Solutions

J1. Solve in real numbers the system of equations:

$$\begin{cases} x^4 + 2y^3 - x = -\frac{1}{4} + 3\sqrt{3} \\ y^4 + 2x^3 - y = -\frac{1}{4} - 3\sqrt{3}. \end{cases}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Bogdan Enescu, "B.P. Hasdeu" National College, Romania

Adding up the two equations yields

$$x^4 + 2x^3 - x + \frac{1}{4} + y^4 + 2y^3 - y + \frac{1}{4} = 0,$$

or, equivalently,

$$\left(x^2 + x - \frac{1}{2}\right)^2 + \left(y^2 + y - \frac{1}{2}\right)^2 = 0.$$

We deduce that $x, y \in \left\{\frac{-1-\sqrt{3}}{4}, \frac{-1+\sqrt{3}}{4}\right\}$. Observe that if α is a root of the equation $t^2 + t - \frac{1}{2} = 0$, then $\alpha^2 = -\alpha + \frac{1}{2}$, $\alpha^3 = -\alpha^2 + \frac{\alpha}{2} = \frac{3\alpha - 1}{2}$, and, finally, $\alpha^4 = -2\alpha + \frac{3}{4}$. Replacing in the first equation of the system gives

$$-x + y = \sqrt{3},$$

hence we must have $x = \frac{-1-\sqrt{3}}{4}$ and $y = \frac{-1+\sqrt{3}}{4}$.

Also solved by Bin Zhao, YunYuan HuaZhong University of Technology and Science,Wuhan,CHINA and the proposer.

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J3. Consider the sequence

$$a_n = \sqrt{1 + \left(1 + \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 - \frac{1}{n}\right)^2}, \quad n \geq 1.$$

Prove that $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_{20}}$ is an integer.

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain

Since

$$\frac{1}{a_n} = \frac{1}{\sqrt{1+\left(1+\frac{1}{n}\right)^2}+\sqrt{1+\left(1-\frac{1}{n}\right)^2}}$$
$$= \frac{n}{4}\left[\sqrt{1+\left(1+\frac{1}{n}\right)^2}-\sqrt{1+\left(1-\frac{1}{n}\right)^2}\right],$$

then

$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_{20}} = \left(\frac{\sqrt{5}}{4}-\frac{1}{4}\right)+\left(\frac{\sqrt{13}}{4}-\frac{\sqrt{5}}{4}\right)+\left(\frac{5}{4}-\frac{\sqrt{13}}{4}\right)$$
$$+\left(\frac{\sqrt{41}}{4}-\frac{5}{4}\right)+\left(\frac{\sqrt{61}}{4}-\frac{\sqrt{41}}{4}\right)+\ldots$$
$$+\left(\frac{\sqrt{685}}{4}-\frac{\sqrt{613}}{4}\right)+\left(\frac{\sqrt{761}}{4}-\frac{\sqrt{685}}{4}\right)+\left(\frac{29}{4}-\frac{\sqrt{761}}{4}\right)$$
$$= \frac{29}{4}-\frac{1}{4} = 7$$

and we are done.

Also solved by Bin Zhao, YunYuan HuaZhong University of Technology and Science,Wuhan,CHINA and the proposer.

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J5. Let x, y, z be positive real numbers such that $xyz = 1$. Show that the following inequality holds:

$$\frac{1}{(x+1)^2+y^2+1}+\frac{1}{(y+1)^2+z^2+1}+\frac{1}{(z+1)^2+x^2+1} \le \frac{1}{2}.$$

Proposed by Dr. Cristinel Mortici, Valahia University, Târgovişte, Romania

Solution by Bin Zhao, YunYuan HuaZhong University of Technology and Science,Wuhan,CHINA

We have

$$\frac{1}{(x+1)^2+y^2+1} = \frac{1}{2+x^2+y^2+2x} \le \frac{1}{2(1+x+xy)}$$

Since $xyz = 1$, we substitute

$$x = \frac{b}{a}, y = \frac{c}{b}, z = \frac{a}{c}$$

to obtain

$$\frac{1}{2\left(1+x+xy\right)}=\frac{a}{2\left(a+b+c\right)}.$$

Writing the other two similar inequalities and adding them up gives the desired result.

Also solved by the proposer.

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S1. Prove that the triangle ABC is right-angled if and only if

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{2}.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Daniel Campos Salas

Let $\dfrac{A}{2}=\dfrac{\pi}{4}+\rho$, with $|\rho|<\dfrac{\pi}{4}$. Then the given identity turns into

$$\left(\frac{\sqrt{2}}{2}\cos\rho-\frac{\sqrt{2}}{2}\sin\rho\right)\cos\frac{B}{2}\cos\frac{C}{2}-\left(\frac{\sqrt{2}}{2}\cos\rho+\frac{\sqrt{2}}{2}\sin\rho\right)\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{2}.$$

Rearranging the terms it follows that this result holds if and only if

$$\cos\rho\left(\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{B}{2}\sin\frac{C}{2}\right)-\sin\rho\left(\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\right)=\frac{\sqrt{2}}{2}$$

$$\Leftrightarrow\cos\rho\cos\left(\frac{B+C}{2}\right)-\sin\rho\cos\left(\frac{B-C}{2}\right)=\frac{\sqrt{2}}{2}$$

$$\Leftrightarrow\cos\rho\cos\left(\frac{\pi}{4}-\rho\right)-\sin\rho\cos\left(\frac{B-C}{2}\right)=\frac{\sqrt{2}}{2}$$

$$\Leftrightarrow\frac{\sqrt{2}}{2}\left(\cos^2\rho+\cos\rho\sin\rho\right)-\sin\rho\cos\left(\frac{B-C}{2}\right)=\frac{\sqrt{2}}{2}$$

Using the identity $\cos^2\alpha=1-\sin^2\alpha$, it follows that the equation holds if and only if

$$\sin\rho\left(\frac{\sqrt{2}}{2}\left(\cos\rho-\sin\rho\right)-\cos\left(\frac{B-C}{2}\right)\right)=0$$

$$\Leftrightarrow\sin\rho\left(\cos\left(\frac{\pi}{4}+\rho\right)-\cos\left(\frac{B-C}{2}\right)\right)=0$$

$$\Leftrightarrow \sin\rho\left(\cos\frac{A}{2}-\cos\left(\frac{B-C}{2}\right)\right)=0$$

$$\Leftrightarrow \sin\rho\sin\left(\frac{A+B-C}{4}\right)\sin\left(\frac{-A+B-C}{4}\right)=0$$

$$\Leftrightarrow \sin\left(\frac{\pi-2A}{4}\right)\sin\left(\frac{\pi-2B}{4}\right)\sin\left(\frac{\pi-2C}{4}\right)=0,$$

which gives the desired result.

Also solved by Bin Zhao, YunYuan HuaZhong University of Technology and Science,Wuhan,CHINA and the proposer. Partially solved by Karsten Bohlen.

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S5. Let a and b be two real numbers such that $a^p - b^p$ is a positive integer for each prime number p. Prove that a and b are integer numbers.

Proposed by Nairi Sedrakian, Yierevan, Armenia.

Solution by the author

First we will prove that numbers a, b are rational. The case $ab = 0$ is simple. Consider now $ab \neq 0$. Then we have

$$\left(a^5-b^5\right)^2-\left(a^7-b^7\right)\left(a^3-b^3\right)=a^3b^3\left(a^2-b^2\right)^2,$$

hence a^3b^3 is a rational number. From the latter and the identity

$$\left(a^7-b^7\right)^2-\left(a^{11}-b^{11}\right)\left(a^3-b^3\right)=a^3b^3\left(a^2-b^2\right)^2\left(a^2+b^2\right)^2$$

we deduce, that the number $\left(a^2+b^2\right)^2=\left(a^2-b^2\right)^2+4a^2b^2$ is a rational number, and so a^2b^2 is rational, from which follows that $ab=\dfrac{a^3b^3}{a^2b^2}$ is also rational number.

Since

$$\left(a^5-b^5\right)\left(a^{11}-b^{11}\right)-\left(a^{13}-b^{13}\right)\left(a^3-b^3\right)=a^3b^3\left(a^2-b^2\right)^2\left(a^2+b^2\right)\left(a^4+b^4\right),$$

then

$$\left(a^2+b^2\right)\left(a^4+b^4\right)=\left(a^3-b^3\right)^2+2a^3b^3+a^2b^2\left(a^2+b^2\right)$$

is a rational number, from which we deduce that a^2+b^2 is a rational number. Finally the numbers $a-b=\dfrac{a^3-b^3}{a^2+ab+b^2}$ and $a+b=\dfrac{a^2-b^2}{a-b}$ are rational numbers, and hence a and b are rational.

Let $a = \dfrac{m}{n}$ and $b = \dfrac{k}{n}$, where $m, k \in \mathbb{Z}$, $n \in \mathbb{N}$ and n is minimal. Then for any prime number p, n^pdivides $m^p - k^p$. Now since n is minimal we find that $(m, n) = (k, n) = 1$. If $n > 1$, then there exists a prime number q so that q divides n. We have

$$\left(m^3 - k^3\right)\left(m^2 + k^2\right) - \left(m^3 + k^3\right)\left(m^2 - k^2\right) = 2m^2k^2(m - k),$$

consequently q^2 divides $2(m - k)$, that is, q divides $m - k$.

If $p > q$, then $m^p - k^p = (m-k)\left(m^{p-1} + m^{p-2}k + \cdots + k^{p-1}\right)$ is divisible by q^p and $\left(m^{p-1} + m^{p-2}k + \cdots + k^{p-1}\right) \equiv pk^{p-1} (\bmod q)$, so that $\left(m^{p-1} + m^{p-2}k + \cdots + k^{p-1}\right)$ isn't divided by q, hence q^p divides $m - k$, which is impossible because p, $(p > q)$ is any prime number, while $m - k \neq 0$.

Thus $n = 1$, and hence a and b are integers.

Also solved by Iurie Boreico, student, Chisinau, Moldova

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S6. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

1. If $a \leq b \leq 1 \leq c$, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1};$$

2. If $a \leq 1 \leq b \leq c$, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1};$$

Proposed by Marian Tetiva, Bârlad, Romania

Solution by the author

Using the condition $a + b + c = 3$, we have

$$\begin{aligned} &\sum \left(\frac{1}{a+b} - \frac{1}{c+1} \right) \\ &= \sum \frac{c + 1 - (a+b)}{(a+b)(c+1)} \\ &= 2 \sum \frac{c-1}{(a+b)(c+1)}. \end{aligned}$$

But

$$\begin{aligned}
&\frac{a-1}{(b+c)(a+1)}+\frac{b-1}{(a+c)(b+1)}+\frac{c-1}{(a+b)(c+1)}\\
&=\frac{a-1}{(b+c)(a+1)}+\frac{b-1}{(a+c)(b+1)}-\frac{a+b-2}{(a+b)(c+1)}\\
&=\frac{a-1}{(b+c)(a+1)}-\frac{a-1}{(a+b)(c+1)}+\frac{b-1}{(a+c)(b+1)}-\frac{b-1}{(a+b)(c+1)}\\
&=\frac{(a-1)(b-1)(c-a)}{(b+c)(a+1)(a+b)(c+1)}+\frac{(a-1)(b-1)(c-b)}{(a+c)(b+1)(a+b)(c+1)}.
\end{aligned}$$

The conclusion follows observing that $c-a\geq 0$, $c-b\geq 0$ and if $a\leq b\leq 1$, then $(a-1)(b-1)\geq 0$ while if $a\leq 1\leq b$, then $(a-1)(b-1)\leq 0$.

Also solved by Bin Zhao, YunYuan HuaZhong University of Technology and Science,Wuhan,CHINA

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U1. Evaluate

$$\int_0^1 \sqrt[3]{2x^3-3x^2-x+1}dx.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by Bin Zhao, YunYuan HuaZhong University of Technology and Science,Wuhan,CHINA

Let

$$f(x)=\sqrt[3]{2x^3-3x^2-x+1}$$

First we will prove:

$$f(x)=-f(1-x)$$

This is because

$$\sqrt[3]{2x^3-3x^2-x+1}=-\sqrt[3]{2(1-x)^3-3(1-x)^2-(1-x)+1}$$

$$\Longleftrightarrow 2x^3-3x^2-x+1=-\Big(2(1-x)^3-3(1-x)^2-(1-x)+1\Big),$$

which is obvious by expanding.

Thus:

$$2\int_0^1 \sqrt[3]{2x^3-3x^2-x+1}dx=2\int_0^1 f(x)dx$$

$$= \int_0^1 f(x)dx + \int_0^1 f(1-x)dx = \int_0^1 (f(x) + f(1-x))\, dx = 0$$

So

$$\int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1} dx = 0$$

Also solved by Iurie Boreico, student, Chisinau, Moldova, and the author.

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U2. Solve in real numbers the equation

$$6^x + 1 = 8^x - 27^{x-1}.$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas

Solution by the author
If we denote $a = 1, b = -2^x, c = 3^{x-1}$, the equation becomes

$$a^3 + b^3 + c^3 - 3abc = 0.$$

The latter factors as

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)\left(a^2 + b^2 + c^2 - ab - bc - ac\right),$$

and the second paranthesis equals zero only if $a = b = c$, which is impossible. Therefore the initial equation is equivalent to

$$1 - 2^x + 3^{x-1} = 0.$$

Write this as

$$3^{x-1} - 2^{x-1} = 2^{x-1} - 1,$$

and for each x consider the function $f(t) = t^{x-1}$, $t > 0$. It follows from Lagrange's theorem the existence of the numbers $\alpha \in (2, 3)$ and $\beta \in (1, 2)$ such that $f(3) - f(2) = f'(\alpha)$ and $f(2) - f(1) = f'(\beta)$. Since $f'(t) = (x-1)\, t^{x-2}$, we obtain

$$(x-1)\, \alpha^{x-2} = (x-1)\, \beta^{x-2}.$$

Clearly, this implies either $x = 1$ or (since $\alpha \neq \beta$) $x = 2$.

Also solved by Iurie Boreico, student, Chisinau, Moldova

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U5. Let $s \geq 2$ be a positive integer. Prove that there is no rational function $R(x)$ such that

$$\frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} = R\left(n\right),$$

for all positive integers $n \geq 2$.

Proposed by Mihai Piticari, Câmpulung, and Dorin Andrica, Babeş-Bolyai University, Cluj Napoca, Romania

Solution by Iurie Boreico, student, Chisinau, Moldova

Suppose there exists such a function $R(n)$. Let

$$S(n) = \sum_{i=2}^{\infty} \frac{1}{i^s} - R(n),$$

thus $S(n) = \sum_{i=n+1}^{\infty} \frac{1}{i^s}$. Let $S(x) = \frac{P(x)}{Q(x)}, gcd(P,Q) = 1$.

Then $S(n-1) - S(n) = \frac{1}{n^s}$, or

$$n^s(P(n-1)Q(n) - P(n)Q(n-1) = Q(n-1)Q(n)).$$

This relation must therefore hold for any x, as it holds for infinitely many x. Particularly, if $S(x)$ is not defined, then $S(x-1)$ is also not defined so if x is a root of Q then so is $x-1$. This would mean that Q has infinitely many roots, which is not the case so Q is constant. However in this case S is a polynomial, but every non-constant polynomial is unbounded, which is not the case for $S(n)$.

Also solved by the authors

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O1. A circle centered at O is tangent to all sides of the convex quadrilateral $ABCD$. The rays BA and CD intersect at K, the rays AD and BC intersect at L. The points X, Y are considered on the line segments OA, OC, respectively. Prove that $\angle XKY = \frac{1}{2}\angle AKC$ if and only if $\angle XLY = \frac{1}{2}\angle ALC$.

Proposed by Pavlo Pylyavskyy, graduate student, MIT

Solution by Iurie Boreico, student, Chisinau, Moldova

Since we have

$$\angle AKO = \angle CKO = \frac{1}{2}\angle AKC,$$

the equality $\angle XKY = \frac{1}{2}\angle AKC$ is equivalent to $\angle XKO = \angle YKC$ and equivalent to

$$\frac{\sin\angle AKX}{\sin\angle XKO} = \frac{\sin\angle OKY}{\sin\angle YKC}.$$

Using the sine law in triangles AXK, XOK, OKY, YKC the latter is equivalent to

$$\frac{XO}{XA}\frac{YO}{YC}\frac{AK\cdot CK}{KO^2}=1.$$

Analogously, $\angle XLY = \frac{1}{2}\angle ALC$ is equivalent to

$$\frac{XO}{XA}\frac{YO}{YC}\frac{AL\cdot CL}{LO^2}=1.$$

Thus, to prove the conclusion we need to prove that

$$\frac{AK\cdot CK}{KO^2}=\frac{AL\cdot CL}{LO^2}.$$

But this follows from the sine law applied in triangles KAO, KOC, LAO, and LOC.

Also solved by Yufei Zhao, Don Mills Collegiate Institute, Toronto, Canada, and the author.

★

O2. Find all positive integers n such that the set

$$A=\{1,3,5,\ldots,2n-1\}$$

can be partitioned into 12 subsets, the sum of elements in each subset being the same.

Proposed by Marian Tetiva, Bârlad, Romania

Solution by Yufei Zhao, Don Mills Collegiate Institute, Toronto, Canada

The sum of the elements in A is $1+3+5+\cdots+(2n-1)=n^2$. Thus, in each of the 12 subsets, the sum of the elements is $n^2/12$. In order for $n^2/12$ to be an integer, n has to be a multiple of 6. So let $n=6k$, where k is a positive integer. Then, the sum of elements in each of the 12 subsets is $3k^2$.

Note that since the element $2n-1=12k-1$ has to be in one of the subsets, and the sum of all the elements in that subset is $3k^2$, we must have $12k-1\leq 3k^2$. Thus $3(k-2)^2\geq 11$, implying that $k\geq 4$.

First, let us show that $k\neq 5$. Suppose that $k=5$, then $A=\{1,3,5,\ldots,59\}$ has 30 elements, and the sum of elements in each of the 12 subsets must be 75. Thus each subset must have more than one element. Considering the parity, we see that each subset must have an odd number of elements. Therefore, each of the 12 subsets must contain at least 3 elements, which is impossible since $|A|=30$.

Now, we will show examples of the partition for $k=4,6,7,9$.

When $k=4$, $A=\{1,3,\ldots,47\}$. We can take the subsets $B_i=\{2i-1,49-2i\}$, $i=1,2,\ldots,12$. Each subset has a sum of 48.

When $k=6$, $A=\{1,3,\dots,71\}$. We can take the subsets $B_i=\{35+2i, 73-2i\}$, $i=1,2,\dots,9$, $B_{10}=\{23,25,29,31\}$, $B_{11}=\{19,21,33,35\}$, and $B_{12}=\{1,3,5,7,9,11,13,15,17,27\}$. Each subset has a sum of 108.

When $k=7$, $A=\{1,3,\dots,83\}$. We can take the subsets $B_i=\{4i-3, 65-2i, 85-2i\}$, $i=1,2,\dots,10$, $B_{11}=\{3,27,35,39,43\}$ and $B_{12}=\{7,11,15,19,23,31,41\}$. Each subset has a sum of 147.

When $k=9$, $A=\{1,3,\dots,107\}$. We can take the subsets $B_i=\{23+8i, 109-4i, 111-4i\}$, $i=1,2,\dots,7$, $B_8=\{1,41,65,67,69\}$, $B_9=\{3,15,73,75,77\}$, $B_{10}=\{29,37,57,59,61\}$, $B_{11}=\{5,9,17,23,25,27,33,51,53\}$, $B_{12}=\{7,11,13,19,21,35,43,45,49\}$. Each subset has a sum of 243.

Now, we show that if $k=k_0$ yields a partition, then so does $k=k_0+4$. Suppose that B_i, $i=1,2,\dots,12$ is a partition for $A=\{1,3,\dots,12k_0-1\}$. Then the new set $A'=\{1,3,5,\dots,12(k_0+4)-1\}$ can be partitioned into the following 12 subsets: $B_i'=B_i\cup\{12k_0-1+2i, 12k_0+49-2i\}$, $i=1,2,\dots,12$. For each i, the sum of the elements in B_i' is greater than that of B_i by $24k_0+48$. Since all the B_i's have the same sum as each other, so do the B_i''s.

Combining the above results, we see that k can be any number of the form $4+4m$, $6+4m$, $7+4m$, or $9+4m$, where m is a non-negative integer. That is, k can be any positive integer other than $1,2,3,5$ (and we have shown that these values are invalid).

Therefore, $n=6k$, where k is either 4 or any positive integer greater than 5.

Also solved by Iurie Boreico, student, Chisinau, Moldova and the author.

★

O3. Prove that there are infinitely many prime numbers p with the following property: in the main period of the decimal representation of $\frac{1}{p}$, the number of 1's plus the number of 3's equals the number of 6's plus the number of 8's.

Proposed by Adrian Zahariuc, student, Romania

Solution by Yufei Zhao, Don Mills Collegiate Institute, Toronto, Canada

Let p be a prime divisor of 10^n+1, where $n\in\mathbb{N}$. We will show that in the main period of the decimal expansion of $1/p$, the number of times that the digit i appears equals to the number of times that the digit $(9-i)$ appears, for each $i=0,1,2,\dots,9$.

Let $10^n+1=pa$, where a is a positive integer. Note that a has at most n digits. Let the decimal representation of a be $a_1a_2a_3\cdots a_n$ (fill with leading zeros if necessary). Note that $a_n\neq 0$ since 10^n+1 is not divisible by 10. Now,

$$\frac{1}{p}=\frac{a(10^n-1)}{10^{2n}-1}=\frac{[10^n(a-1)]+[(10^n-1)-(k-1)]}{10^{2n}-1}$$

Observe that the decimal representation of $[10^n(a-1)]+[(10^n-1)-(k-1)]$ is

$$a_1a_2\cdots a_{n-1}(a_n-1)(9-a_1)(9-a_2)\cdots(9-a_{n-1})(10-a_n)$$

Thus, the decimal expansion of $1/p$ is

$$0.\overline{a_1 a_2 \cdots a_{n-1}(a_n-1)(9-a_1)(9-a_2)\cdots(9-a_{n-1})(10-a_n)}$$

(Although the above representation may include a multiple of the minimal period, it does not affect our purpose in comparing the number of times that each digit appears). Note that for each appearance of the digit i, the digit $9-i$ appears in the corresponding position in the other "half" of the period. Thus there is a one-to-one correspondence between the appearances of the digit i and that of $9-i$ in the period. It follows that those two digits always appear an equal number of times.

Now, we need to show that there are infinitely many primes that is the divisor of 10^n+1 for some $n \in \mathbb{N}$. Consider the set A containing all numbers of the form 10^n+1 where n is a prime greater than 11. Suppose that $10^m+1, 10^n+1 \in A$, and let p (if it exists) be a prime common divisor of the two numbers. Then $10^m \equiv -1 \pmod{p}$ and $10^n \equiv -1 \pmod{p}$. It follows that $10^{2m} \equiv 1 \pmod{p}$ and $10^{2n} \equiv 1 \pmod{p}$. Since $\gcd(2m, 2n) = 2$, we have $1 \equiv 10^{\gcd(2m,2n)} \equiv 100 \pmod{p}$. So $p \mid 99$. Note that 3 does not divide 10^n+1 (since the sum of its digits is 2). Therefore, we can only have $p = 11$. So the only prime that can divide into more than one element in A is 11.

Let $10^n+1 \in A$. Note that n is odd. Now,

$$\begin{aligned}\frac{10^n+1}{10+1} &= 10^{n-1} - 10^{n-2} + 10^{n-3} - \cdots + 10^2 - 10 + 1 \\ &\equiv (-1)^{n-1} - (-1)^{n-2} + (-1)^{n-3} - \cdots + (-1)^2 - (-1) + 1 \pmod{11} \\ &\equiv n \pmod{11}\end{aligned}$$

Since n is a prime greater than 11, we see that $\frac{10^n+1}{11}$ is not divisible by 11.

It follows that the elements of $B = \{\frac{10^n+1}{11} \mid n > 11 \text{ is prime}\}$ are mutually relatively prime. Therefore, there are infinitely many prime numbers that divide into some element of B. Hence, there are infinitely many prime numbers that divide 10^n+1 for some positive integer n.

The result follows immediately.

Also solved by Iurie Boreico, Chisinau, Moldova, and the author.

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O4. Let AB be a diameter of the circle Γ and let C be a point on the circle, different from A and B. Denote by D the projection of C on AB and let ω be a circle tangent to AD, CD, and Γ, touching Γ at X. Prove that the angle bisectors of $\angle AXB$ and $\angle ACD$ meet on AB.

Proposed by Liubomir Chiriac, student, Moldova

Solution by Iurie Boreico, student, Chisinau, Moldova

If R is the point of tangency of the small circle with AD then a well-known states that XR bisects $\angle AXB$. Therefore we must prove that CR bisects $\angle ACD$ which is equivalent to

$$RD = \frac{CDAD}{CD+AC} = \frac{CDAD(AC-CD)}{AC^2-CD^2} = \frac{CD(AC-CD)}{AD}.$$

With no loss of generality, let $D \in (AO)$ where O is the center of the big circle, and let $AO = BO = 1$, $AD = m$. If J is the center of the small circle and r is its radius then $RD = r$. Since the two circles are tangent, we get $OJ^2 = (1-r)^2$ or $r^2 + (r+1-m)^2 = (1-r)^2$. Solving the quadratic equation, we get $RD = r = \sqrt{2(2-m)} - (2-m)$. From the other side,

$$CD = \sqrt{ADBD} = \sqrt{m(2-m)}, AC = \sqrt{ABAD} = \sqrt{2m}$$

so

$$\frac{CD(AC-CD)}{AD} = \frac{\sqrt{m(2-m)}(\sqrt{2m} - \sqrt{m(2-m)})}{m} = \sqrt{2(2-m)} - (2-m) = DR.$$

Also solved by Yufei Zhao, Don Mills Collegiate Institute, Toronto, Canada, and the author.

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O6. Let x, y, z be nonnegative real numbers. Prove the inequality

$$x^4(y+z) + y^4(z+x) + z^4(x+y) \leq \frac{1}{12}(x+y+z)^5.$$

Proposed by Vasile Cârtoaje, Ploieşti, Romania

Solution by Yufei Zhao, Don Mills Collegiate Institute, Toronto, Canada

If $x = y = z = 0$, then the inequality obviously holds. So assume that not all of the three variables are zero. Since the inequality is homogeneous and symmetric, we may assume wolog that $x + y + z = 1$ and $x \geq y \geq z$. It is now equivalent to prove the inequality

$$x^4(1-x) + y^4(1-y) + z^4(1-z) \leq \frac{1}{12}(x+y+z)^5$$

Let $f(x) = x^4(1-x)$. Note that its second derivative $f''^2 - 20x^3$ is positive in $(0, \frac{4}{5})$. Thus $f(x)$ is convex in $[0, \frac{4}{5}]$. Since $x \geq y \geq z$ and $x + y + z = 1$, we must have $\frac{1}{2} \geq y + z \geq y \geq z$. Hence $y, z, y + z \in [0, \frac{4}{5}]$. By Karamata's Majorization Inequality,

$$y^4(1-y) + z^4(1-z) = f(y) + f(z) \leq f(y+z) + f(0) = (y+z)^4(1-y-z) = (1-x)^4 x$$

Therefore,

$$\begin{aligned}
x^4(1-x)+y^4(1-y)+z^4(1-z) &\le x^4(1-x)+(1-x)^4x \\
&= x(1-x)((1-x)^3+x^3) \\
&= x(1-x)(1-3x+3x^2) \\
&= \frac{1}{3}\left[3x(1-x)\right]\left[1-3x(1-x)\right] \\
&\le \frac{1}{3}\cdot\frac{1}{4} \\
&= \frac{1}{12}(x+y+z)^5
\end{aligned}$$

The last inequality was due to AM-GM. And the original inequality is proven.

To find the equality conditions, we just need $3x(1-x)=1-3x(1-x)$ (for the AM-GM step to have equality). It follows that the equality condition is $(x,y,z)=\left(\frac{1}{2}+\frac{\sqrt{3}}{6},\frac{1}{2}-\frac{\sqrt{3}}{6},0\right)$ (or some permutation/dilation thereof).

Also solved by Iurie Boreico, student, Chisinau, Moldova, and the author.