which is the Bevan point of XYZ) is on line OT. But since O is the orthocenter of $A'B'C'$, H is on line OT, which is therefore the Euler line of ABC. Finally, since the incenter O of XYZ and its Bevan point O' are on OT, so is, by the Lemma, their midpoint K which is the circumcircle of XYZ. The conclusion follows.

Also solved by Daniel Vacaru, Pitesti, Romania

S232. Let x, y, z be real numbers such that $x + y + z = 0$ and $xy + yz + zx = -3$. Determine the extreme values of $x^4y + y^4z + z^4x$.

Proposed by Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain
Define $s = \frac{x+y}{2}$ and $c = \frac{x-y}{2\sqrt{3}}$, or $s^2 - 3c^2 = xy$, $x^2 + y^2 = 2s^2 + 6c^2$. Note that

$$z^2 = -z(x + y) = xy + 3 = (x + y)^2 = 4s^2, \qquad 3s^2 + 3c^2 = x^2 + xy + y^2 = 3.$$

It follows that s, c are respectively the sine and cosine of a certain angle α, and

$$x^4y + y^4z + z^4x = x^4y + y^3(x^2 - 3) + x(xy + 3)^2 = 9x^2y - 3y^3 + 9x =$$

$$= -15(3s - 4s^3) - 9\sqrt{3}(4c^3 - 3c) = -15\sin(3\alpha) - 9\sqrt{3}\cos(3\alpha),$$

where we have used the De Moivre formulae,

$$\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha, \qquad \sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha.$$

Now, since $\cos^2(3\alpha) + \sin^2(3\alpha) = 1$, it follows that $|\cos(3\alpha) + \sin(3\alpha)| \le 1$, with equality iff $\cos(3\alpha) = \pm 1$ and $\sin(3\alpha) = 0$ or *vice versa*. Moreover, $9\sqrt{3} > 15$ because $3 \cdot 9^2 = 243 > 225 = 15^2$, or the extrema happen when $\cos(3\alpha) = \pm 1$ and $\sin(3\alpha) = 0$. Now, $\sin(3\alpha) = 0$ is equivalent to $s = 0$ or $s = \pm\frac{\sqrt{3}}{2}$. In the first case, $y = -x$ for $x^2 = y^2 = \sqrt{3}$, and $x = -y = \sqrt{3}$, $z = 0$ for a minimum $-9\sqrt{3}$, or $x = -y = -\sqrt{3}$, $z = 0$ for a maximum $9\sqrt{3}$. In the second case, $x + y = \pm\sqrt{3}$, for $(x + y)^2 = x^2 + xy + y^2$, resulting in $xy = 0$, and analyzing each one of the resulting cases analogously yields the same maximum and minimum, for cyclic permutations of the already found values of x, y, z.

We conclude that

$$-9\sqrt{3} \le x^4y + y^4z + z^4x \le 9\sqrt{3}.$$

The maximum happens for $(x, y, z) = (-\sqrt{3}, \sqrt{3}, 0)$ or one of its cyclic permutations, and the minimum occurs for $(x, y, z) = (\sqrt{3}, -\sqrt{3}, 0)$ or one of its cyclic permutations.

Also solved by Arkady Alt, San Jose, California, USA.

S233. In triangle ABC with $\angle C = 60^\circ$, let AA' and BB' be the angle bisectors of $\angle A$ and $\angle B$. Prove that

$$\frac{a+b}{A'B'} \le \left(1+\frac{c}{a}\right)\left(1+\frac{c}{b}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA

Since $CA' = \dfrac{ab}{b+c}, CB' = \dfrac{ab}{a+c}$, and $\angle C = 60^\circ$, the Law of Cosines yields

$$A'B'^2 = \frac{a^2b^2}{(a+c)^2} + \frac{a^2b^2}{(b+c)^2} - \frac{a^2b^2}{(a+c)(b+c)},$$

and, therefore,

$$\frac{a+b}{A'B'} \le \left(1+\frac{c}{a}\right)\left(1+\frac{c}{b}\right) \text{ which rewrites as } ab(a+b) \le A'B'(a+c)(b+c)$$

becomes equivalent to proving that

$$a^2b^2(a+b)^2 \le \left(\frac{a^2b^2}{(a+c)^2} + \frac{a^2b^2}{(b+c)^2} - \frac{a^2b^2}{(a+c)(b+c)}\right)(a+c)^2(b+c)^2,$$

i.e.

$$(a+b)^2 \le (b+c)^2 + (a+c)^2 - (a+c)(b+c).$$

Since $c^2 = a^2 + b^2 - ab$ (the Law of Cosines), then

$$\begin{aligned}(b+c)^2 + (a+c)^2 - (a+c)(b+c) - (a+b)^2 &= bc + ac + c^2 - 3ab = bc + ac + a^2 + b^2 - 4ab \\ &= bc + ac - 2ab + (a-b)^2.\end{aligned}$$

But $c^2 = a^2 + b^2 - ab \ge ab$, $(b+c)^2 \ge 4ab$; whence $(b+c)^2 c^2 \ge 4a^2b^2$ i.e. $bc + ac \ge 2ab$, and thus

$$(bc + ac - 2ab) + (a-b)^2 \ge 0.$$

This completes the proof. □

Also solved by Ioan Viorel Codreanu, Satulung, Maramure, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

S234. Let ABC be a triangle. Denote by D, E, F the feet of the internal angle bisectors such that $D \in (BC), E \in (CA), F \in (AB)$ and by $(I_a, r_a), (I_b, r_b), (I_c, r_c)$ its three excircles. If τ denotes the Feuerbach point of triangle ABC, prove that there is a choice of signs $+$ and $-$ such that the following equality holds

$$\pm D\tau \cdot \frac{I_aI}{I_aD} \cdot \sqrt{R+2r_a} \pm E\tau \cdot \frac{I_bI}{I_bE} \cdot \sqrt{R+2r_b} \pm F\tau \cdot \frac{I_cI}{I_cF} \cdot \sqrt{R+2r_c} = 0.$$

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

It is relatively well known that D, E, F, τ are concyclic. For a proof of this fact, note that four points are concyclic iff they are not collinear and their trilinear coordinates (α, β, γ) satisfy that the four corresponding vectors $(a\beta\gamma + b\gamma\alpha + c\alpha\gamma, \alpha, \beta, \gamma)$ are linearly dependent. Inserting $\tau \equiv \left(\cos^2\frac{B-C}{2}, \cos^2\frac{C-A}{2}, \cos^2\frac{A-B}{2}\right)$, $D \equiv (0,1,1)$, $E \equiv (1,0,1)$ and $F \equiv (1,1,0)$, after some algebra we find that this is indeed the case.

By the Cosine Law, and using that $AE = \frac{bc}{c+a}$ and $AF = \frac{bc}{a+b}$, we find

$$\frac{EF^2}{b^2c^2} = \frac{AE^2 + AF^2 - 2AE \cdot AF \cos A}{b^2c^2} = \frac{1}{(a+b)^2} + \frac{1}{(c+a)^2} - \frac{b^2+c^2-a^2}{bc(a+b)(c+a)} =$$

$$= \frac{a^2(R+2r_a)}{R(a+b)^2(c+a)^2},$$

where we have used Heron's formula for the area $S = \frac{abc}{4R} = \frac{r(a+b+c)}{2}$ of ABC, and the well-known relation $\frac{r_a}{a+b+c} = \frac{r}{b+c-a}$ arising from incircle-excircle similarities. It follows that

$$EF = K(b+c)\sqrt{R+2r_a}, \qquad \text{where} \qquad K = \frac{abc}{(a+b)(b+c)(c+a)\sqrt{R}},$$

and K is clearly unchanged by cyclic permutation of A, B, C. Moreover, $D = I_aI \cap BC$, or by similarity we find

$$\frac{I_aI}{I_aD} = \frac{r_a + r}{r_a} = 2\frac{b+c}{a+b+c},$$

finally yielding

$$D\tau \cdot \frac{I_aI}{I_aD} \cdot \sqrt{R+2r_a} = \frac{2}{K(a+b+c)} \cdot D\tau \cdot EF,$$

and similarly for its cyclic permutations, where the prefactor $\frac{2}{K(a+b+c)}$ is clearly unchanged. The problem is then equivalent to showing that there is a choice of signs $+$ and $-$ such that

$$\pm D\tau \cdot EF \pm E\tau \cdot FD \pm F\tau \cdot DE = 0,$$

clearly true by Ptolemy's theorem since D, E, F, τ are concyclic. The conclusion follows.

U229. Does the sequence $(x_n)_{n\geq 1}$ defined by $x_n = \{\log_n n!\}$ converge? Here $\{x\}$ denotes the fractional part of the real number x.

Proposed by Cezar Lupu, University of Pittsburgh, USA

Solution by Paolo Perfetti, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy

We will prove that the answer is NO.

We write

$$\log_n n! = \frac{1}{\ln n}\ln\Big[\big(\frac{n}{e}\big)^n\sqrt{2\pi n}(1+o(1))\Big] = n - \frac{n}{\ln n} + \frac{1}{2} + \frac{\ln(\sqrt{2\pi})}{\ln n} + \frac{o(1)}{\ln n}$$

thus

$$\{\log_n n!\} = \left\{-\frac{n}{\ln n} + \frac{1}{2} + \frac{\ln(\sqrt{2\pi})}{\ln n} + \frac{o(1)}{\ln n}\right\}$$

Now we employ a proposition in the book by G.Pólya, G.Szegö, *Problem and Theorem in Analysis, I*, Springer–Verlag, fourth edition, p.90, problem 174. The proposition is

Proposition *Let* $g\colon [1,\infty)$ *such that:*
1) g *is continuously differentiable,*
2) g *is monotone increasing,*
3) g' *is monotone decreasing to* 0 *as* $t\to\infty$,
4) $tg'(t)$ *tends to* ∞ *as* t *tends to* ∞.

Then the numbers $\{g(n)\}$ *are equidistributed on the interval* $[0,1]$.

In particular they are dense and this excludes the existence of the convergence of the sequence $\{g(n)\}$. All we need to do is to apply the proposition to our case. We put a minus sign in front of x_n getting

$$g(t) = \frac{t}{\ln t} - \frac{1}{2} - \frac{\ln(\sqrt{2\pi})}{\ln t} - \frac{o(1)}{\ln t}$$

$$g'(t) = \frac{1}{\ln t} - \frac{1}{\ln^2 t} + \frac{\ln\sqrt{2\pi}}{t\ln^2 t} + \frac{o(1)}{t\ln t}$$

1)–4) are satisfied and the conclusion follows.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Arkady Alt , San Jose ,California, USA.

U230. Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers such that ${a_n}^2 \geq a_{n-1}a_{n+1}$, for all $n \geq 1$. Prove that $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Francesco Bonesi and Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

Since the terms of the sequence arepositive reals, we can consider the sequence $x_n = \log(a_n)$. By hypothesis, $2x_n \geq x_{n+1} + x_{n-1}$, which is equivalent to $x_{n+1} - x_n \leq x_n - x_{n-1}$ that is the sequence $x_{n+1} - x_n$ is not increasing. Hence, there exists the limit

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{n+1-n} = \lim_{n\to\infty} (x_{n+1} - x_n) = l \in [-\infty, +\infty).$$

and, by Stolz-Cesaro, there exists the limit

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{\frac{x_n}{n}} = \lim_{n\to\infty} e^{\frac{x_{n+1}-x_n}{n+1-n}} = e^l.$$

Also solved by Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland; Paolo Perfetti, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U231. Define a sequence of maps on $[0,1]$ by $f_0(x) = 0$ and

$$f_{n+1}(x) = f_n(x) + \frac{x - f_n(x)^2}{2}.$$

It is well-known that f_n converges uniformly to the function $x \to \sqrt{x}$ on $[0,1]$. Prove that there exists $c \in (0,\infty)$ such that

$$\lim_{n\to\infty} n \cdot \max_{x\in[0,1]} |f_n(x) - \sqrt{x}| = c.$$

Gabriel Dospinescu, Ecole Polytechnique, France

Solution by the author

Define $g_n(x) = x - f_n(x^2)$ for $x \in [0,1]$ and $n \geq 0$. Then $g_0(x) = x$ and an easy computation shows that

$$g_{n+1}(x) = g_n(x)(1 - x + \frac{g_n(x)}{2}).$$

An immediate induction shows that $x \geq g_n(x) > 0$ for all $x \in (0,1]$ and $n \geq 0$.

We will need a few estimates, some of which are the object of the following

Lemma 1. a) For all $t > 0$ we have $\frac{e^t+1}{2} \geq \frac{e^t-1}{t}$.

b) For all $t \in (0,1)$ we have

$$\frac{e^{\frac{t}{1-t}} - 1}{t} \geq e^{\frac{t}{1-t}}.$$

Proof. For the first one, simply use the Taylor expansion $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + ...$ and the fact that $2 \cdot (n-1)! \leq n!$ for all $n \geq 2$. For the second one, let $x = \frac{t}{1-t} > 0$, so $t = \frac{x}{1+x}$. The inequality becomes $(x+1)\frac{e^x-1}{x} \geq e^x$ and this also follows immediately from Taylor's formula.

$\square$

The crucial estimate is contained in the following

Lemma 2. For all $t \in [0,1)$ and $n \geq 0$ we have

$$\frac{2t}{1 + e^{\frac{nt}{1-t}}} \leq g_n(t) \leq \frac{2t}{e^{nt}+1}.$$

Proof. We will prove this by induction on n. For $n = 0$ the inequalities are clear. Assuming that they hold for n, it is enough to check that

$$\frac{1}{e^{nt}+1}(1 - t + \frac{t}{e^{nt}+1}) \leq \frac{1}{e^{(n+1)t}+1}$$

and

$$\frac{1}{1+e^{\frac{nt}{1-t}}}(1 - t + \frac{t}{1+e^{\frac{nt}{1-t}}}) \geq \frac{1}{1+e^{\frac{(n+1)t}{1-t}}}.$$

After simple algebraic manipulations, these are equivalent to

$$\frac{e^t - 1}{t} \leq \frac{e^{(n+1)t}+1}{e^{nt}+1} \quad \text{and} \quad \frac{1 + e^{(n+1)\frac{t}{1-t}}}{1 + e^{n\frac{t}{1-t}}} \leq \frac{e^{\frac{t}{1-t}} - 1}{t}.$$

Next, the Cauchy-Schwarz inequality implies that the sequence $\frac{u^{n+1}+1}{u^n+1}$ is increasing when $u > 1$, therefore we have

$$\frac{u+1}{2} \leq \frac{u^{n+1}+1}{u^n+1} \leq u$$

for all n. Using this observation, the desired inequalities follow from the previous lemma.

$\square$

Coming back to the solution, let u be the unique positive solution of the equation $1+e^{-x}=x$. It is easy to see that $1<u<2$ and that

$$\max_{x>0} \frac{2x}{1+e^x} = 2e^{-u} = 2(u-1),$$

attained for $x=u$. It follows that for $n>2$ we have

$$\max_{t\in[0,1]} \frac{2t}{1+e^{nt}} = \frac{2}{n}(u-1).$$

So

$$\max_{x\in[0,1]} g_n(x) \leq \frac{2}{n}(u-1).$$

On the other hand, an easy computation shows that

$$\sup_{t\in(0,1)} \frac{2t}{1+e^{\frac{nt}{1-t}}} = \sup_{x>0} \frac{2x}{(1+x)(1+e^{nx})} = \frac{2}{n}e^{-nx_n}(1+x_n)^2,$$

where x_n is the unique positive solution of the equation

$$1+e^{-nx} = nx(1+x).$$

Combining the previous results it is enough to check that $nx_n \to u$ for $n\to\infty$. But we easily obtain $0<nx_n<2$, so the sequence $y_n = nx_n$ is bounded. On the other hand, the relation

$$1+e^{-y_n} = y_n(1+\frac{y_n}{n})$$

shows that the only limit point of $(y_n)_n$ is u. Hence the sequence $(y_n)_n$ converges to u and this finishes the proof.

U232. Let $\alpha > 0$ be any non-algebraic number. Prove that there is a funtion f with period 1 and a countable set A such that

$$x = f(\alpha x) - f(x) \qquad \text{for all } x \in \mathbb{R}\backslash A.$$

Proposed by Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Given a fixed non-algebraic number α, we define A_α as the set of all reals which can be written in the form

$$n_{-u}\alpha^{-u} + n_{-u+1}\alpha^{-u+1} + \cdots + n_v\alpha^v,$$

where u, v are non-negative integers, and the n_i are integers. For each non-algebraic number $\beta \notin A_\alpha$, we define A_β as the set resulting of taking any number in A_α, and adding one term of the form $\beta\alpha^i$.

Assume that $\gamma \in A_\beta$. Then,

$$\gamma = n_{-u}\alpha^{-u} + n_{-u+1}\alpha^{-u+1} + \cdots + n_v\alpha^v + \beta\alpha^i,$$

$$\beta = \gamma\alpha^{-i} - n_{-u}\alpha^{-u-i} - n_{-u-i+1}\alpha^{-u+1} - \cdots - n_v\alpha^{v-i},$$

and $\beta \in A_\gamma$. It follows that we may establish an equivalence relation in the non-algebraic numbers such that $\beta \equiv \gamma$ iff $\beta \in A_\gamma$ iff $\gamma \in A_\beta$. We conclude that a subset of the reals which includes at least all non-algebraic numbers is the disjoint union of A_α and all the A_β for $\beta \in \mathcal{B}$, where $\mathcal{B}$ is a set formed by exactly one representative of each one of the A_β such that $\beta \notin A_\alpha$. Since the algebraic numbers are countable, clearly $A_\alpha \cup \left(\bigcup_\beta A_\beta\right) = \mathbb{R} - A$, where A is a subset of the algebraic numbers, thus a countable subset of the reals.

Assume now that two reals, $x \in A_\beta$ and $y \in A_\gamma$, differ by an integer. Then,

$$x - y = n_{-u}\alpha^{-u} + \cdots + \beta\alpha^i + \cdots + \gamma\alpha^j + \cdots + n_v\alpha^v = n \in \mathbb{Z}$$

$$\beta = n\alpha^{-i} - n_{-u}\alpha^{-u-i} - \cdots - \gamma\alpha^{j-i} - \cdots - n_v\alpha^{v-i},$$

and $\beta \in A_\gamma$. Moreover, if $x \in A_\beta$, then claerly $\alpha x \in A_\beta$ for any β, or it is sufficient to define a function f that is periodic and satisfies $x = f(\alpha x) - f(x)$ within A_α, and within each one of the A_β, since these sets *do not mix*, in the sense that any two distinct sets have neither one element each, such that their difference is a period of f, nor one element each, such that their ratio is α.

Note now that defining $f(n) = 0$ for each integer 0, and taking

$$f\left(n_{-u}\alpha^{-u} + \cdots + n_v\alpha^v\right) = n_{-u}\frac{\alpha^{-u} - 1}{\alpha - 1} + n_{-u+1}\frac{\alpha^{-u+1} - 1}{\alpha - 1} + \cdots + n_v\frac{\alpha^v - 1}{\alpha - 1},$$

defines f in a consistent and periodic fashion within A_α (easily proved by induction on u, v). Similarly,

$$f\left(n_{-u}\alpha^{-u} + \cdots + n_v\alpha^v + \beta\alpha^i\right) =$$

$$= n_{-u}\frac{\alpha^{-u} - 1}{\alpha - 1} + n_{-u+1}\frac{\alpha^{-u+1} - 1}{\alpha - 1} + \cdots + n_v\frac{\alpha^v - 1}{\alpha - 1} + \beta\frac{\alpha^i - 1}{\alpha - 1} + f(\beta).$$

allows us to extend f into each one of the A_β (easily proved by induction on u, v, i). Since $\beta + n \notin A_\gamma$ for any $\gamma \neq \beta$ (including $\gamma = \alpha$), we may assign whichever value we'd like to $f(\beta)$ for each $\beta \in \mathcal{B}$. The conclusion follows.

U233. Let X be a random variable with a mean μ, variance σ^2, and median m. Denote by $MAD =$ median $|X - m|$ the median of the absolute deviations from the median of X. Prove that $MAD \leq 2\sigma$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Markov's inequality states that

$$\Pr\left(g(X) \leq \Delta\right) \geq 1 - \frac{\mathrm{E}\left[g(X)\right]}{\Delta},$$

where $g(X)$ is a non-negative function of random variable X, $\Delta > 0$ and $\mathrm{E}[Y]$ denotes the expectation value of random variable Y. Apply this inequality to $g(X) = |X - m|$ with $\Delta = 2\sigma$, or we obtain

$$\Pr\left(|X - m| \leq 2\sigma\right) \geq 1 - \frac{\mathrm{E}\left[|X - m|\right]}{2\sigma}.$$

Assume that this probability is smaller than $\frac{1}{2}$, then

$$\mathrm{E}\left[|X - m|\right] > \sigma = \sqrt{\mathrm{E}\left[X^2 - \mu^2\right]} = \sqrt{\mathrm{E}\left[(X - \mu)^2\right]} \geq \mathrm{E}\left[|X - \mu|\right] \geq \mathrm{E}\left[|X - m|\right],$$

absurd. Note that the first inequality is the assumption, the second inequality is a consequence of the AM-QM inequality, and the third inquality is a consequence of a well-known property of the median, namely the median is the value such that the expectation value of the absolute differences of the variable with respect to said value is minimum. It follows that

$$\Pr\left(|X - m| \leq 2\sigma\right) \geq \frac{1}{2} = \Pr\left(|X - m| \leq MAD\right),$$

where the equality stems from the definition of the median and of the MAD. The conclusion follows.

U234. Consider the set of matrices $A \in M_n(\mathbb{R})$ whose coefficients are -1 or 1. What is the average value of $\det A^2$ when A runs through the elements of this set?

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland

Let K_n denotes the set of all $n \times n$ matrices with entries in $\{-1, 1\}$.

Let's define the function $F : K_n \times S_n \mapsto \{-1, 1\}$ in the following way:

$$F([a_{ij}], \sigma) = a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

Obviously, for any $A \in K_n$:

$$\det A^2 = (\det A)^2 = \left(\sum_{\sigma \in S_n} sgn(\sigma) \cdot F(A, \sigma) \right)^2$$

Using the identity: $(\sum_i a_i)^2 = \sum_i a_i^2 + \sum_{i \neq j} a_i a_j$ we get:

$$\begin{aligned} \det A^2 &= \sum_{\sigma \in S_n} (sgn(\sigma) \cdot F(A, \sigma))^2 + \sum_{\sigma \neq \tau} sgn(\sigma) sgn(\tau) \cdot F(A, \sigma) F(A, \tau) = \\ &= \sum_{\sigma \in S_n} 1 + \sum_{\sigma \neq \tau} sgn(\sigma) sgn(\tau) \cdot F(A, \sigma) F(A, \tau) = \\ &= n! + \sum_{\sigma \neq \tau} sgn(\sigma) sgn(\tau) \cdot F(A, \sigma) F(A, \tau) \end{aligned}$$

And hence:

$$\sum_{A \in K_n} \det A^2 = |K_n| \cdot n! + \sum_{A \in K_n} \sum_{\sigma \neq \tau} sgn(\sigma) sgn(\tau) \cdot F(A, \sigma) F(A, \tau) =$$

$$= |K_n| \cdot n! + \sum_{\sigma \neq \tau} sgn(\sigma) sgn(\tau) \cdot \left(\sum_{A \in K_n} F(A, \sigma) F(A, \tau) \right)$$

Now we'll show that if $\sigma \neq \tau$ then:

$$\sum_{A \in K_n} F(A, \sigma) F(A, \tau) = 0$$

Indeed, let's assume that: $\sigma(m) \neq \tau(m)$ for some $m \in \{1, 2, \dots, n\}$.
Let the function $g : K_n \mapsto K_n$ will be defined as follows:

$$g(A) = [b_{ij}] \qquad \text{where}$$

$$b_{ij} = \begin{cases} -a_{ij} & \text{if } i = m \text{ and } j = \sigma(m) \\ a_{ij} & \text{otherwise} \end{cases}$$

It's easy to check that $F(g(A), \sigma) = -F(A, \sigma)$ and $F(g(A), \tau) = F(A, \tau)$ for any $A \in K_n$.

Moreover g is a bijection and hence:

$$\sum_{A \in K_n} F(A, \sigma) F(A, \tau) = \sum_{A \in K_n} F(g(A), \sigma) F(g(A), \tau) = -\sum_{A \in K_n} F(A, \sigma) F(A, \tau)$$

which yields the equality $\sum_{A \in K_n} F(A, \sigma) F(A, \tau) = 0$. Hence:

$$\frac{\sum_{A \in K_n} \det A^2}{|K_n|} = n!$$

Also solved by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

Olympiad problems

O229. Are there rational numbers a, b, c such that $a^2b + b^2c + c^2a = ab^2 + bc^2 + ca^2 = 20.11$?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Note that the hypothesis $a^2b + b^2c + c^2a = ab^2 + bc^2 + ca^2$ is equivalent to $(a-b)(b-c)(c-a) = 0$. Assume that $b = c = 1$; then it suffices to find $a \in \mathbb{Q}$ such that

$$0 = a^2 + a + 1 - 20.11 = a^2 + a - \frac{1911}{100}.$$

This yields two rational solutions $a = 39/10$ and $a = -49/10$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O230. Let ABC be a triangle with incenter I and let X, Y, Z be points lying on the internal angle bisectors AI, BI, CI. Furthermore, let M, N, P be the midpoints of the sides BC, CA, AB, and let D, E, F be the tangency points of the incircle of ABC with these sides. Prove that if MX, NY, PZ are parallel, then DX, EY, FZ are concurrent on the incircle of ABC.

This generalizes Problem S228 from the previous issue.

Proposed by Luis Gonzalez, Maracaibo, Venezuela and Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume that a point X is given on the internal bisector of A. This determines exactly two points of intersection of MX and the incircle. Choose one of them, denote it by T, and draw lines ET, FT, who intersect respectively the internal bisectors of B, C at two points uniquely determined by T, which we call Y, Z. If NY, PZ are parallel to MX, then the proposed result is clearly true, since starting backwards, we could draw parallel lines to MX through N, P, which would determine the same points Y, Z, and DX, EY, FZ would concur at T on the incircle. Or, the problem may be restated as follows: given any point T on the incircle, draw lines DT, ET, FT which intersect the internal bisectors of A, B, C respectively at X, Y, Z. Prove that MX, NY, PZ are parallel. This is the version of the problem that is solved next.

Given ABC and point T on the incircle, choose a cartesian coordinate system such that $I \equiv (0,0)$, the incircle has equation $x^2 + y^2 = 1$, and $T \equiv (1,0)$. Let angles α, β, γ be such that

$$D \equiv (\cos(2\alpha), \sin(2\alpha)), \qquad E \equiv (\cos(2\beta), \sin(2\beta)), \qquad F \equiv (\cos(2\gamma), \sin(2\gamma)).$$

Clearly α, β, γ are distinct and no two of them can differ by 90° (otherwise ABC would be degenerate), and wlog $0 < \alpha, \beta, \gamma < 180^\circ$, since if one of them is 0, T coincides with one of D, E, F, and the problem cannot be well defined. The slope of side BC is clearly $-\frac{1}{\tan(2\alpha)}$, and since $D \in BC$, we can easily find the equation for line BC. Solving the system formed by this equation and its cyclic permutations, we obtain

$$A \equiv \left(\frac{\cos(\beta+\gamma)}{\cos(\beta-\gamma)}, \frac{\sin(\beta+\gamma)}{\cos(\beta-\gamma)} \right), \qquad AI \equiv y = x\tan(\beta+\gamma), \qquad DT \equiv y = \frac{1-x}{\tan\alpha},$$

and solving the system formed by the last two equations,

$$X \equiv \left(\frac{\cos\alpha\cos(\beta+\gamma)}{\cos(\beta+\gamma-\alpha)}, \frac{\cos\alpha\sin(\beta+\gamma)}{\cos(\beta+\gamma-\alpha)} \right).$$

Finally, since the coordinates of B, C can be easily found by cyclic permutation of α, β, γ in the coordinates of A, we also have

$$M \equiv \left(\frac{\cos(\alpha+\gamma)\cos(\alpha-\beta) + \cos(\alpha+\beta)\cos(\alpha-\gamma)}{2\cos(\alpha-\gamma)\cos(\alpha-\beta)}, \right.$$
$$\left. \frac{\sin(\alpha+\gamma)\cos(\alpha-\beta) + \sin(\alpha+\beta)\cos(\alpha-\gamma)}{2\cos(\alpha-\gamma)\cos(\alpha-\beta)} \right).$$

The slope m of MX is then found as $\frac{y_M - y_X}{x_M - x_X}$, where $M \equiv (x_M, y_M)$ and $X \equiv (x_X, y_X)$. Substitution yields $m = \frac{n}{d}$, where the numerator and denominator can be written as

$$n = (\sin(\alpha+\gamma)\cos(\alpha-\beta) + \sin(\alpha+\beta)\cos(\alpha-\gamma))\cos(\beta+\gamma-\alpha) -$$
$$-2\cos(\alpha-\gamma)\cos(\alpha-\beta)\cos\alpha\sin(\beta+\gamma),$$
$$d = (\cos(\alpha+\gamma)\cos(\alpha-\beta) + \cos(\alpha+\beta)\cos(\alpha-\gamma))\cos(\beta+\gamma-\alpha) -$$
$$-2\cos(\alpha-\gamma)\cos(\alpha-\beta)\cos\alpha\cos(\beta+\gamma).$$

After some algebra, we find

$$n = \sin(\beta+\gamma-2\alpha)\left(\cos(\alpha+\beta+\gamma) - 2\cos\alpha\cos\beta\cos\gamma\right),$$

$$d = 2\sin\alpha\sin\beta\sin\gamma\sin(\beta+\gamma-2\alpha).$$

Note that $\beta+\gamma-2\alpha=0$ iff ABC is isosceles in A, in which case the problem cannot be defined, since $M=D$ and $AI=MX$ is the internal bisector of A. We may therefore simplify, and conclude that the slope of MX is

$$m = \frac{\cos(\alpha+\beta+\gamma) - 2\cos\alpha\cos\beta\cos\gamma}{2\sin\alpha\sin\beta\sin\gamma}.$$

Since this expression is invariant under cyclic permutation of α, β, γ, this is also the slope of lines NY, PZ. The conclusion follows.

O231. Let a, b, c, d be real numbers such that $a+b+c+d=2$. Prove that

$$\frac{a}{a^2-a+1}+\frac{b}{b^2-b+1}+\frac{c}{c^2-c+1}+\frac{d}{d^2-d+1}\leq\frac{8}{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Substituting $x=a-\frac{1}{2}$, $y=b-\frac{1}{2}$, $z=c-\frac{1}{2}$, $w=d-\frac{1}{2}$, the inequality becomes

$$\frac{4x+2}{4x^2+3}+\frac{4y+2}{4y^2+3}+\frac{4z+2}{4z^2+3}\leq\frac{8}{3},$$

with $x+y+z+w=0$, which is equivalent to

$$\frac{(2x-1)^2}{4x^2+3}+\frac{(2y-1)^2}{4y^2+3}+\frac{(2z-1)^2}{4z^2+3}\geq\frac{4}{3}.$$

However, $4x^2=3x^2+(y+z+w)^2\leq 3x^2+3(y^2+z^2+w^2)$, thus we have that

$$\frac{(2x-1)^2}{4x^2+3}\geq\frac{(2x-1)^2}{3(x^2+y^2+z^2+w^2+1)},$$

with equality if and only if $x=\frac{1}{2}$ or $y=z=w$. Summing up with the analogous inequalities the conclusion follows.

The equality holds if and only if $x=y=z=w=0$ or $x=y=z=\frac{1}{2}$ and $w=-\frac{3}{2}$, up to permutation. This means that $a=b=c=d=\frac{1}{2}$ or three of the numbers a, b, c, d are equal to 1 and the other to -1.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; there were 3 incorrect solutions.

O232. Let $ABCDE$ be a convex pentagon with area S. The area of the pentagon formed by the intersections of its diagonals is equal to S'. Consider the statement $S' < cS$, where c is constant. Prove that the statement holds when $c = \frac{1}{2}$. Try to find the best constant you can achieve. For example, it can be proved that the statement still holds for $c = 2 - \sqrt{3}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA, Nicusor Zlota, Focsani, Romania.

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Consider triangle ACD with unit area, and for any points P, Q respectively on its sides AC, AD, denote $p = \frac{AP}{AC}$ and $q = \frac{AQ}{AD}$. Assume that C, Q, E and D, P, B are collinear, such that $x = \frac{DP}{BD}$ and $y = \frac{CQ}{CE}$. Let BE intersect AC, AD at U, V respectively, and let BD, CE meet at W. Then $S' = [PUVQW]$ and $S = [ABCDE]$. Note that $[ABCD] = \frac{BD}{PD}$, and $[ACDE] = \frac{CE}{CQ}$, or

$$S = [ABCDE] = [ABCD] + [ACDE] - 1 = \frac{x+y}{xy} - 1 = \frac{1-a}{xy},$$

where $a = (1-x)(1-y)$. Note that S is minimum for a fixed when $x = y$. Denote $u = \frac{AU}{AC}$ and $v = \frac{AV}{AD}$. By Menelaus' theorem,

$$\frac{u(1-x)(1-v)}{v(p-u)} = \frac{AU}{UP} \cdot \frac{PB}{BD} \cdot \frac{DV}{VA} = 1 = \frac{AV}{VQ} \cdot \frac{QE}{EC} \cdot \frac{CU}{UA} = \frac{v(1-y)(1-u)}{u(q-v)},$$

or after some algebra,

$$u = \frac{pq-a}{qx+y(1-x)}, \qquad v = \frac{pq-a}{py+x(1-y)}.$$

Similarly, define $w = \frac{CW}{CE}$ and $t = \frac{DW}{DB}$, or by Menelaus' theorem,

$$\frac{p(1-q)w}{(1-p)(y-w)} = \frac{CW}{WQ} \cdot \frac{QD}{DA} \cdot \frac{AP}{PC} = 1 = \frac{DW}{WP} \cdot \frac{PC}{CA} \cdot \frac{AQ}{QD} = \frac{q(1-p)t}{(1-q)(x-t)},$$

yielding

$$w = \frac{(1-p)y}{1-pq}, \qquad t = \frac{(1-q)x}{1-pq}.$$

Now, $[PQVU] = [APQ] - [AUV] = pq - uv$ and

$$[PQW] = \frac{PW \cdot QW}{CQ \cdot DP}[CDQP] = \frac{(DP-DW)(CQ-CW)}{CQ \cdot DP}(1 - [APQ]) =$$

$$= \frac{(x-t)(y-w)(1-pq)}{xy} = \frac{pq(1-p)(1-q)}{1-pq}.$$

It follows that

$$S' = \frac{pq(2-p-q)}{1-pq} - \frac{(pq-a)^2}{(py+x(1-y))\,(qx+y(1-x))}.$$

Now, applying the AM-GM inequality, we have

$$(py+x(1-y))\,(qx+y(1-x)) \leq \frac{xy}{4}\left(p+q-2+\frac{(1-a)(1+xy-a)}{xy}\right)^2,$$

with equality iff $py + x(1-y) = qx + y(1-x)$, which holds for example whenever $x = y$ and $p = q$. Denote $K = pq$, $L = 2-p-q$, $M = 1-a$, $N = xy$, or

$$S = \frac{M}{N}, \qquad S' \leq T = \frac{KL}{1-K} - \frac{4(K+M-1)^2}{(M^2+MN-LN)^2}.$$

Note that L is maximum for given K when $p = q$, and that given $M = x + y - xy \geq 2\sqrt{N} - N$, or $N \leq (1-\sqrt{1-M})^2$, or for given M, N is maximum when $x = y$.

Note that $\frac{S'}{S}$ will be at most $\frac{S'}{S}$, so we will try to maximize $\frac{T}{S}$, and then check whether $\frac{S'}{S}$ can reach this upper bound. Taking the derivative of $\frac{T}{S}$ with respect to L while keeping K constant, we may have different cases:

- The derivative is not zero within the allowed range of values for L, in which case the maximum of $\frac{T}{S}$ happens either when $L=0$ or when $p=q$.
- The derivative is zero within the allowed range of values of L, but it happens either at a local minimum, or at a local maximum that is not higher than the value for $L=0$ or for $p=q$.
- The derivative is zero within the allowed range of values of L, and it marks a local maximum that is higher than the values for $L=0$ and for $p=q$.

Note also that, for $L=0$, we must have $p=q=1$, in which case $P=C$ and $Q=D$, or $S'=0$, clearly not a maximum of $\frac{T}{S}$, or either the maximum of $\frac{T}{S}$ happens, for given K,M,N, either when the first derivative of $\frac{T}{S}$ with respect to L is zero and at the same time $py+x(1-y)=qx+y(1-x)$, or when $p=q$. Similarly, the maximum of $\frac{T}{S}$ happens, for given K,L,M, either when the first derivative of $\frac{T}{S}$ with respect to N is zero, or when $x=y$.

The first derivative of $\frac{T}{S}$ with respect to L is zero when

$$\frac{K}{1-K}=\frac{8N(K+M-1)^2}{\left(M^2+MN-LN\right)^3},$$

in which case

$$\frac{T}{S}=\frac{K}{1-K}(M+N)-3\frac{\sqrt[3]{N}}{M}\sqrt[3]{\frac{(K+M-1)^2K^2}{(1-K)^2}}.$$

Note now that the second derivative of $\frac{T}{S}$ with respect to N would be

$$\frac{2}{3MN\sqrt[3]{N^2}}\sqrt[3]{\frac{(K+M-1)^2K^2}{(1-K)^2}},$$

clearly positive, or the maximum of $\frac{T}{S}$ would happen if $x=y$ if it happens for $p\neq q$. Similarly, the first derivative of $\frac{T}{S}$ with respect to N is zero when

$$\left(M^2+MN-LN\right)^3=\frac{4(K+M-1)^2(1-K)}{KL}(M^2-MN+LN),$$

for

$$\frac{T}{S}=\frac{KLN^2(L-M)}{M(1-K)(MN-(M-N)(L-M))}.$$

Now, $M-N\geq 2\sqrt{N}(1-\sqrt{N})$, clearly positive, or $L-M$ must be positive and maximum for $\frac{T}{S}$ to be maximum, ie if the maximum of $\frac{T}{S}$ happens when $x\neq y$, then it happens when $p=q$.

We have thus proved that the maximum of $\frac{T}{S}$ happens, either when $x=y$, or when $p=q$. Moreover, S' only reaches its maximum value T when $py+x(1-y)=qx+y(1-x)$. This condition is satisfied, when $x=y$ iff $p=q$, and when $p=q$, iff $(x-y)(1-p)=0$. Since $p=q=1$ produces $S'=0$, then $S'=T$ when $p=q$ or $x=y$, iff $p=q$ and $x=y$. This seems to indicate that $\frac{S'}{S}$ will be maximum if $x=y$ and $p=q$. Rather lengthy (and somewhat nasty) calculations (which the author of this solution refuses to reproduce here, both for his sake and for the reader's), confirm this suspicion, or we need "only" to maximize the expression $R=\frac{S'}{S}$, when $x=y$ and $p=q$, or we need to maximize with respect to x,p the expression

$$R=\frac{2p^2x^2-(1+p)(p-1+x)^2}{(1+p)x(2-x)}$$

Note first that for $x=p=\frac{\sqrt{5}-1}{2}$, we obtain $R=\frac{7-3\sqrt{5}}{2}$, so this is our "target to beat". Clearly, when $x=0$ or $p=0$, the expression is non-positive. If $p=1$, we obtain $R=0$, while if $x=1$ we find

$$R=\frac{p^2(1-p)}{1+p},$$

whose maximum we can find by derivation with respect to p. The first derivative is zero (excluding the case $p=0$ which we know leads nowhere good) iff $p^2+p=1$, for $p=\frac{\sqrt{5}-1}{2}$ (the negative root does not make sense in this problem), and

$$R=\frac{5\sqrt{5}-11}{2},$$

and this is larger than the target to beat iff $\sqrt{80}=4\sqrt{5}>9=\sqrt{81}$, clearly false. It follows that the maximum will occur only if the derivatives of R with respect to x and p are simultaneously zero, or

$$p(2+p)x^2-(1+p)^2x+(1+p)^2(1-p)=0, \qquad px^2+(1+p)(1-p)x-(1+p)(1-p)=0.$$

Combining these equations to get rid of x yields

$$(3-p^2)x=(1-p)(3+2p),$$

or

$$0=6-3p-10p^2-2p^3+p^4=(1-p-p^2)(6+3p-p^2),$$

and since $6+3p-p^2$ is clearly not possible for $p\in(0,1)$, we find that indeed p satisfies $p^2+p-1=0$, or $p=\frac{\sqrt{5}-1}{2}$, for

$$x=\frac{(1-p)(3+2p)}{3-p^2}=\frac{(3-\sqrt{5})^2(2+\sqrt{5})}{4}=\frac{\sqrt{5}-1}{2},$$

or we cannot beat the target, and the maximum of $\frac{S'}{S}$ occurs when $x=y=p=q=\frac{\sqrt{5}-1}{2}$. We conclude that $S'\leq\frac{7-3\sqrt{5}}{2}S$, or $S'<cS$ for any $c>\frac{7-3\sqrt{5}}{2}$.

O233. Let ABC be a triangle with $\angle BAC = 90^\circ$. Let D be a point on hypotenuse BC and let the parallel lines through D to the legs AC and AB meet (for the second time) the circles with diameters DC and DB at U and V, respectively. Furthermore, if P is the second intersection of AD with the circumcircle of triangle ABC, prove that $\angle UPV = 90^\circ$.

Proposed by Luis Gonzalez, Maracaibo, Venezuela

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let A' be the point diametrally opposite A in the circle with diameter BC. Clearly $\angle DUC = 90^\circ$ because U is on the circle with diameter CD, or U is also on the perpendicular to AC through C, which is clearly CA' because AA' is a diameter of the circumcircle of ABC, which is also the circle with diameter BC. It follows that U is the intersection point of perpendicular lines CA' and DU. Similarly, V is the intersection point of perpendicular lines DV and BA'. But $BA' \perp CA'$ because A' in on the circle with diameter BC. Hence $DUA'V$ is a rectangle whose circumcircle has diameters UV and DA'. Let Q be the second point of intersection of the circumcircles of $DUA'V$ and ABC. Clearly, $\angle A'QD = \angle A'QA = 90^\circ$, or A, D, Q are collinear, and $Q = P$. But $P = Q$ is then on a circle with diameter UV. The conclusion follows.

Also solved by Zolbayar Shagdar, Ulaanbaatar, Mongolia, Daniel Vacaru, Pitesti, Romania

O234. Let f be a polynomial of degree 4 with integral coefficients. Prove that there are infinitely many positive integers n such that $2^n - 1$ does not divide $f(n!)$.

Bonus Does the result hold without the hypothesis $\deg(f) = 4$?

Gabriel Dospinescu, Ecole Polytechnique, France

Solution by the author

We will assume that $2^n - 1$ divides $f(n!)$ for all sufficiently large n and derive a contradiction. First, we claim that $f(0) = 0$. Indeed, fix $k \geq 1$ and observe that for n large enough $2^{2^k} + 1$ divides $2^{2^n} - 1$, thus it also divides $f((2^n)!)$. For for n large enough (compared to k), $2^{2^k} + 1$ also divides $(2^n)!$. Putting these two observations together we deduce that $2^{2^k} + 1$ divides $f(0)$. As k was arbitrary, we must have $f(0) = 0$.

Next, we prove that $f(-1) = 0$. Indeed, for large primes p we have $p|2^{p-1} - 1$, so that p also divides $f((p-1)!)$. But since $(p-1)! = -1 \pmod p$ (by Wilson's theorem) and since $a - b$ divides $f(a) - f(b)$ for any integers a, b, we also have $f((p-1)!) = f(-1) \pmod p$. We deduce that p divides $f(-1)$. Since p was arbitrary (large enough), we must have $f(-1) = 0$.

The previous two paragraphs show that we can write

$$f(X) = X(X+1)(aX^2 + bX + c)$$

for some integers a, b, c, with $a \neq 0$. We know that there are infinitely many primes $p = 1 \pmod 4$ such that $2^{\frac{p-1}{2}} = 1 \pmod p$. Actually, using the fact that Legendre's symbol $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$, we can take any $p = 1 \pmod 8$ and it is well-known that there are infinitely many such p (weak version of Dirichlet's theorem). But we can also prove this in a much more elementary way: define $a_k = 2(k^2 - k + 1)$ and choose a sequence of different primes p_j such that each p_j divides some $a_k^2 - 2$ (it is easy to see that it is possible to do this). Then by construction $2^{\frac{p_j-1}{2}} = a_k^{p_j-1} = 1 \pmod{p_j}$ and also p_j divides $a_j^2 - 2 = (a_j-1)^2 + (2k-1)^2$. Thus we must have $p_j = 1 \pmod 4$. This being said, fix such a (large) prime p and let $x = \left(\frac{p-1}{2}\right)!$. Then Wilson's theorem combined with the fact that $p = 1 \pmod 4$ shows that $p|x^2 + 1$. But since p divides $2^{\frac{p-1}{2}} - 1$, which divides $f(x)$, we also have $p|f(x)$. As $p|x^2+1$, we clearly cannot have $p|x$ or $p|x+1$. Thus necessarily $p|ax^2 + bx + c$ and so $p|bx + c - a$. Thus $bx = a - c \pmod p$ and since $x^2 = -1 \pmod p$, we also have $b^2 + (a-c)^2 = 0 \pmod p$. Since this happens for infinitely many such p, we must have $b = 0$ and $a = c$.

Thus we can write $f(X) = aX(X+1)(X^2+1)$. Now comes the most subtle part of the solution, based on the following

Lemma. There are infinitely many primes q for which $\frac{2^q+1}{3}$ has a prime factor $p = -1 \pmod 4$ such that $p \geq 3q + 1$.

Proof of the lemma There are infinitely many primes $q = 1 \pmod 6$. Pick any such prime and note that $\frac{2^q+1}{3} = -1 \pmod 4$ and that $\frac{2^q+1}{3}$ is not a multiple of 3. Thus $\frac{2^q+1}{3}$ has a prime factor $p > 3$ congruent to $-1 \pmod 4$. We claim that $p \geq 3q + 1$. But since p divides $2^{2q} - 1$, the order of 2 modulo p divides $p - 1$ and also $2q$. But this order cannot be q (as p divides $2^q + 1$) nor 1 or 2, so that the order is actually $2q$. Since we cannot have $2q = p - 1$ (otherwise p would be a multiple of 3, as $q = 1 \pmod 6$) and since $2q$ divides $p - 1$, we certainly have $p \geq 3q + 1$ (even $4q + 1$, actually). This ends the proof of this lemma.

Back to the problem, choose large primes p, q as in the previous lemma. As p divides $2^q + 1$, p also divides $2^{2q} - 1$ and so it divides $f((2q)!)$. Remember that $f(X) = aX(X+1)(X^2+1)$. We cannot have $p|(2q)!^2 + 1$, as $p = -1 \pmod 4$. Also, $p > 2q$, so that p cannot divide $(2q)!$. Thus p must divide $a((2q)! + 1)$. Note that $p - 1 - 2q \geq q$ is large when q is large and that p also divides $2^{p-1-2q} - 1$ (since p divides $2^{p-1} - 1$ and also $2^{2q} - 1$). Thus p divides $f((p-1-2q)!)$, too. The same argument as above shows that p divides $a((p-1-2q)! + 1)$. Let $x = (2q)!$ and $y = (p-1-2q)!$. We know that $p|a(x+1)$ and $p|a(y+1)$. Moreover, by Wilson's theorem we have

$$-1 = (p-1)! = y(p-2q)...(p-1) = xy \pmod p.$$

Then $p|xa(y+1) = a(xy+1) + xa - a$ and $p|xa + a$ and finally $p|xy + 1$. Thus p divides $2a$. But remember that p was arbitrarily large, so that necessarily $a = 0$, a (so) desired contradiction!