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FORUM GEOM

Friendship Among Triangle Centers

Floor van Lamoen

Abstract. If we erect on the sides of a scalene triangle three squares, then at the vertices of the triangle we find new triangles, the *flanks*. We study pairs of triangle centers X and Y such that the triangle of Xs in the three flanks is perspective with ABC at Y, and vice versa. These centers X and Y we call *friends*. Some examples of friendship among triangle centers are given.

1. Flanks

Given a triangle ABC with side lengths $BC = a$, $CA = b$, and $AB = c$. By erecting squares AC_aC_bB, BA_bA_cC, and CB_cB_aA externally on the sides, we form new triangles AB_aC_a, BC_bA_b, and CA_cB_c, which we call the *flanks* of ABC. See Figure 1.

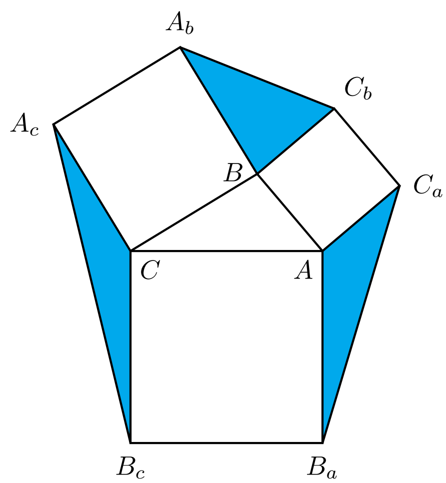

Figure 1

If we rotate the A-flank (triangle AB_aC_a) by $\frac{\pi}{2}$ about A, then the image of C_a is B, and that of B_a is on the line CA. Triangle ABC and the image of the A-flank form a larger triangle in which BA is a median. From this, ABC and the A-flank have equal areas. It is also clear that ABC is the A-flank triangle of the A-flank triangle. These observations suggest that there are a close relationship between ABC and its flanks.

2. Circumcenters of flanks

If P is a triangle center of ABC, we denote by P_A, P_B, and P_C the *same* center of the A-, B-, and C- flanks respectively.

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Let O be the circumcenter of triangle ABC. Consider the triangle $O_AO_BO_C$ formed by the circumcenters of the flanks. By the fact that the circumcenter is the intersection of the perpendicular bisectors of the sides, we see that $O_AO_BO_C$ is homothetic (parallel) to ABC, and that it bisects the squares on the sides of ABC. The distances between the corresponding sides of ABC and $O_AO_BO_C$ are therefore $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{c}{2}$.

3. Friendship of circumcenter and symmedian point

Now, homothetic triangles are perspective at their center of similitude. The distances from the center of similitude of ABC and $O_AO_BO_C$ to the sides of ABC are proportional to the distances between the corresponding sides of the two triangles, and therefore to the sides of ABC. This perspector must be the *symmedian point* K. [1]

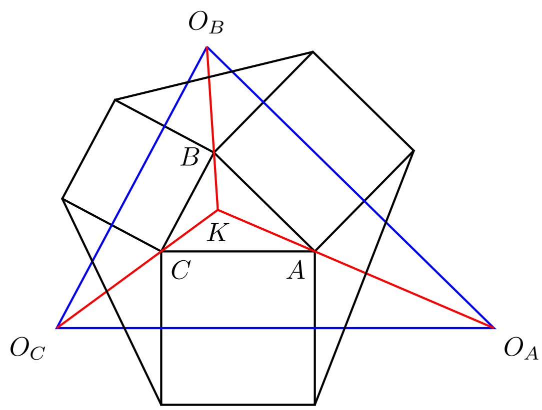

Figure 2

The triangle $O_AO_BO_C$ of *circumcenters* of the flanks is perspective with ABC at the *symmedian point* K of ABC. In particular, the A-Cevian of K in ABC (the line AK) is the same line as the A-Cevian of O_A in the A-flank. Since ABC is the A-flank of triangle AB_aC_a, the A-Cevian of K_A in the A-flank is the same line as the A-Cevian of O in ABC as well. Clearly, the same statement can be made for the B- and C-flanks. The triangle $K_AK_BK_C$ of *symmedian points* of the flanks is perspective with ABC at the *circumcenter* O.

For this relation we call the triangle centers O and K *friends*. See Figure 3. More generally, we say that P *befriends* Q if the triangle $P_AP_BP_C$ is perspective with ABC at Q. Such a friendship relation is always symmetric since, as we have remarked earlier, ABC is the A-, B-, C-flank respectively of its A-, B-, C-flanks.

[1]This is X_6 in [2, 3].

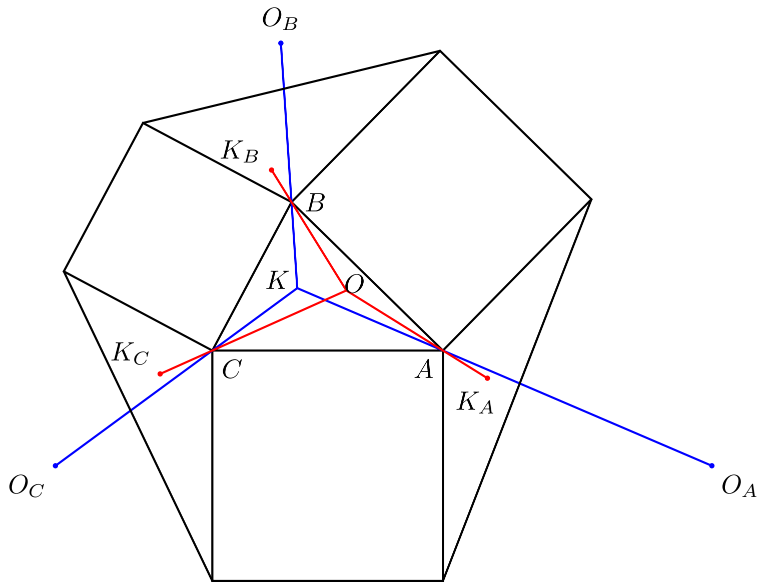

Figure 3

4. Isogonal conjugacy

It is easy to see that the bisector of an angle of ABC also bisects the corresponding angle of its flank. The incenter of a triangle, therefore, *befriends* itself.

Consider two friends P and Q. By reflection in the bisector of angle A, the line PAQ_A is mapped to the line joining the isogonal conjugates of P and Q_A.[2] We conclude:

Proposition. If two triangle centers are friends, then so are their isogonal conjugates.

Since the centroid G and the orthocenter H are respectively the isogonal conjugates of the symmedian point K and the circumcenter O, we conclude that G and H are friends.

5. The Vecten points

The centers of the three squares AC_aC_bB, BA_bA_cC and CB_cB_aA form a triangle perspective with ABC. The perspector is called the *Vecten point* of the triangle. [3] By the same token the centers of three squares constructed *inwardly* on the three sides also form a triangle perspective with ABC. The perspector is called the *second Vecten point*. [4] We show that each of the Vecten points befriends itself.

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[2]For Q_A, this is the same line when isogonal conjugation is considered both in triangle ABC and in the A-flank.

[3]This is the point X_{485} of [3].

[4]This is the point X_{486} of [3], also called the *inner* Vecten point.

6. The Second Vecten points

O. Bottema [1] has noted that the position of the midpoint M of segment B_cC_b depends only on B, C, but not on A. More specifically, M is the apex of the isosceles right triangle on BC pointed towards A. [5]

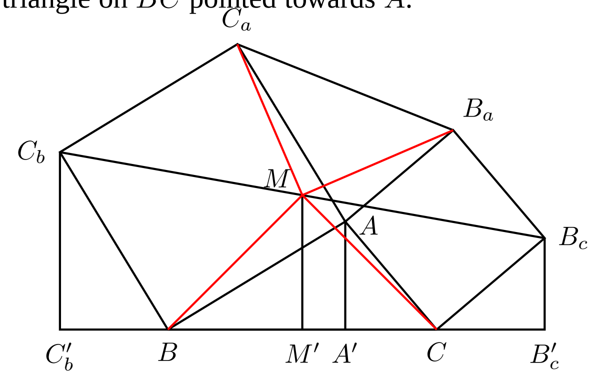

Figure 4

To see this, let A', M', B_c' and C_b' be the orthogonal projections of A, M, B_c and C_b respectively on the line BC. See Figure 4. Triangles $AA'C$ and $CB_c'B_c$ are congruent by rotation through $\pm\frac{\pi}{2}$ about the center of the square CB_cB_aA. Triangles $AA'B$ and $BC_b'C_b$ are congruent in a similar way. So we have $AA' = CB_c' = BC_b'$. It follows that M' is also the midpoint of BC. And we see that $C_b'C_b + B_c' + B_c = BA' + A'C = a$ so $MM' = \frac{a}{2}$. And M is as desired.

By symmetry M is also the apex of the isosceles right triangle on B_aC_a pointed towards A.

We recall that the triangle of apexes of similar isosceles triangles on the sides of ABC is perspective with ABC. The triangle of apexes is called a *Kiepert triangle*, and the *Kiepert perspector* $K(\phi)$ depends on the base angle ϕ (mod π) of the isosceles triangle. [6]

We conclude that AM is the A-Cevian of $K(-\frac{\pi}{4})$, also called the *second Vecten point* of both ABC and the A-flank. From similar observations on the B- and C-flanks, we conclude that the second Vecten point befriends itself.

7. Friendship of Kiepert perspectors

Given any real number t, Let X_t and Y_t be the points that divide CB_c and BC_b such that $CX_t : CB_c = BY_t : BC_b = t : 1$, and let M_t be their midpoint. Then BCM_t is an isosceles triangle, with base angle $\arctan t = \angle BAY_t$. See Figure 5.

Extend AX_t to X_t' on B_aB_c, and AY_t to Y_t' on C_aC_b and let M_t' be the midpoint of $X_t'Y_t'$. Then $B_aC_aM_t'$ is an isosceles triangle, with base angle $\arctan\frac{1}{t} = \angle Y_t'AC_a = \frac{\pi}{2} - \angle BAY_t$. Also, by the similarity of triangles AX_tY_t and $AX_t'Y_t'$

[5] Bottema introduced this result with the following story. Someone had found a treasure and hidden it in a complicated way to keep it secret. He found three marked trees, A, B and C, and thought of rotating BA through 90 degrees to BC_b, and CA through -90 degrees to CB_c. Then he chose the midpoint M of C_bB_c as the place to hide his treasure. But when he returned, he could not find tree A. He decided to guess its position and try. In a desperate mood he imagined numerous

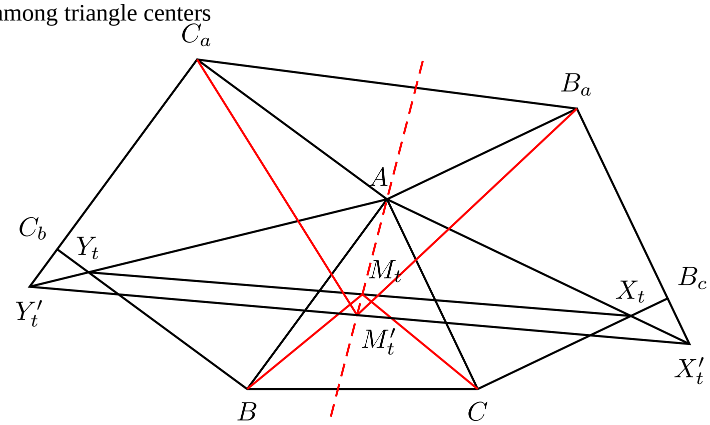

Figure 5

we see that A, M_t and M_t' are collinear. This shows that the Kiepert perspectors $K(\phi)$ and $K(\frac{\pi}{2} - \phi)$ are friends.

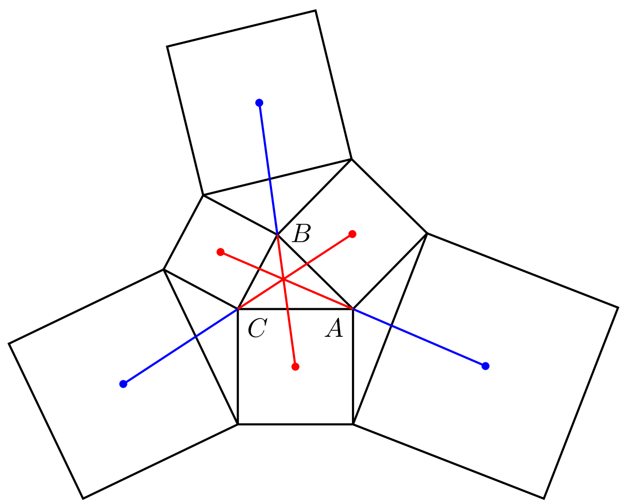

Figure 6

In particular, the first Vecten point $K(\frac{\pi}{4})$ also befriends itself. See Figure 6. The Fermat points $K(\pm\frac{\pi}{3})$ [7] are friends of the the Napoleon points $K(\frac{\pi}{6})$. [8]

Seen collectively, the *Kiepert hyperbola*, the locus of Kiepert perspectors, befriends itself; so does its isogonal transform, the Brocard axis OK.

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diggings without result. But, much to his surprise, he was able to recover his treasure on the very first try!

[6]By convention, ϕ is positive or negative according as the isosceles triangles are pointing outwardly or inwardly.

[7]These are the points X_{13} and X_{14} in [2, 3], also called the isogenic centers.

[8]These points are labelled X_{17} and X_{18} in [2, 3]. It is well known that the Kiepert triangles are equilateral.

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Another Proof of the Erdős-Mordell Theorem

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Abstract. We give a proof of the famous Erdős-Mordell inequality using Ptolemy's theorem.

The following neat inequality is well-known:

Theorem. If from a point O inside a given triangle ABC perpendiculars OD, OE, OF are drawn to its sides, then $OA+OB+OC \geq 2(OD+OE+OF)$. Equality holds if and only if triangle ABC is equilateral.

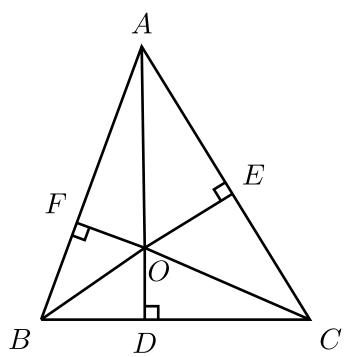

Figure 1

This was conjectured by Paul Erdős in 1935, and first proved by Louis Mordell in the same year. Several proofs of this inequality have been given, using Ptolemy's theorem by André Avez [5], angular computations with similar triangles by Leon Bankoff [2], area inequality by V. Komornik [6], or using trigonometry by Mordell and Barrow [1]. The purpose of this note is to give another elementary proof using Ptolemy's theorem.

Proof. Let HG denote the orthogonal projections of BC on the line FE. See Figure 2. Then, we have $BC \geq HG = HF + FE + EG$. It follows from $\angle BFH = \angle AFE = \angle AOE$ that the right triangles BFH and AOE are similar and $HF = \dfrac{OE}{OA}BF$. In a like manner we find that $EG = \dfrac{OF}{OA}CE$. Ptolemy's theorem applied to $AFOE$ gives

$$OA \cdot FE = AF \cdot OE + AE \cdot OF \quad \text{or} \quad FE = \frac{AF \cdot OE + AE \cdot OF}{OA}.$$

Combining these, we have

$$BC \geq \frac{OE}{OA}BF + \frac{AF \cdot OE + AE \cdot OF}{OA} + \frac{OF}{OA}CE,$$

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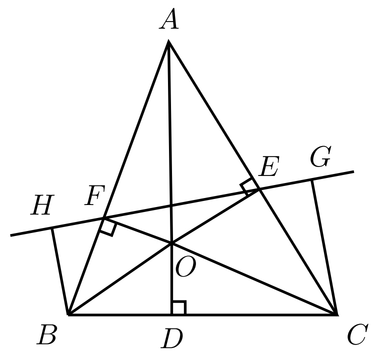

Figure 2

or

$$BC \cdot OA \geq OE \cdot BF + AF \cdot OE + AE \cdot OF + OF \cdot CE = OE \cdot AB + OF \cdot AC.$$

Dividing by BC, we have $OA \geq \frac{AB}{BC}OE + \frac{AC}{BC}OF$.

Applying the same reasoning to other projections, we have

$$OB \geq \frac{BC}{CA}OF + \frac{BA}{CA}OD \quad \text{and} \quad OC \geq \frac{CA}{AB}OD + \frac{CB}{AB}OE.$$

Adding these inequalities, we have

$$OA + OB + OC \geq (\frac{BA}{CA} + \frac{CA}{AB})OD + (\frac{AB}{BC} + \frac{CB}{AB})OE + (\frac{AC}{BC} + \frac{BC}{CA})OF.$$

It follows from this and the inequality $\frac{x}{y} + \frac{y}{x} \geq 2$ (for positive real numbers x, y) that

$$OA + OB + OC \geq 2(OD + OE + OF).$$

It is easy to check that equality holds if and only if $AB = BC = CA$ and O is the circumcenter of ABC. □

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Perspective Poristic Triangles

Edward Brisse

Abstract. This paper answers a question of Yiu: given a triangle ABC, to construct and enumerate the triangles which share the same circumcircle and incircle and are perspective with ABC. We show that there are exactly three such triangles, each easily constructible using ruler and compass.

1. Introduction

Given a triangle ABC with its circumcircle $O(R)$ and incircle $I(r)$, the famous Poncelet - Steiner porism affirms that there is a continuous family of triangles with the same circumcircle and incircle [1, p.86]. Every such triangle can be constructed by choosing an arbitrary point A' on the circle (O), drawing the two tangents to (I), and extending them to intersect (O) again at B' and C'. Yiu [3] has raised the enumeration and construction problems of poristic triangles perspective with triangle ABC, namely, those poristic triangles $A'B'C'$ with the lines AA', BB', CC' intersecting at a common point. We give a complete solution to these problems in terms of the limit points of the coaxial system of circles generated by the circumcircle and the incircle.

Theorem 1. *The only poristic triangles perspective with ABC are:*

(1) *the reflection of ABC in the line OI, the perspector being the infinite point on a line perpendicular to OI,*
(2) *the circumcevian triangles of the two limit points of the coaxial system generated by the circumcircle and the incircle.*

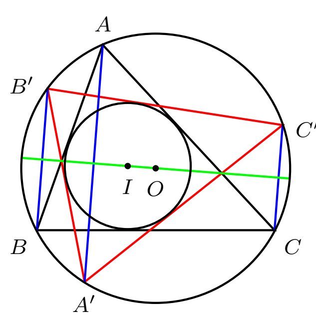

Figure 1

In (1), the lines AA', BB', CC' are all perpendicular to the line OI. See Figure 1. The perspector is the infinite point on a line perpendicular to OI. One such line

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is the trilinear polar of the incenter $I = (a : b : c)$, with equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$$

in homogeneous barycentric coordinates. The perspector is therefore the point $(a(b-c) : b(c-a) : c(a-b))$. We explain in §§2, 3 the construction of the two triangles in (2), which are symmetric with respect to the line OI. See Figure 2. In §4 we justify that these three are the only poristic triangles perspective with ABC.

2. Poristic triangles from an involution in the upper half-plane

An easy description of the poristic triangles in Theorem 1(2) is that these are the circumcevian triangles of the common poles of the circumcircle and the incircle. There are two such points; each of these has the same line as the polar with respect the circumcircle and the incircle. These common poles are symmetric with respect

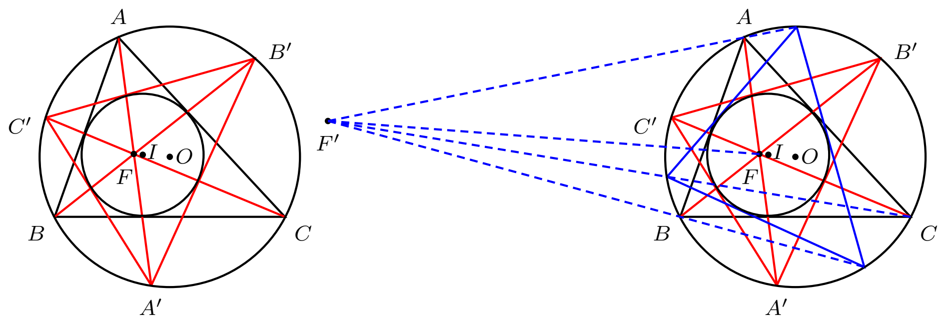

Figure 2

to the radical axis of the circles (O) and (I), and are indeed the *limit points* of the coaxial system of circles generated by (O) and (I).[1] This is best explained by the introduction of an involution of the upper half-plane. Let $a > 0$ be a fixed real number. Consider in the upper half-plane $\mathcal{R}^2_+ := \{(x, y) : y > 0\}$ a family of circles

$$\mathcal{C}_b : \qquad x^2 + y^2 - 2by + a^2 = 0, \qquad b \geq a.$$

Each circle $\mathcal{C}_b$ has center $(0, b)$ and radius $\sqrt{b^2 - a^2}$. See Figure 3. Every point in $\mathcal{R}^2_+$ lies on a unique circle $\mathcal{C}_b$ in this family. Specifically, if

$$b(x, y) = \frac{x^2 + y^2 + a^2}{2y},$$

the point (x, y) lies on the circle $\mathcal{C}_{b(x,y)}$. The circle $\mathcal{C}_a$ consists of the single point $F = (0, a)$. We call this the limit point of the family of circles. Every pair of circles in this family has the x-axis as radical axis. By reflecting the system of circles about the x-axis, we obtain a complete coaxial system of circles. The reflection of F, namely, the point $F' = (0, -a)$, is the other limit point of this system. Every circle through F and F' is orthogonal to every circle $\mathcal{C}_b$.

[1]The common polar of each one of these points with respect to the two circles passes through the other.

Consider a line through the limiting point F, with slope m, and therefore equation $y = mx + a$. This line intersects the circle $\mathcal{C}_b$ at points whose y-coordinates are the roots of the quadratic equation

$$(1+m^2)y^2 - 2(a+bm^2)y + a^2(1+m^2) = 0.$$

Note that the two roots multiply to a^2. Thus, if one of the intersections is (x, y), then the other intersection is $(-\frac{ax}{y}, \frac{a^2}{y})$. See Figure 4. This defines an involution on the upper half plane:

$$P^* = (-\frac{ax}{y}, \frac{a^2}{y}) \quad \text{for} \quad P = (x, y).$$

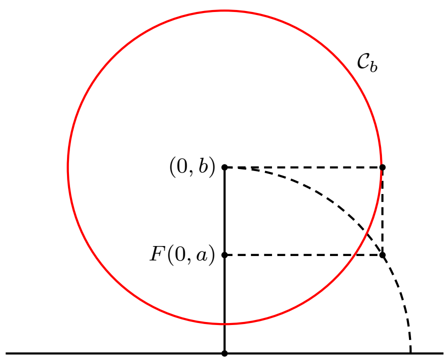

Figure 3

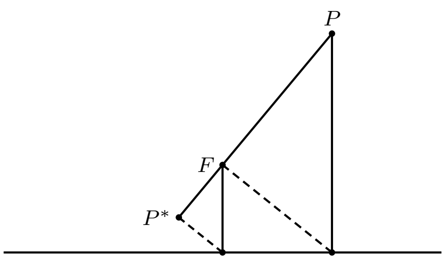

Figure 4

Proposition 2. *(1) $P^{**} = P$.*
(2) P and P^* belong to the same circle in the family $\mathcal{C}_b$. In other words, if P lies on the circle $\mathcal{C}_b$, then the line FP intersects the same circle again at P^*.
(3) The line PF' intersects the circle $\mathcal{C}_b$ at the reflection of P^* in the y-axis.

Proof. (1) is trivial. (2) follows from $b(P) = b(P^*)$. For (3), the intersection is the point $(\frac{ax}{y}, \frac{a^2}{y})$. □

Lemma 3. *Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points on the same circle $\mathcal{C}_b$. The segment AB is tangent to a circle $\mathcal{C}_{b'}$ at the point whose y-coordinate is $\sqrt{y_1 y_2}$.*

Proof. This is clear if $y_1 = y_2$. In the generic case, extend AB to intersect the x-axis at a point C. The segment AB is tangent to a circle $\mathcal{C}_{b'}$ at a point P such that $CP = CF$. It follows that $CP^2 = CF^2 = CA \cdot CB$. Since C is on the x-axis, this relation gives $y^2 = y_1 y_2$ for the y-coordinate of P. □

Theorem 4. *If a chord AB of $\mathcal{C}_b$ is tangent to $\mathcal{C}_{b'}$ at P, then the chord A^*B^* is tangent to the same circle $\mathcal{C}_{b'}$ at P^*.*

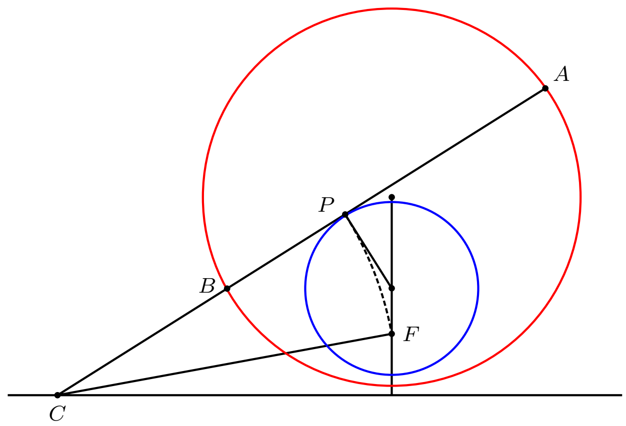

Figure 5

Proof. That P and P^* lie on the same circle is clear from Proposition 2(1). It remains to show that P^* is the correct point of tangency. This follows from noting that the y-coordinate of P^*, being $\frac{a^2}{\sqrt{y_1 y_2}}$, is the geometric mean of those of A^* and B^*. □

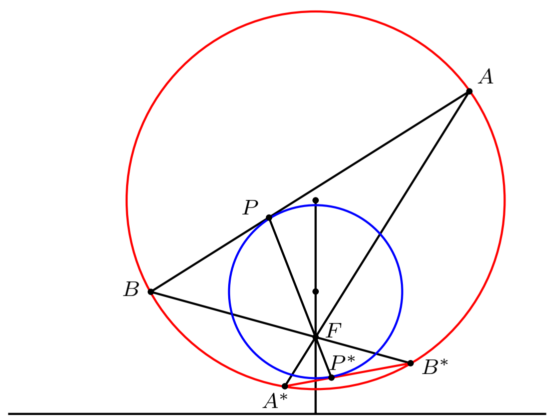

Figure 6

Consider the circumcircle and incircle of triangle ABC. These two circles generate a coaxial system with limit points F and F'.

Corollary 5. *The triangle $A^*B^*C^*$ has $I(r)$ as incircle, and is perspective with ABC at F.*

Corollary 6. *The reflection of the triangle $A^*B^*C^*$ in the line OI also has $I(r)$ as incircle, and is perspective with ABC at the point F'.*

Proof. This follows from Proposition 2 (3). □

It remains to construct the two limit points F and F', and the construction of the two triangles in Theorem 1(2) would be complete.

Proposition 7. *Let XY be the diameter of the circumcircle through the incenter I. If the tangents to the incircle from these two points are XP, XQ', YQ, and YP' such that P and Q are on the same side of OI, then PP' intersects OI at F (so does QQ'), and PQ intersects OI at F' (so does $P'Q'$).*

Proof. This follows from Theorem 4 by observing that Y = X*. □

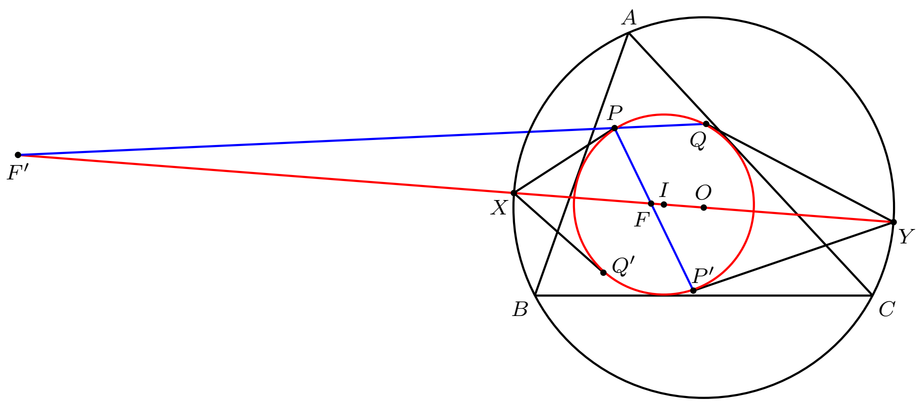

Figure 7

3. Enumeration of perspective poristic triangles

In this section, we show that the poristic triangles constructed in the preceding sections are the only ones perspective with ABC. To do this, we adopt a slightly different viewpoint, by searching for circumcevian triangles which share the same incircle with ABC. We work with homogeneous barycentric coordinates. Recall that if a, b, c are the lengths of the sides BC, CA, AB respectively, then the circumcircle has equation

$$a^2yz + b^2zx + c^2xy = 0,$$

and the incircle has equation

$$\begin{aligned}(s-a)^2x^2 + (s-b)^2y^2 + (s-c)^2z^2 - 2(s-b)(s-c)yz\\ -2(s-c)(s-a)zx - 2(s-a)(s-b)xy = 0,\end{aligned}$$

where $s = \frac{1}{2}(a+b+c)$.

We begin with a lemma.

Lemma 8. *The tangents from a point $(u : v : w)$ on the circumcircle (O) to the incircle (I) intersect the circumcircle again at two points on the line*

$$\frac{(s-a)u}{a^2}x + \frac{(s-b)v}{b^2}y + \frac{(s-c)w}{c^2}z = 0.$$

Remark: This line is tangent to the incircle at the point

$$\left(\frac{a^4}{(s-a)u^2} \ : \ \frac{b^4}{(s-b)v^2} \ : \ \frac{c^4}{(s-c)w^2}\right).$$

Given a point $P = (u : v : w)$ in homogeneous barycentric coordinates, the circumcevian triangle $A'B'C'$ is formed by the *second* intersections of the lines AP, BP, CP with the circumcircle. These have coordinates

$$A' = (\frac{-a^2vw}{b^2w+c^2v} : v : w), \quad B' = (u : \frac{-b^2wu}{c^2u+a^2w} : w), \quad C' = (u : v : \frac{-c^2uv}{a^2v+b^2u}).$$

Applying Lemma 8 to the point A', we obtain the equation of the line $B'C'$ as

$$\frac{-(s-a)vw}{b^2w+c^2v}x + \frac{(s-b)v}{b^2}y + \frac{(s-c)w}{c^2}z = 0.$$

Since this line contains the points B' and C', we have

$$-\frac{(s-a)uvw}{b^2w+c^2v} - \frac{(s-b)uvw}{c^2u+a^2w} + \frac{(s-c)w^2}{c^2} \quad = \quad 0, \tag{1}$$

$$-\frac{(s-a)uvw}{b^2w+c^2v} + \frac{(s-b)v^2}{b^2} - \frac{(s-c)uvw}{b^2u+a^2v} \quad = \quad 0. \tag{2}$$

The difference of these two equations gives

$$\frac{a^2vw+b^2wu+c^2uv}{b^2c^2(b^2u+a^2v)(c^2u+a^2w)} \cdot f = 0, \tag{3}$$

where

$$f = -b^2c^2(s-b)uv + b^2c^2(s-c)wu - c^2a^2(s-b)v^2 + a^2b^2(s-c)w^2.$$

If $a^2vw + b^2wu + c^2uv = 0$, the point $(u : v : w)$ is on the circumcircle, and both equations (1) and (2) reduce to

$$\frac{s-a}{a^2}u^2 + \frac{s-b}{b^2}v^2 + \frac{s-c}{c^2}w^2 = 0,$$

clearly admitting no real solutions. On the other hand, setting the quadratic factor f in (3) to 0, we obtain

$$u = \frac{-a^2}{b^2c^2} \cdot \frac{c^2(s-b)v^2 - b^2(s-c)w^2}{(s-b)v-(s-c)w}.$$

Substitution into equation (1) gives

$$\frac{vw(c(a-b)v - b(c-a)w)}{b^2c^2(c^2v^2-b^2w^2)(v(s-b)-w(s-c))} \cdot g = 0, \tag{4}$$

where

$$\begin{aligned} g \quad = \quad & c^3(s-b)(a^2+b^2-c(a+b))v^2 + b^3(s-c)(c^2+a^2-b(c+a))w^2 \\ & +2bc(s-b)(s-c)(b^2+c^2-a(b+c))vw. \end{aligned}$$

There are two possibilities.

(i) If $c(a-b)v - b(c-a)w = 0$, we obtain $v : w = b(c-a) : c(a-b)$, and consequently, $u : v : w = a(b-c) : b(c-a) : c(a-b)$. This is clearly an infinite point, the one on the line $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$, the trilinear polar of the incenter. This line is perpendicular to the line OI. This therefore leads to the triangle in Theorem 1(1).

(ii) Setting the quadratic factor g in (4) to 0 necessarily leads to the two triangles constructed in §2. The corresponding perspectors are the two limit points of the coaxial system generated by the circumcircle and the incircle.

4. Coordinates

The line OI has equation

$$\frac{(b-c)(s-a)}{a}x + \frac{(c-a)(s-b)}{b}y + \frac{(a-b)(s-c)}{c}z = 0.$$

The radical axis of the two circles is the line

$$(s-a)^2x + (s-b)^2y + (s-c)^2z = 0.$$

These two lines intersect at the point

$$\left(\frac{a(a^2(b+c) - 2a(b^2 - bc + c^2) + (b-c)^2(b+c))}{b+c-a} : \cdots : \cdots\right),$$

where the second and third coordinates are obtained from the first by cyclic permutations of a, b, c. This point is not found in [2].

The coordinates of the common poles F and F' are

$$(a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)) + t(a : b : c)$$

where

$$t = \frac{1}{2}\left(-2abc + \sum_{\text{cyclic}}(a^3 - bc(b+c))\right) \pm 2\triangle\sqrt{2ab + 2bc + 2ca - a^2 - b^2 - c^2}, \tag{5}$$

and $\triangle$ = area of triangle ABC. This means that the points F and F' divide harmonically the segment joining the incenter $I(a : b : c)$ to the point whose homogeneous barycentric coordinates are

$$\begin{aligned}&(a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2))\\ +\;\; &\frac{1}{2}\left(-2abc + \sum_{\text{cyclic}}(a^3 - bc(b+c))\right)(a : b : c).\end{aligned}$$

This latter point is the triangle center

$$X_{57} = \left(\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{c}{a+b-c}\right)$$

in [2], which divides the segment OI in the ratio $OX_{57} : OI = 2R + r : 2R - r$. The common poles F and F', it follows from (5) above, divide the segment IX_{57} harmonically in the ratio $2R - r : \pm\sqrt{(4R+r)r}$.

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Heron Triangles: A Gergonne-Cevian-and-Median Perspective

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Abstract. We give effective constructions of Heron triangles by considering the intersection of a median and a cevian through the Gergonne point.

1. Introduction

Heron gave the triangle area formula in terms of the sides a, b, c:

$$(*) \qquad \triangle = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s = \frac{1}{2}(a+b+c).$$

He is further credited with the discovery of the integer sided and integer area triangle (13,14,15;84). Notice that this is a non-Pythagorean triangle, *i.e.*, it does not contain a right angle. We might as well say that with this discovery he challenged us to determine triangles having integer sides and area, *Heron triangles*. Dickson [2] sketches the early attempts to meet this challenge. The references [1, 3, 4, 5, 6, 7, 9, 10] describe recent attempts in that direction. The present discussion uses the intersection point of a Gergonne cevian (the line segment between a vertex and the point of contact of the incircle with the opposite side) and a median to generate Heron triangles. Why do we need yet another description? The answer is simple: Each new description provides new ways to solve, and hence to acquire new insights into, earlier Heron problems. More importantly, they pose new Heron challenges. We shall illustrate this. Dickson uses the name Heron triangle to describe one having rational sides and area. However, these rationals can always be rendered integers. Therefore for us a Heron triangle is one with integer sides and area except under special circumstances.

We use the standard notation: a, b, c for the sides BC, CA, AB of triangle ABC. We use the word side also in the sense of the length of a side. Furthermore, we assume $a \geq c$. No generality is lost in doing so because we may relabel the vertices if necessary.

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2. A preliminary result

We first solve this problem: Suppose three cevians of a triangle concur at a point. How does one determine the ratio in which the concurrence point sections one of them? The answer is given by

Theorem 1. *Let the cevians AD, BE, CF of triangle ABC concur at the point S. Then*

$$\frac{AS}{SD} = \frac{AE}{EC} + \frac{AF}{FB}.$$

Proof. Let $[T]$ denote the area of triangle T. We use the known result: if two triangles have a common altitude, then their areas are proportional to the corresponding bases. Hence, from Figure 1,

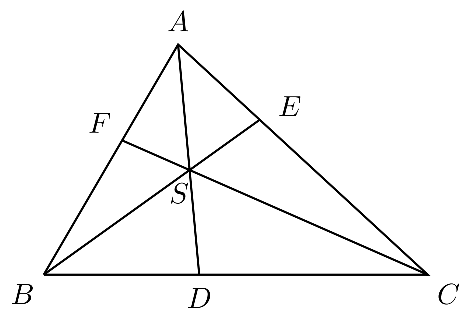

Figure 1

$$\frac{AS}{SD} = \frac{[ABS]}{[SBD]} = \frac{[ASC]}{[SDC]} = \frac{[ABS]+[ASC]}{[SBD]+[SDC]} = \frac{[ABS]}{[SBC]} + \frac{[ASC]}{[SBC]}. \quad (1)$$

But

$$\frac{AE}{EC} = \frac{[ABE]}{[EBC]} = \frac{[ASE]}{[ESC]} = \frac{[ABE]-[ASE]}{[EBC]-[ESC]} = \frac{[ABS]}{[SBC]}, \quad (2)$$

and likewise,

$$\frac{AF}{FB} = \frac{[ASC]}{[SBC]}. \quad (3)$$

Now, (1), (2), (3) complete the proof. □

In the above proof we used a property of equal ratios, namely, if $\frac{p}{q} = \frac{r}{s} = k$, then $k = \frac{p \pm q}{r \pm s}$. From Theorem 1 we deduce the following corollary that is important for our discussion.

Corollary 2. *In Figure 2, let AD denote the median, and BE the Gergonne cevian. Then* $\frac{AS}{SD} = \frac{2(s-a)}{s-c}$.

Proof. The present hypothesis implies $BD = DC$, and E is the point where the incircle is tangent with AC. It is well - known that $AE = s - a$, $EC = s - c$. Now, Ceva's theorem, $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$, yields $\frac{AF}{FB} = \frac{s-a}{s-c}$. Then Theorem 1 upholds the claim of Corollary 2. □

In the case of a Heron triangle, a, b, c and s are natural numbers. Therefore, $\frac{AS}{SD} = \frac{2(s-a)}{s-c} = \lambda$ is a rational ratio. Of course this will be true more generally even if $\triangle$ is not an integer; but that is beside the main point. Also, $a \geq c$ implies that $0 < \lambda \leq 2$. Next we show how each rational number λ generates an infinite family, a $\lambda-$family of Heron triangles.

3. Description of $\lambda-$family of Heron triangles

Theorem 3 gives expressions for the sides of the Heron triangle in terms of λ. At present we do not transform these rational sides integral. However, when we specify a rational number for λ then we do express a, b, c integral such that $\gcd(a,b,c) = 1$. An exception to this common practice may be made in the solution of a Heron problem that requires $\gcd(a,b,c) > 1$, in (*D1*) later, for example.

Theorem 3. *Let λ be a rational number such that $0 < \lambda \leq 2$. The $\lambda-$family of Heron triangles is described by*

$$(a,b,c) = (2(m^2 + \lambda^2 n^2),\ (2+\lambda)(m^2 - 2\lambda n^2),\ \lambda(m^2 + 4n^2)),$$

m, n *being relatively prime natural numbers such that* $m > \sqrt{2\lambda} \cdot n$.

Proof. From the definition we have

$$\frac{2(s-a)}{s-c} = \lambda \quad \text{or} \quad b = \frac{2+\lambda}{2-\lambda}(a-c).$$

If $\lambda \neq 2$, we assume $a - c = (2-\lambda)p$. This gives $b = (2+\lambda)p$. If $\lambda = 2$, then we define $b = 4p$. The rest of the description is common to either case. Next we calculate

$$\begin{aligned} a &= (2-\lambda)p + c, \quad s = c + 2p, \quad \text{and from } (*), \\ \triangle^2 &= 2\lambda p^2 (c+2p)(c-\lambda p). \end{aligned} \tag{4}$$

To render (a,b,c) Heron we must have $(c+2p)(c-\lambda p) = 2\lambda q^2$. There is no need to distinguish two cases: 2λ itself a rational square or not. This fact becomes clearer later when we deduce Corollary 5. With the help of a rational number $\frac{m}{n}$ we may write down

$$c + 2p = \frac{m}{n} q, \quad \text{and} \quad c - \lambda p = \frac{n}{m}(2\lambda q).$$

We solve the above simultaneous equations for p and c:

$$p = \frac{m^2 - 2\lambda n^2}{(2+\lambda)mn} \cdot q, \qquad c = \frac{\lambda(m^2 + 4n^2)}{(2+\lambda)mn} \cdot q.$$

This yields

$$\frac{p}{m^2 - 2\lambda n^2} = \frac{q}{(2+\lambda)mn} = \frac{c}{\lambda(m^2+4n^2)}.$$

Since p, q, c, λ, m, n are positive we must have $m > \sqrt{2\lambda} \cdot n$. We may ignore the constant of proportionality so that

$$p = m^2 - 2\lambda n^2, \qquad q = (2+\lambda)mn \qquad c = \lambda(m^2+4n^2).$$

These values lead to the expressions for the sides a, b, c in the statement of Theorem 3. Also, $\triangle = 2\lambda(2+\lambda)mn(m^2 - 2\lambda n^2)$, see (4), indicates that the area is rational. □

Here is a numerical illustration. Let $\lambda = 1$, $m = 4$, $n = 1$. Then Theorem 3 yields $(a, b, c) = (34, 42, 20)$. Here $\gcd(a, b, c) = 2$. In the study of Heron triangles often gcd$(a, b, c) > 1$. In such a case we divide the side length values by the gcd to list primitive values. Hence, $(a, b, c) = (17, 21, 10)$.

Now, suppose $\lambda = \frac{3}{2}$, $m = 5$, $n = 2$. Presently, Theorem 3 gives $(a, b, c) = (68, \frac{91}{2}, \frac{123}{2})$. As it is, the sides b and c are not integral. In this situation we render the sides integral (and divide by the gcd if it is greater than 1) so that $(a, b, c) = (136, 91, 123)$.

We should remember that Theorem 3 yields the same Heron triangle more than once if we ignore the order in which the sides appear. This depends on the number of ways in which the sides a, b, c may be permuted preserving the constraint $a \geq c$. For instance, the $(17, 21, 10)$ triangle above for $\lambda = 1$, $m = 4$, $n = 1$ may also be obtained when $\lambda = \frac{3}{7}$, $m = 12$, $n = 7$, or when $\lambda = \frac{6}{7}$, $m = 12$, $n = 7$. The verification is left to the reader. It is time to deduce a number of important corollaries from Theorem 3.

Corollary 4. *Theorem 3 yields the Pythagorean triangles* $(a, b, c) = (u^2+v^2, u^2-v^2, 2uv)$ *for* $\lambda = \dfrac{2v}{u}$, $m = 2$, $n = 1$.

Incidentally, we observe that the famous generators u, v of the Pythagorean triples/triangles readily tell us the ratio in which the Gergonne cevian BE intersects the median AD. Similar observation may be made throughout in an appropriate context.

Corollary 5. *Theorem 3 yields the isosceles Heron triangles* $(a, b, c) = (m^2 + n^2, 2(m^2 - n^2), m^2 + n^2)$ *for* $\lambda = 2$.

Actually, $\lambda = 2$ yields $(a, b, c) = (m^2+4n^2, 2(m^2-4n^2), m^2+4n^2)$. However, the transformation $m \mapsto 2m$, $n \mapsto n$ results in the more familiar form displayed in Corollary 5.

Corollary 6. *Theorem 3 describes the complete set of Heron triangles.*

This is because the Gergonne cevian BE must intersect the median AD at a unique point. Therefore for all Heron triangles $0 < \lambda \leq 2$. Suppose first we fix λ at such a rational number. Then Theorem 3 gives the entire $\lambda-$family of Heron triangles each member of which has BE intersecting AD in the same ratio, that is

λ. Next we vary λ over rational numbers $0 < \lambda \leq 2$. By successive applications of the preceding remark the claim of Corollary 6 follows.

Corollary 7. *[Hoppe's Problem] Theorem 3 yields Heron triangles* $(a, b, c) = (m^2 + 9n^2, 2(m^2 + 3n^2), 3(m^2 + n^2))$ *having the sides in arithmetic progression for* $\lambda = \dfrac{m^2}{6n^2}$.

Here too a remark similar to the one following Corollary 5 applies. Corollaries 4 through 7 give us the key to the solution, often may be partial solutions of many Heron problems: Just consider appropriate $\lambda-$family of Heron triangles. We will continue to amplify on this theme in the sections to follow. To richly illustrate this we prepare a table of $\lambda-$families of Heron triangles. In Table 1, π denotes the perimeter of the triangle.

Table 1. $\lambda-$families of Heron triangles

λ	a	b	c	π	$\triangle$
1	$2(m^2+n^2)$	$3(m^2-2n^2)$	m^2+4n^2	$6m^2$	$6mn(m^2-2n^2)$
$1/2$	$4m^2+n^2$	$5(m^2-n^2)$	m^2+4n^2	$10m^2$	$10mn(m^2-n^2)$
$1/3$	$2(9m^2+n^2)$	$7(3m^2-2n^2)$	$3(m^2+4n^2)$	$42m^2$	$42mn(3m^2-2n^2)$
$2/3$	$9m^2+4n^2$	$4(3m^2-4n^2)$	$3(m^2+4n^2)$	$24m^2$	$24mn(3m^2-4n^2)$
$1/4$	$16m^2+n^2$	$9(2m^2-n^2)$	$2(m^2+4n^2)$	$36m^2$	$36mn(2m^2-n^2)$
$3/4$	$16m^2+9n^2$	$11(2m^2-3n^2)$	$6(m^2+4n^2)$	$44m^2$	$132mn(2m^2-3n^2)$
$1/5$	$2(25m^2+n^2)$	$11(5m^2-2n^2)$	$5(m^2+4n^2)$	$110m^2$	$110mn(5m^2-2n^2)$
$2/5$	$25m^2+4n^2$	$6(5m^2-4n^2)$	$5(m^2+4n^2)$	$60m^2$	$60mn(5m^2-4n^2)$
$3/5$	$2(25m^2+9n^2)$	$13(5m^2-6n^2)$	$15(m^2+4n^2)$	$130m^2$	$390mn(5m^2-6n^2)$
$4/5$	$25m^2+16n^2$	$7(5m^2-8n^2)$	$10(m^2+4n^2)$	$70m^2$	$140mn(5m^2-8n^2)$
$3/2$	$4m^2+9n^2$	$7(m^2-3n^2)$	$3(m^2+4n^2)$	$14m^2$	$42mn(m^2-3n^2)$
$4/3$	$9m^2+16n^2$	$5(3m^2-8n^2)$	$6(m^2+4n^2)$	$30m^2$	$60mn(3m^2-8n^2)$
$5/3$	$2(9m^2+25n^2)$	$11(3m^2-10n^2)$	$15(m^2+4n^2)$	$66m^2$	$330mn(3m^2-10n^2)$
$5/4$	$16m^2+25n^2$	$13(2m^2-5n^2)$	$10(m^2+4n^2)$	$52m^2$	$260mn(2m^2-5n^2)$
$7/4$	$16m^2+49n^2$	$15(2m^2-7n^2)$	$14(m^2+4n^2)$	$60m^2$	$420mn(2m^2-7n^2)$

4. Heron problems and solutions

In what follows we omit the word "determine" from each problem statement. "Heron triangles" will be contracted to HT, and we do *not* provide solutions in detail.

A. Involving sides. A1. HT in which two sides differ by a desired integer. In fact finding one such triangle is equivalent to finding an infinity! This is because they depend on the solution of the so-called Fermat-Pell equation $x^2 - dy^2 = e$, where e is an integer and d not an integer square. It is well-known that Fermat-Pell equations have an infinity of solutions (x, y) (i) when $e = 1$ and (ii) when $e \neq 1$ if there is one. The solution techniques are available in an introductory number theory text, or see [2].

HT in which the three sides are consecutive integers are completely given by Corollary 7. For example, $m = 3$, $n = 1$ gives the (3,4,5); $m = 2$, $n = 1$, the (13,14,15), and so on. Here two sides differ by 1 and incidentally, two sides by 2. However, there are other HT in which two sides differ by 1 (or 2). For another partial solution, consider $\lambda = 1$ family from Table 1. Here $a - c = 1 \iff m^2 - 2n^2 = 1$. $m = 3$, $n = 2$ gives the $(26, 3, 25)$. $m = 17$, $n = 12$, the (866,3,865) triangle and so on. We observe that 3 is the common side of an infinity of HT. Actually, it is known that *every* integer greater than 2 is a common side of an infinity of HT [1, 2].

To determine a HT in which two sides differ by 3, take $\lambda = \frac{1}{2}$ family and set $b - a = 3$. This leads to the equation $m^2 - 6n^2 = 3$. The solution $(m, n) = (3, 1)$ gives $(a, b, c) = (37, 40, 13)$; $(m, n) = (27, 11)$ gives $(3037, 3040, 1213)$ and so on. This technique can be extended.

A2. A pair of HT having a common side. Consider the pairs $\lambda = 1$, $\lambda = \frac{1}{2}$; $\lambda = \frac{1}{3}$, $\lambda = \frac{2}{3}$; or some two distinct $\lambda-$families that give identical expressions for a particular side. For instance, $m = 3$, $n = 1$ in $\lambda = \frac{1}{3}$ and $\lambda = \frac{2}{3}$ families yields a pair $(164, 175, 39)$ and $(85, 92, 39)$. It is now easy to obtain as many pairs as one desires. This is a quicker solution than the one suggested by *A1*.

A3. A pair of HT in which a pair of corresponding sides are in the ratio $1 : 2$, $1 : 3$, $2 : 3$ etc. The solution lies in the column for side c.

A4. A HT in which two sides sum to a square. We consider $\lambda = \frac{1}{2}$ family where $a + c = 5(m^2 + n^2)$ is made square by $m = 11$, $n = 2$; $(488, 585, 137)$. It is now a simple matter to generate any number of them.

B. Involving perimeter. The perimeter column shows that it is a function of the single parameter m. This enables us to pose, and solve almost effortlessly, many perimeter related problems. To solve such problems by traditional methods would often at best be extremely difficult. Here we present a sample.

B1. A HT in which the perimeter is a square. A glance at Table 1 reveals that $\pi = 36m^2$ for $\lambda = \frac{1}{4}$ family. An infinity of primitive HT of this type is available.

B2. A pair of HT having equal perimeter. An infinity of solution is provided by the $\lambda = \frac{2}{5}$ and $\lambda = \frac{7}{4}$ families. All that is needed is to substitute identical values for m and suitable values to n to ensure the outcome of primitive HT.

B3. A finite number of HT all with equal perimeter. The solution is unbelievably simple! Take *any* λ family and put sufficiently large constant value for m and then vary the values of n only.

A pair of HT in which one perimeter is twice, thrice, ... another, or three or more HT whose perimeters are in arithmetic progression, or a set of four HT such that the sum or the product of two perimeters equals respectively the sum or the product of the other two perimeters are simple games to play. More extensive tables of $\lambda-$family HT coupled with a greater degree of observation ensures that ingenius problem posing solving activity runs wild.

C. Involving area. The $\lambda = \frac{1}{2}$ family has $\triangle = 10mn(m^2 - n^2)$. Now, $mn(m^2 - n^2)$ gives the area of the Pythagorean triangle $(m^2 - n^2, 2mn, m^2 + n^2)$. Because of this an obvious problem has posed and solved itself:

C1. Given a Pythagorean triangle there exists a non-Pythagorean Heron triangle such that the latter area is ten times the former.

It may happen that sometimes one of them may be primitive and the other not, or both may not be primitive. Also, for $m = 2$, $n = 1$, both are Pythagorean. However, there is the $(6, 25, 29)$ Heron triangle with $\triangle = 60$. This close relationship should enable us to put known vast literature on Pythagorean problems to good use, see the following problem for example.

C2. Two Heron triangles having equal area; two HT having areas in the ratio $r : s$.

In [2], pp. 172 – 175, this problem has been solved for right triangles. The primitive solutions are not guaranteed.

D. Miscellaneous problems. In this section we consider problems involving both perimeter and area.

D1. HT in which perimeter equals area. This is such a popular problem that it continues to resurface. It is known that there are just five such HT. The reader is invited to determine them. Hint: They are in $\lambda = \frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{5}$, 1 and $\frac{4}{3}$ families. Possibly elsewhere too, see the remark preceding Corollary 4.

D2. HT in which π and $\triangle$ are squares. In $\lambda = \frac{1}{4}$ family we put $m = 169$, $n = 1$.

D3. Pairs of HT with equal perimeter and equal area in each pair. An infinity of such pairs may be obtained from $\lambda = \frac{1}{2}$ family. We put $m = u^2 + uv + v^2$, $n_1 = u^2 - v^2$ and $m = u^2 + uv + v^2$, $n_2 = 2uv + v^2$. For instance, $u = 3$, $v = 1$ *i.e.*, $m = 13$, $n_1 = 8$, $n_2 = 7$ produces a desired pair $(148, 105, 85)$ and $(145, 120, 73)$. They have $\pi_1 = \pi_2 = 338$ and $\triangle_1 = \triangle_2 = 4368$.

If we accept pairs of HT that may not be primitive then we may consider $\lambda = \frac{2}{3}$ family. Here, $m = p^2 + 3q^2$, $n_1 = p^2 - 3q^2$ and $m = p^2 + 3q^2$, $n_2 = \frac{1}{2}(-p^2 + 6pq + 3q^2)$.

E. Open problems. We may look upon the problem (D3) as follows: $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = 1$. This immediately leads to the following

Open problem 1. Suppose two HTs have perimeters π_1, π_2 and areas $\triangle_1$, $\triangle_2$ such that $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = \dfrac{p}{q}$, a rational number. Prove or disprove the existence of an infinity of HT such that for each pair $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = \dfrac{p}{q}$ holds.

For instance, $\lambda_1 = \frac{1}{5}$, (odd) $m_1 > 4k$, $n_1 = 4k$ and $\lambda_2 = \frac{4}{5}$, $m_2 > 4k$ (again odd), $m_2 = m_1$, $n_2 = 2k$ yield $\dfrac{\pi_2}{\pi_1} = \dfrac{\triangle_2}{\triangle_1} = \dfrac{11}{7}$ for $k = 1, 2, 3, \ldots$.

With some effort it is possible to find an infinity of pairs of HT such that for each pair, $\dfrac{\triangle_2}{\triangle_1} = e \cdot \dfrac{\pi_2}{\pi_1}$ for certain natural numbers e. This leads to

Open problem 2. Let e be a given natural number. Prove or disprove the existence of an infinity of pairs of HT such that for each pair $\dfrac{\triangle_2}{\triangle_1} = e \cdot \dfrac{\pi_2}{\pi_1}$ holds.

5. Conclusion

The present description of Heron triangles did provide simple solutions to certain Heron problems. Additionally it suggested new ones that arose naturally in our discussion. The reader is encouraged to try other $\lambda-$families for different solutions from the presented ones. There is much scope for problem posing and solving activity. Non-standard problems such as: find three Heron triangles whose perimeters (areas) are themselves the sides of a Heron triangle or a Pythagorean triangle. Equally important is to pose unsolved problems. A helpful step in this direction would be to consider Heron analogues of the large variety of existing Pythagorean problems.

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Equilateral Triangles Intercepted by Oriented Parallelians

Sabrina Bier

Abstract. Given a point P in the plane of triangle ABC, we consider rays through P parallel to the side lines. The intercepts on the sidelines form an equilateral triangle precisely when P is a Brocardian point of one of the Fermat points. There are exactly four such equilateral triangles.

1. Introduction

The construction of an interesting geometric figure is best carried out after an analysis. For example, given a triangle ABC, how does one construct a point P through which the parallels to the three sides make equal intercepts? A very simple analysis of this question can be found in [6, 7]. It is shown that there is only one such point P, [1] which has homogeneous barycentric coordinates

$$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}:\frac{1}{c}+\frac{1}{a}-\frac{1}{b}:\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\sim(ca+ab-bc:ab+bc-ca:bc+ca-ab).$$

This leads to a very easy construction of the point [2] and its three equal parallel intercepts. See Figure 1. An interesting variation is to consider equal "semi-parallel

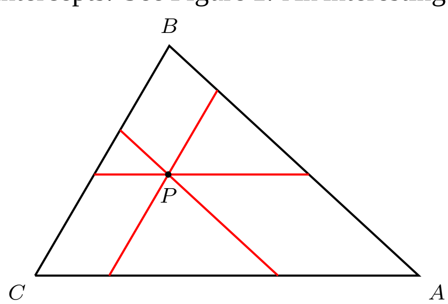

Figure 1

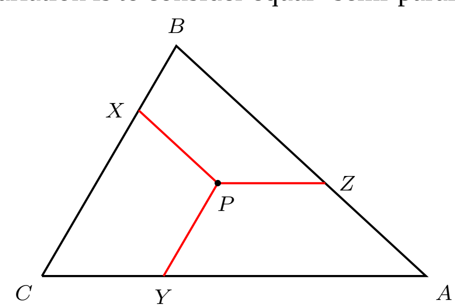

Figure 2

intercepts". Suppose through a point P in the plane of triangle ABC, parallels to the sides AB, BC, CA intersect BC, CA, AB are X, Y, Z respectively. How should one choose P so that the three "semi-parallel intercepts" PX, PY, PZ have equal lengths? (Figure 2). A simple calculation shows that the only point satisfying this requirement, which we denote by $L_{\rightarrow}$, has coordinates $(\frac{1}{c}:\frac{1}{a}:\frac{1}{b})$.

Publication Date: February 19, 2001. Communicating Editor: Paul Yiu.

The results in this paper were obtained in the fall semester, 2000, in a Directed Independent Study under Professor Paul Yiu. This paper was prepared with the assistance of Professor Yiu, who also contributed §5 and the Appendix.

[1]In [3], this is the equal-parallelian point X_{192}. In [7], this is called the equal-intercept point.

[2]If G is the centroid and I' the isotomic conjugate of the incenter of triangle ABC, then $I'P = 3\cdot I'G$.

If we reverse the orientations of the parallel rays, we obtain another point $J_{\leftarrow}$ with coordinates $(\frac{1}{b}:\frac{1}{c}:\frac{1}{a})$. See Figure 5. These two points are called the Jerabek points; they can be found in [2, p.1213]. For a construction, see §4.

2. Triangles intercepted by forward parallelians

Given a triangle ABC, we mean by a *parallelian* a directed ray parallel to one of the sides, *forward* if it is along the direction of AB, BC, or CA, and *backward* if it is along BA, CB, or AC. In this paper we study the question: how should one choose the point P so that so that the *triangle XYZ intercepted by forward parallelians through P* is equilateral? See Figure 2. [3] We solve this problem by performing an analysis using homogeneous barycentric coordinates. If $P=(u:v:w)$, then X, Y, and Z have coordinates

$$X=(0:u+v:w),\quad Y=(u:0:v+w),\quad Z=(w+u:v:0).$$

The lengths of AY and AZ are respectively $\frac{(v+w)b}{u+v+w}$ and $\frac{vc}{u+v+w}$. By the law of cosines, the square length of YZ is

$$\frac{1}{(u+v+w)^2}((v+w)^2b^2+v^2c^2-(v+w)v(b^2+c^2-a^2)).$$

Similarly, the square lengths of ZX and XY are respectively

$$\frac{1}{(u+v+w)^2}((w+u)^2c^2+w^2a^2-(w+u)w(c^2+a^2-b^2))$$

and

$$\frac{1}{(u+v+w)^2}((u+v)^2a^2+u^2b^2-(u+v)u(a^2+b^2-c^2)).$$

The triangle XYZ is equilateral if and only if

$$\begin{aligned} & (v+w)^2b^2+v^2c^2-(v+w)v(b^2+c^2-a^2) \\ = \; & (w+u)^2c^2+w^2a^2-(w+u)w(c^2+a^2-b^2) \\ = \; & (u+v)^2a^2+u^2b^2-(u+v)u(a^2+b^2-c^2). \end{aligned} \tag{1}$$

By taking differences of these expressions, we rewrite (1) as a system of two homogeneous quadratic equations in three unknowns:

$$\mathcal{C}_1: \qquad a^2v^2-b^2w^2-((b^2+c^2-a^2)w-(c^2+a^2-b^2)v)u=0,$$

and

$$\mathcal{C}_2: \qquad b^2w^2-c^2u^2-((c^2+a^2-b^2)u-(a^2+b^2-c^2)w)v=0.$$

[3]Clearly, a solution to this problem can be easily adapted to the case of "backward triangles", as we shall do at the end §4.

3. Intersections of two conics

3.1. *Representation by symmetric matrices.* We regard each of the two equations $\mathcal{C}_1$ and $\mathcal{C}_2$ as defining a conic in the plane of triangle ABC. The question is therefore finding the intersections of two conics. This is done by choosing a suitable combination of the two conics which degenerates into a pair of straight lines. To do this, we represent the two conics by symmetric 3×3 matrices

$$M_1 = \begin{pmatrix} 0 & c^2+a^2-b^2 & -(b^2+c^2-a^2) \\ c^2+a^2-b^2 & 2a^2 & 0 \\ -(b^2+c^2-a^2) & 0 & -2b^2 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} -2c^2 & -(c^2+a^2-b^2) & 0 \\ -(c^2+a^2-b^2) & 0 & a^2+b^2-c^2 \\ 0 & a^2+b^2-c^2 & 2b^2 \end{pmatrix},$$

and choose a combination $M_1 - tM_2$ whose determinant is zero.

3.2. *Reduction to the intersection with a pair of lines.* Consider, therefore, the matrix

$$M_1 - tM_2 = \begin{pmatrix} 2tc^2 & (1+t)(c^2+a^2-b^2) & -(b^2+c^2-a^2) \\ (1+t)(c^2+a^2-b^2) & 2a^2 & -t(a^2+b^2-c^2) \\ -(b^2+c^2-a^2) & -t(a^2+b^2-c^2) & -2(1+t)b^2 \end{pmatrix}. \tag{2}$$

Direct calculation shows that the matrix $M_1 - tM_2$ in (2) has determinant

$$-32\triangle^2((b^2-c^2)t^3 - (c^2+a^2-2b^2)t^2 - (c^2+a^2-2b^2)t - (a^2-b^2)),$$

where $\triangle$ denotes the area of triangle ABC. The polynomial factor further splits into

$$((b^2-c^2)t - (a^2-b^2))(t^2+t+1).$$

We obtain $M_1 - tM_2$ of determinant zero by choosing $t = \dfrac{a^2-b^2}{b^2-c^2}$. This matrix represents a quadratic form which splits into two linear forms. In fact, the combination $(b^2-c^2)\mathcal{C}_1 - (a^2-b^2)\mathcal{C}_2$ leads to

$$((a^2-b^2)u + (b^2-c^2)v + (c^2-a^2)w)(c^2u + a^2v + b^2w) = 0.$$

From this we see that the intersections of the two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ are the same as those of any one of them with the pairs of lines

$$\ell_1: \quad (a^2-b^2)u + (b^2-c^2)v + (c^2-a^2)w = 0,$$

and

$$\ell_2: \quad c^2u + a^2v + b^2w = 0.$$

3.3. *Intersections of $\mathcal{C}_1$ with ℓ_1 and ℓ_2.* There is an easy parametrization of points on the line ℓ_1. Since it clearly contains the points $(1:1:1)$ (the centroid) and $(c^2:a^2:b^2)$, every point on ℓ_1 is of the form $(c^2+t:a^2+t:b^2+t)$ for some real number t. Direct substitution shows that this point lies on the conic $\mathcal{C}_1$ if and only if

$$3t^2+3(a^2+b^2+c^2)t+(a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2)=0.$$

In other words,

$$\begin{aligned} t &= \frac{-(a^2+b^2+c^2)}{2} \pm \frac{1}{2\sqrt{3}}\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4} \\ &= \frac{-(a^2+b^2+c^2)}{2} \pm \frac{2\triangle}{\sqrt{3}}. \end{aligned}$$

From these, we conclude that the conic $\mathcal{C}_1$ and the line ℓ_1 intersect at the points

$$P^{\pm} = \left(\frac{a^2+b^2-c^2}{2} \pm \frac{2\triangle}{\sqrt{3}} : \frac{b^2+c^2-a^2}{2} \pm \frac{2\triangle}{\sqrt{3}} : \frac{c^2+a^2-b^2}{2} \pm \frac{2\triangle}{\sqrt{3}}\right). \quad (3)$$

The line ℓ_2, on the other hand, does not intersect the conic $\mathcal{C}_1$ at real points. [4] It follows that the conics $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect only at the two real points $P^{\pm}$ given in (3) above. [5]

4. Construction of the points $P^{\pm}$

The coordinates of $P^{\pm}$ in (3) can be rewritten as

$$\begin{aligned} P^{\pm} &= (ab\cos C \pm \frac{1}{\sqrt{3}}ab\sin C : bc\cos A \pm \frac{1}{\sqrt{3}}bc\sin A : ca\cos B \pm \frac{1}{\sqrt{3}}ca\sin B) \\ &= (\frac{2ab}{\sqrt{3}}\sin(C \pm \frac{\pi}{3}) : \frac{2bc}{\sqrt{3}}\sin(A \pm \frac{\pi}{3}) : \frac{2ca}{\sqrt{3}}\sin(B \pm \frac{\pi}{3})) \\ &\sim (\frac{1}{c}\cdot\sin(C \pm \frac{\pi}{3}) : \frac{1}{a}\cdot\sin(A \pm \frac{\pi}{3}) : \frac{1}{b}\cdot\sin(B \pm \frac{\pi}{3})). \end{aligned}$$

A simple interpretation of these expressions, via the notion of Brocardian points [5], leads to an easy construction of the points $P^{\pm}$.

Definition. The Brocardian points of a point $Q=(x:y:z)$ are the two points

$$Q_{\rightarrow} = (\frac{1}{z}:\frac{1}{x}:\frac{1}{y}) \quad \text{and} \quad Q_{\leftarrow} = (\frac{1}{y}:\frac{1}{z}:\frac{1}{x}).$$

We distinguish between these two by calling $Q_{\rightarrow}$ the *forward* Brocardian point and $Q_{\leftarrow}$ the *backward* one, and justify these definitions by giving a simple construction.

Proposition 1. *Given a point Q, construct through the traces A_Q, B_Q, C_Q forward parallelians to AB, BC, CA, intersecting CA, AB, BC at Y, Z and X respectively. The lines AX, BY, CZ intersect at $Q_{\rightarrow}$. On the other hand, if the*

[4] Substitution of $u=\frac{-(a^2v+b^2w)}{c^2}$ into $(\mathcal{C}_1)$ gives $a^2v^2+(a^2+b^2-c^2)vw+b^2w^2=0$, which has no real roots since $(a^2+b^2-c^2)^2-4a^2b^2=a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2=-16\triangle^2<0$.

[5] See Figure 9 in the Appendix for an illustration of the conics and their intersections.

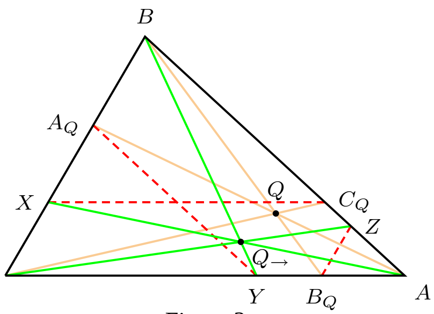

Figure 3a

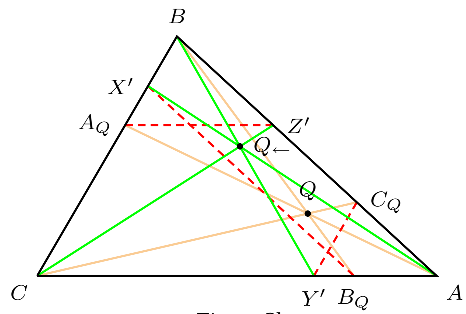

Figure 3b

backward parallelians through A_Q, B_Q, C_Q to CA, AB, BC, intersect AB, BC, CA at Z', X', Y' respectively, then, the lines AX', BY', CZ' intersect at $Q_{\leftarrow}$.

Proof. Suppose $Q = (x : y : z)$ in homogeneous barycentric coordinates. In Figure 3a, $BX : XC = BC_Q : C_QA = x : y$ since $C_Q = (x : y : 0)$. It follows that $X = (0 : y : x) \sim (0 : \frac{1}{x} : \frac{1}{y})$. Similarly, $Y = (\frac{1}{z} : 0 : \frac{1}{x})$ and $Z = (\frac{1}{z} : \frac{1}{y} : 0)$. From these, the lines AX, BY, and CZ intersect at the point $(\frac{1}{z} : \frac{1}{x} : \frac{1}{y})$, which we denote by $Q_{\rightarrow}$. The proof for $Q_{\leftarrow}$ is similar; see Figure 3b. □

Examples. If $Q = K = (a^2 : b^2 : c^2)$, the symmedian point, the Brocardian points $K_{\rightarrow}$ and $K_{\leftarrow}$ are the Brocard points [6] satisfying

$$\angle K_{\rightarrow}BA = \angle K_{\rightarrow}CB = \angle K_{\rightarrow}AC = \omega = \angle K_{\leftarrow}CA = \angle K_{\leftarrow}AB = \angle K_{\leftarrow}BC,$$

where ω is the Brocard angle given by $\cot\omega = \cot A + \cot B + \cot C$. These points lie on the circle with OK as diameter, O being the circumcenter of triangle ABC. See Figure 4.

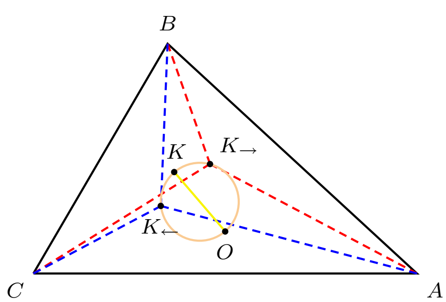

Figure 4

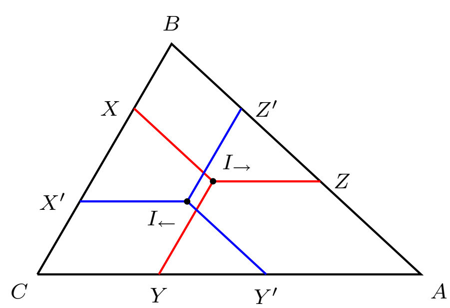

Figure 5

On the other hand, the Brocardian points of the incenter $I = (a : b : c)$ are the Jerabek points $I_{\rightarrow}$ and $I_{\leftarrow}$ mentioned in §1. See Figure 5.

[6]These points are traditionally labelled Ω (for $K_{\rightarrow}$) and Ω' (for $K_{\leftarrow}$) respectively. See [1, pp.274–280.]

Proposition 2. *The points $P^{\pm}$ are the* forward *Brocardian points of the Fermat points* [7]

$$F^{\pm} = \left(\frac{a}{\sin(A \pm \frac{\pi}{3})} : \frac{b}{\sin(B \pm \frac{\pi}{3})} : \frac{c}{\sin(C \pm \frac{\pi}{3})} \right).$$

By reversing the orientation of the parallelians, we obtain two more equilateral triangles, corresponding to the *backward* Brocardian points of the same two Fermat points $F^{\pm}$.

Theorem 3. *There are exactly four equilateral triangles intercepted by oriented parallelians, corresponding to the four points $F^{\pm}_{\rightarrow}$ and $F^{\pm}_{\leftarrow}$.*

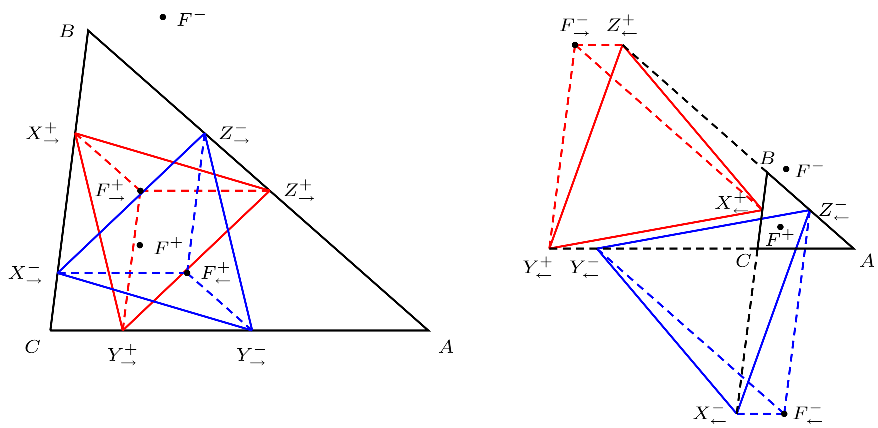

Figure 6a Figure 6b

5. Some further results

The two equilateral triangles $X^+_{\rightarrow}Y^+_{\rightarrow}Z^+_{\rightarrow}$ and $X^+_{\leftarrow}Y^+_{\leftarrow}Z^+_{\leftarrow}$ corresponding to the Fermat point F^+ are congruent; so are $X^-_{\rightarrow}Y^-_{\rightarrow}Z^-_{\rightarrow}$ and $X^-_{\leftarrow}Y^-_{\leftarrow}Z^-_{\leftarrow}$. In fact, they are homothetic at the common midpoint of the segments $X^+_{\rightarrow}Y^+_{\leftarrow}$, $Y^+_{\rightarrow}Z^+_{\leftarrow}$, and $Z^+_{\rightarrow}X^+_{\leftarrow}$, and their sides are parallel to the corresponding cevians of the Fermat point. This is indeed a special case of the following proposition.

Proposition 4. *For every point Q not on the side lines of triangle ABC, the triangle intercepted by the forward parallelians through $Q_{\rightarrow}$ and that by the backward parallelians through $Q_{\leftarrow}$ are homothetic at $(u(v+w) : v(w+u) : w(u+v))$, with ratio* $1 : -1$. *Their corresponding sides are parallel to the cevians AQ, BQ, and CQ respectively.*

[7] The Fermat point F^+ (respectively F^-) of triangle ABC is the intersection of the lines AX, BY, CZ, where XBC, YCA and ZAB are equilateral triangles constructed externally (respectively internally) on the sides BC, CA, AB of the triangle. This is the point X_{13} (respectively X_{14}) in [3].

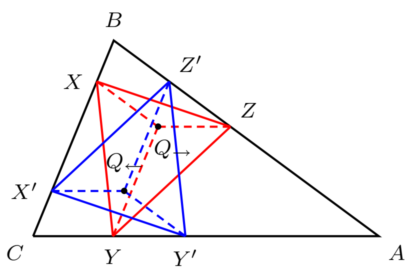

Figure 7

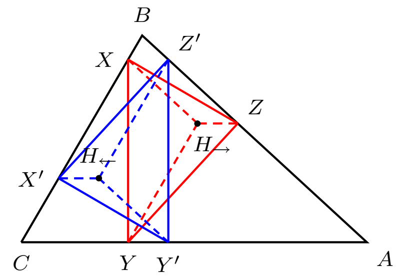

Figure 8

These two triangles are the only inscribed triangles whose sides are parallel to the respective cevians of Q. See Figure 7. They are the Bottema triangles in [4]. Applying this to the orthocenter H, we obtain the two congruent inscribed triangles whose sides are perpendicular to the sides of ABC (Figure 8).

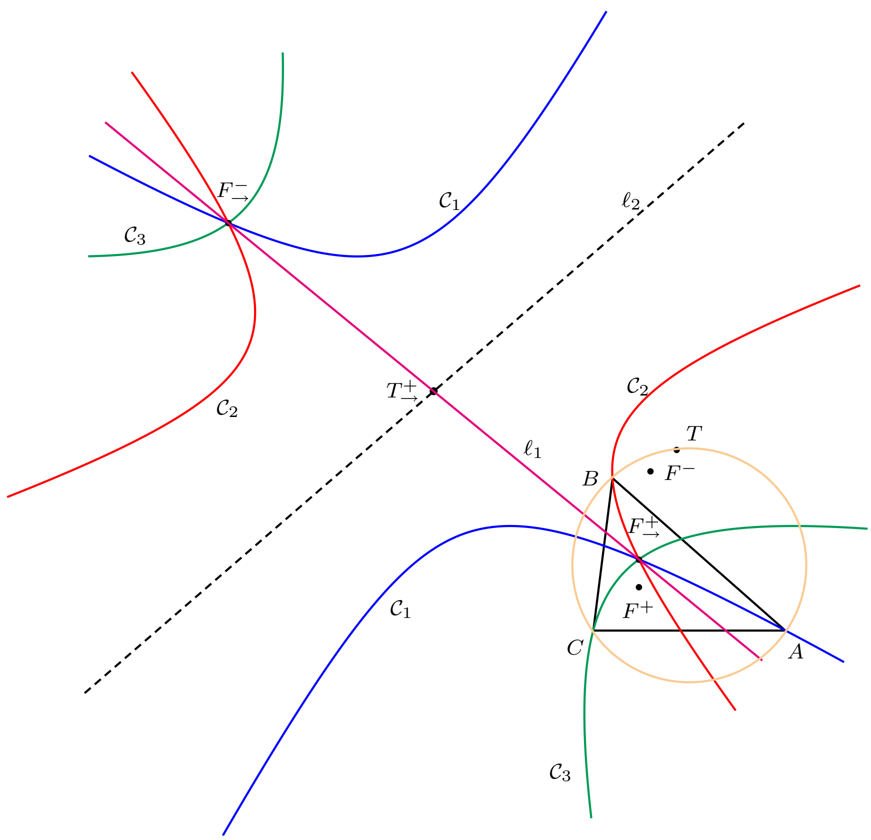

Figure 9

Appendix

Figure 9 illustrates the intersections of the two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ in §2, along with a third conic $\mathcal{C}_3$ which results from the difference of the first two expressions in (1), namely,

$$\mathcal{C}_3 : \qquad c^2u^2 - a^2v^2 - ((a^2 + b^2 - c^2)v - (b^2 + c^2 - a^2)u)w = 0.$$

These three conics are all hyperbolas, and have a common center $T^+_{\rightarrow}$, which is the forward Brocardian point of the Tarry point T, and is the midpoint between the common points $F^+_{\rightarrow}$ and $F^-_{\rightarrow}$. In other words, $F^+_{\rightarrow}F^-_{\rightarrow}$ is a common diameter of the three hyperbolas. We remark that the Tarry point T is the point X_{98} of [3], and is the fourth intersection of the Kiepert hyperbola and the circumcircle of triangle ABC. The fact that ℓ_1 and ℓ_2 intersect at $T_{\rightarrow}$ follows from the observation that these lines are respectively the loci of the forward Brocardians of points on the Kiepert hyperbola $\frac{b^2-c^2}{u} + \frac{c^2-a^2}{v} + \frac{a^2-b^2}{w} = 0$ and the circumcircle $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$ respectively. The tangents to the hyperbolas $\mathcal{C}_1$ at A, $\mathcal{C}_2$ at B, and $\mathcal{C}_3$ at C intersect at the point $H_{\rightarrow}$, the forward Brocardian of the orthocenter.

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The Isogonal Tripolar Conic

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Abstract. In trilinear coordinates with respect to a given triangle ABC, we define the isogonal tripolar of a point $P(p, q, r)$ to be the line p: $p\alpha + q\beta + r\gamma = 0$. We construct a unique conic Φ, called the isogonal tripolar conic, with respect to which p is the polar of P for all P. Although the conic is imaginary, it has a real center and real axes coinciding with the center and axes of the real orthic inconic. Since ABC is self-conjugate with respect to Φ, the imaginary conic is harmonically related to every circumconic and inconic of ABC. In particular, Φ is the reciprocal conic of the circumcircle and Steiner's inscribed ellipse. We also construct an analogous isotomic tripolar conic Ψ by working with barycentric coordinates.

1. Trilinear coordinates

For any point P in the plane ABC, we can locate the right projections of P on the sides of triangle ABC at P_1, P_2, P_3 and measure the distances PP_1, PP_2 and PP_3. If the distances are directed, i.e., measured positively in the direction of each vertex to the opposite side, we can identify the distances $\underline{\alpha} = \overrightarrow{PP_1}$, $\underline{\beta} = \overrightarrow{PP_2}$, $\underline{\gamma} = \overrightarrow{PP_3}$ (Figure 1) such that

$$a\underline{\alpha} + b\underline{\beta} + c\underline{\gamma} = 2\triangle$$

where a, b, c, $\triangle$ are the side lengths and area of triangle ABC. This areal equation for all positions of P means that the ratio of the distances is sufficient to define the *trilinear coordinates* of $P(\alpha, \beta, \gamma)$ where

$$\alpha : \beta : \gamma = \underline{\alpha} : \underline{\beta} : \underline{\gamma}.$$

For example, if we consider the coordinates of the vertex A, the incenter I, and the first excenter I_1, we have absolute $\underline{\alpha\beta\gamma}-$coordinates : $A(h_1, 0, 0)$, $I(r, r, r)$, $I_1(-r_1, r_1, r_1)$, where h_1, r, r_1 are respectively the altitude from A, the inradius and the first exradius of triangle ABC. It follows that the trilinear $\alpha\beta\gamma-$coordinates in their simplest form are $A(1, 0, 0)$, $I(1, 1, 1)$, $I_1(-1, 1, 1)$. Let R be the circumradius, and h_1, h_2, h_3 the altitudes, so that $ah_1 = bh_2 = ch_3 = 2\triangle$. The absolute coordinates of the circumcenter O, the orthocenter H, and the median point [1] G are $O(R\cos A, R\cos B, R\cos C)$, $H(2R\cos B\cos C, 2R\cos C\cos A,$

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[1]The median point is also known as the centroid.

$2R\cos A\cos B$), and $G(\frac{h_1}{3}, \frac{h_2}{3}, \frac{h_3}{3})$, giving trilinear coordinates: $O(\cos A, \cos B, \cos C)$, $H(\sec A, \sec B, \sec C)$, and $G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$.

2. Isogonal conjugate

For any position of P we can define its isogonal conjugate Q such that the directed angles $(AC, AQ) = (AP, AB) = \theta_1$, $(BA, BP) = (BQ, BC) = \theta_2$, $(CB, CP) = (CQ, CA) = \theta_3$ as shown in Figure 1. If the absolute coordinates of Q are $\underline{\alpha}' = \overrightarrow{QQ_1}$, $\underline{\beta}' = \overrightarrow{QQ_2}$, $\underline{\gamma}' = \overrightarrow{QQ_3}$, then

$$\frac{PP_2}{PP_3} = \frac{AP\sin(A-\theta_1)}{AP\sin\theta_1} \quad \text{and} \quad \frac{QQ_2}{QQ_3} = \frac{AQ\sin\theta_1}{AQ\sin(A-\theta_1)}$$

so that that $PP_2 \cdot QQ_2 = PP_3 \cdot QQ_3$, implying $\underline{\beta\beta'} = \underline{\gamma\gamma'}$. Similarly, $\underline{\alpha\alpha'} = \underline{\beta\beta'}$ and $\underline{\gamma\gamma'} = \underline{\alpha\alpha'}$, so that $\underline{\alpha\alpha'} = \underline{\beta\beta'} = \underline{\gamma\gamma'}$. Consequently, $\alpha\alpha' = \beta\beta' = \gamma\gamma'$.

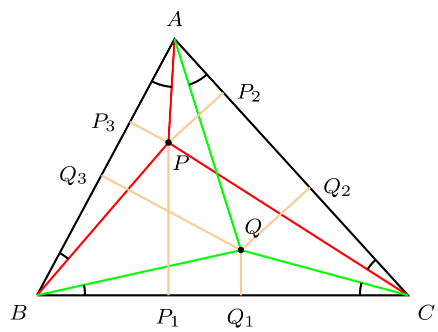

Figure 1

Hence, Q is the triangular inverse of P; i.e., if P has coordinates (α, β, γ), then its isogonal conjugate Q has coordinates $(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma})$. It will be convenient to use the notation $\hat{P}$ for the isogonal conjugate of P. We can immediately note that $O(\cos A, \cos B, \cos C)$ and $H(\sec A, \sec B, \sec C)$ are isogonal conjugates. On the other hand, the symmedian point K, being the isogonal conjugate of $G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$, has coordinates $K(a, b, c)$, i.e., the distances from K to the sides of triangle ABC are proportional to the side lengths of ABC.

3. Tripolar

We can now define the *line coordinates* (l, m, n) of a given line ℓ in the plane ABC, such that any point P with coordinates (α, β, γ) lying on ℓ must satisfy the linear equation $l\alpha + m\beta + n\gamma = 0$. In particular, the side lines BC, CA, AB have line coordinates (1,0,0), (0,1,0), (0,0,1), with equations $\alpha = 0$, $\beta = 0$, $\gamma = 0$ respectively.

A specific line that may be defined is the harmonic or trilinear polar of Q with respect to ABC, which will be called the *tripolar* of Q.

In Figure 2, $L'M'N'$ is the tripolar of Q, where LMN is the diagonal triangle of the quadrangle $ABCQ$; and $L'M'N'$ is the axis of perspective of the triangles ABC and LMN. Any line through Q meeting two sides of ABC at U, V and meeting $L'M'N'$ at W creates an harmonic range $(UV;QW)$. To find the line coordinates of $L'M'N'$ when Q has coordinates (p',q',r'), we note $L = AQ \cap BC$ has coordinates $(0,q',r')$, since $\frac{LL_2}{LL_3} = \frac{QQ_2}{QQ_3}$. Similarly for $M(p',0,r')$ and $N(p',q',0)$. Hence the equation of the line MN is

$$\frac{\alpha}{p'} = \frac{\beta}{q'} + \frac{\gamma}{r'} \tag{1}$$

since the equation is satisfied when the coordinates of M or N are substituted for α, β, γ in (1). So the coordinates of $L' = MN \cap BC$ are $L'(0,q',-r')$. Similarly for $M'(p',0,-r')$ and $N'(p',-q',0)$, leading to the equation of the line $L'M'N'$:

$$\frac{\alpha}{p'} + \frac{\beta}{q'} + \frac{\gamma}{r'} = 0. \tag{2}$$

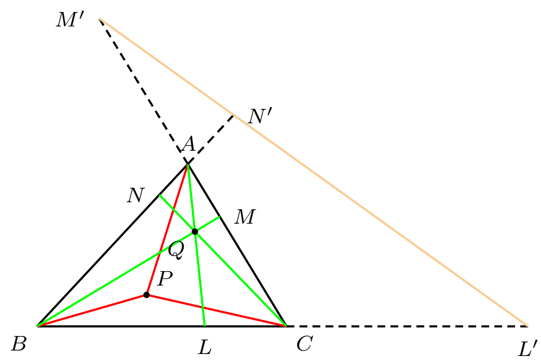

Figure 2

Now from the previous analysis, if $P(p,q,r)$ and $Q(p',q',r')$ are isogonal conjugates then $pp' = qq' = rr'$ so that from (2) the equation of the line $L'M'N'$ is $p\alpha + q\beta + r\gamma = 0$. In other words, the line coordinates of the tripolar of Q are the trilinear coordinates of P. We can then define the *isogonal tripolar* of $P(p,q,r)$ as the line $L'M'N'$ with equation $p\alpha + q\beta + r\gamma = 0$.

For example, for the vertices $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, the isogonal tripolars are the corresponding sides BC ($\alpha = 0$), CA ($\beta = 0$), AB ($\gamma = 0$). For the notable points $O(\cos A, \cos B, \cos C)$, $I(1,1,1)$, $G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$, and $K(a,b,c)$, the corresponding isogonal tripolars are

$$\begin{aligned}
\mathsf{o}: &\quad \alpha\cos A + \beta\cos B + \gamma\cos C = 0,\\
\mathsf{i}: &\quad \alpha + \beta + \gamma = 0,\\
\mathsf{g}: &\quad \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0,\\
\mathsf{k}: &\quad a\alpha + b\beta + c\gamma = 0.
\end{aligned}$$

Here, o, i, g, k are respectively the orthic axis, the anti-orthic axis, Lemoine's line, and the line at infinity, i.e., the tripolars of H, I, K, and G. Clark Kimberling has assembled a catalogue of notable points and notable lines with their coordinates in a contemporary publication [3].

4. The isogonal tripolar conic Φ

Now consider a point $P_2(p_2, q_2, r_2)$ on the isogonal tripolar of $P_1(p_1, q_1, r_1)$, i.e., the line

$$\mathsf{p}_1 : \qquad p_1\alpha + q_1\beta + r_1\gamma = 0.$$

Obviously P_1 lies on the isogonal tripolar of P_2 since the equality $p_1p_2 + q_1q_2 + r_1r_2 = 0$ is the condition for both incidences. Furthermore, the line P_1P_2 has equation

$$(q_1r_2 - q_2r_1)\alpha + (r_1p_2 - r_2p_1)\beta + (p_1q_2 - p_2q_1)\gamma = 0,$$

while the point $\mathsf{p}_1 \cap \mathsf{p}_2$ has coordinates $(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1)$. It follows that $\mathsf{t} = P_1P_2$ is the isogonal tripolar of $T = \mathsf{p}_1 \cap \mathsf{p}_2$. These isogonal tripolars immediately suggest the classical polar reciprocal relationships of a geometrical conic. In fact, the triangle P_1P_2T has the analogous properties of a self-conjugate triangle with respect to a conic, since each side of triangle P_1P_2T is the isogonal tripolar of the opposite vertex. This means that a significant conic could be drawn self-polar to triangle P_1P_2T. But an infinite number of conics can be drawn self-polar to a given triangle; and a further point with its polar are required to identify a unique conic [5]. We can select an arbitrary point P_3 with its isogonal tripolar p_3 for this purpose. Now the equation to the general conic in trilinear coordinates is [4]

$$\mathcal{S} : \qquad l\alpha^2 + m\beta^2 + n\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0$$

and the polar of $P_1(p_1, q_1, r_1)$ with respect to $\mathcal{S}$ is

$$\mathsf{s}_1 : \quad (lp_1 + hq_1 + gr_1)\alpha + (hp_1 + mq_1 + fr_1)\beta + (gp_1 + fq_1 + nr_1)\gamma = 0.$$

By definition we propose that for $i = 1, 2, 3$, the lines p_i and s_i coincide, so that the line coordinates of p_i and s_i must be proportional; i.e.,

$$\frac{lp_i + hq_i + gr_i}{p_i} = \frac{hp_i + mq_i + fr_i}{q_i} = \frac{gp_i + fq_i + nr_i}{r_i}.$$

Solving these three sets of simultaneous equations, after some manipulation we find that $l = m = n$ and $f = g = h = 0$, so that the equation of the required conic is $\alpha^2 + \beta^2 + \gamma^2 = 0$. This we designate the *isogonal tripolar conic* Φ.

From the analysis Φ is the unique conic which reciprocates the points P_1, P_2, P_3 to the lines p_1, p_2, p_3. But any set of points P_i, P_j, P_k with the corresponding isogonal tripolars p_i, p_j, p_k could have been chosen, leading to the same equation for the reciprocal conic. We conclude that *the isogonal tripolar of any point P in the plane ABC is the polar of P with respect to* Φ. Any triangle $P_iP_jT_k$ with $T_k = \mathsf{p}_i \cap \mathsf{p}_j$ is self-conjugate with respect to Φ. In particular, the basic triangle ABC is self-conjugate with respect to Φ, since each side is the isogonal tripolar of its opposite vertex.

From the form of the equation $\alpha^2 + \beta^2 + \gamma^2 = 0$, the isogonal tripolar conic Φ is obviously an imaginary conic. So the conic exists on the complex projective plane. However, it will be shown that the imaginary conic has a real center and real axes; and that Φ is the reciprocal conic of a pair of notable real conics.

5. The center of Φ

To find the center of Φ, we recall that the polar of the center of a conic with respect to that conic is the line at infinty ℓ_∞ which we have already identified as $\mathsf{k}:\ a\alpha+b\beta+c\gamma=0$, the isogonal tripolar of the symmedian point $K(a,b,c)$. So the center of Φ and the center of its director circle are situated at K. From Gaskin's Theorem, the director circle of a conic is orthogonal to the circumcircle of every self-conjugate triangle. Choosing the basic triangle ABC as the self-conjugate triangle with circumcenter O and circumradius R, we have $\rho^2+R^2=OK^2$, where ρ is the director radius of Φ. But it is known [2] that $R^2-OK^2=3\mu^2$, where $\mu=\dfrac{abc}{a^2+b^2+c^2}$ is the radius of the cosine circle of ABC. From this,

$$\rho=i\sqrt{3}\mu=i\sqrt{3}\cdot\frac{abc}{a^2+b^2+c^2}.$$

6. Some lemmas

To locate the axes of Φ, some preliminary results are required which can be found in the literature [1] or obtained by analysis.

Lemma 1. *If a diameter of the circumcircle of ABC meets the circumcircle at X, Y, then the isogonal conjugates of X and Y (designated $\hat{X}$, $\hat{Y}$) lie on the line at infinity; and for arbitrary P, the line $P\hat{X}$ and $P\hat{Y}$ are perpendicular.*

Here is a special case.

Lemma 2. *If the chosen diameter is the Euler line OGH, then $\hat{X}\hat{Y}$ lie on the asymptotes of Jerabek's hyperbola $\mathcal{J}$, which is the locus of the isogonal conjugate of a variable point on the Euler line OGH (Figure 3).*

Lemma 3. *If the axes of a conic $\mathcal{S}$ with center Q meets ℓ_∞ at E, F, then the polars of E, F with respect to $\mathcal{S}$ are the perpendicular lines QF, QE; and E, F are the only points on ℓ_∞ with this property.*

Lemma 4. *If UGV is a chord of the circumcircle Γ through G meeting Γ at U, V, then the tripolar of U is the line $K\hat{V}$ passing through the symmedian point K and the isogonal conjugate of V.*

7. The axes of Φ

To proceed with the location of the axes of Φ, we start with the conditions of Lemma 2 where X, Y are the common points of OGH and Γ.

From Lemma 4, since XGY are collinear, the tripolars of X, Y are respectively $K\hat{Y}$, $K\hat{X}$, which are perpendicular from Lemma 1. Now from earlier definitions, the tripolars of X, Y are the isogonal tripolars of $\hat{X}$, $\hat{Y}$, so that the isogonal tripolars of $\hat{X}$, $\hat{Y}$ are the perpendiculars $K\hat{Y}$, $K\hat{X}$ through the center of Φ. Since $\hat{X}\hat{Y}$ lie on ℓ_∞, $K\hat{X}$, $K\hat{Y}$ must be the axes of Φ from Lemma 3. And these axes are parallel to the asymptotes of $\mathcal{J}$ from Lemma 2.

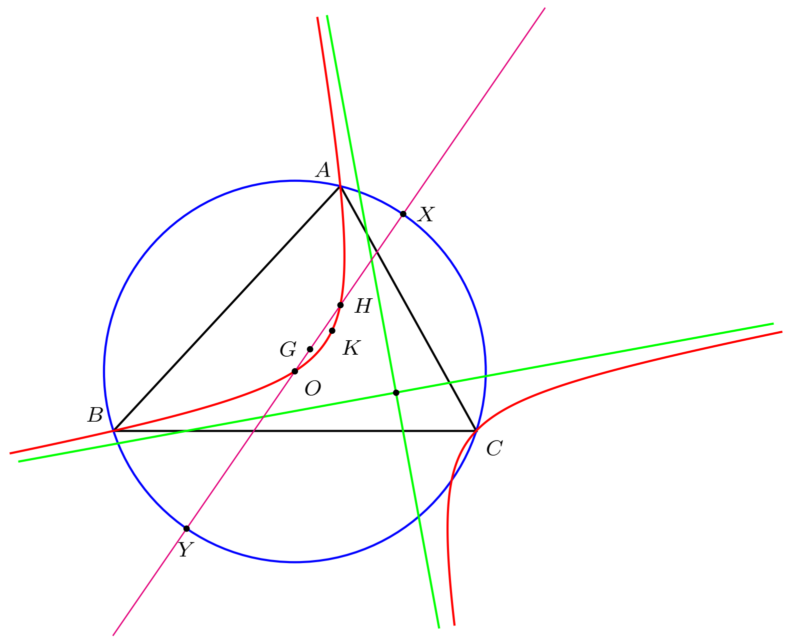

Figure 3. The Jerabek hyperbola

Now it is known [1] that the asymptotes of $\mathcal{J}$ are parallel to the axes of the orthic inconic (Figure 4). The orthic triangle has its vertices at H_1, H_2, H_3 the feet of the altitudes AH, BH, CH. The orthic inconic has its center at K and touches the sides of triangle ABC at the vertices of the orthic triangle. So the axes of the imaginary conic Φ coincide with the axes of the real orthic inconic.

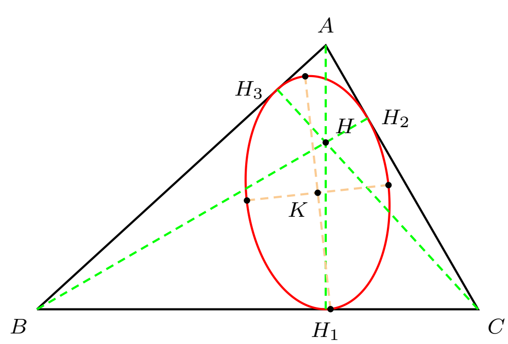

Figure 4. The orthic inconic

8. Φ as a reciprocal conic of two real conics

Although the conic Φ is imaginary, every real point P has a polar p with respect to Φ. In particular if P lies on the circumcircle Γ, its polar p touches Steiner's inscribed ellipse σ with center G. This tangency arises from the known theorem

[1] that the tripolar of any point on ℓ_∞ touches σ. From Lemma 1 this tripolar is the isogonal tripolar of the corresponding point of Γ. Now the basic triangle ABC (which is self-conjugate with respect to Φ) is inscribed in Γ and tangent to σ, which touches the sides of ABC at their midpoints (Figure 5). In the language of classical geometrical conics, the isogonal tripolar conic Φ is harmonically inscribed to Γ and harmoncially circumscribed to σ. From the tangency described above, Φ is the reciprocal conic to $\Gamma \rightleftharpoons \sigma$. Furthermore, since ABC is self-conjugate with respect to Φ, an infinite number of triangles $P_iP_jP_k$ can be drawn with its vertices inscribed in Γ, its sides touching σ, and self-conjugate with respect to Φ. Since Φ is the reciprocal conic of $\Gamma \rightleftharpoons \sigma$, for any point on σ, its polar with respect to Φ (i.e., its isogonal tripolar) touches Γ. In particular, if the tangent p_i touches σ at $T_i(u_i, v_i, w_i)$ for $i = 1, 2, 3$, then t_i, the isogonal tripolar of T_i, touches Γ at P_i (Figure 5).

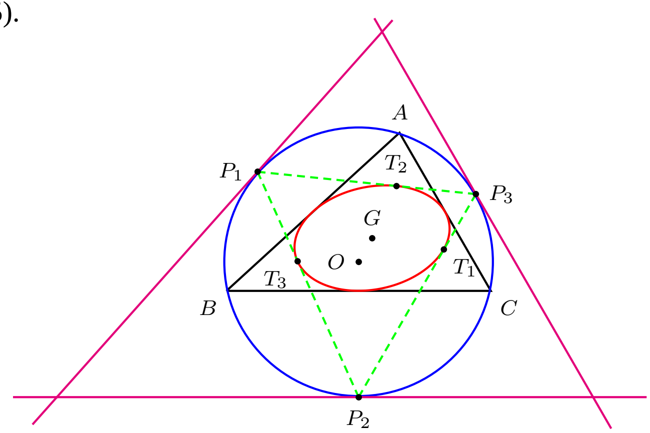

Figure 5

Now, the equation to the circumcircle Γ is $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$. The equation of the tangent to Γ at $P_i(p_i, q_i, r_i)$ is

$$(cq_i + br_i)\alpha + (ar_i + cp_i)\beta + (bp_i + aq_i)\gamma = 0.$$

If this tangent coincides with t_i, the isogonal tripolar of T_i, then the coordinates of T_i are

$$u_i = cq_i + br_i, \qquad v_i = ar_i + cp_i, \qquad w_i = bp_i + aq_i. \tag{3}$$

So, if t_i is the tangent at $P_i(p_i, q_i, r_i)$ to Γ, and simultaneously the isogonal tripolar of T_i, then the coordinates of T_i are as shown in (3). But this relationship can be generalized for any P_i in the plane of ABC, since the equation to the polar of P_i with respect to Γ is identical to the equation to the tangent at P_i (in the particular case that P_i lies on Γ). In other words, the isogonal tripolar of $T_i(u_i, v_i, w_i)$ with the coordinates shown at (3) is the polar of $P_i(p_i, q_i, r_i)$ with respect to Γ, for any P_i, T_i in the plane of ABC.

9. The isotomic tripolar conic Ψ

To find an alternative description of the transformation $P \mapsto T$, we define the *isotomic conjugate* and the *isotomic tripolar*.

In the foregoing discussion we have used trilinear coordinates (α, β, γ) to define the point P and its isogonal tripolar p. However, we could just as well use *barycentric* (areal) coordinates (x, y, z) to define P. With $\underline{x}$ = area (PBC), $\underline{y}$ = area (PCA), $\underline{z}$ = area (PAB), and $\underline{x} + \underline{y} + \underline{z}$ = area (ABC), comparing with trilinear coordinates of P we have

$$a\underline{\alpha} = 2\underline{x}, \qquad b\underline{\beta} = 2\underline{y}, \qquad c\underline{\gamma} = 2\underline{z}.$$

Using directed areas, i.e., positive area (PBC) when the perpendicular distance PP_1 is positive, the ratio of the areas is sufficient to define the (x, y, z) coordinates of P, with $x : \underline{x} = y : \underline{y} = z : \underline{z}$. The absolute coordinates $(\underline{x}, \underline{y}, \underline{z})$ can then be found from the areal coordinates (x, y, z) using the areal identity $\underline{x} + \underline{y} + \underline{z} = \triangle$. For example, the barycentric coordinates of A, I, I_1, O, H, G, K are $A(1, 0, 0)$, $I(a, b, c)$, $I_1(-a, b, c)$, $O(a\cos A, b\cos B, c\cos C)$, $H(a\sec A, b\sec B, c\sec C)$, $G(1, 1, 1)$, $K(a^2, b^2, c^2)$ respectively.

In this barycentric system we can identity the coordinates (x', y', z') of the isotomic conjugate $\overline{P}$ of P as shown in Figure 6, where $\overrightarrow{BL}=\overrightarrow{L'C}$, $\overrightarrow{CM}=\overrightarrow{M'A}$, $\overrightarrow{AN}=\overrightarrow{N'B}$. We find by the same procedure that $xx' = yy' = zz'$ for P, $\overline{P}$, so that the areal coordinates of $\overline{P}$ are $(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$, explaining the alternative description that $\overline{P}$ is the triangular reciprocal of P.

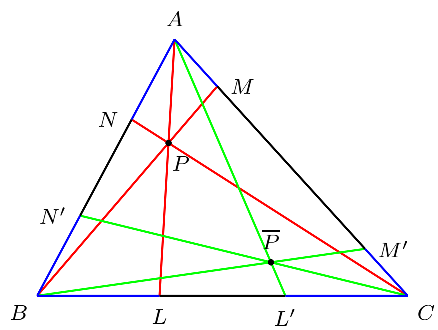

Figure 6

Following the same argument as heretofore, we can define the *isotomic tripolar* of $P(p, q, r)$ as the tripolar of $\overline{P}$ with barycentric equation $px + qy + rz = 0$, and then identify the imaginary *isotomic tripolar conic* Ψ with equation $x^2 + y^2 + z^2 = 0$. The center of Ψ is the median point $G(1, 1, 1)$ since the isotomic tripolar of G is the ℓ_∞ with barycentric equation $x + y + z = 0$. By analogous procedure we can find the axes of Ψ which coincide with the real axes of Steiner's inscribed ellipse σ.

Again, we find that the basic triangle ABC is self conjugate with respect to Ψ, and from Gaskin's Theorem, the radius of the imaginary director circle d is given by $d^2 + R^2 = OG^2$. From this, $d^2 = OG^2 - R^2 = -\frac{1}{9}(a^2 + b^2 + c^2)$, giving

$$d = \frac{i}{3}\sqrt{a^2 + b^2 + c^2}.$$

In the analogous case to Figure 5, we find that in Figure 7, if P is a variable point on Steiner's circum-ellipse θ (with center G), then the isotomic tripolar of P is tangent to σ, and Ψ is the reciprocal conic of $\theta \rightleftharpoons \sigma$. Generalizing this relationship as before, we find that the polar of P (pqr) with respect to θ is the isotomic tripolar of T with barycentric coordinates $(q+r, r+p, p+q)$. Furthermore, we can describe the transformation $P \mapsto T$ in vector geometry as $\overrightarrow{PG} = 2\ \overrightarrow{GT}$, or more succinctly that T is the complement of P [2]. The inverse transformation $T \mapsto P$ is given by $\overrightarrow{TG} = \frac{1}{2}\ \overrightarrow{GP}$, where P is the anticomplement of T. So the transformation of point T to the isotomic tripolar t can be described as

$$\begin{aligned} \mathsf{t} &= \text{isotomic tripole of } T \\ &= \text{polar of } T \text{ with respect to } \Psi \\ &= \text{polar of } P \text{ with respect to } \theta, \end{aligned}$$

where $\overrightarrow{PG} = 2\ \overrightarrow{GT}$. In other words, the transformation of a point $P(p, q, r)$ to its isotomic tripolar $px + qy + rz = 0$ is a dilatation $(G, -2)$ followed by polar reciprocation in θ, Steiner's circum-ellipse.

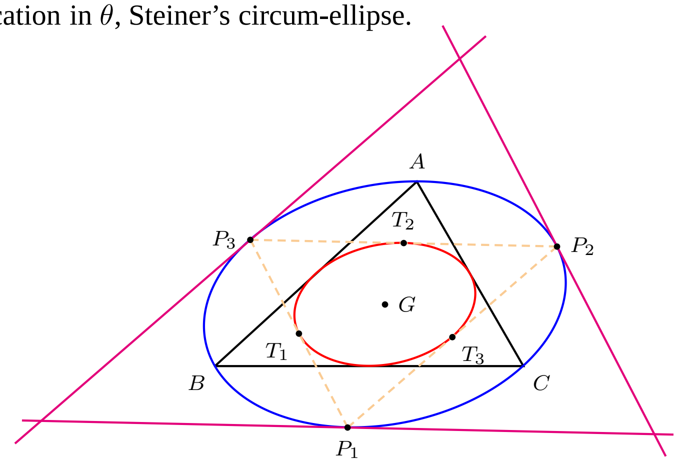

Figure 7

To find the corresponding transformation of a point to its isogonal tripolar, we recall that the polar of $P(p, q, r)$ with respect to Γ is the isogonal tripolar of T, where T has trilinear coordinates $(cq + br, ar + cp, bp + aq)$ from (3). Now, $\overline{\hat{P}}$, the isotomic conjugate of the isogonal conjugate of P, has coordinates $(\frac{p}{a^2}, \frac{q}{b^2}, \frac{r}{c^2})$ [3]. Putting $R = \overline{\hat{P}}$, the complement of R has coordinates $(cq + br, ar + cp, bp + aq)$, which are the coordinates of T. So the transformation of point T to its isogonal tripolar t can be described as

$$\begin{aligned} \mathsf{t} &= \text{isogonal tripolar of } T \\ &= \text{polar of } T \text{ with respect to } \Phi \\ &= \text{polar of } P \text{ with respect to } \Gamma, \end{aligned}$$

where $\overrightarrow{RG} = 2\ \overrightarrow{GT}$, and $P = \hat{\overline{R}}$, the isogonal conjugate of the isotomic conjugate of R. In other words, the transformation of a point P with trilinear coordinates

(p, q, r) to its isogonal tripolar $(p\alpha + q\beta + r\gamma = 0)$ is a dilatation $(G, -2)$, followed by isotomic transformation, then isogonal transformation, and finally polar reciprocation in the circumcircle Γ.

We conclude with the remark that the two well known systems of homogeneous coordinates, viz. trilinear (α, β, γ) and barycentric (x, y, z), generate two analogous imaginary conics Φ and Ψ, whose real centers and real axes coincide with the corresponding elements of notable real inconics of the triangle. In each case, the imaginary conic reciprocates an arbitrary point P to the corresponding line p, whose line coordinates are identical to the point coordinates of P. And in each case, reciprocation in the imaginary conic is the equivalent of well known transformations of the real plane.

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The Malfatti Problem

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Abstract. A solution is given of Steiner's variation of the classical Malfatti problem in which the triangle is replaced by three circles mutually tangent to each other externally. The two circles tangent to the three given ones, presently known as Soddy's circles, are encountered as well.

In this well known problem, construction is sought for three circles C_1', C_2' and C_3', tangent to each other pairwise, and of which C_1' is tangent to the sides A_1A_2 and A_1A_3 of a given triangle $A_1A_2A_3$, while C_2' is tangent to A_2A_3 and A_2A_1 and C_3' to A_3A_1 and A_3A_2. The problem was posed by Malfatti in 1803 and solved by him with the help of an algebraic analysis. Very well known is the extraordinarily elegant geometric solution that Steiner announced without proof in 1826. This solution, together with the proof Hart gave in 1857, one can find in various textbooks.[1] Steiner has also considered extensions of the problem and given solutions. The first is the one where the lines A_2A_3, A_3A_1 and A_1A_2 are replaced by circles. Further generalizations concern the figures of three circles on a sphere, and of three conic sections on a quadric surface. In the nineteenth century many mathematicians have worked on this problem. Among these were Cayley (1852) [2], Schellbach (who in 1853 published a very nice goniometric solution), and Clebsch (who in 1857 extended Schellbach's solution to three conic sections on a quadric surface, and for that he made use of elliptic functions). If one allows in Malfatti's original problem also escribed and internally tangent circles, then there are a total of 32 (real) solutions. One can find all these solutions mentioned by Pampuch (1904).[3] The generalizations mentioned above even have, as appears from investigation by Clebsch, 64 solutions.

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Translation by Floor van Lamoen from the Dutch original of O. Bottema, Het vraagstuk van Malfatti, *Euclides*, 25 (1949-50) 144–149. Permission by Kees Hoogland, Chief Editor of *Euclides*, of translation into English is gratefully acknowledged.

The present article is one, *Verscheidenheid* XXVI , in a series by Oene Bottema (1901-1992) in the periodical *Euclides* of the Dutch Association of Mathematics Teachers. A collection of articles from this series was published in 1978 in form of a book [1]. The original article does not contain any footnote nor bibliography. All annotations, unless otherwise specified, are by the translator. Some illustrative diagrams are added in the Appendix.

[1] See, for examples, [3, 5, 7, 8, 9].

[2] Cayley's paper [4] was published in 1854.

[3] Pampuch [11, 12].

The literature about the problem is so vast and widespread that it is hardly possible to consult completely. As far as we have been able to check, the following special case of the generalization by Steiner has not drawn attention. It is attractive by the simplicity of the results and by the possibility of a certain stereometric interpretation.

The problem of Malfatti-Steiner is as follows: Given are three circles C_1, C_2 and C_3. Three circles C_1', C_2' and C_3' are sought such that C_1' is tangent to C_2, C_3, C_2' and C_3', the circle C_2' to C_3, C_1, C_3' and C_1', and, C_3' to C_1, C_2, C_1' and C_2'. Now we examine the special case, where the *three given circles* C_1, C_2, C_3 *are pairwise tangent as well*.

This problem certainly can be solved following Steiner's general method. We choose another route, in which the simplicity of the problem appears immediately. If one applies an *inversion* with center the point of tangency of C_2 and C_3, then these two circles are transformed into two parallel lines ℓ_2 and ℓ_3, and C_1 into a circle K tangent to both (Figure 1). In this figure the construction of the required circles K_i is very simple. If the distance between ℓ_2 and ℓ_3 is $4r$, then the radii of K_2 and K_3 are equal to r, that of K_1 equal to $2r$, while the distance of the centers of K and K_1 is equal to $4r\sqrt{2}$. Clearly, the problem always has *two* (real) solutions.[4]

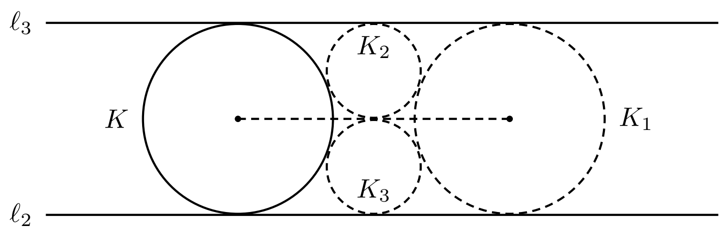

Figure 1

Our goal is the computation of the radii R_1', R_2' and R_3' of C_1', C_2' and C_3' if the radii R_1, R_2 and R_3 of the given circles C_1, C_2 and C_3 (which fix the figure of these circles) are given. For this purpose we let the objects in Figure 1 undergo an arbitrary inversion. Let O be the center of inversion and we choose a rectangular grid with O as its origin and such that ℓ_2 and ℓ_3 are parallel to the $x-$axis. For the power of inversion we can without any objection choose the unit. The inversion is then given by

$$x' = \frac{x}{x^2+y^2}, \qquad y' = \frac{y}{x^2+y^2}.$$

From this it is clear that the circle with center (x_0, y_0) and radius ρ is transformed into a circle of radius

$$\left|\frac{\rho}{x_0^2+y_0^2-\rho^2}\right|.$$

[4]See Figure 2 in the Appendix, which we add in the present translation.

If the coordinates of the center of K are (a,b), then those of K_1 are $(a+4r\sqrt{2},\ b)$. From this it follows that

$$R_1 = \left|\frac{2r}{a^2+b^2-4r^2}\right|, \qquad R_1' = \left|\frac{2r}{(a+4r\sqrt{2})^2+b^2-4r^2}\right|.$$

The lines ℓ_2 and ℓ_3 are inverted into circles of radii

$$R_2 = \frac{1}{2|b-2r|}, \qquad R_3 = \frac{1}{2|b+2r|}.$$

Now we first assume that O is chosen between ℓ_2 and ℓ_3, and outside K. The circles C_1, C_2 and C_3 then are pairwise tangent *externally*. One has $b-2r<0$, $b+2r>0$, and $a^2+b^2>4r^2$, so that

$$R_2 = \frac{1}{2(2r-b)}, \qquad R_3 = \frac{1}{2(2r+b)}, \qquad R_1 = \frac{2r}{a^2+b^2-4r^2}.$$

Consequently,

$$a = \pm\frac{1}{2}\sqrt{\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}}, \quad b = \frac{1}{4}\left(\frac{1}{R_3}-\frac{1}{R_2}\right), \quad r = \frac{1}{8}\left(\frac{1}{R_3}+\frac{1}{R_2}\right),$$

so that one of the solutions has

$$\frac{1}{R_1'} = \frac{1}{R_1}+\frac{2}{R_2}+\frac{2}{R_3}+2\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},$$

and in the same way

$$\begin{aligned}\frac{1}{R_2'} &= \frac{2}{R_1}+\frac{1}{R_2}+\frac{2}{R_3}+2\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},\\ \frac{1}{R_3'} &= \frac{2}{R_1}+\frac{2}{R_2}+\frac{1}{R_3}+2\sqrt{2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)},\end{aligned} \tag{1}$$

while the second solution is found by replacing the square roots on the right hand sides by their opposites and then taking absolute values. The first solution consists of three circles which are pairwise tangent externally. For the second there are different possibilities. It may consist of three circles tangent to each other externally, or of three circles, two tangent externally, and with a third circle tangent internally to each of them.[5] One can check the correctness of this remark by choosing O outside each of the circles K_1, K_2 and K_3 respectively, or inside these. According as one chooses O on the circumference of one of the circles, or at the point of tangency of two of these circles, respectively one , or two, straight lines[6] appear in the solution.

Finally, if one takes O outside the strip bordered by ℓ_2 and ℓ_3, or inside K, then the resulting circles have two internal and one external tangencies. If the circle C_1 is tangent *internally* to C_2 and C_3, then one should replace in solution (1) R_1 by $-R_1$, and the same for the second solution. In both solutions the circles are tangent

[5] See Figures 2 and 3 in the Appendix.

[6] See Figures 4, 5, and 6 in the Appendix.

to each other externally.[7] Incidentally, one can take (1) and the corresponding expression, where the sign of the square root is taken oppositely, as the general solution for each case, if one agrees to accept also negative values for a radius and to understand that two externally tangent circles have radii of equal signs and internally tangent circles of opposite signs.

There are two circles that are tangent to the three given circles. [8] This also follows immediately from Figure 1. In this figure the radii of these circles are both $2r$, the coordinates of their centers $(a \pm 4r,\ b)$. After inversion one finds for the radii of these 'inscribed' circles of the figure C_1, C_2, C_3:

$$\frac{1}{\rho_{1,2}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \pm 2\sqrt{\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}}, \tag{2}$$

expressions showing great analogy to (1). One finds these already in Steiner [9] (*Werke* I, pp. 61 – 63, with a clarifying remark by Weierstrass, p.524). [10] While ρ_1 is always positive, $\frac{1}{\rho_2}$ can be greater than, equal to, or smaller than zero. One of the circles is tangent to all the given circles externally, the other is tangent to them all externally, or all internally, (or in the transitional case a straight line). One can read these properties easily from Figure 1 as well. Steiner proves (2) by a straightforward calculation with the help of a formula for the altitude of a triangle.

From (1) and (2) one can derive a large number of relations among the radii R_i of the given circles, the radii R_i' of the Malfatti circles, and the radii ρ_i of the tangent circles. We only mention

$$\frac{1}{R_1} + \frac{1}{R_1'} = \frac{1}{R_2} + \frac{1}{R_2'} = \frac{1}{R_3} + \frac{1}{R_3'}.$$

About the formulas (1) we want to make some more remarks. After finding for the figure $\mathcal{S}$ of given circles C_1, C_2, C_3 one of the two sets $\mathcal{S}'$ of Malfatti circles C_1', C_2', C_3', clearly one may repeat the same construction to $\mathcal{S}'$. One of the two sets of Malfatti circles that belong to $\mathcal{S}'$ clearly is $\mathcal{S}$. Continuing this way in two directions *a chain of triads of circles* arises, with the property that each of two consecutive triples is a Malfatti figure of the other.

By *iteration* of formula (1) one can express the radii of the circles in the n^{th} triple in terms of the radii of the circles one begins with. If one applies (1) to $\frac{1}{R_i'}$, and chooses the negative square root, then one gets back $\frac{1}{R_i}$. For the new set we find

$$\frac{1}{R_1''} = \frac{17}{R_1} + \frac{16}{R_2} + \frac{16}{R_3} + 20\sqrt{2\left(\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}\right)}$$

[7] See Figure 7 in the Appendix.

[8] See Figure 8 in the Appendix.

[9] Steiner [15].

[10] This formula has become famous in modern times since the appearance of Soddy [5]. See [6]. According to Boyer and Merzbach [2], however, an equivalent formula was already known to René Descartes, long before Soddy and Steiner.

and cyclic permutations. For the next sets,

$$\frac{1}{R_1^{(3)}} = \frac{161}{R_1} + \frac{162}{R_2} + \frac{162}{R_3} + 198\sqrt{2\left(\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}\right)}$$

$$\frac{1}{R_1^{(4)}} = \frac{1601}{R_1} + \frac{1600}{R_2} + \frac{1600}{R_3} + 1960\sqrt{2\left(\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}\right)}$$

If one takes

$$\frac{1}{R_1^{(2p)}} = \frac{a_{2p}+1}{R_1} + \frac{a_{2p}}{R_2} + \frac{a_{2p}}{R_3} + b_{2p}\sqrt{2\left(\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}\right)}$$

$$\frac{1}{R_1^{(2p+1)}} = \frac{a_{2p+1}+1}{R_1} + \frac{a_{2p+1}+2}{R_2} + \frac{a_{2p+1}+2}{R_3} + b_{2p+1}\sqrt{2\left(\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}\right)},$$

then one finds the recurrences [11]

$$\begin{aligned} a_{2p+1} &= 10a_{2p} - a_{2p-1}, \\ a_{2p} &= 10a_{2p-1} - a_{2p-2} + 16, \\ b_k &= 10b_{k-1} - b_{k-2}, \end{aligned}$$

from which one can compute the radii of the circles in the triples.

The figure of three pairwise tangent circles C_1, C_2, C_3 forms with a set of Malfatti circles C_1', C_2', C_3' a configuration of six circles, of which each is tangent to four others. If one maps the circles of the plane to points in a three dimensional projective space, where the point-circles correspond with the points of a quadric surface Ω, then the configuration matches with an octahedron, of which the edges are tangent to Ω. The construction that was under discussion is thus the same as the following problem: *around a quadric surface Ω (for instance a sphere) construct an octahedron, of which the edges are tangent to Ω, and the vertices of one face are given.* This problem therefore has two solutions. And with the above chain corresponds a chain of triangles, all circumscribing Ω, and having the property that two consecutive triangles are opposite faces of a circumscribing octahedron.

From the formulas derived above for the radii it follows that these are decreasing if one goes in one direction along the chain, and increasing in the other direction, a fact that is apparent from the figure. Continuing in one direction, the triple of circles will eventually converge to a single point. With the question of how this point is positioned with respect to the given circles, we wish to end this modest contribution to the knowledge of the curious problem of Malfatti.

[11] These are the sequences A001078 and A053410 in N.J.A. Sloane's *Encyclopedia of Integer Sequences* [13].

Appendix

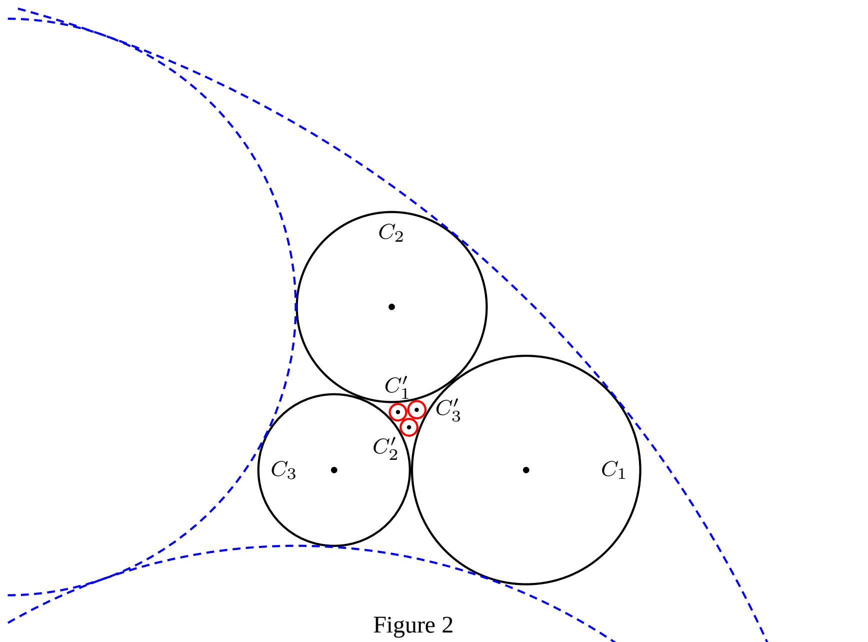

Figure 2

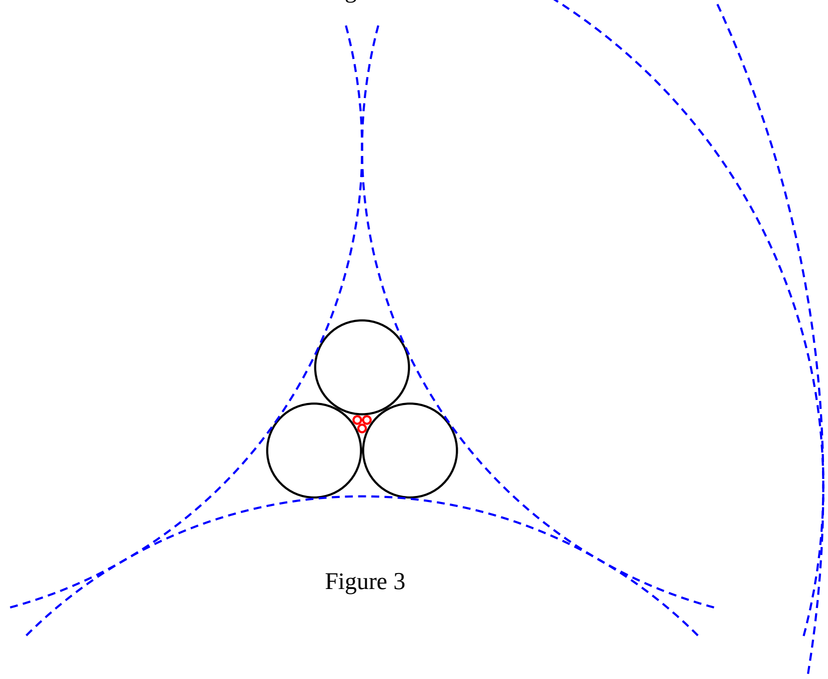

Figure 3

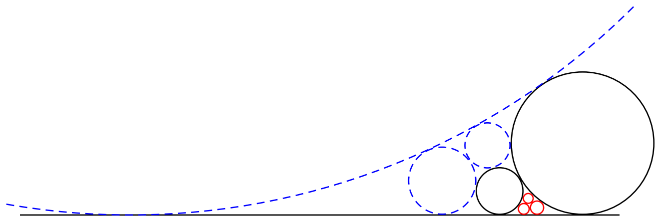

Figure 4

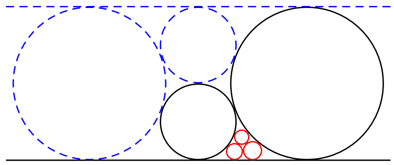

Figure 5

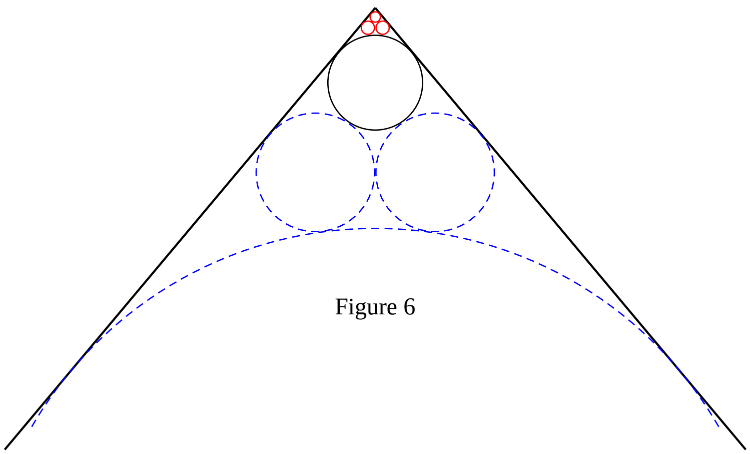

Figure 6

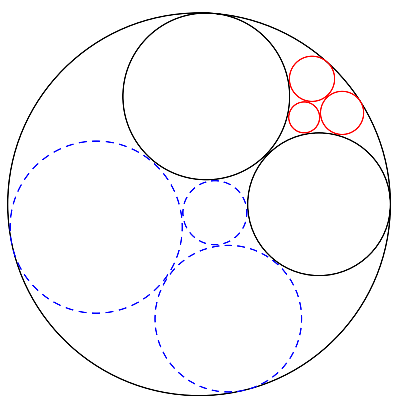

Figure 7

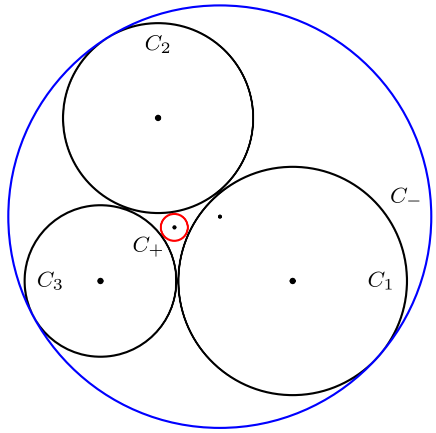

Figure 8

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A Morley Configuration

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Abstract. Given a triangle, the isogonal conjugates of the infinite points of the side lines of the Morley (equilateral) triangle is an equilateral triangle PQR inscribed in the circumcircle. Their isotomic conjugates form another equilateral triangle $P'Q'R'$ inscribed in the Steiner circum-ellipse, homothetic to PQR at the Steiner point. We show that under the one-to-one correspondence $P \mapsto P'$ between the circumcircle and the Steiner circum-ellipse established by isogonal and then isotomic conjugations, this is the only case when both PQR and $P'Q'R'$ are equilateral.

1. Introduction

Consider the Morley triangle $M_aM_bM_c$ of a triangle ABC, the equilateral triangle whose vertices are the intersections of pairs of angle trisectors adjacent to a side. Under *isogonal* conjugation, the infinite points of the Morley lines M_bM_c, M_cM_a, M_aM_b correspond to three points G_a, G_b, G_c on the circumcircle. These three points form the vertices of an equilateral triangle. This phenomenon is true for any three lines making 60° angles with one another.[1]

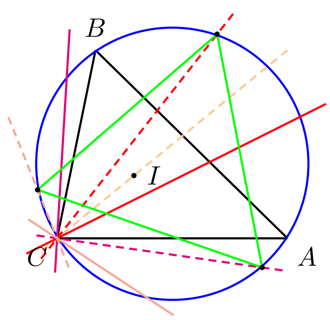

Figure 1

Under *isotomic* conjugation, on the other hand, the infinite points of the same three Morley lines correspond to three points T_a, T_b, T_c on the Steiner circum-ellipse. It is interesting to note that these three points also form the vertices of an equilateral triangle. Consider the mapping that sends a point P to P', the isotomic conjugate of the isogonal conjugate of P. This maps the circumcircle onto the Steiner circum-ellipse. The main result of this paper is that $G_aG_bG_c$ is the only equilateral triangle PQR for which $P'Q'R'$ is also equilateral.

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[1]In Figure 1, the isogonal conjugates of the infinite points of the three lines through A are the intersections of the circumcircle with their reflections in the bisector of angle A.

Main Theorem. Let PQR be an equilateral triangle inscribed in the circumcircle. The triangle $P'Q'R'$ is equilateral if and only if P, Q, R are the isogonal conjugates of the infinite points of the Morley lines.

Before proving this theorem, we make some observations and interesting applications.

2. Homothety of $G_aG_bG_c$ and $T_aT_bT_c$

The two equilateral triangles $G_aG_bG_c$ and $T_aT_bT_c$ are homothetic at the Steiner point S, with ratio of homothety $1 : 4\sin^2\Omega$, where Ω is the Brocard angle of triangle ABC. The circumcircle of the equilateral triangle $T_aT_bT_c$ has center at the third Brocard point [2], the isotomic conjugate of the symmedian point, and is tangent to the circumcircle of ABC at the Steiner point S. In other words, the circle centered at the third Brocard point and passing through the Steiner point intersects the Steiner circum-ellipse at three other points which are the vertices of an equilateral triangle homothetic to the Morley triangle. This circle has radius $4R\sin^2\Omega$ and is smaller than the circumcircle, except when triangle ABC is equilateral.

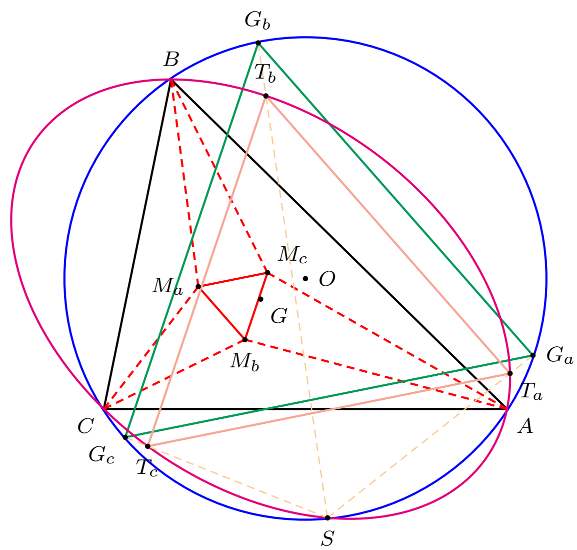

Figure 2

The triangle $G_aG_bG_c$ is the circum-tangential triangle in [3]. It is homothetic to the Morley triangle. From this it follows that the points G_a, G_b, G_c are the points of tangency with the circumcircle of the deltoid which is the envelope of the axes of inscribed parabolas.[3]

[2]This point is denoted by X_{76} in [3].

[3]The axis of an inscribed parabola with focus F is the perpendicular from F to its Simson line, or equivalently, the homothetic image of the Simson line of the antipode of F on the circumcircle, with homothetic center G and ratio -2. In [5], van Lamoen has shown that the points of contact of Simson lines tangent to the nine-point circle also form an equilateral triangle homothetic to the Morley triangle.

3. Equilateral triangles inscribed in an ellipse

Let $\mathcal{E}$ be an ellipse centered at O, and U a point on $\mathcal{E}$. With homothetic center O, ratio $-\frac{1}{2}$, maps U to u. Construct the parallel through u to its polar with respect to $\mathcal{E}$, to intersect the ellipse at V and W. The circumcircle of UVW intersects $\mathcal{E}$ at the Steiner point S of triangle UVW. Let M be the third Brocard point of UVW. The circle, center M, passing through S, intersects $\mathcal{E}$ at three other points which form the vertices of an equilateral triangle. See Figure 3.

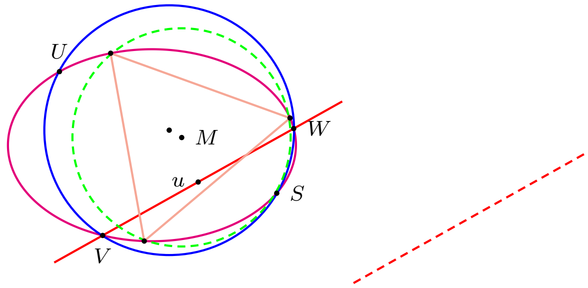

Figure 3

From this it follows that the locus of the centers of equilateral triangles inscribed in the Steiner circum-ellipse of ABC is the ellipse

$$\sum_{\text{cyclic}} a^2(a^2+b^2+c^2)x^2 + (a^2(b^2+c^2) - (2b^4+b^2c^2+2c^4))yz = 0$$

with the same center and axes.

4. Some preliminary results

Proposition 1. *If a circle through the focus of a parabola has its center on the directrix, there exists an equilateral triangle inscribed in the circle, whose side lines are tangent to the parabola.*

Proof. Denote by p the distance from the focus F of the parabola to its directrix. In polar coordinates with the pole at F, let the center of the circle be the point $(\frac{p}{\cos\alpha}, \alpha)$. The radius of the circle is $R = \frac{p}{\cos\alpha}$. See Figure 4. If this center is at a distance d to the line tangent to the parabola at the point $(\frac{p}{1+\cos\theta}, \theta)$, then

$$\frac{d}{R} = \left|\frac{\cos(\theta-\alpha)}{2\cos\frac{\theta}{2}}\right|.$$

Thus, for $\theta = \frac{2}{3}\alpha$, $\frac{2}{3}(\alpha+\pi)$ and $\frac{2}{3}(\alpha-\pi)$, we have $d = \frac{R}{2}$, and the lines tangent to the parabola at these three points form the required equilateral triangle. $\square$

Proposition 2. *If P lies on the circumcircle, then the line PP' passes through the Steiner point S.* [4]

[4] More generally, if $u+v+w=0$, the line joining $(\frac{p}{u} : \frac{q}{v} : \frac{r}{w})$ to $(\frac{l}{u} : \frac{m}{v} : \frac{n}{w})$ passes through the point $(\frac{1}{qn-rm} : \frac{1}{rl-pn} : \frac{1}{pm-ql})$ which is the fourth intersection of the two circumconics $\frac{p}{u}+\frac{q}{v}+\frac{r}{w}=0$ and $\frac{l}{u}+\frac{m}{v}+\frac{n}{w}=0$.

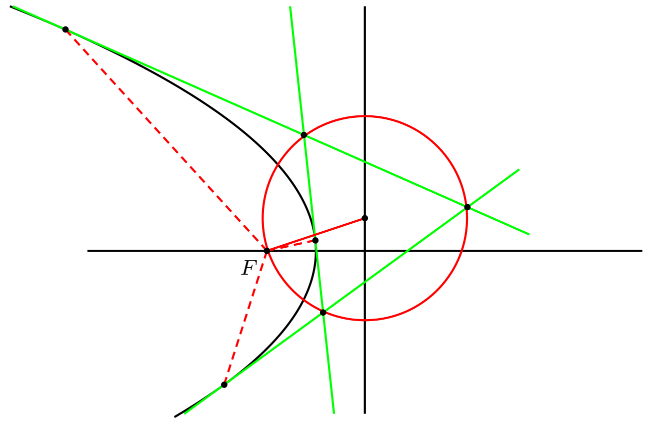

Figure 4

It follows that a triangle PQR inscribed in the circumcircle is always perspective with $P'Q'R'$ (inscribed in the Steiner circum-ellipse) at the Steiner point. The perspectrix is a line parallel to the tangent to the circumcircle at the focus of the Kiepert parabola. [5]

We shall make use of the Kiepert parabola

$$\mathcal{P}: \qquad \sum (b^2-c^2)^2x^2 - 2(c^2-a^2)(a^2-b^2)yz = 0.$$

This is the inscribed parabola with perspector the Steiner point S, focus $\mathcal{S} = \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$,[6] and the Euler line as directrix. For more on inscribed parabolas and inscribed conics in general, see [1].

Proposition 3. *Let PQ be a chord of the circumcircle. The following statements are equivalent:* [7]

(a) PQ and $P'Q'$ are parallel.

(b) The line PQ is tangent to the Kiepert parabola $\mathcal{P}$.

(c) The Simson lines $s(P)$ and $s(Q)$ intersect on the Euler line.

Proof. If the line PQ is $ux+vy+wz=0$, then $P'Q'$ is $a^2ux+b^2vy+c^2wz=0$. These two lines are parallel if and only if

$$\frac{b^2-c^2}{u} + \frac{c^2-a^2}{v} + \frac{a^2-b^2}{w} = 0, \tag{1}$$

which means that PQ is tangent to the the Kiepert parabola.

The common point of the Simson lines $s(P)$ and $s(Q)$ is $(x : y : z)$, where

$$\begin{aligned} x \;=\; & (2b^2(c^2+a^2-b^2)v + 2c^2(a^2+b^2-c^2)w - (c^2+a^2-b^2)(a^2+b^2-c^2)u) \\ & \cdot((a^2+b^2-c^2)v + (c^2+a^2-b^2)w - 2a^2u), \end{aligned}$$

---

[5]This line is also parallel to the trilinear polars of the two isodynamic points.

[6]This is the point X_{110} in [3].

[7]These statements are also equivalent to (d): The orthopole of the line PQ lies on the Euler line.

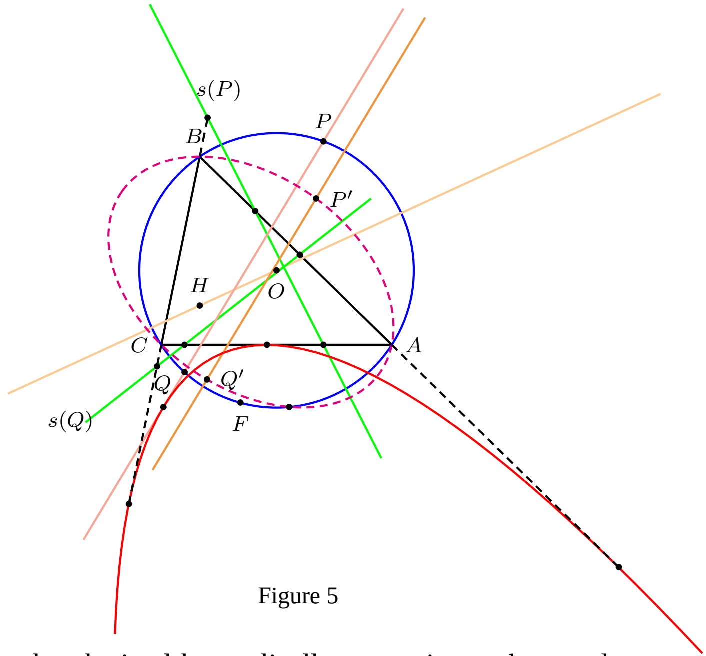

Figure 5

and y and z can be obtained by cyclically permuting a, b, c, and u, v, w. This point lies on the Euler line if and only if (1) is satisfied. □

In the following proposition, (ℓ_1, ℓ_2) denotes the directed angle between two lines ℓ_1 and ℓ_2. This is the angle through which the line ℓ_1 must be rotated in the positive direction in order to become parallel to, or to coincide with, the line ℓ_2. See [2, §§16–19.].

Proposition 4. *Let P, Q, R be points on the circumcircle. The following statements are equivalent.*

(a) The Simson lines $s(P)$, $s(Q)$, $s(R)$ are concurrent.

(b) $(AB, PQ) + (BC, QR) + (CA, RP) = 0 \pmod{\pi}$.

(c) $s(P)$ and QR are perpendicular; so are $s(Q)$ and RP; $s(R)$ and PQ.

Proof. See [4, §§2.16–20]. □

Proposition 5. *A line ℓ is parallel to a side of the Morley triangle if and only if*

$$(AB, \ell) + (BC, \ell) + (CA, \ell) = 0 \pmod{\pi}.$$

Proof. Consider the Morley triangle $M_aM_bM_c$. The line BM_c and CM_b intersecting at P, the triangle PM_bM_c is isoceles and $(M_cM_b, M_cP) = \frac{1}{3}(B + C)$. Thus, $(BC, M_bM_c) = \frac{1}{3}(B - C)$. Similarly, $(CA, M_bM_c) = \frac{1}{3}(C - A) + \frac{\pi}{3}$, and $(AB, M_bM_c) = \frac{1}{3}(A - B) - \frac{\pi}{3}$. Thus

$$(AB, M_bM_c) + (BC, M_bM_c) + (CA, M_bM_c) = 0 \pmod{\pi}.$$

There are only three directions of line ℓ for which $(AB, \ell)+(BC, \ell)+(CA, \ell) = 0$. These can only be the directions of the Morley lines. □

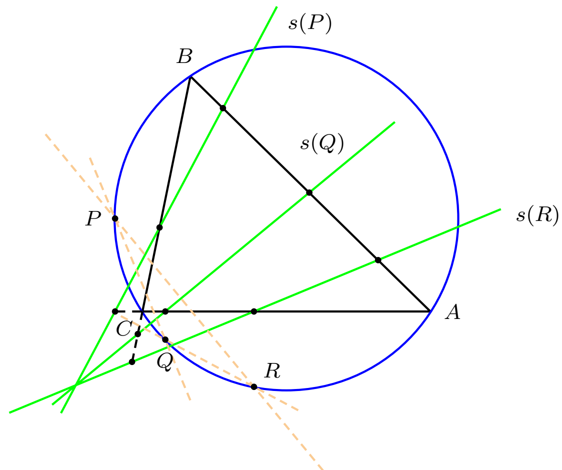

Figure 6

5. Proof of Main Theorem

Let $\mathcal{P}$ be the Kiepert parabola of triangle ABC. By Proposition 1, there is an equilateral triangle PQR inscribed in the circumcircle whose sides are tangent to $\mathcal{P}$. By Propositions 2 and 3, the triangle $P'Q'R'$ is equilateral and homothetic to PQR at the Steiner point S. By Proposition 3 again, the Simson lines $s(P)$, $s(Q)$, $s(R)$ concur. It follows from Proposition 4 that $(AB, PQ) + (BC, QR) + (CA, RP) = 0 \pmod{\pi}$. Since the lines PQ, QR, and RP make 60° angles with each other, we have

$$(AB, PQ) + (BC, PQ) + (CA, PQ) = 0 \pmod{\pi},$$

and PQ is parallel to a side of the Morley triangle by Proposition 5. Clearly, this is the same for QR and RP. By Proposition 4, the vertices P, Q, R are the isogonal conjugates of the infinite points of the Morley sides.

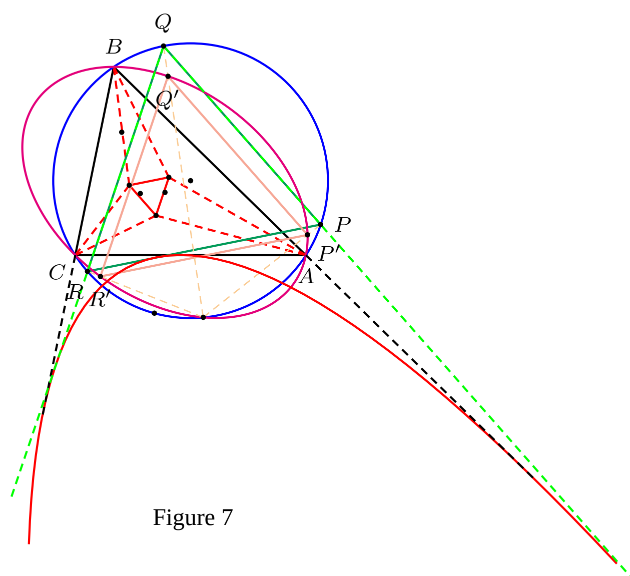

Figure 7

Uniqueness: For $M(x:y:z)$, let
$$f(M) = \frac{x+y+z}{\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}}.$$
The determinant of the affine mapping $P \mapsto P'$, $Q \mapsto Q'$, $R \mapsto R'$ is
$$\frac{f(P)f(Q)f(R)}{a^2b^2c^2}.$$
This determinant is positive for P, Q, R on the circumcircle, which does not intersect the Lemoine axis $\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=0$. Thus, if both triangles are equilateral, the similitude $P \mapsto P'$, $Q \mapsto Q'$, $R \mapsto R'$ is a *direct* one. Hence,
$$(SP', SQ') = (SP, SQ) = (RP, RQ) = (R'P', R'Q'),$$
and the circle $P'Q'R'$ passes through S. Now, through any point on an ellipse, there is a unique circle intersecting the ellipse again at the vertices of an equilateral triangle. This establishes the uniqueness, and completes the proof of the theorem.

6. Concluding remarks

We conclude with a remark and a generalization.

(1) The reflection of $G_aG_bG_c$ in the circumcenter is another equilateral triangle PQR (inscribed in the circumcircle) whose sides are parallel to the Morley lines.[8] This, however, does not lead to an equilateral triangle inscribed in the Steiner circum-ellipse.

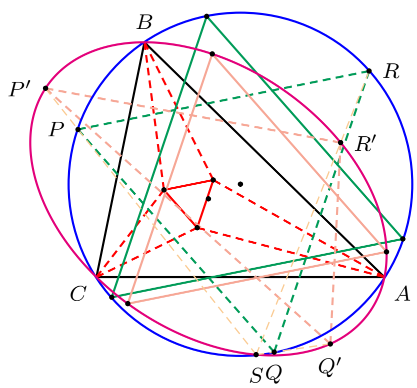

Figure 8

(2) Consider the circum-hyperbola $\mathcal{C}$ through the centroid G and the symmedian point K.[9] For any point P on $\mathcal{C}$, let $\mathcal{C}_P$ be the circumconic with perspector P, intersecting the circumcircle again at a point S_P. [10] For every point M on the

[8]This is called the circumnormal triangle in [3].

[9]The center of this hyperbola is the point $(a^4(b^2-c^2)^2 : b^4(c^2-a^2)^2 : c^4(a^2-b^2)^2)$.

[10]The perspector of a circumconic is the perspector of the triangle bounded by the tangents to the conic at the vertices of ABC. If $P = (u : v : w)$, the circumconic $\mathcal{C}_P$ has center $(u(v+w-u) : v(w+u-v) : w(u+v-w))$, and S_P is the point $\left(\frac{1}{b^2w-c^2v} : \frac{1}{c^2u-a^2w} : \frac{1}{a^2v-b^2u}\right)$. See Footnote 4.

circumcircle, denote by M' the second common point of $\mathcal{C}_U$ and the line MS_P. Then, if G_a, G_b, G_c are the isogonal conjugates of the infinite points of the Morley lines, $G'_aG'_bG'_c$ is homothetic to $G_aG_bG_c$ at S_U. The reason is simple: Proposition 3 remains true. For $U = G$, this gives the equilateral triangle $T_aT_bT_c$ inscribed in the case of the Steiner circum-ellipse. Here is an example. For $U = (a(b+c) : b(c+a) : c(a+b))$, [11] we have the circumellipse with center the Spieker center $(b+c : c+a : a+b)$. The triangles $G_aG_bG_c$ and $G'_aG'_bG'_c$ are homothetic at $X_{100} = (\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b})$, and the circumcircle of $G'_aG'_bG'_c$ is the incircle of the anticomplementary triangle, center the Nagel point, and ratio of homothety $R : 2r$.

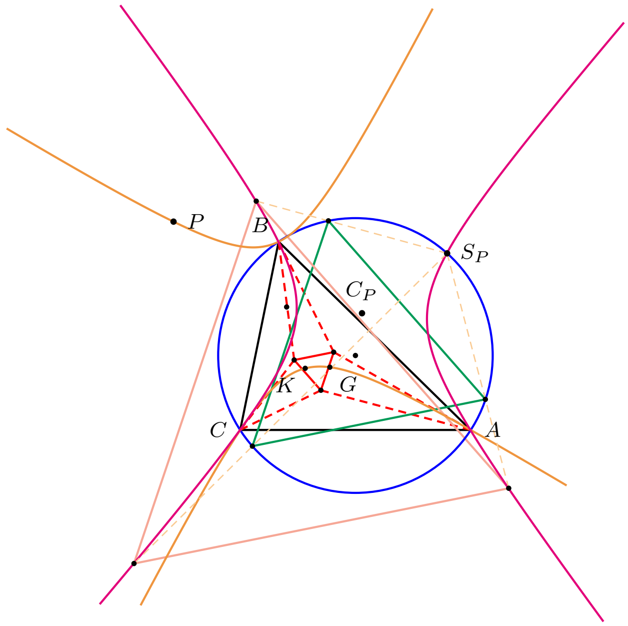

Figure 9

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[11] This is the point X_{37} in [3].

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Concurrency of Four Euler Lines

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Abstract. Using tripolar coordinates, we prove that if P is a point in the plane of triangle ABC such that the Euler lines of triangles PBC, APC and ABP are concurrent, then their intersection lies on the Euler line of triangle ABC. The same is true for the Brocard axes and the lines joining the circumcenters to the respective incenters. We also prove that the locus of P for which the four Euler lines concur is the same as that for which the four Brocard axes concur. These results are extended to a family $\mathcal{L}_n$ of lines through the circumcenter. The locus of P for which the four $\mathcal{L}_n$ lines of ABC, PBC, APC and ABP concur is always a curve through 15 finite real points, which we identify.

1. Four line concurrency

Consider a triangle ABC with incenter I. It is well known [13] that the Euler lines of the triangles IBC, AIC and ABI concur at a point on the Euler line of ABC, the Schiffler point with homogeneous barycentric coordinates[1]

$$\left(\frac{a(s-a)}{b+c}:\frac{b(s-b)}{c+a}:\frac{c(s-c)}{a+b}\right).$$

There are other notable points which we can substitute for the incenter, so that a similar statement can be proven relatively easily. Specifically, we have the following interesting theorem.

Theorem 1. *Let P be a point in the plane of triangle ABC such that the Euler lines of the component triangles PBC, APC and ABP are concurrent. Then the point of concurrency also lies on the Euler line of triangle ABC.*

When one tries to prove this theorem with homogeneous coordinates, calculations turn out to be rather tedious, as one of us has noted [14]. We present an easy analytic proof, making use of tripolar coordinates. The same method applies if we replace the Euler lines by the Brocard axes or the OI-lines joining the circumcenters to the corresponding incenters.

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[1]This appears as X_{21} in Kimberling's list [7]. In the expressions of the coordinates, s stands for the semiperimeter of the triangle.

2. Tripolar coordinates

Given triangle ABC with $BC = a$, $CA = b$, and $AB = c$, consider a point P whose distances from the vertices are $PA = \lambda$, $PB = \mu$ and $PC = \nu$. The precise relationship among λ, μ, and ν dates back to Euler [4]:

$$\begin{aligned}&(\mu^2+\nu^2-a^2)^2\lambda^2+(\nu^2+\lambda^2-b^2)^2\mu^2+(\lambda^2+\mu^2-c^2)^2\nu^2\\&-(\mu^2+\nu^2-a^2)(\nu^2+\lambda^2-b^2)(\lambda^2+\mu^2-c^2)-4\lambda^2\mu^2\nu^2=0.\end{aligned}$$

See also [1, 2]. Geometers in the 19th century referred to the triple (λ, μ, ν) as the *tripolar* coordinates of P. A comprehensive introduction can be found in [12]. [2] This series begins with the following easy theorem.

Proposition 2. *An equation of the form $\ell\lambda^2 + m\mu^2 + n\nu^2 + q = 0$ represents a circle or a line according as $\ell + m + n$ is nonzero or otherwise.*

The center of the circle has homogeneous barycentric coordinates $(\ell : m : n)$. If $\ell + m + n = 0$, the line is orthogonal to the direction $(\ell : m : n)$. Among the applications one finds the equation of the Euler line in tripolar coordinates [op. cit. §26]. [3]

Proposition 3. *The tripolar equation of the Euler line is*

$$(b^2-c^2)\lambda^2+(c^2-a^2)\mu^2+(a^2-b^2)\nu^2=0. \tag{1}$$

We defer the proof of this proposition to §5 below. Meanwhile, note how this applies to give a simple proof of Theorem 1.

3. Proof of Theorem 1

Let P be a point with tripolar coordinates (λ, μ, ν) such that the Euler lines of triangles PBC, APC and ABP intersect at a point Q with tripolar coordinates (λ', μ', ν'). We denote the distance PQ by ρ.

Applying Proposition 3 to the triangles PBC, APC and ABP, we have

$$\begin{aligned}(\nu^2-\mu^2)\rho^2&+(\mu^2-a^2)\mu'^2+(a^2-\nu^2)\nu'^2=0,\\(b^2-\lambda^2)\lambda'^2&+(\lambda^2-\nu^2)\rho^2+(\nu^2-b^2)\nu'^2=0,\\(\lambda^2-c^2)\lambda'^2&+(c^2-\mu^2)\mu'^2+(\mu^2-\lambda^2)\rho^2=0.\end{aligned}$$

Adding up these equations, we obtain (1) with λ', μ', ν' in lieu of λ, μ, ν. This shows that Q lies on the Euler line of ABC.

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[2] [5] and [8] are good references on tripolar coordinates.

[3] The tripolar equations of the lines in §§5 – 7 below can be written down from the barycentric equations of these lines. The calculations in these sections, however, do not make use of these barycentric equations.

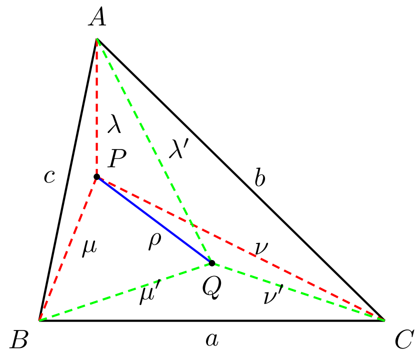

Figure 1

4. Tripolar equations of lines through the circumcenter

O. Bottema [2, pp.37–38] has given a simple derivation of the equation of the Euler line in tripolar coordinates. He began with the observation that since the point-circles

$$\lambda^2 = 0, \qquad \mu^2 = 0, \qquad \nu^2 = 0,$$

are all orthogonal to the circumcircle,[4] for arbitrary t_1, t_2, t_3, the equation

$$t_1\lambda^2 + t_2\mu^2 + t_3\nu^2 = 0 \tag{2}$$

represents a circle orthogonal to the circumcircle. By Proposition 2, this represents a line through the circumcenter if and only if $t_1 + t_2 + t_3 = 0$.

5. Tripolar equation of the Euler line

Consider the centroid G of triangle ABC. By the Apollonius theorem, and the fact that G divides each median in the ratio $2:1$, it is easy to see that the tripolar coordinates of G satisfy

$$\lambda^2 : \mu^2 : \nu^2 = 2b^2 + 2c^2 - a^2 : 2c^2 + 2a^2 - b^2 : 2a^2 + 2b^2 - c^2.$$

It follows that the Euler line OG is defined by (2) with t_1, t_2, t_3 satisfying

$$\begin{array}{rcrcrcl} t_1 & + & t_2 & + & t_3 & = & 0, \\ (2b^2+2c^2-a^2)t_1 & + & (2c^2+2a^2-b^2)t_2 & + & (2a^2+2b^2-c^2)t_3 & = & 0, \end{array}$$

or

$$t_1 : t_2 : t_3 = b^2 - c^2 : c^2 - a^2 : a^2 - b^2.$$

This completes the proof of Proposition 3.

[4]These point-circles are evidently the vertices of triangle ABC.

6. Tripolar equation of the OI-line

For the incenter I, we have

$$\lambda^2 : \mu^2 : \nu^2 = \csc^2 \frac{A}{2} : \csc^2 \frac{B}{2} : \csc^2 \frac{C}{2} = \frac{s-a}{a} : \frac{s-b}{b} : \frac{s-c}{c},$$

where $s = \frac{a+b+c}{2}$. The tripolar equation of the OI-line is given by (2) with t_1, t_2, t_3 satisfying

$$t_1 + t_2 + t_3 = 0, \qquad \frac{s-a}{a}t_1 + \frac{s-b}{b}t_2 + \frac{s-c}{c}t_3 = 0.$$

From these, $t_1 : t_2 : t_3 = \frac{1}{b} - \frac{1}{c} : \frac{1}{c} - \frac{1}{a} : \frac{1}{a} - \frac{1}{b}$, and the tripolar equation of the OI-line is

$$\left(\frac{1}{b} - \frac{1}{c}\right)\lambda^2 + \left(\frac{1}{c} - \frac{1}{a}\right)\mu^2 + \left(\frac{1}{a} - \frac{1}{b}\right)\nu^2 = 0.$$

The same reasoning in §3 yields Theorem 1 with the Euler lines replaced by the OI-lines.

7. Tripolar equation of the Brocard axis

The Brocard axis is the line joining the circumcenter to the symmedian point. Since this line contains the two isodynamic points, whose tripolar coordinates, by definition, satisfy

$$\lambda : \mu : \nu = \frac{1}{a} : \frac{1}{b} : \frac{1}{c},$$

it is easy to see that the tripolar equation of the Brocard axis is[5]

$$\left(\frac{1}{b^2} - \frac{1}{c^2}\right)\lambda^2 + \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\mu^2 + \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\nu^2 = 0.$$

The same reasoning in §3 yields Theorem 1 with the Euler lines replaced by the Brocard axes.

8. The lines $\mathcal{L}_n$

The resemblance of the tripolar equations in §§5 – 7 suggests consideration of the family of lines through the circumcenter:

$$\mathcal{L}_n : \qquad (b^n - c^n)\lambda^2 + (c^n - a^n)\mu^2 + (a^n - b^n)\nu^2 = 0,$$

for nonzero integers n. The Euler line, the Brocard axis, and the OI-line are respectively $\mathcal{L}_n$ for $n = 2$, -2, and -1. In homogeneous barycentric coordinates,

---

[5]The same equation can be derived directly from the tripolar distances of the symmedian point: $AK^2 = \frac{b^2c^2(2b^2+2c^2-a^2)}{(a^2+b^2+c^2)^2}$ etc. This can be found, for example, in [11, p.118].

the equation of $\mathcal{L}_n$ is[6]

$$\sum_{\text{cyclic}} (a^n(b^2-c^2)-(b^{n+2}-c^{n+2}))x = 0.$$

The line $\mathcal{L}_1$ contains the points [7]

$$(2a+b+c : a+2b+c : a+b+2c)$$

and

$$(a(b+c)-(b-c)^2 : b(c+a)-(c-a)^2 : c(a+b)-(a-b)^2).$$

Theorem 1 obviously applies when the Euler lines are replaced by $\mathcal{L}_n$ lines for a fixed nonzero integer n.

9. Intersection of the $\mathcal{L}_n$ lines

It is known that the locus of P for which the Euler lines ($\mathcal{L}_2$) of triangles PBC, APC and ABP are concurrent is the union of the circumcircle and the Neuberg cubic.[8] See [10, p.200]. Fred Lang [9] has computed the locus for the Brocard axes ($\mathcal{L}_{-2}$) case, and found exactly the same result. The coincidence of these two loci is a special case of the following theorem.

Theorem 4. *Let n be a nonzero integer. The $\mathcal{L}_n$ lines of triangles PBC, APC and ABP concur (at a point on $\mathcal{L}_n$) if and only if the $\mathcal{L}_{-n}$ lines of the same triangles concur (at a point on $\mathcal{L}_{-n}$).*

Proof. Consider the component triangles PBC, APC and ABP of a point P. If P has tripolar coordinates (L, M, N), then the $\mathcal{L}_n$ lines of these triangles have tripolar equations

$$\mathcal{L}_n(PBC): \qquad (N^n-M^n)\rho^2 + (M^n-a^n)\mu^2 + (a^n-N^n)\nu^2 = 0,$$

$$\mathcal{L}_n(APC): \qquad (b^n-L^n)\lambda^2 + (L^n-N^n)\rho^2 + (N^n-b^n)\nu^2 = 0,$$

$$\mathcal{L}_n(ABP): \qquad (L^n-c^n)\lambda^2 + (c^n-M^n)\mu^2 + (M^n-L^n)\rho^2 = 0,$$

where ρ is the distance between P and a variable point (λ, μ, ν).[9] These equations can be rewritten as

---

[6]This can be obtained from the tripolar equation by putting

$$\lambda^2 = \frac{1}{(x+y+z)^2}(c^2y^2 + (b^2+c^2-a^2)yz + b^2z^2)$$

and analogous expressions for μ^2 and ν^2 obtained by cyclic permutations of a, b, c and x, y, z.

[7]These are respectively the midpoint between the incenters of ABC and its medial triangle, and the symmedian point of the excentral triangle of the medial triangle.

[8]The Neuberg cubic is defined as the locus of points P such that the line joining P to its isogonal conjugate is parallel to the Euler line.

[9]See Figure 1, with λ, μ, ν replaced by L, M, N, and λ', μ', ν' by λ, μ, ν respectively.

$$\begin{array}{rcrcrcl} & - & (M^n-a^n)(\rho^2-\mu^2) & + & (N^n-a^n)(\rho^2-\nu^2) & = & 0,\\ (L^n-b^n)(\rho^2-\lambda^2) & & & - & (N^n-b^n)(\rho^2-\nu^2) & = & 0,\\ -(L^n-c^n)(\rho^2-\lambda^2) & + & (M^n-c^n)(\rho^2-\mu^2) & & & = & 0. \end{array} \tag{3}$$

One trivial solution to these equations is $\rho=\lambda=\mu=\nu$, which occurs only when the variable point is the circumcenter O, with P on the circumcircle. In this case the $\mathcal{L}_n$ lines all concur at the point O, for all n. Otherwise, we have a solution to (3) with at least one of the values $\rho^2-\lambda^2$, $\rho^2-\mu^2$, and $\rho^2-\nu^2$ being non-zero. And the condition for a solution of this kind is

$$(L^n-b^n)(M^n-c^n)(N^n-a^n)=(L^n-c^n)(M^n-a^n)(N^n-b^n). \tag{4}$$

This condition is clearly necessary. Conversely, take P satisfying (4). This says that (3), as linear homogeneous equations in $\rho^2-\lambda^2$, $\rho^2-\mu^2$, and $\rho^2-\nu^2$, have a nontrivial solution (u,v,w), which is determined up to a scalar multiple. Then the equations of the $\mathcal{L}_n$ lines of triangles ABP and PBC can be rewritten as $(\frac{1}{u}-\frac{1}{v})XP^2-\frac{1}{u}XA^2+\frac{1}{v}XB^2=0$ and $(\frac{1}{v}-\frac{1}{w})XP^2-\frac{1}{v}XB^2+\frac{1}{w}XC^2=0$. If X is a point common to these two lines, then it satisfies

$$\frac{XP^2-XA^2}{u}=\frac{XP^2-XB^2}{v}=\frac{XP^2-XC^2}{w}$$

and also lies on the $\mathcal{L}_n$ line of triangle APC.

Note that (4) is clearly equivalent to

$$\left(\frac{1}{L^n}-\frac{1}{b^n}\right)\left(\frac{1}{M^n}-\frac{1}{c^n}\right)\left(\frac{1}{N^n}-\frac{1}{a^n}\right)=\left(\frac{1}{L^n}-\frac{1}{c^n}\right)\left(\frac{1}{M^n}-\frac{1}{a^n}\right)\left(\frac{1}{N^n}-\frac{1}{b^n}\right),$$

which, by exactly the same reasoning, is the concurrency condition for the $\mathcal{L}_{-n}$ lines of the same triangles. $\square$

Corollary 5. *The locus of P for which the Brocard axes of triangles PBC, APC and ABP are concurrent (at a point on the Brocard axis of triangle ABC) is the union of the circumcircle and the Neuberg cubic.*

Let $\mathcal{C}_n$ be the curve with tripolar equation

$$(\lambda^n-b^n)(\mu^n-c^n)(\nu^n-a^n)=(\lambda^n-c^n)(\mu^n-a^n)(\nu^n-b^n),$$

so that together with the circumcircle, it constitutes the locus of points P for which the four $\mathcal{L}_n$ lines of triangles PBC, APC, ABP and ABC concur.[10] The symmetry of equation (4) leads to the following interesting fact.

Corollary 6. *If P lies on the $\mathcal{C}_n$ curve of triangle ABC, then A (respectively B, C) lies on the $\mathcal{C}_n$ curve of triangle PBC (respectively APC, ABP).*

Remark. The equation of $\mathcal{C}_n$ can also be written in one of the following forms:

$$\sum_{\text{cyclic}}(b^n-c^n)(a^n\lambda^n+\mu^n\nu^n)=0$$

[10] By Theorem 4, it is enough to consider n positive.

or

$$\begin{vmatrix} \lambda^n + a^n & \mu^n + b^n & \nu^n + c^n \\ a^n\lambda^n & b^n\mu^n & c^n\nu^n \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

10. Points common to $\mathcal{C}_n$ curves

Proposition 7. *A complete list of finite real points common to all $\mathcal{C}_n$ curves is as follows:*

(1) the vertices A, B, C and their reflections on the respective opposite side,
(2) the apexes of the six equilateral triangles erected on the sides of ABC,
(3) the circumcenter, and
(4) the two isodynamic points.

Proof. It is easy to see that each of these points lies on $\mathcal{C}_n$ for every positive integer n. For the isodynamic points, recall that $\lambda : \mu : \nu = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$. We show that $\mathcal{C}_1$ and $\mathcal{C}_2$ meet precisely in these 15 points. From their equations

$$(\lambda - b)(\mu - c)(\nu - a) = (\lambda - c)(\mu - a)(\nu - b) \tag{5}$$

and

$$(\lambda^2 - b^2)(\mu^2 - c^2)(\nu^2 - a^2) = (\lambda^2 - c^2)(\mu^2 - a^2)(\nu^2 - b^2). \tag{6}$$

If both sides of (5) are zero, it is easy to list the various cases. For example, solutions like $\lambda = b, \mu = a$ lead to a vertex and its reflection through the opposite side (in this case C and its reflection in AB); solutions like $\lambda = b$, $\nu = b$ lead to the apexes of equilateral triangles erected on the sides of ABC (in this case on AC). Otherwise we can factor and divide, getting

$$(\lambda + b)(\mu + c)(\nu + a) = (\lambda + c)(\mu + a)(\nu + b).$$

Together with (5), this is easy to solve. The only solutions in this case are $\lambda = \mu = \nu$ and $\lambda : \mu : \nu = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, giving respectively $P = O$ and the isodynamic points. □

Remarks. (1) If P is any of the points listed above, then this result says that the triangles ABC, PBC, APC, and ABP have concurrent $\mathcal{L}_n$ lines, for all non-zero integers n. There is no degeneracy in the case where P is an isodynamic point, and we then get an infinite sequence of four-fold concurrences.

(2) The curve $\mathcal{C}_4$ has degree 7, and contains the two circular points at infinity, each of multiplicity 3. These, together with the 15 finite real points above, account for all 21 intersections of $\mathcal{C}_2$ and $\mathcal{C}_4$.

11. Intersections of Euler lines and of Brocard axes

For $n = \pm 2$, the curve $\mathcal{C}_n$ is the Neuberg cubic

$$\sum_{\text{cyclic}} ((b^2 - c^2)^2 + a^2(b^2 + c^2) - 2a^4)x(c^2y^2 - b^2z^2) = 0$$

in homogeneous barycentric coordinates. Apart from the points listed in Proposition 7, this cubic contains the following notable points: the orthocenter, incenter and excenters, the Fermat points, and the Parry reflection point.[11] A summary of interesting properties of the Neuberg cubic can be found in [3]. Below we list the corresponding points of concurrency, giving their coordinates. For points like the Fermat points and Napoleon points resulting from erecting equilateral triangles on the sides, we label the points by $\epsilon = +1$ or -1 according as the equilateral triangles are constructed exterior to ABC or otherwise. Also, Δ stands for the area of triangle ABC. For functions like F_a, F_b, F_c indexed by a, b, c, we obtain F_b and F_c from F_a by cyclic permutations of a, b, c.

P	Intersection of Euler lines	Intersection of Brocard axes
Circumcenter	Circumcenter	Circumcenter
Reflection of vertex on opposite side	Intercept of Euler line on the side line	Intercept of Brocard axis on the side line
Orthocenter	Nine-point center	Orthocenter of orthic triangle
Incenter	Schiffler point	Isogonal conjugate of Spieker center
Excenters $I_a = (-a : b : c)$ $I_b = (a : -b : c)$ $I_c = (a : b : -c)$	 $\left(\frac{as}{b+c} : \frac{b(s-c)}{c-a} : \frac{c(s-b)}{-a+b}\right)$ $\left(\frac{a(s-c)}{-b+c} : \frac{bs}{c+a} : \frac{c(s-a)}{a-b}\right)$ $\left(\frac{a(s-b)}{b-c} : \frac{b(s-a)}{-c+a} : \frac{cs}{a+b}\right)$	 $\left(\frac{a^2}{b+c} : \frac{b^2}{c-a} : \frac{c^2}{-a+b}\right)$ $\left(\frac{a^2}{-b+c} : \frac{b^2}{c+a} : \frac{c^2}{a-b}\right)$ $\left(\frac{a^2}{b-c} : \frac{b^2}{-c+a} : \frac{c^2}{a+b}\right)$
ϵ-Fermat point	centroid	Isogonal conjugate of $(-\epsilon)$-Napoleon point
ϵ-isodynamic point		Isogonal conjugate of ϵ-Napoleon point

Apexes of ϵ-equilateral triangles erected on the sides of ABC. Let P be the apex of an equilateral triangle erected the side BC. This has coordinates

$$\left(-2a^2 : a^2+b^2-c^2+\epsilon\cdot\frac{4}{\sqrt{3}}\Delta : c^2+a^2-b^2+\epsilon\cdot\frac{4}{\sqrt{3}}\Delta\right).$$

The intersection of the Euler lines has coordinates

$$\begin{aligned}\Big(-a^2(a^2-b^2)(a^2-c^2) \quad &: \quad (a^2-b^2)(a^2b^2+\epsilon\cdot\frac{4}{\sqrt{3}}\Delta(a^2+b^2-c^2)) \\ &: \quad (a^2-c^2)(a^2c^2+\epsilon\cdot\frac{4}{\sqrt{3}}\Delta(c^2+a^2-b^2))\Big),\end{aligned}$$

and the Brocard axis intersection is the point

$$\begin{aligned}&\Big(a^2(a^2-b^2)(a^2-c^2)(-\epsilon(b^2+c^2-a^2)+4\sqrt{3}\Delta) \\ : \quad & b^2(a^2-b^2)(-\epsilon(a^4+2b^4+3c^4-5b^2c^2-4c^2a^2-3a^2b^2)+4\sqrt{3}\Delta(c^2+a^2)) \\ : \quad & c^2(a^2-c^2)(-\epsilon(a^4+3b^4+2c^4-5b^2c^2-3c^2a^2-4a^2b^2)+4\sqrt{3}\Delta(a^2+b^2))\Big).\end{aligned}$$

[11] Bernard Gibert has found that the Fermat points of the anticomplementary triangle of ABC also lie on the Neuberg cubic. These are the points X_{616} and X_{617} in [7]. Their isogonal conjugates (in triangle ABC) clearly lie on the Neuberg cubic too. Ed.

Isodynamic points. For the ϵ-isodynamic point, the Euler line intersections are

$$\begin{aligned}&(a^2(\sqrt{3}b^2c^2+\epsilon\cdot 4\Delta(b^2+c^2-a^2))\\&:b^2(\sqrt{3}c^2a^2+\epsilon\cdot 4\Delta(c^2+a^2-b^2))\\&:c^2(\sqrt{3}a^2b^2+\epsilon\cdot 4\Delta(a^2+b^2-c^2))).\end{aligned}$$

These points divide the segment GO harmonically in the ratio $8\sin A\sin B\sin C : 3\sqrt{3}$.[12] The Brocard axis intersections for the Fermat points and the isodynamic points are illustrated in Figure 2.

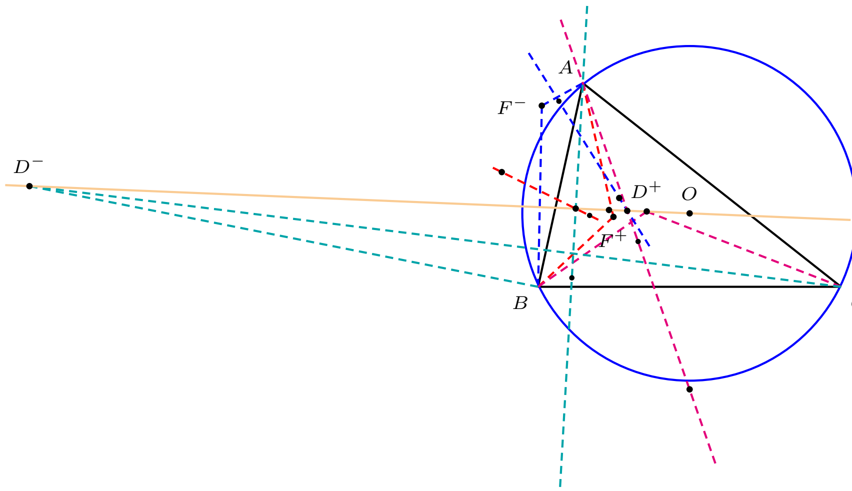

Figure 2

The Parry reflection point. This is the reflection of the circumcenter in the focus of the Kiepert parabola. [13] Its coordinates, and those of the Euler line and Brocard axis intersections, can be described with the aids of three functions.

(1) Parry reflection point: $(a^2P_a : b^2P_b : c^2P_c)$,
(2) Euler line intersection: $(a^2P_af_a : b^2P_bf_b : c^2P_cf_c)$,
(3) Brocard axis intersection: $(a^2f_ag_a : b^2f_bg_b : c^2f_cg_c)$, where

$$\begin{aligned}P_a &= a^8-4a^6(b^2+c^2)+a^4(6b^4+b^2c^2+6c^4)\\&\quad -a^2(b^2+c^2)(4b^4-5b^2c^2+4c^4)+(b^2-c^2)^2(b^4+4b^2c^2+c^4),\\ f_a &= a^6-3a^4(b^2+c^2)+a^2(3b^4-b^2c^2+3c^4)-(b^2-c^2)^2(b^2+c^2),\\ g_a &= 5a^8-14a^6(b^2+c^2)+a^4(12b^4+17b^2c^2+12c^4)\\&\quad -a^2(b^2+c^2)(2b^2+c^2)(b^2+2c^2)-(b^2-c^2)^4.\end{aligned}$$

[12] These coordinates, and those of the Brocard axis intersections, can be calculated by using the fact that triangle PBC has $(-\epsilon)$-isodynamic point at the vertex A and circumcenter at the point

$(a^2((b^2+c^2-a^2)-\epsilon\cdot 4\sqrt{3}\Delta) : b^2((c^2+a^2-b^2)+\epsilon\cdot 4\sqrt{3}\Delta) : c^2((a^2+b^2-c^2)+\epsilon\cdot 4\sqrt{3}\Delta))$.

[13] The Parry reflection point is the point X_{399} in [6]. The focus of the Kiepert parabola is the point on the circumcircle with coordinates $(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2})$.

This completes the identification of the Euler line and Brocard axis intersections for points on the Neuberg cubic. The identification of the locus for the $\mathcal{L}_{\pm 1}$ problems is significantly harder. Indeed, we do not know of any interesting points on this locus, except those listed in Proposition 7.

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Some Remarkable Concurrences

Bruce Shawyer

Abstract. In May 1999, Steve Sigur, a high school teacher in Atlanta, Georgia, posted on The Math Forum, a notice stating that one of his students (Josh Klehr) noticed that

> *Given a triangle with mid-point of each side. Through each mid-point, draw a line whose slope is the reciprocal to the slope of the side through that mid-point. These lines concur.*

Sigur then stated that "we have proved this". Here, we extend this result to the case where the slope of the line through the mid-point is a constant times the reciprocal of the slope of the side.

In May 1999, Steve Sigur, a high school teacher in Atlanta, Georgia, posted on The Math Forum, a notice stating that one of his students (Josh Klehr) noticed that

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Sigur then stated that "we have proved this".

There was a further statement that another student (Adam Bliss) followed up with a result on the concurrency of reflected line, with the point of concurrency lying on the nine-point circle. This was subsequently proved by Louis Talman [2]. See also the variations, using the feet of the altitudes in place of the mid-points and different reflections in the recent paper by Floor van Lamoen [1].

Here, we are interested in a generalization of Klehr's result.

At the mid-point of each side of a triangle, we construct the line such that the product of the slope of this line and the slope of the side of the triangle is a fixed constant. To make this clear, the newly created lines have slopes of the fixed constant times the reciprocal of the slopes of the sides of the triangle with respect to a given line (parallel to the x–axis used in the Cartesian system). We show that the three lines obtained are always concurrent.

Further, the locus of the points of concurrency is a rectangular hyperbola. This hyperbola intersects the side of the triangles at the mid-points of the sides, and each side at another point. These three other points, when considered with the vertices of the triangle opposite to the point, form a Ceva configuration. Remarkably, the point of concurrency of these Cevians lies on the circumcircle of the original triangle.

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Since we are dealing with products of slopes, we have restricted ourselves to a Cartesian proof.

Suppose that we have a triangle with vertices $(0,0)$, $(2a,2b)$ and $(2c,2d)$.

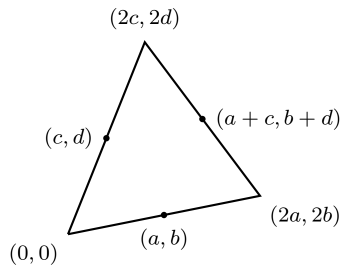

Figure 1

In order to ensure that the triangle is not degenerate, we assume that $ad-bc \neq 0$. For ease of proof, we also take $0 \neq a \neq c \neq 0$ and $0 \neq b \neq d \neq 0$ to avoid division by zero. However, by continuity, the results obtained here can readily be extended to include these cases.

At the mid-point of each side, we find the equations of the new lines:

Mid-point	Slope	Equation
(a,b)	$\frac{\lambda a}{b}$	$y = \frac{\lambda a}{b}x + \frac{b^2-\lambda a^2}{b}$
(c,d)	$\frac{\lambda c}{d}$	$y = \frac{\lambda c}{d}x + \frac{d^2-\lambda c^2}{d}$
$(a+c,b+d)$	$\frac{\lambda(c-a)}{d-b}$	$y = \frac{\lambda(c-a)}{d-b}x + \frac{(a^2-c^2)\lambda+(d^2-b^2)}{d-b}$

With the aid of a computer algebra program, we find that the first two lines meet at

$$\left(\frac{\lambda(a^2d-bc^2)+bd(d-b)}{\lambda(ad-bc)}, \frac{\lambda ac(a-c)+(ad^2-b^2c)}{(ad-bc)}\right),$$

which it is easy to verify lies on the third line.

By eliminating λ from the equations

$$x = \frac{\lambda(a^2d-bc^2)+bd(d-b)}{\lambda(ad-bc)}, \qquad y = \frac{\lambda ac(a-c)+(ad^2-b^2c)}{(ad-bc)},$$

we find that the locus of the points of concurrency is

$$y = \frac{abcd(a-c)(d-b)}{ad-bc} \cdot \frac{1}{(ad-bc)x-(a^2d-bc^2)} + \frac{ad^2-b^2c}{ad-bc}.$$

This is a rectangular hyperbola, with asymptotes

$$x = \frac{a^2d-bc^2}{ad-bc}, \qquad y = \frac{ad^2-b^2c}{ad-bc}.$$

Now, this hyperbola meets the sides of the given triangle as follows:

from	to	mid-point	new-point
$(0,0)$	$(2a,2b)$	(a,b)	$\left(\dfrac{ad+bc}{b}, \dfrac{ad+bc}{a}\right)$
$(0,0)$	$(2c,2d)$	(c,d)	$\left(\dfrac{ad+bc}{d}, \dfrac{ad+bc}{c}\right)$
$(2a,2b)$	$(2c,2d)$	$(a+c,b+d)$	$\left(\dfrac{ad-bc}{d-b}, \dfrac{ad-bc}{a-c}\right)$

The three lines joining the three points (new-point, in each case) to the vertices opposite are concurrent! (Again, easily shown by computer algebra.) The point of concurrency is

$$\left(2(a-c)\left(\frac{ad+bc}{ad-bc}\right),\ 2(d-b)\left(\frac{ad+bc}{ad-bc}\right)\right).$$

It is easy to check that this point is not on the hyperbola. However, it is also easy to check that this point lies on the circumcircle of the original triangle. (Compare this result with the now known result that the point on the hyperbola corresponding to $\lambda = 1$ lies on the nine-point circle. See [2].)

In Figures 2, 3, 4 below, we illustrate the original triangle ABC, the rectangular hyperbola $YWLPXQVZ$ (where $\lambda < 0$) and $MSOUN$ (where $\lambda > 0$), the asymptotes (dotted lines), the circumcircle and the nine-point circle, and the first remarkable point K.

Figure 2 shows various lines through the mid-points of the sides being concurrent on the hyperbola, and also the concurrency of AX, BY, CZ at K.

Figure 3 shows the lines concurrent through the second remarkable point J, where we join points with parameters λ and $-\lambda$. This point J is indeed the center of the rectangular hyperbola.

Figure 4 shows the parallel lines (or lines concurrent at infinity), where we join points with parameters λ and $-\frac{1}{\lambda}$.

Now, this is a purely Cartesian demonstration. As a result, there are several questions that I have not (yet) answered:

(1) Does the Cevian intersection point have any particular significance?
(2) What significant differences (if any) would occur if the triangle were to be rotated about the origin?
(3) Are there variations of these results along the lines of Floor van Lamoen's paper [1]?

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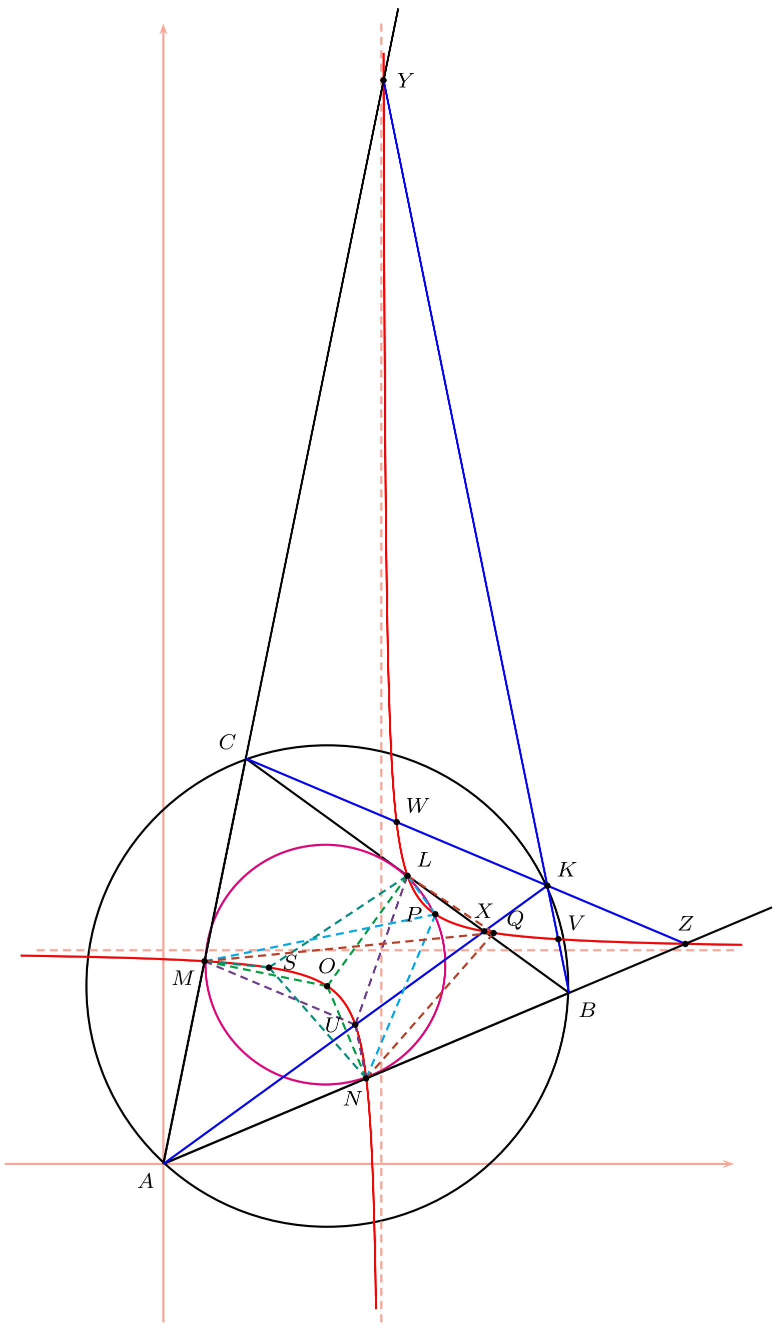

Figure 2

λ	1	$\frac{ac}{bd}$	$\frac{bd}{ac}$	$\frac{b(b-d)}{a(a-c)}$	$\frac{d(b-d)}{c(a-c)}$	$\frac{(b-d)^2}{(a-c)^2}$	$\frac{d^2}{c^2}$	$\frac{b^2}{a^2}$
Point from λ	P	Q	L	M	N	X	Y	Z
Point from $-\lambda$	O	S	U	V	W			

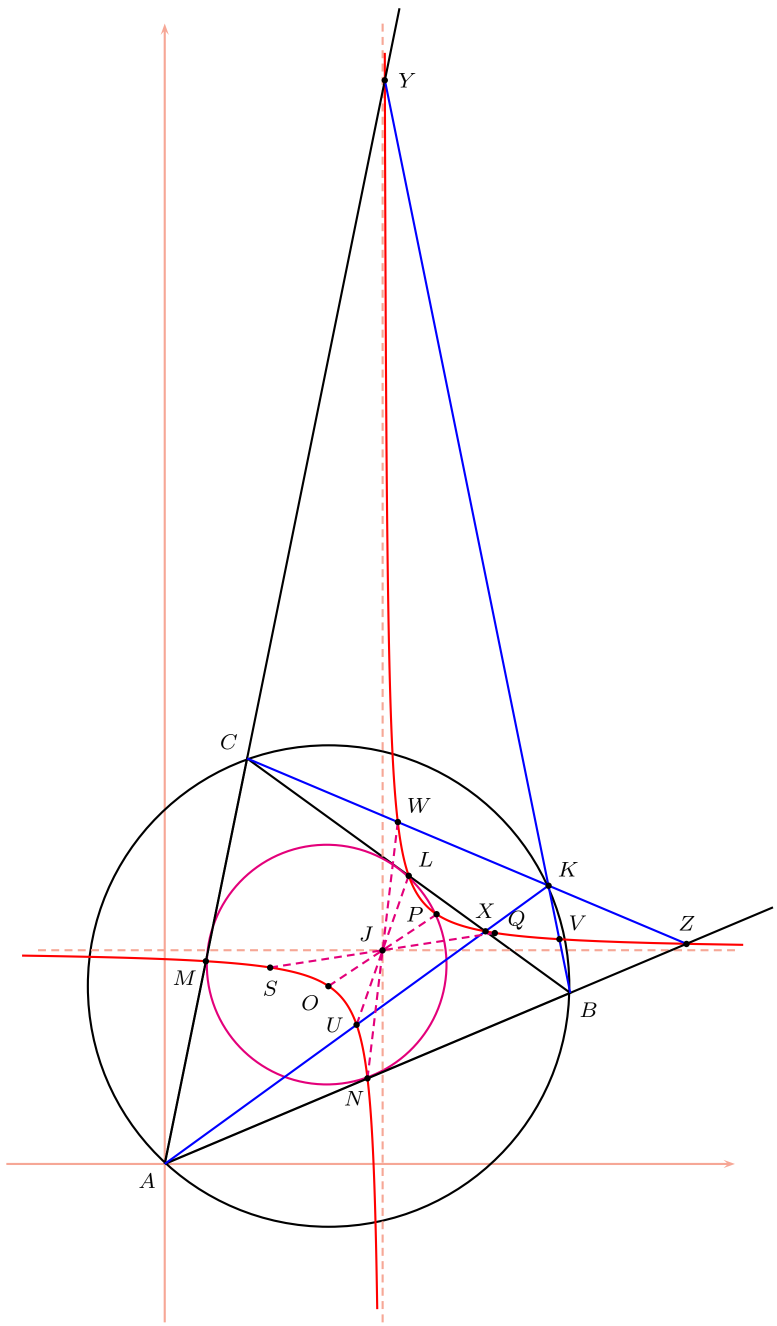

Figure 3

λ	1	$\frac{ac}{bd}$	$\frac{bd}{ac}$	$\frac{b(b-d)}{a(a-c)}$	$\frac{d(b-d)}{c(a-c)}$	$\frac{(b-d)^2}{(a-c)^2}$	$\frac{d^2}{c^2}$	$\frac{b^2}{a^2}$
Point from λ	P	Q	L	M	N	X	Y	Z
Point from $-\lambda$	O	S	U	V	W			

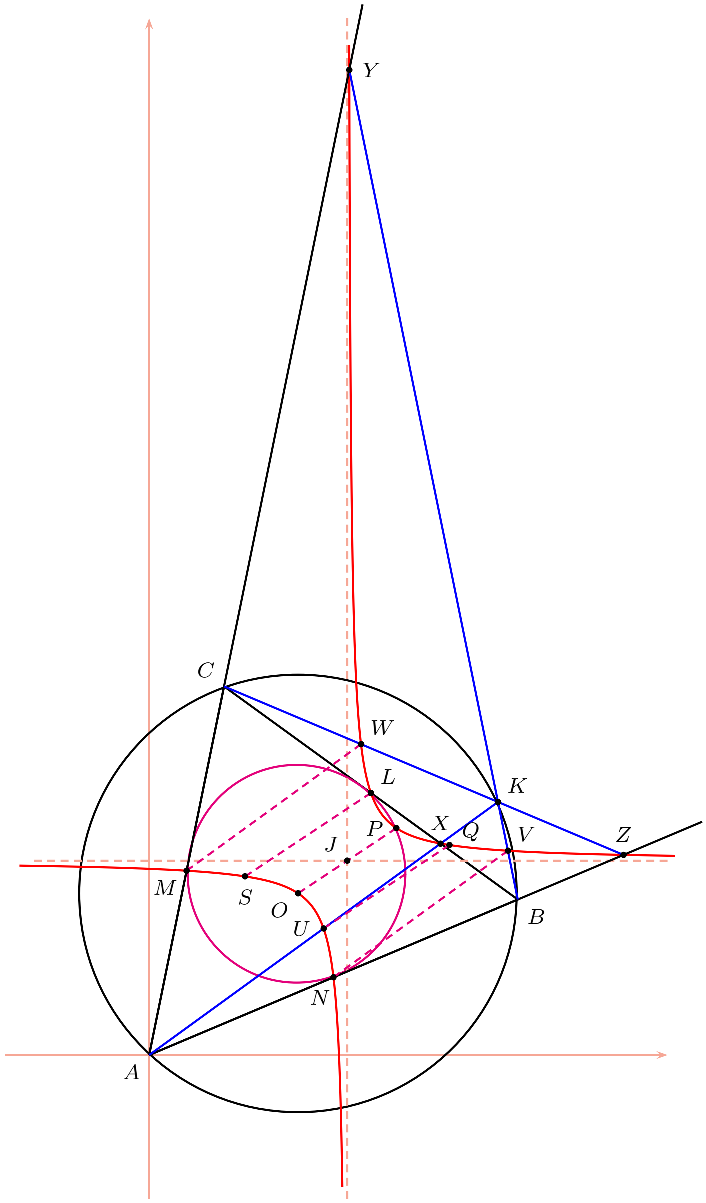

Figure 4

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The Gergonne problem

Nikolaos Dergiades

Abstract. An effective method for the proof of geometric inequalities is the use of the dot product of vectors. In this paper we use this method to solve some famous problems, namely Heron's problem, Fermat's problem and the extension of the previous problem in space, the so called Gergonne's problem. The solution of this last is erroneously stated, but not proved, in F.G.-M.

1. Introduction

In this paper whenever we write AB we mean the length of the vector $\mathbf{AB}$, i.e. $AB = |\mathbf{AB}|$. The method of using the dot product of vectors to prove geometric inequalities consists of using the following well known properties:

(1) $\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$.

(2) $\mathbf{a} \cdot \mathbf{i} \leq \mathbf{a} \cdot \mathbf{j}$ if $\mathbf{i}$ and $\mathbf{j}$ are unit vectors and $\angle(\mathbf{a}, \mathbf{i}) \geq \angle(\mathbf{a}, \mathbf{j})$.

(3) If $\mathbf{i} = \frac{\mathbf{AB}}{|\mathbf{AB}|}$ is the unit vector along $\mathbf{AB}$, then the length of the segment AB is given by

$$AB = \mathbf{i} \cdot \mathbf{AB}.$$

2. The Heron problem and the Fermat point

2.1. *Heron's problem.* A point O on a line XY gives the smallest sum of distances from the points A, B (on the same side of XY) if $\angle XOA = \angle BOY$.

Proof. If M is an arbitrary point on XY (see Figure 1) and $\mathbf{i}, \mathbf{j}$ are the unit vectors of $\mathbf{OA}, \mathbf{OB}$ respectively, then the vector $\mathbf{i} + \mathbf{j}$ is perpendicular to XY since it bisects the angle between $\mathbf{i}$ and $\mathbf{j}$. Hence $(\mathbf{i} + \mathbf{j}) \cdot \mathbf{OM} = 0$ and

$$\begin{aligned}
OA + OB &= \mathbf{i} \cdot \mathbf{OA} + \mathbf{j} \cdot \mathbf{OB} \\
&= \mathbf{i} \cdot (\mathbf{OM} + \mathbf{MA}) + \mathbf{j} \cdot (\mathbf{OM} + \mathbf{MB}) \\
&= (\mathbf{i} + \mathbf{j}) \cdot \mathbf{OM} + \mathbf{i} \cdot \mathbf{MA} + \mathbf{j} \cdot \mathbf{MB} \\
&= \mathbf{i} \cdot \mathbf{MA} + \mathbf{j} \cdot \mathbf{MB} \\
&\leq |\mathbf{i}||\mathbf{MA}| + |\mathbf{j}||\mathbf{MB}| \\
&= MA + MB.
\end{aligned}$$

□

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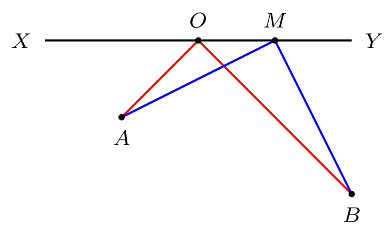

Figure 1

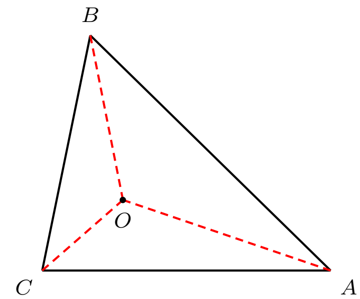

Figure 2

2.2. *The Fermat point.* If none of the angles of a triangle ABC exceeds $120°$, the point O inside a triangle ABC such that $\angle BOC = \angle COA = \angle AOB = 120°$ gives the smallest sum of distances from the vertices of ABC. See Figure 2.

Proof. If M is an arbitrary point and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors of $\mathbf{OA}$, $\mathbf{OB}$, $\mathbf{OC}$, then $\mathbf{i}+\mathbf{j}+\mathbf{k}=\mathbf{0}$ since this vector does not changes by a $120°$ rotation. Hence,

$$\begin{aligned}
OA+OB+OC &= \mathbf{i}\cdot\mathbf{OA}+\mathbf{j}\cdot\mathbf{OB}+\mathbf{k}\cdot\mathbf{OC}\\
&= \mathbf{i}\cdot(\mathbf{OM}+\mathbf{MA})+\mathbf{j}\cdot(\mathbf{OM}+\mathbf{MB})+\mathbf{k}\cdot(\mathbf{OM}+\mathbf{MC})\\
&= (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{OM}+\mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC}\\
&= \mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC}\\
&\le |\mathbf{i}||\mathbf{MA}|+|\mathbf{j}||\mathbf{MB}|+|\mathbf{k}||\mathbf{MC}|\\
&= MA+MB+MC.
\end{aligned}$$

$\square$

3. The Gergonne problem

Given a plane π and a triangle ABC not lying in the plane, the Gergonne problem [3] asks for a point O on a plane π such that the sum $OA+OB+OC$ is minimum. This is an extention of Fermat's problem to 3 dimensions. According to [2, pp. 927–928], [1] this problem had hitherto been unsolved (for at least 90 years). Unfortunately, as we show in §4.1 below, the solution given there, for the special case when the planes π and ABC are parallel, is erroneous. We present a solution here in terms of the centroidal line of a trihedron. We recall the definition which is based on the following fact. See, for example, [1, p.43].

Proposition and Definition. *The three planes determined by the edges of a trihedral angle and the internal bisectors of the respective opposite faces intersect in a line. This line is called the centroidal line of the trihedron.*

Theorem 1. *If O is a point on the plane π such that the centroidal line of the trihedron $O.ABC$ is perpendicular to π, then $OA+OB+OC \le MA+MB+MC$ for every point M on π.*

[1] Problem 742-III, especially 1901 c3.

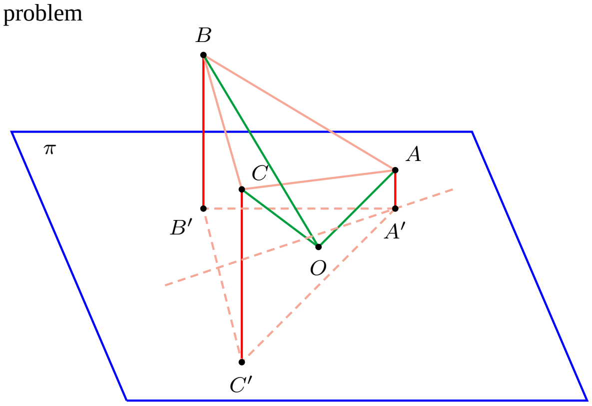

Figure 3

Proof. Let M be an arbitrary point on π, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors along $\mathbf{OA}$, $\mathbf{OB}$, $\mathbf{OC}$ respectively. The vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is parallel to the centroidal line of the trihedron $O.ABC$. Since this line is perpendicular to π by hypothesis we have

$$(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{OM}=0. \tag{1}$$

Hence,

$$\begin{aligned}
OA+OB+OC &= \mathbf{i}\cdot\mathbf{OA}+\mathbf{j}\cdot\mathbf{OB}+\mathbf{k}\cdot\mathbf{OC}\\
&= \mathbf{i}\cdot(\mathbf{OM}+\mathbf{MA})+\mathbf{j}\cdot(\mathbf{OM}+\mathbf{MB})+\mathbf{k}\cdot(\mathbf{OM}+\mathbf{MC})\\
&= (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{OM}+\mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC}\\
&= \mathbf{i}\cdot\mathbf{MA}+\mathbf{j}\cdot\mathbf{MB}+\mathbf{k}\cdot\mathbf{MC}\\
&\le |\mathbf{i}||\mathbf{MA}|+|\mathbf{j}||\mathbf{MB}|+|\mathbf{k}||\mathbf{MC}|\\
&= MA+MB+MC.
\end{aligned}$$

□

4. Examples

We set up a rectangular coordinate system such that A, B, C, are the points $(a,0,p)$, $(0,b,q)$ and $(0,c,r)$. Let A', B', C' be the orthogonal projections of A, B, C on the plane π. Write the coordinates of O as $(x,y,0)$. The x- and y-axes are the altitude from A' and the line $B'C'$ of triangle $A'B'C'$ in the plane π. Since

$$\begin{aligned}
\mathbf{i} &= \frac{-1}{\sqrt{(x-a)^2+y^2+p^2}}(x-a,y,-p),\\
\mathbf{j} &= \frac{-1}{\sqrt{x^2+(y-b)^2+q^2}}(x,y-b,-q),\\
\mathbf{k} &= \frac{-1}{\sqrt{x^2+(y-c)^2+r^2}}(x,y-c,-r),
\end{aligned}$$

it is sufficient to put in (1) for $\mathbf{OM}$ the vectors $(1,0,0)$ and $(0,1,0)$. From these, we have

$$\begin{aligned}\frac{x-a}{\sqrt{(x-a)^2+y^2+p^2}}+\frac{x}{\sqrt{x^2+(y-b)^2+q^2}}+\frac{x}{\sqrt{x^2+(y-c)^2+r^2}} &= 0,\\ \frac{y}{\sqrt{(x-a)^2+y^2+p^2}}+\frac{y-b}{\sqrt{x^2+(y-b)^2+q^2}}+\frac{y-c}{\sqrt{x^2+(y-c)^2+r^2}} &= 0.\end{aligned} \tag{2}$$

The solution of this system cannot in general be expressed in terms of radicals, as it leads to equations of high degree. It is therefore in general not possible to construct the point O using straight edge and compass. We present several examples in which O is constructible. In each of these examples, the underlying geometry dictates that $y = 0$, and the corresponding equation can be easily written down.

4.1. *π parallel to ABC.* It is very easy to mistake for O the Fermat point of triangle $A'B'C'$, as in [2, loc. cit.]. If we take $p = q = r = 3$, $a = 14$, $b = 2$, and $c = -2$, the system (2) gives $y = 0$ and

$$\frac{x-14}{\sqrt{(x-14)^2+9}}+\frac{2x}{\sqrt{x^2+13}}=0, \qquad x>0.$$

This leads to the quartic equation

$$3x^4-84x^3+611x^2+364x-2548=0.$$

This quartic polynomial factors as $(x-2)(3x^3-78x^2+455x+1274)$, and the only positive root of which is $x = 2$.[2] Hence $\angle B'OC' = 90°$, $\angle A'OB' = 135°$, and $\angle A'OC' = 135°$, showing that O is not the Fermat point of triangle $A'B'C'$.[3]

4.2. *ABC isosceles with A on π and BC parallel to π.* In this case, $p = 0$, $q = r$, $c = -b$, and we may assume $a > 0$. The system (2) reduces to $y = 0$ and

$$\frac{x-a}{|x-a|}+\frac{2x}{\sqrt{x^2+b^2+q^2}}=0.$$

Since $0 < x < a$, we get

$$(x,y)=\left(\sqrt{\frac{b^2+q^2}{3}},\ 0\right)$$

with $b^2 + q^2 < 3a^2$. Geometrically, since $OB = OC$, the vectors $\mathbf{i}$, $\mathbf{j} - \mathbf{k}$ are parallel to π. We have

$$\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=0, \qquad (\mathbf{i}+\mathbf{j}+\mathbf{k})(\mathbf{j}-\mathbf{k})=0.$$

Equivalently,

$$\mathbf{i}\cdot\mathbf{j}+\mathbf{i}\cdot\mathbf{k}=-1, \qquad \mathbf{i}\cdot\mathbf{j}-\mathbf{i}\cdot\mathbf{k}=0.$$

Thus, $\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = -\frac{1}{2}$ or $\angle AOB = \angle AOC = 120°$, a fact that is a generalization of the Fermat point to 3 dimensions.

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[2]The cubic factor has one negative root ≈ -2.03472, and two non-real roots. If, on the other hand, we take $p = q = r = 2$, the resulting equation becomes $3x^4-84x^3+596x^2+224x-1568 = 0$, which is irreducible over rational numbers. It roots are not constructible using ruler and compass. The positive real root is $x \approx 1.60536$. There is a negative root ≈ -1.61542 and two non-real roots.

[3]The solution given in [2] assumes erroneously OA, OB, OC equally inclined to the planes π and of triangle ABC.

If $b^2+q^2 \geq 3a^2$, the centroidal line cannot be perpendicular to π, and Theorem 1 does not help. In this case we take as point O to be the intersection of x-axis and the plane MBC. It is obvious that

$$MA+MB+MC \geq OA+OB+OC = |x-a| + 2\sqrt{x^2+b^2+q^2}.$$

We write $f(x) = |x-a| + 2\sqrt{x^2+b^2+q^2}$.

If $0 < a < x$, then $f'(x) = 1 + \frac{2x}{\sqrt{x^2+b^2+q^2}} > 0$ and f is an increasing function. For $x \leq 0$, $f'(x) = -1 + \frac{2x}{\sqrt{x^2+b^2+q^2}} < 0$ and f is a decreasing function.

If $0 < x \leq a \leq \sqrt{\frac{b^2+q^2}{3}}$, then $4x^2 \leq x^2+b^2+q^2$ so that $f'(x) = -1 + \frac{2x}{\sqrt{x^2+b^2+q^2}} \leq 0$ and f is a decreasing function. Hence we have minimum when $x = a$ and $O \equiv A$.

4.3. *B, C on π.* If the points B, C lie on π, then the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is perpendicular to the vectors $\mathbf{j}$ and $\mathbf{k}$. From these, we obtain the interesting equality $\angle AOB = \angle AOC$. Note that they are not neccesarily equal to $120°$, as in Fermat's case. Here is an example. If $a = 10$, $b = 8$, $c = -8$, $p = 3$, $q = r = 0$ the system (2) gives $y = 0$ and

$$\frac{x-10}{\sqrt{(x-10)^2+9}} + \frac{2x}{\sqrt{x^2+64}} = 0, \qquad 0 < x < 10,$$

which leads to the equation

$$3x^4 - 60x^3 + 272x^2 + 1280x - 6400 = 0.$$

This quartic polynomial factors as $(x-4)(3x^3 - 48x^2 + 80x + 1600)$. It follows that the only positive root is $x = 4$.[4] Hence we have

$$\angle AOB = \angle AOC = \arccos(-\frac{2}{5}) \qquad \text{and} \qquad \angle BOC = \arccos(-\frac{3}{5}).$$

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[2] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, 1991, Paris.
[3] J.-D. Gergonne, *Annales mathématiques de Gergonne*, 12 (1821-1822) 380.

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[4]The cubic factor has one negative root ≈ -4.49225, and two non-real roots.

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Pedal Triangles and Their Shadows

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Abstract. The pedal triangle of a point P with respect to a given triangle ABC casts equal shadows on the side lines of ABC if and only if P is the internal center of similitude of the circumcircle and the incircle of triangle ABC or the external center of the circumcircle with one of the excircles. We determine the common length of the equal shadows. More generally, we construct the point the shadows of whose pedal triangle are proportional to given p, q, r. Many interesting special cases are considered.

1. Shadows of pedal triangle

Let P be a point in the plane of triangle ABC, and $A'B'C'$ its pedal triangle, *i.e.*, A', B', C' are the pedals (orthogonal projections) of A, B, C on the side lines BC, CA, AB respectively. If B_a and C_a are the pedals of B' and C' on BC, we call the segment B_aC_a the *shadow* of $B'C'$ on BC. The shadows of $C'A'$ and $A'B'$ are segments C_bA_b and A_cB_c analogously defined on the lines CA and AB. See Figure 1.

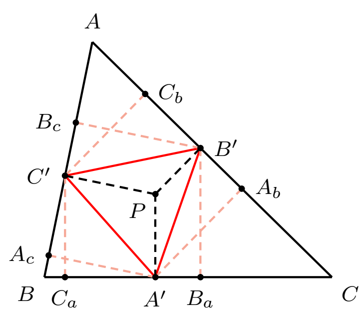

Figure 1

In terms of the *actual* normal coordinates x, y, z of P with respect to ABC,[1] the length of the shadow C_aB_a can be easily determined:

$$C_aB_a = CaA' + A'B_a = z\sin B + y\sin C. \tag{1}$$

In Figure 1, we have shown P as interior point of triangle ABC. For generic positions of P, we regard C_aB_a as a directed segment so that its length given by (1) is signed. Similarly, the shadows of $C'A'$ and $A'B'$ on the respective side lines have signed lengths $x\sin C + z\sin A$ and $y\sin A + x\sin B$.

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[1]Traditionally, normal coordinates are called trilinear coordinates. Here, we follow the usage of the old French term *coordonnées normales* in F.G.-M. [1], which is more suggestive. The actual normal (trilinear) coordinates of a point are the *signed* distances from the point to the three side lines.

Theorem 1. *The three shadows of the pedal triangle of P on the side lines are equal if and only if P is the internal center of similitude of the circumcircle and the incircle of triangle ABC, or the external center of similitude of the circumcircle and one of the excircles.*

Proof. These three shadows are equal if and only if

$$\epsilon_1(y\sin C + z\sin B) = \epsilon_2(z\sin A + x\sin C) = \epsilon_3(x\sin B + y\sin A)$$

for an appropriate choice of signs $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$ subject to the convention

$(\star)$ *at most one of* ϵ_1, ϵ_2, ϵ_3 *is negative.*

It follows that

$$\begin{array}{rcrcrcl} x\epsilon_2\sin C & - & y\epsilon_1\sin C & + & z(\epsilon_2\sin A - \epsilon_1\sin B) & = & 0, \\ x(\epsilon_3\sin B - \epsilon_2\sin C) & + & y\epsilon_3\sin A & - & z\epsilon_2\sin A & = & 0. \end{array}$$

Replacing, by the law of sines, $\sin A$, $\sin B$, $\sin C$ by the side lengths a, b, c respectively, we have

$$\begin{aligned} x : y : z &= \begin{vmatrix} -\epsilon_1 c & \epsilon_2 a - \epsilon_1 b \\ \epsilon_3 a & -\epsilon_2 a \end{vmatrix} : - \begin{vmatrix} \epsilon_2 c & \epsilon_2 a - \epsilon_1 b \\ \epsilon_3 b - \epsilon_2 c & -\epsilon_2 a \end{vmatrix} : \begin{vmatrix} \epsilon_2 c & -\epsilon_1 c \\ \epsilon_3 b - \epsilon_2 c & \epsilon_3 a \end{vmatrix} \\ &= a(\epsilon_3\epsilon_1 b + \epsilon_1\epsilon_2 c - \epsilon_2\epsilon_3 a) : b(\epsilon_1\epsilon_2 c + \epsilon_2\epsilon_3 a - \epsilon_3\epsilon_1 b) : c(\epsilon_2\epsilon_3 a + \epsilon_3\epsilon_1 b - \epsilon_1\epsilon_2 c) \\ &= a(\epsilon_2 b + \epsilon_3 c - \epsilon_1 a) : b(\epsilon_3 c + \epsilon_1 a - \epsilon_2 b) : c(\epsilon_1 a + \epsilon_2 b - \epsilon_3 c). \end{aligned} \tag{2}$$

If $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, this is the point X_{55} in [4], the internal center of similitude of the circumcircle and the incircle. We denote this point by T. See Figure 2A. We show that if one of ϵ_1, ϵ_2, ϵ_3 is negative, then P is the external center of similitude of the circumcircle and one of the excircles.

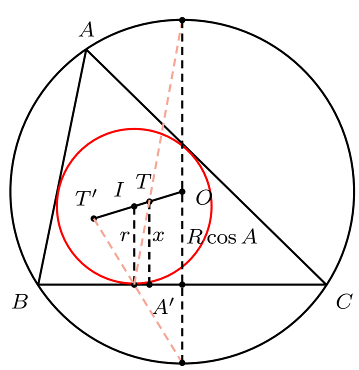

Figure 2A

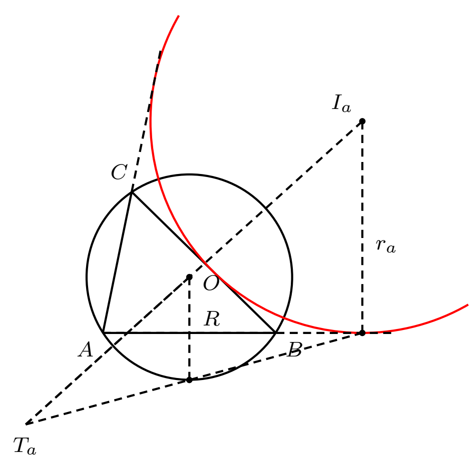

Figure 2B

Let R denote the circumradius, s the semiperimeter, and r_a the radius of the A-excircle. The actual normal coordinates of the circumcenter are $R\cos A$, $R\cos B$,

$R\cos C$, while those of the excenter I_a are $-r_a, r_a, r_a$. See Figure 2B. The external center of similitude of the two circles is the point T_a dividing I_aO in the ratio $I_aT_a : T_aO = r_a : -R$. As such, it is the point $\frac{1}{r_a-R}(r_a \cdot O - R \cdot I_a)$, and has normal coordinates

$$\begin{aligned} & -(1+\cos A) : 1-\cos B : 1-\cos C \\ = \; & -\cos^2\frac{A}{2} : \sin^2\frac{B}{2} : \sin^2\frac{C}{2} \\ = \; & -a(a+b+c) : b(a+b-c) : c(c+a-b). \end{aligned}$$

This coincides with the point given by (2) for $\epsilon_1 = -1$, $\epsilon_2 = \epsilon_3 = 1$. The cases for other choices of signs are similar, leading to the external centers of similitude with the other two excircles. □

Remark. With these coordinates, we easily determine the common length of the equal shadows in each case. For the point T, this common length is

$$\begin{aligned} y\sin C + z\sin B &= \frac{Rr}{R+r}((1+\cos B)\sin C + (1+\cos C)\sin B) \\ &= \frac{Rr}{R+r}(\sin A + \sin B + \sin C) \\ &= \frac{1}{R+r}\cdot\frac{1}{2}(a+b+c)r \\ &= \frac{\triangle}{R+r}, \end{aligned}$$

where $\triangle$ denotes the area of triangle ABC. For T_a, the common length of the equal shadows is $\left|\frac{\triangle}{r_a-R}\right|$; similarly for the other two external centers of similitudes.

2. Pedal triangles with shadows in given proportions

If the signed lengths of the shadows of the sides of the pedal triangle of P (with normal coordinates $(x : y : z)$) are proportional to three given quantities p, q, r, then

$$\frac{cy+bz}{p} = \frac{az+cx}{q} = \frac{bx+ay}{r}.$$

From these, we easily obtain the normal of coordinates of P:

$$(a(-ap+bq+cr) : b(ap-bq+cr) : c(ap+bq-cr)). \tag{3}$$

This follows from a more general result, which we record for later use.

Lemma 2. *The solution of*

$$f_1x + g_1y + h_1z = f_2x + g_2y + h_2z = f_3x + g_3y + h_3z$$

is

$$x : y : z = \begin{vmatrix} 1 & g_1 & h_1 \\ 1 & g_2 & h_2 \\ 1 & g_3 & h_3 \end{vmatrix} : \begin{vmatrix} f_1 & 1 & h_1 \\ f_2 & 1 & h_2 \\ f_3 & 1 & h_3 \end{vmatrix} : \begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix}.$$

Proof. since there are two linear equations in three indeterminates, solution is unique up to a proportionality constant. To verify that this is the correct solution, note that for $i = 1, 2, 3$, substitution into the i-th linear form gives

$$-\begin{vmatrix} 0 & f_i & g_i & h_i \\ 1 & f_1 & g_1 & h_1 \\ 1 & f_2 & g_2 & h_2 \\ 1 & f_3 & g_3 & h_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & f_1 & g_1 & h_1 \\ 1 & f_2 & g_2 & h_2 \\ 1 & f_3 & g_3 & h_3 \end{vmatrix} = \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}$$

up to a constant. □

Proposition 3. *The point the shadows of whose pedal triangle are in the ratio $p : q : r$ is the perspector of the cevian triangle of the point with normal coordinates $(\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$ and the tangential triangle of ABC.*

Proof. If Q is the point with normal coordinates $(\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$, then P, with coordinates given by (3), is the Q-Ceva conjugate of the symmedian point $K = (a : b : c)$. See [3, p.57]. □

If we assume p, q, r positive, there are four points satisfying

$$\frac{cy + bz}{\epsilon_1 p} = \frac{az + cx}{\epsilon_2 q} = \frac{bx + ay}{\epsilon_3 r},$$

for signs ϵ_1, ϵ_2, ϵ_3 satisfying $(\star)$. Along with P given by (3), there are

$$\begin{aligned} P_a &= (-a(ap + bq + cr) : b(ap + bq - cr) : c(ap - bq + cr), \\ P_b &= (a(-ap + bq + cr) : -b(ap + bq + cr) : c(-ap + bq + cr), \\ P_c &= (a(ap - bq + cr) : b(-ap + bq + cr) : -c(ap + bq + cr). \end{aligned}$$

While it is clear that $P_aP_bP_c$ is perspective with ABC at

$$\left(\frac{a}{-ap + bq + cr} : \frac{b}{ap - bq + cr} : \frac{c}{ap + bq - cr}\right),$$

the following observation is more interesting and useful in the construction of these points from P.

Proposition 4. *$P_aP_bP_c$ is the anticevian triangle of P with respect to the tangential triangle of ABC.*

Proof. The vertices of the tangential triangle are

$$A' = (-a : b : c), \qquad B' = (a : -b : c), \qquad C' = (a : b : -c).$$

From

$$\begin{aligned} & (a(-ap + bq + cr),\ b(ap - bq + cr),\ c(ap + bq - cr)) \\ = \ & ap(-a, b, c) + (a(bq + cr),\ -b(bq - cr),\ c(bq - cr)), \end{aligned}$$

and

$$\begin{aligned} & (-a(ap + bq + cr),\ b(ap + bq - cr),\ c(ap - bq + cr)) \\ = \ & ap(-a, b, c) - (a(bq + cr),\ -b(bq - cr),\ c(bq - cr)), \end{aligned}$$

we conclude that P and P_a divide A' and $A'' = (a(bq+r) : -b(bq-cr) : c(bq-cr))$ harmonically. But since

$$(a(bq+cr),\ -b(bq-cr),\ c(bq-cr)) = bq(a,-b,c) + cr(a,b,-c),$$

the point A'' is on the line $B'C'$. The cases for P_b and P_c are similar, showing that triangle $P_aP_bP_c$ is the anticevian triangle of P in the tangential triangle. $\square$

3. Examples

3.1. *Shadows proportional to side lengths.* If $p : q : r = a : b : c$, then P is the circumcenter O. The pedal triangle of O being the medial triangle, the lengths of the shadows are halves of the side lengths. Since the circumcenter is the incenter or one of the excenters of the tangential triangle (according as the triangle is acute- or obtuse-angled), the four points in question are the circumcenter and the excenters of the tangential triangle. [2]

3.2. *Shadows proportional to altitudes.* If $p : q : r = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, then P is the symmedian point $K = (a : b : c)$. [3] Since K is the Gergonne point of the tangential triangle, the other three points, with normal coordinates $(3a : -b : -c)$, $(-a : 3b : -c)$, and $(-a : -b : 3c)$, are the Gergonne points of the excircles of the tangential triangle. These are also the cases when the shadows are inversely proportional to the distances from P to the side lines, or, equivalently, when the triangles PB_aC_a, PC_aB_a and PA_cB_c have equal areas. [4]

3.3. *Shadows inversely proportional to exradii.* If $p : q : r = \frac{1}{r_a} : \frac{1}{r_b} : \frac{1}{r_c} = b+c-a : c+a-b : a+b-c$, then P is the point with normal coordinates $(\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{c}{a+b-c}) = (ar_a : br_b : cr_c)$. This is the *external* center of similitude of the circumcircle and the incircle, which we denote by T'. See Figure 2A. This point appears as X_{56} in [4]. The other three points are the *internal* centers of similitude of the circumcircle and the three excircles.

3.4. *Shadows proportional to exradii.* If $p : q : r = r_a : r_b : r_c = \tan\frac{A}{2} : \tan\frac{B}{2} : \tan\frac{C}{2}$, then P has normal coordinates

$$\begin{aligned}
&a(b\tan\frac{B}{2} + c\tan\frac{C}{2} - a\tan\frac{A}{2}) : b(c\tan\frac{C}{2} + a\tan\frac{A}{2} - b\tan\frac{B}{2}) : c(a\tan\frac{A}{2} + b\tan\frac{B}{2} - c\tan\frac{C}{2})\\
\sim&2a(\sin^2\frac{B}{2} + \sin^2\frac{C}{2} - \sin^2\frac{A}{2}) : 2b(\sin^2\frac{C}{2} + \sin^2\frac{A}{2} - \sin^2\frac{B}{2}) : 2c(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2})\\
\sim&a(1+\cos A - \cos B - \cos C) : b(1+\cos B - \cos C - \cos A) : c(1+\cos C - \cos A - \cos B).
\end{aligned} \tag{4}$$

[2] If ABC is right-angled, the tangential triangle degenerates into a pair of parallel lines, and there is only one finite excenter.

[3] More generally, if $p : q : r = a^n : b^n : c^n$, then the normal coordinates of P are

$$(a(b^{n+1} + c^{n+1} - a^{n+1}) : b(c^{n+1} + a^{n+1} - b^{n+1}) : c(a^{n+1} + b^{n+1} - c^{n+1})).$$

[4] For signs $\epsilon_1, \epsilon_2, \epsilon_3$ satisfying $(\star)$, the equations $\epsilon_1 x(cy+bz) = \epsilon_2 y(az+cx) = \epsilon_3 z(bx+ay)$ can be solved for $yz : zx : xy$ by an application of Lemma 2. From this it follows that $x : y : z = (\epsilon_2+\epsilon_3-\epsilon_1)a : (\epsilon_3+\epsilon_1-\epsilon_2)b : (\epsilon_1+\epsilon_2-\epsilon_3)c$.

This is the point X_{198} of [4]. It can be constructed, according to Proposition 3, from the point with normal coordinates $(\frac{1}{r_a} : \frac{1}{r_b} : \frac{1}{r_c}) = (s-a : s-b : s-c)$, the Mittenpunkt. [5]

4. A synthesis

The five triangle centers we obtained with special properties of the shadows of their pedal triangles, namely, O, K, T, T', and the point P in §3.4, can be organized together in a very simple way. We take a closer look at the coordinates of P given in (4) above. Since

$$1 - \cos A + \cos B + \cos C = 2 - 4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 2 - \frac{r_a}{R},$$

the normal coordinates of P can be rewritten as

$$(a(2R - r_a) : b(2R - r_b) : c(2R - r_c)).$$

These coordinates indicate that P lies on the line joining the symmedian point $K(a : b : c)$ to the point $(ar_a : br_b : cr_c)$, the point T' in §3.3, with division ratio

$$\begin{aligned} T'P : PK &= 2R(a^2+b^2+c^2) : -(a^2r_a + b^2r_b + c^2r_c) \\ &= R(a^2+b^2+c^2) : -2(R-r)s^2. \end{aligned} \tag{5}$$

To justify this last expression, we compute in two ways the distance from T' to the line BC, and obtain

$$\frac{2\triangle}{a^2r_a + b^2r_b + c^2r_c} \cdot ar_a = \frac{Rr}{R-r}(1 - \cos A).$$

From this,

$$\begin{aligned} a^2r_a + b^2r_b + c^2r_c &= \frac{2\triangle(R-r)}{Rr} \cdot \frac{ar_a}{1 - \cos A} \\ &= \frac{2\triangle(R-r)}{Rr} \cdot \frac{4R\sin\frac{A}{2}\cos\frac{A}{2} \cdot s\tan\frac{A}{2}}{2\sin^2\frac{A}{2}} \\ &= 4(R-r)s^2. \end{aligned}$$

This justifies (5) above.

Consider the intersection X of the line TP with OK. See Figure 3. Applying Menelaus' theorem to triangle OKT' with transversal TXP, we have

$$\frac{OX}{XK} = -\frac{OT}{TT'} \cdot \frac{T'P}{PK} = \frac{R-r}{2r} \cdot \frac{R(a^2+b^2+c^2)}{2(R-r)s^2} = \frac{R(a^2+b^2+c^2)}{4\triangle s}.$$

This expression has an interesting interpretation. The point X being on the line OK, it is the isogonal conjugate of a point on the Kiepert hyperbola. Every point on this hyperbola is the perspector of the apexes of similar isosceles triangles constructed on the sides of ABC. If this angle is taken to be $\arctan\frac{s}{R}$, and the

[5] This appears as X_9 in [4], and can be constructed as the perspector of the excentral triangle and the medial triangle, *i.e.*, the intersection of the three lines each joining an excenter to the midpoint of the corresponding side of triangle ABC.

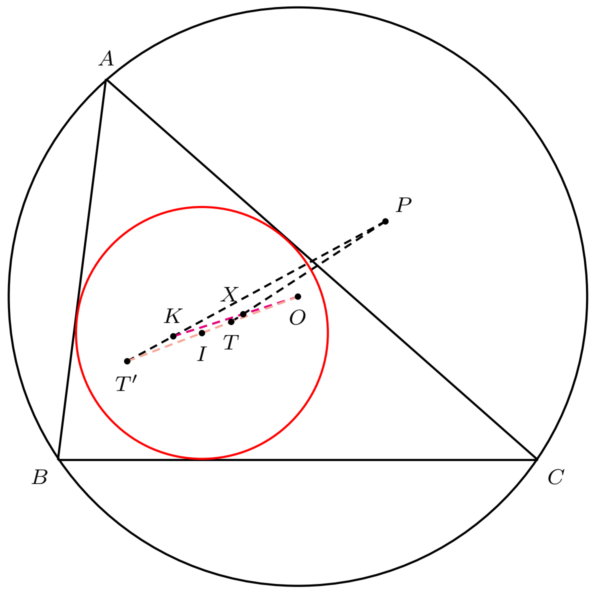

Figure 3

isosceles triangles constructed externally on the sides of triangle ABC, then the isogonal conjugate of the perspector is precisely the point X.

This therefore furnishes a construction for the point P.[6]

5. Two more examples

5.1. *Shadows of pedal triangle proportional to distances from circumcenter to side lines.* The point P is the perspector of the tangential triangle and the cevian triangle of $\left(\frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}\right)$, which is the orthic triangle of ABC. The two triangles are indeed homothetic at the Gob perspector on the Euler line. See [2, pp.259–260]. It has normal coordinates $(a \tan A : b \tan B : c \tan C)$, and appears as X_{25} in [4].

5.2. *Shadows of pedal triangles proportional to distances from orthocenter to side lines.* In this case, P is the perspector of the tangential triangle and the cevian triangle of the circumcenter. This is the point with normal coordinates

$$(a(-\tan A+\tan B+\tan C) : b(\tan A-\tan B+\tan C) : c(\tan A+\tan B-\tan C)),$$

and is the centroid of the tangential triangle. It appears as X_{154} in [4]. The other three points with the same property are the vertices of the anticomplementary triangle of the tangential triangle.

[6]The same P can also be constructed as the intersection of KT' and the line joining the incenter to Y on OK, which is the isogonal conjugate of the perspector (on the Kiepert hyperbola) of apexes of similar isosceles triangles with base angles $\arctan \frac{s}{2R}$ constructed externally on the sides of ABC.

6. The midpoints of shadows as pedals

The midpoints of the shadows of the pedal triangle of $P = (x : y : z)$ are the pedals of the point

$$P' = (x+y\cos C+z\cos B : \ y+z\cos A+x\cos C : \ z+x\cos B+y\cos A) \quad (6)$$

in normal coordinates. This is equivalent to the concurrency of the perpendiculars from the midpoints of the sides of the pedal triangle of P to the corresponding sides of ABC.[7] See Figure 4.

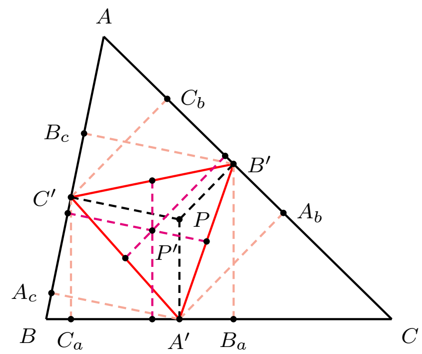

Figure 4

If P is the symmedian point, with normal coordinates $(\sin A : \sin B : \sin C)$, it is easy to see that P' is the same symmedian point.

Proposition 5. *There are exactly four points for each of which the midpoints of the sides of the pedal triangle are equidistant from the corresponding sides of ABC.*

Proof. The midpoints of the sides of the pedal triangle have *signed* distances

$$x+\frac{1}{2}(y\cos C+z\cos B), \quad y+\frac{1}{2}(z\cos A+x\cos C), \quad z+\frac{1}{2}(x\cos B+y\cos A)$$

from the respective sides of ABC. The segments joining the midpoints of the sides and their shadows are equal in length if and only if

$$\epsilon_1(2x+y\cos C+z\cos B) = \epsilon_2(2y+z\cos A+x\cos C) = \epsilon_3(2z+x\cos B+y\cos A)$$

for ϵ_1, ϵ_2, ϵ_3 satisfying $(\star)$. From these, we obtain the four points.

For $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, this gives the point

$$\begin{aligned} M = &((2-\cos A)(2+\cos A-\cos B-\cos C) \\ &:(2-\cos B)(2+\cos B-\cos C+\cos A) \\ &:(2-\cos C)(2+\cos C-\cos A+\cos B)) \end{aligned}$$

in normal coordinates, which can be constructed as the incenter-Ceva conjugate of

$$Q = (2-\cos A : 2-\cos B : 2-\cos C),$$

[7] If x, y, z are the actual normal coordinates of P, then those of P' are halves of those given in (6) above, and P' is $\frac{x}{2}$, $\frac{y}{2}$, and $\frac{z}{2}$ below the midpoints of the respective sides of the pedal triangle.

See [3, p.57]. This point Q divides the segments OI externally in the ratio $OQ : QI = 2R : -r$. See Figures 5A and 5B.

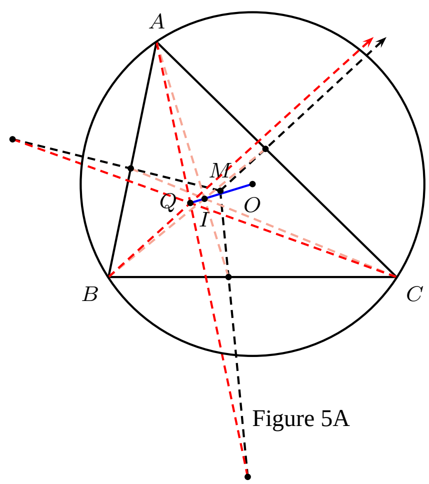

Figure 5A

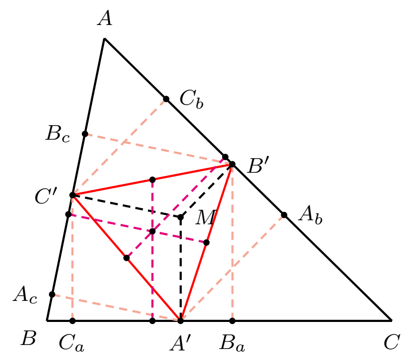

Figure 5B

There are three other points obtained by choosing one negative sign among ϵ_1, ϵ_2, ϵ_3. These are

$$\begin{aligned} M_a = (&-(2-\cos A)(2+\cos A+\cos B+\cos C) \\ &:(2+\cos B)(2-\cos A-\cos B+\cos C) \\ &:(2+\cos C)(2-\cos A+\cos B-\cos C)), \end{aligned}$$

and M_b, M_c whose coordinates can be written down by appropriately changing signs. It is clear that $M_aM_bM_c$ and triangle ABC are perspective at

$$M' = \left(\frac{2+\cos A}{2+\cos A-\cos B-\cos C} : \frac{2+\cos B}{2-\cos A+\cos B-\cos C} : \frac{2+\cos C}{2-\cos A-\cos B+\cos C}\right).$$

□

The triangle centers Q, M, and M' in the present section apparently are not in [4].

Appendix: Pedal triangles of a given shape

The side lengths of the pedal triangle of P are given by $AP \cdot \sin A$, $BP \cdot \sin B$, and $CP \cdot \sin C$. [2, p.136]. This is similar to one with side lengths $p : q : r$ if and only if the *tripolar* coordinates of P are

$$AP : BP : CP = \frac{p}{a} : \frac{q}{b} : \frac{r}{c}.$$

In general, there are two such points, which are common to the three generalized Apollonian circles associated with the point $(\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$ in *normal* coordinates. See, for example, [5]. In the case of equilateral triangles, these are the isodynamic points.

Acknowledgement. The authors express their sincere thanks to the Communicating Editor for valuable comments that improved this presentation.

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Some Properties of the Lemoine Point

Alexei Myakishev

Abstract. The Lemoine point, K, of $\triangle ABC$ has special properties involving centroids of pedal triangles. These properties motivate a definition of Lemoine field, F, and a coordinatization of the plane of $\triangle ABC$ using perpendicular axes that pass through K. These principal axes are symmetrically related to two other lines: one passing through the isodynamic centers, and the other, the isogonic centers.

1. Introduction

Let $A'B'C'$ be the pedal triangle of an arbitrary point Z in the plane of a triangle ABC, and consider the vector field $\mathbf{F}$ defined by $\mathbf{F}(Z) = \mathbf{ZA'} + \mathbf{ZB'} + \mathbf{ZC'}$. It is well known that $\mathbf{F}(Z)$ is the zero vector if and only if Z is the Lemoine point, K, also called the symmedian point. We call $\mathbf{F}$ the *Lemoine field* of $\triangle ABC$ and K the *balance point* of $\mathbf{F}$.

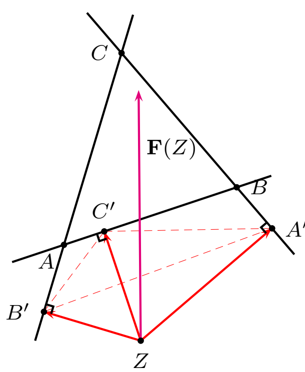

Figure 1

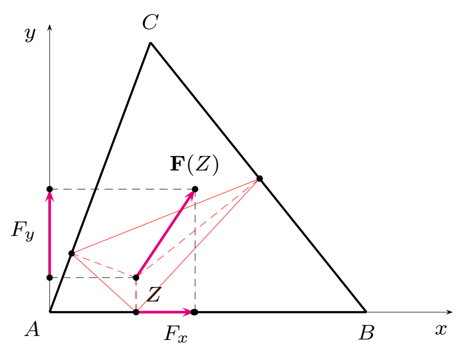

Figure 2

The Lemoine field may be regarded as a physical force field. Any point Z in this field then has a natural motion along a certain curve, or trajectory. See Figure 1. We shall determine parametric equations for these trajectories and find, as a result, special properties of the lines that bisect the angles between the line of the isogonic centers and the line of the isodynamic centers of $\triangle ABC$.

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Я благодарю дорогую Лену за оказанную мне моралную поддержку. Также я очен признателен профессору Кимберлингу за разрешение многочисленных проблем, касающихся англииского языка. The author dedicates his work to *Helen* and records his appreciation to the Communicating Editor for assistance in translation.

2. The Lemoine equation

In the standard cartesian coordinate system, place $\triangle ABC$ so that $A = (0,0)$, $B = (c,0)$, $C = (m,n)$, and write $Z = (x,y)$. For any line $Px + Qy + R = 0$, the vector H from Z to the projection of Z on the line has components

$$h_x = \frac{-P}{P^2+Q^2}(Px+Qy+R), \qquad h_y = \frac{-Q}{P^2+Q^2}(Px+Qy+R).$$

From these, one find the components of the three vectors whose sum defines $\mathbf{F}(Z)$:

vector	$x-\text{component}$	$y-\text{component}$
$\mathbf{ZA}'$	$\frac{-n(nx+y(c-m)-cn)}{n^2+(c-m)^2}$	$\frac{(m-c)(nx+y(c-m)-cn)}{n^2+(c-m)^2}$
$\mathbf{ZB}'$	$\frac{-n(nx-my)}{m^2+n^2}$	$\frac{m(nx-my)}{m^2+n^2}$
$\mathbf{ZC}'$	0	$-y$

The components of the Lemoine field $\mathbf{F}(Z) = \mathbf{ZA}' + \mathbf{ZB}' + \mathbf{ZC}'$ are given by

$$F_x = -(\alpha x + \beta y) + d_x, \qquad F_y = -(\beta x + \gamma y) + d_y,$$

where

$$\alpha = \tfrac{n^2}{m^2+n^2} + \tfrac{n^2}{n^2+(c-m)^2}, \qquad \beta = \tfrac{-mn}{m^2+n^2} + \tfrac{n(c-m)}{n^2+(c-m)^2},$$
$$\gamma = 1 + \tfrac{m^2}{m^2+n^2} + \tfrac{(c-m)^2}{n^2+(c-m)^2};$$
$$d_x = \tfrac{cn^2}{n^2+(c-m)^2}, \qquad d_y = \tfrac{cn(c-m)}{n^2+(c-m)^2}.$$

See Figure 2. Assuming a unit mass at each point Z, Newton's Second Law now gives a system of differential equations:

$$x'' = -(\alpha x + \beta y) + d_x, \qquad y'' = -(\beta x + \gamma y) + d_y,$$

where the derivatives are with respect to time, t. We now translate the origin from $(0,0)$ to the balance point (d_x, d_y), which is the Lemoine point K, thereby obtaining the system

$$x'' = -(\alpha x + \beta y), \qquad y'' = -(\beta x + \gamma y),$$

which has the matrix form

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -M \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

where $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$. We shall refer to (1) as the *Lemoine equation.*

3. Eigenvalues of the matrix M

In order to solve equation (1), we first find eigenvalues λ_1 and λ_2 of M. These are the solutions of the equation $|M - \lambda I| = 0$, *i.e.*, $(\alpha - \lambda)(\gamma - \lambda) - \beta^2 = 0$, or

$$\lambda^2 - (\alpha + \gamma)\lambda + (\alpha\gamma - \beta^2) = 0.$$

Thus

$$\lambda_1+\lambda_2=\alpha+\gamma=1+\frac{m^2+n^2}{m^2+n^2}+\frac{n^2+(c-m)^2}{n^2+(c-m)^2}=3.$$

Writing a, b, c for the sidelengths $|BC|$, $|CA|$, $|AB|$ respectively, we find the determinant

$$|M|=\alpha\gamma-\beta^2=\frac{n^2}{a^2b^2}(a^2+b^2+c^2)>0.$$

The discriminant of the characteristic equation $\lambda^2-(\alpha+\gamma)\lambda+(\alpha\gamma-\beta^2)=0$ is given by

$$D=(\alpha+\gamma)^2-4(\alpha\gamma-\beta^2)=(\alpha-\gamma)^2+4\beta^2\geq 0. \tag{2}$$

Case 1: equal eigenvalues $\lambda_1=\lambda_2=\frac{3}{2}$. In this case, $D=0$ and (2) yields $\beta=0$ and $\alpha=\gamma$. To reduce notation, write $p=c-m$. Then since $\beta=0$, we have $\frac{m}{m^2+n^2}=\frac{p}{p^2+n^2}$, so that

$$(m-p)(mp-n^2)=0. \tag{3}$$

Also, since $\alpha=\gamma$, we find after mild simplifications

$$n^4-(m^2+p^2)n^2-3m^2p^2=0. \tag{4}$$

Equation (3) imples that $m=p$ or $mp=n^2$. If $m=p$, then equation (4) has solutions $n=\sqrt{3}m=\sqrt{3}p$. Consequently, $C=\left(\frac{1}{2}c,\ \frac{\sqrt{3}}{2}c\right)$, so that $\triangle ABC$ is equilateral. However, if $mp=n^2$, then equation (4) leads to $(m+p)^2=0$, so that $c=0$, a contradiction. Therefore from equation (3) we obtain this conclusion: *if the eigenvalues are equal, then $\triangle ABC$ is equilateral.*

Case 2: distinct eigenvalues $\lambda_{1,2}=\frac{3\pm\sqrt{D}}{2}$. Here $D>0$, and $\lambda_{1,2}>0$ according to (2). We choose to consider the implications when

$$\beta=0, \qquad \alpha\neq\gamma. \tag{5}$$

We omit an easy proof that these conditions correspond to $\triangle ABC$ being a right triangle or an isosceles triangle. In the former case, write $c^2=a^2+b^2$. Then the characteristic equation yields eigenvalues α and γ, and

$$\alpha=\frac{n^2}{b^2}+\frac{n^2}{a^2}=\frac{n^2(a^2+b^2)}{a^2b^2}=\frac{n^2c^2}{a^2b^2}=1,$$

since $ab=nc=$ twice the area of the right triangle. Since $\alpha+\gamma=3$, $\gamma=2$.

4. General solution of the Lemoine equation

According to a well known theorem of linear algebra, rotation of the coordinate system about K gives the system $x''=-\lambda_1x$, $y''=-\lambda_2y$. Let us call the axes of this coordinate system the *principal axes* of the Lemoine field.

Note that if $\triangle ABC$ is a right triangle or an isosceles triangle (cf. conditions (5)), then the angle of rotation is zero, and K is on an altitude of the triangle. In this case, one of the principal axes is that altitude, and the other is parallel to the corresponding side. Also if $\triangle ABC$ is a right triangle, then K is the midpoint of that altitude.

In the general case, the solution of the Lemoine equation is given by

$$x = c_1 \cos \omega_1 t + c_2 \sin \omega_2 t, \qquad y = c_3 \cos \omega_1 t + c_4 \sin \omega_2 t, \tag{6}$$

where $\omega_1 = \sqrt{\lambda_1}$, $\omega_2 = \sqrt{\lambda_2}$. Initial conditions $x(0) = x_0$, $y(0) = y_0$, $x'(0) = 0$, $y'(0) = 0$ reduce (6) to

$$x = x_0 \cos \omega_1 t, \qquad y = y_0 \cos \omega_2 t, \tag{7}$$

with $\omega_1 > 0$, $\omega_2 > 0$, $\omega_1^2 + \omega_2^2 = 3$. Equations (7) show that each trajectory is bounded. If $\lambda_1 = \lambda_2$, then the trajectory is a line segment; otherwise, (7) represents a Lissajous curve or an almost-everywhere rectangle-filling curve, according as $\frac{\omega_1}{\omega_2}$ is rational or not.

5. Lemoine sequences and centroidal orbits

Returning to the Lemoine field, **F**, suppose Z_0 is an arbitrary point, and G_{Z_0} is the centroid of the pedal triangle of Z_0. Let Z_0' be the point to which **F** translates Z_0. It is well known that G_{Z_0} lies on the line Z_0Z_0' at a distance $\frac{1}{3}$ of that from Z_0 to Z_0'. With this in mind, define inductively the *Lemoine sequence* of Z_0 as the sequence $(Z_0, Z_1, Z_2, \ldots)$, where Z_n, for $n \geq 1$, is the centroid of the pedal triangle of Z_{n-1}. Writing the centroid of the pedal triangle of Z_0 as $Z_1 = (x_1, y_1)$, we obtain $3(x_1 - x_0) = -\lambda_1 x_0$ and

$$x_1 = \frac{1}{3}(3 - \lambda_1)x_0 = \frac{1}{3}\lambda_2 x_0; \qquad y_1 = \frac{1}{3}\lambda_1 y_0.$$

Accordingly, the Lemoine sequence is given with respect to the principal axes by

$$Z_n = \left(x_0 \left(\frac{\lambda_2}{3} \right)^n, \ y_0 \left(\frac{\lambda_1}{3} \right)^n \right). \tag{8}$$

Since $\frac{1}{3}\lambda_1$ and $\frac{1}{3}\lambda_2$ are between 0 and 1, the points Z_n approach $(0, 0)$ as $n \to \infty$. That is, the Lemoine sequence of every point converges to the Lemoine point.

Representation (8) shows that Z_n lies on the curve $(x, y) = (x_0 u^t, y_0 v^t)$, where $u = \frac{1}{3}\lambda_2$ and $v = \frac{1}{3}\lambda_1$. We call this curve the *centroidal orbit* of Z_0. See Figure 3. Reversing the directions of axes if necessary, we may assume that $x_0 > 0$ and $y_0 > 0$, so that elimination of t gives

$$\frac{y}{y_0} = \left(\frac{x}{x_0} \right)^k, \qquad k = \frac{\ln v}{\ln u}. \tag{9}$$

Equation (9) expresses the centroidal orbit of $Z_0 = (x_0, y_0)$. Note that if $\omega_1 = \omega_2$, then $v = u$, and the orbit is a line. Now let X_Z and Y_Z be the points in which line ZG_Z meets the principal axes. By (8),

$$\frac{|ZG_Z|}{|G_Z X_Z|} = \frac{\lambda_2}{\lambda_1}, \qquad \frac{|ZG_Z|}{|G_Z Y_Z|} = \frac{\lambda_1}{\lambda_2}. \tag{10}$$

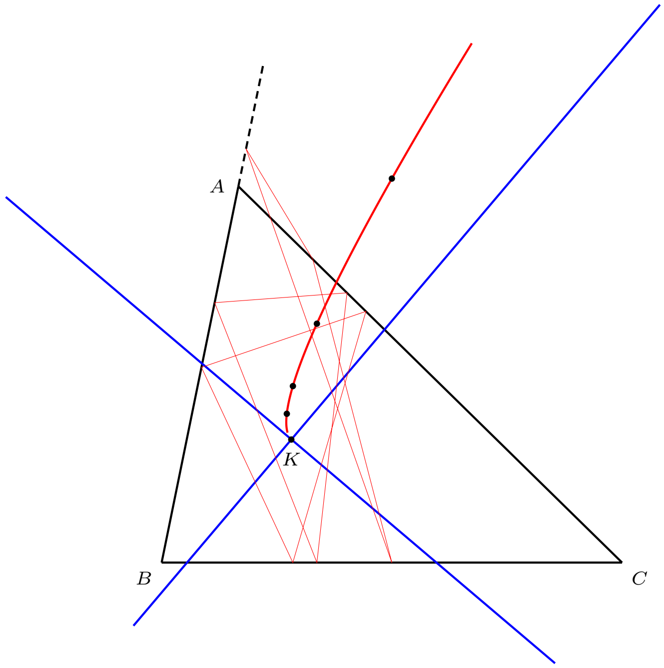

Figure 3

These equations imply that if $\triangle ABC$ is equilateral with center O, then the centroid G_Z is the midpoint of segment OG_Z.

As another consequence of (10), suppose $\triangle ABC$ is a right triangle; let H be the line parallel to the hypotenuse and passing through the midpoint of the altitude H' to the hypotenuse. Let X and Y be the points in which line ZG_Z meets H and H', respectively. Then $|ZG_Z| : |XG_Z| = |YG_Z| : |ZG_Z| = 2 : 1$.

6. The principal axes of the Lemoine field

Physically, the principal axes may be described as the locus of points in the plane of $\triangle ABC$ along which the “direction” of the Lemoine sequence remains constant. That is, if Z_0 lies on one of the principal axes, then all the points $Z_1, Z_2, \dots$ lie on that axis also.

In this section, we turn to the geometry of the principal axes. Relative to the coordinate system adopted in §5, the principal axes have equations $x = 0$ and $y = 0$. Equation (8) therefore shows that if Z_0 lies on one of these two perpendicular lines, then Z_n lies on that line also, for all $n \geq 1$.

Let A_1, A_2 denote the isodynamic points, and F_1, F_2 the isogonic centers, of $\triangle ABC$. Call lines A_1A_2 and F_1F_2 the *isodynamic axis* and the *isogonic axis* respectively. [1]

Lemma 1. *Suppose Z and Z' are a pair of isogonal conjugate points. Let O and O' be the circumcircles of the pedal triangles of Z and Z'. Then $O = O'$, and the center of O is the midpoint between Z and Z'.*

[1]The points F_1, F_2, A_1, A_2 are indexed as X_{13}, X_{14}, X_{15}, X_{16} and discussed in [2].

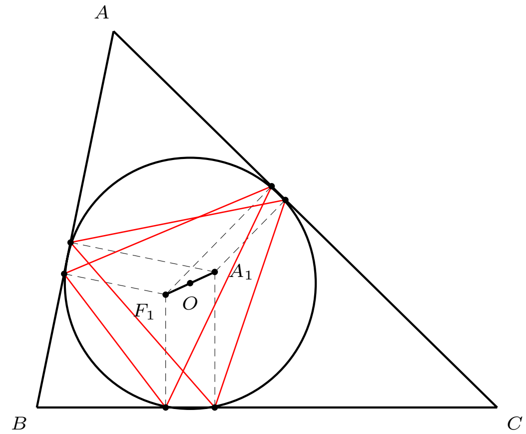

Figure 4

A proof is given in Johnson [1, pp.155–156]. See Figure 4.

Now suppose that $Z = A_1$. Then $Z' = F_1$, and, according to Lemma 1, the pedal triangles of Z and Z' have the same circumcircle, whose center O is the midpoint between A_1 and F_1. Since the pedal triangle of A_1 is equilateral, the point O is the centroid of the pedal triangle of A_1.

Next, suppose L is a line not identical to either of the principal axes. Let L' be the reflection of L about one of the principal axes. Then L' is also the reflection of L about the other principal axis. We call L and L' a *symmetric pair of lines*.

Lemma 2. *Suppose that G_P is the centroid of the pedal triangle of a point P, and that Q is the reflection of P in G_P. Then there exists a symmetric pair of lines, one passing through P and the other passing through Q.*

Proof. With respect to the principal axes, write $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$. Then $G_P = (\frac{1}{3}\lambda_2 x_P,\ \frac{1}{3}\lambda_1 y_P)$, and $\frac{2}{3}\lambda_2 x_P = x_P + x_Q$, so that

$$x_Q = \left(\frac{2}{3}\lambda_2 - 1\right) x_P = \frac{1}{3}(2\lambda_2 - (\lambda_1 + \lambda_2))x_P = \frac{1}{3}(\lambda_2 - \lambda_1)x_P.$$

Likewise, $y_Q = \frac{1}{3}y_P(\lambda_1 - \lambda_2)$. It follows that $\frac{x_P}{y_P} = -\frac{x_Q}{y_Q}$. This equation shows that the line $y = \frac{y_P}{x_P} \cdot x$ passing through P and the line $y = \frac{y_Q}{x_Q} \cdot x$ passing through Q are symmetric about the principal axes $y = 0$ and $x = 0$. See Figure 5. □

Theorem. *The principal axes of the Lemoine field are the bisectors of the angles formed at the intersection of the isodynamic and isogonic axes in the Lemoine point.*

Proof. In Lemma 2, take $P = A_1$ and $Q = F_1$. The symmetric pair of lines are then the isodynamic and isogonic axes. Their symmetry about the principal axes is equivalent to the statement that these axes are the asserted bisectors. □

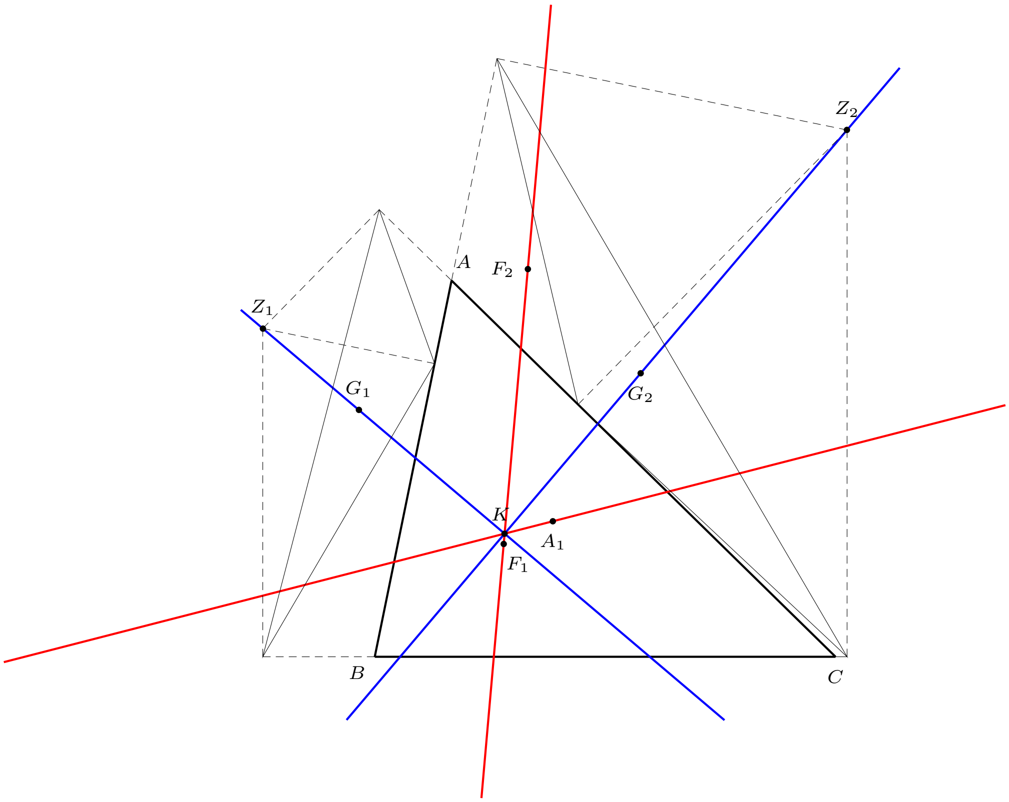

Figure 5

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Multiplying and Dividing Curves by Points

Clark Kimberling

Abstract. Pointwise products and quotients, defined in terms of barycentric and trilinear coordinates, are extended to products $P \cdot \Gamma$ and quotients Γ/P, where P is a point and Γ is a curve. In trilinears, for example, if Γ_0 denotes the circumcircle, then $P \cdot \Gamma_0$ is a parabola if and only if P lies on the Steiner inscribed ellipse. Barycentric division by the triangle center X_{110} carries Γ_0 onto the Kiepert hyperbola Γ'; if P is on Γ_0, then the point $P' = P/X_{110}$ is the point, other than the Tarry point, X_{98}, in which the line PX_{98} meets Γ', and if Ω_1 and Ω_2 denote the Brocard points, then $|P'\Omega_1|/|P'\Omega_2| = |P\Omega_1|/|P\Omega_2|$; that is, P' and P lie on the same Apollonian circle with respect to Ω_1 and Ω_2.

1. Introduction

Paul Yiu [7] gives a magnificent construction for a product $P \cdot Q$ of points in the plane of triangle ABC. If

$$P = \alpha_1 : \beta_1 : \gamma_1 \ \text{ and } \ Q = \alpha_2 : \beta_2 : \gamma_2 \tag{1}$$

are representations in homogeneous barycentric coordinates, then the Yiu product is given by

$$P \cdot Q = \alpha_1\alpha_2 : \beta_1\beta_2 : \gamma_1\gamma_2 \tag{2}$$

whenever $\{\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2\} \neq \{0\}$.

Cyril Parry [3] constructs an analogous product using trilinear coordinates. In view of the applicability of both the Yiu and Parry products, the notation in equations (1) and (2) will represent general homogeneous coordinates, as in [6, Chapter 1], unless otherwise noted. We also define the quotient

$$P/Q := \alpha_1\beta_2\gamma_2 : \beta_1\gamma_2\alpha_2 : \gamma_1\alpha_2\beta_2$$

whenever $Q \notin \{A, B, C\}$. Specialization of coordinates will be communicated by phrases such as those indicated here:

$$\left\{ \begin{array}{c} \text{barycentric} \\ \text{trilinear} \end{array} \right\} \left\{ \begin{array}{c} \text{multiplication} \\ \text{product} \\ \text{division} \\ \text{quotient} \end{array} \right\}.$$

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If S is a set of points, then $P \cdot S := \{P \cdot Q : \ Q \in S\}$. In particular, if S is a curve Γ, then $P \cdot \Gamma$ and Γ/P are curves, except for degenerate cases, such as when $P \in \{A, B, C\}$.

In all that follows, suppose $P = p : q : r$ is a point not on a sideline of triangle ABC, so that $pqr \neq 0$, and consequently, $U/P = \frac{u}{p} : \frac{v}{q} : \frac{w}{r}$ for all $U = u : v : w$.

Example 1. If Γ is a line $\ell\alpha + m\beta + n\gamma = 0$, then $P \cdot \Gamma$ is the line $(\ell/p)\alpha + (m/q)\beta + (n/r)\gamma = 0$ and Γ/P is the line $p\ell\alpha + qm\beta + rn\gamma = 0$. Given the line QR of points Q and R, it is easy to check that $P \cdot QR$ is the line of $P \cdot Q$ and $P \cdot R$. In particular, $P \cdot \triangle ABC = \triangle ABC$, and if T is a cevian triangle, then $P \cdot T$ is a cevian triangle.

2. Conics and Cubics

Each conic Γ in the plane of triangle ABC is given by an equation of the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0. \tag{3}$$

That $P \cdot \Gamma$ is the conic

$$(u/p^2)\alpha^2 + (v/q^2)\beta^2 + (w/r^2)\gamma^2 + 2(f/qr)\beta\gamma + 2(g/rp)\gamma\alpha + 2(h/pq)\alpha\beta = 0 \tag{4}$$

is clear, since $\alpha : \beta : \gamma$ satifies (3) if and only if $p\alpha : q\beta : r\gamma$ satisfies (4). In the case of a circumconic Γ given in general form by

$$\frac{f}{\alpha} + \frac{g}{\beta} + \frac{h}{\gamma} = 0, \tag{5}$$

the product $P \cdot \Gamma$ is the circumconic

$$\frac{pf}{\alpha} + \frac{qg}{\beta} + \frac{rh}{\gamma} = 0.$$

Thus, if X is the point such that $X \cdot \Gamma$ is a given circumconic $\frac{u}{\alpha} + \frac{v}{\beta} + \frac{w}{\gamma} = 0$, then $X = \frac{u}{f} : \frac{v}{g} : \frac{w}{h}$.

Example 2. In trilinears, the circumconic Γ in (5) is the isogonal transform of the line L given by $f\alpha + g\beta + h\gamma = 0$. The isogonal transform of $P \cdot L$ is Γ/P.

Example 3. Let $U = u : v : w$. The conic $W(U)$ given in [1, p. 238] by

$$u^2\alpha^2 + v^2\beta^2 + w^2\gamma^2 - 2vw\beta\gamma - 2wu\gamma\alpha - 2uv\alpha\beta = 0$$

is inscribed in triangle ABC. The conic $P \cdot W(U)$ given by

$$(u/p)^2\alpha^2 + (v/q)^2\beta^2 + (w/r)^2\gamma^2 - 2(vw/qr)\beta\gamma - 2(wu/rp)\gamma\alpha - 2(uv/pq)\alpha\beta = 0$$

is the inscribed conic $W(U/P)$. In trilinears, we start with Γ = incircle, given by

$$u = u(a, b, c) = a(b + c - a), v = u(b, c, a), w = u(c, a, b),$$

and find [1]

[1]The conics in Example 3 are discussed in [1, p.238] as examples of a type denoted by $W(X_i)$, including incircle = $W(X_{55})$, Steiner inscribed ellipse = $W(X_6)$, Kiepert parabola = $W(X_{512})$, and Yff parabola = $W(X_{647})$. A list of X_i including trilinears, barycentrics, and remarks is given in [2].

Conic	Trilinear product	Barycentric product
Steiner inscribed ellipse	$X_9 \cdot \Gamma$	$X_8 \cdot \Gamma$
Kiepert parabola	$X_{643} \cdot \Gamma$	$X_{645} \cdot \Gamma$
Yff parabola	$X_{644} \cdot \Gamma$	$X_{646} \cdot \Gamma$

Example 4. Here we combine notions from Examples 1-3. The circumcircle, Γ_0, may be regarded as a special circumconic, and every circumconic has the form $P \cdot \Gamma_0$. We ask for the locus of a point P for which the circumconic $P \cdot \Gamma_0$ is a parabola. As such a conic is the isogonal transform of a line tangent to Γ_0, we begin with this statement of the problem: find $P = p : q : r$ (trilinears) for which the line L given by $\frac{p}{\alpha} + \frac{q}{\beta} + \frac{r}{\gamma} = 0$ meets Γ_0, given by $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0$ in exactly one point. Eliminating γ leads to

$$\frac{\alpha}{\beta} = \frac{cr - ap - bq \pm \sqrt{(ap + bq - cr)^2 - 4abpq}}{2bp}.$$

We write the discriminant as

$$\Phi(p, q, r) = a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr - 2carp - 2abpq.$$

In view of Example 3 and [5, p.81], we conclude that if $W(X_6)$ denotes the Steiner inscribed ellipse, with trilinear equation $\Phi(\alpha, \beta, \gamma) = 0$, then

$$P \cdot \Gamma_0 \text{ is a } \left\{ \begin{array}{c} \text{hyperbola} \\ \text{parabola} \\ \text{ellipse} \end{array} \right\} \text{ according as } P \text{ lies } \left\{ \begin{array}{rl} \text{inside} & W(X_6) \\ \text{on} & W(X_6) \\ \text{outside} & W(X_6) \end{array} \right\}. \tag{6}$$

Returning to the case that L is tangent to Γ_0, it is easy to check that the point of tangency is $(X_1/P)©X_6$. (See Example 7 for Ceva conjugacy, denoted by ©.)

If the method used to obtain statement (6) is applied to barycentric multiplication, then a similar conclusion is reached, in which the role of $W(X_6)$ is replaced by the inscribed conic whose barycentric equation is

$$\alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta = 0,$$

that is, the ellipse $W(X_2)$.

Example 5. Suppose points P and Q are given in trilinears: $P = p : q : r$, and $U = u : v : w$. We shall find the locus of a point $X = \alpha : \beta : \gamma$ such that $P \cdot X$ lies on the line UX. This on-lying is equivalent to the determinant equation

$$\begin{vmatrix} u & v & w \\ \alpha & \beta & \gamma \\ p\alpha & q\beta & r\gamma \end{vmatrix} = 0,$$

expressible as a circumconic:

$$\frac{u(q - r)}{\alpha} + \frac{v(r - p)}{\beta} + \frac{w(p - q)}{\gamma} = 0. \tag{7}$$

One may start with the line X_1P, form its isogonal transform Γ, and then recognize (7) as $U \cdot \Gamma$. For example, in trilinears, equation (7) represents the hyperbolas of

Kiepert, Jerabek, and Feuerbach according as $(P, U) = (X_{31}, X_{75})$, (X_6, X_{48}), and (X_1, X_3); or, in barycentrics, according as $(P, U) = (X_6, X_{76})$, (X_1, X_3), and (X_2, X_{63}).

Example 6. Again in trilinears, let Γ be the self-isogonal cubic $Z(U)$ given in [1, p. 240] by

$$u\alpha(\beta^2 - \gamma^2) + v\beta(\gamma^2 - \alpha^2) + w\gamma(\alpha^2 - \beta^2) = 0.$$

This is the locus of points X such that X, X_1/X, and U are collinear; the point U is called the *pivot* of $Z(U)$. The quotient Γ/P is the cubic

$$up\alpha(q^2\beta^2 - r^2\gamma^2) + vq\beta(r^2\gamma^2 - p^2\alpha^2) + wr\gamma(p^2\alpha^2 - q^2\beta^2) = 0.$$

Although Γ/P is not generally self-isogonal, it is self-conjugate under the P^2-isoconjugacy defined (e.g., [4]) by $X \to X_1/(X \cdot P^2)$.

Example 7. Let $X©P$ denote the X-Ceva conjugate of P, defined in [1, p.57] for $X = x : y : z$ and $P = p : q : r$ by

$$X©P = p(-\frac{p}{x} + \frac{q}{y} + \frac{r}{z}) : q(-\frac{q}{y} + \frac{r}{z} + \frac{p}{x}) : r(-\frac{r}{z} + \frac{p}{x} + \frac{q}{y}).$$

Assume that $X \neq P$. It is easy to check that the locus of a point X for which $X©P$ lies on the line XP is given by

$$\frac{\alpha}{p}(\frac{\beta^2}{q^2} - \frac{\gamma^2}{r^2}) + \frac{\beta}{q}(\frac{\gamma^2}{r^2} - \frac{\alpha^2}{p^2}) + \frac{\gamma}{r}(\frac{\alpha^2}{p^2} - \frac{\beta^2}{q^2}) = 0. \tag{8}$$

In trilinears, equation (8) represents the product $P \cdot \Gamma$ where Γ is the cubic $Z(X_1)$. The locus of X for which $P©X$ lies on XP is also the cubic (8).

3. Brocard Points and Apollonian Circles

Here we discuss some special properties of the triangle centers X_{98} (the Tarry point) and X_{110} (the focus of the Kiepert parabola). X_{98} is the point, other than A, B, C, that lies on both the circumcircle and the Kiepert hyperbola.

Let ω be the Brocard angle, given by

$$\cot\omega = \cot A + \cot B + \cot C.$$

In trilinears,

$$\begin{aligned} X_{98} &= \sec(A+\omega) : \sec(B+\omega) : \sec(C+\omega), \\ X_{110} &= \frac{a}{b^2 - c^2} : \frac{b}{c^2 - a^2} : \frac{c}{a^2 - b^2}. \end{aligned}$$

Theorem. *Barycentric division by X_{110} carries the circumcircle Γ_0 onto the Kiepert hyperbola Γ'. For every point P on Γ_0, the line joining P to the Tarry point X_{98} (viz., the tangent at X_{98} if $P = X_{98}$) intersects Γ' again at $P' = P/X_{110}$. Furthermore, P/X_{110} lies on the Apollonian circle of P with respect to the two Brocard points Ω_1 and Ω_2; that is*

$$\frac{|P'\Omega_1|}{|P'\Omega_2|} = \frac{|P\Omega_1|}{|P\Omega_2|}. \tag{9}$$

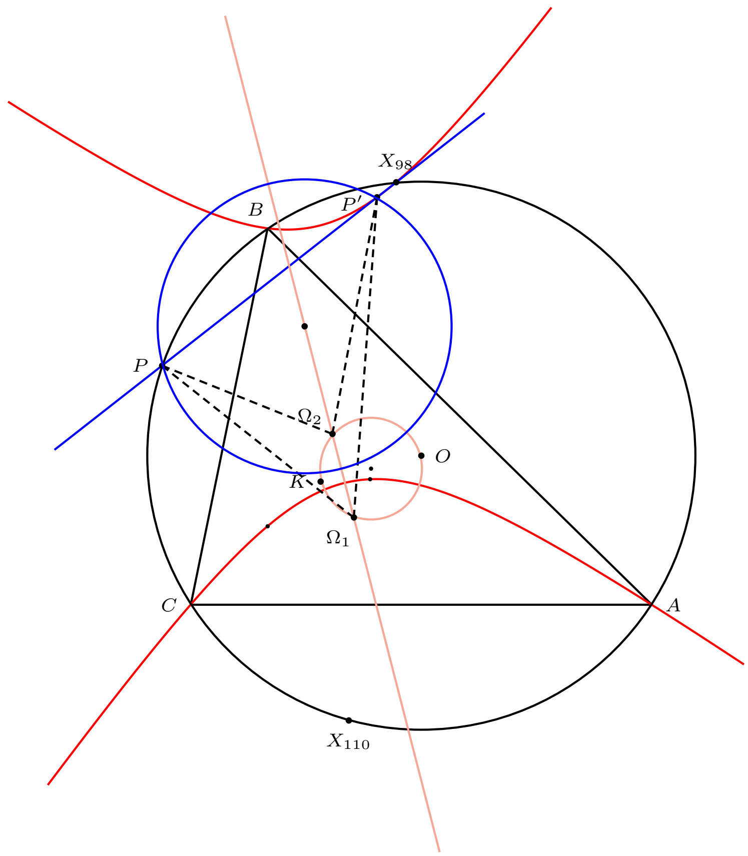

Figure 1

Proof. In barycentrics, Γ_0 and Γ' are given by

$$\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}=0 \text{ and } \frac{b^2-c^2}{\alpha}+\frac{c^2-a^2}{\beta}+\frac{a^2-b^2}{\gamma}=0,$$

and, also in barycentrics,

$$X_{110}=\frac{a}{b^2-c^2}:\frac{b}{c^2-a^2}:\frac{c}{a^2-b^2}$$

so that $\Gamma'=\Gamma_0/X_{110}$.

For the remainder of the proof, we use trilinears. A parametric representation for Γ_0 is given by

$$P=P(t)=a(1-t):bt:ct(t-1), \tag{10}$$

for $-\infty<t<\infty$, and the barycentric product P/X_{110} is given in trilinears by

$$(1-t)\frac{b^2-c^2}{a}:t\frac{c^2-a^2}{b}:t(t-1)\frac{a^2-b^2}{c}.$$

That this point lies on line PX_{98} is equivalent to the following easily verified identity:

$$\begin{vmatrix} (1-t)\frac{b^2-c^2}{a} & t\frac{c^2-a^2}{b} & t(t-1)\frac{a^2-b^2}{c} \\ a(1-t) & bt & ct(t-1) \\ \sec(A+\omega) & \sec(B+\omega) & \sec(C+\omega) \end{vmatrix} = 0.$$

We turn now to a formula [1, p.31] for the distance between two points expressed in normalized[2] trilinears (α, β, γ) and $(\alpha', \beta', \gamma')$:

$$\frac{1}{2\sigma}\sqrt{abc[a\cos A(\alpha-\alpha')^2 + b\,\cos B(\beta-\beta')^2 + c\,\cos C(\gamma-\gamma')^2]}, \qquad (11)$$

where σ denotes the area of triangle ABC. Let

$$\begin{aligned} D &= c^2t^2 - (c^2+a^2-b^2)t + a^2, \\ S &= a^2b^2 + b^2c^2 + c^2a^2. \end{aligned}$$

Normalized trilinears for (10) and the two Brocard points follow:

$$P = ((1-t)ha, thb, t(t-1)hc),$$

where $h = \frac{2\sigma}{D}$, and

$$\Omega_1 = \left(\frac{h_1c}{b}, \frac{h_1a}{c}, \frac{h_1b}{a}\right), \qquad \Omega_2 = \left(\frac{h_1b}{c}, \frac{h_1c}{a}, \frac{h_1a}{b}\right),$$

where and $h_1 = \frac{2abc\sigma}{S}$.

Abbreviate $a\cos A$, $b\cos B$, $c\cos C$, and $1-t$ as a', b', c', and t' respectively, and write

$$E = a'\left(\frac{t'ha - h_1c}{b}\right)^2 + b'\left(\frac{thb - h_1a}{c}\right)^2 + c'\left(\frac{tt'hc - h_1b}{a}\right)^2, \quad (12)$$

$$F = a'\left(\frac{t'ha - h_1b}{c}\right)^2 + b'\left(\frac{thb - h_1c}{a}\right)^2 + c'\left(\frac{tt'hc - h_1a}{b}\right)^2. \quad (13)$$

Equation (11) then gives

$$\frac{|P\Omega_1|^2}{|P\Omega_2|^2} = \frac{E}{F}. \qquad (14)$$

In (12) and (13), replace $\cos A$ by $(b^2+c^2-a^2)/2bc$, and similarly for $\cos B$ and $\cos C$, obtaining from (14) the following:

$$\frac{|P\Omega_1|^2}{|P\Omega_2|^2} = \frac{t^2a^2 - t(a^2+b^2-c^2) + b^2}{t^2b^2 - t(b^2+c^2-a^2) + c^2}.$$

[2] Sometimes trilinear coordinates are called normal coordinates. We prefer "trilinears", so that we can say "normalized trilinears," not "normalized normals." One might say that the latter double usage of "normal" can be avoided by saying "actual normal distances", but this would be unsuitable for normalization of points at infinity. Another reason for retaining "trilinear" and "quadriplanar"— not replacing both with "normal"— is that these two terms distinguish between lines and planes as the objects with respect to which normal distances are defined. In discussing points relative to a tetrahedron, for example, one could have both trilinears and quadriplanars in the same sentence.

Note that if the numerator in the last fraction is written as $f(t,a,b,c)$, then the denominator is $t^2 f(\frac{1}{t},c,b,a)$. Similarly,

$$\frac{|P'\Omega_1|^2}{|P'\Omega_2|^2} = \frac{g(t,a,b,c)}{t^4 g(\frac{1}{t},c,b,a)},$$

where

$$g(t,a,b,c) = t^4 e_4 + t^3 e_3 + t^2 e_2 + t e_1 + e_0,$$

and

$$\begin{aligned}
e_4 &= a^4b^2(a^2-b^2)^2,\\
e_3 &= a^2(a^2-b^2)(b^6+c^6+2a^2b^2c^2-2a^4b^2-2a^2c^4-2b^2c^4+a^4c^2+a^2b^4),\\
e_2 &= b^2c^2(b^2-c^2)^3+a^2c^2(c^2-a^2)^3+a^2b^2(a^6+2b^6-3a^2b^4)\\
&\quad + a^2b^2c^2(b^4+c^4-2a^4-4b^2c^2+2a^2c^2+2a^2b^2),\\
e_1 &= b^2(c^2-b^2)(a^6+c^6-3b^2c^4+2b^4c^2-2a^4b^2-2a^4c^2+2a^2b^2c^2+a^2b^4),\\
e_0 &= b^4c^2(b^2-c^2)^2.
\end{aligned}$$

One may now verify directly, using a computer algebra system, or manually with plenty of paper, that

$$t^2 f(t,a,b,c) g(\frac{1}{t},c,b,a) = f(\frac{1}{t},c,b,a) g(t,a,b,c),$$

which is equivalent to the required equation (9). □

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The Simson cubic

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Abstract. The Simson cubic is the locus of the trilinear poles of the Simson lines. There exists a conic such that a point M lies on the Simson cubic if and only if the line joining M to its isotomic conjugate is tangent to this conic. We also characterize cubics which admit pivotal conics for a given isoconjugation.

1. Introduction

Antreas P. Hatzipolakis [2] has raised the question of the locus of points for which the triangle bounded by the pedal cevians is perspective. More precisely, given triangle ABC, let $A_{[P]}B_{[P]}C_{[P]}$ be the pedal triangle of a point P, and consider the intersection points

$$Q_a := BB_{[P]} \cap CC_{[P]}, \qquad Q_b := CC_{[P]} \cap AA_{[P]}, \qquad Q_c := AA_{[P]} \cap BB_{[P]}.$$

We seek the locus of P for which the triangle $Q_aQ_bQ_c$ is perspective with ABC. See Figure 1. This is the union of

(1a) the Darboux cubic consisting of points whose pedal triangles are cevian,[1]
(1b) the circumcircle together with the line at infinity.

The loci of the perspector in these cases are respectively

(2a) the Lucas cubic consisting of points whose cevian triangles are pedal,[2]
(2b) a cubic related to the Simson lines.

We give an illustration of the Darboux and Lucas cubics in the Appendix. Our main interest is in the singular case (2b) related to the Simson lines of points on the circumcircle. The curve in (2b) above is indeed the locus of the tripoles[3] of the Simson lines. Let P be a point on the circumcircle, and $\mathsf{t}(P) = (u : v : w)$ the tripole of its Simson line $\mathsf{s}(P)$. This means that the perpendicular to the sidelines at the points

$$U = (0 : v : -w), \quad V = (-u : 0 : w), \quad W = (u : -v : 0) \tag{1}$$

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[1]This is the isogonal cubic with pivot the de Longchamps point, the reflection of the orthocenter in the circumcenter. A point P lies on this cubic if and only if its the line joining P to its isogonal conjugate contains the de Longchamps point.

[2]This is the isotomic cubic with pivot $\mathsf{i}(H)$, the isotomic conjugate of the orthocenter. A point P lies on this cubic if and only if its the line joining P to its isotomic conjugate contains the point $\mathsf{i}(H)$.

[3]We use the term tripole as a short form of trilinear pole.

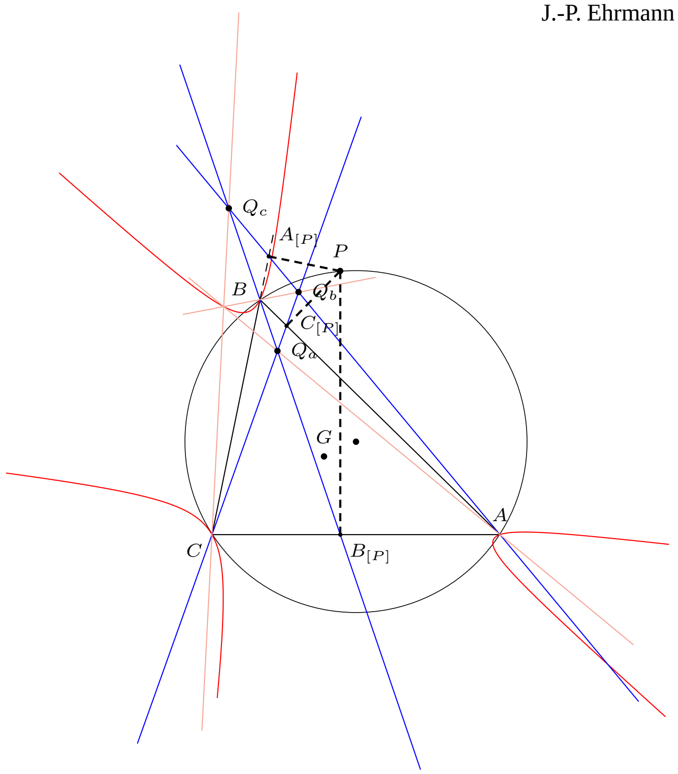

Figure 1

are concurrent (at P on the circumcircle). In the notations of John H. Conway,[4] the equations of these perpendiculars are

$$\begin{array}{ccccccc}(S_Bv+S_Cw)x & + & a^2vy & + & a^2wz & = & 0,\\ b^2ux & + & (S_Cw+S_Au)y & + & b^2wz & = & 0,\\ c^2ux & + & c^2vy & + & (S_Au+S_Bv)z & = & 0.\end{array}$$

Elimination of x, y, z leads to the cubic

$$\mathcal{E}: \ S_Au(v^2+w^2)+S_Bv(w^2+u^2)+S_Cw(u^2+v^2)-(a^2+b^2+c^2)uvw=0.$$

This is clearly a self-isotomic cubic, *i.e.*, a point P lies on the cubic if and only if its isotomic conjugate does. We shall call $\mathcal{E}$ the *Simson cubic* of triangle ABC.

2. A parametrization of the Simson cubic

It is easy to find a rational parametrization of the Simson cubic. Let P be a point on the circumcircle. Regarded as the isogonal conjugate of the infinite point of a

[4]For a triangle ABC with side lengths a, b, c, denote

$$S_A=\frac{1}{2}(b^2+c^2-a^2), \quad S_B=\frac{1}{2}(c^2+a^2-b^2), \quad S_C=\frac{1}{2}(a^2+b^2-c^2).$$

These satisfy a number of basic relations. We shall, however, only need the obvious relations

$$S_B+S_C=a^2, \qquad S_C+S_A=b^2, \quad S_A+S_B=c^2.$$

line $px + qy + rz = 0$, the point P has homogeneous barycentric coordinates

$$\left(\frac{a^2}{q-r} : \frac{b^2}{r-p} : \frac{c^2}{p-q}\right).$$

The pedals of P on the side lines are the points U, V, W in (1) with

$$\begin{aligned} u &= \frac{1}{q-r}(-a^2p + S_Cq + S_Br), \\ v &= \frac{1}{r-p}(S_Cp - b^2q + S_Ar), \\ w &= \frac{1}{p-q}(S_Bp + S_Aq - c^2r). \end{aligned} \tag{2}$$

This means that the tripole of the Simson line $\mathsf{s}(P)$ of P is the point $\mathsf{t}(P) = (u : v : w)$. The system (2) therefore gives a rational parametrization of the Simson cubic. It also shows that $\mathcal{E}$ has a singularity, which is easily seen to be an isolated singularity at the centroid. [5]

3. Pivotal conic of the Simson cubic

We have already noted that $\mathcal{E}$ is a self-isotomic cubic. In fact, the isotomic conjugate of $\mathsf{t}(P)$ is the point $\mathsf{t}(P')$, where P' is the antipode of P (with respect to the circumcircle). [6]

It is well known that the Simson lines of antipodal points intersect (orthogonally) on the nine-point circle. As this intersection moves on the nine-point circle, the line joining the tripoles $\mathsf{t}(P)$, $\mathsf{t}(P')$ of the orthogonal Simson lines $\mathsf{s}(P)$, $\mathsf{s}(P')$ envelopes the conic $\mathcal{C}$ dual to the nine-point circle. This conic has equation[7]

$$\sum_{\text{cyclic}} (b^2 - c^2)^2x^2 - 2(b^2c^2 + c^2a^2 + a^2b^2 - a^4)yz = 0,$$

and is the inscribed ellipse in the anticomplementary triangle with center the symmedian point of triangle ABC, $K = (a^2 : b^2 : c^2)$. The Simson cubic $\mathcal{E}$ can therefore be regarded as an isotomic cubic with the ellipse $\mathcal{C}$ as pivot. See Figure 2.

Proposition 1. *The pivotal conic $\mathcal{C}$ is tritangent to the Simson cubic $\mathcal{E}$ at the tripoles of the Simson lines of the isogonal conjugates of the infinite points of the Morley sides.*

---

[5] If P is an infinite point, its pedals are the infinite points of the side lines. The triangle $Q_aQ_bQ_c$ in question is the anticomplementary triangle, and has perspector at the centroid G.

[6] The antipode of P has coordinates

$$\left(\frac{a^2}{-a^2p + S_Cq + S_Br} : \frac{b^2}{S_Cp - b^2q + S_Ar} : \frac{c^2}{S_Bp + S_Aq - c^2r}\right).$$

[7] The equation of the nine-point circle is $\sum_{\text{cyclic}} S_Ax^2 - a^2yz = 0$. We represent this by a symmetric matrix A. The dual conic is then represented by the adjoint matrix of A.

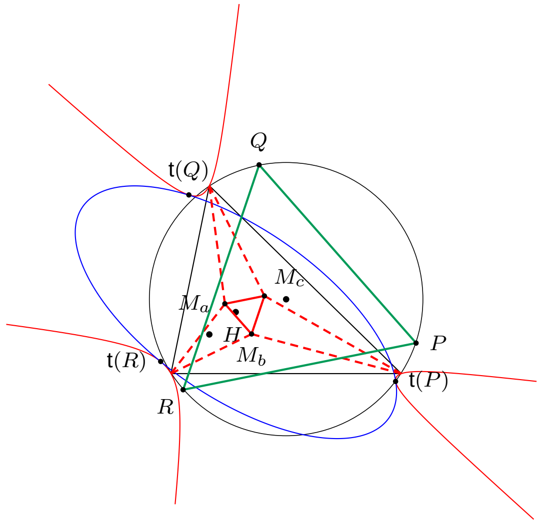

Figure 2

Proof. Since $\mathcal{C}$ is the dual of the nine-point circle, the following statements are equivalent:

(1) $\mathsf{t}(P)$ lies on $\mathcal{C} \cap \mathcal{E}$.

(2) $\mathsf{s}(P')$ is tangent to the nine-point circle.

(3) $\mathsf{s}(P)$ passes through the nine-point center.

Thus, $\mathcal{C}$ and $\mathcal{E}$ are tangent at the three points $\mathsf{t}(P)$ for which the Simson lines $S(P)$ pass through the nine-point center. If P, Q, R are the isogonal conjugates of the infinite points of the side lines of the Morley triangle, then PQR is an equilateral triangle and the Simson lines $\mathsf{s}(P)$, $\mathsf{s}(Q)$, $\mathsf{s}(R)$ are perpendicular to QR, RP, PQ respectively. See [1]. Let H be the orthocenter of triangle ABC, and consider the midpoints P_1, Q_1, R_1 of HP, HQ, HR. Since $\mathsf{s}(P)$, $\mathsf{s}(Q)$, $\mathsf{s}(R)$ pass through P_1, Q_1, R_1 respectively, these Simson lines are the altitudes of the triangle $P_1Q_1R_1$. As this triangle is equilateral and inscribed in the nine-point circle, the Simson lines $\mathsf{s}(P)$, $\mathsf{s}(Q)$, $\mathsf{s}(R)$ pass through the nine-point center. □

Remarks. (1) The triangle PQR is called the circum-tangential triangle of ABC in [3].

(2) The ellipse $\mathcal{C}$ intersects the Steiner circum - ellipse at the four points

$$\left(\tfrac{1}{b-c} : \tfrac{1}{c-a} : \tfrac{1}{a-b}\right), \qquad \left(\tfrac{1}{b-c} : \tfrac{1}{c+a} : \tfrac{-1}{a+b}\right),$$
$$\left(\tfrac{-1}{b+c} : \tfrac{1}{c-a} : \tfrac{1}{a+b}\right), \qquad \left(\tfrac{1}{b+c} : \tfrac{-1}{c+a} : \tfrac{1}{a-b}\right).$$

These points are the perspectors of the four inscribed parabolas tangent respectively to the tripolars of the incenter and of the excenters. In Figure 3, we illustrate the parabolas for the incenter and the B-excenter. The foci are the isogonal conjugates of the infinite points of the lines $\pm ax \pm by \pm cz = 0$, and the directrices are the corresponding lines of reflections of the foci in the side lines of triangle ABC.

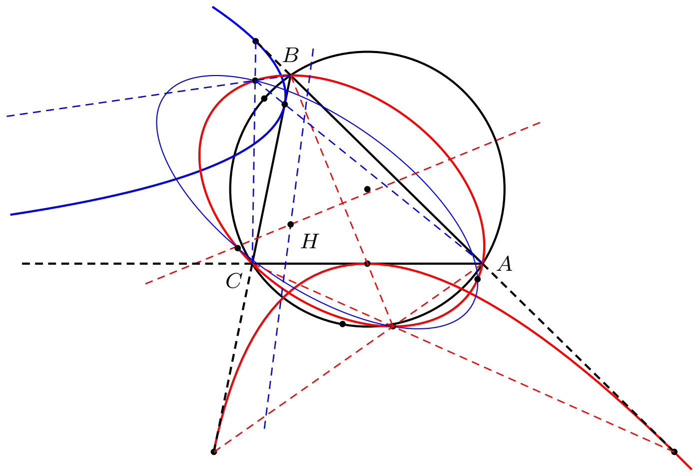

Figure 3

4. Intersection of $\mathcal{E}$ with a line tangent to $\mathcal{C}$

Consider again the Simson line of P and P' intersecting orthogonally at a point N on the nine-point circle. There is a third point Q on the circumcircle whose Simson line $\mathsf{s}(Q)$ passes through N.

- Q is the intersection of the line HN with the circumcircle, H being the orthocenter.
- The line $\mathsf{t}(P)\mathsf{t}(P')$ intersects again the cubic at $\mathsf{t}(Q')$.
- The tangent lines at $\mathsf{t}(P)$ and $\mathsf{t}(P')$ to the cubic intersect at $\mathsf{t}(Q)$ on the cubic.

If the line $\mathsf{t}(P)\mathsf{t}(P')$ touches $\mathcal{C}$ at S, then

(i) S and $\mathsf{t}(Q')$ are harmonic conjugates with respect to $\mathsf{t}(P)$ and $\mathsf{t}(P')$;

(ii) the isotomic conjugate of S is the tripole of the line tangent at N to the nine-point circle.

See Figure 4.

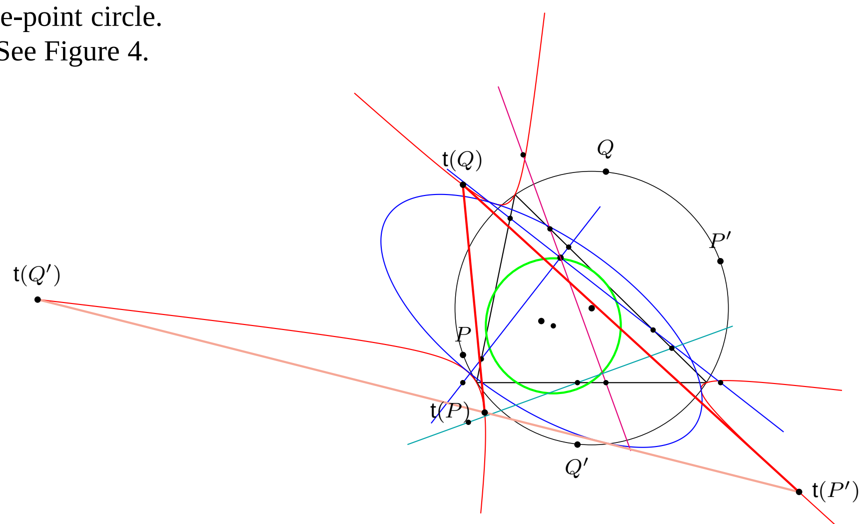

Figure 4

5. Circumcubics invariant under a quadratic transformation

Let $\mathcal{E}$ be a circumcubic invariant under a quadratic transformation τ defined by

$$\tau(x:y:z) = \left(\frac{f^2}{x} : \frac{g^2}{y} : \frac{h^2}{z}\right).$$

The fixed points of τ are the points $(\pm f : \pm g : \pm h)$, which form a harmonic quadruple.

Consider a circumcubic $\mathcal{E}$ invariant under τ. Denote by U, V, W the "third" intersections of $\mathcal{E}$ with the side lines. Then, either U, V, W lie on same line or UVW is perspective with ABC.

The latter case is easier to describe. If UVW is perspective with ABC at P, then $\mathcal{E}$ is the $\tau-$cubic with pivot P, *i.e.*, a point Q lies on $\mathcal{E}$ if and only if the line joining Q and $\tau(Q)$ passes through P.

On the other hand, if U, V, W are collinear, their coordinates can be written as in (1) for appropriate choice of u, v, w, so that the line containing them is the tripolar of the point $(u:v:w)$. In this case, then the equation of $\mathcal{E}$ is

$$\sum_{\text{cyclic}} f^2yz(wy+vz) + txyz = 0$$

for some t.

(a) If $\mathcal{E}$ contains exactly one of the fixed points $F = (f:g:h)$, then

$$t = -2(ghu + hfv + fgw).$$

In this case, $\mathcal{E}$ has a singularity at F. If $M = (x:y:z)$ in barycentric coordinates with respect to ABC, then with respect to the precevian triangle of F (the three other invariant points), the tangential coordinates of the line joining M to $\tau(M)$ are

$$(p:q:r) = \left(\frac{gz-hy}{(g+h-f)(gz+hy)} : \frac{hx-fz}{(h+f-g)(hx+fz)} : \frac{fy-gx}{(f+g-h)(fy+gx)}\right).$$

As the equation of $\mathcal{E}$ can be rewritten as

$$\frac{p_0}{p} + \frac{q_0}{q} + \frac{r_0}{r} = 0,$$

where

$$p_0 = \frac{f(hv+gw)}{g+h-f}, \quad q_0 = \frac{g(fw+hu)}{h+f-g}, \quad r_0 = \frac{h(gu+fv)}{f+g-h},$$

it follows that the line $M\tau(M)$ envelopes a conic inscribed in the precevian triangle of F.

Conversely, if $\mathcal{C}$ is a conic inscribed in the precevian triangle $A^FB^FC^F$, the locus of M such as the line $M\tau(M)$ touches $\mathcal{C}$ is a $\tau-$cubic with a singularity at F. The tangent lines to $\mathcal{E}$ at F are the tangent lines to $\mathcal{C}$ passing through P.

Note that if F lies on $\mathcal{C}$, and T the tangent to $\mathcal{C}$ at P, then $\mathcal{E}$ degenerates into the union of T and T^*.

(b) If $\mathcal{E}$ passes through two fixed points F and, A^F, then it degenerates into the union of FA^F and a conic.

(c) If the cubic $\mathcal{E}$ contains none of the fixed points, each of the six lines joining two of these fixed points contains, apart from a vertex of triangle ABC, a pair of points of $\mathcal{E}$ conjugate under τ. In this case, the lines $M\tau(M)$ cannot envelope a conic, because this conic should be tangent to the six lines, which is clearly impossible.

We close with a summary of the results above.

Proposition 2. *Let $\mathcal{E}$ be a circumcubic and τ a quadratic transformation of the form*

$$\tau(x:y:z) = (f^2yz : g^2zx : h^2xy).$$

The following statements are equivalent.

(1) $\mathcal{E}$ is $\tau-$invariant with pivot a conic.

(2) $\mathcal{E}$ passes through one and only one fixed points of τ, has a singularity at this point, and the third intersections of $\mathcal{E}$ with the side lines lie on a line ℓ.

In this case, if $\mathcal{E}$ contains the fixed point $F = (f : g : h)$, and if ℓ is the tripolar of the point $(u : v : w)$, then the equation of $\mathcal{E}$ is

$$-2(ghu + hfv + fgw)xyz + \sum_{\text{cyclic}} ux(h^2y^2 + g^2z^2) = 0. \tag{3}$$

The pivotal conic is inscribed in the precevian triangle of F and has equation[8]

$$\sum_{\text{cyclic}} (gw - hv)^2x^2 - 2(ghu^2 + 3fu(hv + gw) + f^2vw)yz = 0.$$

Appendix

Proposition 3. *Let ℓ be the tripolar of the point $(u : v : w)$, intersecting the sidelines of triangle ABC at U, V, W with coordinates given by (1), and $F = (f : g : h)$ a point not on ℓ nor the side lines of the reference triangle. The locus of M for which the three intersections $AM \cap FU$, $BM \cap FV$ and $CM \cap FW$ are collinear is the cubic $\mathcal{E}$ defined by (3) above.*

Proof. These intersections are the points

$$\begin{aligned}
AM \cap FU &= (f(wy + vz) : (hv + gw)y : (hv + gw)z),\\
BM \cap FV &= ((fw + hu)x : g(uz + wx) : (fw + hu)z),\\
CM \cap FW &= ((gu + fv)x : (gu + fv)y : h(vx + uy)).
\end{aligned}$$

The corresponding determinant is $(fvw+gwu+huv)R$ where R is the expression on the left hand side of (3). $\square$

The Simson cubic is the particular case $F = G$, the centroid, and ℓ the line

$$\frac{x}{S_A} + \frac{y}{S_B} + \frac{z}{S_C} = 0,$$

which is the tripolar of the isotomic conjugate of the orthocenter H.

[8]The center of this conic is the point $(f(v+w-u)+u(g+h-f) : g(w+u-v)+v(h+f-g) : h(u+v-w)+w(f+g-h))$.

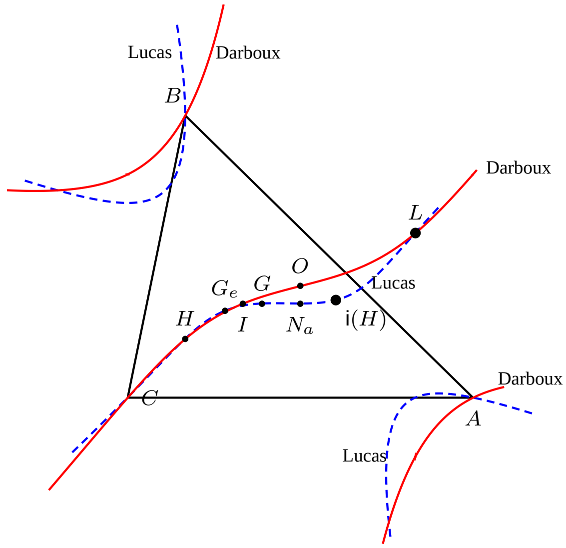

Figure 5. The Darboux and Lucas cubics

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Geometric Construction of Reciprocal Conjugations

Keith Dean and Floor van Lamoen

Abstract. Two conjugation mappings are well known in the geometry of the triangle: the isogonal and isotomic conjugations. These two are members of the family of reciprocal conjugations. In this paper we provide an easy and general construction for reciprocal conjugates of a point, given a pair of conjugate points. A connection is made to desmic configurations.

1. Introduction

Let ABC be a triangle. To represent a point in the plane of ABC we make use of homogeneous coordinates. Two such coordinate systems are well known, barycentric and normal (trilinear) coordinates. See [1] for an introduction on normal coordinates,[1] and [4] for barycentric coordinates. In the present paper we work with homogeneous barycentric coordinates exclusively.

Consider a point $X = (x : y : z)$. The isogonal conjugate X^* of X is represented by $(a^2yz : b^2xz : c^2yz)$, which we loosely write as $(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z})$ for X outside the sidelines of ABC (so that $xyz \neq 0$). In the same way the isotomic conjugate $X^\bullet$ of X is represented by $(\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$. For both X^* and $X^\bullet$ the coordinates are the products of the reciprocals of those of X and the constant 'coordinates' from a certain homogeneous triple, $(a^2 : b^2 : c^2)$ and $(1 : 1 : 1)$ respectively. [2]

With this observation it is reasonable to generalize the two famous conjugations to *reciprocal conjugations*, where the homogeneous triple takes a more general form $(\ell : m : n)$ with $\ell mn \neq 0$. By the $(\ell : m : n)$*-reciprocal conjugation* or simply $(\ell : m : n)$*-conjugation*, we mean the mapping

$$\tau : (x : y : z) \mapsto \left(\frac{\ell}{x} : \frac{m}{y} : \frac{n}{z}\right).$$

It is clear that for any point X outside the side lines of ABC, $\tau(\tau(X)) = X$. A reciprocal conjugation is uniquely determined by any one pair of conjugates: if $\tau(x_0 : y_0 : z_0) = (x_1 : y_1 : z_1)$, then $\ell : m : n = x_0x_1 : y_0y_1 : z_0z_1$. It is convenient to regard $(\ell : m : n)$ as the coordinates of a point P_0, which we call the *pole* of the conjugation τ. The poles of the isogonal and isotomic conjugations,

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[1] What we call normal coordinates are traditionally called *trilinear* coordinates; they are the ratio of signed distances of the point to the side lines of the reference triangle.

[2] Analogous results hold when normal coordinates are used instead of barycentrics.

for example, are the symmedian point and the centroid respectively. In this paper we address the questions of (i) construction of τ given a pair of points P and Q conjugate under τ, (ii) construction of $\tau(P)$ given the pole R_0.

2. The parallelogram construction

2.1. *Isogonal conjugation.* There is a construction of the isogonal conjugate of a point that gives us a good opportunity for generalization to all reciprocal conjugates.

Proposition 1. *Let P be a point and let $A'B'C'$ be its pedal triangle. Let A'' be the point such that $B'PC'A''$ is a parallelogram. In the same way construct B'' and C''. Then the perspector of triangles ABC and $A''B''C''$ is the isogonal conjugate P^* of P.*

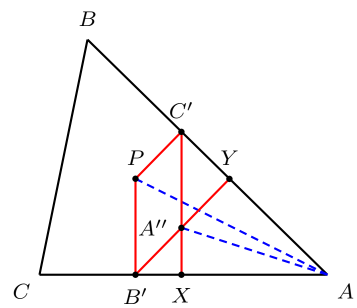

Figure 1

Proof. This is equivalent to the construction of the isogonal conjugate of P as the point of concurrency of the perpendiculars through the vertices of ABC to the corresponding sides of the pedal triangle. Here we justify it directly by noting that AP and AA'' are isogonal lines. Let $C'A''$ and $B'A''$ intersect AC and AB at X and Y respectively. Clearly,

$$B'P : YA'' = A''C' : YA'' = B'A : YA.$$

From this we conclude that triangles APB' and $AA''Y$ are similar, so that AP and AA'' are indeed isogonal lines. □

2.2. *Construction of $(\ell : m : n)$-conjugates.* Observe that the above construction depends on the 'altitudes' forming the pedal triangle $A'B'C'$. When these altitudes are replaced by segments parallel to the cevians of a point $H = (f : g : h)$, we obtain a generalization to the reciprocal conjugation

$$\tau : (x : y : z) \mapsto \left(\frac{f(g+h)}{x} : \frac{g(f+h)}{y} : \frac{h(f+g)}{z}\right).$$

In this way we get the complete set of reciprocal conjugations . In particular, given $(\ell : m : n)$, by choosing H to be the point with (homogeneous barycentric) coordinates

$$\left(\frac{1}{m+n-\ell} : \frac{1}{n+\ell-m} : \frac{1}{\ell+m-n}\right),$$

this construction gives the $(\ell : m : n)$-conjugate of points.[3]

3. The perspective triangle construction

The parallelogram construction depended on a triple of directions of cevians. These three directions can be seen as a degenerate triangle on the line at infinity, perspective to ABC. We show that this triangle can be replaced by any triangle $A_1B_1C_1$ perspective to ABC, thus making the notion of reciprocal conjugation *projective*.

Proposition 2. *A triangle $A_1B_1C_1$ perspective with ABC induces a reciprocal conjugation : for every point M not on the side lines of ABC and $A_1B_1C_1$, construct*

$$\begin{array}{lll} A' = A_1M \cap BC, & B' = B_1M \cap CA, & C' = C_1M \cap AB; \\ A'' = B_1C' \cap C_1B', & B'' = C_1A' \cap A_1C', & C'' = A_1B' \cap B_1A'. \end{array}$$

Triangle $A''B''C''$ is perspective with ABC at a point N, and the correspondence $M \mapsto N$ is a reciprocal conjugation.

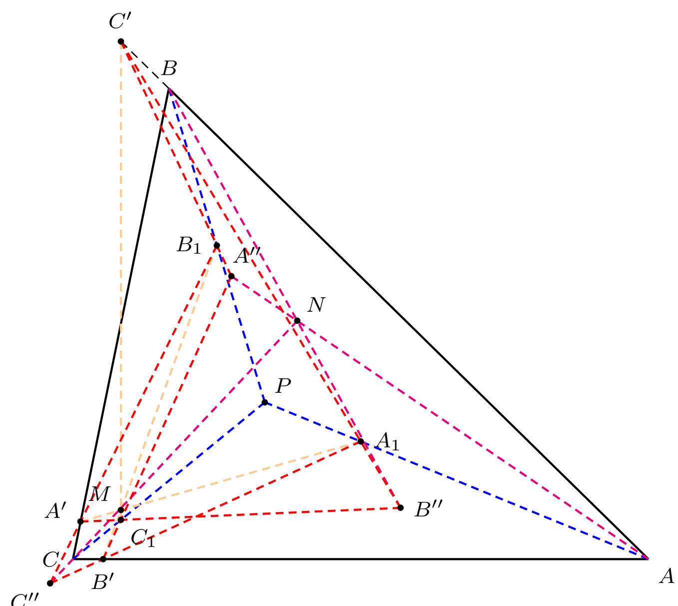

Figure 2

Proof. Since $A_1B_1C_1$ is perspective with ABC, we may write the coordinates of its vertices in the form

$$A_1 = (U : v : w), \qquad B_1 = (u : V : w), \qquad C_1 = (u : v : W). \tag{1}$$

The perspector is $P = (u : v : w)$. Let $M = (f : g : h)$ be a point outside the sidelines of ABC and $A_1B_1C_1$. Explicitly,

$$A' = (0 : gU - fv : hU - fw) \quad \text{and} \quad B' = (fV - gu : 0 : hV - gw).$$

---

[3]Let X be the point with coordinates $(\ell : m : n)$. The point H can be taken as the isotomic conjugate of the point Y which divides the segment XG in the ratio $XG : GY = 1 : 2$.

The lines A_1B' and B_1A' are given by [4]

$$\begin{aligned}(gw-hV)vx+(hUV-gwU-fwV+guw)y+(fV-gu)vz &= 0,\\ (gwU-fvw-hUV+fwV)x+(hU-fw)uy+(fv-gU)uz &= 0.\end{aligned}$$

These lines intersect in the point C'' with coordinates

$$(guU:fvV:fwV+gwU-hUV)\sim\left(\frac{uU}{f}:\ \frac{vV}{g}:\ \frac{fwV+gwU-hUV}{fg}\right).$$

With similar results for A'' and B'' we have the perspectivity of $A''B''C''$ and ABC at the point

$$N=\left(\frac{uU}{f}:\frac{vV}{g}:\frac{wW}{h}\right).$$

The points M and N clearly correspond to one another under the reciprocal $(uU:vV:wW)$-conjugation. □

Theorem 3. *Let P, Q, R be collinear points. Denote by X, Y, Z the traces of R on the side lines BC, CA, AB of triangle ABC, and construct triangle $A_1B_1C_1$ with vertices*

$$A_1=PA\cap QX,\quad B_1=PB\cap QY,\quad C_1=PC\cap QZ. \tag{2}$$

Triangle $A_1B_1C_1$ is perspective with ABC, and induces the reciprocal conjugation under which P and Q correspond.

Proof. If $P=(u:v:w)$, $Q=(U:V:W)$, and $R=(1-t)P+tQ$ for some $t\neq 0,1$, then

$$A_1=\left(\frac{-tU}{1-t}:v:w\right),\qquad B_1=\left(u:\frac{-tV}{1-t}:w\right),\qquad C_1=\left(u:v:\frac{-tW}{1-t}\right).$$

The result now follows from Proposition 2 and its proof. □

Proposition 2 and Theorem 3 together furnish a construction of $\tau(M)$ for an arbitrary point M (outside the side lines of ABC) under the conjugation τ defined by two distinct points P and Q. In particular, the pole P_0 can be constructed by applying to the triangle $A_1B_1C_1$ in (2) and M the centroid of ABC in the construction of Proposition 2.

Corollary 4. *Let P_0 be a point different from the centroid G of triangle ABC, regarded as the pole of a reciprocal conjugation τ. To construct $\tau(M)$, apply the construction in Theorem 3 to $(P,Q)=(G,P_0)$. The choice of R can be arbitrary, say, the midpoint of GP_0.*

Remark. This construction does not apply to isotomic conjugation, for which the pole is the centroid.

[4]Here we have made use of the fact that the line B_1C_1 is given by the equation

$$w(v-V)x+w(u-U)y+(UV-uv)z=0,$$

so that we indeed can divide by $fw(v-V)+gw(u-U)+h(UV-uv)$.

4. Desmic configuration

We take a closer look of the construction in Proposition 2. Given triangle $A_1B_1C_1$ with perspector P, it is known that the triangle $A_2B_2C_2$ with vertices

$$A_2 = BC_1 \cap CB_1, \qquad B_2 = AC_1 \cap CA_1, \qquad C_2 = AB_1 \cap BA_1,$$

is perspective to both ABC and $A_1B_1C_1$, say at points Q and R respectively, and that the perspectors P, Q, R are collinear. See, for example, [2]. Indeed, if the vertices of $A_1B_1C_1$ have coordinates given by (1), those of $A_2B_2C_2$ have coordinates

$$A_2 = (u : V : W), \quad B_2 = (U : v : W), \qquad C_2 = (U : V : w).$$

From these, it is clear that

$$Q = (U : V : W) \qquad \text{and} \qquad R = (u + U : v + V : w + W).$$

Triangle $A_2B_2C_2$ is called the *desmic mate* of triangle $A_1B_1C_1$. The three triangles, their perspectors, and the connecting lines form a *desmic configuration, i.e.*, each line contains 3 points and each point is contained in 4 lines. This configuration also contains the three desmic quadrangles: $ABCR$, $A_1B_1C_1Q$ and $A_2B_2C_2P$.

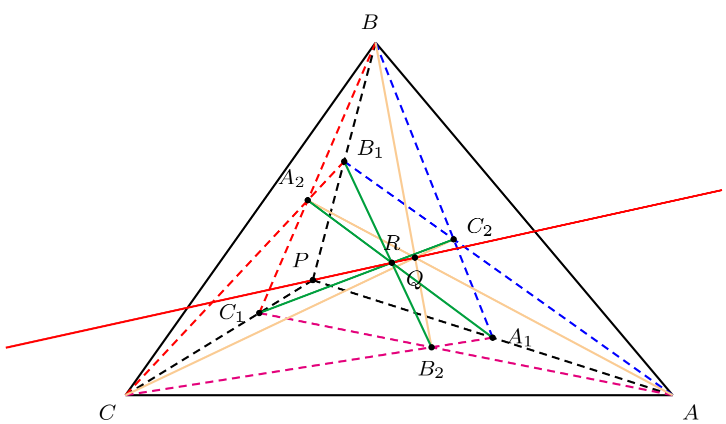

Figure 3

The construction in the preceding section shows that given collinear points P, Q, and R, there is a desmic configuration as above in which the quadrangles $A_1B_1C_1Q$ and $A_2B_2C_2P$ are perspective at R. The reciprocal conjugations induced by $A_1B_1C_1$ and $A_2B_2C_2$ are the same, and is independent of the choice of R.

Barry Wolk [3] has observed that these twelve points all lie on the *iso*-$(uU : vV : wW)$ cubic with pivot R: [5]

$$x(u+U)(wWy^2 - vVz^2) + y(v+V)(uUz^2 - wWx^2) + z(w+W)(vVx^2 - uUy^2) = 0.$$

[5] An iso-$(\ell : m : n)$ cubic with pivot R is the locus of points X for which X and its iso-$(\ell : m : n)$-conjugate are collinear with R.

Since reciprocal conjugates link the vertices of ABC to their opposite sides, clearly the traces of R are also on the cubic. By the symmetry of the desmic configurations, the traces of Q in $A_1B_1C_1$ and P in $A_2B_2C_2$ are also on the desmic cubic.

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A Note on the Feuerbach Point

Lev Emelyanov and Tatiana Emelyanova

Abstract. The circle through the feet of the internal bisectors of a triangle passes through the Feuerbach point, the point of tangency of the incircle and the nine-point circle.

The famous Feuerbach theorem states that the nine-point circle of a triangle is tangent internally to the incircle and externally to each of the excircles. Given triangle ABC, the Feuerbach point F is the point of tangency with the incircle. There exists a family of cevian circumcircles passing through the Feuerbach point. Most remarkable are the cevian circumcircles of the incenter and the Nagel point.[1] In this note we give a geometric proof in the incenter case.

Theorem. *The circle passing through the feet of the internal bisectors of a triangle contains the Feuerbach point of the triangle.*

The proof of the theorem is based on two facts: the triangle whose vertices are the feet of the internal bisectors and the Feuerbach triangle are (a) similar and (b) perspective.

Lemma 1. *In Figure 1, circle $O(R)$ is tangent externally to each of circles $O_1(r_1)$ and $O_2(r_2)$, at A and B respectively. If A_1B_1 is a segment of an external common tangent to the circles (O_1) and (O_2), then*

$$AB = \frac{R}{\sqrt{(R+r_1)(R+r_2)}} \cdot A_1B_1. \tag{1}$$

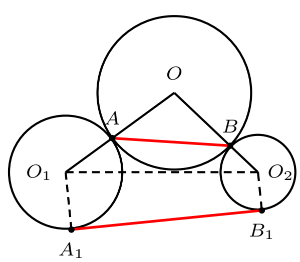

Figure 1

Proof. In the isosceles triangle AOB, $\cos AOB = \frac{2R^2-AB^2}{2R^2} = 1-\frac{AB^2}{2R^2}$. Applying the law of cosines to triangle O_1OO_2, we have

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[1]The cevian feet of the Nagel point are the points of tangency of the excircles with the corresponding sides.

$$\begin{aligned} O_1O_2^2 &= (R+r_1)^2+(R+r_2)^2-2(R+r_1)(R+r_2)\left(1-\frac{AB^2}{2R^2}\right) \\ &= (r_1-r_2)^2+(R+r_1)(R+r_2)\left(\frac{AB}{R}\right)^2. \end{aligned}$$

From trapezoid $A_1O_1O_2B_1$, $O_1O_2^2 = (r_1-r_2)^2 + A_1B_1^2$. Comparison now gives A_1B_1 as in (1). $\square$

Consider triangle ABC with side lengths $BC = a$, $CA = b$, $AB = c$, and circumcircle $O(R)$. Let $I_3(r_3)$ be the excircle on the side AB.

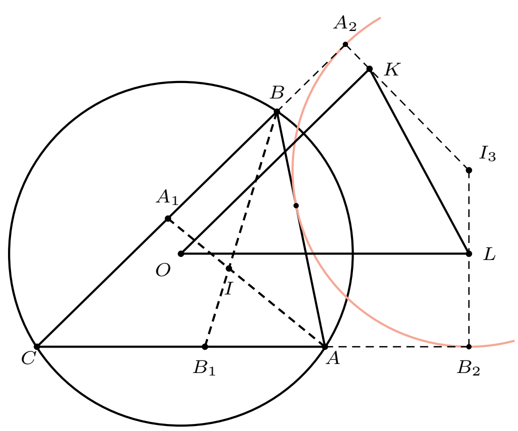

Figure 2

Lemma 2. *If A_1 and B_1 are the feet of the internal bisectors of angles A and B, then*

$$A_1B_1 = \frac{abc\sqrt{R(R+2r_3)}}{(c+a)(b+c)R}. \tag{2}$$

Proof. In Figure 2, let K and L be points on I_3A_2 and I_3B_2 such that $OK//CB$, and $OL//CA$. Since $CA_2 = CB_2 = \frac{a+b+c}{2}$,

$$OL = \frac{a+b+c}{2} - \frac{b}{2} = \frac{c+a}{2}, \qquad OK = \frac{a+b+c}{2} - \frac{a}{2} = \frac{b+c}{2}.$$

Also,

$$CB_1 = \frac{ba}{c+a}, \qquad CA_1 = \frac{ab}{b+c},$$

and

$$\frac{CB_1}{CA_1} = \frac{b+c}{c+a} = \frac{OK}{OL}.$$

Thus, triangle CA_1B_1 is similar to triangle OLK, and

$$\frac{A_1B_1}{LK} = \frac{CB_1}{OK} = \frac{2ab}{(c+a)(b+c)}. \tag{3}$$

Since OI_3 is a diameter of the circle through O, L, K, by the law of sines,

$$LK = OI_3 \cdot \sin LOK = OI_3 \cdot \sin C = OI_3 \cdot \frac{c}{2R}. \tag{4}$$

Combining (3), (4) and Euler's formula $OI_3^2 = R(R+2r_3)$, we obtain (2). □

Now, we prove the main theorem.

(a) Consider the nine-point circle $N(\frac{R}{2})$ tangent to the A- and B-excircles. See Figure 3. The length of the external common tangent of these two excircles is

$$XY = AY + BX - AB = \frac{a+b+c}{2} + \frac{a+b+c}{2} - c = a+b.$$

By Lemma 1,

$$F_1F_2 = \frac{(a+b)\cdot\frac{R}{2}}{\sqrt{(\frac{R}{2}+r_1)(\frac{R}{2}+r_2)}} = \frac{(a+b)R}{\sqrt{(R+2r_1)(R+2r_2)}}.$$

Comparison with (2) gives

$$\frac{A_1B_1}{F_1F_2} = \frac{abc\sqrt{R(R+2r_1)(R+2r_2)(R+2r_3)}}{(a+b)(b+c)(c+a)R^2}.$$

The symmetry of this ratio in a, b, c and the exradii shows that

$$\frac{A_1B_1}{F_1F_2} = \frac{B_1C_1}{F_2F_3} = \frac{C_1A_1}{F_3F_1}.$$

It follows that the triangles $A_1B_1C_1$ and $F_1F_2F_3$ are similar.

(b) We prove that the points F, B_1 and F_2 are collinear. By the Feuerbach theorem, F is the homothetic center of the incircle and the nine-point circle, and F_2 is the internal homothetic center of the nine-point circle and the B- excircle. Note that B_1 is the internal homothetic center of the incircle and the B-excircle. These three homothetic centers divide the side lines of triangle I_2NI in the ratios

$$\frac{NF}{FI} = -\frac{R}{2r}, \qquad \frac{IB_1}{B_1I_2} = \frac{r}{r_2}, \qquad \frac{I_2F_2}{F_2N} = \frac{2r_2}{R}.$$

Since

$$\frac{NF}{FI} \cdot \frac{IB_1}{B_1I_2} \cdot \frac{I_2F_2}{F_2N} = -1,$$

by the Menelaus theorem, F, B_1, and F_2 are collinear. Similarly F, C_1, F_3 are collinear, as are F, A_1, F_1. This shows that triangles $A_1B_1C_1$ and $F_1F_2F_3$ are perspective at F.

From (a) and (b) it follows that

$$\angle C_1FA_1 + \angle C_1B_1A_1 = \angle F_3FF_1 + \angle F_3F_2F_1 = 180^\circ,$$

i.e., the circle $A_1B_1C_1$ contains the Feuerbach point F.

This completes the proof of the theorem.

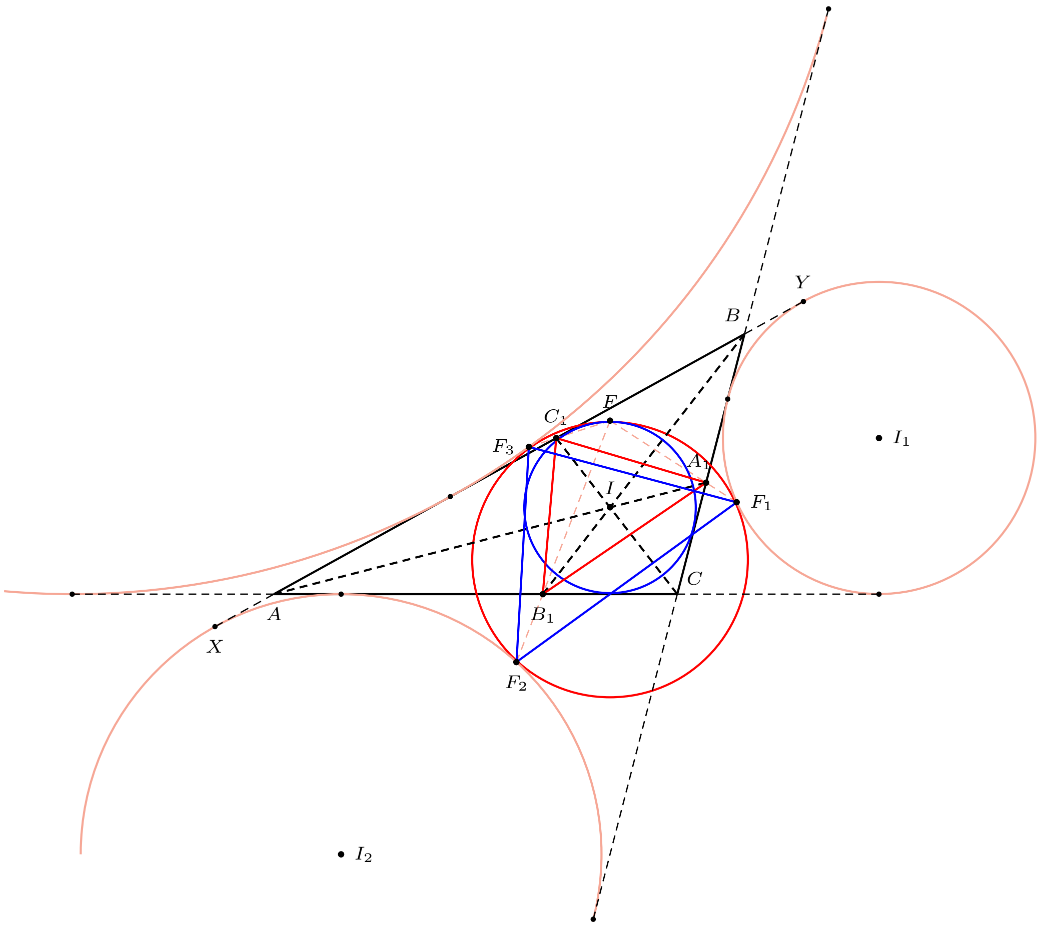

Figure 3

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The Kiepert Pencil of Kiepert Hyperbolas

Floor van Lamoen and Paul Yiu

Abstract. We study Kiepert triangles $\mathcal{K}(\phi)$ and their iterations $\mathcal{K}(\phi,\psi)$, the Kiepert triangles $\mathcal{K}(\psi)$ relative to Kiepert triangles $\mathcal{K}(\phi)$. For arbitrary ϕ and ψ, we show that $\mathcal{K}(\phi,\psi)=\mathcal{K}(\psi,\phi)$. This iterated Kiepert triangle is perspective to each of ABC, $\mathcal{K}(\phi)$, and $\mathcal{K}(\psi)$. The Kiepert hyperbolas of $\mathcal{K}(\phi)$ form a pencil of conics (rectangular hyperbolas) through the centroid, and the two infinite points of the Kiepert hyperbola of the reference triangle. The centers of the hyperbolas in this Kiepert pencils are on the line joining the Fermat points of the medial triangle of ABC. Finally we give a construction of the degenerate Kiepert triangles. The vertices of these triangles fall on the parallels through the centroid to the asymptotes of the Kiepert hyperbola.

1. Preliminaries

Given triangle ABC with side lengths a, b, c, we adopt the notation of John H. Conway. Let S denote *twice* the area of the triangle, and for every θ, write $S_\theta = S\cdot\cot\theta$. In particular, from the law of cosines,

$$S_A=\frac{b^2+c^2-a^2}{2},\quad S_B=\frac{c^2+a^2-b^2}{2},\quad S_C=\frac{a^2+b^2-c^2}{2}.$$

The sum $S_A+S_B+S_C=\frac{1}{2}(a^2+b^2+c^2)=S_\omega$ for the Brocard angle ω of the triangle. See, for example, [2, p.266] or [3, p.47]. For convenience, a product $S_\phi\cdot S_\psi\cdots$ is simply written as $S_{\phi\psi\cdots}$. We shall make use of the following fundamental formulae.

Lemma 1 (Conway)**.** *The following relations hold:*

(a) $a^2=S_B+S_C$, $b^2=S_C+S_A$, *and* $c^2=S_A+S_B$;
(b) $S_A+S_B+S_C=S_\omega$;
(c) $S_{AB}+S_{BC}+S_{CA}=S^2$;
(d) $S_{ABC}=S^2\cdot S_\omega-a^2b^2c^2$.

Proposition 2 (Distance formula)**.** *The square distance between two points with* absolute *barycentric coordinates* $P=x_1A+y_1B+z_1C$ *and* $Q=x_2A+y_2B+z_2C$ *is given by*

$$|PQ|^2=S_A(x_1-x_2)^2+S_B(y_1-y_2)^2+S_C(z_1-z_2)^2. \tag{1}$$

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Proposition 3 (Conway). *Let P be a point such that the directed angles PBC and PCB are respectively ϕ and ψ. The homogeneous barycentric coordinates of P are*

$$(-a^2 \ : \ S_C + S_\psi \ : \ S_B + S_\phi).$$

Since the cotangent function has period π, we may always choose ϕ and ψ in the range $-\frac{\pi}{2} < \phi, \ \psi \leq \frac{\pi}{2}$. See Figure 1.

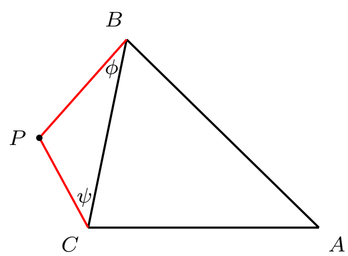

Figure 1

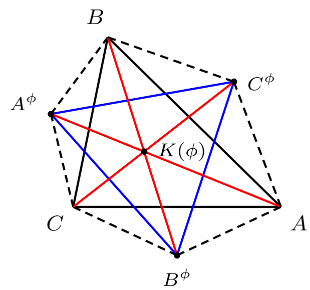

Figure 2

2. The Kiepert triangle $\mathcal{K}(\phi)$

Given an angle ϕ, let A^ϕ, B^ϕ, C^ϕ be the apexes of isosceles triangles on the sides of ABC with base angle ϕ. These are the points

$$\begin{array}{rcl} A^\phi & = & (-a^2 : S_C + S_\phi : S_B + S_\phi), \\ B^\phi & = & (S_C + S_\phi : -b^2 : S_A + S_\phi), \\ C^\phi & = & (S_B + S_\phi : S_A + S_\phi : -c^2). \end{array} \tag{2}$$

They form the *Kiepert triangle* $\mathcal{K}(\phi)$, which is perspective to ABC at the *Kiepert perspector*

$$K(\phi) = \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi}\right). \tag{3}$$

See Figure 2. If $\phi = \frac{\pi}{2}$, this perspector is the orthocenter H. The Kiepert triangle $\mathcal{K}(\frac{\pi}{2})$ is one of three degenerate Kiepert triangles. Its vertices are the infinite points in the directions of the altitudes. The other two are identified in §2.3 below.

The Kiepert triangle $\mathcal{K}(\phi)$ has the same centroid $G = (1 : 1 : 1)$ as the reference triangle ABC. This is clear from the coordinates given in (2) above.

2.1. *Side lengths.* We denote by a_ϕ, b_ϕ, and c_ϕ the lengths of the sides $B^\phi C^\phi$, $C^\phi A^\phi$, and $A^\phi B^\phi$ of the Kiepert triangle $\mathcal{K}(\phi)$. If $\phi \neq \frac{\pi}{2}$, these side lengths are given by

$$\begin{array}{rcl} 4S_\phi^2 \cdot a_\phi^2 & = & a^2 S_\phi^2 + S^2(4S_\phi + S_\omega + 3S_A), \\ 4S_\phi^2 \cdot b_\phi^2 & = & b^2 S_\phi^2 + S^2(4S_\phi + S_\omega + 3S_B), \\ 4S_\phi^2 \cdot c_\phi^2 & = & c^2 S_\phi^2 + S^2(4S_\phi + S_\omega + 3S_C). \end{array} \tag{4}$$

Here is a simple relation among these side lengths.

Proposition 4. *If $\phi \neq \frac{\pi}{2}$,*

$$b_\phi^2 - c_\phi^2 = \frac{1 - 3\tan^2\phi}{4} \cdot (b^2 - c^2);$$

similarly for $c_\phi^2 - a_\phi^2$ and $a_\phi^2 - b_\phi^2$.

If $\phi = \pm\frac{\pi}{6}$, we have $b_\phi^2 = c_\phi^2 = a_\phi^2$, and the triangle is equilateral. This is Napoleon's theorem.

2.2. *Area.* Denote by S' twice the area of $\mathcal{K}(\phi)$. If $\phi \neq \frac{\pi}{2}$,

$$S' = \frac{S}{(2S_\phi)^3} \begin{vmatrix} -a^2 & S_C + S_\phi & S_B + S_\phi \\ S_C + S_\phi & -b^2 & S_A + S_\phi \\ S_B + S_\phi & S_A + S_\phi & -c^2 \end{vmatrix} = \frac{S}{4S_\phi^2}(S_\phi^2 + 2S_\omega S_\phi + 3S^2). \tag{5}$$

2.3. *Degenerate Kiepert triangles.* The Kiepert triangle $\mathcal{K}(\phi)$ degenerates into a line when $\phi = \frac{\pi}{2}$ as we have seen above, or $S' = 0$. From (5), this latter is the case if and only if $\phi = \omega_\pm$ for

$$\cot\omega_\pm = -\cot\omega \pm \sqrt{\cot^2\omega - 3}. \tag{6}$$

See §5.1 and Figures 8A,B for the construction of the two finite degenerate Kiepert triangles.

2.4. *The Kiepert hyperbola.* It is well known that the locus of the Kiepert perspectors is the Kiepert hyperbola

$$\mathcal{K}: \quad (b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy = 0.$$

See, for example, [1]. In this paper, we are dealing with the Kiepert hyperbolas of various triangles. This particular one (of the reference triangle) will be referred to as the *standard* Kiepert hyperbola. It is the rectangular hyperbola with asymptotes the Simson lines of the intersections of the circumcircle with the Brocard axis OK (joining the circumcenter and the symmedian point). Its center is the point $((b^2 - c^2)^2 : (c^2 - a^2)^2 : (a^2 - b^2)^2)$ on the nine-point circle. The asymptotes, regarded as infinite points, are the points $K(\phi)$ for which

$$\frac{1}{S_A + S_\phi} + \frac{1}{S_B + S_\phi} + \frac{1}{S_C + S_\phi} = 0.$$

These are $I_\pm = K(\frac{\pi}{2} - \omega_\pm)$ for $\omega_\pm$ given by (6) above.

3. Iterated Kiepert triangles

Denote by A', B', C' the magnitudes of the angles A^ϕ, B^ϕ, C^ϕ of the Kiepert triangle $\mathcal{K}(\phi)$. From the expressions of the side lengths in (4), we have

$$S'_{A'} = \frac{1}{4S_\phi^2}(S_A S_\phi^2 + 2S^2 S_\phi + S^2(2S_\omega - 3S_A)) \tag{7}$$

together with two analogous expressions for $S'_{B'}$ and $S'_{C'}$.

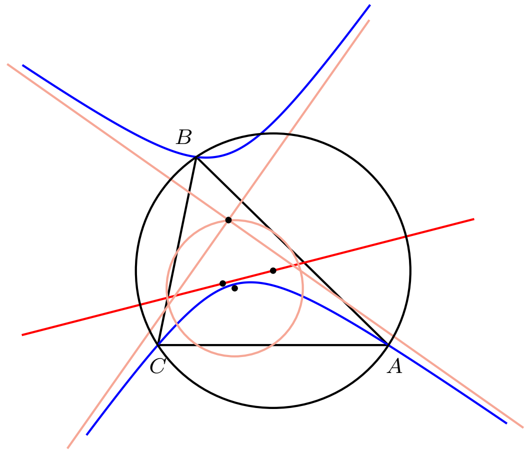

Figure 3

Consider the Kiepert triangle $\mathcal{K}(\psi)$ of $\mathcal{K}(\phi)$. The coordinates of the apex $A^{\phi,\psi}$ with respect to $\mathcal{K}(\phi)$ are $(-a_\phi^2 : S'_{C'} + S'_\psi : S'_{B'} + S'_\psi)$. Making use of (5) and (7), we find the coordinates of the vertices of $\mathcal{K}(\phi,\psi)$ with reference to ABC, as follows.

$$\begin{aligned}
A^{\phi,\psi} &= (-(2S^2 + a^2(S_\phi + S_\psi) + 2S_{\phi\psi}) : S^2 - S_{\phi\psi} + S_C(S_\phi + S_\psi) : S^2 - S_{\phi\psi} + S_B(S_\phi + S_\psi)),\\
B^{\phi,\psi} &= (S^2 - S_{\phi\psi} + S_C(S_\phi + S_\psi) : -(2S^2 + b^2(S_\phi + S_\psi) + 2S_{\phi\psi}) : S^2 - S_{\phi\psi} + S_A(S_\phi + S_\psi)),\\
C^{\phi,\psi} &= (S^2 - S_{\phi\psi} + S_B(S_\phi + S_\psi) : S^2 - S_{\phi\psi} + S_A(S_\phi + S_\psi) : -(2S^2 + c^2(S_\phi + S_\psi) + 2S_{\phi\psi})).
\end{aligned}$$

From these expressions we deduce a number of interesting properties of the iterated Kiepert triangles.

1. The symmetry in ϕ and ψ of these coordinates shows that the triangles $\mathcal{K}(\phi,\psi)$ and $\mathcal{K}(\psi,\phi)$ coincide.

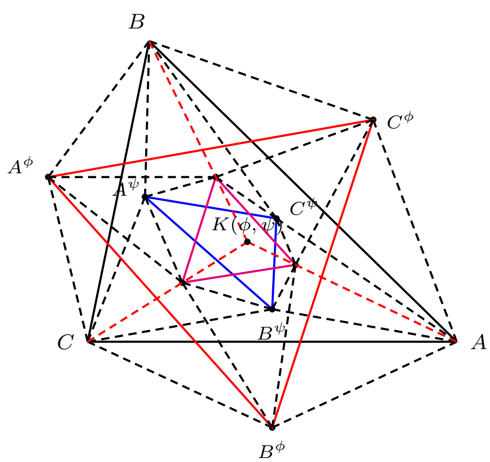

Figure 4

2. It is clear that the iterated Kiepert triangle $\mathcal{K}(\phi,\psi)$ is perspective with each of $\mathcal{K}(\phi)$ and $\mathcal{K}(\psi)$, though the coordinates of the perspectors $K_\phi(\psi)$ and $K_\psi(\phi)$ are very tedious. It is interesting, however, to note that $\mathcal{K}(\phi,\psi)$ is also perspective with ABC. See Figure 4. The perspector has relatively simple coordinates:

$$K(\phi,\psi)=\left(\frac{1}{S^2+S_A(S_\phi+S_\psi)-S_{\phi\psi}}:\frac{1}{S^2+S_B(S_\phi+S_\psi)-S_{\phi\psi}}:\frac{1}{S^2+S_C(S_\phi+S_\psi)-S_{\phi\psi}}\right).$$

3. This perspector indeed lies on the Kiepert hyperbola of ABC; it is the Kiepert perspector $K(\theta)$, where

$$\cot\theta=\frac{1-\cot\phi\cot\psi}{\cot\phi+\cot\psi}=-\cot(\phi+\psi).$$

In other words,

$$K(\phi,\psi)=K(-(\phi+\psi)). \tag{8}$$

From this we conclude that the Kiepert hyperbola of $\mathcal{K}(\phi)$ has the same infinite points of the standard Kiepert hyperbola, *i.e.*, their asymptotes are parallel.

4. The triangle $\mathcal{K}(\phi,-\phi)$ is homothetic to ABC at G, with ratio of homothety $\frac{1}{4}(1-3\tan^2\phi)$. Its vertices are

$$\begin{aligned}
A^{\phi,-\phi} &= (-2(S^2-S_\phi^2):S^2+S_\phi^2:S^2+S_\phi^2),\\
B^{\phi,-\phi} &= (S^2+S_\phi^2:-2(S^2-S_\phi^2):S^2+S_\phi^2),\\
C^{\phi,-\phi} &= (S^2+S_\phi^2:S^2+S_\phi^2:-2(S^2-S_\phi^2)).
\end{aligned}$$

See also [4].

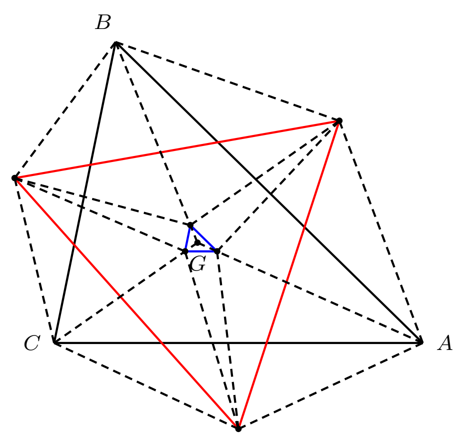

Figure 5

4. The Kiepert hyperbola of $\mathcal{K}(\phi)$

Since the Kiepert triangle $\mathcal{K}(\phi)$ has centroid G, its Kiepert hyperbola $\mathcal{K}_\phi$ contains G. We show that it also contains the circumcenter O.

Proposition 5. *If* $\phi\neq\frac{\pi}{2},\ \pm\frac{\pi}{6}$, $O=K_\phi(-(\frac{\pi}{2}-\phi))$.

Proof. Let $\psi = -(\frac{\pi}{2} - \phi)$, so that $S_\psi = -\frac{S^2}{S_\phi}$. Note that

$$A^{\phi,\psi} = \left(-a^2 : \frac{2S^2 S_\phi}{S^2 + S_\phi^2} + S_C : \frac{2S^2 S_\phi}{S^2 + S_\phi^2} + S_B\right),$$

while

$$A^\phi = (-a^2 : S_C + S_\phi : S_B + S_\phi).$$

These two points are distinct unless $\phi = \frac{\pi}{2}, \pm\frac{\pi}{6}$. Subtracting these two coordinates we see that the line $\ell_a := A^\phi A^{\phi,\psi}$ passes through $(0 : 1 : 1)$, the midpoint of BC. This means, by the construction of A^ϕ, that ℓ_a is indeed the perpendicular bisector of BC, and thus passes through O. By symmetry this proves the proposition. □

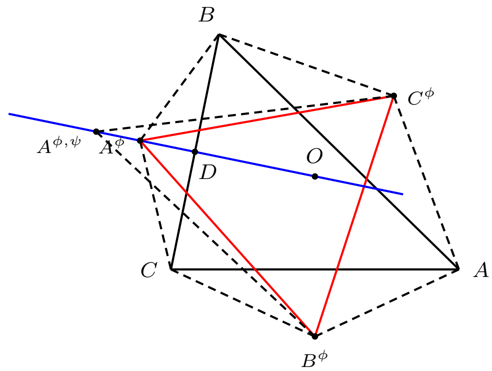

Figure 6

The Kiepert hyperbolas of the Kiepert triangles therefore form the pencil of conics through the centroid G, the circumcenter O, and the two infinite points of the standard Kiepert hyperbola. The Kiepert hyperbola $\mathcal{K}_\phi$ is the one in the pencil that contains the Kiepert perspector $K(\phi)$, since $K(\phi) = K_\phi(-2\phi)$ according to (8). Now, the line containing $K(\phi)$ and the centroid has equation

$$(b^2 - c^2)(S_A + S_\phi)x + (c^2 - a^2)(S_B + S_\phi)y + (a^2 - b^2)(S_C + S_\phi)z = 0.$$

It follows that the equation of $\mathcal{K}_\phi$ is of the form

$$\sum_{\text{cyclic}} (b^2 - c^2)yz + \lambda(x + y + z)(\sum_{\text{cyclic}} (b^2 - c^2)(S_A + S_\phi)x) = 0,$$

where λ is determined by requiring that the conic passes through the circumcenter $O = (a^2 S_A : b^2 S_B : c^2 S_C)$. This gives $\lambda = \frac{1}{2S_\phi}$, and the equation of the conic can be rewritten as

$$2S_\phi(\sum_{\text{cyclic}} (b^2 - c^2)yz) + (x + y + z)(\sum_{\text{cyclic}} (b^2 - c^2)(S_A + S_\phi)x) = 0.$$

Several of the hyperbolas in the pencil are illustrated in Figure 7.

The locus of the centers of the conics in a pencil is in general a conic. In the case of the Kiepert pencil, however, this locus is a line. This is clear from Proposition 4 that the center of $\mathcal{K}_\phi$ has coordinates

$$((b^2 - c^2)^2 : (c^2 - a^2)^2 : (a^2 - b^2)^2)$$

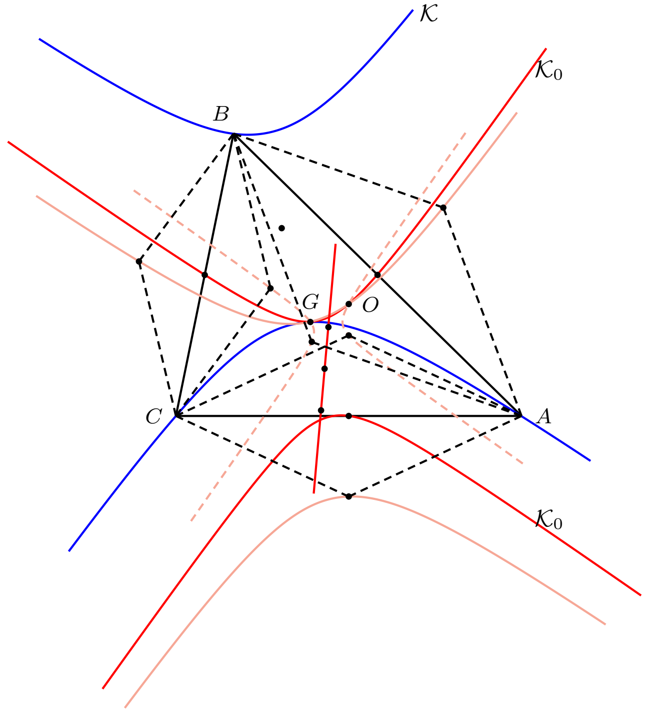

Figure 7

relative to $A^\phi B^\phi C^\phi$, and from (2) that the coordinates of A^ϕ, B^ϕ, C^ϕ are linear functions of S_ϕ. This is the line joining the Fermat points of the medial triangle.

5. Concluding remarks

5.1. *Degenerate Kiepert conics.* There are three degenerate Kiepert triangles corresponding to the three degenerate members of the Kiepert pencil, which are the three pairs of lines connecting the four points G, O, $I_\pm = K(\frac{\pi}{2} - \omega_\pm)$ defining the pencil. The Kiepert triangles $\mathcal{K}(\omega_\pm)$ degenerate into the straight lines $GI_\mp$. The vertices are found by intersecting the line with the perpendicular bisectors of the sides of ABC. The centers of these degenerate Kiepert conics are also on the circle with OG as diameter.

5.2. *The Kiepert hyperbolas of the Napoleon triangles.* The Napoleon triangles $\mathcal{K}(\pm\frac{\pi}{6})$ being equilateral do not posses Kiepert hyperbolas, the centroid being the only finite Kiepert perspector. The rectangular hyperbolas $\mathcal{K}_{\pm\pi/6}$ in the pencil are the circumconics through this common perspector G and O. The centers of these rectangular hyperbolas are the Fermat points of the medial triangle.

5.3. *Kiepert coordinates.* Every point outside the standard Kiepert hyperbola $\mathcal{C}$, and other than the circumcenter O, lies on a unique member of the Kiepert pencil, *i.e.*, it can be *uniquely* written as $K_\phi(\psi)$. As an example, the symmedian point

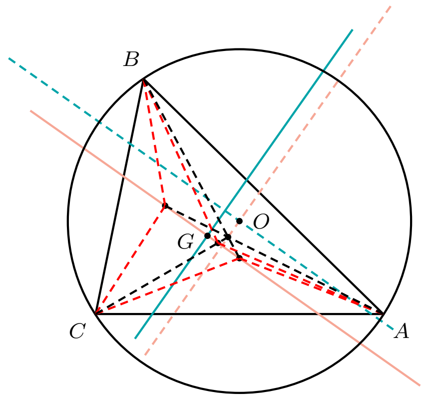

Figure 8A

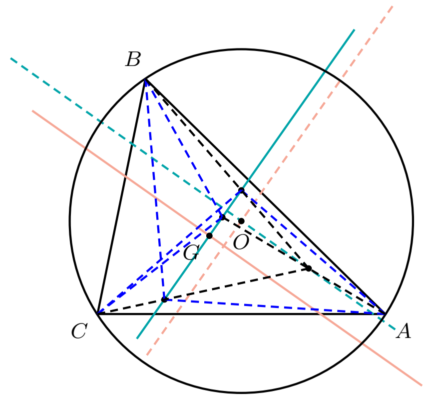

Figure 8B

$K = K_\phi(\psi)$ for $\phi = \omega$ (the Brocard angle) and $\psi = \text{arccot}(\frac{1}{3}\cot\omega)$. We leave the details to the readers.

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Simple Constructions of the Incircle of an Arbelos

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Abstract. We give several simple constructions of the incircle of an arbelos, also known as a shoemaker's knife.

Archimedes, in his *Book of Lemmas*, studied the arbelos bounded by three semicircles with diameters AB, AC, and CB, all on the same side of the diameters.[1] See Figure 1. Among other things, he determined the radius of the incircle of the arbelos. In Figure 2, GH is the diameter of the incircle parallel to the base AB, and G', H' are the (orthogonal) projections of G, H on AB. Archimedes showed that $GHH'G'$ is a square, and that AG', $G'H'$, $H'B$ are in geometric progression. See [1, pp. 307–308].

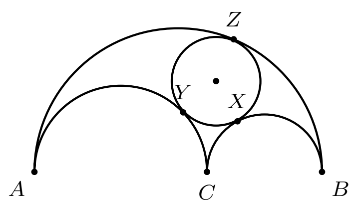


Figure 1

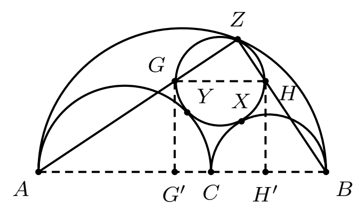
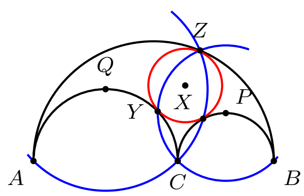
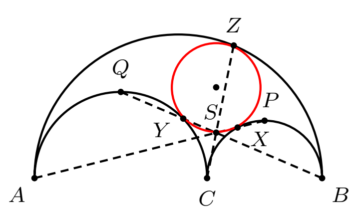


Figure 2

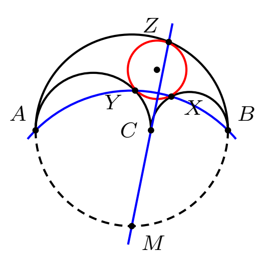
In this note we give several simple constructions of the incircle of the arbelos. The elegant Construction 1 below was given by Leon Bankoff [2]. The points of tangency are constructed by drawing circles with centers at the midpoints of two of the semicircles of the arbelos. In validating Bankoff's construction, we obtain Constructions 2 and 3, which are easier in the sense that one is a ruler-only construction, and the other makes use only of the midpoint of one semicircle.



Construction 1



Construction 2



Construction 3

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[1]The arbelos is also known as the shoemaker's knife. See [3].

Theorem 1 (Bankoff [2])**.** *Let P and Q be the midpoints of the semicircles (BC) and (AC) respectively. If the incircle of the arbelos is tangent to the semicircles (BC), (AC), and (AB) at X, Y, Z respectively, then*

(i) A, C, X, Z lie on a circle, center Q;

(ii) B, C, Y, Z lie on a circle, center P.

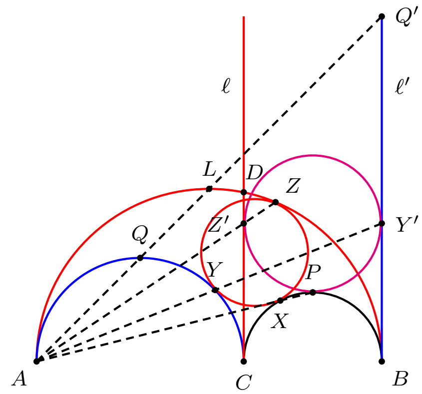


Figure 3

Proof. Let D be the intersection of the semicircle (AB) with the line perpendicular to AB at C. See Figure 3. Note that $AB \cdot AC = AD^2$ by Euclid's proof of the Pythagorean theorem.[2] Consider the inversion with respect to the circle $A(D)$. This interchanges the points B and C, and leaves the line AB invariant. The inversive images of the semicircles (AB) and (AC) are the lines ℓ and ℓ perpendicular to AB at C and B respectively. The semicircle (BC), being orthogonal to the invariant line AB, is also invariant under the inversion. The incircle XYZ of the arbelos is inverted into a circle tangent to the semicircle (BC), and the lines ℓ, ℓ, at P, Y', Z' respectively. Since the semicircle (BC) is invariant, the points A, X, and P are collinear. The points Y' and Z' are such that BPZ' and CPY' are lines making 45° angles with the line AB. Now, the line BPZ' also passes through the midpoint L of the semicircle (AB). The inversive image of this line is a circle passing through A, C, X, Z. Since inversion is conformal, this circle also makes a 45° angle with the line AB. Its center is therefore the midpoint Q of the semicircle (AC). This proves that the points X and Z lies on the circle $Q(A)$.

The same reasoning applied to the inversion in the circle $B(D)$ shows that Y and Z lie on the circle $P(B)$. $\square$

Theorem 1 justifies Construction 1. The above proof actually gives another construction of the incircle XYZ of the arbelos. It is, first of all, easy to construct the circle $PY'Z'$. The points X, Y, Z are then the intersections of the lines AP, AY', and AZ' with the semicircles (BC), (CA), and (AB) respectively. The following two interesting corollaries justify Constructions 2 and 3.

---

[2] *Euclid's Elements*, Book I, Proposition 47.

Corollary 2. *The lines AX, BY, and CZ intersect at a point S on the incircle XYZ of the arbelos.*

Proof. We have already proved that A, X, P are collinear, as are B, Y, Q. In Figure 4, let S be the intersection of the line AP with the circle XYZ. The inversive image S' (in the circle $A(D)$) is the intersection of the same line with the circle $PY'Z'$. Note that

$$\angle AS'Z' = \angle PS'Z' = \angle PY'Z' = 45^\circ = \angle ABZ'$$

so that A, B, S', Z' are concyclic. Considering the inversive image of this circle, we conclude that the line CZ contains S. In other words, the lines AP and CZ intersect at the point S on the circle XYZ. Likewise, BQ and CZ intersect at the same point. □

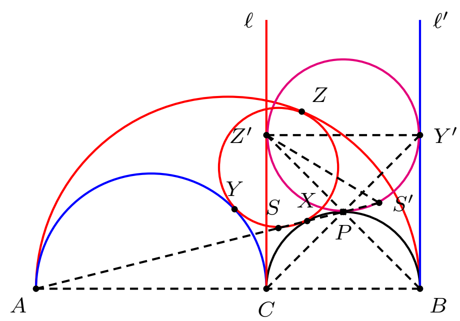


Figure 4

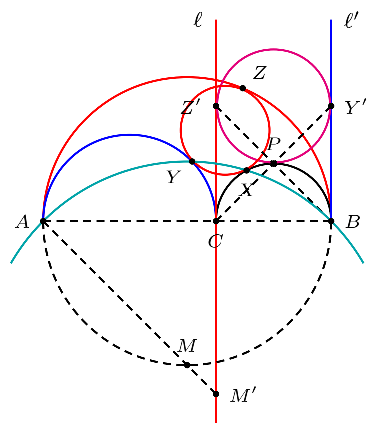


Figure 5

Corollary 3. *Let M be the midpoint of the semicircle (AB) on the opposite side of the arbelos.*

(i) The points A, B, X, Y lie on a circle, center M.

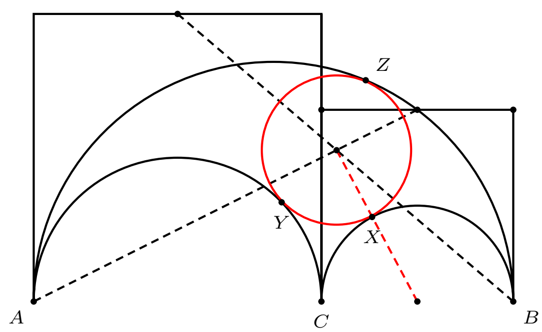
(ii) The line CZ passes through M.

Proof. Consider Figure 5 which is a modification of Figure 3. Since C, P, Y' are on a line making a 45° angle with AB, its inversive image (in the circle $A(D)$) is a circle through A, B, X, Y, also making a 45° angle with AB. The center of this circle is necessarily the midpoint M of the semicircle AB on the opposite side of the arbelos.

Join A, M to intersect the line ℓ at M'. Since $\angle BAM' = 45^\circ = \angle BZ'M'$, the four points A, Z', B, M' are concyclic. Considering the inversive image of the circle, we conclude that the line CZ passes through M. □

The center of the incircle can now be constructed as the intersection of the lines joining X, Y, Z to the centers of the corresponding semicircles of the arbelos.

However, a closer look into Figure 4 reveals a simpler way of locating the center of the incircle XYZ. The circles XYZ and $PY'Z'$, being inversive images, have the center of inversion A as a center of similitude. This means that the center of the incircle XYZ lies on the line joining A to the midpoint of $Y'Z'$, which is the opposite side of the square erected on BC, on the same side of the arbelos. The same is true for the square erected on AC. This leads to the following Construction 4 of the incircle of the arbelos:



Construction 4

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Euler's Formula and Poncelet's Porism

Lev Emelyanov and Tatiana Emelyanova

1. Introduction

It is well known [2, p. 187] that two intersecting circles $O(R)$ and $O_1(R_1)$ are the circumcircle and an excircle respectively of a triangle if and only if the Euler formula

$$d^2 = R^2 + 2RR_1, \tag{1}$$

where $d = |OO_1|$, holds. We present a possibly new proof and an application to the Poncelet porism.

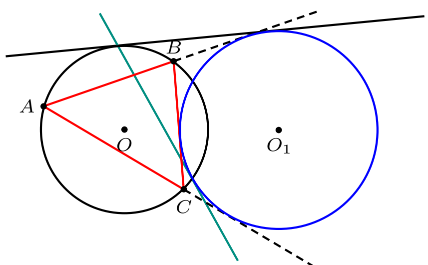

Figure 1

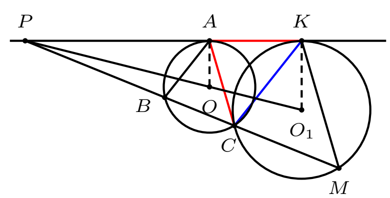

Figure 2

Theorem 1. *Intersecting circles (O) and (O_1) are the circumcircle and an excircle of a triangle if and only if the tangent to (O_1) at an intersection of the circles meets (O) again at the touch point of a common tangent.*

Proof. (Sufficiency) Let $O(R)$ and $O_1(R_1)$ be intersecting circles. (These circles are not assumed to be related to a triangle as in Figure 1.) Of the two lines tangent to both circles, let AK be one of them, as in Figure 2. Let $P = AK \cap OO_1$. Of the two points of intersection of (O) and (O_1), let C be the one not on the same side of line OO_1 as point A. Line AC is tangent to circle $O_1(R_1)$ if and only if $|AC| = |AK|$. Let B and M be the points other than C where line PC meets circles $O(R)$ and $O_1(R_1)$, respectively. Triangles ABC and KCM are homothetic with ratio $\frac{R}{R_1}$, so that $\dfrac{|AB|}{|CK|} = \dfrac{R}{R_1}$. Also, triangles ABC and CAK are similar, since $\angle ABC = \angle CAK$ and $\angle BAC = \angle ACK$. Therefore, $\dfrac{|AB|}{|AC|} = \dfrac{|AC|}{|CK|}$, so that $\dfrac{|CK|}{|AC|} \cdot \dfrac{R}{R_1} = \dfrac{|AC|}{|CK|}$, and

$$|CK| = |AC|\sqrt{\frac{R_1}{R}}. \tag{2}$$

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Also,

$$\begin{aligned}|AK| &= |AC|\cos(\angle CAK) + |CK|\cos(\angle CKA) \\ &= |AC|\sqrt{1-\frac{|AC|^2}{4R^2}} + |CK|\sqrt{1-\frac{|CK|^2}{4R_1^2}}.\end{aligned} \tag{3}$$

If $|AC| = |AK|$, then equations (2) and (3) imply

$$|AK| = |AK|\sqrt{1-\frac{|AK|^2}{4R^2}} + |AK|\sqrt{\frac{R_1}{R}-\frac{|AK|^2}{4R^2}},$$

which simplifies to $|AK|^2 = 4RR_1 - R_1^2$. Since $|AK|^2 = d^2 - (R-R_1)^2$, where $d = |OO_1|$, we have the Euler formula given in (1). □

We shall prove the converse below from Poncelet's porism.

2. Poncelet porism

Suppose triangle ABC has circumcircle $O(R)$ and incircle $I(r)$. The Poncelet porism is the problem of finding all triangles having the same circumcircle and incircle, and the well known solution is an infinite family of triangles. Unless triangle ABC is equilateral, these triangles vary in shape, but even so, they may be regarded as "rotating" about a fixed incircle and within a fixed circumcircle.

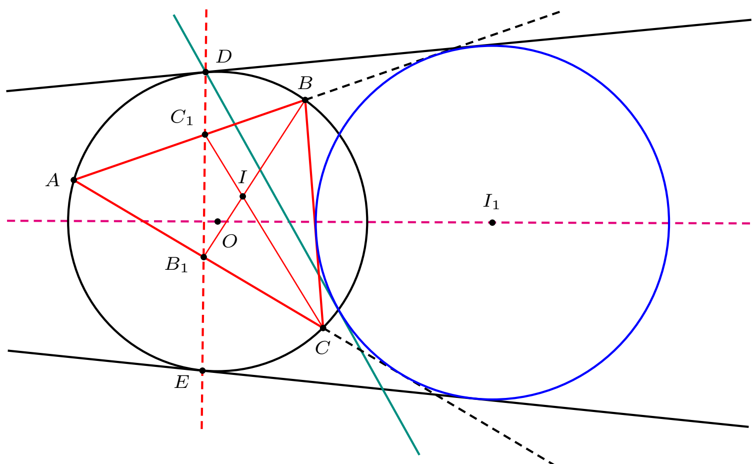

Figure 3

Continuing with the proof of the necessity part of Theorem 1, let $I_1(r_1)$ be the excircle corresponding to vertex A. Since Euler's formula holds for this configuration, the conditions for the Poncelet porism (e.g. [2, pp. 187-188]) hold. In the family of rotating triangles ABC there is one whose vertices A and B coincide in a point, D, and the limiting line AB is, in this case, tangent to the excircle. Moreover, lines CA and BC coincide as the line tangent to the excircle at a point of intersection of the circles, as in Figure 3. This completes the proof of Theorem 1.

Certain points of triangle ABC, other than the centers of the two fixed circles, stay fixed during rotation ([1, p.16-19]). We can also find a fixed line in the Poncelet porism.

Theorem 2. *For each of the rotating triangles ABC with fixed circumcircle and excircle corresponding to vertex A, the feet of bisectors BB_1 and CC_1 traverse line DE, where E is the touch point of the second common tangent.*

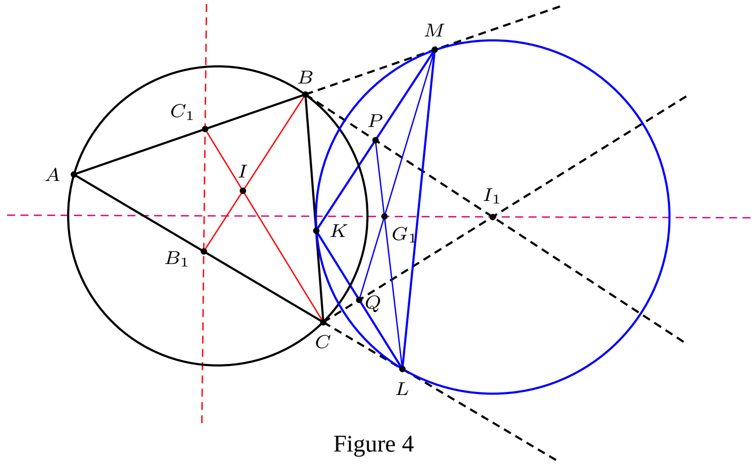

Figure 4

3. Proof of Theorem 2

We begin with the pole-polar correspondence between points and lines for the excircle with center I_1, as in Figure 4.

The polars of A, B, C are LM, MK, KL, respectively, where ΔKLM is the A-extouch triangle. As BB_1 is the internal bisector of angle B and BI_1 is the external bisector, we have $BB_1 \perp BI_1$, and the pole of BB_1 lies on the polar of B, namely MK. Therefore the pole of BB_1 is the midpoint P of segment MK. Similarly, the pole of the bisector CC_1 is the midpoint Q of segment KL. The polar of B_1 is the line passing through the poles of BB_1 and LB_1, i.e. line PL. Likewise, MQ is the polar of C_1, and the pole of B_1C_1 is centroid of triangle KLM, which we denote as G_1.

We shall prove that G_1 is fixed by proving that the orthocenter H_1 of triangle KLM is fixed. (Gallatly [1] proves that the orthocenter of the intouch triangle stays fixed in the Poncelet porism with fixed circumcircle and incircle; we offer a different proof, which applies also to the circumcircle and an excircle.)

Lemma 3. *The orthocenter H_1 of triangle KLM stays fixed as triangle ABC rotates.*

Proof. Let KLM be the extouch triangle of triangle ABC, let RST be the orthic triangle of triangle KLM, and let H_1 and E_1 be the orthocenter and nine-point center, respectively, of triangle KLM, as in Figure 5.

(1) The circumcircle of triangle RST is the nine-point circle of triangle KLM, so that its radius is equal to $\frac{1}{2}R_1$, and its center E_1 is on the Euler line I_1H_1 of triangle KLM.

(2) It is known that altitudes of an obtuse triangle are bisectors (one internal and two external) of its orthic triangle, so that H_1 is the R-excenter of triangle RST.

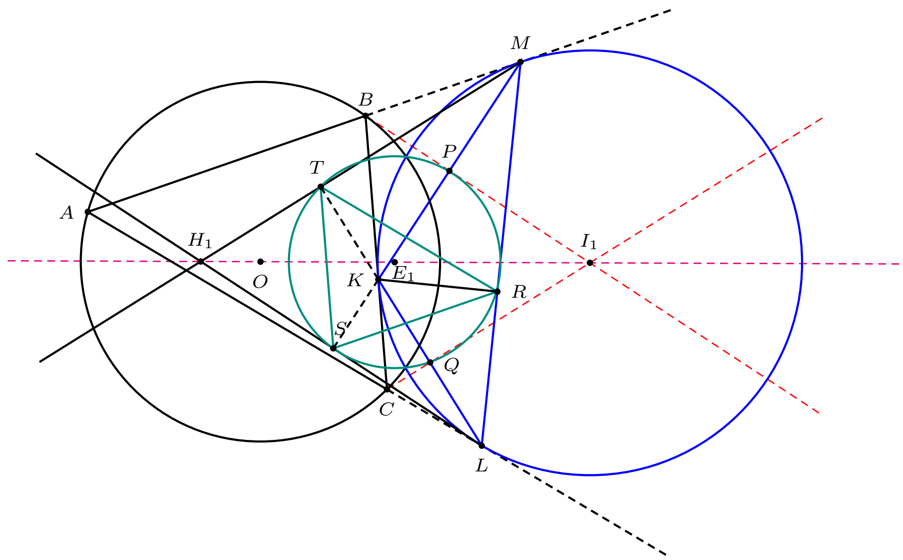

Figure 5

(3) Triangle RST and triangle ABC are homothetic. To see, for example, that $AB||RS$, we have $\angle KRL = \angle KSL = 90°$, so that L, R, S, K are concyclic. Thus, $\angle KLR = \angle KSR = \angle RSM$. On the other hand, $\angle KLR = \angle KLM = \angle KMB$ and $\angle RSM = \angle SMB$. Consequently, $AB||RS$.

(4) The ratio k of homothety of triangle ABC and triangle RST is equal to the ratio of their circumradii, i.e. $k = \frac{2R}{R_1}$. Under this homothety, $O \to E_1$ (the circumcenters) and $I_1 \to H_1$ (the excenter). It follows that $OI_1||E_1H_1$. Since E_1, I_1, H_1 are collinear, O, I_1, H_1 are collinear. Thus OI_1 is the fixed Euler line of every triangle KLM.

The place of H stays fixed on OI because $EH = \frac{OI}{k}$ remains constant. Therefore the centroid of, triangle KLM also stays fixed. □

To complete the proof of Theorem 2, note that by Lemma 3, G_1 is fixed on line OI_1. Therefore, line B_1C_1, as the polar of G_1, is fixed. Moreover, $B_1C_1 \perp OI_1$. Considering the degenerate case of the Poncelet porism, we conclude that B_1C_1 coincides with DE, as in Figure 3.

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Conics Associated with a Cevian Nest

Clark Kimberling

Abstract. Various mappings in the plane of $\triangle ABC$ are defined in the context of a cevian nest consisting of $\triangle ABC$, a cevian triangle, and an anticevian triangle. These mappings arise as Ceva conjugates, cross conjugates, and cevapoints. Images of lines under these mappings and others, involving trilinear and conic-based poles and polars, include certain conics that are the focus of this article.

1. Introduction

Suppose L is a line in the plane of $\triangle ABC$, but not a sideline BC, CA, AB, and suppose a variable point Q traverses L. The isogonal conjugate of Q traces a conic called the isogonal transform of L, which, as is well known, passes through the vertices A, B, C. In this paper, we shall see that for various other transformations, the transform of L is a conic. These include Ceva and cross conjugacies, cevapoints, and pole-to-pole mappings[1]. Let

$$P = p_1 : p_2 : p_3 \tag{1}$$

be a point[2] not on a sideline of $\triangle ABC$. Suppose

$$U = u_1 : u_2 : u_3 \quad \text{and} \quad V = v_1 : v_2 : v_3 \tag{2}$$

are distinct points on L. Then L is given parametrically by

$$Q_t = u_1 + v_1 t : u_2 + v_2 t : u_3 + v_3 t, \; -\infty < t \leq \infty, \tag{3}$$

where $Q_\infty := V$. The curves in question can now be represented by the form $P * Q_t$ (or $P_t * Q$), where $*$ represents any of the various mappings to be considered. For any such curve, a parametric representation is given by the form

$$x_1(t) : x_2(t) : x_3(t),$$

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[1]The cevian triangle of a point P not on a sideline of ABC is the triangle $A'B'C'$, where $A' = PA \cap BC$, $B' = PB \cap CA$, $CB' = PC \cap AB$. The name *cevian* (pronounced cheh´vian) honors Giovanni Ceva (pronounced Chay´va). We use a lower case c in adjectives such as *anticevian* (cf. *nonabelian*) and a capital when the name stands alone, as in *Ceva conjugate.* The name *anticevian* derives from a special case called the *anticomplementary triangle,* so named because its vertices are the anticomplements of A, B, C.

[2]Throughout, coordinates for points are homogeneous trilinear coordinates.

where the coordinates are polynomials in t having no common nonconstant polynomial factor. The degree of the curve is the maximum of the degrees of the polynomials. When this degree is 2, the curve is a conic, and the following theorem (e.g. [5, pp. 60–65]) applies.

Theorem 1. *Suppose a point $X = x_1 : x_2 : x_3$ is given parametrically by*

$$x_1 = d_1t^2 + e_1t + f_1 \tag{4}$$
$$x_2 = d_2t^2 + e_2t + f_2 \tag{5}$$
$$x_3 = d_3t^2 + e_3t + f_3, \tag{6}$$

where the matrix

$$M = \begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}$$

is nonsingular with adjoint (cofactor) matrix

$$M^{\#} = \begin{pmatrix} D_1 & D_2 & D_3 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

Then X lies on the conic:

$$(E_1\alpha + E_2\beta + E_3\gamma)^2 = (D_1\alpha + D_2\beta + D_3\gamma)(F_1\alpha + F_2\beta + F_3\gamma). \tag{7}$$

Proof. Since M is nonsingular, its determinant δ is nonzero, and $M^{-1} = \frac{1}{\delta}M^{\#}$. Let

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix},$$

so that $X = MT$ and $M^{-1}X = T$. This second equation is equivalent to the system

$$\begin{array}{ccccccc} D_1x_1 & + & D_2x_2 & + & D_3x_3 & = & \delta t^2 \\ E_1x_1 & + & E_2x_2 & + & E_3x_3 & = & \delta t \\ F_1x_1 & + & F_2x_2 & + & F_3x_3 & = & \delta. \end{array}$$

The equal quotients $\delta t^2/\delta t$ and $\delta t/\delta$ yield

$$\frac{D_1x_1 + D_2x_2 + D_3x_3}{E_1x_1 + E_2x_2 + E_3x_3} = \frac{E_1x_1 + E_2x_2 + E_3x_3}{F_1x_1 + F_2x_2 + F_3x_3}.$$

□

For a first example, suppose $Q = q_1 : q_2 : q_3$ is a point not on a sideline of $\triangle ABC$, and let L be the line $q_1\alpha + q_2\beta + q_3\gamma = 0$. The P-isoconjugate of Q, is (e.g., [4, Glossary]) the point

$$P * Q = \frac{1}{p_1q_1} : \frac{1}{p_2q_2} : \frac{1}{p_3q_3}.$$

The method of proof of Theorem 1 shows that the P-isoconjugate of L (i.e., the set of points $P * R$ for R on L) is the circumconic

$$\frac{q_1}{p_1\alpha} + \frac{q_2}{p_2\beta} + \frac{q_3}{p_3\gamma} = 0.$$

We shall see that the same method applies to many other configurations.

2. Cevian nests and two conjugacies

A fruitful configuration in the plane of $\triangle ABC$ is the cevian nest, consisting of three triangles T_1, T_2, T_3 such that T_2 is a cevian triangle of T_1, and T_3 is a cevian triangle of T_2. In this article, $T_2 = \triangle ABC$, so that T_1 is the anticevian triangle of some point P, and T_3 is the cevian triangle of some point Q. It is well known (e.g. [1, p.165]) that if any two pairs of such triangles are perspective pairs, then the third pair are perspective also[3]. Accordingly, for a cevian nest, given two of the perspectors, the third may be regarded as the value of a binary operation applied to the given perspectors. There are three such pairs, hence three binary operations. As has been noted elsewhere ([2, p. 203] and [3, Glossary]), two of them are involutory: Ceva conjugates and cross conjugates.

2.1. *Ceva conjugate.* The *P-Ceva conjugate of Q*, denoted by $P©Q$, is the perspector of the cevian triangle of P and the anticevian triangle of Q; for $P = p_1 : p_2 : p_3$ and $Q = q_1 : q_2 : q_3$, we have

$$P©Q = q_1(-\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}) : q_2(\frac{q_1}{p_1} - \frac{q_2}{p_2} + \frac{q_3}{p_3}) : q_3(\frac{q_1}{p_1} + \frac{q_2}{p_2} - \frac{q_3}{p_3}).$$

Theorem 2. *Suppose P, U, V, Q_t are points as in (1)-(3); that is, Q_t traverses line UV. The locus of $P©Q_t$ is the conic*

$$\frac{\alpha^2}{p_1q_1}+\frac{\beta^2}{p_2q_2}+\frac{\gamma^2}{p_3q_3}-(\frac{1}{p_2q_3}+\frac{1}{p_3q_2})\beta\gamma-(\frac{1}{p_3q_1}+\frac{1}{p_1q_3})\gamma\alpha-(\frac{1}{p_1q_2}+\frac{1}{p_2q_1})\alpha\beta = 0, \quad (8)$$

where $Q := q_1 : q_2 : q_3$, the trilinear pole of the line UV, is given by

$$q_1 : q_2 : q_3 = \frac{1}{u_2v_3 - u_3v_2} : \frac{1}{u_3v_1 - u_1v_3} : \frac{1}{u_1v_2 - u_2v_1}.$$

This conic[4] passes through the vertices of the cevian triangles of P and Q.

Proof. First, it is easy to verify that equation (8) holds for $\alpha : \beta : \gamma$ equal to any of these six vertices:

$$0 : p_2 : p_3, \quad p_1 : 0 : p_3, \quad p_1 : p_2 : 0, \quad 0 : q_2 : q_3, \quad q_1 : 0 : q_3, \quad q_1 : q_2 : 0 \qquad (9)$$

[3]Peter Yff has observed that in [1], Court apparently overlooked the fact that $\triangle ABC$ and any inscribed triangle are triply perspective, with perspectors A, B, C. For these cases, Court's result is not always true. It seems that he intended his inscribed triangles to be cevian triangles.

[4]The general equation (8) for the circumconic of two cevian triangles is one of many interesting equations in Peter Yff's notebooks.

A conic is determined by any five of its points, so it suffices to prove that the six vertices are of the form $P©Q_t$. Putting $x_1 = 0$ in (4) gives roots

$$t_a = \frac{-e_1 \pm \sqrt{e_1^2 - 4d_1 f_1}}{2d_1}, \quad (10)$$

where

$$\begin{aligned} d_1 &= v_1(-\frac{v_1}{p_1} + \frac{v_2}{p_2} + \frac{v_3}{p_3}), && (11)\\ e_1 &= -\frac{2u_1v_1}{p_1} + \frac{u_1v_2 + u_2v_1}{p_2} + \frac{u_1v_3 + u_3v_1}{p_3}, && (12)\\ f_1 &= u_1(-\frac{u_1}{p_1} + \frac{u_2}{p_2} + \frac{u_3}{p_3}). && (13)\end{aligned}$$

The discriminant in (10) is a square, and t_a simplifies:

$$t_a = \frac{-e_1p_2p_3q_2q_3 \pm (p_3q_2 - p_2q_3)}{2d_1p_2p_3q_2q_3}.$$

If the numerator is $-e_1p_2p_3q_2q_3 + (p_3q_2 - p_2q_3)$, then (5) and (6), and substitutions for $d_2, e_2, f_2, d_3, e_3, f_3$ obtained cyclically from (11)-(13), give $x_2/x_3 = p_2/p_3$, so that $P©Q_{t_a} = 0 : p_2 : p_3$. On the other hand, if the numerator is $-e_1p_2p_3q_2q_3 - (p_3q_2 - p_2q_3)$, then $x_2/x_3 = q_2/q_3$ and $P©Q_{t_a} = 0 : q_2 : q_3$. Likewise, the roots t_b and t_c of (5) and (6) yield a proof that the other four vertices in (9) are of the form $P©Q_t$. □

Corollary 2.1. *Suppose $P = p_1 : p_2 : p_3$ is a point and L given by $\ell_1\alpha + \ell_2\beta + \ell_3\gamma = 0$ is a line. Suppose the point Q_t traverses L. The locus of $P©Q_t$ is the conic*

$$\frac{\ell_1\alpha^2}{p_1} + \frac{\ell_2\beta^2}{p_2} + \frac{\ell_3\gamma^2}{p_3} - (\frac{\ell_3}{p_2} + \frac{\ell_2}{p_3})\beta\gamma - (\frac{\ell_1}{p_3} + \frac{\ell_3}{p_1})\gamma\alpha - (\frac{\ell_2}{p_1} + \frac{\ell_1}{p_2})\alpha\beta = 0. \quad (14)$$

Proof. Let U, V be distinct points on L, and apply Theorem 2. □

Corollary 2.2. *The conic (14) is inscribed to $\triangle ABC$ if and only if the line $L = UV$ is the trilinear pole of P.*

Proof. In this case, $\ell_1 : \ell_2 : \ell_3 = 1/p_1 : 1/p_2 : 1/p_3$, so that $P = Q$. The cevian triangles indicated by (9) are now identical, and the six pass-through points are three tangency points. □

One way to regard Corollary 2.2 is to start with an inscribed conic Γ. It follows from the general equation for such a conic (e.g., [2, p.238]) that the three touch points are of the form $0 : p_2 : p_3, p_1 : 0 : p_3, p_1 : p_2 : 0$, for some $P = p_1 : p_2 : p_3$. Then Γ is the locus of $P©Q_t$ as Q_t traverses L.

Example 1. Let $P =$ centroid and $Q =$ orthocenter. Then line UV is given by

$$(\cos A)\alpha + (\cos B)\beta + (\cos C)\gamma = 0,$$

and the conic (8) is the nine-point circle. The same is true for $P =$ orthocenter and $Q =$ centroid.

Example 2. Let P = orthocenter and $Q = X_{648}$, the trilinear pole of the Euler line, so that UV is the Euler line. The conic (8) passes through the vertices of the orthic triangle, and X_4, X_{113}, X_{155}, X_{193}, which are the P-Ceva conjugates of X_4, X_{30}, X_3, X_2, respectively.[5]

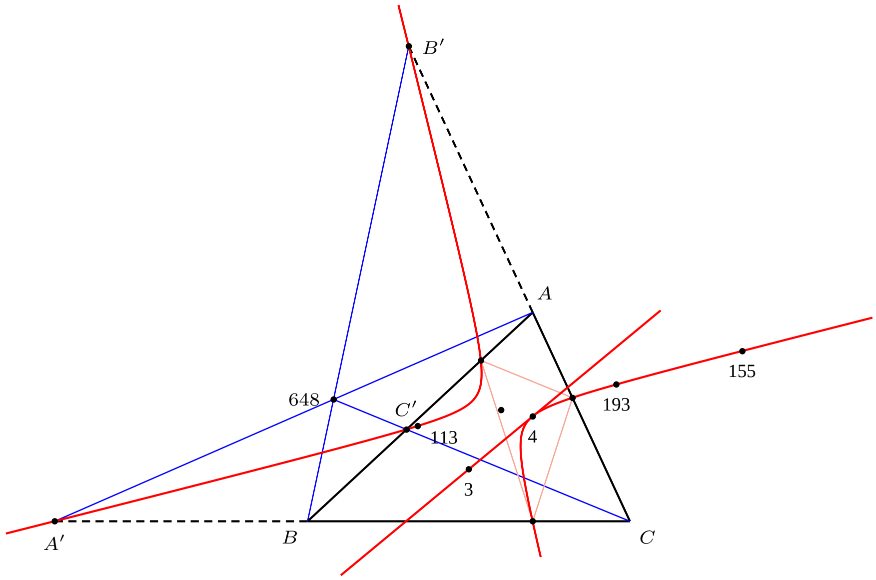

Figure 1

2.2. *Cross conjugate.* Along with Ceva conjugates, cevian nests proffer cross conjugates. Suppose $P = p_1 : p_2 : p_3$ and $Q = q_1 : q_2 : q_3$ are distinct points, neither lying on a sideline of $\triangle ABC$. Let $A'B'C'$ be the cevian triangle of Q. Let

$$A'' = PA' \cap B'C', \quad B'' = PB' \cap C'A', \quad C'' = PC' \cap A'B',$$

so that $A''B''C''$ is the cevian triangle (in $\triangle A'B'C'$) of P. The *cross conjugate* $P \otimes Q$ is the perspector of $\triangle ABC$ and $\triangle A''B''C''$. It has coordinates

$$\frac{q_1}{-\frac{p_1}{q_1}+\frac{p_2}{q_2}+\frac{p_3}{q_3}} : \frac{q_2}{-\frac{p_2}{q_2}+\frac{p_3}{q_3}+\frac{p_1}{q_1}} : \frac{q_3}{-\frac{p_3}{q_3}+\frac{p_1}{q_1}+\frac{p_2}{q_2}}.$$

It is easy to verify directly that $\otimes$ is a conjugacy; i.e., $P \otimes (P \otimes Q) = Q$, or to reach the same conclusion using the identity

$$X \otimes P = (X^{-1} \copyright P^{-1})^{-1},$$

where $(\ )^{-1}$ signifies isogonal conjugation.

The locus of $P \otimes Q_t$ is generally a curve of degree 5. However, on switching the roles of P and Q, we obtain a conic, as in Theorem 3. Specifically, let $Q = q_1 : q_2 : q_3$ remain fixed while

$$P_t = u_1 + v_1 t : u_2 + v_2 t : u_3 + v_3 t, \ -\infty < t \le \infty,$$

ranges through the line UV.

[5]Indexing of triangle centers is as in [3].

Theorem 3. *The locus of the $P_t \otimes Q$ is the circumconic*

$$(\frac{p_3}{q_2}+\frac{p_2}{q_3})\beta\gamma+(\frac{p_1}{q_3}+\frac{p_3}{q_1})\gamma\alpha+(\frac{p_2}{q_1}+\frac{p_1}{q_2})\alpha\beta=0, \tag{15}$$

where line UV is represented as

$$p_1\alpha+p_2\beta+p_3\gamma=(u_2v_3-u_3v_2)\alpha+(u_3v_1-u_1v_3)\beta+(u_1v_2-u_2v_1)\gamma=0.$$

Proof. Following the proof of Theorem 1, let

$$u_1'=-\frac{u_1}{q_1}+\frac{u_2}{q_2}+\frac{u_3}{q_3},\ v_1'=-\frac{v_1}{q_1}+\frac{v_2}{q_2}+\frac{v_3}{q_3},$$

and similarly for u_2', u_3', v_2', v_3'. Then

$$d_1=q_1v_2'v_3', \quad e_1=q_1(u_2'v_3'+u_3'v_2'), \quad f_1=q_1u_2'u_3',$$

and similarly for $d_i,\ e_i,\ f_i,\ i=2,3$. The nine terms $d_i,\ e_i,\ f_i$, yield the nine cofactors D_i, E_i, F_i, which then yield 0 for the coefficients of $\alpha^2, \beta^2, \gamma^2$ in (7) and the other three coefficients as asserted in (15). □

Example 3. Regarding the conic (15), suppose $P=p_1:p_2:p_3$ is an arbitrary triangle center and Γ is an arbitrary circumconic $\ell/\alpha+m/\beta+n/\gamma=0$. Let

$$\begin{aligned} Q &= q_1:q_2:q_3 \\ &= \frac{1}{p_1(-p_1\ell+p_2m+p_3n)}:\frac{1}{p_2(-p_2m+p_3n+p_1\ell)}:\frac{1}{p_3(-p_3n+p_1\ell+p_2m)}. \end{aligned}$$

For P_t ranging through the line L given by $p_1\alpha+p_2\beta+p_3\gamma=0$, the locus of $P_t\otimes Q$ is then Γ, since

$$\frac{p_3}{q_2}+\frac{p_2}{q_3}:\frac{p_1}{q_3}+\frac{p_3}{q_1}:\frac{p_2}{q_1}+\frac{p_1}{q_2}=\ell:m:n.$$

In other words, given P and L, there exists Q such that $P_t\otimes Q$ ranges through any prescribed circumconic. In fact, Q is the isogonal conjugate of P©L', where L' denotes the pole of line L. Specific cases are summarized in the following table.

P	Q	ℓ	pass-through points, X_i, for $i=$
X_1	X_1	1	$88, 100, 162, 190$ (Steiner ellipse)
X_1	X_2	$b+c$	$80, 100, 291$ (ellipse)
X_1	X_6	$a(b+c)$	$101, 190, 292$ (ellipse)
X_1	X_{57}	a	$74, 98, 99, ..., 111, 112,$ ...(circumcircle)
X_1	X_{63}	$\sin 2A$	$109, 162, 163, 293$ (ellipse)
X_1	X_{100}	$b-c$	$1, 2, 28, 57, 81, 88, 89, 105,$ ...(hyperbola)
X_1	X_{101}	$a(b-c)(b+c-a)$	$6, 9, 19, 55, 57, 284, 333,$ (hyperbola)
X_1	X_{190}	$a(b-c)$	$1, 6, 34, 56, 58, 86, 87, 106,$ ...(hyperbola)

3. Poles and polars

In this section, we shall see that, in addition to mappings discussed in §2, certain mappings defined in terms of poles and polars are nicely represented in terms of Ceva conjugates and cross conjugates

We begin with definitions. Suppose $A'B'C'$ is the cevian triangle of a point P not on a sideline of $\triangle ABC$. By Desargues's Theorem, the points $BC \cap B'C'$, $CA \cap C'A'$, $AB \cap A'B'$ are collinear. Their line is the *trilinear polar of* P. Starting with a line L, the steps reverse, yielding the *trilinear pole of* L. If L is given by $x\alpha + y\beta + z\gamma = 0$ then the trilinear pole of L is simply $1/x : 1/y : 1/z$.

Suppose Γ is a conic and X is a point. For each U on Γ, let V be the point other than U in which the line UX meets Γ, and let X' be the harmonic conjugate of X with respect to U and V. As U traverses Γ, the point X' traverses a line, the polar of X with respect to Γ, or Γ-*based polar of* X. Here, too, as with the trilinear case, for given line L, the steps reverse to define the Γ-*based pole of* L.

In §2, two mappings were defined in the context of a cevian nest. We return now to the cevian nest to define a third mapping. Suppose $P = p : q : r$ and $X = x : y : z$ are distinct points, neither lying on a sideline of $\triangle ABC$. Let $A''B''C''$ be the anticevian triangle of X. Let

$$A' = PA'' \cap BC, \quad B' = PB'' \cap CA, \quad C' = PC'' \cap AB.$$

The *cevapoint* of P and X is the perspector, R, of triangles ABC and $A'B'C'$. Trilinears are given by

$$R = \frac{1}{qz + ry} : \frac{1}{rx + pz} : \frac{1}{py + qx}. \tag{16}$$

It is easy to verify that $P = R \copyright X$.

The general conic Γ is given by the equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2p\beta\gamma + 2q\gamma\alpha + 2r\alpha\beta = 0,$$

and the Γ-based polar of $X = x : y : z$ is given (e.g., [5]) by

$$(ux + ry + qz)\alpha + (vy + pz + rx)\beta + (wz + qx + py)\gamma = 0. \tag{17}$$

Example 4. Let Γ denote the circumconic $p/\alpha + q/\beta + r/\gamma = 0$, that is, the circumconic having as pivot the point $P = p : q : r$. The Γ-based polar of X is the trilinear polar of the cevapoint of P and X, given by

$$(qz + ry)\alpha + (rx + pz)\beta + (py + qx)\gamma = 0.$$

In view of (16), (trilinear polar of X) = (Γ-based polar of $X \copyright P$).

Example 5. Let Γ denote conic determined as in Theorem 2 by points P and Q. The conic is inscribed in $\triangle ABC$ if and only if $P = Q$, and in this case, the Γ-based polar of X is given by

$$\frac{1}{p}\left(-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}\right)\alpha + \frac{1}{q}\left(\frac{x}{p} - \frac{y}{q} + \frac{z}{r}\right)\beta + \frac{1}{r}\left(\frac{x}{p} + \frac{y}{q} - \frac{z}{r}\right)\gamma = 0.$$

In other words, (Γ-based polar of X) $=$ (trilinear polar of $X \otimes P$). In particular, choosing $P = X_7$, we obtain the incircle-based polar of X:

$$f(A,B,C)\alpha + f(B,C,A)\beta + f(C,A,B)\gamma = 0,$$

where

$$f(A,B,C) = \frac{\sec^2 \frac{A}{2}}{-x\cos^2\frac{A}{2} + y\cos^2\frac{B}{2} + z\cos^2\frac{C}{2}}.$$

Suppose now that Γ is a conic and L a line. As a point

$$X = p_1 + q_1 t : p_2 + q_2 t : p_3 + q_3 t \tag{18}$$

traverses L, a mapping is defined by the trilinear pole of the Γ-based polar of X. This pole has trilinears found directly from (17):

$$\frac{1}{g_1(t)} : \frac{1}{g_2(t)} : \frac{1}{g_3(t)},$$

where $g_1(t) = u(p_1+q_1t)+r(p_2+q_2t)+q(p_3+q_3t)$, and similarly for $g_2(t)$ and $g_3(t)$. The same pole is given by

$$g_2(t)g_3(t) : g_3(t)g_1(t) : g_1(t)g_2(t), \tag{19}$$

and Theorem 1 applies to form (19). With certain exceptions, the resulting conic (7) is a circumconic; specifically, if $uq_1 + rq_2 + qq_3 \neq 0$, then $g_1(t)$ has a root for which (19) is vertex A, and similarly for vertices B and C.

Example 6. For $P = u : v : w$, let $\Gamma(P)$ be the circumconic $u\beta\gamma + v\gamma\alpha + w\alpha\beta = 0$. Assume that at least one point of $\Gamma(P)$ lies inside $\triangle ABC$; in other words, assume that $\Gamma(P)$ is not an ellipse. Let $\widehat{\Gamma}(P)$ be the conic[6]

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0. \tag{20}$$

For each $\alpha : \beta : \gamma$ on the line $u\alpha + v\beta + w\gamma = 0$ and inside or on a side of $\triangle ABC$, let $P = p : q : r$, with $p \geq 0$, $q \geq 0$, $r \geq 0$, satisfy

$$\alpha = p^2, \quad \beta = q^2, \quad \gamma = r^2,$$

and define

$$\sqrt{P} := \sqrt{p} : \sqrt{q} : \sqrt{r} \tag{21}$$

and

$$P_A := -\sqrt{p} : \sqrt{q} : \sqrt{r}, \quad P_B := \sqrt{p} : -\sqrt{q} : \sqrt{r}, \quad P_C := \sqrt{p} : \sqrt{q} : -\sqrt{r} \tag{22}$$

Each point in (21) and (22) satisfies (20), and conversely, each point satisfying (20) is of one of the forms in (21) and (22). Therefore, the conic (20) consists

[6]Let $\Phi = vwa^2 + wub^2 + uvc^2$. Conic (20) is an ellipse, hyperbola, or parabola according as $\Phi > 0$, $\Phi < 0$, or $\Phi = 0$. Yff [6, pp.131-132], discusses a class of conics of the form (20) in connection with self-isogonal cubics and orthocentric systems.

of all points as in (21) and (22). Constructions[7] for $\sqrt{P}$ are known, and points P_A, P_B, P_C are constructible as harmonic conjugates involving $\sqrt{P}$ and vertices A, B, C; e.g., P_A is the harmonic conjugate of P with respect to A and the point $BC \cap AP$. Now suppose that L is a line, given by $\ell\alpha + m\beta + n\gamma = 0$. For X as in (18) traversing L, we have $g_1(t) = u(p_1 + q_1 t)$, leading to nine amenable coefficients in (4)-(6) and on to amenable cofactors, as indicated by

$$D_1 = up_1^2 r_1, \qquad E_1 = -up_1 q_1 r_1, \qquad F_1 = uq_1^2 r_1,$$

where $r_1 = p_2 q_3 - p_3 q_2$. The nine cofactors and (7) yield this conclusion: the Γ-based pole of X traverses the circumconic

$$\frac{\ell}{u\alpha} + \frac{m}{v\beta} + \frac{n}{w\gamma} = 0. \tag{23}$$

For example, taking line $u\alpha + v\beta + w\gamma = 0$ to be the trilinear polar of X_{100} and L that of X_{101}, the conic (23) is the Steiner circumellipse. In this case, the conic (20) is the hyperbola passing through X_i for $i = 1, 43, 165, 170, 365,$ and 846. Another notable choice of (20) is given by $P = X_{798}$, which has first trilinear $(\cos^2 B - \cos^2 C)\sin^2 A$. Points on this hyperbola include X_i for $i = 1, 2, 20, 63, 147, 194, 478, 488, 616, 617, 627,$ and 628.

Of course, for each $X = x : y : z$ on a conic $\widehat{\Gamma}(P)$, the points

$$-x : y : z, \qquad x : -y : z, \qquad x : y : -z$$

are also on $\widehat{\Gamma}(P)$, and if X also lies inside $\triangle ABC$, then X_1/X^2 lies on $\Gamma(P)$.

Example 7. Let Γ be the circumcircle, given by $a/\alpha + b/\beta + c/\gamma = 0$, and let L be the Brocard axis, which is the line passing through the points $X_6 = a : b : c$ and $X_3 = \cos A : \cos B : \cos C$. Using notation in Theorem 1, we find

$$d_1 = bc, \qquad e_1 = 2a(b^2 + c^2), \qquad f_1 = 4a^2 bc$$

and

$$D_1 = 8ab^2c^2(c^2 - b^2), \qquad E_1 = 4a^2bc(b^2 - c^2), \qquad F_1 = 2a^3(c^2 - b^2),$$

leading to this conclusion: the circumcircle-based pole of X traversing the Brocard axis traverses the circumhyperbola

$$\frac{a(b^2 - c^2)}{\alpha} + \frac{b(c^2 - a^2)}{\beta} + \frac{c(a^2 - b^2)}{\gamma} = 0,$$

namely, the isogonal transform of the trilinear polar of the Steiner point.

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[7] The trilinear square root is constructed in [4]. An especially attractive construction of barycentric square root in [7] yields a second construction of trilinear square root. We describe the latter here. Suppose $P = p : q : r$ in trilinears; then in barycentric, $P = ap : bq : cr$, so that the barycentric square root of P is $\sqrt{ap} : \sqrt{bq} : \sqrt{cr}$. Barycentric multiplication (as in [7]) by $\sqrt{a} : \sqrt{b} : \sqrt{c}$ gives $a\sqrt{p} : b\sqrt{q} : c\sqrt{r}$, these being barycentrics for the trilinear square root of P, which in trilinears is $\sqrt{p} : \sqrt{q} : \sqrt{r}$.

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$P\ell$-Perpendicularity

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Abstract. It is well known that perpendicularity yields an involution on the line at infinity $\mathcal{L}^\infty$ mapping perpendicular directions to each other. Many notions of triangle geometry depend on this involution. Since in projective geometry the perpendicular involution is not different from other involutions, theorems using standard perpendicularity in fact are valid more generally.

In this paper we will classify alternative perpendicularities by replacing the orthocenter H by a point P and $\mathcal{L}^\infty$ by a line ℓ. We show what coordinates undergo with these changes and give some applications.

1. Introduction

In the Euclidean plane we consider a reference triangle ABC. We shall perform calculations using homogeneous barycentric coordinates. In these calculations $(f : g : h)$ denotes the barycentrics of a point, while $[l : m : n]$ denotes the line with equation $lx + my + nz = 0$. The line at infinity $\mathcal{L}^\infty$, for example, has coordinates $[1 : 1 : 1]$.

Perpendicularity yields an involution on the line at infinity, mapping perpendicular directions to each other. We call this involution *the standard perpendicularity*, and generalize it by replacing the orthocenter H by another point P with coordinates $(f : g : h)$, stipulating that the cevians of P be "perpendicular" to the corresponding sidelines of ABC. To ensure that P is outside the sidelines of ABC, we assume $fgh \neq 0$.

Further we let the role of $\mathcal{L}^\infty$ be taken over by another line $\ell = [l : m : n]$ not containing P. To ensure that ℓ does not pass through any of the vertices of ABC, we assume $lmn \neq 0$ as well. We denote by $[L]^\ell$ the intersection of a line L with ℓ. When we replace H by P and $\mathcal{L}^\infty$ by ℓ, we speak of *$P\ell$-perpendicularity*.

Many notions of triangle geometry, like rectangular hyperbolas, circles, and isogonal conjugacy, depend on the standard perpendicularity. Replacing the standard perpendicularity by $P\ell$-perpendicularity has its effects on these notions. Also, with the replacement of the line of infinity $\mathcal{L}^\infty$, we have to replace affine notions like midpoint and the center of a conic by their projective generalizations. So it may seem that there is a lot of triangle geometry to be redone, having to prove many generalizations. Nevertheless, there are at least two advantages in making

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calculations in generalized perpendicularities. (1) Calculations using coordinates in P-perpendicularity are in general easier and more transparent than when we use specific expressions for the orthocenter H. (2) We give ourselves the opportunity to work with different perpendicularities simultaneously. Here, we may find new interesting views to the triangle in the Euclidean context.

2. $P\ell$-Perpendicularity

In the following we assume some basic results on involutions. These can be found in standard textbooks on projective geometry, such as [2, 3, 8].

2.1. *$P\ell$-rectangular conics.* We generalize the fact that all hyperbolas from the pencil through A, B, C, H are rectangular hyperbolas. Let $\mathcal{P}$ be the pencil of circumconics through P. The elements of $\mathcal{P}$ we call $P\ell$-rectangular conics. According to Desargues' extended Involution Theorem (see, for example, [2, §16.5.4], [8, p.153], or [3, §6.72]) each member of $\mathcal{P}$ must intersect a line ℓ in two points, which are images under an involution $\tau_{P\ell}$. This involution we call the $P\ell$-perpendicularity.

Since an involution is determined by two pairs of images, $\tau_{P\ell}$ can be defined by the degenerate members of the pencil, the pairs of lines (BC, PA), (AC, PB), and (AB, PC). Two of these pairs are sufficient.

If two lines L and M intersect ℓ in a pair of images of $\tau_{P\ell}$, then we say that they are $P\ell$-perpendicular, and write $L \perp_{P\ell} M$. Note that for any ℓ, this perpendicularity replaces the altitudes of a triangle by the cevians of P as lines $P\ell$-perpendicular to the corresponding sides.

The involution $\tau_{P\ell}$ has two fixed points J_1 and J_2, real if $\tau_{P\ell}$ is hyperbolic, and complex if $\tau_{P\ell}$ is elliptic.

Again, by Desargues' Involution Theorem, every nondegenerate triangle $P_1P_2P_3$ has the property that the lines through the vertices $P\ell$-perpendicular to the opposite sides are concurrent at a point. We call this point of concurrence the $P\ell$-orthocenter of the triangle.

Remark. In order to be able to make use of the notions of parallelism and midpoints, and to perform calculations with simpler coordinates, it may be convenient to only replace H by P, but not $\mathcal{L}^\infty$ by another line. In this case we speak of P-perpendicularity. Each $P\ell$-perpendicularity corresponds to the Q-perpendicularity for an appropriate Q by the mappings $(x : y : z) \leftrightarrow (lx : my : nz)$.[1]

2.2. *Representation of $\tau_{P\ell}$ in coordinates.*

Theorem 1. *For $P = (f : g : h)$ and $\ell = [l : m : n]$, the $P\ell$-perpendicularity is given by*

$$\tau_{P\ell} : (f_L : g_L : h_L) \mapsto \left(\frac{f(gh_L - hg_L)}{l} : \frac{g(hf_L - fh_L)}{m} : \frac{h(fg_L - gf_L)}{n} \right). \tag{1}$$

[1] These mappings can be constructed by the $(l : m : n)$-reciprocal conjugacy followed by isotomic conjugacy and conversely, as explained in [4].

Proof. Let L be a line passing through C with $[L]^\ell = (f_L : g_L : h_L)$, and let $B_L = L \cap AB = (f_L : g_L : 0)$. We will consider triangle AB_LC. We have noted above that the $P\ell$-altitudes of triangle AB_LC are concurrent. Two of them are very easy to identify. The $P\ell$-altitude from C simply is $CP = [-g : f : 0]$. On the other hand, since $[BP]^\ell = (fm : -lf - hn : hm)$, the $P\ell$-altitude from B_L is $[-hmg_L : hmf_L : fmg_L + flf_L + hnf_L]$. These two $P\ell$-altitudes intersect in the point: [2]

$$X = (f(fmg_L + flf_L + hnf_L) : gn(hf_L - fh_L) : hm(fg_L - gf_Lg)).$$

Finally, we find that the third $P\ell$-altitude meets ℓ in

$$[AX]^\ell = \left(\frac{f(gh_L - hg_L)}{l} : \frac{g(hf_L - fh_L)}{m} : \frac{h(fg_L - gf_L)}{n}\right),$$

which indeed satisfies (1). □

3. $P\ell$-circles

Generalizing the fact that in the standard perpendicularity, all circles pass through the two circular points at infinity, we define a $P\ell$-circle to be any conic through the fixed points J_1 and J_2 of the involution $\tau_{P\ell}$. This viewpoint leads to another way of determining the involution, based on the following well known fact, which can be found, for example, in [2, §5.3]:

> Let a conic $\mathcal{C}$ intersect a line L in two points I and J. The involution τ on L with fixed points I and J can be found as follows: Let X be a point on L, then $\tau(X)$ is the point of intersection of L and the polar of X with respect to $\mathcal{C}$.

It is clear that applying this to a $P\ell$-circle with line ℓ we get the involution $\tau_{P\ell}$. In particular this shows us that in any $P\ell$-circle $\mathcal{C}$ a radius and the tangent to $\mathcal{C}$ through its endpoint are P-perpendicular. Knowing this, and restricting ourselves to P-circles, i.e. $\ell = \mathcal{L}^\infty$, we can conclude that all P-circles are homothetic in the sense that parallel radii of two P-circles have parallel tangents, or equivalently, that two parallel radii of two P-circles have a ratio that is independent of its direction.[3]

We now identify the most important $P\ell$-circle.

Theorem 2. *The conic $\mathcal{O}_{P\ell}$:*

$$f(gm + hn)yz + g(fl + hn)xz + h(fl + gm)xy = 0 \tag{2}$$

is the $P\ell$-circumcircle.

Proof. Clearly A, B and C are on the conic given by the equation. Let $J = (f_1 : g_1 : h_1)$, then with (1) the condition that J is a fixed point of $\tau_{P\ell}$ gives

$$\left(\frac{fgh_1 - fg_1h}{l} : \frac{f_1gh - fgh_1}{m} : \frac{fg_1h - f_1gh}{n}\right) = (f_1 : g_1 : h_1)$$

[2] In computing the coordinates of X, we have have used of the fact that $lf_L + mg_L + nh_L = 0$.

[3] Note here that the ratio might involve a real radius and a complex radius. This happens for instance when we have in the real plane two hyperbolas sharing asymptotes, but on alternative sides of these asymptotes.

which, under the condition $f_1 l + g_1 m + h_1 n = 0$, is equivalent to (2). This shows that the fixed points J_1 and J_2 of τ_P lie on $\mathcal{C}_P$ and proves the theorem. □

As the 'center' of $\mathcal{O}_{P\ell}$ we use the pole of ℓ with respect to $\mathcal{O}_{P\ell}$. This is the point

$$O_{P\ell} = \left(\frac{mg + nh}{l} : \frac{lf + nh}{m} : \frac{lf + mg}{n}\right).$$

3.1. *$P\ell$-Nine Point Circle.* The 'centers' of $P\ell$-rectangular conics, *i.e.*, elements of the pencil $\mathcal{P}$ of conics through A, B, C, P, form a conic through the traces of P, [4] the 'midpoints' [5] of the triangle sides, and also the 'midpoints' of AP, BP and CP. This conic $\mathcal{N}_{P\ell}$ is an analogue of the nine-point conics, its center is the 'midpoint' of P and $O_{P\ell}$.

The conic through A, B, C, P, and J_1 (or J_2) clearly must be tangent to ℓ, so that J_1 (J_2) is the 'center' of this conic. So both J_1 and J_2 lie on $\mathcal{N}_{P\ell}$, which makes it a $P\ell$-circle.

4. $P\ell$-conjugacy

In standard perpendicularity we have the isogonal conjugacy τ_H as the natural (reciprocal) conjugacy. It can be defined by combining involutions on the pencils of lines through the vertices of ABC. The involution that goes with the pencil through A is defined by two pairs of lines. The first pair is AB and AC, the second pair is formed by the lines through A perpendicular to AB and to AC. Of course this involution maps to each other lines through A making opposite angles to AB and AC respectively. Similarly we have involutions on the pencil through B and C. The isogonal conjugacy is found by taking the images of the cevians of a point P under the three involutions. These images concur in the isogonal conjugate of P.

This isogonal conjugacy finds its P-perpendicular cognate in the following reciprocal conjugacy:

$$\tau_{P\ell c} : (x : y : z) \mapsto \left(\frac{f(mg + nh)}{lx} : \frac{g(lf + nh)}{my} : \frac{h(lf + mg)}{nz}\right), \qquad (3)$$

which we will call the $P\ell$-conjugacy. This naming is not unique, since for each line ℓ' there is a point Q so that the $P\ell$- and $Q\ell'$-conjugacies are equal. In particular, if $\ell = \mathcal{L}^\infty$, this reciprocal conjugacy is

$$(x : y : z) \mapsto \left(\frac{f(g + h)}{x} : \frac{g(h + f)}{y} : \frac{h(f + g)}{z}\right).$$

[4]These are the 'centers' of the degenerate elements of $\mathcal{P}$.

[5]The 'midpoint' of XY is the harmonic conjugate of $[XY]^\ell$ with respect to X and Y. The 'midpoints' of the triangle sides are also the traces of the trilinear pole of ℓ.

Clearly the $P\ell$-conjugacy maps P to $O_{P\ell}$. This provides us with a construction of the conjugacy. See [4]. [6] From (2) it is also clear that this conjugacy transforms $\mathcal{C}_{P\ell}$ into ℓ and back.

Now we note that any reciprocal conjugacy maps any line to a circumconic of ABC, and conversely. In particular, any line through $O_{P\ell}$ is mapped to a conic from the pencil $\mathcal{P}$, a $P\ell$-rectangular conic. This shows that $\tau_{P\ell c}$ maps the $P\ell$-perpendicularity to the involution on $\mathcal{O}_{P\ell}$ mapping each point X to the second point of intersection of $O_{P\ell}X$ with $\mathcal{C}_{P\ell}$.

The four points

$$\left(\pm\sqrt{\frac{f(mg+nh)}{l}} : \pm\sqrt{\frac{g(lf+nh)}{m}} : \pm\sqrt{\frac{h(lf+mg)}{n}}\right)$$

are the fixed points of the $P\ell$-conjugacy. They are the centers of the $P\ell$-circles tritangent to the sidelines of ABC.

5. Applications of P-perpendicularity

As mentioned before, it is convenient not to change the line at infinity $\mathcal{L}^{\infty}$ into ℓ and speak only of P-perpendicularity. This notion is certainly less general. Nevertheless, it works with simpler coordinates and it allows one to make use of parallelism and ratios in the usual way. For instance, the Euler line is generalized quite easily, because the coordinates of O_P are $(g+h : f+h : f+g)$, so that it is easy to see that $PG : GO_P = 2 : 1$.

We give a couple of examples illustrating the convenience of the notion of P-perpendicularity in computations and understanding.

5.1. *Construction of ellipses.* Note that the equation (1) does not change when we exchange $(f : g : h)$ and $(x : y : z)$. So we have:

Proposition 3. *P lies on $\mathcal{O}_{Q\ell}$ if and only if Q lies on $\mathcal{O}_{P\ell}$.*

When we restrict ourselves to P-perpendicularity, Proposition 3 is helpful in finding the axes of a circumellipse of a triangle. Let's say that the ellipse is $\mathcal{O}_P$. [7] If we find the fourth intersection X of a circumellipse and the circumcircle, then the X-circumcircle $\mathcal{O}_X$ passes through H as well as P, and thus it is a rectangular hyperbola as well as a P-rectangular conic. This means that the asymptotes of $\mathcal{O}_X$ must correspond to the directions of the axes of the ellipse. These yield indeed the only diameters of the ellipse to which the tangents at the endpoints are (standard) perpendicular. Note also that this shows that all conics through A, B, C, X, apart from the circumcircle have parallel axes. Figure 1 illustrates the case when $P = G$, the centroid, and X = Steiner point. Here, $\mathcal{O}_X =$ is the Kiepert hyperbola.

The knowledge of P-perpendicularity can be helpful when we try to draw conics in dynamic geometry software. This can be done without using foci.

[6] In [4] we can find more ways to construct the $P\ell$-conjugacy, for instance, by using the degenerate triangle where AP, BP and CP meet ℓ.

[7] When we know the center O_P of $\mathcal{O}_P$, we can find P by the ratio $O_PG : GP = 1 : 2$.

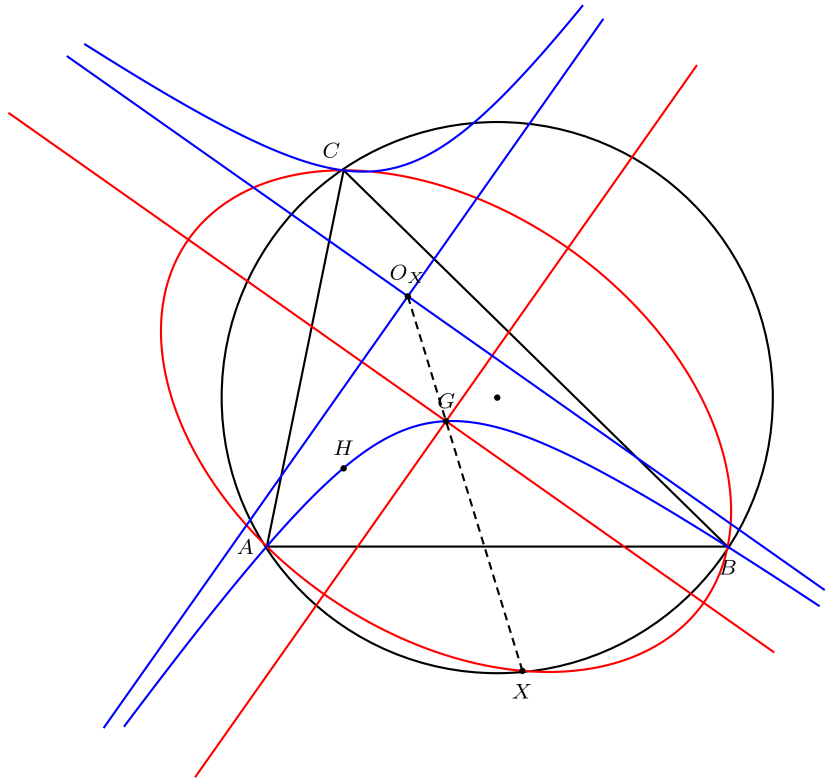

Figure 1

If we have the center O_P of a conic through three given points, say ABC, we easily find P as well. Also by reflecting one of the vertices, say A, through O_P we have the endpoints of a diameter, say AA_r. Then if we let a line m go through A, and a line n which is P-perpendicular to m through A_r. Their point of intersection lies on the P-circle through ABC. See Figure 2.

5.2. *Simson-Wallace lines.* Given a generic finite point $X = (x : y : z)$, let $A' \in BC$ be the point such that $A'X \parallel AP$, and let B' and C' be defined likewise, then we call $A'B'C'$ the *triangle of P-traces* of X. This triangle is represented by the following matrix:

$$M = \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} 0 & gx + (g+h)y & hx + (g+h)z \\ fy + (f+h)x & 0 & hy + (f+h)z \\ fz + (f+g)x & gz + (f+g)y & 0 \end{pmatrix} \quad (4)$$

We are interested in the conic that plays a role similar to the circumcircle in the occurrence of Simson-Wallace lines.[8] To do so, we find that $A'B'C'$ is degenerate

[8]In [5] Miguel de Guzmán generalizes the Simson-Wallace line more drastically. He allows three arbitrary directions of projection, with the only restriction that these directions are not all equal, each not parallel to the side to which it projects.

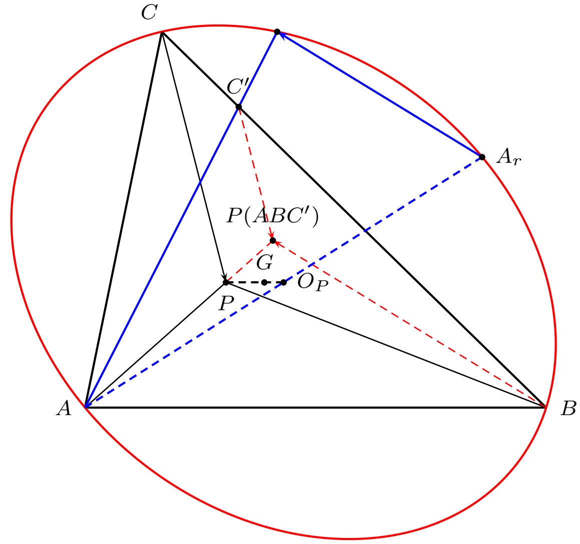

Figure 2

iff determinant $|M| = 0$, which can be rewritten as

$$(f+g+h)(x+y+z)(\tilde{f}yz+\tilde{g}xz+\tilde{h}xy)=0, \tag{5}$$

where

$$\tilde{f}=f(g+h), \qquad \tilde{g}=g(f+h), \qquad \tilde{h}=h(f+g).$$

Using that X and P are finite points, (5) can be rewritten into (2), so that the locus is the P-circle $\mathcal{C}_P$. See Figure 3.

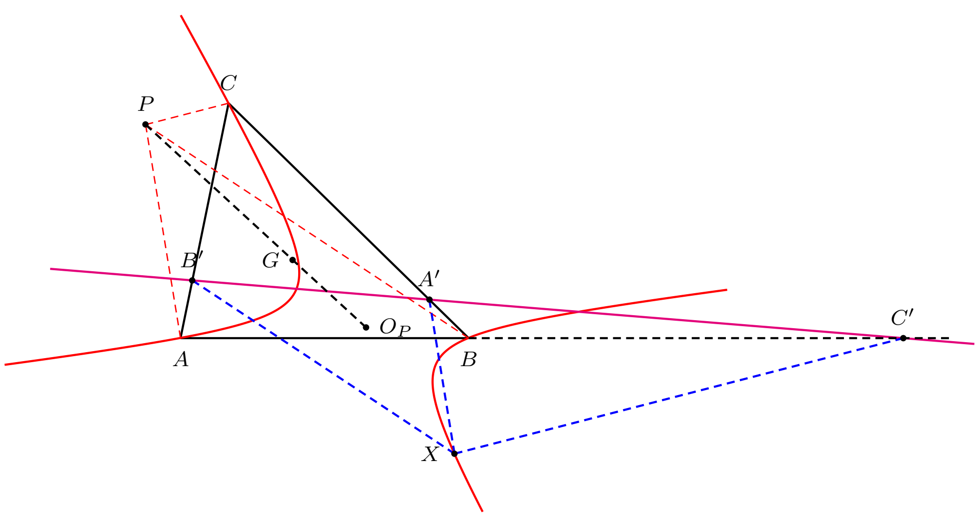

Figure 3

We further remark that since the rows of matrix M in (4) add up to $(f+g+h)X+(x+y+z)P$, the P-Simson-Wallace line $A'B'C'$ bisects the segment XP when $X \in \mathcal{C}_P$. Thus, the point of intersection of $A'B'C'$ and XP lies on $\mathcal{N}_P$.

5.3. *The Isogonal Theorem.* The following theorem generalizes the Isogonal Theorem. [9] We shall make use of the involutions τ_{PA}, τ_{PB} and τ_{PC} that the P-conjugacy causes on the pencil of lines through A, B and C respectively.

Theorem 4. *For $I \in \{A,B,C\}$, consider lines l_I and l'_I unequal to sidelines of ABC that are images under τ_{PI}. Let $A_1 = l_B \cap l'_C$, $B_1 = l_C \cap l'_A$ and $C_1 = l_A \cap l'_B$. We call $A_1B_1C_1$ a P-*conjugate triangle*. Then triangles ABC and $A_1B_1C_1$ are perspective.*

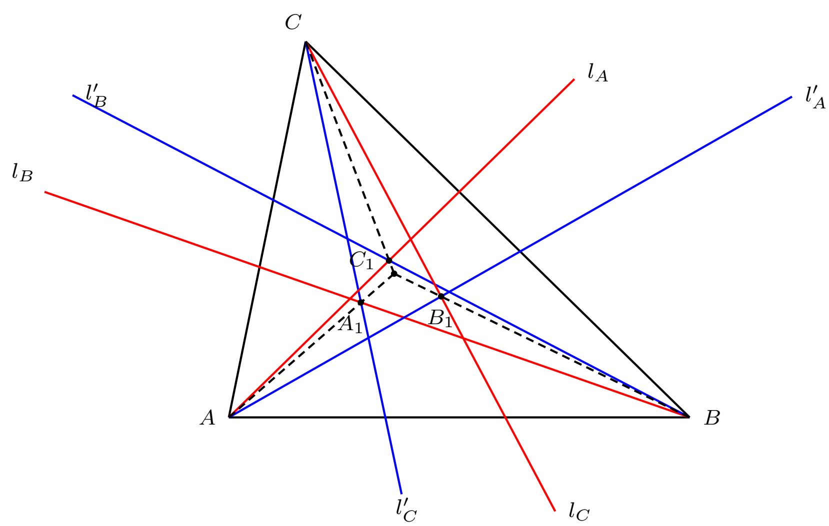

Figure 4

Proof. For $I \in \{A,B,C\}$, let $P_I = (x_I : y_I : z_I) \in l_I$ be a point different from I. We find, for instance, $l_A = [0 : z_A : -y_A]$ and $l'_C = [\tilde{g}/y_C : -\tilde{f}/x_C : 0]$. Consequently $B_1 = (\tilde{f}y_A/x_C : \tilde{g}y_A/y_C : \tilde{g}z_A/y_C)$. In the same way we find coordinates for A_1 and C_1 so that the P-conjugate triangle $A_1B_1C_1$ is given by

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} \tilde{f}x_C/x_B & \tilde{f}y_C/x_B & \tilde{h}x_C/z_B \\ \tilde{f}y_A/x_C & \tilde{g}y_A/y_C & \tilde{g}z_A/y_C \\ \tilde{h}x_B/z_A & \tilde{g}z_B/y_A & \tilde{h}z_B/z_A \end{pmatrix}.$$

With these coordinates it is not difficult to verify that $A_1B_1C_1$ is perspective to ABC. This we leave to the reader. □

[9] This theorem states that a triangle $A_1B_1C_1$ with $\angle BAC_1 = \angle CAB_1$, $\angle CBA_1 = \angle ABC_1$ and $\angle ACB_1 = \angle BCA_1$ is perspective to ABC. See [1, p.55], also [6, 9], and [7, Theorem 6D].

Interchanging the lines l_I and l'_I in Theorem 4 above, we see that the P-conjugates of $A_1B_1C_1$ form a triangle $A_2B_2C_2$ perspective to ABC as well. This is its desmic mate. [10] Now, each triangle perspective to ABC is mapped to its desmic mate by a reciprocal conjugacy. From this and Theorem 4 we see that the conditions 'perspective to ABC' and 'desmic mate is also an image under a reciprocal conjugacy' are equivalent.

5.3.1. Each P-conjugate triangle can be written in coordinates as

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} = M_1 = \begin{pmatrix} \tilde{f} & w & v \\ w & \tilde{g} & u \\ v & u & \tilde{h} \end{pmatrix}.$$

Let a second P-conjugate triangle be given by

$$\begin{pmatrix} A_2 \\ B_2 \\ C_2 \end{pmatrix} = M_2 = \begin{pmatrix} \tilde{f} & W & V \\ W & \tilde{g} & U \\ V & U & \tilde{h} \end{pmatrix}.$$

Considering linear combinations $tM_1 + uM_2$ it is clear that the following proposition holds.

Proposition 5. *Let $A_1B_1C_1$ and $A_2B_2C_2$ be two distinct P-conjugate triangles. Define $A' = A_1A_2 \cap BC$ and B', C' analogously. Then $A'B'C'$ is a cevian triangle. In fact, if $A''B''C''$ is such that the cross ratios $(A_1A_2A'A'')$, $(B_1B_2B'B'')$ and $(C_1C_2C'C'')$ are equal, then $A''B''C''$ is perspective to ABC as well.*

The following corollary uses that the points where the cevians of P meet $\mathcal{L}^\infty$ is a P-conjugate triangle.

Corollary 6. *Let $A_1B_1C_1$ be a P-conjugate triangle. Let A' be the P-perpendicular projections of A_1 on BC, B_1 on CA, and C_1 on AB respectively. Let $A''B''C''$ be such that $A'A_1 : A''A_1 = B'B_1 : B''B_1 = C'C_1 : C''C_1 = t$, then $A''B''C''$ is perspective to ABC. As t varies, the perspector traverses the P-rectangular circumconic through the perspector of $A_1B_1C_1$.*

5.4. *The Darboux cubic.* We conclude with an observation on the analogues of the Darboux cubic. It is well known that the locus of points X whose pedal triangles are perspective to ABC is a cubic curve, the Darboux cubic. We generalize this to triangles of P-traces.

First, let us consider the lines connecting the vertices of ABC and the triangle of P-traces of X given in (4). Let μ_{ij} denote the entry in row i and column j of (4), then we find as matrix of coefficients of these lines

$$N = \begin{pmatrix} 0 & -\mu_{13} & \mu_{12} \\ \mu_{23} & 0 & -\mu_{21} \\ -\mu_{32} & \mu_{31} & 0 \end{pmatrix}. \qquad (6)$$

[10] See for instance [4].

These lines concur iff $\det N = 0$. This leads to the cubic equation

$$(-f+g+h)x(\tilde{h}y^2-\tilde{g}z^2)+(f-g+h)y(\tilde{f}z^2-\tilde{h}x^2)+(f+g-h)z(\tilde{g}x^2-\tilde{f}y^2)=0. \tag{7}$$

We will refer to this cubic as the *P-Darboux cubic*. The cubic consists of the points Q such that Q and its P-conjugate are collinear with the point $(-f+g+h : f-g+h : f+g-h)$, which is the reflection of P in O_P.

It is seen easily from (4) and (6) that if we interchange $(f : g : h)$ and $(x : y : z)$, then (7) remains unchanged. From this we can conclude:

Proposition 7. *For two points P and Q be two points not on the sidelines of triangle ABC, P lies on the Q-Darboux cubic if and only if Q lies on the P-Darboux cubic.*

This example, and others in §5.1, demonstrate the fruitfulness of considering different perpendicularities simultaneously.

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Cubics Associated with Triangles of Equal Areas

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Abstract. The locus of a point X for which the cevian triangle of X and that of its isogonal conjugate have equal areas is a cubic that passes through the 1st and 2nd Brocard points. Generalizing from isogonal conjugate to P-isoconjugate yields a cubic $Z(U,P)$ passing through U; if X is on $Z(U,P)$ then the P-isoconjugate of X is on $Z(U,P)$ and this point is collinear with X and U. A generalized equal areas cubic $\Gamma(P)$ is presented as a special case of $Z(U,P)$. If $\sigma = \text{area}(\triangle ABC)$, then the locus of X whose cevian triangle has prescribed oriented area $K\sigma$ is a cubic $\Lambda(P)$, and P is determined if K has a certain form. Various points are proved to lie on $\Lambda(P)$.

1. Introduction

For any point $X = \alpha : \beta : \gamma$ (homogeneous trilinear coordinates) not a vertex of $\triangle ABC$, let

$$T = \begin{pmatrix} 0 & \beta & \gamma \\ \alpha & 0 & \gamma \\ \alpha & \beta & 0 \end{pmatrix} \qquad \text{and} \qquad T' = \begin{pmatrix} 0 & \gamma & \beta \\ \gamma & 0 & \alpha \\ \beta & \alpha & 0 \end{pmatrix},$$

so that T is the cevian triangle of X, and T' is the cevian triangle of the isogonal conjugate of X. Let σ be the area of $\triangle ABC$, and assume that X does not lie on a sideline $\triangle ABC$. Then oriented areas are given (e.g. [3, p.35]) in terms of the sidelengths a, b, c by

$$\text{area}(T) = \frac{abc}{8\sigma^2}\begin{vmatrix} 0 & k_1\beta & k_1\gamma \\ k_2\alpha & 0 & k_2\gamma \\ k_3\alpha & k_3\beta & 0 \end{vmatrix}, \qquad \text{area}(T') = \frac{abc}{8\sigma^2}\begin{vmatrix} 0 & l_1\gamma & l_1\beta \\ l_2\gamma & 0 & l_2\alpha \\ l_3\beta & l_3\alpha & 0 \end{vmatrix},$$

where k_i and l_i are normalizers. [1] Thus,

$$\text{area}(T) = \frac{k_1k_2k_3\alpha\beta\gamma abc}{4\sigma^2} \qquad \text{and} \qquad \text{area}(T') = \frac{l_1l_2l_3\alpha\beta\gamma abc}{8\sigma^2},$$

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[1] If $P = \alpha : \beta : \gamma$ is not on the line $\mathcal{L}^\infty$ at infinity, then the normalizer h makes $h\alpha, h\beta, h\gamma$ the directed distances from P to sidelines BC, CA, AB, respectively, and $h = 2\sigma/(a\alpha + b\beta + c\gamma)$. If P is on $\mathcal{L}^\infty$ and $\alpha\beta\gamma \neq 0$, then the normalizer is $h := 1/\alpha + 1/\beta + 1/\gamma$; if P is on $\mathcal{L}^\infty$ and $\alpha\beta\gamma = 0$, then $h := 1$.

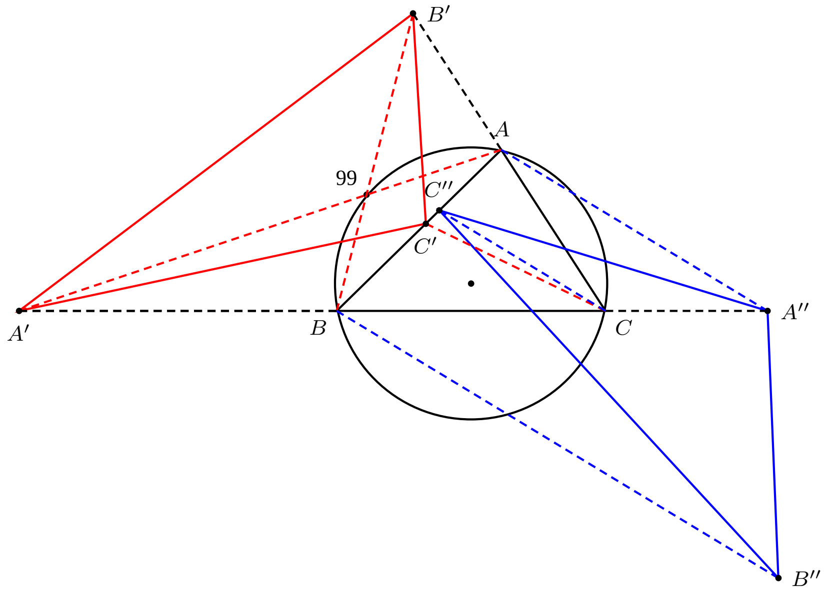

Figure 1. Triangles $A'B'C'$ and $A''B''C''$ have equal areas

so that area$(T) =$ area(T') if and only if $k_1k_2k_3 = l_1l_2l_3$. Substituting yields

$$\frac{1}{b\beta + c\gamma} \cdot \frac{1}{c\gamma + a\alpha} \cdot \frac{1}{a\alpha + b\beta} = \frac{1}{b\gamma + c\beta} \cdot \frac{1}{c\alpha + a\gamma} \cdot \frac{1}{a\beta + b\alpha},$$

which simplifies to

$$a(b^2 - c^2)\alpha(\beta^2 - \gamma^2) + b(c^2 - a^2)\beta(\gamma^2 - \alpha^2) + c(a^2 - b^2)\gamma(\alpha^2 - \beta^2) = 0. \quad (1)$$

In the parlance of [4, p.240], equation (1) represents the self-isogonal cubic $Z(X_{512})$, and, in the terminology of [1, 2], the auto-isogonal cubic having pivot X_{512}. [2] It is easy to verify that the following 24 points lie on this cubic:[3]

vertices A, B, C,
incenter X_1 and excenters,
Steiner point X_{99} and its isogonal conjugate X_{512} (see Figure 1),
vertices of the cevian triangle of X_{512},
1st and 2nd Brocard points Ω_1 and Ω_2,
$X_{512}©X_1$ and $X_{512}©X_{99}$, where © denotes Ceva conjugate,
$(X_{512}©X_1)^{-1}$ and $(X_{512}©X_{99})^{-1}$, where $()^{-1}$ denotes isogonal conjugate,
vertices of triangle T_1 below,
vertices of triangle T_2 below.

---

[2] X_i is the ith triangle center as indexed in [5].

[3] This "equal-areas cubic" was the subject of a presentation by the author at the CRCC geometry meeting hosted by Douglas Hofstadter at Indiana University, March 23-25, 1999.

The vertices of the bicentric[4] triangle T_1 are

$$-ab : a^2 : bc, \qquad ca : -bc : b^2, \qquad c^2 : ab : -ca, \tag{2}$$

and those of T_2 are

$$-ac : bc : a^2, \qquad b^2 : -ba : ca, \qquad ab : c^2 : -cb. \tag{3}$$

Regarding (2), $-ab : a^2 : bc$ is the point other than A and Ω_1 in which line $A\Omega_1$ meets $Z(X_{512})$. Similarly, lines $A\Omega_1$ and $C\Omega_1$ meet $Z(X_{512})$ in the other two points in (2). Likewise, the points in (3) lie on lines $A\Omega_2$, $B\Omega_2, C\Omega_2$. The points in (3) are isogonal conjugates of those in (2).

Vertex $A' := -ab : a^2 : bc$ is the intersection of the C-side of the anticomplementary triangle and the B-exsymmedian, these being the lines $a\alpha + b\beta = 0$ and $c\alpha + a\gamma = 0$. The other five vertices are similarly constructed.

Other descriptions of $Z(X_{512})$ are easy to check: (i) the locus of a point Q collinear with its isogonal conjugate and X_{512}, and (ii) the locus of Q for which the line joining Q and its isogonal conjugate is parallel to the line $\Omega_1\Omega_2$.

2. Isoconjugates and reciprocal conjugates

In the literature, isoconjugates are defined in terms of trilinears and reciprocal conjugates are defined in terms of barycentrics. We shall, in this section, use the notations $(x : y : z)_t$ and $(x : y : z)_b$ to indicate trilinears and barycentrics, respectively. [5]

Definition 1. [6] Suppose $P = (p : q : r)_t$ and $X = (x : y : z)_t$ are points, neither on a sideline of $\triangle ABC$. The P-isoconjugate of X is the point

$$(P \cdot X)_t^{-1} = (qryz : rpzx : pqxy)_t.$$

On the left side, the subscript t signifies trilinear multiplication and division.

Definition 2. [3] Suppose $P = (p : q : r)_b$ and $X = (x : y : z)_b$ are points not on a sideline of $\triangle ABC$. The P-reciprocal conjugate of X is the point

$$(P/X)_b = (pyz : qzx : rxy)_b.$$

In keeping with the meanings of "iso-" and "reciprocal",

$$\begin{aligned} X\text{-isoconjugate of } P \quad &= \quad P\text{-isoconjugate of } X, \\ X\text{-reciprocal conjugate of } P \quad &= \quad \frac{G}{P\text{-reciprocal conjugate of } X}, \end{aligned}$$

where G, the centroid, is the identity corresponding to barycentric division.

---

[4]Definitions of bicentric triangle, bicentric pair of points, and triangle center are given in [5, Glossary]. If $f(a,b,c) : g(a,b,c) : h(a,b,c)$ is the A-vertex of a bicentric triangle, then the B-vertex is $h(b,c,a) : f(b,c,a) : g(b,c,a)$ and the C-vertex is $g(c,a,b) : h(c,a,b) : f(c,a,b)$.

[5]A point X with trilinears $\alpha : \beta : \gamma$ has barycentrics $a\alpha : b\beta : c\gamma$. For points not on $\mathcal{L}^\infty$, trilinears are proportional to the directed distances between X and the sidelines BC, CA, AB, respectively, whereas barycentrics are proportional to the oriented areas of triangles XBC, XCA, XAB, respectively.

3. The cubic $Z(U, P)$

In this section, all coordinates are trilinears; for example, $(\alpha : \beta : \gamma)_t$ appears as $\alpha : \beta : \gamma$. Suppose $U = u : v : w$ and $P = p : q : r$ are points, neither on a sideline of $\triangle ABC$. We generalize the cubic $Z(U)$ defined in [4, p.240] to a cubic $Z(U, P)$, defined as the locus of a point $X = \alpha : \beta : \gamma$ for which the points U, X, and the P-isoconjugate of X are collinear. This requirement is equivalent to

$$\begin{vmatrix} u & v & w \\ \alpha & \beta & \gamma \\ qr\beta\gamma & rp\gamma\alpha & pq\alpha\beta \end{vmatrix} = 0, \tag{4}$$

hence to

$$up\alpha(q\beta^2 - r\gamma^2) + vq\beta(r\gamma^2 - p\alpha^2) + wr\gamma(q\alpha^2 - r\beta^2) = 0.$$

Equation (4) implies these properties:

(i) $Z(U, P)$ is self P-isoconjugate;
(ii) $U \in Z(U, P)$;
(iii) if $X \in Z(U, P)$, then X, U, and $(P \cdot X)_t^{-1}$ are collinear.

The following ten points lie on $Z(U, P)$:

the vertices A, B, C;
the vertices of the cevian triangle of U, namely,

$$0 : v : w, \quad u : 0 : w, \qquad u : v : 0; \tag{5}$$

and the points invariant under P-isoconjugation:

$$\frac{1}{\sqrt{p}} : \frac{1}{\sqrt{q}} : \frac{1}{\sqrt{r}}, \tag{6}$$

$$\frac{-1}{\sqrt{p}} : \frac{1}{\sqrt{q}} : \frac{1}{\sqrt{r}}, \quad \frac{1}{\sqrt{p}} : \frac{-1}{\sqrt{q}} : \frac{1}{\sqrt{r}}, \quad \frac{1}{\sqrt{p}} : \frac{1}{\sqrt{q}} : \frac{-1}{\sqrt{r}}. \tag{7}$$

As an illustration of (i), the cubics $Z(U, X_1)$ and $Z(U, X_{31})$ are self-isogonal conjugate and self-isotomic conjugate, respectively. Named cubics of the type $Z(U, X_1)$ include the Thomson ($U = X_2$), Darboux ($U = X_{20}$), Neuberg ($U = X_{30}$), Ortho ($U = X_4$), and Feuerbach ($U = X_5$). The Lucas cubic is $Z(X_{69}, X_{31})$, and the Spieker, $Z(X_8, X_{58})$. Table 1 offers a few less familiar cubics.

It is easy to check that the points

$$U \copyright X_1 = -u + v + w : u - v + w : u + v - w,$$
$$U \copyright U^{-1} = \frac{1}{u}(-\frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{w^2}) : \frac{1}{v}(\frac{1}{u^2} - \frac{1}{v^2} + \frac{1}{w^2}) : \frac{1}{w}(\frac{1}{u^2} + \frac{1}{v^2} - \frac{1}{w^2})$$

lie on $Z(U)$. Since their isogonal conjugates also lie on $Z(U)$, we have four more points on $Z(U, P)$ in the special case that $P = X_1$.

U	P	Centers on cubic $Z(U,P)$
X_{385}	X_1	$X_1, X_2, X_6, X_{32}, X_{76}, X_{98}, X_{385}, X_{511}, X_{694}$
X_{395}	X_1	$X_1, X_2, X_6, X_{14}, X_{16}, X_{18}, X_{62}, X_{395}$
X_{396}	X_1	$X_1, X_2, X_6, X_{13}, X_{15}, X_{17}, X_{61}, X_{396}$
X_{476}	X_1	$X_1, X_{30}, X_{74}, X_{110}, X_{476}, X_{523}, X_{526}$
X_{171}	X_2	$X_2, X_{31}, X_{42}, X_{43}, X_{55}, X_{57}, X_{81}, X_{171}, X_{365}, X_{846}, X_{893}$
X_{894}	X_6	$X_6, X_7, X_9, X_{37}, X_{75}, X_{86}, X_{87}, X_{192}, X_{256}, X_{366}, X_{894}, X_{1045}$
X_{309}	X_{31}	$X_2, X_{40}, X_{77}, X_{189}, X_{280}, X_{309}, X_{318}, X_{329}, X_{347}, X_{962}$
X_{226}	X_{55}	$X_2, X_{57}, X_{81}, X_{174}, X_{226}, X_{554}, X_{559}, X_{1029}, X_{1081}, X_{1082}$
X_{291}	X_{239}	$X_1, X_6, X_{42}, X_{57}, X_{239}, X_{291}, X_{292}, X_{672}, X_{894}$
X_{292}	X_{238}	$X_1, X_2, X_{37}, X_{87}, X_{171}, X_{238}, X_{241}, X_{291}, X_{292}$

Table 1

4. Trilinear generalization: $\Gamma(P)$

Next we seek the locus of a point $X = \alpha : \beta : \gamma$ (trilinears) for which the cevian triangle T and the cevian triangle

$$\widehat{T} = \begin{pmatrix} 0 & r\gamma & q\beta \\ r\gamma & 0 & p\alpha \\ q\beta & p\alpha & 0 \end{pmatrix}$$

of the P-isoconjugate of X have equal areas. For this, the method leading to (1) yields a cubic denoted by $\Gamma(P)$:

$$ap(rb^2-qc^2)\alpha(q\beta^2-r\gamma^2)+bq(pc^2-ra^2)\beta(r\gamma^2-p\alpha^2)+cr(qa^2-pb^2)\gamma(p\alpha^2-q\beta^2)=0, \tag{8}$$

except for $P = X_{31} = a^2 : b^2 : c^2$; that is, except when P-isoconjugation is isotomic conjugation, for which the two triangles have equal areas for all X. The cubic (8) is $Z(U,P)$ for

$$U = U(P) = a(rb^2 - qc^2) : b(pc^2 - ra^2) : c(qa^2 - pb^2),$$

a point on $\mathcal{L}^\infty$. As in Section 3, the vertices A, B, C and the points (5)-(7) lie on $\Gamma(P)$.

Let U^* denote the P-isoconjugate of U. This is the trilinear pole of the line XX_2, where $X = \frac{a}{p} : \frac{b}{q} : \frac{c}{r}$, the P-isoconjugate of X_2. Van Lamoen has noted that since U lies on the trilinear polar, L, of the P-isoconjugate of the centroid (i.e., L has equation $\frac{p\alpha}{a} + \frac{q\beta}{b} + \frac{r\gamma}{c} = 0$), and U also lies on $\mathcal{L}^\infty$, we have U^* lying on the Steiner circumellipse and on the conic

$$\frac{pa}{\alpha} + \frac{qb}{\beta} + \frac{rc}{\gamma} = 0, \tag{9}$$

this being the P-isoconjugate of $\mathcal{L}^\infty$.

Theorem 1. *Suppose P_1 and P_2 are distinct points, collinear with but not equal to X_{31}. Then $U(P_2) = U(P_1)$.*

Proof. Write $P_1 = p_1 : q_1 : r_1$ and $P_2 = p_2 : q_2 : r_2$. Then for some $s = s(a,b,c) \neq 0$,

$$a^2 = sp_1 + p_2, \qquad b^2 = sq_1 + q_2, \qquad c^2 = sr_1 + r_2,$$

so that for $f(a,b,c) := a[(c^2 - sr_1)b^2 - (b^2 - sq_1)c^2]$, we have

$$\begin{aligned} U(P_2) &= f(a,b,c) : f(b,c,a) : f(c,a,b) \\ &= a(sc^2q_1 - sb^2r_1) : b(sa^2r_1 - sc^2p_1) : c(sb^2p_1 - sa^2q_1) \\ &= U(P_1). \end{aligned}$$

□

Example 1. For each point P on the line $X_1X_{31,}$ the pivot $U(P)$ is the isogonal conjugate (X_{512}) of the Steiner point (X_{99}). Such points P include the Schiffler point (X_{21}), the isogonal conjugate (X_{58}) of the Spieker center, and the isogonal conjugate (X_{63}) of the Clawson point.

The cubic $\Gamma(P)$ meets $\mathcal{L}^\infty$ in three points. Aside from U, the other two are where $\mathcal{L}^\infty$ meets the conic (9). If (9) is an ellipse, then the two points are nonreal. In case P is the incenter, so that the cubic is the equal areas cubic, the two points are given in [6, p.116] by the ratios [6]

$$e^{\pm iB} : e^{\mp iA} : -1.$$

Theorem 2. *The generalized Brocard points defined by*

$$\frac{qc}{b} : \frac{ra}{c} : \frac{pb}{a} \qquad \text{and} \qquad \frac{rb}{c} : \frac{pc}{a} : \frac{qa}{b} \tag{10}$$

lie on $\Gamma(P)$.

Proof. Writing ordered triples for the two points, we have

$$\begin{aligned} &(a(rb^2 - qc^2),\ b(pc^2 - ra^2),\ c(qa^2 - pb^2)) \\ = \quad & abc(\frac{qc}{b}, \frac{ra}{c}, \frac{pb}{a}) + abc(\frac{rb}{c}, \frac{pc}{a}, \frac{qa}{b}), \end{aligned}$$

showing U as a linear combination of the points in (10). Since those two are isogonal conjugates collinear with U, they lie on $\Gamma(P)$. □

If P is a triangle center, then the generalized Brocard points (10) comprise a bicentric pair of points. In §8, we offer geometric constructions for such points.

[6]The pair is also given by $-1 : e^{\pm iC} : e^{\mp iB}$ and by $e^{\mp iC} : -1 : e^{\pm iA}$. Multiplying the three together and then by -1 gives cubes in "central form" with first coordinates

$$\cos(B - C) \pm i \sin(B - C).$$

The other coordinates are now given from the first by cyclic permutations.

5. Barycentric generalization: $\hat{\Gamma}(P)$

Here, we seek the locus of a point $X = \alpha : \beta : \gamma$ (barycentrics) for which the cevian triangle of the P-reciprocal conjugate of X and that of X have equal areas. The method presented in §1 yields a cubic that we denote by $\hat{\Gamma}(P)$:

$$p(q-r)\alpha(r\beta^2-q\gamma^2)+q(r-p)\beta(p\gamma^2-r\alpha^2)+r(p-q)\gamma(q\alpha^2-p\beta^2)=0, \tag{11}$$

In particular, the equal areas cubic (1) is given by (11) using

$$(p:q:r)_b=(a^2:b^2:c^2)_b.$$

In contrast to (11), if equation (1) is written as $s(a,b,c,\alpha,\beta,\gamma)=0$, then

$$s(\alpha,\beta,\gamma,a,b,c)=s(a,b,c,\alpha,\beta,\gamma),$$

a symmetry stemming from the use of trilinear coordinates and isogonal conjugation.

6. A sextic

For comparison with the cubic $\Gamma(P)$ of §4, it is natural to ask about the locus of a point X for which the anticevian triangle of X and that of its isogonal conjugate have equal areas. The result is easily found to be the self-isogonal sextic

$$\begin{aligned} & \alpha\beta\gamma(-a\alpha+b\beta+c\gamma)(a\alpha-b\beta+c\gamma)(a\alpha+b\beta-c\gamma) \\ = \quad & (-a\beta\gamma+b\gamma\alpha+c\alpha\beta)(a\beta\gamma-b\gamma\alpha+c\alpha\beta)(a\beta\gamma+b\gamma\alpha-c\alpha\beta), \end{aligned}$$

on which lie A, B, C, the incenter, excenters, and the two Brocard points. Remarkably, the vertices A, B, C are triple points of this sextic.

7. Prescribed area cubic: $\Lambda(P)$

Suppose $P = p : q : r$ (trilinears) is a point, and let $K\sigma$ be the oriented area of the cevian triangle of P. The method used in §1 shows that if $X = \alpha : \beta : \gamma$, then the cevian triangle of X has area $K\sigma$ if

$$k_1k_2k_3abc\alpha\beta\gamma = 8K\sigma^3, \tag{12}$$

where $k_1 = \frac{2\sigma}{b\beta+c\gamma}$ and k_2 and k_3 are obtained cyclically. Substituting into (12) and simplifying gives

$$K = 2\cdot\frac{pa}{bq+cr}\cdot\frac{qb}{cr+ap}\cdot\frac{rc}{ap+bq}. \tag{13}$$

The locus of X for which (13) holds is therefore given by the equation

$$(bq+cr)(cr+ap)(ap+bq)\alpha\beta\gamma - pqr(b\beta+c\gamma)(c\gamma+a\alpha)(a\alpha+b\beta)=0. \tag{14}$$

We call this curve the *prescribed area cubic for* P (or for K) and denote it by $\Lambda(P)$. One salient feature of $\Lambda(P)$, easily checked by substituting

$$\frac{1}{a^2\alpha},\ \frac{1}{b^2\beta},\ \frac{1}{c^2\gamma}$$

for α, β, γ, respectively, into the left side of (14), is that $\Lambda(P)$ is self-isotomic. That is, if X lies on $\Lambda(P)$ but not on a sideline of $\triangle ABC$, then so does its isotomic conjugate, which we denote by $\widetilde{X}$. (Of course, we already know that $\Lambda(P)$ is self-isotomic, by the note just after (8)).

If $(bq-cr)(cr-ap)(ap-bq) \neq 0$, then the line $P\widetilde{P}$ meets $\Lambda(P)$ in three points, namely P, $\widetilde{P}$, and the point

$$P' := \frac{a^2p^2 - bcqr}{a^2p(bq-cr)} \ : \ \frac{b^2q^2 - carp}{b^2q(cr-ap)} \ : \ \frac{c^2r^2 - abpq}{c^2r(ap-bq)}.$$

If P is a triangle center on $\Lambda(P)$, then $\widetilde{P}$, P', and $\widetilde{P'}$ are triangle centers on $\Lambda(P)$. Since $\widetilde{P'}$ is not collinear with the others, three triangle centers on $\Lambda(P)$ can be found as points of intersection of $\Lambda(P)$ with the lines joining $\widetilde{P'}$ to P, $\widetilde{P}$, and P'. Then more central lines are defined, bearing triangle centers that lie on $\Lambda(P)$, and so on. Some duplication of centers thus defined inductively can be expected, but one wonders if, for many choices of P, this scheme accounts for infinitely many centers lying on $\Lambda(P)$.

It is easy to check that $\Lambda(P)$ meets the line at infinity in the following points:

$$A' := 0 : c : -b, \qquad B' := -c : 0 : a, \qquad C' := b : -a : 0.$$

Three more points are found by intersecting lines PA', PB', PC' with $\Lambda(P)$:

$$A'' := bcp : c^2r : b^2q, \quad B'' := c^2r : caq : a^2p, \quad C'' := b^2q : a^2p : abr.$$

A construction for A'' is given by the equation $A'' = PA' \cap \widetilde{P}A$.

Line AP meets $\Lambda(P)$ in the collinear points A, P, and, as is easily checked, the point

$$\frac{bcqr}{pa^2} : q : r.$$

Writing this and its cyclical cousins integrally, we have these points on $\Lambda(P)$:

$$bcqr : a^2pq : a^2rp, \qquad b^2pq : carp : b^2qr, \qquad c^2rp : c^2qr : abpq.$$

We have seen for given P how to form K. It is of interest to reverse these. Suppose a prescribed area is specified as $K\sigma$, where K has the form

$$k(a,b,c)k(b,c,a)k(c,a,b)$$

in which $k(a,b,c)$ is homogeneous of degree zero in a, b, c.[7] We abbreviate the factors as k_a, k_b, k_c and seek a point $P = p : q : r$ satisfying

$$K = k_a k_b k_c = \frac{2abcpqr}{(bq+cr)(cr+ap)(ap+bq)}.$$

Solving the system obtained cyclically from

$$k_a = \frac{\sqrt[3]{2}ap}{bq+cr} \tag{15}$$

yields

$$p : q : r = \frac{k_a}{a(\sqrt[3]{2}+k_a)} : \frac{k_b}{b(\sqrt[3]{2}+k_b)} : \frac{k_c}{c(\sqrt[3]{2}+k_c)}$$

[7] That is, $k(ta,tb,tc) = k(a,b,c)$, where t is an indeterminate.

except for $k_a = -\sqrt[3]{2}$, which results from (15) with $P = X_{512}$. The following table offers a variety of examples:

P	$k_a/\sqrt[3]{2}$
X_1	$\frac{a}{b+c}$
X_2	1
X_3	$\frac{\sin 2A}{\sin 2B+\sin 2C}$
X_4	$\frac{\tan A}{\tan B+\tan C}$
X_{10}	$\frac{b+c}{a}$
X_{57}	$-\frac{a}{b+c}$
X_{870}	$\frac{bc}{b^2+c^2}$
X_{873}	$\frac{2bc}{b^2+c^2}$

Table 2

Next, suppose $U = u : v : w$ is a point, not on a sideline of $\triangle ABC$, and let

$$P = \frac{vc}{b} : \frac{wa}{c} : \frac{ub}{a}.$$

Write out K as in (13), and use not (15), but instead, put

$$k_a = \frac{\sqrt[3]{2}a^2u}{b^2w + c^2v},$$

corresponding to the point $U \cdot X_6 = ua : vb : wc$, in the sense that the cevian triangle of $U \cdot X_6$ and that of P have equal areas. Likewise, the cevian triangle of the point

$$P' = \frac{wb}{c} : \frac{uc}{a} : \frac{va}{b},$$

has the same area, $K\sigma$. The points P and P' are essentially those of Theorem 2.

Three special cases among the cubics $\Lambda(P)$ deserve further comment. First, for $K = 2$, corresponding to $P = X_{512}$, equation (14) takes the form

$$(a\alpha + b\beta + c\gamma)(bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta) = 0. \tag{16}$$

Since $\mathcal{L}^\infty$ is given by the equation $a\alpha + b\beta + c\gamma = 0$ and the Steiner circumellipse is given by

$$bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta = 0,$$

the points satisfying (16) occupy the line and the ellipse together. J.H. Weaver [8] discusses the cubic.

Second, when $K = \frac{1}{4}$, the cubic $\Lambda(P)$ is merely a single point, the centroid. Finally, we note that $\Lambda(X_6)$ passes through these points:

$$a : b : c, \quad a : c : b, \quad b : c : a, \quad b : a : c, \quad c : a : b, \quad c : b : a. \tag{17}$$

8. Constructions

In the preceding sections, certain algebraically defined points, as in (17), have appeared. In this section, we offer Euclidean constructions for such points.For given $U = u : v : w$ and $X = x : y : z$ and let us begin with the trilinear product, quotient, and square root, denoted respectively by $U \cdot X$, U/X, and $\sqrt{X}$.

Constructions for closely related barycentric product, quotient, and square root are given in [9], and these constructions are easily adapted to give the trilinear results.

We turn now to a construction from X of the point $x : z : y$. In preparation, decree as *positive* the side of line AB that contains C, and also the side of line CA that contains B. The opposite sides will be called *negative*.

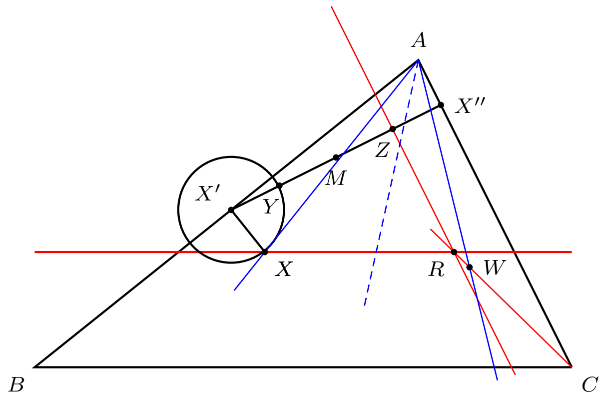

Figure 2. Construction of $W = x : z : y$ from $X = x : y : z$

Let X' be the foot of the perpendicular from X on line AB, and let X'' be the foot of the perpendicular from X' on line CA. Let M be the midpoint of segment $X'X''$, and let $\mathcal{O}$ be the circle centered at X' and passing through X. Line $X'X''$ meets circle $\mathcal{O}$ in two points; let Y be the one closer to M, as in Figure 2, and let Z' be the reflection of Y in M. If X is on the positive side of AB and Z' is on the positive side of CA, or if X is on the negative side of AB and Z' is on the negative side of CA, then let $Z = Z'$; otherwise let Z be the reflection of Z' in line CA.

Now line L through Z parallel to line CA has directed distance kz from line CA, where kx is the directed distance from line BC of the line L' through X parallel to BC. Let $R = L \cap L'$. Line CR has equation $z\alpha = x\beta$. Let L'' be the reflection of line AX about the internal angle bisector of $\angle CAB$. This line has equation $y\beta = z\gamma$. Geometrically and algebraically, it is clear that $x : z : y = CR \cap L''$, labeled W in Figure 2.

One may similarly construct the point $y : z : x$ as the intersection of lines $x\beta = z\gamma$ and $z\alpha = y\beta$. Then any one of the six points

$$x : y : z, \quad x : z : y, \quad y : z : x, \quad y : x : z, \quad z : x : y, \quad z : y : x,$$

can serve as a starting point for constructing the other five. (A previous appearance of these six points is [4, p.243], where an equation for the Yff conic, passing through the six points, is given.)

The methods of this section apply, in particular, to the constructing of the generalized Brocard points (10); e.g., for given $P = p : q : r$, construct $P' := q : r : p$, and then construct $P' \cdot \Omega_1$.

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A Feuerbach Type Theorem on Six Circles

Lev Emelyanov

According to the famous Feuerbach theorem there exists a circle which is tangent internally to the incircle and externally to each of the excircles of a triangle. This is the nine-point circle of the triangle. We obtain a similar result by replacing the excircles with circles each tangent internally to the circumcircle and to the sides at the traces of a point. We make use of Casey's theorem. See, for example, [1, 2].

Theorem (Casey). *Given four circles $\mathcal{C}_i$, $i = 1, 2, 3, 4$, let t_{ij} be the length of a common tangent between $\mathcal{C}_i$ and $\mathcal{C}_j$. The four circles are tangent to a fifth circle (or line) if and only if for appropriate choice of signs,*

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0.$$

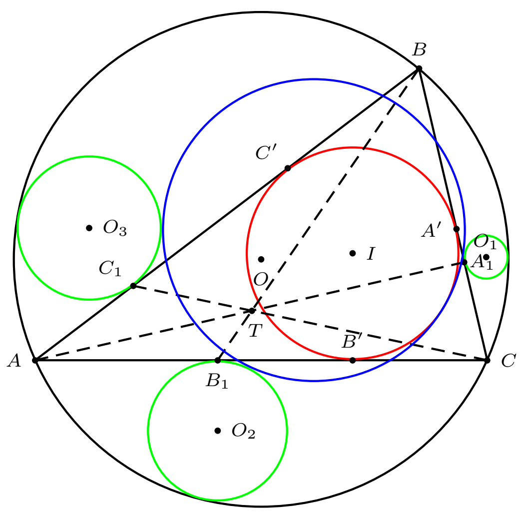

Figure 1

In this note we establish the following theorem. Let ABC be a triangle of side lengths $BC = a$, $CA = b$, and $AB = c$.

Theorem. *Let points A_1, B_1 and C_1 be on the sides BC, CA and AB respectively of triangle ABC. Construct three circles (O_1), (O_2) and (O_3) outside the triangle which is tangent to the sides of ABC at A_1, B_1 and C_1 respectively and also tangent to the circumcircle of ABC. The circle tangent externally to these three circles is also tangent to the incircle of triangle ABC if and only if the lines AA_1, BB_1 and CC_1 are concurrent.*

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Proof. Let in our case $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ be the circles (O_1), (O_2), (O_3) and the incircle respectively. With reference to Figure 1, we show that

$$t_{12}t_{34} - t_{13}t_{42} - t_{14}t_{23} = 0, \tag{1}$$

where t_{12}, t_{13} and t_{23} are the lengths of the common extangents, t_{34}, t_{24} and t_{14} are the lengths of the common intangents.

Let (A) be the degenerate circle $A(0)$ (zero radius) and $t_i(A)$ be the length of the tangent from A to $\mathcal{C}_i$. Similar notations apply to vertices B and C. Applying Casey's theorem to circles (A), (B), (O_1) and (C), which are all tangent to the circumcircle, we have

$$t_1(A) \cdot a = c \cdot CA_1 + b \cdot A_1B.$$

From this we obtain $t_1(A)$, and similarly $t_2(B)$ and $t_3(C)$:

$$t_1(A) = \frac{c \cdot CA_1 + b \cdot A_1B}{a}, \tag{2}$$

$$t_2(B) = \frac{a \cdot AB_1 + c \cdot B_1C}{b}, \tag{3}$$

$$t_3(C) = \frac{b \cdot BC_1 + a \cdot C_1A}{c}. \tag{4}$$

Applying Casey's theorem to circles (B), (C), (O_2) and (O_3), we have

$$t_2(B)t_3(C) = a \cdot t_{23} + CB_1 \cdot C_1B.$$

Using (3) and (4), we obtain t_{23}, and similarly, t_{13} and t_{12}:

$$t_{23} = \frac{a \cdot C_1A \cdot AB_1 + b \cdot AB_1 \cdot BC_1 + c \cdot AC_1 \cdot CB_1}{bc}, \tag{5}$$

$$t_{13} = \frac{b \cdot A_1B \cdot BC_1 + c \cdot BC_1 \cdot CA_1 + a \cdot BA_1 \cdot AC_1}{ca}, \tag{6}$$

$$t_{12} = \frac{c \cdot B_1C \cdot CA_1 + a \cdot CA_1 \cdot AB_1 + b \cdot CB_1 \cdot BA_1}{ab}. \tag{7}$$

In the layout of Figure 1, with A', B', C' the touch points of the incircle with the sides, the lengths of the common tangents of the circles (O_1), (O_2), (O_3) with the incircle are

$$t_{14} = A_1A' = -CA_1 + CA' = -CA_1 + \frac{a+b-c}{2}, \tag{8}$$

$$t_{24} = B_1B' = -AB_1 + AB' = -AB_1 + \frac{b+c-a}{2}, \tag{9}$$

$$t_{34} = C_1C' = BC_1 - BC' = BC_1 - \frac{c+a-b}{2}. \tag{10}$$

Substituting (5)-(10) into (1) and simplifying, we obtain

$$t_{12}t_{34} - t_{13}t_{24} - t_{14}t_{23} = \frac{F(a,b,c)}{abc} \cdot (AB_1 \cdot BC_1 \cdot CA_1 - A_1B \cdot B_1C \cdot C_1A),$$

where

$$F(a,b,c) = 2bc + 2ca + 2ab - a^2 - b^2 - c^2.$$

Since $F(a, b, c)$ can be rewritten as

$$(c + a - b)(a + b - c) + (a + b - c)(b + c - a) + (b + c - a)(c + a - b),$$

it is clearly nonzero. It follows that $t_{12}t_{34} - t_{13}t_{24} - t_{14}t_{23} = 0$ if and only if

$$AB_1 \cdot BC_1 \cdot CA_1 - A_1B \cdot B_1C \cdot C_1A = 0. \tag{11}$$

By the Ceva theorem, (11) is the condition for the concurrency of AA_1, BB_1 and CC_1. It is clear that for different positions of the touch points of circles (O_1), (O_2) and (O_3) relative to those of the incircle, the proofs are analogous. □

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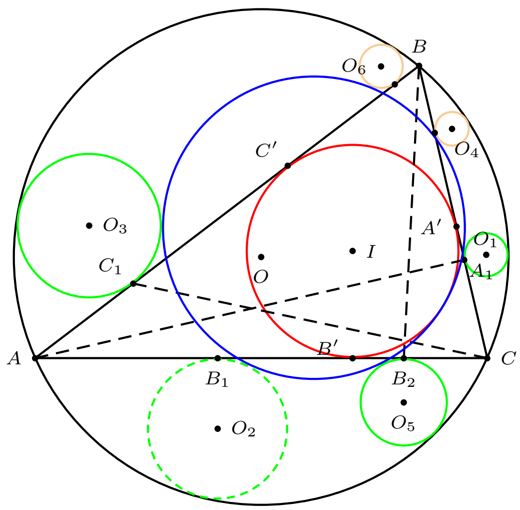
Correction to

A Feuerbach Type Theorem on Six Circles

Lev Emelyanov

Floor van Lamoen has kindly pointed out that the necessity part of the main theorem of [1] does not hold. In the layout of Figure 1 there, it is possible to have a circle (O_5) outside the triangle, tangent to both the circumcircle and the "new" circle, but to AC at a point B_2 between B' and C. The points of tangency of the circles (O_1), (O_5) and (O_3) with the sides of triangles do not satisfy Ceva's theorem. Likewise, it is also possible to place circles (O_4) and (O_6) on the sides BC and AB so that the points of tangency do not satisfy Ceva's theorem.



We hereby modify the statement of the theorem as follows.

Theorem. *Let A_1, B_1, C_1 be the traces of an interior point T on the side lines of triangle ABC. Construct three circles (O_1), (O_2) and (O_3) outside the triangle which are tangent to the sides at A_1, B_1, C_1 respectively and also tangent to the circumcircle of ABC. The circle tangent externally to these three circles is also tangent to the incircle of triangle ABC.*

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