Mathematical Excalibur

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Olympiad Corner

1997 Chinese Mathematical Olympiad:

Part I (8:00-12:30, January 13, 1997)

Problem 1. Let $x_1, x_2, \ldots, x_{1997}$ be real numbers satisfying the following two conditions:

(1) $-\dfrac{1}{\sqrt{3}} \le x_i \le \sqrt{3}$ $(i = 1, 2, \ldots, 1997)$;

(2) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$.

Find the maximum value of

$$x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}.$$

Problem 2. Let $A_1B_1C_1D_1$ be an arbitrary convex quadrilateral. Let P be a point inside the quadrilateral such that the segments from P to each vertex form acute angles with the two sides through the vertex. Recursively define A_k, B_k, C_k and D_k as the points symmetric to P with respect to the lines $A_{k-1}B_{k-1}$, $B_{k-1}C_{k-1}$, $C_{k-1}D_{k-1}$ and $D_{k-1}A_{k-1}$, respectively $(k = 2, 3, \cdots)$.

(*continued on page 4*)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is Apr. 5, 1997.

For individual subscription for the remaining three issues for the 96-97 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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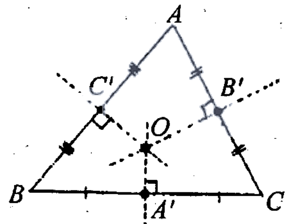
老師不教的幾何（二）

張百康

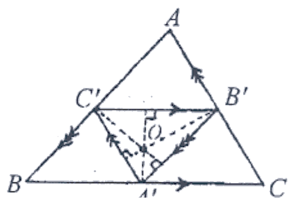
十八世紀時，瑞士出了一位大數學家歐拉 (Euler)。他雖然在二十多歲時已一目失明，但畢生努力從事數學研究，著作豐富。數學上不少定理、公式和方法都是他發現、證明或發明的。我們現在要介紹的是歐拉線 (Euler Line) —— 一條貫穿三角形幾個重要的點的直線。

對一任意的三角形 ABC，通過它的三條邊的中點 (mid-points) A'，B' 和 C' 分別作出這三條邊的垂直平分線 (perpendicular bisectors)。我們知道：這三條垂直平分線相交於同一點，即圖一的點 O。這點 O 就是三角形 ABC 的外接圓心 (circumcentre)，道理相信大家已知道。



（圖一）

另一方面，三角形 $A'B'C'$ 和 ABC 不但相似，而且對應邊平行。這個邊長縮小一半的三角形 $A'B'C'$ 稱爲三角形 ABC 的中點三角形 (medial triangle)。它的三條高 (altitudes) 剛好就落在 OA'，OB' 和 OC' 上，因此 O 點也扮演了中點三角形 $A'B'C'$ 的垂心 (orthocentre) 角色（圖二）。



（圖二）

歐拉發現任何三角形的外接圓心 (O)、重心 (G) 和垂心 (H) 共線，他的證明如下（圖三）：

由於三角形 ABC 的高 AH 和邊 BC 的垂直平分線 OA' 平行，因此

$$\angle HAG = \angle OA'G$$

並且，AH 和 $A'O$ 分別是相似三角形 ABC 和 $A'B'C'$ 的對應線，所以

$$AH : A'O = BC : B'C' = 2 : 1$$

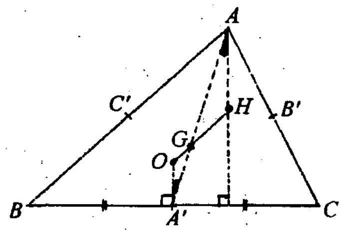
恰巧地，重心 G 也把中線 AA' 分成

$$AG : A'G = 2 : 1$$

因此，三角形 HAG 和 $OA'G$ 相似。由此推知

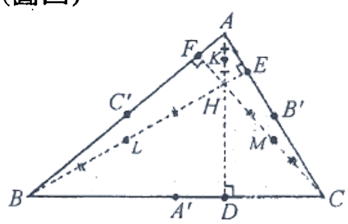
$$\angle HGA = \angle OGA'$$

所以 O、G、H 成一直線，稱爲歐拉線，並且 $OG : GH = 1 : 2$。



（圖三）

歐拉線 OH 的中點絕不平凡，它是著名的九點圓 (Nine-point circle) 的圓心。所謂九點圓是指一個通過三角形 ABC 的三邊的中點 A'、B'、C'，三高的垂足 D、E、F 以及三頂點和垂心間的中點 K、L、M 的圓（圖四）。



（圖四）

有關這九點爲甚麼共圓的完整證明是數學家彭賽列 (Poncelet) 於 1821 年首先給出的，他將 A'，B'，C'，K，L，M 六點分成互有重覆四點組合，然後證明每個組合的四點共圓，再利用這三個組合的重覆性證明這三個圓實質上是同一個圓，最後證明 D，E，F 也在這圓上。讓我們看看他的證法：

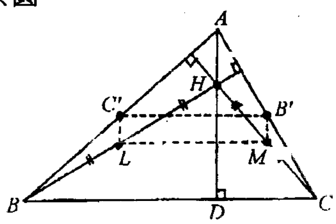
先考慮 B'，C'，L，M 四點（圖五）。

(*continued on page 2*)

老師不教的幾何 (二)

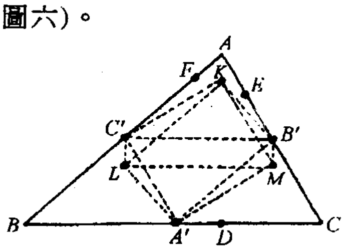
(*continued from page 1*)

在三角形 ABH 中，C' 和 L 分別是邊 AB 和 HB 的中點，因此 $C'L$ 平行 AH。同理，在三角形 ACH 中，$B'M$ 平行 AH。所以 $C'L$ 平行 $B'M$。再考慮三角形 ABC 和 HBC，利用同樣的中點定理，可知 $B'C'$ 平行 ML 和 CB。由於 AD 垂直 BC，因此 $B'C'LM$ 是個矩形。矩形的頂點當然共圓。



(圖五)

重覆同樣的論証於 $A'C'KM$ 和 $A'B'KL$ 可推證它們也是矩形，因此分別共圓。但這三個矩形兩兩有共同對角線，即外接圓 (circumcircle) 的直徑 (圖六)。

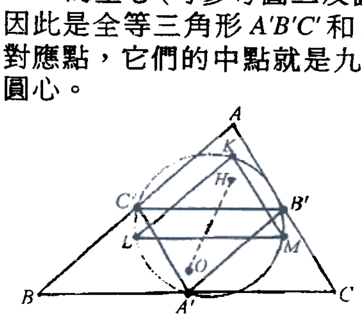


(圖六)

不同的圓不可能有共同直徑，因此 A'，B'，C'，K，L，M 六點共圓。另一方面，$\angle A'DK$ 是直角(圖四)，而 $A'K$ 是前述六點圓的直徑，因此 D 也在此六點圓上。同理，E 和 F 也在此六點圓上，所以九點共圓。

九點圓和歐拉線有甚麼關係？

大家不妨細心比較兩個頂點都在九點圓上的三角形 $A'B'C'$ 和 KLM (圖七)。由於 KA'，LB' 和 MC' 是九點圓的直徑，因此三角形 KLM 繞九點圓的圓心旋轉 180° 可得三角形 $A'B'C'$。三角形 ABC 的歐拉線 OH 兩端恰巧正分別是三角形 $A'B'C'$ 和 KLM 的垂心(可參考圖二及圖四)，因此是全等三角形 $A'B'C'$ 和 KLM 的對應點，它們的中點就是九點圓的圓心。



(圖七)

歐拉線眞不簡單，它一線穿四心，說它是三角形的脊骨一點也不過份。

$\sqrt{2}$ 是無理數的六個證明

香港大學數學系
蕭文強

「如何證明 $\sqrt{2}$ 是無理數呢？」

「那還不容易！設 $\sqrt{2}=m/n$，可當 m 和 n 不全是偶數。由於 $m^2=2n^2$，m 必是偶數，寫作 $2k$，則 $4k^2=2n^2$，$2k^2=n^2$，故 n 亦是偶數，矛盾！」

上述證明，只用到奇偶性質，來源已不可稽考。亞里士多德（ARISTOTLE）在公元前 330 年左右把它（以幾何形式）寫下來，用作反證法的示範，可見在那個時候這回事已是衆所週知了。不過由於這證明是如此簡潔，很多數學史家都相信那不是這回事的發現經過，而是「事後孔明」的解釋。

在這個證明中，2 沒有什麼特別，換了是另一個質數，同樣的思路仍可沿用，只是單憑奇偶性質不足夠，需要用到質因子唯一分解性質。再推廣少許，我們還能夠證明若 P_1、$\cdots$、P_s 是 s 個不同的質數，則 $\sqrt{P_1\cdots P_s}$ 是無理數。因此，若 H 不是完全平方，則 $\sqrt{H}$ 是無理數。其實，如果我們願意運用質因子唯一分解性質，還有另一個證明辦法，即是數一數 $m^2=Hn^2$ 兩邊中某質因子出現的次數，一奇一偶，矛盾！

讓我們來看第三個證明。設 $\sqrt{H}=m/n$，可當 m 和 n 無公共因子。由於 $m^2=Hn^2=n(Hn)$，n 必須是 1 或 -1，即是說 H 是個完全平方，矛盾！這個證明跟前兩個證明有一點不相同，它能推廣至頗一般的情況，證明了若有理數是代數整數，則它必是整數。（代數整數是指首一整數系數多項式方程 $x^N+c_{N-1}x^{N-1}+\cdots+c_1x+c_0=0$ 的根，例如 $\sqrt{H}$ 是 $x^2-H=0$ 的根。請讀者試自行證明這回事吧。）

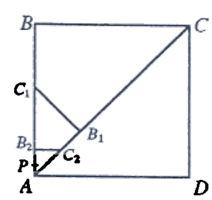
現在再看一個十分簡捷的證明：若 $\sqrt{2}$ 是有理數，取最小正整數 k 使 $k\sqrt{2}$ 是整數，則 $m=k\sqrt{2}-k=k(\sqrt{2}-1)$ 是一個較 k 更小的正整數，但 $m\sqrt{2}=2k-k\sqrt{2}$ 仍是整數，這與 k 的選取矛盾！（把 2 換作一個非完全平方 H，類似的證明適用。）

上述證明是數論專家埃斯特曼（THEODOR ESTERMANN）在 1975 年一則短文的內容，巧妙簡捷，兼而有之。後來有人讚曰：「如同所有精采念頭，一經指出即明顯不過，但這個精采念頭卻要等到畢達哥拉斯 (PYTHAGORAS) 二千多年後才給指出來！」如果我們試圖追尋如何選取 m 的線索，自然會問到它的幾何詮釋，這個幾何詮釋，說不定正是二千多年前希臘數學家發現正方形的對角線和邊是不可公度量的經過呢！不可公度量，是指不存在一公共度量，使對角線和邊各自是該公共度量的若干整數倍，也就是說，$\sqrt{2}$ 不是有理數。（以下敘述，取材於 H. EVES 的著作 "AN INTRODUCTION TO THE HISTORY OF MATHEMATICS" 的第 3 章，3rd edition，1969。）在下圖中設 AP 是正方形的對角線 AC 和邊 AB 的公共度量，即有 $AC=jAP$ 和 $AB=kAP$。構作 B_1C_1 使 B_1C_1 垂直於 AC，也使 $CB=CB_1$。不難知道 $BC_1=B_1C_1=AB_1$，因此

$$AC_1=AB-AB_1=AB-(AC-AB)=2AB-AC=(2k-j)AP,$$

$$AB_1=AC-AB=(j-k)AP。$$

注意：AC_1 和 AB_1 是一個較小的正方形的對角線和邊，那個較小的正方形的邊 AB_1 小於原正方形的邊 AB 的一半。按此步驟重複下去，必得到一個足夠小的正方形，它的邊 AB_t 小於 AP，但 AB_t 卻仍然是 AP 的若干整數倍，豈非矛盾！（有些數學史家認爲古代希臘數學家曾企圖以此方法研究不可公度量理論，相當於企圖發展今天稱作連分數展開式的研究。可惜當時的數學家無功而退，只遺留下來蛛絲馬跡，在古希臘數學名著《歐幾里得原本》(EUCLID'S ELEMENTS) 的章節間依稀可見！）



請注意：$AC_1/AB_1=(2k-j)/(j-k)$，而 $m=j-k$ 正是埃斯特曼的短小精悍證明中的 m。因爲 $AC_1/AB_1=\sqrt{2}$，便有 $(2k-j)/m=\sqrt{2}$，即是 $m\sqrt{2}=2k-j$ 是整數了。當我們了解埃斯特曼證明的背後的幾何詮釋，我們可以把它重寫成第六個證明：若 $\sqrt{2}=j/k$ 是最簡的分數式，則有 $\sqrt{2}=(2k-j)/(j-k)$（這是因爲 $j\sqrt{2}-k\sqrt{2}=2k-j$），但 $k<j<2k$（因爲 $1<\sqrt{2}<2$），故 $2k-j<j$ 和 $j-k<k$，這與 j 和 k 的選取矛盾！

請讀者想一想，上面討論的六個證明，眞的是六個不同的證明嗎？還是六個相同的證明呢？

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is Apr. 5, 1997.

Problem 51. Is there a positive integer n such that $\sqrt{n-1}+\sqrt{n+1}$ is a rational number?

Problem 52. Let a, b, c be distinct real numbers such that $a^3 = 3(b^2+c^2) - 25$, $b^3 = 3(c^2+a^2) - 25$, $c^3 = 3(a^2+b^2) - 25$. Find the value of abc.

Problem 53. For ΔABC, define A' on BC so that $AB + BA' = AC + CA'$ and similarly define B' on CA and C' on AB. Show that AA', BB', CC' are concurrent. (The point of concurrency is called the *Nagel point* of ΔABC.)

Problem 54. Let R be the set of real numbers. Find all functions $f: R \to R$ such that

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy$$

for all $x, y \in R$. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

Problem 55. In the beginning, 65 beetles are placed at different squares of a 9×9 square board. In each move, every beetle creeps to a horizontal or vertical adjacent square. If no beetle makes either two horizontal moves or two vertical moves in succession, show that after some moves, there will be at least two beetles in the same square. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

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Solutions

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Problem 46. For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$? (Source: 1963 Putnam Exam.)

Solution: CHEUNG Tak Fai (Valtorta College, Form 6) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4).

Suppose

$$x^{13} + x + 90 = (x^2 - x + a)q(x),$$

where $q(x)$ is a polynomial with integer coefficients. Taking $x = -1, 0, 1$, we get

$$88 = (2+a)q(-1),$$
$$90 = aq(0)$$
and
$$92 = aq(1).$$

Since a divides 90, 92 and $a+2$ divides 88, a can only be 2 or -1. Now $x^2 - x - 1$ has a positive root, but $x^{13} + x + 90$ cannot have a positive root. So a can only be 2. We can check by long division that $x^2 - x + 2$ divides $x^{13} + x + 90$ or observe that if w is any of the two roots of $x^2 - x + 2$, then $w^2 = w - 2$, $w^4 = -3w + 2$, $w^8 = -3w - 14$, $w^{12} = 45w - 46$ and $w^{13} + w + 90 = 0$.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHAN Wing Sum** (HKUST) and **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6).

Problem 47. If x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$, then show that $x + y + z \le xyz + 2$.

Solution: CHAN Ming Chiu (La Salle College, Form 6).

If one of x, y, z is nonpositive, say z, then

$$2 + xyz - x - y - z = (2-x-y) - z(1-xy) \ge 0$$

because

$$x + y \le \sqrt{2(x^2+y^2)} \le 2$$

and

$$xy \le (x^2 + y^2)/2 \le 1$$

So we may assume x, y, z are positive, say $0 < x \le y \le z$. If $z \le 1$, then

$$2 + xyz - x - y - z = (1-x)(1-y) + (1-z)(1-xy) \ge 0.$$

If $z > 1$, then

$$(x + y) + z \le \sqrt{2\left((x+y)^2 + z^2\right)} = 2\sqrt{xy+1} \le xy + 2 \le xyz + 2.$$

Comments: This was an unused problem in the 1987 IMO and later appeared as a problem on the 1991 Polish Mathematical Olympiad.

Problem 48. Squares $ABDE$ and $BCFG$ are drawn outside of triangle ABC. Prove that triangle ABC is isosceles if DG is parallel to AC.

Solution: Henry NG Ka Man (STFA Leung Kau Kui College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4) and **YUNG Fai** (CUHK).

From B, draw a perpendicular line to AC (and hence also perpendicular to DG.) Let it intersect AC at X and DG at Y. Since $\angle ABX = 90° - \angle DBY = \angle BDY$ and $AB = BD$, the right triangles ABX and BDY are congruent and $AX = BY$. Similarly, the right triangles CBX and BGY are congruent and $BY = CX$. So $AX = CX$, which implies $AB = CB$.

Comments: This was a problem on the 1988 Leningrad Mathematical Olympiad. Most solvers gave solutions using pure geometry or a bit of trigonometry. The editor will like to point out there is also a simple vector solution. Set the origin O at the midpoint of AC. Let $\vec{OC} = m$, $\vec{OB} = n$ and k be the unit vector perpendicular to the plane. Then $\vec{AB} = n + m$, $\vec{CB} = n - m$, $\vec{BD} = -(n + m) \times k$, $\vec{BG} = (n - m) \times k$ and $\vec{DG} = \vec{BG} - \vec{BD} = 2n \times k$. If DG is parallel to AC, then $n \times k$ is a multiple of m and so $m = \vec{OC}$ and $n = \vec{OB}$ are perpendicular. Therefore, triangle ABC is isosceles.

Other commended solvers: **CHAN Wing Chiu** (La Salle College, Form 4), **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 5), **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6), **Yves CHEUNG Yui Ho** (S.T.F.A. Leung Kau Kui College, Form 5), **CHING Wai Hung** (S.T.F.A. Leung Kau Kui College, Form 5), **Alan LEUNG Wing Lun** (STFA Leung Kau Kui College, Form 5), **OR Fook Sing** & **WAN Tsz Kit** (Valtorta College, Form 6), **TSANG Sai Wing** (Valtorta College, Form 6), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 5), **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

Problem 49. Let u_1, u_2, u_3, ... be a sequence of integers such that $u_1 = 29$, $u_2 = 45$ and $u_{n+2} = u_{n+1}^2 - u_n$ for $n = 1, 2, 3, \ldots$. Show that 1996 divides infinitely many terms of this sequence. (Source: 1986 Canadian Mathematical Olympiad with modification)

Solution: William CHEUNG Pok-man (STFA Leung Kau Kui College, Form 6) and **YUNG Fai** (CUHK).

Let U_n be the remainder of u_n upon division by 1996, i.e.,

$$U_n \equiv u_n \pmod{1996}.$$

Consider the sequence of pairs (U_n, U_{n+1}). There are at most 1996^2 distinct pairs. So let $(U_p, U_{p+1}) = (U_q, U_{q+1})$ be the first repetition with $p < q$. If $p > 1$, then the recurrence relation implies $(U_{p-1}, U_p) = (U_{q-1}, U_q)$ resulting in an earlier repetition. So $p = 1$ and the sequence of pairs (U_n, U_{n+1}) is periodic with period $q - 1$. Since $u_3 = 1996$, we have $0 = U_3 = U_{3+k(q-1)}$ and so 1996 divides $u_{3+k(q-1)}$ for every positive integer k.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHAN Wing Sum** (HKUST) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4).

Problem 50. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$). Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ are primes? (Source: unused problem in the 1996 IMO.)

Solution 1: Henry NG Ka Man (STFA Leung Kau Kui College, Form 6) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4).

If the initial numbers are $a = w$, $b = x$, $c = y$, $d = z$, then after 4 steps, the numbers will be

$$a = 2(w - 2x + 3y - 2z),$$
$$b = 2(x - 2y + 3z - 2w),$$
$$c = 2(y - 2z + 3w - 2x),$$
$$d = 2(z - 2w + 3y - 2z).$$

From that point on, a, b, c, d will always be even, so $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ will always be divisible by 4.

Solution 2: Official Solution.

After $n \geq 1$ steps, the sum of the integers will be 0. So $d = -a - b - c$. Then

$$\begin{aligned} bc - ad &= bc + a(a + b + c) \\ &= (a + b)(a + c). \end{aligned}$$

Similarly,

$$ac - bd = (a + b)(b + c)$$

and

$$ab - cd = (a + c)(b + c).$$

Finally $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ cannot all be prime because their product is the square of $(a+b)(a+c)(b+c)$.

Other commended solvers: **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 5) and **William CHEUNG Pok-man** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(*continued from page 1*)

Consider the sequence of quadrilaterals

$$A_jB_jC_jD_j \quad (j = 1, 2, \cdots).$$

(1) Determine which of the first 12 quadrilaterals are similar to the 1997th quadrilateral.

(2) If the 1997th quadrilateral is cyclic, determine which of the first 12 quadrilaterals are cyclic.

Problem 3. Prove that there are infinitely many natural numbers n such that

$$1, 2, \cdots, 3n$$

can be put into an array

$$\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{array}$$

satisfying the following two conditions:

(1) $a_1+b_1+c_1 = a_2+b_2+c_2 = \cdots = a_n+b_n+c_n$ and the sum is a multiple of 6;

(2) $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n = c_1+c_2+\cdots+c_n$ and the sum is a multiple of 6.

Part II (8:00-12:30, January 14, 1997)

Problem 4. Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the two tangents QE and QF to the circle where E and F are the points of tangency. Prove that the three points P, E, F are collinear.

Problem 5. Let $A = \{1, 2, 3, \cdots, 17\}$ For a mapping $f: A \to A$, denote

$$f^{[1]}(x) = f(x),$$
$$f^{[k+1]}(x) = f\left(f^{[k]}(x)\right) \quad (k = 1, 2, 3,$$

Consider one-to-one mappings f from A to A satisfying the condition: there exists a natural number M such that

(1) for $m < M$, $1 \leq i \leq 16$,

$$f^{[m]}(i+1) - f^{[m]}(i) \not\equiv \pm 1 \pmod{17},$$
$$f^{[m]}(1) - f^{[m]}(17) \not\equiv \pm 1 \pmod{17};$$

(2) for $1 \leq i \leq 16$,

$$f^{[M]}(i+1) - f^{[M]}(i) \equiv 1 \text{ or } -1 \pmod{17},$$
$$f^{[M]}(1) - f^{[M]}(17) \equiv 1 \text{ or } -1 \pmod{17}.$$

For all mappings f satisfying the above condition, determine the largest possible value of the corresponding M's.

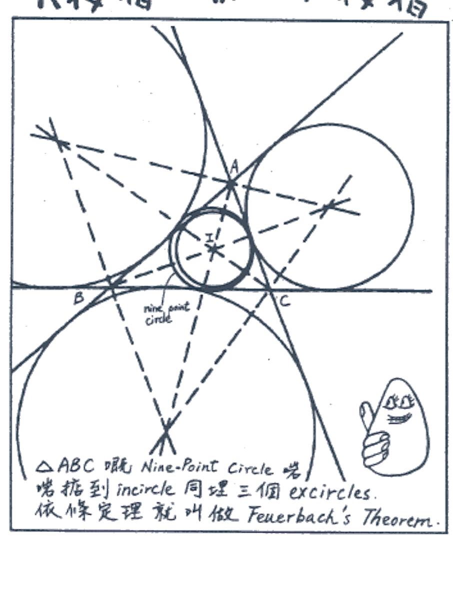
Problem 6. Consider a sequence of nonnegative real numbers a_1, a_2, ... satisfying the condition

$$a_{n+m} \leq a_n + a_m, \quad m, n \in N.$$

Prove that for any $n \geq m$,

$$a_n \leq m a_1 + \left(\frac{n}{m} - 1\right) a_m.$$

大拇指 與 小拇指



Mathematical Excalibur

Volume 3, Number 2 March-May, 1997

Olympiad Corner

The Ninth Asian Pacific Mathematics Olympiad, March 1997:

Time Allowed: 4 hours.
Each question is worth 7 points.

Problem 1. Given

$$S=1+\frac{1}{1+\frac{1}{3}}+\frac{1}{1+\frac{1}{3}+\frac{1}{6}}+\cdots+\frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}},$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers. Prove that $S>1001$.

Problem 2. Find an integer n, with $100\le n\le 1997$, such that $\frac{2^n+2}{n}$ is also an integer.

Problem 3. Let ABC be a triangle inscribed in a circle and let

$$l_a=\frac{m_a}{M_a},\ l_b=\frac{m_b}{M_b},\ l_c=\frac{m_c}{M_c},$$

(*continued on page 4*)

Editors: CHEUNG Pak-Hong, Curr. Studies, HKU
KO Tsz-Mei, EEE Dept, HKUST
LEUNG Tat-Wing, Appl. Math Dept, HKPU
LI Kin-Yin, Math Dept, HKUST
NG Keng Po Roger, ITC, HKPU

Artist: YEUNG Sau-Ying Camille, MFA, CU

Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is July 10, 1997.

For individual subscription for the remaining issue for the 96-97 academic year, send us a stamped self-addressed envelope. Send all correspondence to:

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由圓周率到四年一閏

香港道教聯合會青松中學
梁子傑

有時，一個簡單的電腦程序，就可以令我們發現不少有趣的數學現象，以下便是一個好例子：

```
PROGRAM rational_real; {written in MS QuickPascal}
VAR
  numer, denom : LongInt;
  devi, min, real_no : Double;
BEGIN
  min := 10; real_no := pi;
  FOR denom := 1 TO 50000 DO
    BEGIN
      numer := round(real_no * denom);
      devi := abs(numer/denom - real_no);
      IF devi < min THEN
        BEGIN
          min := devi;
          writeln (numer, '/', denom)
        END
    END
END.
```

這個程序是用來尋找圓周率π的有理數近似值的。程序令分母(denom)由1開始，先計算出最接近的分子(numer)的數值，然後計算出這個有理數近似值(numer/denom)跟圓周率的偏差(devi)，如果偏差比以前的小，就將該有理數印出來。

不出兩秒鐘，電腦就會計算出結果：
3/1, 13/4, 16/5, 19/6, 22/7, 179/57, 201/64, 223/71, 245/78, 267/85, 289/92, 311/99, 333/106, 355/113, 52163/16604, 52518/16717, 52873/16830, 53228/16943, …

從以上的結果，我們不難發現這個現象：並不是每當分母增加時，所計算出來的有理數近似值就一定較準確，好似當分母介乎於8至56之間時，以7作爲分母的近似值就比它們準確了。

如果大家細心地觀察一會，相信亦會發現在這些分母之中，出現了一些「跳躍」的現象，好似由7跳至57、由113跳至16604等。再留心看看，每次跳躍之後，分母增加的幅度亦有關係：7之後的57、64、71等，就相隔7；113之後的16604、16717等，就相隔113。同時分子亦有相類似的關係。奇怪嗎？爲甚麼會有這個現象出現呢？

要解釋以上的現象，我們就需要認識一個很特別的表達數值的方法，它就是「連分數」(continued fraction)。所謂「連分數」就是利用一連串的倒數來表達一個數的數值，例如：

$$\frac{1057}{498}=2+\frac{61}{498}=2+\frac{1}{\frac{498}{61}}$$

$$=2+\frac{1}{8+\frac{10}{61}}=2+\frac{1}{8+\frac{1}{6+\frac{1}{10}}}.$$

我們並且用這個記號來表示以上的結果：$\frac{1057}{498}=[2;8,6,10]$。

到了今天，數學家經已發現了很多有關連分數的性質，其中一項就是「漸近分數」的現象。以上述數字爲例，如果我們逐次選取連分數中部份的數字，即 $[2]=2$，$[2;8]=2+\frac{1}{8}=\frac{17}{8}$，$[2;8,6]=2+\frac{1}{8+\frac{1}{6}}=\frac{104}{49}$，$[2;8,6,10]=\frac{1057}{498}$，所得到的一列分數，就叫做「部份連分數」。我們可以證明「部份連分數」是一列越來越漸近原本數值的分數，換句話講，在數列中每一個分數都可以表示爲原本數值的約數，而且後者的準確性會比前者佳；當然，後者分母的數值卻比前者的大，應用起來就不及前者方便了。

回到上面電腦程序的結果，我們發現如果將圓周率π表示成連分數的話，我們有 $\pi=[3;7,15,1,292,1,1,\ldots]$。（因爲圓周率是一個無理數，它的連分數表達式自然是無窮盡的。）寫出π的部份連分數，我們有 $[3]=3$，$[3;7]=\frac{22}{7}$，$[3;7,15]=\frac{333}{106}$，$[3;7,15,1]=\frac{355}{113}$，$[3;7,15,1,292]=\frac{103993}{33102}$，……等等。而這些部份連分數，不是和電腦程序計算出來的結果相同嗎？

不過，電腦程序計算出來的結果卻比「部份連分數」爲多，這是因爲部份連分數的現象衹是一個充分條件，而不是一個必要條件，所以「部份連分數」並不是一個完整的漸近分數的數列。雖然如此，一個完整的數列亦可以利用「連分數」來表達出來。留意 $\frac{13}{4}=[3;4]$，$\frac{16}{5}=[3;5]$，$\frac{19}{6}=[3;6]$，$\frac{22}{7}=[3;7]$；另外，$\frac{179}{57}=[3;7,8]$，$\frac{201}{64}=[3;7,9]$……等等；不難看出，每當分母增加至和「部份連分數」相同的數值時，下一個漸近分數

就將會由下一個「部份連分數」的一半開始。例如，$\frac{355}{113}=[3;7,15,1]$，下一個部份連分數是[3;7,15,1,292]，而292的一半等於146，故此$\frac{355}{113}$之後的漸近分數應等於$[3;7,15,1,146]=\frac{52163}{16604}$。

以上的現象同時解釋了，爲何在數列中，分母數值的間隔會和「跳躍」前的分母數值相等的現象。大家衹要將[3;7,8]和[3;7,9]計算一次，就會明白爲何分子的差距和分母的差距，剛好等於[3;7]的分子和分母了。

在未討論下一個問題前，值得指出的是，在電腦程序計算出的漸近分數中，有幾個數值是十分「著名的」。例如：$\frac{3}{1}$，這是人類對圓周率最早期的約數，中國古籍中就有「徑一周三」的記載。另外$\frac{22}{7}$更毋須多介紹了。$\frac{223}{71}$是古希臘數學家阿基米德提出的約數；而$\frac{355}{113}$就最先由中國南北朝時代的數學家祖沖之提出的。當知道下一個漸近分數的分母將會高達16604時，相信大家都會明白爲何祖沖之計算出圓周率的七位小數約數之後，要經過多達一千年的時間，才有人能夠計到更佳的圓周率近似值！

其實漸近分數並不是一些數字的玩意，它有不少實際的應用價值，其中一項就是閏年的計算。跟據資料顯示，地球環繞太陽一周需要365日5小時48分46秒，化爲分數就即是有$365\frac{10463}{43200}$日。其中的365日當然沒有問題，但剩下的$\frac{10463}{43200}$日又如何處理呢？完全不理，那麼衹要每過5年，就會有一整天的時間差距了。（又或者每43200年，就會相差了10463日。）因此我們在曆法上就訂立了閏年的制度：即每隔一些年份就在那年增加一日，藉此保持曆法上不會出現偏差。但是，到底要經過多少年才有一次閏年呢？我們不可能經過4萬多年後，才增加萬多日來補償。不過我們可以通過計算$\frac{10463}{43200}$的漸近分數來獲得答案。

首先我們將本文開始的電腦程序中的real_no句子改爲：real_no := 10463/43200; 執行後，可得到結果：

0/1, 1/3, 1/4, 4/17, 5/21, 6/25, 7/29, 8/33, 23/95, 31/128, 101/417, 132/545, ...

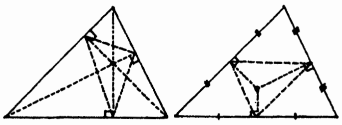
從結果中的$\frac{1}{4}$可知，我們應該每4年就要多加一日。按此比例，每100年就應有25個閏年。但由$\frac{23}{95}$和$\frac{31}{128}$可以知道，每100年其實衹需要24個閏年。所以如果我們衹跟著「四年一閏」的方法來編寫曆

(continued on page 4)

老師不教的幾何（三）

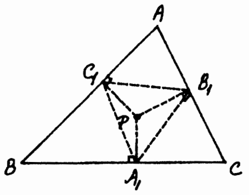
張百康

我們較早前蹚過的垂足三角形(orthic triangle)和中點三角形(medial triangle)有甚麼共同的性質呢?大家請重溫一下這兩個三角形（圖一）。



（圖一）

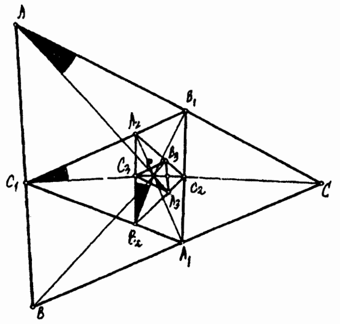
其實，它們可以看成是更一般情況的兩個特例：令P爲已知三角形ABC內任意一點。從P作垂直線垂直於這三角形的三邊，連這三垂線的垂足可得另一三角形$A_1B_1C_1$，稱爲三角形ABC相對於踏板點(pedal point) P的踏板三角形(pedal triangle)（圖二）。



（圖二）

分別以三角形ABC的垂心和外心作踏板點便可得垂足三角形和中點三角形作爲ABC的踏板三角形。踏板三角形有甚麼有趣的性質呢？在1892年出版的一本幾何書中，編輯J. Neuberg提出及證明了踏板三角形的一個周期性現象：

以P作爲三角形$A_1B_1C_1$的踏板點，可以得到$A_1B_1C_1$的踏板三角形$A_2B_2C_2$。繼續這作法以P爲踏板點，可以得到另一個踏板三角形$A_3B_3C_3$，如此類推(圖三)。



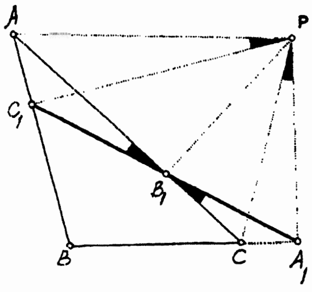
（圖三）

由於各有一對對角是直角，因此下列的四邊形都是外接四邊形：AC_1PB_1、$C_1B_2PA_2$和$B_2A_3PC_3$。因此$\angle B_1AP$、$\angle B_1C_1P$ $(=\angle A_2C_1P)$、$\angle A_2B_2P$ $(=\angle C_3B_2P)$和$\angle C_3A_3P$依次兩兩爲等弧上的圓周角。所以$\angle B_1AP=\angle C_3A_3P$；同理，$\angle C_1AP=\angle B_3A_3P$。這兩結果告訴我們：$\angle BAC=\angle B_3A_3C_3$；同理，$\angle ACB=\angle A_3C_3B_3$和$\angle ABC=\angle A_3B_3C_3$。因此三角形$ABC$和$A_3B_3C_3$相似。顯而易見，踏板三角形有下列周期性：

$$\Delta ABC\sim\Delta A_3B_3C_3\sim\Delta A_6B_6C_6\sim\cdots,$$
$$\Delta A_1B_1C_1\sim\Delta A_4B_4C_4\sim\Delta A_7B_7C_7\sim\cdots$$
和$\Delta A_2B_2C_2\sim\Delta A_5B_5C_5\sim\Delta A_8B_8C_8\sim\cdots$。

如果踏板點在三角形ABC外部時，這周期性還成立嗎？答案在一般情況下是肯定的，大家可參考上述證明加以修改便可。有沒有例外情況？如果大家懂得用互動幾何軟件如Cabri Geometry或Geometer's Sketchpad，這是一個有意義的探究活動。通過探究，大家應該發現：如果踏板點P在三角形ABC的外接圓上，則踏板三角形$A_1B_1C_1$退化爲一直線，也沒有其他的踏板點使踏板三角形退化爲直線。這直線稱爲辛姆生線(Simson Line)。辛姆生(Robert Simson)是十七、八世紀的數學家，但後人在他的著作中找不到這性質的證明，反而是William Wallace在1797年發表了下列證明：

設A_1、B_1、C_1共線（圖四），則$\angle AB_1C_1$和$\angle A_1B_1C$爲對頂角，所以相等。已知$\angle PA_1C=\angle PB_1C=90°$及$\angle PB_1A=\angle PC_1A=90°$，所以$P$、$A_1$、$C$、$B_1$四點共圓，且$P$、$B_1$、$C_1$、$A$也共圓。由此推出$\angle A_1PC=\angle A_1B_1C=\angle AB_1C_1=\angle APC_1$。並且，$\angle PA_1B=\angle PC_1B=90°$，所以$P$、$A_1$、$B$、$C_1$四點也共圓，因此，$\angle A_1PC_1$和$\angle C_1BA_1$互補。但$\angle A_1PC_1=\angle A_1PC+\angle CPC_1=\angle APC_1+\angle CPC_1=\angle APC$，所以$\angle APC$和$\angle ABC$ $(=\angle C_1BA_1)$也互補。故A、B、C、P四點共圓，即P點在三角形ABC的外接圓上。



（圖四）

(continued on page 4)

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is July 10, 1997.

Problem 56. Find all prime numbers p such that $2^p + p^2$ is also prime.

Problem 57. Prove that for real numbers $x, y, z > 0$,

$$\frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+x}\geq\frac{x+y+z}{2}.$$

Problem 58. Let ABC be an acute-angled triangle with $BC > CA$. Let O be its circumcenter, H its orthocenter, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$. (*Source*: unused problem in the 1996 IMO.)

Problem 59. Let n be a positive integer greater than 2. Find all real number solutions $(x_1, x_2, \cdots, x_n)$ to the equation

$$(1-x_1)^2 + (x_1-x_2)^2 + \cdots + (x_{n-1}-x_n)^2 + x_n{}^2 = \frac{1}{n+1}.$$

(Source: 1975 British Mathematical Olympiad)

Problem 60. Find (without calculus) a fifth degree polynomial $p(x)$ such that $p(x) + 1$ is divisible by $(x - 1)^3$ and $p(x) - 1$ is divisible by $(x + 1)^3$.

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Solutions

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Problem 51. Is there a positive integer n such that $\sqrt{n-1}+\sqrt{n+1}$ is a rational number?

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 4).

Assume there is a positive integer n such that

$$\sqrt{n-1}+\sqrt{n+1}=r$$

is rational. Squaring and simplifying, we get

$$\sqrt{n^2-1}=\frac{r^2-2n}{2}$$

is also rational. However, for $n > 1$, if $\sqrt{n^2-1} = a/b$ for some positive integers a, b having no common factor greater than 1, then $a^2 = b^2(n^2-1)$, which implies b also divides a. So b must be 1. Now for $n > 1$,

$$n^2 > n^2 - 1 = a^2 > (n-1)^2$$

is impossible. So $n = 1$, but then

$$\sqrt{n-1}+\sqrt{n+1}=\sqrt{2}$$

is irrational. Therefore, no such n exists.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHAN Wing Sum** (HKUST), **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College, Form 6), **CHOI Wing Shan Winnie** (St. Stephen's Girls' College, Form 6), **LEUNG Shun Ming** (La Salle College, Form 4), **LIU Wai Kwong** (Pui Tak Canossian College), **TSE Wing Ho** (Ho Fung College, Form 5), **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4) and **YUNG Fai** (CUHK).

Problem 52. Let a, b, c be distinct real numbers such that $a^3 = 3(b^2+c^2) - 25$, $b^3 = 3(c^2+a^2) - 25$, $c^3 = 3(a^2+b^2) - 25$. Find the value of abc.

Solution: CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College, Form 6), **YEUNG Yi Pok** (Pui Shing Catholic Secondary School, Form 7) and **YUNG Fai** (CUHK).

Let a, b, c be roots of

$$x^3 - px^2 + qx - r = 0.$$

Then $p = a + b + c$, $q = ab + bc + ca$ and $r = abc$. Since $a^2 + b^2 + c^2 = p^2 - 2q$, so

$$a^3 = 3(b^2 + c^2) - 25 = 3(p^2-2q-a^2) - 25.$$

This is equivalent to $a^3 + 3a^2 + (25 + 6q - 3p^2) = 0$. Then a is a root of $x^3 + 3x^2 + (25 + 6q - 3p^2) = 0$. Similarly, b and c are roots of this equation. Comparing coefficients of the two equations, we get $p = -3$, $q = 0$ and $abc = r = -(25 + 6q - 3p^2) = 2$.

Other commended solvers: **LIU Wai Kwong** (Pui Tak Canossian College), **TSE Wing Ho** (Ho Fung College, Form 5) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4) ∘

Problem 53. For ΔABC, define A' on BC so that $AB + BA' = AC + CA'$ and similarly define B' on CA and C' on AB. Show that AA', BB', CC' are concurrent. (The point of concurrency is called the *Nagel point* of ΔABC.)

Solution: CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College, Form 6), **LIU Wai Kwong** (Pui Tak Canossian College) and **YEUNG Yi Pok** (Pui Shing Catholic Secondary School, Form 7)

Let $a = BC$, $b = CA$, $c = AB$ and $s = (AB + BC + CA)/2$. Since $AB + BA' = s = AC + CA'$, we have $BA' = s - c$ and $CA' = s - b$. Similarly, $CB' = s - a$, $AB' = s - c$, $AC' = s - b$ and $BC' = s - a$. Then

$$(CA'/BA')(AB'/CB')(BC'/AC') = 1.$$

So by the converse of Ceva's theorem, AA', BB', CC' are concurrent.

Other commended solvers: **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

Problem 54. Let R be the set of real numbers. Find all functions $f : R \to R$ such that

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy$$

for all $x, y \in R$. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

Solution: YUNG Fai (CUHK).

Putting $y = 0$, we get

$$f(f(x)) = [1 + f(0)]f(x).$$

Replacing x by $x + y$, we get

$$[1+f(0)]f(x+y) = f(f(x+y)) = f(x+y) + f(x)f(y) - xy,$$

which simplifies to

$$f(0)f(x+y) = f(x)f(y) - xy.$$

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

Putting $y = 1$, we get

$$f(0)f(x+1) = f(x)f(1) - x.$$

Putting $y = -1$ and replacing x by $x+1$, we get

$$f(0)f(x) = f(x+1)f(-1) + x + 1.$$

Eliminating $f(x+1)$ in the last two equations, we get

$$[f^2(0) - f(1)f(-1)]f(x) = [f(0) - f(-1)]x + f(0).$$

If $f^2(0) - f(1)f(-1) \neq 0$, then $f(x)$ is linear. If $f^2(0) - f(1)f(-1) = 0$, then putting $x = 0$ in the last equation, we get $f(0) = 0$. In this case, the displayed equation above implies $f(x)f(y) = xy$. Then $f(x)f(1) = x$ for all $x \in R$. So $f(1) \neq 0$ and $f(x)$ is linear.

Finally, substituting $f(x) = ax + b$ into the original equation, since $f(x)$ cannot be constant, we find $a = 1$ and $b = 0$, i.e., $f(x) = x$ for all $x \in R$.

Other commended solvers: **CHAN Wing Sum** (HKUST) and **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College, Form 6).

Problem 55. In the beginning, 65 beetles are placed at different squares of a 9×9 square board. In each move, every beetle creeps to a horizontal or vertical adjacent square. If no beetle makes either two horizontal moves or two vertical moves in succession, show that after some moves, there will be at least two beetles in the same square. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

Solution: William CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College, Form 6) and **YUNG Fai** (CUHK).

Assign an ordered pair (a,b) to each square with $a, b = 1, 2, \ldots, 9$. Divide the 81 squares into 3 types. Type A consists of squares with both a and b odd, type B consists of squares with both a and b even and type C consists of the remaining squares. The numbers of squares of the types A, B and C are 25, 16 and 40, respectively.

Assume no collision occurs. After two successive moves, beetles in type A squares will be in type B squares. So the number of beetles in type A squares are at most 16 at any time. Then there are at most 32 beetles in type A or type B squares at any time. Also, after one move, beetles in type C squares will go to type A or type B squares. So there are at most 32 beetles in type C squares at any time. Hence there are at most 64 beetles on the board, a contradiction.

Other commended solvers: **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4.

Olympiad Corner

(*continued from page 1*)

where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3,$$

and that equality holds iff ABC is equilateral.

Problem 4. Triangle $A_1A_2A_3$ has a right angle at A_3. A sequence of points is now defined by the following iterative process, where n is a positive integer. From A_n ($n \geq 3$), a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1}.

(a) Prove that if this process were continued indefinitely, then one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n$, $n \geq 3$.

(b) Let A_1 and A_3 be fixed points. By considering all possible locations of A_2 on the plane, find the locus of P.

Problem 5. Suppose that n persons A_1, A_2, …, A_n ($n \geq 3$) are seated in circle and that A_i has a_i objects such that

$$a_1 + a_2 + \cdots + a_n = nN$$

where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give or to receive a certain number of objects to or from its two neighbours A_{i-1} and A_{i+1}, where A_{n+1} means A_1 and A_0 means A_n. How should this distribution be performed so that the total numbers of objects transferred is minimum?

由圓周率到四年一閏

(*continued from page 1*)

法，一百年後就會多了一日。因此在今天我們使用的曆法之中，年份能夠被4整除的，例如1996年，就定爲閏年，但如果年份能夠被100整除的話，例如1900年，就不是閏年了。

再算一算，就知道每400年就有96個閏年，416年就應有96+4=100個閏年。不過，這結果又不乎合 $\frac{101}{417}$ 這個條件！故此，曆法上又需要在每400年中增加一日，就好似2000年，因爲這數字能被400整除，這年又變回一年閏年了！

公元2000年快到了，大家渴望見一見這400年才有一次的閏年嗎？

老師不教的幾何（三）

(*continued from page 1*)

把上述推理逆轉過來，恰巧也成立，因此只有外接圓上的點能使踏板三角形退化爲辛姆生線。

踏板三角形的周期性可否推廣到n邊形呢？大家不妨先用四邊形來試試。B. M. Stewart 在1940年證明了：n邊形的第n個踏板n邊形相似於原n邊形（刊於 American Mathematical Monthly 第七卷第462-466頁）。

台灣師範大學附屬中學初中二年級的孫君儀同學最近以踏板多邊形作爲研究課題，獲得1997年台灣科學展覽第三名。她借助 Geometer's Sketchpad發現了一些有趣性質並加以證明，大家不妨試試探討，甚至再推廣。這些性質包括：

- 對於凹n邊形和自交n邊形，第n個踏板n邊形是否和原n邊形相似？
- 踏板點在n邊形外部，類似性質是否存在？有甚麼條件會使踏板n邊形不存在？
- 第n個踏板n邊形和原n邊形的面積比是多少？
- 垂足改爲夾$x°$角時，類似性質是否存在？
- 踏板點在何處可使第三垂足三角形的面積最大？

Mathematical Excalibur

Volume 3, Number 3 June-August, 1997

Olympiad Corner

The 38th International Mathematical Olympiad, Mar del Plata, Argentina:

First day (July 24, 1997)
Each problem is worth 7 points.
Time Allowed: $4\frac{1}{2}$ hours.

Problem 1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

$$f(m,n) = |S_1 - S_2|.$$

(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

(*continued on page 4*)

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Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is September 30, 1997.

For individual subscription for the five issues for the 97-98 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Error Correcting Codes (Part II)

Tsz-Mei Ko

In Part I, we introduced the family of Hamming codes. In particular, the (7,4) Hamming code encodes 4-bit messages $p_1p_2p_3p_4$ into 7-bit codewords $p_1p_2p_3p_4p_5p_6p_7$ by appending three parity bits

$$p_5 = p_1 + p_2 + p_4 \pmod 2,$$
$$p_6 = p_1 + p_3 + p_4 \pmod 2,$$
$$p_7 = p_2 + p_3 + p_4 \pmod 2,$$

to the original message. Figure 1 shows the 16 possible codewords for the (7,4) Hamming code. To convey the message 0100, as an example, the sender would send 0100101. If there is a transmission error in position 4 so that the received sequence becomes 0101101, the receiver would still be able to recover the error by decoding the received sequence as the closest codeword. (Note that 0100101 is different from 0101101 in only one position while all other codewords are different from 0101101 in more than one position.)

Now, if we group the first six bits of a (7,4) Hamming codeword into two-bit pairs (p_1p_2, p_3p_4, p_5p_6) and use an arithmetic system called a 4-element field (Figure 2), we observe something interesting: the three points (1, p_1p_2), (2, p_3p_4) and (3, p_5p_6) form a straight line! For example, the first 6 bits of the codeword 0100101 forms the ordered triple (01, 00, 10) = (1, 0, 2) and (1,1), (2,0), (3,2) are three consecutive points on the straight line $f(x) = 2x + 3$ since

$$f(1) = 2(1) + 3 = 2 + 3 = 1;$$
$$f(2) = 2(2) + 3 = 3 + 3 = 0;$$
$$f(3) = 2(3) + 3 = 1 + 3 = 2;$$

by using the addition and multiplication tables given in Figure 2. This fact is also true for the other 15 codewords and their corresponding straight lines $f(x)$ are listed in Figure 3.

This "straight line" property can be utilized for decoding. As an example, assume that the received sequence is 0101101. The first 6 bits form the ordered triple (01, 01, 10) = (1, 1, 2). We observe that a straight line passing through (1,1) and (2,1) should pass through (3,1). That is (1,1), (2,1) and (3,2) do not lie on a straight line and thus there is a transmission error. For the (7,4) Hamming code which is capable of correcting one error, we

message $p_1p_2p_3p_4$	codeword $p_1p_2p_3p_4p_5p_6p_7$
0000	0000000
0001	0001111
0010	0010011
0011	0011100
0100	0100101
0101	0101010
0110	0110110
0111	0111001
1000	1000110
1001	1001001
1010	1010101
1011	1011010
1100	1100011
1101	1101100
1110	1110000
1111	1111111

Figure 1. The (7,4) Hamming Code.

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Figure 2. Arithmetic Tables for a 4-Element Field.

codeword	p_1p_2	p_3p_4	p_5p_6	$f(x)$
0000000	0	0	0	0
0001111	0	1	3	$2x+2$
0010011	0	2	1	$3x+3$
0011100	0	3	2	$x+1$
0100101	1	0	2	$2x+3$
0101010	1	1	1	1
0110110	1	2	3	x
0111001	1	3	0	$3x+2$
1000110	2	0	3	$3x+1$
1001001	2	1	0	$x+3$
1010101	2	2	2	2
1011010	2	3	1	$2x$
1100011	3	0	1	$x+2$
1101100	3	1	2	$3x$
1110000	3	2	0	$2x+1$
1111111	3	3	3	3

Figure 3. The (7,4) Hamming Codewords form Straight Lines $f(x)$.

assume that only one of the three points is incorrect. That is, the original "straight line" $f(x)$ should pass through (1,1) and (2,1); (1,1) and (3,2); or (2,1) and (3,2) corresponding to $f(x) = 1$; $f(x) = 2x + 3$; or $f(x) = 3x$ respectively. Then the first 6 bits for the original codeword should be 010101, 010010 or 110110. Among these three possible solutions, only 010010 satisfies the equation for the last parity bit $p_7 = p_2 + p_3 + p_4 \pmod 2$. Thus we decode the received sequence 0101101 as 0100101 corresponding to the message 0100.

The above decoding procedure seems to be quite complicated. However, it can be generalized to construct (and decode) multiple-error correcting codes by using "polynomials" instead of "straight lines". Suppose we would like to transmit a message that contains k symbols $s_1 s_2 \cdots s_k$. We may use these k symbols to form a kth degree polynomial $f(x)$ such that $f(i) = s_i$ $(1 \le i \le k)$. To construct a code that can correct t errors, we may append $2t$ symbols $f(k+1)$, $f(k+2)$, $\ldots$, $f(k+2t)$ to the original message so that the encoded sequence contains $k + 2t$ symbols corresponding to $k + 2t$ consecutive points on a kth degree polynomial (Figure 4). If there are less than or equal to t errors during transmission, at least $k + t$ symbols would be received correctly. Then the receiver may simply check which $k + t$ symbols lie on a kth degree polynomial to decode the received sequence.

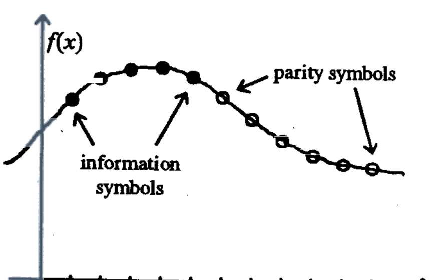

Figure 4. A Polynomial Code.

We use a (21,9) double error correcting code to illustrate the idea. Assume we would like to send a 9 bit message, say 101010100. We may first group the information bits into 3-bit symbols as (101, 010, 100) = (5, 2, 4). (In general, we may group the information bits into m-bit symbols where m cannot be too small. Otherwise, we cannot construct the polynomial $f(x)$. Why? Also m should not be too large to reduce the number of parity bits.) Then we use the three message symbols (5, 2, 4) to form a second degree polynomial $f(x)$ such that $f(1) = 5, f(2) = 2$ and $f(3) = 4$. That is

$$f(x) = \frac{5(x-2)(x-3)}{(1-2)(1-3)} + \frac{2(x-1)(x-3)}{(2-1)(2-3)} + \frac{4(x-1)(x-2)}{(3-1)(3-2)}.$$

Note that we have 8 kinds of symbols (since we group the bits into 3-bit symbols) and thus we need an 8-element field for our arithmetic. (Basically, a field is an arithmetic system that allows us to add, subtract, multiply and divide.) By using the 8-element field given in Figure 5, we can simplify $f(x)$ to obtain

$$f(x) = x^2 + 7x + 5.$$

Note that $f(1) = 5$, $f(2) = 2$ and $f(3) = 4$ as desired.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	3	1	7	5
3	0	3	6	5	7	4	1	2
4	0	4	3	7	6	2	5	1
5	0	5	1	4	2	7	3	6
6	0	6	7	1	5	3	2	4
7	0	7	5	2	1	6	4	3

Figure 5. Arithmetic Tables for an 8-Element Field.

Now suppose we would like to construct a code that can correct two errors. We can append

$$f(4) = 4^2 + 7(4) + 3 = 6 + 1 + 3 = 4;$$
$$f(5) = 5^2 + 7(5) + 3 = 7 + 6 + 3 = 2;$$
$$f(6) = 6^2 + 7(6) + 3 = 2 + 4 + 3 = 5;$$
$$f(7) = 7^2 + 7(7) + 3 = 3 + 3 + 3 = 3;$$

to the message symbols. That is, we would transmit a 21 bit sequence (5,2,4,4,2,5,3) = 101010100100010101011. If there are transmission errors, say at positions 5 and 15, the received sequence becomes 101000100100011101011 = (5,0,4,4,3,5,3). (This code is actually capable of correcting two symbol errors instead of two bit errors.) Then the receiver would search for the 5 received symbols that are not corrupted. Among the $\binom{7}{5} = 21$ cases, only $f(1) = 5$, $f(3) = 4$, $f(4) = 4$, $f(6) = 5$, $f(7) = 3$, form a second degree polynomial. So the receiver uses these five points to reconstruct $f(x) = x^2 + 7x + 5$ and decode the received message as $(f(1), f(2), f(3)) = (5, 2, 4) = 101010100$.

The above idea, using polynomials to construct codes, was first proposed by Reed and Solomon in 1960. It is now widely used in electronics and communication systems including our compact discs.

38th IMO

Kin-Yin Li

For the first time in history, the International Mathematical Olympiad (IMO) was held in the southern hemisphere. Teams representing a record 82 countries and regions participated in the event at Mar del Plata, Argentina this year from July 18 to 31. The site was at a resort area bordered by the beautiful Atlantic Ocean. All through the period, the weather was nice and cool.

The Hong Kong team, like many southeast Asia teams, had to overcome thirty plus hours of flight time to arrive Argentina. With two short days of rest, the team members wrote the exams with jet lag. This year the team consisted of

Chan Chung Lam (*Bishop Hall Jubilee School*)
Cheung Pok Man (*STFA Leung Kau Kui College*)
Lau Lap Ming (*St. Paul's College*)
Leung Wing Chung (*Queen Elizabeth School*)
Mok Tze Tao (*Queen's College*)
Yu Ka Chun (*Queen's College*)

brought home 5 bronze medals and came in one mark behind Canada and one mark ahead of France. The top team was China with 6 gold, followed by Hungary, Iran, USA and Russia. As usual, problem 6 was the most difficult with 73% of the contestants getting zero, 90% getting less than half of the score for the problem.

The excursions were good. The hospitality was superb!!! The team members had a wild time playing the indoor games the day before the closing ceremony. One member of the team even admitted it was the best he has participated in three years. There were many fond memories.

There was a surprise ending on the way back. Due to the typhoon weather in Hong Kong, the team was stranded in Los Angeles for a day. Yes, the team took full advantage to tour the city, Hollywood, Beverly Hills, Rodeo Drive, in particular. The next day the team was stranded again in Taipei. It was unbelievably fortunate to have a chance to see these cities. What a bonus for a year's hard work!

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is September 30, 1997.

Problem 61. Find the smallest positive integer which can be written as the sum of nine, the sum of ten and the sum of eleven consecutive positive integers.

Problem 62. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the midpoint of PQ. (*Source*: 1996 Australian Mathematical Olympiad.)

Problem 63. Show that for $n \ge 2$, there is a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that $|a_k - k| = |a_1 - 1| \ne 0$ for $k = 2, 3, \ldots, n$ if and only if n is even.

Problem 64. Show that it is impossible to place 1995 different positive integers along a circle so that for every two adjacent numbers, the ratio of the larger to the smaller one is a prime number.

Problem 65. All sides and diagonals of a regular 12-gon are painted in 12 colors (each segment is painted in one color). Is it possible that for any three colors there exist three vertices which are joined with each other by segments of these colors?

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Solutions

****

Problem 56. Find all prime numbers p such that $2^p + p^2$ is also prime.

Solution: CHAN Lung Chak (St. Paul's Co-ed. College, Form 4), **CHAN Wing Sum** (HKUST), **LAW Ka Ho** (Queen Elizabeth School, Form 4), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Man Wai** (St. Paul's College, Form 4), **TAM Siu Lung** (Queen Elizabeth School, Form 4), **WONG Chun Wai** (SKH Kei Hau Secondary School, Form 4), **Alan WONG Tak Wai** (University of Waterloo, Canada), **WONG Sui Kam** (Queen Elizabeth School, Form 4) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

For $p = 2$, $2^p + p^2 = 8$ is not prime. For $p = 3$, $2^p + p^2 = 17$ is prime. For prime $p = 3n \pm 1 > 3$, we see that

$$2^p + p^2 = (3-1)^p + (3n \pm 1)^2$$

is divisible by 3 (after expansion) and is greater than 3. So $p = 3$ is the only such prime.

Problem 57. Prove that for real numbers $x, y, z > 0$,

$$\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+x} \ge \frac{x+y+z}{2}.$$

Solution 1: Note that

$$\begin{aligned} 4x^2 &= ((x+y)+(x-y))^2 \\ &= (x+y)^2 + 2(x+y)(x-y) + (x-y)^2 \\ &\ge (x+y)^2 + 2(x+y)(x-y). \end{aligned}$$

Dividing both sides by $4(x+y)$, we obtain

$$\frac{x^2}{x+y} \ge \frac{x+y}{4} + \frac{x-y}{2}.$$

In place of x, y, similar inequalities for y, z and z, x can be obtained. Adding these inequalities give the desired inequality.

Solution 2: Venus CHU Choi Yam (St. Paul's Co-ed. College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Man Wai** (St. Paul's College, Form 4), **Alan WONG Tak Wai** (University of Waterloo, Canada).

The Cauchy-Schwarz inequality asserts that

$$\left(a_1^2 + a_2^2 + \cdots + a_k^2\right)\left(b_1^2 + b_2^2 + \cdots + b_k^2\right) \ge \left(a_1 b_1 + a_2 b_2 + \cdots + a_k b_k\right)^2$$

with equality if and only if $a_i b_j = a_j b_i$ for all i, j such that $1 \le i < j \le k$. Taking $k = 3$,

$$a_1 = \sqrt{x+y},\ a_2 = \sqrt{y+z},\ a_3 = \sqrt{z+x},$$

$$b_1 = \frac{x}{\sqrt{x+y}},\ b_2 = \frac{y}{\sqrt{y+z}},\ b_3 = \frac{z}{\sqrt{z+x}},$$

then dividing both sides by $2(x + y + z)$, we get the desired inequality.

Other commended solvers: **CHAN Wing Sum** (HKUST), **Alex CHUENG King Chung** (Po Leung Kuk 1983 Board of Director's College, Form 6), **Yves CHEUNG Yui Ho** (STFA Leung Kau Kui College, Form 5), **TAM Siu Lung** (Queen Elizabeth School, Form 4), and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

Problem 58. Let ABC be an acute-angled triangle with $BC > CA$. Let O be its circumcenter, H its orthocenter, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$. (*Source*: unused problem in the 1996 IMO.)

Solution: Official Solution.

Let Y be the midpoint of AC. Since $\angle OFP = \angle OYP = 90°$, points F, P, Y, O lie on a circle Γ_1 with center at the midpoint Q of OP. Now the nine point circle Γ_2 of ΔABC also passes through F and Y and has center at the midpoint N of OH. So FY is perpendicular to NQ. Since NQ is parallel to HP by the midpoint theorem, FY is perpendicular to HP. Then $\angle FHP = 90° - \angle YFH = 90° - \angle YCH = \angle BAC$.

Problem 59. Let n be a positive integer greater than 2. Find all real number solutions $(x_1, x_2, \cdots, x_n)$ to the equation

$$(1-x_1)^2 + (x_1-x_2)^2 + \cdots + (x_{n-1}-x_n)^2 + x_n^2 = \frac{1}{n+1}$$

(Source: 1975 British Mathematical Olympiad)

Solution 1: Official Solution.

Let
$$1 - x_1 = \frac{1}{n+1} + z_1,$$
$$x_1 - x_2 = \frac{1}{n+1} + z_2, \ \ldots$$
$$x_{n-1} - x_n = \frac{1}{n+1} + z_n,$$
$$x_n = \frac{1}{n+1} + z_{n+1}.$$

Adding the above $n + 1$ equations, we get

$$z_1 + z_2 + \cdots + z_{n+1} = 0.$$

(continued on page 4)

Problem Corner

(*continued from page 3*)

In terms of z_i, the given equation can then be simplified to

$$z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 0.$$

So all $z_i = 0$, which implies

$$x_i = \frac{n+1-i}{n+1} \text{ for } i = 1, 2, \ldots, n.$$

Solution 2: Venus CHU Choi Yam (St. Paul's Co-ed. College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4) and **POON Man Wai** (St. Paul's College, Form 4).

We use the Cauchy-Schwarz inequality as stated in Problem 57 Solution 2. Taking $k = n + 1$,

$$a_1 = 1 - x_1,\ a_2 = x_1 - x_2,\ \ldots,\ a_n = x_{n-1} - x_n,\ a_{n+1} = x_n,$$

$$b_1 = b_2 = \cdots = b_{n+1} = 1,$$

we see that we have equality. So $a_1 = a_2 = \cdots = a_{n+1}$ yielding the unique solution

$$x_i = \frac{n+1-i}{n+1} \text{ for } i = 1, 2, \ldots, n.$$

Problem 60. Find (without calculus) a fifth degree polynomial $p(x)$ such that $p(x) + 1$ is divisible by $(x - 1)^3$ and $p(x) - 1$ is divisible by $(x + 1)^3$.

Solution: LAW Ka Ho (Queen Elizabeth School, Form 4), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Man Wai** (St. Paul's College, Form 4) and **TAM Siu Lung** (Queen Elizabeth School, Form 4).

Note that $(x - 1)^3$ divides $p(x) + 1$ and $p(-x) - 1$; so $(x - 1)^3$ divides their sum $p(x) + p(-x)$. Also $(x + 1)^3$ divides $p(x) - 1$ and $p(-x) + 1$; so $(x + 1)^3$ divides $p(x) + p(-x)$. Then $(x - 1)^3(x + 1)^3$ divides $p(x) + p(-x)$, which is of degree at most 5. So $p(x) + p(-x) = 0$ for all x. Then the even degree term coefficients of $p(x)$ are zero. Now

$$p(x) + 1 = (x - 1)^3(Ax^2 + Bx - 1).$$

Comparing the degree 2 and 4 coefficients, we get $3 + 3B - A = 0$ and $B - 3A = 0$, which implies $A = -3/8$ and $B = -9/8$. This yields

$$p(x) = -\frac{3}{8}x^5 + \frac{5}{4}x^3 - \frac{15}{8}x.$$

Other commended solvers: **CHAN Wing Sum** (HKUST), **OR Kin** (SKH Bishop Mok Sau Tseng Secondary School, Form 3), **SIN Ka Fai** (STFA Leung Kau Kui College, Form 4) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

Olympiad Corner

(*continued from page 1*)

Problem 2. Angle A is the smallest in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that

$$AU = TB + TC.$$

Problem 3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions:

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and $|x_i| \le \dfrac{n+1}{2}$ for $i = 1, 2, \ldots, n.$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

Second day (July 25, 1997)
Each problem is worth 7 points.
Time Allowed: $4\frac{1}{2}$ hours.

Problem 4. An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n - 1\}$ is called a *silver* matrix if, for each $i = 1, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

Problem 5. Find all pairs (a,b) of integers $a \ge 1$, $b \ge 1$ that satisfy the equation

$$a^{b^2} = b^a$$

Problem 6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways:

$$4;\ 2 + 2;\ 2 + 1 + 1;\ 1 + 1 + 1 + 1.$$

Prove that, for any integer $n \ge 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

Above: A photo of the Hong Kong Team taken in front of the IMO97 score board. From left to right are: LEUNG Wing Chung, CHEUNG Pok Man, YU Ka Chun, LAU Lap Ming, CHAN Chung Lam, MOK Tze Tao, LUK Mee Lin (La Salle College, Deputy Leader), LI Kin Yin (HKUST Math Dept, Team Leader).

Mathematical Excalibur

Volume 3, Number 4 September-November, 1997

Olympiad Corner

British Mathematical Olympiad:

Round 1 (January 15, 1997)
Time Allowed: $3\frac{1}{2}$ hours.

Problem 1. N is a four-digit integer, not ending in zero, and $R(N)$ is the four-digit integer obtained by reversing the digits of N; for example, $R(3275) = 5723$. Determine all such integers N for which $R(N) = 4N + 3$.

Problem 2. For positive integers n, the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is defined by

$$a_1 = 1,\ a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + \cdots + a_{n-1}),\ n > 1.$$

Determine the value of a_{1997}.

Problem 3. The Dwarfs in the Land-under-the-Mountain have just adopted a completely decimal currency system based on the *Pippin*, with gold coins to the value of 1 *Pippin*, 10 *Pippins*, 100 *Pippins* and 1000 *Pippins*.
In how many ways is it possible for a Dwarf to pay, in exact coinage, a bill of 1997 *Pippins*?

(*continued on page 4*)

Editors: 張百康 (CHEUNG Pak-Hong), Curr. Studies, HKU
高子眉 (KO Tsz-Mei), EEE Dept, HKUST
梁達榮 (LEUNG Tat-Wing), Appl. Math Dept, HKPU
李健賢 (LI Kin-Yin), Math Dept, HKUST
吳鏡波 (NG Keng Po Roger), ITC, HKPU

Artist: 楊秀英 (YEUNG Sau-Ying Camille), MFA, CU

Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is January 10, 1998.

For individual subscription for the four remaining issues for the 97-98 academic year, send us four stamped self-addressed envelopes. Send all correspondence to:

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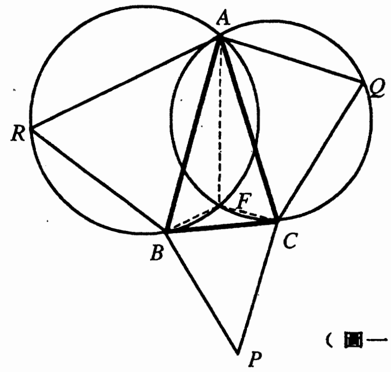
老師不教的幾何（四）

張百康

在任意的三角形的三邊上作另一些三角形，只要滿足一些簡單的條件，卻常常可以得到一些美妙的結果。

圖一的三角形 ABC 的三條邊外側分別隨意作了三個三角形 ABR、BCP 和 CAQ，並同時作三角形 ABR 和 CAQ 的外接圓。連此兩外接圓的一交點 F 至 A、B 及 C。



（圖一）

$$\begin{aligned}\angle BFC &= 360° - \angle AFB - \angle AFC\\ &= 360° - (180° - \angle ARB)\\ &\qquad - (180° - \angle AQC)\\ &= \angle ARB + \angle AQC.\end{aligned}$$

如果條件

$$\angle BPC + \angle ARB + \angle CQA = 180°$$

成立，$\angle BPC$ 和 $\angle BFC$ 互補，因此三角形 BCP 的外接圓也通過點 F。這個條件並不難得，下列兩種情況都是它的特例：

(1) 三角形 ABR、CPB 和 QCA 相似；

(2) A、B 和 C 分別是三角形 PQR 的邊 QR、RP 和 PQ 上的點。

如果三角形 ABR、CPB 和 QCA 相似，則它們的外接圓心 O_3、O_1 和 O_2 所組成的三角形也和它們相似（圖二），道理如下：

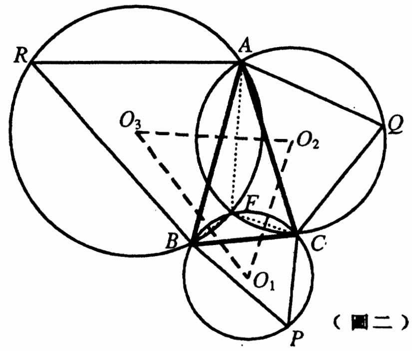
由於 O_1O_2 和 O_1O_3 是圓心連線，所以它們分別垂直公共弦 CF 和 BF，因此

$$\begin{aligned}\angle O_2O_1O_3 &= 360° - 90° - 90° - \angle BFC\\ &= 180° - \angle BFC\\ &= \angle CPB\ (=\angle ABR = \angle QCA)。\end{aligned}$$

同理

$$\angle O_1O_3O_2 = \angle BRA\ (=\angle CAQ = \angle PBC),$$
$$\angle O_3O_2O_1 = \angle AQC\ (=\angle RAB = \angle BCP),$$

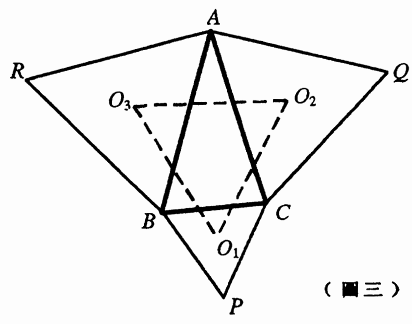
證畢。



（圖二）

拿破侖(Napoleon)是一位大家都知道的大將軍，但你可知道他對數學，尤其是幾何，有濃厚興趣？我現在要介紹的一類三角形，據說是他發現的，所以後人將這種三角形命名為拿破侖三角形。

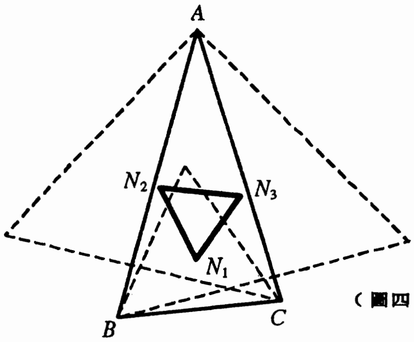
在任意的一個三角形 ABC 的三條邊上，分別向外側作三等邊三角形 ABR、BCP 和 CAQ（圖三）。這三個等邊三角形的心 O_1、O_2、O_3 可連成一三角形 $O_1O_2O_3$，稱



（圖三）

為外拿破侖三角形。由前述結果可推知外拿破侖三角形也是等邊三角形。

如果我們改變一下上述作法，把三個等邊三角形作於三角形 ABC 三條邊的內側，可以得到如圖四所示的另一三角形 $N_1N_2N_3$，稱為內拿破侖三角形。



（圖四）

俄羅斯數學家I.M. Yaglom 巧妙地證明內拿破侖三角形也是等邊三角形：

應用餘弦公式於圖三的三角形 AO_3O_2 可得

$$(O_2O_3)^2=\frac{b^2}{3}+\frac{c^2}{3}-2\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}\cos(A+60°),$$

這裏我們利用了

$$AO_2=\frac{b}{\sqrt{3}}\text{，}AO_3=\frac{c}{\sqrt{3}}$$

和 $\angle O_3AO_2 = A + 60°$

等簡單事實，請同學們自行驗證。

類似手法再應用於三角形 AN_3N_2 可得

$$(N_2N_3)^2=\frac{b^2}{3}+\frac{c^2}{3}-2\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}\cos(60°-A).$$

將上述兩等式同側相減可得

$$\begin{aligned}(O_2O_3)^2-(N_2N_3)^2 &= \frac{2bc}{3}(\cos(60°-A)-\cos(A+60°))\\ &= \frac{2}{\sqrt{3}}bc\sin A\\ &= \frac{4}{\sqrt{3}}\times\Delta ABC\text{ 的面積。}\end{aligned}$$

此處的簡化過程從略。

由於 $O_2O_3 = O_3O_1 = O_1O_2$，因此 $N_2N_3 = N_3N_1 = N_1N_2$。這證明的巧

(*continued on page 4*)

Inverse Sequences and Complementary Sequences

Yau Kwan Kiu Garry
Form 7, Queen's College

Editor's Note: This article is modified and shortened by the editors.

Consider the sequence

$$f(n) = 0, 0, 0, 1, 2, 3, 3, 4, 5, 6, 7, 10, \ldots,$$

i.e., $f(1) = 0, f(2) = 0, f(3) = 0, f(4) = 1$, etc. We can construct another sequence $f^*(n)$ according to the definition

$$f^*(n) = k, \text{ where } f(k) < n \le f(k+1).$$

For our example,

$$f^*(n) = 3, 4, 5, 7, 8, 9, 10, 11, 11, 11, \ldots.$$

Note that $f^*(n)$ can also be referred as the "frequency distribution function" of $f(n)$ since $f^*(n)$ is the number of terms in the sequence f that are less than n.

Figure 1 shows the two functions $f(n)$ and $f^*(n)$. We note something interesting: f^* is a mirror image of f. If we compute the frequency distribution of $f^*(n)$, we obtain $f(n)$ again. That is, $f^{**}(n) = f(n)$. The sequences $f(n)$ and $f^*(n)$ are called inverse sequences.

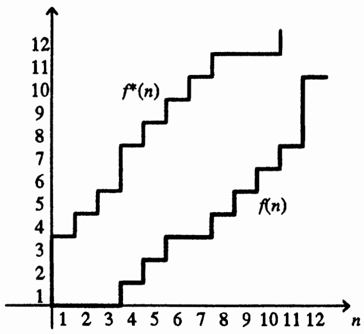


Figure 1. The functions $f(n)$ and $f^*(n)$.

Now we construct two other sequences

$$F(n) = f(n) + n \text{ and } G(n) = f^*(n) + n.$$

For our example,

$$F(n) = 1, 2, 3, 5, 7, 9, 10, 12, 14, \ldots;$$
$$G(n) = 4, 6, 8, 11, 13, 15, 17, 19, 20, \ldots.$$

Notice anything? The two sequences $F(n)$ and $G(n)$ together contain each natural number exactly once. This fact and its converse were first discovered and proved by mathematicians Lambek and Moser in 1954 (c.f. American Mathematical Monthly, vol. 61, p. 454, 1954). The sequences $F(n)$ and $G(n)$ are called complementary sequences.

Theorem (Lambek and Moser). $f(n)$ and $f^*(n)$ are inverse sequences if and only if $F(n) = f(n) + n$ and $G(n) = f^*(n) + n$ are complementary sequences (with the minor conditions that (i) $f(n)$ and $f^*(n)$ are non-decreasing sequences of non-negative integers; (ii) $F(n)$ and $G(n)$ are strictly increasing sequences of positive integers.)

If a formula for the nth term of a sequence is known, the theorem of Lambek and Moser can be used to find a general formula for the complementary sequence. The following example illustrates the idea.

Example. We can separate the natural numbers into two sequences $F(n)$ and $G(n)$ that contain squares and non-squares as follows.

$$F(n) = 1, 4, 9, 16, 25, 36, 49, 64, 81, \ldots,$$
$$G(n) = 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, \ldots.$$

We know that a formula for the nth square is $F(n) = n^2$. Can we find a formula for the nth non-square $G(n)$?

We note that $F(n)$ and $G(n)$ are complementary and thus the sequences

$$f(n) = F(n) - n = 0, 2, 6, 12, 20, \ldots,$$
$$f^*(n) = G(n) - n = 1, 1, 2, 2, 2, 2, 3, \ldots,$$

are inverse sequences. Now

$$f(n) = F(n) - n = n^2 - n.$$

Therefore, $f^*(n) = k$ where

$$f(k) < n \le f(k+1),$$
$$k^2 - k < n \le (k+1)^2 - (k+1) = k^2 + k.$$

Since both k and n are integers,

$$k^2 - k + \tfrac{1}{4} < n < k^2 + k + \tfrac{1}{4},$$
$$(k - \tfrac{1}{2})^2 < n < (k + \tfrac{1}{2})^2,$$
$$k - \tfrac{1}{2} < \sqrt{n} < k + \tfrac{1}{2},$$
$$\sqrt{n} - \frac{1}{2} < k < \sqrt{n} + \frac{1}{2}.$$

Consequently,

$$f^*(n) = k = \left[\sqrt{n} + \tfrac{1}{2}\right]$$

and

$$G(n) = f^*(n) + n = n + \left[\sqrt{n} + \tfrac{1}{2}\right].$$

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is January 10, 1998.

Problem 66.

(a) Find the first positive integer whose square ends in three 4′s.

(b) Find all positive integers whose squares end in three 4′s.

(c) Show that no perfect square ends with four 4′s.

(*Source:* 1995 British Mathematical Olympiad.)

Problem 67. Let Z and R denote the integers and real numbers, respectively. Find all functions $f: Z \to R$ such that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{2}$$

for all integers x, y such that $x + y$ is divisible by 3. (*Source:* a modified problem from the 1995 Iranian Mathematical Olympiad.)

Problem 68. If the equation

$$ax^2 + (c-b)x + (e-d) = 0$$

has real roots greater than 1, show that the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

has at least one real root. (*Source*: 1995 Greek Mathematical Olympiad.)

Problem 69. $ABCD$ is a quadrilateral such that $AB = AD$ and $\angle B = \angle D = 90°$. Points F and E are chosen on BC and CD, respectively, so that $DF \perp AE$. Prove that $AF \perp BE$. (*Source:* 1995 Russian Mathematical Olympiad.)

Problem 70. Lines $l_1, l_2, \cdots, l_k$ are on a plane such that no two are parallel and no three are concurrent. Show that we can label the C_2^k intersection points of these lines by the numbers $1, 2, \cdots, k-1$ so that in each of the lines $l_1, l_2, \cdots, l_k$ the numbers $1, 2, \cdots, k-1$ appear exactly once if and only if k is even. (*Source:* a modified problem from the 1995 Greek Mathematical Olympiad.)

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Solutions

****

Due to the large number of solutions received by the editors, we will first acknowledge the solvers by their schools and grade levels. The numbers following a solver's name are the number of the problems which the solver submitted correct solutions.

Bishop Hall Jubilee School: (Form 4) CHAN Kin Hang (61, 63, 64, 65). **Cheung Chuk Shan College**: (Form 5) CHOW King Fun (61). **Heep Woh College**: (Form 7) KU Wah Kwan (61, 63). **Ho Fung College**: (Form 6) TSE Wing Ho (61, 64). **HK Taoist Association Ching Chung Secondary School**: (Form 7) LI Fung (61, 62). **HKUST**: CHAN Wing Sum (61, 63). **La Salle College**: (Form 3) CHAN Ernest Eason (61); (Form 5) Vincent LUNG (61). **N.T. Heung Yee Kuk Yuen Long District Secondary School**: (Form 7) CHU Kai Mun (61, 63, 64). **Queen Elizabeth School**: (Form 4) LAI Chi Fung Brian (61), LAW Ka Ho (61, 62, 63, 64, 65). **Saint Louis School**: (Form 7) SHAM Wing Hang (61). **St. Paul's Co-educational College**: (Form 5) CHAN Lung Chak (61, 62), MAK Shiu Ting (61), NGAN Chung Wai Hubert (61, 62, 63, 64, 65), SHEK Ka Wai Wilson (62); (Form 7) CHU Choi Yam Venus (61). **St. Stephen's Girls' College**: (Form 6) WAN Hoi Wah (61). **SKH Kei Hau Secondary School**: (Form 4) WONG Chun Wai (61, 62, 63, 64, 65). **Shi Hui Wen Secondary School**: (Form 6) Jimmy KONG Ka Ho (61, 62, 64, 65). **STFA Leung Kau Kui College**: (Form 5) CHU Chun Yiu (61, 63), IP Man Wai (61), Gary NG Ka Wing (61, 62, 63, 64, 65), SIN Ka Fai (61, 62, 64), YUEN Man Long (61, 62, 63, 64, 65); (Form 6) Yves CHEUNG Yui Ho (61, 62, 63, 64), CHING Wai Hung (61, 62, 64), WONG Hau Lun (61, 62, 63, 64, 65); (Form 7) William CHEUNG Pok Man (62, 63, 64). **Valtorta College**: (Form 6) CHANG Pui Kwan (61), KO Tsz Wan (61), Ryan LAI (61), LAM Wai Hung (61), LIN Kai Shuen (61), NG Lai Ha (61), TAM Ka Kwong (61), TANG Ka Wai (61), WONG Shu Fai (61); (Form 7) KWAN Yee Kin (61), LEUNG Pak Keung (62), TSANG Sai Wing (62), WAN Tsz Kit (61, 62, 64).

Problem 61. Find the smallest positive integer which can be written as the sum of nine, the sum of ten and the sum of eleven consecutive positive integers.

Solution:

Let n be the smallest such positive integer. Then

$$n = a + (a+1) + \cdots + (a+8) = 9a + 36,$$
$$n = b + (b+1) + \cdots + (b+9) = 10b + 45,$$
$$n = c + (c+1) + \cdots + (c+10) = 11c + 55.$$

These imply n is divisible by

$$9 \times 5 \times 11 = 495.$$

So $n \ge 495$. Letting $a = 51$, $b = 45$, $c = 40$, we see that 495 is possible. So $n = 495$.

Problem 62. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the midpoint of PQ. (*Source*: 1996 Australian Mathematical Olympiad.)

Solution: WONG Chun Wai.

Let $[XYZ]$ denote the area of ΔXYZ. Then

$$\frac{MP}{MQ} = \frac{[PAC]}{[QAC]} = \frac{\frac{AP}{AB}[ABC]}{\frac{AQ}{AD}[ADC]} = \frac{CD \cdot AD \cdot [ABC]}{AB \cdot BC \cdot [ADC]} = \frac{[ADC] \cdot [ABC]}{[ABC] \cdot [ADC]} = 1.$$

Problem 63. Show that for $n \ge 2$, there is a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that $|a_k - k| = |a_1 - 1| \ne 0$ for $k = 2, 3, \ldots, n$ if and only if n is even.

Solution: LAW Ka Ho.

Suppose for some n, the condition is possible. Let $d = |a_1 - 1|$, p be the number of times $a_k > k$ and q be the number of times $a_k < k$. Then $p + q = n$ and

$$0 = (a_1 - 1) + (a_2 - 2) + \cdots + (a_n - n) = pd - qd.$$

So $p = q$ and n is even. If n is even, then the permutation $2, 1, 4, 3, \ldots, n, n-1$ satisfies the condition with $|a_1 - 1| = 1$.

Comments: This was a problem on the 1996 Australian Mathematical Olympiad.

Problem 64. Show that it is impossible to place 1995 different positive integers

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

along a circle so that for every two adjacent numbers, the ratio of the larger to the smaller one is a prime number.

Solution: William CHEUNG Pok Man.

Suppose this is possible. Let a_1, a_2, ..., a_{1995} be the numbers in the clockwise direction. Then a_{k-1}/a_k is a prime or the reciprocal of a prime for k = 1, 2, ..., 1995 with $a_0 = a_{1995}$. Suppose m of these are primes and 1995 − m of these are reciprocals of primes. Since

$$\left(\frac{a_0}{a_1}\right)\left(\frac{a_1}{a_2}\right)\cdots\left(\frac{a_{1994}}{a_{1995}}\right)=1,$$

this means the product of m primes will equal to a product of 1995 − m primes. Unique prime factorization implies m = 1995 − m, which is impossible as 1995 is odd.

Comments: This was a problem on the 1995 Russia Mathematical Olympiad.

Problem 65. All sides and diagonals of a regular 12-gon are painted in 12 colors (each segment is painted in one color). Is it possible that for any three colors there exist three vertices which are joined with each other by segments of these colors?

Solution: LAW Ka Ho.

There are 12 sides and 54 diagonals. With 12 colors, there is a color, say X, which is used to paint at most 5 of these segments. For each X colored segment, 10 triangles can be formed having this segment as a side (using the remaining 10 vertices). So there are at most 50 triangles with at least one side colored X. However, if any three colors are the colors of the sides of a triangle, there would be C_2^{11} = 55 triangles having at least one side colored X, a contradiction.

Comments: This was also a problem on the 1995 Russia Mathematical Olympiad.

Olympiad Corner

(*continued from page 1*)

Problem 4. Let $ABCD$ be a convex quadrilateral. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively. Given that the quadrilateral $PQRS$ has area 1, prove that the area of the quadrilateral $ABCD$ is 2.

Problem 5. Let x, y and z be positive real numbers.

(i) If $x + y + z \geq 3$, is it necessarily true that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\leq 3$?

(ii) If $x + y + z \leq 3$, is it necessarily true that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 3$?

Round 2 (February 27, 1997)
Time Allowed: $3\frac{1}{2}$ hours.

Problem 1. Let M and N be two 9-digit positive integers with the property that if any one digit of M is replaced by the digit of N in the corresponding place (e.g., the 'tens' digit of M replaced by the 'tens' digit of N) then the resulting integer is a multiple of 7.

Prove that any number obtained by replacing a digit of N by the corresponding digit of M is also a multiple of 7.

Find an integer $d > 9$ such that the above result concerning divisibility by 7 remains true when M and N are two d-digit positive integers.

Problem 2. In the acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

Problem 3. Find the number of polynomials of degree 5 with distinct coefficients from the set {1, 2, 3, 4, 5, 6, 7, 8} that are divisible by $x^2 - x + 1$.

Problem 4. The set

$$S = \{1/r : r = 1, 2, 3, \ldots\}$$

of reciprocals of the positive integers contains arithmetic progressions of various lengths. For instance, 1/20, 1/8, 1/5 is such a progression, of length 3 (and common difference 3/40). Moreover, this is a *maximal progression* in S of length 3 since it cannot be extended to the left or right within S (−1/40 and 11/40 not being members of S).

(i) find a maximal progression in S of length 1996.

(ii) Is there a maximal progression in S of length 1997?

老師不教的幾何（四）

(*continued from page 2*)

妙處在於它帶給我們另一個美麗而意想不到的結果：

外拿破侖三角形的面積
− 內拿破侖三角形的面積
= 三角形ABC的面積，

同學們請自己驗證便可。

在任意三角形ABC的三邊外側作等邊三角形後，還有另一個美妙的特性是十七世紀數學家費馬(Fermat)所發現的：

圖五中三角形ABR、BCP和CAQ都是等邊的，所以ΔARC繞點A旋轉60°可得ΔABQ。因此

$$RC = BQ$$

及 $$\angle RFB = 60°。$$

同理，$PA = CR$。所以

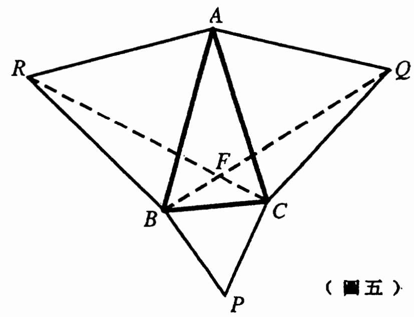
$$AP = BQ = CR。$$

再者，

$$\angle RFB = 60° = \angle RAB$$

及

$$\angle CFQ = 60° = \angle CAQ。$$



（圖五）

因此$ARBF$和$CQAF$都是圓外接四邊形。由於$\angle BFC$ = 120°，而$\angle CPB$ = 60°，可以推知$BPCF$也是圓外接四邊形。這三個圓於F共點，稱為費馬點。由F原是BQ和CR的交點，從對稱觀點可知F也在AP上。