Junior problems

J253. Prove that if $a, b, c > 0$ satisfy $abc = 1$, then

$$\frac{1}{ab+a+2} + \frac{1}{bc+b+2} + \frac{1}{ca+c+2} \le \frac{3}{4}.$$

Proposed by Marcel Chirita, Bucharest, Romania

Solution by Alessandro Ventullo, Milan, Italy

By the AM-HM Inequality, we have

$$\frac{1}{ab+1+a+1} \le \frac{1}{4}\left(\frac{1}{ab+1} + \frac{1}{a+1}\right) = \frac{1}{4}\left(\frac{c}{c+1} + \frac{1}{a+1}\right)$$

$$\frac{1}{bc+1+b+1} \le \frac{1}{4}\left(\frac{1}{bc+1} + \frac{1}{b+1}\right) = \frac{1}{4}\left(\frac{a}{a+1} + \frac{1}{b+1}\right)$$

$$\frac{1}{ca+1+c+1} \le \frac{1}{4}\left(\frac{1}{ca+1} + \frac{1}{c+1}\right) = \frac{1}{4}\left(\frac{b}{b+1} + \frac{1}{c+1}\right).$$

Summing up the three inequalities, we obtain

$$\frac{1}{ab+a+2} + \frac{1}{bc+b+2} + \frac{1}{ca+c+2} \le \frac{1}{4}\left(\frac{a+1}{a+1} + \frac{b+1}{b+1} + \frac{c+1}{c+1}\right) = \frac{3}{4}.$$

Also solved by Arkady Alt, San Jose, California, USA; Toan Pham Quang, Dang Thai Mai Secondary School, Vinh city, Vietnam; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Sayan Das, Kolkata, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Peter Shim, The Pingry School, New Jersey, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Albert Stadler, Switzerland; Zolbayar Shagdar, Orchlon International School, Ulaanbaatar, Mongolia; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

J254. Solve the following equation $F_{a_1} + F_{a_2} + \ldots + F_{a_k} = F_{a_1+a_2+\ldots+a_k}$, where F_i is the i-th Fibonacci number.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

Solution by Alessandro Ventullo, Milan, Italy

If $k = 1$ the equation has infinitely many solutions. Let $k \geq 2$ and suppose that $1 \leq a_1 \leq a_2 \leq \ldots \leq a_k$. We have three cases.

(i) $k = 2$. Suppose that $a_1 + a_2 \leq 4$. It's easy to see that $(1, 2)$ and $(1, 3)$ are solutions. Now, suppose that $a_1 + a_2 \geq 5$. If $a_1 = 1$, then $a_2 \geq 4$ and $F_{a_1+a_2-2} > F_{a_1}$. If $a_1 > 1$, then $a_2 > 2$, so $F_{a_1+a_2-1} > F_{a_2}$ and

$$F_{a_1+a_2} = F_{a_1+a_2-1} + F_{a_1+a_2-2} > F_{a_2} + F_{a_1},$$

i.e. the equation has no solution if $a_1 + a_2 \geq 5$.

(ii) $k = 3$. There are no solutions when $a_3 = 1$. If $a_3 = 2$, we have the solution $(1, 1, 2)$. It's easy to see that there are no more solutions such that $a_2 = 1$ and $a_3 > 2$. Suppose that $a_3 \geq 2$ and $a_2 \geq 2$. If $a_3 = 2$, we get $F_{a_1+a_2+a_3-1} > F_{a_3}$ and if $a_3 > 2$, we get $F_{a_1+a_2+a_3-2} > F_{a_1+a_2}$, so

$$F_{a_1+a_2+a_3} = F_{a_1+a_2+a_3-1} + F_{a_1+a_2+a_3-2} > F_{a_3} + F_{a_1+a_2} \geq F_{a_3} + F_{a_2} + F_{a_1},$$

i.e. no solution if $a_3 > 2$.

(iii) $k = 4$. A simple check shows that there are no solutions if $a_4 = 1$. If $a_4 \geq 2$ and $a_3 = 1$ there are no solutions, and if $a_3 \geq 2$, we can argue similarly to the previous case and conclude that

$$F_{a_1+a_2+a_3+a_4} > F_{a_4} + F_{a_1+a_2+a_3} \geq F_{a_4} + F_{a_3} + F_{a_2} + F_{a_1},$$

i.e. no solution in this case.

(iv) $k = 5$. If $a_5 = 1$, we immediately see that $(1, 1, 1, 1, 1)$ is a solution. If $a_5 \geq 2$ and $a_4 = 1$, there are no solutions and if $a_4 \geq 2$ we get

$$F_{a_1+a_2+a_3+a_4+a_5} > F_{a_5} + F_{a_1+a_2+a_3+a_4} \geq F_{a_5} + F_{a_4} + F_{a_3} + F_{a_2} + F_{a_1}.$$

(v) $k > 5$. Clearly, $a_1 + a_2 + \ldots + a_k \geq k$. If $a_k = 1$, there are no solutions since $a_1 + a_2 + \ldots + a_k = k$ and $F_k > k$. If $a_k \geq 2$ and $a_{k-1} = 1$, then $a_1 + a_2 + \ldots + a_{k-1} = k - 1$ and

$$\begin{aligned} F_{a_1+a_2+\ldots+a_k} &= F_{a_1+a_2+\ldots+a_k-1} + F_{a_1+a_2+\ldots+a_k-2} \\ &> F_{a_k} + F_{a_1+a_2+\ldots+a_{k-1}} = F_{a_k} + F_{k-1} \\ &\geq F_{a_k} + k - 1 \\ &= F_{a_k} + F_{a_{k-1}} + \ldots + F_{a_1}. \end{aligned}$$

If $a_k \geq 2$ and $a_{k-1} \geq 2$, we have

$$\begin{aligned} F_{a_1+a_2+\ldots+a_k} &> F_{a_k} + F_{a_1+a_2+\ldots+a_{k-1}} \\ &> F_{a_k} + F_{a_{k-1}} + F_{a_1+a_2+\ldots+a_{k-2}} \\ &\geq F_{a_k} + F_{a_{k-1}} + F_{a_{k-2}} + F_{a_1+a_2+\ldots+a_{k-3}} \\ &\vdots \quad \vdots \\ &\geq F_{a_k} + F_{a_{k-1}} + \ldots + F_{a_2} + F_{a_1} - 1, \end{aligned}$$

which gives $F_{a_1+a_2+\ldots+a_k} > F_{a_k} + F_{a_{k-1}} + \ldots + F_{a_2} + F_{a_1}$, so there are no solutions in this case.

Also solved by Radouan Boukharfane, Polytechnique de Montreal, Canada; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

J255. Consider the points in a plane at a distance of at least 1 from each other. Prove that there exist two points located at a distance of at least $\dfrac{\sqrt{5}+1}{2}$ from each other.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Radouan Boukharfane, Polytechnique de Montreal, Canada

Denote A_1, A_2, A_3, A_4 and A_5 the five points and without losing the generality let's suppose that $\theta = \angle A_2A_1A_3$ is maximal $\mod \pi$. We will work in the reference $\mathbb{R} = \left(A_1, \vec{A_1A_2}, \vec{A_1A_3}\right)$. By construction, all the points A_i have the same sign. Suppose that one of the five points A_j has a negative composant in $\mathbb{R}$, so $\angle A_jA_1A_3 + \angle A_jA_1A_2 = 2\pi - \theta$, which implies that one of the two angles $\angle A_jA_1A_3$ or $\angle A_jA_1A_2$ is greater than $\pi - \frac{\theta}{2}$.

If $\theta > \frac{2\pi}{3}$, we denote r, s and t as the lenghts of the triangle $\triangle A_1A_2A_3$ where t does not contain A_1. We know that $2rs\cos\theta = r^2+s^2-t^2$, and $\theta > \frac{2\pi}{3} \implies |\cos\theta| \geq \frac{1}{2}$, so $t^2 = r^2+s^2+rs \geq 3$ and finally $t \geq \sqrt{3}$.

If $\theta \leq \frac{2\pi}{3}$, one of the angles $\angle A_jA_1A_3$ and $\angle A_jA_1A_2$ is greater that θ, which contradicts the construction. We infer from this that it suffices to consider the case where all the points belong to $A_1 + \mathbb{R}^+\vec{A_1A_2}\mathbb{R}^+\vec{A_1A_3}$.

Now suppose that all the points have a length less that ϕ which implies that $\theta \leq \frac{3\pi}{5}$, in fact $2\cos\theta = \frac{r^2+s^2-t^2}{rs} > 2 - \phi^2 = 1 - \phi = 2\cos\frac{2\pi}{5}$. We cut up $A_1 + \mathbb{R}^+\vec{A_1A_2}\mathbb{R}^+\vec{A_1A_3}$ in three regions limited by trisection angle of θ. Pigeonhole principle implies the exsitence of two points A_p and A_q belonging to the same region. We have proven the existence of an angle $\theta' = \angle A_pA_1A_q \leq \frac{\pi}{5}$. Let r', s' and t' the lengths of this triangle. We have now $t'^2 = r'^2 + s'^2 - 2r's'\cos\theta' < r'^2 + s'^2 - \phi r's'$, we will find a contradiction if $t'^2 < 1$. We consider the function $f(r') = 1 + \phi r's' - r'^2 - s'^2$, this function is decreasing for $r' > \frac{\phi s'}{2}$ and increasing for $r' < \frac{\phi s'}{2} < \phi$. Because $f(1) > 0$ and $r' \leq \phi$ it suffices to verify that $f(\phi) > 0$ which is equivalent to $(s'-1)(\phi^2 - s' - 1) = (s'-1)(\phi - s') > 0$ which is true. Therefore $t' < 1$ which gives us the contradiction that we look for.

Also solved by Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

J256. Evaluate

$$1^2 2! + 2^2 3! + \cdots + n^2(n+1)!.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Albert Stadler, Switzerland

Notice that

$$j^2(j+1)! = (j+2)(j-1)(j+1)! - (j+1)(j-2)j!$$

This implies that:

$$\sum_{j=1}^{n} j^2(j+1)! = (n+2)(n-1)(n+1)! - (1+1)(1-2)1! = (n-1)(n+2)! + 2$$

Also solved by Sayan Das, Kolkata, India; G.R.A.20 Problem Solving Group, Roma, Italy; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Zolbayar Shagdar, Orchlon International School, Ulaanbaatar, Mongolia; Corneliu Mănescu-Avram, Technological Transportation High School, Ploieşti, Romania; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Radouan Boukharfane, Polytechnique de Montreal, Canada; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Alessandro Ventullo, Milan, Italy; Arkady Alt, San Jose, California, USA; Prithwijit De, HBCSE, Mumbai, India; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

J257. Let BC be a fixed chord of the circle ω and let A vary on the major arc BC of ω.

a) Show that the mirror images of H over the A-angle bisector also run along a circle (possibly with zero radius).

b) Show that the projections of H on the A-angle bisector run along a circle.

Proposed by Michal Rolinek, Charles University, Czech Republic

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let O be the center of ω, and let H' be the symmetric of H with respect to the A-angle bisector. It is well known that the A-angle bisector is also the bisector of $\angle HAO$ (since $\angle BAH = 90° - B$ and $\angle AOC = 2BN$, triangle AOC being isosceles at O). It is also known (by angle chasing and simple trigonometry) that $AH = 2R\cos A$, R being the circumradius of ABC. It follows that $AH' = AH = 2R\cos A$, where H' is on ray AO. Therefore, $|OH'| = |AH' - AO| = R|2\cos A - 1|$, ie H' is on a circle with center O and radius $R|2\cos A - 1|$. Note that when $A = N$ is the midpoint of the major arc BC in ω, the altitude from A coincides with the A-angle bisector, or $H' = H$, ie H' runs along the circle with center O through the orthocenter K of triangle BNC. The circle has zero radius when $O = K$, ie when BNC is equilateral, or when BC is a chord of ω with length $\sqrt{3}R$.

Since $OK = OH'$ and $AH = AH'$ with $OK \parallel AH$ and $OH' \parallel AH'$, it follows that triangles OKH' and AHH' (isosceles respectively at O, A) are similar, hence homothetic. Therefore H' is the center of homothety, or H, K, H' are collinear on some line ℓ, hence the projection H'' of H onto the A-angle bisector (ie the midpoint of HH') is also on ℓ. Now, since $OM = R\cos A = \frac{AH}{2}$, we have

$$\frac{KH'}{H''H'} = 2\frac{KH'}{HH'} = 2\frac{OK}{AH} = \frac{OK}{OM},$$

or since K, H', H'' are collinear, triangles MKH'' and OKH' are similar, hence MKH'' is isosceles at M, or H'' runs along a circle with center M through K.

J258. Let x, y, z be positive real numbers such that $x \leq 1$, $y \leq 2$ and $x+y+z=6$.Prove that

$$(x+1)(y+1)(z+1) \geq 4xyz$$

Proposed by Marius Stanean, Zalau, Romania

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

From the given condition $x+y+z=6$, our inequality is equivalent to

$$3(x+y)(2+xy)+7 \geq 19xy+x^2+y^2.$$

On the other hand from the condition $x \leq 1, y \leq 2$, we have

$$(1-x)(2-y) \geq 0 \Leftrightarrow 2+xy \geq 2x+y,$$

$4 \geq 2xy$ and $3 \geq 3x^2$. Using the above inequalities and the AM-GM inequality, we have

$$\begin{aligned}
3(x+y)(2+xy)+7 &\geq 3(x+y)(2x+y)+7 \\
&\geq 3(x+y)(2x+y)+2xy+3x^2 \\
&= (x^2+y^2)+11xy+8\left(x^2+\frac{y^2}{4}\right) \\
&\geq (x^2+y^2)+11xy+8xy = 19xy+x^2+y^2.
\end{aligned}$$

Equality holds if and only if $x=1, y=2$, and $z=3$. The proof is completed.

Also solved by Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Radouan Boukharfane, Polytechnique de Montreal, Canada; Arkady Alt, San Jose, California, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Switzerland;Daniel Lasaosa, Universidad Pública de Navarra, Spain.

Senior problems

S253. Solve in positive real numbers the system of equations:

$$(2x)^{2013} + (2y)^{2013} + (2z)^{2013} = 3$$
$$xy + yz + zx + 2xyz = 1.$$

Proposed by Roberto Bosch Cabrera, Havana, Cuba

Solution by Christopher Wiriawan, Jakarta, Indonesia

From the condition $xy + yz + zx + 2xyz = 1$, we write$x = \frac{a}{b+c}$, $y = \frac{b}{c+a}$, $z = \frac{c}{a+b}$ for some positive reals a, b, c.
Hence, $(2x)^{2013} + (2y)^{2013} + (2z)^{2013} = 2^{2013} \sum_{cyc} \frac{a^{2013}}{(b+c)^{2013}}$. By Holder's Inequality, we have,

$$\sum_{cyc} \frac{a^{2013}}{(b+c)^{2013}} \cdot (2a + 2b + 2c)^{2012} \cdot (2ab + 2bc + 2ca) \geq (a + b + c)^{2014}.$$

This implies that,

$$\sum_{cyc} \frac{a^{2013}}{(b+c)^{2013}} \geq \frac{(a+b+c)^2}{2^{2013}(ab+bc+ca)} \geq \frac{3(ab+bc+ca)}{2^{2013}(ab+bc+ca)} = \frac{3}{2^{2013}}.$$

Therefore,

$$(2x)^{2013} + (2y)^{2013} + (2z)^{2013} = 2^{2013} \sum_{cyc} \frac{a^{2013}}{(b+c)^{2013}} \geq 3.$$

From the problem's condition, we have the equality case of the above inequality, hence we must have $a = b = c$, which means $x = y = z = \frac{1}{2}$, and so this is the only solution for our system.

Also solved by Albert Stadler, Switzerland; Corneliu Mănescu-Avram, Technological Transportation High School, Ploieşti, Romania; Alessandro Ventullo, Milan, Italy; Arkady Alt, San Jose, California, USA; Sayan Das, Kolkata, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Radouan Boukharfane, Polytechnique de Montreal, Canada; Prithwijit De, HBCSE, Mumbai, India; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania;Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S254. Let G be a graph with n vertices, where $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. Half of the edges of G are colored in red and half of the edges are colored in blue. Denote by $T_{r,r,b}$ the number of triangles with exactly two red edges and by $T_{b,b,r}$ the number of triangles with exactly two blue edges. Prove that if the number of red monochromatic triangles is equal to the number of blue monochromatic triangles, then $T_{r,r,b} = T_{b,b,r}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note that the graph must be complete, otherwise the result is not necessarily true. Consider a quadrilateral $ABCD$ where both diagonals are red, both sides AB and BC are blue, and the other two edges are not present. This graph clearly satisfies all other conditions of the problem ($n = 4$, 2 blue edges and 2 red edges, no monochromatic triangles of either color), but there is clearly only one triangle ABC, for $1 = T_{b,b,r} \neq T_{r,r,b} = 0$.

Assuming therefore that the graph is complete, note that there are $\binom{n}{3}$ triangles, and each edge appears in $n-2$ triangles (as many as points can be added to the two ends of the given edge). Add, for all $\binom{n}{3}$ triangles, all red edges. The sum will be $3u + 2T_{r,r,b} + T_{b,b,r}$, where each red edge is counted $n-2$ times (as many as triangles where it appears), and u is the number of monochromatic red triangles. There are therefore exactly

$$\frac{3u + 2T_{r,r,b} + T_{b,b,r}}{n-2}$$

red edges in the graph. Similarly, there are exactly

$$\frac{3u + 2T_{b,b,r} + T_{r,r,b}}{n-2}$$

blue edges in the graph, since the number of monochromatic blue triangles equals the number u of monochromatic red triangles. But since there are as many red edges as blue edges, we conclude that $T_{b,b,r} = T_{r,r,b}$, qed.

S255. Solve in real numbers the equation

$$2^x + 2^{-x} + 3^x + 3^{-x} + \left(\frac{2}{3}\right)^x + \left(\frac{2}{3}\right)^{-x} = 9x^4 - 7x^2 + 6.$$

Proposed by Mihaly Bencze, Brasov, Romania

Solution by Alessandro Ventullo, Milan, Italy

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Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S256. Two congruent circles ω_1 and ω_2 are both tangent to line ℓ from the same side. Common internal tangent k of ω_1 and ω_2 intersects ℓ at B. Points D and E are chosen on k (and on the same side of ℓ as the circles). Draw a tangent from D to ω_1 and from E to ω_2 (other than k in both cases) and denote their intersection with ℓ as A and C, respectively. Show that if A, B, and C lie on ℓ in this order and satisfy $\frac{BE}{BD} = \frac{BA}{BC}$ then the line joining the B-excenters of triangles ABD and ΔEBC is perpendicular to k.

Proposed by Michal Rolinek, Charles University, Czech Republic

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let T, U be the respective points where ω_1, ω_2 touch ℓ, and let F, G be the respective points where ω_1, ω_2 touch k. Denote by F', G' the points where lines AD, CE touch respectively ω_1, ω_2. Since we may invert A, C and all their associate points, assume wlog that B is closer to T than to U, or $\angle ABD = 2\beta < 90°$, and $\angle CBE = 180° - 2\beta > 90°$. Denote further $2\alpha = \angle DAB = 2\angle O_1AT$ where O_1 is the center of ω_1, and $2\gamma = \angle ECB = 2\angle O_2CU$ where O_2 is the center of ω_2. After some angle chasing, we find that $\angle TO_1B = \beta$ and $\angle FO_1D = 90° - \alpha - \beta$, or since triangles TO_1B and FO_1D are rectangle at T, F respectively, we find $BT = r\tan\beta$, $r = AT\tan\alpha$, and after some algebra, we find

$$DF = \frac{r}{\tan(\alpha+\beta)} = \frac{BU \cdot AB}{TU + AB},$$

where we have used that $r = BU\tan\beta$ or $BU \cdot BT = r^2$, and $TU = BT + BU$. Now, since $BT = BF'$ and $DF = DF'$, with $BD = BF' + DF'$, we find

$$BD = BT + \frac{BU \cdot AB}{TU + AB} = AT \cdot \frac{TU}{TU + AB}.$$

Similarly, we can find that $\angle EO_2G = \beta - \gamma$, with $r = CU\tan\gamma$, for

$$EG = r\tan(\beta - \gamma) = \frac{BT \cdot BC}{TU + BC}, \qquad BE = BU + EG = CU \cdot \frac{TU}{TU + BC}.$$

We conclude that

$$\frac{BE}{BD} = \frac{CU}{AT} \cdot \frac{TU + AB}{TU + BC},$$

where TU is the distance between the tangency points of both circles with ℓ, which clearly coincides with the distance between the centers of ω_1 and ω_2.

Since both B-excircles mentioned in the problem statement touch line k, the line joining their centers will be perpendicular to k iff their tangency points with this line coincide, ie iff the semiperimeter of triangles ABD and EBC are equal. But the semiperimeter of triangle ABD is also the distance AT from A to the point where the A-excircle ω_1 touches AB, and the semiperimeter of triangle EBC is also the distance CU from C to the point where the C-excircle ω_2 touches BC. We conclude that the line joining the B-excenters is perpendicular to k iff $AT = CU$, ie, iff

$$\frac{BE}{BD} = \frac{TU + AB}{TU + BC}.$$

The conclusion follows.

Note: This condition is different from the one given in the problem statement, and they are equivalent iff $AB = BC$, which together with $AT = CU$ yields $BT = BU$, absurd since B cannot be the midpoint of TU unless ω_1 and ω_2 are externally tangent, in which case it can be proved that $\frac{BE}{BD} = \frac{BA}{BC}$ is equivalent to D, E being coincident, and in turn equivalent to the B-excircles being symmetric with respect to k, hence equivalent to the line joining their centers being perpendicular to k.

S257. Find all complex numbers z for which

$$(z - z^2)(1 - z + z^2)^2 = \frac{1}{7}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

From the well-known identity

$$(x + y)^7 = x^7 + y^7 + 7xy(x + y)(x^2 + xy + y^2)^2,$$

we deduce

$$(1 - z)^7 = 1 - z^7 - 7z(1 - z)(1 - z + z^2)^2.$$

Hence our equation is equivalent to $(1 - z)^7 = -z^7$, that is $\left(-\frac{1}{z} + 1\right)^7 = 1$

It follows that

$$-\frac{1}{z_k} + 1 = \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7},$$

for $k = 0, 1, \cdots, 6$.

This reduces to:

$\frac{1}{z_k} = 1 - \cos\frac{2k\pi}{7} - i\sin\frac{2k\pi}{7} = 2\sin^2\frac{k\pi}{7} - 2i\sin\frac{k\pi}{7}\cos\frac{k\pi}{7}$, which is equivalent to

$$z_k = \frac{1}{-2i\sin\frac{k\pi}{7}\left(\cos\frac{k\pi}{7} - i\sin\frac{k\pi}{7}\right)} = \frac{\cos\frac{k\pi}{7} + i\sin\frac{k\pi}{7}}{-2i\sin\frac{k\pi}{7}}$$

$$= \frac{1}{2}\left(-1 + i\cot\frac{k\pi}{7}\right),$$

for $k = 0, 1, \cdots, 6$.

Also solved by Radouan Boukharfane, Polytechnique de Montreal, Canada, Daniel Lasaosa, Universidad Pública de Navarra, Spain; Albert Stadler, Switzerland; Moubinool Omarjee Lycée Henri IV , Paris, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S258. Which rational numbers can be written as the sum of the inverses of finitely many pairwise distinct triangular numbers?

Proposed by Gabriel Dospinescu, Lyon, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Since the n-th triangular number is $t_n = \binom{n+1}{2} = \frac{(n+1)n}{2}$, the sum of the inverses of all triangular numbers is

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 2 - 2\lim_{n\to\infty}\frac{1}{n+1} = 2,$$

or the sum of the inverses of finitely many pairwise distinct triangular numbers is necessarily a rational number less than 2.

Let $\frac{u}{v}$ be any positive rational number less than 2. We proceed as follows:

i) We find the least positive integer m_0 such that $\frac{u}{v} \geq \frac{2}{m_0(m_0+1)}$, and if $\frac{u}{v} \neq \frac{2}{m_0(m_0+1)}$, we then find the greatest integer k such that $\frac{u}{v} \geq \frac{2k}{m_0(m_0+k)}$; if $\frac{u}{v} = \frac{2}{m_0(m_0+1)}$ or $\frac{u}{v} = \frac{2k}{m_0(m_0+k)}$, we are done. Otherwise, we define $\frac{u_1}{v_1} = \frac{u}{v} - \frac{2k}{m_0(m_0+k)}$.

ii) While $\frac{u_i}{v_i}$ is nonzero, we find the least positive integer m_i such that $\frac{u_i}{v_i} \geq \frac{1}{m_i}$, then define $\frac{u_{i+1}}{v_{i+1}} = \frac{u_i}{v_i} - \frac{1}{m_i}$, until $\frac{u_i}{v_i} = \frac{1}{m_i}$ for some i and some positive integer m_i.

We will show that 1) the process is feasible and ends after a finite number of steps, ie $\frac{u}{v} = \frac{2k}{m_0(m_0+k)} + \sum_{i=1}^{I}\frac{1}{m_i}$, where I is a finite integer; 2) $m_1 > m_0 + k$, and for all $i \geq 1$, $m_{i+1} > 2m_i$; and 3) each one of the terms $\frac{2k}{m_0(m_0+k)}$ and $\frac{1}{m_i}$ can be written as the sum of a finite number of distinct consecutive triangular numbers, so that the sets of triangular numbers used to write each one of these terms are finite and disjoint.

1) Clearly $(0,1)$ is the disjoint union of $\left[\frac{2}{k(k+1)}, \frac{2}{k(k-1)}\right)$ for $k = 2, 3, \dots$, or any rational in $(0,1)$ will be inside one of these intervals for some value of k, and $m_0 = k$. If the rational is in $[1,2)$, then $m_0 = 1$. Now, $\frac{2k}{m_0(m_0+k)} = \frac{2}{m_0} - \frac{2}{m_0+k}$ clearly increases with k, having a limit of $\frac{2}{m_0}$, or if $\frac{u}{v} > \frac{2}{m_0(m_0+1)}$ is in $(0,1)$, then $\frac{u}{v} < \frac{2}{m_0(m_0-1)} \leq \frac{2}{m_0}$, and for sufficiently large k, $\frac{2k}{m_0(m_0+k)}$ will be larger than $\frac{u}{v}$, hence k as defined exists. If $2 > \frac{u}{v} \geq 1$, then for $m_0 = 1$, the limit of $\frac{2k}{m_0(m_0+k)}$ when k grows is 2, or again eventually $\frac{2k}{m_0(m_0+k)}$ will become larger than $\frac{u}{v}$, and k as defined exists. Assume now that we we are not finished in step i). Then, $\frac{2k}{m_0(m_0+k)} < \frac{u}{v} < \frac{2(k+1)}{m_0(m_0+k+1)}$, and $0 < \frac{u_1}{v_1} < \frac{2(k+1)}{k+2} - \frac{2k}{k+1} = \frac{2}{m_0(m_0+k)(m_0+k+1)}$. Note that for any integer m_1 such that $\frac{u_1}{v_1} > \frac{1}{m_1}$, we have $m_1 > \frac{m_0(m_0+k)(m_0+k+1)}{2} \geq \frac{3}{2}(m_0+k) > m_0 + k$, where we have used that $m_0, k \geq 1$. Note that in any case $m_1 > 3$. Now, as long as the process is not finished, in step ii) note that m_i is defined in such a way that $\frac{1}{m_i} < \frac{u_i}{v_i} < \frac{1}{m_i - 1}$, or $\frac{u_{i+1}}{v_{i+1}} = \frac{m_i u_i - v_i}{m_i v_i}$. Now, by definition of m_i we have $0 < m_i u_i - v_i < u_i$, or the numerator, which is an integer, clearly decreases each step, or after a finite number of steps, the numerator will be 0 or 1, and we will be finished.

2) We have already shown that $m_1 > m_0 + k$. Now, since $\frac{1}{m_i} < \frac{u_i}{v_i} < \frac{1}{m_i - 1}$, we have $0 \leq \frac{u_{i+1}}{v_{i+1}} < \frac{1}{m_i(m_i-1)}$, or if $\frac{u_{i+1}}{v_{i+1}} > \frac{1}{m_{i+1}}$, then $m_{i+1} > m_i(m_i - 1) \geq 2m_i$, since $m_1 \geq 3$ and the sequence of m_i's is clearly increasing.

3) Note finally that

$$\frac{1}{t_{m_0}} + \frac{1}{t_{m_0+1}} + \dots + \frac{1}{t_{m_0+k-1}} = \sum_{n=m_0}^{m_0+k-1} \frac{2}{n(n+1)} =$$

$$= 2\sum_{n=m_0}^{m_0+k-1}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{2}{m_0} - \frac{2}{m_0+k} = \frac{2k}{m_0(m_0+k)},$$

while

$$\frac{1}{t_{m_i}} + \frac{1}{t_{m_i+1}} + \dots + \frac{1}{t_{2m_i-1}} = 2\sum_{n=m_i}^{2m_i-1}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{2}{m_i} - \frac{2}{2m_i} = \frac{1}{m_i}.$$

The first sum ends with t_{m_0+k-1}, the second one begins with t_{m_1} for $m_1 > m_0 + k$, and for every $i \geq 1$, the sum for $\frac{1}{m_i}$ ends with t_{2m_i-1}, and the next one begins with $t_{m_{i+1}}$ for $m_{i+1} > 2m_i$, or the sums do not share terms.

Since this construction can be applied to any rational in $(0, 2)$, the sum of inverses of finitely many pairwise distinct triangular numbers cannot reach 2, and it can be zero iff there are zero terms in the sum, we conclude that exactly all rationals in $[0, 2)$ can be written as the sum of the inverses of finitely many (zero for zero sum) pairwise distinct triangular numbers.

Undergraduate problems

U253. Evaluate

$$\sum_{n>1} \frac{3n^2+1}{(n^3-n)^3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

Note that $2(3n^2+1) = (n+1)^3 - (n-1)^3$, so the sum becomes

$$\sum_{n\geq 2} \frac{1}{2}(n(n-1))^3 - \frac{1}{2}((n+1)n)^3,$$

which is equal to $\dfrac{1}{16}$.

Second solution by Angel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain

By partial fraction decomposition

$$\begin{aligned} \frac{3n^2+1}{(n^3-n)^3} &= \frac{1/2}{(n-1)^3} - \frac{1}{n^3} + \frac{1/2}{(n+1)^3} \\ &\quad - \frac{3/2}{(n-1)^2} + \frac{-3/2}{(n+1)^2} \\ &\quad + \frac{3}{n-1} - \frac{6}{n} + \frac{3}{n+1}. \end{aligned}$$

That is, the given series may be written as the sum of three telescopic series and therefore

$$\begin{aligned} \sum_{n>1} \frac{3n^2+1}{(n^3-n)^3} &= \frac{1}{2} + \frac{1/2}{8} - \frac{1}{8} - \frac{3}{2} - \frac{3/2}{4} + 3 + \frac{3}{2} - \frac{6}{2} \\ &= \frac{1}{16}. \end{aligned}$$

Also solved by G. C. Greubel, Old Dominion University, Norfolk, VA, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee Lycée Henri IV , Paris, France; Problem Solving Group of Qafqaz University, Baku, Azerbaijan; Tsouvalas Konstantinos, University of Athens, Athens, Greece; Harun Immanuel, ITS Surabaya; Radouan Boukharfane, Polytechnique de Montreal, Canada; Albert Stadler, Switzerland; Alessandro Ventullo, Milan, Italy; Emiliano Torti, Università di Roma "Tor Vergata", Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, California, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy;Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong.

U254. Let A be an $n \times n$ matrix with entries $a_{i,j} \in \mathbb{R}$. Let ${\sigma_i}^2$ and ${\tau_j}^2$ be the variance of entries in row i and column j, respectively. Denote by ${\sigma_A}^2$ the variance of all entries in A and let $B = \frac{1}{n}\sum_{i=1}^{n} {\sigma_i}^2$ and $C = \frac{1}{n}\sum_{j=1}^{n} {\tau_j}^2$. Prove that

$$\max(B, C) \leq {\sigma_A}^2 \leq B + C.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let μ_i, ν_j be the respective averages of the entries in the i-th row and the j-th column, let m_A be the average of all entries in the matrix, and let ${\sigma_\mu}^2, {\sigma_\nu}^2$ be the respective variances of the μ_i's and the ν_j's. Clearly the average of the μ_i's, and the average of the ν_j's, both are equal to m_A. It is well known that the average of the squares of the elements of a set equals their average squared plus their variance. Denoting by Q_A the quadratic mean of the entries in A, by Q_i, R_j the quadratic means of the i-th row and j-th column respectively, it follows that ${Q_A}^2 = {m_A}^2 + {\sigma_A}^2$, ${Q_i}^2 = {\mu_i}^2 + {\sigma_i}^2$ and ${R_j}^2 = {\nu_j}^2 + {\tau_j}^2$, while

$$\frac{1}{n}\sum_{i=1}^{n} {\mu_i}^2 = {m_A}^2 + {\sigma_\mu}^2, \qquad \frac{1}{n}\sum_{j=1}^{n} {\nu_j}^2 = {m_A}^2 + {\sigma_\nu}^2.$$

It follows that

$$B = \frac{1}{n}\sum_{i=1}^{n} {\sigma_i}^2 = \frac{1}{n}\sum_{i=1}^{n} {Q_i}^2 - \frac{1}{n}\sum_{i=1}^{n} {\mu_i}^2 = {Q_A}^2 - {m_A}^2 - {\sigma_\mu}^2 = {\sigma_A}^2 - {\sigma_\mu}^2,$$

or $B \leq {\sigma_A}^2$, with equality iff all rows have equal average. Similarly, $C \leq {\sigma_A}^2$, with equality iff all columns have equal average.

Moreover,

$$B + C - {\sigma_A}^2 = {\sigma_A}^2 - {\sigma_\mu}^2 - {\sigma_\nu}^2 = {Q_A}^2 + {m_A}^2 - \frac{1}{n}\sum_{i=1}^{n} {\mu_i}^2 - \frac{1}{n}\sum_{i=1}^{n} {\nu_j}^2,$$

or multiplying both sides by n^4, we find that $B + C - {\sigma_A}^2$ has the same sign as

$$n^2 \sum_{i=1}^{n}\sum_{j=1}^{n} {a_{i,j}}^2 + \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\right)^2 - n\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{i,j}\right)^2 - n\sum_{j=1}^{n}\left(\sum_{n=1}^{n} a_{i,j}\right)^2.$$

Note now that the respective coefficients, in each one of the sums, are:

- for ${a_{i,j}}^2$, $n^2 + 1 - n - n = (n-1)^2$,
- for $a_{i,j}a_{i',j}$ with $i \neq i'$, $0 + 2 + 0 - 2n = -2(n-1)$,
- for $a_{i,j}a_{i,j'}$ with $j \neq j'$, $0 + 2 - 2n + 0 = -2(n-1)$,
- for $a_{i,j}a_{i',j'}$ with $i \neq i'$ and $j \neq j'$, $0 + 2 + 0 + 0 = 2$.

For any two pairs of indices i, j and i', j' with $i \neq i'$ and $j \neq j'$, note that

$$0 \leq \left(a_{i,j} + a_{i',j'} - a_{i,j'} - a_{i',j}\right)^2 = {a_{i,j}}^2 + {a_{i,j'}}^2 + {a_{i',j}}^2 + {a_{i',j'}}^2 + 2a_{i,j}a_{i',j'} + 2a_{i',j}a_{i,j'} -$$

$$-2a_{i,j}a_{i',j} - 2a_{i,j}a_{i,j'} - 2a_{i',j}a_{i',j'} - 2a_{i,j'}a_{i',j'}.$$

Fix i, j and let i', j' take any other of the $(n-1)^2$ possible pairs of values such that $i \neq i'$ and $j \neq j'$. Adding over all such pairs i', j', note that there are $(n-1)^2$ appearances of ${a_{i,j}}^2$ (one for each pair), one appearance of $2a_{i,j}a_{i',j'}$ (when i', j' are the appropriate ones), $n-1$ appearances of $-2a_{i,j}a_{i,j'}$ (when j' is the appropriate one and for $n-1$ possible values of i'), and likewise $n-1$ appearances of $-2a_{i,j}a_{i',j}$. Therefore,

$$n^4\left(B + C - {\sigma_A}^2\right)^2 = \sum \left(a_{i,j} + a_{i',j'} - a_{i,j'} - a_{i',j}\right)^2 \geq 0,$$

where the sum is carried out over all non-ordered pairs (i, i') and (j, j'), such that $i, j, i', j' \in \{1, 2, \ldots, n\}$ and $i \neq i'$, $j \neq j'$. The conclusion follows. Note that equality is reached in this case when $a_{i,j} - a_{i,j'} = a_{i',j} - a_{i',j'}$ and $a_{i,j} - a_{i',j} = a_{i,j'} - a_{i',j'}$, ie when the difference between any two rows is a row vector with all equal entries, and the difference between any two columns is a column with all equal entries.

Note: Such a matrix can be easily construced as follows: take any row n-vector $\vec{v} \equiv (v_1, v_2, \ldots, v_n)$, and take row vectors $\vec{v} + k_2, \vec{v} + k_3, \ldots, \vec{v} + k_n$ for arbitrary constants $k_2, k_3, \ldots, k_n$, where $\vec{v} + k_i \equiv (v_1 + k_i, v_2 + k_i, \ldots, v_n + k_i)$, and stack all such vectors to form an $n \times n$ matrix. Note that the difference between rows i, i' is $k_i - k_{i'}$, where $k_1 = 0$, and the difference between columns j, j' is $v_j - v_{j'}$. Denoting by m_v, m_k the respective averages of the v_j's and the k_i's and by ${\sigma_v}^2, {\sigma_k}^2$ their variances, we find that $\mu_i = m_v + k_i$, $\nu_j = m_k + v_j$, ${Q_i}^2 = (m_k + v_n)^2 + {\sigma_k}^2$ for ${\sigma_i}^2 = {\sigma_k}^2$ for all i, and similarly ${\tau_j}^2 = {\sigma_v}^2$ for all j. It follows that $B + C = {\sigma_i}^2 + {\sigma_j}^2$. Analogously, $m_A = m_v + m_k$, while ${Q_A}^2 = (m_v + m_k)^2 + {\sigma_v}^2 + {\sigma_k}^2$, for ${\sigma_A}^2 = {\sigma_v}^2 + {\sigma_k}^2 = B + C$. Or any matrix so constructed, as expected from the previous solution, satisfies $B + C = {\sigma_A}^2$, and any matrix that satisfies this equality, can in turn be constructed in this way.

U255. Let S_n be the group of permutations of $\{1, 2, \ldots, n\}$. If $d > 1$ is an integer, let H_d be the set of those $\sigma \in S_n$ for which there are $k \geq 1$ and $\sigma_1, \ldots, \sigma_k \in S_n$ with $\sigma = \sigma_1^d \cdots \sigma_k^d$. Find H_2 and H_3.

Proposed by Mihai Piticari and Sorin Radulescu, Romania

Solution by Alessandro Ventullo, Milan, Italy

If $n = 1, 2$, clearly $H_2 = H_3 = S_n$. Let $n \geq 3$. We first prove that H_d is a group for every integer $d > 1$. Indeed, if $\sigma, \tau \in H_d$, clearly $\sigma\tau \in H_d$. The associative property follows from the associativite property of the product of permutations. Moreover, $\mathrm{id} \in S_n$ and $\mathrm{id} = \mathrm{id}^d$, so $\mathrm{id} \in H_d$ for every $d > 1$. Finally if $\sigma_1, \ldots, \sigma_k \in H_d$ and $\sigma = \sigma_1^d \cdots \sigma_k^d$ for some $k \geq 1$, we have that $\sigma_1^{-1}, \ldots, \sigma_k^{-1} \in S_n$ and $\tau = (\sigma_k^{-1})^d \cdots (\sigma_1^{-1})^d$ is the inverse of σ. Now, let $d = 2$. It is clear that H_2 is a subgroup of A_n since every permutation in H_2 is a product of even permutations and so is an even permutation. For every $a_1, a_2, a_3 \in \{1, \ldots, n\}$, we have also that $(a_1, a_2, a_3) = (a_1, a_3, a_2)^2$ so every 3-cycle belongs to H_2. Since A_n is generated by its 3-cycles, it follows that A_n is a subgroup of H_2, from which $H_2 = A_n$. Now, let $d = 3$. Obviously, H_3 is a subgroup of S_n. Moreover, for every $a_1, a_2 \in \{1, \ldots, n\}$ we have $(a_1, a_2) = (a_1, a_2)^3$, so every 2-cycle belongs to H_3. Since S_n is generated by its 2-cycles, it follows that S_n is a subgroup of H_3, which gives $H_3 = S_n$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

U256. Let G be a finite group and H be a subgroup of G which has index p, for some prime p. Suppose that the order of H and $p-1$ are relatively prime. Prove that H is normal.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by the author

First, let us note that the Sylow p-subgroups of G are cyclic of order p. Indeed, by induction on $|G|$, we see that we can assume without loss of generality that there is no non-trivial normal subgroup N of G contained in H. This yields that the action of G on the p conjugates of H gives an embedding of G into S_p, of order $p!$. Thus $|H|$ divides $(p-1)!$, and so p does not divide $|H|$. Thus the Sylow p-subgroups of G are cyclic of order p.

To proceed, let's us recall the

Burnside Transfer Theorem. Let G be a finite group and P be a p-Sylow subgroup. If P is contained in the center of its normalizer, then G contains a normal subgroup N such that $NP = G$ and $N \cap P$ is trivial.

This is a result is actually a pretty deep result from group theory and it requires some extra care. We refer to *Representations and Characters of Finite Groups* by M. J. Collins for a rigorous treatment.

Now if any $h \in H$ normalized a Sylow p-subgroup P, then h would map into $Aut(P)$, which is the automorphism group of the cyclic group of order p (by the above), so it is actually the unit group; thus, this image should be trivial (by hypothesis). That is, any $h \in H$ normalizing P also centralizes P, or in other words $N_G(P) = C_G(P)$. This means, in particular, that P is contained in the center of its normalizer, so, by the Burnside Transfer theorem stated above, G has a normal subgroup M such that $MP = G$ and $M \cap P$ trivial. Hence, this subgroup has index p (given that P has order p). Moreover, any Sylow q-subgroup of H is then contained in M, and so H is contained in M. But these groups have the same index, so $H = M$, which implies that H is normal, as claimed.

U257. a) Let p and q be distinct primes and let G be a non-commutative group with pq elements. Prove that the center of G is trivial.

b) Let p, q, r be pairwise distinct primes and let G be a non-commutative group with pqr elements. Prove that the number of elements of the center of G is either 1 or a prime number.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Emiliano Torti, Università di Roma "Tor Vergata", Roma, Italy

Let $Z(G)$ be the center of G. It is known that $Z(G)$ is normal subgroup of G and the quotient group $G/Z(G)$ is isomorphic to the group of inner automorphisms of G. In our case, since G is non-commutative it follows that $Z(G)$ is a proper subgroup of G.

a) If $|G| = pq$ then $|Z(G)| \in \{1, p, q\}$. Assume by contradiction that $|Z(G)| = p$ (the case $|Z(G)| = q$ is similar), then $|G/Z(G)| = q$ and $G/Z(G)$ is cyclic because q is prime. This is in contradiction with the fact that G is non-commutative.

b) If $|G| = pqr$ then $|Z(G)| \in \{1, p, q, pq, pr, qr\}$. Assume by contradiction that $|Z(G)| = pq$ (the other cases $|Z(G)| = qr$ and $|Z(G)| = pr$ are similar), then $|G/Z(G)| = r$ and $G/Z(G)$ is cyclic because r is prime. This is in contradiction with the fact that G is non-commutative.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo, Milan, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada; Tsouvalas Konstantinos, University of Athens, Athens, Greece; Harun Immanuel, ITS Surabaya.

U258. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ having the intermediate value property and satisfying $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Proposed by Marius and Sorin Radulescu, Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $y = 0$, or for all $x \in \mathbb{R}$, $f(0) = f(x)f(0)$, hence if $f(x) \neq 1$ for some $x \in \mathbb{R}$, we have $f(0) = 0$. Similarly, let $y = 1$, or for all $x \in \mathbb{R}$, $f(x) = f(x)f(1)$, hence if $f(x) \neq 0$ for some $x \in \mathbb{R}$, we have $f(1) = 1$. Therefore, there are three possible types of solutions: constant functions $f(x) = 0$ and $f(x) = 1$, and functions f such that $f(0) = 0$ and $f(1) = 1$. Constant functions trivially satisfy the intermediate value property, or we need to study only the latter. Assume further that $f(x) = 0$ for some $x \neq 0$. Then, for any real z, taking $y = \frac{z}{x}$, we have $f(z) = f\left(\frac{z}{x}\right) f(x) = 0$, or for any nonconstant solution, we must have $f(x) = 0$ iff $x = 0$.

Let z be any positive real, or taking $x = y = \sqrt{z} \neq 0$, we have $f(z) = (f(x))^2 > 0$. Let now $x = y = -1$, or $(f(-1))^2 = f(1) = 1$, for either $f(-1) = -1$ or $f(1) = 1$. Note therefore that, for all $x \in \mathbb{R}^+$, in the first case $f(-x) = -f(x)$, while in the second case $f(-x) = f(x)$. Therefore, any solution satisfies either $f(x) > 0$ for all $x \neq 0$ with $f(-x) = f(x)$, or $f(x) > 0$ for all $x > 0$ and $f(x) < 0$ for all $x < 0$ with $f(-x) = -f(x)$. Let g be the restriction of f to the non-negative reals. Clearly, if f has the intermediate value property, so does g. Reciprocally, if g defined from non-negative reals into non-negative reals has the intermediate value property and $g(x) = 0$ iff $x = 0$, then $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = g(x)$ for non-negative x, and either $f(x) = -g(-x)$ for all negative x, or $f(x) = g(-x)$ for all negative x, clearly satisfies also the intermediate value property; it trivially does for all pairs of non-negative x, y, and by symmetry it clearly does for all pairs of non-positive x, y. If $x > 0 > y$ where $f(y) < 0$, then every value in $[0, f(x))$ is $f(z)$ for some $z \in [0, x)$, and every value in $(f(y), 0]$ is $f(z)$ for some $z \in (y, 0]$, whereas if $x > 0 > y$ with $f(y) > 0$, and $M = \max\{f(x), f(y)\}$, then all values in $(f(x), f(y)) \subset (0, M)$ are $f(z)$ for some $z \in (0, x)$ or some $z \in (y, 0)$. It therefore suffices to find all functions $g(x)$ defined in the non-negative reals such that $g(xy) = g(x)g(y)$, $g(0) = 0$, and g has the intermediate value property.

Let finally $h : \mathbb{R} \to \mathbb{R}$ be defined in such a way that $h(x) = \ln(g(e^x))$. Since $\ln(x), e^x$ are strictly increasing functions, $h(x)$ has the intermediate value property iff $g(x)$ has the intermediate value property. Moreover, if $g(xy) = g(x)g(y)$ for all non-negative reals x, y, then

$$h(x+y) = \ln\left(g\left(e^{x+y}\right)\right) = \ln\left(g\left(e^x\right) g\left(e^y\right)\right) = \ln\left(g\left(e^x\right)\right) \ln\left(g\left(e^y\right)\right) = h(x) + h(y),$$

and the reciprocal is similarly proved. Now, it is well known (or easily provable by induction) that for any solution of this equation, a real m exists such that, for all $x \in \mathbb{Q}$, we have $f(x) = mx$, where $m = f(1)$. Let now $r \in \mathbb{R} \backslash \mathbb{Q}$. By the principle of nested intervals, two sequences of rationals exist, $x_1 < x_2 < \cdots < r$ and $y_1 > y_2 > \cdots > r$, such that $r \in (x_n, y_n)$ for all n, and $y_n - x_n \to 0$ when $n \to \infty$. The intersection of all (x_n, y_n) is $\{r\}$ when $n \to \infty$. Now, since h has the intermediate value property, for each n, a real z exists in (x_n, y_n) such that $h(z) = u$ for all $u \in (mx_n, my_n)$, or since by the principle of nested intervals, the intersection of all $(mx_n, my_n) = mr$ is $\{mr\}$, we have that one element in $\{r\}$ must have by h an image with value mr, or $h(r) = mr$ for all real r, ie, all functions $h : \mathbb{R} \to \mathbb{R}$ with the intermediate value property and such that $h(x+y) = h(x) + h(y)$, are exactly all functions $h(x) = mx$ for some real m.

Retracing back our steps, we must have $g(x) = x^m$, and since $g(0) = 0$ and g has the intermediate value property, it must be $m > 0$. Therefore, all solutions to the proposed problem are $f(x) = 0$ for all x, $f(x) = 1$ for all x, and for any positive real m, $f(x) = x^m$ for all x.

O253. Find the least multiple n of 2013 for which the system of equations

$$(x^2+y^2)(y^2+z^2)(z^2+x^2) = x^6+y^6+z^6+4n^2,\ xyz = n$$

has a solution.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Albert Stadler, Switzerland

We have

$$0 = x^6+y^6+z^6+4(xyz)^2-(x^2+y^2)(y^2+z^2)(z^2+x^2) = (x^2-y^2-z^2)(x^2+y^2-z^2)(x^2-y^2+z^2)$$

So we deduce that either $x^2-y^2-z^2=0$ or $x^2+y^2-z^2=0$ or $x^2-y^2+z^2=0$.

This means that (x,y,z) is a Pythagorean triple and is therefore of the form $(u^2-v^2, 2uv, u^2+v^2)$ for certain integers u and v.

So $xyz = n = k\cdot 2013 = k\cdot 3\cdot 11\cdot 61 = 2uv(u-v)(u+v)(u^2+v^2)$.

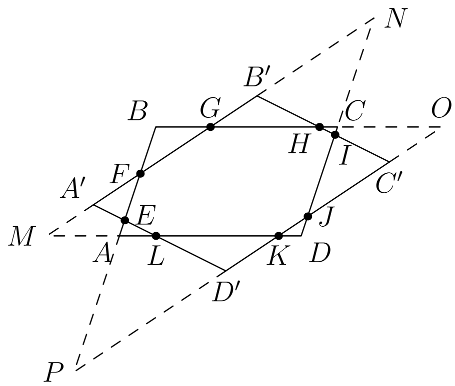
One easily verifies that the least multiple of 2013 for which above equation holds is $n = 20\cdot 2013$. We then have $u=6$, $v=5$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O254. Consider two parallelograms that intersect exactly in eight points. Prove that the common area of these parallelograms is greater than or equal to half of the area of one of them.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Li Zhou, Polk State College, Florida, USA



In the figure above, $ABCD$ and $A'B'C'D'$ are the two parallelograms. Assume that $[ABCD] \le [A'B'C'D']$, where $[\cdot]$ denotes area.

First, locate A'' and B'' on MN so that $A''A \parallel A'D' \parallel B''C$. Then there exists $t \in (0,1)$ such that $A'' = tM + (1-t)F$ and $B'' = tG + (1-t)N$. Hence

$$A'B' < A''B'' \le tMG + (1-t)FN \le \max\{MG, FN\}.$$

Therefore, $[A'B'C'D'] < \max\{[MGOK], [FNJP]\}$.

Next, for the purpose of contradiction, suppose that both $FB + DJ > AB$ and $BG + KD > BC$. Then $[FBG] > [JCO]$ since $FB > JC$, and $[FBG] > [PAK]$ since $BG > AK$. Likewise, $[KJD] > [MFA]$ and $[KJD] > [GNC]$. Hence, $[ABCD] > \max\{[MGOK], [FNJP]\} > [A'B'C'D']$, a contradiction. Therefore, $FB + DJ \le AB$ or $BG + KD \le BC$. Hence, the reflections of $\triangle FBG$ across FG and $\triangle KJD$ across KJ cannot overlap. Similarly, the reflections of $\triangle AEL$ across EL and $\triangle CIH$ across IH cannot overlap. Therefore, $[EFGHIJKL] > \frac{1}{2}[ABCD]$.

O255. A set of positive integers is called *good* if $ab+1$ is a square for every pair of distinct elements a and b of the set. A *good* set S is called *maximal* if $S \cup \{n\}$ is not *good* for every positive integer n.

a) Show that there are not *maximal* sets with cardinality 1, 2, 3.

b) Show that we can find infinitely many *maximal* sets with cardinality 4.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

No solutions have been yet received for this problem.

O256. Let $A_1..A_n$ be a regular polygon and let M be a point inside it. Prove that

$$\sin\angle A_1MA_2+\sin\angle A_2MA_3+\cdots+\sin\angle A_nMA_1>\sin\frac{2\pi}{n}+(n-2)\sin\frac{\pi}{n}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by the author

We start with $\angle A_iMA_{i+1}=\alpha_i>\frac{\cup A_iA_{i+1}}{2}=\frac{\pi}{n}$, $i=1,2,\cdots,n$, $A_{n+1}\equiv A_1$ and $\alpha_1+\alpha_2+\cdots+\alpha_n=2\pi$.

We will prove that if $\frac{\pi}{n}<\alpha_i<\pi$, $i=1,2,\cdots,n$ and $\alpha_1+\alpha_2+\cdots+\alpha_n=2\pi$, then

$$\sin\alpha_1+\sin\alpha_2+\cdots+\sin\alpha_n>\sin\frac{2\pi}{n}+(n-2)\sin\frac{\pi}{n}.$$

It's sufficient to notice that if $\alpha>\frac{\pi}{n}$, $\beta>\frac{\pi}{n}$ and $\alpha+\beta<2\pi$, then

$$\sin\alpha+\sin\beta>\sin\frac{\pi}{n}+\sin(\alpha+\beta-\frac{\pi}{n})$$

follows.

Let $\alpha_n=\max(\alpha_1,\alpha_2,\cdots,\alpha_n)$. We have

$$\sin\alpha_1+\sin\alpha_2+\cdots+\sin\alpha_{n-1}>(n-2)\sin\frac{\pi}{n}+\sin\left(\alpha_1+\alpha_2+\cdots+\alpha_{n-1}-(n-2)\frac{\pi}{n}\right)$$

Consequently, we notice that $\sin\left(\alpha_1+\alpha_2+\cdots+\alpha_{n-1}-(n-2)\frac{\pi}{n}\right)+\sin\alpha_n>\sin\frac{2\pi}{n}+\sin\pi=\sin\frac{2\pi}{n}$ and the conclusion follows.

O257. Let p be an odd prime. Put $N_k := \sum_{j=1}^{\frac{p-1}{2}} \tan^{2k}\left(\frac{j\pi}{p}\right)$. Prove the following statements:

a) N_k is an integer.

b) N_k is divisible by $p^{1+\left[2\frac{k-1}{p-1}\right]}$

c) $N_{\frac{p-1}{2}k}$ is not divisible by p^{k+1}.

Proposed by Albert Stadler, Switzerland

Solution by the author

(a) We have for $1 \le j \le (p-1)/2$,

$$\left(\frac{-1+\tan\left(\frac{\pi j}{p}\right)}{-1-\tan\left(\frac{\pi j}{p}\right)}\right)^p = \left(\frac{\cos\left(\frac{\pi j}{p}\right)+i\sin\left(\frac{\pi j}{p}\right)}{\cos\left(\frac{\pi j}{p}\right)-i\sin\left(\frac{\pi j}{p}\right)}\right)^p = \left(\frac{e^{\frac{i\pi j}{p}}}{e^{-\frac{i\pi j}{p}}}\right)^p = 1.$$

So $0 = \left(-i+\tan\left(\frac{\pi j}{p}\right)\right)^p - \left(-i-\tan\left(\frac{\pi j}{p}\right)\right)^p = \left(i+\tan\left(\frac{\pi j}{p}\right)\right)^p + \left(-i+\tan\left(\frac{\pi j}{p}\right)\right)^p$

$= \sum_{k=0}^{p}\binom{p}{k}\tan^{p-k}\left(\frac{\pi j}{p}\right)(i^k+(-i)^k) = 2\tan\left(\frac{\pi j}{p}\right)\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{p}{2k}\tan^{p-2k-1}\left(\frac{\pi j}{p}\right)$.

Put $x := \tan^2\left(\frac{\pi j}{p}\right)$, $1 \le j \le (p-1)/2$.

We conclude that x_j, $1 \le j \le (p-1)/2$, are the roots of $\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{p}{2k}x^{\frac{p-1}{2}-k} = 0$.

So $\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{p}{2k}x^{\frac{p-1}{2}-k} = \prod_{j=1}^{\frac{p-1}{2}}(x-x_j) = \sum_{k=0}^{\frac{p-1}{2}}(-1)^k e_k(x_1,x_2,\cdots,x_{\frac{p-1}{2}})x^{\frac{p-1}{2}-k}$, **(1)**

where for $k \ge 0$, $e_k = e_k(x_1,x_2,\cdots,x_{\frac{p-1}{2}})$ denotes the k-th elementary symmetric polynomial given by $e_k(x_1,x_2,\cdots,x_{\frac{p-1}{2}}) = \sum_{1\le j_1<\ldots<j_k\le\frac{p-1}{2}} x_{j_1},x_{j_2},\cdots,x_{j_k}$.

In particular, $e_0(x_1,x_2,\cdots,x_{\frac{p-1}{2}}) = 1$,

$e_1(x_1,x_2,\cdots,x_{\frac{p-1}{2}}) = x_1+x_2+..+x_{\frac{p-1}{2}}$,

$e_2(x_1,x_2,\cdots,x_{\frac{p-1}{2}}) = \sum_{1\le j<k\le\frac{p-1}{2}} x_jx_k$,

$\cdots$

$e_{\frac{p-1}{2}}(x_1,x_2,\cdots,x_{\frac{p-1}{2}}) = x_1\cdot x_2\cdots x_{\frac{p-1}{2}}$,

$e_k(x_1,x_2,\cdots,x_{\frac{p-1}{2}}) = 0$, for $k > (p-1)/2$.

We compare coefficients in **(1)** and get $e_j(x_1,x_2,\cdots,x_{\frac{p-1}{2}}) = \binom{p}{2j}$, $0 \le j \le (p-1)/2$. **(2)**

Obviously $N_k = \sum_{j=1}^{\frac{p-1}{2}}\tan^{2k}\left(\frac{j\pi}{p}\right) = \sum_{j=1}^{\frac{p-1}{2}}{x_j}^k$ is a k-th power sum.

Note that $ke_k = \sum_{j=1}^{k}(-1)^{j-1}e_{k-j}N_j$ is valid for all $k \ge 1$, thus it follows that $N_k = (-1)^{k-1}ke_k - \sum_{j=1}^{k-1}(-1)^je_jN_{k-j} = (-1)^{k-1}k\binom{p}{2k} - \sum_{j=1}^{k-1}(-1)^j\binom{p}{2j}N_{k-j}$, **(3)** In particular, by **(2)** and**(3)**,

$$N_1=\sum_{j=1}^{\frac{p-1}{2}}\tan^2\left(\frac{j\pi}{p}\right)=e_1=\binom{p}{2}=\frac{p(p-1)}{2},$$

$$N_2=\sum_{j=1}^{\frac{p-1}{2}}\tan^4\left(\frac{j\pi}{p}\right)=-2e_2+e_1N_1=-2\binom{p}{4}+\binom{p}{2}^2=\frac{p(p-1)(p^2+p-3)}{6},$$

$$N_3=\sum_{j=1}^{\frac{p-1}{2}}\tan^6\left(\frac{j\pi}{p}\right)=3e_3+e_1N_2-e_2N_1=3\binom{p}{6}+\binom{p}{2}\frac{p(p-1)(p^2+p-3)}{6}-\binom{p}{4}\frac{p(p-1)}{2}=$$

$$=\frac{p(p-1)(2p^4+2p^3-8p^2-8p+15)}{30}.$$

From the formula $N_1=\binom{p}{2}$ and the recurrence relation **(3)** we see that all the numbers N_k are integers.

(b) We claim that $p^{1+\left[2\frac{k-1}{p-1}\right]}$ divides N_k. **(4)**

We proceed by induction. By **(3)** the statement **(4)** is true or $k=1,2,\cdots,(p-1)/2$, since p divides $\binom{p}{2j}$ for $1\le j\le (p-1)/2$. Suppose that $n\ge (p-1)/2$ and that the statement **(4)** holds true for all $k\le n$. If $k=n+1$ then $k\ge (p+1)/2$ and by **(3)**,

$$N_k=-\sum_{j=1}^{\frac{p-1}{2}}(-1)^j\binom{p}{2j}N_{k-j}.$$

By induction hypothesis this number is divisiple by $p^{1+1+\left[2\frac{k-\frac{p-1}{2}-1}{p-1}\right]}=p^{1+\left[2\frac{k-1}{p}-1\right]}$.
So the statement holds true for $k=n+1$, as claimed.

(c) We claim that p^{k+1} does not divide $N_{\frac{p-1}{2}k}$, **(5)**

For the proof we proceed by induction as before. By **(2)**,

$$N_{\frac{p-1}{2}}=(-1)^{\frac{p-1}{2}-1}\frac{p-1}{2}\binom{p}{p-1}-\sum_{j=1}^{\frac{p-1}{2}-1}(-1)^j\binom{p}{2j}N_{\frac{p-1}{2}-j}\equiv(-1)^{\frac{p+1}{2}}\binom{p}{2}(\text{mod }p^2)$$

So the statement **(5)** holds true for $k=1$. Suppose the statement holds true for $k=n$. Then we have for $k=n+1\ge 2$:

$$N_{\frac{p-1}{2}k}=\sum_{j=1}^{\frac{p-1}{2}}(-1)^{j-1}\binom{p}{2j}N_{\frac{p-1}{2}k-j}=(-1)^{\frac{p+1}{2}}pN_{\frac{p-1}{2}(k-1)}+\sum_{j=1}^{\frac{p-1}{2}-1}(-1)^{j-1}\binom{p}{2j}N_{\frac{p-1}{2}k-j}\equiv$$

$\equiv(-1)^{\frac{p+1}{2}}pN_{\frac{p-1}{2}(k-1)}(\text{mod }p^{k+1})$. By induction hypothesis $N_{\frac{p-1}{2}(k-1)}$ is not divisible by p^k. Therefore $N_{\frac{p-1}{2}k}$ is not divisible by p^{k+1} and we conclude the proof.

O258. Let $\sigma(n)$ be the sum of all positive divisors of n. Prove that for all $n > 1$,

$$\sum_{k=0}^{n-1}(-1)^k(2k+1)\sigma\left(\frac{n^2+n}{2}-\frac{k^2+k}{2}\right)=(-1)^{n-1}\frac{n(n+1)(2n+1)}{6}$$

Proposed by Gabriel Dospinesu, Lyon, France

Solution by Albert Stadler, Switzerland

By a Theorem of Jacobi, (see for example K. Chandrasekharan, *Elliptic Functions*, Grundlehren der mathematischen Wissenschaften, Vol. 281, Springer Verlag, Chapter VIII, p. 125, Corollary 4.),

$$\prod_{n=1}^{\infty}(1-x^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{1}{2}n(n+1)},\ |x|<1$$

Furthermore,

$$\sum_{n=1}^{\infty}\sigma(n)x^n=\sum_{n=1}^{\infty}\sum_{m|n}mx^n=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}mx^{mn}=\sum_{m=1}^{\infty}m\frac{x^m}{1-x^m}=-x\frac{d}{dx}\sum_{m=1}^{\infty}\log(1-x^m)=-x\frac{d}{dx}\log\prod_{m=1}^{\infty}(1-x^m)=$$

$$=-x\frac{\frac{d}{dx}\prod_{m=1}^{\infty}(1-x^m)}{\prod_{m=1}^{\infty}(1-x^m)}.$$

So,

$$\sum_{n=1}^{\infty}\sigma(n)x^n\cdot\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{1}{2}n(n+1)}=-x\prod_{m=1}^{\infty}(1-x^m)^2\frac{d}{dx}\prod_{m=1}^{\infty}(1-x^m)=$$

$$=-\frac{x}{3}\frac{d}{dx}\prod_{m=1}^{\infty}(1-x^m)^3=-\frac{1}{6}\sum_{n=0}^{\infty}(-1)^n n(n+1)(2n+1)x^{\frac{1}{2}n(n+1)}.$$

We compare the coefficients of $x^{\frac{1}{2}n(n+1)}$ on the both sides of the equation and the claimed statement follows.