Mathematical Excalibur

Volume 18, Number 1 — May-August 2013

Olympiad Corner

Below are the problems of the 2013 Asia Pacific Mathematical Olympiad.

Problem 1. Let ABC be an acute triangle with altitudes AD, BE and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

Problem 2. Determine all positive integers n for which

$$\frac{n^2+1}{[\sqrt{n}]^2+2}$$

is an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Problem 3. For $2k$ real numbers, a_1, a_2, …, a_k, b_1, b_2, …, b_k define the sequence of numbers X_n by

$$X_n=\sum_{i=1}^{n}[a_i n+b_i] \quad (n=1,2,\ldots).$$

If the sequence X_n forms an arithmetic progression, show that $\sum_{i=1}^{k} a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 1, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Ptolemy's Inequality

Nguyen Ngoc Giang, M.Sc.

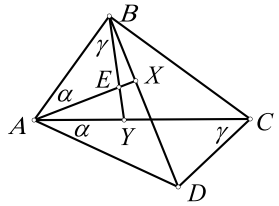
Ptolemy's inequality is a beautiful inequality, but it is rather difficult to see when or how it can be applied to geometry problems. This inequality has often been used in gifted student selection exams in various places.

In this article we will first look at how this inequality is derived.

Theorem (Ptolemy's Inequality) Let $ABCD$ be a quadrilateral. We have

$$AB\times CD+DA\times BC\geq AC\times BD$$

with equality if and only if $ABCD$ is a cyclic quadrilateral.



Proof. From A and from B draw two rays to cut diagonals BD and AC at X and at Y respectively such that $\angle XAB = \angle DAC$ and $\angle YBA = \angle DCA$.

Suppose AX cut BY at E. Then $\angle BAC = \angle EAD$. We have $\Delta ABE \sim \Delta ACD$. So

$$\frac{AB}{AC}=\frac{BE}{CD}=\frac{AE}{AD}\Rightarrow AB\times CD=AC\times BE \quad (1)$$

and we also have $\Delta AED \sim \Delta ABC$. It follows

$$\frac{AD}{AC}=\frac{ED}{BC}\Rightarrow AD\times BC=AC\times ED. \quad (2)$$

Adding the equations (1) and (2), we have

$$AB\times CD+DA\times BC=AC\times(BE+ED)\geq AC\times BD.$$

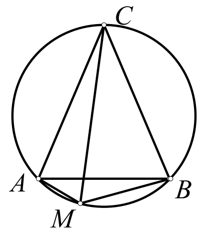
Thus, for arbitrary quadrilateral $ABCD$, we have $AB\times CD+DA\times BC \geq AC\times BD$. Equality holds if and only if E belongs to BD. In that case $\angle ABD = \angle ACD$ or $ABCD$ is a cyclic quadrilateral.

This inequality can also be proved in a different way (cf *vol. 2, no. 4* of *Math Excalibur*).

Next we will look at applications of the theorem and Ptolemy's inequality.

Example 1 An isosceles triangle ABC (with $CA=CB$) is inscribed in a circle with center O and M is an arbitrary point lying on the minor arc BC. Prove that

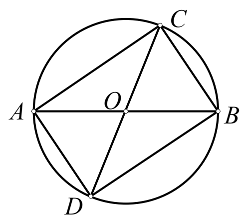
$$\frac{MA+MB}{MC}=\frac{AB}{AC}.$$



Solution. Applying Ptolemy's theorem to $AMBC$, we have $MA\times BC+MB\times CA = MC\times AB$. From $CA=CB$, we have

$$\frac{MA+MB}{MC}=\frac{AB}{AC}.$$

Example 2 (*Pythagorean Theorem*) For right ΔABC with $\angle ACB$=90°, we have $BC^2+AC^2=AB^2$.



Solution. Draw a circle with midpoint O of side AB as center and radius $AB/2$. Let ray CO intersect the circle at D. Applying Ptolemy's theorem to $ACBD$, we have $AD\times BC+BD\times AC = AB\times CD$. Since $AD = BC$, $BD=AC$ and $CD=AB$, we get

$$BC^2+AC^2=AB^2.$$

Example 3 Let a and b be acute angles. Prove that

$$\sin(a+b) = \sin a\cos b + \sin b\cos a.$$

Solution. Let's draw a circle with diameter $AC = 1$. Construct the rays AB and AD lying on opposite sides of the diameter AC such that $\angle CAB = a$ and $\angle CAD = b$. Also draw diameter BE as shown in the next figure.

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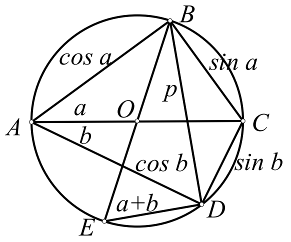


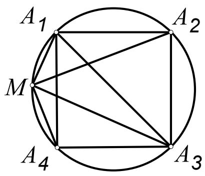
Since AC and BE are diameters, $\angle ABC$, $\angle ADC$ and $\angle BDE$ are right angles, we see that $AB = \cos a$, $BC = \sin a$, $CD = \sin b$, $DA = \cos b$ respectively. Also, $\angle BED = \angle BAD = a+b$ and $BD = p = \sin(a+b)$.

Applying Ptolemy's theorem to $ABCD$, we have $AC\times BD = BC\times DA + CD\times AB$, which is

$$\sin(a+b) = \sin a \cos b + \sin b \cos a.$$

Example 4 If an arbitrary point M lies on the circle circumscribed about square $A_1A_2A_3A_4$, then we have the relation

$$MA_1^2 + MA_3^2 = MA_2^2 + MA_4^2.$$



Solution. Without loss of generality, assume that M lies on minor arc A_1A_4. Let $MA_1 = x_1$, $MA_2 = x_2$, $MA_3 = x_3$, $MA_4 = x_4$ and let the square have side a. Applying Ptolemy's theorem to $MA_1A_2A_3$ and $MA_1A_3A_4$, we have

$$x_1a + x_3a = x_2a\sqrt{2},\quad x_1a + x_4a\sqrt{2} = x_3a.$$

Cancelling a in both equations and rewriting the second equation as

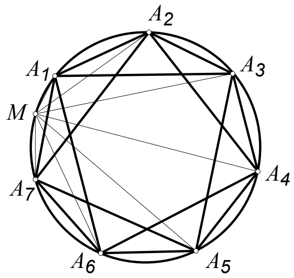
$$x_3 - x_1 = x_4\sqrt{2},$$

we get $(x_1+x_3)^2 + (x_3-x_1)^2 = 2x_2^2 + 2x_4^2$. Expanding the equation and cancelling 2 on both sides, we get

$$x_1^2 + x_3^2 = x_2^2 + x_4^2,$$

which is the desired conclusion.

Example 5 Let $A_1A_2\ldots A_n$ be a regular polygon that has an _odd_ number of sides. Let M be a point on the minor arc A_1A_n of the circle circumscribed about it. Prove that the sum of the distances from the point M to the vertices A_i (i being odd) is equal to the sum of the distances from the point M to the vertices A_k (k being even).



Solution. Let each side of the polygon have length a. Draw the diagonals A_1A_3, A_2A_4, …, A_nA_2, which have a common length b. Next, draw the chords MA_1, MA_2, …, MA_n and let MA_i have length d_i. Applying Ptolemy's theorem to $MA_1A_2A_3$, $MA_2A_3A_4$, …, $MA_{n-1}A_nA_1$ and $MA_nA_1A_2$, we have

$$ad_1+ad_3=bd_2,\quad bd_3=ad_2+ad_4,$$
$$ad_3+ad_5=bd_4,\quad bd_5=ad_4+ad_6,\ \ldots,$$
$$ad_{n-2}+ad_n=bd_{n-1},\quad bd_n+ad_1=ad_{n-1},$$
$$ad_n+bd_1=ad_2.$$

Adding these equations, we get

$$(2a+b)(d_1+d_3+\cdots+d_n) = (2a+b)(d_2+d_4+\cdots+d_{n-1}).$$

Cancelling $2a+b$, this becomes

$$d_1 + d_3 + \cdots + d_n = d_2 + d_4 + \cdots + d_{n-1},$$

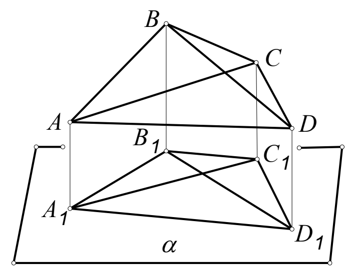
which is the desired conclusion.

Next we will generalize Ptolemy's inequality to higher dimensions.

Example 6 Let $ABCD$ be a tetrahedron in the three dimensional space. Prove that

$$AB\times CD+DA\times BC > AC\times BD$$

Solution. Draw a plane α which is parallel to lines AC and BD. Let A_1, B_1, C_1, D_1 be the feet of perpendiculars from A, B, C, D to plane α respectively. We have $A_1C_1=AC$ and $B_1D_1 = BD$. However, $A_1B_1 \le AB$, $B_1C_1 \le BC$, $C_1D_1 \le CD$, $D_1A_1 \le DA$ and at least one of these is strict since A,B,C,D are not on the same plane.



Using the inequalities and equations above as well as Ptolemy's inequality applied to quadrilateral $A_1B_1C_1D_1$ on the plane α, we have

$$\begin{aligned} & AB\times CD + DA\times BC \\ > \ & A_1B_1\times C_1D_1 + D_1A_1\times B_1C_1 \\ \ge \ & A_1C_1\times B_1D_1 \\ = \ & AC\times BD. \end{aligned}$$

Alternatively, consider the Cartesian coordinate system. Say $A = (a_1,a_2,a_3)$, $B = (b_1,b_2,b_3)$, $C = (c_1,c_2,c_3)$ and $D = (d_1,d_2,d_3)$. Then Ptolemy's inequality in 3-dimensional space is

$$\left(\sum_{i=1}^{3}(a_i-b_i)^2\sum_{i=1}^{3}(c_i-d_i)^2\right)^{1/2}$$
$$+\left(\sum_{i=1}^{3}(a_i-d_i)^2\sum_{i=1}^{3}(b_i-c_i)^2\right)^{1/2}$$
$$\ge\left(\sum_{i=1}^{3}(a_i-c_i)^2\sum_{i=1}^{3}(b_i-d_i)^2\right)^{1/2}. \qquad (*)$$

To prove this, let $x_i = a_i - b_i$, $y_i = a_i - d_i$, $z_i = a_i - c_i$. Also, let

$$\alpha = \sum_{i=1}^{3} y_i^2,\quad \beta = \sum_{i=1}^{3} z_i^2,\quad \gamma = \sum_{i=1}^{3} x_i^2.$$

If α or β or γ is 0, then (*) is obvious. Otherwise, none of them is 0. Dividing both sides by $(\alpha\beta\gamma)^{1/2}$, (*) becomes

$$\left(\sum_{i=1}^{3} p_i^2\right)^{1/2} + \left(\sum_{i=1}^{3} q_i^2\right)^{1/2} \ge \left(\sum_{i=1}^{3}(q_i-p_i)^2\right)^{1/2},$$

where $p_i = \dfrac{y_i}{\alpha} - \dfrac{z_i}{\beta}$ and $q_i = \dfrac{x_i}{\gamma} - \dfrac{z_i}{\beta}$.

The last inequality is known as Minkowski's inequality. By squaring and cancelling $\sum_{i=1}^{3} p_i^2 + \sum_{i=1}^{3} q_i^2$ from both sides, we arrive at the Cauchy-Schwarz inequality, which is a well-known inequality.

Remark. In (*) and its proof, 3 can be replaced by any positive integer n and that gives Ptolemy's inequality in n-dimensional space.

The following are some exercises for the readers.

Exercise 1 A quadrilateral $ABCD$ is inscribed in a circle with center O and $\angle ABC = \angle ADC = 90°$. Prove that $BD = AC \sin\angle BAD$.

Exercise 2 Quadrilateral $ABCD$ is convex. Its sides are $AB=a$, $BC=b$, $CD=c$, $DA=d$ and its diagonals are $AC=m$, $BD=n$. Let $\varphi=\angle A+\angle C$. Prove that $m^2n^2 = a^2c^2 + b^2d^2 - 2abcd\cos\varphi$.

References

[1] Le Quoc Han (2007), *Inside Ptolemy's Theorem* (*Vietnamese*) Education Publishing House.

[2] Vietnamese Mathematics and Youth Magazine.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***October 1, 2013.***

Problem 421. For every acute triangle ABC, prove that there exists a point P inside the circumcircle ω of ΔABC such that if rays AP, BP, CP intersect ω at D, E, F, then $DE: EF: FD = 4:5:6$.

Problem 422. Real numbers a_1, a_2, a_3, … satisfy the relations

$$a_{n+1}a_n + 3a_{n+1} + a_n + 4 = 0$$

and $a_{2013} \le a_n$ for all positive integer n. Determine (with proof) all the possible values of a_1.

Problem 423. Determine (with proof) the largest positive integer m such that a $m\times m$ square can be divided into seven rectangles with no two having any common interior point and the lengths and widths of these rectangles form the sequence 1,2,3,4,5,6,7,8,9,10, 11,12,13,14.

Problem 424. (*Due to Prof. Marcel Chirita, Bucuresti, Romania*) In ΔABC, let a=BC, b=CA, c=AB and R be the circumradius of ΔABC. Prove that

$$\max(a^2+bc, b^2+ca, c^2+ab) \ge \frac{2\sqrt{3}abc}{3R}.$$

Problem 425. Let p be a prime number greater than 10. Prove that there exist distinct positive integers a_1, a_2, …, a_n such that $n \le (p+1)/4$ and

$$\frac{(p-a_1)(p-a_2)\cdots(p-a_n)}{a_1 a_2 \cdots a_n}$$

is a positive integral power of 2.

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Solutions

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Problem 416. If $x_1 = y_1 = 1$ and for $n>1$,

$$x_n = -3x_{n-1} - 4y_{n-1} + n$$
and
$$y_n = x_{n-1} + y_{n-1} - 2,$$

then find x_n and y_n in terms of n only.

Solution. **CHEUNG Wai Lam** (Queen Elizabeth School, Form 3), **F5D** (Carmel Alison Lam Foundation Secondary School), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, Andhra Pradesh, India), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Writing out many terms, we may observe that $\{x_{2n-1}\}$, $\{x_{2n}\}$, $\{y_{2n-1}\}$, $\{y_{2n}\}$ are arithmetic progressions. In fact, for n = 1, 2, 3, …, we can claim

$$x_{2n-1} = 17n-16, \quad y_{2n-1} = -8n+9,$$
$$x_{2n} = -17n+12, \quad y_{2n} = 9n-9.$$

We will prove these by induction. The case n = 1 follows from $x_1 = y_1 = 1$, $x_2=-3-4+2=-5$, $y_2=1+1-2=0$, agreeing with the claim. If the case n is true, then by the definition of x_n and y_n,

$$\begin{aligned} x_{2n+1} &= -3(-17n+12)-4(9n-9)+(2n+1) \\ &= 17(n+1)-16, \end{aligned}$$

$$\begin{aligned} y_{2n+1} &= (-17n+12)+(9n-9)-2 \\ &= -8(n+1)+9, \end{aligned}$$

$$\begin{aligned} x_{2n+2} &= -3(17(n+1)-16)-4(-8(n+1)+9) \\ &\quad +(2n+2) \\ &= -17(n+1)+12, \end{aligned}$$

$$\begin{aligned} y_{2n+2} &= (17(n+1)-16)+(-8(n+1)+9)-2 \\ &= 9(n+1)-9 \end{aligned}$$

and the induction is complete.

Other commended solvers: **Radouan BOUKHARFANE, CHAN Long Tin** (Diocesan Boys' School), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **SHUM Tsz Hin** (alumni of City University of Hong Kong), **Simon YAU C. K.** and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Problem 417. Prove that there does not exist a sequence p_0, p_1, p_2, … of prime numbers such that for all positive integer k, p_k is either $2p_{k-1}+1$ or $2p_{k-1}-1$.

Solution. **Radouan BOUKHARFANE, CHEUNG Wai Lam** (Queen Elizabeth School, Form 3), **F5D** (Carmel Alison Lam Foundation Secondary School) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Assume such sequence exists. (The cases p_0 = 2 or 3 can only yield no more than five terms, then contradiction arises.) For $k \ge 2$, we have $p_k > 3$. Then either $p_k \equiv 1$ or -1 (mod 6).

In the former case, $2p_k+1\equiv 3$ (mod 6) cannot be prime. So $p_{k+1}=2p_k-1\equiv 1$ (mod 6). Then repeating the same reason, we can only have $p_n=2p_{n-1}-1$ for all $n > k$. By induction, we have $p_n=2^{n-k}(p_k-1)+1$ for $n > k$. In the case $n=k+p_k-1$, we get

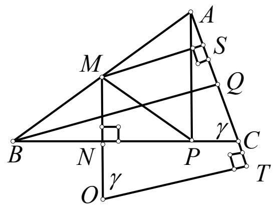
$$2^{p_k-1} \equiv 1 \pmod{p_k}$$

by Fermat's little theorem. Then $p_n \equiv 0$ (mod p_k) contradicting p_n is prime.

In the latter case, $2p_k-1 \equiv 3$ (mod 6) cannot be prime. Similarly, this leads to $p_n=2^{n-k}(p_k+1)-1$ for $n > k$. Again, by Fermat's little theorem, the case $n=k+p_k-1$ leads to contradiction.

Problem 418. Point M is the midpoint of side AB of acute ΔABC. Points P and Q are the feet of perpendicular from A to side BC and from B to side AC respectively. Line AC is tangent to the circumcircle of ΔBMP. Prove that line BC is tangent to the circumcircle of ΔAMQ.

Solution 1. **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).



Let O be the center of the circumcircle Γ of ΔBMP and S, T be the projections of M, O onto line AC. Let OM intersect BP at N. Now OM is the perpendicular bisector of BP. So $\angle MNC=\angle MSC=90°$ implies $MNCS$ is cyclic. Since $MS \parallel OT$, $\gamma=\angle ACB=180°-\angle OMS=\angle MOT$.

By the extended sine law, the chord MP of Γ satisfies

$$\frac{AB}{2} = MP = 2OM\sin\angle ABC = 2OM\frac{AP}{AB}.$$

Hence $OM=AB^2/(4AP)$. Now, line AC is tangent to the circumcircle of ΔBMP if and only if $OM=OT$ if and only if

$$\cos\angle ACB = \cos\angle MOT = \frac{OT-MS}{OM}$$

$$= \frac{\dfrac{AB^2}{4AP} - \dfrac{BQ}{2}}{\dfrac{AB^2}{4AP}} = 1 - \frac{2AP\cdot BQ}{AB^2}.$$

Similarly, line BC is tangent to the circumcircle of $\triangle AMQ$ if and only if

$$\cos\angle BCA = 1 - \frac{2BQ\cdot AP}{BA^2}.$$

The desired conclusion follows.

Solution 2. **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)).

Since $AP\perp BC$ and $BQ\perp AC$, points A,Q,P,B lie on the circle with M as center and AB as diameter. Consider inversion with respect to this circle. Let X' be the image of X under this inversion. We have $A'=A$, $Q'=Q$, $P'=P$, $B'=B$. Since line AC is tangent to the circumcircle of $\triangle BMP$, so the image of line AC and the image of the circumcircle of $\triangle BMP$ are tangent. Since the circumcircle of $\triangle BMP$ passes through M, the image of this circumcircle is the line $B'P'$, which is line BC. Also, the image of line AC is the circle through A', Q', C', M, which is the circumcircle of $\triangle AMQ$. So line BC is tangent to the circumcircle of $\triangle AMQ$.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Problem 419. Let $n\ge 4$. M is a subset of $\{1,2,\ldots,2n-1\}$ with n elements. Prove that M has a nonempty subset, the sum of all its elements is divisible by $2n$.

Solution 1. **Juan G. ALONSO** and **Ángel PLAZA** (Garoé Atlantic School & Universidad de Las Palmas de Gran Canaria, Spain), **F5D** (Carmel Alison Lam Foundation Secondary School), **KWOK Man Yi** (S2, Baptist Lui Ming Choi Secondary School), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM) and **Aliaksei SEMCHANKAU** (School 41 named after Serebryany, Minsk, Belarus).

If n is not in M, then by the pigeonhole principle, since M has n elements, it must contain at least one of the pairs $\{1,2n-1\}$, $\{2,2n-2\}$, …, $\{n-1, n+1\}$. This pair is a subset of M that we want.

If n is in M, then let $a_1, a_2, \ldots, a_n$ be the elements of M. We may let $a_n = n$ and a_1 be the minimum of $a_1, a_2, \ldots, a_{n-1}$. Since M has at least 4 elements, we may assume a_2 is not a_1+n (otherwise replace a_2 by a_3). Consider the n numbers $a_2, S_1, S_2, S_3, \ldots, S_{n-1}$, where S_i is the sum of $a_1, a_2, \ldots, a_i$. By the pigeonhole principle and the fact $a_2\ne a_1+n$, two of the n numbers are congruent (mod n) and the two numbers are not a_1, a_2. Hence, their difference (which is a sum of the a_i's) is equal to jn for some positive integer j. If j is even, then the set T of the a_i's in the sum is a desired subset of M. Otherwise j is odd. We can add $a_n = n$ to T to get a desired subset of M.

Solution 2. **CHEUNG Wai Lam** (Queen Elizabeth School, Form 3) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

We will say $\{a,b,c\}$ is <u>bad</u> if $a,b,c\in M$ and $2n$ divides $a+b+c$. Assume the contrary (which implies M has no bad subsets). Then M contains <u>*exactly one*</u> element in each of the sets $\{1,2n-1\}$, $\{2,2n-2\}$, …, $\{n-1, n+1\}$, $\{n\}$. In particular, n is in M.

Now $1\in M \Rightarrow n-1\notin M$ (otherwise $\{1, n-1, n\}$ is bad) $\Rightarrow n+1\in M \Rightarrow n-2\notin M$ (otherwise $\{1, n-2, n+1\}$ is bad) $\Rightarrow n+2\in M \Rightarrow \cdots \Rightarrow n+(n-1)=2n-1\in M \Rightarrow 1\notin M$, contradiction.

Next $1\notin M \Rightarrow 2n-1\in M \Rightarrow n+1\notin M$ (otherwise $\{n, n+1, 2n-1\}$ is bad) $\Rightarrow n-1\in M \Rightarrow n+2\notin M$ (otherwise $\{n-1, n+2, 2n-1\}$ is bad) $\Rightarrow n-2\in M \Rightarrow \cdots \Rightarrow n-(n-1) = 1\in M$, contradiction.

Other commended solvers: **Radouan BOUKHARFANE.**

Problem 420. Find (with proof) all positive integers x and y such that $2x^2y+xy^2+8x$ is divisible by xy^2+2y.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM) and **Aliaksei SEMCHANKAU** (School 41 named after Serebryany, Minsk, Belarus).

Let $a \mid b$ denote a divides b. Suppose $xy^2+2y \mid 2x^2y+xy^2+8x$. Then $xy^2+2y \mid$

$$y(2x^2y+xy^2+8x) - (2x+y)(xy^2+2y) = 4xy-2y^2.$$

Since $y > 0$, we get $xy+2 \mid 4x-2y$.

If $4x-2y < 0$, then $xy+2 \mid 2y-4x$. Now $2y > 2y-4x \ge xy+2 > xy$ implies $2 > x$. Hence $x = 1$. Then $y+2 \mid 2y-4 = 2(y+2)-8$. So $y+2 \mid 8$, which implies $y = 2$ or 6. We can check $(x,y) = (1,2)$ is a solution, but $(1,6)$ is not.

If $4x-2y = 0$, then $(x,y) = (k,2k)$ for some positive integer k and we have $xy^2+2y=4k^3+4k \mid 2x^2y+xy^2+8x=8k^3+8k$ for all positive integer k.

If $4x-2y > 0$, then $xy+2 \mid 4x-2y$. Now $4x > 4x-2y \ge xy+2 > xy$. So $y < 4$.

The case $y = 1$ leads to $x+2 \mid 4x-2 = 4(x+2)-10$. Hence, $x+2 \mid 10$. Then $(x,y) = (3,1)$ or $(8,1)$. Both can easily be checked to be solutions. The case $y = 2$ leads to $2x+2 \mid 4x-4 = 2(2x+2)-8$. Hence, $x+1 \mid 4$. Then $(x,y) = (1,2)$ or $(3,2)$, which are not solutions. The cases $y = 3$ leads to $3x+2 \mid 4x-6$. Then $3x+2 \mid 4(3x+2) - 3(4x-6) = 26$, which leads to $(x,y) = (8,3)$, but it is not a solution.

So the solutions are $(x,y) = (3,1)$, $(8,1)$ and $(k, 2k)$ for all positive integer k.

Other commended solvers: **Radouan BOUKHARFANE, F5D** (Carmel Alison Lam Foundation Secondary School), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania), **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Olympiad Corner

(continued from page 1)

Problem 4. Let a and b be positive integers, and let A and B be finite sets of positive integers satisfying:

(i) A and B are disjoint;

(ii) if an integer i belongs either to A or to B, then either $i+a$ belongs to A or $i-b$ belongs to B.

Prove that $a|A|=b|B|$. (Here $|X|$ denotes the number of elements in the set X.)

Problem 5. Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

Mathematical Excalibur

Volume 18, Number 2 *September-October 2013*

Olympiad Corner

Below are the problems of the 2013 International Mathematical Olympiad.

Problem 1. Prove that for any pair of positive integers k and n, there exist k positive integers m_1, m_2, …, m_k (not necessarily different) such that

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

Problem 2. A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

• no line passes through any point of the configuration;
• no region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 8, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2013 – Leader Report (I)

Leung Tat-Wing

The 54th International Mathematical Olympiad (IMO) was held in Santa Marta, Colombia from July 18th to July 28th, 2013. It took me 40 hours of flight and waiting time to travel from Hong Kong to Amsterdam, then to Panama City, and then to Barranquilla, Colombia (where the leaders stayed before they met the contestants in Santa Marta after two days of 4½-hour contests held on the mornings of 23rd and 24th of July). Tired and exhausted, I were picked up in the airport of Barranquilla and delivered to Hotel El Prada. We managed to settle down and be prepared for the next two days' Jury meetings. Our team arrived at Santa Marta, three days later, safe and intact, luckily. The next day they still had to travel two hours from Santa Marta to Barranquilla, participating in another opening ceremony, then another two hours back to Santa Marta. It was tough for them. Accommodation was fine though. Contestants stayed in a nice seaside resort hotel (Iratoma), while leaders stayed in a hotel in Barranquilla. They would join the contestants after the two day contests.

Jury meetings were chaired by Maria Losada, a long time veteran of IMO activities. She was very experienced and chaired the meetings well. Interesting to note, she kept on reminding us (leaders) that we should try to form the best possible paper, a paper that can provide intellectual challenge to contestants, that has some aesthetic sense and that allows every contestant to achieve the most. We were also supposed to work out as many possible solutions as possible. We should be able to tell whether a problem is easy, medium and/or hard. Really sometimes I did not know how the goals may be attained or even verified. She also reminded us ethically we should keep the problems with strict security, not to disclose any information to any contestant beforehand, etc. Indeed the Jury meetings were very educational.

After the two days' contests students enjoyed a break. Leaders and deputy leaders had to check the solutions of the contestants, discussed or argued with coordinators and sorted out how many points should be award to contestants. (This process is called coordination). Luckily this year many coordinators were again very experienced. Many of them are old time leaders from Europe and are experienced problem solvers. They were able to discern mistakes made by the contestants (trivial, small or big) and were able to award points accurately. Personally I recognized many of them and I think I have known many of them for at least more than 10 years. That is why little trouble was observed during the coordination process.

The awards (closing) ceremony was held near a historical site, 45 minute drive from the hotel. We were delivered to the site around 7:00 pm. Then the ceremony lasted for more than two hours. Participants were than sent back to the resort for the banquet. That night was surely hectic. The next day we started our trip home. When we arrived at Bogota, we found that the flight from Bogota to Paris was overbooked. Eventually two of us (deputy leader and a member of the team) had to take another flight from Bogota to Frankfurt, then back to Hong Kong, about 10 hours late. Air France is famous (notorious) in terms of scheduling, here is another example. All in all, we did not get delayed too much and we eventually returned home safely. Lucky! Lucky!

Talking about organization of the event, personally I have no problem with the Jury meeting and/or coordination. Accommodation was very nice. However anything concerned a coach (transportation) was simply not good enough. Say, what is the point of waiting for several hours for a bus, then

(continued on page 2)

visit an old town or take a short walk for less than an hour, and then heading back? ? I do not mean to blame the host country. Indeed I want only to illustrate the point that it is such a gigantic and complicated task to host an IMO!

Our team brought home 1 silver and 5 bronze medals. Among 97 teams, we ranked 31. I cannot say that our team did badly. Indeed all our team members managed to get medals, indicating they achieved certain standard. However in these few years, we trailed behind teams like Singapore, Canada, Australia and other teams, not to say the even stronger teams such as China, USA, Korea and Russia, etc. Do we want to do better? Can we recruit better team members? Can we afford time and energy to do that? We have to think about these problems. I can identify some weak points for our team. For example, our team members simply don't like to do geometry and/or combinatorics problems. Our team members usually get stuck in harder problems, presentations and other things. Or perhaps our team members are too much occupied also by other contests? I know for sure IMO team members of teams such as USA, Australia and Canada would not be allowed to compete in other contests such as IOI or IPhO in the same year. Another suggestion is that we do not train our team enough, we have no intensive camp before IMO (compared with China, USA or UK), and perhaps we should start an intensive camp that will also used as a selection criterion of our team. This idea comes from none other than our old team members! We should pause to think about all these for a while, I suppose.

On the other hand, in this IMO, we confirmed that we will host IMO2016, so in 2016, IMO will be held in Hong Kong. Now we just have to do it, and do it right.

I shall discuss the problems of this IMO. First let us see how they were selected. Indeed the host country (Problem Selection Committee) shortlisted about 30 problems from hundred or so problems submitted by various countries. In the last few years, the Jury first chose an easy pair (problem 1 and 4), then a hard pair (problem 3 and 6), then a medium pair (problem 2 and 5). The 6 selected problems will be then juggled to form the papers. However this year, it was proposed (and accepted) 4 easy problems in algebra, combinatorics, geometry and number theory were selected. Likewise 4 medium problems again from the different topics were selected. Then two easy problems were selected from the 4 easy problems, say problems of algebra and conbinatorics were selected. The medium problems of other topics (geometry and number theory) were automatically selected as the medium pair. The idea is to guarantee problems of all topics be selected either as an easy problem or a medium problem. After that it doesn't matter what problems were selected as the hard pair. However, perhaps the end result was not as ideal as we wanted. Eventually in this IMO there are two synthetic geometry problems (Problem 3 and 4). Problem 2, which was supposed to be a combinatorics problem, is actually a problem of combinatorial geometry. Problem 6, which is a combinatorics problem, also has some geometry favor. Problem 1, which was supposed to be a number theory problem, is more like an algebra problem (no prime numbers, no factorization of integers, merely algebraic manipulation and some induction). And finally of course problem 5 is a problem of functional inequalities. So this paper is very much skewed to geometry and with no number theory. Can we say it is balanced? Really at the very beginning, the problems selected were not quite balanced. The problem selection committee suggested there were no easy combinatorics problems and no hard geometry problems! In short, Jury members tended to select problems that demand "ad hoc" considerations, no need to resort to more advanced techniques and/or theorems.

(For the statement of the problems, please see the Olympiad Corner on page 1-*Ed.*)

Problem 1: Problem 1 and 4 (easy pair) turned out to be too easy. Many strong teams get full score in these two problems. For $k = 1$, we have

$$1+\frac{2^1-1}{n}=1+\frac{1}{n},$$

and it is already of the required form. Hence it is natural to solve the problem using some kind of induction procedure. Essentially all of us did the problem using iterations. One of our team members did the problem as follows. Denote the statement that $1+(2^k-1)/n$ is of the form

$$\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right)$$

by $S(n,k)$. Note that

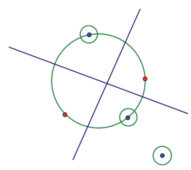
$$1+\frac{2^{k+1}-1}{2n}=\left(1+\frac{1}{2n+2^{k+1}-2}\right)\left(1+\frac{2^k-1}{n}\right),$$

hence if $S(n,k)$ is valid, so is $S(2n,k+1)$. Likewise

$$1+\frac{2^{k+1}-1}{2n-1}=\left(1+\frac{1}{2n-1}\right)\left(1+\frac{2^k-1}{n}\right).$$

Hence if $S(n,k)$ is valid, so is $S(2n–1,k+1)$. Clearly the cases $S(n,1)$ or $S(1,k)$ are valid. Hence by reducing the cases $S(2n,k)$ to $S(n,k–1)$, or $S(2n–1,k)$ to $S(n,k–1)$, (odd or even cases), one can always obtain the cases $S(p,1)$ or $S(1,q)$, and we are done.

Problem 2: All our members guessed the correct answer. The trouble is how to present a proof that is complete (no missing cases). Jury members also worried students didn't realize the minimum value of k should work for all possible configurations. Thus they defined the term "Colombian". (Another definition is the "beautiful" labeling in problem 6. In my opinion it was quite unnecessary.) First we show $k \geq 2013$. Indeed we mark 2013 red points and 2013 blue points alternately on a circle, (and another blue point elsewhere), then there are 4026 arcs formed. All these arcs have two endpoints of different colors and there must be a line passing through an arc to separate the two points, also each line passing through an arc will meet another arc only once, so we see at least 4026/2=2013 lines are needed.



A case of 2 red points and 3 blue points

Now we have to show $k = 2013$ is indeed enough. The official solution goes as follows. First if there are two points of the same color, say A and B, then one can draw two lines parallel to AB, and are sufficiently close and there are only two points between these lines, namely A and B. This statement is intuitively clear. Draw the convex hull P of the points, and there are two cases.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 8, 2013.***

Problem 426. Real numbers a, b, x, y satisfy the property that for all positive integers n, $ax^n+by^n=1+2^{n+1}$. Determine (with proof) the value of x^a+y^b.

Problem 427. Determine all (m,n,k), where m, n, k are integers greater than 1, such that $1!+2!+\cdots+m!=n^k$.

Problem 428. Let $A_1A_2A_3A_4$ be a convex quadrilateral. Prove that the nine point circles of $\Delta A_1A_2A_3$, $\Delta A_2A_3A_4$, $\Delta A_3A_4A_1$ and $\Delta A_4A_1A_2$ pass through a common point.

Problem 429. Inside ΔABC, there is a point P such that $\angle APB=\angle BPC=\angle CPA$. Let $PA=u$, $PB=v$, $PC=w$, $BC=a$, $CA=b$ and $AB=c$. Prove that

$$(u+v+w)^2\le ab+bc+ca\\ -\left(\sqrt{a(b+c-a)}-\sqrt{b(c+a-b)}\right)^2.$$

Problem 430. Prove that among any $2n+2$ people, there exist two of them, say A and B, such that there exist n of the remaining $2n$ people, each either knows both A and B or does not know A nor B. Here, *x knows y* does not necessarily imply *y knows x*.

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Solutions

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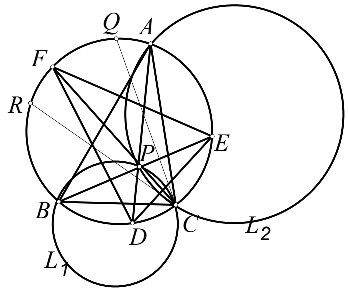
Problem 421. For every acute triangle ABC, prove that there exists a point P inside the circumcircle ω of ΔABC such that if rays AP, BP, CP intersect ω at D, E, F, then $DE: EF: FD = 4:5:6$.

Solution. **Jon GLIMMS** (Vancouver, Canada), **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **William PENG**.

For such a point P, let us apply the exterior angle theorem to ΔABP and ΔACP. Then we have

$$\angle BPC=\angle BAC+\angle ABE+\angle ACF\\ =\angle BAC+\angle FDE.$$

Similarly, $\angle CPA=\angle CBA+\angle DEF$.



To get such a point P, we first draw ΔXYZ with $XY=4$, $YZ=5$ and $ZX=6$. Let $\alpha=\angle ZXY$ and $\beta=\angle XYZ$. Next we consider the locus L_1 of point P such that $\angle BPC=\angle BAC+\alpha$, which is a circle through B and C. Also, let L_2 be the locus of point P such that $\angle CPA=\angle CBA+\beta$, which is a circle through C and A.

Let the tangents to L_1 and L_2 at C intersect ω at Q and R. Then

$$\angle QCB+\angle RCA\\ =180^\circ-(\angle BAC+\alpha)+180^\circ-(\angle CBA+\beta)\\ =\angle ACB+\angle YZX>\angle ACB.$$

This implies L_1 and L_2 intersect at a point P inside ω. Define D, E, F as in the statement of the problem. From the last two paragraphs, we get $\angle ZXY=\alpha=\angle FDE$ and $\angle XYZ=\beta=\angle DEF$. These imply ΔDEF and ΔXYZ are similar. Therefore, $DE: EF: FD = 4:5:6$.

Problem 422. Real numbers $a_1, a_2, a_3, \ldots$ satisfy the relations

$$a_{n+1}a_n+3a_{n+1}+a_n+4=0$$

and $a_{2013}\le a_n$ for all positive integer n. Determine (with proof) all the possible values of a_1.

Solution. **CHEUNG Wai Lam** (Queen Elizabeth School, Form 4), **Jon GLIMMS** (Vancouver, Canada), **William PENG** and **TAM Pok Man** (Sing Yin Secondary School, Form 6).

The recurrence relation can be written as $(a_{n+1}+2)(a_n+2)=(a_n+2)-(a_{n+1}+2)$. If $a_i=-2$ for some i, then all $a_n=-2$ by induction. So $a_1=-2$ is a possible value. Suppose no $a_i=-2$. Then

$$\frac{1}{a_{n+1}+2}=1+\frac{1}{a_n+2}.$$

Letting $b_n=1/(a_n+2)$, we easily get $b_n=n-1+b_1\neq 0$ for all positive integer n. Then $b_1\neq 0,-1,-2,\ldots$ and $a_n=-2+1/(n-1+b_1)$. Now for positive integer n, a_n is least when $n-1+b_1<0$ and nearest 0, i.e.

$$n-1+b_1<0<n+b_1.$$

Setting $n=2013$ and $b_1=1/(a_1+2)$, we can solve the inequality to get

$$-\frac{4025}{2012}<a_1<-\frac{4027}{2013}.$$

Other commended solvers: **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM).

Problem 423. Determine (with proof) the largest positive integer m such that a $m\times m$ square can be divided into seven rectangles with no two having any common interior point and the lengths and widths of these rectangles form the sequence 1,2,3,4,5,6,7,8,9,10, 11,12,13,14.

Solution. **Jon GLIMMS** (Vancouver, Canada), **William PENG** and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Let $a_1, a_2, a_3, a_4, \ldots, a_{2n-1}, a_{2n}$ be a permutation of $1, 2, 3, 4, \ldots, 2n-1, 2n$. We claim the maximum of $a_1a_2+a_3a_4+\cdots+a_{2n-1}a_{2n}$ is $S_n=1\times2+3\times4+\cdots+(2n-1)\times 2n$. The cases $n=1$ or 2 can be checked. Suppose cases 1 to n are true. For the case $n+1$, if $(2n+1)(2n+2)$ is one of the term, then we can switch it with the last term and apply the case n to get

$$a_1a_2+a_3a_4+\cdots+a_{2n-1}a_{2n}+(2n+1)(2n+2)\\ \le S_n+(2n+1)\times(2n+2)=S_{n+1}.$$

Otherwise, $2n+1$ and $2n+2$ are in different terms. We can switch terms so that $a_{2n-1}=2n+1$ and $a_{2n+1}=2n+2$. If we try switching $(2n+1)a_{2n}+(2n+2)a_{2n+2}$ to $a_{2n}a_{2n+2}+(2n+1)(2n+2)$, then since a_{2n} and a_{2n+2} are at most $2n$, we have

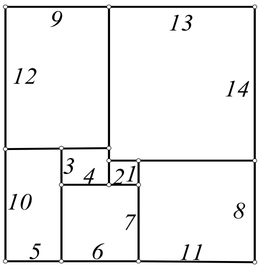
$$[(2n+2)-a_{2n}][(2n+1)-a_{2n+2}]>0.$$

Expanding, we see

$$a_{2n}a_{2n+2}+(2n+1)(2n+2)\\ >(2n+1)a_{2n}+(2n+2)a_{2n+2}\\ =a_{2n-1}a_{2n}+a_{2n+1}a_{2n+2}.$$

Adding $a_1a_2+a_3a_4+\cdots+a_{2n-3}a_{2n-2}$ and using case $n-1$, we see S_{n+1} again is the maximum.

For the problem, the claim implies $m^2\le S_7=1\times2+3\times4+\cdots+13\times14=504$. Then $m\le 22$. To finish, we show a 22×22 square which can be so divided.



Other commended solvers: **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM).

Problem 424. (*Due to Prof. Marcel Chirita, Bucuresti, Romania*) In ΔABC, let $a=BC$, $b=CA$, $c=AB$ and R be the circumradius of ΔABC. Prove that

$$\max(a^2+bc, b^2+ca, c^2+ab) \geq \frac{2\sqrt{3}abc}{3R}.$$

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **TAM Pok Man** (Sing Yin Secondary School, Form 6), **Alex TUNG Kam Chuen** (La Salle College), **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia) and **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

By the extended sine law, $c/\sin C = 2R$. Let $[ABC]$ denote the area of ΔABC. Then $[ABC] = ½ab \sin C = abc/(4R)$. So $ab = 2[ABC]/\sin C$. Using these below, we have

$$\begin{aligned}
&3\max(a^2+bc, b^2+ca, c^2+ab)\\
&\geq a^2+bc+b^2+ca+c^2+ab\\
&\geq 2(ab+bc+ca)\\
&= 4[ABC]\left(\frac{1}{\sin C}+\frac{1}{\sin A}+\frac{1}{\sin B}\right)\\
&\geq 4[ABC]\frac{3}{\sin((A+B+C)/3)}\\
&= \frac{2\sqrt{3}abc}{R},
\end{aligned}$$

where the second inequality is by expanding $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$ and the third inequality is by applying Jensen's inequality to $f(x)=1/\sin x$.

Other commended solvers: **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania) and **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, Form 2).

Problem 425. Let p be a prime number greater than 10. Prove that there exist distinct positive integers $a_1, a_2, \ldots, a_n$ such that $n \leq (p+1)/4$ and

$$\frac{(p-a_1)(p-a_2)\cdots(p-a_n)}{a_1a_2\cdots a_n}$$

is a positive integral power of 2.

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Alex TUNG Kam Chuen** (La Salle College) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

More generally, we prove this is true for all odd integers $p \geq 3$. Let

$$X = \frac{(p-a_1)(p-a_2)\cdots(p-a_n)}{a_1a_2\cdots a_n}$$

If $p \equiv 1 \pmod 4$, then let $n=(p-1)/4$ and for $i=1,2,\ldots,n$, let $a_i = 2i-1$. We have

$$\begin{aligned}
X &= \frac{4n(4n-2)\cdots(2n+2)}{1\cdot 3\cdots(2n-1)}\\
&= \frac{4n(4n-2)\cdots(2n+2)}{1\cdot 3\cdots(2n-1)}\times\frac{2\cdot 4\cdots(2n)}{2\cdot 4\cdots(2n)}\\
&= 2^{2n}.
\end{aligned}$$

If $p \equiv 3 \pmod 4$, then let $n=(p+1)/4$ and for $i=1,2,\ldots,n$, let $a_i = 2i-1$. We have

$$\begin{aligned}
X &= \frac{(4n-2)(4n-4)\cdots(2n)}{1\cdot 3\cdots(2n-1)}\\
&= \frac{(4n-2)(4n-4)\cdots(2n)}{1\cdot 3\cdots(2n-1)}\times\frac{2\cdot 4\cdots(2n-2)}{2\cdot 4\cdots(2n-2)}\\
&= 2^{2n-1}.
\end{aligned}$$

Olympiad Corner

(continued from page 1)

Problem 3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Problem 4. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y, H are collinear.

Problem 5. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let f: $\mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x,y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;
(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x)+f(y)$;
(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Problem 6. Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0,1,\ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a+d = b+c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x,y) of positive integers such that $x+y \leq n$ and $gcd(x,y)=1$. Prove that $M = N+1$.

IMO 2013–Leader Report (I)

(continued from page 2)

<u>Case 1.</u> If there is a red point A on the convex hull P, we can draw a line separating A draw all other points. Then we pair up the remaining 2012 red points into 1006 pairs, and as remarked, draw 1006 pairs of parallel lines (2012 lines), separating each pair of red points from all other points. Thus 2012+1=2013 lines are needed.

<u>Case 2.</u> All vertices of the convex hull P are blue. Take any pair of consecutive blue points A and B, separating them from all other points by a line (one line) parallel to AB. Then pair up the remaining 2012 blue points into 1006 pairs as before, separating each pair from all other points by 1006 pairs of parallel lines (2012 lines). Thus again 2013 lines are used.

(To be continued)

Mathematical Excalibur

Volume 18, Number 3 *November 2013*

Olympiad Corner

Below are the problems of the North Korean Team Selection Test for IMO 2013.

Problem 1. The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, CA, AB at A_1, B_1, C_1 respectively. The line AI meets the circumcircle of ABC at A_2. The line B_1C_1 meets the line BC at A_3 and the line A_2A_3 meets the circumcircle of ABC at A_4 ($\neq A_2$). Define B_4, C_4 similarly. Prove that the lines AA_4, BB_4, CC_4 are concurrent.

(continued on page 4)

IMO 2016 Logo Design Competition

Hong Kong will host the 57th International Mathematical Olympiad (IMO) in July 2016. The Organising Committee now holds the IMO 2016 Logo Design Competition and invites all secondary school students in Hong Kong to submit logo designs for the event. Your design may win you \$7,000 book coupons and become the official logo of IMO 2016! For details, please visit

www.imohkc.org.hk.

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 21, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Sequences

Kin Y. Li

Sequence problems occur often in math competitions. Below we will look at some of these problems involving limits in their solutions.

Example 1. (1980 *British Math Olympiad*) Find all real a_0 such that the sequence defined by $a_{n+1}=2^n-3a_n$ for n=0,1,2,… satisfies $a_0<a_1<a_2<\cdots$.

Solution. We have

$$\begin{aligned} a_{n+1} &= 2^n-3a_n = 2^n-3\times 2^{n-1}+3^2a_{n-1} \\ &= \cdots = 2^n-\sum_{j=1}^{n}(-3)^j2^{n-j}+(-3)^{n+1}a_0 \\ &= \frac{2^{n+1}}{5}+\left(a_0-\frac{1}{5}\right)(-3)^{n+1}. \end{aligned}$$

If a_0=1/5, then it is good. If $a_0\neq 1/5$, then since $(2/3)^n$ goes to 0 as $n\to\infty$, so $a_n/3^n$ will have the same sign as $(a_0-1/5)(-1)^n$ when n is large. Hence, $a_n<a_{n+1}$ will not hold, contradiction.

Example 2. (1971-1972 *Polish Math Olympiad*) Prove that when n tends to infinity, the sum of the digits of 1972^n in base 10 will go to infinity.

Solution. Let a_i be the i-th digit of 1972^n from right to left in base 10. For $1\le k\le n/4$, we claim that among $a_{k+1}, a_{k+2},\ldots, a_{4k}$, at least one of them is nonzero.

Assume not. Then let

$$C = a_1+a_2\times 10+\cdots+a_k\times 10^{k-1}.$$

We have 1972^n-C divisible by 10^{4k}. Since $4k\le n$, so C is divisible by $2^{4k}=16^k>10^k>C$, contradiction.

From the claim, we get at least one digit in each of the following m+1 groups of digits will not be zero

$$\begin{aligned} &a_2, a_3, a_4, \\ &a_5, a_6, a_7, \ldots, a_{16}, \\ &\cdots \\ &a_{j+1}, a_{j+2}, a_{j+3}, \ldots, a_{4j}, \end{aligned}$$

where $n/16<j=4^m\le n/4$. The digit sum of 1972^n is at least $m+1>(\log_4 n)-1$. So, the digit sum of 1972^n goes to infinity.

Example 3. Let a_1, a_2, a_3, … be a sequence of positive numbers. Prove that there exists infinitely many n such that $1+a_n>2^{1/n}a_{n-1}$.

Solution. Assume not. Then there is a M such that for all $n>M$, we have $1+a_n\le 2^{1/n}a_{n-1}$. Since $(1+1/n)^n\ge 2$, we have

$$a_n\le 2^{1/n}a_{n-1}-1\le((n+1)/n)a_{n-1}-1. \quad (*)$$

We claim that for $k\ge M$,

$$a_k\le(k+1)\left(\frac{a_M+1}{M+1}-\sum_{j=M+1}^{k+1}\frac{1}{j}\right).$$

The case $k=M$ is true as the right side is a_M. Suppose case k is true. By (*),

$$\begin{aligned} a_{k+1} &\le \frac{k+2}{k+1}a_k-1=\frac{k+2}{k+1}a_k-\frac{k+2}{k+2} \\ &\le(k+2)\left(\frac{a_M+1}{M+1}-\sum_{j=M+1}^{k+2}\frac{1}{j}\right). \end{aligned}$$

This concludes the induction. As $k\to\infty$, the above sum of $1/j$ goes to infinity, hence some $a_{k+1}<0$, contradiction.

Example 4. (2007 *Chinese Math Olympiad*) Let $\{a_n\}_{n\ge1}$ be a bounded sequence satisfying

$$a_n<\sum_{k=n}^{2n+2006}\frac{a_k}{k+1}+\frac{1}{2n+2007},\quad n=1,2,3,\ldots.$$

Prove that $a_n<1/n$ for $n=1,2,3,\ldots$.

Solution. Let $b_n=a_n-1/n$. Then for $n\ge1$,

$$b_n<\sum_{k=n}^{2n+2006}\frac{b_k}{k+1}. \qquad (*)$$

It suffices to show $b_n<0$. Since a_n is bounded, so there is a constant M such that $b_n<M$. For $n>100{,}000$, we have

$$\begin{aligned} b_n &< \sum_{k=n}^{2n+2006}\frac{b_k}{k+1}<M\sum_{k=n}^{2n+2006}\frac{1}{k+1} \\ &= M\sum_{k=n}^{[3n/2]}\frac{1}{k+1}+M\sum_{k=[3n/2]+1}^{2n+2006}\frac{1}{k+1} \\ &< \frac{M}{2}+M\frac{2006+n/2}{1+3n/2}<\frac{6}{7}M. \end{aligned}$$

Repeating this m times, if $n>100{,}000$, then $b_n<(6/7)^mM$. Letting $m\to\infty$, we get $b_n\le 0$ for $n>100{,}000$. Using (*), we see if for $n\ge N+1$, we have $b_n<0$, then $b_N<0$. This gives $b_n<0$ for $n\ge 1$.

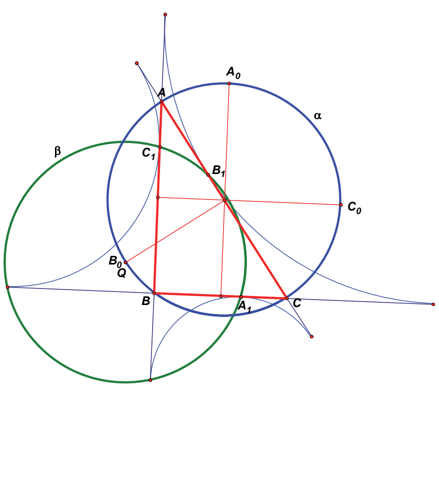
IMO 2013 - Leader Report(II)

Leung Tat-Wing

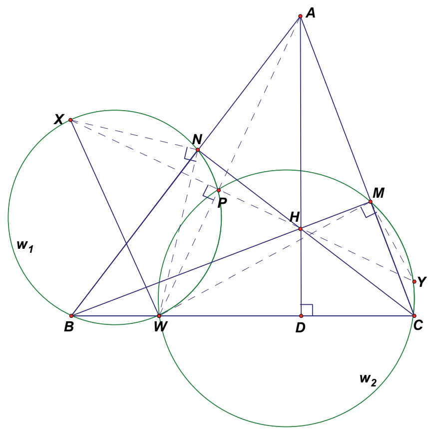
We will continue with our discussion on the IMO 2013 problems, which can be found in the Olympiad Corner of the last issue of *Math Excalibur*.

Problem 3: The problem was selected in the very last minute of the Jury meetings. Indeed another geometry problem concerning properties of hexagons was initially selected as a member of the hard pair. It was however discovered the problem was similar to an USAMO problem. I myself also recalled several similar problems. So the problem was rejected and replaced by this problem 3. After the selection process, it was announced both problem 3 and 6 come from Russia, indeed a problem similar to problem 4 was also found in a Russian geometry problem book. Truly the Russians are masters of posing problems!

Despite being a difficult problem (solved by 40 contestants), problem 3 is indeed a pure geometry problem and can be solved by pure synthetic geometry method. Indeed denote the circumcircles of ABC and $A_1B_1C_1$ by α and β respectively and let Q be the centre of the circumcircle of $A_1B_1C_1$. Let A_0 be the midpoint of arc BC containing A, and define B_0 and C_0 respectively. Then one can check $A_0B_1=A_0C_1$ and A,A_0,B_1,C_1 concyclic. (Likewise $B_0C_1=B_0C_1$ and B,B_0,C_1,A_1 concyclic; $C_0A_1=C_0B_1$ and C,C_0,B_1,A_1 concyclic.) One then consider the largest angle of $A_1B_1C_1$, say B_1, and if Q is on α, then Q must coincide with B_0, and hence $\angle B$=90°, not easy though!



Problem 4: There are more than 19 different solutions and surely there are more. It is possible to solve the problem using complicated angle chasings and/or coordinate geometry. But of course the basic or most natural approach is to look at the radical axis of the two circles. The following proof is given by Lau Chun Ting, a team member of ours.



Suppose ω_1 and ω_2 meet at another point P ($\neq W$). Since $\angle WPX = \angle WPY = 90°$, so X, P and Y are collinear. To show H lies on XY, (X, Y, H collinear), it suffices to show $\angle HPW$ = 90°. Suppose now AH meets BC at D. Now B, N, M, C are concyclic (since $\angle BNC = \angle BMC$ = 90°), we have $AN\times AB=AM\times AC$. So the powers of the point A with respect to the circles ω_1 and ω_2 are the same, that means A lies on the radical axis WP, or A, P, W collinear (radical axis theorem). Now note that H, M, C, D are also concyclic, hence $AH\times AD=AM\times AC$ (quite a few concyclic conditions). As before

$$AM\times AC=AN\times AB=AP\times AW,$$

we get $AP\times AW=AH\times AD$. Therefore, W, P, H, D are concyclic and we get $\angle HPW$ = 90°, as required.

Using coordinate attack, we may let A=(a_1,a_2), B=(–b,0), C=(c,0) and W=(0,0). By computing slopes and equations of lines, (complicated but still manageable), one eventually gets the coordinates of X, H and Y. Hence can verify X, H and Y collinear by calculating slopes of XH and HY.

Problem 5: For problem of this kind, one can try many things to obtain partial results. But the essential (crucial) part of this problem is actually how to make use of condition 3. Indeed if this condition is released, then the function $f(x) = bx^2$, with $b \geq 1$, will satisfy the first and second condition. Now see what we can get by putting different values of x and y into the equations. For examples, put $x = a$ and y = 1, one gets $a\,f(1) = f(a)\,f(1) \geq f(a) = a$, hence $f(1) \geq 1$. We let $f(1) = c \geq 1$. By induction, one can then show $f(n) \geq nc$, for all natural numbers n. So in particular $f(n)$ is positive. Now we show $f(x)$ is strictly increasing. Indeed if $f(x+\Delta x) \leq f(x)$ for some positive rational numbers x and Δx, then

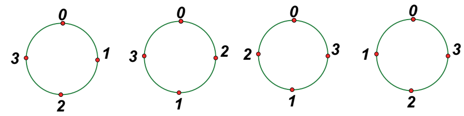
$$f(x) \geq f(x+\Delta x) \geq f(x) + f(\Delta x),$$

therefore $f(\Delta x) \leq 0$. However, we also have $f(n)\,f(\Delta x) \geq f(n\Delta x)$. Now since $f(\Delta x) \leq 0$, so we must also have $f(n\Delta x) \leq 0$ for all n, however surely we can find n so that $n\Delta x$ is a natural number and $f(n\Delta x)$ is positive, a contradiction. Using the same argument, we can show $f(x) > 0$ for all positive rational numbers. One then proceeds to show $f(1)$ = 1. Hence $f(x) = x$ for all positive rationals. I am not going to produce all the details here. Suffices to say, we often need to expand a positive rational number in terms of a, say for a rational number $b < a$, it is of the form

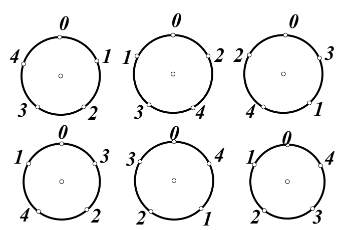
$$k_0 + \frac{k_1}{a} + \frac{k_2}{a^2} + \cdots \quad \text{(finite sum)},$$

some kind of a-adic expansion!

Problem 6: Problem 6 is even harder than problem 3, only 7 contestants solved it. A nice point of the problem is that it links a geometric fact (intersecting chords) to a certain number property, and the relation is an exact relation (M=N+1). For n=3, the beautiful labellings are given below (we always label 0 at the top).



The pairs of positive integers satisfying the stated property are (1,1), (1,2) and (2,1). For n=4, to complete the list of integers with the stated property, we just have to consider those x and y satisfying x+y=4. Indeed we get two more pairs (1,3) and (3,1). Indeed the six beautiful labellings are



(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***December 21, 2013.***

Problem 431. There are 100 people, composed of 2 people from 50 distinct nations. They are seated in a round table. Two people sitting next to each other are _neighbors_.

Prove that it is possible to divide the 100 people in two groups of 50 people so that no 2 people from the same nation are in the same group and each person in a group has at most one neighbor in the group.

Problem 432. Determine all prime numbers p such that there exist integers a,b,c satisfying $a^2 + b^2 + c^2 = p$ and $a^4+b^4+c^4$ is divisible by p.

Problem 433. Let P_1, P_2 be two points inside ΔABC. Let $BC = a$, $CA = b$ and $AB = c$. For $i = 1,2$, let $P_iA = a_i$, $P_iB = b_i$ and $P_iC = c_i$. Prove that

$$aa_1a_2+bb_1b_2+cc_1c_2 \geq abc.$$

Problem 434. Let O and H be the circumcenter and orthocenter of ΔABC respectively. Let D be the foot of perpendicular from C to side AB. Let E be a point on line BC such that $ED \perp OD$. If the circumcircle of ΔBCH intersects side AB at F, then prove that points E, F, H are collinear.

Problem 435. Let $n > 1$ be an integer that is not a power of 2. Prove that there exists a permutation $a_1, a_2,\ldots, a_n$ of $1,2,\ldots, n$ such that

$$\sum_{k=1}^{n} a_k \cos\frac{2k\pi}{n} = 0.$$

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Solutions

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Problem 426. Real numbers a, b, x, y satisfy the property that for all positive integers n, $ax^n+by^n=1+2^{n+1}$. Determine (with proof) the value of x^a+y^b.

Solution. **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain).

Considering the generating functions of the left and right sides of $ax^n+by^n=1+2^{n+1}$, we have

$$\sum_{n=1}^{\infty} ax^n z^{n-1} + \sum_{n=1}^{\infty} by^n z^{n-1} = \sum_{n=1}^{\infty} z^{n-1} + \sum_{n=1}^{\infty} 2^{n+1} z^{n-1}.$$

For $|z| < \min\{1/2, 1/|x|, 1/|y|\}$, using the geometric series formula, we have

$$\frac{ax}{1-xz} + \frac{by}{1-yz} = \frac{1}{1-z} + \frac{4}{1-2z}.$$

The right side is a rational function of z. By the uniqueness of the partial fraction decomposition, either $ax=1$, $x=1$, $by=4$, $y=2$ or $ax=4$, $x=2$, $by=1$, $y=1$. In both cases, $x^a+y^b = 1^1+2^2 = 5$.

Other commended solvers: **CHAN Long Tin** (Cambridge University, Year 1), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **KIM Minsuk Luke** (The South Island School, Hong Kong, Year 13), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, Form 2), **LO Wang Kin**(Wah Yan College, Kowloon), **Math Group** (Carmel Alison Lam Foundation Secondary School), **Alice WONG Sze Nga** (Diocesan Girls' School, Form 6) and **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 427. Determine all (m,n,k), where m, n, k are integers greater than 1, such that $1! + 2! + \cdots + m! = n^k$.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **KIM Minsuk Luke** (The South Island School, Hong Kong, Year 13), **LO Wang Kin** (Wah Yan College, Kowloon), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania School, Ploieşti, Romania), **Math Group** (Carmel Alison Lam Foundation Secondary School) and **William PENG.**

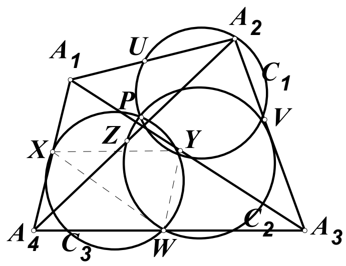
Let $S(m)=1! + 2! + \cdots + m!$. Then $S(2)=3$, $S(3) = 9 = 3^2$, $S(4) = 33 = 3\times 11$, $S(5) = 153 = 3^2\times 17$, $S(6) = 873 = 3^2\times 97$, $S(7) = 5913=3^4\times 73$, $S(8)=46233 = 3^2\times 11\times 467$.

For $m > 8$, since $9!\equiv 0 \pmod{3^3}$, so $S(m) \equiv S(8) \equiv 0 \pmod{3^2}$ and $S(m) \equiv S(8) \not\equiv 0 \pmod{3^3}$. These imply that if $S(m)=n^k$ and $k > 1$, then $k = 2$.

Since $S(4)=33\equiv 3 \pmod 5$, $S(m)\equiv 3 \pmod 5$. Now $n^2 \equiv 0, 1, 4 \pmod 5$. So $S(m)\neq n^2$. We have the only solution is $(m,n,k)=(3,3,2)$.

Problem 428. Let $A_1A_2A_3A_4$ be a convex quadrilateral. Prove that the nine point circles of $\Delta A_1A_2A_3$, $\Delta A_2A_3A_4$, $\Delta A_3A_4A_1$ and $\Delta A_4A_1A_2$ pass through a common point.

Solution. **HOANG Nguyen Viet** (Hanoi, Vietnam), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania School, Ploieşti, Romania), **Apostolis MANOLOUDIS, Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School, Form 6).



Let C_1, C_2, C_3, C_4 be the nine point circles of $\Delta A_1A_2A_3, \Delta A_2A_3A_4, \Delta A_3A_4A_1$, $\Delta A_4A_1A_2$ respectively. Let U, V, W, X, Y, Z be the midpoints of A_1A_2, A_2A_3, A_3A_4, A_4A_1, A_1A_3, A_2A_4 respectively. Let C_1 and C_3 intersect at Y and P (in case C_1, C_3 are tangent, P will be the same as Y). We claim P is on C_2. For that it suffices to show P,V,W,Z are concyclic.

By the midpoint theorem, $XY = ½A_4A_3 = WA_3$ and $XW = ½A_1A_3 = YA_3$. So we have (1) $WXYA_3$ is a parallelogram. Similarly, (2) $YUVA_3$ and (3) $WZVA_3$ are also parallelograms. Now (4) P,U,V,Y are on C_1 and (5) P,X,W,Y are on C_3. We have

$$\begin{aligned} \angle VPW &= \angle YPV + \angle YPW \\ &= \angle YUV + \angle YXW \quad \text{by (4), (5)} \\ &= \angle YA_3V + \angle YA_3W \quad \text{by (2), (1)} \\ &= \angle VA_3W \\ &= \angle VZW \quad \text{by (3).} \end{aligned}$$

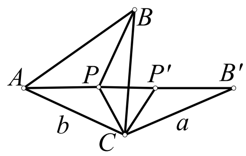
So P is on C_2. Similarly, P is on C_4.

Other commended solvers: **William FUNG, Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania).

Problem 429. Inside ΔABC, there is a point P such that $\angle APB = \angle BPC = \angle CPA$. Let $PA = u$, $PB = v$, $PC = w$, $BC = a$, $CA = b$ and $AB = c$. Prove that

$$(u+v+w)^2 \le ab+bc+ca \\ -\left(\sqrt{a(b+c-a)}-\sqrt{b(c+a-b)}\right)^2.$$

Solution. **LO Wang Kin** (Wah Yan College, Kowloon).



Rotate ΔABC about C by 60° away from A. Let the images of B, P be B', P' respectively. As $\angle PCP' = 60° = \angle BCB'$, so $\Delta PCP'$ and $\Delta BCB'$ are equilateral. As $\angle B'PC = \angle CPA = 120°$, A, P, P', B' are collinear. So $AB' = AP+PP'+P'B' = u+w+v$. By the cosine law, $AB'^2 = a^2+b^2-2ab\cos(C+60°)$.

After expansion and cancellation, the right side of the desired inequality becomes

$$a^2+b^2-ab+2\sqrt{ab(b+c-a)(c+a-b)}.$$

Now

$$\begin{aligned}&\sqrt{ab(b+c-a)(c+a-b)}\\ &= ab\sqrt{2\left(1-\frac{a^2+b^2-c^2}{2ab}\right)}\\ &= ab\sqrt{2(1-\cos C)}.\end{aligned}$$

Using these, the right side minus the left side of the desired inequality is

$$\begin{aligned}&ab(-1+2\sqrt{2(1-\cos C)}+2\cos(C+60°))\\ &= ab(-1+2\sqrt{2(1-\cos C)}+\cos C-\sqrt{3}\sin C)\\ &= 2ab\sqrt{1-\cos C}(\sqrt{2}-\sqrt{1+\cos(C-60°)}) \ge 0\end{aligned}$$

and we are done.

Other commended solvers: **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **T. W. LEE** (Alumni of New Method College), **Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School, Form 6).

Problem 430. Prove that among any $2n+2$ people, there exist two of them, say A and B, such that there exist n of the remaining $2n$ people, each either knows both A and B or does not know A nor B. Here, *x knows y* does not necessarily imply *y knows x*.

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **Math Group** (Carmel Alison Lam Foundation Secondary School).

Take a person P out of the $2n+2$ people. Suppose among the remaining $2n+1$ people, he knows k of them and does not know $2n+1-k$ of them. Among these $2n+1$ people, there are ${}_{2n+1}C_2 = n(2n+1)$ pairs. Call a pair <u>*good*</u> if P knows both of them or does not know both of them, <u>*bad*</u> if P knows one, but not both. By the AM-GM inequality, there are at most $\lceil k(2n+1-k)\rceil \le \lceil (n+½)^2\rceil = n^2+n$ bad pairs. Adding up all the bad pairs for all $2n+2$ people, the number is at most $(2n+2)(n^2+n) = 2n(n+1)^2$. There are ${}_{2n+2}C_2=(n+1)(2n+1)$ pairs altogether. Since the average

$$\frac{2n(n+1)^2}{(n+1)(2n+1)} = \frac{2n(n+1)}{2n+1} < n+1,$$

some pair $\{A, B\}$ will be a bad pair for at most n of the remaining $2n$ people. Then at least n other people will call $\{A,B\}$ a good pair and we are done.

Olympiad Corner

(Continued from page 1)

Problem 2. Let $a_1, a_2, \ldots, a_k$ be numbers such that $a_i \in \{0,1,2,3\}$, $i=1$ to k and $z = (x_k, x_{k-1}, \ldots, x_1)_4$ be a base 4 expansion of $z \in \{0, 1, 2, \ldots, 4^k-1\}$. Define A as follows:

$A=\{z \mid p(z)=z, z=0,1,2,\ldots,4^k-1\}$, where

$$p(z) = \sum_{i=1}^{k} a_i x_i 4^{i-1}.$$

Prove that $|A|$ is a power of 2. ($|X|$ denotes the number of elements in X).

Problem 3. Find all $a,b,c \in \mathbb{Z}$, $c \ge 0$ such that $(a^n+2^n) \mid b^n+c$ for all positive integers n, where $2ab$ is non-square.

Problem 4. Positive integers 1 to 9 are written in each square of a 3×3 table. Let us define an operation as follows: Take an arbitrary row or column and replace these numbers a, b, c with either non-negative numbers $a-x$, $b-x$, $c+x$ or $a+x$, $b-x$, $c-x$, where x is a positive number and can vary in each operation.

1) Does there exist a series of operations such that all 9 numbers turn out to be equal from the following initial arrangement a) ?, b) ?

1	2	3
4	5	6
7	8	9

a)

2	8	5
9	3	4
6	7	1

b)

2) Determine the maximum value which all 9 numbers turn out to be equal to after some steps.

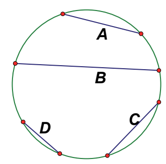
Problem 5. The incircle ω of a quadrilateral $ABCD$ touches AB, BC, CD, DA at E, F, G, H, respectively. Choose an arbitrary point X on the segment AC inside ω. The segments XB, XD meet ω at I, J respectively. Prove that FJ, IG, AC are concurrent.

Problem 6. Show that $x^3+x+a^2=y^2$ has at least one pair of positive integer solution (x,y) for each positive integer a.

IMO 2013–Leader Report (II)

(continued from page 2)

The problem is how to connect the geometry and the number theory information. In general, how to get started? I can only describe it roughly from the official solution. Call three chords *aligned* if one of them separates the other two. For more than three chords, they are aligned if any three of them aligned.



In the figure the chords A, B and C are aligned (the line formed by B separated the two chords A and C; while B, C and D are not aligned (none of the lines formed by B, C or D separates the other two chords). Now call a chord a *k-chord* if the sum of its two endpoints is k (the chord may be degenerated into a point of value k). The crucial observation is: in a beautiful labeling, the k-chords are aligned for any k. To prove this claim, one proceeds by induction. Indeed the only case is when there are three chords not aligned and such that one of the chords has endpoints 0 and n. After the claim is proved, one proceeds again using delicate induction arguments to show $M=N+1$. Indeed the beautiful labellings are eventually divided into classes. Elements of the first class are as before in the induction step. Elements of the second class correspond precisely with the pairs of positive integers satisfying $x+y=n$ and $\gcd(x,y)=1$, (which correspond exactly to the elements $\{x \mid 1 \le x \le n, \gcd(x,y) = 1\}$ with size $\varphi(n)$. Tough!

Mathematical Excalibur

Volume 18, Number 4 *December 2013 – January 2014*

Olympiad Corner

Below are the problems of the Dutch Team Selection Test for IMO 2013.

Problem 1. Show that

$$\sum_{n=0}^{2013} \frac{4026!}{\left(n!(2013-n)!\right)^2}$$

is the square of an integer.

Problem 2. Let P be the intersection of the diagonals of a convex quadrilateral $ABCD$. Let X, Y and Z be points on the interior of AB, BC and CD respectively such that

$$\frac{AX}{XB}=\frac{BY}{YC}=\frac{CZ}{ZD}=2.$$

Suppose moreover that XY is tangent to the circumcircle of ΔBXY. Show that $\angle APD=\angle XYZ$.

Problem 3. Fix a sequence a_1, a_2, a_3, … of integers satisfying the following condition: for all prime numbers p and all positive integers k, we have

$$a_{pk+1}=pa_k-3a_p+13.$$

Determine all possible values of a_{2013}.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 25, 2014***.

For individual subscription for the next five issues for the 13-14 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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PUMaC 2013

Andy Loo (Princeton University)

In the United States there are several annual math competitions organized by undergraduate students at different universities for high school enthusiasts, including the Harvard-MIT Math Tournament (HMMT), the Stanford Math Tournament (SMT), and, last but not least, the Princeton University Mathematics Competition (PUMaC). Started in 2006, PUMaC has grown into an international event in which high schoolers across America are joined by teams from as far away as Bulgaria and China on Princeton campus each year.

PUMaC 2013 was held on November 16, engaging over 600 participants, and I was honored to serve as Problem Tsar (academic coordinator who heads the problem writing team). The responsibility of creating, grading and defending the problems and solutions of a competition of such scale and repute gave me an inspiring learning experience.

The competition is split into Division A (more challenging) and Division B (for less experienced contestants). Each team consists of eight students. In the morning, each contestant takes two out of four one-hour answer-only individual tests (Algebra, Geometry, Combinatorics and Number Theory, eight problems each) of his/her choice, followed by the one-hour Team Round, where members of the same team may discuss and work together (each team enjoying a separate room!).

The top 10 performers on each individual test (possibly with nonempty intersection) qualify for the Individual Finals, a one-hour proof-based test with three problems. I personally feel that an average Individual Finals problem lies somewhere near an IMO problem 1 or 4 in terms of difficulty. Remarkably, in PUMaC 2013, two contestants got a perfect score on the Division A Individual Finals despite the time pressure! Also worth mentioning is the Power Round, which is a relatively long series of problems revolving around a central theme – knot theory in 2013 – released one week before the competition day for the teams to work on and turn in on competition day. (Teams may also enroll on a Power Round-only basis.) It usually takes frantic grading to determine the individual and team rankings in time for the award ceremony in the late afternoon, while mini-events such as Math (quiz) Bowl and Rubik's cube as well as a lecture by a Princeton professor keep the participants entertained.

I would like to discuss a few problems in PUMaC 2013, not necessarily because they are the hardest, but mostly because they bring out certain lessons of problem solving we can learn.

<u>Individual Finals B1.</u>

Let a_1 = 2013 and a_{n+1} = 2013^{a_n} for all positive integers n. Let b_1 = 1 and b_{n+1} = 2013^{2012b_n} for all positive integers n. Prove that $a_n > b_n$ for all positive integers n.

At first sight, one natural reaction to this problem would be to do induction. However, we would quickly realize that the assumption $a_n > b_n$ does not imply $a_{n+1} > b_{n+1}$, as it does not imply $2013^{a_n} > 2013^{2012b_n}$. Many contestants performed pages of tedious calculations in vain. Are we doomed? It turns out that a clever little tweak to the induction idea would lead us to a crisp and compact solution:

Instead of $a_n > b_n$, we shall prove $a_n \geq 2013b_n$ for all positive integers n. This is clearly true for $n = 1$. If $a_k \geq 2013b_k$ for

(continued on page 2)

some positive integer k, then

$$\begin{aligned} a_{k+1} &= 2013^{a_k} \\ &\geq 2013^{2013b_k} \\ &= 2013^{b_k} \cdot 2013^{2012b_k} \\ &\geq 2013b_{k+1}. \end{aligned}$$

There is something intriguing about this seemingly easy proof: if we cannot even prove just the original result, how come we can miraculously prove a stronger result? The answer to this paradox lies in the nature of mathematical induction: when we use induction, our task is essentially to prove the original statement about an arbitrary positive integer but equipped with an additional tool – the assumption that the statement is true for the preceding positive integer(s). If the statement is strengthened, what we need to prove becomes more demanding but the inductive hypothesis that we can use also gets more powerful. In the case of this problem, since the recurrence relations are exponential, the upgrade of the inductive hypothesis outweighs the increase in difficulty of the desired result.

Individual Finals A1.

Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}$$
$$\leq \frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$

for any positive real numbers a, b and c satisfying $a^2 + b^2 + c^2 = 1$.

The usual first step in proving such a symmetric inequality is to use the given condition to *homogenize* the inequality, i.e. to make the terms carry equal degrees. Afterwards, various inequality theorems can be applied. Here we first write the left-hand side as

$$\frac{1}{3a^2+2b^2+2c^2}+\frac{1}{3b^2+2c^2+2a^2}+\frac{1}{3c^2+2a^2+2b^2}$$

and note that by the AM-GM inequality, $3a^2 + 3b^2 \geq 6ab$ and analogous inequalities hold. So

$$\frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$
$$\geq \frac{1}{3a^2+3b^2+c^2}+\frac{1}{3b^2+3c^2+a^2}+\frac{1}{3c^2+3a^2+b^2}.$$

It suffices to prove the following inequality

$$\frac{1}{3x+2y+2z}+\frac{1}{3y+2z+2x}+\frac{1}{3z+2x+2y}$$
$$\leq \frac{1}{3x+3y+z}+\frac{1}{3y+3z+x}+\frac{1}{3z+3x+y}$$

where x, y and z are positive real numbers.

At this stage, one may resort to passionate expansion and then apply Muirhead's inequality and/or Schur's inequality, or, alternatively, factorization and completing the square.

But I wish to share a solution using the *majorization inequality* (see Math Excalibur, vol. 5, no. 5, p.2): Without loss of generality we may assume $x \geq y \geq z$. Then

$(3x + 3y + z, 3y + 3z + x, 3z + 3x + y)$

majorizes

$(3x + 2y + 2z, 3y + 2z + 2x, 3z + 2x + 2y)$.

Due to the convexity of the function $f(t) = 1/t$, the desired inequality follows by the majorization inequality.

Readers may also be interested in an alternative solution involving calculus: First, by Muirhead's inequality (see *Mathematical Excalibur*, vol. 11, no. 1), we have

$$u^3v^3w + v^3w^3u + w^3u^3v$$
$$\geq u^3v^2w^2 + v^3w^2u^2 + w^3u^2v^2$$

for any positive u, v, w. Letting

$$u = t^{x-1/7},\ v = t^{y-1/7} \text{ and } w = t^{z-1/7}$$

where $0 < t < 1$, we get

$$t^{3x+3y+z-1} + t^{3y+3z+x-1} + t^{3z+3x+y-1}$$
$$\geq t^{3x+2y+2z-1} + t^{3y+2z+2x-1} + t^{3z+2x+2y-1}.$$

Now, integrating both sides with respect to t from 0 to 1, we obtain nothing but the desired inequality!

Lastly I encourage all readers to try out the following problem which only one out of the 123 contestants attempting Combinatorics A got right. This is really my favorite problem in PUMaC 2013 because I love eating sushi and find the setting very interesting:

Combinatorics A8.

Eight different pieces of sushi are placed evenly around a round table which can rotate about its center. Eight people sit evenly around the table. Each person has one favorite piece of sushi among the eight, and their favorites are all distinct. Sadly, they find that no matter how they rotate the table, there are never more than three people who have their favorite sushi in front of them simultaneously.

How many possible arrangements of the eight pieces of sushi are there? (Two arrangements that differ by a rotation are considered the same.)

In 1908, a classic Chinese newspaper article famously raised three questions for the country: When can China first send an individual athlete to the Olympic Games? When can China first send a delegation to the Olympic Games? When can China first host the Olympic Games?

In closing, I would also like to ask three questions: When can Hong Kong first take part in the Power Round of PUmaC? When can Hong Kong first send a team to Princeton to join the main competition of PUMaC? When can a university in Hong Kong first host a math competition run by undergraduates for secondary school students?

As Dr. Kin Li (editor of *Math Excalibur*) observes, Hong Kong students need more opportunities to participate in different competitions and broaden their horizons. They will also be able to experience a beautiful university, make friends with some of the most brilliant brains from around the world, and learn team spirit especially through the Power Round and Team Round. With optimism, I hope my three questions will find answers before long.

For further information and past papers, please visit PUMaC's website http://www.pumac.princeton.edu/

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 25, 2014.***

Problem 436. Prove that for every positive integer n, there exists a positive integer $p(n)$ such that the interval $[1, p(n)]$ can be divided into n pairwise disjoint intervals with each contains at least one integer and the sum of the integers in each of these intervals is the square of some integer.

Problem 437. Determine all real numbers x satisfying the condition that $\cos x$, $\cos 2x$, $\cos 4x$, …, $\cos 2^n x$, … are all negative.

Problem 438. Suppose $P(x)$ is a polynomial with integer coefficients such that for every integer n, $P(n)$ is divisible by at least one of the positive integers $a_1, a_2,\ldots, a_m$. Prove that there exists one of the a_i such that for all integer n, $P(n)$ is divisible by that a_i.

Problem 439. In acute triangle ABC, T is a point on the altitude AD (with D on side BC). Lines BT and AC intersect at E, lines CT and AB intersect at F, lines EF and AD intersect at G. A line ℓ passing through G intersects side AB, side AC, line BT, line CT at M, N, P, Q respectively.

Prove that $\angle MDQ = \angle NDP$.

Problem 440. There are n schools in a city. The i-th school will send C_i students to watch a performance at a field. It is <u>known</u> that $0 \le C_i \le 39$ for i=1, 2, …, n and $C_1+C_2+\cdots+C_n$=1990. The seats will be put in a rectangle arrangement with each row having 199 seats. Determine the least number of rows needed to satisfy the condition that all students from the same school must sit in the same row for all possibilities of the known conditions above.

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Solutions

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Problem 431. There are 100 people, composed of 2 people from 50 distinct nations, are seated in a round table. Two people sitting next to each other are <u>neighbors</u>.

Prove that it is possible to divide the 100 people in two groups of 50 people so that no 2 people from the same nation are in the same group and each person in a group has at most one neighbor in the group.

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **Math Group** (Carmel Alison Lam Foundation Secondary School) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Suppose these 100 people $V_1, V_2, \ldots, V_{100}$ are seated in a round table in clockwise order. For $n = 1,2,\ldots, 50$, call $\{V_{2n-1}, V_{2n}\}$ a <u>partner pair</u>. We color V_1 in black and color the person with the same nation as him, say V_r, in white. If V_r's partner is not yet colored, then we color V_r's partner, say V_s, in black (this completes the coloring of the partner pair $\{V_r, V_s\}$) and go on to color the person with the same nation as V_s in white. Repeat this process until we reach a V_r whose partner V_s was colored already, then V_r=V_2 and V_s=V_1 since the only partner pair not yet completing the coloring is $\{V_1,V_2\}$ with V_1 black and V_2 waiting to be colored. This gives the first cycle. Then we start to form another cycle with a remaining partner pair. Since there are 100 people, we will eventually stop. At the end, there are two groups with 50 black's and 50 white's and the required conditions are satisfied.

Problem 432. Determine all prime numbers p such that there exist integers a,b,c satisfying $a^2 + b^2 + c^2 = p$ and $a^4+b^4+c^4$ is divisible by p.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **KIM Minsuk Luke** (The South Island School, Hong Kong, Year 13), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **Math Center** (Carmel Alison Lam Foundation Secondary School) and **O Kin Chit Alex** (G.T. (Ellen Yeung) College).

Without loss of generality, we may assume $a \ge b \ge c \ge 0$. Then

$$\begin{aligned}0 &\equiv (p - b^2 - c^2)^2 + b^4 + c^4\\ &\equiv (b^2 + c^2)^2 + b^4 + c^4 = 2(b^4 + b^2c^2 + c^4)\\ &= 2(b^2 - bc + c^2)(b^2 + bc + c^2) \pmod p.\end{aligned}$$

Next,

$$\begin{aligned}0 \le bc &\le b^2 - bc + c^2\\ &\le b^2 + bc + c^2\\ &\le a^2 + b^2 + c^2 = p.\end{aligned}$$

Since $a \ge b \ge c \ge 0$, if $bc=a^2$, then $a=b=c$ and p being prime implies $a = 1$ and $p = 3$. Otherwise $bc < a^2$ leads to $b^2 + bc + c^2 = 0$ or 1. If $b = 0$, then $a^2 = p$ contradicts p is prime. Then $c = 0$, $b = 1$ and $a^2 + 1 = p$, which leads to

$$0 \equiv a^4+b^4+c^4 = a^4+1 \equiv 2 \pmod p.$$

Then p=2 and a=b=1, c=0. Therefore, the only solutions are $p = 2$ or 3.

Problem 433. Let P_1, P_2 be two points inside ΔABC. Let $BC = a$, $CA = b$ and $AB = c$. For $i = 1,2$, let $P_iA = a_i$, $P_iB = b_i$ and $PC_i = c_i$. Prove that

$$aa_1a_2+bb_1b_2+cc_1c_2 \ge abc.$$

Solution. **Math Group** (Carmel Alison Lam Foundation Secondary School).

Let the complex numbers $\alpha, \beta, \gamma, \mu, \nu$ correspond to the points A, B, C, P_1, P_2 in the complex plane respectively. By expansion, we have

$$\frac{(\mu-\alpha)(\nu-\alpha)}{(\beta-\alpha)(\gamma-\alpha)}+\frac{(\mu-\beta)(\nu-\beta)}{(\alpha-\beta)(\gamma-\beta)}+\frac{(\mu-\gamma)(\nu-\gamma)}{(\alpha-\gamma)(\beta-\gamma)}=1.$$

Then

$$\frac{a_1a_2}{cb}+\frac{b_1b_2}{ca}+\frac{c_1c_2}{ba}$$

$$=\left|\frac{(\mu-\alpha)(\nu-\alpha)}{(\beta-\alpha)(\gamma-\alpha)}\right|+\left|\frac{(\mu-\beta)(\nu-\beta)}{(\alpha-\beta)(\gamma-\beta)}\right|+\left|\frac{(\mu-\gamma)(\nu-\gamma)}{(\alpha-\gamma)(\beta-\gamma)}\right|$$

$$\ge\left|\frac{(\mu-\alpha)(\nu-\alpha)}{(\beta-\alpha)(\gamma-\alpha)}+\frac{(\mu-\beta)(\nu-\beta)}{(\alpha-\beta)(\lambda-\beta)}+\frac{(\mu-\gamma)(\nu-\gamma)}{(\alpha-\gamma)(\beta-\gamma)}\right|$$

= 1.

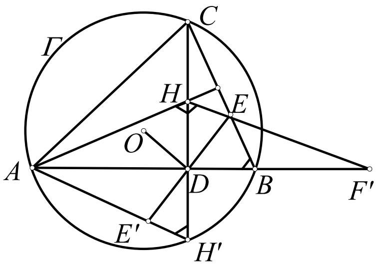
Multiplying both sides by abc, we get the desired result.

Problem 434. Let O and H be the circumcenter and orthocenter of ΔABC respectively. Let D be the foot of perpendicular from C to side AB. Let E be a point on line BC such that $ED \perp OD$. If the circumcircle of ΔBCH intersects line AB at F, then prove that points E, F, H are collinear.

Solution 1. **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **T. W. LEE** (Alumni of New Method College).

Let lines HE and AB intersect at F'. Let Γ be the circumcircle of ΔABC. Let H'

be the intersection of line CD and Γ different from C. Let E' be the intersection of lines DE and AH'.

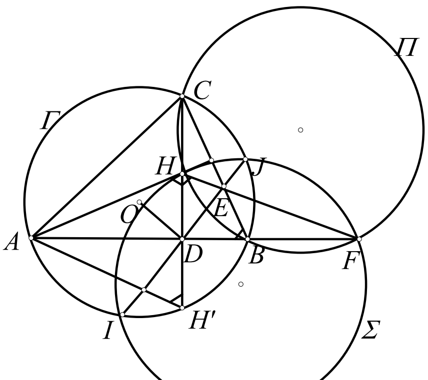


Observe that since $\angle H'DE' = \angle HDE$, $\angle AH'C = \angle ABC = 90^\circ - \angle BAH = \angle AHD$ implies $H'D=HD$ and the butterfly $H'CBAH'$ on Γ gives $E'D=ED$ as $OD\perp DE$, we have $\triangle H'E'D \cong \triangle HED$. Then

$$\angle F'HD = \angle EHD = \angle E'H'D = \angle AH'C = \angle DBC.$$

It follows $\angle CHF' = \angle CBF'$. Then F' is on the line AB and the circumcircle of $\triangle BCH$. Therefore, $F'=F$ and E, F, H are collinear.

Solution 2. **Jerry AUMAN, Georgios BATZOLIS** (*Mandoulides High School, Thessaloniki, Greece*) and **Jon GLIMMS** (Vancouver, Canada).



Let Π be the circle passing through C, H, B, F and let Γ be the circumcircle of $\triangle ABC$. Let line DE meet Γ at I and J. Since $OD\perp DE$, D bisects chord IJ. Next,

$$\angle DCF = \angle DBH = 90^\circ - \angle BAC = \angle DCA$$

implies D bisects AF. Hence $AIFJ$ is a parallelogram. Then $\angle IFJ = \angle IAJ$.

Let H' be the intersection point (different from C) of line CD and Γ. Then D bisects HH' (see solution 1 -- *Ed.*) and $IHJH'$ is a parallelogram. So $\angle IHJ = \angle IH'J$. Then

$$\angle IFJ + \angle IHJ = \angle IAJ + \angle IH'J = 180^\circ.$$

So I,F,J,H lies on a circle Σ.

Finally, the radical axis of Γ and Π is line BC, while the radical axis of Γ and Σ is line IJ. So the radical center of Γ, Π, Σ is the intersection of lines BC and IJ, which is E. Therefore, E is also on the radical axis of Π and Σ, which is line HF.

Comments: One can also solve via coordinate geometry by assigning lines AB and CD as the x-axis and y-axis respectively.

Other commended solvers: **Math Group** (Carmel Alison Lam Foundation Secondary School), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, India) and **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania).

Problem 435. Let $n > 1$ be an integer that is not a power of 2. Prove that there exists a permutation $a_1, a_2,\ldots, a_n$ of $1,2,\ldots, n$ such that

$$\sum_{k=1}^{n} a_k \cos\frac{2k\pi}{n} = 0.$$

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **Math Center** (Carmel Alison Lam Foundation Secondary School).

For integer $n > 1$, let $c_k = \cos(2k\pi/n)$ for $k = 1, 2, \ldots, n$. We have $c_n = 1$, $c_k = c_{n-k}$ and

$$\sum_{k=1}^{n} c_k = \operatorname{Re}\sum_{k=1}^{n}\omega^k = \operatorname{Re}\frac{1-\omega^n}{1-\omega} = 0, \qquad (*)$$

where $\omega = e^{2\pi i/n}$.

Suppose $n = 2m+1$, where $m = 1,2,3,\ldots$. We have $c_1+c_2+\cdots+c_m = -1/2$ (using $c_n = 1$ and $c_k = c_{n-k}$). Hence

$$(2m+2)(c_1+c_2+\cdots+c_m) = -(m+1)c_{2m+1}.$$

Since $c_k = c_{2m+1-k}$, we have

$$(2m+1)c_1+2mc_2+\cdots+(m+2)c_m = (m+2)c_{m+1}+\cdots+2mc_{2m-1}+(2m+1)c_{2m}.$$

Subtracting the two displayed equations above and transposing all terms to the left, we get

$$\sum_{k=1}^{m} kc_k + \sum_{k=m+1}^{2m}(k+1)c_k + (m+1)c_{2m+1} = 0.$$

This solves the cases $n = 3,5,7,\ldots$.

Next, assuming the case n is true, we will show the case $2n$ is also true. Let $d_m = \cos(m\pi/n)$ for $m = 1,2,\ldots,2n$. The case n gives us an equation of the form

$$a_1d_2 + a_2d_4 + \cdots + a_nd_{2n} = 0, \qquad (**)$$

where $a_1, a_2, \ldots, a_{2n}$ is a permutation of $1,2,\ldots,n$.

Using (*), we have

$$d_1 + d_2 + \cdots + d_{2n} = \sum_{k=1}^{2n}\cos\frac{2k\pi}{2n} = 0$$

and

$$d_2 + d_4 + \cdots + d_{2n} = \sum_{k=1}^{n}\cos\frac{2k\pi}{n} = 0.$$

Subtracting these equations, we have $d_1+d_3+\cdots+d_{2n-1}=0$. For $k=1,3,\ldots,2n-1$, we have

$$d_{2n-k} = \cos((2n-k)\pi/n) = \cos(k\pi/n) = d_k.$$

Using this, $d_1+3d_3+\cdots+(2n-1)d_{2n-1} = d_{2n-1}+3d_{2n-3}+\cdots+(2n-1)d_1$. Adding the left and right sides, we get the equation $2n(d_1+d_3+\cdots+d_{2n-1}) = 0$. So

$$d_1+3d_3+\cdots+(2n-1)d_{2n-1} = 0. \qquad (***)$$

Finally, taking twice the equation in (**) and adding it to the equation in (***), we solve the case $2n$.

Comments: **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania) pointed out that Problem 435 is the same as Problem 26753 in the Romanian Mathematical Gazette (G.M.-B) and a solution was appeared in G.M-B, No. 10, 2013, pp. 468-469.

Olympiad Corner

(*Continued from page 1*)

Problem 4. Determine all positive integers $n \ge 2$ satisfying

$$i + j \equiv \binom{n}{i} + \binom{n}{j} \pmod{2}$$

for all i and j such that $0 \le i \le j \le n$.

Problem 5. Let $ABCDEF$ be a cyclic hexagon satisfying $AB\perp BD$ and $BC=EF$. Let P be the intersection of lines BC and AD and let Q be the intersection of lines EF and AD. Assume that P and Q are on the same side of D and that A is on the opposite side. Let S be the midpoint of AD. Let K and L be the centres of the incircles of ΔBPS and ΔEQS respectively. Prove that $\angle KDL = 90^\circ$.

Mathematical Excalibur

Volume 18, Number 5 *February 2014 – March 2014*

Olympiad Corner

Below are the problems of the Fourth Round of the 53^rd Ukrainian National Math Olympiad for 10-th Graders.

Problem 1. Suppose that for real x,y,z,t the following equalities hold: $\{x+y+z\} = \{y+z+t\} = \{z+t+x\} = \{t+x+y\} = 1/4$. Find all possible values of $\{x+y+z+t\}$. (Here $\{x\}=x-[x]$.)

Problem 2. Let M be the midpoint of the side BC of $\triangle ABC$. On the side AB and AC the points F and E are chosen. Let K be the point of the intersection of BF and CE and L be chosen in a way that $CL \| AB$ and $BL \| CE$. Let N be the point of intersection of AM and CL. Show that KN is parallel to FL.

Problem 3. It is known that for natural numbers a,b,c,d and n the following inequalities hold: $a+c < n$ and $a/b+c/d < 1$. Prove that $a/b+c/d < 1-1/n^3$.

Problem 4. There are 100 cards with numbers from 1 to 100 on the table. Andriy and Nick took the same number of cards in a way that the following condition holds: if Andriy has a card with a number n then Nick has a card with a number $2n+2$. What is the maximal number of cards could be taken by the two guys?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 12, 2014***.

For individual subscription for the next five issues for the 13-14 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Using Tangent Lines to Prove Inequalities (Part II)

Ibragim Ibatulin and Adilsultan Lepes

The Republican Specialized Physics Mathematics Secondary Boarding School
Named after O. Zhautykov, Almaty, Kazakhstan

We offer a continuation of the paper by Kin-Yin Li (cf. Math Excalibur, *vol.* 10, *no.* 5) where he considers using tangent lines to prove inequalities.

Example 1. Suppose that a, b, and c are positive real numbers satisfying $a+b+c=3$. Find the minimum of the expression $a^4+2b^4+3c^4$.

Solution. Let $f_k(x)=kx^4$, where $x\in(0,3)$, $k = 1, 2, 3$. As $f_k''(x) = 12kx^2 > 0$, where $x>0$, so functions f_k are convex, which means that their graphs do not fall below their tangents drawn at any point $x_k\in(0,3)$ (k=1,2,3). Points x_1, x_2 and x_3 are chosen such that $f_1'(x_1) = f_2'(x_2) = f_3'(x_3)$ and $x_1+x_2+x_3=3$. That is,

$$4x_1^3=8x_2^3=12x_3^3 \text{ and } x_1+x_2+x_3=3.$$

Hence,

$$x_1=\frac{3\sqrt[3]{6}}{\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{6}},\ x_2=\frac{x_1}{\sqrt[3]{2}},\ x_3=\frac{x_1}{\sqrt[3]{3}}$$

and for any $x\in(0,3)$, we have the inequalities (k = 1,2,3, see Fig. 1)

$$kx^4 \geq f_k(x_k)+f_k'(x_k)(x-x_k). \qquad (1)$$

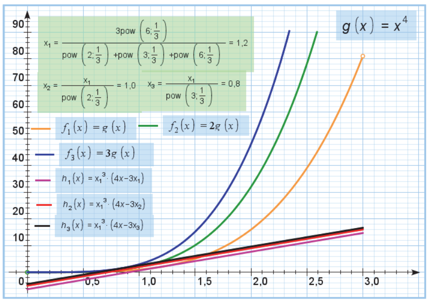

Fig. 1

Adding inequalities (1) for x equals a, b and c, we obtain

$$\begin{aligned} &a^4+2b^4+3c^4 \\ &\geq x_1^4\left(1+\frac{1}{2\sqrt[3]{2}}+\frac{1}{3\sqrt[3]{3}}\right)+f_1'(x_1)(3-\sum_{k=1}^{3}x_k) \\ &=\frac{81(6\sqrt[3]{3}+3\sqrt[3]{3}+2\sqrt[3]{2})}{(\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{6})^4}, \end{aligned}$$

which is the minimum (with equality holding at $a=x_1$, $b=x_2$ and $c=x_3$).

Example 2. Let a, b, c > 0 be real numbers such that $ab+bc+ca = 1$. Prove the inequality

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{\sqrt{3}}{2}.$$

Solution. Let $S=a+b+c$. Based on the inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$, which is equivalent to $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$, we find that $S \geq \sqrt{3}$.

Let $f(x) = x^2/(S-x)$ for $x\in(0,S)$. Let us construct the tangent equation at the point $x_0=S/3$ (see Fig. 2a,b):

$$y=f\left(\frac{S}{3}\right)+f'\left(\frac{S}{3}\right)\left(x-\frac{S}{3}\right)=\frac{S}{6}+\frac{5}{4}\left(x-\frac{S}{3}\right)=\frac{5x-S}{4}.$$

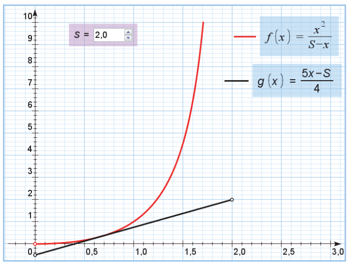

Fig. 2a

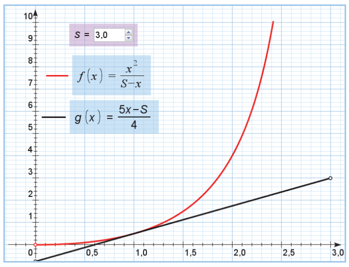

Fig. 2b

Since the inequality $x^2/(S-x) \geq (5x-S)/4$ is equivalent to $(S-3x)^2 \geq 0$ on the interval $(0,S)$, applying it thrice, based on the previously proved inequality $S \geq \sqrt{3}$, we find that

$$\begin{aligned} &\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \\ &=\frac{a^2}{S-a}+\frac{b^2}{S-b}+\frac{c^2}{S-c} \\ &\geq\frac{5(a+b+c)-3S}{4}=\frac{S}{2}\geq\frac{\sqrt{3}}{2}. \end{aligned}$$

(continued on page 2)

Example 3. Let $a,\ b,\ c \ge 0$ be real numbers. Prove the inequality

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1} \ge \sqrt{6(a+b+c)}.$$

Solution. Assume that $S=a+b+c$ and $f(x)=\sqrt{x^2+1}$ for $x\in(0,S)$. We form the tangent equation at the point $x_0=S/3$:

$$y=f\left(\frac{S}{3}\right)+f'\left(\frac{S}{3}\right)\left(x-\frac{S}{3}\right)$$
$$=\frac{\sqrt{S^2+9}}{3}+\frac{S}{\sqrt{S^2+9}}\left(x-\frac{S}{3}\right)$$
$$=\frac{Sx+3}{\sqrt{S^2+9}}.$$

Since on the interval $(0,S)$, the inequality

$$\sqrt{x^2+1} \ge \frac{Sx+3}{\sqrt{S^2+9}} \qquad (2)$$

is equivalent to the inequality $(x-S/3)^2 \ge 0$, we find that

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}$$
$$\ge \sqrt{S^2+9}+\frac{S}{\sqrt{S^2+9}}(a+b+c-S)$$
$$=\sqrt{S^2+9}$$
$$\ge \sqrt{6S}=\sqrt{6(a+b+c)}.$$

Example 4. Let a, b and c be positive real numbers such that $a+2b+3c \ge 20$. Prove the inequality

$$a+b+c+\frac{3}{a}+\frac{9}{2b}+\frac{4}{c} \ge 13.$$

Solution. Note that if $a=2$, $b=3$, $c=4$, the inequality becomes equality. Let $f(x)=1/x$ for $x>0$. Then f is convex in the interval $(0,+\infty)$. Hence the graph of the function f does not go below the tangent line drawn at any point $x_0>0$. Thus, the following inequalities are valid (see Fig. 3):

$$\frac{1}{a} \ge \frac{1}{2}-\frac{1}{4}(a-2)=1-\frac{a}{4},$$
$$\frac{1}{b} \ge \frac{1}{3}-\frac{1}{9}(b-3)=\frac{2}{3}-\frac{b}{9},$$
$$\frac{1}{c} \ge \frac{1}{4}-\frac{1}{16}(c-4)=\frac{1}{2}-\frac{c}{16}.$$

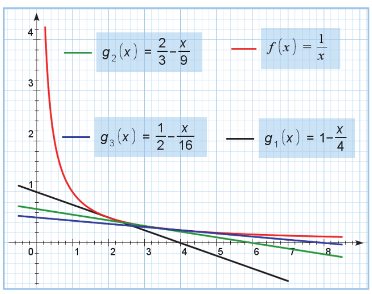

Fig. 3

As given in the statement of the problem, we find that

$$a+b+c+\frac{3}{a}+\frac{9}{2b}+\frac{4}{c}$$
$$\ge a+b+c+3-\frac{3a}{4}+3-\frac{b}{2}+2-\frac{c}{4}$$
$$=8+\frac{a+2b+3c}{4} \ge 8+\frac{20}{4}=13.$$

Example 5. (Pham Kim Hung) Let a, b and c be positive real numbers such that $a^2+b^2+c^2=3$. Prove the inequality

$$\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c} \ge 3.$$

Solution. Note that when $a=b=c=1$, the inequality becomes an equality. Consider $f(x)=1/(2-x)$ and $g(x)=kx^2+m$, where $x\in(0,\sqrt{3})$. The numbers k and m are to be chosen so that $f(1)=g(1)$ and $f'(1)=g'(1)$. That is, $1=k+m$ and $1=2k$. Hence, $k=m=1/2$ and $g(x)=(x^2+1)/2$. Since the inequality $1/(2-x) \ge (x^2+1)/2$ is equivalent to $x(x-1)^2 \ge 0$, it is true for any $x\in(0,\sqrt{3})$ (see Fig. 4). Hence,

$$\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c} \ge \frac{a^2+b^2+c^2+3}{2}=3.$$

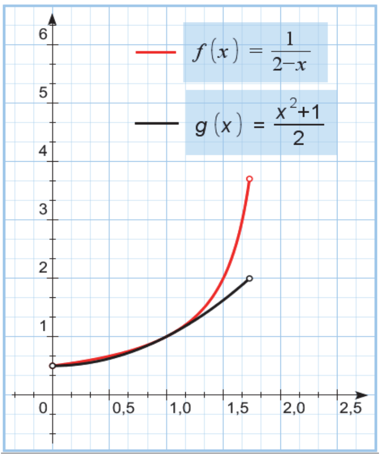

Fig. 4

Example 6. Let a, b and c be positive real numbers. Prove the inequality

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \ge (a+b+c)^3.$$

Solution. Note that when $a=b=c=1$, the inequality becomes an equality. Consider $f(x)=x^5-x^2+3$ and $g(x)=kx^3+m$, where $x>0$. The numbers k and m are to be chosen so that $f(1)=g(1)$ and $f'(1)=g'(1)$. That is, $3=k+m$ and $3=3k$. Hence, $k=1$, $m=1/2$ and $g(x)=x^3+2$. The inequality (see Fig. 5)

$$x^5-x^2+3 \ge x^3+2 \qquad (3)$$

is true for any $x>0$ as it can be represented in the form $(x-1)^2(x^3+2x^2+2x+1) \ge 0$.

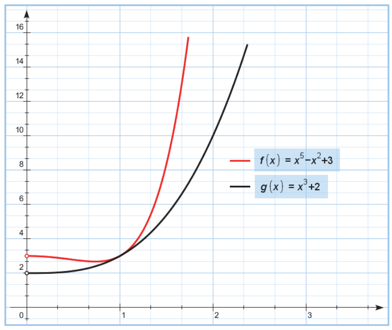

Fig. 5

Example 7. Let a, b, c, d and e be positive real numbers such that

$$\frac{1}{4+a}+\frac{1}{4+b}+\frac{1}{4+c}+\frac{1}{4+d}+\frac{1}{4+e}=1.$$

Prove the inequality

$$\frac{a}{4+a^2}+\frac{b}{4+b^2}+\frac{c}{4+c^2}+\frac{d}{4+d^2}+\frac{e}{4+e^2} \le 1.$$

Solution. Consider $f(x)=x/(4+x^2)$ and $g(x)=m+k/(4+x)$, where $x \ge 0$. The numbers k and m are to be chosen so that $f(1)=g(1)$ and $f'(1)=g'(1)$. Hence $k=-3$ and $m=4/5$. Since the inequality

$$\frac{x}{4+x^2} \le \frac{4}{5}-\frac{3}{4+x}$$

is equivalent to $(x-1)^2(x+1) \ge 0$, it is true for any $x \ge 0$ (see Fig. 6).

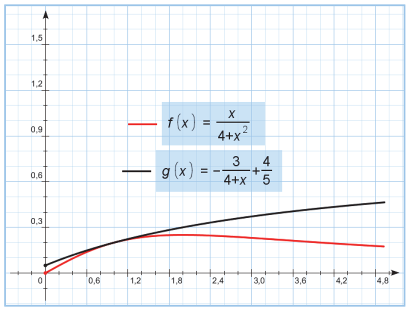

Fig. 6

Applying this inequality, we have

$$\frac{a}{4+a^2}+\frac{b}{4+b^2}+\frac{c}{4+c^2}+\frac{d}{4+d^2}+\frac{e}{4+e^2}$$
$$\le 4-3\left(\frac{1}{4+a}+\frac{1}{4+b}+\frac{1}{4+c}+\frac{1}{4+d}+\frac{1}{4+e}\right)$$
$$=1.$$

Finally, we have some exercises for the readers.

Exercise 1. (Gabriel Dospinescu) Let a_1, a_2, …, a_n be positive real numbers such that $a_1a_2\cdots a_n=1$. Prove that

$$\sqrt{1+a_1^2}+\sqrt{1+a_2^2}+\cdots+\sqrt{1+a_n^2}$$
$$\le \sqrt{2}(a_1+a_2+\cdots+a_n).$$

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 12, 2014.***

Problem 441. There are six circles on a plane such that the center of each circle lies outside of the five other circles. Prove there is no point on the plane lying inside all six circles.

Problem 442. Prove that if $n > 1$ is an integer, then n^5+n+1 has at least two distinct prime divisors.

Problem 443. Each pair of n ($n\geq 6$) people play a game resulting in either a win or a loss, but no draw. If among every five people, there is one person beating the other four and one losing to the other four, then prove that there exists one of the n people beating all the other n–1 people.

Problem 444. Let D be on side BC of equilateral triangle ABC. Let P and Q be the incenters of ΔABD and ΔACD respectively. Let E be the point so that ΔEPQ is equilateral and D, E are on opposite sides of line PQ. Prove that lines BC and DE are perpendicular.

Problem 445. For each positive integer n, prove there exists a polynomial $p(x)$ of degree n with integer coefficients such that $p(0)$, $p(1)$, …, $p(n)$ are distinct and each is of the form $2\times 2014^k+3$ for some positive integer k.

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Solutions

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Problem 436. Prove that for every positive integer n, there exists a positive integer $p(n)$ such that the interval $[1, p(n)]$ can be divided into n pairwise disjoint intervals with each contains at least one integer and the sum of the integers in each of these intervals is the square of some integer.

Solution. **Jerry AUMAN, Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **Jon GLIMMS** (Vancouver, Canada) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

We look for a pattern. Since $1=1^2$, let $p(1)=1$. Since $2+3+4=3^2$, let $p(2)=1+3=4$ and divide $[1,4]$ into $[1,1]$ and $(1,4]$. Since

$$5+6+7+8+9+10+11+12+13 = 9^2,$$

let $p(3) = 1+3+9=13$ and divide $[1,13]$ into $[1,1]$, $(1,4]$, $(4,13]$.

This suggests we let $p(n) = 1 + 3 + 3^2 + \cdots + 3^{n-1} = (3^n-1)/2$ and divide $[1, p(n)]$ into $[1, p(1)]$, $(p(1), p(2)]$, …, $(p(n-1), p(n)]$. The integers in $(p(k), p(k+1)]$ are from $(3^k+1)/2$ to $(3^{k+1}-1)/2$, which sums to 3^{2k}. So we are done.

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India) and **SP47** (Hanoi, Vietnam).

Problem 437. Determine all real numbers x satisfying the condition that $\cos x$, $\cos 2x$, $\cos 4x$, …, $\cos 2^nx$, … are all negative.

***Solution* 1. Jerry AUMAN, T. W. LEE** (Alumni of New Method College) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School)**.**

For such x, we have $2^nx=2\pi(k_n+\theta_n)$, where $k_n\in\mathbb{Z}$ and $1/4 < \theta_n < 3/4$. In base 2 this is $.01_2< \theta_0=.d_1d_2d_3\ldots_2<.10111\ldots_2$. No d_nd_{n+1} can be 00 or 11, otherwise $\theta_{n-1}=.00\ldots_2$ or $.11\ldots_2$ would not be in (1/4,3/4). So $\theta_0 = .010101\ldots_2=1/3$ or $.101010\ldots_2=2/3$. Then $x=2\pi(k_0+1/3)$ or $2\pi(k_0+2/3)$ and for all $n = 0,1,2,\cdots$, $\cos 2^nx = -1/2$.

Solution 2. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania) and **GLIMMS** (Vancouver, Canada).

Let $t=\cos 2\theta$. Suppose $\cos\theta$, $\cos 2\theta$ and $\cos 4\theta$ are negative. Then $t<0$ and $2t^2-1<0$ imply $-\sqrt{2}/2<t=2\cos^2\theta-1<0$. We get

$$\cos\theta<-\frac{\sqrt{2-\sqrt{2}}}{2}<-\frac{1}{4}.$$

Suppose $s_n = \cos 2^nx < 0$ for $n = 0,1,2,3,\cdots$. Then $s_n\in[-1, -1/4)$. So $|s_n-1/2| > 3/4$. Using this and $s_{n+1}=2s_n^2-1$, we have

$$\left|s_{n+1}+\frac{1}{2}\right|=2\left|s_n^2-\frac{1}{4}\right|=2\left|\left(s_n-\frac{1}{2}\right)\left(s_n+\frac{1}{2}\right)\right|$$
$$\geq\frac{3}{2}\left|s_n+\frac{1}{2}\right|.$$

Repeating this, since $-1\leq s_{n+1}<0$, we get

$$\frac{1}{2}\geq\left|s_{n+1}+\frac{1}{2}\right|\geq\left(\frac{3}{2}\right)^n\left|s_0+\frac{1}{2}\right|.$$

Then $|s_0+1/2| < (2/3)^n$. Taking limit, we see $\cos x = s_0= -1/2$, i.e. $x = \pm 2\pi/3+2k\pi$, where k is integer. Conversely, $s_0= -1/2$ implies $s_n= -1/2$ for $n = 1,2,3,\cdots$.

Other commended solvers: **Henry LEUNG Kai Chung** (Graduate of HKUST Maths).

Problem 438. Suppose $P(x)$ is a polynomial with integer coefficients such that for every integer n, $P(n)$ is divisible by at least one of the positive integers $a_1, a_2,\ldots, a_m$. Prove that there exists one of the a_i such that for all integer n, $P(n)$ is divisible by that a_i.

Solution. **Jerry AUMAN, Jon GLIMMS** (Vancouver, Canada) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Assume the contrary that for each a_i, there exists integer n_i such that $P(n_i)$ is not divisible by a_i. Consider the prime factorizations of a_i and $|P(n_i)|$. Then there exists a prime divisor p_i of a_i such that $d_i = p_i^{e_i}$ is the greatest power of p_i dividing a_i, however d_i does not divide $|P(n_i)|$. If two of the d_i's are powers of the same prime, then eliminate the one with the larger exponent. (In this way, each of $a_1, a_2,\ldots, a_m$ is still divisible by one of the remaining d_i's.)

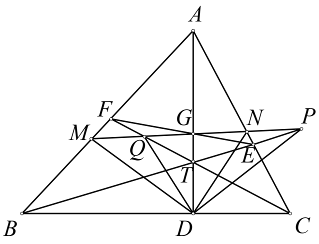
By the Chinese remainder theorem, there exist integers n such that $n\equiv n_i \pmod{d_i}$ for the remaining d_i's. Now $P(n)–P(n_i)$ is divisible by $n–n_i$, which is divisible by d_i. Since $P(n_i)$ is is not divisible by d_i. So $P(n)$ is not divisible by any d_i's, contradicting $P(n)$ is divisible by at least one of the positive integers $a_1, a_2,\ldots, a_m$, hence also divisible by at least one d_i.

Problem 439. In acute triangle ABC, T is a point on the altitude AD (with D on side BC). Lines BT and AC intersect at E, lines CT and AB intersect at F, lines EF and AD intersect at G. A line ℓ passing through G intersects side AB, side AC, line BT, line CT at M, N, P, Q respectively.

Prove that $\angle MDQ = \angle NDP$.

Solution. **William FUNG** and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School)**.**

Set the origin at D and A, B, C at $(0,a)$, $(b,0)$, $(c,0)$ respectively.



Let T be at $(0,1)$. The equations of the lines BT, CT, AB, AC are

$$y = -(x/b) + 1,\quad y = -(x/c) + 1,$$

$$y = -(ax/b) + a,\quad y = -(ax/c) + a$$

respectively. Since $E = BT \cap AC$ and $F = CT \cap AB$, we can solve the equations of the lines to get

$$E = \left(\frac{(a-1)bc}{ab-c}, \frac{a(b-c)}{ab-c}\right)$$

and $$F = \left(\frac{(a-1)cb}{ac-b}, \frac{a(c-b)}{ac-b}\right).$$

From the y-intercept of line EF, we get $G=(0, 2a/(a+1))$. Let the equation of ℓ be $y=mx+2a/(a+1)$. Then $M = \ell \cap AB$ is at

$$\left(\frac{a(a-1)b}{(a+mb)(a+1)}, \frac{a(2a+(a+1)mb)}{(a+mb)(a+1)}\right).$$

Using role symmetry of B and C, we can replace b by c in the coordinates of M to get coordinates of N. Similarly, $P = \ell \cap BT$ is at

$$\left(\frac{-(a-1)b}{(a+mb)(a+1)}, \frac{2a+(a+1)mb}{(a+mb)(a+1)}\right).$$

The coordinates of Q can be found by replacing b by c in the coordinates of P.

Since D is the origin, the slopes of lines DM and DP can be found by taking the y-coordinates of M and P dividing by their respective x-coordinates, which turn out to be the negative of each other! So lines DM and DP are symmetric with respect to the y-axis! Similarly, lines DN and DQ are symmetric with respect to the y-axis. Therefore, $\angle MDQ = \angle NDP$.

Comments: There is a pure geometry solution using a number of equations from applying Menelaus' theorem to different triangles. There is also a solution using harmonic division and cross-ratios from projective geometry.

Other commended solvers: **Georgios BATZOLIS** (Mandoulides High School, Thessaloniki, Greece), **Andrea FANCHINI** (Cantu, Italy), **T. W. LEE** (Alumni of New Method College), **SP47** (Hanoi, Vietnam), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania).

Problem 440. There are n schools in a city. The i-th school will send C_i students to watch a performance at a field. It is <u>known</u> that $0 \le C_i \le 39$ for i=1, 2, …, n and $C_1+C_2+\cdots+C_n=1990$. The seats will be put in a rectangle arrangement with each row having 199 seats. Determine the least number of rows needed to satisfy the condition that all students from the same school must sit in the same row for all possibilities of the known conditions above.

Solution. **Adnan ALI** (9th Grade, Atomic Energy Central School 4 (AECS4), Mumbai, India), **Jerry AUMAN** and **Jon GLIMMS** (Vancouver, Canada).

Let k be the minimal number of rows needed. For m =1, 2,…, k, let there be a_m students in row m. If there are no more than 160 students in some row, then since each school sends at most 39 students, we can put in students from one more school in that row. So we may assume $a_m \ge 161$. Now

$$1990 = a_1 + a_2 + \cdots + a_k \ge 161k,$$

which implies $k \le 12$.

Next, we show 11 rows may not be enough. Suppose there are n = 80 schools with C_i = 25 for i = 1, 2, …, 79 and C_{80} = 15. This totals to 1990 students. Then there can only be one row with $25\times7+15 = 190$ students and the other 10 rows with $25\times7=175$ students. This only totals to 1940 students.

So the least number of rows needed to satisfy the condition that all students from the same school must sit in the same row for all possibilities of the known conditions is 12.

Other commended solvers: **T. W. LEE** (Alumni of New Method College) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Olympiad Corner

(*Continued from page 1*)

Problem 5. Find the values of x such that the following inequality holds

$$\min\{\sin x, \cos x\} < \min\{1-\sin x, 1-\cos x\}.$$

Problem 6. Find all pairs of prime numbers p and q that satisfy the following equation

$$3p^q - 2q^{p-1} = 19.$$

Problem 7. Is it possible to choose 24 points in the space, such that no three of them lie on the same line and choose 2013 planes in a way that each plane passes through at least 3 of the chosen points and each triple of points belongs to at least one of the chosen planes?

Problem 8. Let M be the midpoint of the internal bisector AD of $\triangle ABC$. Circle ω_1 with diameter AC intersects BM at E and circle ω_2 with diameter AB intersects CM at F. Show that B, E, F, C belong to the same circle.

Using Tangent Lines …

(*Continued from page 2*)

<u>Exercises 2.</u> Let a, b and c be non-negative real numbers. Prove that

$$\frac{a}{b^2+c^2+d^2} + \frac{b}{c^2+d^2+a^2} + \frac{c}{d^2+a^2+b^2} + \frac{d}{a^2+b^2+c^2} \ge \frac{3\sqrt{3}}{2}\cdot\frac{1}{\sqrt{a^2+b^2+c^2+d^2}}.$$

<u>Exercise 3.</u> Let a, b and c be positive real numbers. Determine the minimal value of

$$\frac{3a}{b+c} + \frac{4b}{c+a} + \frac{5c}{a+b}.$$

<u>Exercise 4.</u> Let a, b and c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$(a^7 - a^4 + 3)(b^5 - b^2 + 3)(c^4 - c + 3) \ge 27.$$

<u>References</u>

[1] Chetkovski, Z., <u>Inequalities. Theorems, Techniques and Problems.</u> Springer Verlag, Berlin Heidelberg, (2012).

[2] Pham Kim Hung, <u>Secrets in Inequalities (volume 1).</u> Editura Gil, Zalău, (2007).