On a Mixtilinear Coaxality

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Abstract

It is known that the three Apollonius circles, each passing through a vertex and its corresponding feet of the internal and external angle bisectors, have two common points. We study a similar configuration with the tangency points of the mixtilinear incircles [1, 2] with the circumcircle, instead of with the feet of the external angle bisectors.

The term *mixtilinear incircles* of a triangle (L. Bankoff [1]) names the three circles each tangent to two sides and to the circumcircle internally. In [2], P. Yiu establishes one of the fundamental results regarding these circles:

Theorem 1 (Yiu). The three lines each joining a vertex to the point of contact of the circumcircle with the respective mixtilinear incircle are concurrent at the external center of similitude of the circumcircle and incircle.

Consider D, E, F the points of tangency of the incircle with the triangle sides BC, CA, AB, respectively. Let X, Y, Z be the feet of the internal angle bisectors corresponding to the vertices A, B, C, respectively and A', B', C' be the points of tangency of the circumcircle $\mathcal{O}$ with each mixtilinear incircle. In this paper, we prove synthetically that the circles AXA', BYB', CZC' are coaxal and that the external center of similitude of the circumcircle and incircle lies on their common radical axis. We make use of inversion with respect to the incircle.

We begin with two concyclities. Although at first sight unrelated to the coaxality, they prove themselves useful in future arguments.

Theorem 2. The points B', C', X, D lie on the same circle. Analogously, the points C', A', Y, E, respectively A', B', Z, F are concyclic.

Proof of Theorem 2. Consider the inversion Ψ with center I and power r^2, where r is the inradius lenght. The points D, E, F remain invariant, because they are on the inversion circle. The images of A, B, C under Ψ are the midpoints A_1, B_1, C_1 of the sides EF, FD, DE. Since the sidelines BC, CA, AB are tangent to the circle of inversion, their images are the congruent circles Γ_a, Γ_b, Γ_c of diameters r. The image X_1 of X is the second intersection of the line A_1I with Γ_a and analogously the images Y_1, Z_1 of points Y, Z are the second intersections of the lines B_1I, C_1I with Γ_b, Γ_c, respectively. Before proceeding in establishing the images of A', B', C', we first remind the following result, which although it is known, we will not omit its simple proof:

Lemma 1. The line $A'I$ bisects the angle $\angle BA'C$.

Proof of Lemma 1. Let M be the midpoint of the arc BC not containing the vertex A, M' its antipode with respect to $\mathcal{O}$ and S, T the points of contact of the mixtilinear incircle in angle $\angle A$ with the sides AB, AC, respectively. Since I is the midpoint of ST [1] and because of the collinearity of the midpoint N of the arc CA not containing B with the points A' and T,

$$\angle TIC = \angle AIC - 90^\circ = \frac{\angle ABC}{2} = \angle TA'C.$$

Thus, the quadrilateral A_1CTI is cyclic and

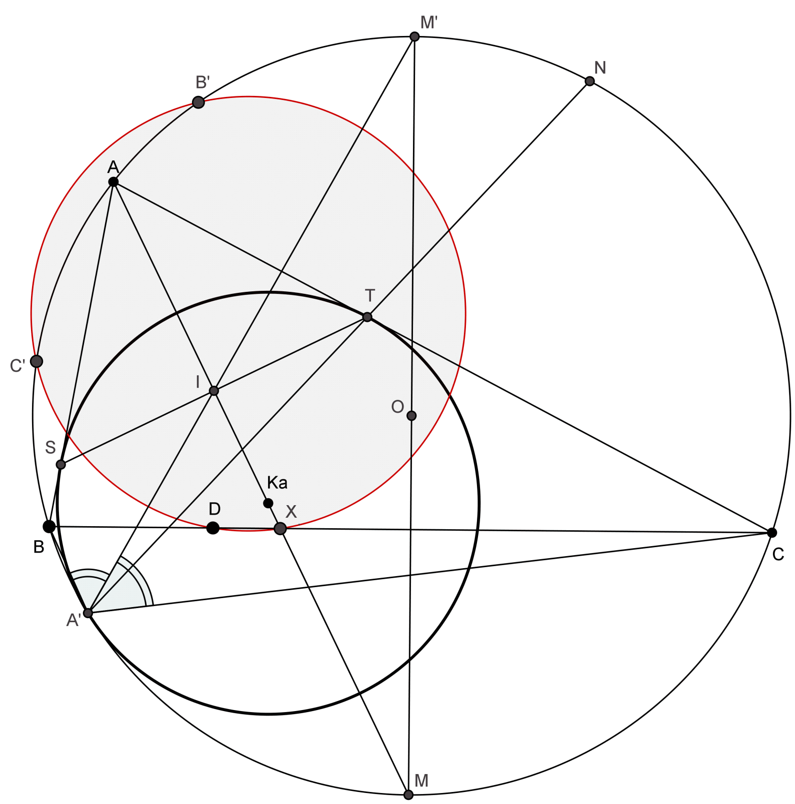
$$\angle IA'C = \angle ITA = 90^\circ - \frac{\angle CAB}{2} = \angle M'A'C.$$

Hence, the points A', I, M' are collinear. This proves Lemma 1.

Since M' is the antipode of M with respect to $\mathcal{O}$, $\angle MA'M' = 90^\circ \mod 180^\circ$. The image M_1 of the point M under Ψ is the second intersection of the line A_1I with $\mathcal{O}_1$. Therefore,

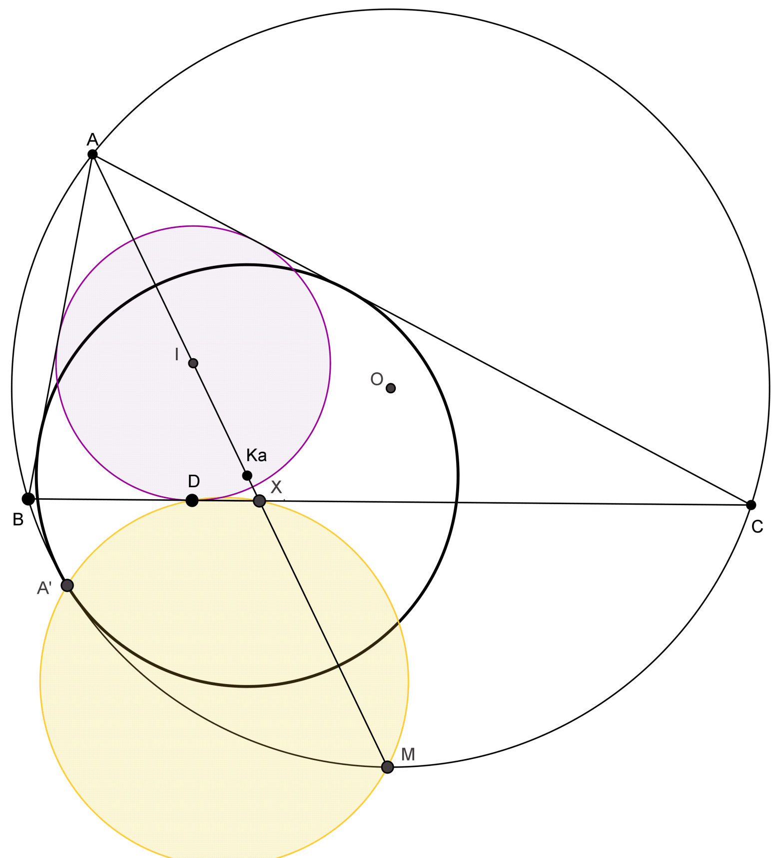
$$\angle A_1M_1A_1' = \angle IM_1A_1' = \angle MA'I = \angle MA'M' = 90^\circ \mod 180^\circ.$$

We now conclude that the images A_1', B_1', C_1' of the points A', B', C' under Ψ are the antipodes of A_1, B_1, C_1 in the image $\mathcal{O}_1$ of the circumcircle of triangle ABC, also being the nine-point circle of triangle DEF. The line $B_1'C_1'$ is parallel to the line C_1B_1, because B_1B_1', C_1C_1' are diameters of the circle $\mathcal{O}_1$, and therefore the quadrilateral $B_1C_1B_1'C_1'$ is a rectangle. Since ID is a diameter of the circle Γ_a, the line X_1D, being perpendicular to A_1I, is also parallel to B_1C_1. The line B_1C_1 is the radical axis of the circles $\mathcal{O}_1$ and Γ_a, which contain the parallel chords $B_1'C_1'$, X_1D, respectively. Consequently, the quadrilateral, $B_1'X_1DC_1'$ is an isosceles trapezium, and hereby it is cyclic. Thus, the quadrilateral $B'XDC'$ is also cyclic. This proves Theorem 2. $\square$



Theorem 3. Let M, N, P be the midpoints of the arcs BC, CA, AB, not containing the opposite vertices. The points A', M, X, D lie on the same circle. Analogously, the points B', N, Y, E, respectively C', P, Z, F are concyclic.

Proof of Theorem 3. Since A_1' is the antipode of A with respect to $\mathcal{O}_1$, the lines A_1M_1 and M_1A_1' are perpendicular. Due to the perpendicularity of A_1I and X_1D, the lines M_1A_1' and X_1D are also parallel. This yields that the quadrilateral $DX_1M_1A_1'$ is a rectangle, and, therefore, it is cyclic. Thus, the quadrilateral $DXMA'$ is also cyclic. This proves Theorem 3. $\square$



We can now proceed in proving the main result:

Theorem 4. The circles AXA', BYB', CZC' are coaxal. The external center of similitude of the circumcircle and incircle lies on their common radical axis.

In [3], A. P. Hatzipolakis et al. prove the concurrency of the Brocard axes of the triangles ABC, IBC, ICA, IAB at the Kimberling center X_{58} (C. Kimberling [4]), using trilinear coordinates. In [5], P. Yiu shows, using barycentric coordinates, that these circles are coaxal and that the Kimberling center X_{58} also lies on their radical axis.

Proof of Theorem 4. We will first prove a preliminary result, which partially follows from Theorem 1 and Theorem 2.

Lemma 2. The lines $B'C'$, $A'M$, BC concur at a point U, on the circle AXA'. Analogously, the lines $C'A'$, $B'N$, CA and $A'B'$, $C'P$, AB concur at points V, W, on the circles BYB', CZC', respectively.

Proof of Lemma 2. By Theorem 1 and Theorem 2, the lines $C'B'$, MA', BC concur at the radical center U of circumcircle of ABC, the circle $B'XDC'$ and the circle $DXMA'$. Since the image of A' under Ψ is the antipode A_1' of A_1,

$$\angle MA'I = \angle IM_1A_1' = 90^\circ \mod 180^\circ,$$

which yields that

$$\angle IA'U = \angle IDU = 90^\circ \mod 180^\circ.$$

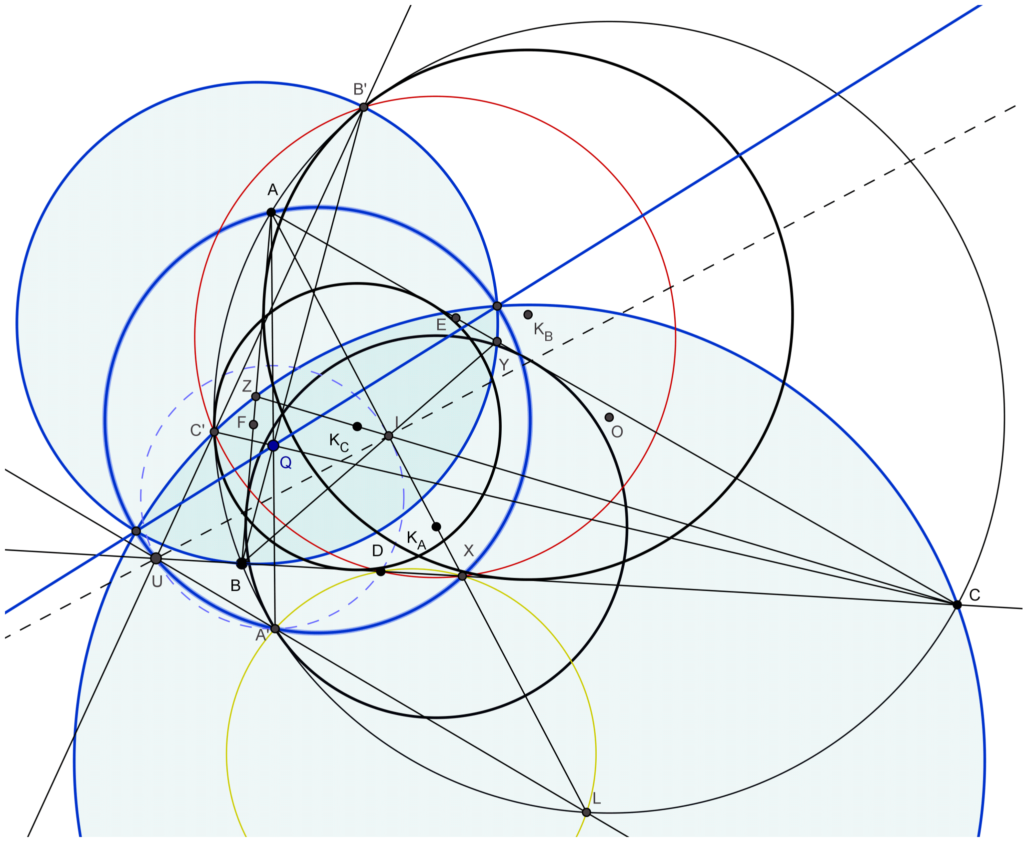
This means that the quadrilateral $IDA'U$ is cyclic and therefore, the images D, A_1', U_1 of D, A', U are collinear. The circle $A_1B_1C_1$ is the nine-point circle of the triangle DEF and consequently, the line joining D and the antipode A_1' of A_1 with respect to the nine-point circle is the altitude of D in the triangle DEF. Thus, U_1 lies on the line through D perpendicular to the line EF and parallel to the line A_1I. Hence,

$$\angle(DU_1, EF) = 90^\circ,$$

and

$$\angle AIU = \angle A_1IU_1 = \angle DU_1I = \angle IDU = 90^\circ \mod 180^\circ.$$

Therefore, U lies on the perpendicular through I to the line AI, which means that the lines EF, IU_1 are parallel. The image U_1 of the point U is the second intersection of the line IU with the circle Γ_a. Because of the parallelism of the lines IU_1, B_1C_1, X_1D with the sideline EF and of the congruence of the circles $\mathcal{O}_1$ and Γ_a, the quadrilateral $A_1X_1A_1'U_1$ is an isosceles trapezoid and hereby, it is cyclic. Thus, the quadrilateral $AXA'U$ is also cyclic. This proves Lemma 2.



On other hand, since

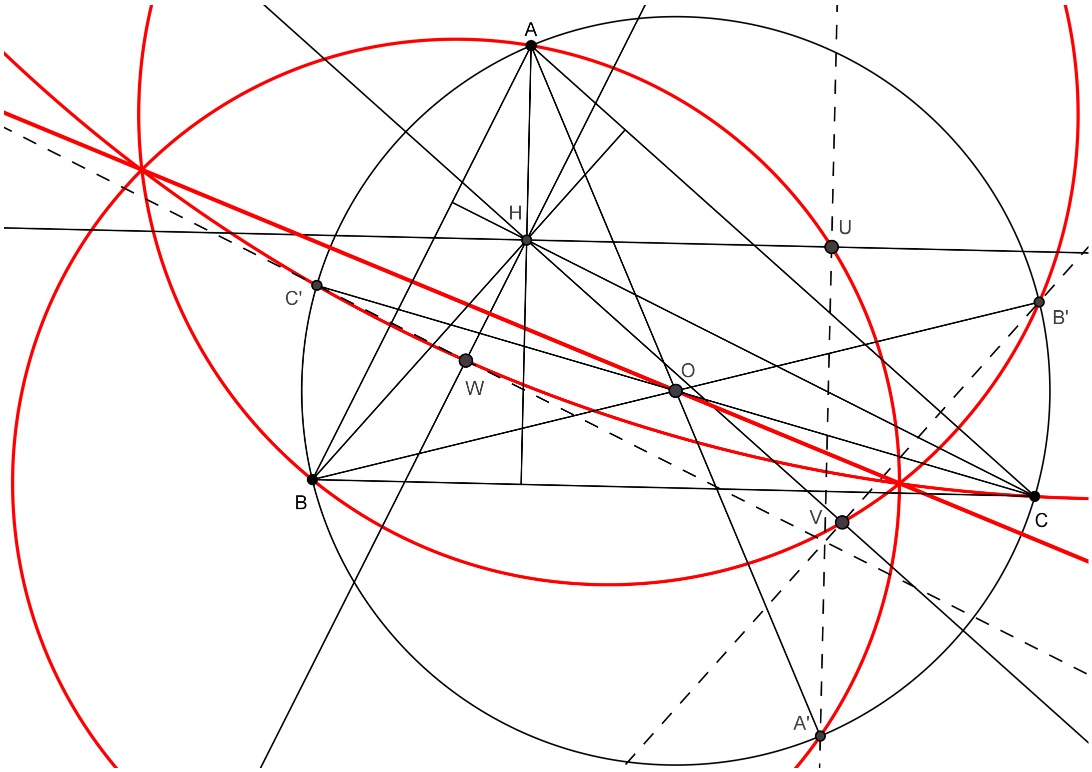
$$\angle(U_1D, EF) = \angle(V_1E, FD) = \angle(W_1F, DE) = 90^\circ,$$

and

$$\angle A_1IU_1 = \angle B_1IV_1 = \angle C_1IW_1 = 90^\circ \mod 180^\circ,$$

the images U_1, V_1, W_1 of the points U, V, W under Ψ are the orthogonal projections of the ortocenter I of triangle $A_1B_1C_1$ on the perpendiculars through A_1', B_1', C_1' to the lines B_1C_1, C_1A_1, A_1B_1. Hereby, the coaxality of the circles $A_1X_1A_1'U_1$, $B_1YB_1'V_1$, $C_1ZC_1'W_1$ can be restated as follows:

Lemma 3. Let the points A', B', C' be the antipodes of the vertices A, B, C with respect to the circumcircle $\mathcal{O}$. Consider the lines d_a, d_b, d_c through A', B', C' perpendicular to the sidelines BC, CA, AB, respectively and let U, V, W be the orthogonal projections of the orthocenter H of triangle ABC on d_a, d_b, d_c respectively. The circles AUA', BVB', CWC' are coaxal.



Proof of Lemma 3. Since the reflections of the orthocenter H in the sidelines BC, CA, AB lie on the circumcircle of triangle ABC, the points U, V, W are the reflections of A', B', C' in the sides of triangle ABC. Hence, the circumcenters O_a, O_b, O_c of triangles AUA', BVB', CWC' are intersections of the perpendicular bisectors of the circumcircle diameters AA', BB', CC' with the sidelines BC, CA, AB. Moreover, if B_a, C_a are the orthogonal projections of the vertices B, C on AA',

$$\frac{\overline{O_aB}}{\overline{O_aC}} = \frac{\overline{OB_a}}{\overline{OC_a}} = \frac{R\cos AOB}{R\cos COA} = \frac{\cos 2C}{\cos 2B}.$$

Analogously, if C_b, A_b and A_c, B_c are the orthogonal projections of the vertices C, A, and A, B on the diameters BB', CC', respectively,

$$\frac{\overline{O_bC}}{\overline{O_bA}} = \frac{\overline{OC_b}}{\overline{OA_b}} = \frac{\cos 2A}{\cos 2C}, \qquad \frac{\overline{O_cA}}{\overline{O_cB}} = \frac{\overline{OA_c}}{\overline{OB_c}} = \frac{\cos 2B}{\cos 2A}.$$

Hence,

$$\frac{\overline{O_aB}}{\overline{O_aC}} \cdot \frac{\overline{O_bC}}{\overline{O_bA}} \cdot \frac{\overline{O_cA}}{\overline{O_cB}} = 1,$$

and therefore, by Menelaus' theorem, the points O_a, O_b, O_c are collinear. Consequently, the three pairwise radical axes of the circles AUA', BVB', CWC' are either parallel or they coincide. On other hand, since

$$\overline{OA} \cdot \overline{OA'} = \overline{OB} \cdot \overline{OB'} = \overline{OC} \cdot \overline{OC'} = -R^2,$$

the circumcenter O lies on all three axes, and therefore, the circles AUA', BVB', CWC' are coaxal. This proves Lemma 3.

Combining with Lemma 2, this yields that the circles $A_1X_1A_1'U_1$, $B_1YB_1'V_1$, $C_1ZC_1'W_1$ are coaxal. Since these three circles are the images of $AXA'U$, $BYB'V$, $CZC'W$ under Ψ, and since any inversion carries a pencil of circles or lines into a pencil of circles or lines, it follows that the circles $AXA'U$, $BYB'V$, $CZC'W$ are also coaxal.

According to Theorem 1, the lines AA', BB', CC' concur at the external center of similitude Q of the circumcircle and incircle (ETC center X_{56} [4]). The power of Q with respect to the circumcircle $\mathcal{O}$ is

$$\overline{QA} \cdot \overline{QA'} = \overline{QB} \cdot \overline{QB'} = \overline{QC} \cdot \overline{QC'},$$

which means that Q lies on the common radical axis of the coaxal circles AXA', BYB', CZC'. This completes the proof of Theorem 4. $\square$

References

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