Junior problems

J217. If a, b, c are integers such that $a^2 + 2bc = 1$ and $b^2 + 2ca = 2012$, find all possible values of $c^2 + 2ab$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Subtracting the two given equations, we have $(b-a)(b+a-2c) = 2011$, so we get

$$\begin{cases} b &= a \pm 1 \\ 2c &= b + a \mp 2011 \end{cases} \qquad \begin{cases} b &= a \pm 2011 \\ 2c &= b + a \mp 1. \end{cases}$$

Substituting these values into the equation $a^2 + 2bc = 1$, we obtain the four equations

$$3a^2 - 2008a - 2011 = 0 \tag{1}$$

$$3a^2 + 2008a - 2011 = 0 \tag{2}$$

$$3a^2 + 6032a + 2010 \cdot 2011 - 1 = 0 \tag{3}$$

$$3a^2 - 6032a + 2010 \cdot 2011 - 1 = 0. \tag{4}$$

Equation (1) gives $(a+1)(3a-2011) = 0$, so $a = -1, b = 0, c = -1006$.
Equation (2) gives $(a-1)(3a+2011) = 0$, so $a = 1, b = 0, c = 1006$.
Equations (3) and (4) have no solution since their discriminant is negative.
In conclusion, $c^2 + 2ab = 1006^2 = 1012036$.

Also solved by Sarah Nale, Auburn University Montgomery; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Albert Stadler, Switzerland; Ercole Suppa, Teramo, Italy; Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzau, Romania; Jojhan Victor Malqui Valera, Peru; Matteo Elia, Università di Roma "Tor Vergata", Roma, Italy

J218. Prove that in any triangle with sides of lenghts a, b, c, circumradius R, and inradius r, the following inequality holds

$$\frac{\sqrt{ab}}{a+b-c}+\frac{\sqrt{bc}}{b+c-a}+\frac{\sqrt{ca}}{c+a-b}\leq 1+\frac{R}{r}.$$

Proposed by Cezar Lupu, University of Pittsburgh, USA, and Virgil Nicula, Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Using well-known relations for the area S of ABC, and denoting by s the semiperimeter of ABC, we find

$$\frac{R}{r}=\frac{abcs}{4S^2}=\frac{abc}{4(s-a)(s-b)(s-c)}.$$

Now recall that

$$(s-a)(s-b)(s-c)=(ab+bc+ca)s-s^3-abc,$$

$$a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=2s^3-2(ab+bc+ca)s+3abc.$$

Thus, it follows that

$$\frac{4R}{r}+4=\frac{abc}{(s-a)(s-b)(s-c)}+4=\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}+6=\frac{b+c}{s-a}+\frac{c+a}{s-b}+\frac{a+b}{s-c},$$

or equivalently,

$$1+\frac{R}{r}=\frac{b+c}{2(b+c-a)}+\frac{c+a}{2(c+a-b)}+\frac{a+b}{2(a+b-c)}.$$

But $a+b\geq 2\sqrt{ab}$ by the AM-GM inequality, with equality iff $a=b$, and similarly for its cyclic permutations. The conclusion follows, equality holds iff ABC is equilateral.

Also solved by Arkady Alt, San Jose, California, USA; Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzau, Romania; Albert Stadler, Switzerland; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Sandu and Mihai Sandu, Bucharest, Romania; Ercole Suppa, Teramo, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Nicu Zlota, Focsani, Romania

J219. Trying to solve a problem, Jimmy used the following "formula": $\log_{ab} x = \log_a x \log_b x$, where a, b, x are positive real numbers different from 1. Prove that this is correct only if x is a solution to the equation $\log_a x + \log_b x = 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, Teramo, Italy

Setting $u = \log_{ab} x$, $v = \log_a x$, $w = \log_b x$, we get that $\log_{ab} x = \log_a x \log_b x$ holds if and only if

$$x = (ab)^u = a^u b^u = a^{vw} b^{vw} = (a^v)^w (b^w)^v = x^w x^v = x^{v+w},$$

i.e. $v + w = 1$, or in other words $\log_a x + \log_b x = 1$. This proves the problem.

Also solved by Albert Stadler, Switzerland; Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzau, Romania; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Antonio Cirulli, Francesco Bonesi, Lorenzo Luzzi, Matteo Elia and Edoardo Bruno, Università di Roma "Tor Vergata", Roma, Italy; Alessandro Ventullo, Milan, Italy; Vicente Vicario Garcia, Huelva, Spain; Daniel Lasaosa, Universidad Pública de Navarra, Spain

J220. Find the least prime p for which $p = {a_k}^2 + k{b_k}^2$, $k = 1, \ldots, 5$, for some (a_k, b_k) in $\mathbb{Z} \times \mathbb{Z}$.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

First, note that $p > 5$. Also,

$$p = a_3^2 + 3b_3^2,\ \text{so } p \equiv a_3^2 \pmod 3 \text{ i.e. } p \equiv 1 \pmod 3$$

and

$$p = a_4^2 + 4b_4^2,\ \text{so } p \equiv a_4^2 \pmod 4 \text{ i.e. } p \equiv 1 \pmod 4.$$

Hence, we know that $p \equiv 1 \pmod{12}$. However,

$$p = a_5^2 + 5b_5^2,\ \text{so } p \equiv a_5^2 \pmod 5 \text{ i.e. } p \equiv 1 \pmod 5 \text{ or } p \equiv 4 \pmod 5.$$

Hence we get that $p \equiv 1 \pmod{60}$ or $p \equiv 1 \pmod{49}$. A quick verification of the numbers $61, 181, 241, 109, 229$ which satisfy the above congruences shows that $p = 241$ is the minimal solution.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Vicente Vicario Garcia, Huelva, Spain; Albert Stadler, Switzerland; Alessandro Ventullo, Milan, Italy

J221. Solve in integers the system of equations

$$\begin{aligned} xy - \tfrac{z}{3} &= xyz + 1, \\ yz - \tfrac{x}{3} &= xyz - 1, \\ zx - \tfrac{y}{3} &= xyz - 9. \end{aligned}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Albert Stadler, Switzerland

We have

$$(x - z)(1 + 3y) = 3(xy - \frac{z}{3}) - 3(yz - \frac{x}{3}) = 6.$$

So

$$(x - z, 1 + 3y) \in (-6, -1), (-3, -2), (-2, -3), (-1, -6), (6, 1), (3, 2), (2, 3), (1, 6).$$

y is an integer. So

$$(x - z, 1 + 3y) \in (-3, -2), (6, 1)$$

and

$$y \in 0, -1$$

If $y = 0$ then $z = -3$ from the first equation and $x = 3$ from the second equation and we verify that $(x, y, z) = (3, 0, -3)$ is a solution to the system of equations. If $y = -1$ then $xz + \frac{1}{3} = -xz - 9$ from the third equation. So $xz = -\frac{14}{3}$ and we get $-x - z/3 = -xz + 1 = \frac{17}{3}$ and $-z - \frac{x}{3} = -xz - 1 = \frac{11}{3}$ or equivalently

$$3x + z = -17 \text{ and } 3z + x = -11$$

which has solution $(x, z) = (-5, -2)$.

However, we can check that $(x, y, z) = (-5, -1, -2)$ is not a solution for the system of equations; thus $(x, y, z) = (3, 0, -3)$ is the only solution in integers to the given system of equations.

Second solution by Mary M. Harris - Auburn University Montgomery

Case	m	n	p	x	y	z
(i)	10	-8	1	-3	3	0
(ii)	10	1	-8	-3	0	3
(iii)	-8	10	1	3	-3	0
(iv)	-8	1	10	3	0	-3
(v)	1	10	-8	0	-3	3
(vi)	1	-8	10	0	3	-3
(vii)	-80	1	1	27	0	0
(viii)	1	-80	1	0	27	0
(ix)	1	1	-80	0	0	27

From the equations, the solutions x, y, and z must be multiples of 3. Multiplying each original equation by 9 and then summing them up together gives

$$9xy + 9yz + 9zx - 3z - 3x - 3y - 27xyz = -81.$$

After adding 1 to both sides of the equation, it can be factored as

$$(1 - 3x)(1 - 3y)(1 - 3z) = -80.$$

Let $m = 1 - 3x$, $n = 1 - 3y$, and $p = 1 - 3z$. Since x, y and z are multiples of 3, we can write $x = 3a$, $y = 3b$, and $z = 3c$ where $a, b, c \in \mathbb{Z}$. Then $m - 1 = -9a$, $n - 1 = -9b$, and $p - 1 = -9c$. So $m - 1$, $n - 1$, and $p - 1$ are divisible by 9.

Since $mnp = -80$, then either one, or all of the integers m, n, and p will be negative. Therefore, the possibilities for m, n, and p, with consideration of x, y, and z, are as given in the table above.

A routine computation shows that case (iv) is the only solution. Consequently, we conclude that the only integer solution is $x = 3$, $y = 0$, and $z = -3$.

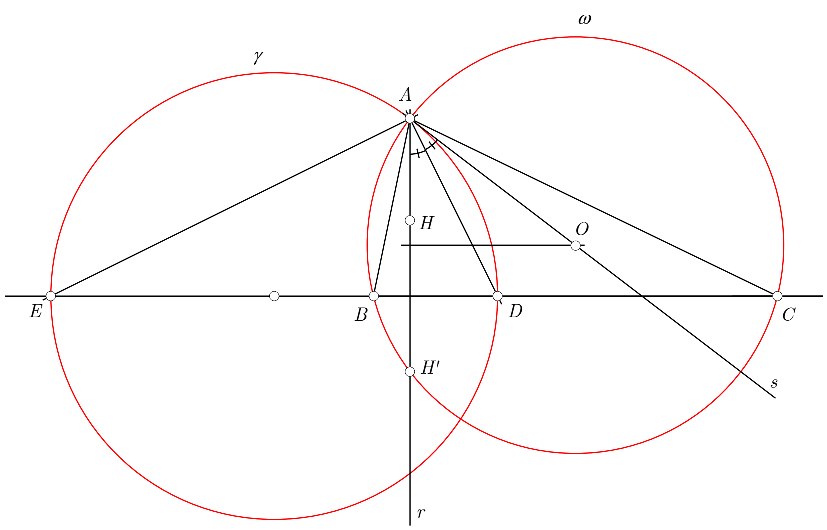
Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzau, Romania; Matteo Elia and Lorenzo Luzzi, Università di Roma "Tor Vergata", Roma, Italy; Alessandro Ventullo, Milan, Italy

J222. Give a ruler and straightedge construction of a triangle ABC given its orthocenter and the intersection points of the internal and external angle bisectors of one of its angles with the corresponding opposite side.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Ercole Suppa, Teramo, Italy

ANALYSIS. Let ABC be the required triangle and denote by D, E the intersection points of the internal and external angle bisectors of $\angle BAC$ with BC. Let H be the orthocenter of $\triangle ABC$ and draw a figure in order to find some relation between the given elements and the unknown parts.



Observe that the vertices B, C lie on the line DE, whereas the vertex A is the meeting point of the line through H perpendicular to DE with the circle having for diameter DE (since AE and AD are perpendicular).

Let O be the circumcentre of $\triangle ABC$ and note that $\angle BAH = 90^\circ - B = \angle OAC$, so $\angle HAD = \angle DAO$. Therefore O belong to the line symmetric of AH with respect to AD.

Let H' be the symmetric of H with respect to DE. It is well known that H' lies on the circumcircle (O). Hence O belong to the mediator of segment AH'.

The vertices B, C are the intersection points of (O) with DE.

CONSTRUCTION. The vertices A, B, C of the required triangle can be constructed in the following way:

- draw the line DE;
- draw the circle γ having for diameter DE;
- draw the line r perpendicular to DE through H;
- construct the point $A = r \cap \gamma$;
- draw the line s symmetric of r wrt AD;
- construct the point H' symmetric of H wrt DE;
- draw the line a, mediator of the segment AH';
- construct the point $O = a \cap s$;

- draw the circle ω through A and having center O;
- construct the points $\{B, C\} = \omega \cap DE$.

Also solved by Saturnino Campo Ruiz, Salamanca, Spain; Vicente Vicario Garcia, Huelva, Spain

S217. Find all integer solutions of the equation $2x^2 - y^{14} = 1$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

We have

$$2x^2 - y^{14} = 1 \Leftrightarrow 2x^2 = (y^2+1)((y^2)^6 - (y^2)^5 + \cdots - y^2 + 1).$$

If there exists prime number p such that $p \mid y^2+1$, then we have $p \not| (y^2)^6 - (y^2)^5 + \cdots - y^2 + 1$. Therefore y^2+1 and $(y^2)^6 - (y^2)^5 + \cdots - y^2 + 1$ are relatively prime. Thus the case $y^2+1 = a^2$ leads to $y = 0$, no solution. Then we must have $y^2 + 1 = 2a^2$. This is a Pell equation with infinitely many solutions.

Denote by $t = y^2 > 0$ and $t^6 - t^5 + \cdots - t + 1 = b^2$. Hence $x = ab$. For $t \geq 4$, we have

$$\begin{aligned} (16b)^2 &= (16t^3 - 8t^2 + 6t - 5)^2 + 140t^2 - 196t + 231 \\ &> (16t^3 - 8t^2 + 6t - 5)^2. \end{aligned}$$

On the other hand

$$\begin{aligned} (16b)^2 &= (16t^3 - 8t^2 + 6t - 4)^2 - (32t^3 - 156t^2 + 208t - 240) \\ &> (16t^3 - 8t^2 + 6t - 4)^2. \end{aligned}$$

Finally, its only remains to check the values $t = 1, 2, 3$. But $t = y^2$ is a square, so the only possibility is $t = 1$, yielding $x, y \in \{-1, 1\}$.

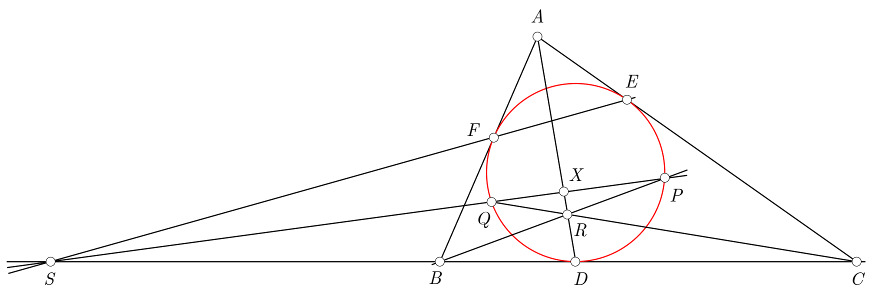
S218. Let ABC be a triangle with incircle $\mathcal{C}$ and incenter I. Let D, E, F be the tangency points of $\mathcal{C}$ with the sides BC, CA, and AB, respectively, and furthermore, let S be the intersection of BC and EF. Let P, Q be the intersection points of SI with $\mathcal{C}$ such that P, Q lie on the small arcs DE and FD respectively. Prove that the lines AD, BP, CQ are concurrent.

Proposed by Marius Stanean, Zalau, Romania

Solution by Ercole Suppa, Teramo, Italy

We will prove that the result is true if the point I is replaced with any point X on the segment AD, i.e.:

Let P, Q be the intersection points of SX with $\mathcal{C}$ such that P, Q lie on the small arcs DE and FD respectively. Prove that the lines AD, BP, CQ are concurrent (see figure).



Let $a = BC$, $b = CA$, $c = AB$ and let s be the semiperimeter of $\triangle ABC$. It is well known that $AF = FE = s-a$, $BD = BF = s-b$, $CD = CE = s-c$, so

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{s-a}{s-b} \cdot \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} = 1. \tag{1}$$

Applying Menealaus theorem to $\triangle ABC$ with transversal EFS we have

$$\frac{AF}{FB} \cdot \frac{BS}{SC} \cdot \frac{CE}{EA} = 1. \tag{2}$$

From (1), (2) it follows that

$$\frac{AF}{FB} \cdot \frac{DB}{DC} \cdot \frac{CE}{EA} = \frac{AF}{FB} \cdot \frac{BS}{SC} \cdot \frac{CE}{EA} \qquad \text{so} \qquad \frac{BD}{DC} = -\frac{BS}{SC},$$

i.e. the division (S, D, B, C) is harmonic.

Let $R = AD \cap CQ$ and let $P' = BR \cap SX$. Since $(SDBC) = -1$ it follows that $R(SDBC)$ is an harmonic boundle, so

$$(SXP'Q) = -1 \qquad \text{so} \qquad (SXQP') = -1. \tag{3}$$

Since S lies on the polar of A wrt $\mathcal{C}$, A lies on the polar of S wrt $\mathcal{C}$. Therefore AD is the polar of S with respect to $\mathcal{C}$, so

$$(SXQP) = -1. \tag{4}$$

Taking into account of (3) and (4), the uniqueness of fourth harmonic yields $P' = P$ and this prove that AD, BP, CQ are concurrent.

Also solved by Saturnino Campo Ruiz, Salamanca, Spain

S219. Let $ABCD$ be a quadrilateral and let $\{P\} = AC \cap BD$, $\{E\} = AD \cap BC$, and $\{F\} = AB \cap CD$. Denote by $\operatorname{isog}_{XYZ}(P)$ the isogonal conjugate of P with respect to triangle XYZ. Prove that $\operatorname{isog}_{ABE}(P) = \operatorname{isog}_{CDE}(P) = \operatorname{isog}_{ADF}(P) = \operatorname{isog}_{BCF}(P)$ if and only if AC and BD are perpendicular.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by the author

Recall the following result about isogonal conjugates with respect to a triangle.

Lemma. Let ABC be a triangle and let P and Q be distinct points in its plane. Then, the circumcircles of the pedal triangles of these two points are the same if and only if P and Q are isogonally conjugated with respect to ABC.

For the proof one can see Darij Grinberg, *Isogonal Conjugation with respect to a Triangle*, available at http://www.cip.ifi.lmu.de/ grinberg/geometry2.html.

Now, using this, it is straightforward to see that the problem becomes equivalent to the following result about quadrilaterals.

Claim. Let $ABCD$ be a quadrilateral. The diagonals AC and BD are perpendicular at point O. The perpendiculars from O on the sides of the quadrilateral meet AB, BC, CD, DA at M, N, P, Q, respectively, and meet again CD, DA, AB, BC at M', N', P', Q', respectively. Then, the points M, N, P, Q, M', N', P', Q' are all concyclic.

This is a rather easy angle chase; we proceed as follows. First, note that $MNPQ$ is cyclic. For this, notice that the quadrilaterals $BMON$, $CNOP$, $DPOQ$ and $AQOP$ have a pair of opposite right angles, hence all of them are cyclic. It follows that $\angle OMN = \angle OBN$, $\angle OMQ = \angle OAQ$, $\angle OPQ = \angle ODQ$ and $\angle OPN = \angle OCN$. Summing up yields $\angle QMN = \angle QPN$, so $MNPQ$ is indeed cyclic.

Next, we show that M', N', P', Q' lie on the circumcicle $\mathcal{K}$ of $MNPQ$. As an external angle, we get $\angle NQ'O = \angle Q'CO + \angle COQ'$. Since $CNOP$ is cyclic, then $\angle Q'CO = \angle NPO$. On the other hand we have $\angle Q'OP = \angle ODQ$, because the sides are perpendicular and $\angle ODQ = \angle OPQ$ (as $DPOQ$ is cyclic). Therefore $\angle NQ'O = \angle NPO + \angle OPQ = \angle NPQ$, implying that Q' belongs to the circumcircle of $MNPQ$. In the same manner we find that M', N', P' lie on $\mathcal{K}$, which proves the claim and the problem.

S220. Let a, b, c be nonnegative real numbers. Prove that

$$\sqrt[3]{a^3+b^3+c^3-\frac{1}{2}\left(ab(a+b)+bc(b+c)+ca(c+a)\right)} \geq \sqrt{a^2+b^2+c^2-ab-bc-ca}.$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote by $2x = (a-b)^2$, $2y = (b-c)^2$, $2z = (c-a)^2$. The term under the radical in the RHS is $x+y+z$, whereas the term under the one from the LHS is $(a+b)x + (b+c)y + (c+a)z$. The proposed inequality is then equivalent to

$$(x+y+z)^3 \leq ((a+b)x + (b+c)y + (c+a)z)^2.$$

Now, since $|a|, |b| \geq |a-b|$ because a, b are both non-negative, we have $a^2, b^2 \geq (a-b)^2 = 2x$, or

$$(a+b)^2 = (a^2+b^2-x) \geq 3x, \qquad a+b \geq \sqrt{3x}.$$

Therefore,

$$((a+b)x + (b+c)y + (c+a)z)^2 \geq 3\left(x^3+y^3+z^3\right) + 6\left(xy\sqrt{xy} + yz\sqrt{yz} + zx\sqrt{zx}\right),$$

and since $2xy\sqrt{xy} + 2yz\sqrt{yz} + 2zx\sqrt{zx} \geq 6xyz$ by the AM-GM inequality, it suffices to show that

$$x^3 - 3x^2y + 4xy\sqrt{xy} - 3xy^2 + y^3 \geq 0,$$

or calling $x = u^2$, $y = v^2$, it suffices to show that

$$0 \leq u^6 - 3u^4v^2 + 4u^3v^3 - 3u^2v^4 + v^6 = (u-v)^2(u^4 + 2u^3v + 2uv^3 + v^4),$$

clearly true, and with equality iff $x = y$. Therefore, $x = y = z$ is a necessary condition for equality to hold in the proposed inequality. Now, if $x \neq 0$, it must be $a^2 = b^2 = 3x$, or $a^2 + b^2 = 3(a-b)^2$, yielding $a^2 + b^2 - 3ab = 0$, or $2(a-b)^2 - (a+b)^2 = 0$, ie $x = (a+b)^2 = 3x$, or $x = 0$. Equality is then reached in the proposed inequality iff $x = y = z = 0$, or iff $a = b = c$.

Also solved by Arkady Alt, San Jose, California, USA; Nicu Zlota, Focsani, Romania; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy

S221. Let ABC be a triangle with centroid G and let F be a point that minimizes the quantity $PA+PB+PC$ over all points P lying in the plane of ABC. Prove that

$$FG \leq \min(AG, BG, CG).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Arkady Alt, San Jose, California, USA

There are two cases. First, if ABC has an obtuse angle greater than or equal to 120°, say $\angle A$, then $F = A$, and since $\min\{AG, BG, CG\} = AG$, we have $FG = \min\{AG, BG, CG\}$. If all angles are less than 120°, then F lies in the interior of the triangle and $\angle AFB = \angle BFC = \angle CFA$.

Let a, b, c be the sidelengths of triangle ABC, F its area and x, y, z the lengths of the segments FA, FB, and FC, respectively. Then

$$F = [BFC] + [CFA] + [AFB] = \frac{\sqrt{3}}{4}(xy + yz + zx),$$

yielding $xy + yz + zx = \dfrac{4F}{\sqrt{3}}$.

We have $a^2 = y^2 + z^2 + yz$, $b^2 = z^2 + x^2 + zx$, $c^2 = x^2 + y^2 + xy$, so

$$2\left(x^2 + y^2 + z^2\right) + xy + yz + zx = a^2 + b^2 + c^2,$$

which gives

$$x^2 + y^2 + z^2 = \frac{a^2 + b^2 + c^2 - \dfrac{4F}{\sqrt{3}}}{2} = \frac{3\left(a^2 + b^2 + c^2\right) - 4\sqrt{3}F}{6}.$$

However,

$$\sum_{cyc} GA^2 = \sum_{cyc}\left(\frac{2}{3}m_a\right)^2 = \frac{4}{9}\sum_{cyc}\frac{2\left(b^2 + c^2\right) - a^2}{4} = \frac{a^2 + b^2 + c^2}{3},$$

By Lagrange's Formula we get

$$\begin{aligned}
FG^2 &= \frac{1}{3}\sum_{cyc}\left(FA^2 - GA^2\right) \\
&= \frac{1}{3}\left(x^2 + y^2 + z^2 - \frac{a^2 + b^2 + c^2}{3}\right) \\
&= \frac{1}{3}\left(\frac{3\left(a^2 + b^2 + c^2\right) - 4\sqrt{3}F}{6} - \frac{a^2 + b^2 + c^2}{3}\right) \\
&= \frac{a^2 + b^2 + c^2 - 4\sqrt{3}F}{18}.
\end{aligned}$$

Thus we are left to show that

$$a^2 + b^2 + c^2 - 4\sqrt{3}F \leq 18\min\left\{AG^2, BG^2, CG^2\right\},$$

which is equivalent to showing the following system of inequalities

$$\begin{cases} a^2 + b^2 + c^2 - 4\sqrt{3}F \leq 4\left(b^2 + c^2\right) - 2a^2 \\ a^2 + b^2 + c^2 - 4\sqrt{3}F \leq 4\left(c^2 + a^2\right) - 2b^2 \\ a^2 + b^2 + c^2 - 4\sqrt{3}F \leq 4\left(a^2 + b^2\right) - 2c^2 \end{cases} \iff \begin{cases} -4F \leq \sqrt{3}\left(b^2 + c^2 - a^2\right) \\ -4F \leq \sqrt{3}\left(c^2 + a^2 - b^2\right) \\ -4F \leq \sqrt{3}\left(a^2 + b^2 - c^2\right) \end{cases}$$

Now, if triangle ABC isn't obtuse then all three inequalities obviously holds. Thus it remains to consider the case, when one of angles is obtuse. Assume WLOG that $\angle A$ is such that $90^\circ < \angle A < 120^\circ$. In this case $-\dfrac{1}{2} < \cos A$ and $\frac{\sqrt{3}}{2} < \sin A$. Thus

$$\begin{aligned} a^2 - b^2 - c^2 = -2bc\cos A < bc \quad &\Longrightarrow \quad \frac{\sqrt{3}}{4}\left(a^2 - b - c^2\right) < \frac{\sqrt{3}}{4}bc < \frac{bc\sin A}{2} = F \\ &\Longrightarrow \quad -4F < \sqrt{3}\left(b^2 + c^2 - a^2\right). \end{aligned}$$

Since angles $\angle B$ and $\angle C$ are acute, we have $c^2 + a^2 - b^2 > 0$ and $a^2 + b^2 - c^2 > 0$. So two other inequalities clearly hold. This completes the proof.

Also solved by Vicente Vicario Garcia, Huelva, Spain; Daniel Lasaosa, Universidad Pública de Navarra, Spain

S222. Solve the equation $3\phi(n) = 4\tau(n)$ where $\phi(n)$ is the Euler totient function and $\tau(n)$ is the number of divisors of n.

Proposed by Roberto Bosch Cabrera, Florida, USA

Solution by Alessio Podda and Antonio Cirulli, Università di Roma "Tor Vergata", Roma, Italy.

We consider the function

$$f_p(\alpha) = \frac{\varphi(p^\alpha)}{\tau(p^\alpha)} = \frac{(p-1)p^{\alpha-1}}{\alpha+1},$$

where p is a prime and α is a positive integer. Then it is easy to verify that

$$f_p(\alpha) < f_p(\alpha+1) \text{ and } f_p(\alpha) < f_q(\alpha) \text{ for } p < q.$$

Hence $f_p(\alpha) \geq f_p(1) = (p-1)/2 \geq 1$ as soon as $p > 2$ or $p = 2$ and $\alpha > 2$. Moreover $f_2(1) = 1/2$, $f_2(2) = 2/3$. Now we show the following claims:

i) If $p \geq 7$ is a prime which divides n then

$$\frac{\varphi(n)}{\tau(n)} \geq f_2(1)f_p(1) \geq f_2(1)f_7(1) = \frac{3}{2} > \frac{4}{3}.$$

ii) If 5^2 divides n then

$$\frac{\varphi(n)}{\tau(n)} \geq f_2(1)f_5(2) = \frac{10}{3} > \frac{4}{3}.$$

iii) If 3^3 divides n then

$$\frac{\varphi(n)}{\tau(n)} \geq f_2(1)f_3(3) = \frac{9}{4} > \frac{4}{3}.$$

iv) If 2^4 divides n then

$$\frac{\varphi(n)}{\tau(n)} \geq f_2(4) = \frac{8}{5} > \frac{4}{3}.$$

Therefore a possible solution of our equation has the form $2^a 3^b 5^c$ for $a = 0, 1, 2, 3$, $b = 0, 1, 2$ and $c = 0, 1$. An exhaustive search in this finite set yields 20, 36 and 60.

Also solved by Albert Stadler, Switzerland; Daniel Lasaosa, Universidad Pública de Navarra, Spain

U217. Define an increasing sequence $(a_k)_{k\in\mathbb{Z}^+}$ to be *attractive* if $\sum_{k=1}^{\infty} \dfrac{1}{a_k}$ diverges and $\sum_{k=1}^{\infty} \dfrac{1}{a_k^2}$ converges. Prove that there is an attractive sequence a_k such that $a_k\sqrt{k}$ is also an attractive sequence.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA.

Solution by Alessandro Ventullo, Milan, Italy

We claim that the sequence $a_k = \sqrt{k}\log k$, $k \geq 3$ satisfies the conditions. Indeed, $a_k = \sqrt{k}\log k$ is obviously increasing for all $k \geq 3$. Moreover, since $\dfrac{1}{k} \leq \dfrac{1}{\sqrt{k}\log k}$ for all $k \geq 3$ and $\displaystyle\int_1^{\infty} \frac{1}{x\log^2 x}\,dx$ converges we obtain that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{k}\log k} = \infty, \qquad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k\log^2 k} = M < \infty$$

as a consequence of the Comparison Test and the Integral Test. Finally, $a_k\sqrt{k} = k\log k$ is also attractive. As a matter of fact, $k\log k$ is increasing for all $k \geq 3$, $\displaystyle\int_1^{\infty} \frac{1}{x\log x}\,dx$ diverges and $\dfrac{1}{k^2\log^2 k} \leq \dfrac{1}{k^2}$ for all $k \geq 3$, therefore

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k\log k} = \infty, \qquad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2\log^2 k} = N < \infty.$$

Also solved by Albert Stadler, Switzerland; Vicente Vicario Garcia, Huelva, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Alessio Podda, Matteo Elia and Edoardo Bruno, Università di Roma "Tor Vergata", Roma, Italy; Arkady Alt, San Jose, California, USA; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy

U218. Let $*$ be an associative and "totally non-commutative" ($x \neq y$ implies $x * y \neq y * x$) binary operation on a set S. Prove that $x * y * z = x * z$ for all x, y, z in S.

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania.

Solution by Adenilson Arcanjo, PECI – "Special Program for International Competitions", Rio de Janeiro, Brazil

Suppose that $x * x \neq x$ thus by the totally non-commutativity $(x * x) * x \neq x * (x * x)$ which is an absurd because $*$ is associative. Similarly suppose that $x * y * x \neq x$; this gives $x * y * x = (x * y * x) * x \neq x * (x * y * x) = x * y * x$, which is again absurd! Finally suppose that $x * y * z \neq x * z$. In this case, $x * y * z = (x * z * x) * (y * z) = (x * z) * (x * y * z) \neq (x * y * z) * (x * z) = x * y * (z * x * z) = x * y * z$. This is again impossible, so we conclude that $x * y * z = x * z$, completing the proof.

Also solved by Lorenzo Luzzi and Cesare Gallozzi, Università di Roma "Tor Vergata", Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Albert Stadler, Switzerland; Johan Gunardi, Indonesia

U219. a) Let $f : [0, \infty) \to \mathbb{R}$ be a convex differentiable function with $f(0) = 0$. Prove that

$$\int_0^x f(t)dt \le \frac{x^2}{2} f'(x) \qquad \text{for all } x \in [0, \infty).$$

b) Find all differentiable functions $f : [0, \infty) \to \mathbb{R}$ for which we have equality in the above inequality.

Proposed by Dorin Andrica, Babes-Bolyai University of Cluj-Napoca, and Mihai Piticari National College "Dragos Voda" Campulung Moldovenesc, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

a) Let

$$F(x) = \int_0^x f(t)dt,$$

which is clearly a differentiable function satisfying $F'(x) = f(x)$ and $F''(x) = f'(x)$, where $F(0) = 0$ and $F'(0) = f(0) = 0$. The Maclaurin series for $F(x)$ (including the error term) is

$$F(x) = F(0) + F'(0)x + F''(c)\frac{x^2}{2} = f'(c)\frac{x^2}{2},$$

for some real c such that $c \in (0, x)$. Now, since f is convex, $f'(c) \le f'(x)$. The conclusion follows.

b) Note that equality holds iff $f'(c) = f'(x)$. Since $f'(x)$ is continuous and monotonic because f is differentiable and convex, the subset $C \subset [0, x]$ such that $f'(c) = f(x)$ has a minimum. Assume that this minimum is $m > 0$, and since equality holds, a $c \in (0, m)$ exists such that $f'(c) = f'(m) = f'(x)$, contradiction. Hence $m = 0$, and $f'(c)$ is constant for all $c \in [0, x]$, for all $x \in \mathbb{R}^+$. It follows that $f(x) = ax + b$ for appropriately chosen reals a, b. But $f(0) = 0$, hence $f(x) = ax$ for any real a, with $f'(x) = a$, and for all such functions, equality clearly holds in the proposed inequality.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain; Albert Stadler, Switzerland; Arkady Alt, San Jose, California, USA; Johan Gunardi, Indonesia; Moubinool Omarjee , Paris, France; G.R.A.20 Problem Solving Group, Roma, Italy; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland

U220. Evaluate

$$\lim_{n\to\infty}\left((n+1)\sqrt[n+1]{\Gamma\left(\frac{1}{n+1}\right)}-n\sqrt[n]{\Gamma\left(\frac{1}{n}\right)}\right),$$

where Γ is the classical gamma function.

Proposed by Cezar Lupu, University of Pittsburgh, USA and Moubinool Omarjee, Lycee Jean Murcat, Paris, France

Solution by Matteo Elia, Università di Roma "Tor Vergata", Roma, Italy.

First, note that for $|z|<1$ we have

$$\begin{aligned} z\Gamma(z) &= e^{-\gamma z}\prod_{n=1}^{\infty}\frac{e^{z/n}}{1+z/n} \\ &= \left(1-\gamma z+\frac{\gamma^2 z^2}{2}+o(z^2)\right)\prod_{n=1}^{\infty}\left(1+\frac{z}{n}+\frac{z^2}{2n^2}+o(z^2)\right)\left(1-\frac{z}{n}+\frac{z^2}{n^2}+o(z^2)\right) \\ &= 1-\gamma z+\frac{\gamma^2 z^2}{2}+\frac{z^2}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}+o(z^2)=1-\gamma z+\left(\frac{\gamma^2}{2}+\frac{\pi^2}{12}\right)z^2+o(z^2). \end{aligned}$$

Hence

$$\begin{aligned} n\Gamma(1/n)^{1/n} &= n^{1+1/n}\left(1-\frac{\gamma}{n}+O(1/n^2)\right)^{1/n} \\ &= n\left(1+\frac{\log(n)}{n}+O((\log(n)/n)^2)\right)(1+O(1/n^2)) \\ &= n+\log(n)+o(1). \end{aligned}$$

Therefore

$$(n+1)\sqrt[n+1]{\Gamma(1/(n+1))}-n\sqrt[n]{\Gamma(1/n)}=n+1+\log(n+1)-n-\log(n)+o(1)\to 1.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Albert Stadler, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Anastasios Kotronis, Athens, Greece; Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy

U221. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous periodic function such that $f(2012)\neq 0$. Prove that there exists $c>0$ such that for all $n\geq 2012$ we have

$$\sum_{k=1}^{n}\frac{|f(k)|}{k}>c\cdot\ln n.$$

Proposed by Gabriel Dospinescu, Ecole Polytechnique, Paris, France

Solution by Perfetti Paolo, Università degli studi di Tor Vergata Roma, Italy

Since the function is continuous, it has a minimum period unless f is constant. Let's suppose indeed that it does not have a minimum period. This means that for any $\varepsilon>0$, there exists $0<T\leq\varepsilon$ such that $f(x+T)=f(x)$ for any x. This implies that $f(x)=2012$ for a dense set of values. The continuity implies $f(x)\equiv 2012\doteq f_0$. In such a case we would have

$$\sum_{k=1}^{n}\frac{|f(k)|}{k}=\sum_{k=1}^{n}\frac{|f_0|}{k}=|f_0|(\ln n+\gamma+o(1))>c\ln n$$

as soon as $c<|f_0|$. We have used the well known $\sum_{k=1}^{n}\frac{1}{k}=\ln n+\gamma+o(1)$ and γ is clearly the Euler–Mascheroni constant

Let f be not constant. If $n=2012$ we have

$$\sum_{k=1}^{n}\frac{|f(k)|}{k}\geq\frac{f_0}{2012}>c'\ln(2012)$$

and we may take $c'=\frac{f_0}{2012\ln(2012)}$.

Suppose that f is not constant and has rational period $T=\frac{N}{M}$ where $T,M\in\mathbb{N}$. By definition $f(2012+pMT)=f(2012)=f_0$ for any $p\in\mathbb{Z}$ and let A_n be the set of values $2012+pMT$ between 1 and n with $n\geq 2012$ and $p\geq 0$. The set $\{1,\dots,n\}\backslash A$ is called B. The condition $2012+pMT\leq n$ gives $p\leq\left[\frac{n-2102}{N}\right]\doteq p_0$ and we may suppose $p_0\geq c_1n/N$ with c_1 depending just on N. We have

$$\sum_{k=1}^{n}\frac{|f(k)|}{k}\geq\sum_{k=1,\ k\in A_n}^{n}\frac{|f(k)|}{k}=\sum_{p=1}^{p_0}\frac{|f_0|}{2012+pN}>|f_0|\int_0^{p_0}\frac{dx}{2012+xN}=$$
$$\frac{|f_0|}{N}\ln\left(1+\frac{p_0N}{2012}\right)\geq\frac{|f_0|}{N}\ln(c_1n)$$

and we want

$$\frac{|f_0|}{N}\ln(\frac{c_1n}{2012})\geq c''\ln n\ \forall\ n>2012\iff c''=\inf_{n>2012}\frac{\frac{|f_0|}{N}(\ln c_1+\ln n-\ln(2012))}{\ln n}.$$

The constant c is equal to $c=\min\{c',c''\}$.

Suppose that f has irrational period $\alpha>0$ and $|f(k)|$ is also α–periodic. By Weyl theorem we know that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|f(k)|=\int_x^{x+\alpha}|f(x)|dx\doteq I,\quad\forall\ x$$

and $I>0$ since the $|f(k)|\geq 0$ and $f(2012)\neq 0$. The continuity of f implies it is different from zero in a neighborhood of 2012 and this in turn implies the integral is different from zero.

Now we employ "Abel–summation–by–parts"

$$\sum_{k=1}^{n}\frac{|f(k)|}{k}=\frac{1}{n}\sum_{k=1}^{n}|f(k)|+\sum_{k=1}^{n-1}(\frac{1}{k}-\frac{1}{k+1})\sum_{j=1}^{k}|f(j)|=$$
$$I+o(1)+\sum_{k=1}^{n-1}\frac{1}{k(k+1)}k(I+o(1)).$$

Moreover for any $n>n_0$ we have

$$I+o(1)+\sum_{k=1}^{n-1}\frac{1}{k(k+1)}k(I+o(1))\geq\frac{I}{2}+\frac{I}{2}\sum_{k=1}^{n-1}\frac{1}{k+1}\geq\frac{I}{2}(1+\ln n)\geq c_1\ln n$$

for a suitable c_1. We have used $\sum_{k=1}^{n-1}\frac{1}{k+1}\geq\int_1^{n-1}\frac{dx}{x}=\ln n$. If $n_0\leq 2012$ we are done and c_1 is the constant c required in the statement. If $n_0>2012$, besides c_1 we consider the constants c_i $i=2102,\ldots,n_0$ defined by

$$c_i\doteq\frac{1}{\ln i}\sum_{k=1}^{i}\frac{|f(k)|}{k}$$

and then we take $c_2\doteq\min_{2012\leq i\leq n_0}\{c_i\}$. Finally, constant c is the minimum between c_1 and c_2.

Also solved by Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Albert Stadler, Switzerland

U222. Let p and q be distinct odd primes and let d be a divisor of $q-1$. Prove that

a) $\mathbb{Q}[\zeta_q]$ has a unique subfield K_d that has degree d over $\mathbb{Q}$, where ζ_q denotes a primitive q-th root of unity.

b) p splits completely in K_d if and only if q splits completely in $\mathbb{Q}\left[\sqrt[d]{p}\right]$.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by the author

(a) This is immediate from the Fundamental Theorem of Galois Theory. Indeed, we know that the Galois group G of $\mathbb{Q}[\zeta_q]$ over $\mathbb{Q}$ is cyclic of order $q-1$; hence there is an unique subgroup of G with index d for every divisor d of $q-1$; hence, via the Galois correspondence, we get an unique subfield $\mathbb{F}_d \subset \mathbb{Q}[\zeta_q]$ having degree d over $\mathbb{Q}$.

(b) This is in turn is a combination of two classical results from algebraic number theory. First, note that q splits completely in $\mathbb{Q}\left[\sqrt[d]{p}\right]$ if and only if the Eisenstein polynomial x^d-p factors completely modulo q, i.e. p is a cube modulo q. This is indeed true according to the following Dedekind-Kummer theorem that gives a method of factorizing ideals of $\mathcal{O}_F$ - ring of integers (given an extension of number fields K/F). More precisely:

Theorem 1. Let K/F be an extension of number fields and suppose that $\mathcal{O}_K = \mathcal{O}_F[\alpha]$. Let $f(x)$ be the irreducible polynomial of α over$F$, and let p be a prime ideal in $\mathcal{O}_F$. Set $\mathbb{F}_p = \mathcal{O}_F/p$ and denote the image of f in $\mathbb{F}_p[x]$ by $\overline{f}$. Suppose that $\overline{f}$ is given in $\mathbb{F}_p[x]$ by

$$\overline{f(x)} = \overline{p_1(x)}^{e_1} \cdots \overline{p_g(x)}^{e_g},$$

where the $\overline{p_j}$'s are distinct monic irreducible polynomials in $\mathbb{F}_p$. For each j, let $p_j(x)$ be a monic lift of the corresponding $\overline{p_j(x)}$ to $\mathcal{O}_F[x]$ and let β_j be the ideal of $\mathcal{O}_K$ generated by p and $p_j(\alpha)$. Then, we have that

$$p\mathcal{O}_k = \beta_1^{e_1} \ldots \beta_g^{e_g}.$$

For a proof and more details about the terminology and where it comes from, see Nancy Childress, *Class Field Theory*, Springer.

Returning to the problem, we just need to show now that p splits completely in K_d if and only if p is a cube modulo q. This is way easier and follows immediately from a group theoretic argument which we leave for the reader.

See also Theorem 30 in Daniel Marcus, *Number Fields*, Springer, pp. 106.

O217. Equilateral triangles ACB' and BDC' are drawn on the diagonals of a convex quadrilateral $ABCD$ so that B and B' are on the same side of AC and C and C' are on the same side of BD. Find $\angle BAD + \angle CDA$ if $B'C' = AB + CD$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

First solution by Li Zhou, Polk State College, USA

We use complex numbers. First, set $A = 0$, $B = z$, $C = w$, and $D = 1$ (with $\mathrm{Im}(z)$, $\mathrm{Im}(w) > 0$). Then $B' = we^{i\pi/3}$ and $C' = 1 + (z-1)e^{-i\pi/3} = e^{i\pi/3} + ze^{-i\pi/3}$. The given condition $AB + CD = B'C'$ becomes

$$|z| + |w-1| = \left|(w-1)e^{i\pi/3} - ze^{-i\pi/3}\right|,$$

which is true if and only if $(w-1)e^{i\pi/3}$ and $ze^{-i\pi/3}$ have opposite directions. Therefore, $\angle BAD + \angle CDA = 2\pi/3$.

Second solution by Li Zhou, Polk State College, USA

Construct the point E so that triangle BCE is equilateral and E and AD are on opposite sides of BC. Then $B'EC$ is a rotation of ABC and thus the two triangles are congruent. Likewise, the triangles $C'EB$ and DCB are congruent, so it follows that $B'C' = AB + CD$ implies that $B'C' = B'E + EC'$, i.e. B', E, C' are collinear. Therefore,

$$\angle ABC + \angle BCD = \angle B'EC + \angle BEC' = \pi + \frac{\pi}{3},$$

and hence $\angle BAD + \angle CDA = 2\pi/3$.

Also solved by Preudtanan Sriwongleang, Ramkamhaeng University, Bangkok, Thailand

O218. Find all integers n such that $2^n + 3^n + 13^3 - 14^n$ is the cube of an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that n cannot be negative. Indeed, if $n = -m$ is a negative integer, then $3^m \left(2^n + 3^n + 13^3 - 14^n\right)$ is the sum of terms which are multiples of 3, except one of them which equals 1, hence the sum is not a multiple of 3, or $2^n + 3^n + 13^3 - 14^n$ is not an integer. For $n = 0$, we have $13^3 < 2^n + 3^n + 13^3 - 14^n = 13^3 + 1 < 14^3$, clearly not the cube of an integer. For $n = 1, 2$, we have $13^3 > 2^n + 3^n + 13^3 - 14^n > 13^3 - 14^2 = 2001 > 1728 = 12^3$, again clearly not the cube of an integer. Therefore no solution exists for $n \leq 2$.

For $n = 3$ we have $2^n + 3^n + 13^3 - 14^n = -512 = (-8)^3$, or $n = 3$ is a solution. No other solution exists for $n = 3m$ a multiple of 3, since

$$14^{3m} > 14^{3m} - 13^3 - 3^{3m} - 2^{3m} > (14^m - 1)^3,$$

because clearly $14^{2m} = 196^m > 27^m + 8^m, 3 \cdot 14^m$ for all $m \geq 1$, while for all $m \geq 2$, $14^{2m} \geq 14^4 > 13^3$.

Assume that $n \geq 4$ is a solution, where 3 does not divide n. Note first that $2^n - 14^n$ is a multiple of 8, and $3^n + 13^3$ is even, or it must be a multiple of 8. But $13^3 = 2197 \equiv 5 \pmod 8$, or $3^n \equiv 3 \pmod 8$, and n must be odd. Therefore, $n \equiv \pm 1 \pmod 6$, and we study two cases:

- If $n \equiv 1 \pmod 6$ with $n \geq 4$, note that $13^3 = 2197 \equiv -1 \pmod 7$, while $2^6 \equiv 3^6 \equiv 1 \pmod 7$, or $2^n + 3^n + 13^3 - 14^n \equiv 4 \pmod 7$. But the remainders modulo 7 of all perfect cubes must be 0, 1 or 6 (easily checked), contradiction.
- If $n \equiv 5 \pmod 6$, note that $13^3 = 2197 \equiv 1 \pmod 9$, while $2^6 \equiv 14^6 \equiv 1 \pmod 9$, or $2^n + 3^n + 13^3 - 14^n \equiv 32 + 1 - 14^5 \equiv 4 \pmod 9$. But the remainders modulo 9 of all perfect cubes must be 0, 1 or 8 (easily checked), contradiction.

Therefore, there is only one solution, $n = 3$.

Also solved by Albert Stadler, Switzerland

O219. Let a, b, c, d be positive real numbers that satisfy

$$\frac{1-c}{a}+\frac{1-d}{b}+\frac{1-a}{c}+\frac{1-b}{d} \geq 0.$$

Prove that $a(1-b)+b(1-c)+c(1-d)+d(1-a) \geq 0$.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, Paris, France

First solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

First, note that proving $\sum_{\text{cyc}} a(1-b) \geq 0$ is equivalent to showing that

$$\sum_{\text{cyc}} a(1-b) \cdot \sum_{\text{cyc}} \frac{1-c}{a} \geq 0,$$

which in turn is equivalent to

$$\sum_{\text{cyc}} acd(1-d) \geq 0 \quad \text{i.e.} \quad abcd \sum_{\text{cyc}} \frac{1-d}{b} \geq 0,$$

and we are done.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note that the proposed inequality may be arranged into

$$(a+c)+(b+d) \geq ab+bc+cd+da=(a+c)(b+d),$$

or equivalently,

$$\frac{1}{a+c}+\frac{1}{b+d} \geq 1.$$

Now, if either $a+c$ or $b+d$ are no larger than 1, the result is trivially true, with strict inequality. Otherwise, note that we may arrange the given condition into

$$0 \leq \frac{a+c-a^2-c^2}{ac}+\frac{b+d-b^2-d^2}{bd}=\frac{(a+c)(1-a-c)}{ac}+\frac{(b+d)(1-b-d)}{bd}+4,$$

or equivalently,

$$\frac{(a+c)(a+c-1)}{4ac}+\frac{(b+d)(b+d-1)}{4bd} \leq 1,$$

Now, by the AM-GM inequality, $4ac \leq (a+c)^2$, or

$$1 \geq \frac{a+c-1}{a+c}+\frac{b+d-1}{b+d}=2-\frac{1}{a+c}-\frac{1}{b+d},$$

equivalent to the proposed inequality. Note that equality holds iff $a=c$ and $b=d$, but no additional condition is necessary, since both $a=c$ and $b=d$ are necessary, and when they hold, the condition given in the problem statement rewrites as $\frac{1}{a}+\frac{1}{b} \geq 2$, while the proposed inequality becomes $a+b \geq 2ab$, both being clearly equivalent.

O220. Let $A_1, \ldots, A_n$ be distinct points in the plane and let G be their center of gravity. Consider a point F in plane for which $A_1F + \cdots + A_nF$ is minimal. Prove that

$$\sum_{i=1}^{n} A_iG \leq 2\left(1 - \frac{1}{n}\right)\sum_{i=1}^{n} A_iF.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by the author

We need the following two lemmas:

Lemma 1. Let ABC be a triangle and let M be a point inside it. Then $AB + AC > MB + MC$.

Proof of Lemma 1. Denote by N the intersection of BM with AC. By triangle inequality we have $AB+AN > BN = MB+MN$ and $MN+NC > MC$. Thus summing them up we get $AB+AC > MB+MC$.

Lemma 2. Let $A_1, \ldots, A_n$ be points in the plane and let G be their center of gravity. Then for any point P we have $A_1P + \cdots + A_nP \geq nGP$.

Proof of Lemma 2. The distance $d(X)$ from point P to any point X in the plane is a convex function. Let $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ be two points in the plane and let $M = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ be their midpoint. Then clearly $d(X)+d(Y) \geq 2d(M)$, for any X and Y, yielding $d(X)$ must be convex. Consider $A_i = (x_i, y_i)$, $i \in [1, n]$, then $G = (\frac{x_1+\cdots+x_n}{n}, \frac{y_1+\cdots+y_n}{n})$. By Jensen inequality we have

$$\frac{d(A_1) + \cdots + d(A_n)}{n} \geq d(G),$$

as desired.

Returning to initial problem, observe that F and G lie inside the convex hull of points $A_1, \ldots, A_n$. Then there exist two vertices, say A_{n-1} and A_n, such that G lies inside the triangle $FA_{n-1}A_n$. Using Lemma 1, we have $A_{n-1}G + A_nG \leq A_{n-1}F + A_nF$. For the remaining set of indices, $i \in [1, n-2]$, by the triangle inequality we have $A_iG \leq A_iF + FG$. Summing them up, we get

$$A_1G + \cdots + A_nG \leq A_1F + \cdots + A_nF + (n-2)FG.$$

From Lemma 2 we know that $A_1F + \cdots + A_nF \geq nFG$ and we conclude that

$$A_1G + \cdots + A_nG \leq 2\left(1 - \frac{1}{n}\right)(A_1F + \cdots + A_nF).$$

O221. Suppose that an ant starts at a vertex of complete graph K_4 and moves on edges with probability $\frac{1}{3}$. Determine the probability that the ant returns to the original vertex within n moves.

Proposed by Antonio Blanca Pimentel, UC Berkeley and Roberto Bosch Cabrera, Florida, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

After the first move, we consider a Markov chain with two states A and B. State A means that the ant is in the original vertex. State B means that the ant is in a vertex different from the original one. The transition probabilities are: $p_{(A,A)} = 1$ and $p_{(A,B)} = 0$ (state A is absorbing), whereas $p_{(B,A)} = 1/3$ and $p_{(B,B)} = 2/3$. So the probability that the ant returns to the original vertex within n moves is equal to 1 minus the probability that the system stays in the state B from the second step till the n-th one. Hence the required probability is:

$$1 - p_{(B,B)}^{n-1} = 1 - \left(\frac{2}{3}\right)^{n-1}.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Albert Stadler, Switzerland

O222. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be nonnegative reals, where n is a positive integer. Let σ be a permutation of $\{1, 2, \dots, n\}$. For every $k \in \{1, 2, \dots, n\}$, let

$$c_k = \max\left(\{a_1b_k, a_2b_k, \dots, a_kb_k\} \cup \{a_kb_1, a_kb_2, \dots, a_kb_k\}\right).$$

Prove that

$$a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + \cdots + a_nb_{\sigma(n)} \le c_1 + c_2 + \cdots + c_n.$$

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Claim: For any given set of values for $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$, the sum $c_1 + c_2 + \cdots + c_n$ is minimum when $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$.

Proof: Assume that $a_k > a_{k+1}$, and denote

$$A_k = \{a_kb_1, a_kb_2, \dots, a_kb_{k-1}\}, \qquad A_{k+1} = \{a_{k+1}b_1, a_{k+1}b_2, \dots, a_{k+1}b_{k-1}\},$$

$$B_k = \{a_1b_k, a_2b_k, \dots, a_{k-1}b_k\}, \qquad B_{k+1} = \{a_1b_{k+1}, a_2b_{k+1}, \dots, a_{k-1}b_{k+1}\},$$

$$c_k = \max(A_k \cup B_k \cup \{a_kb_k\}), \qquad c_{k+1} = \max(A_{k+1} \cup B_{k+1} \cup \{a_kb_{k+1}, a_{k+1}b_k, a_{k+1}b_{k+1}\}).$$

Consider now two cases:

Case 1: If $b_k \le b_{k+1}$, we exchange only a_k and a_{k+1}, hence the new values d_k, d_{k+1} taken respectively by c_k, c_{k+1} are $d_k = \max(A_{k+1} \cup B_k \cup \{a_{k+1}b_k\})$ and $d_{k+1} = \max(A_k \cup B_{k+1} \cup \{a_kb_k, a_kb_{k+1}, a_{k+1}b_{k+1}\})$. Note that $\max(A_{k+1} \cup \{a_{k+1}b_k\}) \le \max(A_k \cup \{a_kb_k\})$, with equality iff $b_1 = b_2 = \cdots = b_n = 0$, and $\max(B_{k+1} \cup \{a_kb_{k+1}\}) \ge \max(B_k \cup \{a_kb_k\})$, with equality iff $b_k = b_{k+1}$ because $a_k > a_{k+1} \ge 0$. Therefore, $c_{k+1} \ge d_k$. Moreover, $c_k \ge d_{k+1}$, unless $c_k \in (B_k \cup \{a_kb_k\}) > \max(A_k) \ge \max(A_{k+1})$, in which case $d_k \le c_k$, and moreover $c_{k+1} = \max(B_{k+1} \cup \{a_kb_{k+1}, a_{k+1}b_k\})$ and $d_{k+1} = \max(B_{k+1} \cup \{a_kb_{k+1}\})$, for $d_{k+1} \le c_{k+1}$.

Case 2: If $b_k > b_{k+1}$, we exchange a_k and a_{k+1}, and also b_k and b_{k+1}, hence the new values d_k, d_{k+1} taken respectively by c_k, c_{k+1} are $d_k = \max(A_{k+1} \cup B_{k+1})$ and $d_{k+1} = \max(A_k \cup B_{k+1} \cup \{a_kb_{k+1}, a_{k+1}b_{k+1}\})$. Analogously as in Case 1, we find that indeed $c_k + c_{k+1} \ge d_k + d_{k+1}$.

Assume that a_1 is not the minimum of $a_1, a_2, \dots, a_n$. After at most $n-1$ exchanges as described above, the minimum of the a_i can be moved to a_1, and the value of the sum $c_1 + c_2 + \cdots + c_n$ is not larger than at the starting point. By symmetry, proceed similarly with the minimum of $b_1, b_2, \dots, b_n$. Now, assume that a_2 is not the second smallest of $a_1, a_2, \dots, a_n$, and so on. It follows that we may order the a_i and b_i non-decreasingly by performing exchanges as described above, while the sum $c_1 + c_2 + \cdots + c_n$ never increases. The Lemma follows.

Rename the a_i's and b_i's so that they are ordered non-decreasingly, or the original RHS is at least equal to the RHS after the reordering. Since the actual values of the a_i's and b_i's remain unchanged, and only their ordering is altered, the LHS is the same, it just suffices to pick a different permutation $\sigma(i)$. Therefore, it suffices to show that the inequality holds when the a_i's and b_i's are ordered non-decreasingly, in which case clearly $c_k = a_kb_k$ for all $k = 1, 2, \dots, n$, and it suffices to show that, for any permutation $\sigma(i)$ of $\{1, 2, \dots, n\}$, the following inequality holds:

$$a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + \cdots + a_nb_{\sigma(n)} \le a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

But since the a_i's and b_i's are ordered non-decreasingly, this is clearly true because of the reordering inequality. The conclusion follows.

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