Junior problems

J277. Is there an integer n such that $4^{5^n} + 5^{4^n}$ is a prime?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA
The answer is – No because $4^{5^n} + 5^{4^n}$ can be represented in the form:

$$a^4 + 4b^4 = \left(a^2 - 2ab + 2b^2\right)\left(a^2 + 2ab + 2b^2\right).$$

Indeed, since $5^n - 1 = (5-1)\left(5^{n-1} + 5^{n-2} + ... + 5 + 1\right) = 4\left(5^{n-1} + 5^{n-2} + ... + 5 + 1\right)$
then $5^{4^n} + 4^{5^n} = \left(5^{4^{n-1}}\right)^4 + 4 \cdot \left(4^{5^{n-1}+5^{n-2}+...+5+1}\right)^4 = a^4 + 4b^4$, where

$$a := 5^{4^{n-1}} \text{and } b := 4^{5^{n-1}+5^{n-2}+...+5+1}$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo, Milan, Italy; Arbër Avdullahu,Kosovo; Corneliu Mănescu-Avram, "Henri Mathias Berthelot" Secondary School, Ploieşti, Romania; Mathematical Group "Galaktika shqiptare", Albania; Daniele Mastrostefano, Università di Roma "Tor Vergata", Roma, Italy; Polyahedra, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ritaman Ghosh, West Bengal, India; David Xu, Charter School of Wilmington, USA.

J278. Find all positive integers n for which

$$\{\sqrt[3]{n}\} \le \frac{1}{n},$$

where $\{x\}$ denotes the fractional part of x.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Michelle Andersen, College at Brockport, SUNY

The inequality obviously has every cube as solution. We will look for solutions of the type

$$n = k^3 + p$$

where $1 \le p \le 3k^2 + 3k$.

In this case

$$\{\sqrt[3]{n}\} = \sqrt[3]{n} - \lfloor \sqrt[3]{n} \rfloor = \sqrt[3]{k^3+p} - k$$

and thus the inequality is

$$\sqrt[3]{k^3+p} - k < \frac{1}{k^3+p} \iff \sqrt[3]{k^3+p} < k + \frac{1}{k^3+p}$$

$$\iff k^3 + p < k^3 + \frac{3k^2}{k^3+p} + \frac{3k}{(k^3+p)^2} + \frac{1}{(k^3+p)^3} \iff p < \frac{3k^2}{k^3+p} + \frac{3k}{(k^3+p)^2} + \frac{1}{(k^3+p)^3}$$

$$\frac{3k^2}{k^3+p} < \frac{3k^2}{k^3} = \frac{3}{k} \le 3.$$

$$\frac{3k}{(k^3+p)^2} = \frac{3k}{k^6 + 2pk^3 + p^2} < \frac{3k}{2k^3 p} \le \frac{3}{2}.$$

$$\frac{1}{(k^3+p)^3} = \frac{1}{k^9 + 3pk^6 + 3p^2k^3 + 1} < \frac{1}{3pk^6} \le \frac{1}{3}.$$

This implies that

$$p < 3 + \frac{3}{2} + \frac{1}{3} < 5$$

and so $p \in \{1, 2, 3, 4\}$.

Now we go back to

$$p < \frac{3k^2}{k^3+p} + \frac{3k}{(k^3+p)^2} + \frac{1}{(k^3+p)^3} \iff p(k^3+p)^3 < 3k^2(k^3+p)^2 + 3k(k^3+p) + 1$$

We will show that if $k \ge 4$ then

$$3k^2(k^3+p)^2 + 3k(k^3+p) + 1 \le p(k^3+p)^3$$

$$p(k^3+p)^3 \ge (k^3+1)^3 = k^9 + 3k^6 + 3k^3 + 1$$

$$3k^2(k^3+p)^2 + 3k(k^3+p) + 1 \le 3k^2(k^3+4)^2 + 3k(k^3+4) + 1 = 3k^8 + 24k^5 + 3k^4 + 48k^2 + 12k + 1$$

Then it suffices to show that

$$k^9 + 3k^6 + 3k^3 + 1 \ge 3k^8 + 24k^5 + 3k^4 + 48k^2 + 12k + 1$$

for $k \ge 4$.

This is equivalent to

$$k^8 + 3k^5 + 3k^2 \ge 3k^7 + 24k^4 + 3k^3 + 48k + 12$$

Since

$$k^8 + 3k^5 + 3k^2 \geq k^8 + 3k^4 + 3k$$

it suffices to show that

$$k^8 + 3k^4 + 3k \geq 3k^7 + 24k^4 + 3k^3 + 48k + 12 \iff k^8 \geq 3k^7 + 21k^4 + 3k^3 + 45k + 12$$

$$\begin{aligned} 3k^7 + 21k^4 + 3k^3 + 45k + 12 &\leq 3k^7 + 21k^4 + 3k^3 + 45k + 3k = 3k^7 + 21k^4 + 3k^3 + 48k \\ &\leq 3k^7 + 21k^4 + 3k^3 + k^4 = 3k^7 + 22k^4 + 3k^3 \leq 3k^7 + 22k^4 + k^4 = 3k^7 + 23k^4 \\ &\leq 3k^7 + k^7 = 4k^7 \leq k^8 \end{aligned}$$

Therefore $k \in \{1, 2, 3, 4\}$ and thus we have to check if for any of the 16 cases corresponding to the possible ranges for p and k the inequality

$$p(k^3 + p)^3 < 3k^2(k^3 + p)^2 + 3k(k^3 + p) + 1$$

is true.

If $k = 1$ we get

$$p(1 + p)^3 < 3(1 + p)^2 + 3(1 + p) + 1 \iff p^4 + 3p^3 + 3p^2 + p < 3p^2 + 9p + 7 \iff p^4 + 3p^3 < 8p + 7$$

with only solution $p = 1$ and thus $n = k^3 + p = 2$.

If $k = 2$ we get

$$p(8 + p)^3 < 12(8 + p)^2 + 6(8 + p) + 1 \iff p^4 + 24p^3 + 192p^2 + 512p < 12p^2 + 22p + 817$$

$$\iff p^4 + 24p^3 + 180p + 490p < 817$$

with only solution $p = 1$ and hence $n = k^3 + p = 9$.

If $k = 3$ we get

$$p(27 + p)^3 < 27(27 + p)^2 + 9(27 + p) + 1 \iff p^4 + 81p^3 + 2187p^2 + 19683p < 27p^2 + 1467p + 19927$$

$$\iff p^4 + 81p^3 + 2160p^2 + 18216p < 19927$$

which does not have solutions.

If $k = 4$ we het

$$p(64 + p)^3 < 48(64 + p)^2 + 12(64 + p) + 1$$

which does not have solutions because

$$48(64 + p)^2 + 12(64 + p) + 1 < 48(64 + p)^2 + 12(64 + p) + 64 + p = 48(64 + p)^2 + 13(64 + p)$$

$$< 48(64 + p)^2 + (64 + p)^2 = 49(64 + p)^2 < (64 + p)^3 \leq p(64 + p)^3$$

Therefore the only solutions of the inequality are $n = k^3$, $n = 2$ and $n = 9$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Polyahedra, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Albert Stadler, Switzerland; Alessandro Ventullo, Milan, Italy; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Mathematical Group "Galaktika shqiptare", Albania; Julien Portier, Vitry-le-François,France; Mengyue Sun Lansheng Fudan Middle School Shanghai China; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Daniele Mastrostefano, and Daniele Di Tullio Università di Roma "Tor Vergata", Roma, Italy; Stanescu Florin, Serban Cioculescu school, Gaesti, Romania.

J279. Find all triples (p, q, r) of primes such that $pqr = p + q + r + 2000$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Assume without loss of generality that $p \le q \le r$. The given equality can be rewritten as

$$(rq - 1)(p - 1) + (r - 1)(q - 1) = 2002. \tag{1}$$

If p is an odd prime, then q and r are odd prime also, but this means that the LHS is divisible by 4 and the RHS is not divisible by 4, a contradiction. Thus, $p = 2$ and equation (1) becomes

$$(2q - 1)(2r - 1) = 4005 = 3^2 \cdot 5 \cdot 89.$$

Since $2q - 1 \le 2r - 1$, then $(2q - 1)^2 \le 4005$, i.e. $2q - 1 \le 63$. This means that $2q - 1 \in \{1, 3, 5, 9, 15, 45\}$. Clearly, $2q - 1 \ne 1, 15$, therefore we have the four systems of equations

$$\begin{aligned} 2q - 1 &= 3 \\ 2r - 1 &= 1335, \end{aligned} \qquad \begin{aligned} 2q - 1 &= 5 \\ 2r - 1 &= 801, \end{aligned} \qquad \begin{aligned} 2q - 1 &= 9 \\ 2r - 1 &= 445, \end{aligned} \qquad \begin{aligned} 2q - 1 &= 45 \\ 2r - 1 &= 89. \end{aligned}$$

It's easy to see that the first and the last system have no solution in primes, and the other two systems give $q = 3, r = 401$ and $q = 5, r = 223$. Therefore, $(2, 3, 401)$ and $(2, 5, 223)$ are two solutions to the given problem and by symmetry all the solutions are

$$(2, 3, 401), (2, 401, 3), (3, 2, 401), (3, 401, 2), (401, 2, 3), (401, 3, 2),$$

$$(2, 5, 223), (2, 223, 5), (5, 2, 223), (5, 223, 2), (223, 2, 5), (223, 5, 2).$$

Also solved by Faraz Masroor, Miami, FL, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Polyahedra, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Albert Stadler, Switzerland; Anthony Morse and Aouiza Ochi, students, College at Brockport, SUNY; Arbër Avdullahu,Kosovo; Arpan Sadhukhan, Indian Statistical Institute, Kolkata, India; Julien Portier, Vitry-le-François,France; Neculai Stanciu, Buzău and Titu Zvonaru, Comănești, Romania; Daniele Mastrostefano, Università di Roma "Tor Vergata", Roma, Italy; Stanescu Florin, Serban Cioculescu school, Gaești, Romania.

J280. Let a, b, c, d be positive real numbers. Prove that

$$2(ab+cd)(ac+bd)(ad+bc) \geq (abc+bcd+cda+dab)^2.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain
Performing the products and reorganizing terms yields the equivalent inequality

$$2abcd(a^2+b^2+c^2+d^2-ab-bc-cd-da)+a^2c^2(b-d)^2+b^2d^2(a-c)^2 \geq 0.$$

The second and third terms in the LHS are non-negative, being zero iff $b=d$ and $a=c$ respectively, whereas the first term is clearly non-negative by the scalar product inequality applied to vectors (a,b,c,d) and (b,c,d,a), with equality iff $a=b=c=d$, which is clearly also sufficient for the second and third term to be zero. The conclusion follows, equality holds iff $a=b=c=d$.

Also solved by Albert Stadler, Switzerland; Kastriot Jashari, Sami Frasheri High School ,Kumanovo, Macedonia; Polyahedra, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ritaman Ghosh, West Bengal, India; David Xu, Charter School of Wilmington, USA; Alessandro Ventullo, Milan, Italy; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Arkady Alt , San Jose ,California, USA; Mathematical Group "Galaktika shqiptare", Albania; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Julien Portier, Vitry-le-François,France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Stanescu Florin, Serban Cioculescu school, Gaesti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

J281. Solve the equation

$$x+\sqrt{(x+1)(x+2)}+\sqrt{(x+2)(x+3)}+\sqrt{(x+3)(x+1)}=4.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Let $f(x)$ be the expression on the left side of the equation. Then the domain of $f(x)$ is $(-\infty,-3]\cup[-1,\infty)$. Consider $x\geq -1$ first. Let $u=\sqrt{x+1}+\sqrt{x+2}$, $v=\sqrt{x+2}+\sqrt{x+3}$, and $w=\sqrt{x+3}+\sqrt{x+1}$. Then $uv=f(x)+2=6$, $vw=f(x)+3=7$, and $wu=f(x)+1=5$. Hence, $u^2=\frac{30}{7}$, $v^2=\frac{42}{5}$, and $w^2=\frac{35}{6}$, from which we get $2\sqrt{(x+1)(x+2)}=\frac{30}{7}-2x-3$, $2\sqrt{(x+2)(x+3)}=\frac{42}{5}-2x-5$, and $2\sqrt{(x+3)(x+1)}=\frac{35}{6}-2x-4$. Thus, the original equation is equivalent to

$$2x+\frac{30}{7}+\frac{42}{5}+\frac{35}{6}-6x-12=8,$$

which yields $x=-\frac{311}{840}\in[-1,\infty)$.

Now consider $x\leq -3$. Let $u=\sqrt{-x-1}+\sqrt{-x-2}$, $v=\sqrt{-x-2}+\sqrt{-x-3}$, and $w=\sqrt{-x-3}+\sqrt{-x-1}$. Then $uv=f(x)-2x-2=2-2x$, $vw=f(x)-2x-3=1-2x$, and $wu=f(x)-2x-1=3-2x$. Hence,

$$u^2=\frac{(2-2x)(3-2x)}{1-2x},\quad v^2=\frac{(1-2x)(2-2x)}{3-2x},\quad w^2=\frac{(3-2x)(1-2x)}{2-2x},$$

from which we get

$$2\sqrt{(x+1)(x+2)}=\frac{2}{1-2x}+7,\quad 2\sqrt{(x+2)(x+3)}=\frac{2}{3-2x}+5,$$

$$2\sqrt{(x+3)(x+1)}=-\frac{1}{2-2x}+6.$$

Thus, the original equation is equivalent to

$$2x+\frac{2}{1-2x}+\frac{2}{3-2x}-\frac{1}{2-2x}+18=8,$$

which becomes, upon clearing denominators, $16x^4+32x^3-208x^2+232x-73=0$. By the rational-root theorem, the solutions of this equation are not rational. By the quartic formula, we get $x_1=0.9567\cdots$, $x_2=1.5763\cdots$, $x_3=0.5904\cdots$, and $x_4=-5.1235\cdots$, of which only x_4 is in the interval of $(-\infty,-3]$.

Also solved by Kastriot Jashari, Sami Frasheri High School ,Kumanovo, Macedonia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee , Lycée Henri IV , Paris France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Jan Jurka; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Arkady Alt , San Jose ,California, USA; Alessandro Ventullo, Milan, Italy; Albert Stadler, Switzerland; Ritaman Ghosh, West Bengal, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Anthony Morse, College at Brockport, SUNY.

J282. Given an $m \times n$ board, k cells are painted such that if the centers of four cells are the vertices of a quadrilateral with parallel sides to the borders of the board then at most two must be painted. Find the greatest value of k.

Proposed by Roberto Bosch Cabrera, Texas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The proposed condition has the following immediate consequence: if two painted cells are in the same row or in the same column, then there are no more painted cells in the same column or in the same row, respectively. If this condition holds, ie if for every pair of painted cells which share a row or a column, no more cells are painted in their columns or rows, respectively, then clearly the condition given in the problem statement also holds, hence they are equivalent. Assume wlog (we may exchange rows and columns by rotating the board without altering the problem) that $m \geq n$. Then $k \leq m$ when $n \in \{1, 2\}$, and $k \leq m + n - 2$ when $n \geq 2$. We prove this result by considering increasing values of n.

If $n = 1$, the board contains exactly m cells and all m cells can clearly be painted.

If $n = 2$, assume that $p > m+n-2 = m$ cells can be painted. Therefore, a row (the i-th one) exists such that both its cells are painted. No more cells can be painted in either column, or $k \leq 2 \leq m$, contradiction. However, $m = m + n - 2$ cells can be painted by painting exactly a full column.

If $n \geq 3$, assume that $p \geq m + n - 2$ cells can be painted. Choose the row that has most cells painted, and assume that it has u cells painted. Then $u \geq 2$ because there are more than m cells painted but only m rows. Clearly, no other cells are painted in the u columns for which a painted cell exists in the selected row. Consider the board resulting from erasing the row and the u columns that contain these u painted cells, which clearly satisfies the stated condition. It has $p - u$ painted cells, $m - 1$ rows, and $n - u$ columns. Clearly, $p - u \geq m + n - 2 - u = (m - 1) + (n - u) - 1$. If we perform v such steps, by trivial induction we obtain a $m - v \times n - U$ board with at least $(m - v) + (n - U) - 2 + v$ painted cells, where U is the sum of the u's in each one of these v steps. While $n - U > 2$, we can continue performing this process, until either $n - U = 2$ or $n - U = 1$. In the first case, we would have a $m - v \times 2$ board with at least $(m - v) + v > m - v$ painted cells, contradiction. In the second case, we would have a $m - v \times 1$ board with at least $m - v + (v - 1) \geq m - v$ painted cells, possible iff $v = 1$, with $U = u = n - 1$. In other words, when $n \geq 3$, at most $m + n - 2$ cells can be painted, iff we paint all cells in one row, and all cells in one column, except for the cell at their intersection which is left unpainted.

Restoring generality, we conclude that $k \leq m + n - 2$ when $\min\{m, n\} \geq 2$ by painting a full row and a full column, except for the cell that is at their intersection, and $k \leq \max\{m, n\}$ when $\min\{m, n\} \leq 2$ by painting all cells in a row (when $n \geq m$) or in a column (when $m \geq n$).

Also solved by Polyahedra, Polk State College, FL, USA.

Senior problems

S277. Let a, b, c be positive real numbers such that

$$\frac{1}{a^3+b^3}+\frac{1}{b^3+c^3}+\frac{1}{c^3+a^3}\leq\frac{3}{a+b+c}.$$

Prove that

$$2(a^2+b^2+c^2)+(a-b)^2+(b-c)^2+(c-a)^2\geq 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

Without loss of generality, assume that $a \leq b \leq c$. Then the given condition yields that

$$\begin{aligned}
1 &\geq \frac{1}{6}\left[(a+b)+(b+c)+(c+a)\right]\left(\frac{1}{a^3+b^3}+\frac{1}{b^3+c^3}+\frac{1}{c^3+a^3}\right)\\
&\geq \frac{1}{2}\left(\frac{a+b}{a^3+b^3}+\frac{b+c}{b^3+c^3}+\frac{c+a}{c^3+a^3}\right)\\
&= \frac{1}{2}\left(\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\right)\\
&\geq \frac{9}{4\left(a^2+b^2+c^2\right)-2(ab+bc+ca)},
\end{aligned}$$

where the second and third inequalities are by virtue of the Chebyshev and the AM-HM inequalities, respectively. Therefore,

$$2\left(a^2+b^2+c^2\right)+(a-b)^2+(b-c)^2+(c-a)^2=4\left(a^2+b^2+c^2\right)-2(ab+bc+ca)\geq 9.$$

Also solved by Kastriot Jashari, Sami Frasheri High School ,Kumanovo, Macedonia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Mathematical Group "Galaktika shqiptare", Albania; Arkady Alt , San Jose ,California, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

S278. Let a,b,c be complex numbers such that $|a|=|b|=|c|=1$. If there is a positive integer n such that $|a+b|^{2^n}+|b+c|^{2^n}+|c+a|^{2^n}\leq 3$, prove that a,b,c are the affixes of the vertices of an equilateral triangle.

Proposed by Marcel Chirita, Bucharest, Romania

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Each of the terms $|a+b|$, $|b+c|$, $|c+a|$ may be written in function of the respective angles betwen the complex number a and b, b and c, and c and a. For example, if $\angle(a,b)=\alpha$, then $|a+b|=\sqrt{(1+\cos\alpha)^2+\sin^2\alpha}=\sqrt{2+2\cos\alpha}=2\left|\cos\dfrac{\alpha}{2}\right|$. Anologously, $|b+c|=2\left|\cos\frac{\beta}{2}\right|$, and $|c+a|=2\left|\cos\frac{\gamma}{2}\right|$.

Applying the Chebyshev'sum inequality, if there is a positive integer $n>$ such that $\dfrac{|a+b|^{2^n}+|b+c|^{2^n}+|c+a|^{2^n}}{3}\leq 1$, then

$$\begin{aligned}\frac{|a+b|^{2^n}+|b+c|^{2^n}+|c+a|^{2^n}}{3} &\geq \left(\frac{|a+b|^{2^{n-1}}+|b+c|^{2^{n-1}}+|c+a|^{2^{n-1}}}{3}\right)^2\\ &\geq \left(\frac{|a+b|^{2^{n-2}}+|b+c|^{2^{n-2}}+|c+a|^{2^{n-2}}}{3}\right)^{2^2}\\ &\geq \left(\frac{|a+b|^{2}+|b+c|^{2}+|c+a|^{2}}{3}\right)^{2^{n-1}}\end{aligned}$$

and, therefore

$$\left|\cos\frac{\alpha}{2}\right|+\left|\cos\frac{\beta}{2}\right|+\left|\cos\frac{\gamma}{2}\right|\leq\frac{3}{4}$$

but, since in any triangle with angles $\dfrac{\alpha}{2}$, $\dfrac{\beta}{2}$, $\dfrac{\gamma}{2}$:

$$\left|\cos\frac{\alpha}{2}\right|+\left|\cos\frac{\beta}{2}\right|+\left|\cos\frac{\gamma}{2}\right|\geq\frac{3}{4},$$

we have the equality, which holds if and only if the triangle is equilateral.

Also solved by Moubinool Omarjee Lycée Henri IV , Paris France; Li Zhou, Polk State College, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

S279. Solve in integers the equation

$$(2x+y)(2y+x) = 9\min(x,y).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniele Di Tullio and Daniele Mastrostefano, Università di Roma "Tor Vergata", Roma, Italy
By symmetry, we suppose $y \geq x$. So we want to find the integer solutions of

$$(2x+y)(2y+x) = 9x,$$

that is

$$2x^2 + x(5y-9) + 2y^2 = 0$$

which yields

$$x = \frac{9-5y \pm 3\sqrt{(y-1)(y-9)}}{4}.$$

We note that $\Delta = (y-1)(y-9)$ have to be a perfect square.
Now $\Delta = 0$ iff $y = 1$ or $y = 9$, and the solutions are $(1,1)$ and $(-9,9)$.
Assume that $\Delta > 0$. So $g = \gcd(y-1, y-9) = \gcd(y-1, 8) \in \{1,2,4,8\}$ and $\gcd((y-1)/g, (y-9)/g) = 1$. Hence $g^2((y-1)/g)((y-9)/g)$ is a perfect square iff $(y-1)/g$ and $(y-9)/g$ are both perfect squares or both the opposite of perfect squares. Since there is a unique positive integer solution of $m^2 - n^2 = 8$, that is $(3,1)$, it follows that for $g = 1$ there are only two cases:

1) $y-1 = 9$, $y-9 = 1$, so $y = 10$ which implies $x = -8$,

2) $y-1 = -1$, $y-9 = -9$, so $y = 0$ which implies $x = 0$.

For $g = 2, 4, 8$ there are no solutions because there do not exist non-zero perfect squares whose difference is 4, 2 or 1. $\square$

Solution proposed by G.R.A.20 Problem Solving Group, Roma, Italy.

We show that the solutions are $(0,0)$, $(1,1)$, $(-9,9)$, $(9,-9)$, $(-8,10)$, $(10,-8)$.

By symmetry, we assume that $y \geq x$.
Let $x = u-4$ and $y = v+5$ then the diophantine equation becomes

$$(2u+v)(2v+u) = -18.$$

Hence, we have to solve the system

$$\begin{cases} 2u+v = a, \\ u+2v = b \end{cases}$$

with $(a,b) \in \{(\pm 18, \mp 1), (\pm 1, \mp 18), (\pm 9, \mp 2), (\pm 2, \mp 9)(\pm 6, \mp 3), (\pm 3, \mp 6)\}$.
It is easy to see that $(u,v) = ((2a-b)/3, (2b-a)/3) \in \mathbb{Z}$ iff

$$\begin{aligned}
(a,b) = (6,-3) &\to (u,v) = (5,-4) \to (x,y) = (1,1), \\
(a,b) = (-6,3) &\to (u,v) = (-5,4) \to (x,y) = (-9,9), \\
(a,b) = (-3,6) &\to (u,v) = (-4,5) \to (x,y) = (-8,10), \\
(a,b) = (3,-6) &\to (u,v) = (4,-5) \to (x,y) = (0,0).
\end{aligned}$$

Also solved by Moubinool Omarjee Lycée Henri IV , Paris France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Stephanie Lash and Jessica Schuler, College at Brockport, SUNY; Kastriot Jashari, Sami Frasheri High School ,Kumanovo, Macedonia; Julien Portier, Vitry-le-François,France; Arkady Alt , San Jose ,California, USA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Alessandro Ventullo, Milan, Italy; Ritaman Ghosh, West Bengal, India; Li Zhou, Polk State College, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

S280. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$x^4y^4z^4(x^3+y^3+z^3) \leq 3.$$

Proposed by Sayan Das, ISI Kolkata, India

Solution by Arkady Alt , San Jose ,California, USA
Since homogeneous form of original inequality is
(1) $\quad 3^{14}x^4y^4z^4(x^3+y^3+z^3) \leq (x+y+z)^{15}$
then assuming $x+y+z=1$ and denoting

$$t := 3\left(xy+yz+zx\right), q := xyz$$

we obtain (1) $\iff 3^{14}q^4\left(1-t+3q\right) \leq 1 \iff 1-3^{14}q^4\left(1-t\right)-3^{15}q^5 \geq 0.$
Since

$$3xyz\left(x+y+z\right) \leq \left(xy+yz+zx\right)^2 \iff q \leq \frac{t^2}{3^3} \Rightarrow$$

$$\Rightarrow 1-3^{14}q^4\left(1-3p\right)-3^{15}q^5 \geq 1-3^{14}\left(\frac{t^2}{3^3}\right)^4\left(1-t\right)-3^{15}\left(\frac{p^2}{3^3}\right)^5 =$$

$$1-9t^8\left(1-t\right)-t^{10} = \left(1-t\right)\left(t^9+\left(1+t+t^2+t^3+t^4+t^5+t^6+t^7-8t^8\right)\right) =$$

$$t^9\left(1-t\right)+\left(1-t\right)^2\left(1+2t+3t^2+4t^3+5t^4+6t^5+7t^6+8t^7\right) \geq 0$$

because

$$t = 3\left(xy+yz+zx\right) \leq \left(x+y+z\right)^2 = 1.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Daniele Mastrostefano, and Daniele Di Tullio Università di Roma "Tor Vergata", Roma, Italy; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Kastriot Jashari, Sami Frasheri High School ,Kumanovo, Macedonia; Mathematical Group "Galaktika shqiptare", Albania.

S281. Let n be an integer greater than 1. For $a \in \mathbb{C}\backslash\mathbb{R}$ with $|a| = 1$, consider the equation

$$\sum_{k=0}^{n} \binom{n}{k} (a^k + 1)x^k = 0.$$

Prove that
(a) All roots of the equation lie on a line d_a.
(b) Lines d_a and d_b are perpendicular if and only if $a + b = 0$.

Proposed by Dorin Andrica, Babes Bolyai University, Cluj-Napoca, Romania

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain
By the binomial theorem the given equation is written as

$$(1 + ax)^n + (1 + x)^n = 0$$

from where, the roots may be obtained easily as $x = \dfrac{e_k - 1}{-e_k + a}$, where $e_k = e^{\left(\frac{\pi}{n} + \frac{2k\pi}{n}\right)i}$ for $0 \le k \le n - 1$ are the n roots of -1. Function $f_a(z) = \dfrac{z - 1}{-z + a}$ is a Möbius transformation since $a - 1 \neq 0$. Since $f_a(1) = 0$, $f_a(a) = \infty$, and $f_a(-1) = \dfrac{-2}{1 + a}$, the image by f of the unit circle is a line, and therefore all the roots of the given equation lie on a line d_a. This proves part (a).

For part (b) since both lines d_a and d_b pass through the origen, it is enough to prove that if numbers $f_a(-1) = \dfrac{-2}{1 + a}$ and $f_b(-1) = \dfrac{-2}{1 + b}$ are perpendicular then $a + b = 0$.

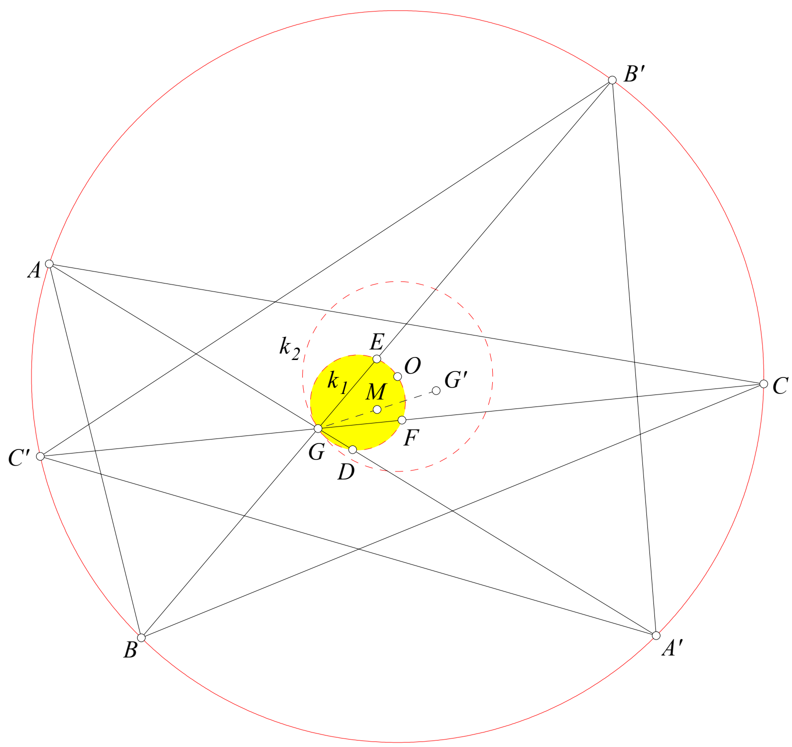
Note that $\dfrac{-2}{1 + a}$ and $\dfrac{-2}{1 + b}$ are perpendicular if and only if its quotient is pure imaginary: $\dfrac{1 + a}{1 + b} \in \Im$. Now, for fixed complex $a \in \mathbb{C}\backslash\mathbb{R}$ with $|a| = 1$, consider the Möbius transformation $g(z) = \dfrac{1 + a}{1 + z}$. Since $g(-1) = \infty$ and $g(1) = \dfrac{1 + a}{2}$, g transforms the unit circle in a line which intersect the imaginary axes in a unique point. Since $g(-a) \in \Im$, this proves part (b).

Also solved by Moubinool Omarjee Lycée Henri IV , Paris; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo, Milan, Italy; Li Zhou, Polk State College, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

S282. Let ABC be a triangle, G its centroid, and O its circumcenter. Lines AG, BG, CG intersect the circumcircle of triangle ABC at A', B', C'. Denote by G' the centroid of triangle $A'B'C'$. Prove that $OG \geq OG'$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Li Zhou, Polk State College, USA



Let D, E, F be the midpoints of AA', BB', CC', respectively. Then $OD \perp GD$, $OE \perp GE$, and $OF \perp GF$. Hence, D, E, F are on the circle k_1 of diameter OG. Let M be the midpoint of GG', then M is the centroid of the hexagon $AC'BA'CB'$, thus the centroid of $\triangle DEF$ as well. Thus, M is in the interior of k_1, which implies that G' is in the interior of the circle k_2 centered at O and of radius OG. Therefore, $OG \geq OG'$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Titu Zvonaru, Comănești and Neculai Stanciu, Buzău, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

U277. For $n \in \mathbb{N}$, $n \geq 2$, find the greatest integer less than $2\left(e^{\frac{1}{n+1}} + \cdots + e^{\frac{1}{n+n}}\right)$.

Proposed by Marius Cavachi, Constanta, Romania

Solution by Konstantinos Tsouvalas, University of Athens, Athens, Greece

We have that:

$$2\sum_{k=n+1}^{2n} e^{1/k} \leq 2\sum_{k=n+1}^{2n}\int_{k-1}^{k} e^{1/x}dx = 2\int_{n}^{2n} e^{1/x}dx$$

Moreover,

$$2\int_{n}^{2n} e^{1/x}dx = 2n + 2\ln 2 + \sum_{k=2}^{\infty}\frac{2}{(k-1)k!}\left(\frac{1}{n^{k-1}} - \frac{1}{(2n)^{k-1}}\right)$$

$$\sum_{k=2}^{\infty}\frac{2}{(k-1)k!}\left(\frac{1}{n^{k-1}} - \frac{1}{(2n)^{k-1}}\right) \leqslant \sum_{k=2}^{\infty}\frac{4}{k!2^k} = 4\left(e^{1/2} - 3/2\right)$$

and furthermore

$$2\ln 2 + 4\left(e^{1/2} - 3/2\right) < 2,$$

which gives

$2n+1 \leqslant 2\int_n^{2n} e^{1/x}dx < 2n+2 \Rightarrow \left[2\int_n^{2n} e^{1/x}dx\right] = 2n+1$ (1)

We also observe using the inequality $e^x \geq x+1, x>0$:

$$2\sum_{k=n+1}^{2n} e^{1/k} \geq 2n + 2\sum_{k=n+1}^{2n}\frac{1}{k} \geq_{C.S} 2n + \frac{2n^2}{n^2 + \frac{n^2-n}{2}} \geqslant 2n + 4/3$$

which implies:

$$\left[2\sum_{k=n+1}^{2n} e^{1/k}\right] \geqslant 2n+1,$$

and using (1)

we conclude that:

$$\left[2\sum_{k=n+1}^{2n} e^{1/k}\right] = 2n+1$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee Lycée Henri IV , Paris France; Albert Stadler, Switzerland; Alessandro Ventullo, Milan, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt , San Jose ,California, USA; Georgios Dasoulas, National Technical University of Athens, Greece; Nikolaos Zarifis, National Technical University of Athens, Greece; Daniele Mastrostefano, Università di Roma "Tor Vergata", Roma, Italy; Pierre Cote, Lycee Henri IV, Paris; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

U278. Evaluate

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{1}{(kn+1)k!}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note that

$$\frac{1}{kn+1}=\int_0^1 x^{kn}dx,$$

or by independence of variables x, k, n, we have

$$\sum_{k=0}^{\infty}\frac{1}{(kn+1)k!}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_0^1 x^{kn}dx\right)=\int_0^1\left(\sum_{k=0}^{\infty}\frac{(x^n)^k}{k!}\right)dx=\int_0^1 e^{x^n}dx.$$

When $n\to\infty$ and for all $x\in[0,1)$, we have $x^n\to 0$ and hence $e^{x^n}\to 1$, or for sufficiently large values of n, the integrand is arbitrarily close to 1 in all the integration interval, except at a set of points $\{1\}$ with zero measure, where the integral evaluates to the finite value e. The proposed limit is therefore 1.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Pierre Cote, Lycee Henri IV, Paris; Albert Stadler, Switzerland; Alessandro Ventullo, Milan, Italy; Georgios Dasoulas, National Technical University of Athens, Greece; Konstantinos Tsouvalas, University of Athens, Athens, Greece; Michelle Andersen, College at Brockport, SUNY; Moubinool Omarjee Lycée Henri IV , Paris France; Nikolaos Zarifis, National Technical University of Athens, Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Daniele Di Tullio, Università di Roma "Tor Vergata", Roma, Italy.

U279. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with lateral limits at every point. Prove that there is some real number x_0 such that

$$\lim_{x \to x_0, x > x_0} f(x) \le x_0 \le \lim_{x \to x_0, x < x_0} f(x).$$

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Tommaso Cornelis Rosati and Daniele Di Tullio, Università di Roma "Tor Vergata", Roma, Italy

Let $A = \left\{ x_0 \in \mathbb{R} : x_0 \le \lim_{x \to x_0^-} f(x) \right\}$ and let $s = \sup(A)$. Since f is bounded, it follows that A is not empty and $s \in \mathbb{R}$. Now we show that s is the number we are looking for.

i) We first prove that $s \le \lim_{x \to s^-} f(x)$, that is $s \in A$.

Suppose that $\lim_{x \to s^-} f(x) = L < s$. So for $\varepsilon = \frac{s-L}{2} > 0$, there is $0 < \delta < \frac{s-L}{2}$ such that for all $x \in (s - \delta, s)$, $|f(x) - L| < \varepsilon$, which implies $f(x) < \frac{s+L}{2}$. According to the definition of least upper bound, since we are assuming that $s \notin A$, it follows that there is $x_0 \in A \cap (s - \delta, s)$. Therefore

$$\lim_{x \to x_0^-} f(x) \le \frac{s+L}{2} < s - \delta < x_0,$$

which is a contradiction because $x_o \in A$.

ii) Now we shall prove that $\lim_{x \to s^+} f(x) \le s$.

Let us suppose it is not, that is $\lim_{x \to s^+} f(x) = L > s$. By the definition of limit, for $\varepsilon = \frac{L-s}{2} > 0$, there is $0 < \delta < \frac{L-s}{2}$, such that for all $x \in (s, s + \delta)$, $|f(x) - L| < \varepsilon$, which implies $\frac{L+s}{2} < f(x)$ and

$$\lim_{x \to s+\delta^-} f(x) \ge \frac{L+s}{2} > s + \delta.$$

This means that $s + \delta \in A$ which is in contradiction with the fact that $s = \sup(A)$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

U280. Let S be an uncountable set of circles in the plane. Prove that there is an uncountable subset S' of S such that all the circles in S' have a common interior point.

Proposed by Marius Cavachi, Constanta, Romania

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let $S = \{C((x_i, y_i), r_i) : i \in I\}$, where I is an uncountable index set and $C((x, y), r)$ is the circle of center (x, y) and radius $r > 0$. Since

$$\mathbb{R}^2 = \bigcup_{n,m\in\mathbb{Z}} [n, n+1] \times [m, m+1],$$

it follows that there are $n_0, m_0 \in \mathbb{Z}$ such that

$$[n_0, n_0+1] \times [m_0, m_0+1] \cap \{(x_i, y_i) : i \in I\}$$

is uncountable, because the countable union of countable sets is countable. Let $I_1 \subset I$ the corresponding index set. For the same reason,

$$(0, +\infty) = \bigcup_{j\in\mathbb{N}^+} \left(\frac{1}{j}, +\infty\right)$$

implies that there is $j_0 > 0$ such that

$$\left(\frac{1}{j_0}, +\infty\right) \cap \{r_i : i \in I_1\}$$

is uncountable. Let $I_2 \subset I_1$ the corresponding index set.

Now, let $Q_0 = [n_0, n_0+1] \times [m_0, m_0+1]$ and for $l \geq 1$ we define recursively Q_l as one of the closed squares obtained from Q_{l-1} by dividing it in four equal parts, such that

$$Q_l \cap \{(x_i, y_i) : i \in I_2\}$$

is uncountable.

By Bolzano-Weierstrass Theorem, the intersection of all these compact sets Q_l is not empty. Let $(x_0, y_0) \in \bigcap_{l\geq 0} Q_l$. Then there is a positive integer l_0 such that $\operatorname{diam}(Q_{l_0}) = \sqrt{2}/2^{l_0} < 1/j_0 < r_i$ for any $i \in I_2$, that is (x_0, y_0) is an interior point of an uncountable subset of circles of the set $\{C((x_i, y_i), r_i) : i \in I_2\}$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Michelle Andersen, College at Brockport, SUNY; Konstantinos Tsouvalas, University of Athens, Athens, Greece.

U281. Let G be a graph on n vertices so that for every connected subgraph H of G, the graph $G - H$ is connected. Prove that G is either a cycle or a complete graph

Proposed by Cosmin Pohoata, USA

Solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy

We show that if G is not complete than it is a cycle. If it is not complete then there are v_1, v_2 distinct vertices not connected. Then it follows by the hypothesis that $G/\{v_1, v_2\}$ is not connected. So let $H_1, \ldots, H_k$ its connected components, with $k > 1$. Since the trivial graph with one vertex is connected, it follows that by removing v_1 or v_2, that both v_1 and v_2 are connected to at least one vertex of each H_i. Now assume that $k > 2$. Then v_1, v_2, H_1 are connected and there remain at least $k - 1 > 2$, $H_2, \ldots, H_k$, connected components which is in contradiction with the hypothesis. Hence $k = 2$. Now since v_1, v_2, H_1 and v_1, v_2, H_2, are connected, let P_1 a simple path in H_1 connecting v_1 with v_2, and let P_2 a simple path in H_2 connecting v_1 and v_2. Now it's clear that $P_1/\{v_1, v_2\} = H_1$, $P_2/\{v_1, v_2\} = H_2$, otherwise by removing P_1 then the remaining graph is not connected (H_1 and H_2 are connected just by v_1 and v_2), the same argument for P_2.

So we have found that the vertices of G lie on a cycle (the two paths P_1 and P_2 attached at their ends). So we have to exclude that there are other edges between vertices of P_1, and other edges between vertices of P_2. If you have a simple path with some additional edges, than you can always find one vertex that once removed leaves the graph connected: indeed if $(a_1, \ldots, a_n)$ is a simple path and a_i and a_j are connected with $|i - j| > 1$, than any a_k with k in (i, j) does this job. So if there is another edge between two vertices of P_1, call w such a point, and consider $P_1/\{w\}$ which is connected. Now the complementary is w and H_2 which is not connected, and we have a contradiction. The same argument can be used for P_2. So there are no other edges, and G is exactly a cycle.

U282. Let $P = \{2, 3, 5, 7, 11, \dots\}$ denote the set of all primes less than 2^{100}. Prove that $\sum_{p\in P} \frac{1}{p} < 8$.

Proposed by Marius Cavachi, Constanta, Romania

First solution by the proposer

Let $P^* = P \cup \{1\}$. Notice that the products of the form $p_0 p_1 \cdots p_5$, where p_i run through P^* subject to $p_0 \le p_1 \le \cdots \le p_5$, are pairwise distinct integers of the range $1, 2, \dots 2^{600} - 1$.

Recall that $\sum_{k=1}^{2^n-1} \dfrac{1}{k} < n, n \ge 2 \Rightarrow$

$$\left(1 + \sum_{p\in P} \frac{1}{p}\right)^6 = \left(\sum_{p\in P^*} \frac{1}{p}\right)^6 < 6! \sum_{p_0\le p_1\le\cdots\le p_5} \frac{1}{p_0 p_1 \cdots p_5}$$

$$< 6! \sum_{k=1}^{2^{600}-1} \frac{1}{k} < 6! \cdot 600 = 2^7 \cdot 3^3 \cdot 5^3 < 3^{12}.$$

The conclusion follows.

Second solution by Albert Stadler, Switzerland

By theorem 5 of the seminal paper of *Rosser and Schoenfeld* entitled *Approximate formulas for some functions of prime numbers* we have:

$$\log\log x + B - \frac{1}{2\log^2 x} < \sum_{p\le x} \frac{1}{p} < \log\log x + B + \frac{1}{2\log^2 x},$$

$x \ge 286$, where $B = 0.261497212847643$

Setting $x = 2^{100}$ and get

$$4.50005 < \sum_{p\in P} \frac{1}{p} < 4.50026$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Konstantinos Tsouvalas, University of Athens, Athens, Greece; G.R.A.20 Problem Solving Group, Roma, Italy.

O277. Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 1$. Prove that

$$\frac{a+b}{\sqrt{ab+c}}+\frac{b+c}{\sqrt{bc+a}}+\frac{c+a}{\sqrt{ca+b}} \geq 3\sqrt[6]{abc}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the proposer

We have

$$\frac{a+b}{\sqrt{ab+c}}=\frac{(a+b)\cdot\sqrt{c}}{\sqrt{abc+c^2}} \geq \frac{(a+b)\cdot\sqrt{c}}{\sqrt{ab+bc+ca+c^2}}=$$

$$=\frac{(a+b)\cdot\sqrt{c}}{\sqrt{(b+c)\cdot(c+a)}}$$

and the analogous.

Now the conclusion follows from the AM-GM inequality.

Also solved by Semchankau Aliaksei Moscow Institute Physics and Technique, Russia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongoli; Arkady Alt , San Jose ,California, USA; Kastriot Jashari, Sami Frasheri High School ,Kumanovo, Macedonia; Sayan Das, Indian Statistical Institute, Kolkata, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Li Zhou, Polk State College, USA; Ajat Adriansyah, Surya Institute.

O278. Find all primes p, q, r such that $p|2qr + r$, $q|2rp + p$, and $r|2pq + q$.

Proposed by Roberto Bosch Cabrera, Texas, USA

Solution by Daniele Mastrostefano, Università di Roma "Tor Vergata", Roma, Italy

If one of the primes is 2, say p, then $p \mid (2q + 1)r$ implies that $p = r = 2$ and $r \mid (2p + 1)q$ implies that $r = q = 2$.

We assume now that the primes are all odd. Then $p \mid (2q + 1)r$ implies that

$$1)\ p = r \quad \text{or} \quad 2)\ p = 2q + 1 \quad \text{or} \quad 3)\ p \mid (2q + 1) \text{ and } p \leq (2q + 1)/3.$$

Similar options hold for q and r:

$$1)\ q = p \quad \text{or} \quad 2)\ q = 2r + 1 \quad \text{or} \quad 3)\ q \mid (2r + 1) \text{ and } q \leq (2r + 1)/3,$$

$$1)\ r = q \quad \text{or} \quad 2)\ r = 2p + 1 \quad \text{or} \quad 3)\ r \mid (2p + 1) \text{ and } r \leq (2p + 1)/3.$$

We divide the rest of the proof in several cases.

i) If at least two conditions of type 1) hold then $p = q = r$.

ii) If we have one condition of type 1), say $p = r$, then $r \mid (2p + 1)$ is impossible.

iii) If we have all conditions of type 3) then

$$p \leq (2q + 1)/3 \leq (4r + 5)/9 \leq (8p + 19)/27$$

that is $p \leq 1$ which is impossible.

iv) If we have all conditions of type 2) then $p = 2q + 1 = 4r + 3 = 8p + 7$, that is $p = -1$ which is impossible.

v) If we have one condition of type 2), say $p = 2q + 1$, and two conditions of type 3) then

$$q \leq (2r + 1)/3 \leq (4p + 5)/9 = (8q + 9)/9$$

that is $q \leq 9$, which implies that $q = 3, 5, 7$, $p = 7, 11, 15$, $r \mid 15, 23, 31$ and $q \mid 31, 47, 63$ which is impossible.

vi) If we one condition of type 3), say $p \mid (2q + 1)$ and $p \leq (2q + 1)/3$, and two conditions of type 2), that is $q = 2r + 1$, $r = 2p + 1$, then p divides

$$2q + 1 = 4r + 3 = 8p + 7$$

that is $p = 7$ and $r = 15$ which is impossible.

Also solved by Adnan Ibric, University of Tuzla, Bosnia and Herzegovina; Albert Stadler, Switzerland; Ajat Adriansyah, Surya Institute; Daniel Lasaosa, Universidad Pública de Navarra, Spain.

O279. One hundred boys and one hundred girls go to prom. Knowing that each boy dances with a girl at most once and that there are 1050 couples dancing, prove that there are two boys and two girls who dance with both of the boys.

Proposed by Marius Cavachi, Constanta, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Number the girls from 1 to 100, and denote by u_i the number of boys with which the i-th girl danced. The number of couples dancing is clearly $u_1 + u_2 + \cdots + u_{100} = C$. The number of pairs of boys with which the i-th girl danced is clearly $\binom{u_i}{2} = \frac{{u_i}^2 - u_i}{2}$, or the total number of pairs of boys with which any one of the girls danced is

$$\sum_{i=1}^{100} \binom{u_i}{2} = \frac{1}{2}\sum_{i=1}^{100} {u_i}^2 - \frac{1}{2}\sum_{i=1}^{100} u_i \geq 50\left(\frac{1}{100}\sum_{i=1}^{100} u_i\right)^2 - \frac{C}{2} =$$

$$= \frac{C(C-100)}{200} = \frac{9975}{2},$$

where we have applied the AM-QM inequality. Note now that there are $\binom{100}{2} = \frac{9900}{2}$ possible pairs of boys, or by the Pigeonhole Principle, at least one pair of boys appears twice, ie, there are at least two girls that danced with the same pair of boys. The conclusion follows.

Also solved by Ajat Adriansyah, Surya Institute; David Xu, Charter School of Wilmington, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

O280. Find all positive integers that can be written as

$$\frac{\left(3a_1{}^2+2a_1-4\right)\left(3a_2{}^2+2a_2-4\right)\cdots\left(3a_k{}^2+2a_k-4\right)}{\left(3b_1{}^2+2b_1-4\right)\left(3b_2{}^2+2b_2-4\right)\cdots\left(3b_k{}^2+2b_k-4\right)}$$

for some positive integers a_k, b_k and some $k \in \mathbb{N}^*$.

Proposed by Vlad Matei, University of Wisconsin, Madison, USA

Partial solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $u_n = 3n^2 + 2n - 4$. We say that a positive integer is *good* if it can be expressed in the form

$$\frac{u_{a_1} \cdot u_{a_2} \cdot \cdots \cdot u_{a_k}}{u_{b_1} \cdot u_{b_2} \cdot \cdots \cdot u_{b_k}}$$

for positive integers $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$, and the problem asks us to find all good integers. 1 is clearly good, since it suffices to take $k = 1$ and $a_1 = b_1$. Note that any good integer can be expressed as the product of good integers by separating equal numbers of terms in the numerator and the denominator, and reciprocally the product of good integers is good. Moreover, if one good integer A divides another good integer B, then $\frac{B}{A}$ is also good; it suffices to exchange in the expression for A the u_{a_i}'s and the u_{b_i}'s, multiplying the result by the expression for B.

O281. Let $a_1, a_2 \ldots, a_n$ be a decreasing sequence of positive numbers. Prove that

$$\sqrt{a_1^2+\cdots+a_n^2} \le a_1+\frac{a_2}{\sqrt{2}+1}+\cdots+\frac{a_n}{\sqrt{n}+\sqrt{n-1}} \le \sqrt{\left(1+\frac{1}{4}\ln n\right)(a_1^2+\cdots+a_n^2)}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

The left-hand side inequality may be proved by induction. For $n=1$ trivially we have an identity. Let us suppose the inequality holds for $n-1$. Then for n we have

$$\begin{aligned}\sqrt{a_1^2+\cdots+a_{n-1}^2+a_n^2} &= \sqrt{a_1^2+\cdots+a_{n-1}^2\left(1+a_n^2/a_{n-1}^2\right)}\\ &\le a_1+\frac{a_2}{\sqrt{2}+1}+\cdots+\frac{a_{n-1}}{\sqrt{n-1}+\sqrt{n-2}}\left(\sqrt{1+a_n^2/a_{n-1}^2}\right)\end{aligned}$$

by hypothesis of induction. Therefore it is enough to prove that

$$\begin{aligned}\frac{a_{n-1}}{\sqrt{n-1}+\sqrt{n-2}}\left(\sqrt{1+a_n^2/a_{n-1}^2}\right) &\le \frac{a_{n-1}}{\sqrt{n-1}+\sqrt{n-2}}+\frac{a_n}{\sqrt{n}+\sqrt{n-1}}\\ \sqrt{a_{n-1}^2+a_n^2} &\le a_{n-1}+\frac{\sqrt{n-1}+\sqrt{n-2}}{\sqrt{n}+\sqrt{n-1}}a_n.\end{aligned}$$

Considering new variable $x=a_n/a_{n-1}$, and if $\lambda=\frac{\sqrt{n-1}+\sqrt{n-2}}{\sqrt{n}+\sqrt{n-1}}$, last inequality is equivalent to

$$\begin{aligned}x^2 &\le \lambda^2x^2+2\lambda x\\ (1-\lambda^2)x^2-2\lambda x &\le 0\end{aligned}$$

which is true since the roots of the polynomial are $x_0=0$, and $x_1\frac{2\lambda}{1-\lambda^2}$, and $\frac{1}{\sqrt{2}}\le\lambda\le 1$ and therefore $x_1>1$, while $x=a_n/a_{n-1}<1$.

For the right-hand side inequality, note that $\frac{1}{2\sqrt{k}}<\frac{1}{\sqrt{k}+\sqrt{k-1}}<\frac{1}{2\sqrt{k-1}}$, and then $\frac{1}{4k}<\left(\frac{1}{\sqrt{k}+\sqrt{k-1}}\right)^2<\frac{1}{4(k-1)}$. By the Cauchy-Schwarz inequality

$$\begin{aligned}a_1+\frac{a_2}{\sqrt{2}+1}+\cdots+\frac{a_n}{\sqrt{n}+\sqrt{n-1}} &\le \sqrt{\left(\sum_{k=1}^n a_k^2\right)\left(1+\frac{1}{4\cdot 2}+\cdots+\frac{1}{4\cdot(n-1)}\right)}\\ &\le \sqrt{\left(\sum_{k=1}^n a_k^2\right)\left(1+\frac{1}{4}\cdot\ln n\right)},\end{aligned}$$

where for the last inequality comes from the $\sum_{k=2}^{n-1}\frac{1}{k}\le\int_1^n\frac{1}{x}\,dx=\ln n$.

Also solved by Arkady Alt , San Jose ,California, USA; Li Zhou, Polk State College, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

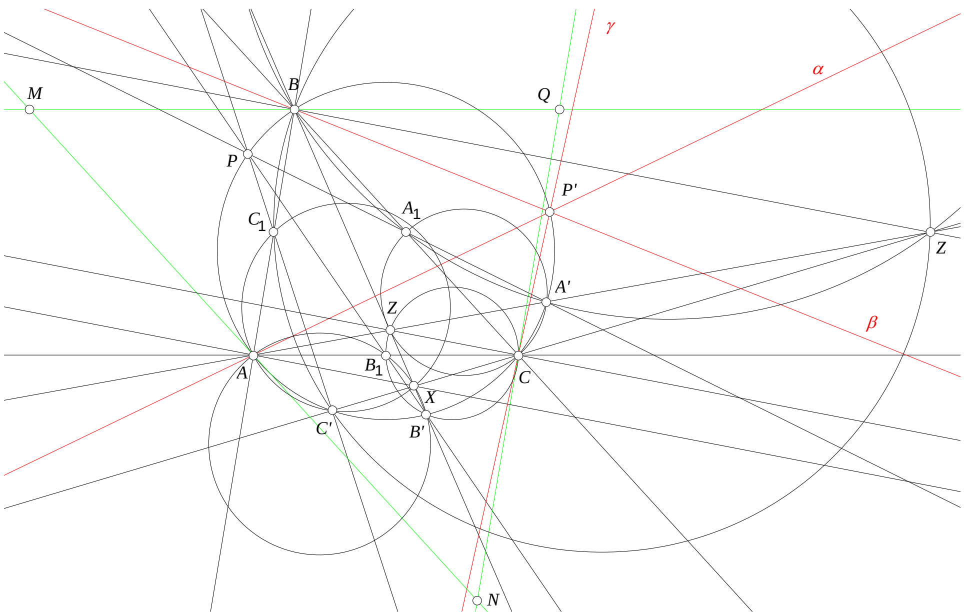
O282. Let ABC be a triangle and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle of triangle ABC. Lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C', respectively and X, Y, Z are the intersections $BB' \cap CC'$, $CC' \cap AA'$, and $AA' \cap BB'$. Parallels through A, B, C to BC, CA, AB determine a triangle MNQ and α, β, γ are the reflections of AX, BY, CZ into NQ, QM, MN, respectively. Prove that:

a) α, β, γ concur on the circumcircle of triangle ABC;

b) $AX \parallel BY \parallel CZ \parallel PP'$, where P' is the concurrence point from part a).

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Sebastiano Mosca, Pescara, Italy



For the Pascal theorem C_1B_1X, C_1A_1Y and B_1ZA_1 are on the same line.

Due to C_1X is parallel to BC we have:

$$\angle C_1XC' = \angle BCC' \quad , \quad \angle BCC' + \angle BAC' = 180^\circ$$

so the quadrilateral AC_1XC' is cyclic.

Quadrilateral $AB_1B'X$ is also cyclic because:

$$\angle AB'B = \angle ACB = \angle C_1B_1A = \angle B'B_1C = 180^\circ - \angle XB_1A$$

so it follow $\angle AB'B + \angle XB_1A = 180^\circ$.

In analogous way we can prove that quadrilaterals $B'B_1ZC$, CZA_1A', $YA'A_1B$ and $YC'C_1B$ are cyclic. It follow that:

$$\angle BYC_1 = \angle BC'C_1 = \angle PB'B = \angle B_1CZ = \angle XAB_1$$

The segment C_1A_1 is parallel to AC so it means that segments AX, BY and CZ have the same inclination with respect to the segment AC so they are parallel. Let call P_1 the intersection of the line α and the line β. We have:

$$\angle CBP_1 = \angle CBY - \angle P_1BY \quad , \quad \angle CAP_1 = \angle XAP_1 - \angle XAC$$

but $\angle CBY = \angle NAX = \angle XAP_1$ and $\angle P_1BY = \angle YBQ = \angle XAC$ so it follow that $\angle CBP_1 = \angle CAP_1$. It means that

$$\angle ABP_1 = \angle ABC + \angle CBP_1 \quad , \quad \angle BAP_1 = \angle BAC - \angle CAP_1$$

so

$$\angle AP_1B = 180^\circ - \angle ABP_1 - \angle BAP_1 = 180^\circ - \angle ABC - \angle BAC = \angle BCA$$

It means that P_1 is on the circumscribed circle of triangle ABC. In the same way we can prove that the line α and the line γ, the line β and the line γ, intersect on the circumscribed circle of triangle ABC in a point P_2 and P_3.

But, if three line passing from the vertices of a triangle with them intersection taken two by two intersect on circumcircle, the three line intersect on the same point, so $P_1 = P_2 = P_3 = P'$.

P' is on the circumscribed circle of triangle ABC, whence

$$\angle BP'P = \angle BB'P = \angle XAB_1 = \angle YBQ = \angle YBP_1$$

so PP' have the same inclination with respect to the segment BP' of the segment BY; it means that PP' and BY are parallel and this conclude the proof.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Solución de Saturnino Campo Ruiz, Profesor de Matemáticas jubilado, de Salamanca.