TWO PROBLEMS AND THEIR GENERALIZATION

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Abstract. In this note we study two problems from Mathematical Olympiads using a novel approach and we consider their generalization vinculated with programs of simbolic computation.

The following problem was proposed in the XXVI Brazilian Undergraduate Mathematical Olympiad (see solution in [1]):
Let

$$S_n = \sum_{k=0}^{\infty} \frac{1}{(nk+1)(nk+2)\cdots(nk+n)}.$$

Evaluate $S_3 = \sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}$.
The problem of finding S_4 was proposed in IMC 2010 and it appears in [2] with two solutions. Our approach to these problems is novel because we include the Gamma function which allows the study of sums involving reciprocals of binomial coefficients and we rewrite the original sum as an integral. For $n \geq 5$ these integrals are calculated using programs of simbolic computation such as Maple or Mathematica.

Consider $S_4 = \sum_{n=0}^{\infty} \frac{1}{(4n+1)(4n+2)(4n+3)(4n+4)}$.
We have

$$\begin{aligned} S_4 &= \sum_{n=0}^{\infty} \frac{(4n)!}{(4n+4)!} \\ &= \frac{1}{6} \sum_{n=0}^{\infty} \frac{\Gamma(4n+1)\Gamma(4)}{\Gamma(4n+5)} \\ &= \frac{1}{6} \sum_{n=0}^{\infty} \beta(4n+1,4). \end{aligned}$$

Recall that

$$\begin{aligned} \Gamma(a) &= \int_0^{\infty} x^{a-1} e^{-x} dx \quad a>0 \\ \Gamma(n+1) &= n! \\ \beta(a,b) &= \int_0^1 x^{a-1}(1-x)^{b-1} dx \quad a>0,\ b>0 \\ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} &= \beta(a,b). \end{aligned}$$

It follows that

$$\begin{aligned} S_4 &= \frac{1}{6}\sum_{n=0}^{\infty}\int_0^1 x^{4n}(1-x)^3dx \\ &= \frac{1}{6}\int_0^1\sum_{n=0}^{\infty} x^{4n}(1-x)^3dx \\ &= \frac{1}{6}\int_0^1(1-x)^3\sum_{n=0}^{\infty} x^{4n}dx. \end{aligned}$$

But

$$\sum_{n=0}^{\infty} x^{4n} = \frac{1}{1-x^4},$$

hence

$$S_4 = \frac{1}{6}\int_0^1 \frac{(1-x)^3}{1-x^4}dx = \frac{1}{6}\int_0^1 \frac{(1-x)^2}{(1+x)(1+x^2)}dx.$$

Because

$$\frac{(1-x)^2}{(1+x)(1+x^2)} = \frac{2}{1+x} - \frac{1}{1+x^2} - \frac{x}{1+x^2},$$

$$\begin{aligned} \int_0^1 \frac{(1-x)^2}{(1+x)(1+x^2)}dx &= 2\int_0^1 \frac{1}{1+x}dx - \int_0^1 \frac{1}{1+x^2}dx - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx \\ &= 2\ln(1+x)\ |_0^1 - \arctan(x)\ |_0^1 - \frac{1}{2}\ln(1+x^2)\ |_0^1 \\ &= 2\ln 2 - \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{3}{2}\ln 2 - \frac{\pi}{4}, \end{aligned}$$

so finally

$$S_4 = \frac{\ln 2}{4} - \frac{\pi}{24}.$$

Similarly,

$$S_n = \frac{1}{(n-1)!}\int_0^1 \frac{(1-x)^{n-1}}{1-x^n}dx.$$

Trying to calculate this integral without the use of a computer we can see that the "easy" cases are $n = 2, 3, 4$. Let us see some values of S_n by using Maple:

$$\begin{aligned} S_6 &= -\frac{7\sqrt{3}}{4320}\pi - \frac{3}{160}\ln 3 + \frac{2}{45}\ln 2 \\ S_8 &= \frac{17}{6720}\ln 2 + \frac{11}{40320}\pi - \frac{17}{40320}\sqrt{2}\ln(2+\sqrt{2}) \\ &+ \frac{31}{40320}\sqrt{2}\ln(2-\sqrt{2}) - \frac{1}{4032}\sqrt{2}\pi - \frac{1}{5760}\sqrt{2}\ln 2 \end{aligned}$$

We invite the readers to evaluate S_5, S_7, S_9, S_{10}. To conclude, we would like to mention that it is an open problem to show that S_n is irrational for all n.

References:

[1] Eureka 22. Olimpiada Brasilera de Matemática
[2]`http://www.imc-math.org.uk/imc2010/imc2010-day1-solutions.pdf`

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