Mathematical Excalibur

Volume 7, Number 1 *March 2002 – April 2002*

Olympiad Corner

The 32nd Austrian Mathematical Olympiad 2001.

Problem 1. Prove that

$$\frac{1}{25}\sum_{k=0}^{2001}\left[\frac{2^k}{25}\right]$$

is an integer. ($[x]$ denotes the largest integer less than or equal to x.)

Problem 2. Determine all triples of positive real numbers x, y and z such that both $x + y + z = 6$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 2-\frac{4}{xyz}$ hold.

Problem 3. We are given a triangle ABC and its circumcircle with mid-point U and radius r. The tangent c' of the circle with mid-point U and radius $2r$ is determined such that C lies between $c = AB$ and c', and a' and b' are defined analogously, yielding the triangle $A'B'C'$. Prove that the lines joining the mid-points of corresponding sides of ΔABC and $\Delta A'B'C'$ pass through a common point.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 15, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

對 數 表 的 構 造

李 永 隆

在現今計算工具發達的年代，要找出如 $ln2$ 這個對數值只需一指之勞。但是大家有沒有想過，在以前計算機尚未出現的時候，那些厚厚成書的對數表是如何精確地構造出來的？ 當然，在歷史上曾出現很多不同的構造方法，各有其所長，但亦各有其所限。下面我們將會討論一個比較有系統的方法，它只需要用上一些基本的微積分技巧，就能夠有效地構造對數表到任意的精確度。

首先注意，$ln(xy) = ln\,x + ln\,y$, 所以我們只需求得所有質數 p 的對數值便可以由此算得其他正整數的對數值。 由 $ln(1 + t)$ 的微分運算和幾何級數公式直接可得

$$\frac{d}{dt}\ln(1+t)=\frac{1}{1+t}=1-t+t^2-t^3+\cdots+(-1)^{n-1}t^{n-1}+\frac{(-1)^n t^n}{1+t}$$

運用微積分基本定理（亦即微分和積分是兩種互逆的運算），即得下式：

$$\ln(1+x)=\int_0^x\frac{1}{1+t}dt=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}+\int_0^x\frac{(-1)^n t^n}{1+t}dt$$

能夠對於所有正整數 n 皆成立。 現在我們去估計上式中的積分餘項的大小。 設 $|x|<1$，則有：

$$\left|\int_0^x\frac{(-1)^n t^n}{1+t}dt\right|\le\left|\int_0^x\left|\frac{t^n}{1+t}\right|dt\right|\le\left|\int_0^x\frac{t^n}{1-|x|}dt\right|=\frac{|x|^{n+1}}{(n+1)(1-|x|)}$$

由此可見，這個餘項的絕對值會隨著 n 的增大而趨向 0。換句話說，只要 n 選得足夠大， $\ln(1+x)$ 和 $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}$ 之間的誤差就可以小到任意小，所以我們不妨改用下式表達這個情況：

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\quad(|x|<1)$$

總之 n 的選取總是可以讓我們忽略兩者的誤差。 把上式中的 x 代以 $-x$ 然後將兩式相減，便可以得到下面的公式：

$$\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right)\qquad(*)$$

可惜的是若直接代入 $x=\frac{p-1}{p+1}$ 使得 $\frac{1+x}{1-x}=p$ 時，$(*)$-式並不能有效地計算 $\ln p$。例如取 $p=29$，則 $x=\frac{29-1}{29+1}=\frac{14}{15}$，在此時即使計算了 100 項至 $\frac{2x^{199}}{199}\approx 1.1\times10^{-8}$，$\ln p$ 的數值還未必能準確至第 8 個小數位（嚴格來說，應該用 $(*)$-式的積分餘項來做誤差估計，不過在這裏我們只是想大約知道其大小）; 又例如取 $p=113$，則 $x=\frac{56}{57}$ 而 $\frac{2x^{199}}{199}\approx 3\times10^{-4}$，$\ln p$ 的準確度則更差。 但是我們可以取 $x=\frac{1}{2p^2-1}$，則有

$$\ln\left(\frac{1+x}{1-x}\right)=\ln\frac{2p^2-1+1}{2p^2-1-1}=\ln\frac{p^2}{(p+1)(p-1)}=2\ln p-\ln(p+1)(p-1)$$

而當質數 $p > 2$ 時，$(p + 1)$ 和 $(p - 1)$ 的質因數都必定小於 p，所以如果我們已算得小於 p 的質數的對數值，就可以用上式來計算 $\ln p$ 的值：

$$2\ln p=\ln\left(\frac{1+x}{1-x}\right)+\ln(p+1)+\ln(p-1)$$

而未知的 $\ln\left(\frac{1+x}{1-x}\right)$ 是能夠有效計算的，因為現在所選的 x 的絕對值很小。例如當 $p = 29$ 時，$x=\frac{1}{2\cdot 29^2-1}$ $=\frac{1}{1681}$，所以只需計算到 $\frac{2x^5}{5}\approx 3\times 10^{-17}$，便能夠準確至十多個小數位了。

經過上面的討論，假設現在我們想構造一個 8 位對數表，則可以依次序地求 2, 3, 5, 7, 11, 13, … 的對數值，而後面質數的對數值都可以用前面的質數的對數值來求得。由此可見，在開始時的 $\ln 2$ 是需要算得準確一些：

$$\ln 2=\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)\approx 2\left(\frac{1}{3}+\frac{\left(\frac{1}{3}\right)^3}{3}+\frac{\left(\frac{1}{3}\right)^5}{5}+\ldots+\frac{\left(\frac{1}{3}\right)^{21}}{21}\right)=0.6931471805589\ldots$$

這個和確實數值

$$\ln 2 = 0.693147180559945\ldots$$

相比其精確度已到達第 11 位小數。

接著便是要計算 $\ln 3$。取 $x=\frac{1}{2\cdot 3^2-1}$ $=\frac{1}{17}$，則有

$$\ln\left(\frac{1+\frac{1}{17}}{1-\frac{1}{17}}\right)\approx 2\left(\frac{1}{17}+\frac{\left(\frac{1}{17}\right)^3}{3}+\frac{\left(\frac{1}{17}\right)^5}{5}+\frac{\left(\frac{1}{17}\right)^7}{7}\right)=0.117783035654504\ldots$$

注意 $\frac{2\left(\frac{1}{17}\right)^9}{9}\approx 1.9\times 10^{-12}$，在 8 位的精確度之下大可以不用考慮。 所以

$$\ln 3\approx\frac{1}{2}\left(0.11778303565\ldots+\ln 4+\ln 2\right)=1.098612288635\ldots$$

（續於第四頁）

Pell's Equation (II)

Kin Y. Li

For a fixed nonzero integer N, as the case $N = -1$ shows, the generalized equation $x^2-dy^2=N$ may not have a solution. If it has a least positive solution (x_1, y_1), then $x^2-dy^2=N$ has infinitely many positive solutons given by (x_n, y_n), where

$$x_n+y_n\sqrt{d}=(x_1+y_1\sqrt{d})(a+b\sqrt{d})^{n-1}$$

and (a, b) is the least positive solution of $x^2-dy^2=1$. However, in general these do not give all positive solutions of $x^2-dy^2=N$ as the following example will show.

Example 9. Consider the equation $x^2-23y^2=-7$. It has $(x_1, y_1) = (4, 1)$ as the least positive solution. The next two solutions are (19, 4) and (211, 44). Now the least positive solution of $x^2-23y^2=1$ is $(a, b) = (24, 5)$. Since $(4+\sqrt{23})(24+5\sqrt{23}) = 211+44\sqrt{23}$, the solution (19, 4) is skipped by the formula above.

In case $x^2-dy^2=N$ has positive solutions, how do we get them all? A solution (x, y) of $x^2-dy^2=N$ is called *primitive* if x and y (and N) are relatively prime. For $0 \le s < |N|$, we say the solution belong to class C_s if $x \equiv sy \pmod{|N|}$. As x, y are relatively prime to N, so is s. Hence, there are at most $\phi(|N|)$ classes of primitive solutions, where $\phi(k)$ is *Euler's ϕ-function* denoting the number of positive integers $m \le k$ that are relatively prime to k. Also, for such s, $(s^2-d)y^2\equiv x^2-dy^2\equiv 0 \pmod{|N|}$ and y, N relatively prime imply $s^2\equiv d \pmod{|N|}$.

Theorem. *Let (a_1, b_1) be a C_s primitive solutions of $x^2-dy^2=N$. A pair (a_2, b_2) is also a C_s primitive solution of $x^2-dy^2=N$ if and only if $a_2+b_2\sqrt{d}=(a_2-b_2\sqrt{d})/(a_1-b_1\sqrt{d})$. Multiplying these two equations, we get $u^2-dv^2=N/N=1$.*

To see u, v are integers, note $a_1a_2-db_1b_2\equiv(s^2-d)b_1b_2\equiv 0 \pmod{|N|}$, which implies u is an integer. Since $a_1b_2-b_1a_2\equiv sb_1b_2-b_1sb_2=0 \pmod{|N|}$, v is also an integer.

For the converse, multiplying the equation with its conjugate shows (a_2, b_2) solves $x^2-dy^2=N$. From $a_2=ua_1+dvb_1$ and $b_2=ub_1+va_1$, we get $a_2=ua_2-dvb_2$ and $b_1=ub_2-va_2$. Hence, common divisors of a_2, b_2 are also common divisors a_1, b_1. So a_2, b_2 are relatively prime. Finally, $a_2-sb_2\equiv(usb_1+dvb_1)-s(ub_1+vsb_1)=(d-s^2)vb_1\equiv 0 \pmod{|N|}$ concludes the proof.

Thus, all primitive solutions of $x^2-dy^2=N$ can be obtained by finding a solution (if any) in each class, then multiply them by solutions of $x^2-dy^2=1$. For the nonprimitive solutions, we can factor the common divisors of a and b to reduce N.

Example 10. (1995 IMO proposal by USA leader T. Andreescu) Find the smallest positive integer n such that $19n + 1$ and $95n + 1$ are both integer squares.

Solution. Let $95n+1=x^2$ and $19n+1=y^2$, then $x^2-5y^2=-4$. Now $\phi(4)=2$ and (1, 1), (11, 5) are C_1, C_3 primitive solutions, respectively. As (9, 4) is the least positive solution of $x^2-5y^2=1$ and $9+4\sqrt{5}=(2+\sqrt{5})^2$, so the primitive positive solutions are pairs (x, y), where $x+y\sqrt{5}=(1+\sqrt{5})(2+\sqrt{5})^{2n-2}$ or $(11+5\sqrt{5})(2+\sqrt{5})^{2n-2}$.

Since the common divisors of x, y divide 4, the nonprimitive positive solutions are the cases x and y are even. This reduces to considering $u^2-5v^2=-1$, where we take $u = x/2$ and $v = y/2$. The least positive solution for $u^2-5v^2=-1$ is (2, 1). So $x+y\sqrt{5}=2(u+v\sqrt{5})=2(2+\sqrt{5})^{2n-1}$.

In attempt to combine these solutions, we look at the powers of $1+\sqrt{5}$ coming from the least positive solutions (1, 1).

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***May 15, 2002***.

Problem 146. Is it possible to partition a square into a number of congruent right triangles each containing an 30° angle? (*Source: 1994 Russian Math Olympiad, 3rd Round*)

Problem 147. Factor x^8+4x^2+4 into two nonconstant polynomials with integer coefficients.

Problem 148. Find all distinct prime numbers p, q, r, s such that their sum is also prime and both p^2+qs, p^2+qr are perfect square numbers. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Problem 149. In a 2000×2000 table, every square is filled with *a* 1 or -1. It is known that the sum of these numbers is nonnegative. Prove that there are 1000 columns and 1000 rows such that the sum of the numbers in these intersection squares is at least 1000. (*Source: 1994 Russian Math Olympiad, 5th Round*)

Problem 150. Prove that in a convex n-sided polygon, no more than n diagonals can pairwise intersect. For what n, can there be n pairwise intersecting diagonals? (Here intersection points may be vertices.) (*Source: 1962 Hungarian Math Olympiad*)

****

Solutions

****

Problem 141. Ninety-eight points are given on a circle. Maria and José take turns drawing a segment between two of the points which have not yet been joined by a segment. The game ends when each point has been used as the endpoint of a segment at least once. The winner is the player who draws the last segment. If José goes first, who has a winning strategy? (*Source: 1998 Iberoamerican Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), 何思銳 (大角嘴天主教小學, Primary 5), **LAM Sze Yui** (Carmel Divine Grace Foundation Secondary School, Form 4), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), LEUNG Chi Man (Cheung Sha Wan Catholic Secondary School, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Yiu Keung** (HKUST, Math Major, Year 1), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **Ricky TANG** (La Salle College, Form 4), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

José has the following winning strategy. He will let Maria be the first person to use the ninety-sixth unused point. Since there are $C_2^{95}=4465$ segments joining pairs of the first ninety-five points, if Maria does not use the ninety-sixth point, José does not have to use it either. Once Maria starts using the ninety-sixth point, José can win by joining the ninety-seventh and ninety-eighth points.

Problem 142. *ABCD* is a quadrilateral with $AB \parallel CD$. *P* and *Q* are on sides *AD* and *BC* respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Prove that *P* and *Q* are equal distance from the intersection point of the diagonals of the quadrilateral. (*Source: 1994 Russian Math Olympiad, Final Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

Let *O* be the intersection point of the diagonals. Since $\Delta AOB, \Delta COD$ are similar, *AO:CO = AB:CD = BO:DO*. By sine law,

$$\frac{AB}{BP}=\frac{\sin\angle APB}{\sin\angle BAP}=\frac{\sin\angle CPD}{\sin\angle CDP}=\frac{CD}{CP}.$$

So *AB:CD = BP:CP*. Let *S* be on *BC* so that $SP\perp AD$ and *R* be on *AD* so that $RQ\perp BC$. Then *SP* bisects $\angle BPC$, *BS:CS = BP:CP = AB:CD = AO:CO*. This implies $OS \parallel AB$. Then *AB:OS = CA:CO*. Similarly, *AB:RO = DB:DO*. However,

$$\frac{CA}{CO}=1+\frac{AO}{CO}=1+\frac{BO}{DO}=\frac{DB}{DO}.$$

So *OS = RO*. Since *O* is the midpoint of *RS* and ΔSPR, ΔRQS are right triangles, *PO = OS = QO*.

Other commended solvers: **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Problem 143. Solve the equation cos cos cos cos *x* = sin sin sin sin *x*. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7).

Let $f(x)$ = sin sin x and $g(x)$ = cos cos x. Now

$$\begin{aligned} g(x)-f(x) &= \sin\left(\frac{\pi}{2}-\cos x\right)-\sin\sin x \\ &= 2\cos\left(\frac{\pi}{4}-\frac{\cos x}{2}+\frac{\sin x}{2}\right) \\ &\quad\times\sin\left(\frac{\pi}{4}-\frac{\cos x}{2}-\frac{\sin x}{2}\right) \end{aligned}$$

and

$$\left|\frac{\cos x\pm\sin x}{2}\right|=\frac{\sqrt{2}\left|\sin(x\pm\pi/4)\right|}{2}<\frac{\pi}{4}.$$

So $g(x)-f(x)>0$ (hence $g(x) > f(x)$) for all x. Since $\sin x$, $f(x)$, $g(x) \in [-1,1] \subset [-\pi/2, \pi/2]$ and sin x is strictly increasing in $[-\pi/2, \pi/2]$, so $f(x)$ is strictly increasing in $[-\pi/2, \pi/2]$ and

$$f(f(x))<f(g(x))<g(g(x))$$

for all x. Therefore, the equation has no solution.

Other commended solvers: **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **OR Kin** (HKUST, Year 1) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Problem 144. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all (non-degenerate) triangles *ABC* with consecutive integer sides *a, b, c* and such that $C=2A$.

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 4), **LAM Wai Pui Billy** (STFA Leung

Kau Kui College, Form 4), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **YEUNG Wing Fung** (STFA Leung Kau Kui College).

Let a=*BC*, b=*CA*, c=*AB*. By sine and cosine laws,

$$\frac{c}{a}=\frac{\sin C}{\sin A}=2\cos A=\frac{b^2+c^2-a^2}{bc}.$$

This gives $bc^2=ab^2+ac^2-a^3$. Factoring, we get $(a-b)(c^2-a^2-ab)=0$. Since the sides are consecutive integers and $C>A$ implies $c>a$, we have $(a, b, c)=(n, n-1, n+1)$, $(n-1, n+1, n)$ or $(n-1, n, n+1)$ for some positive integer $n>1$. Putting these into $c^2-a^2-ab=0$, the first case leads to $-n^2+3n+1=0$, which has no integer solution. The second case leads to $2n-n^2=0$, which yields a degenerate triangle with sides 1, 2, 3. The last case leads to $5n-n^2=0$, which gives $(a, b, c)=(4, 5, 6)$.

Other commended solvers: **CHENG Ka Wai** (STFA Leung Kau Kui College, Form 4), **Clark CHONG Fan Fei** (Queen's College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 4).

Problem 145. Determine all natural numbers $k>1$ such that, for some distinct natural numbers m and n, the numbers k^m+1 and k^n+1 can be obtained from each other by reversing the order of the digits in their decimal representations. (*Source: 1992 CIS Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **Ricky TANG** (La Salle College, Form 4) and **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

Without loss of generality, suppose such numbers exist and $n>m$. By the required property, both numbers are not power of 10. So k^n and k^m have the same number of digits. Then $10>\frac{k^n}{k^m}=k^{n-m}\geq k$. Since every number and the sum of its digits are congruent (mod 9), we get $k^n+1\equiv k^m+1 \pmod 9$. Then $k^n-k^m=k^m(k^{n-m}-1)$ is divisible by 9. Since the two factors are relatively prime, $10>k$ and $9>k^{n-m}-1$, we can only have k = 3, 6 or 9.

Now $3^3+1=28$ and $3^4+1=82$ show k = 3 is an answer. The case k = 6 cannot work as numbers of the form 6^i+1 end in 7 so that both k^m+1 and k^n+1 would begin and end with 7, which makes $k^n/k^m\geq k$ impossible. Finally, the case k = 9 also cannot work as numbers of the form 9^i+1 end in 0 or 2 so that both numbers would begin and end with 2, which again makes $k^n/k^m\geq k$ impossible.

Other commended solvers: **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 4. Determine all real valued functions $f(x)$ in one real variable for which

$$f(f(x)^2+f(y))=xf(x)+y$$

holds for all real numbers x and y.

Problem 5. Determine all integers m for which all solutions of the equation

$$3x^3-3x^2+m=0$$

are rational.

Problem 6. We are given a semicircle with diameter *AB*. Points *C* and *D* are marked on the semicircle, such that *AC* = *CD* holds. The tangent of the semicircle in *C* and the line joining *B* and *D* interect in a point *E*, and the line joining *A* and *E* intersects the semicircle in a point *F*. Show that *CF* < *FD* must hold.

對數表的構造

（續第二頁）

這個和確實數值

$$\ln 3=1.09861228866811\ldots$$

相比其精確度也到達第 10 位小數。 讀者不妨自行試算 ln 5, ln 7 等等的數值，然後再和計算機所得的作一比較。

回看上述極為巧妙的計算方法，真的令人佩服當年的數學家們對於數字關係和公式運算的那種創意與觸覺！

【參考文獻】:

項武義教授分析學講座筆記第三章

http://ihome.ust.hk/~malung/391.html

Pell's Equation (II)

(continued from page 2)

The powers are $1+\sqrt{5}$, $6+2\sqrt{5}$, $16+8\sqrt{5}=8(2+\sqrt{5})$, $56+24\sqrt{5}$, $176+80\sqrt{5}=16(11+5\sqrt{5})$, … . Thus, the primitive positive solutions are (x, y) with $x+y\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^{6n-5}$ or $2\left(\frac{1+\sqrt{5}}{2}\right)^{6n-1}$. The nonprimitive positive solutions are (x, y) with $x+y\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^{6n-3}$. So the general positive solutions are (x, y) with

$$x+y\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^k \text{ for odd } k.$$

Then

$$y=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right)=F_k,$$

where F_k is the k-th term of the famous *Fibonacci sequence*. Finally, $y^2\equiv 1 \pmod{19}$ and k should be odd. The smallest such $y=F_{17}=1597$, which leads to $n=\left(F_{17}^2-1\right)/19=134232$.

Comments: For the readers not familiar with the Fibonacci sequence, it is defined by $F_1=1$, $F_2=1$ and $F_{n+1}=F_n+F_{n-1}$ for $n>1$. By math induction, we can check that they satisfy *Binet's formula* $F_n=\left(r_1^n-r_2^n\right)/\sqrt{5}$, where $r_1=(1+\sqrt{5})/2$ and $r_2=(1-\sqrt{5})/2$ are the roots of the *characteristic equation* $x^2=x+1$. (Check cases n = 1, 2 and in the induction step, just use $r_i^{n+1}=r_i^n+r_i^{n-1}$.)

Mathematical Excalibur

Volume 7, Number 2 *May 2002 – June 2002*

Olympiad Corner

The 31st United States of America Mathematical Olympiad 2002

Problem 1. Let S be a set with 2002 elements, and let N be an integer with $0 \le N \le 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:

(a) the union of any two white subsets is white;

(b) the union of any two black subsets is black;

(c) there are exactly N white subsets.

Problem 2. Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2+\left(2\cot\frac{B}{2}\right)^2+\left(3\cot\frac{C}{2}\right)^2=\left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor and determine these integers.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***Sep 20, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Problem Solving I

Kin-Yin LI

George Polya's famous book *How to Solve It* is a book we highly recommend every student who is interested in problem solving to read. In solving a difficult problem, Polya teaches us to ask the following questions. What is the condition to be satisfied? Have you seen a similar problem? Can you restate the problem in another way or in a related way? Where is the difficulty? If you cannot solve it, can you solve a part of the problem if the condition is relaxed. Can you solve special cases? Is there any pattern you can see from the special cases? Can you guess the answer? What clues can you get from the answer or the special cases? Below we will provide some examples to guide the student in analyzing problems.

Example 1. (*Polya, How to Solve It, pp. 23-25*) Given ΔABC with AB the longest side. Construct a square having two vertices on side AB and one vertex on each of sides BC and CA using a compass and a straightedge (i.e. a ruler without markings).

Analysis. (*Where is the difficulty?*) The difficulty lies in requiring all four vertices on the sides of the triangle. If we relax *four* to *three*, the problem becomes much easier. On CA, take a point P close to A. Draw the perpendicular from P to AB and let the foot be Q. With Q as center and PQ as radius, draw a circle and let it intersect AB at R. Draw the perpendicular line to AB through R and let S be the point on the line which is PQ units from R and on the same side of AB as P. Then $PQRS$ is a square with P on CA and Q, R on AB.

(*What happens if you move the point P on side CA?*) You get a square similar to $PQRS$. (*What happens in the special case P = A?*) You get a point. (*What happens to S if you move P from A toward C?*) As P moves along AC, the triangles APQ will be similar to each other. Then the triangles APS will also be similar to each other and S will trace a line segment from A. This line AS intersects BC at a point S', which is the fourth vertex we need. From S', we can find the three other vertices dropping perpendicular lines and rotating points.

Example 2. (*1995 Russian Math Olympiad*) There are $n > 1$ seats at a merry-go-around. A boy takes n rides. Between each ride, he moves clockwise a certain number (less than n) of places to a new horse. Each time he moves a different number of places. Find all n for which the boy ends up riding each horse.

Analysis. (*Can you solve special cases?*) The cases n = 2, 4, 6 work, but the cases n = 3, 5 do not work. (*Can you guess the answer?*) The answer should be n is even. (*What clues can you get from the special cases?*) From experimenting with cases, we see that if $n > 1$ is odd, then the last ride seems to always repeat the first horse. (*Why?*) From the first to the last ride, the boy moved $1 + 2 + \cdots + (n-1) = n(n-1)/2$ places. If $n > 1$ is odd, this is a multiple of n and so we repeat the first horse.

(Is there any pattern you can see from the special cases when n is even?) Name the horses 1, 2, …, n in the clockwise direction. For n = 2, we can ride horses 1, 2 in that order and the move sequence is 1. For n = 4, we can ride horses 1, 2, 4, 3 in that order and the move sequence is 1, 2, 3. For n = 6, we can ride horses 1, 2, 6, 3, 5, 4 and the

move sequence is 1, 4, 3, 2, 5. Then for the general even cases n, we can ride horses 1, 2, n, 3, $n-1$, …, $(n/2)+1$ in that order with move sequence 1, $n-2$, 3, $n-4$, …, 2, $n-1$. The numbers in the move sequence are all distinct as it is the result of merging odd numbers 1, 3, …, $n-1$ with even numbers $n-2$, $n-4$, …, 2.

Example 3. (*1982 Putnam Exam*) Let $K(x, y, z)$ be the area of a triangle with sides x, y, z. For any two triangles with sides a, b, c and a', b', c' respectively, show that

$$\sqrt{K(a,b,c)}+\sqrt{K(a',b',c')} \le \sqrt{K(a+a',b+b',c+c')}$$

and determine the case of equality.

Analysis. (*Can you restate the problem in another way?*) As the problem is about the area and sides of a triangle, we bring out Heron's formula, which asserts the area of a triangle with sides x, y, z is given by

$$K(x,y,z)=\sqrt{s(s-x)(s-y)(s-z)},$$

where s is half the perimeter, i.e. $s=\frac{1}{2}(x+y+z)$. Using this formula, the problem becomes showing

$$\sqrt[4]{stuv}+\sqrt[4]{s't'u'v'} \le \sqrt[4]{(s+s')(t+t')(u+u')(v+v')},$$

where $s=\frac{1}{2}(a+b+c)$, $t=s-a$, $u=s-b$, $v=s-c$ and similarly for s',t',u',v'.

(*Have you seen a similar problem or can you relax the condition?*) For those who saw the forward-backward induction proof of the AM-GM inequality before, this is similar to the proof of the case $n=4$ from the case $n=2$. For the others, having groups of four variables are difficult to work with. We may consider the more manageable case $n=2$. If we replace 4 by 2, we get a simpler inequality

$$\sqrt{xy}+\sqrt{x'y'} \le \sqrt{(x+x')(y+y')}.$$

This is easier. Squaring both sides, canceling common terms, then factoring, this turns out to be just $\left(\sqrt{xy'}-\sqrt{x'y}\right)^2 \ge 0$. Equality holds if and only if $x:x'=y:y'$. Applying this simpler inequality twice, we easily get the required inequality

$$\begin{aligned}&\sqrt[4]{stuv}+\sqrt[4]{s't'u'v'}\\ &\le \sqrt{(\sqrt{st}+\sqrt{s't'})(\sqrt{uv}+\sqrt{u'v'})}\\ &\le \sqrt{\sqrt{(s+s')(t+t')}\sqrt{(u+u')(v+v')}}.\end{aligned}$$

Tracing the equality case back to the simpler inequality, we see equality holds if and only if $a:b:c=a':b':c'$, i.e. the triangles are similar.

Example 4. Is there a way to pack 250 $1\times1\times4$ bricks into a $10\times10\times10$ box?

Analysis. (*Where is the difficulty?*) 10 is large for a 3 dimensional cube. We can relax the problem a bit by considering a two dimensional analogous problem with smaller numbers, say 1×2 cards pack into a 8×8 board. This is clearly possible. (*What if we relax the board to be a square, say by taking out two squares from the board?*) This may become impossible. For example, if the 8×8 board is a checkerboard and we take out two black squares, then since every 1×2 card covers exactly one white and one black square, any possible covering must require the board to have equal number of white and black squares.

(*What clue can you get from the special cases?*) Coloring a board can help to solve the problem. (*Can we restate the problem in a related way?*) Is it possible to color the cubes of the $10\times10\times10$ box with four colors in such a way that in every four consecutive cubes each color occurs exactly once, where consecutive cubes are cubes sharing a common face? Yes, we can put color 1 in a corner cube, then extend the coloring to the whole box by putting colors 1, 2, 3, 4 periodically in each of the three perpendicular directions parallel to the edges of the box. However, a counting shows that for the $10\times10\times10$ box, there are 251 color 1 cubes, 251 color 2 cubes, 249 color 3 cubes and 249 color 4 cubes. So the required packing is impossible.

Example 5. (*1985 Moscow Math Olympiad*) For every integer $n\ge3$, show that $2^n=7x^2+y^2$ for some odd positive integers x and y.

Analysis. (*cf. Arthur Engel, Problem-Solving Strategies, pp. 126-127*) (*Can you solve special cases?*) For $n=3, 4, \ldots, 10$, we have the table:

n	3	4	5	6	7	8	9	10
$x=x_n$	1	1	1	3	1	5	7	3
$y=y_n$	1	3	5	1	11	9	13	31

(*Is there any pattern you can see from the special cases?*) In cases $n=3, 5, 8$, it seems that x_{n+1} is the average of x_n and y_n. For cases $n=4, 6, 7, 9, 10$, the average of x_n and y_n is even and it seems that $|x_n-y_n|=2x_{n+1}$. (*Can you guess the answer?*) The answer should be

$$x_{n+1}=\begin{cases}\frac{1}{2}(x_n+y_n) & \text{if } \frac{1}{2}(x_n+y_n) \text{ is odd}\\ \frac{1}{2}|x_n-y_n| & \text{if } \frac{1}{2}(x_n+y_n) \text{ is even}\end{cases}$$

and

$$\begin{aligned}&y_{n+1}\\ &=\sqrt{2^{n+1}-7x_{n+1}^2}\\ &=\sqrt{2(7x_n^2+y_n^2)-7x_{n+1}^2}\\ &=\begin{cases}\frac{1}{2}|7x_n-y_n| & \text{if } \frac{1}{2}(x_n+y_n) \text{ is odd}\\ \frac{1}{2}(7x_n+y_n) & \text{if } \frac{1}{2}(x_n+y_n) \text{ is even}\end{cases}.\end{aligned}$$

(*Is this correct?*) The case $n=3$ is correct. If $2^n=7x_n^2+y_n^2$, then the choice of y_{n+1} will give $2^{n+1}=7x_{n+1}^2+y_{n+1}^2$. (*Must x_{n+1} and y_{n+1} be odd positive integers?*) Yes, this can be checked by writing x_n and y_n in the form $4k\pm1$.

IMO 2002

IMO 2002 will be held in Glasgow, United Kingdom from July 19 to July 30 this summer. Based on the selection test performances, the following students have been chosen to represent Hong Kong:

CHAO Khek Lun (*St. Paul's College*)
CHAU Suk Ling (*Queen Elizabeth School*)
CHENG Kei Tsi (*La Salle College*)
IP Chi Ho (*St. Joseph College*)
LEUNG Wai Ying (*Queen Elizabeth School*)
YU Hok Pun (*SKH Bishop Baker Secondary Sch*)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***September 20, 2002***.

Problem 151. Every integer greater than 2 can be written as a sum of distinct positive integers. Let $A(n)$ be the maximum number of terms in such a sum for n. Find $A(n)$. (*Source: 1993 German Math Olympiad*)

Problem 152. Let $ABCD$ be a cyclic quadrilateral with E as the intersection of lines AD and BC. Let M be the intersection of line BD with the line through E parallel to AC. From M, draw a tangent line to the circumcircle of $ABCD$ touching the circle at T. Prove that $MT = ME$. (*Source: 1957 Nanjing Math Competition*)

Problem 153. Let R denote the real numbers. Find all functions $f: R \to R$ such that the equality $f(f(x) + y) = f(x^2 - y) + 4f(x)y$ holds for all pairs of real numbers x, y. (*source: 1997 Czech-Slovak Match*)

Problem 154. For nonnegative numbers a, d and positive numbers b, c satisfying $b + c \geq a + d$, what is the minimum value of $\frac{b}{c+d} + \frac{c}{a+b}$?

(*Source: 1988 All Soviet Math Olympiad*)

Problem 155. We are given 1997 distinct positive integers, any 10 of which have the same least common multiple. Find the maximum possible number of pairwise relatively prime numbers among them. (*Source: 1997 Hungarian Math Olympiad*)

****

Solutions

****

Problem 146. Is it possible to partition a square into a number of congruent right triangles each containing a 30° angle? (*Source: 1994 Russian Math Olympiad, 3rd Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHEUNG Chung Yeung** (STFA Leung Kau Kui College, Form 4), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **WONG Wing Hong** (La Salle College, Form 4) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 4).

Without loss of generality, let the sides of the triangles be 2, 1, $\sqrt{3}$. Assume n such triangles can partition a square. Since the sides of the square are formed by sides of these triangles, so the sides of the square are of the form $a + b\sqrt{3}$, where a, b are nonnegative integers. Considering the area of the square, we get $(a+b\sqrt{3})^2 = \frac{n\sqrt{3}}{2}$, which is the same as $2(a^2+3b^2) = (n-4ab)\sqrt{3}$. Since a, b are integers and $\sqrt{3}$ is irrational, we must have $a^2+3b^2 = 0$ and $n - 4ab = 0$. The first equation implies $a = b = 0$, which forces the sides of the square to be 0, a contradiction.

Other commended solver: **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7).

Problem 147. Factor x^8+4x^2+4 into two nonconstant polynomials with integer coefficients.

Solution. **CHENG Ka Wai** (STFA Leung Kau Kui College, Form 4), **CHEUNG CHUNG YEUNG** (STFA Leung Kau Kui College, Form 4), **FUNG Yi** (La Salle College, Form 4), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **TANG Sze Ming** (STFA Leung Kau Kui College, Form 4).

$$\begin{aligned}
&x^8+4x^2+4\\
&=(x^8+4x^6+8x^4+8x^2+4)\\
&\qquad-(4x^6+8x^4+4x^2)\\
&=(x^4+2x^2+2)^2-(2x^3+2x)^2\\
&=(x^4+2x^3+2x^2+2x+2)\\
&\qquad\times(x^4-2x^3+2x^2+2).
\end{aligned}$$

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **HUI Chun Yin John** (Hong Kong Chinese Women's Club College, Form 6), **LAW Siu Lun Jack** (CCC Ming Kei College, Form 7), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **Tak Wai Alan WONG** (University of Toronto, Canada), **WONG Wing Hong** (La Salle College, Form 4) & **YEUNG Kai Tsz Max** (Ju Ching Chu Secondary School, Form 5).

Problem 148. Find all distinct prime numbers p, q, r, s such that their sum is also prime and both $p^2 + qs$, $p^2 + qr$ are perfect square numbers. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHEUNG CHUNG YEUNG** (STFA Leung Kau Kui College, Form 4), **LAW Siu Lun Jack** (CCC Ming Kei College, Form 7), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TANG Chun Pong Ricky** (La Salle College, Form 4), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **WONG Wing Hong** (La Salle College, Form 4), **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 4) and **YUEN Ka Wai** (Carmel Divine Grace Foundation Secondary School, Form 6).

Since the sum of the primes p, q, r, s is a prime greater than 2, one of p, q, r, s is 2. Suppose $p \neq 2$. Then one of q, r, s is 2 so that one of $p^2 + qs$, $p^2 + qr$ is of the form $(2m+1)^2 + 2(2n + 1) = 4(m^2 + m + n) + 3$, which cannot be a perfect square as perfect squares are of the form $(2k)^2 = 4k^2$ or $(2k+1)^2 = 4(k^2+k) + 1$. So $p = 2$.

Suppose $2^2 + qs = a^2$, then q, s odd implies a odd and $qs = (a + 2)(a - 2)$. Since q, s are prime, the smaller factor $a - 2 = 1$, q or s. In the first case, $a = 3$ and $qs = 5$, which is impossible. In the remaining two cases, either $q = a-2$, $s = a+2 = q+4$ or $s = a-2$, $q = a+2 = s+4$. Next $2^2+qr = b^2$ will similarly implies q, r differe by 4. As q, r, s are distinct primes, one of r, s is $q-4$ and the other is $q = 4$. Note that $q - 4$, q, $q + 4$ have different remainders when they are divided by 3. One of them is 3 and it must be $q-4$. Thus there are two solutions $(p, q, r, s) = (2, 7, 3, 11)$ or $(2, 7, 11, 3)$. It is easy to check both solutions satisfy all conditions.

Other commended solvers: **WONG Wai Yi** (True Light Girl's College, Form 4)

Problem 149. In a 2000 × 2000 table, every square is filled with $a + 1$ or $a - 1$. It is known that the sum of these numbers is nonnegative. Prove that there are 1000 columns and 1000 rows such that the sum of the numbers in these intersection squares is at least 1000. (*Source: 1994 Russian Math Olympiad, 5th Round*)

Solution 1. **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Since the numbers have a nonnegative sum, there is a column with a nonnegative sum. Hence there are at least one thousand squares in that column filled with +1. Thus, without loss of generality we may assume the squares in rows 1 to 1000 of column 1 are filled with +1. Evaluate the sums of the numbers in the squares of rows 1 to 1000 for each of the remaining columns. Pick the 999 columns with the largest sums in these evaluations. If these 999 columns have a nonnegative total sum S, then we are done (simply take rows 1 to 1000 and the first column with these 999 columns). Otherwise, $S < 0$ and at least one of the 999 columns has a negative sum. Since the sum of the first 100 squares in each column must be even, the sum of the first 100 squares in that column is at most –2. Then the total sum of all squares in rows 1 to 1000 is at most $1000 + S + (-2)1000 < -1000$.

Since the sum of the whole table is nonnegative, the sum of all squares in rows 1001 to 2000 would then be greater than 1000. Then choose the squares in these rows and the 1000 columns with the greatest sums. If these squares have a sum at least 1000, then we are done. Otherwise, assume the sum is less than 1000, then at least one of these 1000 columns will have a nonpositive sum. Thus, the remaining 1000 columns will each have a nonpositive sum. This will lead to the sum of all squares in rows 1001 to 2000 be less than 1000 + (0)1000 = 1000, a contradiction.

Solution 2. **CHAO Khek Lun Harold** (St. Paul's College, Form 7).

We first prove that for a $n \times n$ square filled with +1 and -1 and the sum is at least m, where m, n are of the same parity and $m < n$, there exists a $(n-1) \times (n-1)$ square the numbers there have a sum at least $m + 1$. If the sum of the numbers in the $n \times n$ square is greater than m, we may convert some of the +1 squares to -1 to make the sum equal m. Let the sum of the numbers in rows 1 to n be $r_1, \ldots, r_n$. Since $r_1 + \cdots + r_n = m < n$, there is a $r_j \le 0$. For each square in row j, add up the numbers in the row and column on which the square lies. Let them be $a_1, \ldots, a_n$. Now $a_1 + \cdots + a_n = m + (n-1)r_j \le m < n$. Since a_i is the sum of the numbers in $2n - 1$ squares, each a_i is odd. So there exists some $a_k \le -1$. Removing row j and column k, the sum of the numbers in the remaining $(n-1) \times (n-1)$ square is $m - a_k \ge m + 1$. Finally convert back the –1 squares to +1 above and the result follows.

For the problem, start with $n = 2000$ and $m = 0$, then apply the result above 1000 times to get the desired statement.

Problem 150. Prove that in a convex n-sided polygon, no more than n diagonals can pairwise intersect. For what n, can there be n pairwise intersecting diagonals? (Here intersection points may be vertices.) (*Source: 1962 Hungarian Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **TANG Sze Ming** (STFA Leung Kau Kui College, Form 4.

For $n = 3$, there is no diagonal and for $n = 4$, there are exactly two intersecting diagonals. So let $n \ge 5$. Note two diagonals intersect if and only if the pairs of vertices of the diagonals share a common vertex or separate each other on the boundary. Thus, without loss of generality, we may assume the polygon is regular. For each diagonal, consider its perpendicular bisector. If n is odd, the perpendicular bisectors are exactly the n lines joining a vertex to the midpoint of its opposite side. If n is even, the perpendicular bisectors are either lines joining opposite vertices or lines joining the midpoints of opposite edges and again there are exactly n such lines. Two diagonals intersect if and only if their perpendicular bisectors do not coincide. So there can be no more than n pairwise intersecting diagonals. For $n \ge 5$, since there are exactly n different perpendicular bisectors, so there are n pairwise intersecting diagonals.

Other commended solvers: **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Problem 4. Let R be the set of real numbers. Determine all functions $f: R \to R$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all real numbers x and y.

Problem 5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each $i (1 \le i < k)$.

Problem 6. I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that

$$\frac{1}{7}n^2 - cn \le b(n) \le \frac{1}{5}n^2 + dn$$

for all $n > 0$.

Mathematical Excalibur

Volume 7, Number 3 *July 2002 – October 2002*

Olympiad Corner

The 43^{rd} International Mathematical Olympiad 2002.

Problem 1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is colored red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \le x$ and $y' \le y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y-set to be a set of n blue points having distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

Problem 2. Let BC be a diameter of the circle Γ with center O. Let A be a point on Γ such that $0° < \angle AOB < 120°$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incentre of the triangle CEF.

Problem 3. Find all pairs of integers such that there exist infinitely many positive integers a for which

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 2, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Mathematical Games (I)

Kin Y. Li

An *invariant* is a quantity that does not change. A *monovariant* is a quantity that keeps on increasing or keeps on decreasing. In some mathematical games, winning often comes from understanding the invariants or the monovariants that are controlling the games.

Example 1. (*1974 Kiev Math Olympiad*) Numbers $1, 2, 3, \ldots, 1974$ are written on a board. You are allowed to replace any two of these numbers by one number, which is either the sum or the difference of these numbers. Show that after 1973 times performing this operation, the only number left on the board cannot be 0.

Solution. There are 987 odd numbers on the board in the beginning. Every time the operation is performed, the number of odd numbers left either stay the same (when the numbers taken out are not both odd) or decreases by two (when the numbers taken out are both odd). So the number of odd numbers left on the board after each operation is always odd. Therefore, when one number is left, it must be odd and so it cannot be 0.

Example 2. In an 8×8 board, there are 32 white pieces and 32 black pieces, one piece in each square. If a player can change all the white pieces to black and all the black pieces to white in any row or column in a single move, then is it possible that after finitely many moves, there will be exactly one black piece left on the board?

Solution. No. If there are exactly k black pieces in a row or column before a move is made to that row or column, then after the moves, the number of black pieces in the row or in the column will become $8 - k$, a change of $(8 - k) - k = 8 - 2k$ black pieces on the board. Since $8 - 2k$ is even, the parity of the number of black pieces stay the same before and after the move. Since at the start, there are 32 black pieces, there cannot be 1 black piece left at any time.

Example 3. Four x's and five o's are written around the circle in an arbitrary order. If two consecutive symbols are the same, then insert a new x between them. Otherwise insert a new o between them. Remove the old x's and o's. Keep on repeating this operation. Is it possible to get nine o's?

Solution. If we let $x = 1$ and $o = -1$, then note that consecutive symbols are replaced by their product. If we consider the product P of the nine values before and after each operation, we will see that the new P is the square of the old P. Hence, P will always equal 1 after an operation. So nine o's yielding $P = -1$ can never happen.

Example 4. There are three piles of stones numbering 19, 8 and 9, respectively. You are allowed to choose two piles and transfer one stone from each of these two piles to the third piles. After several of these operations, is it possible that each of the three piles has 12 stones?

Solution. No. Let the number of stones in the three piles be a, b and c, respectively. Consider (mod 3) of these numbers. In the beginning, they are 1, 2, 0. After one operation, they become 0, 1, 2 no matter which two piles have stones transfer to the third pile. So the remainders are always 0, 1, 2 in some order. Therefore, all piles having 12 stones are impossible.

Example 5. Two boys play the following game with two piles of candies. In the first pile, there are 12 candies and in the second pile, there are 13 candies. Each boy takes turn to make a move consisting of eating two candies from one of the piles or transferring a candy from the first pile to the second. The boy who cannot make a move loses. Show that the boy who played second cannot lose. Can he win?

Solution. Consider S to be the number of candies in the second pile minus the first. Initially, $S = 13 - 12 = 1$. After each move, S increases or decreases by 2. So S (mod 4) has the pattern 1, 3, 1, 3, … . Every time after the boy who played first made a move, S (mod 4) would always be 3. Now a boy loses if and only if there are no candies left in the second pile, then $S = 1 - 0 = 1$. So the boy who played second can always make a move, hence he cannot lose.

Since either the total number of candies decreases or the number of candies in the first pile decreases, so eventually the game must stop, so the boy who played second must win.

Example 6. Each member of a club has at most three enemies in the club. (Here enemies are mutual.) Show that the members can be divided into two groups so that each member in each group has at most one enemy in the group.

Solution. In the beginning, randomly divide the members into two groups. Let S be the sum of the number of the pairs of enemies in each group. If a member has at least two enemies in the same group, then the member has at most one enemy in the other group. Transferring the member to the other group, we will decrease S by at least one. Since S is a nonnegative integer, it cannot be decreased forever. So after finitely many transfers, each member can have at most one enemy in the same group.

(Continued on page 4)

IMO 2002

Kin Y. Li

The International Mathematical Olympiad 2002 was held in Glasgow, United Kingdom from July 19 to 30. There were a total of 479 students from 84 countries and regions participated in the Olympiad.

The Hong Kong team members were

Chao Khek Lun (St. Paul's College)
Chau Suk Ling (Queen Elizabeth School)
Cheng Kei Tsi (La Salle College)
Ip Chi Ho (St. Joseph's College)
Leung Wai Ying (Queen Elizabeth School)
Yu Hok Pun (SKH Bishop Baker Secondary School).

The team leader was *K. Y. Li* and the deputy leaders were *Chiang Kin Nam* and *Luk Mee Lin*.

The scores this year ranged from 0 to 42. The cutoffs for medals were 29 points for gold, 24 points for silver and 14 points for bronze. The Hong Kong team received 1 gold medal (*Yu Hok Pun*), 2 silver medals (*Leung Wai Ying* and *Cheng Kei Tsi*) and 2 bronze medals (*Chao Khek Lun* and *Ip Chi Ho*). There were 3 perfect scores, two from China and one from Russia. After the 3 perfect scores, the scores dropped to 36 with 9 students! This was due to the tough marking schemes, which intended to polarize the students' performance to specially distinguish those who had close to complete solutions from those who should only deserve partial points.

The top five teams are China (212), Russia (204), USA (171), Bulgaria (167) and Vietnam (166). Hong Kong came in 24^{th} (120), ahead of Australia, United Kingdom, Singapore, New Zealand, but behind Canada, France and Thailand this year.

One piece of interesting coincidence deserved to be pointed out. Both Hong Kong and New Zealand joined the IMO in 1988. Both won a gold medal for the first time this year and both gold medallists scored 29 points.

The IMO will be hosted by Japan next year at Keio University in Tokyo and the participants will stay in the Olympic village. Then Greece, Mexico, Slovenia will host in the following years.

Addendum. After the IMO, the German leader Professor Gronau sent an email to inform all leaders about his updated webpage

http://www.Mathematik-Olympiaden.de/

which contains IMO news and facts. Clicking *Internationale Olympiaden* on the left, then on that page, scrolling down and clicking *Top-Mathematikern*, *Die erfolgreichsten IMO-Teilnehmer* in blue on the right, we could find the following past IMO participants who have also won the Fields medals, the Nevanlinna prizes and the Wolf prizes:

Richard Borcherds (1977 IMO silver, 1978 IMO gold, 1998 Fields medal)

Vladmir Drinfeld (1969 IMO gold, 1990 Fields medal)

Tim Gowers (1981 IMO gold, 1998 Fields medal)

Laurent Lafforgue (1984 IMO silver, 1985 IMO silver, 2002 Fields medal)

Gregori Margulis (1959 IMO member, 1962 IMO silver, 1978 Fields medal)

Jean-Christoph Yoccoz (1974 IMO gold, 1994 Fields medal)

Alexander Razborov (1979 IMO gold, 1990 Nevanlinna prize)

Peter Shor (1977 IMO silver, 1998 Nevanlinna prize)

László Lovász (1963 IMO silver, 1964 IMO gold, 1965 IMO gold, 1966 IMO gold, 1999 Wolf prize)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***November 2, 2002***.

Problem 156. If a, b, $c > 0$ and $a^2+b^2+c^2=3$, then prove that

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\ge\frac{3}{2}.$$

Problem 157. In base 10, the sum of the digits of a positive integer n is 100 and of $44n$ is 800. What is the sum of the digits of $3n$?

Problem 158. Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on BC such that $BD = 2DC$ and let P be a point on AD such that $\angle BAC = \angle BPD$. Prove that

$$\angle BAC = 2\ \angle DPC.$$

Problem 159. Find all triples (x, k, n) of positive integers such that

$$3^k - 1 = x^n.$$

Problem 160. We are given 40 balloons, the air pressure inside each of which is unknown and may differ from balloon to balloon. It is permitted to choose up to k of the balloons and equalize the pressure in them (to the arithmetic mean of their respective pressures.) What is the smallest k for which it is always possible to equalize the pressures in all of the balloons?

****

Solutions

****

Problem 151. Every integer greater than 2 can be written as a sum of distinct positive integers. Let $A(n)$ be the maximum number of terms in such a sum for n. Find $A(n)$. (*Source: 1993 German Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Chi Man** (Cheung Sha Wan Catholic Secondary School, Form 6), **Poon Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **Tsui Ka Ho** (CUHK, Year 1), **Tak Wai Alan WONG** (University of Toronto) and **WONG Wing Hong** (La Salle College, Form 5).

Let $a_m = m(m+1)/2$. This is the sum of $1, 2, \ldots, m$ and hence the sequence a_m is strictly increasing to infinity. So for every integer n greater than 2, there is a positive integer m such that $a_m \le n < a_{m+1}$. Then n is the sum of the m positive integers

$$1, 2, \ldots, m-1,\ n-m(m-1)/2.$$

Assume $A(n) > m$. Then

$$a_{m+1} = 1+2+\cdots+(m+1)\le n,$$

a contradiction. Therefore, $A(n) = m$. Solving the quadratic inequality

$$a_m = m(m+1)/2 \le n,$$

we find m is the greatest integer less than or equal to $\left(-1+\sqrt{8n+1}\right)/2$.

Other commended solvers: **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7).

Problem 152. Let $ABCD$ be a cyclic quadrilateral with E as the intersection of lines AD and BC. Let M be the intersection of line BD with the line through E parallel to AC. From M, draw a tangent line to the circumcircle of $ABCD$ touching the circle at T. Prove that $MT = ME$. (*Source: 1957 Nanjing Math Competition*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **Poon Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), TANG Sze Ming (STFA Leung Kau Kui College), **Tsui Ka Ho** (CUHK, Year 1) and **WONG Wing Hong** (La Salle College, Form 5).

Since ME and AC are parallel, we have

$$\angle MEB = \angle ACB = \angle ADB = \angle MDE.$$

Also, $\angle BME = \angle EMD$. So triangles BME and EMD are similar. Then

$$MB / ME = ME / MD.$$

So $ME^2 = MD \cdot MB$. By the intersecting chord theorem, also $MT^2 = MD \cdot MB$. Therefore, $MT = ME$.

Problem 153. Let R denote the real numbers. Find all functions $f: R\to R$ such that the equality

$$f(f(x)+y) = f(x^2-y) + 4f(x)\,y$$

holds for all pairs of real numbers x, y. (*Source: 1997 Czech-Slovak Match*)

Solution. **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12),

Setting $y = x^2$, we have

$$f(f(x)+x^2) = f(0) + 4x^2 f(x).$$

Setting $y = -f(x)$, we have

$$f(0) = f(f(x)+x^2) - 4f(x)^2.$$

Comparing these, we see that for each x, we must have $f(x) = 0$ or $f(x) = x^2$. Suppose $f(a) = 0$ for some nonzero a. Putting $x = a$ into the given equation, we get

$$f(y) = f(a^2 - y).$$

For $y \ne a^2/2$, we have

$$y^2 \ne (a^2-y)^2,$$

which will imply $f(y) = 0$. Finally, setting $x = 2a$ and $y = a^2/2$, we have

$$f(a^2/2) = f(7a^2/2) = 0 \quad \text{if } x\ne y,$$

$$0 = f(9a^2/2) = f(7a^2/2) + 8a^4 \quad \text{if } x = y,$$

contradiction as $a \ne 0$. So either $f(x) = 0$ for all x or $f(x) = x^2$ for all x. We can easily check both are solutions.

Comments: Many solvers submitted incomplete solutions. Most of them got $\forall x\ (f(x) = 0$ or $x^2)$, which is not the same as the desired conclusion that $(\forall x\ f(x) = 0)$ or $(\forall x\ f(x) = x^2)$.

Problem 154. For nonnegative numbers a, d and positive numbers b, c satisfying $b + c \ge a + d$, what is the minimum value of $\dfrac{b}{c+d}+\dfrac{c}{a+b}$? (*Source: 1988 All Soviet Math Olympiad*)

Solution. Without loss of generality, we may assume that $a \ge d$ and $b \ge c$. From $b + c \ge a + d$, we get

$$b+c \geq (a+b+c+d)/2.$$

Now

$$\frac{b}{c+d}+\frac{c}{a+b}$$

$$=\frac{b+c}{c+d}-c\left(\frac{1}{c+d}-\frac{1}{a+b}\right)$$

$$\geq \frac{a+b+c+d}{2(c+d)}$$

$$-(c+d)\left(\frac{1}{c+d}-\frac{1}{a+b}\right)$$

$$=\frac{a+b}{2(c+d)}+\frac{c+d}{a+b}-\frac{1}{2}$$

$$\geq 2\sqrt{\frac{a+b}{2(c+d)}\cdot\frac{c+d}{a+b}}-\frac{1}{2}$$

$$=\sqrt{2}-\frac{1}{2},$$

where the AM-GM inequality was used to get the last inequality. Tracing the equality conditions, we need $b+c=a+d$, $c=c+d$ and $a+b=\sqrt{2}\,c$. So the minimum $\sqrt{2}-1/2$ is attained, for example, when $a=\sqrt{2}+1$, $b=\sqrt{2}-1$, $c=2$, $d=0$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Problem 155. We are given 1997 distinct positive integers, any 10 of which have the same least common multiple. Find the maximum possible number of pairwise relatively prime numbers among them. (*Source: 1997 Hungarian Math Olympiad*)

Solution. **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12) and **WONG Wing Hong** (La Salle College, Form 5).

The answer is 9. Suppose there were 10 pairwise relatively prime numbers $a_1, a_2, \ldots, a_{10}$ among them. Being pairwise relatively prime, their least common multiple is their product M. Then the least common multiple of $b, a_2, \ldots, a_{10}$ for any other b in the set is also M. Since a_1 is relatively prime to each of $a_2, \ldots, a_{10}$, so b is divisible by a_1. Similarly, b is divisible by the other a_i. Hence b is divisible by M. Since M is a multiple of b, so $b = M$, a contradiction to having 1997 distinct integers.

To get an example of 9 pairwise relatively prime integers among them, let p_n be the n-th prime number, $a_i = p_i$ (for $i = 1, 2, \ldots, 8$), $a_9 = p_9 p_{10} \cdots p_{1988}$ and

$$b_i = p_1 p_2 \cdots p_{1988} / p_i$$

for $i = 1, 2, \ldots, 1988$. It is easy to see that the a_i's are pairwise relatively prime and any 10 of these 1997 numbers have the same least common multiple.

Other commended solvers: **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 3. (*cont.*)

$$\frac{a^m+a-1}{a^n+a^2-1}$$

is an integer.

Problem 4. Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$ where

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Define

$$D = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k.$$

(a) Prove that $D < n^2$.

(b) Determine all n for which D is a divisor of n^2.

Problem 5. Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that

$$(f(x)+f(z))(f(y)+f(t)) = f(xy-zt)+f(xt+yz)$$

for all x, y, z, t in $\mathbb{R}$.

Problem 6. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centers by $O_1, O_2, \ldots, O_n$ respectively. Suppose that no line meets more than two of the circles. Prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

Mathematical Games (I)

(Continued from page 2)

Remarks. This method of proving is known as the *method of infinite descent*. It showed that you cannot always decrease a quantity when it can only have finitely many possible values.

Example 7. (*1961 All-Russian Math Olympiad*) Real numbers are written in an $m \times n$ table. It is permissible to reverse the signs of all the numbers in any row or column. Prove that after a number of these operations, we can make the sum of the numbers along each line (row or column) nonnegative.

Solution. Let S be the sum of all the mn numbers in the table. Note that after an operation, each number stay the same or turns to its negative. Hence there are at most 2^{mn} tables. So S can only have finitely many possible values. To make the sum of the numbers in each line nonnegative, just look for a line whose numbers have a negative sum. If no such line exists, then we are done. Otherwise, reverse the sign of all the numbers in the line. Then S increases. Since S has finitely many possible values, S can increase finitely many times. So eventually the sum of the numbers in every line must be nonnegative.

Example 8. Given $2n$ points in a plane with no three of them collinear. Show that they can be divided into n pairs such that the n segments joining each pair do not intersect.

Solution. In the beginning randomly pair the points and join the segments. Let S be the sum of the lengths of the segments. (Note that since there are finitely many ways of connecting $2n$ points by n segments, there are finitely many possible values of S.) If two segments AB and CD intersect at O, then replace pairs AB and CD by AC and BD. Since

$$AB + CD = AO + OB + CO + OD > AC + BD$$

by the triangle inequality, whenever there is an intersection, doing this replacement will always decrease S. Since there are only finitely many possible values of S, so eventually there will not be any intersection.

Mathematical Excalibur

Volume 7, Number 4 *October 2002 – November 2002*

Olympiad Corner

The 2002 Canadian Mathematical Olympiad

Problem 1. Let S be a subset of $\{1, 2, \ldots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5. What is the maximum number of elements that S can contain?

Problem 2. Call a positive integer n **practical** if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

For example, the divisors of 6 are **1**, **2**, **3**, and **6**. Since

$$1 = \mathbf{1},\quad 2 = \mathbf{2},\quad 3 = \mathbf{3},\quad 4 = \mathbf{1} + \mathbf{3},$$
$$5 = \mathbf{2} + \mathbf{3},\quad 6 = \mathbf{6}$$

we see that 6 is practical.

Prove that the product of two practical numbers is also practical.

(continued on page 4)

Editors: 張百康 (CHEUNG Pak-Hong), Munsang College, HK
高子眉 (KO Tsz-Mei)
梁達榮 (LEUNG Tat-Wing)
李健賢 (LI Kin-Yin), Dept. of Math., HKUST
吳鏡波 (NG Keng-Po Roger), ITC, HKPU

Artist: 楊秀英 (YEUNG Sau-Ying Camille), MFA, CU

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 15, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

簡介費馬數

梁達榮

考慮形狀如 $2^m + 1$ 的正整數，如果它是質數，則 m 一定是 2 的正次冪。否則的話，設 $m = 2^n s$，其中 s 是 3 或以上的奇數，我們有 $2^m + 1 = 2^{2^n s} + 1 = (2^{2^n})^s + 1 = (2^{2^n} + 1)((2^{2^n})^{s-1} - (2^{2^n})^{s-2} + \cdots \pm 1)$，容易看到 $2^m + 1$ 分解成兩個正因子的積。 業餘數學家費馬(1601-1665)曾經考慮過以下這些“費馬”整數，設 $F_n = 2^{2^n} + 1$, $n = 0, 1, 2, \ldots$，費馬看到 $F_0 = 2^{2^0} + 1 = 3$，$F_1 = 2^{2^1} + 1 = 5$，$F_2 = 2^{2^2} + 1 = 17$，$F_3 = 2^{2^3} + 1 = 257$，$F_4 = 2^{2^4} + 1 = 65537$，都是質數，(最後一個是質數，需要花些功夫證明)，他據此而猜想，所有形如 $2^{2^n} + 1$ 的正整數都是質數。

不幸的是，大概一百年後，歐拉(1707 – 1783)發現，F_5 不是質數，事實上，直到現在，已知的 F_n，$n \ge 5$，都不是質數。對於 F_5 不是質數，有一個簡單的證明。事實上 $641 = 5^4 + 2^4 = 5 \times 2^7 + 1$，因此 641 整除 $(5^4 + 2^4)2^{28} = 5^4 \times 2^{28} + 2^{32}$。另一方面，由於 $641 = 5 \times 2^7 + 1$，因此 641 也整除 $(5 \times 2^7 + 1)(5 \times 2^7 - 1) = 5^2 \times 2^{14} - 1$，由此，得到 641 整除 $(5^2 \times 2^{14} - 1)(5^2 \times 2^{14} + 1) = 5^4 \times 2^{28} - 1$。最後 641 整除 $5^4 \times 2^{28} + 2^{32}$ 和 $5^4 \times 2^{28} - 1$ 之差，即是 $2^{32} + 1 = F_5$。

這個證明很簡潔，但並不自然，首先，如何知道一個可能的因子是 641，其二，641 能夠寫成兩種和式，實有點幸運。或許可以探究一下，歐拉是怎樣發現 F_5 不是質數。我們相信大概的過程是這樣的，歐拉觀察到，如果 p 是 $F_n = 2^{2^n} + 1$ 的質因子，則 p 一定是 $k \cdot 2^{n+1} + 1$ 的形式。用模算術的言語，如果 p 整除 $2^{2^n} + 1$，則 $2^{2^n} \equiv -1 \pmod{p}$，取平方，得出 $2^{2^{n+1}} \equiv 1 \pmod{p}$。另外，用小費馬定理，(歐拉時已經存在)，知 $2^{p-1} \equiv 1 \pmod{p}$。如果 d 是最小的正整數，使得 $2^d \equiv 1 \pmod{p}$，可以證明(請自証)，d 整除 $p - 1$，也整除 2^{n+1}，但 d 不整除 2^n，(因為 $2^{2^n} \equiv -1 \pmod{p}$)，所以 $d = 2^{n+1}$，再因 d 整除 $p - 1$，所以 $p - 1 = k \cdot 2^{n+1}$，或者 $p = k \cdot 2^{n+1} + 1$。(如果用到所謂的二次互反律，還可以證明，p 實際上是 $k \cdot 2^{n+2} + 1$ 的形式。)

例如考慮 F_4，它的質因子一定是 $32k + 1$ 的形式，取 $k = 1, 2, \ldots$，等，得可能的因子是 97, 193，(小於 $\sqrt{65537}$，以 $32k + 1$ 形式出現的質數)。但 97 和 193 都不整除 65537，所以 65537 是質數。另外，F_5 的質因子一定是 $64k + 1$ 的形式，取 $k = 1, 2, \ldots$，等，得可能的因子是 193, 257, 449, 577, 641, …，經幾次嘗試，得出 $2^{2^5} + 1 = 4294967297 = 641 \times 6700417$，這樣快就找出 F_5 的一個質因子，也算幸運，事實上第二個因子也是質數，不過要證明就比較麻煩。

但是如果試圖用這樣的方法找尋其他費馬數的因子，很快就遇上問題。舉例說，$F_6 = 2^{2^6} + 1$ 是一個二十位數，它的平方根是一個十位數 $(\approx 4.29 \times 10^9)$，其中形狀如 $k \cdot 2^7 + 1 = 128k + 1$ 的數有三百多萬個，要從中找尋 F_6 的因子，可不是易事。讀者可以想像一下，F_5 的完全分解歐拉在 1732 年已找到，而在一百年後 Landry 和 Le Lasseur (1880) 才找到 F_6 的完全分解，再過約一百年， Morrison 和 Brillhart (1970) 發現 F_7 的完全分解，因此找尋費馬數的因子分解肯定不是易事。另一方面由於找尋費馬數不是

易事，Pepin在1877年找到費馬數是否質數的一個判斷：$N>3$是一個形如 $2^{2^n}+1$ 的費馬數，則N是質數的一個充分必須條件是 $3^{\frac{N-1}{2}} \equiv -1 \pmod N$。考慮到 $\frac{N-1}{2}=2^{2^n-1}$，因此是對3不斷取平方，然後求對 N 的摸。近代對求費馬數是否一個質數上，許多都以此為起點。也因此，曾經有一段長時間，已經知道 F_7 不是質數，但它的任一因子都不知道。

再簡述一下近代的結果，現在已知由 F_5 至 F_{11}，都是合數，並且已完全分解。F_{12}, F_{13}, F_{15} 至 F_{19} 是合數，並且知道部分因子。但 F_{14}, F_{20}, F_{22} 等，知道是合數，但一個因子也不知道。最大的費馬合數，並且找到一個因子的是 F_{382447}，讀者可想像一下，如果以十進制形式寫下這個數，它是多少個位數。另外如 F_{33}, F_{34}, F_{35} 等，究竟是合數或質數，一點也不知道。有興趣的話，可參考網頁

http://www.fermatsearch.org/status.htm。

由於費馬數和相關的數有特定的形式，而且具備很多有趣的性質，因此也常在競賽中出現。舉例如下：

例一： 給定費馬數 $F_0, F_1, \ldots, F_n$，有以下的關係 $F_0F_1\cdots F_{n-1}+2=F_n$。

證明： 事實上 $F_n=2^{2^n}+1=2^{2^n}-1+2$ $=2^{2^{n-1}2}-1+2=(2^{2^{n-1}}+1)(2^{2^{n-1}}-1)+2$ $=(2^{2^{n-1}}-1)F_{n-1}+2$。

對於 $2^{2^{n-1}}-1$，可以再分解下去，就可以得到要求的結果。當然嚴格證明可以用歸納法。

例二： 給定費馬數 F_m, F_n, $m>n$，則 F_m, F_n 是互質的。

證明： 因為 $F_m=F_{m-1}\cdots F_n\cdots F_0+2$。設 d 整除 F_m 和 F_n，則 d 也 整除2，所以$d=1$或2。但 $d\neq 2$，因為 F_m, F_n 都是奇數，因此 $d=1$，即 F_m, F_n 互質。

(因此知道，$F_0, F_1, F_2, \ldots$，是互質的，即他們包括無限多數個質因子，引申是有無限多個質數。)

例三： 有無限多個 n，使得 F_n+2 不是質數。

證明： 只要嘗試幾次就可以觀察到 $F_1+2=7$, $F_3+2=259$, 都是7的倍數。事實上，對於$n=0, 1, 2, \ldots$, $2^{2^n}\equiv 2, 4, 2, 4, \ldots \pmod 7$。因此對於奇數$n$, $F_n+2\equiv 2^{2^n}+1+2\equiv 4+1+2\equiv 0 \pmod 7$，所以不是質數。

另一個容易看到的事實是：

例四： 對於 $n>1$, F_n 最尾的數字是 7。

證明: 對於 $n>1$, 2^n 是 4 的倍數，設 $2^n=4k$, 得 $F_n=2^{2^n}+1=2^{4k}+1=(2^4)^k+1\equiv 1^k+1\equiv 2 \pmod 5$。因此 F_n 最尾的數字是 2 或 7，它不可以是 2，因為 F_n 不是偶數。

例五： 證明存在一個正整數 k，使得對任何正整數 n，$k\cdot 2^n+1$ 都不是質數。

（如果n固定，但容許k在正整數中變動，由一個重要的定理 (Dirichlet) 知道在序列中存在無限多個質數。但若果k固定,而n變動,在序列中究竟有多少個質數，是否無限多個，一般都不大清楚。事實上，反可以找到一個k，對於任何正整數n，$k\cdot 2^n+1$ 都不是質數。這原是波蘭數學家Sierpinski (1882-1969)的一個結果，後來演變成美國數學奧林匹克(1982)的一個題目，直到現在，基本是只有一種證明方法，並且與費馬數有關。）

(續於第四頁)

The 2002 Hong Kong IMO team at the Hong Kong Chek Lap Kok Airport taken on August 1, 2002. From left to right, *Chau Suk Ling*, *Chao Khek Lun*, *Cheng Kei Tsi*, *Chiang Kin Nam* (Deputy Leader), *Yu Hok Pun*, *Ip Chi Ho*, *Leung Wai Ying*, *Li Kin Yin* (Leader).

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***December 15, 2002***.

Problem 161. Around a circle are written all of the positive integers from 1 to N, $N \geq 2$, in such a way that any two adjacent integers have at least one common digit in their base 10 representations. Find the smallest N for which this is possible.

Problem 162. A set of positive integers is chosen so that among any 1999 consecutive positive integers, there is a chosen number. Show that there exist two chosen numbers, one of which divides the other.

Problem 163. Let a and n be integers. Let p be a prime number such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be a product of two nonconstant polynomials with integer coefficients.

Problem 164. Let O be the center of the excircle of triangle ABC opposite A. Let M be the midpoint of AC and let P be the intersection of lines MO and BC. Prove that if $\angle BAC = 2\angle ACB$, then $AB = BP$.

Problem 165. For a positive integer n, let $S(n)$ denote the sum of its digits. Prove that there exist distinct positive integers $n_1, n_2, \ldots, n_{50}$ such that

$$n_1 + S(n_1) = n_2 + S(n_2) = \cdots = n_{50} + S(n_{50}).$$

****

Solutions

****

Problem 156. If $a, b, c > 0$ and $a^2 + b^2 + c^2 = 3$, then prove that

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{3}{2}.$$

(*Source: 1999 Belarussian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 5).

By the AM-GM and AM-HM inequalities, we have

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca}$$

$$\geq \frac{1}{1+\frac{a^2+b^2}{2}} + \frac{1}{1+\frac{b^2+c^2}{2}} + \frac{1}{1+\frac{c^2+a^2}{2}}$$

$$\geq \frac{9}{3+a^2+b^2+c^2} = \frac{3}{2}.$$

Other commended solvers: **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 5), **LAM Ho Yin** (South Tuen Mun Government Secondary School, Form 6), **LAM Wai Pui** (STFA Leung Kau Kui College, Form 6), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LO Chi Fai** (STFA Leung Kau Kui College, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 5), **TAM Choi Nang Julian** (SKH Lam Kau Mow Secondary School, teacher), **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6), **TANG Sze Ming** (STFA Leung Kau Kui College, Form 5), **YAU Chun Biu** and **YIP Wai Kiu** (Jockey Club Ti-I College, Form 5) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 5).

Problem 157. In base 10, the sum of the digits of a positive integer n is 100 and of $44n$ is 800. What is the sum of the digits of $3n$? (*Source: 1999 Russian Math Olympiad*)

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LO Chi Fai** (STFA Leung Kau Kui College, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6), and **WONG Wing Hong** (La Salle College, Form 5).

Let $S(x)$ be the sum of the digits of x in base 10. For digits a and b, if $a + b > 9$, then $S(a + b) = S(a) + S(b) - 9$. Hence, if we have to carry in adding x and y, then $S(x + y) < S(x) + S(y)$. So in general, $S(x + y) \leq S(x) + S(y)$. By induction, we have $S(kx) \leq kS(x)$ for every positive integer k. Now

$$\begin{aligned} 800 &= S(44n) = S(40n + n) \\ &\leq S(40n) + S(4n) = 2S(4n) \\ &\leq 8S(n) = 800. \end{aligned}$$

Hence equality must hold throughout and there can be no carry in computing $4n = n + n + n + n$. So there is no carry in $3n = n + n + n$ and $S(3n) = 300$.

Other commended solvers: **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7).

Problem 158. Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on BC such that $BD = 2DC$ and let P be a point on AD such that $\angle BAC = \angle BPD$. Prove that

$$\angle BAC = 2\angle DPC.$$

(*Source: 1999 Turkish Math Olympiad*)

Solution. **LAM Wai Pui** (STFA Leung Kau Kui College, Form 6), **POON Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) , **WONG Wing Hong** (La Salle College, Form 5) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 5).

Let E be a point on AD extended so that $PE=PB$. Since $\angle CAB=\angle EPB$ and $CA/AB=1=EP/PB$, triangles CAB and EPB are similar. Then $\angle ACB=\angle PEB$, which implies A, C, E, B are concyclic. So $\angle AEC=\angle ABC=\angle AEB$. Therefore, AE bisects $\angle CEB$.

Let M be the midpoint of BE. By the angle bisector theorem, $CE/EB = CD/DB = 1/2$. So $CE = ½EB = ME$. Also, $PE = PE$ and PE bisects $\angle CEM$. It follows triangles CEP and MEP are congruent. Then $\angle BAC = \angle BPE = 2\angle MPE = 2\angle CPE = 2\angle DPC$.

Other commended solvers: **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

Problem 159. Find all triples (x, k, n) of positive integers such that

$$3^k - 1 = x^n.$$

(*Source: 1999 Italian Math Olympiad*)

Solution. (*Official Solution*)

For $n = 1$, the solutions are $(x, k, n) = (3^k - 1, k, 1)$, where k is for any positive integer.

For $n > 1$, if n is even, then $x^n + 1 \equiv 1$ or 2 (mod 3) and hence cannot be $3^k \equiv 0$ (mod 3). So n must be odd. Now $x^n + 1$ can be factored as

$$(x+1)(x^{n-1} - x^{n-2} + \cdots + 1).$$

If $3^k = x^n + 1$, then both of these factors are powers of 3, say they are 3^s, 3^t, respectively. Since

$$x + 1 \le x^{n-1} - x^{n-2} + \cdots + 1,$$

so $s \le t$. Then

$$0 \equiv 3^t \equiv (-1)^{n-1} - (-1)^{n-2} + \cdots + 1$$
$$= n \pmod{x+1}$$

implying n is divisible by $x + 1$ (and hence also by 3). Let $y = x^{n/3}$. Then

$$3^k = y^3 + 1 = (y + 1)(y^2 - y + 1).$$

So $y + 1$ is also a power of 3, say it is 3^r. If $r = 1$, then $y = 2$ and $(x, k, n) = (2, 2, 3)$ is a solution. Otherwise, $r > 1$ and

$$3^k = y^3 + 1 = 3^{3r} - 3^{2r+1} + 3^{r+1}$$

is strictly between 3^{3r-1} and 3^{3r}, a contradiction.

Other commended solvers: **LEE Pui Chung** (Wah Yan College, Kowloon, Form 7), **LEUNG Chi Man** (Cheung Sha Wan Catholic Secondary School, Form 6), **POON Ming Fung** (STFA Leung Kau Kui College, Form 5) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Problem 160. We are given 40 balloons, the air pressure inside each of which is unknown and may differ from balloon to balloon. It is permitted to choose up to k of the balloons and equalize the pressure in them (to the arithmetic mean of their respective pressures.) What is the smallest k for which it is always possible to equalize the pressures in all of the balloons?

(*Source:1999 Russian Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

For $k = 5$, it is always possible. We equalize balloons 1 to 5, then 6 to 10, and so on (five at time). Now take one balloon from each of these 8 groups. We have eight balloons, say a, b, c, d, e, f, g, h. We can equalize a, b, c, d, then e, f, g, h, followed by a, b, e, f and finally c, d, g, h. This will equalize all 8 balloons. Repeat getting one balloon from each of the 8 groups for 4 more times, then equalize them similarly. This will make all 40 balloons having the same pressure.

For $k < 5$, it is not always possible. If the i-th balloon has initial pressure $p_i = \pi^i$, then after equalizing operations, their pressures will always have the form $c_1 p_1 + \cdots + c_{40} p_{40}$ for some rational numbers $c_1, \ldots, c_{40}$. The least common multiple of the denominators of these rational numbers will always be of the form $2^r 3^s$ as $k = 1, 2, 3$ or 4 implies we can only change the denominators by a factor of 2, 3 or 4 after an operation. So $c_1, \ldots, c_{40}$ can never all be equal to 1/40.

Olympiad Corner

(continued from page 1)

Problem 3. Prove that for all positive real numbers a, b, and c,

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \ge a + b + c$$

and determine when equality occurs.

Problem 4. Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with center B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let the line CP meet Γ again at Q. Prove that $PQ = r$.

Problem 5. Let $N = \{0, 1, 2, \ldots\}$. Determine all functions $f: N \to N$ such that

$$xf(y) + yf(x) = (x + y) f(x^2 + y^2)$$

for all x and y in N.

簡介費馬數

(續第二頁)

證明：（證明的起點是中國餘式定理，設 $m_1, m_2, \ldots, m_r$ 是互質的正整數，$a_1, a_2, \ldots, a_r$ 是任意整數，則方程組 $x \equiv a_1 \pmod{m_1}, x \equiv a_2 \pmod{m_2}, \ldots, x \equiv a_r \pmod{m_r}$,有解。並且其解對於模 $m = m_1 m_2 \cdots m_r$ 唯一。現在考慮到任意正整數 n，都可以寫成 $2^h q$ 的形式，其中 q 是奇數。如果能夠選擇 k，使得 $k > 1, k \equiv 1 \pmod{2^{2^h}+1}$, 則 $k \cdot 2^n + 1 = k \cdot 2^{2^h q} + 1 \equiv (1)(2^{2^h})^q + 1 \equiv (1)(-1)^q + 1 \equiv (-1) + 1 \equiv 0 \pmod{2^{2^h}+1}$，所以 $k2^n + 1$ 不是質數。留意到這裡用到 q 是奇數的性質。不過，如果這樣做的話，h 會因 n 而變，而 k 隨 h 而變，這是不容許的，k 要在起先之前決定，而不受 n 影響。）

解決的方法是這樣的，我們可以先選擇 k，使得 $k > 1,\ k \equiv 1 \pmod{2^{2^h}+1}$，其中 $h = 0, 1, 2, 3, 4$。這是可能的，因為我們知道 F_0, F_1, F_2, F_3 和 F_4 是不同的質數。這樣的話，可以證明對於所有 $n = 2^h q$，其中 $h < 5$，q 是奇數，$k \cdot 2^n + 1$ 都不是質數。對於 $n = 2^h q$，$h \ge 5$，又可以怎樣處理呢。留意到所有這樣的數，都可以寫成 $n = 2^h q = 2^5 m$ 的形式，其中 m 可以是奇數，也可以是偶數。另一方面，我們知道 $F_5 = 2^{2^5} + 1 = (641) \times (6700417)$，其中 $P = 641$，$Q = 6700417$ 是不同的質數。如果我們選擇 k，使得 $k > 1$，和 $k \equiv -1 \pmod{P}$，$k \equiv 1 \pmod{Q}$，則 $k2^n + 1 = k2^{2^5 m} + 1 \equiv (-1)(2^{2^5})^m + 1 \equiv (-1)(-1)^m + 1 \equiv (-1)^{m+1} + 1 \pmod{P}$，另一方面 $k2^n + 1 = k2^{2^5 m} + 1 \equiv (1)(2^{2^5})^m + 1 \equiv (-1)^m + 1 \pmod{Q}$。如果 m 是偶數，則 $k2^n + 1$ 是 P 的倍數，如果 m 是奇數，則 $k2^n + 1$ 是 Q 的倍數，因此都不是質數。歸納言之，選擇 k，使得 $k \equiv 1 \pmod{x}$，$x = 3, 5, 17, 257, 65537, 6700417$，$k \equiv -1 \pmod{641}$，則對於所有形如 $k2^n + 1$ 的數，都不是質數。（最後要留意的是，這方程組的最小正整數解不可能是 1，因此所有的 $k2^n + 1$，都不是質數。）

Mathematical Excalibur

Volume 7, Number 5 December 2002 – January 2003

Olympiad Corner

The 19th Balkan Mathematical Olympiad was held in Antalya, Turkey on April 27, 2002. The problems are as follow.

Problem 1. Let $A_1, A_2, \ldots, A_n$ $(n \geq 4)$ be points on the plane such that no three of them are collinear. Some pairs of distinct points among $A_1, A_2, \ldots, A_n$ are connected by line segments in such a way that each point is connected to three others. Prove that there exists $k > 1$ and distinct points $X_1, X_2, \ldots, X_{2k}$ in $\{ A_1, A_2, \ldots, A_n \}$ so that for each $1 \leq i \leq 2k-1$, X_i is connected to X_{i+1} and X_{2k} is connected to X_1.

Problem 2. The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by $a_1 = 20$, $a_2 = 30$, $a_{n+2} = 3a_{n+1} - a_n$, $n > 1$. Find all positive integers n for which $1+5a_na_{n+1}$ is a perfect square.

Problem 3. Two circles with different radii intersect at two pints A and B. The common tangents of these circles are MN and ST where the points M, S are on one of the circles and N, T are on the other. Prove that the orthocenters of the triangles AMN, AST, BMN and BST are the vertices of a rectangle.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 26, 2003***.

For individual subscription for the next five issues for the 02-03 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Mathematical Games (II)

Kin Y. Li

There are many mathematical game problems involving strategies to win or to defend. These games may involve number theoretical properties or combinatorial reasoning or geometrical decomposition. Some games may go on forever, while some games come to a stop eventually. A winning strategy is a scheme that allows a player to make moves to win the game no matter how the opponent plays. A defensive strategy cuts off the opponent's routes to winning. The following examples illustrate some standard techniques.

Examples 1. There is a table with a square top. Two players take turn putting a dollar coin on the table. The player who cannot do so loses the game. Show that the first player can always win.

Solution. The first player puts a coin at the center. If the second player can make a move, the first player can put a coin at the position symmetrically opposite the position the second player placed his coin with respect to the center of the table. Since the area of the available space is decreasing, the game must end eventually. The first player will win.

Example 2. (*Bachet's Game*) Initially, there are n checkers on the table, where $n > 0$. Two persons take turn to remove at least 1 and at most k checkers each time from the table. The last person who can remove any checker wins the game. For what values of n will the first person have a winning strategy? For what values of n will the second person have a winning strategy?

Solution. By testing small cases of n, we can easily see that if n is *not* a multiple of $k + 1$ in the beginning, then the first person has a winning strategy, otherwise the second person has a winning strategy.

To prove this, let n be the number of checkers on the table. If $n = (k+1)q + r$ with $0 < r < k + 1$, then the first person can win by removing r checkers each time. (Note $r > 0$ every time at the first person's turn since in the beginning it is so and the second person starts with a multiple of $k + 1$ checkers each time and can only remove 1 to k checkers.)

However, if n is a multiple of $k + 1$, then no matter how many checkers the first person takes, the second person can now win by removing r checkers every time.

Example 3. (*Game of Nim*) There are 3 piles of checkers on the table. The first, second and third piles have x, y and z checkers respectively in the beginning, where $x, y, z > 0$. Two persons take turn choosing one of the three piles and removing at least one to all checkers in that pile each time from the table. The last person who can remove any checker wins the game. Who has a winning strategy?

Solution. In base 2 representations, let

$$x = (a_1a_2\ldots a_n)_2, \quad y = (b_1b_2\ldots b_n)_2,$$
$$z = (c_1c_2\ldots c_n)_2, \quad N = (d_1d_2\ldots d_n)_2,$$

where $d_i \equiv a_i + b_i + c_i \pmod 2$. The first person has a winning strategy if and only if N is not 0, i.e. not all d_i's are 0.

To see this, suppose N is not 0. The winning strategy is to remove checkers so N becomes 0. When the d_i's are not all zeros, look at the smallest i such that $d_i = 1$, then one of a_i, b_i, c_i equals 1, say $a_i = 1$. Then remove checkers from the first pile so that $x = (e_ie_{i+1}\ldots e_n)_2$ checkers are left, where $e_j = a_j$ if $d_j = 0$, otherwise $e_j = 1 - a_j$.

(For example, if $x = (1000)_2$ and $N = (1001)_2$, then change x to $(0001)_2$.) After the move, N becomes 0. So the first person can always make a move. The second person will always have $N = 0$ at his turn and making any move will result in at least one d_i not 0, i.e. $N \neq 0$. As the number of checkers is decreasing, eventually the second person cannot make a move and will lose the game.

Example 4. Twenty girls are sitting around a table and are playing a game with n cards. Initially, one girl holds all the cards. In each turn, if at least one girl holds at least two cards, one of these girls must pass a card to each of her two neighbors. The game ends if and only if each girl is holding at most one card.

(a) Prove that if $n \geq 20$, then the game cannot end.
(b) Prove that if $n < 20$, the game must end eventually.

Solution. (a) If $n > 20$, then by the pigeonhole principle, at every moment there exists a girl holding at least two cards. So the game cannot end.

If $n = 20$, then label the girls $G_1, G_2, \ldots, G_{20}$ in the clockwise direction and let G_1 be the girl holding all the cards initially. Define the current value of a card to be i if it is being held by G_i. Let S be the total value of the cards. Initially, $S = 20$.

Consider before and after G_i passes a card to each of her neighbors. If $i = 1$, then S increases by $-1-1+2+20=20$. If $1 < i < 20$, then S does not change because $-i-i+(i-1)+(i+1)=0$. If $i = 20$, then S decreases by 20 because $-20-20+1+19=-20$. So before and after moves, S is always a multiple of 20. Assume the game can end. Then each girl holds a card and $S = 1 + 2 + \cdots + 20 = 210$, which is not a multiple of 20, a contradiction. So the game cannot end.

(b) To see the game must end if $n < 20$, let's have the two girls sign the card when it is the first time one of them passes card to the other. Whenever one girl passes a card to her neighbor, let's require the girl to use the signed card between the pair if available. So a signed card will be stuck between the pair who signed it. If $n < 20$, there will be a pair of neighbors who never signed any card, hence never exchange any card.

If the game can go on forever, record the number of times each girl passed cards. Since the game can go on forever, not every girl passed card finitely many time. So starting with a pair of girls who have no exchange and moving clockwise one girl at a time, eventually there is a pair G_i and G_{i+1} such that G_i passed cards finitely many times and G_{i+1} passed cards infinitely many times. This is clearly impossible since G_i will eventually stopped passing cards and would keep on receiving cards from G_{i+1}.

Example 5. (*1996 Irish Math Olympiad*) On a 5 × 9 rectangular chessboard, the following game is played. Initially, a number of discs are randomly placed on some of the squares, no square containing more than one disc. A turn consists of moving all of the discs subject to the following rules:

(i) each disc may be moved one square up, down, left or right;
(ii) if a disc moves up or down on one turn, it must move left or right on the next turn, and vice versa;
(iii) at the end of each turn, no square can contain two or more discs.

The game stops if it becomes impossible to complete another turn. Prove that if initially 33 discs are placed on the board, the game must eventually stop. Prove also that it is possible to place 32 discs on the board so that the game can continue forever.

Solution. If 32 discs are placed in the lower right 4 × 8 rectangle, they can all move up, left, down, right, repeatedly and the game can continue forever.

To show that a game with 33 discs must stop eventually, label the board as shown below:

1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1

Note that there are only eight squares labeled with 3's. A disc on 1 goes to a 3 after two moves, a disc on 2 goes to a 1 or 3 immediately, and a disc on 3 goes to a 2 immediately. Thus if k discs start on 1 and $k > 8$, the game stops because there are not enough 3's to accommodate these discs after two moves. Thus we assume $k \leq 8$, in which case there are at most sixteen discs on squares with 1's or 3's at the start, and at least seventeen discs on squares with 2's. Of these seventeen discs, at most eight can move onto squares with 3's after one move, so at least nine end up on squares with 1's. These discs will not all be able to move onto squares with 3's two moves later. So the game must eventually stop.

Example 6. (*1995 Israeli Math Olympiad*) Two players play a game on an infinite board that consists of 1 × 1 squares. Players I chooses a square and marks it with an O. Then, player II chooses another square and marks with an X. They play until one of the players marks a row or a column of five consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning.

Solution: Label the squares as shown below.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	1	2	3	3	1	2	3	3	⋯
⋯	1	2	4	4	1	2	4	4	⋯
⋯	3	3	1	2	3	3	1	2	⋯
⋯	4	4	1	2	4	4	1	2	⋯
⋯	1	2	3	3	1	2	3	3	⋯
⋯	1	2	4	4	1	2	4	4	⋯
⋯	3	3	1	2	3	3	1	2	⋯
⋯	4	4	1	2	4	4	1	2	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Note that each number occurs in a pair. The 1's and 2's are in vertical pairs and the 3's and 4's are in horizontal pairs. Whenever player I marks a square, player II will mark the other square in the pair. Since any five consecutive vertical or horizontal squares must contain a pair of these numbers, so player I cannot win.

Example 7. (*1999 USAMO*) The Y2K Game is played on a 1 × 2000 grid as follow. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell *SOS* wins. If all boxes are filled without producing any *SOS,* then the game is a draw. Prove that the second player has a winning strategy.

Solution. Call an empty square *bad* if playing an S or an O in that square will let the next player gets *SOS* in the next move.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***January 26, 2003***.

Problem 166. (*Proposed by Ha Duy Hung, Hanoi University of Education, Hanoi City, Vietnam*) Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\,[a/b] - [c/a]\,[c/b] \le c\,\min\{1/a, 1/b\}.$$

Problem 167. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all positive integers such that they are equal to the sum of their digits in base 10 representation.

Problem 168. Let AB and CD be nonintersecting chords of a circle and let K be a point on CD. Construct (with straightedge and compass) a point P on the circle such that K is the midpoint of the part of segment CD lying inside triangle ABP.

Problem 169. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 11/2 times any other group.

Problem 170. (*Proposed by Abderrahim Ouardini, Nice, France*) For any (nondegenerate) triangle with sides a, b, c, let $\sum' h(a,b,c)$ denote the sum $h(a,b,c) + h(b,c,a) + h(c,a,b)$. Let $f(a,b,c) = \sum' (a/(b+c-a))^2$ and $g(a,b,c) = \sum' j(a,b,c)$, where $j(a,b,c) = (b+c-a)/\sqrt{(c+a-b)(a+b-c)}$. Show that $f(a,b,c) \ge \max\{3, g(a,b,c)\}$ and determine when equality occurs. (Here $\max\{x,y\}$ denotes the maximum of x and y.)

****

Solutions

****

Problem 161. Around a circle are written all of the positive integers from 1 to N, $N \ge 2$, in such a way that any two adjacent integers have at least one common digit in their base 10 representations. Find the smallest N for which this is possible. (*Source: 1999 Russian Math Olympiad*)

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yan Sau** (True Light Girls' College, Form 6), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **LAM Wai Pui** (STFA Leung Kau Kui College, Form 5), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **LEUNG Chi Man** (Cheung Sha Wan Catholic Secondary School, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 5).

Note one of the numbers adjacent to 1 is at least 11. So $N \ge 11$. Then one of the numbers adjacent to 9 is at least 29. So $N \ge 29$. Finally $N = 29$ is possible by writing 1, 11, 12, 2, 22, 23, 3, 13, 14, 4, 24, 25, 5, 15, 16, 6, 26, 27, 7, 17, 18, 8, 28, 29, 9, 19, 21, 20, 10 around a circle. Therefore, the smallest N is 29.

Problem 162. A set of positive integers is chosen so that among any 1999 consecutive positive integers, there is a chosen number. Show that there exist two chosen numbers, one of which divides the other. (*Source: 1999 Russian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Define $A(1, i) = i$ for i=1,2,…, 1999. For $k \ge 2$, let $B(k)$ be the product of $A(k–1, 1)$, $A(k–1, 2)$, …, $A(k–1, 1999)$ and define $A(k, i) = B(k) + A(k–1, i)$ for i = 1,2,…, 1999. Since $B(k)$ is a multiple of $A(k–1, i)$, so $A(k, i)$ is also a multiple of $A(k–1, i)$. Then $m < n$ implies $A(n, i)$ is a multiple of $A(m, i)$.

Also, by simple induction on k, we can check that $A(k, 1), A(k, 2), \ldots, A(k, 1999)$ are consecutive integers. So for k = 1,2, …, 2000, among $A(k, 1), A(k, 2), \ldots, A(k, 1999)$, there is a chosen number $A(k, i_k)$. Since $1 \le i_k \le 1999$, by the pigeonhole principle, two of the i_k's are equal. Therefore, among the chosen numbers, there are two numbers with one dividing the other.

Comments: The condition "among any 1999 consecutive positive integers, there is a chosen number" is meant to be interpreted as "among any 1999 consecutive positive integers, there exists at least one chosen number." The solution above covered this interpretation.

Other commended solvers: **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

Problem 163. Let a and n be integers. Let p be a prime number such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be the product of two nonconstant polynomials with integer coefficients. (*Source: 1999 Romanian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **TAM Choi Nang Julian** (SKH Lam Kau Mow Secondary School).

Assume we have $f(x) = g(x)\,h(x)$, where $g(x)$ and $h(x)$ are two nonconstant polynomials with integer coefficients. Since $p = f(0) = g(0)\,h(0)$, we have either

$$g(0) = \pm p,\quad h(0) = \pm 1$$
$$\text{or}\quad g(0) = \pm 1,\quad h(0) = \pm p.$$

Without loss of generality, assume $g(0) = \pm 1$. Then $g(x) = \pm x^m + \cdots \pm 1$. Let $z_1, z_2, \ldots, z_m$ be the (possibly complex) roots of $g(x)$. Since $1 = |g(0)| = |z_1|\,|z_2| \cdots |z_m|$, so $|z_i| \le 1$ for some i. Now $0 = f(z_i) = z_i^n + az_i + p$ implies

$$p = -z_i^n - az_i \le |z_i|^n + |a|\,|z_i| \le 1 + |a|,$$

a contradiction.

Other commended solvers: **FOK Kai Tung** (Yan Chai Hospital No. 2 Secondary School, Form 6).

Problem 164. Let O be the center of the excircle of triangle ABC opposite A. Let M be the midpoint of AC and let P be the intersection of lines MO and BC. Prove that if $\angle BAC = 2\angle ACB$, then $AB = BP$. (*Source: 1999 Belarussian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Let AO cut BC at D and AP extended cut OC at E. By Ceva's theorem (ΔAOC and point P), we have

$$\frac{AM}{MC}\times\frac{CE}{EO}\times\frac{OD}{DA}=1.$$

Since $AM = MC$, we get $OD/DA = OE/EC$, which implies $DE \parallel AC$. Then $\angle EDC = \angle DCA = \angle DAC = \angle ODE$, which implies DE bisects $\angle ODC$. In ΔACD, since CE and DE are external angle bisectors at $\angle C$ and $\angle D$ respectively, so E is the excenter of ΔACD opposite A. Then AE bisects $\angle OAC$ so that $\angle DAP = \angle CAP$. Finally,

$$\begin{aligned}\angle BAP &= \angle BAD + \angle DAP\\ &= \angle DCA + \angle CAP\\ &= \angle BPA.\end{aligned}$$

Therefore, $AB = BP$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

Problem 165. For a positive integer n, let $S(n)$ denote the sum of its digits. Prove that there exist distinct positive integers $n_1, n_2, \ldots, n_{50}$ such that

$$\begin{aligned}n_1 + S(n_1) = n_2 + S(n_2) &= \cdots\\ &= n_{50} + S(n_{50}).\end{aligned}$$

(*Source: 1999 Polish Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

We will prove the statement that for $m > 1$, there are positive integers $n_1 < n_2 < \cdots < n_m$ such that all $n_i + S(n_i)$ are equal and n_m is of the form $10\cdots08$ by induction.

For the case $m = 2$, take $n_1 = 99$ and $n_2 = 108$, then $n_i + S(n_i) = 117$.

Assume the case $m = k$ is true and $n_k = 10\cdots08$ with h zeros. Consider the case $m = k + 1$. For $i = 1, 2, \ldots, k$, define

$$N_i = n_i + C,\ \text{where } C = 99\cdots900\cdots0$$

(C has $n_k - 8$ nines and $h + 2$ zeros) and $N_{k+1} = 10\cdots08$ with $n_k - 7 + h$ zeros. Then for $i = 1, 2, \ldots, k$,

$$N_i + S(N_i) = C + n_i + S(n_i) + 9(n_k - 8)$$

are all equal by the case $m = k$. Finally,

$$\begin{aligned}N_k + S(N_k) &= C + n_k + 9 + 9(n_k - 8)\\ &=10\cdots017\ (n_k - 8 + h \text{ zeros})\\ &=10\cdots008 + 9\\ &=N_{k+1} + S(N_{k+1})\end{aligned}$$

completing the induction.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5).

Olympiad Corner

(continued from page 1)

Problem 4. Find all functions $f\colon \mathbb{N}\to\mathbb{N}$ such that for each $n \in \mathbb{N}$,

$$\begin{aligned}2n + 2001 \le\ & f(f(n)) + f(n)\\ \le\ & 2n + 2003.\end{aligned}$$

($\mathbb{N}$ is the set of all positive integers.)

Mathematical Games II

(continued from page 2)

<u>Key Observation</u>: A square is bad if and only if it is in a block of 4 consecutive squares of the form *S****S*, where * denotes an empty square.

(<u>*Proof.*</u> Clearly, the empty squares in *S****S* are bad. Conversely, if a square is bad, then playing an *O* there will allow an *SOS* in the next move by the other player. Thus the bad square must have an *S* on one side and an empty square on the other side. Playing an *S* there will also lose the game in the next move, which means there must be another *S* on the other side of the empty square.)

Now the second player's winning strategy is as follow: after the first player made a move, play an *S* at least 4 squares away from either end of the grid and from the first player's first move. On the second move, the second player will play an *S* three squares away from the second player's first move so that the squares in between are empty. (If the second move of the first player is next to or one square away from the first move of the second player, then the second player will place the second *S* on the other side.) After the second move of the second player, there are 2 bad squares on the board. So eventually somebody will fill these squares and the game will not be a draw.

On any subsequent move, when the second player plays, there will be an odd number of empty squares and an even number of bad squares, so the second player can always play a square that is not bad.

Example 8. (*1993 IMO*) On an infinite chessboard, a game is played as follow. At the start, n^2 pieces are arranged on the chessboard in an $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece that has been jumped over is then removed. Find those values of n for which the game can end with only one piece remaining on the board.

Solution. Let $\mathbb{Z}$ denotes the set of integers. Consider the pieces placed at the *lattice points* $\mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\}$. For $k = 0, 1, 2$, let $C_k = \{(x, y) \in \mathbb{Z}^2 : x+y \equiv k \pmod 3\}$. Let a_k be the number of pieces placed at lattice points in C_k.

A horizontal move takes a piece at (x, y) to an unoccupied point $(x \pm 2, y)$ jumping over a piece at $(x \pm 1, y)$. After the move, each a_k goes up or down by 1. So each a_k changes parity. If n is divisible by 3, then $a_0 = a_1 = a_2 = n^2/3$ in the beginning. Hence at all time, the a_k's are of the same parity. So the game cannot end with one piece left causing two a_k's 0 and the remaining 1.

If n is not divisible by 3, then the game can end. We show this by induction. For $n = 1$ or 2, this is easily seen. For $n \ge 4$, we introduce a trick to reduce the $n \times n$ square pieces to $(n-3) \times (n-3)$ square pieces.

<u>*Trick:*</u> Consider pieces at (0,0), (0,1), (0,2), (1,0). The moves $(1,0) \to (-1,0)$, $(0,2) \to (0,0)$, $(-1,0) \to (1,0)$ remove three consecutive pieces in a column and leave the fourth piece at its original lattice point.

We can apply this trick repeatedly to the $3 \times (n-3)$ pieces on the bottom left part of the $n \times n$ squares from left to right, then the $n \times 3$ pieces on the right side from bottom to top. This will leave $(n-3) \times (n-3)$ pieces. Therefore, the $n \times n$ case follows from the $(n-3) \times (n-3)$ case, completing the induction.

Construction of Logarithm Table

LEE Wing Lung

In this modern age of computers, to find a logarithmic number like $\ln 2$ requires only the touch of a finger. Have you ever wondered, before the appearance of calculators, how the thick logarithm tables were constructed with precisions? Of course, historically there appeared many methods of constructions, each has its advantages, each has its limits. Below we will discuss one such method systematically. It only uses a bit of basic calculus skills and yields an efficient way of constructing a logarithm table to any precision.

First note that since $\ln(xy) = \ln x + \ln y$, we only need to find the logarithm of prime numbers p to get the logarithm of the other positive integers. From the differentiation of $\ln(1+t)$ and the geometric series formula, we get

$$\frac{d}{dt}\ln(1+t) = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^{n-1}t^{n-1} + \frac{(-1)^n t^n}{1+t}.$$

By the fundamental theorem of calculus (that is the inverse relations of differentiation and integration), we get

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-1)^n t^n}{1+t}dt$$

for all positive integer n. Let us now estimate the integral remainder term. For $|x| < 1$, we have

$$\Big|\int_0^x \frac{(-1)^n t^n}{1+t}dt\Big| \le \Big|\int_0^x \Big|\frac{t^n}{1+t}\Big||dt|\Big| \le \Big|\int_0^x \frac{t^n}{1-|x|}dt\Big| = \frac{|x|^{n+1}}{(n+1)(1-|x|)}.$$

From this we can see that as n increases, the absolute value of the remainder term will go to 0. In other words, for n sufficiently large, the difference between $\ln(1+x)$ and $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}$ will become arbitrarily small. Hence we may represent this in the following way:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (|x| < 1).$$

Thus by a choice of n we may ignore the difference. Replacing x by $-x$ and subtracting the two equation, we have the following equation

$$\ln\Big(\frac{1+x}{1-x}\Big) = \ln(1+x) - \ln(1-x) = 2\Big(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\Big). \qquad (*)$$

Unfortunately, if we substitute $x = \frac{p-1}{p+1}$ to get $\frac{1+x}{1-x} = p$, equation (*) will not be able to produce $\ln p$ efficiently. For example, take $p = 29$, then $x = \frac{29-1}{29+1} = \frac{14}{15}$, and compute even to the 100th term $\frac{2x^{199}}{199} \approx 1.1 \times 10^{-8}$, the value of $\ln p$ is still not determined precisely to the 8th decimal place (more rigorously, we should have used the remainder term of the equation (*) to estimate the difference, but here we just want to know the difference approximately); for another example, take $p = 113$, then $x = \frac{56}{57}$ and $\frac{2x^{199}}{199} \approx 3 \times 10^{-4}$, the precision of $\ln p$ is even worse. However, if we use $x = \frac{1}{2p^2-1}$, then

$$\ln\Big(\frac{1+x}{1-x}\Big) = \ln\frac{2p^2-1+1}{2p^2-1-1} = \ln\frac{p^2}{(p+1)(p-1)} = 2\ln p - \ln(p+1)(p-1).$$

For prime $p > 2$, the common prime divisors of $(p+1)$ and $(p-1)$ must be less than p. So if we have computed the logarithm of all prime numbers less than p, then we can use the above equation to compute the value of $\ln p$:

$$2\ln p = \ln\Big(\frac{1+x}{1-x}\Big) + \ln(p+1) + \ln(p-1).$$

Now the term $\ln\Big(\dfrac{1+x}{1-x}\Big)$ can be computed efficiently because the absolute value of x chosen will be small. For example, when $p = 29$, $x = \dfrac{1}{2\cdot 29^2 - 1} = \dfrac{1}{1681}$, so just compute to $\dfrac{2x^5}{5} \approx 3\times 10^{-17}$ we can obtain ten decimal place accuracy.

From the discussion above, suppose we now want to construct a 8 decimal place logarithm table. Then we can compute the logarithm of $2, 3, 5, 7, 11, 13, \ldots$ in order and the logarithm of the numbers can be obtained from the logarithm of the numbers preceding them. From this we see that in the beginning we need to compute $\ln 2$ to a high precision:

$$\ln 2 = \ln\Big(\frac{1-\frac{1}{3}}{1+\frac{1}{3}}\Big) \approx 2\Big(\frac{1}{3} + \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} + \cdots + \frac{(\frac{1}{3})^{21}}{21}\Big) = 0.6931471805589\ldots.$$

This and the actual answer $\ln 2 = 0.693147180559945\ldots$ agree to 11 decimal places. Then we compute $\ln 3$. Taking $x = \dfrac{1}{2\cdot 3^2 - 1} = \dfrac{1}{17}$, we have

$$\ln\Big(\frac{1-\frac{1}{17}}{1+\frac{1}{17}}\Big) \approx 2\Big(\frac{1}{17} + \frac{(\frac{1}{17})^3}{3} + \frac{(\frac{1}{17})^5}{5} + \frac{(\frac{1}{17})^7}{7}\Big) = 0.117783035654504\ldots.$$

Note that since $\dfrac{2(\frac{1}{17})^9}{9} \approx 1.9\times 10^{-12}$, we may ignore this if only 8 decimal place accuracy is desired. So

$$\ln 3 \approx \frac{1}{2}(0.11778303565\ldots + \ln 4 + \ln 2) = 1.098612288635\ldots.$$

This and the actual answer $\ln 3 = 1.09861228866811\ldots$ agree to 10 decimal places. We will leave the computations of $\ln 5, \ln 7$ and so on to the reader and let them compare with the results given by calculators.

Looking back at the above very clever computation method, everyone should respect the mathematicians in the past for their creativity and instinct in the relations of numbers and computations!

References:

Chapter 3 of Professor Wu-Yi Hsiang's Notes for his Lectures on Analysis available at

http://ihome.ust.hk/~malung/391.html

A Brief Introduction to Fermat Numbers

LEUNG Tat-Wing

Consider a positive integer of the form 2^m+1. If it is a prime number, then m must be a power of 2. Otherwise, let $m=2^n s$, where s is an odd number greater than or equal to 3. We have $2^m+1=2^{2^n s}+1=(2^{2^n})^s+1=(2^{2^n}+1)\big((2^{2^n})^{s-1}-(2^{2^n})^{s-2}+\cdots\pm 1\big)$, from which it is easily seen that 2^m+1 decomposed into a product of two positive divisors. The amateur mathematician Fermat (1601-1665) have considered the following "Fermat" numbers. Let $F_n=2^{2^n}+1,\ n=0,1,2,\ldots$. Fermat observed $F_0=2^{2^0}+1=3, F_1=2^{2^1}+1=5, F_2=2^{2^2}+1=17, F_3=2^{2^3}+1=257, F_4=2^{2^4}+1=65537$ are prime numbers (that the last number is a prime takes a bit of work to prove). Because of these, he conjectured all positive integers of the form $2^{2^n}+1$ are prime numbers.

Unfortunately, about a hundred years later, Euler (1707-1783) discovered F_5 is not a prime number. In fact, up to now, all known $F_n, n\geq 5$ are not prime numbers. For F_5 not prime, there is a simple proof. Since $641=5^4+2^4=5\times 2^7+1$, so 641 divides $(5^4+2^4)2^{28}=5^4\times 2^{28}+2^{32}$. Also, since $641=5\times 2^7+1$, so 641 divides $(5\times 2^7+1)(5\times 2^7-1)=5^2\times 2^{14}-1$. Hence, we get 641 divides $(5^2\times 2^{14}-1)(5^2\times 2^{14}+1)=5^4\times 2^{28}-1$. Finally, 641 divides the difference of $5^4\times 2^{28}+2^{32}$ and $5^4\times 2^{28}-1$, which is $2^{32}+1=F_5$.

This proof is very concise, but not natural. First, it is not known how to get 641 is a positive divisor. Second, that 641 can be written as the two sums above is quite fortunate. Let us investigate how Euler discovered F_5 was not a prime number. We believe the process may have been like this. Euler observed that if p is a prime divisor of $F_n=2^{2^n}+1$, then p must be of the form $k\cdot 2^{n+1}+1$. Using the language of modulo arithmetic, if p divides $2^{2^n}+1$, then $2^{2^n}\equiv -1(\text{mod } p)$. Squaring yield $2^{2^{n+1}}\equiv 1(\text{mod } p)$. Next, using Fermat's little theorem (known at Euler's time), we see $2^{p-1}\equiv 1(\text{mod } p)$. If d is the smallest positive integer such that $2^d\equiv 1(\text{mod } p)$, it can be proved (check please) that d divides $p-1$ and 2^{n+1}, but not 2^n (as $2^{2^n}\equiv -1(\text{mod } p)$). Hence $d=2^{n+1}$. Also, since d divides $p-1$, so $p-1=k\cdot 2^{n+1}$ or $p=k\cdot 2^{n+1}+1$. (If the so-called law of quadratic reciprocity is used, it can even be proved that p is of the form $k\cdot 2^{n+2}+1$.)

For example, consider F_4. Its prime divisor must be of the form $32k+1$. Taking $k=1,2,\ldots$, the possible divisors are 97, 193 (the prime numbers of the form $32k+1$ and less than $\sqrt{65537}$). However, 97 and 193 do not divide 65537, so 65537 is prime. Next, the prime divisors of F_5 must be of the form $64k+1$. Taking $k=1,2,\ldots$, the possible divisors are 193, 257, 449, 577, 641, $\cdots$. After a few trials, we get $2^{2^5}+1=4294967297=641\times 6700417$. With a bit of luck, very quickly we found a prime divisor of F_5. In fact, the second factor is also prime, but proving that is a bit more tedious.

However, if we try to use this method to find the divisors of the other Fermat numbers, we will run into problem very quickly. For example, $F_6=2^{2^6}+1$ is a twenty-digit number. Its square root is a ten-digit number ($\approx 4.29\times 10^9$). There are over three million numbers of the form $k\cdot 2^7+1=128k+1$. To find a divisor of F_6 among them is not simple. The readers can think about this. In 1732 Euler found the complete factorization of F_5. It took one hundred years for Landry and Le Lasseur (1880) to find the complete factorization of F_6. Another one hundred years passed before Morrison and Brillhart (1970) discovered the complete factorization of F_7. So to find the prime factorizations of Fermat numbers must not be easy. In another direction, as finding Fermat numbers is not easy, Pepin in 1877 obtained a criterion for a Fermat number to be prime, namely for a Fermat number $N>3$ of the form $2^{2^n}+1$, a necessary and sufficient condition for N to be prime is $3^{\frac{N-1}{2}}\equiv -1(\text{mod } N)$. Considering $\dfrac{N-1}{2}=2^{2^n-1}$, we should start with 3 and keep on squaring, then take (mod N). In recent years, this is the starting point for determining if Fermat numbers are primes or not. Also, for a long time since F_7 was shown to be not prime, nothing was known about any of its divisors.

Briefly we mention some recent results. It is now known that F_5 to F_{11} are composite numbers and their complete factorizations are found. F_{12}, F_{13}, F_{15} and F_{19} are known to be composite with only some divisors found. F_{14}, F_{20} and F_{22} are known to be composite with no divisors found.The largest composite Fermat number with one divisor known is F_{382447}. The reader can imagine if this number is written in base ten how many digits it will have. Next, for F_{33}, F_{34}, F_{35}, nothing is known whether they are composite or prime numbers. For those who are interested, please consult the webpage http:// www.fermatsearch.org/status.htm.

As Fermat numbers and related numbers are of special forms and have many interesting properties, they often appeared in many competitions. Here are some examples.

Example 1. The Fermat numbers $F_0, F_1, \ldots, F_n$ are given. We have the following relation $F_0F_1 \cdots F_{n-1} + 2 = F_n$.

Proof. In fact, $F_n = 2^{2^n} + 1 = 2^{2^n} - 1 + 2 = 2^{2^{n-1}2} - 1 + 2 = (2^{2^{n-1}} + 1)(2^{2^{n-1}} - 1) + 2 = (2^{2^{n-1}} - 1)F_{n-1} + 2$. For $2^{2^{n-1}} - 1$, we can factor further to get the required result. Of course, a rigorous proof can be given bu using mathematical induction.

Example 2. Fermat numbers F_m, F_n, $m > n$, are given. Then F_m, F_n are relatively prime.

Proof. Since $F_m = F_{m-1} \cdots F_n \cdots F_0 + 2$, let d divide F_m and F_n. Then d also divides 2. So $d = 1$ or 2. However, $d \neq 2$ as F_m, F_n are odd. So $d = 1$, i.e. F_m, F_n are relatively prime. (Hence $F_0, F_1, F_2, \ldots$ are pairwise relatively prime. That is, they include infinitely many prime divisors. Consequently, there are infinitely many prime numbers.)

Example 3. There are infinitely many n such that $F_n + 2$ is not prime.

Proof. Experimenting a few times it can be seen that $F_1 + 2 = 7, F_3 + 2 = 259$ are multiple of 7. In fact, for $n = 0, 1, 2, \ldots$, $2^{2^n} \equiv 2, 4, 2, 4, \ldots (\text{mod } 7)$. Since for odd n, $F_n + 2 \equiv 2^{2^n} + 1 + 2 \equiv 4 + 1 + 2 \equiv 0 (\text{mod } 7)$, so it is not prime.

Here is another fact.

Example 4. For $n > 1$, the units digit of F_n is 7.

Proof. For $n > 1$, 2^n is a multiple of 4. Let $2^n = 4k$, then $F_n = 2^{2^n} + 1 = 2^{4k} + 1 = (2^4)^k + 1 \equiv 1^k + 1 \equiv 2 (\text{mod } 5)$. So the units digit of F_n is 2 or 7. It cannot be 2 as F_n is not even.

Example 5. Prove that there exists a positive integer k such that for every positive integer n, $k \cdot 2^n + 1$ is not prime.

(If n is fixed and k is allowed to vary among positive integers, then from an important theorem (Dirichlet), it is known that the series will contain infinitely many primes. However, if k is fixed and n varies, then in general, it is not clear how many primes, infinite or not, there are in the series. In fact, one can find a k such that for every positive integer n, $k \cdot 2^n + 1$ is not prime. Originally this was a result of the Polish mathematician Sierpinski (1882-1969). Later it became a problem on the USA Math Olympiad (1982). Up to now, there is only one method of proving it and it is related too Fermat numbers.)

Proof. (The starting point of the proof is the Chinese remainder theorem. Let $m_1, m_2, \ldots, m_r$ be pairwise relatively prime positive integers and $a_1, a_2, \ldots, a_r$ are arbitrary integers. Then the