Junior problems

J259. Among all triples of real numbers (x, y, z) which lie on a unit sphere $x^2 + y^2 + z^2 = 1$ find a triple which maximizes $\min(|x - y|, |y - z|, |z - x|)$.

Proposed by Arkady Alt, San Jose, California, USA

Solution by Polyahedra, Polk State College, USA

Without loss of generality, assume that $x \le y \le z$. Then

$$m = \min(|x - y|, |y - z|, |z - x|) = \min(y - x, z - y) \le \frac{z - x}{2}.$$

Now $x^2 + y^2 + z^2 = 1$ can be written as $(z - x)^2 = 2 - (z + x)^2 - 2y^2$. Thus $(z - x)^2 \le 2$, with equality if and only if $z + x = 0 = y$. Hence $m \le \frac{\sqrt{2}}{2}$, with the maximum of $\frac{\sqrt{2}}{2}$ attained at $(x, y, z) = \left(\frac{-\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$.

Also solved by Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo, Milan, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada.

J260. Solve in integers the equation

$$x^4 - y^3 = 111.$$

Proposed by José Hernández Santiago, Oaxaca, México

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

It can be easily checked that any perfect cube leaves, modulus 13, a remainder of $-5, -1, 0, 1, 5$, and any perfect fourth power leaves, modulus 13, a remainder of $-4, 0, 1, 3$. But 111 leaves a remainder of $7 \equiv -6$ modulus 13, which can be easily checked that cannot be obtained as the sum of one out of $-4, 0, 1, 3$ and one out of $-5, -1, 0, 1, 5$. It follows that the proposed equation can have no solutions in integers (x, y).

Also solved by Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Andrea Fiacco, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy; Corneliu Mănescu-Avram, Ploieşti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Alessandro Ventullo, Milan, Italy; Sayan Das, Kolkata, India; Radouan Boukharfane, Polytechnique de Montreal, Canada; Polyahedra, Polk State College, USA; George Batzolis, Mandoulides High School, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Robin Park, Thomas Jefferson High School For Science and Technology; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

J261. Let $A_1 \dots A_n$ be a polygon inscribed in a circle with center O and radius R. Find the locus of points M on the circumference such that

$$A_1M^2 + \cdots + A_nM^2 = 2nR^2.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let G be the center of mass of the n-gon, ie, for a coordinate system such that wlog its origin is located at the circumcenter O of the n-gon, we have

$$\overrightarrow{OG} = \frac{\overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n}}{n}.$$

Note now that

$$A_iM^2 = \left(\overrightarrow{OM} - \overrightarrow{OA_i}\right)^2 = 2R^2 - 2\overrightarrow{OM} \cdot \overrightarrow{OA_i},$$

where we have used that $|OM| = |OA_i| = R$ because the vertices of the n-gon, as well as M, are on the circle with center O and radius R. Adding these equations for $i = 1, 2, \dots, n$, we find

$$A_1M^2 + \cdots + A_nM^2 = 2nR^2 - 2n\overrightarrow{OM} \cdot \overrightarrow{OG},$$

ie, the condition given in the problem statement is equivalent to $\overrightarrow{OM} \cdot \overrightarrow{OG} = 0$. We have now two cases:

Case 1: If $G = O$, the condition is satisfied for all M on the circumference, ie, the locus of points M satisfying the condition given in the problem statement is the entire circumference.

Case 2: If $G \neq O$, the condition is equivalent to $\overrightarrow{OG}$ and $\overrightarrow{OM}$ being perpendicular, ie, the locus of points M satisfying the condition given in the problem statement is the two ends of the diameter of the circumference which is perpendicular to OG, G being the center

Also solved by Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; G.R.A.20 Problem Solving Group, Roma, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada; George Batzolis, Mandoulides High School, Greece; Polyahedra, Polk State College, USA.

J262. Find all positive integers m, n such that $\binom{m+1}{n} = \binom{m}{n+1}$.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

Solution by Daniele Fakhoury, and Andrea Fiacco, Università di Roma "Tor Vergata", Roma, Italy
We note that if $m+1 > n \geq 0$ then

$$\binom{m+1}{n} > 0 \qquad \text{and} \qquad \binom{n}{m+1} = 0.$$

On the other hand if $0 \leq m+1 < n$

$$\binom{m+1}{n} = 0 \qquad \text{and} \qquad \binom{n}{m+1} > 0.$$

Finally, if $n = m+1 \geq 0$ then

$$1 = \binom{m+1}{n} = \binom{n}{m+1}.$$

So equality holds iff $n = m+1 \geq 0$.

Also solved by Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Vlad Petrache, Mihaela Petrică and Daniel Văcaru, Colegiul Economic Maria Teiuleanu, Piteşti, Romania; 'Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Polyahedra, Polk State College, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Radouan Boukharfane, Polytechnique de Montreal, Canada; Alessandro Ventullo, Milan, Italy; Robin Park, Thomas Jefferson High School For Science and Technology.

J263. The n-th pentagonal number is given by the formula $p_n = \frac{n(3n-1)}{2}$. Prove that there are infinitely many pentagonal numbers that can be written as a sum of two perfect squares of positive integers.

Proposed by José Hernández Santiago, Oaxaca, México

Solution by Robin Park, Thomas Jefferson High School For Science and Technology

We claim that there exist infinitely many pentagonal numbers p_n that can be written as $m^2 + n^2$, where m and n are positive integers. For this to be true, there must exist infinitely many perfect squares m^2 that can be expressed as $\frac{n(n-1)}{2}$.

Let $m^2 = \frac{n(n-1)}{2}$. This rearranges to $(2n-1)^2 - 8m^2 = 1$, which is a Pell Equation $a^2 - kb^2 = 1$, where $a = 2n-1, k = 8, b = m$, and is known to have an infinite number of solutions. Since k is even, a must be odd, so there exist infinitely many n that satisfy this equation.

Therefore there exist infinitely many pentagonal numbers that can be expressed as $m^2 + n^2$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Andrea Fiacco, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy; Alessandro Ventullo, Milan, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada; Corneliu Mănescu-Avram, Ploieşti, Romania; Polyahedra, Polk State College, FL, USA; George Batzolis, Mandoulides High School, Thessaloniki, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

J264. In triangle ABC, $2\angle A = 3\angle B$. Prove that

$$\left(a^2 - b^2\right)\left(a^2 + ac - b^2\right) = b^2c^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Giulia Giovannotti, Università di Roma "Tor Vergata", Roma, Italy

We note that $A = 3B/2$, $C = \pi - A - B = \pi - 5B/2$, and

$$a = 2R\sin(3B/2), \quad b = 2R\sin(B), \quad c = 2R\sin(5B/2),$$

where R is the radius of the circumcircle. So it suffices to show that

$$(\sin(3B/2)^2 - \sin(B)^2)(\sin(3B/2)^2 + \sin(3B/2)\sin(5B/2) - \sin(B)^2) = \sin(B)^2\sin(5B/2)^2.$$

Since $\sin(x)\sin(y) = (\cos(x-y) - \cos(x+y))/2$ this is equivalent to

$$(\cos(2B) - \cos(3B))(\cos(B) + \cos(2B) - \cos(3B) - \cos(4B)) = (1 - \cos(2B))(1 - \cos(5B)).$$

Now $\cos(x)\cos(y) = (\cos(x-y) + \cos(x+y))/2$, and by expanding the two sides of the above equation we obtain

$$2\text{LHS} = 2 - 2\cos(2B) - 2\cos(5B) + \cos(7B) \quad \text{and} \quad 2\text{RHS} = 2 - 2\cos(2B) - 2\cos(5B) + \cos(7B).$$

So the required equation holds for any B.

Also solved by Polyahedra, Polk State College, FL, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Arkady Alt , San Jose ,California, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada.

S259. Let a, b, c, d, e be integers such that

$$a(b+c)+b(c+d)+c(d+e)+d(e+a)+e(a+b)=0.$$

Prove that $a+b+c+d+e$ divides $a^5+b^5+c^5+d^5+e^5-5abcde$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Suppose that a, b, c, d, e are the five roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of a fifth degree polynomial $P(x)$. Let

$$\sigma_k = \sum_{i=1}^{5} \alpha_i^k, \qquad s_k = \sum_{1 \le j_1 < j_2 < \ldots < j_k \le 5} \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_k}.$$

With this notation, we know that $s_2 = 0$ and we want to prove that σ_1 divides $\sigma_5 - 5s_5$. We have

$$P(x) = x^5 - s_1x^4 + s_2x^3 - s_3x^2 + s_4x - s_5,$$

so

$$\begin{array}{rcl} P(\alpha_1) & = & \alpha_1^5 - s_1\alpha_1^4 + s_2\alpha_1^3 - s_3\alpha_1^2 + s_4\alpha_1 - s_5 = 0 \\ P(\alpha_2) & = & \alpha_2^5 - s_1\alpha_2^4 + s_2\alpha_2^3 - s_3\alpha_2^2 + s_4\alpha_2 - s_5 = 0 \\ P(\alpha_3) & = & \alpha_3^5 - s_1\alpha_3^4 + s_2\alpha_3^3 - s_3\alpha_3^2 + s_4\alpha_3 - s_5 = 0 \\ P(\alpha_4) & = & \alpha_4^5 - s_1\alpha_4^4 + s_2\alpha_4^3 - s_3\alpha_4^2 + s_4\alpha_4 - s_5 = 0 \\ P(\alpha_5) & = & \alpha_5^5 - s_1\alpha_5^4 + s_2\alpha_5^3 - s_3\alpha_5^2 + s_4\alpha_5 - s_5 = 0. \end{array}$$

Summing up the columns, we get

$$\sigma_5 - s_1\sigma_4 + s_2\sigma_3 - s_3\sigma_2 + s_4\sigma_1 - 5s_5 = 0.$$

Since $s_1 = \sigma_1, s_2 = 0$ and $\sigma_2 = \sigma_1^2 - 2s_2 = \sigma_1^2$ we obtatin

$$\sigma_5 - 5s_5 = \sigma_1(\sigma_4 + s_3\sigma_1 - s_4),$$

hence $\sigma_1 | (\sigma_5 - 5s_5)$.

Also solved by Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Albert Stadler, Switzerland; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Corneliu Mănescu-Avram, Ploieşti, Romania; Andrea Fiacco, and Emiliano Torti, Università di Roma "Tor Vergata", Roma, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada; Li Zhou, Polk State College, Florida, USA; Arkady Alt , San Jose ,California, USA.

S260. Let $m < n$ be positive integers and let $x_1, x_2, \ldots, x_n$ be positive real numbers. If A is a subset of $\{1, 2, \ldots, n\}$, define $s_A = \sum_{i \in A} x_i$ and $A^c = \{i \in \{1, 2, \ldots, n\} | i \notin A\}$. Prove that

$$\sum_{|A|=m} \frac{s_A}{s_{A^c}} \geq \frac{m}{n-m} \binom{n}{m},$$

where the sum is taken over all m-element subsets A of $\{1, 2, \ldots, n\}$.

Proposed by Mircea Becheanu, University of Bucharest, Romania

Solution by Carlo Pagano, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy
Let $A, B \subset \{1, 2, \ldots, n\}$. Since

$$s(A) + s(A^c) = s(B) + s(B^c) = \sum_{i=1}^{n} x_i$$

it follows that $s(A) \geq s(B)$ iff $1/s(A^c) \geq 1/s(B^c)$ and by Chebycev's inequality

$$\sum_{|A|=m} \frac{s_A}{s_{A^c}} \geq \frac{1}{\binom{n}{m}} \sum_{|A|=m} s_A \sum_{|A|=m} \frac{1}{s_{A^c}}.$$

Moreover, by AM-HM inequality,

$$\frac{1}{\binom{n}{m}} \sum_{|A|=m} \frac{1}{s_{A^c}} \geq \frac{\binom{n}{m}}{\sum_{|A|=m} s_{A^c}}.$$

Now we note that

$$\sum_{|A|=m} s_A = \binom{n-1}{m-1} \sum_{i=1}^{n} x_i \quad \text{and} \quad \sum_{|A|=m} s_{A^c} = \binom{n-1}{n-m-1} \sum_{i=1}^{n} x_i.$$

Hence

$$\sum_{|A|=m} \frac{s_A}{s_{A^c}} \geq \frac{\binom{n}{m} \sum_{|A|=m} s_A}{\sum_{|A|=m} s_{A^c}} = \frac{\binom{n}{m}\binom{n-1}{m-1}}{\binom{n-1}{n-m-1}} = \frac{m}{n-m} \binom{n}{m}.$$

Also solved by Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Robin Park, Thomas Jefferson High School For Science and Technology; Li Zhou, Polk State College, Florida, USA; George Batzolis, Mandoulides High School, Thessaloniki, Greece; Radouan Boukharfane, Polytechnique de Montreal, Canada.

S261. Let ABC be a triangle with circumcircle Γ and let $\mathcal{K}$ be the circle simultaneously tangent to AB, AC and Γ, internally. Let X be a point on the circumcircle of ABC and let Y, Z be the intersections of Γ with the tangents from X with respect to $\mathcal{K}$. As X varies on Γ, what is the locus of the incenters of triangles XYZ?

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Li Zhou, Polk State College, USA

Let i be the inversion with respect to the circle centered at A and with radius $\sqrt{bc}$, where $b = AC$ and $c = AB$. Let B' be $i(B)$, etc.. Then Γ' is the line $B'C'$, and thus $\mathcal{K}'$ is the excircle of $\triangle AB'C'$ opposite A. Let s and r be the semiperimeter and inradius of $\triangle ABC \cong \triangle AC'B'$. Then

$$AD = \frac{bc}{AD'} = \frac{2[ABC]}{s \sin A} = \frac{2r}{\sin A}.$$

Hence, the distance from F to AC, being half of that from D to AC, is r. This completes the proof of the lemma.

The lemma implies that F and A are images of each other under the inversion j with respect to $\mathcal{K}$. Therefore, the locus of the incenters of $\triangle XYZ$ is the circle $j(\Gamma)$. Of course, whether to include the tangency point of Γ and $\mathcal{K}$ depends on whether $\triangle XYZ$ is allowed to degenerate into a point.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Robin Park, Thomas Jefferson High School For Science and Technology.

S262. Let a, b, c be the sides of a triangle and let m_a, m_b, m_c be the lengths of its medians. Prove that

$$a^2+b^2+c^2-ab-bc-ca \leq 4\left({m_a}^2+{m_b}^2+{m_c}^2-m_am_b-m_bm_c-m_cm_a\right).$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Radouan Boukharfane, Polytechnique de Montreal, Canada
We know that for a triangle ABC, if m_a,m_b,m_c are its medians on BC,CA,AB, so we have :

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}\left(a^2+b^2+c^2\right)$$

Taking account this identity, we are left to prove that :

$$\sum m_bm_c \leq \frac{1}{2}\sum a^2+\frac{1}{4}\sum bc$$

We will prove that :

$$m_bm_c \leq \frac{a^2}{2}+\frac{bc}{b}$$

By squaring both sides of this inequality we get :

$$16m_a^2m_b^2 \leq (2a^2+bc)^2$$

which is equivalent to :

$$16\frac{2(c^2+a^2)-b^2}{4}\cdot\frac{2(a^2+b^2)-c^2}{4} \leq (2a^2+bc)^2 \Leftrightarrow (b-c)^2(a+b+c)(a-b-c) \leq 0$$

Which is true. We used here the identities:

$$m_b^2=\frac{2(c^2+a^2)-b^2}{4}$$

and

$$m_c^2=\frac{2(a^2+b^2)-c^2}{4}$$

We prove similary:

$$m_am_c \leq \frac{b^2}{2}+\frac{ac}{b}$$

and

$$m_am_b \leq \frac{c^2}{2}+\frac{ab}{b}$$

By summing up the last inequalities we proved we get our result.

Also solved by Scott H. Brown, Auburn University, Montgomery, AL; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Li Zhou, Polk State College, USA; Robin Park, Thomas Jefferson High School For Science and Technology; Ioan Viorel Codreanu, Satulung, Maramures, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

S263. Prove that for all $n \geq 2$ and all $1 \leq i \leq n$ we have

$$\sum_{j=1}^{n}(-1)^{n-j}\frac{\binom{n+j}{n}\binom{n}{n-j}}{i+j} = 1.$$

Proposed by Marcel Chirita, Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain
It can be easily found, by taking particular cases with $n = 2, 3$, that the proposed relation is not necessarily true. It is however always true that

$$S(n,i) = \sum_{j=0}^{n}(-1)^{n-j}\frac{\binom{n+j}{n}\binom{n}{n-j}}{i+j} = 0$$

for $1 \leq i \leq n$. In fact, we will prove a more general result, namely that for all positive integer i, and all non-negative integer n, we have

$$S(n,i) = \sum_{j=0}^{n}(-1)^{n-j}\frac{\binom{n+j}{n}\binom{n}{n-j}}{i+j} = \frac{(i-1)(i-2)\cdots(i-n)}{i(i+1)\cdots(i+n)},$$

which clearly results in $S(n,i) = 0$ for all integer $1 \leq i \leq n$. In order to obtain this result, we will use the following

Claim: For all positive integer n, we have

$$T(n) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n+j}{n}\binom{n}{n-j} = 1.$$

Proof: Consider the following way to generate n-element subsets C of set $A = A_1 \cup A_2$ with $A_1 = \{1, 2, \ldots, n\}$, $A_2 = \{n+1, n+2, \ldots, 2n\}$: we first remove $n-j$ elements out of A_2 (which can be done in $\binom{n}{n-j}$ ways), and then choose n elements out of the remaining j elements of A_2, plus the n elements of A_1 (which can be done in $\binom{n+j}{n}$ ways). Given j, we generate $\binom{n}{n-j}\binom{n+j}{n}$ sets, not necessarily distinct. Consider a set $C = C_1 \cup C_2$, where $C_1 = C \cap A_1$ and $C_2 = C \cap A_2$, and C_2 has exactly k elements. As long as $j \geq k$, C can be generated. Out of the $\binom{n}{n-j}$ ways in which we remove $n-j$ elements, there are exactly $\binom{n-k}{n-j}$ ways in which we can choose to remove $n-j$ elements out of A_2 that will leave the k elements of C_2 available for selection into C. Therefore, any given n-subset element C such that C_2 has exactly k elements, can be chosen in $\binom{n-k}{n-j}$ ways. Let us count, for each C, the number of ways in which we can choose it when $n-j$ is even, minus the number of ways in which we can choose it when $n-j$ is odd, which clearly equals $T(n)$ when added over all possible subsets C. For any particular set with $n-1 \geq k \geq 0$, this number of ways is

$$\sum_{j=k}^{n}(-1)^{n-j}\binom{n-k}{n-j} = \sum_{m=0}^{n-k}(-1)^{m}\binom{n-k}{m} = (1-1)^{n-k} = 0,$$

where we have defined $m = n-j$. Therefore, the contribution to $T(n)$ from all sets $C \neq A_2$ is zero. Moreover, $C = A_2$ can be chosen in exactly one way, when $j = n$, ie when no elements of A_2 are removed prior to the selection of the n-element subset. Since $n - j = 0$ is even, this contributes $+1$ to $T(n)$, being the only nonzero term. The Claim follows.

Returning to the proposed problem, note that, since $n+i = (n-j)+(i+j)$, $n+j = (i+j)-(i-n)$, while $\binom{n}{n-j} = \frac{n}{n-j}\binom{n-1}{n-1-j}$ and $\binom{n+j}{n} = \frac{n+j}{n}\binom{n-1+j}{n-1}$, we have

$$(n+i)\frac{\binom{n+j}{n}\binom{n}{n-j}}{i+j} = -(i-n)\frac{\binom{n-1+j}{n-1}\binom{n-1}{n-1-j}}{i+j} + \binom{n+j}{n}\binom{n}{n-j}+$$

$$+\binom{n-1+j}{n-1}\binom{n-1}{n-1-j},$$

or after substitution of this relation, and using the Claim, we have

$$(i+n)S(n,i) = (i-n)S(n-1,i) + T(n) - T(n-1) = (i-n)S(n-1,i).$$

Together with

$$S(1,i) = \sum_{j=0}^{1}(-1)^{1-j}\frac{1+j}{i+j} = -\frac{1}{i} + \frac{2}{i+1} = \frac{i-1}{i(i+1)},$$

the improved, more general result follows after straightforward induction.

Also solved by Anastasios Kotronis, Athens, Greece; Radouan Boukharfane, Polytechnique de Montreal, Canada; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy; Li Zhou, Polk State College, USA; G.R.A.20 Problem Solving Group, Roma, Italy; Albert Stadler, Switzerland; Tsouvalas Konstantinos, University of Athens, Athens, Greece.

S264. Let a, b, c, x, y, z be positive real numbers such that $ab + bc + ca = xy + yz + zx = 1$. Prove that

$$a(y + z) + b(z + x) + c(x + y) \geq 2.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy

Let $A = a^2 + b^2 + c^2$, $X = x^2 + y^2 + z^2$, $P = ab + bc + ca$, $Q = xy + yz + zx$.

The inequality is

$$\sum_{\text{cyc}} a(y + z) = (a + b + c)(x + y + z) \geq \sum_{\text{cyc}} ax + 2$$

Squaring and employing $P = Q = 1$, we get

$$(A + 2)(X + 2) \geq \left(2 + \sum_{\text{cyc}} ax\right)^2$$

Cauchy–Schwarz yields

$$\sum_{\text{cyc}} ax \leq \sqrt{AX}$$

thus we come to

$$(A + 2)(X + 2) \geq \left(2 + \sqrt{AX}\right)^2$$

or

$$AX + 2A + 2X + 4 \geq 4 + \sqrt{AX} + AX \quad \Longleftrightarrow \quad A + X \geq 2\sqrt{AX}$$

which is trivial.

Also solved by Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Semchankau Aliaksei, Minsk, Belarus; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou, Polk State College, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Arkady Alt , San Jose ,California, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

U259. Compute

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n(n+a)}\right)^{n^3}}{\left(1+\frac{1}{n(n+b)}\right)^{n^3}}.$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Giulio Calimici, and Emiliano Torti, Università di Roma "Tor Vergata", Roma, Italy

We note that

$$n^3 \ln\left(1+\frac{1}{n(n+a)}\right) = n^3\left(\frac{1}{n(n+a)} + O(1/n^4)\right) = n\left(\frac{1}{1+a/n} + O(1/n^2)\right)$$
$$= n\left(1-\frac{a}{n} + O(1/n^2)\right) = n - a + O(1/n).$$

Hence

$$\frac{\left(1+\frac{1}{n(n+a)}\right)^{n^3}}{\left(1+\frac{1}{n(n+b)}\right)^{n^3}} = \exp\left(n^3\ln\left(1+\frac{1}{n(n+a)}\right) - n^3\ln\left(1+\frac{1}{n(n+b)}\right)\right)$$
$$= \exp\left(n-a+O(1/n) - n + b + O(1/n)\right) \to \exp(b-a) = e^{b-a}.$$

Also solved by Albert Stadler, Switzerland; Stanescu Florin, Serban Cioculescu school, Gaesti, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee Lycée Henri IV , Paris , France; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Alessandro Ventullo, Milan, Italy; Anastasios Kotronis, Athens, Greece; Robin Park, Thomas Jefferson High School For Science and Technology; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada; Li Zhou, Polk State College, USA.

U260. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ which are derivatives and satisfy

$$f^2 \in \int f(x)\,dx.$$

Proposed by Mihai Piticari, "Dragoş Voda" National College, Romania

Solution by Daniele Fakhoury, and Andrea Fiacco, Università di Roma "Tor Vergata", Roma, Italy

If f' exists in an open interval $I \subset \mathbb{R}$ then, by hypothesis,

$$f(x)(2f'(x)-1)=0 \qquad \forall x \in I.$$

By continuity of f, the above equation implies that

$$f(x)=0 \qquad \forall x\in I \qquad \text{or} \qquad f(x)=\frac{x}{2}+c \qquad \forall x \in I.$$

In the general case, since f is a derivative, by Darboux's Theorem it follows that f has intermediate value property. This implies that our f is continuous and it is non-differentiable in at most two points $a \le b$:

$$f(x)=\begin{cases}\frac{1}{2}(x-b) & x\in(b,+\infty),\\ 0 & x\in[a,b],\\ \frac{1}{2}(x-a) & x\in(-\infty,a).\end{cases}$$

Also solved by Stanescu Florin, Serban Cioculescu school, Gaesti, Romania; Li Zhou, Polk State College, USA.

U261. Let $T_n(x)$ be the sequence of Chebyshev polynomials of the first kind, defined by $T_0(x)=0$, $T_1(x)=x$, and

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$$

for $n\geq 1$. Prove that for all $x\geq 1$ and all positive integers n

$$x\leq \sqrt[n]{T_n(x)}\leq 1+n(x-1).$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain
The formal definition of Chebyshev polynomials is as in the problem statement, but with $T_0(x)=1$. We will actually use this definition in the following solution, since among other things, the definition with $T_0(x)=0$ results after induction in $T_n(1)=n$ for all n, hence the second inequality would not hold. With the definition $T_0(x)=1$, as we will soon see, both inequalities hold.

Assume that x is a fixed value, and for said x, consider the x-dependent sequence $(s_n)_{n\geq 0}$ defined by $s_0=1$, $s_1=x$, and for all $n\geq 2$, $s_n=2xs_{n-1}-s_{n-2}$. Using standard techniques, it follows that

$$s_n=\frac{\left(x+\sqrt{x^2-1}\right)^n+\left(x-\sqrt{x^2-1}\right)^n}{2}\geq\left(\frac{\left(x+\sqrt{x^2-1}\right)+\left(x-\sqrt{x^2-1}\right)}{2}\right)^n=x^n,$$

where we have used the power-mean inequality, which is valid since $x\pm\sqrt{x^2-1}$ are positive reals because $0\leq\sqrt{x^2-1}<x$ for all $x\geq 1$. Note that equality holds iff equality in the power mean inequality holds, ie iff $\sqrt{x^2-1}=0$, for $x=1$, or iff $n=1$ for all x.

The second inequality can be rewritten as

$$T_n(x)\leq(1+n(x-1))^n.$$

For $n=0$, both sides are identically 1, while for $n=1$, both sides are identically x, or the inequality holds with equality for all $x\geq 1$ in these cases. For $n=2$, the inequality rewrites as

$$2x^2-1\leq(2x-1)^2,\qquad\qquad 2(x-1)^2\geq 0,$$

clearly true, with equality iff $x=1$. Now, we will show by induction that, for all $n\geq 2$ and all x, we have

$$T_n(x)=2(x-1)\sum_{k=1}^{n-1}(n-k)T_k(x)+n(x-1)T_0(x)+1.$$

For $n=2$ and $n=3$, this result is respectively equivalent to

$$T_2(x)=2(x-1)T_1(x)+2(x-1)+1=2x^2-1,$$

$$T_3(x)=2(x-1)T_2(x)+4(x-1)T_1(x)+3(x-1)+1=2xT_2(x)-x,$$

clearly true in both cases. These are our base cases, and if the result holds for $n,n-1$, then

$$T_{n+1}(x)=2(x-1)T_n(x)+2T_n(x)-T_{n-1}(x)=$$

$$=2(x-1)T_n(x)+4(x-1)T_{n-1}(x)+2(x-1)\sum_{k=1}^{n-2}(n-k+1)T_k(x)+(n+1)(x-1)T_0(x)+1,$$

where we have used the hypothesis of induction for $n,n-1$, and the result clearly holds for $n+1$ too. Hence it holds for all positive integer n.

Now, this means that, if for some $n \geq 3$ the inequality holds for $1, 2, \ldots, n-1$, then

$$T_n(x) \leq 2(x-1) \sum_{k=1}^{n-1} (n-k) \left(1 + k(x-1)\right)^k + n(x-1) + 1,$$

with equality iff $x = 1$, since $T_2(x) = (1 + 2(x-1))^2$ iff $x = 1$, and if $x = 1$ then both sides are identically 1. Using the expression for the sum of the geometric progression with ratio $1 + n(x-1)$ from 1 to $(1 + n(x-1))^{n-1}$, we obtain

$$(1 + n(x-1))^n - 1 = n(x-1) \sum_{k=0}^{n-1} \left(1 + n(x-1)\right)^k,$$

or it suffices to show that

$$\sum_{k=1}^{n-1} \left(1 + n(x-1)\right)^k \geq \sum_{k=1}^{n-1} \frac{2(n-k)}{n} \left(1 + k(x-1)\right)^k.$$

If n is even, when $k = \frac{n}{2}$ the term in the LHS is $(1 + n(x-1))^{\frac{n}{2}}$, and in the RHS is $\left(1 + \frac{n}{2}(x-1)\right)^{\frac{n}{2}}$, clearly not larger, and equal iff $x = 1$. Whether n is odd or even, every k other than $\frac{n}{2}$ can be grouped in pairs of sum n, or it suffices to show that, for all integer k such that $1 \leq k < \frac{n}{2}$, we have

$$(1 + n(x-1))^k + (1 + n(x-1))^{n-k} \geq \frac{2(n-k)}{n} \left(1 + k(x-1)\right)^k + \frac{2k}{n} \left(1 + (n-k)(x-1)\right)^{n-k}.$$

Now, since $k < n - k < n$ for all such n, we have $1 + k(x-1) \leq 1 + (n-k)(x-1) \leq 1 + n(x-1)$, with equality iff $x = 1$, or it suffices to show that

$$\frac{n-2k}{n} \left(1 + n(x-1)\right)^{n-k} \geq \frac{n-2k}{n} \left(1 + n(x-1)\right)^k,$$

clearly true since $n - 2k > 0$ and $1 + n(x-1) \geq 1$, with equality again iff $x = 1$.

The conclusion follows, equality holds in both equalities iff either $n = 1$ and for all $x \geq 1$, or $x = 1$ for all n.

Also solved by Daniel Văcaru, Colegiul Economic "Maria Teiuleanu", Piteşti, Romania; Radouan Boukharfane, Polytechnique de Montreal, Canada.

U262. Let a and b be positive real numbers. Find $\lim_{n\to\infty} \sqrt[n]{\prod_{i=1}^{n}\left(a+\frac{b}{i}\right)}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let $x_n = \prod_{i=1}^{n}\left(a+\frac{b}{i}\right)$. Then

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\left(a+\frac{b}{n+1}\right) = a,$$

which implies that the sequence $\sqrt[n]{x_n}$ has the same limit.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Robinson Higuita Diego Harvin, Universidad de Antioquia, Colombia; Radouan Boukharfane, Polytechnique de Montreal, Canada; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy; Moubinool Omarjee ,Lycée Henri IV , Paris , France; Stanescu Florin Serban Cioculescu school, Gaesti, Romania; Angel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Konstantinos Tsou- valas, University of Athens, Athens, Greece; Anastasios Kotronis, Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Li Zhou, Polk State College, USA.

U263. Let $n \geq 2$ be an integer. A general $n \times n$ magic square is a matrix $A \in M_n(\mathbb{R})$ such that the sum of the elements in each row of A is the same. Prove that the set of $n \times n$ general magic squares is an $\mathbb{R}$-vector space and find its dimension.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote by $S_n(\mathbb{R})$ the set of general $n \times n$ magic squares. Clearly $S_n(\mathbb{R}) \subset M_n(\mathbb{R})$, and $M_n(\mathbb{R})$ is well-known to be an $\mathbb{R}$-vector space. Let $A, B \in S_n(\mathbb{R})$ and $\lambda, \rho \in \mathbb{R}$, then $\lambda A + \rho B \in S_n(\mathbb{R})$ because the sums of elements in each row of $\lambda A + \rho B$ are all equal to $\lambda s + \rho t$, where s, t are the respective values of the sums of elements in each row for A, B. We conclude that $S_n(\mathbb{R})$ is an $\mathbb{R}$-vector subspace of $M_n(\mathbb{R})$, and it remains only to find its dimension.

Consider any $n \times (n-1)$ matrix A', and any real value s. Clearly, we can generate a matrix $A \in S_n(\mathbb{R})$ as follows: the first $n-1$ elements in row i of A are the elements in row i of A', and the n-th element is s minus the sum of the previous $n-1$ elements. Reciprocally, given a matrix $A \in S_n(\mathbb{R})$ such that the sum of the elements of its rows is s, there is exactly one $n \times (n-1)$ matrix A' resulting of erasing the last column in A. Note moreover that two distinct matrices $A \in S_n(\mathbb{R})$, either they differ in one of the values of the $n-1$ first columns, or they differ in the sum of each row, or they are the same, whereas two distinct $n \times (n-1)$ matrices, and/or two distinct value of the sum s of each row, generate distinct matrices in $S_n(\mathbb{R})$. It follows that we can establish a bijection between the set $S_n(\mathbb{R})$, and the set of pairs formed by one real s, and one $n \times (n-1)$ matrix, or the dimension of $S_n(\mathbb{R})$ is $n(n-1) + 1 = n^2 - n + 1$.

Also solved by Harun Immanuel, ITS Surabaya; Andrea Fiacco, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada; Li Zhou, Polk State College, USA.

U264. Let A be a finite ring such that $1+1=0$. Prove that the equations $x^2=0$ and $x^2=1$ have the same number of solutions in A.

Proposed by Mihai Piticari, "Dragos Voda" National College, Romania

Solution by Daniele Fakhoury, Andrea Fiacco, Emiliano Torti, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy

Let $Z=\{x\in A\,:\,x^2=0\}$ and $U=\{x\in A\,:\,x^2=1\}$ and let us define the map $f:A\to A$ as $f(x)=x+1$ for any $x\in A$. We claim that f is a bijective map from Z onto U. Indeed, if $x\in Z$ then $f(x)\in U$ because

$$f(x)^2=(x+1)^2=x^2+(1+1)x+1=x^2+1=1.$$

If $x_1,x_2\in Z$ and $f(x_1)=f(x_2)$ then

$$x_1=x_1+1+1=f(x_1)+1=f(x_2)+1=x_2+1+1=x_2$$

and it follows that f is injective on Z. If $y\in U$ then $y^2=1$. Let $x=y+1$ and we have that

$$x^2=(y+1)^2=y^2+(1+1)y+1=y^2+1=1+1=0,$$

so $x\in Z$ and $f(x)=y+1+1=y$, that is $F(Z)=U$.

Also solved by Maria Miliaresi and Anastasios Kotronis, Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Harun Immanuel, ITS Surabaya; Corneliu Mănescu-Avram, Ploieşti, Romania.

O259. Solve in integers the equation $x^5 + 15xy + y^5 = 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA
Due to symmetry of equation we may assume that $x \geq y$.
Since in the case $x = y$ we get equation $2x^5 + 15x^2 = 1$ which obviously have no solution in integers then further we also can assume $x > y$.
Consider three cases.
1. $xy = 0$.Then we get solution $(x, y) = (1, 0)$;
2. $xy < 0$.Then due to supposition $x > y$ we have $x > 0,\ y < 0$ and by replacing y in equation with $-y$ we obtain equation $x^5 - 15xy - y^5 = 1,$ where $x,\ y \geq 1$
(because now x, y are positive integers).
Since $x^5 - y^5 = 15xy + 1 > 0$ then $x - y > 0 \iff x \geq y + 1$ yields
$x^5 - y^5 = (x - y)\left(x^4 + y^4 + xy\left(x^2 + y^2\right) + x^2y^2\right) \geq 5(x - y)x^2y^2$ and $xy \geq 2$.Therefore,
$15xy + 1 \geq 5(x - y)x^2y^2 \iff 3xy + \dfrac{1}{3} \geq (x - y)x^2y^2 \implies 3 \geq (x - y)xy \iff \dfrac{3}{xy} \geq x - y$ and
since $1 \leq x - y$ then $x - y = 1$ and $xy \leq 3$.
But system $\begin{cases} x - y = 1 \\ xy = 3 \end{cases}$ have no integer solutions. Then remains system $\begin{cases} x - y = 1 \\ xy = 2 \end{cases}$
which give us solution $x = 2, y = 1$.Since $(x, y) = (2, 1)$ satisfy $x^5 - 15xy - y^5 = 1$
then $(x, y) = (2, -1), (-1, 2)$ are solutions of original equation in the case $xy < 0$.
3. Let $xy > 0$. It is possible if $x, y < 0$.Then by replacing (x, y) in original equation with $(-x, -y)$ we obtain equation $x^5 + y^5 = 15xy - 1$ where $1 \leq x < y$.
Hence, $xy \geq 2$ and $x + y \geq 2x + 1 \geq 3$.Since $x^5 + y^5 = (x + y)\left(x^4 + y^4 - xy\left(x^2 + y^2\right) + x^2y^2\right) =$
$(x + y)\left(\left(xy + x^2 + y^2\right)(x - y)^2 + x^2y^2\right) \geq (x + y)\left(\left(xy + x^2 + y^2\right) + x^2y^2\right) \geq (x + y)\left(3xy + x^2y^2\right) =$
$(x + y)(3 + xy)xy \geq 3 \cdot (3 + 2)xy = 15xy$ then $15xy - 1 = x^5 + y^5 \geq 15xy$,that is the contradiction.
Thus, all solutions of original equation are $(x, y) = (1, 0), (0, 1), (2, -1), (-1, 2)$.

Also solved by Li Zhou, Polk State College, USA; George Batzolis, Mandoulides High School, Thessaloniki, Greece; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Semchankau Aliaksei, Minsk, Belarus; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy.

O260. Let p be a positive real number. Define a sequence $(a_n)_{n\geq 1}$ by $a_1 = 0$ and

$$a_n = \left\lfloor \frac{n+1}{2} \right\rfloor^p + a_{\lfloor \frac{n}{2} \rfloor}$$

for $n \geq 2$. Find the minimum of $\frac{a_n}{n^p-1}$ over all positive integers n.

Proposed by Arkady Alt, San Jose, California, USA

Solution by Radouan Boukharfane, Polytechnique de Montreal, Canada

We observe that :

$$a_{2n} = \left\lfloor \frac{2n+1}{2} \right\rfloor^p + a_{\lfloor \frac{2n}{2} \rfloor} = n^p + a_n$$

$$a_{2n+1} = \left\lfloor \frac{2n+2}{2} \right\rfloor^p + a_{\lfloor \frac{2n+1}{2} \rfloor} = (n+1)^p + a_n$$

therefore we conclude from this two expressions that:

$$a_{2n+1} - a_{2n} = (n+1)^p - n^p$$

that can be rewriten :

$$a_{n+1} - a_n = \left(\frac{n}{2}+1\right)^p - \left(\frac{n}{2}\right)^p$$

from where we deduce that a_n is increasing and that for $n \geq 3$:

$$a_n = 1 + \sum_{k=2}^{n-1} \left(\left(\frac{k}{2}+1\right)^p - \left(\frac{k}{2}\right)^p \right)$$

The sequence $s_n = \frac{a_n}{n^p-1}$ is decreasing and the minimum of s_n over all positive integers n is attained for

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 + \sum_{k=2}^{n-1} \left(\left(\frac{k}{2}+1\right)^p - \left(\frac{k}{2}\right)^p \right)}{n^p - 1}$$

or:

$$\frac{\sum_{k=2}^{n-1} \left(\left(\frac{k}{2}+1\right)^p - \left(\frac{k}{2}\right)^p \right)}{n^p-1} = \frac{1}{2^p} \frac{\sum_{k=2}^{n-1} \left((k+2)^p - k^p\right)}{n^p-1}$$

or:

$$\begin{aligned}
&\sum_{k=2}^{n-1} [(k+2)^p - k^p] = \sum_{k=2}^{n-1} [(k+2)^p - (k+1)^p + (k+1)^p - k^p] \\
&= \sum_{k=2}^{n-1} [(k+2)^p - (k+1)^p] + \sum_{k=2}^{n-1} [(k+1)^p - k^p] = (n+1)^p - 3^p + n^p - 2^p \\
&\Rightarrow \lim_{n\to\infty} \frac{\sum\limits_{k=2}^{n-1} [(k+2)^p - k^p]}{n^p-1} = \lim_{n\to\infty} \frac{(n+1)^p - 3^p + n^p - 2^p}{n^p-1}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{(n+1)^p + n^p}{n^p-1} + \lim_{n\to\infty} \frac{\overbrace{-3^p - 2^p}^{Cste}}{n^p-1} = \lim_{n\to\infty} \frac{(n+1)^p + n^p}{n^p-1} + 0 \\
&= \lim_{n\to\infty} \frac{(n+1)^p + n^p}{n^p-1} = \lim_{n\to\infty} \frac{(n+1)^p}{n^p-1} + \lim_{n\to\infty} \frac{n^p}{n^p-1} \\
&= \lim_{n\to\infty} \frac{\frac{(n+1)^p}{n^p}}{\frac{n^p-1}{n^p}} + \lim_{n\to\infty} \frac{n^p}{n^p-1} = \frac{\lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^p}{1} + 1 = \frac{1}{1} + 1 = 2
\end{aligned}$$

which means that :

$$\lim_{n\to\infty} s_n = \frac{1}{2^{p-1}}$$

which is the minimum we look for. We are done.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Li Zhou, Polk State College, USA

O261. Find all positive integers n for which

$$\sigma(n) - \varphi(n) \le 4\sqrt{n},$$

where $\sigma(n)$ is the sum of positive divisors of n and φ is Euler's totient function.

Proposed by Albert Stadler Buchenrain, Herrliberg, Switzerland

Solution by Andrea Fiacco, Emiliano Torti, and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy

We will show that the inequality holds iff $n \in \{1, 8, p, p^2 \ : \ p \text{ is a prime}\}$.
For $n = 1$ this is trivial. If $n = p$ then

$$\sigma(p) - \varphi(p) = 1 + p - (p-1) = 2 \le 4\sqrt{p}$$

which holds for any prime p. If $n = p^2$ then

$$\sigma(p) - \varphi(p) = 1 + p + p^2 - (p^2 - p) = 1 + 2p \le 4\sqrt{p^2} = 4p$$

which holds for any prime p. Now, we note that

$$\sigma(n) - \varphi(n) = n\left(\sum_{d|n}\frac{1}{d} - \sum_{d|n}\frac{\mu(d)}{d}\right) \ge 2n\sum_{p|n}\frac{1}{p}.$$

If $n = p^k$ with $k \ge 3$ then, by the above inequality,

$$4p^{k/2} \ge \sigma(p^k) - \varphi(p^k) \ge 2p^{k-1}$$

that is $4 \ge p^{k-2}$ which holds iff $n \in \{2^3, 2^4, 3^3\}$. It is easy to verify that the required inequality holds only for $n = 8$.
Finally, if n is divisible by $r \ge 2$ distinct primes then, by AM-GM inequality,

$$\sigma(n) - \varphi(n) \ge 2n\sum_{p|n}\frac{1}{p} > \frac{2nr}{(\prod_{p|n} p)^{1/r}} \ge 2rn^{1-\frac{1}{r}} \ge 4n^{1-\frac{1}{2}} = 4\sqrt{n},$$

and the required inequality does not hold.

Also solved by Semchankau Aliaksei, Minsk, Belarus; George Batzolis, Mandoulides High School, Thessaloniki, Greece; Radouan Boukharfane, Polytechnique de Montreal, Canada; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong; Li Zhou, Polk State College, USA.

O262. Let $n \geq 3$ be an integer. Consider a convex n-gon $A_1 \dots A_n$ for which there is a point P in its interior such that $\angle A_i P A_{i+1} = \frac{2\pi}{n}$ for all $i \in [1, n-1]$. Prove that P is the point which minimizes the sum of distances to the vertices of the n-gon.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Consider a coordinate system with center in P and such that the positive horizontal semiaxis is directed along ray OA_n, or denoting by r_i the distances OA_i, we have $A_i \equiv (r_i \cos(i\alpha), r_i \sin(i\alpha))$ for $i = 1, 2, \dots, n$. Let now Q be any point in the plane, denoted by $Q \equiv (r \cos\beta, r \sin\beta)$. Clearly,

$$QA_i = \sqrt{r^2 + {r_i}^2 - 2rr_i \cos(\beta - i\alpha)} = \sqrt{(r_i - r\cos(\beta - i\alpha))^2 + r^2 \sin^2(\beta - i\alpha)} \geq$$

$$\geq r_i - r\cos(\beta - i\alpha),$$

with equality iff either $r = 0$, or $\beta - i\alpha$ is an integral multiple of π. Adding all such inequalities, and subtracting the sum of all distances to P, we find

$$\sum_{i=1}^{n} QA_i - \sum_{i=1}^{n} PA_i \geq r_i - r\cos(\beta - i\alpha) = -r \sum_{i=1}^{n} \cos(\beta - i\alpha) = 0,$$

where we have used that, as it is well-known, the last sum is zero (it is equivalent to the statement "a regular n-gon exists such that one of its sides forms an angle β with one of the axes"). Note that, since $\beta - i\alpha$ cannot be simultaneously a multiple of π for $i = 1, 2, \dots, n$ (the polygon could then have at most two vertices, absurd), we conclude that equality holds iff $r = 0$, ie iff $Q = P$.

Also solved by Semchankau Aliaksei, Minsk, Belarus; Radouan Boukharfane, Polytechnique de Montreal, Canada; Li Zhou, Polk State College, USA.

O263. A tournament T with a linear order $<$ on its vertices is called an *ordered tournament* and is denoted by $(T, <)$. An ordered tournament $(T, <)$ is said to be an *induced subtournament* of another one, $(T', <')$, if there is a function $f : V(T) \to V(T')$ satisfying the conditions (i) $f(u) <' f(v)$ if and only if $u < v$; (ii) $\overrightarrow{f(u)f(v)} \in E(T')$ if and only if $\overrightarrow{uv} \in E(T)$. Prove that for any ordered tournament $(T, <)$, there exists a tournament T' such that, for every ordering $<'$ of T', $(T, <)$ is an induced subtournament of $(T', <')$.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by the author

The problem at hand comes as a really nice Lemma in a paper by Noga Alon et al on the celebrated

Erdös-Hajnal Conjecture. For every graph H, there exists a constant $\delta(H) > 0$ such that every H-free graph G (in the sense that G does not contain H as an induced subgraph) has either a clique or a stable set of size at least $|V(G)|^{\delta(H)}$.

This is a very active research question in structural graph theory about which not very much is known, and where, even for apparently easy to handle classes of graphs, the proofs that are available to us involve very sophisticated machinery. We refer to M. Chudnovksy, *The Erdös-Hajnal Conjecture - A Survey*, Journal of Graph Theory (to appear, but available at http://www.columbia.edu/ mc2775/EHsurvey.pdf) for a very recent survey.

In *Ramsey type theorems for tournaments and for H-free graphs* (available at http://www.tau.ac.il/ nogaa/PDFS/aps4.pdf), Noga Alon et al opened the door to proving a lot of particular cases of this conjecture by translating the question for graphs above into one for tournaments in the most obvious sense possible.

Erdös-Hajnal Conjecture for Tournaments. For every tournament T, there exists a constant $\epsilon(T) > 0$ such that every T-free tournament with n vertices has a transitive subtournament (i.e. a subtournament with no directed cycle) whose size is at least $n^{\epsilon(T)}$.

The proof of the equivalence is somewhat intricate and uses quite a few careful probabilistic arguments that carry each of the two version above to ordered graphs. As you might guess now, the result from O263 is one of the Lemmas used by the authors in this proof, and unfortunately, it comes itself with a pretty messy probabilistic argument. We were hoping to see an elementary approach but no solutions have been received, so we refer to the aforementioned paper for the details of the original proof.

O264. Let $p > 3$ be a prime. Prove that $2^{p-1} \equiv 1 \pmod{p^2}$ if and only if the numerator of

$$\sum_{k=2}^{(p-1)/2} \frac{1}{k}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k-1}\right)$$

is a multiple of p.

Proposed by Gabriel Dospinescu, Lyon, France

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

So it suffices to show that

$$\sum_{k=2}^{(p-1)/2} \frac{H_{k-1}(1)}{k} = \frac{1}{2}\left(\left(H_{(p-1)/2}(1)\right)^2 - H_{(p-1)/2}(2)\right) \equiv 2q^2 \pmod{p}$$

where $H_{k-1}(d) = 1 + \frac{1}{2^d} + \cdots + \frac{1}{(k-1)^d}$.

Since by Wolstenholme's Theorem $H_{p-1}(1) \equiv H_{p-1}(2) \equiv 0 \pmod{p}$ for any prime $p > 3$, it follows that

$$\begin{aligned} 2q = \frac{2^p - 2}{p} &= \frac{1}{p}\sum_{k=1}^{p-1}\binom{p}{k} \equiv \sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k} \pmod{p} \\ &= \sum_{k=1}^{p-1}\frac{1}{k} - 2\sum_{k=1}^{(p-1)/2}\frac{1}{2k} = H_{p-1}(1) - H_{(p-1)/2}(1) \equiv -H_{(p-1)/2}(1) \pmod{p}, \end{aligned}$$

and

$$H_{(p-1)/2}(2) = \frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{k^2} + \frac{1}{2}\sum_{k=(p+1)/2}^{p-1}\frac{1}{(p-k)^2} \equiv \frac{1}{2}H_{p-1}(2) \equiv 0 \pmod{p}.$$

Hence,

$$\frac{1}{2}\left(\left(H_{(p-1)/2}(1)\right)^2 - H_{(p-1)/2}(2)\right) \equiv \frac{1}{2}\left((-2q)^2 + 0\right) \equiv 2q^2.$$

Also solved by Semchankau Aliaksei, Minsk, Belarus; Radouan Boukharfane, Polytechnique de Montreal, Canada.