Distance Formula for a Point Inside a Triangle

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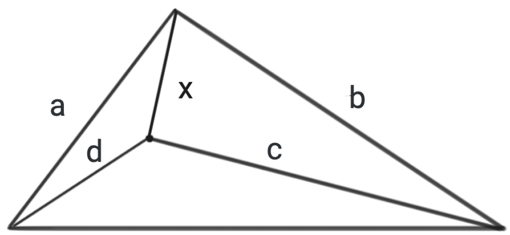
Abstract

Given a point inside a triangle. Let the lengths a, b, c, d, and e be given (see the figure). In this paper, we obtained a novel formula for finding the length x. The motivation for our work was the **Sedrakyan-Mozayeni theorem** [1] for convex quadrilaterals. Moreover, we generalize the Sedrakyan-Mozayeni theorem for any quadrilateral, including the case of a concave quadrilateral. The authors would like to thank *Nairi Sedrakyan* and *A. Mozayeni* for their contributions to this work.

1 Introduction

Assume that the lengths a, b, c, d, e are given (see the figure).



Then, the length x can be found with the following novel formula.

Theorem 1 (**Sedrakyan-Gandhi**).

$$2x^2 = a^2 + b^2 + c^2 + d^2 - e^2 + \frac{(a^2-b^2)(c^2-d^2)}{e^2} - \frac{\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}}{e^2}.$$

A stronger result when **the point is not necessarily inside the triangle and can be anywhere** (Theorem 2 representation) also holds true.

Theorem 2 (**Sedrakyan-Gandhi (stronger result)**).

$$(d^2-x^2)(d^2-c^2)b^2 + (x^2-d^2)(x^2-c^2)e^2 + (c^2-d^2)(c^2-x^2)a^2 + d^2b^2(b^2-a^2-e^2)+$$
$$+x^2e^2(e^2-a^2-b^2) + c^2a^2(a^2-b^2-e^2) + a^2b^2e^2 = 0.$$

2 Main results and proofs

First, let us prove Theorem 2. Consider a point inside a triangle (see above figure).

Theorem 2 (**Sedrakyan-Gandhi (stronger result)**).

$$(d^2-x^2)(d^2-c^2)b^2 + (x^2-d^2)(x^2-c^2)e^2 + (c^2-d^2)(c^2-x^2)a^2 + d^2b^2(b^2-a^2-e^2)+$$
$$+x^2e^2(e^2-a^2-b^2) + c^2a^2(a^2-b^2-e^2) + a^2b^2e^2 = 0.$$

In order to prove Theorem 2, let us prove the following lemma:

Lemma. If numbers α, β, γ are so that either one of them is equal to the sum of the other two, or $\alpha+\beta+\gamma=2\pi$, then

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma.$$

Proof of the lemma. If either for example $\alpha+\beta=\gamma$, or if $\alpha=2\pi-\beta-\gamma$, then we have that

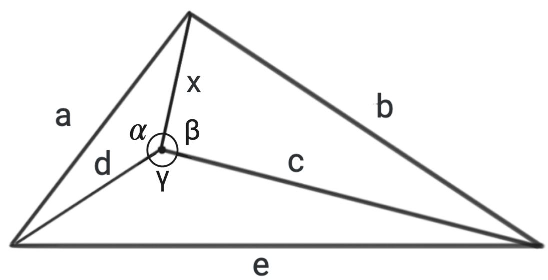
$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+\frac{\cos 2\alpha+\cos 2\beta}{2}+\cos^2\gamma=$$

$$=1+\cos(\alpha+\beta)\cos(\alpha-\beta)+\cos^2\gamma=1+\cos(\gamma)\cdot\cos(\alpha-\beta)+\cos^2\gamma=$$

$$=1+\cos\gamma(\cos(\alpha-\beta)+\cos\gamma)=1+\cos\gamma(\cos(\alpha-\beta)+\cos(\alpha+\beta))=$$

$$=1+\cos\gamma(2\cos\alpha\cos\beta)=1+2\cdot\cos\gamma\cdot\cos\alpha\cdot\cos\beta.$$

This ends the proof of the lemma.
Let us denote by α, β, γ the angles shown in the following figure.



Applying the law of cosines (see the figure) in the triangles with side lengths (d,x,a), (c,x,b) and (d,c,e), we get that

$$\cos\alpha=\frac{d^2+x^2-a^2}{2\cdot a\cdot x},$$

$$\cos\beta=\frac{c^2+x^2-b^2}{2\cdot c\cdot x},$$

$$\cos\gamma=\frac{d^2+c^2-e^2}{2\cdot d\cdot c}.$$

Note that for α, β, γ the conditions of the lemma hold true, as $\alpha+\beta+\gamma=2\pi$ (see the figure). Plugging in the above values of $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ into the following identity (obtained from the lemma)

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma,$$

and simplifying it further (bring fractions to a common denominator), we get

$$(d^2-x^2)(d^2-c^2)b^2+(x^2-d^2)(x^2-c^2)e^2+(c^2-d^2)(c^2-x^2)a^2+d^2b^2(b^2-a^2-e^2)+$$

$$+x^2e^2(e^2-a^2-b^2)+c^2a^2(a^2-b^2-e^2)+a^2b^2e^2=0.$$

This ends the proof of Theorem 2.

Remark. One can easily prove that Theorem 2 holds true for any four points in a plane. Moreover, it holds true for the case when among these four points there are coinciding points.

Proof of Theorem 1. Let us take $y=x^2$. From Theorem 2, we get that

$$(d^2-y)(d^2-c^2)b^2+(y-d^2)(y-c^2)e^2+(c^2-d^2)(c^2-y)a^2+d^2b^2(b^2-a^2-e^2)+$$

$$+ye^2(e^2-a^2-b^2)+c^2a^2(a^2-b^2-e^2)+a^2b^2e^2=0$$

After expanding and simplifying the previous equation, we get the following quadratic equation with respect to variable y:

$$y^2e^2+y\Big(e^2(a^2+b^2+c^2+d^2-e^2)+(a^2-b^2)(c^2-d^2)\Big)+$$

$$+(a^2+c^2-b^2-d^2)(a^2c^2-b^2d^2)+e^2(a^2-d^2)(b^2-c^2)=0.$$

Note that the coefficients of this quadratic equation are

$$A = e^2,$$

$$B = e^2(a^2 + b^2 + c^2 + d^2 - e^2) + (a^2 - b^2)(c^2 - d^2),$$
$$C = (a^2 + c^2 - b^2 - d^2)(a^2c^2 - b^2d^2) + e^2(a^2 - d^2)(b^2 - c^2).$$

Using the following formula of the discriminant D of a quadratic equation:

$$D = \sqrt{B^2 - 4 \cdot A \cdot C},$$

we get that

$$D = \sqrt{(4a^2b^2 - (a^2 + b^2 - e^2)^2)(4c^2d^2 - (c^2 + d^2 - e^2)^2)}.$$

Then, using the quadratic formula

$$y = \frac{-B \pm \sqrt{D}}{2A},$$

we get that

$$y_1 = \frac{e^2(a^2+b^2+c^2+d^2-e^2)+(a^2-b^2)(c^2-d^2)-\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}}{2e^2}.$$

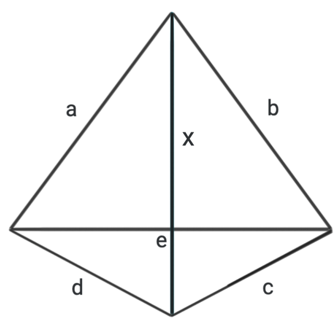
$$y_2 = \frac{e^2(a^2+b^2+c^2+d^2-e^2)+(a^2-b^2)(c^2-d^2)+\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}}{2e^2}.$$

Note that the last two equations can be rewritten in the following forms

$$2y_1 = a^2 + b^2 + c^2 + d^2 - e^2 + \frac{(a^2-b^2)(c^2-d^2)}{e^2} - \frac{\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}}{e^2}.$$

$$2y_2 = a^2 + b^2 + c^2 + d^2 - e^2 + \frac{(a^2-b^2)(c^2-d^2)}{e^2} + \frac{\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}}{e^2}.$$

Note also that, according to **Sedrakyan-Mozayeni theorem** [1] root y_2 corresponds to the case when the point is outside the triangle (see the figure).



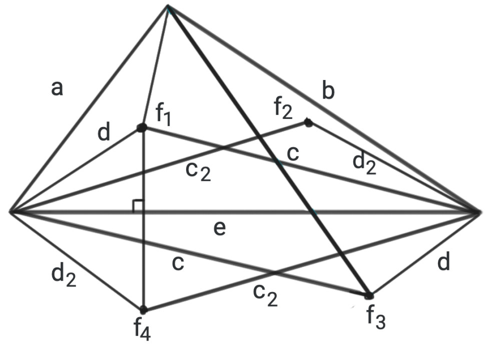
Thus, it follows that root y_1 corresponds to the case when a point is inside a triangle. As $y = x^2$, then we obtain the following theorem:

Theorem 1 (**Sedrakyan-Gandhi**).

$$2x^2 = a^2 + b^2 + c^2 + d^2 - e^2 + \frac{(a^2-b^2)(c^2-d^2)}{e^2} - \frac{\sqrt{(4a^2b^2-(a^2+b^2-e^2)^2)(4c^2d^2-(c^2+d^2-e^2)^2)}}{e^2}.$$

This ends the proof of Theorem 1.
So, Theorem 1 is a consequence of Theorem 2.

Remark. Two possible points are represented by f_1 and f_2 (see the figure) in the case where the original quadrilateral with side lengths a, b, c, and d is concave. Also, there are two possible points expressed by points f_3 and f_4 for when the original quadrilateral is convex. A kite or parallelogram can be constructed using the existing side lengths c and d to obtain the four points.



Let us prove that $c = c_2$. Using the theorem, we get

$$c^2 + d^2 + \sqrt{3(4c^2d^2 - (c^2 + d^2 - e^2)^2)} = c_2^2 + d_2^2 + \sqrt{3(4c_2^2d_2^2 - (c_2^2 + d_2^2 - e^2)^2)}.$$

After substituting d for c, we get,

$$2c^2 + \sqrt{3(4c^2e^2 - e^4)} = 2c_2^2 + \sqrt{3(4c_2^2e^2 - e^4)}.$$

If we assume that $c_2 > c$, then the right-hand-side of the last equation is greater than its left-hand-side (since e is a constant). In a similar way, the case $c_2 < c$ is not possible. So, c must equal c_2. In a similar way, we prove that $d = d_2$.

3 Applications

Application 1 (AMC 12 preparation book, Sedrakyan): Let vertices A, B, C of triangle ABC lie on circles with center O and radii $\sqrt{52}$, 3 and 5, respectively. Given that point O lies in the interior of triangle ABC and $\angle ABC = 120^\circ, \angle BAC = 30^\circ$. What is the value of the length of side AB?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Application 2: Let $ABCD$ be a quadrilateral with side lengths a, b, c, d and diagonals e and f, where $a, b, c, d, e, f \in \mathbb{N}$. Prove that

- at least one of the numbers a, b, c, d, e, f is divisible by 2,
- at least one of these numbers is divisible by 3.

Application 3: Let P be a point inside a square $ABCD$, such that
$PA : PB : PC = 1 : 2 : 3$. Find $\angle APB$.

Application 4: Solve the following equation

$$\sqrt{2x(x-3)} + \sqrt{6x(x-5)} + \sqrt{3x(x-4)} = 6.$$

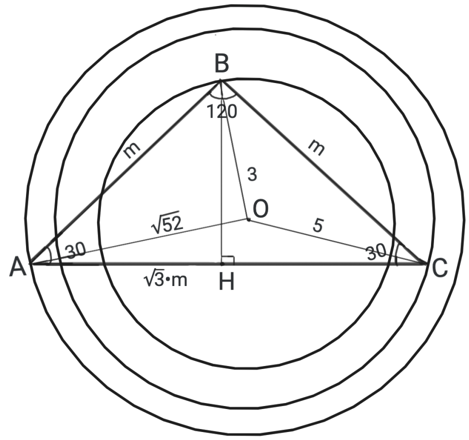
Application 5: Let the side lengths of rectangle $ABCD$ be odd integers. Let P be a point inside the rectangle $ABCD$, prove that the lengths of PA, PB, PC, and PD cannot all at the same time be integers.

4 Solutions of applications

Application 1 (AMC 12 preparation book, Sedrakyan): Let vertices A, B, C of triangle ABC lie on circles with center O and radii $\sqrt{52}$, 3 and 5, respectively. Given that point O lies in the interior of triangle ABC and $\angle ABC = 120^\circ, \angle BAC = 30^\circ$. What is the value of the length of side AB?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Solution: Let us prove that the correct answer is (D).
Let $AB = m$. As $\angle ABC = 120°$ and $\angle BAC = 30°$, then $\angle BCA = 30°$. Thus, $BC = AB = m$. Consider the altitude BH, then triangle BCH is a $30-60-90$ triangle. So $BH = \frac{m}{2}$ and $CH = \frac{\sqrt{3}\cdot m}{2}$. Then $AC = \sqrt{3}\cdot m$.



Using Theorem 2 for $a = m, b = m, c = 5, d = \sqrt{52}, e = m\sqrt{3}, x = 3$, we get

$$(52-9)(52-25)m^2 + (9-52)(9-25)\left(3m^2\right) +$$

$$+(25-52)(25-9)m^2 + (52m^2)\left(m^2 - m^2 - 3m^2\right) +$$

$$+(9\cdot 3m^2)(3m^2 - m^2 - m^2) + (25\cdot m^2)\left(m^2 - m^2 - 3m^2\right) + m^2\cdot m^2\cdot 3m^2 = 0.$$

After simplifying, we get

$$3m^2(m+\sqrt{19})(m-\sqrt{19})(m+7)(m-7) = 0$$

As m must be positive, then either $m = \sqrt{19}$ or $m = 7$. Using the answer choices, we get that the correct answer is (D). Note that even without having the answer choices one can easily prove that the case $m = \sqrt{19}$ is not possible, as in this case by Heron's formula we get that the sum of the areas of triangles ABO, BCO, CAO is not equal to the area of ABC.

Application 2: Let $ABCD$ be a quadrilateral with side lengths a, b, c, d and diagonals e and f, where $a, b, c, d, e, f \in \mathbb{N}$. Prove that

- at least one of the numbers a, b, c, d, e, f is divisible by 2,
- at least one of these numbers is divisible by 3.

Solution:
Divisible by 2:
Proof by a contradiction argument. Assume that a, b, c, d, e, f are odd numbers. Using Theorem 2, we get

$$(d^2-f^2)(d^2-c^2)b^2 + (f^2-d^2)(f^2-c^2)e^2 + (c^2-d^2)(c^2-f^2)a^2 + d^2b^2(b^2-a^2-e^2)+$$

$$+f^2e^2(e^2-a^2-b^2) + c^2a^2(a^2-b^2-e^2) + a^2b^2e^2 = 0.$$

It is a well-known fact that a square of an odd number can only leave a remainder of 1 when divided by 8. Let us look at the remainder of each addend when divided by 8. When divided by 8, the first three addends leave a remainder of 0, the next three addends leave a remainder of -1, and the last addend leaves a remainder of 1. So, the left-hand-side of this equation leaves a remainder of $0+0+0+(-1)+(-1)+(-1)+1 = -2$ when divided by 8. This leads to a contradiction, as on the right-hand side we have 0, so when divided by 8 the remainder must also be 0. Therefore, a, b, c, d, e, f cannot all be odd and at least one of them must be even.

Divisible by 3:
Proof by contradiction argument. Assume that a, b, c, d, e, and f are not divisible by 3. Using Theorem 2, we get

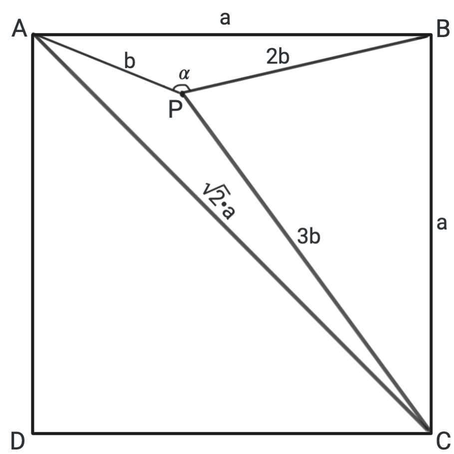
$$(d^2-f^2)(d^2-c^2)b^2 + (f^2-d^2)(f^2-c^2)e^2 + (c^2-d^2)(c^2-f^2)a^2 + d^2b^2(b^2-a^2-e^2)+$$

$$+f^2e^2(e^2-a^2-b^2) + c^2a^2(a^2-b^2-e^2) + a^2b^2e^2 = 0.$$

It is a well-known fact that a square of a number that is not divisible by 3 leaves a remainder of 1 when divided by 3. Let us look at the remainder of each addend when divided by 3. The first three addends leave remainders of 0, the next three addends leave remainders of -1, and the last addend leaves a remainder of 1. So, the left-hand-side of this equation leaves a remainder of $0+0+0+(-1)+(-1)+(-1) = -2$ when divided by 3. This leads to a contradiction, as on the right side we have 0, so when divided by 3 the remainder must be 0. So, at least one of the numbers a, b, c, d, e, f must be divisible by 3.

Application 3: Let P be a point inside a square $ABCD$, such that $PA : PB : PC = 1 : 2 : 3$. Find $\angle APB$.

Solution: Let us consider the figure below.



Let $AB = a$, then $BC = a$, and $AC = \sqrt{2} \cdot a$. Let $PA = b$. So, $PB = 2b$ and $PC = 3b$. Using Theorem 2 for $a = a$, $b = a$, $c = 3b$, $d = b$, $e = \sqrt{2} \cdot a$, $x = 2b$, we obtain

$$(b^2 - (2b)^2)(b^2 - (3b)^2)a^2 + ((2b)^2 - b^2)((2b)^2 - (3b)^2)(\sqrt{2} \cdot a)^2 +$$

$$+((3b)^2 - b^2)((3b)^2 - (2b)^2)a^2 + b^2a^2(a^2 - a^2 - (\sqrt{2} \cdot a)^2) +$$

$$+(2b)^2(\sqrt{2} \cdot a)^2((\sqrt{2} \cdot a)^2 - a^2 - a^2) + (3b)^2a^2(a^2 - a^2 - (\sqrt{2} \cdot a)^2) + a^2a^2(\sqrt{2} \cdot a)^2 = 0.$$

After simplifying, we get

$$a^4 - 10a^2b^2 + 17b^4 = 0.$$

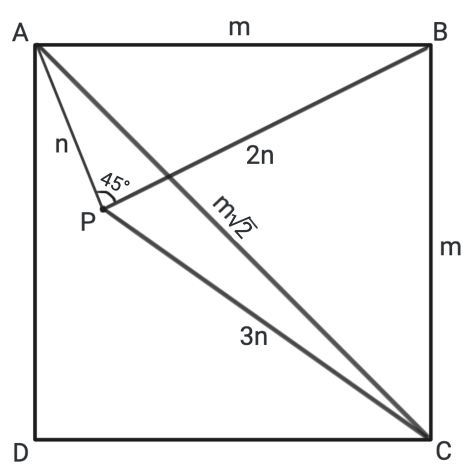
Solving for a^2 in terms of b^2, we get

$$a^2 = 5 \pm 2\sqrt{2} \cdot b^2.$$

Using the law of cosines in triangle APB after substituting a^2 in terms of b, we obtain

$$\pm 2\sqrt{2} \cdot b^2 = b^2 + 4b^2 + 4b^2 \cdot \cos(\alpha)$$

After simplifying, we get $\alpha = 45^\circ$ or $\alpha = 135^\circ$.
Let us consider the case when $\alpha = 45^\circ$ (see the figure).



Note that quadrilateral $ABPC$ is cyclic, as $\angle APB = 45^\circ = \angle ACB$. On the other hand, square $ABCD$ is also cyclic. This means that point P should be on the same circle as points A, B, C, D, as P is inside the square this is not possible. Thus, $\alpha = 45^\circ$ is not possible and the only solution is $\angle APB = 135^\circ$ (one can easily check that this case satisfies the assumptions of the problem).

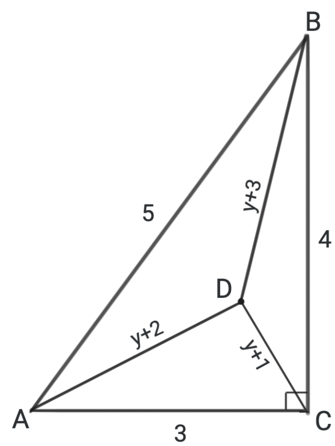
Application 4: Solve the following equation

$$\sqrt{2x(x-3)} + \sqrt{6x(x-5)} + \sqrt{3x(x-4)} = 6.$$

Solution: Note that $x \neq 0$. Note also that x cannot be positive. If $x > 0$, as $x - 5$ is under the square root, then we get $x > 5$. On the other hand, if $x > 5$, then we get that the left side of the equation is greater than 6. Thus, $x < 0$. Denote $x = -y$, where $y > 0$.

$$\sqrt{2y(y+3)} + \sqrt{6y(y+5)} + \sqrt{3y(y+4)} = 6.$$

Let us consider a right triangle ABC, where $AC = 3$, $BC = 4$, $AB = 5$, $AD = y+2$, $BD = y+3$, $CD = y+1$ (see the figure).



We have

$$Area(ABC) = \frac{3 \cdot 4}{2} = 6,$$
$$Area(ACD) = \sqrt{2y(y+3)},$$
$$Area(BCD) = \sqrt{3y(y+4)},$$
$$Area(ABD) = \sqrt{6y(y+5)}.$$

Note that

$$Area(ABC) = Area(ACD) + Area(BCD) + Area(ABD).$$

Thus, it follows that

$$6 = \sqrt{2y(y+3)} + \sqrt{6y(y+5)} + \sqrt{3y(y+4)}.$$

So, we turned the given algebraic equation into a geometry problem.
Using Theorem 2 for $a = 5$, $b = 4$, $c = y+1$, $d = y+2$, $x = y+3$, $e = 3$ and plugging in these values we get

$$\left((y+2)^2 - (y+3)^2\right)\left((y+2)^2 - (y+1)^2\right)4^2 +$$
$$+\left((y+3)^2 - (y+2)^2\right)\left((y+3)^2 - (y+1)^2\right)3^2 +$$
$$+\left((y+1)^2 - (y+2)^2\right)\left((y+1)^2 - (y+3)^2\right)5^2 +$$
$$+(y+2)^2 4^2 \left(4^2 - 5^2 - 3^2\right) + (y+3)^2 3^2 \left(3^2 - 5^2 - 4^2\right) +$$
$$+(y+1)^2 5^2 \left(5^2 - 4^2 - 3^2\right) + 5^2 4^2 3^2 = 0$$

Simplifying this equation, we obtain

$$-23y^2 - 132y + 36 = 0$$

We get $y=-6$ or $y=\frac{6}{23}$. Note that y is positive, so $y=-6$ is not possible. Then, $y=\frac{6}{23}$. One can easily verify that $y=\frac{6}{23}$ satisfies the initial equation. Since $x=-y$, then $x=-\frac{6}{23}$.

Remark. In a similar way, for any positive number a, b, c, one can solve the following equations

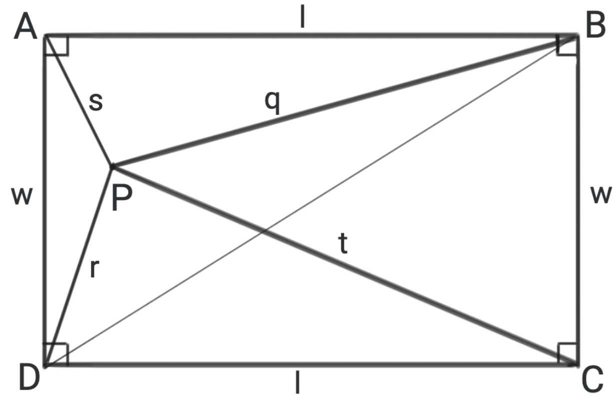
$$\sqrt{abc(a+b+c)}=\sqrt{abx(x-a-b)}+\sqrt{bcx(x-b-c)}+\sqrt{acx(x-a-c)},$$

or

$$\sqrt{abc(a+b+c)}=\sqrt{abx(x+a+b)}+\sqrt{bcx(x+b+c)}+\sqrt{acx(x+a+c)}.$$

Application 5: Let the side lengths of rectangle $ABCD$ be odd integers. Let P be a point inside the rectangle $ABCD$, prove that the lengths of PA, PB, PC, and PD cannot all at the same time be integers.

Solution: Let us call the length l and the width w. By the Pythagorean theorem, we have $DB=\sqrt{l^2+w^2}$. Consider the figure below.



By the British Flag theorem $s^2+t^2=q^2+r^2$. We get three possible cases.

Case 1: *s, t, q, and r are odd.* Since we also know a and b are odd, we are led to a contradiction when applying Theorem 2, where $a=w, b=l, c=q, d=r, x=s$, and $e=\sqrt{l^2+w^2}$. We proved this to be impossible by a contradiction in Application 2. After looking at the remainder of 8 when divided by each of the addends in the theorem, we obtain that $-2=0$. Therefore, s, t, q, and r cannot all be odd.

Case 2: *Two of them are odd and the other two are even. For example, s, t are odd, and q, r are even.* In this case, we get that the left side of Theorem 2 equation is odd, so it cannot be equal to 0.

Case 3: *s, t, q, and r are even.* By considering the remainders when dividing by 4, we get a remainder of 2 on the left side, with the last addend since e^2 is not divisible by 4, only by 2. This leads us to a contradiction (cannot be equal to 0).

References

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