On the continuous functions having bilateral infinite derivative at no point

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Abstract Following Banach's proof of a celebrated result, we prove that the set of continuous functions of one variable having bilateral, infinite derivative at least at one point, is of first Baire's category.

1. Introduction

Let $C(I)$ be the space of the real continuous functions on $I=[a,b]$ equipped with the sup–norm. We define the four Dini–derivatives

$$D^+f(x)=\overline{\lim}_{h\to 0^+}\frac{f(x+h)-f(x)}{h},\qquad D_+f(x)=\underline{\lim}_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$$
$$D^-f(x)=\overline{\lim}_{h\to 0^-}\frac{f(x+h)-f(x)}{h},\qquad D_-f(x)=\underline{\lim}_{h\to 0^-}\frac{f(x+h)-f(x)}{h}$$

$D^+f(x)=D_+f(x)\doteq f'(x^+)$ means that f is differentiable at right and $D^-f(x)=D_-f(x)\doteq f'(x^-)$ at left[1]

A Banach's celebrated theorem states that

Main theorem *As a subset of $C(I)$ it is of first category the set of functions having $-\infty<D_+f(x)\le D^+f(x)<+\infty$ and/or $-\infty<D_-f(x)\le D^-f(x)<+\infty$ at least at one point.*

We recall that a subset S of $C(I)$, is a set of first Baire's category if it can be written as a denumerable union of nowhere dense sets that is $S=\bigcup_{k=1}^{\infty}S_k$ and $\overset{\circ}{\overline{S}}_k=\emptyset$. Thus the set of functions having a *finite* derivative (even unilateral) at least at one point is of first category. The main reference is the original Banach's paper [B]. Other references are [M], [Bo] p.62, [HS] pp.260–262, [K] pp.420–421, [Y] pp.72–73, [O] pp.45–46, [T] pp.71–84

According to [S][2] (see also [Br1] p.680, [O] p.46), following the original Banach's argument in [B], the following result can be proven

Corollary *The set of functions in $C(I)$ having bilateral, infinite, derivative at least at one point is of first category.*

In formulae the set of functions having $f'(x^+)=f'(x^-)=+\infty$ or $-\infty$ at least at one point is of first category in $C(I)$.
As far as we know, the proof of this fact is present in [BG], [Br] p.143 but the authors do not adopt Banach's strategy.

Our aim is to give a proof following Banach's.

As a consequence of the Main theorem and the Corollary, in the sense of category almost all functions of $C(I)$, do not admit unilateral or bilateral finite derivative or bilateral infinite derivative at any point. These non–differentiable functions are usually said of "Weiestrass–type" being Weiertrass the first who provided an example around 1880 (see [H] for an accurate reference as well as for a substantial improvement of the original condition on the non–differentiability). However according to [T], the first example of a continuous nowhere differentiable function may be dated back to 1830 by Bertrand Bolzano although it remained unpublished until 1930.

[1] At $x=a$ we have only the first two limits; the second two limits at $x=b$

[2] Let me remind that Stanislaw Saks was killed by the nazis in 1942 because of his jewish origin. Here http://matwbn.icm.edu.pl/ksiazki/fm/fm33/fm3311.pdf the interested reader may find what happened to many Polish scientists after the Nazi's occupation of Poland and here http://turnbull.dcs.st-and.ac.uk/history/Mathematicians/Saks.html for a biography of Saks.

Contrary to what one might believe, the subset of $C(I)$ of the functions having *unilateral* infinite derivative at least at one point, is *not* of first category . Precisely it is of *second category* in $C(I)$ the set of functions having infinite unilateral derivative at a non–denumerable subset of $[a, b]$ as proved in [S]. Later we will point out why our proof cannot be adapted to this case. Saks' result explains why it is usually difficult to find continuous functions not having derivative, unilateral or bilateral, finite or infinite at any point. This type of functions is said of "Besicovitch–type" being Besicovitch the first who provided an example ([Be], [P], [M], [Si1], [Si2], [Ha]).

2. Main Result

Let $S \subset C(I)$ be the set of functions defined over $I = [a, b]$ having bilateral infinite derivative at least at one point of (a, b).

2.1 Proposition *S is of first Baire–category*

Proof We define the sets $F_{m,n}$ and $E_{m,n}$

$$F_{m,n} = \Big\{ f \in C(I) \mid \exists\, x \in [a + \frac{b-a}{n}, b - \frac{b-a}{n}],\ \exists\, \{\xi_n, \lambda_n\} \textit{ not depending on } f,\ \xi_n > 0$$
$$\lambda_n \leq \frac{b-a}{n},\ \lambda_n - \xi_n \geq \frac{b-a}{2n} :\ \xi_n \leq |h| \leq \lambda_n\ \Rightarrow\ \frac{f(x+h) - f(x)}{h} \geq m \Big\}$$

$$E_{m,n} = \Big\{ f \in C(I) \mid \exists\, x \in [a + \frac{b-a}{n}, b - \frac{b-a}{n}],\ \exists\, \{\xi_n, \lambda_n\} \textit{ not depending on } f,\ \xi_n > 0$$
$$\lambda_n \leq \frac{b-a}{n},\ \lambda_n - \xi_n \geq \frac{b-a}{2n} :\ \xi_n \leq |h| \leq \lambda_n\ \Rightarrow\ \frac{f(x+h) - f(x)}{h} \leq -m \Big\}$$

The first step is $S \subset \bigcup_{n=2}^{\infty} (F_{m,n} \cup E_{m,n})$. This may be seen by observing that if $f \in S$ then for any integer c there exists δ_c such that $\dfrac{f(x+h) - f(x)}{h} > c$ for any $0 < |h| < \delta_c$ (we consider the derivative equal to $+\infty$). Taking

$$\xi_n = \min\{\frac{1}{2}\delta_c, \frac{b-a}{2n}\}, \qquad \lambda_n = \min\{\delta_c, \frac{b-a}{n}\}, \qquad n \geq \frac{b-a}{\delta_c}, \qquad m = c - 1$$

we ensure that $f \in F_{m,n}$.

The second step is to show that: 1) $F_{m,n} = \overline{F}_{m,n}$, 2) $\overset{\circ}{F}_{m,n} = \emptyset$. Of course the set $E_{m,n}$ has the same properties. It follows that $\bigcup_{n=2}^{\infty} (F_{m,n} \cup E_{m,n})$ is of first category and then also S.

We start with 1). The argument here is exactly as that of Banach. Let $f_k \to f$ be a sequence of functions $f_k \in F_{m,n}$ converging to f. To each k corresponds x_k as defined in the definition of $F_{m,n}$ and then a subsequence x_{k_j} such that $x_{k_j} \to x_o$ for $j \to +\infty$, $x_o \in [a + \frac{b-a}{n}, b - \frac{b-a}{n}]$. We have

$$\frac{f(x_o + h) - f(x_o)}{h} = \sum_{k=1}^{5} g_k$$

where

$$g_1 = \frac{f_{k_j}(x_{k_j} + h) - f_{k_j}(x_{k_j})}{h}, \quad g_2 = \frac{f(x_o + h) - f_{k_j}(x_o + h)}{h},$$
$$g_3 = \frac{f_{k_j}(x_o + h) - f_{k_j}(x_{k_j} + h)}{h}, \quad g_4 = \frac{f_{k_j}(x_{k_j}) - f_{k_j}(x_o)}{h}, \quad g_5 = \frac{f_{k_j}(x_o) - f(x_o)}{h}$$

and of course

$$\frac{f(x_o+h)-f(x_o)}{h} \geq g_1 - \sum_{k=2}^{5} |g_k|$$

The continuity of each f_{k_j}, the convergence to f as well as the convergence of x_{k_j} to x_o, imply $|g_k| < \varepsilon$ for $k = 2, 3, 4, 5$ and then

$$\forall\ \varepsilon > 0 \quad \xi_n < |h| < \lambda_n \ \Rightarrow\ \frac{f(x_o+h)-f(x_o)}{h} > m - 4\varepsilon$$

hence

$$\frac{f(x_o+h)-f(x_o)}{h} \geq m.$$

Now we prove 2) that is $\overset{\circ}{F}_{m,n} = \emptyset$ which is implied by $\overline{F^c_{m,n}} = C(I) \qquad (F^c_{m,n} = C(I) \backslash F_{m,n})$.

$$F^c_{m,n} = \Big\{ f \in C(I) : \forall\ x \in [a + \frac{b-a}{n}, b - \frac{b-a}{n}],$$
$$\forall\ 0 < \xi < \lambda < \frac{b-a}{n},\ \lambda - \xi \geq \frac{b-a}{2n},$$
$$\exists\ h_o \in (\xi, \lambda) : \quad \left(m > \frac{f(x+h_o)-f(x)}{h_o} \vee m > \frac{f(x-h_o)-f(x)}{-h_o} \right) \Big\}$$

We prove that the set of continuous piecewise–linear functions, whose slope at the points of differentiability is sufficiently large in absolute value, is a subset of $F^c_{m,n}$. Since the set of the piecewise–linear functions is dense in $C(I)$ (a standard argument which can be found in [HS], [O] for instance), it follows $\overline{F^c_{m,n}} = C(I)$. So we fix $x \in [a + \frac{b-a}{n}, b - \frac{b-a}{n}] \doteq [a_n, b_n]$, $n \geq 2$. Let ξ, λ be such that $0 < \xi < \lambda < \frac{b-a}{n}$, $\lambda - \xi \geq \frac{b-a}{2n}$, and let $p(x)$ be a piecewise linear, continuous periodic functions such that $p(a_n) = p(b_n) = 0$ (otherwise take $p(x) + \alpha x + \beta$ with suitable α, β). Taking the period T of $p(x)$ small enough (say $T < (b-a)/(2n)$), there exists y such that $p(x) = p(y)$ that is $\frac{p(x+h_o)-p(x)}{h_o} = 0 < m$ where $x + h_o = y$ and $|h_o| \in (\xi, \eta)$. As for $E_{m,n}$, we proceed in the same way being $\frac{p(x+h'_o)-p(x)}{h'_o} = 0 > -m$ with h'_o in place of h_o. It is important to note that $\lambda - \xi \geq (b-a)/(2n)$. Q.E.D.

We now explain why our argument does not work if we considered unilateral (right for instance) differentiable functions with $+\infty$ derivative at a point. As for the definition of $F_{m,n}$ and $F^c_{m,n}$ we would have

$$F_{m,n} = \Big\{ f \in C(I) :\ \exists\ x \in [a + \frac{b-a}{n}, b - \frac{b-a}{n}] :$$
$$1)\ \exists\ \{\xi_n, \lambda_n\}\ \textit{not depending on } f,\ \xi_n > 0,\ \lambda_n \leq \frac{b-a}{n},\ \lambda_n - \xi_n \geq \frac{b-a}{2n},$$
$$:\ \xi_n \leq h \leq \lambda_n \Longrightarrow \frac{f(x+h)-f(x)}{h} \geq m$$
$$2)\ \exists\ r :\ \forall\ 0 < \delta < \frac{b-a}{n}\ \exists\ h \in (0, \delta) : \frac{f(x-h)-f(x)}{-h} \leq r \Big\}$$

The set of functions that have derivative at x equal to $+\infty$ at right and do not have derivative equal to $+\infty$ at left (they could not have derivative at all) is a subset of each $F_{m,n}$.

$$F^c_{m,n} = \Big\{ f \in C(I) : \forall\ x \in [a + \frac{b-a}{n}, b - \frac{b-a}{n}] :$$
$$1')\ \forall\ 0 < \xi < \lambda \leq \frac{b-a}{n},\ \lambda - \xi \geq \frac{b-a}{2n},\ \exists\ h_o \in (\xi, \lambda)\ : m > \frac{f(x+h_o)-f(x)}{h_o}$$

or

$$2')\ \forall\ r > 0\ \exists\ 0 < \delta_r < \frac{b-a}{n} :\ 0 < h < \delta_r \implies \frac{f(x-h)-f(x)}{-h} > r \Big\}$$

Now if $1')$ holds but $2')$ does not hold, thus 2) holds, the continuous piecewise linear functions with arbitrarily large slope would be no more a subset of $F^c_{m,n}$. Indeed those functions, call them $p(x)$, cannot satisfy 2) neither at the points of differentiability nor at the points of non–differentiability. Any point x has at left a neighborhood $(x-\delta, x]$ such that $(p(x')-p(x))/(x'-x)$ is large enough $(x-\delta < x' < x)$.

If instead $2')$ holds, since a continuous piecewise linear functions $p(x)$, has a fixed slope at the points of differentiability, it can't satisfy $2')$.

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