Junior problems

J361. Solve in positive integers the equation

$$\frac{x^2-y}{8x-y^2}=\frac{y}{x}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alan Yan, Princeton Junction, NJ, USA

Let $\gcd(x,y)=g$ and $x=ga, y=gb$ where $\gcd(a,b)=1$. Substituting, the equation simplifies to $ga^3+gb^3=9ab$. Taking modulo a and since a,b are relatively prime, we must have $a|g$. By symmetry, we have $b|g$. So we can write $g=kab$ for some positive integer k. So we have

$$kab(a^3+b^3)=9ab \implies k(a^3+b^3)=9.$$

Note that $a^3+b^3\geq 2$, so a^3+b^3 can only be 3 or 9. Obviously $a^3+b^3=3$ has no solutions, so $a^3+b^3=9 \implies a=2,b=1$ or $a=1,b=2$. Then $k=1,g=2$. So we have the solutions $(4,2),(2,4)$ but since $8x-y^2\neq 0$, the second ordered pair is extraneous. Thus, the only solution is $(4,2)$.

Also solved by Arkady Alt, San Jose, CA, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Georgios Tsapakidis, High School "Panagia Prousiotissa", Agrinio, Greece; Polyahedra, Polk State College, FL, USA; Michael Tang, USA; David E. Manes, Oneonta, NY, USA; Vincelot Ravoson, France, Paris, Lycée Henri IV; Arpon Basu, Mumbai, India; Nikos Kalapodis, Patras, Greece; Problem Solving Group, Department of Financial and Management Engineering, University of the Aegean; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA; Yong Xi Wang, Affiliated High School of Shanxi University; Joel Schlosberg, Bayside, NY; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Alessandro Ventullo, Milan, Italy; Tolibjon Ismoilov, academic lyceum named after S.H.Sirojiddinov, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Paul Revenant, Lycee, Champollion, Grenoble, France; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Daniel López-Aguayo, Centro de Ciencias Matemáticas UNAM, Morelia, Mexico; Dimitris Avramidis, Evaggeliki Gymnasium, Athens, Greece; Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Nishant Dhankhar, Delhi, India; Adithya Bhaskar, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Cherlyse Alexander-Reid, College at Brockport, SUNY; Joachim Studnia, Lycee Condorcet, Paris, France; Petros Panigyrakis, Evaggeliki Gymnasium, Athens, Greece; WSA, L.T "Orizont", Moldova; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Neculai Stanciu and Titu Zvonaru, Romania.

J362. Let a, b, c, d be real numbers such that $abcd = 1$. Prove that the following inequality holds:

$$ab + bc + cd + da \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}$$

Proposed by Mircea Becheanu, University of Bucharest, România

Solution by Henry Ricardo, New York Math Circle, Tappan, NY, USA

We use the Cauchy-Schwarz inequality to see that

$$\begin{aligned} ab + bc + cd + da &= \sum_{cyclic} ab = \sum_{cyclic} \frac{abcd}{cd} = \sum_{cyclic} \frac{1}{c} \cdot \frac{1}{d} \leq \left(\sum_{cyclic} \frac{1}{c^2} \right)^{1/2} \left(\sum_{cyclic} \frac{1}{d^2} \right)^{1/2} \\ &= \sum_{cyclic} \frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}. \end{aligned}$$

Equality holds if and only if $a = b = c = d = 1$.

Also solved by Panigyraki Chrysoula, Evaggeliki Gymnasium, Athens, Greece; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Adithya Bhaskar, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Rajarshi Kanta Ghosh, Calcutta Boy's School, Kolkata, India; Nishant Dhankhar, Delhi, India; Joachim Studnia, Lycee Condorcet, Paris, France; WSA, L.T "Orizont", Moldova; Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Alok Kumar, Delhi, India; Mamedov Shatlyk, School of Young Physics and Maths No. 21. Dashoguz, Turkmenistan; Neculai Stanciu and Titu Zvonaru, Romania; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA; Alan Yan, Princeton Junction, New Jersey, United States of America; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Joel Schlosberg, Bayside, NY; Tan Qi Huan, Universiti Sains Malaysia, Malaysia; Yong Xi Wang, Affiliated High School of Shanxi University; Vincelot Ravoson, France, Paris, Lycée Henri IV; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Pamplona, Spain; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Eeshan Banerjee, West Bengal, India; Tolibjon Ismoilov, academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Francisco Javier Martínez Aguianga, student at UCM, Madrid, Spain; Michael Tang, USA; Nikos Kalapodis, Patras, Greece; Polyahedra, Polk State College, FL, USA; Georgios Tsapakidis, High School "Panagia Prousiotissa", Agrinio, Greece; David E. Manes, Oneonta, NY, USA.

J363. Solve in integers the system of equations

$$x^2 + y^2 - z(x+y) = 10$$
$$y^2 + z^2 - x(y+z) = 6$$
$$z^2 + x^2 - y(z+x) = -2$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Michael Tang, USA

Adding the equations, we get

$$2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx = 14 \implies (x-y)^2 + (y-z)^2 + (z-x)^2 = 14.$$

Thus, $|x-y|, |y-z|, |z-x| \leq 3$. Now subtracting the second equation from the first equation, we get

$$x^2 - z^2 - yz + xy = 4 \implies (x-z)(x+y+z) = 4.$$

Subtracting the third equation from the second equation, we get

$$y^2 - x^2 - xz + yz = 8 \implies (y-x)(x+y+z) = 8.$$

Thus, $y - x = 2(x-z)$ and $y - x \mid 8$, $x - z \mid 4$. Along with the fact $|y-x|, |z-x| \leq 3$, this forces $(y-x, x-z) = (2,1)$ or $(-2,-1)$. If $(y-x, x-z) = (2,1)$, then $x+y+z = 4$, and solving this system gives $(x,y,z) = (1,3,0)$, which satisfies the equations. If $(y-x, x-z) = (-2,-1)$, then $x+y+z = -4$, and solving this system gives $(x,y,z) = (-1,-3,0)$, which satisfies the equations. Therefore, the two solutions are $(x,y,z) = (1,3,0)$ and $(x,y,z) = (-1,-3,0)$.

Also solved by Arkady Alt, San Jose, CA, USA; Dimitris Avramidis, Evaggeliki Gymnasium, Athens, Greece; Joachim Studnia, Lycee Condorcet, Paris, France; Panigyraki Chrysoula, Evaggeliki Gymnasium, Athens, Greece; Cherlyse Alexander-Reid, College at Brockport, SUNY; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Albert Stadler, Herrliberg, Switzerland; Adithya Bhaskar, Mumbai, India; Daniel López-Aguayo, Centro de Ciencias Matemáticas UNAM, Morelia, Mexico; Polyahedra, Polk State College, FL, USA; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; David E. Manes, Oneonta, NY, USA; Vincelot Ravoson, France, Paris, Lycée Henri IV; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Alan Yan, Princeton Junction, New Jersey, United States of America; Daniel Lasaosa, Pamplona, Spain; Yong Xi Wang, Affiliated High School of Shanxi University; Neculai Stanciu and Titu Zvonaru, Romania; Alessandro Ventullo, Milan, Italy; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico.

J364. Consider a triangle ABC with circumcircle ω. Let O be the center of ω and let D, E, F be the midpoints of minor arcs BC, CA, AB respectively. Let DO intersect ω again at a point A'. Define B' and C' similarly. Prove that

$$\frac{[ABC]}{[A'B'C']} \leq 1.$$

Note that $[X]$ denotes the area of figure X.

Proposed by Taimur Khalid, Coral Academy of Science, Las Vegas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note that $A'B'C'$ and DEF are the result of a reflection upon O, hence their areas are the same. We will show that the proposed result is true if D, E, F are taken as the midpoints of arcs BC, CA, AB which do not contain A, B, C, the result being otherwise false for obtuse triangles. If ABC is rectangular wlog at A, then there is no minor arc BC, and again the result is true iff we chose the arc BC which does not contain A. We do so by considering three cases:

Case 1: If ABC is acute-angled, the minor arc BC and the arc BC which does not contain A coincide. In this case, and since OD, OE are the perpendicular bisectors of BC, CA, we have

$$\angle DOE = \angle DOC + \angle COE = \frac{1}{2}\angle BOC + \frac{1}{2}\angle COA = A + B,$$

and similarly for the other angles, or

$$[A'B'C'] = [DEF] = [DOE] + [EOF] + [FOD] =$$

$$= \frac{R^2}{2}\left(\sin(A+B) + \sin(B+C) + \sin(C+A)\right),$$

where R is the radius of ω. Similarly

$$[ABC] = [AOB] + [BOC] + [COA] = \frac{R^2}{2}\left(\sin(2A) + \sin(2B) + \sin(2C)\right),$$

and the conclusion follows by trivial application of Jensen's inequality because the sine function is strictly concave in $(0^\circ, 180^\circ)$, equality holding iff $A = B = C$, ie iff ABC is equilateral.

Case 2: If ABC is either right- or obtuse-angled, wlog at A, and D, E, F are taken as the midpoints of arcs BC, CA, AB which do not contain A, B, C, then

$$\angle DOE = \angle DOC + \angle COE = 180^\circ - \frac{1}{2}\angle BOC + \frac{1}{2}\angle COA = A + B,$$

and similarly for $\angle FOD$, whereas $\angle EOF$ is calculated as in Case 1, yielding again

$$[A'B'C'] = \frac{R^2}{2}\left(\sin(A+B) + \sin(B+C) + \sin(C+A)\right),$$

whereas

$$[ABC] = [AOB] - [BOC] + [COA] = \frac{R^2}{2}\left(\sin(2B) + \sin(2C) - \sin(360^\circ - 2A)\right) =$$

$$= \frac{R^2}{2}\left(\sin(2A) + \sin(2B) + \sin(2C)\right).$$

Now, $\sin(2A) \leq 0$ because A is right or obtuse, while $\frac{1}{2}\sin(2B) = \sin B \cos B < \sin B = \sin(A+C)$, and similarly $\frac{1}{2}\sin(2C) < \sin(A+B)$. Applying finally Jensen's inequality we obtain $\frac{1}{2}\sin(2B) + \frac{1}{2}\sin(2C) \leq \sin(B+C)$, or the proposed inequality holds strictly in this case.

Case 3: If ABC is obtuse-angled, wlog at A, and D, E, F are taken as the midpoints of minor arcs BC, CA, AB, or if ABC is right-angled, and D is taken as the midpoint of arc BC which contains A, then $\angle BOD = \angle COD = B + C$, $\angle COE = B$, and $\angle DOE = \angle COD - \angle COE = C$, or $\angle DFE = \frac{C}{2}$, and similarly $\angle DEF = \frac{B}{2}$. Therefore,

$$[A'B'C'] = [DEF] = 2R^2 \sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{B+C}{2} = 2R^2 \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}$$

whereas $[ABC] = 2R^2 \sin A \sin B \sin C$, or

$$\frac{[ABC]}{[A'B'C']} = 8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4\sin^2\frac{A}{2} + 2\cos B + 2\cos C > 1,$$

since as $B + C \leq 90^\circ$, either $\cos B$ or $\cos C$ is at least $\cos 45^\circ = \frac{\sqrt{2}}{2} > \frac{1}{2}$.

We conclude that the proposed inequality is true, with equality iff ABC is equilateral, as long at D, E, F are defined as the midpoints of arcs BC, CA, AB which do not contain A, B, C. It is false when ABC is obtuse if D, E, F are defined as the midpoints of minor arcs BC, CA, AB.

Also solved by Polyahedra, Polk State College, FL, USA; Tolibjon Ismoilov, academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Nikos Kalapodis, Patras, Greece; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Joel Schlosberg, Bayside, NY; Alan Yan, Princeton Junction, New Jersey, United States of America; Neculai Stanciu and Titu Zvonaru, Romania; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA.

J365. Let $x_1, x_2, \cdots, x_n$ be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Find the minimum possible value of

$$\sqrt{x_1+1}+\sqrt{2x_2+1}+\cdots+\sqrt{nx_n+1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

Claim: Let $u, v \geq 0$ and $k \geq 2$ a positive integer. Then,

$$\sqrt{u+1}+\sqrt{kv+1} \geq \sqrt{u+v+1}+1,$$

with equality iff $v = 0$.

Proof: Squaring both sides yields the equivalent inequality

$$(k-1)v + 2\sqrt{(u+1)(kv+1)} \geq 2\sqrt{u+v+1}.$$

Now,

$$(u+1)(kv+1) - u - v - 1 = (ku+k-1)v \geq 0,$$

with equality iff $v = 0$, and clearly $(k-1)v \geq 0$ with equality iff $v = 0$. The Claim follows.

By trivial induction over n, we can now prove that

$$\sqrt{x_1+1}+\sqrt{2x_2+1}+\cdots+\sqrt{nx_n+1} \geq \sqrt{2}+n-1,$$

with equality iff $x_1 = 1$ and $x_2 = x_3 = \cdots = x_n = 0$. Indeed, the result for $n = 2$ is equivalent to the Claim with $k = 2$. If the result is true for $n-1$, note that by the Claim with $k = n$ we have

$$\sqrt{x_1+1}+\sqrt{nx_n+1} \geq \sqrt{(x_1+x_n)+1}+1,$$

with equality iff $x_n = 0$, or renaming $x_1 + x_n$ as x_1 and applying the hypothesis of induction, the conclusion follows.

Also solved by Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Adithya Bhaskar, Mumbai, India; Neculai Stanciu and Titu Zvonaru, Romania; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Paul Revenant, Lycee, Champollion, Grenoble, France; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Polyahedra, Polk State College, FL, USA; Joel Schlosberg, Bayside, NY, USA.

J366. Prove that in any triangle ABC,

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\sqrt{6+\frac{r}{2R}}-1.$$

Proposed by Florin Stănescu, Găesti, România

Solution by Polyahedra, Polk State College, FL, USA

By squaring, we see that the inequality is equivalent to

$$6+\frac{r}{2R}\ge 1+\sum_{cyclic}\sin^2\frac{A}{2}+2\sum_{cyclic}\sin\frac{A}{2}+2\sum_{cyclic}\sin\frac{A}{2}\sin\frac{B}{2}.$$

Let $x=s-a$, $y=s-b$, and $z=s-c$. Then it is well known that

$$\frac{r}{2R}=1-\sum_{cyclic}\sin^2\frac{A}{2},\quad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}=\sqrt{\frac{yz}{(z+x)(x+y)}}.$$

Hence, it suffices to show that

$$3\ge\sum_{cyclic}\frac{yz}{(z+x)(x+y)}+\sum_{cyclic}\sqrt{\frac{yz}{(z+x)(x+y)}}+\sum_{cyclic}\frac{z}{x+y}\sqrt{\frac{xy}{(y+z)(z+x)}},$$

which, upon multiplying by $(x+y)(y+z)(z+x)$, becomes

$$3(x+y)(y+z)(z+x)\ge\sum_{cyclic}yz(y+z)+(x+y+z)\sum_{cyclic}\sqrt{xy(y+z)(z+x)}.$$

Now

$$3(x+y)(y+z)(z+x)-\sum_{cyclic}yz(y+z)=2(x+y+z)(xy+yz+zx),$$

and

$$\sum_{cyclic}\sqrt{xy(y+z)(z+x)}\le\sqrt{3\sum_{cyclic}xy(y+z)(z+x)}\le 2(xy+yz+zx),$$

where the last inequality follows from

$$4(xy+yz+zx)^2-3\sum_{cyclic}xy(y+z)(z+x)=\frac{1}{2}\left[(xy-yz)^2+(yz-zx)^2+(zx-xy)^2\right]\ge 0.$$

This completes the proof.

Also solved by Arkady Alt, San Jose, CA, USA; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; WSA, L.T "Orizont", Moldova; Bobojonova Latofat, academic lycuem S.H.Sirojiddinov, Tashkent, Uzbekistan; Yong Xi Wang, Affiliated High School of Shanxi University; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Nikos Kalapodis, Patras, Greece; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Scott H. Brown, Auburn University Montgomery, AL, USA.

Senior problems

S361. Find all integers n for which there are integers a and b such that $(a+bi)^4 = n + 2016i$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

Notice that by equating the real and imaginary parts we get,

$$a^4 + b^4 - 6a^2b^2 = n$$

and

$$4a^3b - 4ab^3 = 4ab(a+b)(a-b) = 2016.$$

So, we need integer pairs (a,b) such that $ab(a+b)(a-b) = 504$. Notice that if (a,b) is a solution, then so are $(-a,-b),(b,-a),(-b,a)$. So, we'll assume W.L.O.G that $a > b > 0$. Let $d = a-b$. Then $db(d+b)(d+2b) = 504$. Notice that the highest power of three dividing 504 is 2. So clearly, exactly one of the four factors must be divisible by 9. Observe that $db(d+b)(d+2b) = 504 < 546 = 1\cdot 6\cdot(1+6)\cdot(1+12)$. Thus $b \le 5$. Similarly, $db(d+b)(d+2b) = 504 = 7\cdot 1\cdot(7+1)\cdot(7+2)$. Thus $d \le 7$. It is evident that neither d nor b can be a multiple of 9. So, either $d+b = 9$ or $d+2b = 9$ (because $d+2b \le 7+2\cdot 5 = 17 < 18$). So, the possible pairs are $(7,2),(5,4),(4,5)$ for (d,b) when $d+b=9$. But $5 \nmid 504$ and so we reject the last two pairs. It is seen that for $(d,b) = (7,2)$, $d-b = 5$, but clearly $5 \nmid 504$. So, no solutions for the case $d+b=9$. Moving on, for $d+2b=9$, we have the following possible pairs $(7,1),(5,2),(1,4)$. Since $5 \nmid 504$, we reject the middle pair, while $d+b=5$ for $(d,b)=(1,4)$ and since $5 \nmid 504$, we reject the last pair. It is clear that the first pair $(7,1)$ satisfies the equation. This gives us $(a,b)=(8,1)$ and so in summary the possible solutions are $(a,b) = (8,1),(-8,-1),(-1,8),(1,-8)$. Hence the only possible value of n is $8^4+1^4-6(8\cdot 1)^2 = 3713$.

Also solved by Adithya Bhaskar, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Joel Schlosberg, Bayside, NY; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Khurshid Juraev, academic lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain; Daniel López-Aguayo, Centro de Ciencias Matemáticas UNAM, Morelia, Mexico; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Li Zhou, Polk State College, Winter Haven, FL, USA; Alessandro Ventullo, Milan, Italy.

S362. Let $0 < a, b, c, d \leq 1$. Prove that

$$\frac{1}{a+b+c+d} \geq \frac{1}{4} + \frac{64}{27}(1-a)(1-b)(1-c)(1-d).$$

Proposed by Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

By the arithmetic mean - geometric mean inequality,

$$(1-a)(1-b)(1-c)(1-d) \leq \left(\frac{4-a-b-c-d}{4}\right)^4,$$

so it suffices to show that

$$\frac{1}{a+b+c+d} \geq \frac{1}{4} + \frac{1}{108}(4-a-b-c-d)^4.$$

Let $x = a+b+c+d$. Then we must show that

$$\frac{1}{x} \geq \frac{1}{4} + \frac{1}{108}(4-x)^4,$$

for $0 < x \leq 4$. This is equivalent to

$$(x-1)^2(x-4)\left[(x-5)^2+2\right] \leq 0,$$

which is clearly true for $0 < x \leq 4$. Equality holds for $x = 1$ and $x = 4$, which translates to $a = b = c = d = \frac{1}{4}$ and $a = b = c = d = 1$, respectively.

Also solved by Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Yong Xi Wang, Affiliated High School of Shanxi University; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Li Zhou, Polk State College, Winter Haven, FL, USA; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Daniel Lasaosa, Pamplona, Spain; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy.

S363. Determine if there are distinct positive integers $n_1, n_2, ..., n_{k-1}$ such that

$$(3n_1^2 + 4n_2^2 + ... + (k+1)n_{k-1}^2)^3 = 2016(n_1^3 + n_2^3 + ... + n_{k-1}^3)^2$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina

Using Holder's inequality we have

$$\left[3^3 + 4^3 + ... + (k+1)^3\right] \cdot \left[n_1^3 + n_2^3 + ... + n_{k-1}^3\right]^2 \geq \left[3n_1^2 + 4n_2^2 + ... + (k+1)n_{k-1}^2\right]^3 = 2016(n_1^3 + ... + n_{k-1}^3)^2 \Rightarrow$$

$$3^3 + 4^3 + ... + (k+1)^3 \geq 2016 \Leftrightarrow 1^3 + ... + (k+1)^3 \geq 2025 \Leftrightarrow$$

$$\left[\frac{(k+1)(k+2)}{2}\right]^2 \geq 2025 \Leftrightarrow \frac{(k+1)(k+2)}{2} \geq 45 \Rightarrow k \geq 8$$

Equality holds if and only if $k = 8, n_1 = 3, n_2 = 4, ..., n_7 = 9$

Also solved by Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Li Zhou, Polk State College, Winter Haven, FL, USA; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico.

S364. Let a, b, c be nonnegative real numbers such that $a \geq 1 \geq b \geq c$ and $a+b+c=3$. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{2(a^2+b^2+c^2)}{3(ab+bc+ca)}+\frac{5}{6}.$$

Proposed by Marius Stănean, Zalău, România

Solution by Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA

Assign variables $s = ab+bc+ca$ and $p = abc$. So, we have to prove the inequality

$$\frac{3p-2(a+b+c)s+(a+b+c)^3}{(a+b+c)s-p} \geq \frac{2(a+b+c)^2-4s}{3s}+\frac{5}{6},$$

which is equivalent to

$$\frac{3p-6s+27}{3s-p} \geq \frac{18-4s}{3s}+\frac{5}{6}.$$

After simplifying, this inequality becomes $p \geq \dfrac{9s(s-2)}{5s+12}$. Since $a \geq 1 \geq b \geq c$, we have that $(a-1)(1-b)(1-c) \geq 0$. This implies that

$$a+b+c-1-ab-bc-ca+abc \geq 0 \Leftrightarrow p \geq s-2.$$

If $s < 2$, then the result is trivially true. Otherwise, $3s = 3(ab+ac+bc) \leq (a+b+c)^2 = 9$, hence $s \leq 3 \Rightarrow 5s+12 \geq 9s$, so $p \geq s-2 \geq \dfrac{9s(s-2)}{5s+12}$.

Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Daniel Lasaosa, Pamplona, Spain.

S365. Let

$$a_k = \frac{(k^2+1)^2}{k^4+4}, \quad k = 1, 2, 3, \cdots$$

Prove that for every positive integer n,

$$a_1^n a_2^{n-1} a_3^{n-2} \cdots a_n = \frac{2^{n+1}}{n^2+2n+2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let $c_k = k^2+1$ and $d_k = k^4+1$. Then

$$\begin{aligned} c_{k-1}c_{k+1} &= \left[(k-1)^2+1\right]\left[(k+1)^2+1\right] \\ &= (k^2-2k+2)(k^2+2k+2) = k^4+4 = d_k, \end{aligned}$$

so

$$a_k = \frac{c_k^2}{c_{k-1}c_{k+1}}.$$

Then, for any positive integer n,

$$\begin{aligned} a_1^n a_2^{n-1} a_3^{n-2} \cdots a_n &= \prod_{k=1}^{n} a_k^{n+1-k} = \prod_{k=1}^{n} \frac{c_k^{2n+2-k}}{c_{k-1}^{n+1-k}c_{k+1}^{n+1-k}} \\ &= \frac{c_1^{2n-(n-1)}}{c_0^n} \cdot \prod_{k=2}^{n} \frac{c_k^{2n+2-2k}}{c_k^{n+2-k}c_k^{n-k}} \cdot \frac{1}{c_{n+1}} \\ &= \frac{c_1^{n+1}}{c_0^n} \cdot 1^{n-1} \cdot \frac{1}{c_{n+1}} \\ &= \frac{c_1^{n+1}}{c_0^n c_{n+1}} \end{aligned}$$

With $c_0 = 1$, $c_1 = 2$, and $c_{n+1} = (n+1)^2+1 = n^2+2n+2$, it follows that

$$a_1^n a_2^{n-1} a_3^{n-2} \cdots a_n = \frac{2^{n+1}}{n^2+2n+2}.$$

Also solved by Arkady Alt, San Jose, CA, USA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA; Adithya Bhaskar, Mumbai, India; Li Zhou, Polk State College, Winter Haven, FL, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; G. C. Greubel, Newport News, VA; Joel Schlosberg, Bayside, NY; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Daniel Lasaosa, Pamplona, Spain; Moubinool Omarjee Lycée Henri IV, Paris, France; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Mamedov Shatlyk, School of Young Physics and Maths No. 21. Dashoguz, Turkmenistan; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu and Titu Zvonaru, Romania; Alessandro Ventullo, Milan, Italy; WSA, L.T "Orizont", Moldova.

S366. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$. Prove that

$$9+\frac{1}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 \geq \frac{70}{ab+bc+cd+da+ac+bd}.$$

Proposed by Marius Stănean, Zalău, România

Solution by Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina

Let $a+b+c+d=4p, ab+ac+ad+bc+bd+cd=6q^2, abc+acd+bcd+cda=4r^3, abcd=s^4$

$$9+\frac{1}{6}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d})^2 \geq \frac{70}{ab+ac+ad+bc+bd+cd} \Leftrightarrow$$

$$9+\frac{1}{6}\cdot p^2\cdot(\frac{4r^3}{s^4})^2 \geq \frac{70p^2}{6q^2} \Leftrightarrow 27q^2s^8+8p^2\cdot q^2\cdot r^6 \geq 35p^2s^8$$

Using Newton's inequalities, we have

$$q^2 \geq pr, \wedge r^2 \geq qs \Rightarrow q^4 \geq p^2r^2 \geq p^2qs \Rightarrow q^3 \geq p^2s \Rightarrow q \geq p^{\frac{2}{3}}\cdot s^{\frac{1}{3}}$$

$$r^2 \geq qs \geq p^{\frac{2}{3}}\cdot s^{\frac{4}{3}} \Rightarrow r \geq p^{\frac{1}{3}}\cdot s^{\frac{2}{3}}$$

Using these two we get

$$27q^2s^8+8p^2\cdot q^2\cdot r^6 \geq 27\cdot p^{\frac{4}{3}}\cdot s^{\frac{26}{3}}+8\cdot p^{\frac{16}{3}}\cdot s^{\frac{14}{3}}$$

Hence it is suffices to prove

$$27\cdot p^{\frac{4}{3}}\cdot s^{\frac{26}{3}}+8\cdot p^{\frac{16}{3}}\cdot s^{\frac{14}{3}} \geq 35p^2s^8$$

Using AmGm inequality we have

$$27\cdot p^{\frac{4}{3}}\cdot s^{\frac{26}{3}}+8\cdot p^{\frac{16}{3}}\cdot s^{\frac{14}{3}} \geq 35\sqrt[35]{p^{\frac{236}{3}}\cdot s^{\frac{814}{3}}}$$

Hence it is sufficies to prove

$$35\sqrt[35]{p^{\frac{236}{3}}\cdot s^{\frac{814}{3}}} \geq 35p^2s^8 \Leftrightarrow p^{26} \geq s^{26}$$

Which is an immediate consequence of AmGm inequality. Equality holds if and only if $a=b=c=d=1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania.

U361. Consider all possible ways one can assign the numbers 1 through 10 with a nonnegative probability so that the probabilities sum to 1. Let X be the number selected. Suppose that $E[X]^k = E[X^k]$ for a given integer $k \geq 2$. Find the number of possible ways of assigning these probabilities.

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

Solution by Li Zhou, Polk State College, FL, USA

Let $p_1, p_2, \ldots, p_{10}$ be the probabilities assigned to $1, 2, \ldots, 10$, respectively. Then

$$E[X]^k = (p_1 + 2p_2 + \cdots + 10p_{10})^k, \quad E[X^k] = 1^k p_1 + 2^k p_2 + \cdots + 10^k p_{10},$$

which are clearly equal if one of the p_i's is 1 and the others are 0. We show that these are the only possibilities.

If $p_{10} = 1$, then we are done. So consider that $p_{10} < 1$. Let $x = \frac{p_1+2p_2+\cdots+9p_9}{p_1+p_2+\cdots+p_9}$. Then $x \leq 9$. By the convexity of t^k for $t \geq 0$, we have

$$E[X]^k = [(1-p_{10})x + p_{10} \cdot 10]^k \leq (1-p_{10})x^k + p_{10} \cdot 10^k \leq E[X^k],$$

where the first inequality is equality if and only if $p_{10} = 0$. Thus $p_{10} = 0$ and $E[X]^k = x^k$. Repeating this argument, we get $p_9 = 1$ or $p_9 = 0$, and so on. We conclude that there are 10 ways to assign these probabilities.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland.

U362. Let

$$S_n = \sum_{1\le i<j<k\le n} q^{i+j+k},$$

where $q \in (-1,0) \cup (0,1)$. Evaluate $lim_{n\to\infty} S_n$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Albert Stadler, Herrliberg, Switzerland

$$\lim_{n\to\infty} S_n = \sum_{1\le i<j<k} q^{i+j+k} = \sum_{1\le i<j} q^{i+j} \cdot \frac{q^{j+1}}{1-q} = \frac{1}{1-q} \sum_{1\le i<j} q^{i+2j+1} =$$

$$= \frac{1}{1-q} \sum_{1\le i} q^{i+1} \cdot \frac{q^{2i+2}}{1-q^2} = \frac{1}{(1-q)(1-q^2)} \sum_{1\le i} q^{3i+3} = \frac{q^6}{(1-q)(1-q^2)(1-q^3)}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Adnan Ali, A.E.C.S-4, Mumbai, India; Li Zhou, Polk State College, FL, USA; Yong Xi Wang, High School of Shanxi University, China.

U363. Let a be a positive number. Prove that there is a number $\theta = \theta(a)$, $1 < \theta < 2$, such that

$$\sum_{j=0}^{\infty} \left| \binom{a}{j} \right| = 2^a + \theta \left| \binom{a-1}{\lfloor a \rfloor + 1} \right|.$$

Furthermore, prove that

$$\left| \binom{a-1}{\lfloor a \rfloor + 1} \right| \le \frac{|\sin \pi a|}{\pi a}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Li Zhou, Polk State College, FL, USA

By the binomial series,

$$\sum_{j=0}^{\infty} \left| \binom{a}{j} \right| - 2^a = 2 \left| \sum_{i=1}^{\infty} \binom{a}{\lfloor a \rfloor + 2i} \right|.$$

Let $n = \lfloor a \rfloor$ and $x = a - \lfloor a \rfloor \in [0, 1)$. Then

$$\begin{aligned}
\frac{\theta}{2} &= \left| \frac{1}{\binom{a-1}{\lfloor a \rfloor + 1}} \sum_{i=1}^{\infty} \binom{a}{\lfloor a \rfloor + 2i} \right| \\
&= \frac{n+x}{n+2} + \frac{(n+x)(2-x)(3-x)}{(n+2)(n+3)(n+4)} + \frac{(n+x)(2-x)\cdots(5-x)}{(n+2)\cdots(n+6)} + \cdots \\
&= \left(1 - \frac{2-x}{n+2}\right) + \left(\frac{(2-x)(3-x)}{(n+2)(n+3)} - \frac{(2-x)(3-x)(4-x)}{(n+2)(n+3)(n+4)}\right) \\
&\qquad + \left(\frac{(2-x)\cdots(5-x)}{(n+2)\cdots(n+5)} - \frac{(2-x)\cdots(6-x)}{(n+2)\cdots(n+6)}\right) + \cdots \\
&= 1 - \frac{2-x}{n+2}\left(1 - \frac{3-x}{n+3}\right) - \frac{(2-x)(3-x)(4-x)}{(n+2)(n+3)(n+4)}\left(1 - \frac{5-x}{n+5}\right) - \cdots < 1,
\end{aligned}$$

since n and x cannot both be 0. Thus $\theta < 2$.

Next, if $n \ge 2$, then $\frac{n+x}{n+2} \ge \frac{1}{2}$, so $\theta > 1$.

Now consider $n = 0$ and $0 < x < 1$. Then

$$\begin{aligned}
\theta &= 2\left(\frac{x}{2!} + \frac{x(x-2)(x-3)}{4!} + \frac{x(x-2)\cdots(x-5)}{6!} + \cdots\right) \\
&= \frac{(1+1)^x + (1-1)^x - 2}{x-1} = \frac{2-2^x}{1-x} > 1,
\end{aligned}$$

where the inequality follows from $2^x < 1 + x$.

It remains to consider $n = 1$ and $0 \le x < 1$. Then

$$\begin{aligned}
\theta &= 4\left(\frac{x+1}{3!} + \frac{(x+1)(x-2)(x-3)}{5!} + \frac{(x+1)(x-2)\cdots(x-5)}{7!} + \cdots\right) \\
&= \frac{2\left((1+1)^{x+1} - (1-1)^{x+1} - 2(x+1)\right)}{x(x-1)} = \frac{4\left(1 + x - 2^x\right)}{x(1-x)}.
\end{aligned}$$

Let $f(x) = 4\left(1 + x - 2^x\right) - x(1-x)$. Then $f(0) = f(1) = 0$ and $f'(x) = 3 + 2x - 2^{x+2}\ln 2$. Note that $f'(0) = 3 - 4\ln 2 > 0$, $f'(1) = 5 - 8\ln 2 < 0$, and $f'(x)$ has only one 0 for $0 < x < 1$. Hence, for $0 < x < 1$, $f(x) > 0$ and thus $\theta > 1$. For $x = 0$, $\theta = 4(1 - \ln 2) > 1$ by L'Hôpital's rule.

Finally, for the second part, we start with $|\sin \pi a| = \sin \pi x$ and use the well-known fact that $\frac{\pi}{\sin \pi x} = \Gamma(x)\Gamma(1-x)$. Therefore,

$$\frac{\pi a}{\sin \pi x} \left| \binom{a-1}{\lfloor a \rfloor + 1} \right| = \left| \frac{a\Gamma(a)\Gamma(2-x)}{\Gamma(a+2-x)} \right| = a\int_0^1 t^{a-1}(1-t)^{1-x} dt \le a\int_0^1 t^{a-1} dt = 1,$$

completing the proof.

U364. Evaluate

$$\int \frac{5x^2 - x - 4}{x^5 + x^4 + 1} dx.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alok Kumar, Delhi, India

Note that $P(x) = x^5 + x^4 + 1$ factors as

$$x^5 + x^4 + 1 = (x^2 + x + 1)(x^3 - x + 1).$$

An elegant way to see this is to observe that $P(w) = 0$, where w is a primitive cube root of unity.
To split the numerator $5x^2 - x - 4$, as to render the integrand integrable, we rewrite it as:

$$\begin{aligned} 5x^2 - x - 4 = (x-1)(5x+4) & \\ &= x^3(5x+4) - (x^3 - x + 1)(5x+4) \\ &= (5x^4 + 4x^3) - (x^3 - x + 1)(5x+4) \end{aligned}$$

Note that $5x^4 + 4x^3$ is the derivative of $x^5 + x^4 + 1$ with respect to x. Now the stage is set for integration.

$$\begin{aligned} \int \frac{5x^2 - x - 4}{x^5 + x^4 + 1} \mathrm{dx} &= \int \frac{(5x^4 + 4x^3) - (x^3 - x + 1)(5x+4)}{x^5 + x^4 + 1} \mathrm{dx} \\ &= \int \frac{5x^4 + 4x^3}{x^5 + x^4 + 1} \mathrm{dx} - \int \frac{5x + 4}{x^2 + x + 1} \mathrm{dx} \\ &= \ln|x^5 + x^4 + 1| - \frac{1}{2} \int \frac{5(2x+1) + 3}{x^2 + x + 1} \mathrm{dx} \\ &= \ln|x^5 + x^4 + 1| - \frac{5}{2} \int \frac{2x+1}{x^2 + x + 1} \mathrm{dx} - \frac{3}{2} \int \frac{\mathrm{dx}}{(x + \frac{1}{2})^2 + \frac{3}{4}} \\ &= \ln|x^5 + x^4 + 1| - \frac{5}{2} \ln|x^2 + x + 1| - \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \\ &= \ln|x^5 + x^4 + 1| - \frac{5}{2} \ln|x^2 + x + 1| - \sqrt{3} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \lambda, \end{aligned}$$

where λ is an arbitrary constant.

Also solved by Daniel Lasaosa, Pamplona, Spain; Alessandro Ventullo, Milan, Italy; Albert Stadler, Herrliberg, Switzerland; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Moubinool Omarjee Lycée Henri IV , Paris France; Daniel López-Aguayo, Centro de Ciencias Matemáticas UNAM, Morelia, Mexico; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Joel Schlosberg, Bayside, NY, USA; Henry Ricardo, New York Math Circle, USA; G. C. Greubel, Newport News, VA, USA; David E. Manes, Oneonta, NY, USA; Cherlyse Alexander - Reid, College at Brockport, SUNY, USA; Behzod Kurbonboev, National University of Uzbekistan, Tashkent, Uzbekistan; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Li Zhou, Polk State College, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Ravoson Vincelot, Lycee Henri IV, Paris, France; Yong Xi Wang, High School of Shanxi University, China; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA.

U365. Let n be a positive integer. Evaluate

(a) $\int_0^n e^{\lfloor x \rfloor} dx$,

(b) $\int_0^n \lfloor e^x \rfloor dx$,

where $\lfloor a \rfloor$ denotes the integer part of a.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

(a) Clearly, $e^{\lfloor x \rfloor} = e^{n-1}$ when $n \in [n-1, n)$, or

$$\int_0^n e^{\lfloor x \rfloor} dx = \sum_{k=1}^{n} \int_{k-1}^{k} e^{\lfloor x \rfloor} dx = \sum_{k=1}^{n} e^{k-1} \int_{k-1}^{k} dx = \sum_{k=1}^{n} e^{k-1} = \frac{e^n - 1}{e - 1}.$$

(b) Let $N = \lfloor e^n \rfloor$, or $\ln(N) \le n < \ln(N+1)$. Note that $\lfloor e^x \rfloor = k$ for all $\ln(k) \le x < \ln(k+1)$, or

$$\int_0^n \lfloor e^x \rfloor dx = \sum_{k=1}^{N-1} \int_{\ln(k)}^{\ln(k+1)} \lfloor e^x \rfloor dx + \int_{\ln N}^{n} \lfloor e^x \rfloor dx = \sum_{k=1}^{N-1} k \int_{\ln(k)}^{\ln(k+1)} dx + N \int_{\ln N}^{n} dx =$$

$$= Nn - \ln(N) - \ln(N-1) - \cdots - \ln(1) = Nn - \ln(N!) = n\lfloor e^n \rfloor - \ln\left(\lfloor e^n \rfloor !\right).$$

Also solved by Alessandro Ventullo, Milan, Italy; Albert Stadler, Herrliberg, Switzerland; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Moubinool Omarjee Lycée Henri IV , Paris France; Daniel López-Aguayo, Centro de Ciencias Matemáticas UNAM, Morelia, Mexico; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Joel Schlosberg, Bayside, NY, USA; Henry Ricardo, New York Math Circle, USA; G. C. Greubel, Newport News, VA, USA; Problem Solving Group, Department of Financial and Management Engineering, University of the Aegean, Greece; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Li Zhou, Polk State College, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Ravoson Vincelot, Lycee Henri IV, Paris, France; Yong Xi Wang, High School of Shanxi University, China; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Adithya Bhaskar, Mumbai, India.

U366. If $f : [0,1] \to \mathbb{R}$ is a convex and integrable function with $f(0) = 0$, prove that

$$\int_0^1 f(x)dx \geq 4 \int_0^{\frac{1}{2}} f(x)dx.$$

Proposed by Florin Stănescu, Găeşti, România

Solution by Alessandro Ventullo, Milan, Italy

Since f is a convex function, we have

$$\begin{aligned} \int_0^1 f(x)\ dx = \frac{1}{2}\left[\int_0^1 f(x)\ dx + \int_0^1 f(1-x)\ dx\right] &= \int_0^1 \frac{f(x)+f(1-x)}{2}\ dx \\ &\geq \int_0^1 f\left(\frac{x+(1-x)}{2}\right)\ dx \\ &= f\left(\frac{1}{2}\right). \end{aligned}$$

On the other hand, since f is convex and $f(0) = 0$, we have

$$\begin{aligned} f\left(\frac{1}{2}\right) = 2 \cdot \frac{f(0) + f\left(\frac{1}{2}\right)}{2} &= 2\int_0^1 \left[(1-x)f(0) + xf\left(\frac{1}{2}\right)\right]\ dx \\ &\geq 2\int_0^1 f\left((1-x)\cdot 0 + x \cdot \frac{1}{2}\right)\ dx \\ &= 4\int_0^{\frac{1}{2}} f(x)\ dx, \end{aligned}$$

and the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Daniel López-Aguayo, Centro de Ciencias Matemáticas UNAM, Morelia, Mexico; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Joel Schlosberg, Bayside, NY, USA; Henry Ricardo, New York Math Circle, USA; Bekhzod Kurbonboev, National University of Uzbekistan, Tashkent, Uzbekistan; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Li Zhou, Polk State College, FL, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Ravoson Vincelot, Lycee Henri IV, Paris, France; Yong Xi Wang, High School of Shanxi University, China; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Adithya Bhaskar, Mumbai, India.

Olympiad problems

O361. Determine the least integer $n > 2$ such that there are n consecutive integers whose sum of squares is a perfect square.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Li Zhou, Polk State College, FL, USA

Notice that $18^2 + 19^2 + \cdots + 28^2 = 77^2$. We shall show that n cannot be < 11. First,

$$(k-1)^2 + k^2 + (k+1)^2 = 3k^2 + 2 \equiv 2 \pmod 3,$$

$$(k-1)^2 + k^2 + (k+1)^2 + (k+2)^2 = 4k^2 + 4k + 6 \equiv 2 \pmod 4,$$
$$(k-2)^2 + (k-1)^2 + \cdots + (k+3)^2 = 6k(k+1) + 19 \equiv 3 \pmod 4,$$
$$(k-3)^2 + (k-2)^2 + \cdots + (k+4)^2 = 4[2k(k+1) + 11].$$

Because 2 is not a quadratic residue (mod 3), 2 and 3 are not quadratic residues (mod 4), and $2k(k+1)+11 \equiv 3 \pmod 4$, we see that n cannot be $3, 4, 6, 8$. Next,

$$(k-2)^2 + (k-1)^2 + \cdots + (k+2)^2 = 5(k^2 + 2),$$

$$(k-3)^2 + (k-2)^2 + \cdots + (k+3)^2 = 7(k^2 + 4),$$
$$(k-4)^2 + (k-3)^2 + \cdots + (k+4)^2 = 3(3k^2 + 20),$$
$$(k-4)^2 + (k-3)^2 + \cdots + (k+5)^2 = 5(2k^2 + 2k + 17).$$

Because $k^2 + 2 \not\equiv 0 \pmod 5$, $k^2 + 4 \not\equiv 0 \pmod 7$, $20 \not\equiv 0 \pmod 3$, and $2k^2 + 2k + 17 \not\equiv 0 \pmod 5$, we see that n cannot be $5, 7, 9, 10$. Hence, the least such integer is $n = 11$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Khurshid Juraev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; José Hernández Santiago; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; David E. Manes, Oneonta, NY, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Michael Tang, USA; Hyun Min Victoria Woo, Northfield Mount Hermon School, Mount Hermon, MA, USA; Adithya Bhaskar, Mumbai, India.

O362. Let (F_n), $n \geq 0$, with $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$. Prove that the following identities hold:

(a) $\frac{F_{3n}}{F_n} = 2\left({F_{n-1}}^2 + {F_{n+1}}^2\right) - F_{n-1}F_{n+1}$.

(b) $\binom{2n+1}{0}F_{2n+1} + \binom{2n+1}{1}F_{2n-1} + \binom{2n+1}{2}F_{2n-3} + \cdots + \binom{2n+1}{n}F_1 = 5^n$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let (F_n) denote the Fibonacci sequence; that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$. The Binet formula for the nth Fibonacci number is

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad \text{where} \quad \alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Note

$$\alpha\beta = -1, \quad \alpha + \frac{1}{\alpha} = \sqrt{5}, \quad \beta + \frac{1}{\beta} = -\sqrt{5},$$

and

$$\alpha^2 + \frac{1}{\alpha^2} = \beta^2 + \frac{1}{\beta^2} = \alpha^2 + \beta^2 = 3.$$

(a) Using the Binet formula,

$$\frac{F_{3n}}{F_n} = \frac{\alpha^{3n} - \beta^{3n}}{\alpha^n - \beta^n} = \alpha^{2n} + \alpha^n\beta^n + \beta^{2n} = \alpha^{2n} + \beta^{2n} + (-1)^n.$$

Next,

$$\begin{aligned} F_{n-1}^2 &= \frac{1}{5}\left(\alpha^{2n-2} + \beta^{2n-2} + 2(-1)^n\right), \text{ and} \\ F_{n+1}^2 &= \frac{1}{5}\left(\alpha^{2n+2} + \beta^{2n+2} + 2(-1)^n\right), \end{aligned}$$

so that

$$\begin{aligned} 2(F_{n-1}^2 + F_{n+1}^2) &= \frac{2}{5}\left(\alpha^2 + \frac{1}{\alpha^2}\right)\alpha^{2n} + \frac{2}{5}\left(\beta^2 + \frac{1}{\beta^2}\right)\beta^{2n} + \frac{8}{5}(-1)^n \\ &= \frac{6}{5}\alpha^{2n} + \frac{6}{5}\beta^{2n} + \frac{8}{5}(-1)^n. \end{aligned}$$

Moreover,

$$\begin{aligned} F_{n-1}F_{n+1} &= \frac{1}{5}\left(\alpha^{2n} - \alpha n - 1\beta^{n+1} - \alpha^{n+1}\beta^{n-1} + \beta^{2n}\right) \\ &= \frac{1}{5}\left(\alpha^{2n} - \alpha^{n-1}\beta^{n-1}(\alpha^2 + \beta^2) + \beta^{2n}\right) \\ &= \frac{1}{5}\alpha^{2n} + \frac{1}{5}\beta^{2n} + \frac{3}{5}(-1)^n. \end{aligned}$$

Thus,

$$2(F_{n-1}^2 + F_{n+1}^2) - F_{n-1}F_{n+1} = \alpha^{2n} + \beta^{2n} + (-1)^n = \frac{F_{3n}}{F_n}.$$

(b) First note

$$\begin{aligned}\sum_{k=0}^{2n+1}\binom{2n+1}{k}F_{2n+1-2k} &= \sum_{k=0}^{n}\binom{2n+1}{k}F_{2n+1-2k}+\sum_{k=n+1}^{2n+1}\binom{2n+1}{k}F_{2n+1-2k}\\ &= \sum_{k=0}^{n}\binom{2n+1}{k}F_{2n+1-2k}+\sum_{k=0}^{n}\binom{2n+1}{2n+1-k}F_{2k-(2n+1)}\\ &= 2\sum_{k=0}^{n}\binom{2n+1}{k}F_{2n+1-2k},\end{aligned}$$

where we have used the identities

$$F_{-n}=(-1)^{n+1}F_n \quad \text{and} \quad \binom{n}{k}=\binom{n}{n-k}.$$

Now,

$$\begin{aligned}&\sum_{k=0}^{2n+1}\binom{2n+1}{k}F_{2n+1-2k}\\ &= \frac{1}{\sqrt{5}}\left(\sum_{k=0}^{2n+1}\binom{2n+1}{k}\alpha^{2n+1-2k}-\sum_{k=0}^{2n+1}\binom{2n+1}{k}\beta^{2n+1-2k}\right)\\ &= \frac{1}{\sqrt{5}}\left(\alpha^{-(2n+1)}\sum_{k=0}^{2n+1}\binom{2n+1}{k}\alpha^{4n+2-2k}-\beta^{-(2n+1)}\sum_{k=0}^{2n+1}\binom{2n+1}{k}\beta^{4n+2-2k}\right)\\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\alpha^2}{\alpha}\right)^{2n+1}-\left(\frac{1+\beta^2}{\beta}\right)^{2n+1}\right]\\ &= \frac{1}{\sqrt{5}}\left(\sqrt{5}^{2n+1}+\sqrt{5}^{2n+1}\right)=2\cdot 5^n.\end{aligned}$$

Thus,

$$\sum_{k=0}^{n}\binom{2n+1}{k}F_{2n+1-2k}=\frac{1}{2}\sum_{k=0}^{2n+1}\binom{2n+1}{k}F_{2n+1-2k}=5^n.$$

Also solved by Arkady Alt, San Jose, CA, USA; Daniel Lasaosa, Pamplona, Spain; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Jorge Ledesma, Faculty of Sciences UNAM, Mexico City, Mexico; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Li Zhou, Polk State College, FL, USA; Ravoson Vincelot, Lycee Henri IV, Paris, France.

O363. Solve in integers the system of equations

$$x^2+y^2+z^2+\frac{xyz}{3}=2\left(xy+yz+zx+\frac{xyz}{3}\right)=2016.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, FL, USA

It is easy to verify that all six permutations of $\{x,y,z\}=\{12,-12,-24\}$ are solutions. We show that they are the only solutions. By symmetry, it suffices to consider $x \geq y \geq z$. Subtracting the second equation from twice the first, we get $(x-y)^2+(y-z)^2+(z-x)^2=2016$. Because $\{0,1,4,9\}$ is the complete set of quadratic residues (mod 16), $x-y \equiv y-z \equiv z-x \equiv 0 \pmod 4$. Now $2016/16=126$, and we have only $126=11^2+2^2+1^2=10^2+5^2+1^2=9^2+6^2+3^2$. Also, $x-z=(x-y)+(y-z)$. Thus we must have $x-z=36$ and $\{x-y,y-z\}=\{24,12\}$. If $x-y=12$, then $4032=(x+y+z)^2+xyz=(3x-48)^2+x(x-12)(x-36)$, which implies $x=3u$ and yields $0=u^3-13u^2+16u-64$. This has no integer solution by the rational root theorem. If $x-y=24$, then $4032=(3x-60)^2+x(x-24)(x-36)$, which implies $x=3u$ and yields $0=(u-4)(u^2-13u+4)$. Hence, $u=4$ and $(x,y,z)=(12,-12,-24)$, completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Adnan Ali, A.E.C.S-4, Mumbai, India.

O364. (a) If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where p_i are distinct primes, find the value of

$$\sum_{d|n} \frac{n\phi(d)}{d}$$

as a function of $\{p_i\}$ and $\{e_i\}$.

(b) Find the number of integral solutions to $x^x \equiv 1 \pmod{97}$, $1 \le x \le 9312$.

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

Solution by Li Zhou, Polk State College, FL, USA

(a) Since $\frac{\phi(n)}{n}$ is a multiplicative function of n, so is $f(n) = \sum_{d|n} \frac{\phi(d)}{d}$. (See G. H. Hardy & E. M. Wright, *An Intro. to the Theory of Numbers*, 5th ed., Oxford, p. 235.) Now for any prime p,

$$f(p^e) = \sum_{i=0}^{e} \frac{\phi(p^i)}{p^i} = 1 + \sum_{i=1}^{e} \frac{p-1}{p} = 1 + e\left(1 - \frac{1}{p}\right).$$

Hence,

$$\sum_{d|n} \frac{n\phi(d)}{d} = nf(n) = n\prod_{i=1}^{k}\left[1 + e_i\left(1 - \frac{1}{p_i}\right)\right] = p_1^{e_1-1} \cdots p_k^{e_k-1} \prod_{i=1}^{k} \left[p_i(1+e_i) - e_i\right].$$

(b) Notice that $9312 = 97 \cdot 96$ and $x \not\equiv 0 \pmod{97}$. So we consider $x = 97q + r$, with $0 \le q \le 95$ and $1 \le r \le 96$. Then $x^x \equiv r^{97q+r} \equiv r^{q+r} \pmod{97}$. Now for any prime p, it is well known that if $d|(p-1)$, then $r^d \equiv 1 \pmod{p}$ has exactly d roots. (See G. H. Hardy & E. M. Wright, *An Intro. to the Theory of Numbers*, 5th ed., Oxford, p. 85.) Also, recall that the order of r is the least positive integer such that $r^d \equiv 1 \pmod{p}$. Hence, there are exactly $\phi(d)$ roots r of order d. For each such r of order d, there are $96/d$ values of q such that $d|(q+r)$. Therefore, the answer to the question is the special case of (a) for $n = 96 = 2^5 \cdot 3$:

$$\sum_{d|96} \frac{96\phi(d)}{d} = 2^4 \left[2(1+5) - 5\right] \left[3(1+1) - 1\right] = 560.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Joel Schlosberg, Bayside, NY, USA; David E. Manes, Oneonta, NY, USA; Adnan Ali, A.E.C.S-4, Mumbai, India.

O365. Prove or disprove the following statement: there is a non-vanishing polynomial $P(x, y, z)$ with integer coefficients such that $P(\sin u, \sin v, \sin w) = 0$ whenever $u + v + w = \frac{\pi}{3}$.

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Daniel Lasaosa, Pamplona, Spain

Let A, B, C be the angles of a triangle (or for that matter, any three angles, positive or negative, such that their sum is π), and for brevity denote $p = \sin A$, $q = \sin B$ and $r = \sin C$. Expanding $\sin(A + B + C) = 0$ as a function of the trigonometric functions of A, B, C yields

$$pqr - p\cos B\cos C = (q\cos C + r\cos B)\cos A,$$

which after squaring and using $\cos^2 A = 1 - p^2$, $\cos^2 B = 1 - q^2$ and $\cos^2 C = 1 - r^2$, and rearranging terms, results in

$$p^2q^2r^2 + p^2(1 - q^2)(1 - r^2) - q(1 - p^2)(1 - r^2) - r(1 - p^2)(1 - q^2) = 2qr\cos B\cos C.$$

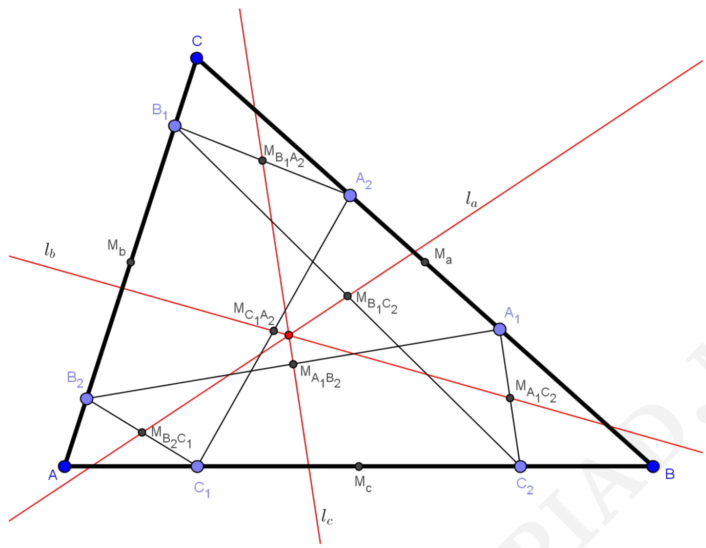
Squaring again allows us to find a non-vanishing expression equal to zero in terms of p, q, r (it suffices to notice that the coefficient of p^2 in the LHS is nonzero, and p does not appear in the RHS). Now, for any $u+v+w = \frac{\pi}{3}$, we can take $A = 3u$, $B = 3v$ and $C = 3w$, or $p = \sin(3u) = 3\cos^2 u\sin u - \sin^3 u = 3\sin u - 4\sin^3 u$, and similarly for q, r. Substitution into the previously found expression on p, q, r allows us to find a non-vanishing polynomial (it suffices to take $v = w = 0$ and let u vary to obtain nonzero values, since we know that terms containing p^2, and hence containing $\sin u$, do not vanish). But this polynomial is zero when $u + v + w = \frac{\pi}{3}$. It follows that the answer is yes, we have proved that such a polynomial exists.

Also solved by Khurshid Juraev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Li Zhou, Polk State College, FL, USA.

O366. In triangle ABC, let A_1, A_2 be two arbitrary isotomic points on BC. We define points $B_1, B_2 \in CA$ and $C_1, C_2 \in AB$ similarly. Let ℓ_a be the line passing through the midpoints of segments (B_1C_2) and (B_2C_1). We define lines ℓ_b and ℓ_c similarly. Prove that all three of these lines are concurrent.

Proposed by Marius Stănean, Zalău, România

Solution by Andrea Fanchini, Cantù, Italy



We use barycentric coordinates and the usual Conway's notations with reference to the triangle ABC. Points A_1, B_1, C_1 have the followings absolute coordinates

$$A_1\left(0, \frac{a-d}{a}, \frac{d}{a}\right), \quad B_1\left(\frac{e}{b}, 0, \frac{b-e}{b}\right), \quad C_1\left(\frac{c-f}{c}, \frac{f}{c}, 0\right)$$

where d, e, f are parameters. Then points A_2, B_2, C_2 have absolute coordinates

$$A_2\left(0, \frac{d}{a}, \frac{a-d}{a}\right), \quad B_2\left(\frac{b-e}{b}, 0, \frac{e}{b}\right), \quad C_2\left(\frac{f}{c}, \frac{c-f}{c}, 0\right)$$

- *Midpoints of segments* (B_1C_2) *and* (B_2C_1) *and the others similarly.*

$$M_{B_1C_2}(bf+ce : b(c-f) : c(b-e)), \qquad M_{B_2C_1}(2bc-bf-ce : bf : ce)$$

$$M_{A_1C_2}(af : 2ac-af-cd : cd), \qquad M_{A_2C_1}(a(c-f) : af+cd : c(a-d))$$

$$M_{A_1B_2}(a(b-e) : b(a-d) : ae+bd), \qquad M_{A_2B_1}(ae : bd : 2ab-ae-bd)$$

- *Equations of lines* l_a, l_b, l_c.

$$l_a : (ec-bf)x + (2bc-3ce-bf)y + (3bf+ce-2bc)z = 0$$

$$l_b : (af+3cd-2ac)x + (af-cd)y + (2ac-3af-cd)z = 0$$

$$l_c : (2ab-ae-3bd)x + (bd+3ae-2ab)y + (bd-ae)z = 0$$

Now the three lines l_a, l_b, l_c are concurrent if and only if

$$\begin{vmatrix} ec-bf & 2bc-3ce-bf & 3bf+ce-2bc \\ af+3cd-2ac & af-cd & 2ac-3af-cd \\ 2ab-ae-3bd & bd+3ae-2ab & bd-ae \end{vmatrix} = 0$$

making calculations we obtain

$$8a^2b^2c^2-12a^2b^2cf-16a^2bc^2e+22a^2bcef+6a^2c^2e^2-8a^2ce^2f-20ab^2c^2d+26ab^2cdf+30abc^2de-28abcdef-4ac^2de^2$$
$$+12b^2c^2d^2-12b^2cd^2f-12bc^2d^2e-8a^2b^2c^2+12a^2b^2cf+20a^2bc^2e-26a^2bcef-12a^2c^2e^2+12a^2ce^2f+16ab^2c^2d-22ab^2cdf$$
$$-30abc^2de+28abcdef+12ac^2de^2-6b^2c^2d^2+8b^2cd^2f+4bc^2d^2e-4a^2bc^2e+4a^2bcef+6a^2c^2e^2-4a^2ce^2f+4ab^2c^2d$$
$$-4ab^2cdf-8ac^2de^2-6b^2c^2d^2+4b^2cd^2f+8bc^2d^2e=0$$

as we can easy check.

Also solved by Daniel Lasaosa, Pamplona, Spain; Khurshid Juraev, Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Li Zhou, Polk State College, FL, USA.