Junior problems

J313. Solve in real numbers the system of equations

$$x(y+z-x^3)=y(z+x-y^3)=z(x+y-z^3)=1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J314. Alice was dreaming. In her dream, she thought that primes of the form $3k+1$ are weird. Then she thought it would be interesting to find a sequence of consecutive integers all of which are greater than 1 and which are not divisible by weird primes. She quickly found five consecutive numbers with this property:

$$8=2^3,\quad 9=3^2,\quad 10=2\cdot 5,\quad 11=11,\quad 12=2^2\cdot 3.$$

What is the length of the longest sequence she can find?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J315. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that

$$\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}\geq\sqrt{5}+2.$$

Proposed by Cosmin Pohoata, Columbia University, USA

J316. Solve in prime numbers the equation

$$x^3+y^3+z^3+u^3+v^3+w^3=53353.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J317. In triangle ABC, the angle-bisector of angle A intersects line BC at D and the circumference of triangle ABC at E. The external angle-bisector of angle A intersects line BC at F and the circumference of triangle ABC at G. Prove that $DG\perp EF$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J318. Determine the functions $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(x-y)-xf(y)\leq 1-x$ for all real numbers x and y.

Proposed by Marcel Chirita, Bucharest, Romania

Senior problems

S313. Let a, b, c be nonnegative real numbers such that $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$. Prove that

$$\sqrt{(a+b+1)(c+2)}+\sqrt{(b+c+1)(a+2)}+\sqrt{(c+a+1)(b+2)}\geq 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S314. Let p, q, x, y, z be real numbers satisfying

$$x^2y+y^2z+z^2x=p \quad \text{and} \quad xy^2+yz^2+zx^2=q.$$

Evaluate $(x^3-y^3)(y^3-z^3)(z^3-x^3)$ in terms of p and q.

Proposed by Marcel Chirita, Bucharest, Romania

S315. Consider triangle ABC with inradius r. Let M and M' be two points inside the triangle such that $\angle MAB=\angle M'AC$ and $\angle MBA=\angle M'BC$. Denote by d_a, d_b, d_c and d'_a, d'_b, d'_c the distances from M and M' to the sides BC, CA, AB, respectively. Prove that

$$d_ad_bd_cd'_ad'_bd'_c\leq r^6.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

S316. Circles $C_1(O_1,R_1)$ and $C_2(O_2,R_2)$ intersect in points U and V. Points A_1, A_2, A_3 lie on C_1 and points B_1, B_2, B_3 lie on C_2 such that A_1B_1, A_2B_2, A_3B_3 are passing through U. Denote by M_1, M_2, M_3 the midpoints of A_1B_1, A_2B_2, A_3B_3. Prove that $M_1M_2M_3V$ is a cyclic quadrilateral.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S317. Let ABC be an acute triangle inscribed in a circle of radius 1. Prove that

$$\frac{\tan A}{\tan^3 B}+\frac{\tan B}{\tan^3 C}+\frac{\tan C}{\tan^3 A}\geq 4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S318. Points $A_1, B_1, C_1, D_1, E_1, F_1$ are lying on the sides AB, BC, CD, DE, EF, FA of a convex hexagon $ABCDEF$ such that

$$\frac{AA_1}{AB}=\frac{AF_1}{AF}=\frac{CC_1}{CD}=\frac{CB_1}{BC}=\frac{ED_1}{ED}=\frac{EE_1}{EF}=\lambda.$$

Prove that A_1D_1, B_1E_1, C_1F_1 are concurrent if and only if $\frac{[ACE]}{[BDF]}=\left(\frac{\lambda}{1-\lambda}\right)^2$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Undergraduate problems

U313. Let X and Y be nonnegative definite Hermitian matrices such that $X - Y$ is also nonnegative definite. Prove that $\mathrm{tr}(X^2) \geq \mathrm{tr}(Y^2)$

Proposed by Radouan Boukharfane, Sidislimane, Morocco

U314. Prove that for any positive integer k,

$$\lim_{n\to\infty}\left(\frac{1+\sqrt[n]{2}+\cdots+\sqrt[n]{k}}{k}\right)^n > \frac{k}{e},$$

where e is Euler constant.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U315. Let X and Y be complex matrices of the same order with $XY^2 - Y^2X = Y$. Prove that Y is nilpotent.

Proposed by Radouan Boukharfane, Sidislimane, Morocco

U316. The sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. For any nonnegative integer m, let $v_2(m)$ be the highest power of 2 dividing m. Prove that there is exactly one positive real number μ such that the equation

$$v_2(\lfloor \mu n \rfloor !) = v_2(F_1 \ldots F_n)$$

is satisfied by infinitely many positive integers n. Find μ.

Proposed by Albert Stadler, Herrliberg, Switzerland

U317. For any positive integers s, t, p, prove that there is a number $M(s,t,p)$ such that every graph G with a matching of size at least $M(s,t,p)$ contains either a complete graph K_s, an induced copy of the complete bipartite graph $K_{t,t}$, or a matching of size p as an induced subgraph. Does the result remain true if we replace the word "matching" by "path"?

Proposed by Cosmin Pohoata, Columbia University, USA

U318. Determine all possible values of $\sum_{k=1}^{\infty} \frac{(-1)^{q(k)}}{k^2}$, where $q(x)$ is a quadratic polynomial that assumes only integer values at integer places.

Proposed by Albert Stadler, Herrliberg, Switzerland

Olympiad problems

O313. Find all positive integers n for which there are positive integers $a_0, a_1, \ldots, a_n$ such that $a_0 + a_1 + \cdots + a_n = 5(n-1)$ and

$$\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} = 2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O314. Prove that every polynomial $p(x)$ with integer coefficients can be represented as a sum of cubes of several polynomials that return integer values for any integer x.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O315. Let a, b, c be positive real numbers. Prove that

$$(a^3 + 3b^2 + 5)(b^3 + 3c^2 + 5)(c^3 + 3a^2 + 5) \geq 27(a + b + c)^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O316. Prove that for all integers $k \geq 2$ there exists a power of 2 such that at least half of the last k digits are nines. For example, for $k = 2$ and $k = 3$ we have $2^{12} = \ldots 96$ and $2^{53} = \ldots 992$.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

O317. Twelve scientists met at a math conference. It is known that every two scientists have a common friend among the rest of the people. Prove that there is a scientist who knows at least five people from the attendees of the conference.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O318. Find all polynomials $f \in \mathbb{Z}[X]$ with the property that for any distinct primes p and q, $f(p)$ and $f(q)$ are relatively prime.

Proposed by Marius Cavachi, Constanta, Romania