Junior problems

J271. Find all positive integers n with the following property: if a, b, c are integers such that n divides $ab + bc + ca + 1$, then n divides $abc(a + b + c + abc)$.

Proposed by Titu Andreescu, University of Texas at Dallas and Gabriel Dospinescu, Ecole Normale Suprieure, Lyon

J272. Let ABC be a triangle with centroid G and circumcenter O. Prove that if BC is its greatest side, then G lies in the interior of the circle of diameter AO.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J273. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\frac{a^3+2}{b^2-b+1}+\frac{b^3+2}{c^2-c+1}+\frac{c^3+2}{a^2-a+1}\geq 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J274. Let p be a prime and let k be a nonnegative integer. Find all positive integer solutions (x, y, z) to the equation

$$x^k(y-z)+y^k(z-x)+z^k(x-y)=p.$$

Proposed by Alessandro Ventullo, Milan, Italy

J275. Let $ABCD$ be a rectangle and let point P lie on side AB. The circle through A, B, and the orthogonal projection of E onto CD intersects AD and BC at X and Y. Prove that XY passes through the orthocenter of triangle CDE.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J276. Find all positive integers m and n such that

$$10^n-6^m=4n^2.$$

Proposed by Tigran Akopyan, Vanadzor, Armenia

Senior problems

S271. Determine if there is an $n \times n$ square with all entries cubes of pairwise distinct positive integers such that the product of entries on each of the n rows, n columns, and two diagonals is 2013^{2013}.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S272. Let $A_1, A_2, \ldots, A_{2n}$ be a polygon inscribed in a circle $C(O, R)$. Diagonals A_1A_{n+1}, $A_2A_{n+2}, \ldots, A_nA_{2n}$ intersect at point P. Let G be the centroid of the polygon. Prove that $\angle OMG$ is acute.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S273. Let a, b, c be positive integers such that $a \geq b \geq c$ and $\frac{a-c}{2}$ is a prime. Prove that if

$$a^2 + b^2 + c^2 - 2(ab + bc + ca) = b,$$

then b is either a prime or a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S274. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{ca+1} + \frac{b}{ab+1} + \frac{c}{bc+1} \leq \frac{1}{2}(a^2 + b^2 + c^2)$$

Proposed by Sayan Das, Kolkata, India

S275. Let ABC be a triangle with incircle $\mathcal{I}$ and incenter I. Let A', B', C' be the intersections of $\mathcal{I}$ with the segments AI, BI, CI, respectively. Prove that

$$\frac{AB}{A'B'} + \frac{BC}{B'C'} + \frac{CA}{C'A'} \geq 12 - 4\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right)$$

Proposed by Marius Stanean, Zalau, Romania

S276. Let a, b, c be real numbers such that

$$\frac{2}{a^2+1} + \frac{2}{b^2+1} + \frac{2}{c^2+1} \geq 3.$$

Prove that $(a-2)^2 + (b-2)^2 + (c-2)^2 \geq 3$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate problems

U271. Let $a > b$ be positive real numbers and let n be a positive integer. Prove that

$$\frac{(a^{n+1}-b^{n+1})^{n-1}}{(a^n-b^n)^n} > \frac{n}{(n+1)^2}\cdot\frac{e}{a-b},$$

where e is the Euler number.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U272. Let a be a positive real number and let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = \sqrt{a}$, $a_{n+1} = \sqrt{a_n + a}$, for all positive integers n. Prove that there are infinitely many irrational numbers among the terms of the sequence.

Proposed by Marius Cavachi, Constanţa, Romania

U273. Let Φ_n be the nth cyclotomic polynomial, defined by

$$\Phi_n(X) = \prod_{1\leq m\leq n, \gcd(m,n)=1} (X - e^{\frac{2i\pi m}{n}}).$$

a) Let k and n be positive integers with k even and $n > 1$. Prove that

$$\pi^{k\varphi(n)}\cdot\prod_p \Phi_n\left(\frac{1}{p^k}\right) \in \mathbf{Q},$$

where the product is taken over all primes and φ is the Euler totient function.

b) Prove that

$$\prod_p\left(1-\frac{1}{p^2}+\frac{1}{p^6}-\frac{1}{p^8}+\frac{1}{p^{10}}-\frac{1}{p^{14}}+\frac{1}{p^{16}}\right) = \frac{192090682746473135625}{3446336510402\pi^{16}}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

U274. Let $A_1, \ldots, A_m \in M_n(\mathbf{C})$ satisfying $A_1 + \ldots + A_m = mI_n$ and $A_1^2 = \ldots = A_m^2 = I_n$. Prove that $A_1 = \ldots = A_m$.

Proposed by Marius Cavachi, Constanţa, Romania

U275. Let m and n be positive integers and let $(a_k)_{k\geq 1}$ be real numbers. Prove that

$$\sum_{d|m,\ e|n,\ g|\gcd(d,e)} \frac{\mu(g)}{g} de\cdot a_{de/g} = \sum_{k|mn} ka_k.$$

Here, μ is the usual Möbius function.

Proposed by Darij Grinberg, Massachusetts Institute of Technology, USA

U276. Let K be a finite field. Find all polynomials $f\in K[X]$ such that $f(X) = f(aX)$ for all $a \in K^*$.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Olympiad problems

O271. Let $(a_n)_{n\geq 0}$ be the sequence given by $a_0 = 0$, $a_1 = 2$ and $a_{n+2} = 6a_{n+1} - a_n$ for $n \geq 0$. Let $f(n)$ be the highest power of 2 that divides n. Prove that $f(a_n) = f(2n)$ for all $n \geq 0$.

Proposed by Albert Stadler, Herrliberg, Switzerland

O272. Let ABC be an acute triangle with orthocenter H and let X be a point in its plane. Let X_a, X_b, X_c be the reflections of X across AH, BH, CH, respectively. Prove that the circumcenters of triangles AHX_a, BXH_b, CXH_c are collinear.

Proposed by Michal Rolinek, Institute of Science and Technology, Vienna and Josef Tkadlec, Charles University, Prague

O273. Let P be a polygon with perimeter L. For a point X, denote by $f(X)$ the sum of the distances to the vertices of P. Prove that for any point X in the interior of P, $f(X) < \frac{n-1}{2}L$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O274. Let a, b, c be positive integers such that a and b are relatively prime. Find the number of lattice points in

$$D = \{(x, y) \mid \; x, y \geq 0, \; bx + ay \leq abc\}.$$

Proposed by Arkady Alt, San Jose, California, USA

O275. Let ABC be a triangle with circumcircle $\Gamma(O)$ and let ℓ be a line in the plane which intersects the lines BC, CA, AB at X, Y, Z, respectively. Let ℓ_A, ℓ_B, ℓ_C be the reflections of ℓ across BC, CA, AB, respectively. Furthermore, let M be the Miquel point of triangle ABC with respect to line ℓ.

a) Prove that lines ℓ_A, ℓ_B, ℓ_C determine a triangle whose incenter lies on the circumcircle of triangle ABC.

b) If S is the incenter from (a) and O_a, O_b, O_c denote the circumcenters of triangles AYZ, BZX, CXY, respectively, prove that the circumcircles of triangles SOO_a, SOO_b, SOO_c are concurrent at a second point, which lies on Γ.

Proposed by Cosmin Pohoata, Princeton University, USA

O276. For a prime p, let $S_1(p) = \{(a, b, c) \in \mathbf{Z}^3, \quad p | a^2b^2 + b^2c^2 + c^2a^2 + 1\}$ and $S_2(p) = \{(a, b, c) \in \mathbf{Z}^3, \quad p | a^2b^2c^2(a^2 + b^2 + c^2 + a^2b^2c^2)\}$. Find all p for which $S_1(p) \subset S_2(p)$.

Proposed by Titu Andreescu, University of Texas at Dallas and Gabriel Dospinescu, Ecole Normale Suprieure, Lyon