Mathematical Excalibur

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Olympiad Corner

Following are the problems of 2004 Estonian IMO team selection contest.

Problem 1. Let $k > 1$ be a fixed natural number. Find all polynomials $P(x)$ satisfying the condition $P(x^k) = (P(x))^k$ for all real number x.

Problem 2. Let O be the circumcentre of the acute triangle ABC and let lines AO and BC intersect at a point K. On sides AB and AC, points L and M are chosen such that $KL = KB$ and $KM = KC$. Prove that segments LM and BC are parallel.

Problem 3. For which natural number n is it possible to draw n line segments between vertices of a regular $2n$-gon so that every vertex is an endpoint for exactly one segment and these segments have pairwise different lengths?

Problem 4. Denote

$$f(m)=\sum_{k=1}^{m}(-1)^k\cos\frac{k\pi}{2m+1}.$$

For which positive integers m is $f(m)$ rational?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 7, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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例析數學競賽中的計數問題(二)

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3　運用算兩次原理與抽屜原理

算兩次原理，就是把一個量從兩個（或更多）方面去考慮它，然後綜合起來得到一個關係式（可以是等式或不等式），或者導出一個矛盾的結論．具體表示爲三步：“一方面（利用一部分條件）……，另一方面（利用另一部分條件）……，綜合這兩個方面 ……”．義大利數學家富比尼（Fubini）首先應用這個思想方法，因此今天我們也稱它爲富比尼原理．

在解這些問題時，要根據問題的特點選擇一個適當的量，再將這個量用兩（或幾）種不同的方法表達出來．

抽屜原理，德國數學家狄利克雷（Dirichlet）提出．對於這個原理的具體解釋，想必很多同學早就知道了，在此不再贅述．

<u>例 6</u>　有 26 個不同國家的集郵愛好者，想通過互相通信的方法交換各國最新發行的紀念郵票，爲了使這 26 人每人都擁有這 26 個國家的一套最新紀念郵票，他們至少要通多少封信？

<u>解答</u>　不妨設這 26 個集郵愛好者中的某一個人爲組長．

一方面，對於組長，要接收到其他 25 個國家的最新紀念郵票，必須從這 25 個集郵愛好者的手中發出（不管他們是否直接發給組長），至少要通 25 封信；同樣地，其他 25 個集郵愛好者分別要接收到組長的一套紀念郵票，必須由組長發出（不管組長是否直接發給這 25 個集郵愛好者），至少要通 25 封信．總計至少要通 50 封信．

另一方面，其餘 25 個集郵愛好者每人將本國的一套最新的紀念郵票 25 份或 26 份發給組長，計 25 封信；組長收到這 25 封信後，再分別給這 25 個集郵愛好者各發去一封信，每封信中含有 25 套郵票（發給某人的信中不含其本國的郵票）或 26 套郵票（發給某人的信中包含其本國的郵票），計 25 封信．總計 50 封信．這就是說通 50 封信可以使這 26 人每人都擁有這 26 個國家的一套最新紀念郵票．

因此他們至少要通 50 封信．

<u>例 7</u>　從 1，2，3，…，1997 這 1997 個數中至多能選出多少個數，使得選出的數中沒有一個是另一個的 19 倍？

<u>解答</u>　因爲 1997÷19=105…2，所以 106，107，…，1997 這 1892 個數中沒有一個是另一個的 19 倍．

又因 106÷19=5…11，故 1，2，3，4，5，106，107，…，1997 這 1897 個數中沒有一個是另一個的 19 倍．

另一方面，從（6，6×19），（7，7×19），…，（105，105×19）這 100 對互異的數中最多可選出 100 個數（每對中至多選 1 個），即滿足題意的數至少剔除 100 個數．

綜上所述，從 1，2，3，…，1997 中至多選出 1897 個數，使得選出的數中沒有一個是另一個的 19 倍．

<u>例 8</u>　在正整數 1，2，3，…，1995，1996，1997 裏，最多能選出多少數，使其中任意兩個數的和不能被這兩個數的差整除．

<u>解答</u>　在所選的數中，不能出現連續自然數、連續奇數或連續偶數，這是由於連續自然數之和必能被其差 1 整除；連續奇數或連續偶數之和是偶數，必能被其差 2 整除．再考慮差值爲 3 的兩數，不能是 3 的倍數，否則其和仍是 3 的倍數，必能被其差 3 整除；而選擇全是 3 除餘 1，或全是 3 除餘 2 的數，注意到各自中任意兩數之和非 3 的倍數，不能被其差 3 的倍

數整除，滿足題意．

另一方面，從(1,2,3)，(4,5,6)，…，(1993,1993,1995)，(1996,1997)中，最多可選出 666 個(每組至多可選一個)，否則會出現連續自然數、連續奇數或連續偶數，而不滿足題意．又間隔 4 的所有數的個數較上述滿足題意的所有數的個數少．

綜上可知，1，4，7，…，1990，1993，1996（666 個）或 2，5，8，…，1991，1994，1997（666 個）均滿足題意．

即最多可選出 666 個，使其中任意兩數之和不能被這兩數之差整除．

<u>例 9</u> 設自然數n有以下性質：從 1，2，…，n中任取 50 個不同的數，這 50 個數中必有兩個數之差等於 7，這樣的n最大的一個是多少？

<u>解答</u> n的最大值是 98．說明如下：

⑴一方面當自然數從 1，2，…，98 中任取 50 個不同的數，必有兩個數之差等於 7．這是因為：

首先將自然數 1，2，…，98 分成 7 組：（1，2，3，4，5，6，7，8，9，10，11，12，13，14），（15，16，17，18，19，20，21，22，23，24，25，26，27，28），…，（85，86，87，88，89，90，91，92，93，94，95，96，97，98）．

考慮取出的數中不出現某兩個數之差等於 7 的情形：由於每組中含有差為 7 的兩數，故每組最多可取出 7 個數（即每組中屬於 7 的同一個剩餘類的兩個數只能取其中的任意一個）．並且如果在第 1 組中取出了 m（m=1，2，…，14），那麼後面的每組分別取出 m+14n（n=1，2，…，6），可使所取數中的任意兩個數之差都不是 7．這樣從上述 7 組數中最多只能取出 7×7=49 個數．

根據抽屜原理，知從 1，2，…，98 中任取 50 個不同的數，必有兩個數之差等於 7．

⑵另一方面當自然數從 1，2，…，99 中任取 50 個不同的數，不能保證必有兩個數之差等於 7．這是因為：

首先將自然數 1，2，…，99 分成 8 組：（1，2，3，4，5，6，7，8，9，10，11，12，13，14），（15，16，17，18，19，20，21，22，23，24，25，26，27，28），…，（85，86，87，88，89，90，91，92，93，94，95，96，97，98），（99）．

比如，取出前 7 組中每組的前 7 個數，第 8 組的 99 這 50 個數，就不含有兩個數之差等於 7．

綜合⑴、⑵，可得 n 的最大值是 98．

<u>例 10</u> 某校組織了 20 次天文觀測活動，每次有 5 名學生參加，任何 2 名學生都至多同時參加過一次觀測．證明：至少有 21 名學生參加過這些觀測活動．

<u>證法 1</u> （反證法）假設至多有 20 名學生參加過這些觀測活動．

每次觀測活動中的 5 名學生中有 $C_5^2=\frac{5\times4}{2\times1}=10$ 個 2 人小組，又由題意知 20 次觀測中 2 人小組各不相同，所以 20 次觀測中 2 人小組總共有 $20\times10=200$ 個．

而另一方面，20 名學生中的 2 人小組最多有 $C_{20}^2=\frac{20\times19}{2\times1}=190$ 個．

兩者自相矛盾．故至少有 21 名學生參加過這些觀測活動．

稍作簡化，即可證明如下：

<u>證法 2</u> （反證法）假設至多有 20 名學生參加過這些觀測活動．

由題意知：⑴共有 20 次觀測；⑵最多有 $\frac{C_{20}^2}{C_5^2}=19$ 次觀測．

兩者自相矛盾．故至少有 21 名學生參加過這些觀測活動．

對於低年級學生，還可作出如下證明：

<u>證法 3</u> 設參加觀測活動次數最多的學生A參加了a次觀測，共有x名學生參加過天文觀測活動．

由於有 A 參加的每次觀測活動中，除了 A，其他學生各不相同（這是因為任何 2 名學生都至多同時參加過一次觀測），故 $x\geq 4a+1$ ．（I）

另一方面，學生 A 參加觀測的次數不小於每名學生平均觀測次數．即

$a\geq\frac{20\times5}{x}$ ．（II）

綜合（I）、（II），得 $x\geq\frac{400}{x}+1$ ，

$x^2-x-400\geq 0$ ．從而 $x\geq 21$ ．

即至少有 21 名學生參加過這些觀測活動．

<u>例 11</u> 2n名選手參加象棋循環賽，每一輪中每個選手與其他 $2n-1$ 人各賽一場，勝得 1 分，平各得 $\frac{1}{2}$ 分，負得 0 分．證明：如果每個選手第一輪總分與第二輪總分至少相差n分，那麼每個選手兩輪總分恰好相差n分．

<u>證明</u> 令集A={第二輪總分>第一輪總分的人}，集B={第二輪總分<第一輪總分的人}，並且|A|=k，|B|=h，k+h=2n．

不妨設 $k\geq n\geq h$．考慮 A 中選手第二輪總分之和 S（若 $h\geq n\geq k$，則考慮 B 中選手第一輪總分之和 T)．另一方面，對於每輪 A 中選手和 B 中選手的 kh 場比賽中，所得總分之和為 kh，充其量全為 A 中選手取勝，則 $S\leq C_k^2+kh$．如 A 中選手第一輪總分之和為 S'，那麼 $S-S'\geq kn$，$C_k^2+kh-kn\geq S-kn\geq S'\geq C_k^2$ ．從而得 $h\geq n$，所以 n=h=k，並且以上不等式均為等式．

所以 A 中每個選手第二輪總分恰比第一輪總分多 n 分，B 中每個選手第一輪總分恰比第二輪總分多 n 分．因此，原命題成立．

(to be continued)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 7, 2005.***

Problem 221. (*Due to Alfred Eckstein, Arad, Romania*) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1}+F_{n-2}$ for $n \ge 2$.

Prove that $7F_{n+2}^3 - F_n^3 - F_{n+1}^3$ is divisible by F_{n+3}.

Problem 222. All vertices of a convex quadrilateral $ABCD$ lie on a circle ω**.** The rays AD, BC intersect in point K and the rays AB, DC intersect in point L.

Prove that the circumcircle of triangle AKL is tangent to ω if and only if the circumcircle of triangle CKL is tangent to ω.
(*Source: 2001-2002 Estonian Math Olympiad, Final Round*)

Problem 223. Let $n \ge 3$ be an integer and x be a real number such that the numbers x, x^2 and x^n have the same fractional parts. Prove that x is an integer.

Problem 224. (*Due to Abderrahim Ouardini*) Let a, b, c be the sides of triangle ABC and I be the incenter of the triangle.

Prove that

$$IA \cdot IB \cdot IC \le \frac{abc}{3\sqrt{3}}$$

and determine when equality occurs.

Problem 225. A luminous point is in space. Is it possible to prevent its luminosity with a finite number of disjoint spheres of the same size?
(*Source: 2003-2004 Iranian Math Olympiad, Second Round*)

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Solutions

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Problem 216. (*Due to Alfred Eckstein, Arad, Romania*) Solve the equation

$$4x^6 - 6x^2 + 2\sqrt{2} = 0.$$

Solution. **Kwok Sze CHAI Charles** (HKU, Math Major, Year 1), **CHAN Tsz Lung**, **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **Badr SBAI** (Morocco), **TAM Yat Fung** (Valtorta College, Form 5), **WANG Wei Hua** and **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 6).

We have $8x^6 - 12x^2 + 4\sqrt{2} = 0$. Let $t = 2x^2$. We get

$$\begin{aligned} 0 &= t^3 - 6t + 4\sqrt{2} \\ &= t^3 - (\sqrt{2})^3 - (6t - 6\sqrt{2}) \\ &= (t - \sqrt{2})(t^2 + \sqrt{2}t - 4) \\ &= (t - \sqrt{2})(t - \sqrt{2})(t + 2\sqrt{2}). \end{aligned}$$

Solving $2x^2 = \sqrt{2}$ and $2x^2 = -2\sqrt{2}$, we get $x = \pm 1/\sqrt[4]{2}$ or $\pm i\sqrt[4]{2}$.

Other commended solvers: **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7), **Kin-Chit O** (STFA Cheng Yu Tung Secondary School) and **WONG Sze Wai** (True Light Girls' College, Form 4).

Problem 217. Prove that there exist infinitely many positive integers which cannot be represented in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11},$$

where x_1, x_2, x_3, x_4, x_5 are positive integers. (*Source:* 2002 *Belarussian Mathematical Olympiad, Final Round*)

Solution. **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **Tak Wai Alan WONG** (Markham, ON, Canada).

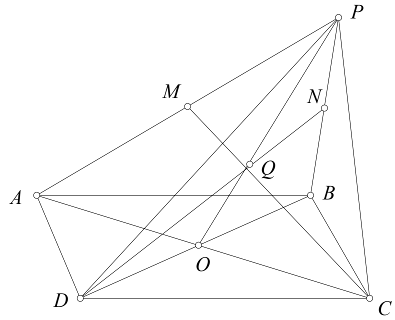
On the interval $[1, n]$, if there is such an integer, then

$$x_1 \le [n^{1/3}], x_2 \le [n^{1/5}], \cdots, x_5 \le [n^{1/11}].$$

So the number of integers in $[1, n]$ of the required form is at most $n^{1/3}n^{1/5}n^{1/7}n^{1/9}n^{1/11} = n^{3043/3465}$. Those not of the form is at least $n - n^{3043/3465}$, which goes to infinity as n goes to infinity.

Problem 218. Let O and P be distinct points on a plane. Let $ABCD$ be a parallelogram on the same plane such that its diagonals intersect at O. Suppose P is not on the reflection of line AB with respect to line CD. Let M and N be the midpoints of segments AP and BP respectively. Let Q be the intersection of lines MC and ND. Prove that P, Q, O are collinear and the point Q does not depend on the choice of parallelogram $ABCD$. (*Source: 2004 National Math Olympiad in Slovenia, First Round*)

Solution. **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).



Let G_1 be the intersection of OP and MC. Since OP and MC are medians of triangle APC, G_1 is the centroid of triangle APC. Hence OG_1=⅓OP. Similarly, let G_2 be the intersection of OP and ND. Since OP and ND are medians of triangle BPD, G_2 is the centroid of triangle BPD. Hence OG_2=⅓OP. So G_1=G_2 and it is on both MC and ND. Hence it is Q. This implies P, Q, O are collinear and Q is the unique point such that OQ=⅓OP, which does not depend on the choice of the parallelogram $ABCD$.

Other commended solvers: **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7) and **CHAN Tsz Lung**, **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6) and **WONG Tsun Yu** (St. Mark's School, Form 5).

Problem 219. (*Due to Dorin Mărghidanu, Coleg. Nat. "A.I. Cuza", Corabia, Romania)* The sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are defined as follows: $a_0, b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n}, \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for $n = 1,2,3,\ldots$. Prove that

$$\max\{a_{2004}, b_{2004}\} > \sqrt{2005}.$$

Solution. **CHAN Tsz Lung**, **Kin-Chit**

O (STFA Cheng Yu Tung Secondary School), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

We have

$$a_{n+1}b_{n+1} = (a_n + \frac{1}{2b_n})(b_n + \frac{1}{2a_n})$$
$$= a_n b_n + \frac{1}{4a_n b_n} + 1$$
$$= a_{n-1}b_{n-1} + \frac{1}{4a_{n-1}b_{n-1}} + \frac{1}{4a_n b_n} + 2$$
$$= \cdots$$
$$= a_0 b_0 + \sum_{i=0}^{n} \frac{1}{4a_i b_i} + n + 1.$$

Then

$$(\max\{a_{2004}, b_{2004}\})^2 \ge a_{2004} \cdot b_{2004}$$
$$> a_0 b_0 + \frac{1}{4a_0 b_0} + 2004$$
$$\ge 2\sqrt{a_0 b_0 \frac{1}{4a_0 b_0}} + 2004$$
$$= 2005$$

and the result follows.

Other commended solvers: **HUDREA Mihail** (High School “Tiberiu Popoviciu” Cluj-Napoca Romania).

Problem 220. (*Due to Cheng HAO, The Second High School Attached to Beijing Normal University*) For $i = 1, 2, \ldots, n$, and $k \ge 4$, let $A_i = (a_{i1}, a_{i2}, \ldots, a_{ik})$ with $a_{ij} = 0$ or 1 and every A_i has at least 3 of the k coordinates equal 1. Define the distance between A_i and A_j to be

$$\sum_{m=1}^{k} |a_{im} - a_{jm}|.$$

If the distance between any A_i and A_j ($i \ne j$) is greater than 2, then prove that

$$n \le 2^{k-3} - 1.$$

Solution.

Let $|A_i - A_j|$ denote the distance between A_i and A_j. We add $A_0 = (0,\ldots,0)$ to the n A_m’s. Then $|A_i - A_j| \ge 3$ still holds for $A_0, A_1, \ldots, A_n$.

Next we put the coordinates of A_0 to A_n into a $(n + 1) \times k$ table with the coordinates of A_i in the $(i + 1)$-st row.

Note if we take any of the k columns and switch all the 0’s to 1’s and 1’s to 0’s, then we get $n + 1$ new ordered k-tuples that still satisfy the condition $|A_i - A_j| \ge 3$. Thus, we may change A_0 to any combination with 0 or 1 coordinates. Then the problem is equivalent to showing $n + 1 \le 2^{k-3}$ for $n + 1$ sets satisfying $|A_i - A_j| \ge 3$, but removing the condition each A_i has at least 3 coordinates equal 1.

For $k = 4$, we have $n + 1 \le 2$. Next, suppose $k > 4$ and the inequality is true for the case $k-1$.

In column k of the table, there are at least $[(n + 2)/2]$ of the numbers which are the same (all 0’s or all 1’s). Next we keep only $[(n+2)/2]$ rows whose k-th coordinates are the same and we remove column k. The condition $|A_i - A_j| \ge 3$ still holds for these new ordered $(k-1)$-tuples. By the case $k - 1$, we get $[(n + 2)/2] + 1 \le 2^{k-4}$. Since $(n + 1)/2 < [(n + 2)/2] + 1$, we get $n + 1 \le 2^{k-3}$ and case k is true.

Generalization of Problem 203

Naoki Sato

We prove the following generalization of problem 203:

Let $a_1, a_2, \ldots, a_n$ be real numbers, and let s_i be the sum of the products of the a_i taken i at a time. If $s_1 \ne 0$, then the equation

$$s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ns_n = 0$$

has only real roots.

<u>*Proof*</u>. Let

$$f(x) = s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ns_n.$$

We can assume that none of the a_i are equal to 0, for if some of the a_i are equal to 0, then rearrange them so that $a_1, a_2, \ldots, a_k$ are nonzero and $a_{k+1}, a_{k+2}, \ldots, a_n$ are 0. Then $s_{k+1} = s_{k+2} = \ldots = s_n = 0$, so

$$f(x) = s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ns_n$$
$$= s_1 x^{n-1} + 2s_2 x^{n-2} + \cdots + ks_k x^{n-k}$$
$$= x^{n-k}(s_1 x^{k-1} + 2s_2 x^{k-2} + \cdots + ks_k).$$

Thus, the problem reduces to proving the same result on the numbers $a_1, a_2, \ldots, a_k$.

Let $g(x) = (a_1x+1)(a_2x+1)\ldots(a_nx+1)$. The roots of $g(x) = 0$ are clearly real, namely $-1/a_1, -1/a_2, \ldots, -1/a_n$. We claim that the roots of $g'(x)=0$ are all real.

Suppose the roots of $g(x) = 0$ are distinct. Let $r_1 < r_2 < \ldots < r_n$ be these roots. Then by Rolle’s theorem, the equation $g'(x) = 0$ has a root in each of the intervals $(r_1,r_2), (r_2,r_3), \ldots, (r_{n-1},r_n)$, so it has $n - 1$ real roots.

Now, suppose the equation $g(x) = 0$ has j distinct roots $r_1 < r_2 < \ldots < r_j$, and root r_i has multiplicity m_i so $m_1+m_2+\ldots+m_j = n$. Then r_i is a root of the equation $g'(x) = 0$ having multiplicity m_i-1. In addition, again by Rolle’s theorem, the equation has a root in each of the interval $(r_1,r_2), (r_2,r_3), \ldots, (r_{j-1},r_j)$, so the equation $g'(x) = 0$ has the requisite

$$(m_1-1)+(m_2-1)+\ldots+(m_j-1)+j-1 = n-1$$

real roots.

Expanding, we have that

$$g(x) = (a_1x+1)(a_2x+1)\ldots(a_nx+1)$$
$$= s_n x^n + s_{n-1}x^{n-1} + \ldots + 1,$$

So $g'(x) = ns_n x^{n-1} + (n-1)s_{n-1}x^{n-2} + \ldots + s_1$. Since $s_1 \ne 0$, 0 is not a root of $g'(x) = 0$. Finally, we get that the polynomial

$$x^{n-1}g'(\frac{1}{x}) = s_1 x^{n-1} + 2s_2 x^{n-1} + \cdots + ns_n$$

has all real roots.

Olympiad Corner

(continued from page 1)

Problem 5. Find all natural numbers n for which the number of all positive divisors of the number lcm$(1,2,\ldots,n)$ is equal to 2^k for some non-negative integer k.

Problem 6. Call a convex polyhedron a *footballoid* if it has the following properties.

(1) Any face is either a regular pentagon or a regular hexagon.

(2) All neighbours of a pentagonal face are hexagonal (a *neighbour* of a face is a face that has a common edge with it).

Find all possibilities for the number of a pentagonal and hexagonal faces of a footballoid.

Mathematical Excalibur

Volume 10, Number 2 *May 2005 – July 2005*

Olympiad Corner

Following are the problems of 2005 Chinese Mathematical Olympiad.

Problem 1. Let $\theta_i \in (-\pi/2, \pi/2)$, i = 1, 2, 3, 4. Prove that there exists $x\in\mathbb{R}$ satisfying the two inequalities

$$\cos^2\theta_1\cos^2\theta_2-(\sin\theta_1\sin\theta_2-x)^2\ge 0$$
$$\cos^2\theta_3\cos^2\theta_4-(\sin\theta_3\sin\theta_4-x)^2\ge 0$$

if and only if

$$\sum_{i=1}^{4}\sin^2\theta_i \le 2\Big(1+\prod_{i=1}^{4}\sin\theta_i+\prod_{i=1}^{4}\cos\theta_i\Big).$$

Problem 2. A circle meets the three sides BC, CA, AB of triangle ABC at points D_1, D_2; E_1, E_2 and F_1, F_2 in turn. The line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the three lines AL, BM and CN are concurrent.

Problem 3. As in the figure, a pond is divided into $2n$ ($n\ge 5$) parts. Two parts are called neighbors if they have a common side or arc. Thus every part has

(continued on page 4)

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On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 10, 2005***.

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例析數學競賽中的計數問題(三)

費振鵬 (江蘇省鹽城市城區永豐中學 224054)

例 12 是否可能將正整數 1，2，3，…，64 分別填入 8×8 的正方形的 64 個小方格內，使得形如圖 1（方向可以任意轉置）的任意四個小方格內數之和總能被 5 整除？試說明理由。

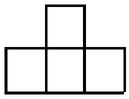

圖 1

解答 不可能。下面用反證法證明：假設圖 2 中 $a,b,c,\cdots,k,l,\cdots$ 就是符合題設填好的數。

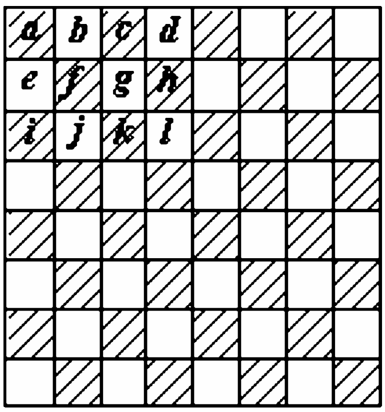

圖 2

因 $5\,|\,b+e+f+g$，$5\,|\,j+e+f+g$，作差有 $5\,|\,b-j$，即 $b\equiv j(\mathrm{mod}\,5)$，設餘數為 r。

同理因 $5\,|\,j+f+b+g$，$5\,|\,e+f+b+g$，故 $5\,|\,j-e$，即 $j\equiv e(\mathrm{mod}\,5)$，顯然其餘數也為 r。

將圖中 64 個小方格染成黑白相間的形式，可得 $b,j,e,g,d,l,\cdots$ 即除角上兩白格中的兩數外，其餘白格中的 $\frac{64}{2}-2=30$ 個數被 5 除都同餘 r。

另一方面，由抽屜原理，1～64 這 64 個正整數中最多有 13 個數被 5 除同餘，與前面得出的結論矛盾！因此，不存在滿足題設的填法。

例 13 平面上給定五點 A、B、C、D、E，其中任何三點不在一直線上。試證：任意地用線段連結某些點（這些線段稱為邊），若所得到的圖形中不出現以這五點中的任何三點為頂點的三角形，則這個圖形不可能有 7 條或更多條邊。

證法 1 （反證法）假設圖形有 7 條或更多條邊，則各點度數和至少是 14。

(1) 若某點度數是 4，則其餘點的度數和至少是 10，由抽屜原理知其中必有一點度數至少是 $\left[\frac{10}{4}\right]+1=3$（度數是 2 就已足夠），故此時必然出現三角形。

(2) 若每點度數至多是 3，由抽屜原理知至少有 4 點的度數是 3，選其中 2 點，不妨設為 A、B，且 A 與 B、C、D 有連線，此時考慮 B 與 A 已有連線，由抽屜原理知 B 必與 C、D 中某一點有連線，這樣也出現了三角形。

而(1)、(2)所得結論都與題設“圖形中不出現以這五點中的任何三點為頂點的三角形”相矛盾，故原命題成立。

證法 2 （反證法）假設圖形有 7 條或更多條邊。

首先我們構造抽屜：每個抽屜裏有三個相異點，共可得 $C_5^3=10$ 個抽屜[注]，又由於同一條邊會在 $5-2=3$ 個抽屜裏出現，則 10 個抽屜裏共有 $7\times 3=21$ 條或更多條邊。

由抽屜原理知，至少有一個抽屜裏有 3 條邊，而每條邊在一個三角形中最多出現一次。這 3 條邊恰好與其中不

共線的相異三點構成一個三角形。而這與題設“圖形中不出現以這五點中的任何三點為頂點的三角形”相矛盾，故原命題成立。

注 對於低年級學生計算構造抽屜的個數，我們可以考慮從A、B、C、D、E五點中任取的三個點與剩下的兩點一一對應，而選擇兩點的情形有：AB、AC、AD、AE、BC、BD、BE、CD、CE、DE，共 10 種。

對這個問題稍作引伸，便得下面的問題：

例 14 平面上給定$n(n>3)$個點，其中任何三點不共線。任意地用線段連結某些點 (這些線段稱為邊)，得到x條邊。

(1) 若確保圖形中出現以給定點為頂點的三角形，求證：

$$x \ge \frac{n(n-1)(n-2)+3}{3(n-2)}\text{。}$$

當$\frac{n(n-1)(n-2)+3}{3(n-2)}$是整數時，求所有$n$連值及對應$x$的最小值；

(2) 若確保圖形中出現以給定點為頂點的m $(m<n)$ 階完全圖(即m點中任何兩點都有邊連接的圖)，求證：

$$x \ge \frac{C_n^m(C_m^2-1)+1}{C_{n-2}^{m-2}}\text{。}$$

證明 (1) Ⅰ．構造抽屜：每個抽屜裏有三個相異點，共可得C_n^3個抽屜。又由於同一條邊會在C_{n-2}^1個抽屜裏出現，根據抽屜原理知，當$x\cdot C_{n-2}^1 \ge 2C_n^3+1$時，才能確保有一個抽屜裏有 3 條邊，而這 3 條邊恰好與其中不共線的相異三點構成一個三角形。

這就是說，確保圖形中出現以給定點為頂點的三角形，則$x \ge \frac{2C_n^3+1}{C_{n-2}^1}$，即

$$x \ge \frac{n(n-1)(n-2)+3}{3(n-2)}\text{。}$$

Ⅱ．顯然n，$n-1$，$n-2$中有且只有一個是 3 的倍數。

(i) 當n或$n-1$是 3 的倍數時，一方面

$$\frac{n(n-1)(n-2)+3}{3(n-2)} = \frac{n(n-1)}{3} + \frac{1}{n-2}$$

是整數，則$\frac{1}{n-2}$是整數；另一方面$n>3$，$n-2>1$，則$\frac{1}{n-2}$是分數。矛盾！此時n無解。

(ii) 當$n-2$是 3 的倍數時，不妨設$n-2=3k$，考慮

$$\begin{aligned}&\frac{n(n-1)(n-2)+3}{3(n-2)}\\&= \frac{3k(3k+1)(3k+2)+3}{3\cdot 3k}\\&= \frac{3k(3k^2+3k+1)+1-k}{3k}\\&= 3k^2+3k+1+\frac{1-k}{3k}\end{aligned}$$

是整數，則$\frac{1-k}{3k}$是整數。

令$\frac{1-k}{3k}=t$，則$k(3t+1)=1$。從而$k=1$，$3t+1=1$即$k=1$，$t=0$。因此

$$n=3k+2=5\text{。}$$

從而$x \ge 3k^2+3k+1$，即$x \ge 7$。因此x的最小值是 7。

綜合(i)、(ii)可知，當

$$\frac{n(n-1)(n-2)+3}{3(n-2)}$$

是整數時，$n=5$，$x_{\min}=7$。

(2) 構造抽屜：每個抽屜裏有m個相異點，共可得C_n^m個抽屜。又由於同一條邊會在C_{n-2}^{m-2}個抽屜裏出現。根據抽屜原理知，當

$$x\cdot C_{n-2}^{m-2} \ge C_n^m(C_m^2-1)+1$$

時，才能確保有一個抽屜裏有C_m^2條邊，而這C_m^2條邊恰好與其中不共線的相異m點構成一個m階完全圖。

這就是說，確保圖形中出現以給定點為頂點的m階完全圖，則

$$x \ge \frac{C_n^m(C_m^2-1)+1}{C_{n-2}^{m-2}}\text{。}$$

注 題中字母k、m、n、t、x都是指整數。

以上解決數學競賽題的思路與方法告訴我們：見多識廣，可以增強領悟能力；博採眾長，才能減少盲目性。解題中的靈感突現，源自平時的日積月累。只有多鑽研，多探索，做題時便能隨機應變，亦或獨闢蹊徑，以致迎刃而解。

你覺得“數學好玩”嗎？只要你有興趣，數學就會變得迥然不同。你就會感受到數學無盡的魅力，就會具有攻無不克的意志力，就會產生無堅不摧的戰鬥力。如果你根本就沒愛上數學，又怎麼可能碰撞出最為絢爛的火花呢？哪怕非常短暫，瞬間即逝。

有很多同學熱愛數學，都為能在數學奧林匹克的賽場上一試身手、摘金奪銀而默默鑽研，苦苦奮鬥。我想學習中保持長久的數學興趣和培養創造性的思維是成功的關鍵，也是將來可持續發展的保障。而汲取眾家之長是創造性思維的源泉，學會獨立思考是提高創造性思維能力的良策。

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 10, 2005.***

Problem 226. Let $z_1, z_2, \ldots, z_n$ be complex numbers satisfying

$$|z_1| + |z_2| + \cdots + |z_n| = 1.$$

Prove that there is a nonempty subset of $\{z_1, z_2, \ldots, z_n\}$ the sum of whose elements has modulus at least 1/4.

Problem 227. For every integer $n \geq 6$, prove that

$$\sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2^{k-1}} \leq \frac{16}{5}.$$

Problem 228. In $\triangle ABC$, M is the foot of the perpendicular from A to the angle bisector of $\angle BCA$. N and L are respectively the feet of perpendiculars from A and C to the bisector of $\angle ABC$. Let F be the intersection of lines MN and AC. Let E be the intersection of lines BF and CL. Let D be the intersection of lines BL and AC.

Prove that lines DE and MN are parallel.

Problem 229. For integer $n \geq 2$, let a_1, a_2, a_3, a_4 be integers satisfying the following two conditions:

(1) for i = 1, 2, 3, 4, the greatest common divisor of n and a_i is 1 and
(2) for every $k = 1, 2, \ldots, n-1$, we have

$$(ka_1)_n + (ka_2)_n + (ka_3)_n + (ka_4)_n = 2n,$$

where $(a)_n$ denotes the remainder when a is divided by n.

Prove that $(a_1)_n, (a_2)_n, (a_3)_n, (a_4)_n$ can be divided into two pairs, each pair having sum equals n.
(*Source: 1992 Japanese Math Olympiad*)

Problem 230. Let k be a positive integer. On the two sides of a river, there are in total at least 3 cities. From each of these cities, there are exactly k routes, each connecting the city to a distinct city on the other side of the river. Via these routes, people in every city can reach any one of the other cities.

Prove that if any one route is removed, people in every city can still reach any one of the other cities via the remaining routes.
(*Source: 1996 Iranian Math Olympiad, Round 2*)

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Solutions

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Due to an editorial mistake in the last issue, solutions to problems 216, 217, 218, 219 by **D. Kipp Johnson** (teacher, Valley Catholic School, Beaverton, Oregon, USA) were overlooked and his name was not listed among the solvers. We express our apology to him.

Problem 221. (*Due to Alfred Eckstein, Arad, Romania*) The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Prove that $7F_{n+2}^3 - F_n^3 - F_{n+1}^3$ is divisible by F_{n+3}.

Solution. **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania) and **Kin-Chit O** (STFA Cheng Yu Tung Secondary School).

As $a = 7F_{n+2}^3 + 7F_{n+1}^3$ is divisible by $F_{n+2} + F_{n+1} = F_{n+3}$ and $b = 8F_{n+1}^3 + F_n^3$ is divisible by $2F_{n+1} + F_n = F_{n+2} + F_{n+1} = F_{n+3}$, so $7F_{n+2}^3 - F_n^3 - F_{n+1}^3 = a - b$ is divisible by F_{n+3}.

Other commended solvers: **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7), **CHAN Tsz Lung**, **CHAN Yee Ling** (Carmel Divine Grace Foundation Secondary School, Form 6), **G.R.A. 20 Math Problem Group** (Roma, Italy), **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 6) and **WONG Kwok Kit** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 222. All vertices of a convex quadrilateral $ABCD$ lie on a circle ω. The rays AD, BC intersect in point K and the rays AB, DC intersect in point L. Prove that the circumcircle of triangle AKL is tangent to ω if and only if the circumcircle of triangle CKL is tangent to ω.
(*Source: 2001-2002 Estonian Math Olympiad, Final Round*)

Solution. **LEE Kai Seng** (HKUST) and **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5).

Let ω_1 and ω_2 be the circumcircles of ΔAKL and ΔCKL respectively. For a point P on a circle Ω, let $\Omega(P)$ denote the tangent line to Ω at P.

Pick D' on $\omega(A)$ so that D and D' are on opposite sides of line BL and pick L' on $\omega_1(A)$ so that L and L' are on opposite sides of line BL.

Next, pick D'' on $\omega(C)$ so that D and D'' are on opposite sides of line BK and pick L'' on $\omega_2(C)$ so that L and L'' are on opposite sides of line BK. Now ω, ω_1 both contain A and ω, ω_2 both contain C. So

$$\begin{aligned} & \omega(A) = \omega_1(A) \\ \Leftrightarrow\ & \angle D'AB = \angle L'AB \\ \Leftrightarrow\ & \angle ADB = \angle ALB \\ \Leftrightarrow\ & BD \parallel LK \\ \Leftrightarrow\ & \angle BDC = \angle KLC \\ \Leftrightarrow\ & \angle BCD'' = \angle KCL'' \\ \Leftrightarrow\ & \omega(C) = \omega_2(C). \end{aligned}$$

Other commended solvers: **CHAN Tsz Lung** and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

Problem 223. Let $n \geq 3$ be an integer and x be a real number such that the numbers x, x^2 and x^n have the same fractional parts. Prove that x is an integer.
(*Source: 1997 Romanian Math Olympiad, Final Round*)

Solution. **G.R.A. 20 Math Problem Group** (Roma, Italy).

By hypotheses, there are integers a, b such that $x^2 = x + a$ and $x^n = x + b$. Since x is real, the discriminant $\Delta = 1 + 4a$ of $x^2 - x - a = 0$ is nonnegative. So $a \geq 0$. If $a = 0$, then $x = 0$ or 1.

If $a > 0$, then define integers c_j, d_j so that $x^j = c_jx + d_j$ for $j \geq 2$ by $c_2 = 1$, $d_2 = a > 0$,

$$x^3 = x^2 + ax = (1 + a)x + a$$

leads to $c_3 = 1 + a$, $d_3 = a$ and for $j > 3$, $x^j = (x + a)x^{j-2} = (c_{j-1} + ac_{j-2})x + (d_{j-1} +$

ad_{j-2}) leads to $c_j = c_{j-1} + ac_{j-2} > c_{j-1} > 1$ and $d_j = d_{j-1} + ad_{j-2}$.

Now $c_nx + d_n = x^n = x + b$ with $c_n > 1$ implies $x = (b - d_n)/(c_n - 1)$ is rational. This along with a being an integer and $x^2 - x - a = 0$ imply x is an integer.

Other commended solvers: **CHAN Tsz Lung**, **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5), and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

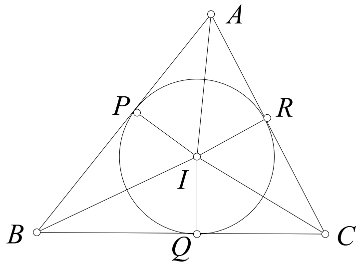
Problem 224. (*Due to Abderrahim Ouardini*) Let a, b, c be the sides of triangle ABC and I be the incenter of the triangle.

Prove that

$$IA \cdot IB \cdot IC \le \frac{abc}{3\sqrt{3}}$$

and determine when equality occurs.

Solution. **CHAN Tsz Lung** and **Kin-Chit O** (STFA Cheng Yu Tung Secondary School).



Let r be the radius of the incircle and s be the semiperimeter $(a + b + c)/2$. The area of $\triangle ABC$ is $(a + b + c)r/2 = sr$ and $\sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula. So

$$r^2 = (s-a)(s-b)(s-c)/s. \qquad (*)$$

Let P, Q, R be the feet of perpendiculars from I to AB, BC, CA. Now $s = AP + BQ + CR = AP + BC$, so $AP = s-a$. Similarly, $BQ = s-b$ and $CR = s-c$. By the *AM-GM* inequality,

$$\begin{aligned} s/3 &= [(s-a)+(s-b)+(s-c)]/3 \\ &\ge \sqrt[3]{(s-a)(s-b)(s-c)}. \qquad (**) \end{aligned}$$

Using Pythagoras' theorem, (*) and (**), we have

$$\begin{aligned} &IA^2 \cdot IB^2 \cdot IC^2 \\ &= [r^2+(s-a)^2][r^2+(s-b)^2][r^2+(s-c)^2] \\ &= [(s-a)bc/s][(s-b)ca/s][(s-c)ab/s] \\ &\le (abc)^2/3^3 \end{aligned}$$

with equality if and only if $a = b = c$. The result follows.

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **KWOK Lo Yan** (Carmel Divine Grace Foundation Secondary School, Form 5), **MA Hoi Sang** (Shun Lee Catholic Secondary School, Form 5) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).

Problem 225. A luminous point is in space. Is it possible to prevent its luminosity with a finite number of disjoint spheres of the same size?
(*Source: 2003-2004 Iranian Math Olympiad, Second Round*)

Official Solution.

Let the luminous point be at the origin. Consider all spheres of radius $r = \sqrt{2}/4$ centered at (i, j, k), where i, j, k are integers (not all zero) and $|i|, |j|, |k| \le 64$. The spheres are disjoint as the radii are less than 1/2. For any line L through the origin, by the symmetries of the spheres, we may assume L has equations of the form $y = ax$ and $z = bx$ with $|a|, |b| \le 1$. It suffices to show L intersects one of the spheres.

We claim that for every positive integer n and every real number c with $|c| \le 1$, there exists a positive integer $m \le n$ such that $|\{mc\}| < 1/n$, where $\{x\} = x - [x]$ is the fractional part of x.

To see this, partition [0,1) into n intervals of length $1/n$. If one of $\{c\}, \{2c\}, \ldots, \{nc\}$ is in $[0,1/n)$, then the claim is true. Otherwise, by the pigeonhole principle, there are $0 < m' < m'' \le n$ such that $\{m'c\}$ and $\{m''c\}$ are in the same interval. Then $|\{m'c\}-\{m''c\}| < 1/n$ implies $|\{mc\}| < 1/n$ for $m = m'' - m' \le n$.

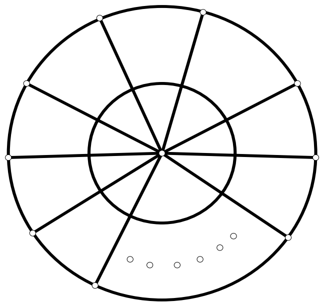
Since $|a| \le 1$, by the claim, there is a positive integer $m \le 16$ such that $|\{ma\}| < 1/16$ and there is a positive integer $n \le 4$ such that $|\{nmb\}| < 1/4$. Now $|\{ma\}| < 1/16$ and $n \le 4$ imply $|\{nma\}| < 1/4$. Then $i = nm \le 64$ and $j = [nma]$, $k = [nmb]$ satisfy $|j-nma| < 1/4$ and $|k-nmb| < 1/4$. So the distance between the point (i, ia, ib) on L and the center (i, j, k) is less than r. Therefore, every line L through the origin will intersect some sphere.

Olympiad Corner

(continued from page 1)

three neighbors. Now there are $4n + 1$ frogs at the pond. If there are three or more frogs at one part, then three of the frogs of the part will jump to the three neighbors respectively.

Prove that at some time later, the frogs at the pond will be uniformly distributed. That is, for any part, either there is at least one frog at the part or there is at least one frog at each of its neighbors.



Problem 4. Given a sequence $\{a_n\}$ satisfying $a_1 = 21/16$ and $2a_n - 3a_{n-1} = 3/2^{n+1}$, $n \ge 2$. Let m be a positive integer, $m \ge 2$.

Prove that if $n \le m$, then

$$\left(a_n + \frac{3}{2^{n+3}}\right)^{1/m}\left(m - \left(\frac{2}{3}\right)^{n(m-1)/m}\right) < \frac{m^2 - 1}{m - n + 1}.$$

Problem 5. Inside and including the boundary of a rectangle $ABCD$ with area 1, there are 5 points, no three of which are collinear.

Find (with proof) the least possible number of triangles having vertices among these 5 points with areas not greater than 1/4.

Problem 6. Find (with proof) all nonnegative integral solutions (x, y, z, w) to the equation

$$2^x \cdot 3^y - 5^z \cdot 7^w = 1.$$

Mathematical Excalibur

Volume 10, Number 3 *August 2005 – September 2005*

Olympiad Corner

The 2005 International Mathematical Olympiad was held in Merida, Mexico on July 13 and 14. Below are the problems.

Problem 1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC; B_1, B_2 on CA; C_1, C_2 on AB. These points are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Problem 2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders on division by n. Prove that each integer occurs exactly once in the sequence.

Problem 3. Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0.$$

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 30, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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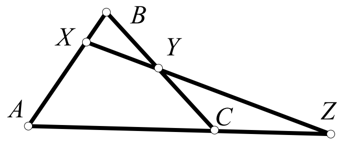
Famous Geometry Theorems

Kin Y. Li

There are many famous geometry theorems. We will look at some of them and some of their applications. Below we will write $P = WX \cap YZ$ to denote P is the point of intersection of lines WX and YZ. If points A, B, C are collinear, we will introduce the *sign convention*: $AB/BC = \overrightarrow{AB}/\overrightarrow{BC}$ (so if B is between A and C, then $AB/BC \geq 0$, *otherwise* $AB/BC \leq 0$).

Menelaus' Theorem Points X, Y, Z are taken from lines AB, BC, CA (which are the sides of $\triangle ABC$ extended) respectively. If there is a line passing through X, Y, Z, then

$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA}=-1.$$



Proof Let L be a line perpendicular to the line through X, Y, Z and intersect it at O. Let A', B', C' be the feet of the perpendiculars from A, B, C to L respectively. Then

$$\frac{AX}{XB}=\frac{A'O}{OB'},\frac{BY}{YC}=\frac{B'O}{OC'},\frac{CZ}{ZA}=\frac{C'O}{OA'}.$$

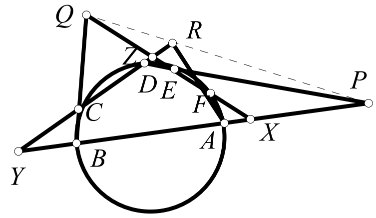
Multiplying these equations together, we get the result.

The converse of Menelaus' Theorem is also true. To see this, let $Z'=XY\cap CA$. Then applying Menelaus theorem to the line through X, Y, Z' and comparing with the equation above, we get $CZ/ZA=CZ'/Z'A$. It follows $Z=Z'$.

Pascal's Theorem Let A, B, C, D, E, F be points on a circle (which are not necessarily in cyclic order). Let

$$P=AB\cap DE,\ Q=BC\cap EF,\ R=CD\cap FA.$$

Then P,Q,R are collinear.



Proof Let $X = EF \cap AB$, $Y = AB \cap CD$, $Z = CD \cap EF$. Applying Menelaus' Theorem respectively to lines BC, DE, FA cutting $\triangle XYZ$ extended , we have

$$\frac{ZQ}{QX}\cdot\frac{XB}{BY}\cdot\frac{YC}{CZ}=-1,$$
$$\frac{XP}{PY}\cdot\frac{YD}{DZ}\cdot\frac{ZE}{EX}=-1,$$
$$\frac{YR}{RZ}\cdot\frac{ZF}{FX}\cdot\frac{XA}{AY}=-1.$$

Multiplying these three equations together, then using the intersecting chord theorem (see *vol* 4, *no*. 3, *p*. 2 of *Mathematical Excalibur*) to get $XA\cdot XB = XE\cdot XF$, $YC\cdot YD = YA\cdot YB$, $ZE\cdot ZF = ZC\cdot ZD$, we arrive at the equation

$$\frac{ZQ}{QX}\cdot\frac{XP}{PY}\cdot\frac{YR}{RZ}=-1.$$

By the converse of Menelaus' Theorem, this implies P, Q, R are collinear.

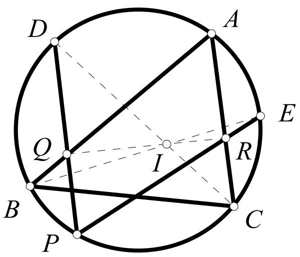
We remark that there are limiting cases of Pascal's Theorem. For example, we may move A to approach B. In the limit, A and B will coincide and the line AB will become the tangent line at B.

Below we will give some examples of using Pascal's Theorem in geometry problems.

Example 1 (*2001 Macedonian Math Olympiad*) For the circumcircle of $\triangle ABC$, let D be the intersection of the tangent line at A with line BC, E be the intersection of the tangent line at B with line CA and F be the intersection of the tangent line at C with line AB. Prove that points D, E, F are collinear.

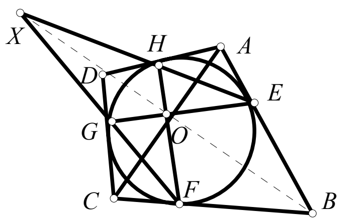
Solution Applying Pascal's Theorem to *A*, *A*, *B*, *B*, *C*, *C* on the circumcircle, we easily get *D*, *E*, *F* are collinear.

Example 2 Let *D* and *E* be the midpoints of the minor arcs *AB* and *AC* on the circumcircle of △ *ABC*, respectively. Let *P* be on the minor arc *BC*, *Q* = *DP* ∩ *BA* and *R* = *PE* ∩ *AC*. Prove that line *QR* passes through the incenter *I* of △*ABC*.



Solution Since *D* is the midpoint of arc *AB*, line *CD* bisects ∠ *ACB*. Similarly, line *EB* bisects ∠*ABC*. So *I* = *CD* ∩ *EB*. Applying Pascal's Theorem to *C, D, P, E, B, A*, we get *I, Q, R* are collinear.

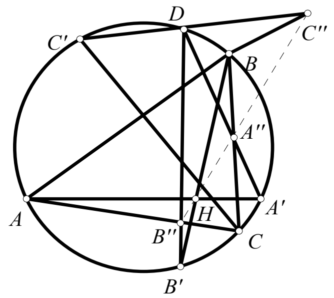
Newton's Theorem A circle is inscribed in a quadrilateral *ABCD* with sides *AB, BC, CD, DA* touch the circle at points *E, F, G, H* respectively. Then lines *AC, EG, BD, FH* are concurrent.



Proof. Let *O* = *EG*∩*FH* and *X* = *EH*∩*FG*. Since *D* is the intersection of the tangent lines at *G* and at *H* to the circle, applying Pascal's Theorem to *E,G,G,F,H,H*, we get *O, D, X* are collinear. Similarly, applying Pascal's Theorem to *E, E, H, F, F, G*, we get *B, X, O* are collinear.

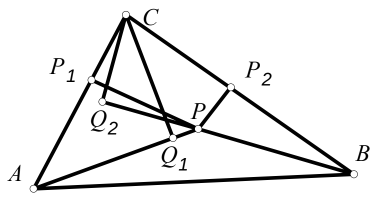
Then *B,O,D* are collinear and so lines *EG, BD, FH* are concurrent at *O*. Similarly, we can also obtain lines *AC, EG, FH* are concurrent at *O*. Then Newton's Theorem follows.

Example 3 (*2001 Australian Math Olympiad*) Let *A, B, C, A', B', C'* be points on a circle such that *AA'* is perpendicular to *BC, BB'* is perpendicular to *CA, CC'* is perpendicular to *AB*. Further, let *D* be a point on that circle and let *DA'* intersect *BC* in *A", DB'* intersect *CA* in *B"*, and *DC'* intersect *AB* in *C"*, all segments being extended where required. Prove that *A", B", C"* and the orthocenter of triangle *ABC* are collinear.



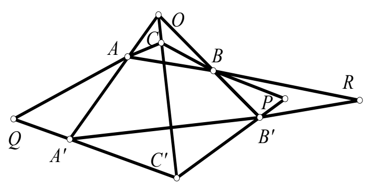
Solution Let *H* be the orthocenter of △ *ABC*. Applying Pascal's theorem to *A, A', D, C', C, B*, we see *H, A", C"* are collinear. Similarly, applying Pascal's theorem to *B', D, C', C, A, B*, we see *B", C", H* are collinear. So *A", B", C"*, *H* are collinear.

Example 4 (*1991 IMO unused problem*) Let *ABC* be any triangle and *P* any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from *P* to the two sides *AC* and *BC*. Draw *AP* and *BP* and from *C* drop perpendiculars to *AP* and *BP*. Let Q_1 and Q_2 be the feet of these perpendiculars. If $Q_2 \neq P_1$ and $Q_1 \neq P_2$, then prove that the lines P_1Q_2, Q_1P_2 and *AB* are concurrent.



Solution Since $\angle CP_1P$, $\angle CP_2P$, $\angle CQ_2P$, $\angle CQ_1P$ are all right angles, we see that the points *C*, Q_1, P_1, *P*, P_2, Q_2 lie on a circle with *CP* as diameter. Note $A = CP_1 \cap PQ_1$ and $B = Q_2P \cap P_2C$. Applying Pascal's theorem to *C*, P_1, Q_2, *P*, Q_1, P_2, we see $X = P_1Q_2 \cap Q_1P_2$ is on line *AB*.

Desargues' Theorem For △*ABC* and △ *A'B'C'*, if lines *AA', BB', CC'* concur at a point *O*, then points *P, Q, R* are collinear, where *P* = *BC* ∩ *B'C'*, *Q* = *CA* ∩ *C'A'*, *R* = *AB* ∩ *A'B'*.



Proof Applying Menelaus' Theorem respectively to line *A'B'* cutting △*OAB* extended, line *B'C'* cutting △ *OBC* extended and the line *C'A'* cutting △*OCA* extended, we have

$$\frac{OA'}{A'A}\cdot\frac{AR}{RB}\cdot\frac{BB'}{B'O}=-1,$$

$$\frac{OB'}{B'B}\cdot\frac{BP}{PC}\cdot\frac{CC'}{C'O}=-1,$$

$$\frac{AA'}{A'O}\cdot\frac{OC'}{C'C}\cdot\frac{CQ}{QA}=-1.$$

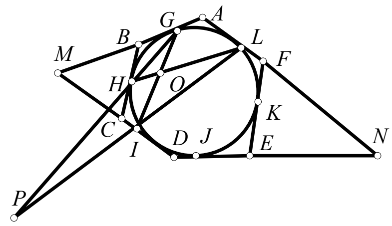
Multiplying these three equations,

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=-1.$$

By the converse of Menelaus' Theorem, this implies *P, Q, R* are collinear.

We remark that the converse of Desargues' Theorem is also true. We can prove it as follow: let *O* = *BB'* ∩ *CC'*. Consider △*RBB'* and △*QCC'*. Since lines *RQ, BC, B'C'* concur at *P*, and *A* = *RB* ∩ *QC*, *O* = *BB'* ∩ *CC'*, *A'* = *BR'* ∩ *C'Q*, by Desargues' Theorem, we have *A,O,A'* are collinear. Therefore, lines *AA', BB', CC'* concur at *O*.

Brianchon's Theorem Lines *AB, BC, CD, DE, EF, FA* are tangent to a circle at points *G, H, I, J, K, L* (not necessarily in cyclic order). Then lines *AD, BE, CF* are concurrent.



Proof Let *M* = *AB* ∩ *CD*, *N* = *DE* ∩ *FA*. Applying Newton's Theorem to quadrilateral *AMDN*, we see lines *AD, IL, GJ* concur at a point *A'*. Similarly, lines *BE, HK, GJ* concur at a point *B'* and lines *CF, HK, IL* concur at a point *C'*. Note line *IL* coincides with line *A'C'*. Next we apply Pascal's Theorem to *G, G, I, L, L, H* and get points *A, O, P* are collinear, where *O* = *GI* ∩ *LH* and *P* = *IL* ∩ *HG*. Applying Pascal's Theorem again to *H, H, L, I, I, G*, we get *C, O, P* are collinear. Hence *A, C, P* are collinear.

Now *G* = *AB* ∩ *A'B'*, *H* = *BC* ∩ *B'C'*, *P* = *CA* ∩ *IL* = *CA* ∩ *C'A'*. Applying the converse of Desargues' Theorem to △*ABC* and △*A'B'C'*, we get lines *AA'* = *AD*, *BB'* = *BE*, *CC'* = *CF* are concurrent.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***October 30, 2005.***

Problem 231. On each planet of a star system, there is an astronomer observing the nearest planet. The number of planets is odd, and pairwise distances between them are different. Prove that at least one planet is not under observation.
(*Source*: *1966 Soviet Union Math Olympiad*)

Problem 232. B and C are points on the segment AD. If $AB = CD$, prove that $PA + PD \ge PB + PC$ for any point P.
(*Source*: *1966 Soviet Union Math Olympiad*)

Problem 233. Prove that every positive integer not exceeding $n!$ can be expressed as the sum of at most n distinct positive integers each of which is a divisor of $n!$.

Problem 234. Determine all polynomials $P(x)$ of the smallest possible degree with the following properties:

a) The coefficient of the highest power is 200.
b) The coefficient of the lowest power for which it is not equal to zero is 2.
c) The sum of all its coefficients is 4.
d) $P(-1) = 0$, $P(2) = 6$ and $P(3) = 8$.

(*Source: 2002 Austrian National Competition*)

Problem 235. Forty-nine students solve a set of three problems. The score for each problem is an integer from 0 to 7. Prove that there exist two students A and B such that, for each problem, A will score at least as many points as B.

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Solutions

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Problem 226. Let $z_1, z_2, \ldots, z_n$ be complex numbers satisfying

$$|z_1| + |z_2| + \cdots + |z_n| = 1.$$

Prove that there is a nonempty subset of $\{z_1, z_2, \ldots, z_n\}$ the sum of whose elements has modulus at least 1/4.

Solution. **LEE Kai Seng** (HKUST).

Let $z_k = a_k + b_k i$ with a_k , b_k real. Then $|z_k| \le |a_k| + |b_k|$. So

$$1 = \sum_{k=1}^{n} |z_k| \le \sum_{k=1}^{n} |a_k| + \sum_{k=1}^{n} |b_k|$$

$$= \sum_{a_k \ge 0} a_k + \sum_{a_k < 0} (-a_k) + \sum_{b_k \ge 0} b_k + \sum_{b_k < 0} (-b_k).$$

Hence, one of the four sums is at least 1/4, say $\sum_{a_k \ge 0} a_k \ge \frac{1}{4}$. Then

$$\left| \sum_{a_k \ge 0} z_k \right| \ge \left| \sum_{a_k \ge 0} a_k \right| \ge \frac{1}{4}.$$

Problem 227. For every integer $n \ge 6$, prove that

$$\sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2^{k-1}} \le \frac{16}{5}.$$

Comments. In the original statement of the problem, the displayed inequality was stated incorrectly. The $<$ sign should be an $\le$ sign.

Solution. **CHAN Pak Woon** (Wah Yan College, Kowloon, Form 7), **Roger CHAN** (Vancouver, Canada) and **LEE Kai Seng** (HKUST).

For $n = 6, 7, \ldots$, let

$$a_n = \sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2^{k-1}}.$$

Then $a_6 = 16/5$. For $n \ge 6$, if $a_n \le 16/5$, then

$$a_{n+1} = \sum_{k=1}^{n} \frac{n+1}{n+1-k} \cdot \frac{1}{2^{k-1}} = \sum_{j=0}^{n-1} \frac{n+1}{n-j} \cdot \frac{1}{2^j}$$

$$= \frac{n+1}{n} + \frac{n+1}{2n} \sum_{j=1}^{n-1} \frac{n}{n-j} \cdot \frac{1}{2^{j-1}}$$

$$= \frac{n+1}{n}\left(1 + \frac{a_n}{2}\right) \le \frac{7}{6}\left(1 + \frac{8}{5}\right) < \frac{16}{5}.$$

The desired inequality follows by mathematical induction.

Problem 228. In $\triangle ABC$, M is the foot of the perpendicular from A to the angle bisector of $\angle BCA$. N and L are respectively the feet of perpendiculars from A and C to the bisector of $\angle ABC$. Let F be the intersection of lines MN and AC. Let E be the intersection of lines BF and CL. Let D be the intersection of lines BL and AC.

Prove that lines DE and MN are parallel.

Solution. **Roger CHAN** (Vancouver, Canada).

Extend AM to meet BC at G and extend AN to meet BC at I. Then $AM = MG$, $AN = NI$ and so lines MN and BC are parallel.

From $AM = MG$, we get $AF = FC$. Extend CL to meet line AB at J. Then $JL = LC$. So lines LF and AB are parallel.

Let line LF intersect BC at H. Then $BH = HC$. In $\triangle BLC$, segments BE, LH and CD concur at F. By Ceva's theorem (see *vol.* 2, *no.* 5, *pp.* 1-2 of *Mathematical Excalibur*),

$$\frac{BH}{HC} \cdot \frac{CE}{EL} \cdot \frac{LD}{DB} = 1.$$

Since $BH = HC$, we get $CE/EL = DB/LD$, which implies lines DE and BC are parallel. Therefore, lines DE and MN are parallel.

Problem 229. For integer $n \ge 2$, let a_1, a_2, a_3, a_4 be integers satisfying the following two conditions:

(1) for $i = 1, 2, 3, 4$, the greatest common divisor of n and a_i is 1 and
(2) for every $k = 1, 2, \ldots, n-1$, we have

$$(ka_1)_n + (ka_2)_n + (ka_3)_n + (ka_4)_n = 2n,$$

where $(a)_n$ denotes the remainder when a is divided by n.

Prove that $(a_1)_n, (a_2)_n, (a_3)_n, (a_4)_n$ can be divided into two pairs, each pair having sum equals n.
(*Source: 1992 Japanese Math Olympiad*)

Solution. (***Official Solution***)

Since n and a_1 are relatively prime, the remainders $(a_1)_n, (2a_1)_n, \ldots, ((n-1)a_1)_n$ are nonzero and distinct. So there is a k among $1, 2, \ldots, n-1$ such that $(ka_1)_n = 1$. Note that such k is relatively prime to n. If $(ka_1)_n + (ka_j)_n = n$, then $ka_1 + ka_j \equiv 0 \pmod n$ so that $a_1 + a_j \equiv 0 \pmod n$ and $(a_1)_n + (a_j)_n = n$. Thus, to solve the problem, we may replace a_i by $(ka_i)_n$ and assume $1 = a_1 \le a_2 \le a_3 \le$

$a_4 \le n - 1$. By condition (2), we have

$$1+a_2+a_3+a_4=2n. \qquad (A)$$

For $k = 1, 2, \ldots, n - 1$, let

$$f_i(k) = [ka_i/n] - [(k-1)a_i/n],$$

then $f_i(k) \le (ka_i/n) + 1 - (k-1)a_i/n = 1 + (a_i/n) < 2$. So $f_i(k) = 0$ or 1. Since $x = [x/n]n + (x)_n$, subtracting the case $x = ka_i$ from the case $x = (k-1)a_i$, then summing $i = 1, 2, 3, 4$, using condition (2) and (A), we get

$$f_1(k) + f_2(k) + f_3(k) + f_4(k) = 2.$$

Since $a_1 = 1$, we see $f_1(k)=0$ and exactly two of $f_2(k), f_3(k), f_4(k)$ equal 1. (B)

Since $a_i<n$, $f_i(2) = [2a_i/n]$. Since $a_2 \le a_3 \le a_4 < n$, we get $f_2(2) = 0$, $f_3(2) = f_4(2) = 1$, i.e. $1 = a_1 \le a_2 < n/2 < a_3 \le a_4 \le n - 1$.

Let $t_2 = [n/a_2] + 1$, then $f_2(t_2) = [t_2a_2/n] - [(t_2 - 1)a_2/n] = 1 - 0 = 1$. If $1 \le k < t_2$, then $k < n/a_2$, $f_2(k) = [ka_2/n] - [(k-1)a_2/n] = 0 - 0 = 0$. Next if $f_2(j) = 1$, then $f_2(k) = 0$ for $j < k < j + t_2 - 1$ and exactly one of $f_2(j + t_2 - 1)$ or $f_2(j + t_2) = 1$. (C)

Similarly, for $i = 3, 4$, let $t_i = [n/(n - a_i)] + 1$, then $f_i(t_i) = 0$ and $f_i(k) = 1$ for $1 \le k < t_i$. Also, if $f_i(j) = 0$, then $f_i(k) = 1$ for $j < k < j + t_i - 1$ and exactly one of $f_i(j + t_i - 1)$ or $f_i(j + t_i) = 0$. (D)

Since $f_3(t_3) = 0$, by (B), $f_2(t_3) = 1$. If $k < t_3 \le t_4$, then by (D), $f_3(k) = f_4(k) = 1$. So by (B), $f_2(k) = 0$. Then by (C), $t_2 = t_3$.

Assume $t_4 < n$. Since $n/2 < a_4 < n$, we get $f_4(n - 1) = (a_4 - 1) - (a_4 - 2) = 1 \ne 0 = f_4(t_4)$ and so $t_4 \ne n-1$. Also, $f_4(t_4) = 0$ implies $f_2(t_4) = f_3(t_4) = 1$ by (B).

Since $f_3(t_3) = 0 \ne 1 = f_3(t_4)$, $t_3 \ne t_4$. Thus $t_2 = t_3 < t_4$. Let $s < t_4$ be the largest integer such that $f_2(s) = 1$. Since $f_2(t_4) = 1$, we have $t_4 = s + t_2 - 1$ or $t_4 = s + t_2$. Since $f_2(s) = f_4(s) = 1$, we get $f_3(s) = 0$. As $t_2 = t_3$, we have $t_4 = s + t_3 - 1$ or $t_4 = s + t_3$. Since $f_3(s) = 0$ and $f_3(t_4) = 1$, by (D), we get $f_3(t_4 - 1) = 0$ or $f_3(t_4 + 1) = 0$. Since $f_2(s) = 1$, $f_2(t_4) = 1$ and $t_2 > 2$, by (C), we get $f_2(s + 1) = 0$ and $f_2(t_4 + 1) = 0$. So $s + 1 \ne t_4$, which implies $f_2(t_4 - 1) = 0$ by the definition of s. Then $k = t_4 - 1$ or $t_4 + 1$ contradicts (B).

So $t_4 \ge n$, then $n - a_4 = 1$. We get $a_1 + a_4 = n = a_2 + a_3$.

Problem 230. Let k be a positive integer. On the two sides of a river, there are in total at least 3 cities. From each of these cities, there are exactly k routes, each connecting the city to a distinct city on the other side of the river. Via these routes, people in every city can reach any one of the other cities.

Prove that if any one route is removed, people in every city can still reach any one of the other cities via the remaining routes.
(*Source: 1996 Iranian Math Olympiad, Round 2*)

Solution. **LEE Kai Seng** (HKUST).

Associate each city with a vertex of a graph. Suppose there are X and Y cities to the left and to the right of the river respectively. Then the number of routes (or edges of the graph) in the beginning is $Xk = Yk$ so that $X = Y$. We have $X + Y \ge 3$.

After one route between city A and city B is removed, assume the cities can no longer be connected via the remaining routes. Then each of the other cities can only be connected to exactly one of A or B. Then the original graph decomposes into two connected graphs G_A and G_B, where G_A has A as vertex and G_B has B as vertex.

Let X_A be the number of cities among the X cities on the left sides of the river that can still be connected to A after the route between A and B was removed and similarly for X_B, Y_A, Y_B. Then the number of edges in G_A is $X_Ak-1 = Y_Ak$. Then $(X_A - Y_A)k = 1$. So $k = 1$. Then in the beginning $X = 1$ and $Y = 1$, contradicting $X + Y \ge 3$.

Olympiad Corner

(continued from page 1)

Problem 4. Consider the sequence a_1, a_2, … defined by

$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \ldots)$$

Determine all positive integers that are relatively prime to every term of the sequence.

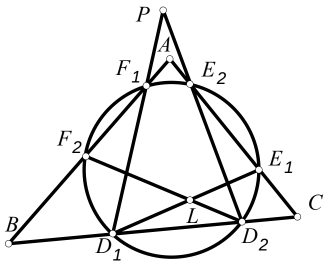
Problem 5. Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

Problem 6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than 2/5 of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

Famous Geometry Theorems

(continued from page 2)

Example 5 (*2005 Chinese Math Olympiad*) A circle meets the three sides BC, CA, AB of triangle ABC at points D_1, D_2; E_1, E_2 and F_1, F_2 in turn. The line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the three lines AL, BM and CN are concurrent.



Solution. Let $P = D_1F_1 \cap D_2E_2$, $Q = E_1D_1 \cap E_2F_2$, $R = F_1E_1 \cap F_2D_2$. Applying Pascal's Theorem to E_2, E_1, D_1, F_1, F_2, D_2, we get A, L, P are collinear. Applying Pascal's Theorem to F_2, F_1, E_1, D_1, D_2, E_2, we get B, M, Q are collinear. Applying Pascal's Theorem to D_2, D_1, F_1, E_1, E_2, F_2, we get C, N, R are collinear.

Let $X = E_2E_1 \cap D_1F_2 = CA \cap D_1F_2$, $Y = F_2F_1 \cap E_1D_2 = AB \cap E_1D_2$, $Z = D_2D_1 \cap F_1E_2 = BC \cap F_1E_2$. Applying Pascal's Theorem to D_1, F_1, E_1, E_2, D_2, F_2, we get P, R, X are collinear. Applying Pascal's Theorem to E_1, D_1, F_1, F_2, E_2, D_2, we get Q, P, Y are collinear. Applying Pascal's Theorem to F_1, E_1, D_1, D_2, F_2, E_2, we get R, Q, Z are collinear.

For $\triangle ABC$ and $\triangle PQR$, we have $X = CA \cap RP$, $Y = AB \cap PQ$, $Z = BC \cap QR$. By the converse of Desargues' Theorem, lines $AP = AL$, $BQ = BM$, $CR = CN$ are concurrent.

Mathematical Excalibur

Volume 10, Number 4 *October 2005 – November 2005*

Olympiad Corner

Below is the Bulgarian selection test for the 46th IMO given on May 18 – 19, 2005.

Problem 1. An acute triangle ABC is given. Find the locus of points M in the interior of the triangle such that $AB-FG = (MF\cdot AG+MG\cdot BF)/CM$, where F and G are the feet of perpendiculars from M to the lines BC and AC, respectively.

Problem 2. Find the number of subsets B of the set $\{1, 2, \ldots, 2005\}$ such that the sum of the elements of B is congruent to 2006 modulo 2048.

Problem 3. Let R_* be the set of non-zero real numbers. Find all functions $f: R_* \to R_*$ such that

$$f(x^2+y)=f^2(x)+\frac{f(xy)}{f(x)}$$

for all $x, y \in R_*$, $y \neq -x^2$.

Problem 4. Let $a_1, a_2, \ldots, a_{2005}, b_1, b_2, \cdots, b_{2005}$ be real numbers such that

$$(a_i x - b_i)^2 \geq \sum_{j=1, j\neq i}^{2005}(a_j x - b_j)$$

for any real number x and $i = 1, 2, \ldots, 2005$. What is the maximal number of positive a_i's and b_i's?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 10, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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The Method of Infinite Descent

Leung Tat-Wing

The technique of infinite descent (*descent infini*) was developed by the great amateur mathematician Pierre de Fermat (1601-1665). Besides using the technique to prove negative results such as the equation $x^4 + y^4 = z^2$ has no nontrivial integer solution, he also used the technique to prove positive results.

For instance, he knew that an odd prime p can be expressed as the sum of two integer squares if and only if p is of the form $4k + 1$. To show that a prime of the form $4k + 3$ is not a sum of two squares is not hard. In fact, every square equals 0 or 1 mod 4, thus no matter what possibilities, the sum of two squares cannot be of the form $4k + 3 \equiv 3 \pmod 4$. To prove a prime of the form $4k + 1$ is the sum of two squares, he assumed that if there is a prime of the form $4k + 1$ which is not the sum of two squares, then there will be another (smaller) prime of the same nature, and hence a third one, and so on. Eventually he would come to the number 5, which should not be the sum of two squares. But we know $5 = 1^2 + 2^2$ a sum of two squares, a contradiction!

The idea of infinite descent may be described as follows. Mainly it is because a finite subset of natural numbers must have a smallest member. So if A is a subset of the natural numbers N, and if we need to prove, for every $a \in A$, the statement $P(a)$ is valid. Suppose by contradiction, the statement is not valid for all $a \in A$, i.e. there exists a non-empty subset of A, denoted by B, and such that $P(x)$ is not true for any $x \in B$. Now because B is non-empty, there exists a smallest element of B, denoted by b and such that $P(b)$ is not valid. Using the given conditions, if we can find a still smaller $c \in A$ ($c < b$), and such that $P(c)$ is not valid, then this will contradict the assumption of b. The conclusion is that $P(a)$ must be valid for all $a \in A$.

There are variations of this scenario. For instance, suppose there is a positive integer a_1 such that $P(a_1)$ is valid, and from this, if we can find a smaller positive integer a_2 such that $P(a_2)$ is valid, then we can find a still smaller positive integer a_3 such that $P(a_3)$ is valid, and so on. Hence we can find an infinite and decreasing chain of positive integers (infinite descent) $a_1 > a_2 > a_3 > \cdots$. This is clearly impossible. So the initial hypothesis $P(a_1)$ cannot be valid.

So the method of descent is essentially another form of induction. Recall that in mathematical induction, we start from a smallest element a of a subset of natural numbers, (initial step), and prove the so-called inductive step. So we can go from $P(a)$ to $P(a + 1)$, then $P(a + 2)$ and so on.

Many problems in mathematics competition require the uses of the method of descent. We give a few examples. First we use the method of infinite descent to prove the well-known result that $\sqrt{2}$ is irrational. Of course the classical proof is essentially a descent argument.

<u>**Example 1**</u>: Show that $\sqrt{2}$ is irrational.

<u>**Solution.**</u> We need to show that there do not exist positive integers x and y such that $x/y = \sqrt{2}$ or by taking squares, we need to show the equation $x^2 = 2y^2$ has no positive integer solution.

Suppose otherwise, let $x = m$, $y = n$ be a solution of the equation and such that m is the *smallest* possible value of x that satisfies the equation. Then $m^2 = 2n^2$ and this is possible only if m is even, hence $m=2m_1$. Thus, $4m_1^2 = (2m_1)^2=2n^2$, so $n^2 = 2m_1^2$. This implies n is also a possible value of x in the equation $x^2 = 2y^2$. However, $n < m$, contradicting the minimality of m.

Example 2 (Hungarian MO 2000): Find all positive primes p for which there exist positive integers x, y and n such that $p^n = x^3 + y^3$.

Solution. Observe $2^1 = 1^3 + 1^3$ and $3^2 = 2^3 + 1^3$. After many trials we found no more primes with this property. So we suspect the only answers are $p = 2$ or $p = 3$. Thus, we need to prove there exists no prime p ($p > 3$) satisfying $p^n = x^3 + y^3$. Clearly we need to prove by contradiction and one possibility is to make use of the descent method. (In this case we make descent on n and it works.)

So we assume $p^n = x^3 + y^3$ with x, y, n positive integers and n of the smallest possible value. Now $p \geq 5$. Hence at least one of x and y is greater than 1. Also

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2),$$

with $x + y \geq 3$ and

$$x^2 - xy + y^2 = (x-y)^2 + xy \geq 2.$$

Hence both $x + y$ and $x^2 - xy + y^2$ are divisible by p. Therefore

$$(x+y)^2 - (x^2 - xy + y^2) = 3xy$$

is also divisible by p. However, 3 is not divisible by p, so at least one of x or y must be divisible by p. As $x + y$ is divisible by p, both x and y are divisible by p. Then $x^3 + y^3 \geq 2p^3$. So we must have $n > 3$ and

$$p^{n-3} = \frac{p^n}{p^3} = \frac{x^3}{p^3} + \frac{y^3}{p^3} = \left(\frac{x}{p}\right)^3 + \left(\frac{y}{p}\right)^3.$$

This contradicts the minimality of n.

Example 3 (Putnam Exam 1973): Let $a_1, a_2, \cdots, a_{2n+1}$ be a set of integers such that, if any one of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove $a_1 = a_2 = \cdots = a_{2n+1}$.

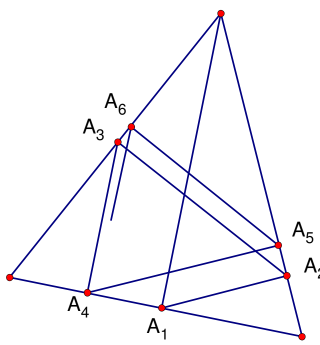
Solution. Assume $a_1 \leq a_2 \leq \cdots \leq a_{2n+1}$. By subtracting the smallest number from the sequence we observe the new sequence still maintain the property. So we may assume $a_1 = 0$. The sum of any $2n$ members equals 0 mod 2, so any two members must be of the same parity, (otherwise we may swap two members to form two groups of $2n$ elements which are of different parity). Therefore

$$0 = a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \pmod 2.$$

Dividing by 2, we note the new sequence will maintain the same property. Using the same reasoning we see that $0 = a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \pmod{2^2}$. We may descent to $0 = a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \pmod{2^m}$ for all $m \geq 1$. This is possible only if the initial numbers are all equal to others.

Example 4: Starting from a vertex of an acute triangle, the perpendicular is drawn, meeting the opposite side (side 1) at A_1. From A_1, a perpendicular is drawn to meet another side (side 2) at A_2. Starting from A_2, the perpendicular is drawn to meet the third side (side 3) at A_3. The perpendicular from A_3 is then drawn to meet side 1 at A_4 and then back to side 2, and so on.

Prove that the points A_1, A_2, … are all distinct.



Solution. First note that because the triangle is acute, all the points A_i, $i \geq 1$ lie on the sides of the triangle, instead of going outside or coincide with the vertices of the triangle. This implies A_i and A_{i+1} will not coincide because they lie on adjacent sides of the triangle. Suppose now A_i coincides with A_j ($i < j$), and i is the smallest index with this property. Then in fact $i = 1$. For otherwise A_{i-1} will coincide with A_{j-1}, contradicting the minimality of i. Finally suppose A_1 coincides with A_j, $j \geq 3$, this happens precisely when A_{j-1} is the vertex of the triangle facing side 1. But we know that no vertices of the triangle are in the list, so again impossible.

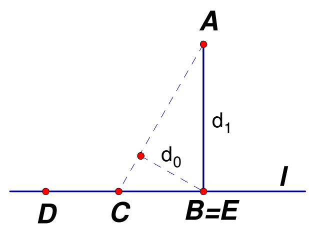
The following example was a problem of Sylvester (1814-1897). Accordingly Sylvester was annoyed to find that he was unable to tackle this deceptively simple problem. It was later solved by the technique of descent. The idea is to consider the smallest possible element with a certain property.

Example 5 (Sylvester's Problem): Given n ($n \geq 3$) points on the plane. If a line passing through any two points also passes through a third point of the set, then prove that all the points lie on the same line.

Solution. We prove an equivalent statement. Namely if there are n ($n \geq 3$) points on the plane and such that they are not on the same line, then there exists a line passing through exactly two points.

Now there are finitely many lines that may be formed by the points of the point set. Given such a line, there is at least one point of the set which does not lie on the line. We then consider the distance between the point and the line. Finally we list all such distances as $d_1 \leq d_2 \leq \cdots \leq d_m$, namely d_1 is the minimum distance between all possible points and all possible lines, say it is the distance between A and the line l. We now proceed to show that l contains exactly two points of the point set.

Suppose not, say points B, C and D of the point set also lie on l. From A, draw the line AE perpendicular to l, with E on l. If E is one of the B, C or D, say E and B coincide, we have the picture



Now $AB = d_1$. However if we draw a perpendicular line from B to AC, then we will get a distance d_0 less than d_1, contradicting its minimality. Similarly if E coincides with C or D, we can also obtain a smaller distance.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for submitting solutions is ***December 10, 2005.***

Problem 236. Alice and Barbara order a pizza. They choose an arbitrary point P, different from the center of the pizza and they do three straight cuts through P, which pairwise intersect at 60° and divide the pizza into 6 pieces. The center of the pizza is not on the cuts. Alice chooses one piece and then the pieces are taken clockwise by Barbara, Alice, Barbara, Alice and Barbara. Which piece should Alice choose first in order to get more pizza than Barbara? (*Source: 2002 Slovenian National Math Olympiad*)

Problem 237. Determine (with proof) all polynomials p with real coefficients such that $p(x)\,p(x+1) = p(x^2)$ holds for every real number x.
(*Source*: *2000 Bulgarian Math Olympiad*)

Problem 238. For which positive integers n, does there exist a permutation $(x_1, x_2, \ldots, x_n)$ of the numbers 1, 2, …, n such that the number $x_1 + x_2 + \cdots + x_k$ is divisible by k for every $k\in\{1,2, \ldots, n\}$?
(*Source: 1998 Nordic Mathematics Contest*)

Problem 239. (*Due to José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) In any acute triangle ABC, prove that

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right)$$
$$\le\frac{\sqrt{2}}{2}\left(\frac{a+b}{\sqrt{a^2+b^2}}+\frac{b+c}{\sqrt{b^2+c^2}}+\frac{c+a}{\sqrt{c^2+a^2}}\right).$$

Problem 240. Nine judges independently award the ranks of 1 to 20 to twenty figure-skaters, with no ties. No two of the rankings awarded to any figure-skater differ by more than 3. The nine rankings of each are added. What is the maximum of the lowest of the sums? Prove your answer is correct.

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Solutions

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Problem 231. On each planet of a star system, there is an astronomer observing the nearest planet. The number of planets is odd, and pairwise distances between them are different. Prove that at least one planet is not under observation.
(*Source*: *1966 Soviet Union Math Olympiad*)

Solution. **CHAN Pak Woon** (HKU Math, Year 1), **LEE Kai Seng** (HKUST), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Let there be n planets. The case of $n = 1$ is clear. For $n \ge 3$, suppose the case $n-2$ is true. For the two closest planets, the astronomers on them observe each other. If any of the remaining $n - 2$ astronomers observes one of these two planets, then we do not have enough astronomers to observe the $n - 2$ remaining planets. Otherwise, we can discard these two closest planets and apply the case $n - 2$.

Commended Solvers: **Roger CHAN** (Vancouver, Canada) and **Anna Ying PUN** (STFA Leung Kau Kui College).

Problem 232. B and C are points on the segment AD. If $AB = CD$, prove that $PA+PD \ge PB+PC$ for any point P.
(*Source*: *1966 Soviet Union Math Olympiad*)

Solution 1. **Anna Ying PUN** (STFA Leung Kau Kui College).

Suppose P is not on line AD. Let P' be such that $PAP'D$ is a parallelogram. Now $AB=CD$ implies $PBP'C$ is a parallelogram. By interchanging B and C, we may assume B is between A and C. Let line PB intersect AP' at F. Then $PA+PD = PA+AP' = PA+AF +FP' > PF + FP' = PB + BF + FP' > PB +BP' = PB + PC$. The case P is on line AD is easy to check.

Solution 2. **LEE Kai Seng** (HKUST).

Consider the complex plane with line AD as the real axis and the origin at the midpoint O of segment AD. Let the complex numbers correspond to A, B, P be a, b, p, respectively. Since $|p \pm a|^2 = |p|^2 \pm 2\mathrm{Re}\, ap + a^2$, so $(PA + PD)^2 = 2(|p|^2 + |p^2-a^2| + a^2)$. Then

$$(PA+PD)^2-(PB+PC)^2$$
$$=2(|p^2-a^2|+a^2-b^2-|p^2-b^2|)\ge 0$$

by the triangle inequality. So $PA+PD \ge PB+PC$.

Also equality holds if and only if the ratio of p^2-a^2 and a^2-b^2 is a nonnegative number, which is the same as $p\ge a$ or $p\le -a$.

Commended Solvers: **CHAN Wai Hung** (Carmel Divine Grace Foundation Secondary School, Form 7), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Problem 233. Prove that every positive integer not exceeding $n!$ can be expressed as the sum of at most n distinct positive integers each of which is a divisor of $n!$.

Solution. **CHAN Ka Lok** (STFA Leung Kau Kui College, Form 6), **G.R.A. 20 Math Problem Group** (Roma, Italy), **LEE Kai Seng** (HKUST) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

We prove by induction on n. The case $n = 1$ is clear. Suppose case $n - 1$ is true. For $n > 1$, let $1 \le k \le n!$ and let q and r be such that $k = qn + r$ with $0 \le r < n$. Then $0 \le q \le (n-1)!$. By the case $n-1$, q can be expressed as $d_1 + d_2 + \cdots + d_m$, where $m \le n - 1$ and d_i is a divisor of $(n - 1)!$ and d_i's are distinct. Omitting r if $r = 0$, we see $d_1n + d_2n + \cdots + d_mn + r$ is a desired expansion of k.

Problem 234. Determine all polynomials $P(x)$ of the smallest possible degree with the following properties:

a) The coefficient of the highest power is 200.
b) The coefficient of the lowest power for which it is not equal to zero is 2.
c) The sum of all its coefficients is 4.
d) $P(-1) = 0$, $P(2) = 6$ and $P(3) = 8$.

(*Source: 2002 Austrian National Competition*)

Solution. **CHAN Pak Woon** (HKU Math, Year 1), **G.R.A. 20 Math Problem Group** (Roma, Italy), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Note c) is the same as $P(1) = 4$. For

$$P(x)=200x(x+1)(x-1)(x-2)(x-3)+2x+2$$
$$=200x^5-1000x^4+1000x^3+1000x^2-1198x+2,$$

all conditions are satisfied. Assume R is another such polynomial with degree at most 5. Then P and R agree at –1, 1, 2, 3. So

$$P(x)–R(x) = (x+1)(x–1)(x–2)(x–3)S(x)$$

with degree of S at most 1. If S is constant, then b) implies $P(0)–R(0)$ is 0 or 2. Then $S(x) = –1/3$ and we get

$$R(x) = P(x)+(x+1)(x–1)(x–2)(x–3)/3 = 200x^5 + \cdots –1196⅓x,$$

which fails b). If S is of degree 1, then a) and b) imply $S(x)=200x–1/3$ and we will get $R(x) =$

$$P(x) – (x+1)(x–1)(x–2)(x–3)(200x–1/3) = x^4/3 + \cdots,$$

which fails a). So no such R exists and P is the unique answer.

Problem 235. Forty-nine students solve a set of three problems. The score for each problem is an integer from 0 to 7. Prove that there exist two students A and B such that, for each problem, A will score at least as many points as B.
(*Source: 29th IMO Unused Problem*)

Solution. **LEE Kai Seng** (HKUST) and **Anna Ying PUN** (STFA Leung Kau Kui College).

For n = 0,1,2,3, let S_n be the set of ordered pairs $(0,n),(1,n),\cdots,(7–n,n)$ and $(7 – n, n + 1),\cdots, (7 – n, 7)$. Let $S_4 = \{(x,y): x=2 \text{ or } 3; y= 4,5,6 \text{ or } 7\}$ and $S_5 = \{(x,y): x=0 \text{ or } 1; y=4,5,6 \text{ or } 7\}$.

For each student, let his/her score on the first problem be x and on the second problem be y. Note if two students have both of their (x,y) pairs in one of S_0, S_1, S_2 or S_3, then one of them will score at least as many point as the other in each of the first two problems.

Of the 49 pairs (x,y), there are $[49/6]+1 = 9$ of them belong to the same S_n. If this S_n is S_4 or S_5, which has 8 elements, then two of the 9 pairs are the same and the two students will satisfy the desired condition. If the S_n is S_0, S_1, S_2 or S_3, then two of these 9 students will have the same score on the third problem and they will satisfy the desired condition by the note in the last paragraph.

Commended Solvers: **CHAN Pak Woon** (HKU Math, Year 1), **LAW Yan Pui** (Carmel Divine Grace Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Olympiad Corner

(continued from page 1)

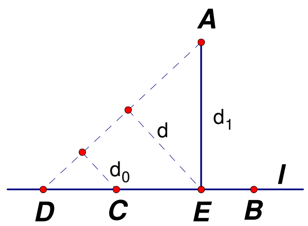
Problem 5. Let ABC be an acute triangle with orthocenter H, incenter I and $AC \neq BC$. The lines CH and CI meet the circumcircle of $\triangle ABC$ for the second time at points D and L, respectively. Prove that $\angle CIH = 90^\circ$ if and only if $\angle IDL = 90^\circ$.

Problem 6. In a group of nine people there are no four every two of which know each other. Prove that the group can be partitioned into four groups such that the people in every group do not know each other.

The Method of Infinite Descent

(continued from page 2)

Now if the perpendicular from A to l does not meet any of B, C or D, then by the pigeonhole principle, there are two points (say C and D) which lie on one side of the perpendicular. Again from the diagram



We draw perpendiculars from E and C to AD, and we observe the distances $d_0 < d < d_1$, again contradicting the minimality of d_1. From the above arguments, we conclude that l contains exactly two points.

From the above example, we have

__Example 6 (Polish MO 1967-68)__: Given n ($n \geq 3$) points on the plane and these points are not on the same line. From any two of these points a line is drawn and altogether k distinct lines are formed. Show that $k \geq n$.

__**Solution.**__ We proceed by induction. Clearly three distinct lines may be drawn from three points not on a line. Hence the statement is true for $n = 3$. Suppose the statement is valid for some $n \geq 3$. Now let $A_1, A_2, \ldots, A_n, A_{n+1}$ be $n + 1$ distinct points which are not on the same line. By Sylvester's "theorem", there exists a line containing exactly two points of the point set, say A_1A_{n+1}.

Let's consider the sets $Z_1 =\{A_1, A_2, \ldots, A_n\}$and $Z_2 =\{A_2, A_3, \ldots, A_n, A_{n+1}\}$ Clearly at least one of the point sets does not lie on a line. If $A_1, A_2, \ldots, A_n$ do not lie on a line, by the inductive hypothesis, we can form at least n lines using these points. As A_{n+1} is not one of the members of Z_1, so A_1A_{n+1} will form a new line, (A_1A_{n+1} contains no other points of the set) and we have at least n + 1 lines. If $A_2, A_3, \ldots, A_n, A_{n+1}$ do not lie on a line, then again we can form at least n lines using these points. As A_1 is not one of the members of Z_2, so A_1A_{n+1} will form a new line, (A_1A_{n+1} contains no other points of the set) and we have at least n + 1 lines.

The method of infinite descent was used to prove a hard IMO problem.

__Example 7 (IMO 1988)__: Prove that if positive integers a and b are such that $ab + 1$ divides $a^2 + b^2$, then $(a^2 + b^2)/(ab + 1)$ is a perfect square.

__**Solution.**__ Assume $(a^2 + b^2)/(ab + 1) = k$ and k is not a perfect square. After rearranging we have $a^2 – kab + b^2 = k$, with $a > 0$ and $b > 0$. Assume now (a_0, b_0) is a solution of the Diophantine equation and such that $a_0 + b_0$ is as *small* as possible. By symmetry we may assume $a_0 \geq b_0 > 0$. Fixing b_0 and k, we may assume a_0 is a solution of the quadratic equation

$$x^2 – kb_0x + b_0^2 – k = 0.$$

Now let the other root of the equation be a'. Using sum and product of roots, we have $a_0 + a' = kb_0$ and $a_0a' = b_0^2 – k$. The first equation implies a' is an integer. The second equation implies $a' \neq 0$, otherwise k is a perfect square, contradicting our hypothesis. Now a' also cannot be negative, otherwise

$$a'^2 – ka'b_0 + b_0^2 \geq a'^2 + k + b_0^2 > k.$$

Hence $a' > 0$. Finally

$$a' = \frac{b_0^2 - k}{a_0} \leq \frac{b_0^2 - 1}{a_0} \leq \frac{a_0^2 - 1}{a_0} < a_0.$$

This implies (a', b_0) is a positive integer solution of $a^2 – kab + b^2 = k$, and $a' + b_0 < a_0 + b_0$, contradicting the minimality of $a_0 + b_0$. Therefore k must be a perfect square.

Mathematical Excalibur

Volume 10, Number 5 *December 2005 – January 2006*

Olympiad Corner

Below is the Czech-Polish-Slovak Match held in Zwardon on June 20-21, 2005.

Problem 1. Let n be a given positive integer. Solve the system of equations

$$x_1 + x_2^2 + x_3^3 + \cdots + x_n^n = n,$$
$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = \frac{n(n+1)}{2}$$

in the set of nonnegative real numbers $x_1, x_2, \ldots, x_n$.

Problem 2. Let a convex quadrilateral $ABCD$ be inscribed in a circle with center O and circumscribed to a circle with center I, and let its diagonals AC and BD meet at a point P. Prove that the points O, I and P are collinear.

Problem 3. Determine all integers $n \ge 3$ such that the polynomial $W(x) = x^n - 3x^{n-1} + 2x^{n-2} + 6$ can be expressed as a product of two polynomials with positive degrees and integer coefficients.

Problem 4. We distribute $n \ge 1$ labelled balls among nine persons $A, B, C, D, E, F, G, H, I$. Determine in how many ways

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 12, 2006***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Using Tangent Lines to Prove Inequalities

Kin-Yin Li

For students who know calculus, sometimes they become frustrated in solving inequality problems when they do not see any way of using calculus. Below we will give some examples, where finding the equation of a tangent line is the critical step to solving the problems.

Example 1. Let a,b,c,d be positive real numbers such that $a + b + c + d = 1$. Prove that

$$6(a^3+b^3+c^3+d^3) \ge (a^2+b^2+c^2+d^2) + 1/8.$$

Solution. We have $0 < a, b, c, d < 1$. Let $f(x) = 6x^3 - x^2$. (Note: Since there is equality when $a = b = c = d = 1/4$, we consider the graph of $f(x)$ and its tangent line at $x = 1/4$. By a simple sketch, it seems the tangent line is below the graph of $f(x)$ on the interval (0,1). Now the equation of the tangent line at $x = 1/4$ is $y = (5x - 1)/8$.) So we claim that for $0 < x < 1$, $f(x) = 6x^3 - x^2 \ge (5x - 1)/8$. This is equivalent to $48x^3 - 8x^2 - 5x + 1 \ge 0$. (Note: Since the graphs intersect at $x = 1/4$, we expect $4x - 1$ is a factor.) Indeed, $48x^3 - 8x^2 - 5x + 1 = (4x - 1)^2 (3x + 1) \ge 0$ for $0 < x < 1$. So the claim is true. Then $f(a) + f(b) + f(c) + f(d) \ge 5(a + b + c + d)/8 - 4/8 = 1/8$, which is equivalent to the required inequality.

Example 2. (*2003 USA Math Olympiad*) Let a,b,c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8.$$

Solution. Setting $a' = a/(a + b + c)$, $b' = b/(a + b + c)$, $c' = c/(a + b + c)$ if necessary, we may assume $0 < a, b, c < 1$ and $a + b + c = 1$. Then the first term on the left side of the inequality is equal to

$$f(a)=\frac{(a+1)^2}{2a^2+(1-a)^2}=\frac{a^2+2a+1}{3a^2-2a+1}.$$

(Note: When $a = b = c = 1/3$, there is equality. A simple sketch of $f(x)$ on [0,1] shows the curve is below the tangent line at $x = 1/3$, which has the equation $y = (12x + 4)/3$.) So we claim that

$$\frac{a^2+2a+1}{3a^2-2a+1}\le\frac{12a+4}{3}$$

for $0 < a < 1$. Multiplying out, we see this is equivalent to $36a^3 - 15a^2 - 2a + 1 \ge 0$ for $0 < a < 1$. (Note: Since the curve and the line intersect at $a = 1/3$, we expect $3a-1$ is a factor.) Indeed, $36a^3 - 15a^2 - 2a + 1 = (3a - 1)^2(4a + 1) \ge 0$ for $0 < a < 1$. Finally adding the similar inequality for b and c, we get the desired inequality.

The next example looks like the last example. However, it is much more sophisticated, especially without using tangent lines. The solution below is due to Titu Andreescu and Gabriel Dospinescu.

Example 3. (*1997 Japanese Math Olympiad*) Let a,b,c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\ge\frac{3}{5}.$$

Solution. As in the last example, we may assume $0 < a, b, c < 1$ and $a + b + c = 1$. Then the first term on the left become $\frac{(1-2a)^2}{(1-a)^2+a^2}=2-\frac{2}{1+(1-2a)^2}$.

Next, let $x_1 = 1 - 2a$, $x_2 = 1 - 2b$, $x_3 = 1 - 2c$, then $x_1 + x_2 + x_3 = 1$, but $-1 < x_1, x_2, x_3 < 1$. In terms of x_1, x_2, x_3, the desired inequality is

$$\frac{1}{1+x_1^2}+\frac{1}{1+x_2^2}+\frac{1}{1+x_3^2}\le\frac{27}{10}.$$

(Note: As in the last example, we consider the equation of the tangent line to $f(x) = 1/(1 + x^2)$ at $x = 1/3$, which is $y = 27(-x + 2)/50$.) So we claim that $f(x) \le 27(-x + 2)/50$ for $-1 < x < 1$. This is equivalent to $(3x - 1)^2(4 - 3x) \ge 0$. Hence the claim is true for $-1 < x < 1$. Then $f(x_1) + f(x_2) + f(x_3) \le 27/10$ and the desired inequality follows.

Schur's Inequality

Kin Yin Li

Sometimes in proving an inequality, we do not see any easy way. It will be good to know some brute force methods in such situation. In this article, we introduce a simple inequality that turns out to be very critical in proving inequalities by brute force.

Schur's Inequality. For any $x, y, z \ge 0$ and any real number r,

$$x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-x)(z-y) \ge 0.$$

Equality holds if and only if $x = y = z$ or two of x, y, z are equal and the third is zero.

Proof. Observe that the inequality is symmetric in x, y, z. So without loss of generality, we may assume $x \ge y \ge z$. We can rewrite the left hand side as $x^r(x-y)^2+(x^r-y^r+z^r)(x-y)(y-z)+z^r(y-z)^2$. The first and third terms are clearly nonnegative. For the second term, if $r \ge 0$, then $x^r \ge y^r$. If $r < 0$, then $z^r \ge y^r$. Hence, $x^r-y^r+z^r \ge 0$ and the second term is nonnegative. So the sum of all three terms is nonnegative. In case $x \ge y \ge z$, equality holds if and only if $x = y$ first and z equals to them or zero.

In using the Schur's inequality, we often expand out expressions. So to simplify writing, we introduce the *symmetric sum* notation $\sum_{sym} f(x,y,z)$ to denote the sum of the six terms $f(x,y,z)$, $f(x,z,y)$, $f(y,z,x)$, $f(y,x,z)$, $f(z,x,y)$ and $f(z,y,x)$. In particular,

$$\sum_{sym} x^3 = 2x^3 + 2y^3 + 2z^3,$$

$$\sum_{sym} x^2y = x^2y+x^2z+y^2z+y^2x+z^2x+z^2y \text{ and}$$

$$\sum_{sym} xyz = 6xyz.$$

Similarly, for a function of n variables, the symmetric sum is the sum of all $n!$ terms, where we take all possible permutations of the n variables.

The $r = 1$ case of Schur's inequality is $x(x-y)(x-z) + y(y-x)(y-z) + z(z-x)(z-y) = x^3 + y^3 + z^3 - (x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) + 3xyz \ge 0$. In symmetric sum notation, it is

$$\sum_{sym}(x^3 - 2x^2y + xyz) \ge 0.$$

By expanding both sides and rearranging terms, each of the following inequalities is equivalent to the $r = 1$ case of Schur's inequality. These are common disguises.

a) $x^3+y^3+z^3+3xyz \ge xy(x+y)+yz(y+z) +zx(z+x)$,

b) $xyz \ge (x+y-z)(y+z-x)(z+x-y)$,

c) $4(x+y+z)(xy+yz+zx) \le (x+y+z)^3+9xyz$.

Example 1. (*2000 IMO*) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$(a-1+\frac{1}{b})(b-1+\frac{1}{c})(c-1+\frac{1}{a}) \le 1.$$

Solution. Let $x = a$, $y = 1$, $z = 1/b = ac$. Then $a = x/y$, $b = y/z$ and $c = z/x$. Substituting these into the desired inequality, we get

$$\frac{(x-y+z)}{y}\frac{(y-z+x)}{z}\frac{(z-x+y)}{x} \le 1,$$

which is disguise b) of the $r = 1$ case of Schur's inequality.

Example 2. (*1984 IMO*) Prove that

$$0 \le yz + zx + xy - 2xyz \le 7/27,$$

where x, y, z are nonnegative real numbers such that $x + y + z = 1$.

Solution. In Schur's inequality, all terms are of the same degree. So we first change the desired inequality to one where all terms are of the same degree. Since $x + y + z = 1$, the desired inequality is the same as

$$0 \le (x+y+z)(yz+zx+xy) - 2xyz \le \frac{7(x+y+z)^3}{27}.$$

Expanding the middle expression, we get $xyz + \sum_{sym} x^2y$, which is clearly nonnegative and the left inequality is proved. Expanding the rightmost expression and subtracting the middle expression, we get

$$\frac{7}{54}\sum_{sym}(x^3 - \frac{12}{7}x^2y + \frac{5}{7}xyz). \qquad (1)$$

By Schur's inequality, we have

$$\sum_{sym}(x^3 - 2x^2y + xyz) \ge 0. \qquad (2)$$

By the AM-GM inequality, we have

$$\sum_{sym} x^2y \ge 6(x^6y^6z^6)^{1/6} = \sum_{sym} xyz,$$

which is the same as

$$\sum_{sym}(x^2y - xyz) \ge 0. \qquad (3)$$

Multiplying (3) by 2/7 and adding it to (2), we see the symmetric sum in (1) is nonnegative. So the right inequality is proved.

Example 3. (*2004 APMO*) Prove that

$$(a^2+2)(b^2+2)(c^2+2) \ge 9(ab+bc+ca)$$

for any positive real numbers a,b,c.

Solution. Expanding and expressing in symmetric sum notation, the desired inequality is

$$(abc)^2 + \sum_{sym}(a^2b^2+2a^2) + 8 \ge \frac{9}{2}\sum_{sym} ab.$$

As $a^2+b^2 \ge 2ab$, we get $\sum_{sym} a^2 \ge \sum_{sym} ab$.

As $a^2b^2 + 1 \ge 2ab$, we get

$$\sum_{sym} a^2b^2 + 6 \ge 2\sum_{sym} ab.$$

Using these, the problem is reduced to showing

$$(abc)^2 + 2 \ge \sum_{sym}(ab - \frac{1}{2}a^2).$$

To prove this, we apply the AM-GM inequality twice and disguise c) of the $r = 1$ case of Schur's inequality as follow:

$$\begin{aligned}(abc)^2+2 &\ge 3(abc)^{2/3}\\ &\ge 9abc/(a+b+c)\\ &\ge 4(ab+bc+ca)-(a+b+c)^2\\ &= 2(ab+bc+ca)-(a^2+b^2+c^2)\\ &= \sum_{sym}(ab - \frac{1}{2}a^2).\end{aligned}$$

Example 4. (*2000 USA Team Selection Test*) Prove that for any positive real numbers a, b, c, the following inequality holds

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \le \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***February 12, 2006.***

Problem 241. Determine the smallest possible value of

$$S = a_1 \cdot a_2 \cdot a_3 + b_1 \cdot b_2 \cdot b_3 + c_1 \cdot c_2 \cdot c_3,$$

if a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 is a permutation of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. (*Source: 2002 Belarussian Math. Olympiad*)

Problem 242. Prove that for every positive integer n, 7 is a divisor of $3^n + n^3$ if and only if 7 is a divisor of $3^n n^3 + 1$. (*Source: 1995 Bulgarian Winter Math Competition*)

Problem 243. Let R^+ be the set of all positive real numbers. Prove that there is no function $f: R^+ \to R^+$ such that

$$(f(x))^2 \geq f(x+y)(f(x)+y)$$

for arbitrary positive real numbers x and y. (*Source: 1998 Bulgarian Math Olympiad*)

Problem 244. An infinite set S of coplanar points is given, such that every three of them are not collinear and every two of them are not nearer than 1cm from each other. Does there exist any division of S into two disjoint infinite subsets R and B such that inside every triangle with vertices in R is at least one point of B and inside every triangle with vertices in B is at least one point of R? Give a proof to your answer. (*Source: 2002 Albanian Math Olympiad*)

Problem 245. *ABCD* is a concave quadrilateral such that $\angle BAD = \angle ABC = \angle CDA = 45^\circ$. Prove that $AC = BD$.

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Solutions

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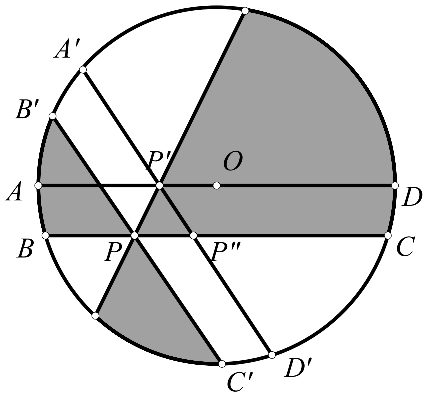
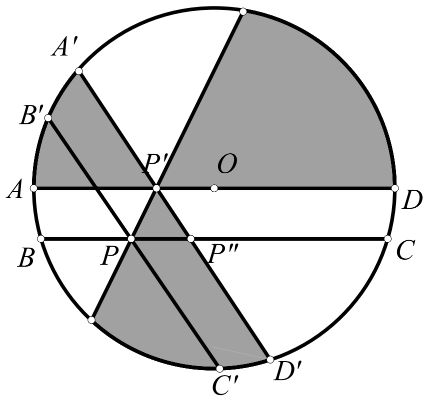
Problem 236. Alice and Barbara order a pizza. They choose an arbitrary point P, different from the center of the pizza and they do three straight cuts through P, which pairwise intersect at 60° and divide the pizza into 6 pieces. The center of the pizza is not on the cuts. Alice chooses one piece and then the pieces are taken clockwise by Barbara, Alice, Barbara, Alice and Barbara. Which piece should Alice choose first in order to get more pizza than Barbara? (*Source: 2002 Slovenian National Math Olympiad*)

Solution. (Official Solution)

Let Alice choose the piece that contains the center of the pizza first. We claim that the total area of the shaded regions below is greater than half of the area of the pizza.



Without loss of generality, we can assume the center of the pizza is at the origin O and one of the cuts is parallel to the x-axis (that is, BC is parallel to AD in the picture). Let P' be the intersection of the x-axis and the 60°-cut. Let $A'D'$ be parallel to the 120°-cut $B'C'$. Let P'' be the intersection of BC and $A'D'$. Then $\Delta PP'P''$ is equilateral. This implies the belts $ABCD$ and $A'B'C'D'$ have equal width. Since $AD > A'D'$, the area of the belt $ABCD$ is greater than the area of the belt $A'B'C'D'$. Now when the area of the belt $ABCD$ is subtracted from the total area of the shaded regions and the area of $A'B'C'D'$ is then added,



we get exactly half the area of the pizza. Therefore, the claim follows.

Problem 237. Determine (with proof) all polynomials p with real coefficients such that $p(x)\,p(x+1) = p(x^2)$ holds for every real number x. (*Source*: *2000 Bulgarian Math Olympiad*)

Solution. **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Let $p(x)$ be such a polynomial. In case $p(x)$ is a constant polynomial, $p(x)$ must be 0 or 1. For the case $p(x)$ is nonconstant, let r be a root of $p(x)$. Then setting $x = r$ and $x + 1 = r$ in the equation, we see r^2 and $(r-1)^2$ are also roots of $p(x)$. Also, r^2 is a root implies $(r^2-1)^2$ is also a root. If $0 < |r| < 1$ or $|r| > 1$, then $p(x)$ will have infinitely many roots r, r^2, r^4, …, a contradiction. So $|r| = 0$ or 1 for every root r.

The case $|r| = 1$ and $|r-1| = 1$ lead to $r = (1 \pm i\sqrt{3})/2$, but then $|r^2 - 1| \neq 0$ or 1, a contradiction. Hence, either $|r| = 0$ or $|r-1| = 0$, that is, $r = 0$ or 1.

So $p(x) = x^m(x-1)^n$ for some nonnegative integers m, n. Putting this into the equation, we find $m = n$. Conversely, $p(x) = x^m(x-1)^m$ is easily checked to be a solution for every nonnegative integer m.

Problem 238. For which positive integers n, does there exist a permutation $(x_1, x_2, \ldots, x_n)$ of the numbers 1, 2, …, n such that the number $x_1 + x_2 + \cdots + x_k$ is divisible by k for every $k \in \{1,2, \ldots, n\}$? (*Source: 1998 Nordic Mathematics Contest*)

Solution. **G.R.A. 20 Math Problem Group** (Roma, Italy), **LEE Kai Seng** (HKUST), **LO Ka Wai** (Carmel Divine Grace Foundation Secondary School, Form 7), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

For a solution n, since $x_1 + x_2 + \cdots + x_n = n(n+1)/2$ is divisible by n, n must be odd. The cases $n = 1$ and $n = 3$ (with permutation (1,3,2)) are solutions.

Assume $n \geq 5$. Then $x_1 + x_2 + \cdots + x_{n-1} = n(n+1)/2 - x_n \equiv 0 \pmod{n-1}$ implies $x_n \equiv (n+1)/2 \pmod{n-1}$. Since $1 \leq x_n \leq n$ and $3 \leq (n+1)/2 \leq n-2$, we get $x_n = (n+1)/2$. Similarly, $x_1 + x_2 + \cdots + x_{n-2} = n(n+1)/2 - x_n - x_{n-1} \equiv 0 \pmod{n-2}$ implies $x_{n-1} \equiv (n+1)/2 \pmod{n-2}$. Then also $x_{n-1} = (n+1)/2$, which leads to $x_n = x_{n-1}$, a contradiction. Therefore, $n = 1$ and 3 are the only solutions.

Problem 239. (*Due to José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) In any acute triangle *ABC*, prove that

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right)$$
$$\le\frac{\sqrt{2}}{2}\left(\frac{a+b}{\sqrt{a^2+b^2}}+\frac{b+c}{\sqrt{b^2+c^2}}+\frac{c+a}{\sqrt{c^2+a^2}}\right).$$

Solution. (Proposer's Solution)

By cosine law and the *AM-GM* inequality,

$$1-2\sin^2\frac{A}{2}=\cos A=\frac{b^2+c^2-a^2}{2bc}$$
$$\ge\frac{b^2+c^2-a^2}{b^2+c^2}=1-\frac{a^2}{b^2+c^2}.$$

So $\sin\frac{A}{2}\le\frac{a}{\sqrt{2(b^2+c^2)}}$.

By sine law and $\cos(A/2) = \sin((B+C)/2)$, we get

$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=$$

$$\frac{2\sin(A/2)\cos(A/2)}{2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})}=\frac{\sin(A/2)}{\cos(\frac{B-C}{2})}.$$

Then

$$\cos(\frac{B-C}{2})=\frac{b+c}{a}\sin\frac{A}{2}\le\frac{\sqrt{2}}{2}\frac{b+c}{\sqrt{b^2+c^2}}.$$

Adding two similar inequalities, we get the desired inequality.

Commended solvers: **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Problem 240. Nine judges independently award the ranks of 1 to 20 to twenty figure-skaters, with no ties. No two of the rankings awarded to any figure-skater differ by more than 3. The nine rankings of each are added. What is the maximum of the lowest of the sums? Prove your answer is correct. (*Source: 1968 All Soviet Union Math Competitions*)

Solution. **WONG Kwok Kit** (Carmel Divine Grace Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Suppose the 9 first places go to the same figure skater. Then 9 is the lowest sum.

Suppose the 9 first places are shared by two figure skaters. Then one of them gets at least 5 first places and that skater's other rankings are no worse than fourth places. So the lowest sum is at most $5 \times 1 + 4 \times 4 = 21$.

Suppose the 9 first places are shared by three figure skaters. Then the other 18 rankings of these figure skaters are no worse than 9 third and 9 fourth places. Then the lowest sum is at most $9(1 + 3 + 4)/3 = 24$.

Suppose the 9 first places are shared by four figure skaters. Then their rankings must be all the first, second, third and fourth places. So the lowest sum is at most $9(1 + 2 + 3 + 4)/4 < 24$.

Suppose the 9 first places are shared by $k > 4$ figure skaters. On one hand, these k skaters have a total of $9k > 36$ rankings. On the other hand, these k skaters can only be awarded first to fourth places, so they can have at most $4 \times 9 = 36$ rankings all together, a contradiction.

Now 24 is possible if skaters *A, B, C* all received 3 first, 3 third and 3 fourth places; skater *D* received 5 second and 4 fifth places; skater *E* received 4 second and 5 fifth places; and skater *F* received 9 sixth places, …, skater *T* received 9 twentieth places. Therefore, 24 is the answer.

Olympiad Corner

(continued from page 1)

Problem 4. (Cont.) it is possible to distribute the balls under the condition that *A* gets the same number of balls as the persons *B, C, D* and *E* together.

Problem 5. Let *ABCD* be a given convex quadrilateral. Determine the locus of the point *P* lying inside the quadrilateral *ABCD* and satisfying

$$[PAB]\cdot[PCD] = [PBC]\cdot[PDA],$$

where $[XYZ]$ denotes the area of triangle *XYZ*.

Problem 6. Determine all pairs of integers (x,y) satisfying the equation

$$y(x + y) = x^3 - 7x^2 + 11x - 3.$$

Schur's Inequality

(continued from page 2)

Solution. From the last part of the solution of example 3, we get

$$3(xyz)^{2/3} \ge 2(xy + yz + zx) - (x^2 + y^2 + z^2)$$

for any $x, y, z > 0$. (Note: this used Schur's inequality.) Setting $x=\sqrt{a}$, $y=\sqrt{b}$ and $z=\sqrt{c}$ and arranging terms, we get

$$a+b+c-3\sqrt[3]{abc}$$
$$\le 2(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca})$$
$$=(\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2+(\sqrt{c}-\sqrt{a})^2$$
$$\le 3\max\{(\sqrt{a}-\sqrt{b})^2,(\sqrt{b}-\sqrt{c})^2,(\sqrt{c}-\sqrt{a})^2\}.$$

Dividing by 3, we get the desired inequality.

Example 5. (*2003 USA Team Selection Test*) Let a,b,c be real numbers in the interval $(0, \pi/2)$. Prove that

$$\frac{\sin a\sin(a-b)\sin(a-c)}{\sin(b+c)}+\frac{\sin b\sin(b-c)\sin(b-a)}{\sin(c+a)}$$
$$+\frac{\sin c\sin(c-a)\sin(c-b)}{\sin(a+b)}\ge 0.$$

Solution. Observe that

$$\sin(u-v)\sin(u+v) = (\cos 2v - \cos 2u)/2 = \sin^2 u - \sin^2 v.$$

Setting $x = \sin^2 a$, $y = \sin^2 b$, $z = \sin^2 c$, in adding up the terms, the left side of the inequality becomes

$$\frac{\sqrt{x}(x-y)(x-z)+\sqrt{y}(y-z)(y-x)+\sqrt{z}(z-x)(z-y)}{\sin(b+c)\sin(c+a)\sin(a+b)}.$$

This is nonnegative by the $r = 1/2$ case of Schur's inequality.

For many more examples on Schur's and other inequalities, we highly recommend the following book.

Titu Andreescu, Vasile Cîrtoaje, Gabriel Dospinescu and Mircea Lascu, Old and New Inequalities, GIL Publishing House, 2004.

Anyone interested may contact the publisher by post to GIL Publishing House, P. O. Box 44, Post Office 3, 450200, Zalau, Romania or by email to gil1993@zalau.astral.ro.