Mathematical Excalibur

Volume 17, Number 1 *May-August 2012*

Olympiad Corner

Below are the problems of the 2012 International Math Olympiad.

Problem 1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This circle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

(The *excircle* of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 2. Let $n \ge 3$ be an integer, and let $a_2,a_3,\ldots,a_n$ be positive real numbers such that $a_2a_3\cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$

Problem 3. The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***September 20, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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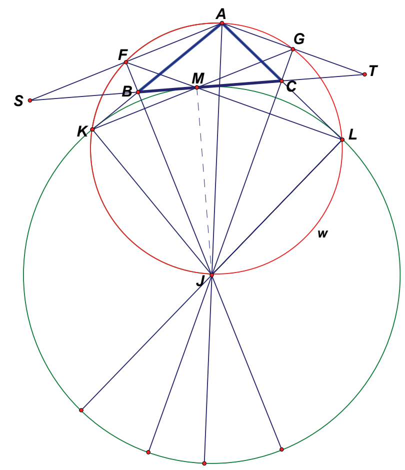
IMO 2012 (Leader Perspective)

Tat-Wing Leung

As leader, I arrived Mar del Plata, Argentina (the IMO 2012 site) four days earlier than the team. Despite cold weather, jet lag and delay of luggage, I managed to get myself involved in choosing the problems for the contest. Once the "easy" pair was selected, the jury did not have much choice but to choose problems of possibly other topics for the "medium" and the "difficult" pairs. The two papers of the contest were then set. We had to decide the various official versions and the marking scheme of the contest. After that, I just had to wait for the contestants to finish the contest and get myself involved in the coordination to decide the points obtained by our team. Here I would like to discuss the problems. (Please see Olympiad Corner for the statements of the problems.)



Problem 1. Really problem 1 is quite easy, merely a lot of angle chasings and many angles of 90° (tangents) and similar triangles, etc, and no extra lines or segments needed to be constructed. First note that $\angle AKJ = \angle ALJ = 90°$, hence A,K,L,J lie on the circle ω with diameter AJ. The idea is to show that F and G also lie on the same circle. Looking at angles around B, we see that $4\angle MBJ + 2\angle ABC = 360°$. Thus $\angle MBJ = 90° - ½\angle ABC$. Also, $\angle BMF = \angle CML = ½\angle ACB$ (as $CM=CL$). Then $\angle LFJ = \angle MBJ - \angle BMF = ½\angle BAC$. Thus $\angle LFJ = \angle LAJ$. Hence, F lies on ω. By the same token, so is G. Now AB and SB are symmetric with respect to the external bisector of $\angle ABC$, so is BK and BM. Now $SM = SB+BM = AB+BK = AK$. Similarly, $TM=AL$. So $SM=TM$.

It is relatively easy to tackle the problem using coordinate geometry. For instance, we can let the excircle be the unit circle with J=(0,0), M=(0,1), BC is aligned so that B=(b,1) and C=(c,1). Coordinates of other points are then calculated to verify the required property. But one must be really careful if he tries to use coordinate method. It was somehow decided that if a contestant cannot get a full solution using coordinate method, then he will be "seriously penalized"!

Problem 2. As it turned out, this problem caused quite a bit of trouble and many students didn't know how to tackle the problem at all. More sophisticated inequalities such as Muirhead do not work, since the expression is not "homogeneous". The Japanese leader called the problem a disaster. There were trivial questions such as "why is there no a_1?" A more subtle issue is how to isolate $a_2,a_3,\ldots,a_n$.

Clearly $(1+a_2)^2 \ge 2^2a_2$ by the AM-GM inequality. But how about $(1+a_3)^3$? Indeed the trick is to apply AM-GM inequality to get for k=2 to $n-1$,

$$(1+a_{k+1})^{k+1} = \left(\frac{1}{k}+\cdots+\frac{1}{k}+a_{k+1}\right)^{k+1}$$

$$\ge \left((k+1)\sqrt[k+1]{\frac{a_{k+1}}{k^k}}\right)^{k+1} = \frac{(k+1)^{k+1}a_{k+1}}{k^k}.$$

By multiplying the inequalities, the constants cancelled out and we get the final inequality. That the inequality is strict is trivial using the conditions of AM equals GM. The above inequality can also be used as the inductive step of proving the equivalent inequality

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^na_2a_3\cdots a_n.$$

Problem 3. Comparing with problem 6, I really found this problem harder to approach! Nevertheless there were still 8 contestants who completely solved the problem. Among them three were from the US team. That was an amazing achievement!

We can deal with this combinatorial probabilistic problem as follows. Ask repeatedly if x is 2^k. If A answers *no* k+1 times in a row, then the answer is honest and $x\neq 2^k$. Otherwise B stops asking about 2^k at the first time answer *yes*. He then asks, for each i=1,2,…,k, if the binary representation of x has a 0 in the i-th digit. Whatever the answer is, they are all inconsistent with a certain number y in the set $\{0,1,2,\cdots, 2^k-1\}$. The answer *yes* to 2^k is also inconsistent with y. Hence $x\neq y$. Otherwise the last k+1 answers are not honest and that is impossible. So we find y and it can be eliminated. Or we can eliminate corresponding numbers with nonzero digits at higher end. Notice we may need to do some re-indexing and asking more questions about the *indices* of the numbers subsequently. With these questions, we can reduce the size of the set that x lies until it lies in a set of size 2^k.

Part 2 makes use of a function so that using the function, A can devise a strategy (to lie or not to lie, but lying not more than k times consecutively) so that no extra information will be provided to B and hence B cannot eliminate anything for sure. Due to limit of space, I cannot provide all details here.

It was decided that part 1 answered correctly alone was worth 3 points and part 2 alone worth 5 points. But altogether a problem is worth at most 7 points. So 3 + 5 = 7! At the end it really did not matter. After all, not too many students did the problem right.

The problem is noted to be related to the Lovasz Local Lemma. See N. Alon et al, The Probabilistic Methods, Wiley, 1992. In the book it seems that there is an example that deals with similar things. One may check how the lemma and the problem are related!

Problem 4. Despite being regarded as an easy problem, this problem is not at all easy. It is much more involved than expected. Also this problem eventually caused more trouble because of the disputes about the marking scheme. First, by putting a=b=c=0, one gets $f(0)$=0. By putting $b=-a$ and c=0, one gets $f(a) = f(-a)$. More importantly, by putting $c=-(a+b)$ and solving $f(a+b) = f(-(a+b))$ as a quadratic equation of $f(a)$ and $f(b)$, one gets

$$f(a+b) = f(a) + f(b) \pm 2\sqrt{f(a)f(b)}.$$

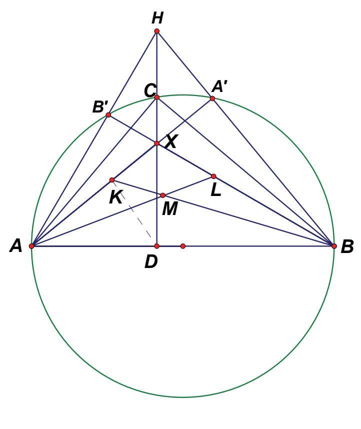
Putting a=b and $c=-2a$ into the original equation, one gets $f(2a)$=0 or $f(2a)$=4$f(a)$. Now the problem becomes getting all possible solutions from these two relations. Using the two conditions, one checks that there are four types of solution:

(i) $f_1(x)\equiv 0$, (ii) $f_2(x) = kx^2$,

(iii) $f_3(x) = \begin{cases} 0, x \text{ even} \\ k, x \text{ odd} \end{cases}$ and

(iv) $f_4(x) = \begin{cases} 0, x\equiv 0 \,(\mathrm{mod}\, 4) \\ k, x\equiv \pm 1 \,(\mathrm{mod}\, 4) \\ 4k, x\equiv 2 \,(\mathrm{mod}\, 4) \end{cases}.$

The "k" in the solutions is essentially $f(1)$. Indeed if $f(1)$=0, then $f(2)$=0, one then show by induction $f(x)$=0 for all x. (Or by showing $f(x)$ is periodic of period 1.) Now if $f(1)$=k, using the condition $f(2a)$=0, one can show again by induction $f(x)$ is k for x odd and is 0 for x even. Now if $f(1)$=k and $f(2)$=4k, then $f(4)$=0 or 16k. In the first case we get a function with period 4 and arrive at the solution $f_4(x)$. In the second case we get $f_2(x)$. (One needs to verify the details.) By checking the values of a, b and c mod 2 or 4, or other possible forms, one can check the solutions are indeed valid. Eventually if a contestant claimed that all the solutions are easy to check, but without checking, one point would be deducted. If a contestant says nothing about the solutions satisfy the functional equation and check nothing, then two points would be deducted!



Problem 5. The following solution was obtained by one of our team members.

Extend AX to meet the circumcircle of ABC at A', likewise extend BX to meet the circle at B'. Now extend AB' and BA' to meet at H, which is exactly the orthocentre of ABX and it lies on the extension of DC.

Since $BK^2 = BC^2 = BD{\cdot}BA$, we have $\Delta ABK{\sim}\Delta KBD$, so $\angle BKD = \angle BAK = \angle BHD$, which implies B, D, K, H concyclic. So $\angle BKH = \angle BDH$=90°. This implies $HK^2 = BH^2 - BK^2 = BH^2 - BD{\cdot}BA = BH^2 - BA'{\cdot}BH = HA'{\cdot}HB$. Similarly $HL^2 = HB'{\cdot}HA$. But $HA'{\cdot}HB = HB'{\cdot}HA$. Hence HK=HL. Using similar arguments as above, we have $\angle ALH = 90°$ ($=\angle BKH$.) Along with HK=HL, we see $\Delta MKH\cong\Delta MLH$. Therefore, MK=ML.

Problem 6. Clearing denominators of

$$\frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \ldots + \frac{n}{3^{a_n}} = 1,$$

one gets $x_1+2x_2+\cdots+nx_n=3^a$, where $x_1,x_2,\ldots,x_n$ are non-negative integer powers of 3. Taking mod 2, one gets $n(n+1)/2 \equiv 1 (\mathrm{mod}\ 2)$. This is the case only when $n \equiv 1, 2 \ (\mathrm{mod}\ 4)$. The hard part is to prove the converse also holds. The cases n=1 or 2 are easy. By trials, for n=5, $(a_1,\ldots,a_5)$=(2,2,2,3,3) works. The official solution gave a systematic analysis of how to obtain solutions by using identities $1/2^a=1/2^{a+1}+1/2^{a+1}$ and $w/3^a=u/3^{a+1}+v/3^{a+1}$, where $u+v=3w$. For $n=4k+1\geq 5$, one can arrive at the solution a_1=2=a_3, a_2=k+1, $a_{4k} = k+2 = a_{4k+1}$ and $a_m = [m/4]+3$ for $4\leq m<4k$. Similarly, for $n=4k+2\geq 6$, one can arrive at the solution $a_1 = 2$, $a_2 = k+1$, $a_3 = a_4 = 3$, $a_{4k+1} = k+2 = a_{4k+2}$ and $a_m = [(m-1)/4]+3$ for $4<m\leq 4k$. One can check these are indeed solutions by math induction on k. In the inductive steps of both cases, just notice a_2, a_{n-1}, a_n are increased by 1 so to balance the new a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4} terms.

This reminds me of the 1978 *USAMO* problem: an integer n is called good if we can write $n=a_1+a_2+\cdots+a_k$, where $a_1,a_2,\ldots,a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

Given 33 to 73 are good, prove that all integer greater than 33 are good. The idea there is to show if n is good, then $2n$+8 and $2n$+9 are good by dividing both sides of the above equation by 2 and adding the terms 1/4+1/4 and 1/3+1/6 respectively.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***September 20, 2012.***

Problem 396. Determine (with proof) all functions $f: \mathbb{R}\to\mathbb{R}$ such that for all real numbers x and y, we have

$$f(x^2+xy+f(y))=(f(x))^2+xf(y)+y.$$

Problem 397. Suppose in some set of 133 distinct positive integers, there are at least 799 pairs of relatively prime integers. Prove that there exist a,b,c,d in the set such that $\gcd(a,b)=\gcd(b,c)=\gcd(c,d)=\gcd(d,a)=1$.

Problem 398. Let k be positive integer and m an odd integer. Show that there exists a positive integer n for which the number n^n-m is divisible by 2^k.

Problem 399. Let ABC be a triangle for which $\angle BAC=60°$. Let P be the point of intersection of the bisector of $\angle ABC$ and the side AC. Let Q be the point of intersection of the bisector of $\angle ACB$ and the side AB. Let r_1 and r_2 be the radii of the incircles of triangles ABC and APQ respectively. Determine the radius of the circumcircle of triangle APQ in terms of r_1 and r_2 with proof.

Problem 400. Determine (with proof) all the polynomials $P(x)$ with real coefficients such that for every rational number r, the equation $P(x)=r$ has a rational solution.

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Solutions

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Problem 391. Let $S(x)$ denote the sum of the digits of the positive integer x in base 10. Determine whether there exist distinct positive integers a, b, c such that $S(a+b)<5$, $S(b+c)<5$, $S(c+a)<5$, but $S(a+b+c)>50$ or not.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **LI Jianhui** (CNEC Christian College, F.5), **LO Shing Fung** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (St. Paul's Co-educational College), **YUEN Wai Kiu** (St. Francis' Canossian College) and **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia).

Yes, we can try a=5,555,554,445 and b=5,554,445,555 and c=4,445,555,555. Then

$$S(a+b)=S(11{,}110{,}000{,}000)=4,$$
$$S(b+c)=S(10{,}000{,}001{,}110)=4,$$
$$S(c+a)=S(10{,}001{,}110{,}000)=4.$$

Finally,

$$S(a+b+c)=S(15{,}555{,}555{,}555)=51.$$

Other commended solvers: **Alice WONG** (Diocesan Girls' School), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 392. Integers $a_0, a_1, \cdots, a_n$ are all greater than or equal to -1 and are not all zeros. If

$$a_0+2a_1+2^2a_2+\cdots+2^na_n=0,$$

then prove that $a_0+a_1+a_2+\cdots+a_n>0$.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Harry NG Ho Man** (La Salle College, Form 5), **SHUM Tsz Hin** (City University of Hong Kong), **Alice WONG** (Diocesan Girls' School), **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia),**Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

For all the conditions to hold, $n\neq 0$. We will prove by mathematical induction. For n=1, if $a_0+2a_1=0$, then the conditions on a_0 and a_1 imply a_0 is an even positive integer. So $a_0+a_1=a_0/2>0$. Suppose the case $n=k$ is true. For the case $n=k+1$, the given equation implies a_0 is even, hence $a_0\geq 0$. So $a_0=2b$, with b a nonnegative integer. Then dividing the equation by 2 on both sides, we get that $(b+a_1)+2a_2+\cdots+2^ka_{k+1}=0$. From the cases $n=k$ and n=1 (in cases $a_2=\cdots=a_{k+1}=0$), we get $a_0+a_1+a_2+\cdots+a_n\geq(b+a_1)+a_2+\cdots+a_n>0$, ending the induction.

Problem 393. Let p be a prime number and $p\equiv 1 \pmod 4$. Prove that there exist integers x and y such that

$$x^2-py^2=-1.$$

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (St. Paul's Co-educational College), **Corneliu MĂNESCU-AVRAM** (Dept of Math, Transportation High School, Ploiesti, Romania), **Alice WONG** (Diocesan Girls' School) and **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia).

Let (m,n) be the fundamental solution (i.e. the least positive integer solution) of the Pell's equation $x^2-py^2=1$ (see Math Excal., *vol.* 6, *no.* 3, *p.*1). Then

$$m^2-n^2\equiv m^2-pn^2=1 \pmod 4.$$

Then m is odd and n is even. Since

$$\frac{m-1}{2}\cdot\frac{m+1}{2}=p\left(\frac{n}{2}\right)^2$$

and $(m-1)/2$, $(m+1)/2$ are consecutive integers (hence relatively prime), either

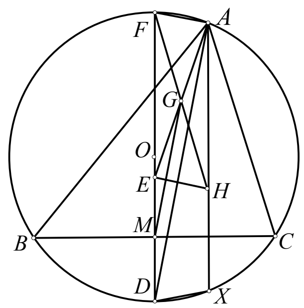
$$\frac{m-1}{2}=pu^2,\frac{m+1}{2}=v^2,n=2uv$$

or $$\frac{m-1}{2}=u^2,\frac{m+1}{2}=pv^2,n=2uv$$

for some positive integers u and v. In the former case, $v^2-pu^2=1$ with $0<v\leq v^2=(m+1)/2<m$ and $0<u=n/(2v)<n$. This contradicts the minimality of (m,n). So the latter case must hold, i.e. $u^2-pv^2=-1$.

Problem 394. Let O and H be the circumcenter and orthocenter of acute $\triangle ABC$. The bisector of $\angle BAC$ meets the circumcircle Γ of $\triangle ABC$ at D. Let E be the mirror image of D with respect to line BC. Let F be on Γ such that DF is a diameter. Let lines AE and FH meet at G. Let M be the midpoint of side BC. Prove that $GM\perp AF$.

Solution **1. AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Kevin LAU** (St. Paul's Co-educational College, S.3), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece), **Mihai STOENESCU** (Bischwiller, France), **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).



As AD bisects $\angle BAC$, D is the midpoint of arc BC. Hence, FD is the perpendicular bisector of BC. Thus, (1) $FE \parallel AH$. Let line AH meet Γ again at X. Since

$$\angle BCX = \angle BAX = 90^\circ - \angle ABC = \angle BCH,$$

H is the mirror image of X with respect to BC. Therefore, $\angle HED = \angle XDE = \angle AFE$. Thus, (2) $AF \parallel HE$. By (1) and (2), $AFEH$ is a parallelogram. Hence, G is the midpoint of AE. As M is also the midpoint of DE, we get $GM \parallel AD$. Since DF is the diameter of Γ, $AD \perp AF$, hence $GM \perp AF$.

***Solution* 2. Andy LOO** (St. Paul's Co-educational College).

Place the figure on the complex plane and let the circumcircle of $\triangle ABC$ be the unit circle centered at the origin. Denote the complex number representing each point by the respective lower-case letter. Without loss of generality we may assume $a = 1$ and that the points A, B and C lie on the circle in anticlockwise order. Let $b = u^2$ and $c = v^2$, where $|u| = |v| = 1$. Then $d = uv$ and hence $f = -uv$. Next, E is the mirror image of D with respect to BC means

$$\frac{e-b}{c-b} = \overline{\left(\frac{d-b}{c-b}\right)},$$

giving $e = u^2 - uv + v^2$. By the Euler line theorem, $h = a+b+c = 1+u^2+v^2$. Now G on lines AE and FH means

$$\frac{g-a}{e-a} = \frac{\overline{g}-\overline{a}}{\overline{e}-\overline{a}} \text{ and } \frac{g-f}{h-f} = \frac{\overline{g}-\overline{f}}{\overline{h}-\overline{f}}.$$

Solving these simultaneously for G, we get $g = (u^2 - uv + v^2 + 1)/2$. Also, $m = (b+c)/2 = (u^2+v^2)/2$.

To show $GM \perp AF$, it suffices to prove that $(m-g)/(f-a)$ is an imaginary number. Indeed, $\frac{m-g}{f-a} = \frac{1}{2}\cdot\frac{1-uv}{1+uv}$ and

$$\overline{\left(\frac{m-g}{f-a}\right)} = \frac{1}{2}\cdot\frac{1-\frac{1}{u}\cdot\frac{1}{v}}{1+\frac{1}{u}\cdot\frac{1}{v}} = \frac{1}{2}\cdot\frac{uv-1}{uv+1} = -\frac{m-g}{f-a}$$

as desired.

Other commended solvers: **Simon LEE** (Carmel Alison Lam Foundation Secondary School), and **Alice WONG** (Diocesan Girls' School).

Problem 395. One frog is placed on every vertex of a $2n$-sided regular polygon, where n is an integer at least 2. At a particular moment, each frog will jump to one of the two neighboring vertices (with more than one frog at a vertex allowed).

Find all n such that there exists a jumping of these frogs so that after the moment, all lines connecting two frogs at different vertices do not pass through the center of the polygon.

Solution. **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **LI Jianhui** (CNEC Christian College, F.5) and **Andy LOO** (St. Paul's Co-educational College).

If $n \equiv 2 \pmod 4$, say $n = 4k+2$, then label the $2n = 8k+4$ vertices from 1 to $8k+4$ in clockwise direction. For $j \equiv 1$ or $2 \pmod 4$, let the frog at vertex j jump in the clockwise direction. For $j \equiv 3$ or $4 \pmod 4$, let the frog at vertex j jump in the counter-clockwise direction. After the jump, the frogs are at vertices 2, 6, …, $8k+2$ and 3, 7, …, $8k+3$. No two of these vertex numbers have a difference of the form $2 \pmod 4$. So no line through two different vertices with frogs will go through the center.

If $n \not\equiv 2 \pmod 4$, then assume there is such a jump. We may exclude the cases all frogs jump clockwise or all frogs jump counter-clockwise, which clearly do not work. Hence, in this jump, there is a frog, say at vertex i, jumps in the counter-clockwise direction, then the frog at vertex $i+m(n-2) \pmod{2n}$ must jump in the same direction as the frog at vertex i for $m = 1, 2, \ldots$.

If n is odd, then $\gcd(n-2, 2n) = 1$. So there are integers a and b such that $a(n-2) + b(2n) = 1$. For every integer q in $[1, 2n]$, letting $m = (q-i)a$, we have $i+m(n-2) \equiv q \pmod{2n}$. This means all frogs jump in the counter-clockwise direction, which does not work.

If n is divisible by 4, then $\gcd(n-2, 2n) = 2$. So there are integers c and d such that $c(n-2) + d(2n) = 2$. Letting $m = nc/2$, we have $i+m(n-2) \equiv i+n \pmod{2n}$. Then frogs at vertices i and $i+n$ jump in the counter-clockwise direction and the line after the jump passes through the center, contradiction.

Therefore, the answer is $n \equiv 2 \pmod 4$.

Other commended solvers: **Alice WONG** (Diocesan Girls' School).

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

1. If $n \ge 2^k$, then B can guarantee a win.
2. For all sufficiently large k, there exists an integer $n \ge 1.99^k$ such that B cannot guarantee a win.

Problem 4. Find all functions $f: Z \to Z$ such that, for all integers a, b, c that satisfy $a+b+c=0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here Z denotes the set of integers.)

Problem 5. Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK. Show that $MK = ML$.

Problem 6. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \ldots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \ldots + \frac{n}{3^{a_n}} = 1.$$

Mathematical Excalibur

Volume 17, Number 2 *October 2012*

Olympiad Corner

Below are the problems of the 2012 IMO Team Selection Test 1 from Saudi Arabia.

Problem 1. In triangle ABC, points D and E lie on sides BC and AC respectively such that $AD\perp BC$ and $DE\perp AC$. The circumcircle of triangle ABD meets segment BE at point F (other than B). Ray AF meets segment DE at point P. Prove that $DP/PE = CD/DB$.

Problem 2. In an $n\times n$ board, the numbers 0 through n^2-1 are written so that the number in row i and column j is equal to $(i-1)+n(j-1)$ where $1\le i,j \le n$. Suppose we select n different cells of the board, where no two cells are in the same row or column. Find the maximum possible product of the numbers in the n cells.

Problem 3. Let $\mathbb{Q}$ be the set of rational numbers. Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that for all rational numbers x,y,

$$f(f(x)+ x f(y)) = x + f(x)\, y.$$

Problem 4. Find all pairs of prime numbers p, q such that $p^2-p-1 = q^3$.

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 20, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2012 (Member Perspective)

Andy Loo

This year's International Mathematical Olympiad (IMO) has been of considerable significance to Hong Kong. At the 1997 IMO held in Mar del Plata, Argentina, shortly after our official transfer of sovereignty, the Hong Kong delegation accomplished the special mission of elucidating Article 149 of its *Basic Law* in light of Annex I of the *Sino-British Joint Declaration*, thereby consolidating the legitimacy of its participation in the IMO. This July, following the 15th anniversary of the establishment of the Special Administrative Region, this annual event returns to Argentina, in exactly the same city as last time's. In addition to battling in the examination hall, the Hong Kong team was endowed with the invigorating task of bringing the IMO to Hong Kong again in 2016.

Joined by 542 young brains from 99 countries, the Hong Kong team comprised the following personnel: Dr. Leung Tat Wing (leader), Mr. Leung Chit Wan (deputy leader) and the team members were Kevin Lau Chun Ting (St. Paul's Co-educational College), Andy Loo (St. Paul's Co-educational College), Albert Li Yau Wing (Ying Wah College), Jimmy Chow Chi Hong (Bishop Hall Jubilee School), Kung Man Kit (SKH Lam Woo Memorial Secondary School) and Alice Wong Sze Nga (Diocesan Girls' School).

This contest bestows certain personal touch upon me, for it not only marks my unprecedented landing on the continent of South America, but is also my first and, in all probability, my last IMO, an ultimate platform for me to display my years of Mathematical Olympiad endeavor in my high school career. Having represented Hong Kong at both the International Physics Olympiad (IPhO) and the IMO is a great responsibility which I feel extremely grateful to have had the unique chance to shoulder.

<u>**July 7 and 8**</u> Our flights from Hong Kong to Frankfurt and from Frankfurt to Buenos Aires, each over 12 hours long, were predominantly occupied by sleep and math exercises, considering the disappointing fact that our planes turned out to be two of the very few Boeing-74748 models of Lufthansa that lack in-flight entertainment systems. Our amazement at a German flight attendant, who spoke more than fluent Mandarin Chinese, as well as a cozy conversation with a Slovakian neighbor, highlighted the otherwise uneventful journey.

We arrived at the Argentinean capital city early in the morning of July 8 (in winter!), and, after being transported to the domestic airport, employed a time-consuming conglomeration of *Google Translate* effort and sign language to manage to purchase a couple of SIM cards at a tiny store, where the shopkeeper knew literally no English. A Maradona-like bus driver kindly offering us a free ride, we embarked on a tour around the city and enjoyed a beef-dominated meal before returning to the airport in the late afternoon to catch our flight to Mar del Plata, on which I, being absolutely exhausted, slept from the first to the last minute.

<u>**July 9**</u> The major event of this day was the Opening Ceremony. It was held in the Radio City. I met British team member Josh Lam and congratulated him on his mother's recent promotion to Chief Secretary of Hong Kong. If I were to describe the entire ceremony in one word it would definitely be "Spanish". Almost all the speeches were delivered in Spanish, albeit accompanied by English interpretation. To most, the more exciting parts of the ceremony included the IMO anthem, the parade of nations and the distant waves from the leaders, who were forbidden to communicate with us before the contest as they took part in problem selection.

July 10 On this first day of the contest, we had 3 problems to solve in 4.5 hours. Because questions could only be raised in the first 30 minutes, I had to understand all the problems quickly.

Problem 1 Given triangle ABC the point J is the center of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

I decided to use my favorite method – complex numbers. Indeed, denote the complex number representing each point by the corresponding small letter. Setting j=0 and m=1, I found s=2k/(k+l) and t=2l/(k+l) after a straightforward computation, and the result followed.

Problem 2 Let $n \geq 3$ be an integer, and let $a_2,a_3,\ldots,a_n$ be positive real numbers such that $a_2a_3\cdots a_n=1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$

Inequalities were once among the hottest topics on the IMO but totally disappeared in the last three years due to the rising popularity of brute force techniques, e.g. Muirhead's inequality and Schur's inequality. But my firm belief in the revival of inequalities has never been shaken, and instead was only strengthened by Problem 5 of APMO 2012. Consequently I had done appreciable preparation in this area before the Olympiad.

In IMO history, this problem was quite unique. For one, it is an n-variable inequality. For the other, it has no equality case. Both features are unparalleled according to my memory.

I spent about an hour attempting to solve the problem using induction or analysis, with no avail. In despair, I took logarithm and applied Jensen's inequality by appealing to concavity of the log function. Miraculously, it gave precisely the inequality in the problem! After checking that equality case cannot satisfy the condition $a_2a_3\cdots a_n$ =1, I was basically done.

Then on a second thought, I realized that I could actually convert my proof into a logarithm-free one that involves the AM-GM inequality only. So I rewrote my solution in this new form and marked the original as an alternative solution. It turned out that Alice was also able to solve this problem with the AM-GM inequality.

Problem 3 The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any k+1 consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

1. If $n \geq 2^k$, then B can guarantee a win.
2. For all sufficiently large k, there exists an integer $n \geq 1.99^k$ such that B cannot guarantee a win.

This problem was not only long, but also terribly difficult. In the end, only 8 contestants managed to solve it. Despite my effort, the only thing I was able to do was proving the k = 1 case in Part 1, with the hope of getting slim partial credits.

Finally Day 1 of the contest was over. Our team aced Problem 1. As for Problem 2, Alice and I should be able to get 7's while Albert's partial analytic solution would be subject to vigorous debate. Kit also finished the k = 1 case in Part 1 of Problem 3. Overall I was satisfied with my performance on Day 1.

July 11 The six IMO problems are usually partitioned into the four categories (algebra, combinatorics, geometry and number theory) in the fashion of {1,5}, {2,4}, {3} and {6} (up to permutation). Judging from this pattern I would face an easy algebraic problem, an intermediate geometric problem and a hard number theoretic problem on Day 2. I figured that I would plausibly get a Gold medal for solving two of them, a Silver medal for one and a Bronze medal for none. My strategy was to guarantee Problem 4 and then aim to get Problem 5 by hook or by crook.

To my astonishment, Problem 4 was much more involved than I had expected. On the other hand I felt I could do Problem 5 with analytic tools:

Problem 5 Let ABC be a triangle with $\angle BCA$ = 90°, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that BK=BC. Let L be the point on the segment BX such that AL=AC. Let M be the point of intersection of AL and BK. Show that MK=ML.

I proceeded to do coordinate geometry, only to find out I was doomed after almost one hour. The reason was as follows. The expressions were quadratic in nature (as lengths took part in the formulation of the problem), leading to the prevalence of square roots. (As a side note, this also deterred me from using complex numbers, where one may have difficulty in selecting the correct roots of the quadratic equations.)

As the old Chinese saying goes, one should "drop his cleaver and become a Buddha (放下屠刀，立地成佛)". I decided to abandon Problem 5 for a moment and to reconsider Problem 4:

Problem 4 Find all functions f: $Z \rightarrow Z$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$\begin{aligned} &f(a)^2+f(b)^2+f(c)^2 \\ &=2f(a)f(b)+2f(b)f(c)+2f(c)f(a). \end{aligned}$$

(Here Z denotes the set of integers.)

This was a problem with unusual answers. It took me quite a while to write up a tidy solution and to ensure that no point could sneak away from my hands. Thus it was 2.5 hours into Day 2. I still had Problems 5 and 6 left.

Problem 6 Find all positive integers n for which there exist non-negative integers $a_1,a_2,\ldots,a_n$ such that

$$\frac{1}{2^{a_1}}+\frac{1}{2^{a_2}}+\ldots+\frac{1}{2^{a_n}}=\frac{1}{3^{a_1}}+\frac{2}{3^{a_2}}+\ldots+\frac{n}{3^{a_n}}=1.$$

I quickly determined that Problem 6 was hopeless. Turning to Problem 5 again, I spent all the remaining time expanding everything. I was finally able to convince myself that my proof was complete.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***November 20, 2012.***

Problem 401. Suppose all faces of a convex polyhedron are parallelograms. Can it have exactly 2012 faces? Please provide an explanation to your answer.

Problem 402. Let S be a 30 element subset of $\{1,2,\ldots,2012\}$ such that every pair of elements in S are relatively prime. Prove that at least half of the elements of S are prime numbers.

Problem 403. On the coordinate plane, 1000 points are randomly chosen. Prove that there exists a way of coloring each of the points either red or blue (but not both) so that on every line parallel to the x-axis or y-axis, the number of red points minus the number of blue points is equal to -1, 0 or 1.

Problem 404. Let I be the incenter of acute $\triangle ABC$. Let Γ be a circle with center I that lies inside $\triangle ABC$. D, E, F are the intersection points of circle Γ with the perpendicular rays from I to sides BC, CA, AB respectively. Prove that lines AD, BE, CF are concurrent.

Problem 405. Determine all functions f,g: $(0,+\infty) \to (0,+\infty)$ such that for all positive number x, we have

$$f(g(x)) = \frac{x}{xf(x)-2} \text{ and } g(f(x)) = \frac{x}{xg(x)-2}.$$

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Solutions

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Problem 396. Determine (with proof) all functions f: $\mathbb{R}\to\mathbb{R}$ such that for all real numbers x and y, we have

$$f(x^2+xy+f(y)) = (f(x))^2 + xf(y) + y.$$

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **CHEUNG Wai Lam** (Queen Elizabeth School), **Dusan DROBNJAK** (Mathematical Grammar School, Belgrade, Serbia), **Kevin LAU** (St. Paul's Co-educational College, S.4), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Mohammad Reza SATOURI** (Persian Gulf University, Bushehr, Iran) and **Maksim STOKIĆ** (Mathematical Grammar School, Belgrade, Serbia).

Call the required equation (*). For x=0, we get $f(f(y))=y+f(0)^2$ for all y. Call this (**). The right side may be any real number, hence f is surjective. By (**), $y = f(f(y)) - f(0)^2$. If $f(y) = f(y')$, then the last equation implies $y=y'$, i.e. f is injective.

Putting $x=-y$ in (*), we get $f(f(y)) = (f(-y))^2 - yf(y)+y$ for all y. Call this (***).

Now f surjective implies there exists z such that $f(z)=0$. Let $x=y=z$, then (*) yields $f(2z^2)=z$. Putting $(x,y)=(0,2z^2)$ in (*), we get $0=2z^2+f(0)^2$. Then $z=0$ and $f(0)=0$. So (**) reduces to $f(f(y))=y$ for all y. Putting $y = 0$ in (*), since $f(0) = 0$, we get $f(x^2) = (f(x))^2$. The last two sentences reduce (***) to $y = (f(y))^2 - yf(y)+y$. This simplifies to $f(y) = 0$ or $f(y) = y$ for every y. Since f is injective and $f(0) = 0$, we get $f(y) = y$ for all y. Conversely, a quick check shows $f(y) = y$ for all y satisfies (*).

Other commended solvers: **Tobi MOEKTIJONO** (National University of Singapore).

Problem 397. Suppose in some set of 133 distinct positive integers, there are at least 799 pairs of relatively prime integers. Prove that there exist a,b,c,d in the set such that $\gcd(a,b) = \gcd(b,c) = \gcd(c,d) = \gcd(d,a) = 1$.

Solution. **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Dusan DROBNJAK** (Mathematical Grammar School, Belgrade, Serbia), **Kevin LAU** (St. Paul's Co-educational College, S.4), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (Princeton University), **Tobi MOEKTIJONO** (National University of Singapore) and **Maksim STOKIĆ** (Mathematical Grammar School, Belgrade, Serbia).

Let $S=\{n_1, n_2,\ldots,n_{133}\}$ be the set of these 133 positive integers. From i=1 to 133, let X_i be the set of all n_k in S such that $k\neq i$ and $\gcd(n_i,n_k)=1$. Denote by $|X|$ the number of elements in set X. For $k\neq i$, $\gcd(n_i,n_k)=1$ implies $n_i\in X_k$ and $n_k\in X_i$. Then $N = |X_1| + |X_2| + \cdots + |X_{133}| \geq 2\times799 = 1598$.

Define $f(x) = x(x-1)/2$. In a set X with j elements, there are exactly $j(j-1)/2 = f(|X|)$ pairs of distinct elements. Since $f(x)$ is concave on $\mathbb{R}$, by Jensen's inequality,

$$\sum_{i=1}^{133} f(|X_i|) \geq 133 f\left(\frac{N}{133}\right) \geq 133 f\left(\frac{1598}{133}\right)$$

$$> 133 f(12) = f(133) = f(|S|).$$

Since every pair of distinct element in X_i is also a pair of distinct element in S, the inequality above implies in counting pairs of distinct elements in the X_i's, there are repetitions, i.e. there are X_i, X_k with $i\neq k$ sharing a common pair of distinct elements a,c. Let $b=n_i$ and $d=n_k$. Then a,b,c,d satisfy $\gcd(a,b) = \gcd(b,c) = \gcd(c,d) = \gcd(d,a) = 1$.

Problem 398. Let k be positive integer and m an odd integer. Show that there exists a positive integer n for which the number $n^n - m$ is divisible by 2^k.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Dusan DROBNJAK** (Mathematical Grammar School, Belgrade, Serbia), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (Princeton University), **Tobi MOEKTIJONO** (National University of Singapore) and **Maksim STOKIĆ** (Mathematical Grammar School, Belgrade, Serbia).

For k=1, let n=1. Suppose it is true for case k (i.e. there exists n such that $2^k \mid n^n - m$). Now m odd implies n odd. For case k+1, if $2^{k+1} \mid n^n - m$, then the same n works for k+1. Otherwise, $n^n - m = 2^k l$ for some odd integer l. Let $v=2^k$. By binomial theorem,

$$\begin{aligned}(n+v)^{n+v} &= n^{n+v} + (n+v)n^{n+v-1}v + v^2x \\ &= n^{n+v} + vn^{n+v} + v^2y\end{aligned}$$

for some integers x,y. By Euler's theorem, since n is odd and $\varphi(2^{k+1})=2^k$,

$$n^v = n^{2^k} \equiv 1 \pmod{2^{k+1}}.$$

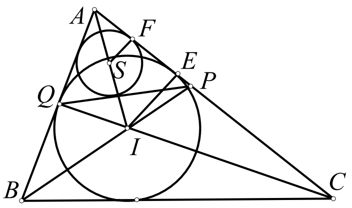
Since $l+n^n$ is even, we have

$$\begin{aligned}(n+v)^{n+v} &= n^{n+v} + vn^{n+v} + v^2y \\ &\equiv n^n + 2^k n^n = m + 2^k(l+n^n) \\ &\equiv m \pmod{2^{k+1}}.\end{aligned}$$

So $n+v$ works for k+1.

Problem 399. Let ABC be a triangle for which $\angle BAC=60°$. Let P be the point of intersection of the bisector of $\angle ABC$ and the side AC. Let Q be the point of intersection of the bisector of $\angle ACB$ and the side AB. Let r_1 and r_2 be the radii of the incircles of triangles ABC and APQ respectively. Find the radius of the circumcircle of triangle APQ in terms of r_1 and r_2 with proof.

Solution. **Dusan DROBNJAK** (Mathematical Grammar School, Belgrade, Serbia), **Kevin LAU** (St. Paul's Co-educational College, S.4), **Andy LOO** (Princeton University), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece), **Tobi MOEKTIJONO** (National University of Singapore) and **Maksim STOKIC** (Mathematical Grammar School, Belgrade, Serbia).



Let I and S be the incenters of $\triangle ABC$ and $\triangle APQ$ respectively. (Note A,S,I are on the bisector of $\angle BAC$.) Now $\angle PIQ = \angle CIB = 180^\circ - (\angle CBI + \angle BCI) = 180^\circ - \frac{1}{2}(\angle CBA + \angle BCA) = 120^\circ$ using $\angle BAC = 60^\circ$. So $APIQ$ is cyclic.

Applying sine law to ΔAPI, we get $IP/(\sin\angle IAP) = 2R$. So $R = IP$. By a well-known property of incenter, we have $IP=IS$ (see *vol.11*, *no.2*, *p.1* of Math Excal.). Let the incircles of ΔABC and ΔAPQ touch AC at E and F respectively. Then $R=IP=IS = AI-AS = IE/\sin 30^\circ - SF/\sin 30^\circ = 2r_1 - 2r_2$.

Other commended solvers: **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramure, Romania), **Simon LEE** (Carmel Alison Lam Foundation Sec. School) and **Mihai STOENESCU** (Bischwiller, France).

Problem 400. Determine (with proof) all the polynomials $P(x)$ with real coefficients such that for every rational number r, the equation $P(x) = r$ has a rational solution.

Solution. **Tobi MOEKTIJONO** (National University of Singapore), **Maksim STOKIC** (Mathematical Grammar School, Belgrade, Serbia) and **TAM Ka Yu** (MIT).

We will show $P(x)$ satisfies the desired condition if and only if $P(x)=ax+b$, where $a,b \in \mathbb{Q}$ and $a \neq 0$. For the *if*-part, $P(x) = r \in \mathbb{Q}$ implies $x = (r-b)/a \in \mathbb{Q}$.

Conversely, let $P(x)$ satisfy the desired condition and let $n=\deg P$. For each $r = 0,1,\ldots, n$, let $P(x_r)=r$ for some $x_r \in \mathbb{Q}$. By the Lagrange interpolation formula,

$$P(x) = \sum_{r=0}^{n} \left(r \prod_{0\le r\le n, r\ne i} \frac{x - x_i}{x_r - x_i} \right).$$

Expanding the right side, we see $P(x)$ has rational coefficients.

Letting M be the product of the denominators, we see $Q(x)=MP(x)$ has integer coefficients. Let k be the leading coefficient of $Q(x)$ and c be the constant term of $P(x)$. Let p_1, p_2, p_3, … be the sequence of prime numbers. Let $P(x)=c+p_i/M$ has solution $t_i \in \mathbb{Q}$. Then $Q(x)-(cM+p_i)$ has k as the leading coefficient and $-p_i$ as constant term. Now $Q(t_i)=0$, which implies $t_i=1/d_i$ or p_i/d_i for some (not necessarily positive) divisor of k. Since $P(t_i)$'s are distinct, so the t_i's are distinct. Hence, $t_i=1/d_i$ for at most as many time as the number of divisors of k. So there must exist a divisor d of k such that there are infinitely many times $t_i=p_i/d$. This imply that $P(x)-(c+dx/M)=0$ has infinitely many solutions. So the left side is the zero polynomial. Then $P(x)=ax+b$ with $a=d/M\neq 0$ and $b=c$ rational.

Other commended solvers: **Simon LEE** (Carmel Alison Lam Foundation Secondary School).

IMO 2012 (Member Perspective)

(continued from page 2)

The arrival of Dr. Leung stirred up much happiness after the contest. We reported on how we did. Albert and Jimmy shone on Day 2, solving Problems 4 and 5. Kit was also comfortable with Problem 4 while Kevin had some technical troubles in one particular case. Nobody achieved anything substantial on Problem 6.

We celebrated that evening at a Chinese restaurant. It was especially memorable that our deputy leader raised a couplet (對聯), which he regarded as an open puzzle for millenniums (千古絕對):

望江樓，望江流
望江樓上望江流
江樓千古，江流千古

It took me nearly an hour to come up with a so-so solution:

觀雨亭，觀雨停
觀雨亭下觀雨停
雨亭四方，雨停四方

July 12 It was the contestants' turn to have fun and the leaders' turn to work hard. At night, Dr. Leung briefed us on the progress of the first day of coordination. In addition to our previous expectations, Albert pocketed one point for proving the necessary condition on Problem 6. Regretfully, Kit lost one point on Problem 4 for not having verified the feasibility of the functions obtained. Dr. Leung had refused to sign Alice's and Kevin's scores on Problem 4 in order to bargain later.

July 13 The marking scheme stipulated that any solution of Problem 5 with coordinate geometry would score a 0 if not a 7. Despite our leaders' relentless effort, the coordinators were able to detect a fatal error of mine. So my Problem 5 was destined to be a 0.

On another note, Dr. Leung succeeded in getting 1 point for Alice on Problem 4, which in his words was "an achievement". Kevin's Problem 4 was finalized with a score of 4.

July 14 We got up early in the morning to enjoy the sunrise scene at the seaside. Kevin had a pitiful blunder. His shoes and trousers were wetted by a sudden strike of waves. That morning the last coordination on Problem 2 was done. Albert was awarded 3 marks for his analytic struggle. The uncertainties of our results then shifted from our actual scores to the medal cutting scores.

We went shopping for souvenirs in the afternoon and as soon as we got back to the hotel, I learned from the Chinese leaders that the cutting scores for Gold, Silver and Bronze Medals were 28, 21 and 14 respectively, all being multiples of 7. I breathed a sigh of relief as my Silver Medal was ultimately secure.

July 15 In the afternoon we had the Closing Ceremony followed by a chain of photo-taking. We won three Silver Medals (Albert, Jimmy and me), one Bronze Medal (Alice) and two Honorable Mentions (Kit and Kevin).

July 16, 17 and 18 The six-hour bus journey from Mar del Plata to Buenos Aires passed rapidly in our dreams. Then after a long flight, we were finally home in one piece and me with several bonus pimples.

In conclusion I shall stress one point – succinctly but with all the strength that I command – one can never pay sufficient tribute to our IMO trainers, who have so selflessly devoted countless hours of their own time to Mathematical Olympiad over the years. I can find no words to thank them the way they truly deserve.

"Ask not what your country can do for you; ask what you can do for your country." With this John F. Kennedy exclamation I urge you all to support the 2016 Hong Kong IMO by whatever means you can, so that together we can make it an all-time success.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 15th Hong Kong China Math Olympiad.

Problem 1. For any positive integer n, let a_1, a_2, …, a_m be all the positive divisors of n, where $m\geq 1$. If there exist m integers b_1, b_2, …, b_m such that

$$n=\sum_{i=1}^{m}(-1)^{b_i}a_i,$$

then we say that n is a *good number*. Prove that there exists a good number with exactly 2013 distinct prime factors.

Problem 2. Some of the lattice points (x,y), with $1\leq x\leq 101$ and $1\leq y\leq 101$ are marked so that no 4 marked points form the vertices of an isosceles trapezoid with bases parallel to the x-axis or the y-axis (a rectangle is counted as an isosceles trapezoid). Determine the maximum number of marked points. (A lattice point is a point with integral coordinates.)

Problem 3. Prove that for every positive integer n and every group of real numbers a_1, a_2, …, $a_n>0$,

$$\sum_{k=1}^{n}\frac{k}{a_1^{-1}+a_2^{-1}+\ldots+a_k^{-1}}\leq 2\sum_{k=1}^{n}a_k.$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 3, 2013***.

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Primes in Arithmetic Progressions

Kin Y. Li

To see there are infinitely many prime numbers, we assume only finitely many of them exist, say $p_1,p_2,\ldots,p_m$. Consider $q=p_1p_2\cdots p_m+1$. Let p be a prime in the prime factorization of q. Then p is one of the p_i's. So p divides q and $q-1$. Then p divides $q-(q-1)=1$, contradiction.

Other than 2, the rest of the prime numbers are in the arithmetic progression $2n+1$, where n denotes a positive integer. It is natural to ask how many prime numbers are in the other arithmetic progressions $an+b$, where a and b are given integers with $a>0$. Certainly, if $\gcd(a,b)>1$, then no primes will be in the sequence $an+b$.

In case $(a,b)=(4,-1)$ we can see the answer is infinitely many by modifying the proof above. Assume $p_1,p_2,\ldots,p_m$ are all the primes of the form $4n-1$. Then let $q=4p_1p_2\cdots p_m-1$. Now $q\equiv -1$ (mod 4). Assume q is a product of primes in the sequence $4n+1$. Then $q\equiv 1$ (mod 4), contradiction. So q must have at least one prime divisor p in the sequence $4n-1$. Then p is one of the p_i's. So p divides q and $q+1$. Then p divides $(q+1)-q=1$, contradiction.

In case $(a,b)=(p,1)$, where p is a prime, we will need facts from number theory.

Fact 1 (Bezout's Theorem). For all positive integers a and b, there exist integers r and s such that $ar+bs=\gcd(a,b)$.

Fact 2 (Euler's Theorem). For positive integer n, let $\varphi(n)$ be the number of integers among $1,2,\ldots,n$ that is relatively prime to n. If $\gcd(a,n)=1$, then $a^{\varphi(n)}\equiv 1$ (mod n). In case n is a prime, we have $\varphi(n)=n-1$ and $a^{n-1}\equiv 1$ (mod n). This case is *Fermat's Little Theorem*.

Example 1 (*2004 Korean Mathematical Olympiad*). Let p be a prime and $f_p(x)=x^{p-1}+x^{p-2}+\cdots+x+1$.

(1) For each integer m divisible by p, is there an integer q such that q divides $f_p(m)$ and $\gcd(q,m(m-1))=1$?

(2) Prove that there are infinitely many integers n such that $pn+1$ is prime.

Solution. (1) Yes. Let q be a prime divisor of $f_p(m)$. As $f_p(m)\equiv 1$ (mod m), we see q does not divide m. Hence $\gcd(m,q)=1$. Assume $m\equiv 1$ (mod q). Then $0\equiv f_p(m)\equiv p$ (mod q), which implies $p=q$. Since p divides m, we get $1\equiv f_p(m)\equiv p$ (mod p), contradiction. Hence q does not divide $m-1$. Then $\gcd(q,m(m-1))=1$.

(2) Assume $p_1,p_2,\ldots,p_k$ are all the primes of the form $pn+1$. Let $m=p_1p_2\cdots p_k\,p$ and q be a prime divisor of $f_p(m)$. By (1), $m\not\equiv 0$ or 1 (mod q), which implies $\gcd(m,q)=1$. By Fermat's little theorem, $m^{q-1}\equiv 1$ (mod q). Now $m^p-1=(m-1)\,f_p(m)$ implies $m^p\equiv 1$ (mod q).

Assume $\gcd(q-1,p)=1$. By Bezout's theorem, there are integers r and s such that $(q-1)r+ps=1$. Then $m=m^{(q-1)r}m^{ps}\equiv 1$ (mod q), contradicting the last underlined expression. Then $\gcd(q-1,p)=p$, i.e. q is of the form $pn+1$. As q divides $f_p(m)$ and $f_p(m)\equiv 1$ (mod p_i), we see $q\neq p_1,p_2,\ldots,p_k$.

In the general case $\gcd(a,b)>1$, we have

Dirichlet's Theorem. If a and b are given integers with $a>0$ and $\gcd(a,b)>1$, then there are infinitely many primes in the arithmetic progression $an+b$.

All known proof of this theorem is beyond the scope of secondary school curriculum. Below we will look at some examples. First we need more facts.

Fact 3 (Chinese Remainder Theorem). If k_1, k_2, …, k_n are pairwise relatively prime positive integers and c_1, c_2, …, c_n are integers, then there exist a unique integer x in the interval $[1, k_1k_2\cdots k_n]$ such that $x\equiv c_i$ (mod k_i) for $i=1,2,\ldots,n$.

Fact 4 (Wilson's Theorem). If p is a prime, then $(p-1)!\equiv -1$ (mod p).

At the end of the article, we will give explanations for facts 1 to 4.

Example 2 (*1996 St Petersburg Math Olympiad*) Prove that there are no positive integers a and b such that for each pair p, q of distinct primes greater than 1000, the number $ap+bq$ is also prime.

Solution. Assume such a and b exist. Let r be a prime number with $\gcd(r,a)=\gcd(r,b)=1$. By Dirichlet's theorem, there exist positive integers x and y such that $p=rx+b$ and $q=ry-a$ are prime numbers greater than 1000. Then $ap+bq=(ax+by)r$ is not prime, contradiction.

Example 3 (*1997 British Mathematical Olympiad*) Let $S=\{1/r : r=1,2,3,\ldots\}$. For all integer $k>1$, prove that there is a k-term arithmetic progression in S such that no addition term in S can be added to it to form a $(k+1)$-term arithmetic progression.

Solution. By Dirichlet's theorem, there exists a positive integer n such that $kn+1$ is prime. Let $a_1=1/(kn)!$ and $d=n/(kn)!$. For $i=2,\ldots,k$, $a_i=a_1+(i-1)d=(1+(i-1)n)/(kn)!$ are in S. However, the term $a_{k+1}=a_1+kd=(kn+1)/(kn)!$ is not in S since $kn+1$ is a prime. So $a_1, a_2, \ldots, a_k$ is such an example.

Example 4 Prove that for every positive integer s, a, b with $\gcd(a,b)=1$, there are infinitely many integers n such that $an+b$ is a product of s pairwise distinct prime numbers.

Solution. The case $s=1$ is Dirichlet's theorem. Suppose the case s is true. Then there exists an integer N such that $aN+b=q_1q_2\cdots q_s$, where $q_1, q_2, \ldots, q_s$. are pairwise distinct primes. Next, by Dirichlet's theorem, there exist infinitely many positive integers n such that $an+1$ is a prime greater than all of $q_1, q_2, \ldots, q_s$. Let $t_n=q_1q_2\cdots q_sn+N$. Then $at_n+b = aq_1q_2\cdots q_sn+aN+b = q_1q_2\cdots q_s(an+1)$ is a product of $s+1$ pairwise distinct prime numbers. This completes the induction.

Example 5 (*2011 Mongolian Math Olympiad Team Selection Test*) Let m be a positive odd integer. Prove that there exist infinitely many positive integer n such that $(2^n-1)/(mn+1)$ is an integer.

Solution. By Dirichlet's theorem, there exist infinitely many primes $p>m$ and $p=\varphi(m)k+1$ for some positive integer k. By Euler's theorem, $2^{\varphi(m)}\equiv 1 \pmod{m}$. Then

$$2^p=2^{\varphi(m)k+1}\equiv 2 \pmod{m}.$$

This leads to $n=(2^p-2)/m$ is an integer. By Fermat's little theorem, p divides 2^p-2. Since $p>m$, we see p divides n. Then $mn+1=2^p-1$ divides 2^n-1. Therefore, $(2^n-1)/(mn+1)$ is an integer.

Example 6 (*American Math Monthly 4772*) Let p_k be the k-th prime number. For every integer N, prove that there exists a positive integer k such that both p_{k-1} and p_{k+1} are not in the interval $[p_k-N, p_k+N]$.

Solution. Let q be a prime number greater than $N+2$. Observe that $a=q!$ and $b=(q-1)!-1$ are relatively prime because the prime divisors of $q!$ are the primes less than or equal to q, however $(q-1)!-1$ is not divisible by any prime number less than q and $(q-1)!-1\equiv -2 \pmod{q}$ by Wilson's theorem.

By Dirichlet's theorem, there is a prime $p_k\equiv(q-1)!-1 \pmod{q!}$. Then $p_k+1\equiv 0 \pmod{(q-1)!}$. Also, by Wilson's theorem, $p_k+2\equiv(q-1)!+1\equiv 0 \pmod{q}$. These showed p_k+1 and p_k+2 are not primes. For $j=2,\ldots,q-1$, we have

$$p_k+1\pm j\equiv p_k+1\equiv(q-1)!\equiv 0 \pmod{j}.$$

So integers in $[p_k-q+2, p_k+q]$ except p_k are not primes. Since $q>N+2$, both p_{k-1} and p_{k+1} cannot be in the $[p_k-N, p_k+N]$.

Example 7 (*American Math Monthly E1632*) Prove that if $f(x)$ is a polynomial with rational coefficients such that $f(p)$ is a prime number for every prime number p, then either $f(x)=x$ for all x or $f(x)$ is the same prime constant for all x.

Solution. Assume the conclusion is false. Let k be the least common multiple of the denominators of the coefficients of $f(x)$ and let $g(x)=kf(x)$. Then $g(x)$ has integer coefficients. Now there must be a prime p such that p and $g(p)$ are relatively prime (otherwise, for the infinitely many primes p that are relatively prime to k, we have $\gcd(p,g(p))=p$, so p divides $g(p)=kf(p)$, hence both primes $f(p)$ and p are equal, which forces $f(x)=x$).

By Dirichlet's theorem, there are infinitely many integers n_i such that $m_i=g(p)n_i+p$ is prime. Now $g(m_i)\equiv g(p)\equiv 0 \pmod{g(p)}$ for all i. Then $kf(p)$ divides $kf(m_i)$. Hence $f(p)$ divides $f(m_i)$. Since $f(p)$ and $f(m_i)$ are primes, we get $f(m_i)=f(p)$ for infinitely many i. This leads to $f(x)$ being the constant polynomial $f(p)$, contradiction.

Example 8 (*American Math Monthly 4524*) Prove that for every pair of positive integers n and N, there are consecutive positive integers $k, k+1, \ldots, k+N$ such that $\varphi(k), \varphi(k+1), \ldots, \varphi(k+N)$ are all divisible by n, where $\varphi(n)$ is as defined in Euler's theorem.

Solution. We need the *fact* that if integer $w=ab$, where $a=p^m$, p is prime and $\gcd(b,p)=1$, then $\varphi(w)$ is divisible by $p-1$.

Granting the *fact*, by Dirichlet's theorem, there are distinct primes $p_0, p_1, \ldots, p_N\equiv 1 \pmod{n}$. By the Chinese remainder theorem, there is an integer k such that $k\equiv 0 \pmod{p_0}$, $k\equiv -1 \pmod{p_1}$,…, $k\equiv -N \pmod{p_N}$. So for $j=0,1,\ldots,N$, the number $k+j$ is divisible by the prime p_j. Then $\varphi(k+j)$ is divisible by p_j-1 by the fact, which is a multiple of n.

For the *fact*, note $\gcd(a,b)=1$. Then $\varphi(ab)=\varphi(a)\varphi(b)$. (This follows from the Chinese remainder theorem, since for every k in $[1,ab]$ with $\gcd(k,ab)=1$, let r and s be the remainders when k is divided by a and b respectively. Now $\gcd(k,ab)=1$ if and only if $\gcd(r,a)=1=\gcd(s,b)$. The Chinese remainder theorem asserts that $x\equiv k \pmod{ab}$ if and only if $x\equiv r \pmod{a}$ and $x\equiv s \pmod{b}$. Thus $x\leftrightarrow(r,s)$ is bijective.) For x in $[1,p^m]$, $\gcd(x,p^m)>1$ if and only if x is a multiple of p. So $\varphi(a)=\varphi(p^m)=p^m-p^{m-1}=p^{m-1}(p-1)$. Then $\varphi(w)=\varphi(a)\varphi(b)$ is divisible by $p-1$.

Example 9 Prove that there are infinitely many positive integers n such that the equation $x^n+y^n=z^n$ has no solution (x,y,z) in integers with $xyz\neq 0$ and $\gcd(n,xyz)=1$. (*These n's may even be chosen to be pairwise relatively prime.*)

(*Remark* Barry Powell published this result in the *American Mathematical Monthly* on November 1978.)

Solution. The case $n=4$ is well-known. Next, suppose $n_1, n_2, \ldots, n_k$ are such n's. By Dirichlet's theorem, there is a prime p such that $p\equiv -1 \pmod{4n_1n_2\cdots n_k}$. We define a new $n=p(p-1)/2$. Note $n\equiv 1 \pmod{4}$. Since $(p-1)/2$, $(p+1)/2$ are consecutive integers and $p>(p+1)/2$, so $\gcd(p(p-1)/2, (p+1)/2)=1$. Hence, $\gcd(n, 4n_1n_2\cdots n_k)=1$. (*In particular, n is relatively prime to every one of $n_1, n_2, \ldots, n_k$.*)

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 3, 2013.***

Problem 406. For every integer $m>2$, let P be the product of all those positive integers that are less than m and relatively prime to m, prove that P^2-1 is divisible by m.

Problem 407. Three circles S, S_1, S_2 are given in a plane. S_1 and S_2 touch each other externally, and both of them touch S internally at A_1 and at A_2 respectively. Let P be one of the two points where the common internal tangent to S_1 and S_2 meets S. Let B_i be the intersection points of PA_i and S_i (i=1,2). Prove that line B_1B_2 is a common tangent to S_1 and S_2.

Problem 408. Let $\mathbb{Q}$ denote the set of all rational numbers. Let $f:\mathbb{Q}\to\{0,1\}$ be a function such that for all x,y in $\mathbb{Q}$ with $f(x)=f(y)$, we have $f((x+y)/2)=f(x)$. If $f(0)=0$ and $f(1)=1$, then prove that $f(x)=1$ for every rational $x>1$.

Problem 409. The population of a city is one million. Every two citizens there know another common citizen (here knowing is mutual). Prove that it is possible to choose 5000 citizens from the city such that each of the remaining citizens will know at least one of the chosen citizens.

Problem 410. (*Due to Titu ZVONARU and Neculai STANCIU, Romania*) Prove that for all positive real x,y,z,

$$\sum_{cyc}(x+y)\sqrt{(x+z)(y+z)} \ge 4(xy+yz+zx)$$

$$+\frac{xy+yz+zx}{3(x^2+y^2+z^2)}((x-y)^2+(y-z)^2+(z-x)^2).$$

Here $\sum_{cyc} f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y)$.

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Solutions

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Problem 401. Suppose all faces of a convex polyhedron are parallelograms. Can it have exactly 2012 faces? Please provide an explanation to your answer.

Solution. **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)) and **F5 Group** (Carmel Alison Lam Foundation Secondary School).

The answer is negative. Let us call a series of faces $F_1, F_2,\ldots, F_k$ a <u>loop</u> if the pairs $(F_1, F_2), (F_2, F_3),\ldots, (F_{k-1}, F_k), (F_k, F_1)$ each have a common edge and all these common edges are parallel. Clearly any two loops have exactly two common faces and conversely each face belongs to exactly two loops. Therefore, if there are n loops, the total number of faces must be $2\ {}_nC_2=n(n-1)$. However, $n(n-1)=2002$ has no solution in integer.

Problem 402. Let S be a 30 element subset of $\{1,2,\ldots,2012\}$ such that every pair of elements in S are relatively prime. Prove that at least half of the elements of S are prime numbers.

Solution. **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **F5 Group** (Carmel Alison Lam Foundation Secondary School), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France), **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia) and **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Assume there are more than 15 elements in S are not prime. Excluding 1, there are at least 15 of them are composite numbers. Each composite number in S has a prime divisor at most $[2012^{1/2}] = 46$. There are 14 prime numbers less than 46. By the pigeonhole principle, two of the 15 composite numbers above will share a common prime divisor, contradiction.

Problem 403. On the coordinate plane, 1000 points are randomly chosen. Prove that there exists a way of coloring each of the points either red or blue (but not both) so that on every line parallel to the x-axis or y-axis, the number of red points minus the number of blue points is equal to -1, 0 or 1.

Solution. **J. S. GLIMMS** (Vancouver, Canada) and **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France).

Replace 1000 by n. We prove by induction on n. The case n=1 is clear. Suppose the case $n=k$ is true. For the case $n=k+1$, we have two cases.

<u>Case A</u> (one of the lines L parallel to the x-axis or the y-axis contains an odd number of the points). Ignore one of the points P on L. By inductive step, there is a desired coloring for the remaining k points. Since there is an even number of point on L now, the number of red and blue points must be the same. Then look at the coloring on the line $L^\perp$ through P perpendicular to L. Color P red if $L^\perp$ is a -1 or 0 case and blue if it is a 1 case.

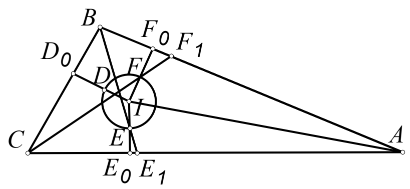
<u>Case B</u> (all lines parallel to the x-axis or y-axis contain an even number of the points). Ignore one of the points P on one of the lines L parallel to the x-axis. By inductive step, there is a desired coloring for the remaining k points. Let $L^\perp$ be the line through P parallel to the y-axis.

Since other than L, the lines parallel to x-axis all contain an even number of the points, they must all be 0 case lines. Ignoring P, if L is a case 1 line, then in the whole plane there is exactly one more red point than blue point. Also, other than $L^\perp$, the lines parallel to y-axis all contain an even number of the points, they must all be 0 case lines. Then $L^\perp$ must also be a case 1 line. We then color P blue so both L and $L^\perp$ become case 0 lines. Similarly, ignoring P, both lines may be 0 cases, then color P red or blue. Otherwise both lines are -1 cases, then color P red.

Other commended solvers: **F5 Group** (Carmel Alison Lam Foundation Secondary School).

Problem 404. Let I be the incenter of acute ΔABC. Let Γ be a circle with center I that lies inside ΔABC. D, E, F are the intersection points of circle Γ with the perpendicular rays from I to sides BC, CA, AB respectively. Prove that lines AD, BE, CF are concurrent.

Solution. **F5 Group** (Carmel Alison Lam Foundation Secondary School) and **J. S. GLIMMS** (Vancouver, Canada).



(Below $P=\alpha\cap\beta$ will mean lines α and β meet at point P, $d(P,\alpha)$ will denote the distance from point P to line α and $[XYZ]$ will denote the area of ΔXYZ.)

Let $D_0 = ID\cap BC$, $E_0 = IE\cap CA$, $F_0 = IF\cap AB$. Since AI bisects $\angle CAB$, IE_0 and IF_0 are symmetric respect to AI. Now $IE=IF$ implies E and F are symmetric respect to AI. Hence, $d(E,AB)=d(F,AC)$. Then

$$\frac{[CFA]}{[AEB]}=\frac{CA\cdot d(F,AC)/2}{AB\cdot d(E,AB)/2}=\frac{CA}{AB}.$$

Similarly,

$$\frac{[BEC]}{[CDA]}=\frac{BC}{CA}\quad and\quad \frac{[ADB]}{[BFC]}=\frac{AB}{BC}.$$

Let $D_1 = AD\cap BC$, $E_1 = BE\cap CA$, $F_1 = CF\cap AB$. We have

$$\frac{AF_1}{F_1B}=\frac{[CF_1A]}{[BF_1C]}=\frac{d(A,CF)}{d(B,CF)}=\frac{[CFA]}{[BFC]}.$$

Similarly,

$$\frac{BD_1}{D_1C}=\frac{[ADB]}{[CDA]}\quad and\quad \frac{CE_1}{E_1A}=\frac{[BEC]}{[AEB]}.$$

From the equations above, we get

$$\frac{AF_1}{F_1B}\frac{BD_1}{D_1C}\frac{CE_1}{E_1A}=\frac{CA}{BC}\frac{AB}{CA}\frac{BC}{AB}=1.$$

By Ceva's theorem, lines AD, BE, CF are concurrent.

Other commended solvers: **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece).

Comment: **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania) mentioned that the problem was well-known and the point of concurrency is called the Kariya point.

Problem 405. Determine all functions f,g: $(0,+\infty)\to(0,+\infty)$ such that for all positive number x, we have

$$f(g(x))=\frac{x}{xf(x)-2}\quad and\quad g(f(x))=\frac{x}{xg(x)-2}.$$

Solution. **F5 Group** (Carmel Alison Lam Foundation Secondary School) and **J. S. GLIMMS** (Vancouver, Canada).

Let $F(x)=xf(x)$ and $G(x)=xg(x)$. For all $x>0$, $f(g(x))>0$ and $g(f(x))>0$ imply $F(x)>2$ and $G(x)>2$. Define $a_1=2$. Now

$$\frac{G(x)}{F(x)-2}=g(x)f(g(x))=F(g(x))>a_1.$$

Then $\quad G(x) > a_1F(x) - 2a_1. \qquad (1)$

Similarly, $\quad F(x) > a_1G(x) - 2a_1. \qquad (2)$

Doing $(1)\times a_1+(2)$ and simplifying, we get

$$F(x)<\frac{2a_1}{a_1-1}=4.$$

Define $b_1=4$. Similarly we get $G(x) < b_1$. Repeating the above steps, but reversing all the inequality signs, we can get

$$F(x)>\frac{2b_1}{b_1-1}=a_2,\quad G(x)>a_2,$$

$$F(x)<\frac{2a_2}{a_2-1}=b_2\quad and\quad G(x)<b_2.$$

This suggest defining

$$a_{n+1}=\frac{2b_n}{b_n-1}\quad and\quad b_n=\frac{2a_n}{a_n-1}$$

for $n=1,2,3,\ldots$. Replacing a_1, b_1, a_2, b_2 by a_n, b_n, a_{n+1}, b_{n+1} and repeating the steps above, we can prove $a_n< F(x)$, $G(x) < b_n$ for $n=1,2,3,\ldots$ by induction on n. Next we will show a_n, b_n have same limit. Now

$$a_{n+1}=\frac{2b_n}{b_n-1}=\frac{4a_n/(a_n-1)}{(a_n+1)/(a_n-1)}=\frac{4a_n}{a_n+1}.$$

Taking reciprocal, we get

$$\frac{1}{a_{n+1}}=\frac{1}{4}+\frac{1}{4}\frac{1}{a_n}.$$

Defining $c_n=1/a_n$, we get $c_{n+1}=(1+c_n)/4$. Subtracting 1/3 from both sides, we get $c_{n+1}-1/3 = (c_n-1/3)/4$. Using this, we get

$$c_{n+1}-\frac{1}{3}=\frac{1}{4^n}(c_1-\frac{1}{3})=\frac{1}{6\cdot4^n}.$$

From this, letting n tends to infinity, we can see c_n has limit 1/3. Then a_n has limit 3. Similarly b_n has limit 3. Thus, for all $x>0$, $F(x)=3=G(x)$, i.e. $f(x)=3/x=g(x)$. Plugging these into the given equations, we see indeed they are solutions.

Olympiad Corner

(continued from page 1)

Problem 3. (*Cont.*) Can "2" immediately to the right of the inequality be replaced by a smaller positive number?

Problem 4. In ΔABC, $AB > AC$, M is the midpoint of BC of its circumcircle containing A. Its incircle with incentre I is tangent to BC at D. The line passing through D and parallel to AI intersects the incircle again at P. Prove that the lines AP and IM intersect at a point on the circumcircle of ΔABC.

Primes in Arithmetic Progressions

(continued from page 2)

Assume there are integers x, y, z satisfying $x^n+y^n=z^n$ with $xyz\neq0$ and $\gcd(n,xyz)=1$. Then $\gcd(p,x) = \gcd(p,y) = \gcd(p,z) = 1$. Let $w = x^{(p-1)/2}$. By Euler's theorem, $w^2=x^{p-1}\equiv1 \pmod p$. Then p divides $w-1$ or $w+1$. Hence $x^{(p-1)/2}= w\equiv\pm1 \pmod p$. Then $x^n\equiv\pm1 \pmod p$. Similarly, y^n, $z^n\equiv\pm1 \pmod p$. But then $x^n+y^n\equiv 0$ or $\pm2 \pmod p$, contradicting $x^n+y^n=z^n$.

Explanations for Facts 1 to 4.

For fact 1, let $n=\min\{a,b\}$. For $n=1$, we may assume $a\geq b=1$ and take $(r,s)=(0,1)$. Suppose cases $n=1$ to k are true. For case $n=k+1$, say $a\geq b=k+1$. Dividing a by b, we can write $a=qb+c$, where $q=[a/b]$ and $0\leq c<b$. If $c=0$, then take $(r,s) = (1,q-1)$ to get $ar+bs = b = \gcd(a,b)$. If $c\geq1$, then since $k+1=b>c\geq1$ and $\gcd(b,c) = \gcd(b,a-qb) = \gcd(b,a)$, we can apply inductive step to get r', s' so that $\gcd(b,c)=br'+cs'$. Then $\gcd(a,b) = br'+(a-qb)s' = as'+b(r'-qs')$.

Remark: In case $\gcd(a,b)=1$, fact 1 gives $ar\equiv1 \pmod b$. We denote this r by a^{-1} in (mod b). Hence we can cancel a in $ax\equiv ay \pmod b$ to get $x\equiv y \pmod b$ by multiplying both sides by a^{-1}.

For fact 2, let $k=\varphi(n)$ and let $r_1, r_2, \ldots, r_k$ be the integers in $[1,n]$ relatively prime to n. If $\gcd(a,n)=1$, then $ar_i\equiv ar_j \pmod n$ implies $r_i=r_j$ by the remark above. Then $ar_1, ar_2, \ldots, ar_k$ is just a permutation of $r_1, r_2, \ldots, r_k \pmod n$. So $(ar_1)(ar_2)\cdots(ar_k)\equiv r_1r_2\cdots r_k \pmod n$. As $\gcd(r_1r_2\cdots r_k,n)=1$, by the remark above we may cancel $r_1r_2\cdots r_k$ to get $a^k\equiv1 \pmod n$, which is Euler's theorem.

For fact 3, let $K= k_1k_2\cdots k_n$ and $M_i =K/k_i$. Then $\gcd(M_i,k_i) =1$ and for $j\neq i$, $M_j\equiv0 \pmod{k_i}$. Let x be the integer in the interval $[1,K]$ such that

$$x\equiv c_1M_1^{\varphi(k_1)}+\cdots+c_nM_n^{\varphi(k_n)} \pmod K.$$

Using Euler's theorem, $x\equiv c_i \pmod{k_i}$. If x' in $[1,K]$ is another solution, then $x-x'\equiv c_i-c_i=0 \pmod{k_i}$ for $i=1,2,\cdots,n$. This leads to $x-x'\equiv0 \pmod K$. As x, x' are both in $[1,K]$, we get $x=x'$.

For fact 4, $p=2$ or 3 cases are clear. For $p>3$, let a be in $[1, p-1]$. If $a\equiv a^{-1} \pmod p$, then $a^2\equiv1 \pmod p$. So p divides $(a-1)(a+1)$. Hence $a=1$ or $p-1$. For a in $[2,p-2]$, we can form $(p-3)/2$ pairs a and a^{-1}. Then $(p-1)! \equiv 1(aa^{-1})^{(p-3)/2}(-1) = -1 \pmod p$.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 28th Italian Math Olympiad.

Problem 1. Let ABC be a triangle with right angle at A. Choose points D, E, F on sides BA, CA, AB respectively so that $AFDE$ is a square. Denote by x the side-length of this square. Prove that

$$\frac{1}{x}=\frac{1}{AB}+\frac{1}{AC}.$$

Problem 2. Determine all positive integers that are 300 times the sum of their digits.

Problem 3. Let n be an integer greater than or equal to 2. There are n persons in a line, and each of these persons is either a villain (and this means that he/she always lies) or a knight (and this means he/she always tells the truth). Apart from the first person in the line, every person indicates one of those before him and declares either "this person is a villain" or "this person is a knight". It is known that the number of villains is greater than the number of knights. Prove that, watching the declarations, it is possible to determine, for each of the n persons, whether he/she is a villain or a knight.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 10, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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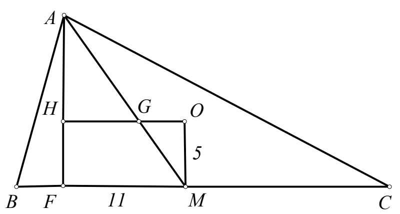
Putnam Exam

Kin Y. Li

The title of our article is an abbreviated name for the famous <u>*William Lowell Putnam Mathematical Competition*</u>. It started in the year 1938. Thousands of students in many US and Canadian universities participate in this competition annually. The top five scorers each year are designated as Putnam Fellows. These Putnam Fellows include the Physics Nobel Laureates Richard Feynman, Kenneth Wilson, the Fields' Medalists John Milnor, David Mumford, Dan Quillen and many other famous celebrities.

Although it is a math competition for undergraduate students, some of the problems may be solved by secondary school students interested in math olympiads. Below we will provide some examples.

<u>Example 1</u> (*1997 Putnam Exam*) A rectangle $HOMF$ has sides HO=11 and OM=5. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC and F the foot of the altitude from A. What is the length of BC?



<u>Solution.</u> Recall the centroid G of ΔABC is on the Euler line OH (see <u>*Math Excalibur*</u>, *vol.* 3, *no.* 1, *p.* 1) and $AG/GM = 2$. As FH, $MO \perp OH$ and $\angle AGH = \angle MGO$, so $\Delta AHG \sim \Delta MOG$. Hence $AH = 2OM = 10$. Then $OC^2 = OA^2 = AH^2 + OH^2 = 221$ and $BC = 2MC = 2(OC^2-OM^2)^{1/2} = 28$.

<u>Example 2</u> (*1991 Putnam Exam*) Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}\equiv 2^p+1 \pmod{p^2}.$$

<u>Solution.</u> Let W be the left side of the equation. Since $\binom{p+j}{j}=\binom{p+j}{p}$, W is the coefficient of x^p in the polynomial

$$\sum_{j=0}^{p}\binom{p}{j}\sum_{k=0}^{p+j}\binom{p+j}{k}x^k$$
$$=\sum_{j=0}^{p}\binom{p}{j}(1+x)^{p+j}$$
$$=(1+x)^p\sum_{j=0}^{p}\binom{p}{j}(1+x)^j$$
$$=(1+x)^p(2+x)^p.$$

Expanding $(1+x)^p(2+x)^p$, we see

$$W=\sum_{k=0}^{p}\binom{p}{k}\binom{p}{p-k}2^k.$$

For $0 < k < p$, p divides $p!$, but not $k!(p-k)!$. So p divides $\binom{p}{k}=\binom{p}{p-k}$.

In (mod p^2) of W, we may ignore the terms with $0 < k < p$ to get

$$W\equiv\binom{p}{0}\binom{p}{p}2^0+\binom{p}{p}\binom{p}{0}2^p=1+2^p \pmod{p^2}.$$

<u>Example 3</u> (*2000 Putnam Exam*) Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form (±1, ±1, …, ±1) in n-dimensional space with $n\geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

<u>Solution.</u> Let S be the set of all points $(x_1, x_2, \ldots, x_n)$ with all $x_i = \pm 1$. For each P in B, let S_P be the set of all points in S which differ from P in exactly one coordinate. Each S_P contains n points. So the union of all S_P's over all P in B (counting points repeated as many times as they appeared in the union) must contain more than 2^{n+1} points. Since this is more than twice 2^n, by the pigeonhole principle, there must exist a point T appeared in at least three of the sets S_P, S_Q, S_R, where P, Q, R are distinct points in B. Then any two of P, Q, R have exactly two different coordinates. Then ΔPQR is equilateral with sides $2^{3/2}$.

Example 4 (*1947 Putnam Exam*) Given $P(z) = z^2 + az + b$, a quadratic polynomial for the complex variable z with complex coefficients a and b. Suppose that $|P(z)| = 1$ for every z such that $|z| = 1$. Prove that $a = b = 0$.

Solution. Let $\omega \neq 1$ be a cube root of unity. Let $\alpha = P(1)$, $\beta = \omega P(\omega)$ and $\gamma = \omega^2 P(\omega^2)$. We have $|\alpha| = |\beta| = |\gamma| = 1$ and $\alpha+\beta+\gamma = 3+a(1+\omega^2+\omega^4)+b(1+\omega+\omega^2) = 3$. Hence, $|\alpha+\beta+\gamma| = |\alpha| + |\beta| + |\gamma|$. By the equality case of the triangle inequality, we get $\alpha = \beta = \gamma = 1$. Then $P(1) = 1$, $P(\omega) = \omega^2$ and $P(\omega^2) = \omega = \omega^4$. Since P is of degree 2 and $P(z)-z^2 = 0$ has three distinct roots 1, ω and ω^2, we get $P(z) = z^2$ for all complex number z.

Example 5 (*1981 Putnam Exam*) Prove that there are infinitely many positive integers n with the property that if p is a prime divisor of n^2+3 then p is also a divisor of k^2+3 for some integer k with $k^2 < n$.

Solution. First we look at the sequence m^2+3 with $m \geq 0$. The terms are 3, 4, 7, 12, 28, 39, 52, 67, 84, …. We can observe that 3×4, 4×7, 7×12,… are also in the sequence. This suggests multiplying $(m^2+3)[(m+1)^2+3]$. By completing square of the result, we see

$$(m^2+3)[(m+1)^2+3] = (m^2+m+3)^2+3.$$

Let $n = (m^2+m+2)(m^2+m+3)+3$. Using the identity above twice, we see $n^2+3 = (m^2+3)[(m+1)^2+3][(m^2+m+2)^2+1]$. So if p is a prime divisor of n^2+3, then p is also a divisor of either m^2+3 or $(m+1)^2+3$ or $(m^2+m+2)^2+1$ and m^2, $(m+1)^2$, $(m^2+m+2)^2 < n$. Letting $m = 1,2,3,\ldots$, we get infinitely many such n.

Example 6 (*1980 Putnam Exam*) Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > ½|X| \geq 5$ for $1 \leq i \leq 1066$. Prove there exists ten elements $x_1, x_2, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, x_2, \ldots, x_{10}$. (Here $|S|$ means the number of elements in the set S.)

Solution. Let $X = \{x_1, x_2, \ldots, x_m\}$ with $m = |X|$ and n_k be the number of i such that x_k is in A_i. We may arrange the x_k's so that n_k is decreasing. For $1 \leq i \leq 1066$ and $1 \leq k \leq m$, let $f(i,k)=1$ if x_k is in A_i and $f(i,k)=0$ otherwise. Then

$$n_1|X| \geq \sum_{k=1}^{m} n_k = \sum_{k=1}^{m}\sum_{i=1}^{1066} f(i,k) = \sum_{j=1}^{1066}\sum_{k=1}^{m} f(i,k)$$
$$= \sum_{i=1}^{1066}|A_i| \geq \sum_{i=1}^{1066}\frac{1}{2}m = 533|X|.$$

Then n_1 is greater than 533, i.e. x_1 is in more than 533 A_i's.

Next let $B_1, B_2, \ldots, B_r$ be those A_i's not containing x_1 and $Y = \{x_2, x_3, \ldots, x_m\}$. Then $r = 1066-n_1 \leq 532$ and each $|B_i| > ½|X| > ½|Y|$. Repeating the reasoning above, we will get $n_2 > r/2$. Let $C_1, C_2, \ldots, C_s$ be those A_i's not containing x_1, x_2 and $Z = \{x_3, x_4, \ldots, x_m\}$. Then $s = r-n_2 < r/2$, i.e. $s \leq 265$. After 532 and 265, repeating the reasoning, we will get 132, 65, 32, 15, 7, 3, 1. Then at most 1 set is left not containing $x_1, x_2, \ldots, x_9$. Finally, we may need to use x_{10} to take care of the last possible set.

Example 7 (*1970 Putnam Exam*) A quadrilateral which can be inscribed in a circle is said to be <u>*inscribable*</u> or <u>*cyclic*</u>. A quadrilateral which can be circumscribed to a circle is said to be <u>*circumscribable*</u>. If a circumscribable quadrilateral of sides a, b, c, d has area $A = \sqrt{abcd}$, then prove that it is also inscribable.

Solution. Since the two tangent segments from a point (outside a circle) to the circle are equal and the quadrilateral is circumscribable, we have $a+c=b+d$. Let k be the length of a diagonal and α, β be opposite angles of the quadrilateral so that

$$a^2+b^2-2ab\cos\alpha = k^2 = c^2+d^2-2cd\cos\beta.$$

Subtracting $(a-b)^2 = (c-d)^2$, we get

$$2ab(1-\cos\alpha) = 2cd(1-\cos\beta). \quad (*)$$

Now $2\sqrt{abcd} = 2A = ab\sin\alpha + cd\sin\beta$. Squaring and using (*) twice, we get

$$\begin{aligned} 4abcd &= a^2b^2(1-\cos^2\alpha)+2abcd\sin\alpha\sin\beta \\ &\quad +c^2d^2(1-\cos^2\beta) \\ &= abcd(1+\cos\alpha)(1-\cos\beta) \\ &\quad +2abcd\sin\alpha\sin\beta \\ &\quad +abcd(1+\cos\beta)(1-\cos\alpha). \end{aligned}$$

Simplifying this, we get $4=2-2\cos(\alpha+\beta)$, i.e. $\alpha+\beta=180°$. Therefore the quadrilateral is cyclic.

Example 8 (*1964 Putnam Exam*) Show that the unit disk in the plane cannot be partitioned into two disjoint congruent subsets.

Solution. Let D be the unit disk, O be its center and $d(X,Y)$ denote the distance between X and Y in D. <u>*Assume*</u> D can be partitioned into two disjoint congruent subsets A and B. Without loss of generality, suppose O is in A. For each X in A, let X^* be the corresponding point in B. Then O^* is in B. For all X, Y in A, $d(X,Y) = d(X^*,Y^*)$.

Since $d(O,X) \leq 1$ for all X in A and the set $B = \{X^* : X \text{ in } A\}$, so $d(O^*,Z) \leq 1$ for all Z in B. Let R and S be the endpoints of the diameter perpendicular to line OO^*. Then $d(O^*,R) = d(O^*,S) > 1$. Hence, R and S are in A. Now $d(R^*,S^*) = d(R,S) = 2$, so R^*S^* is a diameter. Since O is the midpoint of diameter RS in A, O^* must be the midpoint of the diameter R^*S^*. Then $O^*=O$, which contradicts A, B are disjoint.

Example 9 (*1950 Putnam Exam*) In each of N houses on a straight street are one or more boys. At what point should all the boys meet so that the sum of the distances that they walk is as small as possible?

Solution. Think of the street is the real axis. Suppose the i-th boy's house is at x_i so that $x_1 \leq x_2 \leq \cdots \leq x_n$. Suppose they meet at x, the first and the n-th boy together must walk a distance of x_n-x_1 if x is in $[x_1, x_n]$ and more if x is outside $[x_1, x_n]$. This is similar for the second boy and the $(n-1)$-st boy, etc.

If n is even, say $n=2k$, then the least distance all n boys have to walk is

$$(x_n-x_1)+(x_{n-1}-x_2)+\cdots+(x_{k+1}-x_k)$$

with equality if x is in $[x_k, x_{k+1}]$. If n is odd, say $n = 2k - 1$, then the least distance they have to walk is

$$(x_n-x_1)+(x_{n-1}-x_2)+\cdots+(x_{k+1}-x_k)+0$$

with equality if $y = x_k$.

Example 10 (*1956 Putnam Exam*) The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z)+1$ and $Q(z)+1$. Prove that $P(z) \equiv Q(z)$.

Solution. Observe that if $P(z)$ has c as a zero with multiplicities $k > 0$, then the derivative $P'(z)$ has c as a zero with multiplicities $k-1$, which follows from differentiating $P(z) = (z-c)^k R(z)$ on both sides.

Now suppose $P(z)$ has degree m and $Q(z)$ has degree n. By symmetry, we may assume $m \geq n$. Let the distinct zeros of $P(z)$ be $a_1, a_2, \ldots, a_s$ and let the distinct zeros of $P(z)+1$ be $b_1, b_2, \ldots, b_t$. Clearly, $a_1, a_2, \ldots, a_s$, $b_1, b_2, \ldots, b_t$ are all distinct.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***March 10, 2013.***

Problem 411. A and B play a game on a square board divided into 100×100 squares. Each of A and B has a checker. Initially A's checker is in the lower left corner square and B's checker is in the lower right corner square. They take turn to make moves. The rule is that each of them has to move his checker one square up, down, left or right within the board and A goes first. Prove that no matter how B plays, A can always move his checker to meet B's checker eventually.

Problem 412. $\triangle ABC$ is equilateral and points D, E, F are on sides BC, CA, AB respectively. If

$$\angle BAD+\angle CBE+\angle ACF = 120^\circ,$$

then prove that $\triangle BAD$, $\triangle CBE$ and $\triangle ACF$ cover $\triangle ABC$.

Problem 413. Determine (with proof) all integers $n\geq3$ such that there exists a positive integer M_n satisfying the condition for all n positive numbers $a_1, a_2, \ldots, a_n$, we have

$$\frac{a_1+a_2+\cdots+a_n}{\sqrt[n]{a_1a_2\cdots a_n}} \le M_n\left(\frac{a_2}{a_1}+\frac{a_3}{a_2}+\cdots+\frac{a_n}{a_{n-1}}+\frac{a_1}{a_n}\right).$$

Problem 414. Let p be an odd prime number and $a_1, a_2, \ldots, a_{p-1}$ be positive integers not divisible by p. Prove that there exist integers $b_1, b_2, \ldots, b_{p-1}$, each equals 1 or -1 such that

$$a_1b_1+a_2b_2+\cdots+a_{p-1}b_{p-1}$$

is divisible by p.

Problem 415. (*Due to MANOLOUDIS Apostolos, Piraeus, Greece*) Given a triangle ABC such that $\angle BAC=103^\circ$ and $\angle ABC=51^\circ$. Let M be a point inside $\triangle ABC$ such that $\angle MAC=30^\circ$ and $\angle MCA=13^\circ$. Find $\angle MBC$ with proof.

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Solutions

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Problem 406. For every integer $m>2$, let P be the product of all those positive integers that are less than m and relatively prime to m, prove that P^2-1 is divisible by m.

Solution. **Jon GLIMMS** (Vancouver, Canada), **Corneliu MĂNESCU- AVRAM** (Technological Transportation High School, Ploieşti, Romania), **WONG Ka Fai** and **YUNG Fai**.

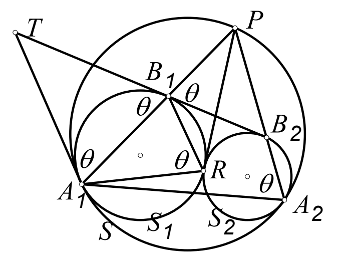
Let a in interval $[1,m)$ be relatively prime to m. By Bezout's theorem, there exists a unique a^{-1} in $[1,m)$ such that $aa^{-1}\equiv1$ (mod m). Then $\gcd(a^{-1},m) = 1$. Since $a^{-1}a \equiv 1$ (mod m), by uniqueness, $(a^{-1})^{-1} = a$.

For those factor a in the product P satisfying $a\neq a^{-1}$, a will be cancelled by a^{-1} (mod m). Thus, P is congruent modulo m to the product of those remaining factor a satisfying $a= a^{-1}$. Now $a = a^{-1}$ implies $a^2 = aa^{-1}\equiv1$ (mod m). It follows $P^2 \equiv 1$ (mod m) and we are done.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 407. Three circles S, S_1, S_2 are given in a plane. S_1 and S_2 touch each other externally, and both of them touch S internally at A_1 and at A_2 respectively. Let P be one of the two points where the common internal tangent to S_1 and S_2 meets S. Let B_i be the intersection points of PA_i and S_i (i=1,2). Prove that line B_1B_2 is a common tangent to S_1 and S_2.

Solution. **F5D** (Carmel Alison Lam Foundation Secondary School), **William FUNG and Jacob HA** and **NGUYEN Van Thien** (Luong The Vinh High School, Dongnai Province, Vietnam)**.**



Let the tangent at A_1 to S (and S_1) and line B_1B_2 meet at T. Let R be the tangent point of S_1 and S_2. By the intersecting chord theorem, we have

$$PB_1\times PA_1 = PR^2 = PB_2\times PA_2.$$

So A_1, B_1, B_2, A_2 are concyclic. Using (1) line TA_1 is tangent to S_1, (2) line TA_1 is tangent to S, (3) A_1, B_1, B_2, A_2 concyclic and (4) vertical angles are congruent in that order, we get

$$\angle B_1RA_1 = \angle B_1A_1T = \angle PA_2A_1 = \angle PB_1B_2 = \angle TB_1A_2.$$

Then line TB_1=B_1B_2 is tangent to S_1 at B_1. Similarly, line B_1B_2 is tangent to S_2 at B_2. Therefore, line B_1B_2 is a common tangent to S_1 and S_2.

Other commended solvers: **Dusan DROBNJAK**(Mathematical Grammar School, Belgrade, Serbia), **Jon GLIMMS** (Vancouver, Canada), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece) and **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, Andhra Pradesh, India).

Problem 408. Let $\mathbb{Q}$ denote the set of all rational numbers. Let f:$\mathbb{Q}\to\{0,1\}$ be a function such that for all x,y in $\mathbb{Q}$ with $f(x)=f(y)$, we have $f((x+y)/2)=f(x)$. If $f(0)=0$ and $f(1)=1$, then prove that $f(x)=1$ for every rational $x > 1$.
(*Source*: 2000 *Indian Math Olympiad*)

Solution. **Ioan Viorel CODREANU,** (Secondary School Satulung, Maramures, Romania) and **Dusan DROBNJAK**(Mathematical Grammar School, Belgrade, Serbia).

We *claim* that if a,b are rational numbers and $f(a)\neq f(b)$, then for all positive integer n, we have $f(a+n(b-a))=f(b)$.

We will prove this by induction on n. The case n=1 is clear. Suppose the case $n=k$ is true. Then we have $f(a+k(b-a))=f(b)$. *Assume* $f(a+(k+1)(b-a)) \neq f(b)$. Since $f(r)$=0 or 1 for all r in $\mathbb{Q}$ and $f(a)\neq f(b)$, we get $f(a+(k+1)(b-a)) = f(a)$. Let $x = a$, $y = a+(k+1)(b-a)$, $x' = b$, $y' = a+k(b-a)$. From above, we have $f(x) = f(y)$ and $f(x') = f(y')$. By the given property of f, since

$$\frac{x+y}{2}=\frac{(k+1)b-(k-1)a}{2}=\frac{x'+y'}{2},$$

we get $f(a)=f(x)=f(x')=f(b)$, contradiction. Hence the case $n=k+1$ is true and we complete the induction.

Now by the claim, since $f(0)=0\neq1=f(1)$, for all positive integer n, we get $f(n) = f(1) =1$. For a rational $r > 1$, let $r-1=p/q$, where p, q are positive integers. *Assume* $f(r)\neq1$. Using the claim with $a = 1$, $b = r$, and $n = q$, we get $f(1+q(r-1)) = f(r)$. But $f(1+q(r-1)) = f(1+p) = 1$, contradiction. So, for all rational $r>1$, $f(r) = 1$.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 409. The population of a city is one million. Every two citizens there know another common citizen (here knowing is mutual). Prove that it is possible to choose 5000 citizens from the city such that each of the remaining citizens will know at least one of the chosen citizens.
(*Source*: 63^{rd} *St. Petersburg Math Olympiad*)

Solution. **Jon GLIMMS** (Vancouver, Canada).

Let $m=10^6$ and $x_1, x_2, \ldots, x_m$ be all the citizens in the city. Let $F(x_i)$ be all the citizens (not including x_i) that x_i knows and $|F(x_i)|$ denote the number of such citizens.

If there exists a x_i with $|F(x_i)|\le 5000$, then let us choose any 5000 citizens including all members of $F(x_i)$. For any x_j not among the chosen 5000 citizens, by the given assumption, x_i and x_j know a common citizen in $F(x_i)$, who is in the chosen 5000 citizens.

Otherwise, we may assume for every x_i, $|F(x_i)| > 5000$. Now there are m^{5000} ordered 5000-tuples $(C_1, C_2, \ldots, C_{5000})$, where each C_k may be any one of the m citizens. For each x_i, let

$$S(x_i)=\{(C_1,C_2,\ldots,C_{5000})\text{: all } C_k\notin F(x_i)\}$$

Now $S(x_i)$ has less than $(m-5000)^{5000}$ members since $|F(x_i)|>5000$. Let S be the union of $S(x_1), S(x_2),\ldots,S(x_m)$. We claim that $m(m-5000)^{5000} < m^{5000}$. The claim means there exists $(C_1, C_2, \ldots, C_{5000})$ not in every $S(x_i)$. That means by choosing $C_1, C_2, \ldots, C_{5000}$, every x_i will know at least one C_k and we are done.

For the claim, using $(1+x)^n \ge 1+nx$ from the binomial theorem, we have the equivalent inequality

$$\left(\frac{m}{m-5000}\right)^{5000}=\left(1+\frac{5000}{m-5000}\right)^{5000}$$
$$>\left(1+\frac{1}{200}\right)^{200\times 25}>2^{25}>(10^3)^{2.5}>m.$$

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 410. (*Due to Titu ZVONARU and Neculai STANCIU, Romania*) Prove that for all positive real x,y,z,

$$\sum_{cyc}(x+y)\sqrt{(x+z)(y+z)}\ge 4(xy+yz+zx)$$
$$+\frac{xy+yz+zx}{3(x^2+y^2+z^2)}((x-y)^2+(y-z)^2+(z-x)^2).$$

Here $\sum_{cyc} f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y)$.

Solution of Proposers.

Observe that $4(xy+yz+zx)$ is the cyclic sum of $x(y+z)+y(x+z)$. Now

$$\begin{aligned}
&(x+y)\sqrt{(x+z)(y+z)}-x(y+z)-y(x+z)\\
&=x\sqrt{y+z}(\sqrt{x+z}-\sqrt{y+z})\\
&\quad+y\sqrt{x+z}(\sqrt{y+z}-\sqrt{x+z})\\
&=(x\sqrt{y+z}-y\sqrt{x+z})(\sqrt{x+z}-\sqrt{y+z})\\
&=\frac{(x^2(y+z)-y^2(x+z))}{(x\sqrt{y+z}+y\sqrt{x+z})}\frac{((x+z)-(y+z))}{(\sqrt{x+z}+\sqrt{y+z})}\\
&=\frac{(xy+yz+zx)(x-y)^2}{(x\sqrt{y+z}+y\sqrt{x+z})(\sqrt{x+z}+\sqrt{y+z})}.
\end{aligned}$$

By the *AM-GM* inequality, we have $(x+y)^2 \le 2(x^2+y^2)$ and $xy+yz+zx \le x^2+y^2+z^2$. Using these, we get

$$\begin{aligned}
&(x\sqrt{y+z}+y\sqrt{x+z})(\sqrt{x+z}+\sqrt{y+z})\\
&=(x+y)\sqrt{(x+z)(y+z)}+x(y+z)+y(x+z)\\
&\le(x+y)\frac{x+y+2z}{2}+2xy+yz+zx\\
&=\frac{(x+y)^2}{2}+2(xy+yz+zx)\\
&\le x^2+y^2+2(x^2+y^2+z^2)\\
&\le 3(x^2+y^2+z^2).
\end{aligned}$$

So it follows that

$$\begin{aligned}
&(x+y)\sqrt{(x+z)(y+z)}-2xy-yz-zx\\
&\ge\frac{xy+yz+zx}{3(x^2+y^2+z^2)}(x-y)^2.
\end{aligned}$$

Rotating x,y,z to y,z,x to z,x,y, we get two other similar inequalities. Adding the three inequalities, we will get the desired inequality. Equality holds if and only if $x=y=z$.

Other commended solvers: **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Comment: The proposers mention that this is a refinement of problem 2 of the 2012 Balkan Math Olympiad.

Olympiad Corner

(continued from page 1)

Problem 4. Let $x_1, x_2, x_3, \ldots$ be the sequence defined by the following recurrence: $x_1=4$ and for $n \ge 1$,

$$x_{n+1} = x_1x_2x_3\cdots x_n + 5.$$

(The first few terms of the sequence are then $x_1=4$, $x_2=4+5=9$, $x_3=4\cdot 9+5=41$, …) Find all pairs $\{a,b\}$ of positive integers such that x_ax_b is a perfect square.

Problem 5. Let $ABCD$ be a square. Find the locus of points P in the plane, different from A, B, C, D such that

$$\angle APB+\angle CPD=180^\circ.$$

Problem 6. Determine all pairs $\{a,b\}$ of positive integers with the following property: for any possible coloring of the set of all positive integers with two colors A and B, there exist either two positive integers colored by A with difference a or two positive integers colored by B with difference b.

Putnam Exam

(continued from page 2)

Now let r_i's be the multiplicities of the a_i's as zeros of $P(z)$, then the sum of the r_i's is m. By the observation above, the multiplicity of a_i as zeros of $P'(z)$ is r_i-1 and these multiplicities sum to $m-s$. Similarly, the sum of the multiplicities of the b_i's as zeros of $P'(z)=(P+1)'(z)$ is $m-t$. So

$$(m-s)+(m-t)\le \deg P'(z) < m.$$

Hence $s+t > m$. However, $a_1, a_2,\ldots,a_s, b_1, b_2, \ldots, b_t$ are zeros of $P(z)-Q(z)$ with degree at most m. So, $P(z)\equiv Q(z)$.

The interested readers are highly encouraged to browse the following books for more problems of the Putnam Exam.

A. M. Gleason, R. E. Greenwood and L. M. Kelly, The William Lowell Putnam Mathematical Competition Problems and Solutions: 1938-1964, MAA, USA, 1980.

G. L. Alexanderson, L. F. Klosinski and L. C. Larson, The William Lowell Putnam Mathematical Competition Problems and Solutions: 1965-1984, MAA, USA, 1985.

K. S. Kedlaya, B. Poonen and R. Vakil, The William Lowell Putnam Mathematical Competition 1985-2000 Problems, Solutions and Commentary, MAA, USA, 2002.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the Final Selection Test for the 2012 Croatian IMO Team.

Problem 1. Determine all functions f: $\mathbb{R}\to\mathbb{R}$ such that for all real numbers x and y holds

$$f(x^2+f(y))=(f(x)+y^2)^2.$$

(*Tonći Kokan*)

Problem 2. Along the coast of an island there are 20 villages. Each village has 20 fighters. Every fighter fights all the fighters from all the other villages. No two fighters have equal strength and the stronger fighter wins the fight.

We say that the village A is *stronger* than the village B if in at least k fights among the fighters from A and B a fighter from the village A wins. It turned out that every village is stronger than its neighbour (in the clockwise direction).

Show that the maximal possible k is 290.

(*Moscow Olympiad 2003, modified*)

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 10, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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The Inequality of A. Oppenheim

Prof. Marcel Chirita, Bucharest, Romania

In this note we establish conditions solving the problems of A. Oppenheim and O. Bothema, then we solve some problems. Below, we let a,b,c,S,s,R,r denote the sides BC, CA, AB, area, semiperimeter, circumradius, inradius of a triangle ABC respectively. In [1], two problems are stated as follow:

Problem 1. (*O. Bothema*) For $\triangle ABC$, give conditions on real numbers x,y,z so

$$yza^2+zxb^2+xyc^2\le R^2(x+y+z)^2 \qquad (1)$$

with equality if and only if

$$\frac{x}{\sin 2A}=\frac{y}{\sin 2B}=\frac{z}{\sin 2C}. \qquad (2)$$

Problem 2. (*A.Oppenheim*) For $\triangle ABC$, give conditions on real numbers x,y,z so

$$xa^2+yb^2+zc^2\ge 4S\sqrt{xy+yz+zx} \qquad (3)$$

with equality if and only if

$$\frac{x}{-a^2+b^2+c^2}=\frac{y}{a^2-b^2+c^2}=\frac{z}{a^2+b^2-c^2}. \qquad (4)$$

The author will solve problem 2, then use it to solve problem 1. It is easy to see these problems are false for some x,y,z. For example, if one of x,y,z is negative, problems 1 and 2 may be false.

Theorem. For $\triangle ABC$, if $x+y>0$, $y+z>0$, $z+x>0$ and $xy+yz+zx>0$, then (3) and (4) hold.

Proof. Let $k=4\sqrt{xy+yz+zx}$. Using $c^2=a^2+b^2-2ab\cos C$ and $S=½ab\sin C$, we can rewrite (3) as

$$2(x+z)a^2+2(y+z)b^2\ge(4z\cos C+k\sin C)ab.$$

By the *AM-GM* inequality, the left side is greater than or equal to

$$4\sqrt{(x+z)(y+z)}ab=\sqrt{16z^2+k^2}ab,$$

which is greater than or equal to the right side by the Cauchy-Schwarz inequality. So (3) is true. Equality holds (from *AM-GM* and Cauchy-Schwarz) if and only if

$$\frac{a^2}{y+z}=\frac{b^2}{z+x}=\frac{c^2}{x+y}.$$

Let t be this ratio. Then $a^2=t(y+z)$, $b^2=t(z+x)$, $c^2=t(x+y)$. So $-a^2+b^2+c^2=2tx$, $a^2-b^2+c^2=2ty$ and $a^2+b^2-c^2=2tz$. This gives (4) and steps can be reversed. Using the cosine law, we can see (4) is equivalent to

$$\frac{xa}{\cos A}=\frac{yb}{\cos B}=\frac{zc}{\cos C}.$$

From (4), we see x,y,z can be all positive or one negative and two positive.

To solve problem 1, in place of x,y,z, we use x/a^2, y/b^2, z/c^2, which also satisfy the conditions of the theorem. Then (3) is

$$x+y+z\ge 4S\sqrt{\frac{xy}{a^2b^2}+\frac{yz}{b^2c^2}+\frac{zx}{c^2a^2}}.$$

Using the formula $S=abc/(4R)$ (which is from $S=½ab\sin C$ and $c/(\sin C)=2R$), the last inequality becomes

$$x+y+z\ge\frac{1}{R}\sqrt{xyc^2+yza^2+zxb^2},$$

which is equivalent to (1). For equality case, observe that using the cosine law and $a/(\sin A)=2R$,

$$\frac{x/a^2}{-a^2+b^2+c^2}=\frac{x}{2a^2bc\cos A}=t\frac{x}{\sin 2A},$$

where $t=1/(2Rabc)$. This gives (2).

Next we give many applications of these inequalities.

Example 1 If we take $x=y=z$ in (3), then we get $a^2+b^2+c^2\ge 4S\sqrt{3}$, which dated back to Ionescu (1897), later to Weitzenböck (1919) and Carlitz (1961).

Example 2 If we take $x=a^2$, $y=b^2$ and $z=c^2$ in (3), then we get

$$a^4+b^4+c^4\ge 4S\sqrt{a^2b^2+b^2c^2+c^2a^2}.$$

Since Heron's formula gives

$$2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)=16S^2$$

(continued on page 2)

and expanding $(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 \ge 0$ leads to

$$a^2b^2+b^2c^2+c^2a^2 \le a^4+b^4+c^4,$$

it follows immediately that

$$a^2b^2+b^2c^2+c^2a^2 \ge 16S^2$$

and hence $a^4+b^4+c^4 \ge 16S^2$.

Example 3 (*a*) If $x = 9$, $y = 5$ and $z = -3$ in (3), then we get $9a^2+5b^2-3c^2 \ge 4S\sqrt{3}$.

(b) If x=27, y=27 and z=−13 in (3), then we get $27a^2+27b^2-13c^2 \ge 12S\sqrt{3}$.

(c) If x=3, y=−1 and z=15 in (3), then we get $3a^2-b^2+15c^2 \ge 12S\sqrt{3}$.

These were exercises proposed in [6] and [9].

Example 4 If we consider $x=bc/a$, $y=ca/b$ and $z=ab/c$ in (3), then we have

$$3abc \ge 4\sqrt{a^2+b^2+c^2}\,S.$$

Taking into account that $4RS=abc$ and $ab+bc+ca \le a^2+b^2+c^2$, we have $ab+bc+ca \le a^2+b^2+c^2 \le 9R^2$.

Example 5 If we consider $x=bc$, $y=ca$ and $z=ab$ in (3), then we have

$$abc(a+b+c) \ge 4S\sqrt{abc(a+b+c)},$$

which implies $abc(a+b+c) \ge 16S^2$. Using $S = ½(a+b+c)r = sr$, we get $abc \ge 8sr^2$. Using abc=4RS=4Rsr, we have $R \ge 2r$.

Example 6 Let $x > 0$. If we consider $2x-1, \dfrac{2}{x}-1$ and 1, then we can easily check that they satisfy the conditions in the theorem. So (3) yields

$$(2x-1)a^2+\left(\frac{2}{x}-1\right)b^2+c^2 \ge 4S\sqrt{3}.$$

This was a proposed exercise of B. Suceavă in [9].

Example 7 If we consider

$$x=\frac{s-a}{a^2}, y=\frac{s-b}{b^2}, z=\frac{s-c}{c^2}$$

in (3), then we get

$$s \ge 4S\sqrt{\frac{(s-a)(s-b)}{a^2b^2}+\frac{(s-b)(s-c)}{b^2c^2}+\frac{(s-c)(s-a)}{c^2a^2}}.$$

Squaring both sides and applying the *AM-GM* inequality on the right side, we get

$$s^2 \ge 48S^2\sqrt[3]{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^4b^4c^4}},$$

which is equivalent to

$$a^4b^4c^4s^6 \ge 48^3S^6(s-a)^2(s-b)^2(s-c)^2.$$

Using abc=4RS=4Rsr on the left and Heron's formula on the right, we can simplify this to $sR^2 \ge 12\sqrt{3}r^3$.

Example 8 Consider

$$x=\frac{s-a}{a}, y=\frac{s-b}{b}, z=\frac{s-c}{c}.$$

Then

$$xa^2+yb^2+zc^2 = a(s-a)+b(s-b)+c(s-c)$$

$$=\frac{2ab+2bc+2ca-(a^2+b^2+c^2)}{2}.$$

From [3], we have $ab+bc+ca=s^2+r^2+4Rr$ and $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$. Putting these into the above equation, we get

$$xa^2+yb^2+zc^2 = 2r^2+8Rr.$$

Recall by cosine law

$$\frac{(s-a)(s-b)}{ab}=\frac{c^2-a^2-b^2+2ab}{4ab}$$

$$=\frac{1-\cos C}{2}$$

$$=\sin^2\frac{C}{2}.$$

Using this and similar equations, we have

$$4S\sqrt{xy+yz+zx}$$

$$=4S\sqrt{\frac{(s-a)(s-b)}{ab}+\frac{(s-b)(s-c)}{bc}+\frac{(s-c)(s-a)}{ca}}$$

$$=4S\sqrt{\sin^2\frac{C}{2}+\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}$$

$$\ge 4S\sqrt{\frac{3}{4}}=2S\sqrt{3},$$

where the last inequality follows by applying Jensen's inequality to $f(x) = \sin^2(x/2)$ on $[0,\pi/2]$. Thus, (3) yields

$$r^2+4Rr \ge S\sqrt{3}.$$

Example 9 If instead of x, y, z, we replace them by $\dfrac{yz}{a^2}, \dfrac{xy}{b^2}, \dfrac{zx}{c^2}$ in (3), then we get after calculations that

$$xa^2+yb^2+zc^2 \le R^2\frac{(xy+yz+zx)^2}{xyz}.$$

Example 10 If instead of x, y, z, we consider yz, zx, xy, then (1) and (3) yield the following inequality

$$4S\sqrt{xyz(x+y+z)} \le a^2yz+b^2zx+c^2xy$$
$$\le (x+y+z)^2R^2,$$

which is the subject of the article "On an inequality in a triangle" from GM 8 in 1984 by Prof. Virgil Nicula.

Example 11 If instead of x,y,z, we consider

$$\frac{p}{q+r},\ \frac{q}{r+p},\ \frac{r}{p+q},$$

where $p,q,r > 0$, then (3) yields

$$\frac{p}{q+r}a^2+\frac{q}{r+p}b^2+\frac{r}{p+q}c^2 \ge 2S\sqrt{3}.$$

This is problem *E3150* proposed by G. Tsintsifas in the American Math. Monthly in 1988.

Example 12 If instead of x,y,z, we consider

$$\frac{b}{a}m,\ \frac{c}{b}n,\ \frac{a}{c}p,$$

where $m, n, p > 0$, then (3) yields

$$mab+nbc+pca \ge 4S\sqrt{\frac{c}{a}mn+\frac{a}{b}np+\frac{b}{c}mp}.$$

By the AM-GM inequality, we have

$$\frac{c}{a}mn+\frac{a}{b}np+\frac{b}{c}mp \ge 3\sqrt[3]{m^2n^2p^2}.$$

Combining the last two inequalities, we get

$$mab+nbc+pca \ge 4S\sqrt{3\sqrt[3]{m^2n^2p^2}}.$$

If we take $m = n = p = 1$, then we get

$$ab+bc+ca \ge 4S\sqrt{3},$$

which is due to V. E. Olhov, see [7] and [8] in the bibliography on page 4.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 10, 2013.***

Problem 416. If $x_1 = y_1 = 1$ and for $n > 1$,

$$x_n = -3x_{n-1} - 4y_{n-1} + n$$
and $$y_n = x_{n-1} + y_{n-1} - 2,$$

then find x_n and y_n in terms of n only.

Problem 417. Prove that there does not exist a sequence $p_0, p_1, p_2, \ldots$ of prime numbers such that for all positive integer k, p_k is either $2p_{k-1}+1$ or $2p_{k-1}-1$.

Problem 418. Point M is the midpoint of side AB of acute $\triangle ABC$. Points P and Q are the feet of perpendicular from A to side BC and from B to side AC respectively. Line AC is tangent to the circumcircle of $\triangle BMP$. Prove that line BC is tangent to the circumcircle of $\triangle AMQ$.

Problem 419. Let $n \geq 4$. M is a subset of $\{1,2,\ldots,2n-1\}$ with n elements. Prove that M has a nonempty subset, the sum of all its elements is divisible by $2n$.

Problem 420. Find (with proof) all positive integers x and y such that $2x^2y+xy^2+8x$ is divisible by xy^2+2y.

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Solutions

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Problem 411. A and B play a game on a square board divided into 100×100 squares. Each of A and B has a checker. Initially A's checker is in the lower left corner square and B's checker is in the lower right corner square. They take turn to make moves. The rule is that each of them has to move his checker one square up, down, left or right within the board and A goes first. Prove that no matter how B plays, A can always move his checker to meet B's checker eventually.

Solution. **Jon GLIMMS** (Vancouver, Canada) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Suppose the squares are unit length. A can apply the following strategy. After B made the n-th move, let $R(n)$ denote the rectangle bounded by the squares in the same row or same column as one of the two squares containing the checkers. Let $a(n)$ be the length (i.e. long side) and $b(n)$ be the width (i.e. short side) of $R(n)$. As $R(0)$ is consisted of the lowest row squares, $a(0)=100$ and $b(0)=1$. Following the rules, A can always make a move to decrease the length of $R(n)$. After B made $n+1$ moves, $a(n+1)+b(n+1)$ will either be $a(n)+b(n)$ or $a(n)+b(n)-2$. In particular, $a(n)+b(n)$ is always odd, non-increasing and $a(n) > b(n)$. Since the side of the board is finite, eventually $a(n)+b(n)$ must decrease to 3 and A can move his checker to meet B's checker in the next move.

Other commended solvers: **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)) and **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 412. $\triangle ABC$ is equilateral and points D, E, F are on sides BC, CA, AB respectively. If

$$\angle BAD + \angle CBE + \angle ACF = 120^\circ,$$

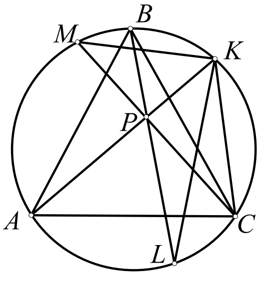
then prove that $\triangle BAD$, $\triangle CBE$ and $\triangle ACF$ cover $\triangle ABC$.
(*Source*: 2006 *Indian Math Olympiad Team Selection Test*)

Solution. **Jon GLIMMS** (Vancouver, Canada) and **William PENG**.

Assume P is in $\triangle ABC$ not covered by $\triangle BAD$, $\triangle CBE$ and $\triangle ACF$. Then $\angle BAD < \angle BAP$, $\angle CBE < \angle CBP$ and $\angle ACF < \angle ACP$. Adding these, we have

$$120^\circ < \angle BAP + \angle CBP + \angle ACP.$$

Now P cannot be the circumcenter of $\triangle ABC$ (otherwise $\angle BAP + \angle CBP + \angle ACP = 90^\circ$ would contradict the inequality above). So PA, PB, PC are not all equal. Suppose $PA > PB$. Let rays AP, BP, CP intersect the circumcircle of $\triangle ABC$ at points K, L, M respectively.



Since $\angle BAP = \angle KLP$ and $\angle ABP = \angle LKP$, $\triangle ABP$ and $\triangle LKP$ are similar. Then $PA > PB$ implies $PL > PK$ and so $\angle BAP = \angle KLP < \angle LKP$. We get

$$\begin{aligned}&\angle BAP + \angle CBP + \angle ACP\\ &= \angle KLP + \angle CKL + \angle AKM\\ &< \angle LKP + \angle CKL + \angle AKM\\ &< \angle BKC = 120^\circ,\end{aligned}$$

which contradicts the inequality above.

Other commended solvers: **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)) and **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France).

Problem 413. Determine (with proof) all integers $n \geq 3$ such that there exists a positive integer M_n satisfying the condition for all n positive numbers $a_1, a_2, \ldots, a_n$, we have

$$\frac{a_1+a_2+\cdots+a_n}{\sqrt[n]{a_1a_2\cdots a_n}} \leq M_n\left(\frac{a_2}{a_1}+\frac{a_3}{a_2}+\cdots+\frac{a_n}{a_{n-1}}+\frac{a_1}{a_n}\right).$$

(*Source*: 2005 *Chinese Taipei Math Olympiad Team Selection Test*)

Solution. **F5D** (Carmel Alison Lam Foundation Secondary School) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

For $n=3$, let $a_1, a_2, a_3 > 0$ and

$$x = \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_1}{a_3}.$$

Suppose $a_3 \geq a_1, a_2$. Then $x > a_2/a_1, a_3/a_2, a_1/a_3$. So $a_2 > a_3/x$ and $a_1 > a_2/x > a_3/x^2$. Hence,

$$\frac{a_1+a_2+a_3}{\sqrt[3]{a_1a_2a_3}} \leq \frac{3a_3}{\sqrt[3]{\frac{a_3}{x^2}\frac{a_3}{x}a_3}} = 3x.$$

So we can take $M_3=3$. For $n>3$, assume there is such M_n. Let $a_1 = c$, $a_2 = c^2, \ldots, a_n = c^n$. Then

$$M_n \geq \frac{c+c^2+\cdots+c^n}{\sqrt[n]{c^{n(n+1)/2}}}\left((n-1)c+\frac{1}{c^{n-1}}\right)^{-1}$$

$$\geq \frac{c^n}{c^{(n+1)/2}}\frac{1}{c((n-1)+c^{-n})} = \frac{c^{(n-3)/2}}{n-1+c^{-n}}.$$

As $c \to \infty$, $c^{(n-3)/2}/(n-1+c^{-n}) \to \infty$. Then M_n cannot be finite, contradiction.

Problem 414. Let p be an odd prime number and $a_1, a_2, \ldots, a_{p-1}$ be positive integers not divisible by p. Prove that there exist integers $b_1, b_2, \ldots, b_{p-1}$, each equals 1 or -1 such that

$$a_1b_1+a_2b_2+\cdots+a_{p-1}b_{p-1}$$

is divisible by p.

Solution. **Jon GLIMMS** (Vancouver, Canada).

For $k = 1, 2, \ldots, p-1$, we will prove the numbers of the form $a_1c_1 + a_2c_2 + \cdots + a_kc_k$ (where each c_i is 0 or 1) when divided by p will yield at least $k+1$ different remainders. For $k = 1$, we are given that $a_1 \not\equiv 0 \pmod p$.

Suppose a case $k < p-1$ is true. For the case k+1, if the numbers $a_1c_1+a_2c_2+\cdots+a_kc_k$ when divided by p yield at least k+2 different remainders, then the case k+1 is also true. Otherwise, there are numbers $m_1, m_2, \ldots, m_{k+1}$ of the form $a_1c_1+a_2c_2+\cdots+a_kc_k$ when divided by p yield exactly k+1 different remainders. Considering (mod p), we see m_1+a_{k+1}, $m_2+a_{k+1}, \ldots, m_{k+1}+a_{k+1}$ also have k+1 different remainders.

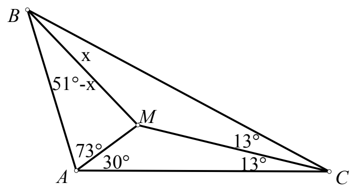
Assume these two groups of k+1 remainders are the same. Then we get $m_1+m_2+\cdots+m_{k+1} \equiv (m_1+a_{k+1}) + (m_2+a_{k+1}) + \cdots +(m_{k+1}+a_{k+1}) \pmod p$. This implies $(k+1)a_{k+1}\equiv 0 \pmod p$, which is not possible as $k+1<p$ and a_{k+1} is not divisible by p. Hence, there must be at least k+2 different remainders among the two groups. So the case k+1 is true.

Let $S=a_1+a_2+\cdots+a_{p-1}$. Since gcd(2,p) = 1, there is an integer r such that $2r \equiv S \pmod p$. From the case $k = p-1$ above, we see there is $a_1c_1 + a_2c_2 + \cdots + a_{p-1}c_{p-1} \equiv r \pmod p$. Let $b_i = 1-2c_i$, then $b_i = \pm 1$ and $a_1b_1 + a_2b_2 + \cdots + a_{p-1}b_{p-1} \equiv S-2r \equiv 0 \pmod p$.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 415. (*Due to MANOLOUDIS Apostolos***,** *Piraeus, Greece*) Given a triangle ABC such that $\angle BAC = 103^\circ$ and $\angle ABC = 51^\circ$. Let M be a point inside $\triangle ABC$ such that $\angle MAC = 30^\circ$ and $\angle MCA = 13^\circ$. Find $\angle MBC$ with proof.

Solution. **F5D** (Carmel Alison Lam Foundation Secondary School), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **Adrian Iain LAM** (St. Paul's College), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, Andhra Pradesh, India), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania)**.**



Let $x = \angle MBC$. By the trigonometric form of Ceva's theorem, we have

$$\frac{\sin 13^\circ}{\sin 13^\circ}\,\frac{\sin 73^\circ}{\sin 30^\circ}\,\frac{\sin x}{\sin(51^\circ - x)} = 1.$$

Then $2\sin 73^\circ = \dfrac{\sin 51^\circ \cos x - \cos 51^\circ \sin x}{\sin x}$

$= \sin 51°\cot x - \cos 51°$.

Using sin73° = cos17°, we get

$\cot x = (2\cos 17°+\cos 51°)/\sin 51°$. (*)

Since cot is strictly decreasing on (0°,51°), there is at most one such x. Now we have

$$\begin{aligned}2\sin y \cos y &= \sin 2y\\ &= \sin(3y-y)\\ &= \sin 3y \cos y - \cos 3y \sin y.\end{aligned}$$

Dividing by sin y leads to

$$2\cos y = \sin 3y \cot y - \cos 3y.$$

Solving for cot y and setting y=17°, we get

$$\cot 17° = (2\cos 17°+\cos 51°)/\sin 51°.$$

Therefore, x = 17°.

Other commended solvers: **Christian Pratama BUNAIDI (**University of Tarumanagara, Jakarta, Indonesia), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Prithwijit DE** (HBCSE, Mumbai, India), **Uma GIRISH** (Vidya Mandir Senior Secondary School, Chennai, India), **KWOK Man Yi** (S2, Baptist Lui Ming Choi Secondary School), **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France), **Mihai STOENESCU** (Bischwiller, France) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Olympiad Corner

(continued from page 1)

Problem 3. Trapzoid $ABCD$ with a longer base AB is inscribed in the circle k. Let A_0, B_0 be respectively the midpoints of segments BC, CA. Let N be the foot of the altitude from the point C to AB, and G the centroid of the trangle ABC. Circle k_1 goes through A_0 and B_0 and touches the circle k in the point X, different than C. Prove that the points D, G, N and X are collinear.

(*IMO Shortlist 2011, modified*)

Problem 4. For a given positive integer k let $S(k)$ denote the sum of all numbers from the set $\{1,2\ldots,k\}$ relatively prime to k. Let m be a positive integer and n an odd positive integer. Prove that there exist positive integers x and y such that m divides x and $2S(x) = y^n$.

(*Columbia 2008*)

The Inequality of A. Oppenheim

(continued from page 2)

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