Junior problems

J283. Let a, b, c be positive real numbers. Prove that

$$\frac{2a+1}{b+c}+\frac{2b+1}{c+a}+\frac{2c+1}{a+b}\geq 3+\frac{9}{2(a+b+c)}.$$

Proposed by Zarif Ibragimov , SamSU, Samarkand, Uzbekistan

Solution by Sayan Das, Indian Statistical Institute, Kolkata, India

First, recall Nesbitt's inequality that

$$\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}\geq 3.$$

Second, note that Cauchy-Schwarz yields

$$\frac{1}{a+b}+\frac{1}{c+a}+\frac{1}{b+c}\geq\frac{(1+1+1)^2}{(a+b)+(c+a)+(b+c)}=\frac{9}{2(a+b+c)},$$

thus, by adding the two inequalities, we get

$$\frac{2a+1}{b+c}+\frac{2b+1}{c+a}+\frac{2c+1}{a+b}\geq 3+\frac{9}{2(a+b+c)},$$

which is precisely what we wanted to show. Equality holds clearly when $a=b=c$.

Also solved by Michel Faleiros Martins, Instituto Tecnológico de Aeronáutica, Brazil; Daniel Lasaosa, Pamplona, Navarra, Spain; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Arkady Alt , San Jose ,California, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Corneliu Mănescu-Avram, "Henri Mathias Berthelot" Secondary School, Ploieşti, Romania; Daniel Văcaru, Piteşti, Romania; Ercole Suppa, Teramo, Italy; Mathematical Group "Galaktika shqiptare", Albania; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Julien Portier, Francois 1er, France; Titu Zvonaru, Comănesti and Neculai Stanciu, Buzău, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sayak Mukherjee, Kolkata, India; Polyahedra, Polk State College, FL, USA; Aymane Maysae; Ioan Viorel Codreanu, Satulung, Maramures, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Stanescu Florin Serban Cioculescu Gaesti school, Dambovita, Romania.

J284. Find the greatest integer that cannot be written as a sum of distinct prime numbers.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

The problem statement may be considered unclear when it comes to expressing primes as a sum of distinct primes. Do we consider that a prime p is itself a sum of (trivially since there is only one) distinct primes? We will show that, if the answer to this question is yes, then the answer to the proposed problem is 6, but if the answer is no and we require the integer to be the sum of at least two primes, all distinct, then the answer to the proposed problem is 11.

Assume that n can be written as a sum of distinct primes. Then, there is at least one way to write $n = a + b$, where a, b are positive integers, such that:

- a, b are distinct primes, or
- a is a prime, and b is a non-prime which can be written as a sum of distinct primes, in which a does not appear, or
- a, b are non-primes, each one of which can be written as a sum of distinct primes, such that no prime appears in both sums.

Using this observation, we can find by brute force all possible ways to write the first few positive integers as a sum of distinct primes, finding

$$5 = 3 + 2, \qquad 7 = 5 + 2, \qquad 8 = 5 + 3, \qquad 9 = 7 + 2, \qquad 10 = 7 + 3 = 5 + 3 + 2,$$

Note that $1, 2, 3, 4$ are less than the sum of the two lowest primes $2 + 3 = 5$, that 6 is a non-prime which cannot be written as a sum of distinct primes, and that 11 cannot be written as a sum of distinct primes because $2, 3$ appear in the only possible such sums for $9 = 11 - 2$ and $8 = 11 - 3$, respectively. We can write a few more integers as a sum of distinct primes:

$$12 = 7 + 5 = 7 + 3 + 2, \qquad 13 = 11 + 2, \qquad 14 = 11 + 3 = 7 + 5 + 2,$$

$$15 = 13 + 2 = 7 + 5 + 3, \qquad 16 = 13 + 3 = 11 + 5 = 11 + 3 + 2, \qquad 17 = 7 + 5 + 3 + 2,$$

$$18 = 13 + 5 = 13 + 3 + 2 = 11 + 7 = 11 + 5 + 2, \qquad 19 = 17 + 2 = 11 + 5 + 3,$$

$$20 = 18 + 2 = 17 + 3 = 13 + 7 = 13 + 5 + 2 = 11 + 7 + 2,$$

$$21 = 19 + 2 = 13 + 5 + 3 = 11 + 7 + 3 = 11 + 5 + 3 + 2.$$

Assume now that we can write all integers larger than 6 and less than 22 as either a prime (in the case of 11) or as a sum of distinct primes (in all other cases). We will show by induction on $n \geq 22$ that it is possible to write n as a sum of (more than one) distinct primes.

An improved version of Bertrand's Postulate due to Andy Loo states that for all integer $k \geq 2$, there is a prime p such that $3k < p < 4k$. For $n \geq 22$, take the smallest integer k such that $3k + 1 \geq \frac{n}{2}$, or $3k + 1 \leq \frac{n+5}{2}$, yielding $4k - 1 \leq \frac{2n}{3} + 1 = n - \frac{n}{3} + 1 < n - 6$. It follows that there is a prime p such that $\frac{n}{2} \leq p < n - 6$, and $\frac{n}{2} \geq n - p \geq 7$. By hypothesis of induction, $n - p$ is either prime 11, or can be written as a sum of distinct primes, clearly all less than $\frac{n}{2}$, hence different from p. It follows that, expressing $n - p$ as either a prime or a sum of distinct primes, and adding p, yields n expressed as a sum of distinct primes. The only exception to this construction is when $n = 22$ and $p = 11$, but in this case it suffices to take $p = 13$, and $22 = 13 + 7 + 2$.

It follows that the largest prime that cannot be written as a sum of more than one prime, all distinct, is 11, while the largest non-prime which cannot be written as a sum of distinct primes is 6.

Note: An argument very similar to the previous one can be carried out using Bertrand's Postulate, ie for every integer $k \geq 4$, there exists a prime p such that $k < p < 2k - 2$, which fails only for odd integers $n = 2m + 1$ such that $m - 1$ and $2m - 5$ are primes, and no other prime exists between them. But any refinement of Bertrand's Postulate rules out this case for sufficiently large m, and the solution is completed by checking a few remaining cases.

Also solved by G.R.A.20 Problem Solving Group, Roma, Italy; Aymane Maysae; Polyahedra, Polk State College, FL, USA.

J285. Let a, b, c be the sidelengths of a triangle. Prove that

$$8 < \frac{(a+b+c)(2ab+2bc+2ca-a^2-b^2-c^2)}{abc} \leq 9.$$

Proposed by Adithya Ganesh, Plano, USA

Solution by Ercole Suppa, Teramo, Italy

By using the Ravi transformation $a = y+z$, $b = z+x$, $c = x+y$, after some algebra, the inequality rewrites as

$$8 < \frac{8(x+y+z)(xy+yz+zx)}{(y+z)(x+z)(x+y)} \leq 9 \qquad (*)$$

For the right-hand side inequality observe that

$$(x+y+z)(xy+xz+yz) - (x+y)(x+z)(y+z) = xyz > 0$$

because $x, y, z > 0$.

The left-hand side inequality becomes

$$9(x+y)(x+z)(y+z) - 8(x+y+z)(xy+xz+yz) \geq 0,$$

which expands to

$$x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 - 6xyz \geq 0$$

which is true by AM-GM inequality, because

$$x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 \geq 6\sqrt[6]{x^6y^6z^6} = 6xyz$$

The equality holds iff $x = y = z$ i.e. iff $a = b = c$.

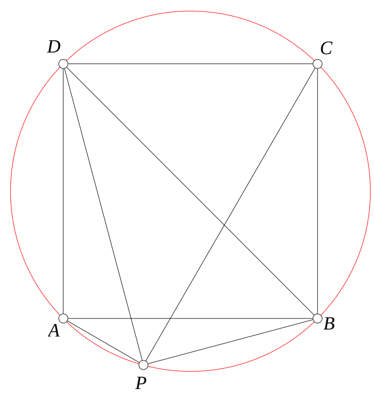
Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Polyahedra, Polk State College, FL, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Stanescu Florin Serban Cioculescu Gaesti school, Dambovita, Romania; Arkady Alt, San Jose, California, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Michel Faleiros Martins, Instituto Tecnológico de Aeronáutica, Brazil; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sayan Das, Indian Statistical Institute, Kolkata, India; Scott H. Brown, Auburn University Montgomery, AL, USA; Zarif Ibragimov, SamSU, Samarkand, Uzbekistan.

J286. Let $ABCD$ be a square inscribed in a circle. If P is a point on the arc AB, find the maximum of the expression

$$\frac{PC \cdot PD}{PA \cdot PB}.$$

Proposed by Panagiote Ligouras, Noci, Italy

Solution by Ercole Suppa, Teramo, Italy



Let $AB = a$, $PA = x$, $PB = y$, $PC = z$, $PD = w$. By applying Ptolemy's theorem on quadrilaterals $APBC$ and $APBD$ we get

$$BC \cdot PA + CA \cdot PB = AB \cdot PC$$

i.e.

$$ax + a\sqrt{2}y = az \quad \Rightarrow \quad z = x + \sqrt{2}y \qquad (1)$$

and

$$BD \cdot PA + AD \cdot PB = AB \cdot PD$$

i.e.

$$a\sqrt{2} + ay = aw \quad \Rightarrow \quad w = \sqrt{2}x + y \qquad (2)$$

By using (1), (2) and the well known inequality $x^2 + y^2 \geq 2xy$ we obtain

$$\frac{PC \cdot PD}{PA \cdot PB} = \frac{\left(x + \sqrt{2}y\right)\left(\sqrt{2}x + y\right)}{xy} = \frac{3xy + \sqrt{2}\left(x^2 + y^2\right)}{xy} \geq 3 + 2\sqrt{2}$$

and the equality holds if and only if $x = y$.

Therefore the minimum of the expression is $3 + 2\sqrt{2}$ and is attained when P is the midpoint the arc AB.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Polyahedra, Polk State College, FL, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt , San Jose ,California, USA.

J287. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers in the interval $\left(0, \frac{1}{n}\right)$. Prove that

$$\log_{1-a_1}(1 - na_2) + \log_{1-a_2}(1 - na_3) + \cdots + \log_{1-a_n}(1 - na_1) \geq n^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

We begin with the following

Lemma: For every $x \in \left(0, \frac{1}{n}\right)$, we have $\frac{\ln(1-nx)}{\ln(1-x)} \geq n$, where ln denotes the natural logarithm, and equality holds iff $n = 1$.

Proof: Since $0 < 1 - nx < 1 - x < 1$ for all $x \in \left(0, \frac{1}{n}\right)$, the natural logarithms exist and are negative reals, or the proposed result is equivalent to $\ln(1 - nx) \leq n \ln(1 - x)$, or to $(1 - nx) \leq (1 - x)^n$, clearly true and with equality for all $x \in \left(0, \frac{1}{n}\right)$ when $n = 1$. If the result is true for n, note that

$$(1 - x)^{n+1} \geq (1 - x)(1 - nx) = 1 - (n + 1)x + nx^2 > 1 - (n + 1)x,$$

or the result is true for $n + 1$, with strict inequality. The Lemma follows.

Since $\log_{1-a_1}(1 - na_2) = \frac{\ln(1-na_2)}{\ln(1-a_1)}$, and similarly for the rest of the terms, note that by the AM-GM inequality, it would suffice to show that

$$\sqrt[n]{\frac{\ln(1 - na_2)\ln(1 - na_3)\ldots\ln(1 - na_1)}{\ln(1 - a_1)\ln(1 - a_2)\ldots\ln(1 - a_n)}} \geq n,$$

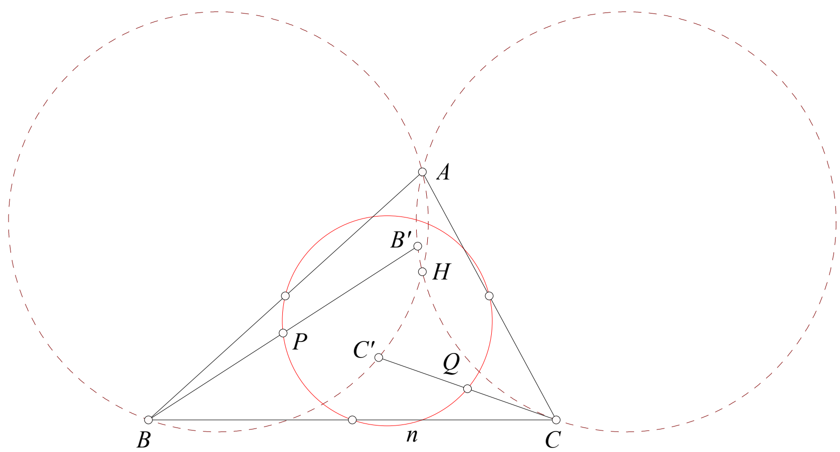
clearly true since the radicand may be written as the product of n terms of the form $\frac{\ln(1-na_i)}{\ln(1-a_i)}$, each one of them larger than or equal to n by the Lemma. The conclusion follows, equality holds iff $n = 1$.

Also solved by Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Polyahedra, Polk State College, FL, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Julien Portier, Francois 1er, France; Mathematical Group "Galaktika shqiptare", Albania; Arkady Alt , San Jose, California, USA.

J288. Four points are given in the plane such that no three of them are collinear. These four points form six segments. Prove that if five of their midpoints lie on a single circle, then the sixth midpoint lies on this circle too.

Proposed by Michal Rolinek and Josef Tkadlec, Czech Republic

Solution by Polyahedra, Polk State College, FL, USA



Let A, B, C be three of the four given points. Then the midpoints of AB, BC, CA lie on the nine-point circle n of $\triangle ABC$. Now let H be the orthocenter of $\triangle ABC$. As the midpoint P of any line segment BB' moves on n, the locus of B' is the circumcircle of $\triangle AHC$. Likewise, as the midpoint Q of any line segment CC' moves on n, the locus of C' is the circumcircle of $\triangle AHB$. Hence, to satisfy the given condition, H must be the fourth one of the four given points. Therefore, the midpoint of AH is on n as well.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Saturnino Campo Ruiz, Salamanca, Spain.

Senior problems

S283. Let a, b, c be positive real numbers greater than or equal to 1, such that

$$5\left(a^2-4a+5\right)\left(b^2-4b+5\right)\left(c^2-4c+5\right) \leq a+b+c-1.$$

Prove that

$$\left(a^2+1\right)\left(b^2+1\right)\left(c^2+1\right) \geq (a+b+c-1)^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Zarif Ibragimov, SamSU, Samarkand, Uzbekistan
Let $a = x+1, b = y+1, c = z+1$. We have $x, y, z \geq 0$ from $a, b, c \geq 1$. Then (1) is equivalent to

$$5(x^2-2x+2)(y^2-2y+2)(z^2-2z+2) \leq x+y+z+2 \tag{1}$$

and we have to prove that

$$(x^2+2x+2)(y^2+2y+2)(z^2+2z+2) \geq (x+y+z+2)^3.$$

We will use proof by contradiction. Let

$$(x^2+2x+2)(y^2+2y+2)(z^2+2z+2) < (x+y+z+2)^3 \tag{2}$$

Then from (2) and (3) we have

$$5((x^2+2)^2-4x^2)((y^2+2)^2-4y^2)((z^2+2)^2-4z^2) < (x+y+z+2)^4$$

which is equivalent to

$$5(x^4+4)(y^4+4)(z^4+4) < (x+y+z+2)^4 \tag{3}$$

From Cauchy-Schwarz inequality we have

$$(x^4+1+1+2)(1+y^4+1+2)(1+1+z^4+2)(1+1+1+2) \geq (x+y+z+2)^4$$

or

$$5(x^4+4)(y^4+4)(z^4+4) \geq (x+y+z+2)^4 \tag{4}$$

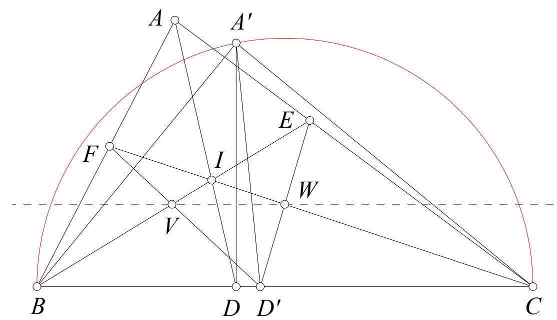
From (4) and (5) we have a contradiction. So, (3) is false and we are done.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; BTR Problem Solving Group; Li Zhou, Polk State College, FL, USA.

S284. Let I be the incenter of triangle ABC and let D, E, F be the feet of the angle bisectors. The perpendicular dropped from D onto BC intersects the semicircle with diameter BC in A', where A and A' lie on the same side of line BC. Then angle bisector of $\angle BA'C$ intersects BC at D'. Denote by $V = BE \cap FD'$ and $W = CF \cap ED'$. Prove that VW is parallel to BC.

Proposed by Ercole Suppa, Teramo, Italy

Solution by Li Zhou, Polk State College, FL, USA



Applying Menelaus' theorem in $\triangle IBC$, to transversals FD' and ED' respectively, we get

$$\frac{IV}{VB} \cdot \frac{BD'}{D'C} \cdot \frac{CF}{FI} = -1, \quad \frac{IW}{WC} \cdot \frac{CD'}{D'B} \cdot \frac{BE}{EI} = -1.$$

Therefore, it suffices to show that

$$\frac{BD'}{D'C} \cdot \frac{CF}{FI} = \frac{CD'}{D'B} \cdot \frac{BE}{EI}.$$

Let $a = BC$, $b = CA$, $c = AB$, and use $[\cdot]$ to denote area. Then

$$\frac{CF}{IF} = \frac{[ABC]}{[ABI]} = \frac{a+b+c}{c}, \quad \frac{BE}{IE} = \frac{[ABC]}{[AIC]} = \frac{a+b+c}{b}.$$

Hence,

$$\left(\frac{BD'}{D'C}\right)^2 = \left(\frac{A'B}{A'C}\right)^2 = \left(\frac{BD}{A'D}\right)^2 = \frac{BD^2}{BD \cdot DC} = \frac{BD}{DC} = \frac{c}{b} = \frac{BE}{EI} \cdot \frac{FI}{CF},$$

completing the proof.

Also solved by Titu Zvonaru and Neculai Stanciu, Romania; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Daniel Lasaosa, Pamplona, Navarra, Spain; BTR Problem Solving Group.

S285. Let be a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\frac{a}{b^2+c^2+2} + \frac{b}{c^2+a^2+2} + \frac{c}{a^2+b^2+2} \le \frac{1}{8}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

Solution by Michel Faleiros Martins, Instituto Tecnológico de Aeronáutica, Brazil First, we homogenize the inequality. Using the condition given, we substitute the denominators of the LHS by

$$a^2 + b^2 + 2 = a^2 + b^2 + 2ab + 2bc + 2ca$$

and its analogous. So, eliminating the denominators and rearranging terms we find

$$\begin{aligned} &2\sum_{sym} a^6b^2 + 9\sum_{sym} a^5b^3 + 7\sum_{sym} a^4b^4 + 13\sum_{sym} a^4b^3c \\ &\quad \ge 2\sum_{sym} a^6bc + 3\sum_{sym} a^5b^2c + 17\sum_{sym} a^3b^3c^2 + 9\sum_{sym} a^4b^2c^2 \end{aligned}$$

Now, it is immediate to use Muirhead's inequality. We have

$$[6,2,0] \succ [6,1,1],\ [5,3,0] \succ [5,2,1],\ [4,4,0] \succ [3,3,2],\ [4,3,1] \succ [4,2,2]$$

Thus, we obtain

$$6\sum_{sym} a^5b^3 + 4\sum_{sym} a^4b^3c \ge 10\sum_{sym} a^3b^3c^2$$

and since

$$[5,3,0], [4,3,1] \succ [3,3,2]$$

the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Li Zhou, Polk State College, FL, USA; Sayan Das, Indian Statistical Institute, Kolkata, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, Buzău, Romania; Arkady Alt, San Jose, California, USA; BTR Problem Solving Group; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

S286. Let ABC be a triangle and let P be a point in its interior. Lines AP, BP, CP intersect BC, CA, AB at A_1, B_1, C_1, respectively. Prove that the circumcenters of triangles $APB_1, APC_1, BPC_1, BPA_1$ are concyclic if and only if lines A_1B_1 and AB are parallel.

Proposed by Mher Mnatsakanyan, Armenia

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

Denote by $O_{BA_1}, O_{AB_1}, O_{AC_1}, O_{BC_1}$ the respective circumcenters of triangles $PBA_1, PAB_1, PAC_1, PBC_1$, denote by M_A, M_B the midpoints of PA, PB, and let Q be the point where the perpendicular bisectors of PA, PB meet. Note that O_{BA_1}, O_{BC_1} are on the perpendicular bisector of PB, and O_{AB_1}, O_{AC_1} are on the perpendicular bisector of PA. Moreover, PM_AQM_B is clearly cyclic with diameter PQ, and $\angle PQM_A + \angle PQM_B = \angle PAB + \angle PBA$, where since $\frac{PM_A}{PA} = \frac{PM_B}{PB} = \frac{1}{2}$, it follows that $\angle PQM_A = \angle PBA$ and $\angle PQM_B = \angle PAB$, or $PM_A = PQ \sin \angle PBA$, $QM_A = PQ \cos \angle PBA$, and similarly for PM_B and QM_B.

Using the well known result that the signed distance from circumcenter to one side of a triangle equals the circumradius times the cosine of the opposite angle, and the Sine Law, we have $QM_AO_{AB_1} = \frac{PA}{2\tan \angle PB_1A}$. Then, using the Sine and Cosine Laws, we find

$$M_AO_{AB_1} = \frac{PA \cos \angle BB_1A}{2\sin \angle BB_1A} = PA\frac{2AB_1 \cdot BB_1 \cos \angle BB_1A}{4AB \cdot BB_1 \sin \angle ABP} =$$

$$= PQ\frac{AB_1{}^2 + BB_1{}^2 - AB^2}{2AB \cdot BB_1} = \frac{PQ \cdot BB_1}{AB} - PQ \cos \angle ABP,$$

or

$$QO_{AB_1} = QM_A + M_AO_{AB_1} = \frac{PQ \cdot BB_1}{AB}, \qquad QO_{BA_1} = \frac{PQ \cdot AA_1}{AB},$$

where the second equality is proved analogously. Similarly, we have $M_AO_{AC_1} = \frac{PA}{2\tan \angle AC_1P}$, or using again the Sine and Cosine Laws,

$$M_AO_{AC_1} = \frac{AC_1{}^2 + PC_1{}^2 - AP^2}{4AC_1 \sin \angle PAB} = \frac{AC_1 - AP \cos \angle PAB}{2\sin \angle PAB},$$

$$QO_{AC_1} = QM_A - M_AO_{AC_1} = PA\frac{\sin \angle APB \sin \angle AC_1P - \sin \angle APC_1 \sin \angle PBA}{2\sin \angle PAB \sin \angle PBA \sin \angle AC_1P} =$$

$$= PA\frac{\sin \angle BPC_1}{2\sin \angle PBA \sin \angle AC_1P} = PA\frac{BC_1}{2PC_1 \sin \angle AC_1P},$$

and similarly

$$QO_{BC_1} = PB\frac{AC_1}{2PC_1 \sin \angle BC_1P}.$$

Now, $\angle AC_1P + \angle BC_1P = 180°$, or

$$\frac{QO_{AB_1} \cdot QO_{AC_1}}{QO_{BA_1} \cdot QO_{BC_1}} = \frac{PA}{PB} \cdot \frac{BC_1}{AC_1} \cdot \frac{BB_1}{AA_1},$$

and this last expression is 1 iff the four circumcenters are concyclic. Now, using Menelaus' theorem, we have

$$\frac{PA}{AA_1} \cdot \frac{A_1B}{BC} \cdot \frac{CC_1}{C_1P} = 1, \qquad \frac{PB}{BB_1} \cdot \frac{B_1A}{AC} \cdot \frac{CC_1}{C_1P} = 1,$$

or the four circumcenters are concyclic iff

$$\frac{A_1C}{BC} \cdot \frac{AC}{B_1C} = \frac{BC_1}{C_1A} \cdot \frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} = 1,$$

where we have used Ceva's theorem, and which is clearly equivalent by Thales' theorem to A_1B_1 being paralle to BC. The conclusion follows.

Also solved by BTR Problem Solving Group; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Li Zhou, Polk State College, FL, USA; Shohruh Ibragimov, Lyceum Nr2 under the SamIES, Samarkand, Uzbekistan; Nikolaos Kolliopoulos, National Technical University of Athens, Greece.

S287. Let m and n be positive integers such that $\gcd(m,n)=1$. Prove that

$$\sum_{k=1}^{m}\phi\left(\left\lfloor\frac{nk}{m}\right\rfloor\right)\left\lfloor\frac{m}{k}\right\rfloor=\sum_{k=1}^{n}\phi\left(\left\lfloor\frac{mk}{n}\right\rfloor\right)\left\lfloor\frac{n}{k}\right\rfloor,$$

where ϕ is the Euler totient function.

Proposed by Marius Cavachi, Romania

Solution by the author

Consider points $O(0,0), A(m,0), B(m,n)$ and $C(0,n)$ which are the verticies of a rectangle and the set of lattice points M, which are situated on the sides or inside of the triangle OAB excluding the origin.

Let point $P\in M$ be visible from the origin if there are no lattice points on the segment (O,P). If $P(k,y)$ is visible , as $\frac{y}{k}\leq\frac{n}{m}$ and $(k,y)=1$, it's clear that y can take $\phi(\lfloor\frac{nk}{m}\rfloor)$ values. On the other hand

$$|M\cap(OP)|=\left\lfloor\frac{m}{k}\right\rfloor.$$

Summing up all the visible points we obtain

$$|M|=\frac{(m+1)(n+1)}{2}=\sum_{k=1}^{m}\phi\left(\left\lfloor\frac{nk}{m}\right\rfloor\right)\cdot\left\lfloor\frac{m}{k}\right\rfloor$$

Analogously we prove that the RHS summation equals to $\frac{(m+1)(n+1)}{2}$ and the conclusion follows.

S288. Consider triangle ABC with vertices A, B, C in the counterclockwise order. On the sides AB and AC construct outwards similar rectangles $ABXY$ and $CAZT$ in the clockwise order. Let D be the second point of intersection of the circumcircle of triangle AYZ and ω, the circumcircle of triangle ABC. Denote by $U = DY \cap \omega$ and $V = DZ \cap \omega$ and let W be the midpoint of UV . Prove that AW is perpendicular to YZ.

Proposed by Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

If $D \neq A$, assume wlog that D is in the arc AU that does not contain V (otherwise, we may invert the roles of B, C and all henceforth defined points without altering the problem). Note therefore that

$$\angle AUV = \angle ADV = 180^\circ - \angle ADZ = \angle AYZ,$$

$$\angle AVU = 180^\circ - \angle ADU = \angle ADY = \angle AZY,$$

or AUV, AYZ are similar. Note that if $D = A$, the previous angle equalities are trivially true.

Using the Sine Law, we have

$$\frac{AU}{AV} = \frac{\sin\angle AVU}{\sin\angle AUV} = \frac{\sin\angle AZY}{\sin\angle AYZ} = \frac{AY}{AZ} = \frac{AB}{AC},$$

this last equality because rectangles $ABXY$ and $CAZT$ are similar. Therefore, applying the Sine Law to triangles AUW and AVW, we obtain

$$\sin\angle UAW = \frac{UW\sin\angle AWU}{AU}, \qquad \sin\angle VAW = \frac{VW\sin\angle AWV}{AV},$$

and since $UW = VW$ because W is the midpoint of UV, and $\sin\angle AWU = \sin\angle AWV$ because both angles add up to 180°, we obtain

$$\frac{\sin\angle UAW}{\sin\angle VAW} = \frac{AV}{AU} = \frac{AC}{AB}.$$

Now, $\angle UAV = \angle UDV = \angle YDZ = \angle YAZ$, where since $\angle BAY = \angle CAZ$, we have $\angle YAZ = 180^\circ - A$, or angles $\angle UAW$, $\angle VAW$ add up to $180^\circ - A$, and their sines are in the same proportion as the sines of B, C. We conclude that $\angle UAW = B$, $\angle VAW = C$, or by the Sine Law, and denoting by A' the second point where AW meets ω, we obtain that $AA'CU$ and $AA'BV$ are isosceles trapezoids, with $AA' \parallel BV \parallel CU$, and AA' is the median from A in triangle ABC.

Let A'' be the intersection of line $AW = AA'$ with YZ. Clearly, $\angle YAA'' = 90^\circ - \angle BAA'$, or since $AW \perp YZ$ is equivalent to $\cos\angle YAA'' = \sin\angle AYA'' = \sin\angle AYZ = \sin\angle AUV$, it suffices to show that $\angle BAA' = \angle AUV$, clearly true since $AV = BA'$ because $AA'BV$ is an isosceles trapezoid with parallel sides AA', BV. The conclusion follows.

Also solved by Li Zhou, Polk State College, FL, USA; BTR Problem Solving Group.

U283. Let a, b, c, d be nonnegative real numbers such that $a+b+c+d=2$. Prove that for any positive integer n,

$$\frac{a^n+b^n+c^n+d^n+2^{3-n}}{3} \geq \left(\frac{2-a}{3}\right)^n+\left(\frac{2-b}{3}\right)^n+\left(\frac{2-c}{3}\right)^n+\left(\frac{2-d}{3}\right)^n.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain
Define

$$f(a,b,c,d)=\frac{a^n+b^n+c^n+d^n+2^{3-n}}{3}-\left(\frac{2-a}{3}\right)^n-\left(\frac{2-b}{3}\right)^n-\left(\frac{2-c}{3}\right)^n-\left(\frac{2-d}{3}\right)^n,$$

and let $\mathcal{R}$ be the region in 4-dimensional space which belongs to the hyperplane $a+b+c+d=2$ and satisfies $a, b, c, d \geq 0$. The problem is equivalent to showing that the minimum of f in $\mathcal{R}$ is 0. Since we may exchange any two variables without altering the problem, we may assume wlog that $a \geq b \geq c \geq d$.

If $a \geq b \geq c \geq d > 0$, we are in the interior of $\mathcal{R}$, and $\mathcal{R}$ satisfies $g(a,b,c,d)=2$ for $g(a,b,c,d)=a+b+c+d$, with constant derivative equal to 1 with respect to any of the variables. By Lagrange's multiplier method, it follows that the partial derivatives of f with respect to any two variables must be equal at any extremum inside $\mathcal{R}$. Equivalently, any minimum such that a, b, c, d are all positive, must happen when

$$h(a)=h(b)=h(c)=h(d) \qquad \text{where} \qquad h(x)=x^{n-1}+\left(\frac{2-x}{3}\right)^{n-1}.$$

If $n=1$, then $h(x)=2$ is a constant, and so is $f(a,b,c,d)=0$. If $n=2$, then $h(x)=2\frac{1+x}{3}$, and $h(a)=h(b)=h(c)=h(d)$ iff $a=b=c=d=\frac{1}{2}$, in which case $f(a,b,c,d)=0$. For $n \geq 3$, note that $h''(x)=(n-1)(n-2)\left(x^{n-3}+\frac{1}{9}\left(\frac{2-x}{3}\right)^{n-3}\right)$ is strictly positive, or for every $a > \frac{1}{2}$, there is at most one value $d < \frac{1}{2}$ such that $h(a)=h(d)$. We can then have three possible cases when a, b, c, d are not all equal:

- $a > \frac{1}{2} > b = c = d$, or $2-a=3d$, and

$$3f(a,d,d,d)=(2-3d)^n+8\left(\frac{1}{2}\right)^n-9\left(\frac{2-d}{3}\right)^n,$$

  and by the inequality between arithmetic and power means, the sum of the first two terms is at least $9\left(\frac{2-3d+8\cdot\frac{1}{2}}{9}\right)^n=9\left(\frac{2-d}{3}\right)^n$, with equality iff $2-3d=\frac{1}{2}$, ie iff $d=\frac{1}{2}$, which cannot occur. It follows that $f(a,d,d,d)>0$ when $a>d$.

- $a = b = c > \frac{1}{2} > d$, which is treated analogously and with the same result, ie $f(a, a, a, d) > 0$ when $a > d$.

- $a = b > \frac{1}{2} > c = d$, or $d = 1 - a$, and

$$\frac{1}{2} f(a, a, d, d) = \frac{a^n + 2\left(\frac{1}{2}\right)^n}{3} + \frac{d^n + 2\left(\frac{1}{2}\right)^n}{3} - \left(\frac{1+a}{3}\right)^n - \left(\frac{1+d}{3}\right)^n.$$

  But again by the inequality between power mean and arithmetic mean, it follows that the first term in the RHS is at least $\left(\frac{a+2\frac{1}{2}}{3}\right)^n = \left(\frac{1+a}{3}\right)^n$, and similarly for the second term, with equality in both cases iff $a = d = \frac{1}{2}$, which cannot occur. We conclude that $f(a, a, d, d) > 0$ when $a > d$.

We conclude that the only possible minimum of f inside $\mathcal{R}$ occurs for $a = b = c = d = \frac{1}{2}$, in which case $f = 0$.

The conclusion follows, equality holds iff either $n = 1$ for any a, b, c, d, or $a = b = c = d = \frac{1}{2}$ for any n.

Also solved by Georgios Dasoulas, National Technical University of Athens, Greece; Qafqaz University Problem Solving Group; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Nikolaos Kolliopoulos, National Technical University of Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Li Zhou, Polk State College, FL, USA; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Michel Faleiros Martins, Instituto Tecnológico de Aeronáutica, Brazil.

U284. Let $a_n = \left\{\sqrt{n^2+1}\right\}$ be the sequence of real numbers, where $\{x\}$ denotes the fractional part of x. Find $\lim_{n\to\infty} na_n$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Arkady Alt , San Jose ,California, USA

Since $n^2 < n^2+1 < (n+1)^2$, we have that $\left\lfloor\sqrt{n^2+1}\right\rfloor = n$; hence

$$a_n = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n}.$$

This gives

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}+n} = \frac{1}{2}.$$

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Pamplona, Navarra, Spain; Stanescu Florin Serban Cioculescu Gaesti school, Dambovita, Romania; Li Zhou, Polk State College, FL, USA; Nikolaos Kolliopoulos, National Technical University of Athens, Greece; Konstantinos Tsouvalas University of Athens, Greece; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Qafqaz University Problem Solving Group; Corneliu Mănescu-Avram, "Henri Mathias Berthelot" Secondary School, Ploieşti, Romania; Daniel Vacaru, Colegiul Economic Maria Teiuleanu, Pitesti; Daniele Di Tullio, Università di Roma "Tor Vergata", Roma, Italy; David Rose, Polk State College, FL, USA; Mathematical Group "Galaktika shqiptare", Albania; Michel Faleiros Martins, Instituto Tecnológico de Aeronáutica, Brazil; Michelle Andersen, College at Brockport, SUNY; Nikolaos Zarifis, National Technical University of Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sayan Das, Indian Statistical Institute, Kolkata, India; Zarif Ibragimov, SamSU, Samarkand, Uzbekistan; Harun Immanuel, ITS Surabaya; Titouan Morvan, Lycée Millet, France.

U285. Let $(a_n)_{n>0}$ be the sequence defined by

$$a_n = \frac{e^{\frac{1}{n+1}}}{n+1} + \frac{e^{\frac{1}{n+2}}}{n+2} + \cdots + \frac{e^{\frac{1}{2n}}}{2n}.$$

Prove that sequence $(a_n)_{n>0}$ is decreasing and find its limit.

Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Konstantinos Tsouvalas, University of Athens, Greece

Initially, $\lim_{x\to 0^+} \frac{e^x-1}{x} = 1$ hence there exists $M > 0$:

$$e^x \leqslant 1 + 2x, \quad x < M$$

There exists $m \in \mathbb{N}$ such that $\frac{1}{m} < M$,and for $n > m$ we have:

$$\sum_{k=0}^{n} \frac{\exp\left(\frac{1}{n+k}\right)}{n+k} \leqslant \sum_{k=0}^{n} \frac{1}{n+k} + \sum_{k=n}^{2n} \frac{2}{k^2}.$$

From, the obvious inequality: $e^x > x+1, x > 0$ we have:

$$\sum_{k=0}^{n} \frac{\exp\left(\frac{1}{n+k}\right)}{n+k} \geqslant \sum_{k=0}^{n} \frac{1}{n+k} + \sum_{k=n}^{2n} \frac{1}{k^2}$$

Since the series $\sum_n n^{-2}$ is convergent, we have

$$\lim_{n\to\infty} \sum_{k=n}^{2n} \frac{1}{k^2} = 0$$

Moreover,

$$\sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \to \int_0^1 \frac{dx}{1+x} = \ln 2.$$

From, the first inequality we see that

$$\limsup_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\exp\left((n+k)^{-1}\right)}{n+k} \right) \leqslant \ln 2,$$

and from the second:

$$\liminf_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\exp\left((n+k)^{-1}\right)}{n+k} \right) \geqslant \ln 2$$

Finally, it is obvious that:

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\exp\left((n+k)^{-1}\right)}{n+k} \right) = \ln 2$$

Also solved by Qafqaz University Problem Solving Group; Daniel Lasaosa, Pamplona, Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Moubinool Omarjee, Lycée Henri IV, Paris, France; Michel Faleiros Martins, Instituto Tecnológico de Aeronáutica, Brazil; Daniele Di Tullio, Università di Roma "Tor Vergata", Roma, Italy; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Nikolaos Kolliopoulos, National Technical University of Athens, Greece; Li Zhou, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U286. Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers such that

$$(x_{n+1}-x_n)(x_{n+1}x_n-1)\leq 0$$

for all n and $\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=1$. Prove that $(x_n)_{n\geq 1}$ is convergent.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Daniele Di Tullio, Università di Roma "Tor Vergata", Roma, Italy

If $\{x_n\}_{n\in\mathbb{N}^+}$ is eventually increasing or decreasing (not necessarily strictly), the problem is trivial. If there exists some n_0 so that $x_{n_0}=1$, it's simple to notice that for all $n\geq n_0$, we have $x_n=1$, in which case the problem is also trivial.

Now suppose to be in the other cases: for some $n_0\in\mathbb{N}^+$, if $x_{n_0}<1$ then $x_n\geq x_{n_0}\forall n\geq n_0$; else if $x_{n_0}>1$ then $x_n\leq x_{n_0}\forall n\geq n_0$.

Denote $C=\{n\in\mathbb{N}^+ : x_n<1\},\ \ D=\{n\in\mathbb{N}^+ : x_n>1\}$ (note that $\mathbb{N}^+=C\cup D$). Let $\{c_n\}_{n\in\mathbb{N}^+}$ the increasing enumeration of C, $\{d_n\}_{n\in\mathbb{N}^+}$ the increasing enumeration of D. Clearly, $\{x_{c_n}\}_{n\in\mathbb{N}^+}$ is increasing and $\{x_{d_n}\}_{n\in\mathbb{N}^+}$ is decreasing, thus $\exists\lim\limits_{n\to\infty}x_{c_n}=L\leq 1$ and $\exists\lim\limits_{n\to\infty}x_{d_n}=M\geq 1$.

However, we can create a subsequence $\{x_{c_{n_k}}\}_{k\in\mathbb{N}^+}\subset\{x_{c_n}\}_{c\in\mathbb{N}^+}$ such that $\forall k\in\mathbb{N}^+\ \ x_{n_{c_k}+1}\in\{x_{d_n}\}_{n\in\mathbb{N}^+}$; we then have

$$\lim_{k\to\infty}\frac{x_{c_k+1}}{x_{c_k}}=\frac{M}{L}.$$

But by hypothesis, $\dfrac{M}{L}=1$, i.e. $L=M$. Hence, it follows that

$$\lim_{n\to\infty}x_n=L=M$$

. This completes the proof.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Konstantinos Tsouvalas University of Athens, Greece; Nikolaos Kolliopoulos, National Technical University of Athens, Greece; Li Zhou, Polk State College, FL, USA.

U287. Let $u, v : [0, +\infty) \to (0, +\infty)$ be differentiable functions such that $u' = v^v$ and $v' = u^u$. Prove that $lim_{x\to\infty}(u(x) - v(x)) = 0$.

Proposed by Aaron Doman, University of California, Berkeley , USA

Solution by Matteo Fiacchi and Tommaso Cornelis Rosati, Università di Roma "Tor Vergata", Roma, Italy

We divide the proof into three steps.

i) $\exists n_0 \in \mathbb{N}$ such that $u(x) > \frac{1}{e}$, $v(x) > \frac{1}{e}$ for $x \geq n_0$.

Such n_0 exists because by the Mean Value Theorem, $u(x) = u(0) + u'(t)x$, for some $t \in (0, x)$. Since $u'(t) = v(t)^{v(t)} \geq \exp(-\exp(-1)) > 0$, $\forall t \in \mathbb{R}$, it follows that as $x \to +\infty$,

$$u(x) \geq u(0) + \exp(-\exp(-1))x \to +\infty.$$

In a similar way, we have that $\lim_{x\to+\infty} v(x)$.

ii) $\forall n \geq n_0$, $\exists x_n \geq n$ such that $u(x_n) = v(x_n)$.

If $u(n) = v(n)$ then let $x_n = n$. Otherwise, let us assume $u(n) > v(n)$. If $u(x) > v(x)$, $\forall x \in [n, \infty)$ then $d(x) > 0$, by i), $d' = v^v - u^u < 0$ (because x^x is strictly increasing for $x \geq \frac{1}{e}$), and $d'' = u^u v^v \ln(\frac{v}{u}) < 0$ where $d(x) := u(x) - v(x)$. This means that the function d is concave and

$$d(x) \leq d(n) + d'(n)(x - n) < 0, \quad \forall x > n - \frac{d(n)}{d'(n)},$$

which contradicts $d(x) > 0$. Hence there exists x_n such that $d(x_n) = 0$. Finally we can rearrange the sequence so that $x_n < x_{n+1}$ without losing the properties we imposed.

iii) $u(x) = v(x), \forall x \geq x_{n_0}$.

Let $n \geq n_0$ then, by ii), $d(x_n) = d(x_{n+1}) = 0$. Let M be the maximum value of $|d(x)|$ in $[x_n, x_{n+1}]$. If $M > 0$ then there is a point $t_M \in (x_n, x_{n+1})$ such that $d'(t_M) = 0$. But $d' = 0$ if and only if $u^u = v^v$. Since x^x is strictly increasing for $x \geq \frac{1}{e}$, we find that, by i), $u(t_M) = v(t_M)$ and $M = 0$. This proves that $d(x) = 0$, $\forall x \in [x_n, x_{n+1}]$.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain; Konstantinos Tsouvalas University of Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; N.J. Buitrago A., Universidade de São Paulo, Brazil; Nikolaos Kolliopoulos, National Technical University of Athens, Greece.

U288. Let $f : [a, b] \to \mathbb{R}$ be an integrable function, which has an antidervative and has the property that for each interval $(m, n) \subset [a, b]$ there is an interval (m', n') contained in (m, n) such that $f(x) \geq 0$ for all x in (m', n'). Prove that $f(x) \geq 0$ for all x in $[a, b]$.

Proposed by Mihai Piticari and Sorin Radulescu, Romania

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

The statement of the problem has been corrected. We argue by contradiction supposing that $f(x) < 0$ somewhere. Clearly the points such that $f(x) < 0$, say D_-, are points of discontinuity because by definition the set $\{x \in [a, b] \colon f(x) \geq 0\}$ is everywhere dense. This means that the Lebesgue–measure of D_- is zero and then

$$F(x) = \int_a^x f(y)dy = \int_{[a,x]-D_-} f(y)dy$$

Since

$$F'(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(y)dy, \qquad y \in ([a, b] - D_-) \cap [x, x + h]$$

and $f \geq 0$ in $[a, b]D_-$ and $F'(x) = f(x)$ for any $x \in [a, b]$, it follows $f(x) \geq 0$ for any $x \in [a, b]$ establishing a contradiction. This completes the proof.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain.

Olympiad problems

O283. Prove that for all positive real numbers $x_1, x_2, \ldots, x_n$ the following inequality holds:

$$\sum_{i=1}^{n} \frac{{x_i}^3}{{x_1}^2 + \cdots + {x_{i-1}}^2 + {x_{i+1}}^2 + \cdots + {x_n}^2} \geq \frac{x_1 + \cdots + x_n}{n-1}.$$

Proposed by Mircea Becheanu, Bucharest, Romania

Solution by Sayan Das, Indian Statistical Institute, Kolkata, India By theCauchy-Schwarz inequality we have

$$\sum_{i=1}^{n} \frac{x_i^3}{\sum\limits_{k=1}^{n} x_k^2 - x_i^2} = \sum_{i=1}^{n} \frac{x_i^5}{x_i^2 \sum\limits_{k=1}^{n} x_k^2 - x_i^4} \geq \frac{\left(\sum\limits_{k=1}^{n} x_k^{\frac{5}{2}}\right)^2}{\sum\limits_{i=1}^{n} \left[x_i^2 \sum\limits_{k=1}^{n} x_k^2 - x_i^4\right]} = \frac{\left(\sum\limits_{k=1}^{n} x_k^{\frac{5}{2}}\right)^2}{\sum\limits_{\substack{i=1 \\ i \neq j}}^{n} \sum\limits_{j=1}^{n} x_i^2 x_j^2}.$$

By the Power Mean inequality we have

$$\left(\sum_{k=1}^{n} x_k^{\frac{5}{2}}\right)^2 \geq \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{5}{2}} \quad \text{and} \quad \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} \geq \frac{1}{n} \sum_{k=1}^{n} x_k$$

Hence it suffices to show that

$$\left(\sum_{k=1}^{n} x_k^2\right)^2 \geq \frac{n}{n-1} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} x_i^2 x_j^2,$$

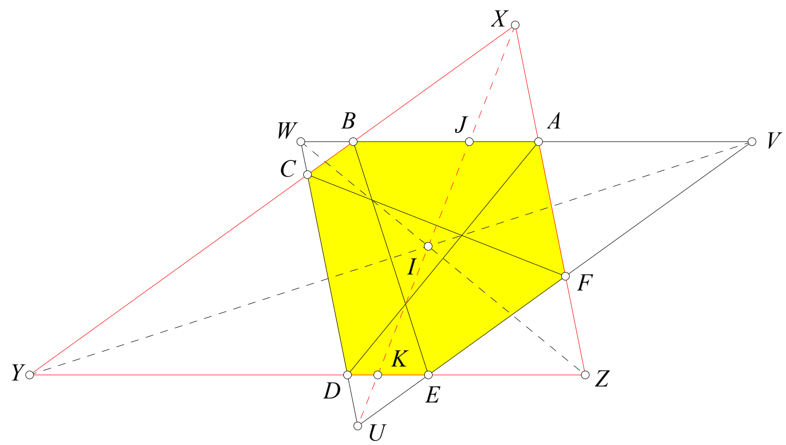
which is equivalent with $\sum\limits_{\substack{i=1 \\ i<j}}^{n} \sum\limits_{j=1}^{n} (x_i - x_j)^2 \geq 0$. This completes the proof. Equality clearly holds when $x_1 = x_2 = \cdots = x_n$.

Also solved by Semchankau Aliaksei, Moscow Institute Physics and Technique, Russia; Daniel Lasaosa, Pamplona, Navarra, Spain; Shohruh Ibragimov, Lyceum Nr2 under the SamIES, Samarkand, Uzbekistan; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, California, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Julien Portier, Francois 1er, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; BTR Problem Solving Group; Li Zhou, Polk State College, FL, USA; Nikolaos Kolliopoulos, National Technical University of Athens, Greece.

O284. Consider a convex hexagon $ABCDEF$ with $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$. The distance between the lines AB and DE is equal to the distance between the lines BC and EF and is equal to the distance between the lines CD and FA. Prove that $AD + BE + CF$ does not exceed the perimeter of the hexagon $ABCDEF$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Li Zhou, Polk State College, FL, USA



Let U, V, W, X, Y, Z be $CD \cap EF$, $EF \cap AB$, $AB \cap CD$, $FA \cap BC$, $BC \cap DE$, $DE \cap FA$, respectively. Then the equidistance condition implies that $UFXC$, $VBYE$, and $WDZA$ are rhombuses. Hence UX, VY, WZ concur at I, the incenter of both $\triangle UVW$ and $\triangle XYZ$. Thus, I is the center of similarity of $\triangle UVW$ and $\triangle XYZ$, with similarity coefficient $k = IU/XI$. Let $x = YZ$, $y = ZX$, $z = XY$, and let r and s be the inradius and semiperimeter of $\triangle XYZ$. Suppose that UX intersects VW at J and YZ at K. Then

$$CF = XU \tan \frac{X}{2} = \frac{r(1+k)XI}{s-x} = \frac{r(1+k)}{\cos(X/2)} = \frac{(1+k)\sqrt{yz(s-y)(s-z)}}{s},$$

where the last equality is by the half-angle formula and then the law of cosines. Likewise,

$$BE = \frac{(1+k)\sqrt{zx(s-z)(s-x)}}{s}, \quad AD = \frac{(1+k)\sqrt{xy(s-x)(s-y)}}{s}.$$

On the other hand,

$$\frac{AB}{x} = \frac{XJ}{XK}, \quad \frac{DE}{kx} = \frac{DE}{WV} = \frac{KU}{JU} = \frac{KU}{kXK}.$$

Hence,

$$AB + DE = \frac{x(XJ + KU)}{XK} = \frac{x(1+k)(XI - IK)}{XK} = \frac{x(1+k)(s-x)}{s}.$$

Likewise,

$$BC + EF = \frac{z(1+k)(s-z)}{s}, \quad CD + FA = \frac{y(1+k)(s-y)}{s}.$$

Finally, by the AM-GM inequality,

$$x(s-x) + y(s-y) + z(s-z) \geq \sqrt{xy(s-x)(s-y)} + \sqrt{yz(s-y)(s-z)} + \sqrt{zx(s-z)(s-x)},$$

completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Navarra, Spain.

O285. Let a_n be the sequence of integers defined as $a_1 = 1$ and $a_{n+1} = 2^n\left(2^{a_n} - 1\right)$ for $n \geq 1$. Prove that $n!$ divides a_n.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

Note first that, given a prime p and an integer $n \geq p$, the multiplicity with which p divides $n!$ is at most $n - p + 1$. Indeed, this multiplicity is

$$\sum_{k=1}^{\infty}\left\lfloor \frac{n}{p^k}\right\rfloor < n\sum_{k=1}^{\infty}\frac{1}{p^k} = \frac{n}{p-1},$$

or it suffices to show that $\frac{n}{p-1} \leq n-p+1$, equivalent to $(n-p)(p-2) \geq 1$, clearly true except when $p = 2$ or $n = p$. If $p = 2$, note that the multiplicity with which 2 divides $n!$ is less than n, ie at most $n-1 = n-p+1$, or the result is also true in this case. Moreover, if $n = p$, clearly the multiplicity with which p divides $n! = p!$ is $1 = n - p + 1$, or the result is also true in this case. Hence the result is always true for any prime p and any integer $n \geq p$.

By the previous result, it suffices to show that for any prime p and any integer $n \geq p$, p divides a_n with multiplicity at least $n - p + 1$. This is trivially true for $p = 2$, since for any $n \geq 2$, $2^{n-1} = 2^{n-2+1}$ divides a_n. The result is easily proved by induction over n for $p = 3$, since for any $n \geq 2$, a_n is even, or 2 divides a_2, hence $2^{a_2} \equiv 1 \pmod 3$, hence 3^{3-3+1} divides a_3, whereas if the result is true for n, then a_n is a multiple of $2 \cdot 3^{n-p+1}$, hence by Euler-Fermat's theorem, $3^{(n+1)-p+1}$ divides 2^{a_n}, or it divides a_{n+1}. We will next complete the proof of this result, and hence the solution, by induction over the set of odd primes.

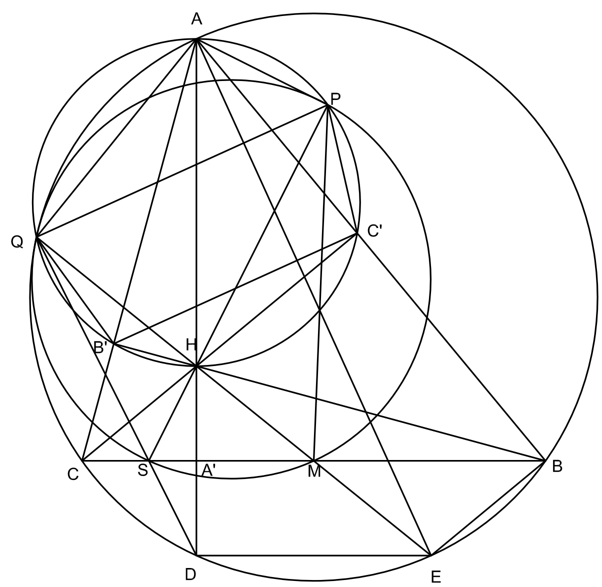
Given any odd prime $p > 3$, assume that the result is true for all primes less than p, or since $p - 1$ is a product of primes less than p, by hypothesis of induction and previously given arguments, $(p - 1)!$ divides a_{p-1}, or in particular $p - 1$ divides a_{p-1}, hence by Euler-Fermat's theorem, $2^{a_{p-1}} - 1$ is divisible by p, or for $n = p$, $a_n = a_p$ is divisible by $p = p^{n-p+1}$. Given now any $n \geq p$, if p^{n-p+1} divides p, and since $p - 1 < n$ then $p-1$ divides $n!$ which divides a_n, it follows that $(p-1)p^{n-p+1}$ divides a_n, or by Euler-Fermat's theorem, $p^{(n+1)-p+1}$ divides $2^{a_n} - 1$, hence it divides a_{n+1}. The conclusion follows.

Also solved by Li Zhou, Polk State College, FL, USA.

O286. Let ABC be a triangle with orthocenter H. Let HM be the median and HS be the symmedian in triangle BHC. Denote by P the orthogonal projection of A onto HS. Prove that the circumcircle of triangle MPS is tangent to the circumcircle of triangle ABC.

Proposed by Marius Stanean, Zalau, Romania

Solution by Sebastiano Mosca, Pescara, Italy



Let D be the intersection of the line passing for AA' with the circumcircle ω of triangle ABC and E the intersection of line passing for D and parallel to CB with ω. We have

$$\angle CHB = 180^\circ - \angle CAB = \angle CDB$$
$$\angle BCH = \angle BAA' = \angle BCD = \angle EBC$$

so $CH = CD = EB$ and EB is parallel to CH.

It means that segment EH pass for M, the midpoint of CB. Let Q be the other intersection point of the line passing for MH and the circumcircle of triangle ABC. Since $\angle HQA = 90^\circ$, Q is on the circle γ passing through A, C', H, B' where C' and B' are the feet of the altitudes from C and B.

Due to the facts that angle $APQ = 90^\circ$, the point P is on γ

$$\angle A'SH = \angle HAP = \angle HQP$$

So the circle ϑ pass from points M, S and P pass also from Q. Segment AE is a diameter of circle ABC because $\angle ADE = 90^\circ$ and due to $\angle EAB = \angle DAC$ and $\angle AC'B' = \angle ABC$ it follows that $AE \perp B'C'$.

Since $\angle BHE = \angle SHC = \angle PHC' = \angle QHB'$ it follows that $QB' = PC'$ and consequently $QP \parallel B'C'$. So segment AE is perpendicular to segment $B'C'$. It means that $\angle ADQ = \angle AEQ = \angle AQP = \angle AHP = \angle SHA'$.

$\angle AEQ = \angle ADS$ but we also know that $\angle SDA' = \angle SHA'$ because $A'H = A'D$. It follows that $\angle SDA' = \angle AEQ$ and consequently points Q, S and E are aligned.

Circle ϑ pass on the same point Q of circle ω and on points S and M who are on the side QD and QE of triangle QED and SM is parallel to DE. It follow that the two circles are tangent to each other in Q. $\square$

Remark. Circle ϑ is also tangent to the nine point circle of triangle ABC. Furthermore

$$\angle MQP = \angle MSP = \angle DSA' = 180^\circ - \angle MSQ = \angle QPM$$

It means that the center of ϑ is on a straigth line ℓ passing to point M and perpendicular to QP. Observe that $B'M = MC = MB = MB = MC'$ so the center of the nine points circles lies on a straigth ℓ' passing for point M perpendicular to $B'C'$. But we already know that segment QP is parallel to segment $B'C'$ so the straigth line ℓ and straigth line ℓ' coincide. In other words ϑ and nine point circle pass for M and their centers are aligned with M which means that the two circles are tangent to each other in M.

Also solved by Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Daniel Lasaosa, Pamplona, Navarra, Spain; BTR Problem Solving Group; Li Zhou, Polk State College, FL, USA.

O287. We are given a 6×6 table with 36 unit cells. A 2×2 square with 4 unit cells is called a *block*.A set of blocks covers the table if each cell of the table is covered by at least one cell of one block in the set. Blocks can overlap with each other. Find the largest integer n such that there is a cover of the table with n blocks, but if we remove any block from the covering, the remaining set of blocks will no longer cover the table.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by BTR Problem Solving Group
Let us enumerate the rows and columns, starting from the upper left corner.We can notice that there are blocks in every of the corners.

We will say that a cell corresponds to a block if when we remove the particular block, the cell remains uncovered.We have to find the maximum number of pairs of corresponding block and a corresponding cell. Obviously each of the cells $(1;1);(1;6);(6;1);(6;6)$ corresponds to the block covering it and the other four triplets of cells covered by the particular blocks do not correspond to any block.

Moreover, if we take the pairs of cells like $(3;1);(3;2)$ and $(1;3);(2;3)$ and the symmetrical to them with respect to the midlines of the square and the diadonal, then there is a block covering both of them, so only one cell of these pairs can correspond to that block.

So we have found 8 pairs, from which we are sure that there is at least one cell that does not correspond to a block and four triplets of the same kind of cells bordering with the cells

$$(1;1);(6;1);(6;6) \text{ and } (1;6)$$

so there are at least 20 bad cells (cells that do not correspond to a block).
An example of exactly 20 bad cells is as follows:

$$(1;1);(1;6);(6;1);(6;6);(3;1);(1;3);(3;3);(4;1);(6;3);(4;3);(1;4);(3;4);(3;6);(4;4);(6;4);(4;6)$$

corresponding to blocks with centres all points not lying on the midlines of the square.

O288. Let $ABCD$ be a square situated in the plane $\mathcal{P}$. Find the minimum and the maximum of the function $f : \mathcal{P} \to \mathbb{R}$ defined by

$$f(P) = \frac{PA + PB}{PC + PD}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

Solution by Daniel Lasaosa, Pamplona, Navarra, Spain

Let P be any point in the plane such that $f(P)$ is maximum, and let, for this point P, $PA + PB = u$ and $PC + PD = v$. The respective loci of points in $\mathcal{P}$ such that $PA+PB = u$ and $PC+PD = v$ are two ellipses with foci A, B and foci C, D, hence both symmetric around the common perpendicular bisector of AB and CD. Assume that there is one point Q on the ellipse with foci A, B which is inside the ellipse with foci C, D. Then, the ellipse with foci C, D through Q is such that $QC+QD < PC+PD$, while $QA+QB = PA+PB$, or $f(P)$ is not maximum because $f(Q) > f(P)$. It follows that the ellipse with foci A, B cannot have any point inside the ellipse with foci C, D, or since P is on both ellipses, they must touch externally at P, which is clearly on the common symmetry axis of both ellipses. Similarly, if a point on the ellipse with foci A, B is outside the ellipse with foci C, D, $f(P)$ cannot be minimum, or again any point P such that $f(P)$ is minimum must be the point where the ellipse with foci A, B is internally tangent to the ellipse with foci C, D, hence on the common perpendicular bisector of AB and CD.

Since the problem is invariant under scalings, consider a coordinate system such that $A \equiv (-1, 1)$, $B \equiv (-1, -1)$, $C \equiv (1, -1)$ and $D \equiv (1, 1)$. By the previous argument, it suffices to find the maximum and minimum of f over the horizontal axis, for which $PA^2 = PB^2 = (x+1)^2+1$, and $PC^2 = PD^2 = (x-1)^2+1$, or for both extrema, we would have

$$g(x) = (f(P))^2 = \frac{PA^2}{PC^2} = \frac{(x+1)^2+1}{(x-1)^2+1},$$

and it suffices to find the extrema of this function. Note first that the limit when $x \to \pm\infty$ is 1, that the expression is continuous and differentiable, or it suffices to check the values of $g(x)$ when its first derivative is zero, which clearly satisfy

$$2(x+1)\left((x-1)^2+1\right) = 2(x-1)\left((x+1)^2+1\right), \qquad x^2 = 2.$$

Taking $x = \sqrt{2}$, we find

$$f(P) = \sqrt{g\left(\sqrt{2}\right)} = \sqrt{\frac{(\sqrt{2}+1)^2+1}{(\sqrt{2}-1)^2+1}} = \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}} = \sqrt{2}+1 > 1,$$

while for $x = -\sqrt{2}$, we have

$$f(P) = \sqrt{g\left(-\sqrt{2}\right)} = \sqrt{\frac{(-\sqrt{2}+1)^2+1}{(-\sqrt{2}-1)^2+1}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{2}-1 < 1.$$

It follows that the maximum and minimum of $f(P)$ are respectively $\sqrt{2}+1$ and $\sqrt{2}-1$, and occur for points P_+, P_-, which are the intersections of the circumcircle of the square and the common perpendicular bisector of AB, CD, such that P_+ is the one closest to side AB, and P_- is the one closest to side CD.

Also solved by AN- anduud Problem Solving Group, Ulaanbaatar, Mongolia; BTR Problem Solving Group; Li Zhou, Polk State College, FL, USA; Nikolaos Kolliopoulos, National Technical University of Athens, Greece.