Mathematical Excalibur

Volume 14, Number 1 *March - April, 2009*

Olympiad Corner

The following were the problems of the 2009 Asia-Pacific Math Olympiad.

Problem 1. Consider the following operation on positive real numbers written on a blackboard: *Choose a number r written on the blackboard, erase that number, and then write a pair of real numbers a and b satisfying the condition $2r^2 = ab$ on the board.*

Assume that you start out with just one positive real number r on the blackboard, and apply this operation k^2-1 times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr.

Problem 2. Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:

$$\frac{a_1}{k^2+1}+\frac{a_2}{k^2+2}+\frac{a_3}{k^2+3}+\frac{a_4}{k^2+4}+\frac{a_5}{k^2+5}=\frac{1}{k^2}$$

for $k = 1,2,3,4,5$. Find the value of

$$\frac{a_1}{37}+\frac{a_2}{38}+\frac{a_3}{39}+\frac{a_4}{40}+\frac{a_5}{41}.$$

(Express the value in a single fraction.)

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 7, 2009***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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A Nice Identity

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There are many methods to prove inequalities. In this paper, we would like to introduce to the readers some applications of a nice identity for solving inequalities.

Theorem 0. Let *a, b, c* be real numbers. Then

$$(a+b)(b+c)(c+a)$$
$$= (a+b+c)(ab+bc+ca) - abc.$$

Proof. This follows immediately by expanding both sides.

Corollary 1. Let *a, b, c* be real numbers. If $abc = 1$, then

$$(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-1.$$

Corollary 2. Let *a, b, c* be real numbers. If $ab + bc + ca = 1$, then

$$(a+b)(b+c)(c+a)=a+b+c-abc\ .$$

Next we will give some applications of these facts. The first example is a useful well-known inequality.

Example 1. Let *a, b, c* be nonnegative real numbers. Prove that

$$(a+b)(b+c)(c+a)\geq\frac{8}{9}(a+b+c)(ab+bc+ca).$$

Solution. By the *AM-GM* inequality,

$$\frac{1}{9}(a+b+c)(ab+bc+ca)-abc$$
$$\geq\frac{1}{9}(3\sqrt[3]{abc})(3\sqrt[3]{a^2b^2c^2})-abc=0.$$

Using Theorem 0, we have

$$(a+b)(b+c)(c+a)\geq\frac{8}{9}(a+b+c)(ab+bc+ca).$$

The next example was a problem on the third team selection test of Romania for the Balkan Mathematical Olympiad 2005. Subsequently, it also appeared in the Croatian Team Selection Test 2006.

Example 2. (*Cezar Lupu, Romania 2005; Croatia TST 2006*) Let *a, b, c* be positive real numbers satisfying $(a+b)(b+c)(c+a) = 1$. Prove that

$$ab+bc+ca \leq 3/4.$$

Solution. By the *AM-GM* inequality,

$$a+b+c=\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}$$
$$\geq 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}=\frac{3}{2}$$

and

$$abc=\sqrt{ab}\sqrt{bc}\sqrt{ca}$$
$$\leq\frac{(a+b)(b+c)(c+a)}{8}=\frac{1}{8}.$$

Using Theorem 0, we get

$$1=(a+b)(b+c)(c+a)$$
$$=(a+b+c)(ab+bc+ca)-abc$$
$$\geq\frac{3}{2}(ab+bc+ca)-\frac{1}{8}.$$

Hence $ab+bc+ca\leq\frac{3}{4}$.

The following example was taken from the Vietnamese magazine, Mathematics and Youth Magazine.

Example 3. (*Proposed by Tran Xuan Dang*) Let *a, b, c* be nonnegative real numbers satisfying $abc = 1$. Prove that

$$(a+b)(b+c)(c+a)\geq 2(1+a+b+c).$$

Solution. Using Corollary 1, this is equivalent to

$$(a+b+c)(ab+bc+ca-2)\geq 3.$$

We can obtain this by the *AM-GM* inequality as follows:

$$(a+b+c)(ab+bc+ca-2)$$
$$\geq(3\sqrt[3]{abc})(3\sqrt[3]{a^2b^2c^2}-2)=3.$$

The inequality in the next example is very hard. It was a problem in the Korean Mathematical Olympiad.

Example 4. (*KMO Winter Program Test 2001*) Let *a, b, c* be positive real numbers. Prove that

$$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)}$$
$$\geq abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}.$$

Solution. Dividing by abc, the given inequality becomes

$$\sqrt{\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{b}+\frac{b}{c}\right)}$$
$$\ge 1+\sqrt[3]{\left(\frac{a^2}{bc}+1\right)\left(\frac{b^2}{ca}+1\right)\left(\frac{c^2}{ab}+1\right)}.$$

After the substitution $x = a/b$, $y = b/c$ and $z = c/a$, we get $xyz = 1$. It takes the form

$$\sqrt{(x+y+z)(xy+yz+zx)}$$
$$\ge 1+\sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

Using Corollary 1, the previous inequality becomes

$$\sqrt{(x+y)(y+z)(z+x)+1}$$
$$\ge 1+\sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

Setting $t=\sqrt[3]{(x+y)(y+z)(z+x)}$, we need to prove that

$$\sqrt{t^3+1}\ge 1+t.$$

By the *AM-GM* inequality, we have

$$t=\sqrt[3]{(x+y)(y+z)(z+x)}$$
$$\ge\sqrt[3]{2\sqrt{xy}\,2\sqrt{yz}\,2\sqrt{zx}}=2.$$

Therefore,

$$\sqrt{t^3+1}=\sqrt{(t+1)(t^2-t+1)}$$
$$\ge\sqrt{(t+1)(2t-t+1)}=1+t.$$

In the next example, we will see a nice inequality. It was from a problem in the 2001 USA Math Olympiad Summer Program.

Example 5. (*MOSP 2001*) Let a, b, c be positive real numbers satisfying abc=1. Prove that

$$(a+b)(b+c)(c+a)\ge 4(a+b+c-1).$$

Solution. Using Corollary 1, it suffices to prove that

$$(a+b+c)(ab+bc+ca)-1$$
$$\ge 4(a+b+c-1)$$

or $$ab+bc+ca+\frac{3}{a+b+c}\ge 4.$$

We will use the inequality

$$(x+y+z)^2\ge 3(xy+yz+zx),\qquad (*)$$

which after expansion and cancelling common terms amounts to

$$x^2+y^2+z^2-xy-yz-zx$$
$$=\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)\ge 0.$$

Using (*), it is easy to see that

$$(ab+bc+ca)^2\ge 3(ab\cdot bc+bc\cdot ca+ca\cdot ab)$$
$$=3(a+b+c).\qquad (**)$$

By the *AM-GM* inequality and (**),

$$ab+bc+ca+\frac{3}{a+b+c}$$
$$=3\left(\frac{ab+bc+ca}{3}\right)+\frac{3}{a+b+c}$$
$$\ge 4\sqrt[4]{\frac{3(ab+bc+ca)^3}{3^3(a+b+c)}}$$
$$\ge 4\sqrt[4]{\frac{3(3\sqrt[3]{a^2b^2c^2})(3(a+b+c))}{3^3(a+b+c)}}=4.$$

Next, we will show some nice trigonometric inequalities can also be proved using Theorem 0.

Example 6. For a triangle ABC, prove that

(i) $\sin A+\sin B+\sin C\le 3\sqrt{3}/2.$

(ii) $\cos A+\cos B+\cos C\le 3/2.$

Solution. By the substitutions

$a=\tan(A/2)$, $b=\tan(B/2)$, $c=\tan(C/2)$,

we get $ab+bc+ca = 1$.

Using the facts $\sin 2x= (2\tan x)/(1+\tan^2 x)$ and $1+a^2=a^2+ab+bc+ca=(a+b)(a+c)$, inequality (i) becomes

$$\frac{a}{1+a^2}+\frac{b}{1+b^2}+\frac{c}{1+c^2}\le\frac{3\sqrt{3}}{4},$$

which is the same as

$$\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)}\le\frac{3\sqrt{3}}{4}.$$

Clearing the denominators, this simplifies to $(a+b)(b+c)(c+a)\ge 8\sqrt{3}/9.$

To prove this, use the *AM-GM* inequality to get

$$1=ab+bc+ca\ge 3\sqrt[3]{a^2b^2c^2},$$

which is

$$abc\le\sqrt{3}/9.\qquad (***)$$

Next, by (*),

$$a+b+c\ge\sqrt{3(ab+bc+ca)}=\sqrt{3}.\qquad (****)$$

Finally, by Corollary 2,

$$(a+b)(b+c)(c+a)=a+b+c-abc$$
$$\ge\sqrt{3}-\frac{\sqrt{3}}{9}=\frac{8\sqrt{3}}{9}.$$

Next, using $\cos 2x=(1-\tan^2 x)/(1+\tan^2 x)$, inequality (ii) becomes

$$\frac{1-a^2}{1+a^2}+\frac{1-b^2}{1+b^2}+\frac{1-c^2}{1+c^2}\le\frac{3}{2}.$$

Using $1+a^2=a^2+ab+bc+ca=(a+b)(a+c)$ in the denominators, doing the addition on the left and applying Corollary 2 in the common denominator, we can see the above inequality is the same as

$$\frac{2(a+b+c)-[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{a+b+c-abc}\le\frac{3}{2}.$$

Observe that $a^2(b+c)+b^2(c+a)+c^2(a+b)=(a+b+c)(ab+bc+ca)-3abc=a+b+c-3abc$. So the inequality becomes

$$\frac{2(a+b+c)-(a+b+c-3abc)}{a+b+c-abc}\le\frac{3}{2},$$

which simplifies to $a+b+c\ge 9abc$. This follows easily from (***) and (****).

Finally, we have some exercises for the readers.

Exercise 1. (*Due to Nguyen Van Ngoc*) Let a, b, c be positive real numbers. Prove that

$$abc(a+b+c)\le\frac{3\left((a+b)(b+c)(c+a)\right)^{4/3}}{16}.$$

Exercise 2. (*Due to Vedula N. Murty*) Let a, b, c be positive real numbers. Prove that

$$\frac{a+b+c}{3}\le\frac{1}{4}\sqrt[3]{\frac{(a+b)^2(b+c)^2(c+a)^2}{abc}}.$$

Exercise 3. (*Carlson's inequality*) Let a, b, c be positive real numbers. Prove that

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}\ge\sqrt{\frac{ab+bc+ca}{3}}.$$

Exercise 4. Let ABC be a triangle. Prove that

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\ge\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\sqrt{3}.$$

References

[1] Hojoo Lee, Topics in Inequalities -Theorems and Techniques, 2007.

[2] Pham Kin Hung, Secrets in Inequalities (in Vietnames), 2006.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***May 7, 2009.***

Problem 321. Let *AA', BB'* and *CC'* be three non-coplanar chords of a sphere and let them all pass through a common point P inside the sphere. There is a (unique) sphere S_1 passing through A, B, C, P and a (unique) sphere S_2 passing through A', B', C', P.

If S_1 and S_2 are externally tangent at P, then prove that $AA'=BB'=CC'$.

Problem 322. (*Due to Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam*) Let a, b, c be positive real numbers satisfying the condition $a+b+c=3$. Prove that

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}\ge 2.$$

Problem 323. Prove that there are infinitely many positive integers n such that 2^n+2 is divisible by n.

Problem 324. $ADPE$ is a convex quadrilateral such that $\angle ADP = \angle AEP$. Extend side AD beyond D to a point B and extend side AE beyond E to a point C so that $\angle DPB = \angle EPC$. Let O_1 be the circumcenter of ΔADE and let O_2 be the circumcenter of ΔABC.

If the circumcircles of ΔADE and ΔABC are not tangent to each other, then prove that line O_1O_2 bisects line segment AP.

Problem 325. On a plane, n distinct lines are drawn. A point on the plane is called a *k-point* if and only if there are exactly k of the n lines passing through the point. Let $k_2, k_3, \ldots, k_n$ be the numbers of 2-points, 3-points, …, n-points on the plane, respectively.

Determine the number of regions the n lines divided the plane into in terms of $n, k_2, k_3, \ldots, k_n$.

(*Source: 1998 Jiangsu Province Math Competition*)

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Solutions

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Problem 316. For every positive integer $n > 6$, prove that in every n-sided convex polygon $A_1A_2\ldots A_n$, there exist $i \ne j$ such that

$$|\cos\angle A_i - \cos\angle A_j| < \frac{1}{2(n-6)}.$$

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5).

Note the sum of all angles is

$$(n-2)180^\circ = 6\times120^\circ+(n-6)180^\circ.$$

So there are at most five angles less than 120°. The remaining angles are in [120°, 180°) and their cosines are in (−1,−1/2]. Divide (−1,−1/2] into $n-6$ left open, right closed intervals with equal length. By the pigeonhole principle, there exist two of the cosines in the same interval, which has length equal to $1/(2n-12)$. The desired inequality follows.

Problem 317. Find all polynomial $P(x)$ with integer coefficients such that for every positive integer n, 2^n-1 is divisible by $P(n)$.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5).

First we prove a fact: for all integers p and n and all polynomials $P(x)$ with integer coefficients, p divides $P(n+p)-P(n)$. To see this, let $P(x) = a_kx^k + \cdots + a_0$. Then

$$P(n+p)-P(n)=\sum_{i=1}^{k}a_i\left[(n+p)^i-n^i\right]$$

$$=\sum_{i=1}^{k}a_i p\left[\sum_{j=0}^{i-1}(n+p)^j n^{i-1-j}\right].$$

Now we claim that the only polynomials $P(x)$ solving the problem are the constant polynomials 1 and −1.

Assume $P(x)$ is such a polynomial and $P(n) \ne \pm1$ for some integer $n > 1$. Let p be a prime which divides $P(n)$, then p divides 2^n-1. So p is odd and $2^n \equiv 1 \pmod p$.

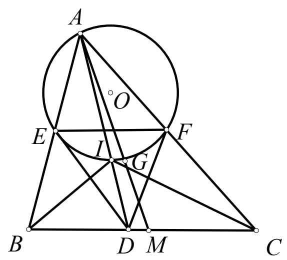
By the fact above, p also divides $P(n+p)-P(n)$. Hence, p divides $P(n+p)$. Since $P(n+p)$ divides $2^{n+p}-1$, p also divides $2^{n+p}-1$. Then $2^p \equiv 2^n2^p = 2^{n+p} \equiv 1 \pmod p$.

By Fermat's little theorem, $2^p \equiv 2 \pmod p$. Thus, $1 \equiv 2 \pmod p$. This leads to p divides 2−1=1, which is a contradiction. Hence, $P(n) = 1$ or −1 for every integer $n > 1$. Then $P(x)-1$ or $P(x)+1$ has infinitely many roots, i.e. $P(x) \equiv 1$ or −1.

Comments: Two readers pointed out that this problem appeared earlier as Problem 252 in vol. 11, no. 2.

Problem 318. In ΔABC, side BC has length equal to the average of the two other sides. Draw a circle passing through A and the midpoints of AB, AC. Draw the tangent lines from the centroid of the triangle to the circle. Prove that one of the points of tangency is the incenter of ΔABC.
(*Source: 2000 Chinese Team Training Test*)

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5).



Let G be the centroid and I be the incenter of ΔABC. Let line AI intersect side BC at D. Let E and F be the midpoints of AB and AC respectively. Let O be the circumcenter of ΔAEF. Let M be the midpoint of side BC.

We claim I is the circumcenter of ΔDEF. To see this, note I is on line AD. So

$$\frac{DB}{2EB}=\frac{DB}{AB}=\frac{DI}{AI}=\frac{DC}{AC}=\frac{DC}{2FC}=\lambda.$$

Also, $DB + DC = BC = (AB + AC)/2 = EB + FC = 2\lambda(DB + DC)$ implies $\lambda=1/2$. Then $DB=EB$ and $DC=FC$. So lines BI and CI are the perpendicular bisectors of DE and DF respectively.

Now we show I is on the circumcircle of ΔAEF. To see this, we compute

$\angle EIF = 2\angle EDB$
$= 2(180^\circ-\angle BDE-\angle CDF)$
$= (180^\circ-2\angle BDE) + (180^\circ-2\angle CDF)$
$= \angle DBE+\angle DCF$
$= 180^\circ-\angle EAF.$

Finally, we show $OI\perp IG$. Since $IE=IF$, $OI\perp EF$. Since $EF\|BC$, we just need to show $IG\|BC$, which follows from $DI/AI = 1/2 = MG/AG$.

Problem 319. For a positive integer n, let S be the set of all integers m such

that $|m| < 2n$. Prove that whenever $2n+1$ elements are chosen from S, there exist three of them whose sum is 0.
(*Source: 1990 Chinese Team Training Test*)

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of Miu Fat Buddhist Monastery) and **Fai YUNG**.

For $n = 1$, $S = \{-1,0,1\}$. If 3 elements are chosen from S, then they are $-1,0,1$, which have zero sum.

Suppose case n is true. For the case $n+1$, S is the union of $S'=\{m: -2n+1 \le m \le 2n-1\}$ and $S''=\{-2n-1,-2n,2n,2n+1\}$. Let T be a $2n+3$ element subset of S.

Case 1: (T contains at most 2 elements of S''). Then T contains $2n+1$ elements of S'. By case n, T has 3 elements with zero sum.

Case 2: (T contains exactly 3 elements of S''.) There are 4 subcases:

Subcase 1: ($\pm 2n$ and $2n+1$ are in T.) If 0 is in T, then $\pm 2n$ and 0 are in T with zero sum. If -1 is in T, then $2n+1$, $-2n$, -1 are in T with zero sum.

Otherwise, the other $2n$ numbers of T are among $1, \pm 2, \pm 3, \ldots, \pm(2n-1)$, which (after removing n) can be divided into the $2n-2$ pairs $\{1, 2n-1\}$, $\{2, 2n-2\}$, …, $\{n-1, n+1\}$, $\{-2, -2n+1\}$, $\{-3, -2n+2\}$, …, $\{-n, -n-1\}$. By the pigeonhole principle, the two numbers in one of the pairs must both be in T. Since the sums for these pairs are either $2n$ or $-2n-1$, we can add the pair to $-2n$ or $2n+1$ to get three numbers in T with zero sum.

Subcase 2: ($2n$ and $\pm(2n+1)$ are in T.) If 0 is in T, then $\pm(2n+1)$ and 0 are in T with zero sum. If 1 is in T, then $-2n-1$, $2n$, 1 are in T with zero sum.

Otherwise, the other $2n$ numbers of T are among $-1, \pm 2, \pm 3, \ldots, \pm(2n-1)$, which (after removing $-n$) can be divided into the $2n-2$ pairs $\{2, 2n-1\}$, $\{3, 2n-2\}$, …, $\{n, n+1\}$, $\{-1, -2n+1\}$, $\{-2, -2n+2\}$, …, $\{-n+1, -n-1\}$. By the pigeonhole principle, the two numbers in one of the pairs must both be in T. Since the sums for these pairs are either $2n+1$ or $-2n$, we can add the pair to $-2n-1$ or $2n$ to get three numbers in T with zero sum.

Subcase 3: ($\pm 2n$ and $-2n-1$ are in T.) This can be handled as in subcase 1.

Subcase 4: ($-2n$ and $\pm(2n+1)$ are in T.) This can be handled as in subcase 2.

Case 3: (T contains S''.) If 0 is in T, then $-2n$, $2n$, 0 are in T with zero sum. If 1 is in T, then $-2n-1$, $2n$, 1 are in T with zero sum. If -1 is in T, then $2n+1$, $-2n$, -1 are in T with zero sum.

Otherwise, the other $2n-1$ numbers of T are among $\pm 2, \pm 3, \ldots, \pm(2n-1)$, which can be divided into the $2n-2$ pairs $\{2, 2n-1\}$, $\{3, 2n-2\}$, …, $\{n, n+1\}$, $\{-2, -2n+1\}$, $\{-3, -2n+2\}$, …, $\{-n, -n-1\}$. By the pigeonhole principle, the two numbers in one of the pairs must both be in T. Since the sums for these pairs are either $2n+1$ or $-2n-1$, we can add the pair to $-2n-1$ or $2n+1$ to get three numbers in T with zero sum.

This completes the induction and we are done.

Problem 320. For every positive integer $k > 1$, prove that there exists a positive integer m such that among the rightmost k digits of 2^m in base 10, at least half of them are 9's.
(*Source: 2005 Chinese Team Training Test*)

Solution. **CHUNG Ping Ngai** (La Salle College, Form 5) and **G.R.A. 20 Problem Solving Group** (Roma, Italy).

We claim $m=2\times 5^{k-1}+k$ works. Let $f(k)=2\times 5^{k-1}$. We check by induction that

$$2^{f(k)} \equiv -1 \pmod{5^k}. \qquad (*)$$

First $f(2)=10$, $2^{10}=1024 \equiv -1 \pmod{5^2}$. Next, suppose case k is true. Then $2^{f(k)} = -1 + 5^k n$ for some integer n. We get

$$\begin{aligned} 2^{f(k+1)} &= (-1+5^k n)^5 \\ &= \sum_{j=0}^{5}\binom{5}{j}(-1)^{5-j}5^{kj}n^j \\ &\equiv -1 \pmod{5^{k+1}}, \end{aligned}$$

completing the induction.

By (*), we get $2^m \equiv -2^k \pmod{5^k}$. Also, clearly $2^m \equiv 0 \equiv -2^k \pmod{2^k}$. Hence,

$$2^m \equiv -2^k \equiv 10^k - 2^k \pmod{10^k}.$$

This implies the k rightmost digits in base 10 of 2^m and $10^k - 2^k$ are the same. For $k > 1$, $2^k < 10^{(k-1)/2}$. So

$$10^k - 1 \ge 10^k - 2^k > 10^k - 10^{(k-1)/2}.$$

The result follows from the fact that the k-digit number $10^k - 10^{(k-1)/2}$ in base 10 has at least half of its digits are 9's on the left.

Olympiad Corner

(continued from page 1)

Problem 3. Let three circles $\Gamma_1, \Gamma_2, \Gamma_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points A_1, B_1, A_2, B_2, A_3, B_3 as follows: For each $i= 1,2,3$, A_i, B_i are distinct points on the circle Γ_i such that the lines PA_i and PB_i are both tangents to Γ_i. Call the point P *exceptional* if, from the construction, three lines A_1B_1, A_2B_2, A_3B_3 are concurrent. Show that every exceptional point of the plane, if exists, lies on the same circle.

Problem 4. Prove that for any positive integer k, there exists an arithmetic sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_k}{b_k}$$

of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \ldots, k$, such that the positive integers $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all distinct.

Problem 5. Larry and Bob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every l kilometer driving from the start; Rob makes a 90° right turn after every r kilometer driving from the start, where l and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction.

Let the car start from Argovia facing towards Zillis. For which choices of the pair (l, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

Mathematical Excalibur

Volume 14, Number 2 *May - September, 2009*

Olympiad Corner

The following were the problems of the first day of the 2008 Chinese Girls' Math Olympiad.

Problem 1. (a) Determine if the set $\{1,2,\cdots,96\}$ can be partitioned into 32 sets of equal size and equal sum.

(b) Determine if the set $\{1,2,\cdots,99\}$ can be partitioned into 33 sets of equal size and equal sum.

Problem 2. Let $\varphi(x) = ax^3+bx^2+cx+d$ be a polynomial with real coefficients. Given that $\varphi(x)$ has three positive real roots and that $\varphi(x) < 0$, prove that $2b^3 + 9a^2d - 7abc \le 0$.

Problem 3. Determine the least real number a greater than 1 such that for any point P in the interior of square $ABCD$, the area ratio between some two of the triangles PAB, PBC, PCD, PDA lies in the interval $[1/a, a]$.

Problem 4. Equilateral triangles ABQ, BCR, CDS, DAP are erected outside the (convex) quadrilateral $ABCD$. Let X, Y, Z, W be the midpoints of the segments PQ, QR, RS, SP respectively. Determine the maximum value of

$$\frac{XZ+YW}{AC+BD}.$$

Acknowledgment: Thanks to Elina Chiu, Dept. of Math., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 3, 2009***.

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Remarks on IMO 2009

Leung Tat-Wing

2009 IMO Hong Kong Team Leader

The 50th International Mathematical Olympiad (IMO) was held in Bremen, Germany from 10th to 22nd July 2009. I arrived Bremen amid stormy and chilly (16°C) weather. Our other team members arrived three days later. The team eventually obtained 1 gold, 2 silver and 2 bronze medals, ranked (unofficially) 29 out of 104 countries/regions. This was the first time more than 100 countries participated. Our team, though not among the strongest teams, did reasonably well. But here I mainly want to give some remarks about this year's IMO, before I forget.

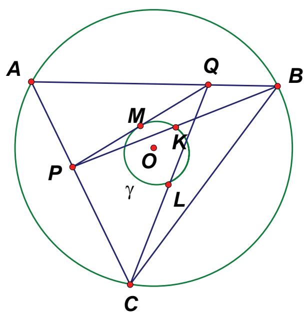
First, the problems of the contest:

Problem 1. Let n be a positive integer and let $a_1, a_2, \ldots, a_k$ $(k \ge 2)$ be distinct integers in the set $\{1,2,\ldots,n\}$ such that n divides $a_i(a_{i+1}-1)$ for $i=1,\ldots,k-1$. Prove that n does not divide $a_k(a_1-1)$.

This nice and easy number theory problem was the only number theory problem in the contest. Indeed it is not easy to find a sequence satisfying the required conditions, especially when k is close to n, or n is prime. Since adding the condition n divides $a_k(a_1-1)$ should be impossible, it was natural to prove the statement by contradiction. Clearly $2 \le k \le n$, and we have $a_1 \equiv a_1a_2 \pmod n$, $a_2 \equiv a_2a_3 \pmod n$, …, $a_{k-1} \equiv a_{k-1}a_k \pmod n$. The extra condition $a_k \equiv a_ka_1 \pmod n$ would in fact "complete the circle". Now $a_1 \equiv a_1a_2 \pmod n$. Using the second condition, we get $a_1 \equiv a_1a_2 \equiv a_1a_2a_3 \pmod n$ and so on, until we get $a_1 \equiv a_1a_2\cdots a_k \pmod n$. However, in a circle every point is a starting point. So starting from a_2, using the second condition we have $a_2 \equiv a_2a_3 \pmod n$. By the third condition, we then have $a_2 \equiv a_2a_3a_4 \pmod n$. As now the circle is complete, we eventually have $a_2 \equiv a_2a_3\cdots a_ka_1 \pmod n$. Arguing in this manner we eventually have $a_1 \equiv a_2 \equiv \cdots \equiv a_k \pmod n$, which is of course a contradiction!

Problem 2. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that PQ is tangent to the circle Γ. Prove that $OP=OQ$.

The nice geometry problem was supposed to be a medium problem, but it turned out it was easier than what the jury had thought. The trick was to understand the relations involved. A very nice solution provided by one of our members went as follows.



As $KM \| BQ$ (midpoint theorem), we have $\angle AQP = \angle QMK$. Since PQ is tangent to Γ, we have $\angle QMK = \angle MLK$ (angle of alternate segment). Therefore, $\angle AQP = \angle MLK$. By the same argument, we have $\angle APQ = \angle MKL$. Hence, $\Delta APQ \sim \Delta MKL$. Therefore,

$$\frac{AP}{AQ} = \frac{MK}{ML} = \frac{2MK}{2ML} = \frac{BQ}{CP}.$$

This implies $AP\cdot PC = AQ\cdot QB$. But by considering the power of P with respect to the circle ABC, we have

$$\begin{aligned} AP\cdot PC &= (R+OP)(R-OP) \\ &= R^2 - OP^2, \end{aligned}$$

where R is the radius of the circumcircle of ΔABC.

Likewise,

$$AQ{\cdot}QB = (R+OQ)(R-OQ)$$
$$= R^2-OQ^2.$$

These force $OP^2 = OQ^2$, or $OP = OQ$, done!

Problem 3. Suppose that s_1, s_2, s_3, … is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \text{ and } s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence s_1, s_2, s_3, … is itself an arithmetic progression.

This was one of the two hard problems (3 and 6). Fortunately, it turned out that it was still within reach.

One trouble is of course the notation. Of course, s_{s_1} stands for the s_1^{th} term of the s_i sequence and so on. Starting from an arithmetic progression (AP) with common difference d, then it is easy to check that both

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \text{ and } s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are APs with common difference d^2. The question is essentially proving the "converse". So the first step is to prove that the common differences of the two APs s_{s_i} and s_{s_i+1} are in fact the same, say s. It is not too hard to prove and is intuitively clear, for two lines of different slopes will eventually meet and cross each other, violating the condition of strictly increasing sequence. The next step is the show the difference between two consecutive terms of s_i is indeed $\sqrt{s}$, (thus s is a square). One can achieve this end by the method of descent, or max/min principle, etc.

Problem 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

This problem was also relatively easy. It is interesting to observe that an isosceles triangle can be the starting point of an IMO problem. With geometric software such as *Sketchpad*, one can easily see that $\angle CAB$ should be 60° or 90°. To prove the statement of the problem, one may either use synthetic method or coordinate method. One advantage of using the coordinate method is after showing the possible values of $\angle CAB$, one can go back to show these values do work by suitable substitutions. Some contestants lost marks either because they missed some values of $\angle CAB$ or forgot to check the two possible cases do work.

Problem 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths a, $f(b)$ and $f(b+f(a)-1)$. (A triangle is *non-degenerate* if its vertices are not collinear.)

The Jury worried if the word <u>triangle</u> may be allowed to be degenerate in some places. But I supposed all our secondary school students would consider only non-degenerate triangles. This was a nice problem in functional inequality (triangle inequality). One proves the problem by establishing several basic properties of f. Indeed the first step is to prove $f(1)=1$, which is not entirely easy. Then one proceeds to show that f is injective and/or $f(f(x)) = x$, etc, and finally shows that the only possible function is the identity function $f(x) = x$ for all x.

Problem 6. Let a_1, a_2, …, a_n be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s=a_1+a_2+\cdots+a_n$. A grasshopper is to jump along the real axis, starting from the point O and making n jumps to the right with lengths a_1, a_2, …, a_n in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

It turned out that this problem was one of the most difficult problems in IMO history. Only three of the 564 contestants received full scores. (Perhaps it was second to problem 3 posed in IMO 2007, for which only 2 contestants received full scores.)

When I first read the solution provided by the Problem Committee, I felt I was reading a paper of analysis. Without reading the solution, of course I would say we could try to prove the problem by induction, as the cases of small n were easy. The trouble was how to establish the induction step. Later the Russians provided a solution by induction, by separating the problem into sub-cases min $M < a_n$ or min $M \geq a_n$, and then applying the principle-hole principle, etc judiciously to solve the problem. Terry Tao said (jokingly) that the six problems were easy. But in his blog, he admitted that he had spent sometime reading the problem and he even wrote an article about it (I have not seen the article.)

The two hard problems (3 and 6) were more combinatorial and/or algebraic in nature. I had a feeling that this year the Jury has been trying to avoid hard number theory problems, which were essentially corollaries of deep theorems (for example, IMO 2003 problem 6 by the Chebotarev density theorem or IMO 2008 problem 3 by a theorem of H. Iwaniec) or hard geometry problem using sophisticated geometric techniques (like IMO 2008 problem 6).

The Germans ran the program vigorously (obstinately). They had an organization (Bildung und Begabung) that looked after the entire event. They had also prepared a very detailed shortlist problem set and afterwards prepared very detailed marking schemes for each problem. The coordinators were very professional and they studied the problems well. Thus, there were not too many arguments about how many points should be awarded for each problem.

Three of the problems (namely 1, 2 and 4) were relatively easy, problems 3 and 5 were not too hard, so although problem 6 was hard, contestants still scored relatively high points. This explained why the cut-off scores were not low, 14 for bronze, 24 for silver and 32 for gold.

It might seem that we still didn't do the hard problems too well. But after I discussed with my team members, I found that they indeed had the potential and aptitude to do the hard problems. What may still be lacking are perhaps more sophisticated skills and/or stronger will to tackle such problems.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 3, 2009.***

Problem 326. Prove that $3^{4^5}+4^{5^6}$ is the product of two integers, each at least 10^{2009}.

Problem 327. Eight pieces are placed on a chessboard so that each row and each column contains exactly one piece. Prove that there is an even number of pieces on the black squares of the board.
(*Source: 1989 USSR Math Olympiad*)

Problem 328. *(Due to Tuan Le, Fairmont High School, Anaheim, Ca., USA)* Let $a,b,c > 0$. Prove that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2}+\frac{\sqrt{b^3+c^3}}{b^2+c^2}+\frac{\sqrt{c^3+a^3}}{c^2+a^2}$$
$$\geq \frac{6(ab+bc+ca)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}.$$

Problem 329. Let $C(n,k)$ denote the binomial coefficient with value $n!/(k!(n-k)!)$. Determine all positive integers n such that for all $k = 1, 2, \cdots, n-1$, we have $C(2n,2k)$ is divisible by $C(n,k)$.

Problem 330. In ΔABC, $AB = AC = 1$ and $\angle BAC = 90°$. Let D be the midpoint of side BC. Let E be a point inside segment CD and F be a point inside segment BD. Let M be the point of intersection of the circumcircles of ΔADE and ΔABF, other than A. Let N be the point of intersection of the circumcircle of ΔACE and line AF, other than A. Let P be the point of intersection of the circumcircle of ΔAMN and line AD, other than A. Determine the length of segment AP with proof.
(*Source: 2003 Chinese IMO team test*)

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Solutions

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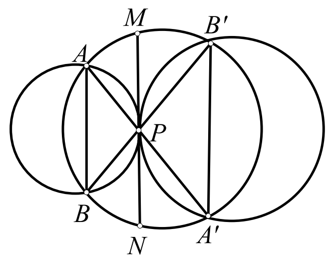
Problem 321. Let *AA', BB'* and *CC'* be three non-coplanar chords of a sphere and let them all pass through a common point P inside the sphere. There is a (unique) sphere S_1 passing through *A, B, C, P* and a (unique) sphere S_2 passing through *A', B', C', P*.

If S_1 and S_2 are externally tangent at P, then prove that *AA'=BB'=CC'*.

Solution. **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **Jim Robert STUDMAN** (Hanford, Washington, USA).

Consider the intersection of the 3 spheres with the plane through *A, A', B, B'* and *P*.



Let *MN* be the common external tangent through *P* to the circle through *A, B, P* and the circle through *A', B'P* as shown above. We have $\angle ABP = \angle APM = \angle A'PN = \angle A'B'P = \angle A'B'B = \angle BAA' = \angle BAP$. Hence, *AP=BP*. Similarly, *A'P = B'P*. So *AA' = AP+A'P = BP+B'P = BB'*. Similarly, *BB' = CC'*.

Other commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 6) and **LAM Cho Ho** (CUHK Math Year 1).

Problem 322. (*Due to Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam*) Let *a, b, c* be positive real numbers satisfying the condition $a+b+c = 3$. Prove that

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}\geq 2.$$

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and the proposer independently**.**

Observe that

$$\frac{a^2(b+1)}{a+b+ab}=a-\frac{ab}{a+b+ab}. \qquad (*)$$

Applying the AM-GM inequality twice, we have

$$\frac{ab}{a+b+ab}\leq\frac{ab}{3\sqrt[3]{a^2b^2}}=\frac{\sqrt[3]{ab}}{3}\leq\frac{a+b+1}{9}.$$

By (*), we have

$$\frac{a^2(b+1)}{a+b+ab}\geq a-\frac{a+b+1}{9}=\frac{8a-b-1}{9}.$$

Adding two other similar inequalities and using $a+b+c = 3$ on the right, we get the desired inequality.

Other commended solvers: **LAM Cho Ho** (CUHK Math Year 1), **Manh Dung NGUYEN** (Special High School for Gifted Students, HUS, Vietnam), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Stefan STOJCHEVSKI** (Yahya Kemal College, Skopje, Macedonia), **Jim Robert STUDMAN** (Hanford, Washington, USA) and **Dimitar TRENEVSKI** (Yahya Kemal College, Skopje, Macedonia).

Problem 323. Prove that there are infinitely many positive integers n such that 2^n+2 is divisible by n.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6), **LAM Cho Ho** (CUHK Math Year 1) and **WONG Ka Fai** (Wah Yan College Kowloon, Form 4).

We will prove the stronger statement that there are infinitely many positive even integers n such that 2^n+2 is divisible by n and also that 2^n+1 is divisible by $n-1$. Call such n a good number. Note $n = 2$ is good. Next, it suffices to prove that if n is good, then the larger integer $m = 2^n+2$ is also good.

Suppose n is good. Since n is even and $m = 2^n+2$ is twice an odd integer, so $m = nj$ for some odd integer j. Also, the odd integer $m-1 = 2^n+1 = (n-1)k$ for some odd integer k. Using the factorization $a^i+1 = (a+1)(a^{i-1}-a^{i-2}+\cdots+1)$ for positive odd integer i, we see that

$$2^m+2 = 2(2^{(n-1)k}+1)$$
$$= 2(2^{n-1}+1)\,(2^{(n-1)(k-1)}-\cdots+1)$$

is divisible by $2(2^{n-1}+1) = m$ and

$$2^m+1 = 2^{nj}+1 = (2^n+1)(2^{n(j-1)}-\cdots+1)$$

is divisible by $2^n+1 = m-1$. Therefore, m is also good.

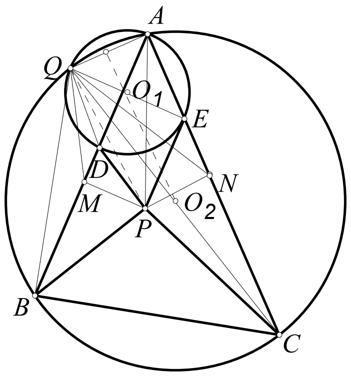
Problem 324. *ADPE* is a convex quadrilateral such that $\angle ADP = \angle AEP$. Extend side *AD* beyond *D* to a point *B* and extend side *AE* beyond *E* to a point *C* so that $\angle DPB = \angle EPC$. Let O_1 be the circumcenter of *ΔADE* and let O_2 be the circumcenter of *ΔABC*.

If the circumcircles of *ΔADE* and *ΔABC* are not tangent to each other, then prove that line O_1O_2 bisects line segment AP.

Solution. **Jim Robert STUDMAN** (Hanford, Washington, USA).

Let the circumcircle of ΔADE and the circumcircle of ΔABC intersect at A and Q.

Observe that line O_1O_2 bisects chord AQ and $O_1O_2 \perp AQ$. Hence, line O_1O_2 bisects line segment AP will follow if we can show that $O_1O_2 \parallel PQ$, or equivalently that $PQ \perp AQ$.



Let points M and N be the feet of perpendiculars from P to lines AB and AC respectively. Since $\angle ANP = 90° = \angle AMP$, points A, N, P, M lie on a circle Γ with AP as diameter. <u>We claim that $\angle MQN = \angle MAN$.</u> This would imply Q is also on circle Γ, and we would have $PQ \perp AQ$ as desired.

Since we are given $\angle ADP = \angle AEP$, we get $\angle BDP = \angle CEP$. This combines with the given fact $\angle DPB = \angle EPC$ imply ΔDPB and ΔEPC are similar, which yields $DB/EC = DP/EP = DM/EN$.

Since A,E,D,Q are concyclic, we have

$$\begin{aligned}\angle BDQ &= 180° - \angle ADQ \\ &= 180° - \angle AEQ = \angle CEQ.\end{aligned}$$

This and $\angle DBQ = \angle ABQ = \angle ACQ = \angle ECQ$ imply ΔDQB and ΔEQC are similar. So we have $QD/QE = DB/EC$. Combining with the equation at the end of the last paragraph, we get

$$QD/QE = DM/EN.$$

Using ΔDQB and ΔEQC are similar, we get $\angle MDQ = \angle BDQ = \angle CEQ = \angle NEQ$. These imply ΔMDQ and ΔNEQ are similar. Then $\angle MQD = \angle NQE$.

Finally, for the claim, we now have

$$\begin{aligned}\angle MQN &= \angle MQD + \angle DQN \\ &= \angle NQE + \angle DQN \\ &= \angle DQE \\ &= \angle DAE \\ &= \angle MAN.\end{aligned}$$

Comments: Some solvers used a bit of homothety to simplify the proof.

Other commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 6), **LAM Cho Ho** (CUHK Math Year 1), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 7).

Problem 325. On a plane, n distinct lines are drawn. A point on the plane is called a <u>*k-point*</u> if and only if there are exactly k of the n lines passing through the point. Let $k_2, k_3, \ldots, k_n$ be the numbers of 2-points, 3-points, …, n-points on the plane, respectively.

Determine the number of regions the n lines divided the plane into in terms of n, $k_2, k_3, \ldots, k_n$.
(*Source: 1998 Jiangsu Province Math Competition*)

Solution. **LAM Cho Ho** (CUHK Math Year 1).

Take a circle of radius r so that all intersection points of the n lines are inside the circle and none of the n lines is tangent to the circle. Now each line intersects the circle at two points. These $2n$ points on the circle are the vertices of a convex $2n$-gon (call it M) as we go around the circle, say clockwise. Let the n lines partition the interior of M into P_3 triangles, P_4 quadrilaterals, ⋯, P_j j-gons, ⋯. These polygonal regions are all convex since the angles of these regions, which were formed by intersecting at least two lines, are all less than 180°. By convexity, no two sides of any polygonal region are parts of the same line. So we have $P_j = 0$ for $j>3n$.

Consider the sum of all the angles of these regions partitioning M. On one hand, it is $180°(P_3+2P_4+3P_5+\cdots)$ by counting region by region. On the other hand, it also equals $360°(k_2+k_3+\cdots+k_n)+(2n-2)180°$ by counting all the angles around each vertices of the regions. Cancelling 180°, we get

$$P_3+2P_4+3P_5+\cdots=2(k_2+k_3+\cdots+k_n)+(2n-2).$$

Next, consider the total number of all the edges of these regions partitioned M (with each of the edges inside M counted twice). On one hand, it is $3P_3+4P_4+5P_5+\cdots$ by counting region by region. On the other hand, it is also $(4k_2+6k_3+\cdots 2nk_n)+4n$ by counting the number of edges around the k-points and around the vertices of M. The $4n$ term is due to the $2n$ edges of M and each vertex of M (being not a k-point) issues exactly one edge into the interior of M. So we have

$$3P_3+4P_4+5P_5+\cdots=4k_2+6k_3+\cdots 2nk_n+4n.$$

Subtracting the last two displayed equations, we can obtain

$$P_3+P_4+P_5+\cdots=k_2+2k_3+(n-1)k_n+n+1.$$

Finally, the number of regions these n lines divided the plane into is the limit case r tends to infinity. Hence, it is exactly $k_2+2k_3+\cdots+(n-1)k_n+n+1$.

Other commended solvers: **CHUNG Ping Ngai** (La Salle College, Form 6) and **YUNG Fai**.

Remarks on IMO 2009

(continued from page 2)

As I found out from the stronger teams (Chinese, Japanese, Korean, or Thai, etc.), they were obviously more heavily or vigorously trained. For instance, a Thai boy/girl had to go through more like 10 tests to be selected as a team member.

Another thing I learned from the meeting was several countries were interested to host the event (South-East Asia countries and Asia-Minor countries). In fact, one country is going to host three international competitions of various subjects in a row for three years. Apparently they think hosting these events is good for gifted education.

The first IMO was held in Romania in 1959. Throughout these 51 years, only one year IMO was not held (1980). To commemorate the fiftieth anniversary of IMO in 2009, six notable mathematicians related to IMO (B. Bollabas, T. Gowers, L. Lovasz, S. Smirnov, T. Tao and J. C. Yoccoz) were invited to talk to the contestants. Of course, Yoccoz, Gowers and Tao were Fields medalists. The afternoon of celebration then became a series of (rather) heavy lectures (not bad). They described the effects of IMOs on them and other things. The effect of IMO on the contestants is to be seen later, of course!

Mathematical Excalibur

Volume 14, Number 3 *October-November, 2009*

Olympiad Corner

The 2009 Czech-Polish-Slovak Math Competition was held on June 21-24. The following were the problems.

Problem 1. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$(1+yf(x))(1-yf(x+y))=1$$

for all $x,y \in \mathbb{R}^+$.

Problem 2. Given positive integers a and k, the sequence a_1, a_2, a_3, … is defined by $a_1=a$ and $a_{n+1}=a_n+k\rho(a_n)$, where $\rho(m)$ stands for the product of the digits of m in its decimal representation (e.g. $\rho(413) = 12$, $\rho(308) = 0$). Prove that there exist positive integers a and k such that the sequence a_1, a_2, a_3, … contains exactly 2009 different numbers.

Problem 3. Given ΔABC, let k be the excircle at the side BC. Choose any line p parallel to BC intersecting line segments AB and AC at points D and E. Denote by ℓ the incircle of ΔADE. The tangents from D and E to the circle k not passing through A intersect at P. The tangents from B and C to the circle ℓ not passing through A intersect at Q. Prove that the line PQ passes through a point independent of p.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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Probabilistic Method

Law Ka Ho

Roughly speaking, the probabilistic method helps us solve combinatorial problems via considerations related to probability.

We know that among any 6 people, there exist 3 who know each other or 3 who don't know each other (we assume if A knows B, then B knows A). When 6 is replaced by 5, this is no longer true, as can be seen by constructing a counter-example. When the numbers get large, constructing counterexamples becomes difficult. In this case the probabilistic method helps.

Example 1. Show that among 2^{100} people, there do not necessarily exist 200 people who know each other or 200 people who don't know each other.

Solution. Assign each pair of people to be knowing each other or not by flipping a fair coin. Among a set of 200 people, the probability that they know each other or they don't know each other is thus $2\times 2^{-C_2^{200}} = 2^{-19899}$. As there are $C_{200}^{2^{100}}$ choices of 200 people, the probability that there exist 200 people who know each other or 200 people who don't know each other is at most

$$C_{200}^{2^{100}} \times 2^{-19899} < \frac{(2^{100})^{200}}{200!}\times 2^{-19899} = \frac{2^{101}}{200!} < 1$$

Hence the probability for the non-existence of 200 people who know each other or 200 people who don't know each other is greater than 0, which implies the result.

Here we see that the general rationale is to show that in a random construction of an example, the probability that it satisfies what we want is positive, which means that there exists such an example. Clearly, the

Example 2. In each cell of a 100×100 table, one of the integers 1, 2, …, 5000 is written. Moreover, each integer appears in the table exactly twice. Prove that one can choose 100 cells in the table satisfying the three conditions below:

(1) Exactly one cell is chosen in each row.

(2) Exactly one cell is chosen in each column.

(3) The numbers in the cells chosen are pairwise distinct.

Solution. Take a random permutation a_1, …, a_{100} of {1, …, 100} and choose the a_i-th cell in the i-th row. Such choice satisfies (1) and (2). For j = 1, …, 5000, the probability of choosing both cells written j is

$$\begin{cases} 0 & \text{they are in the same row or column} \\ \frac{1}{100}\times\frac{1}{99} & \text{otherwise} \end{cases}$$

Hence the probability that such choice satisfies (3) is at least

$$1-5000\times\frac{1}{100}\times\frac{1}{99}>0.$$

Of course, one can easily transform the above two probabilistic solutions to merely using counting arguments (by counting the number of 'favorable outcomes' instead of computing the probabilities), which is essentially the same. But a probabilistic solution is usually neater and more natural.

Another common technique in the probabilistic method is to compute the average (or expected value) – the total is the average times the number of items, and there exists an item which is as good as the average. These are illustrated in the next two examples.

Example 3. (*APMO 1998*) Let F be the set of all n-tuples (A_1, A_2, …, A_n) where each A_i, i = 1, 2, …, n, is a subset of {1, 2, …, 1998}. Let $|A|$ denote the number of elements of the set A. Find the number

$$\sum_{(A_1, A_2, \ldots, A_n)} |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

Solution. (*Due to Leung Wing Chung, 1998 Hong Kong IMO team member*) Note that the set {1, 2, …, 1998} has 2^{1998} subsets because we may choose to include or not to include each of the 1998 elements in a subset. Hence there are altogether 2^{1998n} terms in the summation.

Now we compute the average value of each term. For i = 1, 2, …, 1998, i is an element of $A_1 \cup A_2 \cup \cdots \cup A_n$ if and only if i is an element of at least one of $A_1, A_2, \ldots, A_n$. The probability for this to happen is $1-2^{-n}$. Hence the average value of each term in the summation is $1998(1-2^{-n})$, and so the answer is $2^{1998n} \cdot 1998(1-2^{-n})$.

Example 4. In a chess tournament there are 40 players. A total of 80 games have been played, and every two players compete at most once. For certain integer n, show that there exist n players, no two of whom have competed. (Of course, the larger the n, the stronger the result.)

Solution 4.1. If we use a traditional counting approach, we can prove the case $n=4$. Assume on the contrary that among any 4 players, at least one match is played. Then the number of games played is at least $C_4^{40} \div C_2^{38} = 260$, a contradiction. Note that this approach cannot prove the $n=5$ case since $C_5^{40} \div C_3^{38} = 78 < 80$.

Solution 4.2. We use a probabilistic approach to prove the $n=5$ case. Randomly choose some players such that each player has probability 0.25 to be chosen. Then discard all players who had lost in a match with another chosen player. In this way no two remaining players have played with each other.

What is the average number of players left? On average $40\times 0.25 = 10$ players would be chosen. For each match played, the probability that both players are chosen is 0.25^2, so on average there are $80\times 0.25^2 = 5$ matches played among the chosen players. After discarding the losers, the average number of players left is at least 5 (in fact greater than 5 since the losers could repeat). That means there exists a choice in which we obtain at least 5 players who have not played against each other.

(Note: if we replace 0.25 by p, then the average number of players left would be $40p-80p^2 = 5-80(p-0.25)^2$ and this explains the choice of the number 0.25.)

Solution 4.3. This time we use another probabilistic approach to prove the $n=8$ case. (!!) We assign a random ranking to the 40 players, and we pick those who have only played against players with lower ranking. Note that in this way no two of the chosen players have competed.

Suppose the i-th player has played d_i games. Since 80 games have been played, we have $d_1+d_2+\cdots+d_{40} = 80\times 2$. Also, the i-th player is chosen if and only if he is assigned the highest ranking among himself and the players with whom he has competed, and the probability for this to happen is $1/(d_i+1)$. Hence the average number of players chosen is

$$\frac{1}{d_1+1}+\cdots+\frac{1}{d_{40}+1} \geq \frac{40^2}{(d_1+1)+\cdots+(d_{40}+1)} = \frac{40^2}{160+40} = 8$$

Here we made use of the Cauchy- Schwarz inequality. This means there exists 8 players, no two of whom have competed.

Remark. Solution 4.3 is the best possible result. Indeed, we may divide the 40 players into eight groups of 5 players each. If two players have competed if and only if they are from the same group, then the number of games played will be $8\times C_2^5 = 80$ and it is clear that it is impossible to find 9 players, no two of whom have competed.

The above example shows that the probabilistic method can sometimes be more powerful than traditional methods. We conclude with the following example, which makes use of an apparently trivial property of probability, namely the probability of an event always lies between 0 and 1.

Example 5. In a public examination there are n subjects, each offered in Chinese and English. Candidates may sit for as many (or as few) subjects as they like, but each candidate may only choose one language version for each subject. For any two different subjects, there exists a candidate sitting for different language versions of the two subjects. If there are at most 10 candidates sitting for each subject, determine the maximum possible value of n.

Solution. The answer is 1024. The following example shows that n = 1024 is possible. Suppose there are 10 candidates (numbered 1 to 10), each sitting for all 1024 subjects (numbered 0 to 1023). For student i, the j-th subject is taken in Chinese if the i-th digit from the right is 0 in the binary representation of j, and the subject is taken in English otherwise. In this way it is easy to check that the given condition is satisfied. (The answer along with the example is not difficult to get if one begins by replacing 10 with smaller numbers and then observe the pattern.)

To show that 1024 is the maximum, we randomly assign each candidate to be 'Chinese' or 'English'. Let E_j be the event 'all candidates in the j-th subject are sitting for the language version which matches their assigned identity'. As there are at most 10 candidates in each subject, we have the probability

$$P(E_j) \geq 2^{-10} = \frac{1}{1024}.$$

Since 'for any two different subjects, there exists a candidate sitting for different language versions of the two subjects', no two E_j may occur simultaneously. It follows that

$$\begin{aligned} &P(\text{at least one } E_j \text{ happens}) \\ &= P(E_1)+P(E_2)+\cdots+P(E_n) \\ &\geq \frac{n}{1024} \end{aligned}$$

But since the probability of an event is at most 1, the above gives $1 \geq \frac{n}{1024}$, so we have $n \leq 1024$ as desired!

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***December 1, 2009.***

Problem 331. For every positive integer n, prove that

$$\sum_{k=0}^{n-1}(-1)^k\cos^n(k\pi/n)=\frac{n}{2^{n-1}}.$$

Problem 332. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let BD bisect OC perpendicularly. On diagonal AC, choose the point P such that $PC=OC$. Let line BP intersect line AD and the circumcircle of $ABCD$ at E and F respectively. Prove that PF is the geometric mean of EF and BF in length.

Problem 333. Find the largest positive integer n such that there exist n 4-element sets $A_1, A_2, \ldots, A_n$ such that every pair of them has exactly one common element and the union of these n sets has exactly n elements.

Problem 334. (*Due to FEI Zhenpeng, Northeast Yucai School, China*) Let $x,y \in (0,1)$ and x be the number whose n-th digit after the decimal point is the n^n-th digit after the decimal point of y for all $n=1,2,3,\ldots$. Show that if y is rational, then x is rational.

Problem 335. (*Due to Ozgur KIRCAK, Yahya Kemal College, Skopje, Macedonia*) Find all $a\in\mathbb{R}$ for which the functional equation $f\colon \mathbb{R}\to\mathbb{R}$

$$f(x-f(y))=a(f(x)-x)-f(y)$$

for all $x, y\in\mathbb{R}$ has a unique solution.

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Solutions

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Problem 326. Prove that $3^{4^5}+4^{5^6}$ is the product of two integers, each at least 10^{2009}.

Solution. **CHAN Ho Lam Franco** (GT (Ellen Yeung) College, Form 3), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **Manh Dung NGUYEN** (Hanoi University of Technology, Vietnam), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (GT(Ellen Yeung) College) and **Pedro Henrique O. PANTOJA** (UFRN, Brazil).

Let $a=3^{256}$ and $b=4^{3906}$. Then

$$\begin{aligned}3^{4^5}+4^{5^6}&=a^4+4b^4\\&=(a^4+4a^2b^2+4b^4)-4a^2b^2\\&=(a^2+2b^2+2ab)(a^2+2b^2-2ab).\end{aligned}$$

Note that $a^2+2b^2+2ab > a^2+2b^2-2ab > 2b^2-2ab=2b(b-a)>b>2^{7800}>(10^3)^{780}>10^{2009}$. The result follows.

Problem 327. Eight pieces are placed on a chessboard so that each row and each column contains exactly one piece. Prove that there is an even number of pieces on the black squares of the board.

(*Source: 1989 USSR Math Olympiad*)

Solution. **G.R.A. 20 Problem Solving Group** (Roma, Italy), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM) and **YUNG Fai**.

Without loss of generality, we may assume the square in row 1, column 1 is not black. Then, for all $i, j=1,2,\ldots,8$, the square in row i, column j is black if and only if $i+j\equiv 1 \pmod 2$. Since the pieces are in different columns, the position of the piece contained in the i-th row is in column $p(i)$, where p is some permutation of $\{1,2,\ldots,8\}$. Therefore, the number of pieces on the black squares in mod 2 is congruent to

$$\sum_{i=1}^{8}(i+p(i))=\sum_{i=1}^{8}i+\sum_{i=1}^{8}p(i)=72,$$

which is even.

Other commended solvers: **Abby LEE** (SKH Lam Woo Memorial Secondary School) and **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam).

Problem 328. *(Due to Tuan Le, Fairmont High School, Anaheim, Ca., USA)* Let $a,b,c>0$. Prove that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2}+\frac{\sqrt{b^3+c^3}}{b^2+c^2}+\frac{\sqrt{c^3+a^3}}{c^2+a^2}\ge\frac{6(ab+bc+ca)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}.$$

Solution 1. **Manh Dung NGUYEN** (Hanoi University of Technology, Vietnam), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam),

Below we will use the cyclic notation

$$\sum_{cyc}f(a,b,c)=f(a,b,c)+f(b,c,a)+f(c,a,b).$$

By the Cauchy-Schwarz inequality, we have $(a^3+b^3)(a+b)\ge(a^2+b^2)^2$. Using this, the left side is

$$\sum_{cyc}\frac{\sqrt{a^3+b^3}}{a^2+b^2}\ge\sum_{cyc}\frac{1}{\sqrt{a+b}}=\frac{\sum_{cyc}\sqrt{(a+b)(b+c)}}{\sqrt{(a+b)(b+c)(c+a)}}.$$

So it suffices to show

$$\sum_{cyc}\sqrt{(a+b)(b+c)}\ge\frac{6(ab+bc+ca)}{a+b+c}.\quad(*)$$

First we claim that

$$(a+b)(b+c)(c+a)\ge\frac{8}{9}(a+b+c)(ab+bc+ca)$$

and $(a+b+c)^2\ge 3(ab+bc+ca)$.

These follow from

$$9(a+b)(b+c)(c+a)-8(a+b+c)(ab+bc+ca)$$
$$=a(b-c)^2+b(c-a)^2+c(a-b)^2\ge 0$$

and

$$(a+b+c)^2-3(ab+bc+ca)$$
$$=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}\ge 0.$$

By the AM-GM inequality,

$$\sum_{cyc}\sqrt{(a+b)(b+c)}\ge 3\sqrt[3]{(a+b)(b+c)(c+a)}.$$

To get (*), it remains to show

$$(a+b+c)\sqrt[3]{(a+b)(b+c)(c+a)}\ge 2(ab+bc+ca).$$

This follows by cubing both sides and using the two inequalities in the claim to get

$$(a+b+c)^3(a+b)(b+c)(c+a)$$
$$\ge\frac{8}{9}(a+b+c)^4(ab+bc+ca)$$
$$\ge 8(ab+bc+ca)^3.$$

Solution 2. **LEE Ching Cheong** (HKUST, Year 1).

Due to the homogeneity of the original inequality, without loss of generality we may assume $ab+bc+ca=1$. Then

$(a+b)(b+c) = 1+b^2$. The inequality (*) in solution 1 becomes

$$\sum_{cyc}\sqrt{1+b^2} \ge \frac{6}{a+b+c}.$$

Observe that

$$\sqrt{1+x^2} \ge \frac{1}{2}\left(x-\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}} = \frac{x+\sqrt{3}}{2},$$

which can be checked by squaring both sides and simplified to $(\sqrt{3}x-1)^2 \ge 0$ (or alternatively, $f(x) = \sqrt{1+x^2}$ is a convex function on $\mathbb{R}$ and $y = (x+\sqrt{3})/2$ is the equation of the tangent line to the graph of $f(x)$ at $(1/\sqrt{3}, 2/\sqrt{3})$.)

Now $(a+b+c)^2 \ge 3(ab+bc+ca)$ can be expressed as

$$\sum_{cyc} b = a+b+c \ge \sqrt{3}.$$

Using these, inequality (*) follows as

$$\sum_{cyc}\sqrt{1+b^2} \ge \frac{\sum_{cyc} b + 3\sqrt{3}}{2}$$

$$\ge 2\sqrt{3} \ge \frac{6}{a+b+c}.$$

Other commended solvers: **Salem MALIKIĆ** (Student, University of Sarajevo, Bosnia and Herzegovina) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Problem 329. Let $C(n,k)$ denote the binomial coefficient with value $n!/(k!(n-k)!)$. Determine all positive integers n such that for all $k = 1, 2, \cdots, n-1$, we have $C(2n,2k)$ is divisible by $C(n,k)$.

Solution. **HUNG Ka Kin Kenneth** (Diocesan Boys' School).

For $n < 6$, we can check that $n = 1, 2, 3$ and 5 are the only solutions. For $n \ge 6$, we will show there are no solutions. Observe that after simplification,

$$\frac{C(2n,2k)}{C(n,k)} = \frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{(2k-1)(2k-3)\cdots 1}.$$

Let n be an even integer with $n \ge 6$. Then $n-1 \ge 5$. So $n-1$ has a <u>*prime*</u> factor $p \ge 3$. Now $1 < (p+1)/2 \le n/2 < n-1$. Let $k = (p+1)/2$. Then $p = 2k-1$, but p is not a factor of $2n-1, 2n-3, \ldots, 2n-2k+1$ since the closest consecutive multiples of p are $2n-2k-1 = 2(n-1)-p$ and $2n-2 = 2(n-1)$. Hence, $C(2n, 2k)/C(n, k)$ is not an integer. So such n cannot a solution for the problem.

For an odd integer $n \ge 7$, we divide into three cases.

<u>*Case 1*</u> : ($n-1 \ne 2^a$ for all a=1,2,3,…) Then $n-1$ has a <u>*prime*</u> factor $p \ge 3$. We repeat the argument above.

<u>*Case 2*</u> : ($n-2 \ne 3^b$ for all b=1,2,3,…) Then $n-2$ has a <u>*prime*</u> factor $p \ge 5$. Now $1 < (p+1)/2 \le n/2 < n-1$. Let $k = (p+1)/2$. Then $p=2k-1$, but p is not a factor of $2n-1, 2n-3, \ldots, 2n-2k+1$ since again $2n-2k-3 = 2(n-2)-p$ and $2n-4 = 2(n-2)$ are multiples of p. Hence, $C(2n,2k)/C(n,k)$ is not an integer.

<u>*Case 3*</u> : ($n-1 = 2^a$ and $n-2 = 3^b$ for some positive integers a and b) Then $2^a - 3^b = 1$. Consider mod 3, we see a is even, say $a = 2c$. Then

$$3^b = 2^a-1 = 2^{2c}-1 = (2^c-1)(2^c+1).$$

Since 2^c+1 and 2^c-1 have a difference of 2 and they are powers of 3 by unique prime factorization, we must have $c = 1$. Then $a = 2$ and $n = 5$, which contradicts $n \ge 7$.

Other commended solvers: **G.R.A. 20 Problem Solving Group** (Roma, Italy) and **O Kin Chit Alex** (GT(Ellen Yeung) College).

Problem 330. In ΔABC, $AB = AC = 1$ and $\angle BAC = 90°$. Let D be the midpoint of side BC. Let E be a point inside segment CD and F be a point inside segment BD. Let M be the point of intersection of the circumcircles of ΔADE and ΔABF, other than A. Let N be the point of intersection of the circumcircle of ΔACE and line AF, other than A. Let P be the point of intersection of the circumcircle of ΔAMN and line AD, other than A. Determine the length of segment AP with proof.
(*Source: 2003 Chinese IMO team test*)

Official Solution.

We will show A, B, P, C are concyclic. (Then, by symmetry, AP is a diameter of the circumcircle of ΔABC. We see $\angle ABP = 90°$, $AB = 1$ and $\angle BAP = 45°$, which imply $AP = \sqrt{2}$.)

Consider inversion with center at A and $r = 1$. Let X^* denote the image of point X. Let the intersection of lines XY and WZ be denoted by $XY\cap WZ$. We have $B^* = B$ and $C^* = C$. The line BC is sent to the circumcircle ω of ΔABC. The points F, D, E are sent to the intersection points F^*, D^*, E^* of lines AF, AD, AE with ω respectively.

The circumcircles of ΔADE and ΔABF are sent to lines D^*E^* and BF^*. So $M^* = D^*E^* \cap BF^*$. Also, the circumcircle of ΔACE and line AF are sent to lines CE^* and AF^*. Hence, $N^* = CE^* \cap AF^*$. Next, the circumcircle of ΔAMN and line AD are sent to lines M^*N^* and AD^*. So, $P^* = M^*N^* \cap AD^*$.

Now D^*, E^*, C, B, F^*, A are six points on ω. By Pascal's theorem, $M^* = D^*E^*\cap BF^*$, $N^* = E^*C\cap F^*A$ and $D = CB\cap AD^*$ are collinear. Since $P^* = M^*N^* \cap AD^*$, we get $D = P^*$. Then $P = D^*$ and A, B, P, C are all on ω.

Olympiad Corner

(continued from page 1)

Problem 4. Given a circle k and its chord AB which is not a diameter, let C be any point inside the longer arc AB of k. We denote by K and L the reflections of A and B with respect to the axes BC and AC. Prove that the distance of the midpoints of the line segments KL and AB is independent of the location of point C.

Problem 5. The n-tuple of positive integers $a_1,\ldots,a_n$ satisfies the following conditions:

(*i*) $1 \le a_1 < a_2 < \cdots < a_n \le 50$;

(*ii*) for any n-tuple of positive integers $b_1,\ldots, b_n$, there exist a positive integer m and an n-tuple of positive integers $c_1,\ldots, c_n$ such that

$$mb_i = c_i^{a_i} \text{ for } i = 1,\ldots,n.$$

Prove that $n \le 16$ and find the number of different n-tuples $a_1,\ldots, a_n$ satisfying the given conditions for $n = 16$.

Problem 6. Given an integer $n \ge 16$, consider the set

$$G = \{(x,y): x,y \in \{1,2,\ldots,n\}\}$$

consisting of n^2 points in the plane. Let A be any subset of G containing at least $4n\sqrt{n}$ points. Prove that there are at least n^2 convex quadrangles with all their vertices in A such that their diagonals intersect in one common point.

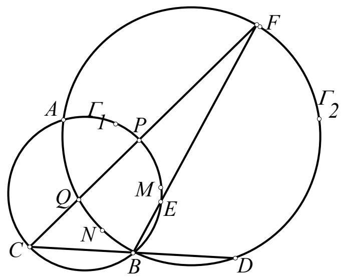
Mathematical Excalibur

Volume 14, Number 4 — *December 2009-March, 2010*

Olympiad Corner

The 2010 Chinese Mathematical Olympiad was held on January. Here are the problems.

Problem 1. As in the figure, two circles Γ_1, Γ_2 intersect at points A, B. A line through B intersects Γ_1, Γ_2 at C, D respectively. Another line through B intersects Γ_1, Γ_2 at E, F respectively. Line CF intersects Γ_1, Γ_2 at P, Q respectively. Let M, N be the midpoints of arcs PB, arc QB respectively. Prove that if $CD = EF$, then C, F, M, N are concyclic.



Problem 2. Let $k \geq 3$ be an integer. Sequence $\{a_n\}$ satisfies $a_k=2k$ and for all $n > k$, $a_n = a_{n-1} + 1$ if a_{n-1} and n are coprime and $a_n=2n$ if a_{n-1} and n are not coprime. Prove that the sequence $\{a_n - a_{n-1}\}$ contains infinitely many prime numbers.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 17, 2010***.

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A Refinement of Bertrand's Postulate

Neculai Stanciu

(Buzău, Romania)

In this article, we give an elementary demonstration of the famous Bertrand's postulate by using a theorem proved by the mathematician M. El Bachraoni in 2006.

Interesting is the distribution of prime numbers among the natural numbers and problems about their distributions have been stated in very simple ways, but they all turned out to be very difficult. The following *open problem* was stated by the Polish mathematician W. Sierpiński in 1958:

For all natural numbers $n > 1$ and $k \leq n$, there is at least one prime in the range $[kn,(k+1)n]$.

The case k=1 (known as Bertrand's postulate) was stated in 1845 by the French mathematician J. Bertrand and was proved by the Russian mathematician P. L. Chebysev. Simple proofs have been given by the Hungarian mathematician P. Erdos in 1932 and recently by the Romanian mathematician M. Tena [3]. The case k=2 was proved in 2006 by M. El Bachraoni (see [1]). His proof was relatively short and not too complicated. It is freely available on the internet [4].

Below we will present a refinement of Bertrand's postulate and it is perhaps the simplest demonstration of the postulate based on the following

Theorem 1. For any positive integer $n > 1$, there is a prime number between $2n$ and $3n$. (For the proof, see [1] or [4].)

The demonstration in [1] was typical of many theorems in number theory and was based on multiple inequalities valid for large values of n which can be calculated effectively. For the rest of the values of n, there are many basic improvisations, some perhaps difficult to follow.

Theorem 2. For $n \geq 1$, there is a prime number p such that $n < p < 3(n+1)/2$. (Since $3(n+1)/2<2n$ for $n > 3$, this is a refinement of the Bertrand's postulate.)

For the proof, the case n=1 follows from $1<p=2<3$. The case n=2 follows from $2<p=3<9/2$. For n even, say $n=2k$, by Theorem 1, we have a prime p such that $n=2k<p<3k<3(2k+1)/2=3(n+1)/2$. Similarly, for n odd, say $n=2k+1$, we have a prime p such that $n = 2k+1 < 2k+2=2(k+1) < p < 3(k+1)=3(n+1)/2$.

Concerning the distribution of prime numbers among the natural numbers, recently (in 2008) Rafael Jakimczuk has proved a formula (see [2] or [4]) for the n-th prime p_n, which provided a better error term than previous known approximate formulas for p_n. His formula is for $n \geq 4$,

$$p_n = n\log n + \log(n\log n)(n - Li(n\log n)) + \sum_{k=2}^{\infty} \frac{(-1)^k Q_{k-1}(\log(n\log n))}{k!\,n^{k-1}\log^{k-1} n}(n - Li(n\log n))^k + O(h(n)),$$ where

$$Li(x) = \int_2^x \frac{dt}{\log t},\quad h(n) = \frac{n\log^2 n}{\exp(d\sqrt{\log n})}$$

and $Q_{k-1}(x)$ are polynomials.

References

[1] M. El Bachraoni, "*Primes in the Interval [2n,3n]*," Int. J. Contemp. Math. Sciences, vol. 1, 2006, no. 13, 617-621.

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Max-Min Inequalities

Pedro Henrique O. Pantoja

(UFRN, NATAL, BRAZIL)

There are many inequalities. In this article, we would like to introduce the readers to some inequalities that involve maximum and minimum.

The first example was a problem from the Federation of Bosnia for Grade 1 in 2008.

Example 1 (*Bosnia-08*) For arbitrary real numbers x, y and z, prove the following inequality:

$$x^2+y^2+z^2-xy-yz-zx \ge \max\left\{\frac{3(x-y)^2}{4},\frac{3(y-z)^2}{4},\frac{3(z-x)^2}{4}\right\}.$$

Solution. Without loss of generality, suppose $x \ge y \ge z$. Then

$$\max\left\{\frac{3(x-y)^2}{4},\frac{3(y-z)^2}{4},\frac{3(z-x)^2}{4}\right\}=\frac{3}{4}(z-x)^2.$$

Let $a = x-y$, $b = y-z$ and $c = z-x$. Then $c = -(a+b)$. Hence, $(z-x)^2 = c^2 = (a+b)^2 = a^2+2ab+b^2$ and

$$\begin{aligned} x^2+y^2+z^2-xy-yz-zx &= \frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2] \\ &= \frac{1}{2}(a^2+b^2+a^2+2ab+b^2) \\ &= a^2+ab+b^2. \end{aligned}$$

So it suffices to show

$$a^2+ab+b^2 \ge \frac{3}{4}(a^2+2ab+b^2),$$

which is equivalent to $(a-b)^2 \ge 0$.

The next example was a problem on the 1998 Iranian Mathematical Olympiad.

Example 2. (Iran-98) Let a, b, c, d be positive real numbers such that $abcd$=1. Prove that

$$a^3+b^3+c^3+d^3 \ge \max\left\{a+b+c+d,\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right\}.$$

Solution. It suffices to show

$$a^3+b^3+c^3+d^3 \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

and

$$a^3+b^3+c^3+d^3 \ge a+b+c+d.$$

For the first inequality, we observe that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{bcd+acd+abd+abc}{abcd} = bcd+acd+abd+abc.$$

Now, by the AM-GM inequality, we have $a^3+b^3+c^3 \ge 3abc$, $a^3+b^3+d^3 \ge 3abd$, $a^3+c^3+d^3 \ge 3acd$ and $b^3+c^3+d^3 \ge 3bcd$. Adding these four inequalities, we get the first inequality.

Next, let S=a+b+c+d. Then we have

$$S=a+b+c+d \ge 4(abcd)^{1/4}=4$$

by the AM-GM inequality and so $S^3 = S^2S \ge 16S$. The second inequality follows by applying the power mean inequality to obtain

$$\frac{a^3+b^3+c^3+d^3}{4} \ge \left(\frac{a+b+c+d}{4}\right)^3=\frac{S^3}{64} \ge \frac{S}{4}.$$

Example 3. Let a, b, c be positive real numbers. Prove that if $x = \max\{a,b,c\}$ and $y = \min\{a,b,c\}$, then

$$\frac{x}{y}+\frac{y}{x} \ge \frac{18abc}{(a+b+c)(a^2+b^2+c^2)}.$$

Solution. Suppose $a \ge b \ge c$. Then $x = a$ and $y = c$. Using the AM-GM inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{a}{c}+\frac{c}{a} &= \frac{a^2+c^2}{ac}=\frac{(a^2+c^2)b}{abc} \\ &\ge \frac{(2ac)b}{[(a+b+c)/3]^3}=\frac{54abc}{(a+b+c)^3} \\ &\ge \frac{54abc}{3(a^2+b^2+c^2)(a+b+c)}. \end{aligned}$$

The next example was problem 4 in the 2009 USA Mathematical Olympiad.

Example 4. (*USAMO-09*) For $n \ge 2$, let a_1, a_2, …, a_n be positive real numbers such that

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \le \left(n+\frac{1}{2}\right)^2.$$

Prove that

$$\max\{a_1,a_2,\ldots,a_n\} \le 4\min\{a_1, a_2,\ldots, a_n\}.$$

Solution. Without loss of generality, we may assume

$$m=a_1 \le a_2 \le \cdots \le a_n = M.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left(n+\frac{1}{2}\right)^2 &\ge (a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \\ &= (m+a_2+\cdots+M)\left(\frac{1}{M}+\frac{1}{a_2}+\cdots+\frac{1}{m}\right) \\ &\ge \left(\sqrt{\frac{m}{M}}+n-2+\sqrt{\frac{M}{m}}\right)^2. \end{aligned}$$

Taking square root of both sides,

$$n+\frac{1}{2} \ge \sqrt{\frac{m}{M}}+n-2+\sqrt{\frac{M}{m}}.$$

Simplifying, we get $2(m+M) \le 5\sqrt{mM}$. Squaring both sides, we can get

$$4M^2-17mM+4m^2 \ge 0.$$

Factoring, we see

$$(4M-m)(M-4m) \ge 0.$$

Since $4M-m \ge 0$, we get $M-4m \ge 0$, which is the desired inequality.

The next example was problem 1 on the 2008 Greek National Math Olympiad.

Example 5. (*Greece-08*) For positive integers a_1, a_2, …, a_n, prove that if k=$\max\{a_1,a_2,\ldots,a_n\}$ and t=$\min\{a_1,a_2,\ldots, a_n\}$, then

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \ge \prod_{i=1}^n a_i,$$

When does equality hold?

Solution. By the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^n a_i\right)^2 \le \sum_{i=1}^n 1^2 \sum_{i=1}^n a_i^2 = n\sum_{i=1}^n a_i^2.$$

Hence,

$$\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i} \ge \frac{\sum_{i=1}^n a_i}{n}.$$

Since each $a_i \ge 1$, the right side of the above inequality is at least one. Also, we have $kn/t \ge n$. So, applying the above inequality and the AM-GM inequality we have

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \ge \left(\frac{\sum_{i=1}^n a_i}{n}\right)^n \ge \prod_{i=1}^n a_i.$$

Equality holds if and only if all a_i's are equal.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 17, 2010.***

Problem 336. (*Due to Ozgur Kircak, Yahya Kemal College, Skopje, Macedonia*) Find all distinct pairs (x,y) of integers satisfying the equation

$$x^3+2009y=y^3+2009x.$$

Problem 337. In triangle ABC, $\angle ABC=\angle ACB=40°$. P and Q are two points inside the triangle such that $\angle PAB=\angle QAC=20°$ and $\angle PCB=\angle QCA=10°$. Determine whether B, P, Q are collinear or not.

Problem 338. Sequences $\{a_n\}$ and $\{b_n\}$ satisfies $a_0=1$, $b_0=0$ and for $n=0,1,2,\ldots$,

$$a_{n+1}=7a_n+6b_n-3,$$
$$b_{n+1}=8a_n+7b_n-4.$$

Prove that a_n is a perfect square for all $n=0,1,2,\ldots$

Problem 339. In triangle ABC, $\angle ACB=90°$. For every n points inside the triangle, prove that there exists a labeling of these points as $P_1, P_2, \ldots, P_n$ such that

$$P_1P_2^2+P_2P_3^2+\cdots+P_{n-1}P_n^2\le AB^2.$$

Problem 340. Let k be a given positive integer. Find the least positive integer N such that there exists a set of $2k+1$ distinct positive integers, the sum of all its elements is greater than N and the sum of any k elements is at most $N/2$.

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Solutions

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Problem 331. For every positive integer n, prove that

$$\sum_{k=0}^{n-1}(-1)^k\cos^n(k\pi/n)=\frac{n}{2^{n-1}}.$$

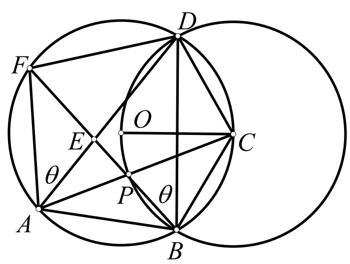
Solution. **Federico BUONERBA** (Università di Roma "Tor Vergata", Roma, Italy), **CHUNG Ping Ngai** (La Salle College, Form 6), **Ovidiu FURDUI** (Campia Turzii, Cluj, Romania), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Let $\omega=\cos(\pi/n)+i\sin(\pi/n)$. Then we have $\omega^n=-1$ and $(\omega^k+\omega^{-k})/2=\cos(k\pi/n)$. So

$$\sum_{k=0}^{n-1}(-1)^k\cos^n(k\pi/n)=\sum_{k=0}^{n-1}\omega^{kn}\left(\frac{\omega^k+\omega^{-k}}{2}\right)^n$$
$$=\frac{1}{2^n}\sum_{k=0}^{n-1}\omega^{kn}\sum_{j=0}^{n}\binom{n}{j}\omega^{k(n-2j)}$$
$$=\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n-1}(\omega^{2n-2j})^k$$
$$=\frac{1}{2^n}\left(\binom{n}{0}n+\binom{n}{n}n\right)$$
$$=\frac{n}{2^{n-1}}.$$

Problem 332. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let BD bisect OC perpendicularly. On diagonal AC, choose the point P such that $PC=OC$. Let line BP intersect line AD and the circumcircle of $ABCD$ at E and F respectively. Prove that PF is the geometric mean of EF and BF in length.

Solution. **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **Abby LEE** (SKH Lam Woo Memorial Secondary School).



Since $PC=OC=BC$ and ΔBCP is similar to ΔAFP, we have $PF=AF$.

Next, $CB=CD=CP$ implies P is the incenter of ΔABD. Then BF bisects $\angle ABD$ yielding $\angle FAD=\angle ADF$, call it θ. (Alternatively, we have $\angle FAD=\angle PBD=½\angle PCD$. Then

$$\begin{aligned}\angle AFD&=180°-\angle ACD\\&=180°-\angle PCD\\&=180°-2\angle PBD\\&=180°-2\theta.\end{aligned}$$

Hence, $\angle ADF=\theta$.) Also, we see $\angle AFE=\angle BFA$ and $\angle EAF=\theta=\angle ADF=\angle ABF$, which imply ΔAFE is similar to ΔBFA. So $AF/EF=BF/AF$. Then

$$PF=AF=\sqrt{EF\times BF}.$$

Comments: For those who are not aware of the incenter characterization used above, they may see <u>*Math Excalibur*</u>, vol. 11, no. 2 for details.

Other commended solvers: **CHOW Tseung Man** (True Light Girls' College), **CHUNG Ping Ngai** (La Salle College, Form 6), **Nicholas LEUNG** (St. Paul's School, London) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM).

Problem 333. Find the largest positive integer n such that there exist n 4-element sets $A_1, A_2, \ldots, A_n$ such that every pair of them has exactly one common element and the union of these n sets has exactly n elements.

Solution. **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM).

Let the n elements be 1 to n. For $i=1$ to n, let s_i denote the number of sets in which i appeared. Then $s_1+s_2+\cdots+s_n=4n$. On average, each i appeared in 4 sets.

Assume there is an element, say 1, appeared in more than 4 sets, say 1 is in $A_1, A_2, \ldots, A_5$. Then other than 1, the remaining $3\times5=15$ elements must all be distinct. Now 1 cannot be in all sets, otherwise there would be $3n+1>n$ elements in the union. So there is a set A_6 not containing 1. Its intersections with each of $A_1, A_2, \ldots, A_5$ must be different, yet A_6 only has 4 elements, contradiction. On the other hand, if there is an element appeared in less than 4 sets, then there would be another element appeared in more than 4 sets, contradiction. Hence, every i appeared in exactly 4 sets.

Suppose 1 appeared in A_1, A_2, A_3, A_4. Then we may assume that $A_1=\{1,2,3,4\}$, $A_2=\{1,5,6,7\}$, $A_3=\{1,8,9,10\}$ and $A_4=\{1,11,12,13\}$. Hence, $n\ge13$. Assume $n\ge14$. Then 14 would be in a set A_5. The other 3 elements of A_5 would come from A_1, A_2, A_3, say. Then A_4 and A_5 would have no common element, contradiction.

Hence, n can only be 13. Indeed, for the $n=13$ case, we can take A_1, A_2, A_3, A_4, as above and $A_5=\{2,5,8,11\}$, $A_6=\{2,6,9,12\}$, $A_7=\{2,7,10,13\}$, $A_8=\{3,5,10,12\}$, $A_9=\{3,6,8,13\}$, $A_{10}=\{3,7,9,11\}$, $A_{11}=\{4,5,9,13\}$, $A_{12}=\{4,6,10,11\}$ and $A_{13}=\{4,7,8,12\}$.

Other commended solvers: **CHUNG**

Ping Ngai (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Problem 324. (*Due to FEI Zhenpeng, Northeast Yucai School, China*) Let $x,y \in (0,1)$ and x be the number whose n-th digit after the decimal point is the n^n-th digit after the decimal point of y for all n =1,2,3,…. Show that if y is rational, then x is rational.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6),

Since the decimal representation of y is eventually periodic, let L be the length of the period and let the decimal representation of y start to become periodic at the m-th digit. Let k be the least common multiple of 1,2,…,L. Let n be any integer at least L and $n^n \geq m$.

By the pigeonhole principle, there exist $i < j$ among 0,1,…,L such that $n^i \equiv n^j$ (mod L). Then for all positive integer d, we have $n^i \equiv n^{i+d(j-i)}$ (mod L). Since k is a multiple of $j-i$ and $n \geq L > i$, so we have $n^n \equiv n^{n+k}$ (mod L). Since k is also a multiple of L, we have $(n+k)^{n+k} \equiv n^{n+k} \equiv n^n$ (mod L). Then the n-th and $(n+k)$-th digit of x are the same. So x is rational.

Other commended solvers: **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Problem 335. (*Due to Ozgur KIRCAK, Yahya Kemal College, Skopje, Macedonia*) Find all $a \in \mathbb{R}$ for which the functional equation $f: \mathbb{R} \to \mathbb{R}$

$$f(x - f(y)) = a(f(x) - x) - f(y)$$

for all $x, y \in \mathbb{R}$ has a unique solution.

Solution. **LE Trong Cuong** (Lam Son High School, Vietnam)

Let $g(x) = f(x)-x$. Then, in terms of g, the equation becomes

$$g(x-y-g(y))=ag(x)-x.$$

Assume $f(y)=y+g(y)$ is not constant. Let r, s be distinct elements in the range of $f(y)=y+g(y)$. For every real x,

$$g(x-r) = ag(x)-x = g(x-s).$$

This implies $g(x)$ is periodic with period $T=|r-s|>0$. Then

$$\begin{aligned} ag(x) - x &= g(x-y-g(y)) \\ &= g(x+T-y-g(y)) \\ &= ag(x+T) - (x+T) \\ &= ag(x)-x-T. \end{aligned}$$

This implies T=0, contradiction. Thus, f is constant, i.e. there exists a real number c so that for all real y, $f(y)=c$. Then the original equation yields $c=a(c-x)-c$ for all real x, which forces a=0 and c=0.

Other commended solvers: **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM).

Olympiad Corner

(continued from page 1)

Problem 3. Let a,b,c be complex numbers such that for every complex number z with $|z| \leq 1$, we have $|az^2+bz+c| \leq 1$. Find the maximum of $|bc|$.

Problem 4. Let m,n be integers greater than 1. Let $a_1 < a_2 < \cdots < a_m$ be integers. Prove that there exists a subset T of the set of all integers such that the number of elements of T, denoted by $|T|$, satisfies

$$|T| \leq 1 + \frac{a_m - a_1}{2n+1}$$

and for every $i \in \{1,2,\cdots,m\}$, there exist $t \in T$ and $s \in [-n,n]$ such that $a_i=t+s$.

Problem 5. For $n \geq 3$, we place a number of cards at points A_1, A_2, ⋯, A_n and O. We can perform the following operations:

(1) if the number of cards at some point A_i is not less than 3, then we can remove 3 cards from A_i and transfer 1 card to each of the points A_{i-1}, A_{i+1} and O (here $A_0=A_n$, $A_{n+1}=A_1$); or

(2) if the number of cards at O is not less than n, then we can remove n cards from O and transfer 1 card to each A_1, A_2, ⋯, A_n.

Prove that if the sum of all the cards placed at these n+1 points is not less than n^2+3n+1, then we can always perform finitely many operations so that the number of cards at each of the points is not less than n+1.

Problem 6. Let a_1, a_2, a_3, b_1, b_2, b_3 be distinct positive integers satisfying

$$(n+1)a_1^n + na_2^n + (n-1)a_3^n \mid (n+1)b_1^n + nb_2^n + (n-1)b_3^n$$

for all positive integer n. Prove that there exists a positive integer k such that $b_i=ka_i$ for i=1,2,3.

Max-Min Inequalities

(continued from page 2)

The inequality in the next example was very hard. It was proposed by Reid Barton and appeared among the 2003 IMO shortlisted problems.

Example 6. Let n be a positive integer and let $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ be two sequences of positive real numbers. Let $(z_1, z_2, \ldots, z_{2n})$ be a sequence of positive real numbers such that for all $1 \leq i, j \leq n$, $z_{i+j}^2 \geq x_i y_j$. Let $M=\max\{z_1, z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M+z_2+\cdots+z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1+\cdots+x_n}{n}\right)\left(\frac{y_1+\cdots+y_n}{n}\right).$$

Solution. (*Due to Reid Barton and Thomas Mildorf*) Let

$$X = \max\{x_1, x_2, \ldots, x_n\}$$

and

$$Y = \min\{x_1, x_2, \ldots, x_n\}.$$

By replacing x_i by $x_i'=x_i/X$, y_i by $y_i'=y_i/Y$ and z_i by $z_i'= z_i/(XY)^{1/2}$, we may assume $X=Y=1$. It suffices to prove

$$M+z_2+\cdots+z_{2n} \geq x_1+\cdots+x_n+y_1+\cdots+y_n. \quad (*)$$

Then

$$\frac{M+z_2+\cdots+z_{2n}}{2n} \geq \frac{1}{2}\left(\frac{x_1+\cdots+x_n}{n}+\frac{y_1+\cdots+y_n}{n}\right),$$

which implies the desired inequality by applying the AM-GM inequality to the right side.

To prove (*), we will *claim* that for any $r \geq 0$, the number of terms greater than r on the left side is at least the number of such terms on the right side. Then the k-th largest term on the left side is greater than the k-th largest term on the right side for each k, proving (*).

For $r \geq 1$, there are no terms greater than 1 on the right side. For $r < 1$, let $A=\{i: x_i>r\}$, $B=\{j: y_j>r\}$, $A+B=\{i+j: i \in A, j \in B\}$ and $C=\{k: k>1, z_k>r\}$. Let $|A|$, $|B|$, $|A+B|$, $|C|$ denote the number of elements in A, B, $A+B$, C respectively.

Since $X=Y=1$, so $|A|$, $|B|$ are at least 1. Now $x_i>r$, $y_j>r$ imply $z_{i+j}>r$. So $A+B$ is a subset of C. If A is consisted of $i_1<\cdots<i_a$ and B is consisted of $j_1<\cdots<j_b$, then $A+B$ contains

$$i_1+j_1 < i_1+j_2 < \cdots < i_1+j_b < i_2+j_b < \cdots < i_a+j_b.$$

Hence, $|C| \geq |A+B| \geq |A|+|B|-1 \geq 1$. So $z_k>r$ for some k. Then $M>r$. So the left side of (*) has $|C|+1 \geq |A|+|B|$ terms greater than r, which finishes the proof of the claim.

Mathematical Excalibur

Volume 14, Number 5 — *March - April, 2010*

Olympiad Corner

Here are the Asia Pacific Math Olympiad problems on March 2010.

Problem 1. Let *ABC* be a triangle with $\angle BAC \neq 90°$. Let *O* be the circumcenter of triangle *ABC* and let *Γ* be the circumcircle of triangle *BOC*. Suppose that *Γ* intersects the line segment *AB* at *P* different from *B*, and the line segment *AC* at *Q* different from *C*. Let *ON* be a diameter of the circle *Γ*. Prove that the quadrilateral *APNQ* is a parallelogram.

Problem 2. For a positive integer *k*, call an integer a *pure k-th power* if it can be represented as m^k for some integer *m*. Show that for every positive integer *n* there exist *n* distinct positive integers such that their sum is a pure 2009-th power, and their product is a pure 2010-th power.

Problem 3. Let *n* be a positive integer. *n* people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 21, 2010***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Ramsey Numbers and Generalizations

Law Ka Ho

The following problem is classical: among any 6 people, there exist 3 who know each other or 3 who don't know each other (we assume if *A* knows *B*, then *B* knows *A*). When 6 is replaced by 5, this is no longer true, as can be seen by constructing a counterexample. We write $R(3,3)=6$ and this is called a *Ramsey number*. In general, $R(m,n)$ denotes the smallest positive integer *k* such that, among any *k* people, there exist *m* who know each other or *n* who don't know each other.

How do we know that $R(m,n)$ exists for all *m*, *n*? A key result is the following.

<u>Theorem 1.</u> For any *m*, *n* > 1, we have $R(m,n) \le R(m-1,n)+R(m,n-1)$.

<u>Proof.</u> Take $R(m-1,n)+R(m,n-1)$ people. We need to show that there exist *m* people who know each other or *n* people who don't know each other. If a person *X* knows $R(m-1,n)$ others, then among the people *X* knows, there exist either $m-1$ who know each other (so that together with *m*, there are *m* people who know each other) or *n* people who don't know each other, so we are done. Similarly, if *X* doesn't know $R(m,n-1)$ others, we are also done. But one of these two cases must occur because the total number of 'others' is $R(m-1,n)+R(m,n-1)-1$.

Using Theorem 1, one can easily show (by induction on $m+n$) that $R(m,n) \le C_{m-1}^{m+n-2}$. This establishes an upper bound on $R(m,n)$. To establish a lower bound, we need a counter-example. While construction of counter-examples is in general very difficult, the probabilistic method (see Vol. 14, No. 3) may be able to help us in getting a non-constructive proof. Yet to get the exact value of a Ramsey number, the lower and upper bounds must match, which is extremely difficult. For *m*, *n* > 3, fewer than 10 values of $R(m,n)$ are known:

$R(3,4)=9$, $R(3,5)=14$, $R(3,6)=18$
$R(3,7)=23$, $R(3,8)=28$, $R(3,9)=36$
$R(4,4)=18$, $R(4,5)=25$

Even $R(5,5)$ is unknown at present. The best lower and upper bounds obtained so far are respectively 43 and 49. Paul Erdös once made the following remark.

> *Suppose an evil alien would tell mankind "Either you tell me [the value of R(5,5)] or I will exterminate the human race." ... It would be best in this case to try to compute it, both by mathematics and with a computer.*
>
> *If he would ask [for the value of R(6,6)], the best thing would be to destroy him before he destroys us, because we couldn't [determine R(6,6)].*

Problems related to the Ramsey numbers occur often in mathematical competitions.

<u>Example 2.</u> (CWMO 2005) There are *n* new students. Among any three of them there exist two who know each other, and among any four of them there exist two who do not know each other. Find the greatest possible value of *n*.

<u>Solution.</u> The answer is 8. First, *n* can be 8 if the 8 students are numbered 1 to 8 and student *i* knows student *j* if and only if $|i-j| \not\equiv 1, 4 \pmod 8$. Next, suppose $n=9$ is possible. Then no student may know 6 others, for among the 6 either 3 don't know each other or 3 know each other (so together with the original student there exist 4 who know each other). Similarly, it cannot happen that a student doesn't know 4 others. Hence each student knows exactly 5 others. But this is impossible, because if we sum the number of others whom each student know, we get $9\times 5=45$, which is odd, yet each pair of students who know each other is counted twice.

Remark. The answer to the above problem is $R(3,4)-1$, as can be seen by comparing with the definition of $R(3,4)$.

The Ramsey number can be generalised in many different directions. One is to increase the number of statuses from 2 (know or don't know) to more than 2, as the following example shows.

Example 3. (IMO 1964) Seventeen people correspond by mail with one another — each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

Solution. Suppose the three topics are A, B and C. Pick any person; he writes to 16 others. By the pigeonhole principle, he writes to 6 others on the same topic, say A. If any two of the 6 people write to each other on A, then we are done. If not, then these 6 people write to each other on B or C. Since $R(3,3)=6$, either 3 of them write to each other on B, or 3 of them write to each other on C. In any case there exist 3 people who write to each other about the same topic.

Remark. The above problem proves $R(3,3,3)\le 17$, where $R(m,n,p)$ is defined analogously as $R(m,n)$ except that there are now three possible statuses instead of two. It can be shown that $R(3,3,3)=17$ by constructing a counterexample when there are only 16 people.

Another direction of generalization is to generalise 'm people who know each other' or 'n people who don't know each other' to other structures. (Technically, the *graph Ramsey number* $R(G,H)$ is the smallest positive integer k such that when every two of k points are joined together by a red or blue edge, there must exist a red copy of G or a blue copy of H. Hence $R(m,n)=R(K_m,K_n)$, where K_m denotes the complete graph on m vertices, i.e. m points among which every two are joined by an edge).

Example 4. N people attend a meeting, and some of them shake hands with each other. Suppose that each person shakes hands with at most 100 other people, and among any 50 people there exist at least two who have shaken hands with each other. Find the greatest possible value of N.

Solution. The answer is 4949. We first show that $N = 4949$ is possible: suppose there are 49 groups of 101 people each, and two people shake hands if and only if they are in the same group. It is easy to check that the requirements of the question are satisfied. Now suppose $N = 4950$ and each person shakes hands with at most 100 others. We will show that there exist 50 people who have not shaken hands with each other, thus contradicting the given condition. To do this, pick a first person P_1 and cross out all those who have shaken hands with him. Then pick P_2 from the rest and again cross out those who have shaken hands with him, and so on. In this way, at most 100 people are crossed out each time. After P_{49} is chosen, at least $4950-49-49\times 100=1$ person remains, so we will be able to choose P_{50}. Because of the 'crossing out' algorithm, we see that no two of P_1, P_2, …, P_{50} have shaken hands with each other.

Remark. By identifying each person with a point and joining two points by a red line if two people have shaken hands and a blue line otherwise, we see that the above problem proves $R(K_{1,100},K_{50})=4950$. Here $K_{1,100}$ is the graph on 101 points by joining 1 point to the other 100 points.

The *Van der Waerden number* $W(r,k)$ is the smallest positive integer N such that if each of 1, 2, …, N is assigned one of r colours, then there exist a monochromatic k-term arithmetic progression. The following example shows that we have $W(2,3)\le 325$.

Example 5. If each of the integers 1, 2, …, 325 is assigned red or blue colour, there exist three integers p, q, r which are assigned the same colour and which form an arithmetic progression.

Solution. Divide the 325 integers into 65 groups $G_1 = \{1, 2, 3, 4, 5\}$, $G_2 = \{6, 7, 8, 9, 10\}$, …, $G_{65} = \{321, 322, 323, 324, 325\}$. There are $2^5 = 32$ possible colour patterns for each group. Hence there exist three groups G_a and G_b, $1\le a<b\le 33$, whose colour patterns are the same. We note that $2b-a\le 65$ and that a, b, $2b-a$ form an arithmetic progression. Now two of the first three numbers of G_a are of the same colour, say, the first and third are red (it can be seen that the proof goes exactly the same way if it is the first and second, or second and third). If the fifth is also red, then we are done. Otherwise, the first and third numbers of both G_a and G_b (recall that they have identical colour patterns) are red while the fifth is blue. If the fifth number of G_{2b-a} is red, then it together with the first number of G_a and the third number of G_b form a red arithmetic progression; if it is blue, then it together with the fifth numbers of G_a and G_b form a blue arithmetic progression.

It can be shown via a two-dimensional inductive argument that $W(r,k)$ exists for all r, k. We see that the existence of Ramsey numbers and van der Waerden numbers are very similar: both say that the desired structure exists in a sufficiently large population.

An analogy to this (though not mathematically rigorous) is that when there are sufficiently many stars in the sky, one can form from them whatever picture one wishes. (This is one of the lines in the movie *A Beautiful Mind*!)

Yet another generalization of the van der Waerden Theorem (which says that $W(r,k)$ exists for all r, k) is the *Hales-Jewett Theorem*. The exact statement of the theorem is rather technical, but we can look at an informal version here. We are familiar with the two-person tic-tac-toe game played on a 3×3 square in two dimensions. We also have the two-person tic-tac-toe game played on a $4\times 4\times 4$ cube in three dimensions (try it out at http://www.mathdb.org/fun/games/tie_toe/e_tie_toe.htm!). Both games can end in a draw. However, it is easy to see that a two-person tic-tac-toe game played on a 2×2 square in two dimensions cannot end in a draw. The Hales-Jewett Theorem says that for any n and k, the k-person tic-tac-toe game played on an $n\times n\times\cdots\times n$ (D factors of n, where D is the dimension) hypercube cannot end in a draw when D is large enough! (For instance, we have just seen that when $n = 2$ and $k = 2$, then $D = 2$ is large enough, while when $n = 3$ and $k = 2$, then $D = 2$ is not large enough.) In case $k = 2$ (i.e. a two-person game) and when D is large enough so that a draw is impossible, it can be shown (via a so-called strategy stealing argument) that the first player has a winning strategy!

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 21, 2010.***

Problem 341. Show that there exists an infinite set S of points in the 3-dimensional space such that every plane contains at least one, but not infinitely many points of S.

Problem 342. Let $f(x)=a_nx^n+\cdots+a_1x+p$ be a polynomial with coefficients in the integers and degree $n>1$, where p is a prime number and

$$|a_n|+|a_{n-1}|+\cdots+|a_1|<p.$$

Then prove that $f(x)$ is not the product of two polynomials with coefficients in the integers and degrees less than n.

Problem 343. Determine all ordered pairs (a,b) of positive integers such that $a\neq b$, $b^2+a=p^m$ (where p is a prime number, m is a positive integer) and a^2+b is divisible by b^2+a.

Problem 344. $ABCD$ is a cyclic quadrilateral. Let M, N be midpoints of diagonals AC, BD respectively. Lines BA, CD intersect at E and lines AD, BC intersect at F. Prove that

$$\left|\frac{BD}{AC}-\frac{AC}{BD}\right|=\frac{2MN}{EF}.$$

Problem 345. Let $a_1, a_2, a_3, \cdots$ be a sequence of integers such that there are infinitely many positive terms and also infinitely many negative terms. For every positive integer n, the remainders of $a_1, a_2, \cdots, a_n$ upon divisions by n are all distinct. Prove that every integer appears exactly one time in the sequence.

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Solutions

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Problem 336. (*Due to Ozgur Kircak, Yahya Kemal College, Skopje, Macedonia*) Find all distinct pairs (x,y) of integers satisfying the equation

$$x^3+2009y=y^3+2009x.$$

Solution. **CHOW Tseung Man** (True Light Girls' College), **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, Oregon, USA), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy), **Pedro Henrique O. PANTOJA** (UFRN, Natal, Brazil), **PUN Ying Anna** (HKU), **TSOI Kwok Wing** (PLK Centenary Li Shiu Chung Memorial College), **Simon YAU Chi-Keung** and **Fai YUNG**.

All pairs (x,x) satisfy the equation. If (x,y) satisfies the equation and $x\neq y$, then

$$x^2+xy+y^2=\frac{x^3-y^3}{x-y}=2009\equiv 2 \pmod 3.$$

However, $x^2+xy+y^2\equiv x^2-2xy+y^2=(x-y)^2\equiv 0$ or 1 (mod 3). So there are no solutions with $x\neq y$.

Problem 337. In triangle ABC, $\angle ABC=\angle ACB=40°$. P and Q are two points inside the triangle such that $\angle PAB=\angle QAC=20°$ and $\angle PCB=\angle QCA=10°$. Determine whether B, P, Q are collinear or not.

***Solution* 1.** **CHUNG Ping Ngai** (La Salle College, Form 6) and **HUNG Ka Kin Kenneth** (Diocesan Boys' School).

Let $\angle PBA=a$, $\angle PBC=b$, $\angle QBA=a'$ and $\angle QBC=b'$. By the trigonometric form of Ceva's theorem, we have

$$1=\frac{\sin\angle PBA}{\sin\angle PBC}\frac{\sin\angle PCB}{\sin\angle PCA}\frac{\sin\angle PAC}{\sin\angle PAB}=\frac{\sin a\sin 10^\circ\sin 80^\circ}{\sin b\sin 30^\circ\sin 20^\circ},$$

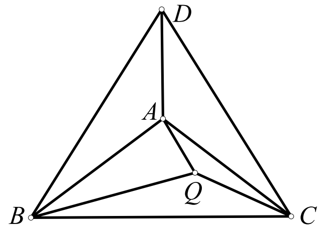
$$1=\frac{\sin\angle QBA}{\sin\angle QBC}\frac{\sin\angle QCB}{\sin\angle QCA}\frac{\sin\angle QAC}{\sin\angle QAB}=\frac{\sin a'\sin 30^\circ\sin 20^\circ}{\sin b'\sin 10^\circ\sin 80^\circ}.$$

As $\sin10°\sin80° = \sin10°\cos10° =½\sin20° = \sin30°\sin20°$, we obtain $\sin a= \sin b$ and $\sin a' = \sin b'$. Since $0<a,b,a',b'<90°$ and $a+b=40°=a'+b'$, we get $a=b=a'=b'=20°$, i.e. $\angle PBA=\angle PBC=\angle QBA=\angle QBC$. Therefore, B, P, Q are collinear.

***Solution* 2.** **LEE Kai Seng**.

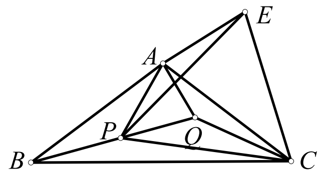
We will show B,P,Q collinear by proving lines BQ and BP bisect $\angle ABC$.

Draw an equilateral triangle BDC with D on the same side of BC as A. Since $\angle ABC=\angle ACB=40°$, $AB=AC$. Then both D and A are equal distance from B and C. So DA bisects $\angle BDC$. We have



$\angle QCD = 60°-\angle BCQ = 30° = \angle ADC$.

Also, $\angle DCA=\angle QCD-\angle QCA=20°=\angle QAC$, which implies $QA\|CD$. Then $AQCD$ is an isosceles trapezoid, so $AD = QC$. This with $BD=BC$ and $\angle BDA = 30° = \angle QCB$ imply $\Delta BDA\cong\Delta QCB$. Then $BA=BQ$. Since $\angle BAQ=\angle BAC-\angle QAC = 100°-20° = 80°$, we get $\angle ABQ = 20° = ½\angle ABC$. So BQ bisects $\angle ABC$.



Extend BA to a point E so that $BE=BC$. Then $\angle BCE = ½(180°-\angle ABC) = 70°$. Next, we will show ΔEPC is equilateral.

We have $\angle PCE=\angle BCE-\angle PCB=60°$, $\angle ACE=\angle BCE-\angle BCA=30°=½\angle PCE$. So CA bisects $\angle PCE$. Next, $\angle CAE = 180°-\angle BAC = 80° = \angle BAC-\angle BAP=\angle CAP$. Then $\Delta CAE\cong\Delta CAP$. So $CE = CP$ and ΔEPC is equilateral. Then B, P are equal distance from E and C. Hence BP bisects $\angle ABC$.

Other commended solvers: **CHAN Chun Wai** (St. Paul's College), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **PUN Ying Anna** (HKU).

Problem 338. Sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_0=1$, $b_0=0$ and for $n=0,1,2,\ldots$,

$$a_{n+1}=7a_n+6b_n-3,$$
$$b_{n+1}=8a_n+7b_n-4.$$

Prove that a_n is a perfect square for all $n=0,1,2,\ldots$

***Solution* 1.** **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, Oregon, USA), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (G.T. (Ellen Yeung) College), **Ercole SUPPA** (Teramo, Italy) and **YEUNG Chun Wing** (St. Paul's College).

Solving for b_n in the first equation and putting it into the second equation, we have

$a_{n+2}=14a_{n+1}-a_n-6$ for $n=0,1,2,\ldots$ (*)

with $a_0=1$ and $a_1=4$. Let $d_n=a_n-½$. Then (*) becomes $d_{n+2}=14d_{n+1}-d_n$. Since the roots of $x^2-14x+1=0$ are $7\pm 4\sqrt{3}$, we get d_n is of the form $\alpha(7-4\sqrt{3})^n+\beta(7+4\sqrt{3})^n$. Using $d_0=½$ and $d_1=3½$, we get $\alpha=¼$ and $\beta=¼$. So

$$a_n = d_n + \frac{1}{2} = \frac{2+(7-4\sqrt{3})^n+(7+4\sqrt{3})^n}{4}.$$

Now, consider the sequence $\{c_n\}$ of positive integers, defined by $c_0=1$, $c_1=2$ and

$c_{n+2}=4c_{n+1}-c_n$ for $n=0,1,2,\ldots.$ (**)

Since the roots of $x^2-4x+1=0$ are $2\pm\sqrt{3}$, as above we get

$$c_n = \frac{(2-\sqrt{3})^n+(2+\sqrt{3})^n}{2}.$$

Squaring c_n, we see $a_n=c_n^2$.

***Solution* 2**. **William CHAN** and **Invisible MAK** (Carmel Alison Lam Foundation Secondary School).

The equations imply

$a_{n+2}=14a_{n+1}-a_n-6$ for $n=0,1,2,\ldots$ (*)

We will prove $a_na_{n+2}=(a_{n+1}+3)^2$ by math induction. The case $n=0$ is $1\times 49=(4+3)^2$. Suppose $a_{n-1}a_{n+1}=(a_n+3)^2$. Then

$$\begin{aligned} & a_na_{n+2}-(a_{n+1}+3)^2 \\ &= a_n(14a_{n+1}-a_n-6)-(a_{n+1}+3)^2 \\ &= 14a_na_{n+1}-a_n^2-6a_n-a_{n+1}^2-6a_{n+1}-9 \\ &= (14a_n-a_{n+1}-6)a_{n+1}-(a_n+3)^2 \\ &= a_{n-1}a_{n+1}-(a_n+3)^2 \\ &= 0. \end{aligned}$$

This completes the induction.

Next, we will show all a_n's are perfect squares. Now $a_0=1^2$ and $a_1=2^2$. Suppose $a_{n-1}=r^2$ and $a_n=s^2$, we get $a_{n+1}=(a_n+3)^2/r^2$ and $a_{n+2}=(a_{n+1}+3)^2/s^2$. Since the square root of a positive integer is an integer or an irrational number, a_{n+1} and a_{n+2} are perfect squares. By mathematical induction, the result follows.

Other commended solvers: **PUN Ying Anna** (HKU), **TSOI Kwok Wing** (PLK Centenary Li Shiu Chung Memorial College).

Problem 339. In triangle ABC, $\angle ACB=90°$. For every n points inside the triangle, prove that there exists a labeling of these points as $P_1, P_2, \ldots, P_n$ such that

$$P_1P_2^2+P_2P_3^2+\cdots+P_{n-1}P_n^2 \le AB^2.$$

Solution. **Federico BUONERBA** (Università di Roma "Tor Vergata", Roma, Italy), **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **PUN Ying Anna** (HKU).

We will prove the following more general result:

Let ABC be a triangle with $\angle ACB=90°$. For every n points inside or on the sides of the triangle, there exists a labeling of these points as $P_1, P_2, \ldots, P_n$ such that

$$AP_1^2+P_1P_2^2+\cdots+P_{n-1}P_n^2+P_nB^2 \le AB^2.$$

We prove this by induction on n. For the case $n=1$, since $\angle AP_1B \ge 90°$, the cosine law gives $AP_1^{\,2}+P_1B^{\,2} \le AB^{\,2}$.

Next we assume all cases less than n are true. For the case n, we can divide the original right triangle into two right triangles by taking the altitude from C to H on the hypotenuse AB. We can assume that the two smaller right triangles AHC and BHC contain $m>0$ and $n-m>0$ points respectively (otherwise, one of these two smaller triangles contains all the points and we keep dividing in the same way the smaller right triangle which contains all the points). Since $m<n$ and $n-m<n$, by the induction hypothesis, there exist a labeling of points in triangle AHC as $P_1, P_2, \ldots, P_m$ such that

$$AP_1^2+P_1P_2^2+\cdots+P_{m-1}P_m^2+P_mC^2 \le AC^2$$

and a labeling of points in triangle BHC as $P_{m+1}, P_{m+2}, \ldots, P_m$ such that

$$CP_{m+1}^2+P_{m+1}P_{m+2}^2+\cdots+P_nB^2 \le CB^2.$$

Since $\angle P_mCP_{m+1} \le 90°$, the cosine law gives $P_mP_{m+1}^{\,2} \le P_mC^2+CP_{m+1}^{\,2}$. Then

$$\begin{aligned} & AP_1^2+P_1P_2^2+\cdots+P_{n-1}P_n^2+P_nB^2 \\ & \le AC^2+CB^2=AB^2. \end{aligned}$$

Problem 340. Let k be a given positive integer. Find the least positive integer N such that there exists a set of $2k+1$ distinct positive integers, the sum of all its elements is greater than N and the sum of any k elements is at most $N/2$.

Solution. **CHAN Chun Wai** (St. Paul's College), **CHOW Tseung Man** (True Light Girls' College), **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **PUN Ying Anna** (HKU).

Let $a_1, a_2, \ldots, a_{2k+1}$ be such a set of $2k+1$ of positive integers arranged in increasing order. We have

$$\sum_{i=1}^{2k+1} a_i \ge N+1 \ge 2\sum_{i=k+2}^{2k+1} a_i + 1.$$

Then

$$\begin{aligned} a_{k+1} &\ge \sum_{i=k+2}^{2k+1} a_i - \sum_{i=1}^{k} a_i + 1 \\ &= \sum_{i=1}^{k}(a_{i+k+1}-a_i)+1 \\ &\ge \sum_{i=1}^{k}(k+1)+1 \\ &= k^2+k+1. \end{aligned}$$

Also,

$$\begin{aligned} \frac{N}{2} &\ge \sum_{i=k+2}^{2k+1} a_i = \sum_{i=k+2}^{2k+1}(a_i-a_{k+1})+\sum_{i=k+2}^{2k+1} a_{k+1} \\ &\ge \sum_{i=k+2}^{2k+1}(i-k-1)+k(k^2+k+1) \\ &= \frac{2k^3+3k^2+3k}{2}. \end{aligned}$$

Now all inequalities above become equality if we take $a_i=k^2+i$ for $i=1, 2, \ldots, 2k+1$. So the least positive value of N is $2k^3+3k^2+3k$.

Olympiad Corner

(continued from page 1)

Problem 4. Let ABC be an acute triangle satisfying the condition $AB>BC$ and $AC>BC$. Denote by O and H the circumcenter and orthocenter, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and that the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcenter of the triangle MNH lies on the line OH.

Problem 5. Find all functions f from the set $\boldsymbol{R}$ of real numbers into $\boldsymbol{R}$ which satisfy for all $x, y, z \in \boldsymbol{R}$ the identity

$$\begin{aligned} & f(f(x)+f(y)+f(z)) \\ &= f(f(x)-f(y))+f(2xy+f(z))+2f(xz-yz). \end{aligned}$$