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Antiparallels and Concurrent Euler Lines

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Abstract. We study the condition for concurrency of the Euler lines of the three triangles each bounded by two sides of a reference triangle and an antiparallel to the third side. For example, if the antiparallels are concurrent at P and the three Euler lines are concurrent at Q, then the loci of P and Q are respectively the tangent to the Jerabek hyperbola at the Lemoine point, and the line parallel to the Brocard axis through the inverse of the deLongchamps point in the circumcircle. We also obtain an interesting cubic as the locus of the point P for which the three Euler lines are concurrent when the antiparallels are constructed through the vertices of the cevian triangle of P.

1. Thébault's theorem on Euler lines

We begin with the following theorem of Victor Thébault [8] on the concurrency of three Euler lines.

Theorem 1 (Thébault). *Let $A'B'C'$ be the orthic triangle of ABC. The Euler lines of the triangles $AB'C'$, $BC'A'$, $CA'B'$ are concurrent at the Jerabek center.* [1]

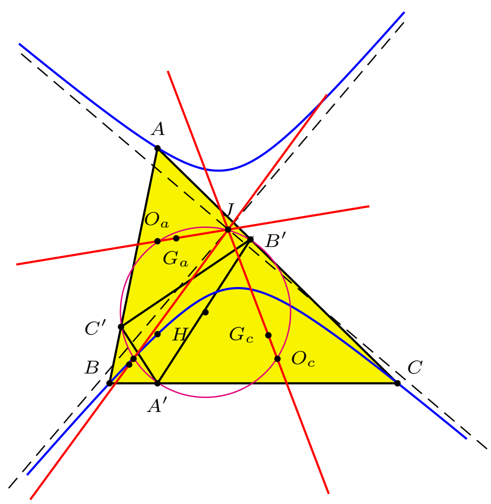

Figure 1. Thébault's theorem on the concurrency of Euler lines

We shall make use of homogeneous barycentric coordinates. With reference to triangle ABC, the vertices of the orthic triangle are the points

$$A' = (0 : S_C : S_B), \qquad B' = (S_C : 0 : S_A), \qquad C' = (S_B : S_A : 0).$$

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[1] Thébault [8] gave an equivalent characterization of this common point. See also [7].

These are the traces of the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$.

The centroid of $AB'C'$ is the point

$$(S_{AA} + 2S_{AB} + 2S_{AC} + 3S_{BC} : S_A(S_C + S_A) : S_A(S_A + S_B)).$$

The circumcenter of $A'BC$, being the midpoint of AH, has coordinates

$$(S_{CA} + S_{AB} + 2S_{BC} : S_{AC} : S_{AB}).$$

It is straightforward to verify that these two points lie on the line

$$S_{AA}(S_B - S_C)(x+y+z) = (S_A+S_B)(S_{AB}+S_{BC}-2S_{CA})y-(S_C+S_A)(S_{BC}+S_{CA}-2S_{AB})z, \tag{1}$$

which is therefore the Euler line of triangle $AB'C'$. Furthermore, the line (1) also contains the point

$$J = (S_A(S_B - S_C)^2 : S_B(S_C - S_A)^2 : S_C(S_A - S_B)^2),$$

which is the center of the Jerabek hyperbola. [2] Similar reasoning gives the equations of the Euler lines of triangles $BC'A'$ and $A'B'C$, and shows that these contain the same point J. This completes the proof of Thébault's theorem.

2. Triangles intercepted by antiparallels

Since the sides of the orthic triangles are antiparallel to the respective sides of triangle ABC, we consider the more general situation when the residuals of the orthic triangle are replaced by triangles intercepted by lines ℓ_1, ℓ_2, ℓ_3 antiparallel to the sidelines of the reference triangle, with the following intercepts on the sidelines

	BC	CA	AB
ℓ_1		B_a	C_a
ℓ_2	A_b		C_b
ℓ_3	A_c	B_c	

These lines are parallel to the sidelines of the orthic triangle $A'B'C'$. We shall assume that they are the images of the lines $B'C'$, $C'A'$, $A'B'$ under the homotheties $\mathsf{h}(A, 1-t_1)$, $\mathsf{h}(B, 1-t_2)$, and $\mathsf{h}(C, 1-t_3)$ respectively. The points B_a, C_a etc. have homogeneous barycentric coordinates

$$\begin{array}{ll} B_a = (t_1S_A + S_C : 0 : (1-t_1)S_A), & C_a = (t_1S_A + S_B : (1-t_1)S_A : 0), \\ C_b = ((1-t_2)S_B : t_2S_B + S_A : 0), & A_b = (0 : t_2S_B + S_C : (1-t_2)S_B), \\ A_c = (0 : (1-t_3)S_C : t_3S_C + S_B), & B_c = ((1-t_3)S_C : 0 : t_3S_C + S_A). \end{array}$$

[2] The point J appears as X_{125} in [4].

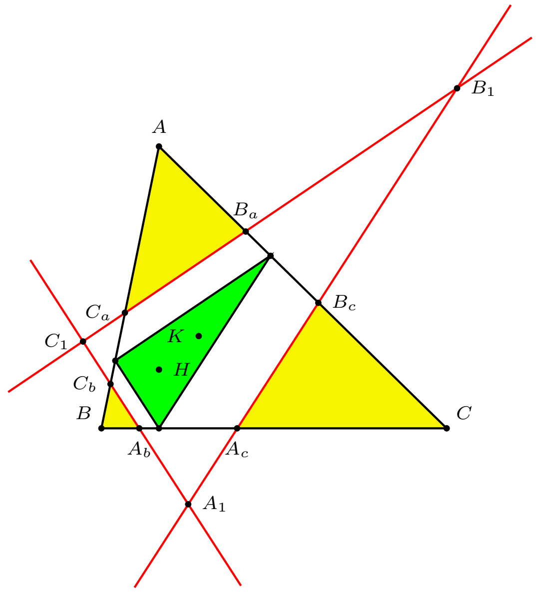

Figure 2. Triangles intercepted by antiparallels

2.1. *The Euler lines* $\mathcal{L}_i$, $i = 1, 2, 3$. Denote by $\mathbf{T}_1$ the triangle AB_aC_a intercepted by ℓ_1; similarly $\mathbf{T}_2$ and $\mathbf{T}_3$. These are oppositely similar to ABC. We shall study the condition of the concurrency of their Euler lines.

Proposition 2. *With reference to triangle ABC, the barycentric equations of the Euler lines of $\mathbf{T}_i$, $i = 1, 2, 3$, are*

$$\begin{aligned}(1-t_1)S_{AA}(S_B-S_C)(x+y+z) =&c^2(S_{AB}+S_{BC}-2S_{CA})y-b^2(S_{BC}+S_{CA}-2S_{AB})z,\\(1-t_2)S_{BB}(S_C-S_A)(x+y+z) =&a^2(S_{BC}+S_{CA}-2S_{AB})z-c^2(S_{CA}+S_{AB}-2S_{BC})x,\\(1-t_3)S_{CC}(S_A-S_B)(x+y+z) =&b^2(S_{CA}+S_{AB}-2S_{BC})x-a^2(S_{AB}+S_{BC}-2S_{CA})y.\end{aligned}$$

Proof. It is enough to establish the equation of the Euler line $\mathcal{L}_1$ of $\mathbf{T}_1$. This is the image of the Euler line $\mathcal{L}_1'$ of triangle $AB'C'$ under the homothety $\mathsf{h}(A, 1-t_1)$. A point $(x : y : z)$ on $\mathcal{L}_1$ corresponds to the point $((1-t_1)x - t_1(y+z) : y : z)$ on $\mathcal{L}_1'$. The equation of $\mathcal{L}_1$ can now by obtained from (1). □

From the equations of these Euler lines, we easily obtain the condition for their concurrency.

Theorem 3. *The three Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent if and only if*

$$t_1a^2(S_B-S_C)S_{AA}+t_2b^2(S_C-S_A)S_{BB}+t_3c^2(S_A-S_B)S_{CC}=0. \qquad (2)$$

Proof. From the equations of $\mathcal{L}_i$, $i = 1, 2, 3$, given in Proposition 2, it is clear that the condition for concurrency is

$$(1-t_1)a^2(S_B-S_C)S_{AA}+(1-t_2)b^2(S_C-S_A)S_{BB}+(1-t_3)c^2(S_A-S_B)S_{CC}=0.$$

This simplifies into (2) above. □

2.2. *Antiparallels with given common point of* $\mathcal{L}_i$, $i = 1, 2, 3$. We shall assume triangle ABC scalene, ***i.e.***, its angles are unequal and none of them is a right angle. For such triangles, the Euler lines of the residuals of the orthic triangle and the corresponding altitudes intersect at finite points.

Theorem 4. *Given a point* Q *in the plane of a scalene triangle* ABC*, there is a unqiue triple of antiparallels* ℓ_i*,* $i = 1, 2, 3$*, for which the Euler lines* $\mathcal{L}_i$*,* $i = 1, 2, 3$*, are concurrent at* Q*.*

Proof. Construct the parallel through Q to the Euler line of $AB'C'$ to intersect the line AH at O_a. The circle through A with center O_a intersects AC and AB at B_a and C_a respectively. The line B_aC_a is parallel to $B'C'$. It follows that its Euler line is parallel to that of $AB'C'$. This is the line O_aQ. Similar constructions give the other two antiparallels with corresponding Euler lines passing through Q. $\square$

We make a useful observation here. From the equations of the Euler lines given in Proposition 2 above, the intersection of any two of them have coordinates expressible in linear functions of t_1, t_2, t_3. It follows that if t_1, t_2, t_3 are linear functions of a parameter t, and the three Euler lines are concurrent, then as t varies, the common point traverses a straight line. In particular, $t_1 = t_2 = t_3 = t$, the Euler lines are concurrent by Theorem 3. The locus of the intersection of the Euler lines is a straight line. Since this intersection is the Jerabek center when $t = 0$ (Thébault's theorem), and the orthocenter when $t = -1$,[3] this is the line

$$\mathcal{L}_{\rm c}: \qquad \sum_{\text{cyclic}} S_{AA}(S_B - S_C)(S_{CA} + S_{AB} - 2S_{BC})x = 0.$$

We give a summary of some of the interesting loci of common points of Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, when the lines ℓ_i, $i = 1, 2, 3$, are subjected to some further conditions. In what follows, $\mathbf{T}$ denotes the triangle bounded by the lines ℓ_i, $i = 1, 2, 3$.

Line	Construction	Condition	Reference
$\mathcal{L}_{\rm c}$	HJ	$\mathbf{T}$ homothetic to orthic triangle at X_{25}	
$\mathcal{L}_{\rm q}$	Remark below	ℓ_i, $i = 1, 2, 3$, concurrent	§3.2
$\mathcal{L}_{\rm t}$	KX_{74}	ℓ_i are the antiparallels of a Tucker hexagon	§6
$\mathcal{L}_{\rm f}$	X_5X_{184}	$\mathcal{L}_i$ intersect on Euler line of $\mathbf{T}$	§7.2
$\mathcal{L}_{\rm r}$	GX_{110}	$\mathbf{T}$ and ABC perspective	§8.3

Remark. $\mathcal{L}_{\rm q}$ can be constructed as the line parallel to the Brocard axis through the intersection of the inverse of the deLongchamps point in the circumcircle.

[3]For $t = 1$, this intersection is the point X_{74} on the circumcircle, the isogonal conjugate of the infinite point of the Euler line.

3. Concurrent antiparallels

In this section we consider the case when the antiparallels ℓ_1, ℓ_2, ℓ_3 all pass through a point $P = (u : v : w)$. In this case,

$$\begin{aligned}
B_a &= ((S_C + S_A)u - (S_B - S_C)v : 0 : (S_A + S_B)v + (S_C + S_A)w),\\
C_a &= ((S_A + S_B)u + (S_B - S_C)w : (S_A + S_B)v + (S_C + S_A)w : 0),\\
C_b &= ((S_B + S_C)w + (S_A + S_B)u : (S_A + S_B)v - (S_C - S_A)w : 0),\\
A_b &= (0 : (S_B + S_C)v + (S_C - S_A)u : (S_B + S_C)w + (S_A + S_B)u),\\
A_c &= (0 : (S_C + S_A)u + (S_B + S_C)v : (S_B + S_C)w - (S_A - S_B)u),\\
B_c &= ((S_C + S_A)u + (S_B + S_C)v : 0 : (S_C + S_A)w + (S_A - S_B)v).
\end{aligned}$$

For example, when $P = K$, these are the vertices of the second cosine circle.

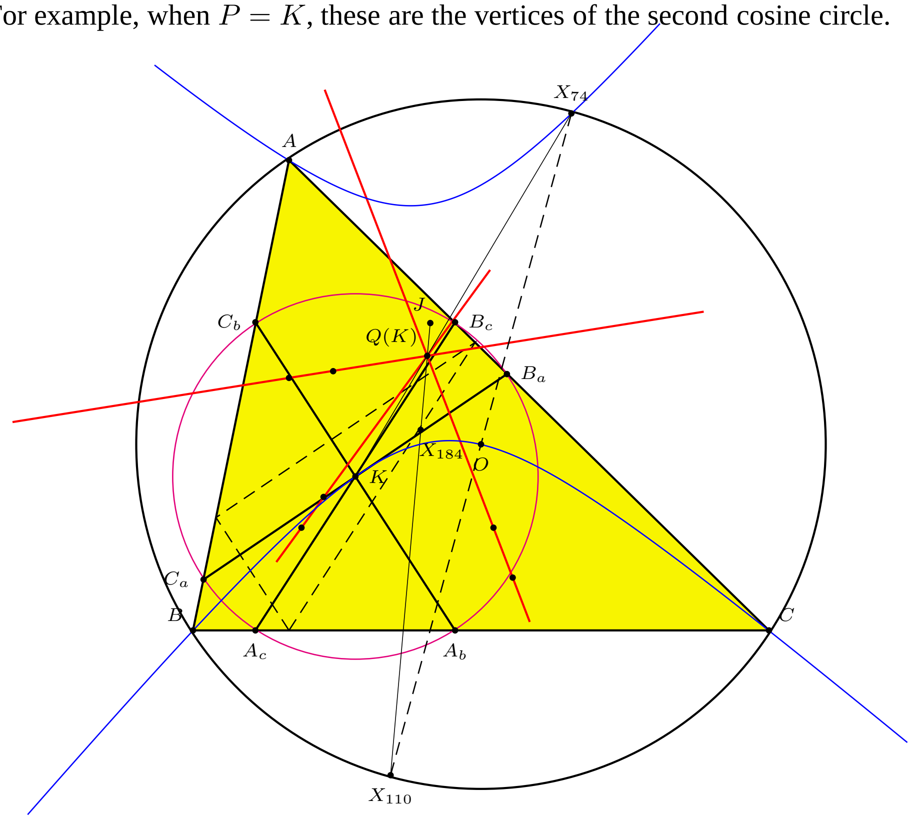

Figure 3. $Q(K)$ and the second Lemoine circle

Proposition 5. *The Euler lines of triangles $\mathbf{T}_i$, $i = 1, 2, 3$, are concurrent if and only if P lies on the line*

$$\mathcal{L}_{\rm p}: \qquad \frac{S_A(S_B - S_C)}{a^2}x + \frac{S_B(S_C - S_A)}{b^2}y + \frac{S_C(S_A - S_B)}{c^2}z = 0.$$

When P traverses $\mathcal{L}_{\rm p}$, the intersection Q of the Euler lines traverses the line

$$\mathcal{L}_{\rm q}: \qquad \sum_{\text{cyclic}} \frac{(b^2 - c^2)(a^2(S_{AA} + S_{BC}) - 4S_{ABC})}{a^2}x = 0.$$

For a point P on the line $\mathcal{L}_{\rm p}$, we denote by $Q(P)$ the corresponding point on $\mathcal{L}_{\rm q}$.

Proposition 6. *For points P_1, P_2, P_3 on $\mathcal{L}_{\rm p}$, $Q(P_1)$, $Q(P_2)$, $Q(P_3)$ are points on $\mathcal{L}_{\rm q}$ satisfying*

$$Q(P_1)Q(P_2) : Q(P_2)Q(P_3) = P_1P_2 : P_2P_3.$$

3.1. *The line $\mathcal{L}_{\rm p}$.* The line $\mathcal{L}_{\rm p}$ contains K and is the tangent to the Jerabek hyperbola at K. See Figure 4. It also contains, among others, the following points.

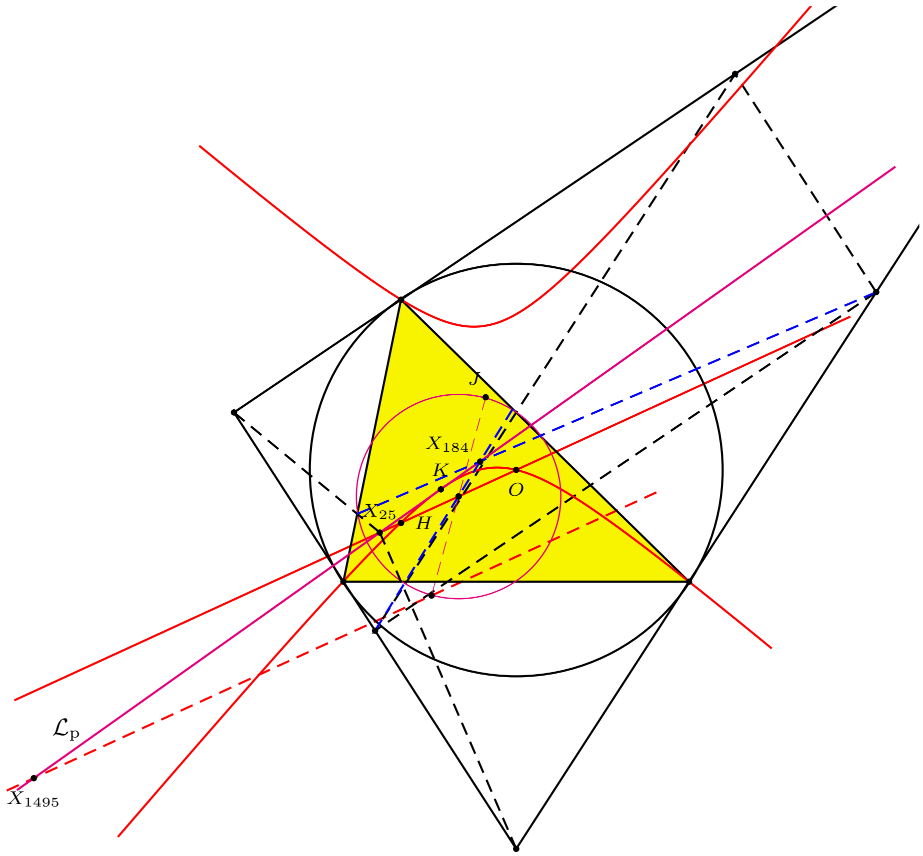

Figure 4. The line $\mathcal{L}_{\rm p}$

(1) $X_{25} = \left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right)$ which is on the Euler line of ABC, and is the homothetic center of the orthic and the tangential triangles,[4]

(2) $X_{184} = (a^4S_A : b^4S_B : c^4S_C)$ which is the homothetic center of the orthic triangle and the medial tangential triangle, [5]

[4]See also §4.1.

[5]For other interesting properties of X_{184}, see [6], where it is named the procircumcenter of triangle ABC.

(3) $X_{1495} = (a^2(S_{CA}+S_{AB}-2S_{BC}) : \cdots : \cdots)$ which lies on the parallel to the Euler line through the antipode of the Jerabek center on the nine-point circle. [6]

3.2. *The line* $\mathcal{L}_{\rm q}$. The line $\mathcal{L}_{\rm q}$ is parallel to the Brocard axis. See Figure 5. It contains the following points.

(1) $Q(K) = (a^2S_A(b^2c^2(S_{BB}-S_{BC}+S_{CC})-2a^2S_{ABC}) : \cdots : \cdots)$. It can be constructed as the intersection of the lines joining K to X_{74}, and J to X_{110}. See Figure 3 and §6 below. The line $\mathcal{L}_{\rm q}$ can therefore be constructed as the parallel through this point to the Brocard axis.

(2) $Q(X_{1495}) = (a^2S_A(a^2S^2 - 6S_{ABC}) : \cdots : \cdots)$, which is on the line joining O to X_{184} (on $\mathcal{L}_{\rm p}$).

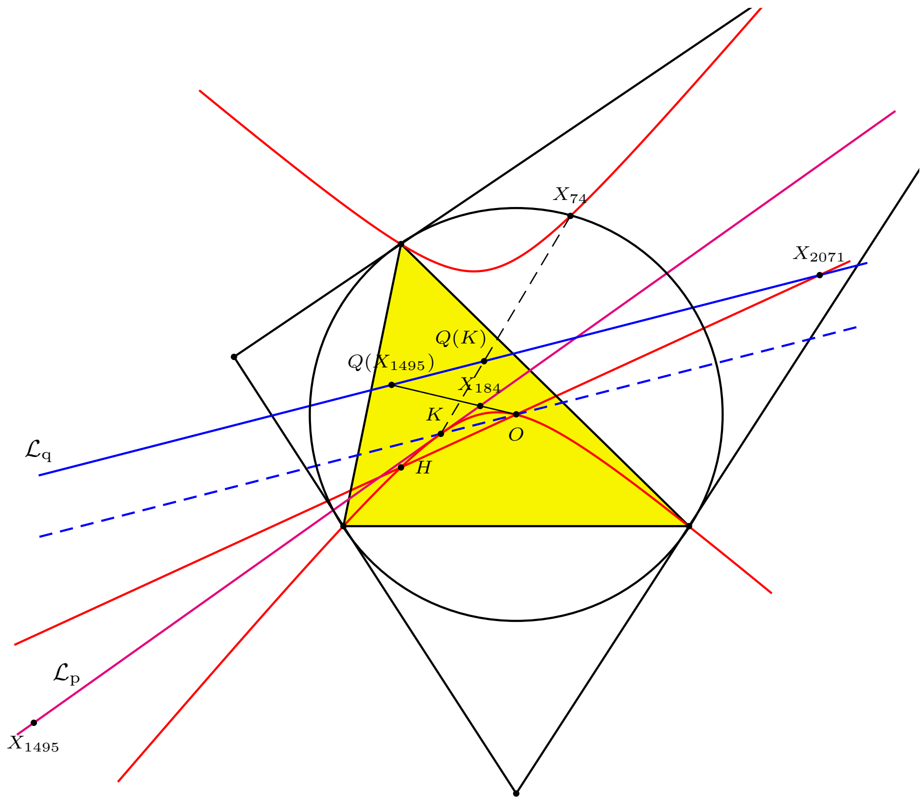

Figure 5. The line $\mathcal{L}_{\rm q}$

The line $\mathcal{L}_{\rm q}$ intersects the Euler line of ABC at the point

$$X_{2071} = (a^2(a^2S_{AAA} + S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - S_{BBCC}) : \cdots : \cdots),$$

[6]This is the point X_{113}.

which is the inverse of the de Longchamps point in the circumcircle. This corresponds to the antiparallels through

$$P_{2071} = (a^4((a^2 S_{AAA} + S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - S_{BBCC}) : \cdots : \cdots)$$

on the line $\mathcal{L}_{\mathrm{p}}$. This point can be constructed by a simple application of Theorem 4 or Proposition 6. (See also Remark 2 following Theorem 12).

3.3. *The intersection of $\mathcal{L}_{\mathrm{p}}$ and $\mathcal{L}_{\mathrm{q}}$.* The lines $\mathcal{L}_{\mathrm{p}}$ and $\mathcal{L}_{\mathrm{q}}$ intersect at the point

$$M = (a^2 S_A(S_{AB} + S_{AC} + S_{BB} - 4S_{BC} + S_{CC}) : \cdots : \cdots).$$

(1) $Q(M)$ is the point on $\mathcal{L}_{\mathrm{q}}$ with coordinates

$$(a^2 S_A(S_{AA}(S_{BB} + S_{CC}) + a^2 S_A(S_{BB} - 3S_{BC} + S_{CC}) + S_{BC}(S_B - S_C)^2) : \cdots : \cdots).$$

(2) The point P on $\mathcal{L}_{\mathrm{p}}$ for which $Q(P) = M$ has coordinates

$$(a^2(a^2(2S_{AA} - S_{BC}) + 2S_A(S_{BB} - 3S_{BC} + S_{CC})) : \cdots : \cdots).$$

4. The triangle $\mathbf{T}$ bounded by the antiparallels

We assume the line ℓ_i, $i = 1, 2, 3$, nonconcurrent so that they bound a nondegenerate triangle $\mathbf{T} = A_1B_1C_1$. Since these lines have equations

$$\begin{aligned}
-t_1 S_A(x + y + z) &= -S_A x + S_B y + S_C z,\\
-t_2 S_B(x + y + z) &= \ \ S_A x - S_B y + S_C z,\\
-t_3 S_C(x + y + z) &= \ \ S_A x + S_B y - S_C z,
\end{aligned}$$

the vertices of $\mathbf{T}$ are the points

$$\begin{aligned}
A_1 =&(-a^2(t_2 S_B + t_3 S_C) : 2S_{CA} + t_2 b^2 S_B + t_3 S_C(S_C - S_A)\\
&: 2S_{AB} + t_2 S_B(S_B - S_A) + t_3 c^2 S_C),\\
B_1 =&(2S_{BC} + t_3 S_C(S_C - S_B) + t_1 a^2 S_A : -b^2(t_3 S_C + t_1 S_A)\\
&: 2S_{AB} + t_3 c^2 S_C + t_1 S_A(S_A - S_B))\\
C_1 =&(2S_{BC} + t_1 a^2 S_A + t_2 S_B(S_B - S_C) : 2S_{CA} + t_1 S_A(S_A - S_C) + t_2 b^2 S_B\\
&: -c^2(t_1 S_A + t_2 S_B)).
\end{aligned}$$

4.1. *Homothety with the orthic triangle* . The triangle $\mathbf{T} = A_1B_1C_1$ is homothetic to the orthic triangle $A'B'C'$. The center of homothety is the point

$$P(\mathbf{T}) = \left(\frac{t_2 S_B + t_3 S_C}{S_A} : \frac{t_3 S_C + t_1 S_A}{S_B} : \frac{t_1 S_A + t_2 S_B}{S_C}\right), \tag{3}$$

and the ratio of homothety is

$$1 + \frac{t_1 a^2 S_{AA} + t_2 b^2 S_{BB} + t_3 c^2 S_{CC}}{2S_{ABC}}.$$

Proposition 7. *If the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent, the homothetic center $P(\mathbf{T})$ of $\mathbf{T}$ and the orthic triangle lies on the line $\mathcal{L}_{\mathrm{p}}$.*

Proof. If we write $P(\mathbf{T}) = (x : y : z)$. From (3), we obtain

$$t_1 = \frac{-xS_A + yS_B + zS_C}{2S_A}, \quad t_2 = \frac{-yS_B + zS_C + xS_A}{2S_B}, \quad t_3 = \frac{-zS_C + xS_A + yS_B}{2S_C}.$$

Substitution in (2) yields the equation of the line $\mathcal{L}_{\rm p}$. □

For example, if $t_1 = t_2 = t_3 = t$, $P(\mathbf{T}) = X_{25} = \left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right)$. [7] If the ratio of homothety is 0, triangle $\mathbf{T}$ degenerates into the point X_{25} on $\mathcal{L}_{\rm p}$. The intersection of $\mathcal{L}_{\rm c}$ and $\mathcal{L}_{\rm q}$ is the point

$$\begin{aligned}Q(X_{25}) =&(a^2 S_A(b^4 S_B^4 + c^4 S_C^4 + a^2 S_{AAA}(S_B - S_C)^2 \\ &- S_{ABC}(4a^2 S_{BC} + 3S_A(S_B - S_C)^2)) : \cdots : \cdots).\end{aligned}$$

Remark. The line $\mathcal{L}_{\rm p}$ is also the locus of the centroid of $\mathbf{T}$ for which the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, concur.

4.2. *Common point of $\mathcal{L}_i$, $i = 1, 2, 3$, on the Brocard axis.* We consider the case when the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, intersect on the Brocard axis. A typical point on the Brocard axis, dividing the segment OK in the ratio $t : 1-t$, has coordinates

$$(a^2(S_A(S_A + S_B + S_C) + (S_{BC} - S_{AA})t) : \cdots : \cdots).$$

This point lies on the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, if and only if we choose

$$\begin{aligned}t_1 =&\frac{-(S_A + S_B + S_C)(S^2 - S_{AA}) + b^2c^2(S_B + S_C - 2S_A)t}{2S_{AA}(S_A + S_B + S_C)},\\ t_2 =&\frac{-(S_A + S_B + S_C)(S^2 - S_{BB}) + c^2a^2(S_C + S_A - 2S_B)t}{2S_{BB}(S_A + S_B + S_C)},\\ t_3 =&\frac{-(S_A + S_B + S_C)(S^2 - S_{CC}) + a^2b^2(S_A + S_B - 2S_C)t}{2S_{CC}(S_A + S_B + S_C)}.\end{aligned}$$

The corresponding triangle $\mathbf{T}$ is homothetic to the orthic triangle at the point

$$(a^2(-(S_A + S_B + S_C) \cdot a^2 S_A + t(-(2S_A + S_B + S_C)S_{BC} + b^2 S_{CA} + c^2 S_{AB}) : \cdots : \cdots),$$

which divides the segment $X_{184}K$ in the ratio $2t : 1-2t$. The ratio of homothety is $-\frac{a^2b^2c^2}{4S_{ABC}}$. These triangles are all directly congruent to the medial tangential triangle of ABC. We summarize this in the following proposition.

Proposition 8. *Corresponding to the family of triangles directly congruent to the medial tangential triangle, homothetic to orthic triangle at points on the line $\mathcal{L}_{\rm p}$, the common points of the Euler lines of $\mathcal{L}_i$, $i = 1, 2, 3$, all lie on the Brocard axis.*

[7] See also §3.1(1). The tangential triangle is $\mathbf{T}$ with $t = 1$.

5. Perspectivity of $\mathbf{T}$ with ABC

Proposition 9. *The triangles* $\mathbf{T}$ *and* ABC *are perspective if and only if*

$$\sum_{\text{cyclic}}(S_B - S_C)(t_1 S_{AA} - t_2 t_3 S_{BC}) = 0. \tag{4}$$

Proof. From the coordinates of the vertices of $\mathbf{T}$, it is straightforward to check that $\mathbf{T}$ and ABC are perspective if and only if

$$t_1 a^2 S_{AA} + t_2 b^2 S_{BB} + t_3 c^2 S_{CC} + 2S_{ABC} = 0$$

or (4) holds. Since the area of triangle $\mathbf{T}$ is

$$\frac{(t_1 a^2 S_{AA} + t_2 b^2 S_{BB} + t_3 c^2 S_{CC} + 2S_{ABC})^2}{a^2 b^2 c^2 S_{ABC}}$$

times that of triangle ABC, we assume $t_1 a^2 S_{AA}+t_2 b^2 S_{BB}+t_3 c^2 S_{CC}+2S_{ABC} \neq 0$ and (4) is the necessary and sufficient condition for perspectivity. $\square$

Theorem 10. *If the triangle* $\mathbf{T}$ *is nondegenerate and is perspective to* ABC*, then the perspector lies on the Jerabek hyperbola of* ABC*.*

Proof. If triangles $A_1B_1C_1$ and ABC are perspective at $P = (x : y : z)$, then

$$A_1 = (u + x : y : z), \quad B_1 = (x : v + y : z), \quad C_1 = (x : y : w + z)$$

for some u, v, w. Since the line B_1C_1 is parallel to $B'C'$, which has infinite point $(S_B - S_C : -(S_C + S_A) : S_A + S_B)$, we have

$$\begin{vmatrix} S_B - S_C & -(S_C + S_A) & S_A + S_B \\ x & y + v & z \\ x & y & z + w \end{vmatrix} = 0,$$

and similarly for the other two lines. These can be rearranged as

$$\begin{aligned}
\frac{(S_C + S_A)x - (S_B - S_C)y}{v} - \frac{(S_B - S_C)z + (S_A + S_B)x}{w} &= S_B - S_C, \\
\frac{(S_A + S_B)y - (S_C - S_A)z}{w} - \frac{(S_C - S_A)x + (S_B + S_C)y}{u} &= S_C - S_A, \\
\frac{(S_B + S_C)z - (S_A - S_B)x}{u} - \frac{(S_A - S_B)y + (S_C + S_A)z}{v} &= S_A - S_B.
\end{aligned}$$

Multiplying these equations respectively by

$$S_A(S_B + S_C)yz, \qquad S_B(S_C + S_A)zx, \qquad S_C(S_A + S_B)xy$$

and adding up, we obtain

$$\left(1 + \frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right) \sum_{\text{cyclic}} S_A(S_{BB} - S_{CC})yz = 0.$$

Since the area of triangle $\mathbf{T}$ is

$$uvw\left(1 + \frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right)$$

times that of triangle ABC, we must have $1 + \frac{x}{u} + \frac{y}{v} + \frac{z}{w} \neq 0$. It follows that

$$\sum_{\text{cyclic}} S_A(S_{BB} - S_{CC})yz = 0.$$

This means that P lies on the Jerabek hyperbola. □

We shall identify the locus of the common points of Euler lines in §8.3 below. In the meantime, we give a construction for the point Q from the perspector on the Jerabek hypebola.

Construction. Given a point P on the Jerabek hyperbola, construct parallels to $A'B'$ and $A'C'$ through an arbitrary point A_1' on the line AP. Let M_1 be the intersection of the Euler lines of the triangles formed by these antiparallels and the sidelines of ABC. With another point A_1'' obtain a point M_2 by the same construction. Similarly, working with two points B_1' and B_1'' on BP, we construct another line M_3M_4. The intersection of M_1M_2 and M_3M_4 is the common point Q of the Euler lines corresponding the antiparallels that bound a triangle perspective to ABC at P.

6. The Tucker hexagons and the line $\mathcal{L}_{\mathrm{t}}$

It is well known that if the antiparallels, together with the sidelines of triangle ABC, bound a Tucker hexagon, the vertices lie on a circle whose center is on the Brocard axis. If this center divides the segment OK in the ratio $t : 1 - t$, the antiparallels pass through the points dividing the symmedians in the same ratio. The vertices of the Tucker hexagon are

$$\begin{aligned}
B_a &= (S_C + (1-t)c^2 : 0 : tc^2), & C_a &= (S_B + (1-t)b^2 : tb^2 : 0),\\
C_b &= (ta^2 : S_A + (1-t)a^2 : 0), & A_b &= (0 : S_C + (1-t)c^2 : tc^2),\\
A_c &= (0 : tb^2 : S_B + (1-t)b^2), & B_c &= (ta^2 : 0 : S_A + (1-t)a^2).
\end{aligned}$$

In this case,

$$1-t_1 = \frac{t \cdot b^2c^2}{S_A(S_A+S_B+S_C)}, \quad 1-t_2 = \frac{t \cdot c^2a^2}{S_B(S_A+S_B+S_C)}, \quad 1-t_3 = \frac{t \cdot a^2b^2}{S_C(S_A+S_B+S_C)}.$$

It is clear that the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent. As t varies, this common point traverses a straight line $\mathcal{L}_{\mathrm{t}}$. We show that this is the line joining K to $Q(K)$.

(1) For $t = 1$, this Tucker circle is the second Lemoine circle with center K, the triangle **T** degenerates into the point K. The common point of the Euler lines is therefore the point $Q(K)$. See §3.2 and Figure 3.

(2) For $t = \frac{3}{2}$, the vertices of the Tucker hexagon are

$$\begin{aligned}
B_a &= (a^2 + b^2 - 2c^2 : 0 : 3c^2), & C_a &= (c^2 + a^2 - 2b^2 : 3b^2 : 0),\\
C_b &= (3a^2 : b^2 + c^2 - 2a^2 : 0), & A_b &= (0 : a^2 + b^2 - 2c^2 : 3c^2),\\
A_c &= (0 : 3b^2 : c^2 + a^2 - 2b^2), & B_c &= (3a^2 : 0 : b^2 + c^2 - 2a^2).
\end{aligned}$$

The triangles $\mathbf{T}_i$, $i = 1, 2, 3$, have a common centroid K, which is therefore the common point of their Euler lines. The corresponding Tucker center is the point X_{576} (which divides OK in the ratio $3 : -1$).

From these, we obtain the equation of the line

$$\mathcal{L}_{\text{t}} : \qquad \sum_{\text{cyclic}} b^2c^2S_A(S_B - S_C)(S_{CA} + S_{AB} - 2S_{BC})x = 0.$$

Remarks. (1) The triangle $\mathbf{T}$ is perspective to ABC at K. See, for example, [5].

(2) The line $\mathcal{L}_{\text{t}}$ also contains X_{74} which we may regard as corresponding to $t = 0$.

For more about Tucker hexagons, see §8.2.

7. Concurrency of four or more Euler lines

7.1. *Common point of $\mathcal{L}_i$, $i = 1, 2, 3$, on the Euler line of ABC.* We consider the case when the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, intersect on the Euler line of ABC. A typical point on the Euler line axis divides the segment OH in the ratio $t : 1 - t$, has coordinates

$$(a^2S_A - (S_{CA} + S_{AB} - 2S_{BC})t) : \cdots : \cdots).$$

This lies on the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, if and only if we choose

$$\begin{aligned} t_1 =& \frac{-(S^2 - S_{AA}) + (S^2 - 3S_{AA})t}{2S_{AA}}, \\ t_2 =& \frac{-(S^2 - S_{BB}) + (S^2 - 3S_{BB})t}{2S_{BB}}, \\ t_3 =& \frac{-(S^2 - S_{CC}) + (S^2 - 3S_{CC})t}{2S_{CC}}. \end{aligned}$$

Independently of t, the corresponding triangle $\mathbf{T}$ is always homothetic to the medial tangential triangle at the point P_{2071} on the line $\mathcal{L}_{\text{p}}$ for which $Q(P_{2071}) = X_{2071}$, the intersection of $\mathcal{L}_{\text{q}}$ with the Euler line. See the end of §3.2 above. The ratio of homothety is $1 + t - \frac{8S_{ABC}}{a^2b^2c^2}t$. We summarize this in the following proposition.

Proposition 11. *Let P_{2071} be the point on $\mathcal{L}_{\text{p}}$ such that $Q(P_{2071}) = X_{2071}$. The Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, corresponding to the sidelines of triangles homothetic at P_{2071} to the medial tangential triangle intersect on the Euler line of ABC.*

7.2. *The line $\mathcal{L}_{\text{f}}$.* The Euler line of triangle $\mathbf{T}$ is the line

$$\begin{aligned} &(x + y + z) \sum_{\text{cyclic}} t_1a^2S_{AA}(S_B - S_C)(S^2 + S_{BC})(S^2 - S_{AA}) \\ =&2S_{ABC} \sum_{\text{cyclic}} (S^2 + S_{CA})(S^2 + S_{AB})x. \end{aligned} \qquad (5)$$

Theorem 12. *The Euler lines of the four triangles* $\mathbf{T}$ *and* $\mathbf{T}_i$*,* $i = 1, 2, 3$*, are concurrent if and only if*

$$t_1 = -\frac{16S^2 \cdot S_{ABC} + t(a^2b^4c^4 - 4S_{ABC}(3S^2 - S_{AA}))}{4S_{AA}(a^2b^2c^2 + 4S_{ABC})},$$
$$t_2 = -\frac{16S^2 \cdot S_{ABC} + t(a^4b^2c^4 - 4S_{ABC}(3S^2 - S_{BB}))}{4S_{BB}(a^2b^2c^2 + 4S_{ABC})},$$
$$t_3 = -\frac{16S^2 \cdot S_{ABC} + t(a^4b^4c^2 - 4S_{ABC}(3S^2 - S_{CC}))}{4S_{CC}(a^2b^2c^2 + 4S_{ABC})},$$

with $t \neq \frac{-24a^2b^2c^2S_{ABC}}{(a^2b^2c^2-8S_{ABC})(3(S_A+S_B+S_C)S^2+S_{ABC})}$. *The locus of the common point of the four Euler lines is the line* $\mathcal{L}_{\mathrm{f}}$ *joining the nine-point center of* ABC *to* X_{184}*, with the intersection with* $\mathcal{L}_{\mathrm{q}}$ *deleted.*

Proof. The equation of the Euler line $\mathcal{L}_i$, $i = 1, 2, 3$, can be rewritten as

$$\begin{aligned} &t_1S_A(S_B - S_C)(x + y + z) + S_{AA}(S_B - S_C)x \\ &+(S_{AB}(S_B - S_C) - (S_{AA} - S_{BB})S_C)y + (S_{AC}(S_B - S_C) + (S_{AA} - S_{CC})S_B)z = 0, \quad (6) \\ &t_2S_A(S_B - S_C)(x + y + z) + S_{BB}(S_C - S_A)y \\ &+(S_{BA}(S_C - S_A) + (S_{BB} - S_{AA})S_C)x + (S_{BC}(S_C - S_A) - (S_{BB} - S_{CC})S_A)z = 0, \quad (7) \\ &t_3S_C(S_A - S_B)(x + y + z) + S_{CC}(S_A - S_B)z \\ &+(S_{CA}(S_A - S_B) - (S_{CC} - S_{AA})S_B)x + (S_{CB}(S_A - S_B) + (S_{CC} - S_{BB})S_A)y = 0. \quad (8) \end{aligned}$$

Multiplying (4), (5), (6) respectively by

$$a^2S_A(S^2+S_{BC})(S^2-S_{AA}), \quad b^2S_B(S^2+S_{CA})(S^2-S_{BB}), \quad c^2S_C(S^2+S_{AB})(S^2-S_{CC}),$$

and adding, we obtain by Theorem 10 the equation of the line

$$\mathcal{L}_{\mathrm{f}}: \qquad \sum_{\text{cyclic}} (S_B - S_C)(S^2(2S_{AA} - S_{BC}) + S_{ABC} \cdot S_A)x = 0$$

which contains the common point of the Euler lines of $\mathbf{T}_i$, $i = 1, 2, 3$, if it also lies on the Euler line $\mathcal{L}$ of $\mathbf{T}$. The line $\mathcal{L}_{\mathrm{f}}$ contains the nine-point center X_5 and $X_{184} = (a^4S_A : b^4S_B : c^4S_C)$. Let Q_t be the point which divides the segment $X_{184}X_5$ in the ratio $t : 1 - t$. It has coordinates

$$\begin{aligned} &((1 - t)4S^2 \cdot a^4S_A + t(a^2b^2c^2 + 4S_{ABC})(S_{CA} + S_{AB} + 2S_{BC}) \\ &: (1 - t)4S^2 \cdot b^4S_B + t(a^2b^2c^2 + 4S_{ABC})(2S_{CA} + S_{AB} + S_{BC}) \\ &: (1 - t)4S^2 \cdot c^4S_C + t(a^2b^2c^2 + 4S_{ABC})(S_{CA} + 2S_{AB} + S_{BC})). \end{aligned}$$

The point Q_t lies on the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, respectively if we choose t_1, t_2, t_3 given above.

If Q lies on $\mathcal{L}_{\rm q}$, then $Q_t = Q(P)$ for some point P on $\mathcal{L}_{\rm p}$. [8] In this case, the triangle $\mathbf{T}$ degenerates into the point $P \neq Q$ and its Euler line is not defined. It should be excluded from $\mathcal{L}_{\rm f}$. The corresponding value of t is as given in the statement above. $\square$

Here are some interesting points on $\mathcal{L}_{\rm f}$.

(1) For $t = 0$, $\mathbf{T}$ is perspective with ABC at X_{74}, and the common point of the four Euler lines is X_{184}. The antiparallels are drawn through the intercepts of the trilinear polars of $X_{186} = \left(\frac{a^2}{S_A(S^2-3S_{AA})} : \cdots : \cdots\right)$, the inversive image of the orthocenter in the circumcircle.

(2) For $t = 1$, this common point is the nine-point of triangle ABC. The triangle $\mathbf{T}$ is homothetic to the orthic triangle at X_{51} and to the medial tangential triangle at the point P_{2071} in §3.2.

(3) $t = -\frac{a^2b^2c^2}{4S_{ABC}}$ gives X_{156}, the nine-point center of the tangential triangle.

In these two cases, we have the concurrency of five Euler lines.

(4) The line $\mathcal{L}_{\rm f}$ intersects the Brocard axis at X_{569}. This corresponds to $t = \frac{2a^2b^2c^2}{3a^2b^2c^2+4S_{ABC}}$.

Proposition 13. *The triangle $\mathbf{T}$ is perspective with ABC and its Euler line contains the common point of the Euler lines of $\mathbf{T}_i$, $i = 1, 2, 3$ precisely in the following three cases.*

(1) $t = 0$, *with perspector X_{74} and common point of Euler line X_{184}.*

(2) $t = \frac{-12a^2b^2c^2S_{ABC}}{a^4b^4c^4-12a^2b^2c^2S_{ABC}-16(S_{ABC})^2}$, *with perspector K.*

Remarks. (1) In the first case,

$$t_1 = \frac{k}{S_{AA}}, \qquad t_2 = \frac{k}{S_{BB}}, \qquad t_3 = \frac{k}{S_{CC}}$$

for $k = -\frac{4S^2 \cdot S_{ABC}}{a^2b^2c^2+4S_{ABC}}$. The antiparallels pass through the intercepts of the trilinear polar of X_{186}, the inversive image of H in the circumcircle.

(2) In the second case, the antiparallels bound a Tucker hexagon. The center of the Tucker circle divides OK in the ratio $t : 1-t$, where

$$t = \frac{S^2(S_A+S_B+S_C)(a^2b^2c^2-16S_{ABC})}{a^4b^4c^4-12a^2b^2c^2S_{ABC}-16(S_{ABC})^2}.$$

It follows that the common point of the Euler lines is the intersection of the lines $\mathcal{L}_{\rm f} = X_5X_{184}$ and $\mathcal{L}_{\rm t}$.

8. Common points of $\mathcal{L}_i$, $i = 1, 2, 3$, when T is perspective

If the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent, then, according to (2) we may put

$$t_1 = \frac{k(\lambda+S_A)}{a^2S_{AA}}, \qquad t_2 = \frac{k(\lambda+S_B)}{b^2S_{BB}}, \qquad t_3 = \frac{k(\lambda+S_C)}{c^2S_{CC}}$$

[8] This point is the intersection of $\mathcal{L}_{\rm p}$ with the line joining the Jerabek center J to X_{323}, the reflection in X_{110} of the inversive image of the centroid in the circumcircle.

for some λ and k. If, also, the $\mathbf{T}$ is perspective, (4) gives

$$k(k\lambda + S_{ABC})(\lambda + S_A + S_B + S_C)(k(3\lambda + S_A + S_B + S_C) + 2S_{ABC}) = 0.$$

If $k = 0$, $\mathbf{T}$ is the orthic triangle. We consider the remaining three cases below.

8.1. *The case* $k(S_A + S_B + S_C + 3\lambda) + 2S_{ABC} = 0$. In this case,

$$t_1 = -\frac{2S_{ABC} + k(S_B + S_C - 2S_A)}{3a^2 S_{AA}},$$
$$t_2 = -\frac{2S_{ABC} + k(S_C + S_A - 2S_B)}{3b^2 S_{BB}},$$
$$t_3 = -\frac{2S_{ABC} + k(S_A + S_B - 2S_C)}{3c^2 S_{CC}}.$$

The antiparallels are concurrent.

8.2. *The case* $k\lambda + S_{ABC} = 0$. In this case,

$$t_1 = \frac{k - S_{BC}}{a^2 S_A}, \qquad t_2 = \frac{k - S_{CA}}{b^2 S_B}, \qquad t_3 = \frac{k - S_{AB}}{c^2 S_C}.$$

In this case, the perspector is the Lemoine point K. The antiparallels bound a Tucker hexagon. The locus of the common point of Euler lines is the line $\mathcal{L}_{\mathrm{t}}$. Here are some more interesting points on this line.

(1) For $k = 0$, we have

$$t_1 = -\frac{S_{BC}}{S_A(S_B + S_C)}, \qquad t_2 = -\frac{S_{CA}}{S_B(S_C + S_A)}, \qquad t_3 = -\frac{S_{AB}}{S_C(S_A + S_B)}.$$

This gives the Tucker hexagon with vertices

$$\begin{array}{ll} B_a = (S_{CC} : 0 : S^2), & C_a = (S_{BB} : S^2 : 0), \\ C_b = (S^2 : S_{AA} : 0), & A_b = (0 : S_{CC} : S^2), \\ A_c = (0 : S^2 : S_{BB}), & B_c = (S^2 : 0 : S_{AA}). \end{array}$$

These are the pedals of A', B', C' on the sidelines. The Tucker circle is the Taylor circle. The triangle $\mathbf{T}$ is the medial triangle of the orthic triangle. The corresponding Euler lines intersect at X_{974}, which is the intersection of $\mathcal{L}_{\mathrm{t}} = KX_{74}$ with X_5X_{125}. See [2].

(2) For $k = \frac{S_{ABC}}{S_A+S_B+S_C}$, we have

$$t_1 = -\frac{S_{BC}}{S_A(S_A + S_B + S_C)}, \qquad t_2 = -\frac{S_{CA}}{S_B(S_A + S_B + S_C)}, \qquad t_3 = -\frac{S_{AB}}{S_C(S_A + S_B + S_C)}.$$

The Tucker circle is the second Lemoine circle, considered in §6.

(3) The line $\mathcal{L}_{\mathrm{t}}$ intersects the Euler line at

$$X_{378} = \left(\frac{a^2(S^2 + 3S_{AA})}{S_A} : \cdots : \cdots\right).$$

The corresponding Tucker circle has center

$$(S^2(S_B+S_C)(S_C-S_A)(S_A-S_B)+3(S_A+S_B)(S_B+S_C)(S_C+S_A)S_{BC}:\cdots:\cdots)$$

which is the intersection of the Brocard axis and the line joining the orthocenter to X_{110}.

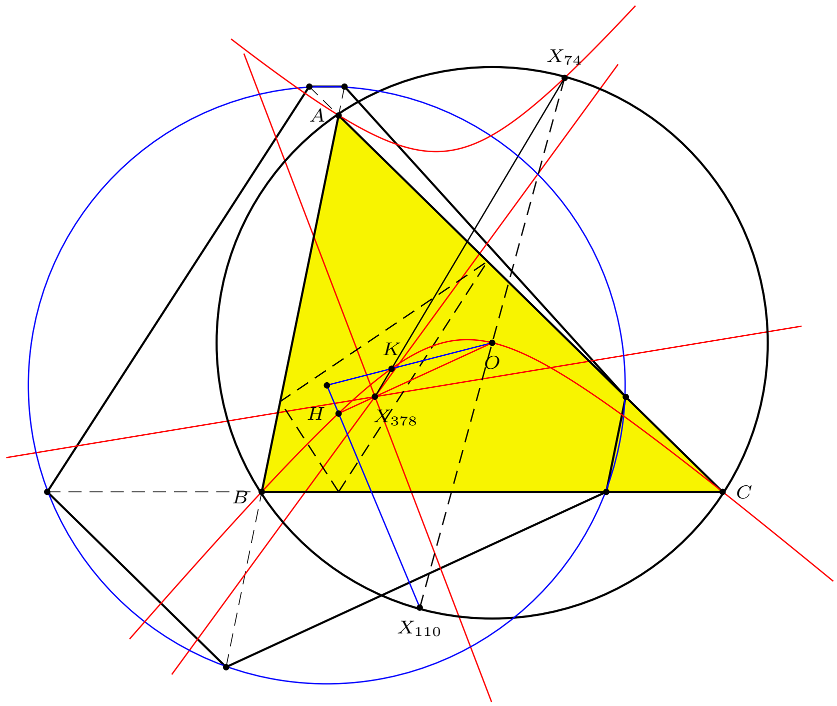

Figure 6. Intersection of 4 Euler lines at X_{378}

8.3. *The case* $\lambda = -(S_A+S_B+S_C)$. In this case, we have

$$t_1 = -\frac{k}{S_{AA}}, \qquad t_2 = -\frac{k}{S_{BB}}, \qquad t_3 = -\frac{k}{S_{CC}}.$$

In this case, the perspector is the point

$$\left(\frac{1}{2S_{ABC}\cdot S_A - k(b^2c^2-2S_{BC})}:\cdots:\cdots\right)$$

on the Jerabek hyperbola. If the point on the Jerabek hyperbola is the isogonal conjugate of the point which divides OH in the ratio $t:1-t$, then

$$k = \frac{4tS^2\cdot S_{ABC}}{a^2b^2c^2(1+t)+4t\cdot S_{ABC}}.$$

The locus of the intersection of the Euler lines $\mathcal{L}_i$, $i=1,2,3$, is clearly a line. Since this intersection is the Jerabek center for $k=0$ (Thébault's theorem) and the centroid for $k = \frac{S^2}{3}$, this is the line

$$\mathcal{L}_{\mathrm{r}}: \qquad \sum_{\text{cyclic}} (S_B - S_C)(S_{BC} - S_{AA})x = 0.$$

This line also contains, among other points, X_{110} and X_{184}. We summarize the general situation in the following theorem.

Theorem 14. *Let P be a point on the Euler line other than the centroid G. The antiparallels through the intercepts of the trilinear polar of P bound a triangle perspective with ABC (at a point on the Jerabek hyperbola). The Euler lines of the triangles $\mathbf{T}_i$, $i = 1, 2, 3$, are concurrent (at a point Q on the line L_{r} joining the centroid G to X_{110}).*

Here are some interesting examples with P easily constructed on the Euler line.

P	Perspector	Q
H	H	X_{125}
O	$X_{64} = X_{20}^*$	X_{110}
X_{30}	X_{2071}^*	G
X_{186}	X_{74}	X_{184}
X_{403}	$X_{265} = X186^*$	X_{1899}
X_{23}	$X_{1177} = X_{858}^*$	X_{182}
X_{858}		X_{1352}
X_{1316}		X_{98}

Remarks. (1) X_{186} is the inversive image of H in the circumcircle.
(2) X_{403} is the midpoint between H and X_{186}.
(3) X_{23} is the inversive image of G in the circumcircle.
(4) X_{858} is the inferior of X_{23}.
(5) X_{182} is the midpoint of OK, the center of the Brocard circle.
(6) X_{1352} is the reflection of K in the nine-point center.
(7) X_{1316} is the intersection of the Euler line and the Brocard circle apart from O.

9. Two loci: a line and a cubic

We conclude this paper with a brief discussion on two locus problems.

9.1. *Antiparallels through the vertices of a pedal triangle.* Suppose the antiparallels ℓ_i, $i = 1, 2, 3$, are constructed through the vertices of the pedal triangle of a finite point P. Then the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent if and only if P lies on the line

$$\sum_{\text{cyclic}} S_A(S_B - S_C)(S_{AA} - S_{BC})x = 0.$$

This is the line containing H and the Tarry point X_{98}. For $P = H$, the common point of the Euler line is

$$X_{185} = (a^2 S_A(S_A(S_{BB} + S_{CC}) + a^2 S_{BC}) : \cdots : \cdots).$$

9.2. *Antiparallels through the vertices of a cevian triangle.* If, instead, the antiparallels ℓ_i, $i = 1, 2, 3$, are constructed through the vertices of the cevian triangle of P, then the locus of P for which the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, are concurrent is the cubic

$$\mathcal{K}: \quad \frac{S_A+S_B+S_C}{S_{ABC}}xyz + \sum_{\text{cyclic}} \frac{x}{S_A(S_B-S_C)}\left(\frac{S_A+S_B}{S_C}y^2 - \frac{S_C+S_A}{S_B}z^2\right) = 0.$$

This can also be written in the form

$$\begin{aligned}&(\sum_{\text{cyclic}}(S_B+S_C)yz)(\sum_{\text{cyclic}} S_A(S_B-S_C)(S_B+S_C-S_A)x)\\ =&(\sum_{\text{cyclic}} S_A(S_B-S_C)x)(\sum_{\text{cyclic}} S_A(S_B+S_C)yz).\end{aligned}$$

From this, we obtain the following points on $\mathcal{K}$:

- the orthocenter H (as the intersection of the Euler line and the line $\sum_{\text{cyclic}} S_A(S_B-S_C)(S_B+S_C-S_A)x) = 0$),
- the Euler reflection point X_{110} (as the "fourth" intersection of the circumcircle and the circumconic $\sum_{\text{cyclic}} S_A(S_B+S_C)yz = 0$ with center K),
- the intersections of the Euler line with the circumcircle, the points X_{1113} and X_{1114}.

Corresponding to $P = X_{110}$, the Euler lines $\mathcal{L}_i$, $i = 1, 2, 3$, intersect at the circumcenter O. On the other hand, X_{1113} and X_{1114} are the points

$$(a^2S_A + \lambda(S_{CA} + S_{AB} - 2S_{BC}) : \cdots : \cdots)$$

for $\lambda = -\frac{abc}{\sqrt{a^2b^2c^2-8S_{ABC}}}$ and $\lambda = \frac{abc}{\sqrt{a^2b^2c^2-8S_{ABC}}}$ respectively. The antiparallels through the traces of each of these points correspond to

$$t_1 = t_2 = t_3 = \frac{\lambda-1}{\lambda+1}.$$

This means that the corresponding intersections of Euler lines lie on the line $\mathcal{L}_c = HJ$ in §2.2.

9.3. *The cubic $\mathcal{K}$.* The infinite points of the cubic $\mathcal{K}$ can be found by rewriting the equation of $\mathcal{K}$ in the form

$$\begin{aligned}&(\sum_{\text{cyclic}} S_A(S_B-S_C)(S_B+S_C)yz)(\sum_{\text{cyclic}}(S_B+S_C)x)\\ =&(x+y+z)(\sum_{\text{cyclic}}(S_B+S_C)(S_B-S_C)(S_A(S_A+S_B+S_C)-S_{BC})yz)\end{aligned}$$

They are the infinite points of the Jerabek hyperbola and the line $(S_B+S_C)x + (S_C+S_A)y + (S_A+S_B)z = 0$. The latter is $X_{523} = (S_B - S_C : S_C - S_A : S_A - S_B)$. The asymptotes of $\mathcal{K}$ are

- the parallels to the asymptotes of Jerabek hyperbola through the antipode the Jerabek center on the nine-point circle, *i.e.*,

$$X_{113} = ((S_{CA}+S_{AB}-2S_{BC})(b^2S_{BB}+c^2S_{CC}-a^2S_{AA}-2S_{ABC}) : \cdots : \cdots),$$

- the perpendicular to the Euler line (of ABC) at the circumcenter O, intersecting $\mathcal{K}$ again at

$$Z = \left(\frac{S_{CA}+S_{AB}-2S_{BC}}{b^2S_{BB}+c^2S_{CC}-a^2S_{AA}-2S_{ABC}} : \cdots : \cdots\right),$$

which also lies on the line joining H to X_{110}. See Figure 7. [9]

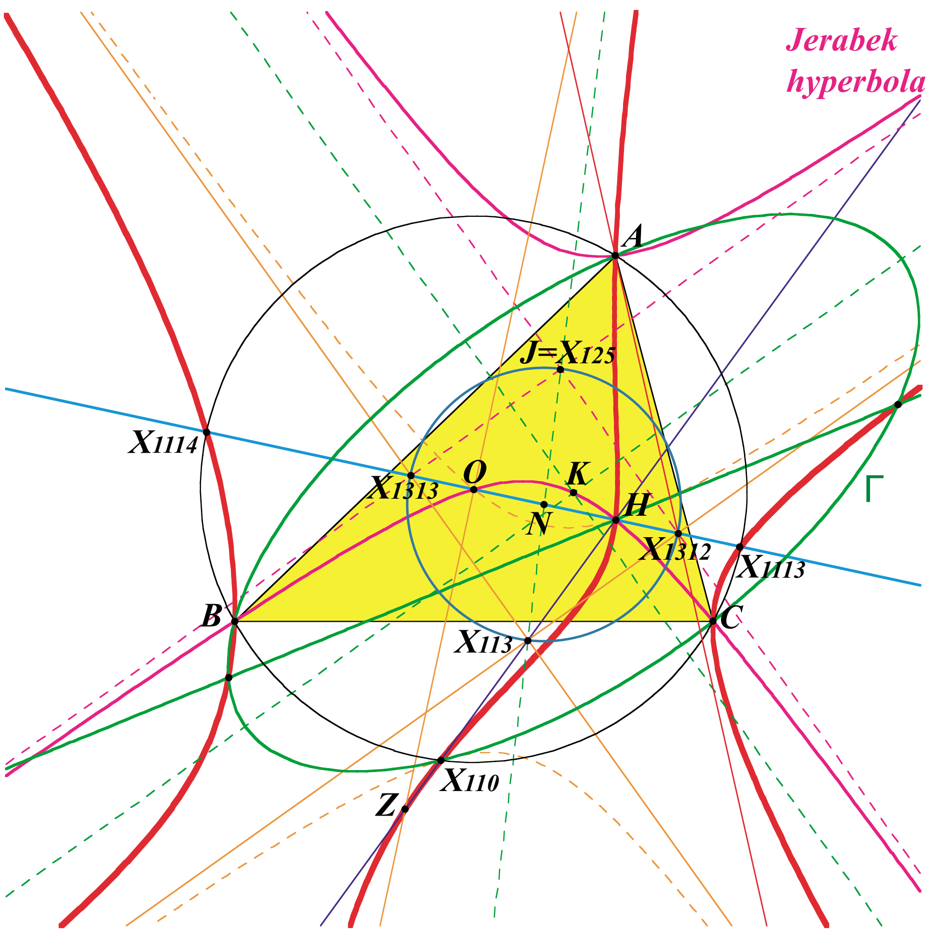

Figure 7. The cubic $\mathcal{K}$

[9] We thank Bernard Gibert for providing the sketch of $\mathcal{K}$ in Figure 7.

Remark. The asymptotes of $\mathcal{K}$ and the Jerabek hyperbola bound a rectangle inscribed in the nine-point circle. Two of the vertices and $J = X_{125}$ and its antipode X_{113}. The other two are the points X_{1312} and X_{1313} on the Euler line.

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Another 5-step Division of a Segment in the Golden Section

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Abstract. We give one more 5-step division of a segment into golden section, using ruler and compass only.

Inasmuch as we have given in [1, 2] 5-step constructions of the golden section we present here another very simple method using ruler and compass only. It is fascinating to discover how simple the golden section appears. For two points P and Q, we denote by $P(Q)$ the circle with P as center and PQ as radius.

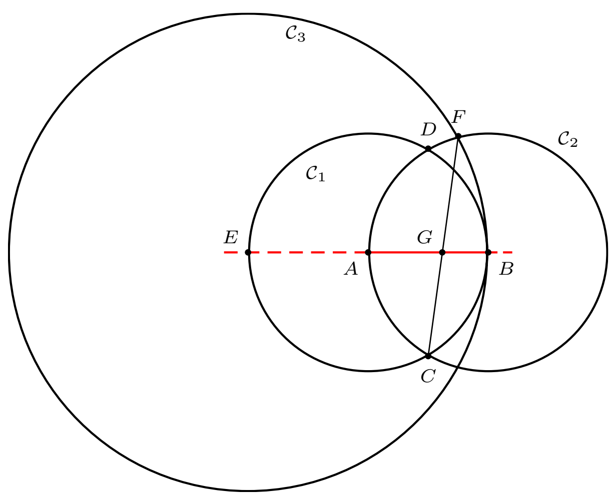

Figure 1

Construction. Given a segment AB, construct

(1) $\mathcal{C}_1 = A(B)$,
(2) $\mathcal{C}_2 = B(A)$, intersecting $\mathcal{C}_1$ at C and D,
(3) the line AB to intersect $\mathcal{C}_1$ at E (apart from B),
(4) $\mathcal{C}_3 = E(B)$ to intersect $\mathcal{C}_2$ at F (so that C and F are on opposite sides of AB),
(5) the segment CF to intersect AB at G.

The point G divides the segment AB in the golden section.

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Proof. Suppose AB has unit length. It is enough to show that $AG = \frac{1}{2}(\sqrt{5}-1)$.

Extend BA to intersect $\mathcal{C}_3$ at H. Let CD intersect AB at I, and let J be the orthogonal projection of F on AB. In the right triangle HFB, $BH = 4$, $BF = 1$. Since $BF^2 = BJ \times BH$, $BJ = \frac{1}{4}$. Therefore, $IJ = \frac{1}{4}$. It also follows that $JF = \frac{1}{4}\sqrt{15}$.

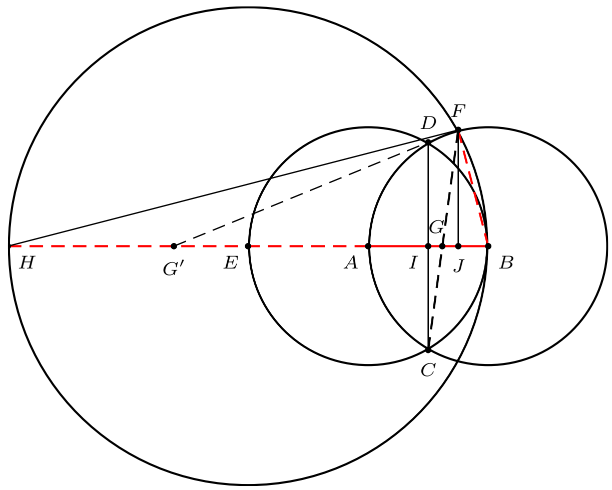

Figure 2

Now, $\frac{IG}{GJ} = \frac{IC}{JF} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{4}\sqrt{15}} = \frac{2}{\sqrt{5}}$. It follows that $IG = \frac{2}{\sqrt{5}+2} \cdot IJ = \frac{\sqrt{5}-2}{2}$, and $AG = \frac{1}{2} + IG = \frac{\sqrt{5}-1}{2}$. This shows that G divides AB in the golden section. □

Remark. If FD is extended to intersect AH at G', then G' is such that $G'A : AB = \frac{1}{2}(\sqrt{5}+1) : 1$.

After the publication of [2], Dick Klingens and Marcello Tarquini have kindly written to point out that the same construction had appeared in [3, p.51] and [4, S.37] almost one century ago.

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Extreme Areas of Triangles in Poncelet's Closure Theorem

Mirko Radić

Abstract. Among the triangles with the same incircle and circumcircle, we determine the ones with maximum and miniumum areas. These are also the ones with maximum and minimum perimeters and sums of altitudes.

Given two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ of radii r and R whose centers are at a distance d apart satisfying Euler's relation

$$R^2 - d^2 = 2Rr, \tag{1}$$

by Poncelet's closure theorem, for every point A_1 on the circle $\mathcal{C}_2$, there is a triangle $A_1A_2A_3$ with incircle $\mathcal{C}_1$ and circumcircle $\mathcal{C}_2$. In this article we determine those triangles with extreme areas, perimeters, and sum of altitudes.

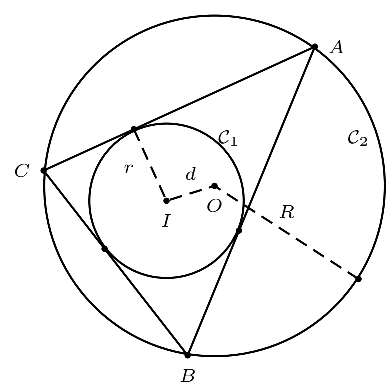

Figure 1a

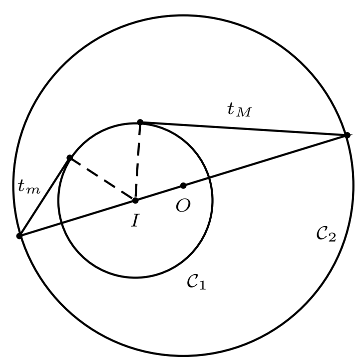

Figure 1b

Denote by t_m and t_M respectively the lengths of the shortest and longest tangents that can be drawn from C_2 to C_1. These are given by

$$t_m = \sqrt{(R-d)^2 - r^2}, \qquad t_M = \sqrt{(R+d)^2 - r^2}. \tag{2}$$

We shall use the following result given in [2, Theorem 2.2]. Let t_1 be any given length satisfying

$$t_m \leq t_1 \leq t_M, \tag{3}$$

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and let t_2 and t_3 be given by

$$t_2 = \frac{2Rrt_1 + \sqrt{D}}{r^2 + t_1^2}, \qquad t_3 = \frac{2Rrt_1 - \sqrt{D}}{r^2 + t_1^2}, \tag{4}$$

where

$$D = 4R^2r^2t_1^2 - r^2(r^2 + t_1^2)(4Rr + r^2 + t_1^2).$$

Then there is a triangle $A_1A_2A_3$ with incircle $\mathcal{C}_1$ and circumcircle $\mathcal{C}_2$ with side lengths

$$a_i = |A_iA_{i+1}| = t_i + t_{i+1}, \qquad i = 1, 2, 3. \tag{5}$$

Here, the indices are taken modulo 3. It is easy to check that

$$\begin{aligned}(t_1 + t_2 + t_3)r^2 =& t_1t_2t_3,\\ t_1t_2 + t_2t_3 + t_3t_1 =& 4Rr + r^2,\end{aligned}$$

and that these are necessary and sufficient for C_1 and C_2 to be the incircle and circumcircle of triangle $A_1A_2A_3$.

Denote by $J(t_1)$ the area of triangle $A_1A_2A_3$. Thus,

$$J(t_1) = r(t_1 + t_2 + t_3). \tag{6}$$

Note that $D = 0$ when $t_1 = t_m$ or $t_1 = t_M$. In these cases,

$$t_2 = t_3 = \begin{cases} \frac{2Rrt_m}{r^2+t_m^2}, & \text{if } t_1 = t_m,\\ \\ \frac{2Rrt_M}{r^2+t_M^2}, & \text{if } t_1 = t_M.\end{cases}$$

For convenience, we shall write

$$\widehat{t_m} = \frac{2Rrt_m}{r^2 + t_m^2} \quad \text{and} \quad \widehat{t_M} = \frac{2Rrt_M}{r^2 + t_M^2}. \tag{7}$$

Theorem 1. *$J(t_1)$ is maximum when $t_1 = t_M$ and minimum when $t_1 = t_m$. In other words, $J(t_m) \le J(t_1) \le J(t_M)$ for $t_m \le t_1 \le t_M$.*

Proof. It follows from (6) and (4) that

$$J(t_1) = r\left(t_1 + \frac{4Rrt_1}{r^2 + t_1^2}\right).$$

From $\frac{d}{dt_1}J(t_1) = 0$, we obtain the equation

$$t_1^4 - 2(2Rr - r^2)t_1^2 + 4Rr^3 + r^4 = 0,$$

and

$$t_1^2 = 2Rr - r^2 \pm 2r\sqrt{R^2 - 2Rr} = 2Rr - r^2 \pm 2rd.$$

Since $4R^2r^2 = (R^2 - d^2)^2$, we have

$$
\begin{aligned}
&2Rr - r^2 + 2rd - \widehat{t_m}^2 \\
=&2Rr - r^2 + 2rd - \frac{(R+d)^2((R-d)^2 - r^2)}{(R-d)^2} \\
=&\frac{(R-d)^2(2Rr - r^2 + 2rd) - (R+d)^2((R-d)^2 - r^2)}{(R-d)^2} \\
=&\frac{((R+d)^2 - (R-d)^2)r^2 + 2r(R+d)(R-d)^2 - (R^2-d^2)^2}{(R-d)^2} \\
=&\frac{4Rdr^2 + 2r(R-d)(2Rr) - (2Rr)^2}{(R-d)^2} \\
=&0.
\end{aligned}
$$

Similarly, $2Rr - r^2 - 2rd - \widehat{t_M}^2 = 0$. It follows that $\frac{d}{dt_1}J(t_1) = 0$ for $t_1 = \widehat{t_m}, \widehat{t_M}$. The maximum of J occurs at $t_1 = t_M$ and $\widehat{t_M}$ while the minimum occurs at $t_1 = t_m$ and $\widehat{t_m}$.

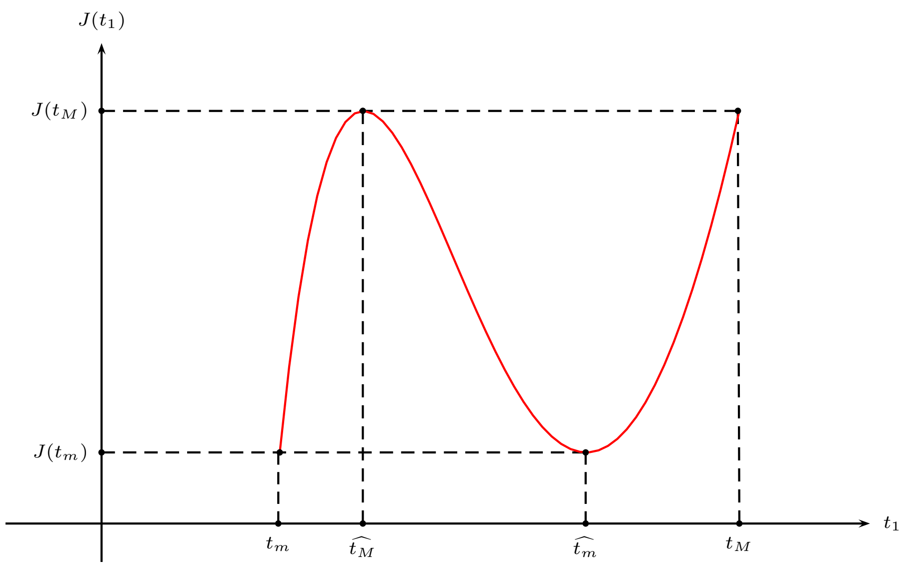

Figure 2

The triangle determined by $\widehat{t_m}$ (respectively $\widehat{t_M}$) is exactly the one determined by t_m (respectively t_M).

$\square$

We conclude with an interesting corollary. Let h_1, h_2, h_3 be the altitudes of the triangle $A_1A_2A_3$. Since

$$2R(h_1 + h_2 + h_3) = a_1a_2 + a_2a_3 + a_3a_1 = (t_1 + t_2 + t_3)^2 + 4Rr + r^2,$$

the following are equivalent:

- the triangle has maximum (respectively minimum) area,
- the triangle has maximum (respectively minimum) perimeter,
- the triangle has maximum (respectively minimum) sum of altitudes.

It follows that these are precisely the two triangles determined by t_M and t_m.

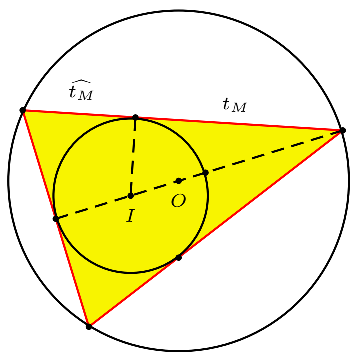

Figure 3a

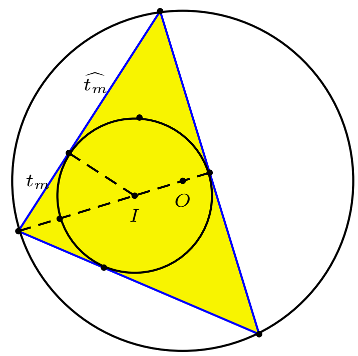

Figure 3b

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The Archimedean Circles of Schoch and Woo

Hiroshi Okumura and Masayuki Watanabe

Abstract. We generalize the Archimedean circles in an arbelos (shoemaker's knife) given by Thomas Schoch and Peter Woo.

1. Introduction

Let three semicircles α, β and γ form an arbelos, where α and β touch externally at the origin O. More specifically, α and β have radii $a, b > 0$ and centers $(a, 0)$ and $(-b, 0)$ respectively, and are erected in the upper half plane $y \geq 0$. The y-axis divides the arbelos into two curvilinear triangles. By a famous theorem of Archimedes, the inscribed circles of these two curvilinear triangles are congruent and have radii $r = \frac{ab}{a+b}$. See Figure 1. These are called the twin circles of Archimedes. Following [2], we call circles congruent to these twin circles Archimedean circles.

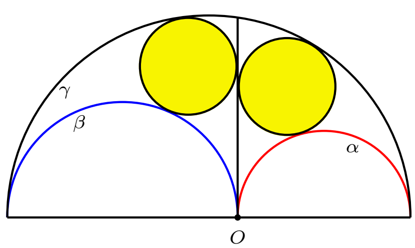

Figure 1

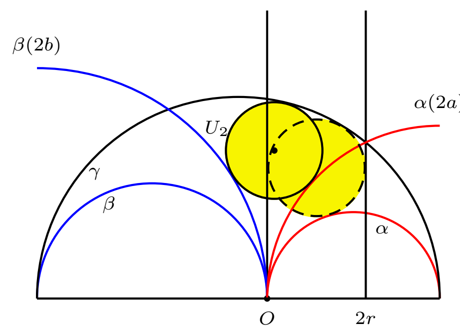

Figure 2

For a real number n, denote by $\alpha(n)$ the semicircle in the upper half-plane with center $(n, 0)$, touching α at O. Similarly, let $\beta(n)$ be the semicircle with center $(-n, 0)$, touching β at O. In particular, $\alpha(a) = \alpha$ and $\beta(b) = \beta$. T. Schoch has found that

(1) the distance from the intersection of $\alpha(2a)$ and γ to the y-axis is $2r$, and
(2) the circle U_2 touching γ internally and each of $\alpha(2a)$, $\beta(2b)$ externally is Archimedean. See Figure 2.

P. Woo considered the Schoch line $\mathcal{L}_s$ through the center of U_2 parallel to the y-axis, and showed that for every nonnegative real number n, the circle U_n with center on $\mathcal{L}_s$ touching $\alpha(na)$ and $\beta(nb)$ externally is also Archimedean. See Figure 3. In this paper we give a generalization of Schoch's circle U_2 and Woo's circles U_n.

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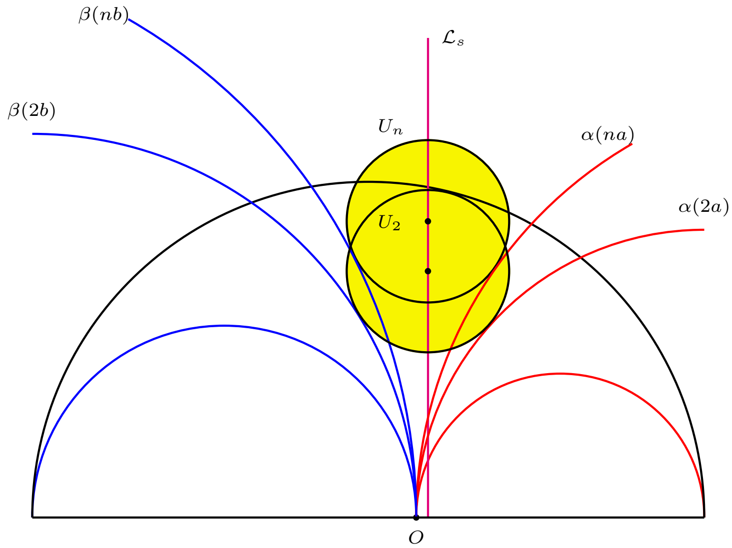

Figure 3

2. A generalization of Schoch's circle U_2

Let a' and b' be real numbers. Consider the semicircles $\alpha(a')$ and $\beta(b')$. Note that $\alpha(a')$ touches α internally or externally according as $a' > 0$ or $a' < 0$; similarly for $\beta(b')$ and β. We assume that the image of $\alpha(a')$ lies on the right side of the image of $\beta(b')$ when these semicircles are inverted in a circle with center O. Denote by $\mathcal{C}(a', b')$ the circle touching γ internally and each of $\alpha(a')$ and $\beta(b')$ at a point different from O.

Theorem 1. *The circle $\mathcal{C}(a', b')$ has radius $\frac{ab(a'+b')}{aa'+bb'+a'b'}$.*

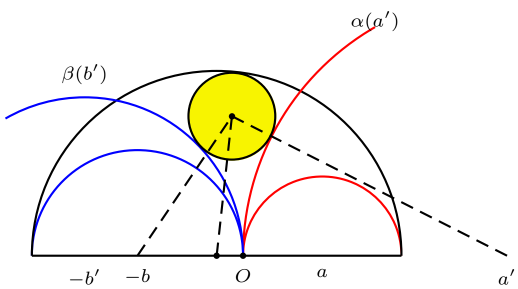

Figure 4a

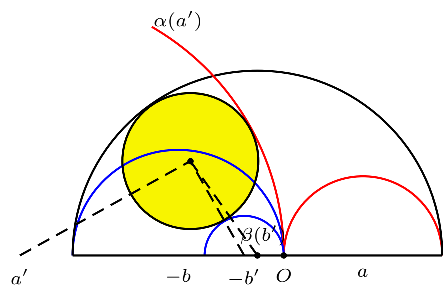

Figure 4b

Proof. Let x be the radius of the circle touching γ internally and also touching $\alpha(a')$ and $\beta(b')$ each at a point different from O. There are two cases in which this circle touches both $\alpha(a')$ and $\beta(b')$ externally (see Figure 4a) or one internally and the other externaly (see Figure 4b). In any case, we have

$$\frac{(a-b+b')^2+(a+b-x)^2-(b'+x)^2}{2(a-b+b')(a+b-x)}$$
$$=-\frac{(a'-(a-b))^2+(a+b-x)^2-(a'+x)^2}{2(a'-(a-b))(a+b-x)},$$

by the law of cosines. Solving the equation, we obtain the radius given above. □

Note that the radius $r=\frac{ab}{a+b}$ of the Archimedean circles can be obtained by letting $a'=a$ and $b'\to\infty$, or $a'\to\infty$ and $b'=b$.

Let $P(a')$ be the external center of similitude of the circles γ and $\alpha(a')$ if $a'>0$, and the internal one if $a'<0$, regarding the two as complete circles. Define $P(b')$ similarly.

Theorem 2. *The two centers of similitude $P(a')$ and $P(b')$ coincide if and only if*

$$\frac{a}{a'}+\frac{b}{b'}=1. \tag{1}$$

Proof. If the two centers of similitude coincide at the point $(t,0)$, then by similarity,

$$a':t-a'=a+b:t-(a-b)=b':t+b'.$$

Eliminating t, we obtain (1). The converse is obvious by the uniqueness of the figure.

□

From Theorems 1 and 2, we obtain the following result.

Theorem 3. *The circle $\mathcal{C}(a',b')$ is an Archimedean circle if and only if $P(a')$ and $P(b')$ coincide.*

When both a' and b' are positive, the two centers of similitude $P(a')$ and $P(b')$ coincide if and only if the three semicircles $\alpha(a')$, $\beta(b')$ and γ share a common external tangent. Hence, in this case, the circle $\mathcal{C}(a',b')$ is Archimedean if and only if $\alpha(a')$, $\beta(b')$ and γ have a common external tangent. Since $\alpha(2a)$ and $\beta(2b)$ satisfy the condition of the theorem, their external common tangent also touches γ. See Figure 5. In fact, it touches γ at its intersection with the y-axis, which is the midpoint of the tangent. The original twin circles of Archimedes are obtained in the limiting case when the external common tangent touches γ at one of the intersections with the x-axis, in which case, one of $\alpha(a')$ and $\beta(b')$ degenerates into the y-axis, and the remaining one coincides with the corresponding α or β of the arbelos.

Corollary 4. *Let m and n be nonzero real numbers. The circle $\mathcal{C}(ma,nb)$ is Archimedean if and only if*

$$\frac{1}{m}+\frac{1}{n}=1.$$

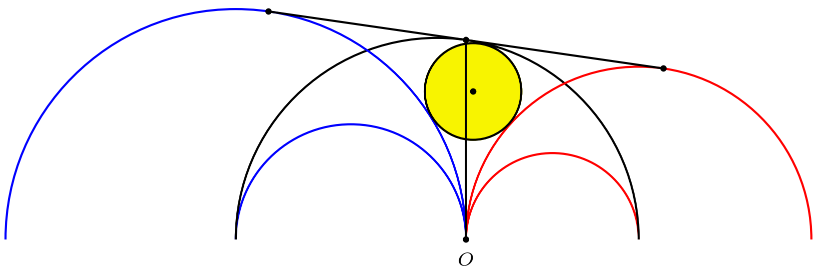

Figure 5

3. Another characterizaton of Woo's circles

The center of the Woo circle U_n is the point

$$\left(\frac{b-a}{b+a}r,\ 2r\sqrt{n+\frac{r}{a+b}}\right). \tag{2}$$

Denote by $\mathcal{L}$ the half line $x = 2r,\ y \geq 0$. This intersects the circle $\alpha(na)$ at the point

$$\left(2r,\ 2\sqrt{r(na-r)}\right). \tag{3}$$

In what follows we consider β as the complete circle with center $(-b, 0)$ passing through O.

Theorem 5. *If T is a point on the line $\mathcal{L}$, then the circle touching the tangents of β through T with center on the Schoch line $\mathcal{L}_s$ is an Archimedean circle.*

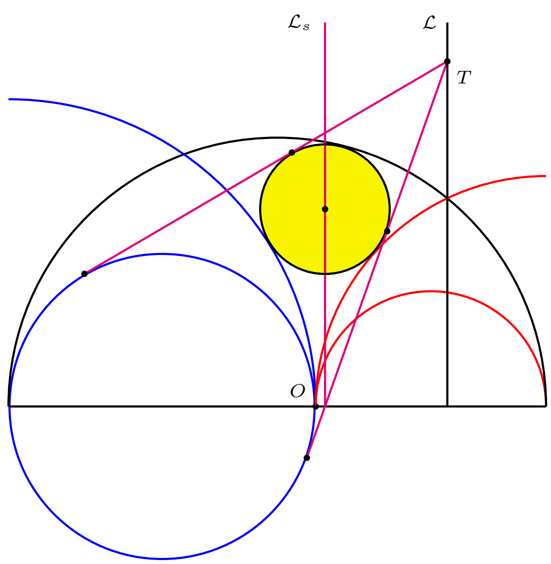

Figure 6

Proof. Let x be the radius of this circle. By similarity (see Figure 6),

$$b + 2r : b = 2r - \frac{b-a}{b+a}r : x.$$

From this, $x = r$. $\square$

The set of Woo circles is a proper subset of the set of circles determined in Theorem 5 above. The external center of similitude of U_n and β has y-coordinate

$$2a\sqrt{n + \frac{r}{a+b}}.$$

When U_n is the circle touching the tangents of β through a point T on $\mathcal{L}$, we shall say that it is determined by T. The y-coordinate of the intersection of α and $\mathcal{L}$ is $2a\sqrt{\frac{r}{a+b}}$. Therefore we obtain the following theorem (see Figure 7).

Theorem 6. *U_0 is determined by the intersection of α and the line $\mathcal{L} : x = 2r$.*

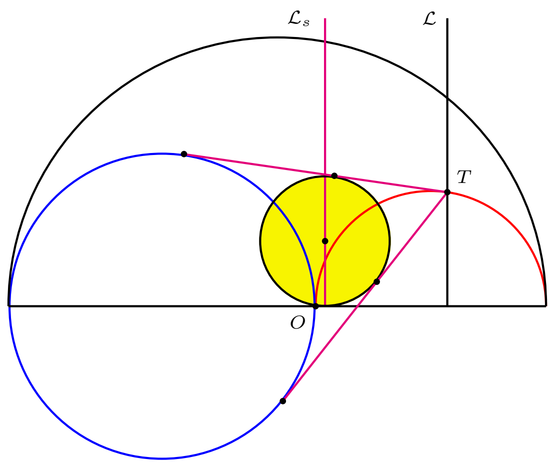

Figure 7

As stated in [2] as the property of the circle labeled as W_{11}, the external tangent of α and β also touches U_0 and the point of tangency at α coincides with the intersection of α and $\mathcal{L}$. Woo's circles are characterized as the circles determined by the points on $\mathcal{L}$ with y-coordinates greater than or equal to $2a\sqrt{\frac{r}{a+b}}$.

4. Woo's circles U_n with $n < 0$

Woo considered the circles U_n for nonnegative numbers n, with U_0 passing through O. We can, however, construct more Archimedean circles passing through points on the y-axis below O using points on $\mathcal{L}$ lying below the intersection with α. The expression (2) suggests the existence of U_n for

$$-\frac{r}{a+b} \leq n < 0. \tag{4}$$

In this section we show that it is possible to define such circles using $\alpha(na)$ and $\beta(nb)$ with negative n satisfying (4).

Theorem 7. *For n satisfying (4), the circle with center on the Schoch line touching $\alpha(na)$ and $\beta(nb)$ internally is an Archimedean circle.*

Proof. Let x be the radius of the circle with center given by (2) and touching $\alpha(na)$ and $\beta(nb)$ internally, where n satisfies (4). Since the centers of $\alpha(na)$ and $\beta(nb)$ are $(na, 0)$ and $(-nb, 0)$ respectively, we have

$$\left(\frac{b-a}{b+a}r - na\right)^2 + 4r^2\left(n + \frac{r}{a+b}\right) = (x+na)^2,$$

and

$$\left(\frac{b-a}{b+a}r + nb\right)^2 + 4r^2\left(n + \frac{r}{a+b}\right) = (x+nb)^2.$$

Since both equations give the same solution $x = r$, the proof is complete. □

5. A generalization of U_0

We conclude this paper by adding an infinite set of Archimedean circles passing through O. Let x be the distance from O to the external tangents of α and β. By similarity,

$$b - a : b + a = x - a : a.$$

This implies $x = 2r$. Hence, the circle with center O and radius $2r$ touches the tangents and the lines $x = \pm 2r$. We denote this circle by $\mathcal{E}$. Since U_0 touches the external tangents and passes through O, the circles U_0, $\mathcal{E}$ and the tangent touch at the same point. We easily see from (2) that the distance between the center of U_n and O is $\sqrt{4n+1}r$. Therefore, U_2 also touches $\mathcal{E}$ externally, and the smallest circle touching U_2 and passing through O, which is the Archimedean circle W_{27} in [2] found by Schoch, and U_2 touches $\mathcal{E}$ at the same point. All the Archimedean circles pass through O also touch $\mathcal{E}$. In particular, Bankoff's third circle [1] touches $\mathcal{E}$ at a point on the y-axis.

Theorem 8. *Let $\mathcal{C}_1$ be a circle with center O, passing through a point P on the x-axis, and $\mathcal{C}_2$ a circle with center on the x-axis passing through O. If $\mathcal{C}_2$ and the vertical line through P intersect, then the tangents of $\mathcal{C}_2$ at the intersection also touches $\mathcal{C}_1$.*

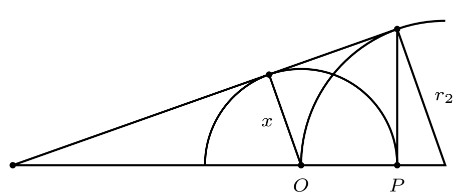

Figure 8a

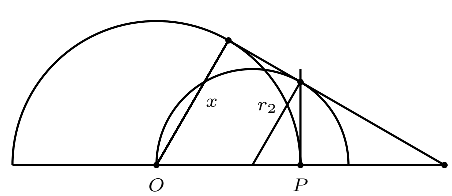

Figure 8b

Proof. Let d be the distance between O and the intersection of the tangent of $\mathcal{C}_2$ and the x-axis, and let x be the distance between the tangent and O. We may assume $r_1 \neq r_2$ for the radii r_1 and r_2 of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$. If $r_1 < r_2$, then

$$r_2 - r_1 : r_2 = r_2 + d = x : d.$$

See Figure 8a. If $r_1 > r_2$, then

$$r_1 - r_2 : r_2 = r_2 : d - r_2 = x : d.$$

See Figure 8b. In each case, $x = r_1$. □

Let t_n be the tangent of $\alpha(na)$ at its intersection with the line $\mathcal{L}$. This is well defined if $n \geq \frac{b}{a+b}$. By Theorem 8, t_n also touches $\mathcal{E}$. This implies that the smallest circle touching t_n and passing through O is an Archimedean circle, which we denote by $\mathcal{A}(n)$. Similarlary, another Archimedean circle $\mathcal{A}'(n)$ can be constructed, as the smallest circle through O touching the tangent t_n' of $\beta(nb)$ at its intersection with the line $\mathcal{L}' : x = -2r$. See Figure 9 for $\mathcal{A}(2)$ and $\mathcal{A}'(2)$. Bankoff's circle is $\mathcal{A}\left(\frac{2r}{a}\right) = \mathcal{A}'\left(\frac{2r}{b}\right)$, since it touches $\mathcal{E}$ at $(0, 2r)$. On the other hand, $U_0 = \mathcal{A}(1) = \mathcal{A}'(1)$ by Theorem 6.

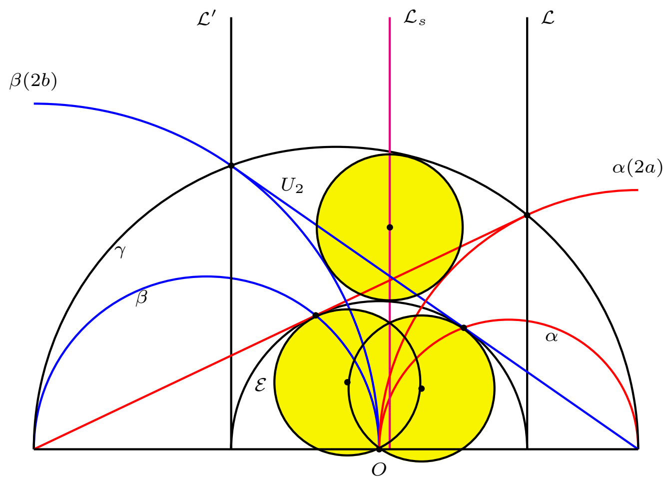

Figure 9

Theorem 9. *Let m and n be positive numbers. The Archimedean circles $\mathcal{A}(m)$ and $\mathcal{A}'(n)$ coincide if and only if m and n satisfy*

$$\frac{1}{ma} + \frac{1}{nb} = \frac{1}{r} = \frac{1}{a} + \frac{1}{b}. \tag{5}$$

Proof. By (3) the equations of the tangents t_m and t_n' are

$$-(ma + (m-2)b)x + 2\sqrt{b(ma + (m-1)b)}y = 2mab,$$
$$(nb + (n-2)a)x + 2\sqrt{a(nb + (n-1)a)}y = 2nab.$$

These two tangents coincide if and only if (5) holds. □

The line t_2 has equation

$$-ax + \sqrt{b(2a+b)}y = 2ab. \tag{6}$$

It clearly passes through $(-2b, 0)$, the point of tangency of γ and β (see Figure 9). Note that the point

$$\left(-\frac{2r}{a+b}a,\ \frac{2r}{a+b}\sqrt{b(2a+b)}\right)$$

lies on $\mathcal{E}$ and the tangent of $\mathcal{E}$ is also expressed by (6). Hence, t_2 touches $\mathcal{E}$ at this point. The point also lies on β. This means that $\mathcal{A}(2)$ touches t_2 at the intersection of β and t_2. Similarly, $\mathcal{A}'(2)$ touches t'_2 at the intersection of α and t'_2. The Archimedean circles $\mathcal{A}(2)$ and $\mathcal{A}'(2)$ intersect at the point

$$\left(\frac{b-a}{b+a}r,\ \frac{r}{a+b}(\sqrt{a(a+2b)} + \sqrt{b(2a+b)})\right)$$

on the Schoch line.

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Steiner's Theorems on the Complete Quadrilateral

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Abstract. We give a translation of Jacob Steiner's 1828 note on the complete quadrilateral, with complete proofs and annotations in barycentric coordinates.

1. Steiner's note on the complete quadrilateral

In 1828, Jakob Steiner published in Gergonne's *Annales* a very short note [9] listing ten interesting and important theorems on the complete quadrilateral. The purpose of this paper is to provide a translation of the note, to prove these theorems, along with annotations in barycentric coordinates. We begin with a translation of Steiner's note.

Suppose four lines intersect two by two at six points.

(1) These four lines, taken three by three, form four triangles whose circumcircles pass through the same point F.
(2) The centers of the four circles (and the point F) lie on the same circle.
(3) The perpendicular feet from F to the four lines lie on the same line $\mathcal{R}$, and F is the only point with this property.
(4) The orthocenters of the four triangles lie on the same line $\mathcal{R}'$.
(5) The lines $\mathcal{R}$ and $\mathcal{R}'$ are parallel, and the line $\mathcal{R}$ passes through the midpoint of the segment joining F to its perpendicular foot on $\mathcal{R}'$.
(6) The midpoints of the diagonals of the complete quadrilateral formed by the four given lines lie on the same line $\mathcal{R}''$ (Newton).
(7) The line $\mathcal{R}''$ is a common perpendicular to the lines $\mathcal{R}$ and $\mathcal{R}'$.
(8) Each of the four triangles in (1) has an incircle and three excircles. The centers of these 16 circles lie, four by four, on eight new circles.
(9) These eight new circles form two sets of four, each circle of one set being orthogonal to each circle of the other set. The centers of the circles of each set lie on a same line. These two lines are perpendicular.
(10) Finally, these last two lines intersect at the point F mentioned above.

The configuration formed by four lines is called a complete quadrilateral. Figure 1 illustrates the first 7 theorems on the complete quadrilateral bounded by the four lines UVW, UBC, AVC, and ABW. The diagonals of the quadrilateral are the

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segments AU, BV, CW. The four triangles ABC, AVW, BWU, and CUV are called the associated triangles of the complete quadrilateral. We denote by

- H, H_a, H_b, H_c their orthocenters,
- Γ, Γ_a, Γ_b, Γ_c their circumcircles, and
- O, O_a, O_b, O_c the corresponding circumcenters.

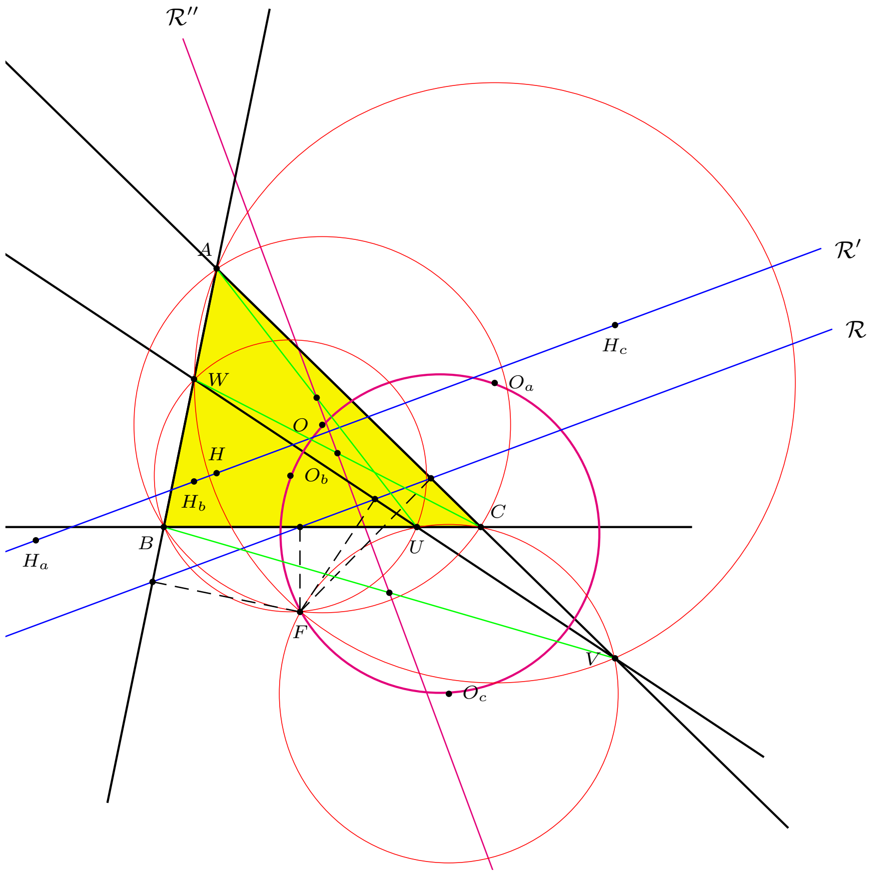

Figure 1.

2. Geometric preliminaries

2.1. *Directed angles.* We shall make use of the notion of *directed* angles. Given two lines ℓ and ℓ', the directed angle (ℓ, ℓ') is the angle through which ℓ must be rotated in the positive direction in order to become parallel to, or to coincide with, the line ℓ'. See [3, §§16–19]. It is defined modulo π.

Lemma 1. (1) $(\ell, \ell'') = (\ell, \ell') + (\ell', \ell'')$.

(2) *Four noncollinear points* P, Q, R, S *are concyclic if and only if* $(PR, PS) = (QR, QS)$.

2.2. *Simson-Wallace lines.* The pedals [1] of a point M on the lines BC, CA, AB are collinear if and only if M lies on the circumcircle Γ of ABC. In this case, the Simson-Wallace line passes through the midpoint of the segment joining M to the orthocenter H of triangle ABC. The point M is the isogonal conjugate (with respect to triangle ABC) of the infinite point of the direction orthogonal to its own Simson-Wallace line.

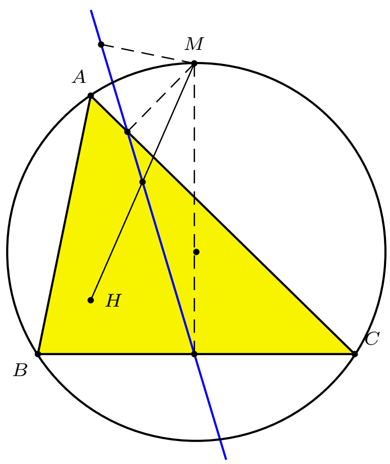

Figure 2

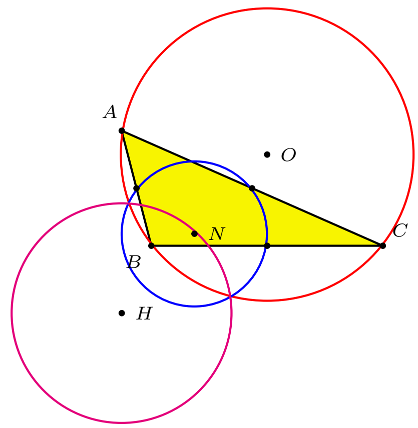

Figure 3

2.3. *The polar circle of a triangle.* There exists one and only one circle with respect to which a given triangle ABC is self polar. The center of this circle is the orthocenter of ABC and the square of its radius is

$$-4R^2 \cos A \cos B \cos C.$$

This *polar* circle is real if and only if ABC is obtuse-angled. It is orthogonal to any circle with diameter a segment joining a vertex of ABC to a point of the opposite sideline. The inversion with respect the polar circle maps a vertex of ABC to its pedal on the opposite side. Consequently, this inversion swaps the circumcircle and the nine-point circle.

2.4. *Center of a direct similitude.* Suppose that a direct similitude with center Ω maps M to M' and N to N', and that the lines MM' and NN' intersect at S. If Ω does not lie on the line MN, then M, N, Ω, S are concyclic; so are M', N', Ω, S. Moreover, if $MN \perp M'N'$, the circles $MN\Omega S$ and $M'N'\Omega S$ are orthogonal.

[1] In this paper we use the word pedal in the sense of orthogonal projection.

3. Steiner's Theorems 1–7

3.1. *Steiner's Theorem 1 and the Miquel point.* Let F be the second common point (apart from A) of the circles Γ and Γ_a. Since

$$(FB, FW) = (FB, FA)+(FA, FW) = (CB, CA)+(VA, VW) = (UB, UW),$$

we have $F \in \Gamma_b$ by Lemma 1(2). Similarly $F \in \Gamma_c$. This proves (1).

We call F the *Miquel point* of the complete quadrilateral.

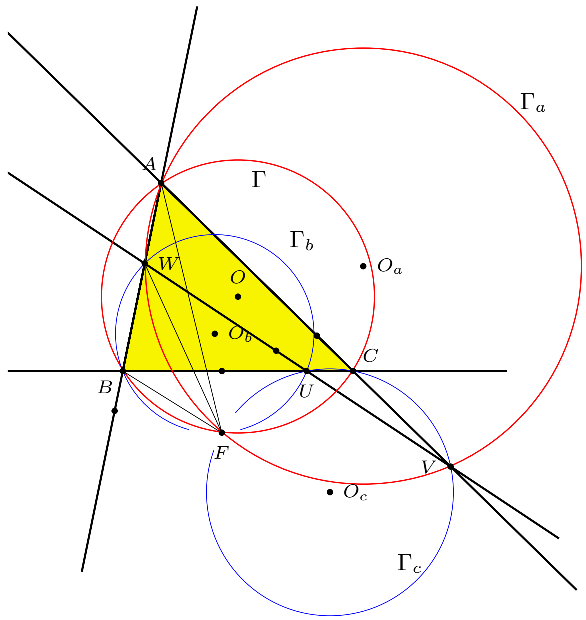

Figure 4.

3.2. *Steiner's Theorem 3 and the pedal line.* The point F has the same Simson-Wallace line with respect to the four triangles of the complete quadrilateral. See Figure 5. Conversely, if the pedals of a point M on the four sidelines of the complete quadrilateral lie on a same line, M must lie on each of the four circumcircles. Hence, $M = F$. This proves (3).

We call the line $\mathcal{R}$ the *pedal line* of the quadrilateral.

3.3. *Steiner's Theorems 4, 5 and the orthocentric line.* As the midpoints of the segments joining F to the four orthocenters lie on $\mathcal{R}$, the four orthocenters lie on a line $\mathcal{R}'$, which is the image of $\mathcal{R}$ under the homothety $\mathsf{h}(F, 2)$. This proves (4) and (5). See Figure 5.

We call the line $\mathcal{R}'$ the *orthocentric line* of the quadrilateral.

Remarks. (1) As U, V, W are the reflections of F with respect to the sidelines of the triangle $O_aO_bO_c$, the orthocenter of this triangle lies on $\mathcal{L}$.

(2) We have $(BC, FU) = (CA, FV) = (AB, FW)$ because, for instance, $(BC, FU) = (UB, UF) = (WB, WF) = (AB, FW)$.

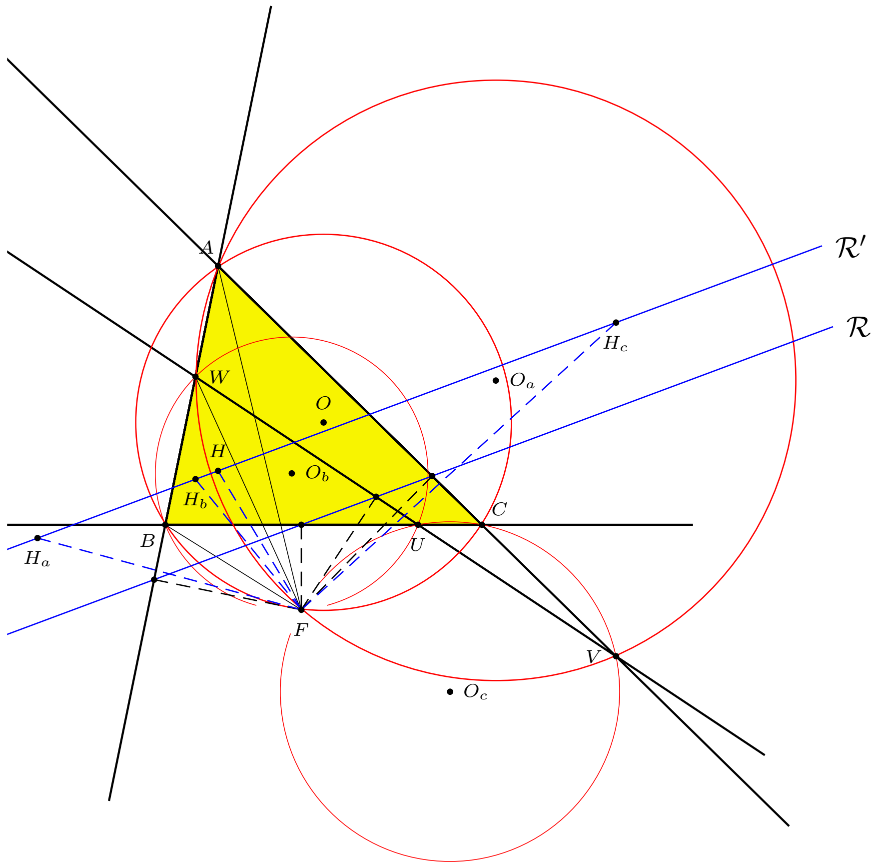

Figure 5.

(3) Let P_a, P_b, P_c be the projections of F upon the lines BC, CA, AB. As P_a, P_b, C, F are concyclic, it follows that F is the center of the direct similitude mapping P_a to U and P_b to V. Moreover, by (2) above, this similitude maps P_c to W.

3.4. *Steiner's Theorem 2 and the Miquel circle.* By Remark (3) above, if F_a, F_b, F_c are the reflections of F with respect to the lines BC, CA, AB, a direct similitude σ with center F maps F_a to U, F_b to V, F_c to W. As A is the circumcenter of FF_bF_c, it follows that $\sigma(A) = O_a$; similarly, $\sigma(B) = O_b$ and $\sigma(C) = O_c$. As A, B, C, F are concyclic, so are O_a, O_b, O_c, F. Hence F and the circumcenters of three associated triangles are concyclic. It follows that O, O_a, O_b, O_c, F lie on the same circle, say, Γ_m. This prove (2).

We call Γ_m the *Miquel circle* of the complete quadrilateral. See Figure 6.

3.5. *The Miquel perspector.* Now, by §2.4, the second common point of Γ and Γ_m lies on the three lines AO_a, BO_b, CO_c. Hence,

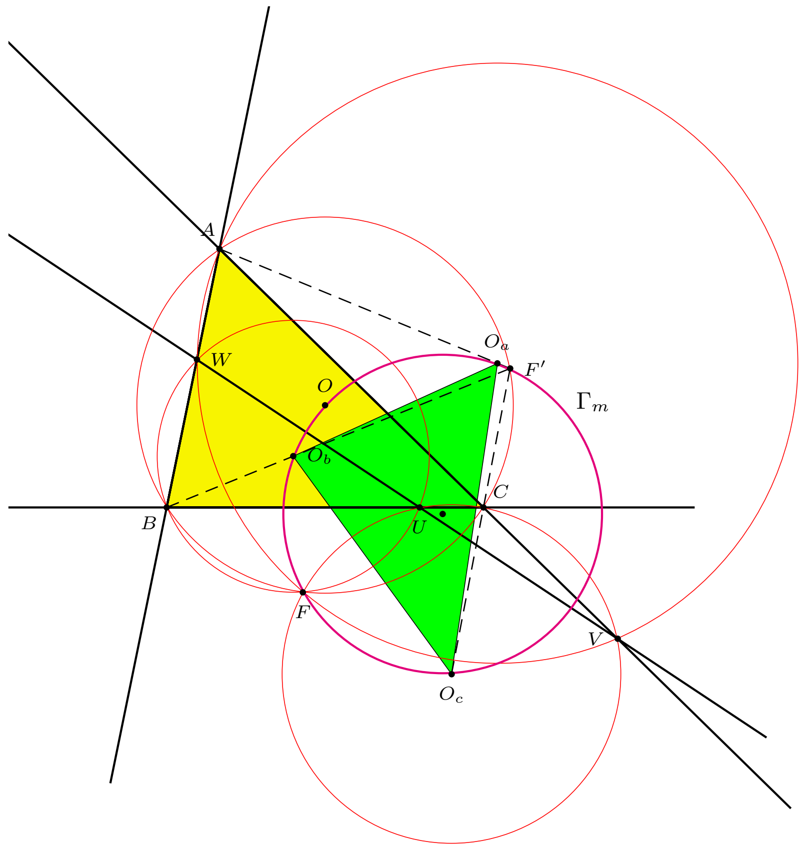

Figure 6.

Proposition 2. *The triangle $O_aO_bO_c$ is directly similar and perspective with ABC. The center of similitude is the Miquel point F and the perspector is the second common point F' of the Miquel circle and the circumcircle Γ of triangle ABC.*

We call F' the *Miquel perspector* of the triangle ABC.

3.6. *Steiner's Theorems 6, 7 and the Newton line.* We call *diagonal triangle* the triangle $A'B'C'$ with sidelines AU, BV, CW.

Lemma 3. *The polar circles of the triangles ABC, AVW, BWU, CUV and the circumcircle of the diagonal triangle are coaxal. The three circles with diameter AU, BV, CW are coaxal. The corresponding pencils of circles are orthogonal.*

Proof. By §2.3, each of the four polar circles is orthogonal to the three circles with diameter AU, BV, CW. More over, as each of the quadruples (A, U, B', C'), (B, V, C', A') and (C, W, A', B') is harmonic, the circle $A'B'C'$ is orthogonal to the three circles with diameter AU, BV and CW. □

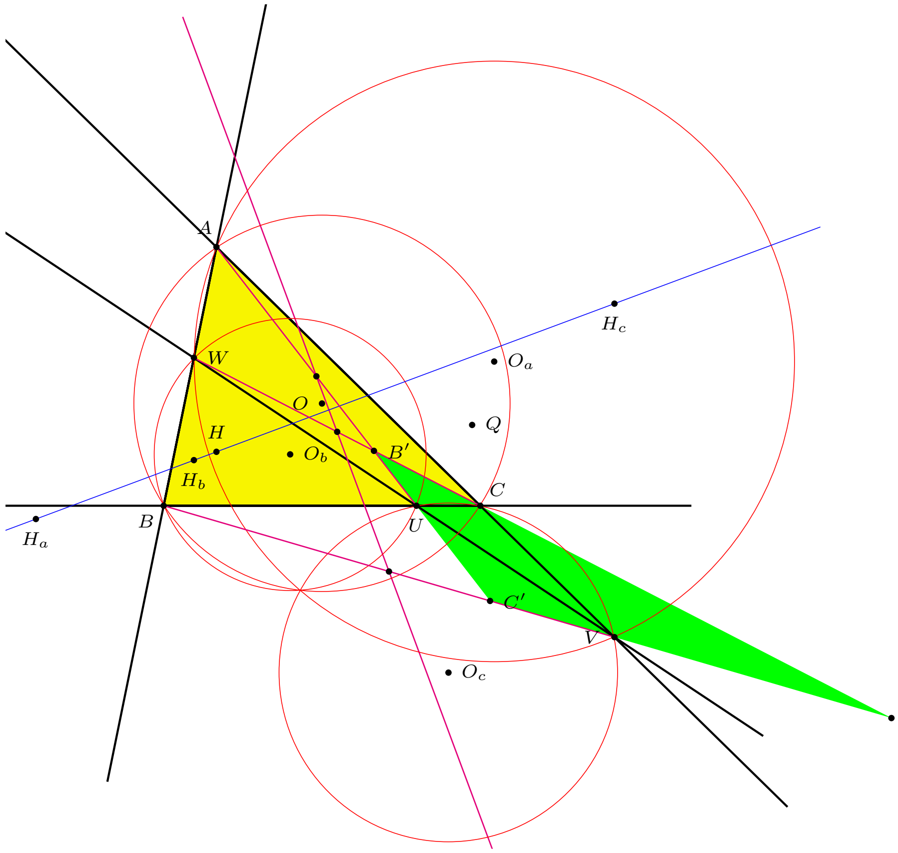

Figure 7.

As the line of centers of the first pencil of circles is the orthocentric line $\mathcal{R}'$, it follows that the midpoints of AU, BV and CW lie on a same line $\mathcal{R}''$ perpendicular to $\mathcal{R}'$. This proves (6) and (7).

4. Some further results

4.1. *The circumcenter of the diagonal triangle.*

Proposition 4. *The circumcenter of the diagonal triangle lies on the orthocentric line.*

This follows from Lemma 3 and §2.3.

We call the line $\mathcal{R}''$ the *Newton line* of the quadrilateral. As the Simson-Wallace line $\mathcal{R}$ of F is perpendicular to $\mathcal{R}''$, we have

Proposition 5. *The Miquel point is the isogonal conjugate of the infinite point of the Newton line with respect to each of the four triangles ABC, AVW, BWU, CUV.*

4.2. *The orthopoles.* Recall that the three lines perpendicular to the sidelines of a triangle and going through the projection of the opposite vertex on a given line go through a same point : the *orthopole* of the line with respect to the triangle.

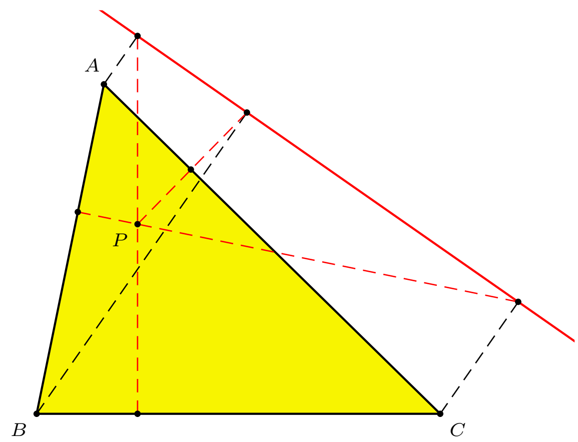

Figure 8

Proposition 6 (Goormaghtigh)**.** *The orthopole of a sideline of the complete quadrilateral with respect to the triangle bounded by the three other sidelines lies on the orthocentric line.*

Proof. See [1, pp.241–242]. □

5. Some barycentric coordinates and equations

5.1. *Notations.* Given a complete quadrilateral, we consider the triangle bounded by three of the four given lines as a reference triangle ABC, and construe the fourth line as the trilinear polar with respect to ABC of a point Q with homogeneous barycentric coordinates $(u : v : w)$, *i.e.*, the line

$$\mathcal{L} : \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

The intercepts of $\mathcal{L}$ with the sidelines of triangle ABC are the points

$$U = (0 : v : -w), \qquad V = (-u : 0 : w), \qquad W = (u : -v : 0).$$

The lines AU, BV, CW bound the diagonal triangle with vertices

$$A' = (-u : v : w), \qquad B' = (u : -v : w), \qquad C' = (u : v : -w).$$

Triangles ABC and $A'B'C'$ are perspective at Q.

We adopt the following notations. If a, b, c stand for the lengths of the sides BC, CA, AB, then

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2), \quad S_B = \frac{1}{2}(c^2 + a^2 - b^2), \quad S_C = \frac{1}{2}(a^2 + b^2 - c^2).$$

We shall also denote by S *twice* of the signed area of triangle ABC, so that

$$S_A = S \cdot \cot A, \quad S_B = S \cdot \cot B, \quad S_C = S \cdot \cot C,$$

and

$$S_{BC} + S_{CA} + S_{AB} = S^2.$$

Lemma 7. (1) *The infinite point of the line $\mathcal{L}$ is the point*

$$(u(v-w) : v(w-u) : w(u-v)).$$

(2) *Lines perpendicular to $\mathcal{L}$ have infinite point $(\lambda_a : \lambda_b : \lambda_c)$, where*

$$\begin{aligned}
\lambda_a &= S_B v(w-u) - S_C w(u-v), \\
\lambda_b &= S_C w(u-v) - S_A u(v-w), \\
\lambda_c &= S_A u(v-w) - S_B v(w-u).
\end{aligned}$$

Proof. (1) is trivial. (2) follows from (1) and the fact that two lines with infinite points $(p : q : r)$ and $(p' : q' : r')$ are perpendicular if and only if

$$S_A pp' + S_B qq' + S_C rr' = 0.$$

Consequently, given a line with infinite point $(p : q : r)$, lines perpendicular to it all have the infinite point $(S_B q - S_C r : S_C r - S_A p : S_A p - S_B q)$. □

5.2. *Coordinates and equations.* We give the barycentric coordinates of points and equations of lines and circles in Steiner's theorems.

(1) The Miquel point:

$$F = \left(\frac{a^2}{v-w} : \frac{b^2}{w-u} : \frac{c^2}{u-v} \right).$$

(2) The pedal line:

$$\mathcal{R}: \quad \frac{v-w}{S_C v + S_B w - a^2 u} x + \frac{w-u}{S_A w + S_C u - b^2 v} y + \frac{u-v}{S_B u + S_A v - c^2 w} z = 0.$$

(3) The orthocentric line:

$$\mathcal{R}': \qquad (v-w) S_A x + (w-u) S_B y + (u-v) S_C z = 0.$$

(4) The Newton line:

$$\mathcal{R}'': \qquad (v+w-u)x + (w+u-v)y + (u+v-w)z = 0.$$

(5) The equation of the Miquel circle:

$$a^2 yz + b^2 zx + c^2 xy + \frac{2R^2(x+y+z)}{(v-w)(w-u)(u-v)} \left(\frac{v-w}{a^2} \lambda_a x + \frac{w-u}{b^2} \lambda_b y + \frac{u-v}{c^2} \lambda_c z \right) = 0.$$

(6) The Miquel perspector, being the isogonal conjugate of the infinite point of the direction orthogonal to $\mathcal{L}$, is

$$F' = \left(\frac{a^2}{\lambda_a} : \frac{b^2}{\lambda_b} : \frac{c^2}{\lambda_c} \right).$$

The Simson-Wallace line of F' is parallel to ℓ.

(7) The orthopole of $\mathcal{L}$ with respect to ABC is the point

$$(\lambda_a(-S_BS_Cvw + b^2S_Bwu + c^2S_Cuv) : \cdots : \cdots).$$

5.3. *Some metric formulas* . Here, we adopt more symmetric notations. Let ℓ_i, $i = 1, 2, , 3, 4$, be four given lines.

- For distinct i and j, $A_{i,j} = \ell_i \cap \ell_j$,
- $\mathcal{T}_i$ the triangle bounded by the three lines other than ℓ_i, O_i its circumcenter, R_i its circumradius.
- $F_i = O_jA_{k,l} \cap O_kA_{l,j} \cap O_lA_{j,k}$ its Miquel perspector, *i.e.*, the second intersection (apart from F) of its circumcircle with the Miquel circle; R_m is the radius of the Miquel circle.

Let d be the distance from F to the pedal line $\mathcal{R}$ and $\theta_i = (\mathcal{R}, \ell_i)$. Up to a direct congruence, the complete quadrilateral is characterized by d, θ_1, θ_2, θ_3, and θ_4.

(1) The distance from F to ℓ_i is $\dfrac{d}{|\cos\theta_i|}$.
(2) $|FA_{i,j}| = \dfrac{d}{|\cos\theta_i\cos\theta_j|}$.
(3) $|A_{k,i}A_{k,j}| = d\left|\dfrac{\sin(\theta_j-\theta_i)}{\cos\theta_i\cos\theta_j\cos\theta_k}\right|$.
(4) The directed angle $(FA_{k,i}, FA_{k,j}) = (\ell_i, \ell_j) = \theta_j - \theta_i \mod \pi$.
(5) $R_m = \dfrac{d}{4\,|\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4|} = \dfrac{R_i}{2\,|\cos\theta_i|}$ for $i = 1, 2, 3, 4$.
(6) $|FA_{1,2}| \cdot |FA_{3,4}| = |FA_{1,3}| \cdot |FA_{2,4}| = |FA_{1,4}| \cdot |FA_{2.3}| = 4dR_m$.
(7) $|FF_i| = 2R_i|\sin\theta_i|$.
(8) The *oriented* angle between the vectors $\mathbf{O_iF}$ and $\mathbf{O_iF_i} = -2\theta_i \mod 2\pi$.
(9) The distance from F to $\mathcal{R}''$ is

$$\frac{d}{2}\left|\tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \tan\theta_4\right|.$$

6. Steiner's Theorems 8 – 10

At each vertex M of the complete quadrilateral, we associate the pair of angle bisectors m and m'. These lines are perpendicular to each other at M. We denote the intersection of two bisectors m and n by $\mathsf{m} \cap \mathsf{n}$.

- $\mathbf{T}(\mathsf{m},\mathsf{n},\mathsf{p})$ denotes the triangle bounded by a bisector at M, one at N, and one at P.
- $\Gamma(\mathsf{m},\mathsf{n},\mathsf{p})$ denotes the circumcircle of $\mathbf{T}(\mathsf{m},\mathsf{n},\mathsf{p})$.

Consider three bisectors $\mathsf{a}, \mathsf{b}, \mathsf{c}$ intersecting at a point J, the incenter or one of the excenters of ABC. Suppose two bisectors v and w intersect on a. Then so do v' and w'. Now, the line joining $\mathsf{b} \cap \mathsf{w}$ and $\mathsf{c} \cap \mathsf{v}$ is a U-bisector. If we denote this line by u, then u' the line joining $\mathsf{b} \cap \mathsf{w}'$ and $\mathsf{c} \cap \mathsf{v}'$.

The triangles $\mathbf{T}(\mathsf{a}',\mathsf{b}',\mathsf{c}')$, $\mathbf{T}(\mathsf{u},\mathsf{v},\mathsf{w})$, and $\mathbf{T}(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ are perspective at J. Hence, by Desargues' theorem, the points $\mathsf{a}' \cap \mathsf{u}$, $\mathsf{b}' \cap \mathsf{v}$, and $\mathsf{c}' \cap \mathsf{w}$ are collinear; so are $\mathsf{a}' \cap \mathsf{u}'$, $\mathsf{b}' \cap \mathsf{v}'$, and $\mathsf{c}' \cap \mathsf{w}'$. Moreover, as the corresponding sidelines of triangles $\mathbf{T}(\mathsf{u},\mathsf{v},\mathsf{w})$, and $\mathbf{T}(\mathsf{u}',\mathsf{v}',\mathsf{w}')$ are perpendicular, it follows from §2.4 that their circumcircles $\Gamma(u, v, w)$, and $\Gamma(u', v', w')$ are orthogonal and pass through J. See Figure 9. [2]

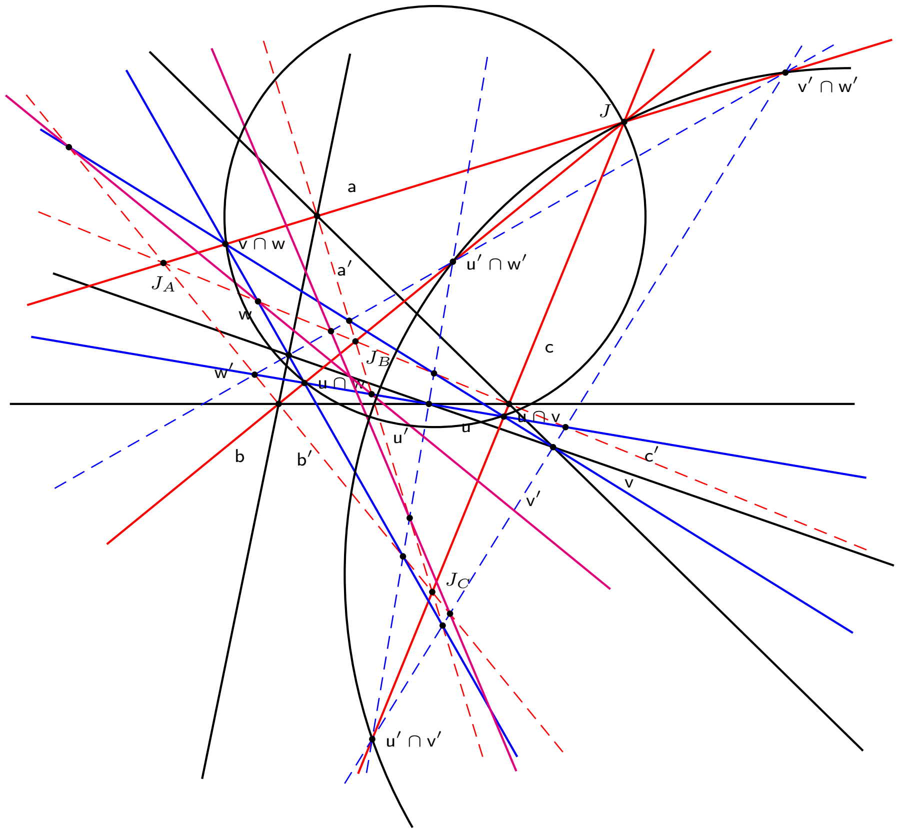

Figure 9

As a intersects the circle $\Gamma(u', v', w')$ at J and $v' \cap w'$ and u' intersects the circle $\Gamma(u', v', w')$ at $u' \cap v'$ and $u' \cap w'$, it follows that the polar line of $a \cap u'$ with respect to $\Gamma(u', v', w')$ passes through $b \cap v'$ and $c \cap w'$. Hence $\Gamma(u', v', w')$ is the polar circle of the triangle with vertices $a \cap u'$, $b \cap v'$, $c \cap w'$. Similarly, $\Gamma(u, v, w)$ is the polar circle of the triangle with vertices $a \cap u$, $b \cap v$, $c \cap w$.

By the same reasoning, we obtain the following.

(a) As the triangles $\mathbf{T}(a', b, c)$, $\mathbf{T}(u, v', w')$, and $\mathbf{T}(u', v, w)$ are perspective at $J_A = a \cap b' \cap c'$, it follows that

- the circles $\Gamma(u, v', w')$ and $\Gamma(u', v, w)$ are orthogonal and pass through J_A,
- the points $a' \cap u$, $b \cap v'$, and $c \cap w'$ are collinear; so are $a' \cap u'$, $b \cap v$, and $c \cap w$,

[2] In Figures 9 and 10, at each of the points A, B, C, U, V, W are two bisectors, one shown in solid line and the other in dotted line. The bisectors in solid lines are labeled a, b, c, u, v, w, and those in dotted line labeled a', b', c', u', v', w'. Other points are identified as intersections of two of these bisectors. Thus, for example, $J = a \cap b$, and $J_A = b' \cap c'$.

- the circle $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}'\cap\mathsf{v}'$, $\mathsf{c}'\cap\mathsf{w}'$, and $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}'\cap\mathsf{v}$, $\mathsf{c}'\cap\mathsf{w}$.

(b) As the triangles $\mathbf{T}(\mathsf{a},\mathsf{b}',\mathsf{c})$, $\mathbf{T}(\mathsf{u}',\mathsf{v},\mathsf{w}')$, and $\mathbf{T}(\mathsf{u},\mathsf{v}',\mathsf{w})$ are perspective at $J_B=\mathsf{a}'\cap\mathsf{b}\cap c'$, it follows that

- the circles $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}')$ and $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w})$ are orthogonal and pass through J_B,
- the points $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}'\cap\mathsf{v}$, and $\mathsf{c}\cap\mathsf{w}'$ are collinear; so are $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}'\cap\mathsf{v}'$, and $\mathsf{c}\cap\mathsf{w}$,
- the circle $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}$, $\mathsf{c}'\cap\mathsf{w}'$, and $\Gamma(\mathsf{u},\mathsf{v}',\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}'$, $\mathsf{c}'\cap\mathsf{w}$.

(c) As the triangles $\mathbf{T}(\mathsf{a},\mathsf{b},\mathsf{c}')$, $\mathbf{T}(\mathsf{u}',\mathsf{v}',\mathsf{w})$, and $\mathbf{T}(\mathsf{u},\mathsf{v},\mathsf{w}')$ are perspective at $J_C=\mathsf{a}'\cap\mathsf{b}'\cap c$, it follows that

- the circles $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w})$ and $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}')$ are orthogonal and pass through J_C,
- the points $\mathsf{a}\cap\mathsf{u}'$, $\mathsf{b}\cap\mathsf{v}'$, and $\mathsf{c}'\cap\mathsf{w}$ are collinear; so are $\mathsf{a}\cap\mathsf{u}$, $\mathsf{b}\cap\mathsf{v}$, and $\mathsf{c}'\cap\mathsf{w}'$,
- the circle $\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w})$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}'$, $\mathsf{b}'\cap\mathsf{v}'$, $\mathsf{c}\cap\mathsf{w}$, and $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}')$ is the polar circle of the triangle with vertices $\mathsf{a}'\cap\mathsf{u}$, $\mathsf{b}'\cap\mathsf{v}$, $\mathsf{c}\cap\mathsf{w}'$.

Therefore, we obtain two new complete quadrilaterals:

(1) $\mathcal{Q}_1$ with sidelines those containing the triples of points

$$(\mathsf{a}'\cap\mathsf{u},\ \mathsf{b}'\cap\mathsf{v},\ \mathsf{c}'\cap\mathsf{w}),\ (\mathsf{a}'\cap\mathsf{u},\ \mathsf{b}\cap\mathsf{v}',\ \mathsf{c}\cap\mathsf{w}'),\ (\mathsf{a}\cap\mathsf{u}',\ \mathsf{b}'\cap\mathsf{v},\ \mathsf{c}\cap\mathsf{w}'),\ (\mathsf{a}\cap\mathsf{u}',\ \mathsf{b}\cap\mathsf{v}',\ \mathsf{c}'\cap\mathsf{w}),$$

(2) $\mathcal{Q}_2$ with sidelines those containing the triples of points

$$(\mathsf{a}'\cap\mathsf{u}',\ \mathsf{b}'\cap\mathsf{v}',\ \mathsf{c}'\cap\mathsf{w}'),\ (\mathsf{a}'\cap\mathsf{u}',\ \mathsf{b}\cap\mathsf{v},\ \mathsf{c}\cap\mathsf{w}),\ (\mathsf{a}\cap\mathsf{u},\ \mathsf{b}'\cap\mathsf{v}',\ \mathsf{c}\cap\mathsf{w}),\ (\mathsf{a}\cap\mathsf{u},\ \mathsf{b}\cap\mathsf{v},\ \mathsf{c}'\cap\mathsf{w}').$$

The polar circles of the triangles associated with $\mathcal{Q}_1$ are

$$\Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v},\mathsf{w}').$$

These circles pass through J, J_A, J_B, J_C respectively.

The polar circles of the triangles associated with $\mathcal{Q}_2$ are

$$\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}).$$

These circles pass through J, J_A, J_B, J_C respectively. Moreover, by §2.4, the circles in the first group are orthogonal to those in the second group. For example, as u and u' are perpendicular to each other, the circles $\Gamma(\mathsf{u},\mathsf{v},\mathsf{w})$ and $\Gamma(\mathsf{u}',\mathsf{v},\mathsf{w})$ are orthogonal. Now it follows from Lemma 3 applied to $\mathcal{Q}_1$ and $\mathcal{Q}_2$ that

Proposition 8 (Mention [4]). *(1) The following seven circles are members of a pencil Φ:*

$$\Gamma(\mathsf{u},\mathsf{v},\mathsf{w}),\ \Gamma(\mathsf{u},\mathsf{v}',\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v},\mathsf{w}'),\ \Gamma(\mathsf{u}',\mathsf{v}',\mathsf{w}),$$

and those with diameters

$$(\mathsf{a}\cap\mathsf{u}')(\mathsf{a}'\cap\mathsf{u}),\ (\mathsf{b}\cap\mathsf{v}')(\mathsf{b}'\cap\mathsf{v}),\ (\mathsf{c}\cap\mathsf{w}')(\mathsf{c}'\cap\mathsf{w}).$$

(2) The following seven circles are members of a pencil Φ':

$$\Gamma(\mathsf{u}', \mathsf{v}', \mathsf{w}'),\ \Gamma(\mathsf{u}', \mathsf{v}, \mathsf{w}),\ \Gamma(\mathsf{u}, \mathsf{v}', \mathsf{w}),\ \Gamma(\mathsf{u}, \mathsf{v}, \mathsf{w}'),$$

and those with diameters

$$(\mathsf{a} \cap \mathsf{u})(\mathsf{a}' \cap \mathsf{u}'),\ (\mathsf{b} \cap \mathsf{v})(\mathsf{b}' \cap \mathsf{v}'),\ (\mathsf{c} \cap \mathsf{w})(\mathsf{c}' \cap \mathsf{w}').$$

(3) The circles in the two pencils Φ and Φ' are orthogonal.

This clearly gives Steiner's Theorems 8 and 9.

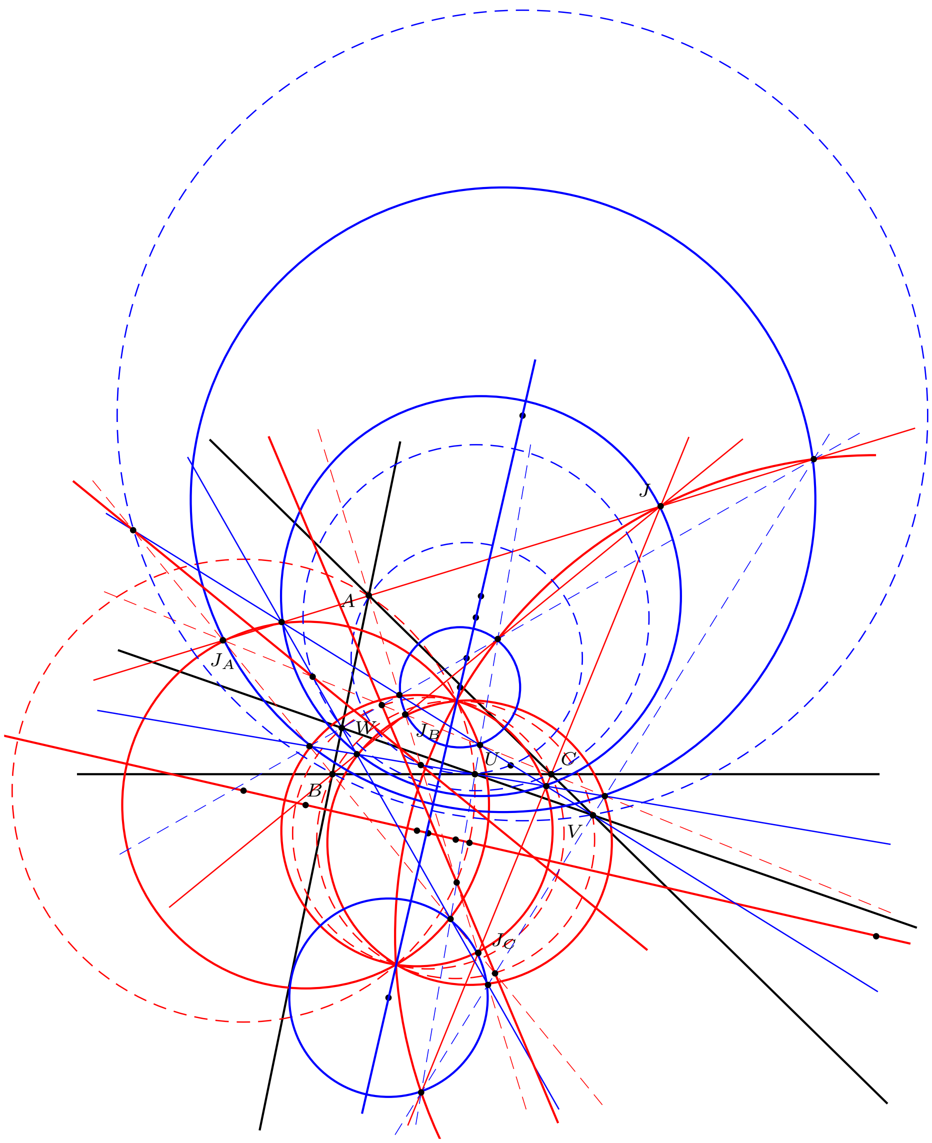

Figure 10

Let P be the midpoint of the segment joining $\mathsf{a}\cap\mathsf{u}'$ and $\mathsf{a}'\cap\mathsf{u}$, and P' the midpoint of the segment joining $\mathsf{a}\cap\mathsf{u}$ and $\mathsf{a}'\cap\mathsf{u}'$. The nine-point circle of the orthocentric system

$$\mathsf{a}\cap\mathsf{u},\quad \mathsf{a}'\cap\mathsf{u}',\quad \mathsf{a}\cap\mathsf{u}',\quad \mathsf{a}'\cap\mathsf{u}$$

is the circle with diameter PP'. This circle passes through A and U. See Figure 11. Furthermore, P and P' are the midpoints of the two arcs AU of this circle. As P is the center of the circle passing through A, U, $\mathsf{a}\cap\mathsf{u}'$ and $\mathsf{a}'\cap\mathsf{u}$, we have

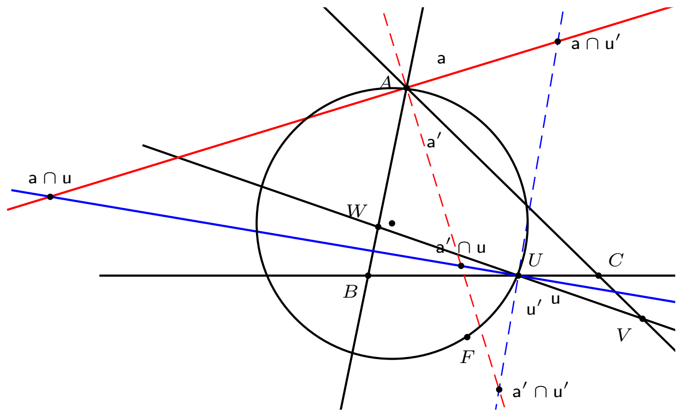

Figure 11.

$$\begin{aligned}(PA,\ PU) =&2((\mathsf{a}\cap\mathsf{u}')A,\ (\mathsf{a}\cap\mathsf{u}')U)\\ =&2((\mathsf{a}\cap\mathsf{u}')A,\ AB)+2(AB,\ UV)+2(UV,\ (\mathsf{a}\cap\mathsf{u}')U)\\ =&(AC,\ AB)+2(AB,\ UV)+(UV,BC)\\ =&(CA,\ CB)+(AB,\ UV)\\ =&(CA,\ CB)+(WB,\ WU)\\ =&(FA,\ FB)+(FB,\ FU)\\ =&(FA,\ FU).\end{aligned}$$

Hence, F lies on the circle with diameter PP', and the lines FP, FP' bisect the angles between the lines FA and FU.

As the central lines of the pencils Φ and Φ' are perpendicular and pass respectively through P and P', their common point lies on the circle FAU. Similarly, this common point must lie on the circles FBV and FCW. Hence, this common point is F. This proves Steiner's Theorem 10 and the following more general result.

Proposition 9 (Clawson)**.** *The central lines of the pencils Φ and Φ' are the common bisectors of the three pairs of lines $(FA,\ FU)$, $(FB,\ FV)$, and $(FC,\ FW)$.*

Note that, as $(FA,\ FB) = (FV,\ FU)\ = (CA,\ CB)$, it is clear that the three pairs of lines $(FA,\ FU)$, $(FB,\ FV)$, $(FC,\ FW)$ have a common pair of bisectors $(\mathsf{f}, \mathsf{f}')$. These bisectors are called the *incentric lines* of the complete quadrilateral. With the notations of §5.3, we have

$$2(\mathcal{R},\ \mathsf{f}) = 2(\mathcal{R},\ \mathsf{f}') = \theta_1 + \theta_2 + \theta_3 + \theta_4 \bmod \pi.$$

7. Inscribed conics

7.1. *Centers and foci of inscribed conics.* We give some classical properties of the conics tangent to the four sidelines of the complete quadrilateral.

Proposition 10. *The locus of the centers of the conics inscribed in the complete quadrilateral is the Newton line $\mathcal{R}''$.*

Proposition 11. *The locus of the foci of these conics is a circular focal cubic* (van Rees focal).

This cubic γ passes through A, B, C, U, V, W, F, the circular points at infinity I_∞, J_∞ and the feet of the altitudes of the diagonal triangle.

The real asymptote is the image of the Newton line under the homothety $\mathsf{h}(F, 2)$, and the imaginary asymptotes are the lines FI_∞ and FJ_∞. In other words, F is the singular focus of γ. As F lies on the γ, γ is said to be *focal*. The cubic γ is self isogonal with respect to each of the four triangles ABC, AVW, BWU, CUV. It has barycentric equation

$$\begin{aligned} &ux\left(c^2y^2 + b^2z^2\right) + vy\left(a^2z^2 + c^2x^2\right) + wz\left(b^2x^2 + a^2y^2\right) \\ &+2\left(S_Au + S_Bv + S_Cw\right)xyz = 0. \end{aligned}$$

If we denote by $\overline{PQRS}$ the van Rees focal of P, Q, R, S, *i.e.*, the locus of M such as $(MP, MQ) = (MR, MS)$, then

$$\gamma = \overline{ABVU} = \overline{BCWV} = \overline{CAUW} = \overline{AVBU} = \overline{BWCV} = \overline{CUAW}.$$

Here is a construction of the cubic γ.

Construction. Consider a variable circle through the pair of isogonal conjugate points on the Newton line. [3] Draw the lines through F tangent to the circle. The locus of the points of tangency is the cubic γ. See Figure 12

7.2. *Orthoptic circles.* Recall that the Monge (or orthoptic) circle of a conic is the locus of M from which the tangents to the conic are perpendicular.

Proposition 12 (Oppermann). *The circles of the pencil generated by the three circles with diameters AU, BV, CW are the Monge circle's of the conics inscribed in the complete quadrilateral.*

Proof. See [5, pp.60–61]. □

[3] These points are not necessarily real.

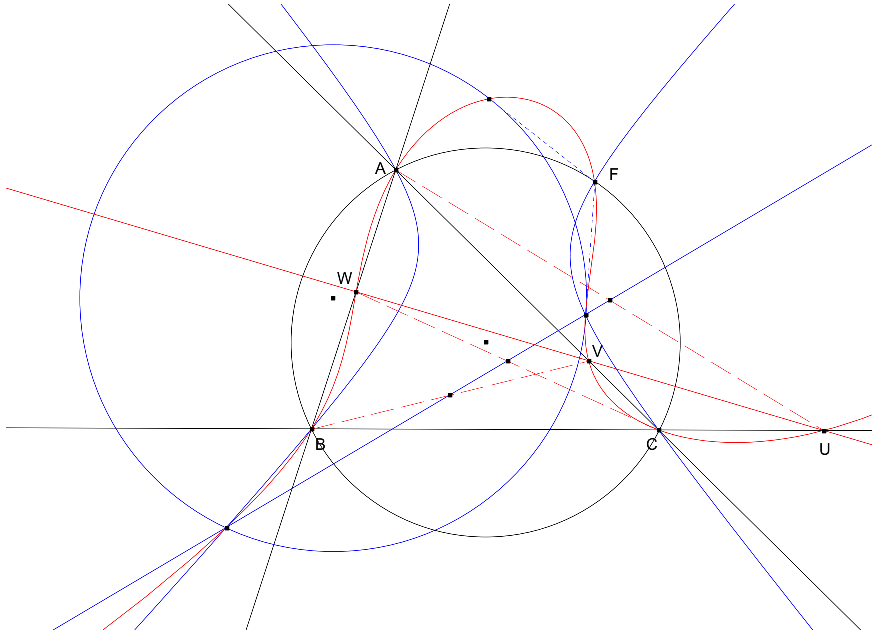

Figure 12.

7.3. *Coordinates and equations.* Recall that the perspector (or Brianchon point) of a conic inscribed in the triangle ABC is the perspector of ABC and the contact triangle. Suppose the perspector is the point $(p : q : r)$.

(1) The center of the conic is the point

$$(p(q+r) : q(r+p) : r(p+q)).$$

(2) The equation of the conic is

$$\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2}-2\frac{xy}{pq}-2\frac{yz}{qr}-2\frac{zx}{rp}=0.$$

(3) The line $\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=0$ is tangent to the conic if and only if $\frac{u}{p}+\frac{v}{q}+\frac{w}{r}=0$.

(4) The equation of the Monge circle of the conic is

$$\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)(a^2yz+b^2zx+c^2xy)=(x+y+z)\left(\frac{S_A}{p}x+\frac{S_B}{q}y+\frac{S_C}{r}z\right).$$

The locus of the perspectors of the conics inscribed in the complete quadrilateral is the circumconic

$$\frac{u}{x}+\frac{v}{y}+\frac{w}{z}=0,$$

i.e., the circumconic with perspector Q.

7.4. *Inscribed parabola.*

Proposition 13. *The only parabola inscribed in the quadrilateral is the parabola with focus F and directrix the orthocentric line $\mathcal{R}'$.*

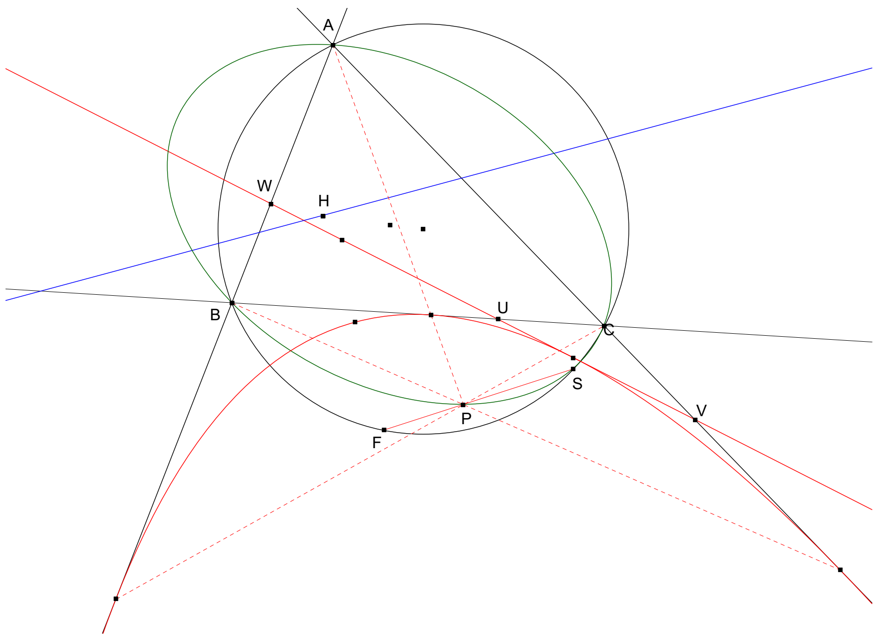

Figure 13

The perspector of the parabola has barycentric coordinates

$$\left(\frac{1}{v-w} : \frac{1}{w-u} : \frac{1}{u-v}\right).$$

This point is the isotomic conjugate of the infinite point of the Newton line. It is also the second common point (apart from the Steiner point S of triangle ABC) of the line SF and the Steiner circum-ellipse.

If a line ℓ' tangent to the parabola intersects the lines BC, CA, AB respectively at U', V', W', we have

$$(FU,\ FU') = (FV,\ FV') = (FW,\ FW') = (\ell,\ \ell').$$

If four points P, Q, R, S lie respectively on the sidelines BC, CA, AB, ℓ and verify

$$(FP,\ BC) = (FQ,\ CA) = (FR,\ AB) = (FS,\ \ell),$$

then these four points lie on the same line tangent to the parabola. This is a generalization of the pedal line.

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Orthopoles and the Pappus Theorem

Atul Dixit and Darij Grinberg

Abstract. If the vertices of a triangle are projected onto a given line, the perpendiculars from the projections to the corresponding sidelines of the triangle intersect at one point, the orthopole of the line with respect to the triangle. We prove several theorems on orthopoles using the Pappus theorem, a fundamental result of projective geometry.

1. Introduction

Theorems on orthopoles are often proved with the help of coordinates or complex numbers. In this note we prove some theorems on orthopoles by using a well-known result from projective geometry, the Pappus theorem. Notably, we need not even use it in the general case. What we need is a simple affine theorem which is a special case of the Pappus theorem. We denote the intersection of two lines g and g′ by g ∩ g′. Here is the Pappus theorem in the general case.

Theorem 1. *Given two lines in a plane, let A, B, C be three points on one line and A', B', C' three points on the other line. The three points*

$$BC' \cap CB', \qquad CA' \cap AC', \qquad AB' \cap BA'$$

are collinear.

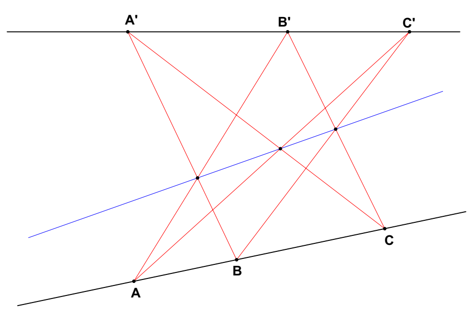

Figure 1

Theorem 1 remains valid if some of the points A, B, C, A', B', C' are projected to infinity, even if one of the two lines is the line at infinity. In this paper, the only case we need is the special case if the points A', B', C' are points at infinity. For the sake of completeness, we give a proof of the Pappus theorem for this case.

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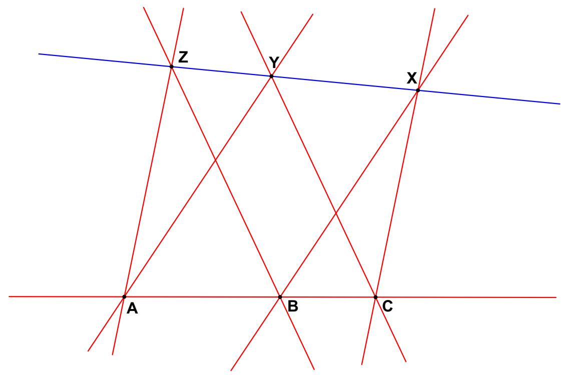

Figure 2

Let $X = BC' \cap CB'$, $Y = CA' \cap AC'$, $Z = AB' \cap BA'$. The points A', B', C' being infinite points, we have $CY \parallel BZ$, $AZ \parallel CX$, and $BX \parallel AY$. See Figure 2. We assume the lines ZX and ABC intersect at a point P, and leave the easy case $ZX \parallel ABC$ to the reader. In Figure 3, let $Y' = ZX \cap AY$. We show that $Y' = Y$. Since $AY \parallel BX$, we have $\frac{PA}{PB} = \frac{PY'}{PX}$ in signed lengths. Since $AZ \parallel CX$, we have $\frac{PC}{PA} = \frac{PX}{PZ}$. From these, $\frac{PC}{PB} = \frac{PY'}{PZ}$, and $CY' \parallel BZ$. Since $CY \parallel BZ$, the point Y' lies on the line CY. Thus, $Y' = Y$, and the points X, Y, Z are collinear.

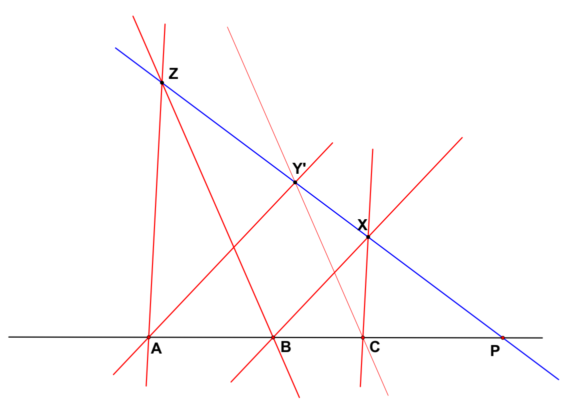

Figure 3

2. The orthocenters of a fourline

We denote by $\Delta\mathsf{abc}$ the triangle bounded by three lines a, b, c. A complete quadrilateral, or, simply, a fourline is a set of four lines in a plane. The fourline consisting of lines a, b, c, d, is denoted by $\square\mathsf{abcd}$. If g is a line, then all lines perpendicular to g have an infinite point in common. This infinite point will be called $\overline{\mathsf{g}}$. With this notation, $P\overline{\mathsf{g}}$ is the perpendicular from P to g. Now, we establish the well-known Steiner's theorem.

Theorem 2 (Steiner)**.** *If* a, b, c, d *are any four lines, the orthocenters of* $\Delta\mathsf{bcd}$, $\Delta\mathsf{acd}$, $\Delta\mathsf{abd}$, $\Delta\mathsf{abc}$ *are collinear.*

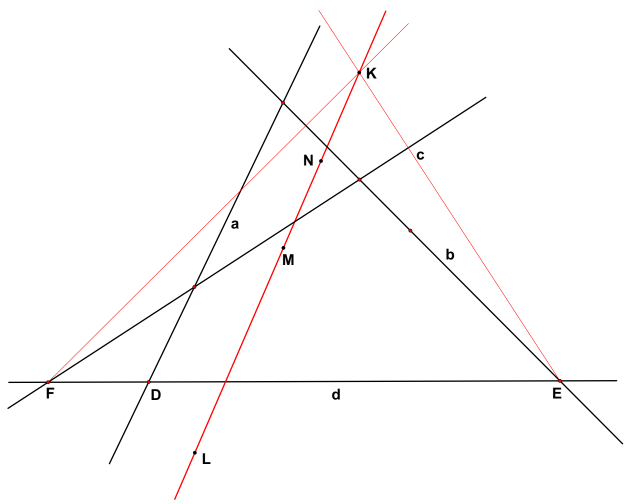

Figure 4

Proof. Let D, E, F be the intersections of d with a, b, c, and K, L, M, N the orthocenters of $\Delta\mathsf{bcd}$, $\Delta\mathsf{acd}$, $\Delta\mathsf{abd}$, and $\Delta\mathsf{abc}$. Note that $K = E\overline{\mathsf{c}} \cap F\overline{\mathsf{b}}$, being the intersection of the perpendiculars from E to c and from F to b. Similarly, $L = F\overline{\mathsf{a}} \cap D\overline{\mathsf{c}}$ and $M = D\overline{\mathsf{b}} \cap E\overline{\mathsf{a}}$. The points D, E, F being collinear and the points $\overline{\mathsf{a}}$, $\overline{\mathsf{b}}$, $\overline{\mathsf{c}}$ being infinite, we conclude from the Pappus theorem that K, L, M are collinear. Similarly, L, M, N are collinear. The four orthocenters lie on the same line. $\square$

The line $KLMN$ is called the Steiner line of the fourline $\square ABCD$. Theorem 2 is usually associated with Miquel points [6, §9] and proved using radical axes. A consequence of such proofs is the fact that the Steiner line of the fourline $\square\mathsf{abcd}$ is the radical axis of the circles with diameters AD, BE, CF, where $A = \mathsf{b} \cap \mathsf{c}$, $B = \mathsf{c} \cap \mathsf{a}$, $C = \mathsf{a} \cap \mathsf{b}$, $D = \mathsf{d} \cap \mathsf{a}$, $E = \mathsf{d} \cap \mathsf{b}$, $F = \mathsf{d} \cap \mathsf{c}$. Also, the Steiner line is the directrix of the parabola touching the four lines a, b, c, d. The Steiner line is also called four-orthocenter line in [6, §11] or the orthocentric line in [5], where it is studied using barycentric coordinates.

3. The orthopole and the fourline

We prove the theorem that gives rise to the notion of orthopole.

Theorem 3. *Let ΔABC be a triangle and* d *a line. If A', B', C' are the pedals of A, B, C on* d*, then the perpendiculars from A', B', C' to the lines BC, CA, AB intersect at one point.*

This point is the orthopole of the line d with respect to ΔABC.

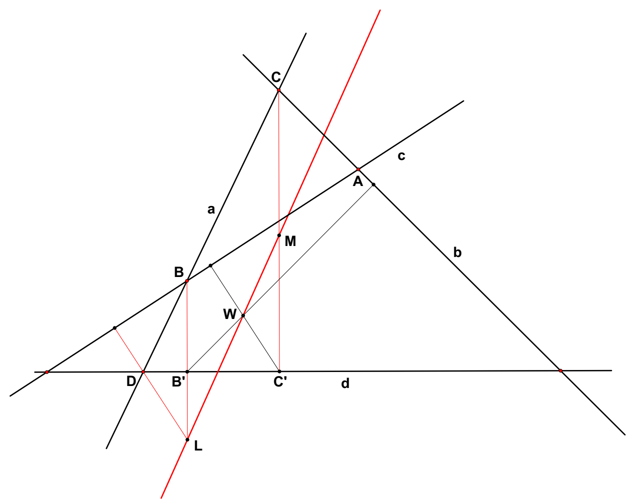

Figure 5

Proof. Denote by a, b, c the lines BC, CA, AB. By Theorem 2, the orthocenters K, L, M, N of triangles $\Delta\mathsf{bcd}$, $\Delta\mathsf{acd}$, $\Delta\mathsf{abd}$, $\Delta\mathsf{abc}$ lie on a line. Let $D = \mathsf{d} \cap \mathsf{a}$, and $W = B'\overline{\mathsf{b}} \cap C'\overline{\mathsf{c}}$. The orthocenter L of $\Delta\mathsf{acd}$ is the intersection of the perpendiculars from D to c and from B to d. Since the perpendicular from B to d is also the perpendicular from B' to d, $L = D\overline{\mathsf{c}} \cap B'\overline{\mathsf{d}}$. Analogously, $M = D\overline{\mathsf{b}} \cap C'\overline{\mathsf{d}}$. By the Pappus theorem, the points W, M, L are collinear. Hence, W lies on the line $KLMN$. Since $W = B'\overline{\mathsf{b}} \cap C'\overline{\mathsf{c}}$, the intersection W of the lines $KLMN$ and $B'\overline{\mathsf{b}}$ lies on $C'\overline{\mathsf{c}}$. Similarly, this intersection W lies on $A'\overline{\mathsf{a}}$. Hence, the point W is the common point of the four lines $A'\overline{\mathsf{a}}$, $B'\overline{\mathsf{b}}$, $C'\overline{\mathsf{c}}$, and $KLMN$. Since $A'\overline{\mathsf{a}}$, $B'\overline{\mathsf{b}}$, $C'\overline{\mathsf{c}}$ are the perpendiculars from A', B', C' to a, b, c respectively, the perpendiculars from A', B', C' to BC, CA, AB and the line $KLMN$ intersect at one point. This already shows more than the statement of the theorem. In fact, we conclude that the orthopole of d with respect to triangle ΔABC lies on the Steiner line of the complete quadrilateral $\square\mathsf{abcd}$. $\square$

The usual proof of Theorem 3 involves similar triangles ([1], [10, Chapter 11]) and does not directly lead to the fourline. Theorem 4 originates from R. Goormaghtigh, published as a problem [7]. It was also mentioned in [5, Proposition 6], with reference to [2]. The following corollary is immediate.

Corollary 4. *Given a fourline* $\square$abcd*, the orthopoles of* a, b, c, d *with respect to* $\triangle$bcd, $\triangle$acd, $\triangle$abd, $\triangle$abc *lie on the Steiner line of the fourline.*

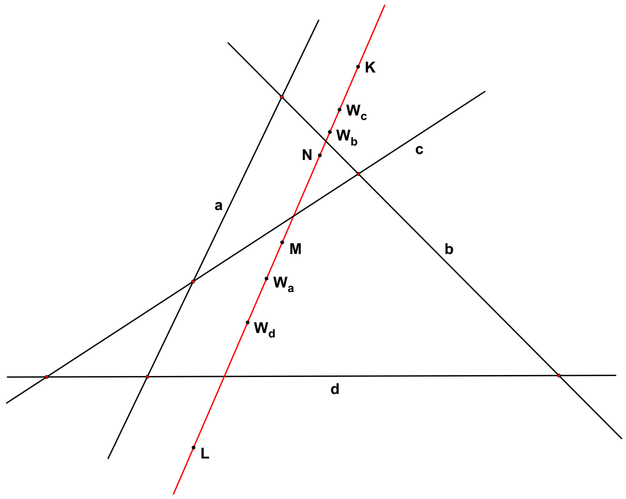

Figure 6

4. Two theorems on the collinarity of quadruples of orthopoles

Theorem 5. *If A, B, C, D are four points and* e *is a line, then the orthopoles of* e *with respect to triangles ΔBCD, ΔCDA, ΔDAB, ΔABC are collinear.*

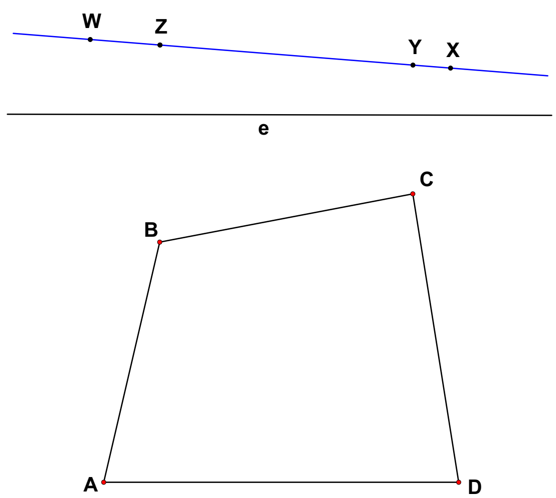

Figure 7

Proof. Denote these orthopoles by X, Y, Z, W respectively. If A', B', C', D' are the pedals of A, B, C, D on e, then $X = B'\overline{CD} \cap C'\overline{BD}$. Similarly, $Y = C'\overline{AD} \cap A'\overline{CD}$, $Z = A'\overline{BD} \cap B'\overline{AD}$. Now, A', B', C' lie on one line, and $\overline{AD}$, $\overline{BD}$, $\overline{CD}$ lie on the line at infinity. By Pappus' theorem, the points X, Y, Z are collinear. Likewise, Y, Z, W are collinear. We conclude that all four points X, Y, Z, W are collinear. □

Theorem 5 was also proved using coordinates by N. Dergiades in [3] and by R. Goormaghtigh in [8, p.178]. A special case of Theorem 5 was shown in [11] using the Desargues theorem.[1] Another theorem surprisingly similar to Theorem 5 was shown in [9] using complex numbers.

Theorem 6. *Given five lines* a, b, c, d, e, *the orthopoles of* e *with respect to* $\triangle\mathsf{bcd}$, $\triangle\mathsf{acd}$, $\triangle\mathsf{abd}$, $\triangle\mathsf{abc}$ *are collinear.*

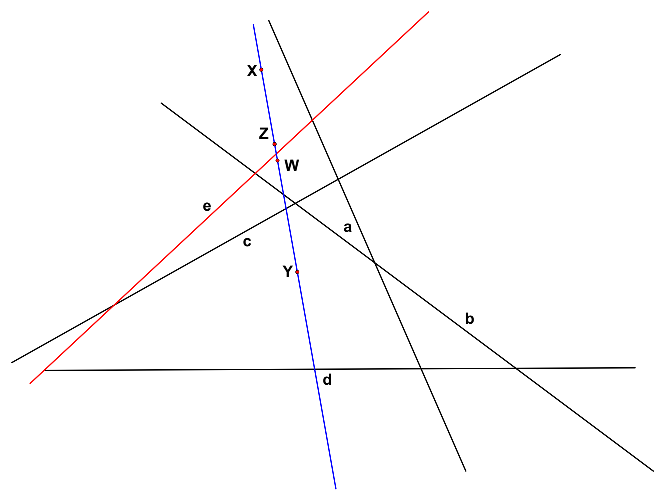

Figure 8

Proof. Denote these orthopoles by X, Y, Z, W respectively. Let the line d intersect a, b, c at D, E, F, and let D', E', F' be the pedals of D, E, F on e.

Since $E = \mathsf{b} \cap \mathsf{d}$ and $F = \mathsf{c} \cap \mathsf{d}$ are two vertices of triangle $\triangle\mathsf{bcd}$, and E' and F' are the pedals of these vertices on e, the orthopole $X = E'\overline{\mathsf{c}} \cap F'\overline{\mathsf{b}}$. Similarly, $Y = F'\overline{\mathsf{a}} \cap D'\overline{\mathsf{c}}$, and $Z = D'\overline{\mathsf{b}} \cap E'\overline{a}$. Since D', E', F' lie on one line, and $\overline{\mathsf{a}}$, $\overline{\mathsf{b}}$, $\overline{\mathsf{c}}$ lie on the line at infinity, the Pappus theorem yields the collinearity of the points X, Y, Z. Analogously, the points Y, Z, W are collinear. The four points X, Y, Z, W are on the same line. □

[1] In [11], Witczyński proves Theorem 5 for the case when A, B, C, D lie on one circle and the line e crosses this circle. Instead of orthopoles, he equivalently considers Simson lines. The Simson lines of two points on the circumcircle of a triangle intersect at the orthopole of the line joining the two points.

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On the Areas of the Intouch and Extouch Triangles

Juan Carlos Salazar

Abstract. We prove an interesting relation among the areas of the triangles whose vertices are the points of tangency of the sidelines with the incircle and excircles.

1. The intouch and extouch triangles

Consider a triangle ABC with incircle touching the sides BC, CA, AB at A_0, B_0, C_0 respectively. The triangle $A_0B_0C_0$ is called the intouch triangle of ABC. Likewise, the triangle formed by the points of tangency of an excircle with the sidelines is called an extouch triangle. There are three of them, the A-, B-, C-extouch triangles, [1] as indicated in Figure 1. For $i = 0, 1, 2, 3$, let T_i denote the area of triangle $A_iB_iC_i$. In this note we present two proofs of a simple interesting relation among the areas of these triangles.

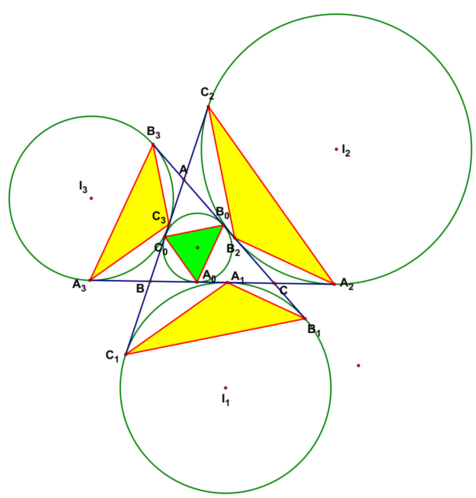

Figure 1

Theorem 1. $\frac{1}{T_0} = \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3}$.

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[1]These qualified extouch triangles are not the same as the extouch triangle in [2, §6.9], which means triangle $A_1B_2C_3$ in Figure 1. For a result on this unqualified extouch triangle, see §3.

Proof. Let I be the incenter and r the inradius of triangle ABC. Consider the excircle on the side BC, with center I_1, tangent to the lines BC, CA, AB at A_1, B_1, C_1 respectively. See Figure 2. It is easy to see that triangles $I_1A_1C_1$ and BA_0C_0 are similar isosceles triangles; so are triangles $I_1A_1B_1$ and CA_0B_0. From these, it easily follows that the angles $B_0A_0C_0$ and $B_1I_1C_1$ are supplementary. It follows that

$$\frac{T_0}{T_1} = \frac{A_0B_0 \cdot A_0C_0}{A_1B_1 \cdot A_1C_1} = \frac{IC}{I_1C} \cdot \frac{IB}{I_1B} = \frac{IB \cdot IC}{I_1B \cdot I_1C}.$$

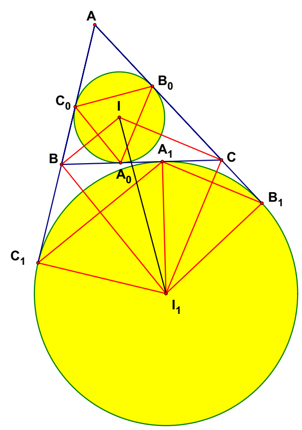

Figure 2

Now, in the cyclic quadrilateral IBI_1C with diameter II_1,

$$IB \cdot IC = IB \cdot II_1 \sin II_1C = II_1 \cdot IA_0 = r \cdot II_1.$$

Similarly, $I_1B \cdot I_1C = II_1 \cdot r_1$, where r_1 is the radius of the A-excircle. It follows that

$$\frac{T_0}{T_1} = \frac{r}{r_1}. \tag{1}$$

Likewise, $\frac{T_0}{T_2} = \frac{r}{r_2}$ and $\frac{T_0}{T_3} = \frac{r}{r_3}$, where r_2 and r_3 are respectively the radii of the B- and C-excircles. From these,

$$\frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} = \left(\frac{r}{r_1} + \frac{r}{r_2} + \frac{r}{r_3}\right)\frac{1}{T_0} = \frac{1}{T_0},$$

since $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$. $\square$

Corollary 2. *Let $ABCD$ be a quadrilateral with an incircle $I(r)$ tangent to the sides at W, X, Y, Z. If the excircles $I_W(r_W)$, $I_X(r_X)$, $I_Y(r_Y)$, $I_Z(r_Z)$ have areas T_W, T_X, T_Y, T_Z respectively, then*

$$\frac{T_W}{r_W} + \frac{T_Y}{r_Y} = \frac{T_X}{r_X} + \frac{T_Z}{r_Z} = \frac{T}{r},$$

where T is the area of the intouch quadrilateral $WXYZ$. See Figure 3.

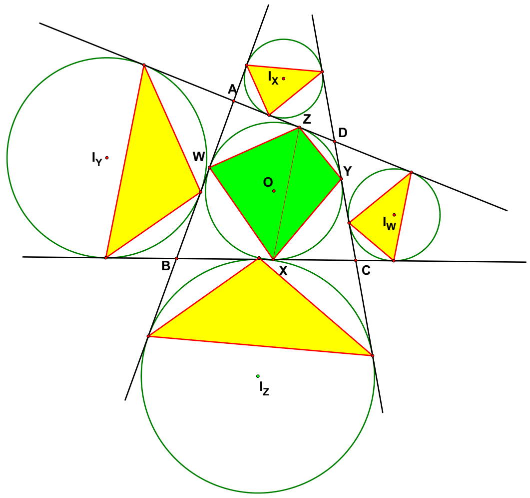

Figure 3

Proof. By (1) above, we have $\frac{T_W}{r_W} = \frac{\text{Area } XYZ}{r}$ and $\frac{T_Y}{r_Y} = \frac{\text{Area } ZWX}{r}$ so that

$$\frac{T_W}{r_W} + \frac{T_Y}{r_Y} = \frac{\text{Area } XYZ + \text{Area } ZWX}{r} = \frac{T}{r}.$$

Similarly, $\frac{T_X}{r_X} + \frac{T_Z}{r_Z} = \frac{T}{r}$. □

2. An alternative proof using barycentric coordinates

The area of a triangle can be calculated easily from its barycentric coordinates. Denote by Δ the area of the reference triangle ABC. The area of a triangle with vertices $A' = (x_1 : y_1 : z_1)$, $B' = (x_2 : y_2 : z_2)$, $C' = (x_3 : y_3 : z_3)$ is given by

$$\frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta}{(x_1+y_1+z_1)(x_2+y_2+z_2)(x_3+y_3+z_3)}. \tag{2}$$

Note that this area is signed. It is positive or negative according as triangle $A'B'C'$ has the same or opposite orientation as the reference triangle. See, for example, [3]. In particular, the area of the cevian triangle of a point with coordinates $(x : y : z)$ is

$$\frac{\begin{vmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{vmatrix} \Delta}{(y+z)(z+x)(z+y)} = \frac{2xyz\Delta}{(y+z)(z+x)(z+y)}. \tag{3}$$

Let s denote the semiperimeter of triangle ABC, *i.e.*, $s = \frac{1}{2}(a+b+c)$.

The barycentric coordinates of the vertices of the intouch triangle are

$$A_0 = (0 : s-c : s-b), \quad B_0 = (s-c : 0 : s-a), \quad C_0 = (s-b : s-a : 0). \tag{4}$$

The area of the intouch triangle is

$$\begin{aligned}T_0 =&\frac{1}{abc}\begin{vmatrix}0 & s-c & s-b\\ s-c & 0 & s-a\\ s-b & s-a & 0\end{vmatrix}\Delta\\ =&\frac{2(s-a)(s-b)(s-c)}{abc}\Delta.\end{aligned}$$

For the A-extouch triangle $A_1B_1C_1$,

$$A_1 = (0 : s-b : s-c), \quad B_1 = (-(s-b) : 0 : s), \quad C_1 = (-(s-c) : s : 0), \tag{5}$$

the area is

$$\frac{1}{abc}\begin{vmatrix}0 & s-b & s-c\\ -(s-b) & 0 & s\\ -(s-c) & s & 0\end{vmatrix}\Delta = \frac{-2s(s-b)(s-c)}{abc}\Delta.$$

Similarly, the areas of the B- and C-extouch triangles are $\frac{-2s(s-c)(s-a)}{abc}\Delta$ and $\frac{-2s(s-a)(s-b)}{abc}\Delta$ respectively. Note that these are all negative. Disregarding signs, we have

$$\begin{aligned}\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3} =&\frac{abc}{2s(s-a)(s-b)(s-c)}\left((s-a)+(s-b)+(s-c)\right)\cdot\frac{1}{\Delta}\\ =&\frac{abc}{2(s-a)(s-b)(s-c)}\cdot\frac{1}{\Delta}\\ =&\frac{1}{T_0}.\end{aligned}$$

3. A generalization

Using the area formula (3) it is easy to see that the (unqualifed) extouch triangle $A_1B_2C_3$ has the same area T_0 as the intouch triangle. This is noted, for example, in [1]. The use of coordinates in §2 also leads to a more general result. Replace the incircle by the inscribed conic with center $P = (p : q : r)$, and the excircles by those with centers

$$P_1 = (-p : q : r), \quad P_2 = (p : -q : r), \quad P_3 = (p : q : -r),$$

respectively. These are the vertices of the anticevian triangle of P, and the four inscribed conics are homothetic. See Figure 4. The coordinates of their points of tangency with the sidelines can be obtained from (4) and (5) by replacing a, b, c by p, q, r respectively. It follows that the areas of intouch and extouch triangles for these conics bear the same relation given in Theorem 1.

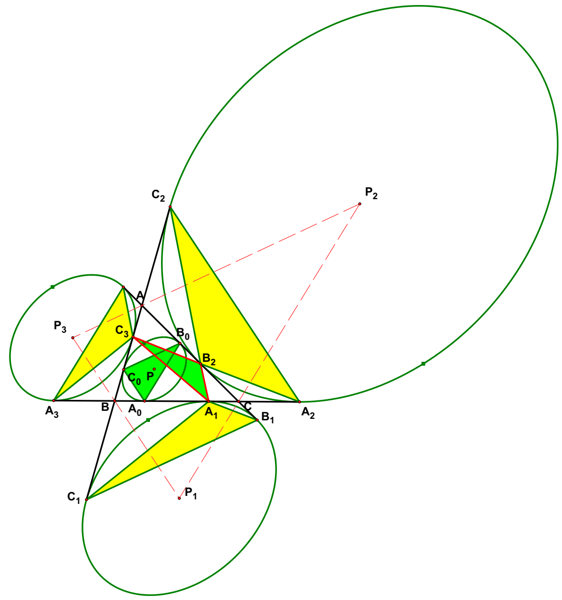

Figure 4

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Signed Distances and the Erdős-Mordell Inequality

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Abstract. Using signed distances from the sides of a triangle we prove an inequality from which we get the Erdős-Mordell inequality as a simple consequence.

Let P be an arbitrary point in the plane of triangle ABC. Denote by x_1, x_2, x_3 the distances of P from the vertices A, B, C, and d_1, d_2, d_3 the *signed* distances of P from the sidelines BC, CA, AB respectively. Let a, b, c be the lengths of these sides. We establish an inequality from which the famous Erdős-Mordell inequality easily follows.

Theorem.

$$x_1 + x_2 + x_3 \geq \left(\frac{b}{c} + \frac{c}{b}\right) d_1 + \left(\frac{c}{a} + \frac{a}{c}\right) d_2 + \left(\frac{a}{b} + \frac{b}{a}\right) d_3; \qquad (1)$$

equality holds if and only if P is the circumcenter of ABC.

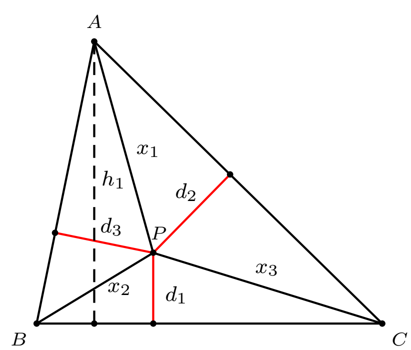

Figure 1

Proof. Let h_1 be the length of the altitude from A to BC, and Δ the area of ABC. Clearly,

$$2\Delta = ah_1 = ad_1 + bd_2 + cd_3.$$

Note that $x_1 + d_1 \geq h_1$. This is true even if $d_1 < 0$, *i.e.*, when P is not an interior point of the triangle. Also, equality holds if and only if P lies on the line containing the A-altitude. We have $ax_1 + ad_1 \geq ah_1 = ad_1 + bd_2 + cd_3$, or

$$ax_1 \geq bd_2 + cd_3. \qquad (2)$$

If we apply inequality (2) to triangle $AB'C'$ symmetric to ABC with respect to the A-bisector of ABC we get

$$ax_1 \geq cd_2 + bd_3$$

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or

$$x_1 \geq \frac{c}{a}d_2 + \frac{b}{a}d_3. \tag{3}$$

Equality holds only when P lies on the A-altitude of $AB'C'$, *i.e.*, the line passing through A and the circumcenter of ABC.

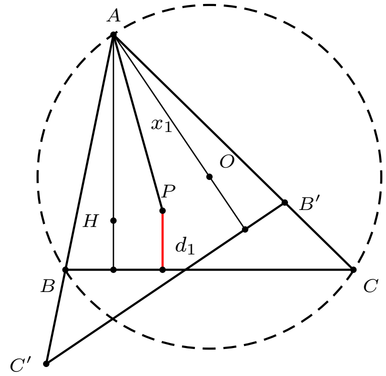

Figure 2

Similarly we get

$$x_2 \geq \frac{a}{b}d_3 + \frac{c}{b}d_1, \tag{4}$$

$$x_3 \geq \frac{b}{c}d_1 + \frac{a}{c}d_2, \tag{5}$$

and by addition of (3), (4), (5), we get the inequality (1). Equality holds only when P is the circumcenter of ABC. □

If P is an internal point of ABC, $d_1,\ d_2,\ d_3 > 0$. Since $\frac{b}{c} + \frac{c}{b} \geq 2$, $\frac{c}{a} + \frac{a}{c} \geq 2$, $\frac{a}{b} + \frac{b}{a} \geq 2$, we have

$$x_1 + x_2 + x_3 \geq 2(d_1 + d_2 + d_3).$$

This is the famous Erdős-Mordell inequality. The equality holds only when $a = b = c$, *i.e.*, ABC is equilateral, and P is the circumcenter of ABC.

There are numerous proofs of the Erdős-Mordell inequality. See, for example, [3] and the bibliography therin. In Mordell's original proof [2], the inequality (1) was established assuming $d_1, d_2, d_3 > 0$. See also [1, §12.13]. Our proof of (1) is more transparent and covers all positions of P.

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A Simple Construction of the Congruent Isoscelizers Point

Eric Danneels

Abstract. We give a very simple construction of the congruent isoscelizers point as an application of the cevian nest theorem.

1. Construction of the congruent isoscelizers point

Given a triangle, an isoscelizer is a segment intercepted in the interior of the triangle by a line perpendicular to an angle bisector. There is a unique point through which the three isoscelizers have equal lengths. This is the congruent isoscelizers points X_{173} of [4]. In this note we present a very simple construction of this triangle center.

Theorem 1. *Let $A'B'C'$ be the intouch triangle of ABC, and $A''B''C''$ the intouch triangle of $A'B'C'$. The triangles $A''B''C''$ and ABC are perspective at the congruent isoscelizers point of ABC.*

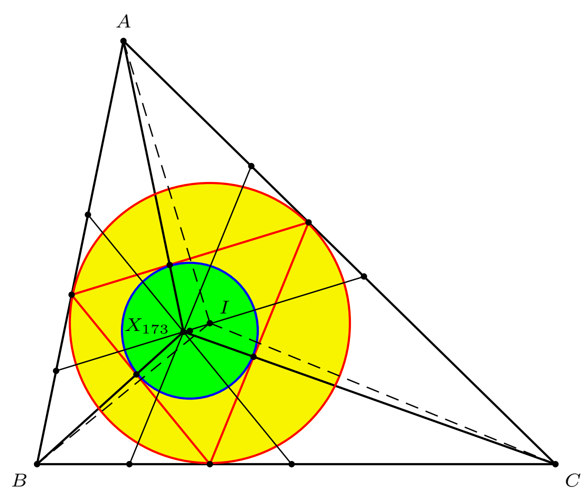

Figure 1

The proof is a simple application of the following cevian nest theorem.[1]

Theorem 2. *Let $A'B'C'$ be the cevian triangle of P in triangle ABC with homogeneous barycentric coordinates $(u : v : w)$ with respect to ABC, and $A''B''C''$*

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[1]Theorem 2 appears in [1, p.165, Supplementary Exercise 7] as follows: The triangle $(Q) = DEF$ is inscribed in the triangle $(P) = ABC$, and the triangle $(K) = KLM$ is inscribed in (Q). Show that if any two of these triangles are perspective to the third, they are perspective to each other.

the cevian triangle of Q in triangle $A'B'C'$, with homogeneous barycentric coordinates $(x : y : z)$ with respect to triangle $A'B'C'$. Triangle $A''B''C''$ is the cevian triangle of

$$Q(P) = \left(\frac{u(v+w)}{x} : \frac{v(w+u)}{y} : \frac{w(u+v)}{z}\right) \tag{1}$$

with respect to triangle ABC.

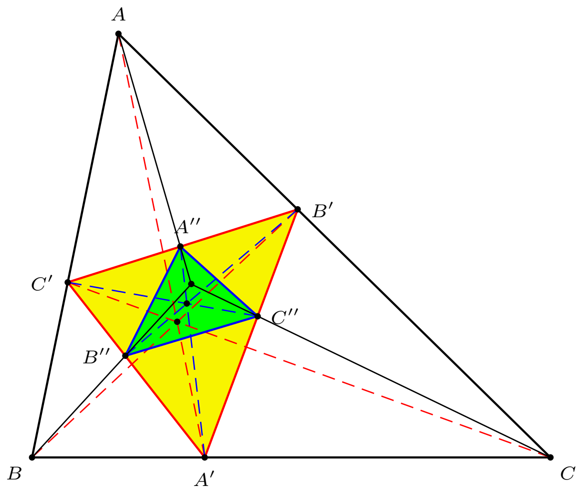

Figure 2

The concurrency of the lines AA'', BB'', CC'' follows from the fact every cevian triangle and every anticevian triangle with respect to $A'B'C'$ are perspective. See, for example, [3, §2.12]. The cevian and anticevian triangles in question are $A''B''C''$ and ABC respectively.

Proof. We compute the absolute barycentric coordinates explicitly.

$$\begin{aligned} A'' =& \frac{yB' + zC'}{y+z} = \frac{y \cdot \frac{wC+uA}{w+u} + z \cdot \frac{uA+vB}{u+v}}{y+z} \\ =& \frac{(y(u+v) + z(w+u))uA + zv(w+u)B + yw(u+v)C}{(y+z)(w+u)(u+v)}. \end{aligned}$$

It is clear that the line AA'' intersects BCat the point with homogeneous barycentric coordinates

$$(0 : zv(w+u) : yw(u+v)) = \left(0 : \frac{v(w+u)}{y} : \frac{w(u+v)}{z}\right).$$

Similarly, the intersections of BB'' with CA, CC'' with AB are the points

$$\left(\frac{u(v+w)}{x} : 0 : \frac{w(u+v)}{z}\right) \quad \text{and} \quad \left(\frac{u(v+w)}{x} : \frac{v(w+u)}{y} : 0\right)$$

respectively. It is clear that the lines AA'', BB'', CC'' intersect at the point given by (1) above. □

2. Proof of Theorem 1

Let P be the Gergonne point, and $A'B'C'$ the intouch triangle. The sidelengths are in the proportions of

$$B'C' : C'A' : A'B' = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}.$$

If Q is the Gergonne point of $A'B'C'$, then we have

$$Q(P) = \left(a\left(-\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) : \cdots : \cdots\right).$$

This is the point X_{173}, the congruent isoscelizers point.

3. Another example

Let P be the incenter of triangle ABC, with cevian triangle $A'B'C'$, and Q the centroid of $A'B'C'$. Then

$$Q(P) = (a(b+c) : b(c+a) : c(a+b)).$$

This is the triangle center X_{37} of [4].

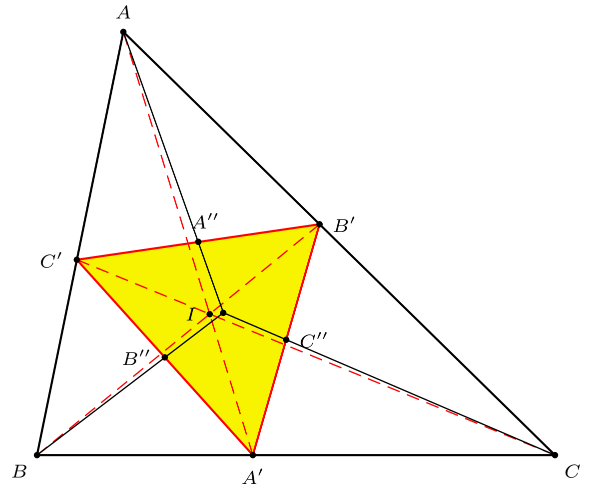

Figure 3

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Triangles with Special Isotomic Conjugate Pairs

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Abstract. We study the condition for the line joining a pair of isotomic conjugates to be parallel to a side of a given triangle. We also characterize triangles in which the line joining a specified pair of isotomic conjugates is parallel to a side.

1. Introduction

Two points in the plane of a given triangle ABC are called isotomic conjugates if the cevians through them divide the opposite sides in ratios that are reciprocals to each other. See [3], also [1]. We study the condition for the line joining a pair of isotomic conjugates to be parallel to a side of a given triangle. We also characterize triangles in which the line joining a specified pair of isotomic conjugates is parallel to a side.

2. Some background material

The standard notation is used throughout: a, b, c for the sides or the lengths of BC, CA, AB respectively of triangle ABC. The median and the altitude through A (and their lengths) are denoted by m_a and h_a respectively. We denote the centroid, the incenter, and the circumcenter by G, I, and O respectively.

2.1. *The orthic triangle.* The triangle formed by the feet of the altitudes is called its orthic triangle. It is the cevian triangle of the orthocenter H. Its sides are easily calculated to be the absolute values of $a\cos A$, $b\cos B$, $c\cos C$.

2.2. *The Gergonne and symmedian points.* The Gergonne point Γ is the concurrence point of the cevians that connect the vertices of triangle ABC to the points of contact of the opposite sides with the incircle.

The symmedian point K is the Gergonne point of the tangential triangle which is bounded by the tangents to the circumcircle at A, B, C.

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2.3. *The Brocard points.* The Crelle-Brocard points Ω_+ and Ω_- are the interior points such that

$$\angle\Omega_+AB = \angle\Omega_+BC = \angle\Omega_+CA = \omega,$$
$$\angle\Omega_-AC = \angle\Omega_-BA = \angle\Omega_-CB = \omega,$$

where ω is the Crelle-Brocard angle.

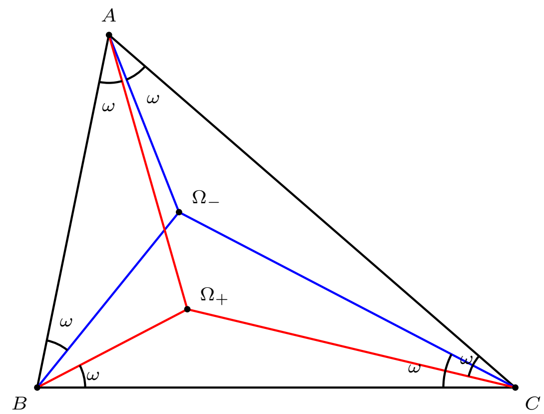

Figure 1

It is known that

$$\cot\omega = \cot A + \cot B + \cot C.$$

See, for example, [3, 5]. According to [4],

$$A + \omega = \frac{\pi}{2} \text{ if and only if } \tan^2 A = \tan B \tan C. \tag{1}$$

2.4. *Self-altitude triangles.* The sides a, b, c of a triangle are in geometric progression if and only if they are proportional to h_a, h_b, h_c in some order. Such a triangle is called a self-altitude triangle in [6]. It has a number of interesting properties. Suppose $a^2 = bc$. Then

(1) Ω_+ and Ω_- are the perpendicular feet of the symmedian point K on the perpendicular bisectors of AC and AB.
(2) The line $\Omega_+\Omega_-$ coincides with the bisector AI.
(3) $B\Omega_+$ and $C\Omega_-$ are tangent to the Brocard circle which has diameter OK.
(4) The median BG and the symmedian CK intersect on AI; so do CG and BK.

See Figure 2.

2.5. *A generalization of a property of equilateral triangles.* An equilateral triangle ABC has this easily provable property: if P is any point on the minor arc BC of the circumcircle of ABC, then $AP = BP + PC$. Surprisingly, however, if triangle ABC is non-isosceles, then there exists a unique point P on the arc BC (not containing the vertex A) such that $AP = BP + PC$ if and only if $a = \frac{mb^2+nc^2}{mb+nc}$.

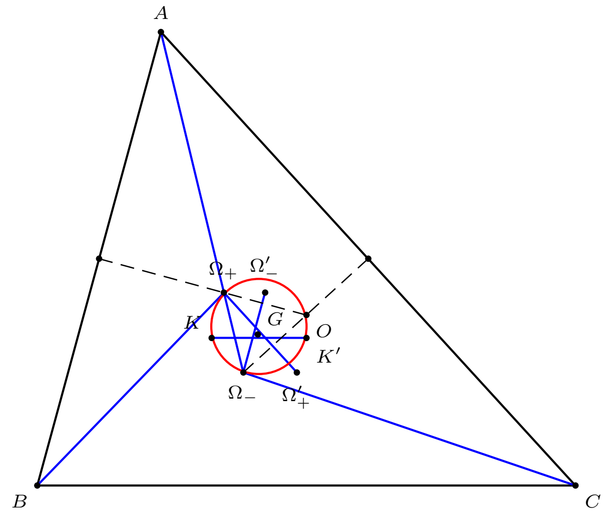

Figure 2

See [8]. Here, $\frac{m}{n}$ is the ratio in which AP divides the side BC. In particular, the extension AP of the median m_a has the preceding property if and only if

$$a = \frac{b^2 + c^2}{b + c}. \tag{2}$$

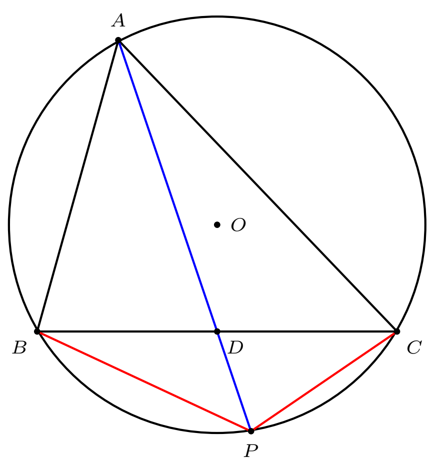

Figure 3.

3. Homogeneous barycentric coordinates

With reference to triangle ABC, every point in the plane is specified by a set of homogeneous barycentric coordinates. See, for example, [9]. If P is a point (not on any of the side lines of triangle ABC) with coordinates $(x : y : z)$, its isotomic conjugate P' has coordinates $\left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. Here are the coordinates of some of classical triangle centers.

Point	Coordinates
centroid G	$(1 : 1 : 1)$
incenter I	$(a : b : c)$
circumcenter O	$(a\cos A : b\cos B : c\cos C)$
orthocenter H	$(\tan A : \tan B : \tan C)$
symmedian point K	$(a^2 : b^2 : c^2)$
Gergonne point Γ	$\left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right)$
Brocard point Ω_+	$\left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$
Brocard point Ω_-	$\left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$

The isotomic conjugate of the Gergonne point is the Nagel point N, which is the concurrence points of the cevians joining the vertices to the point of tangency of its opposite side with the excircle on that side. It has coordinates $(b+c-a : c+a-b : a+b-c)$.

The homogeneous barycentric coordinate of a point can be normalized to give its *absolute* homogeneous barycentric coordinate, provided the sum of the coordinates is nonzero. If $P = (x : y : z)$, we say that in absolute barycentric coordinates,

$$P = \frac{xA + yB + zC}{x + y + z},$$

provided $x + y + z \neq 0$. Points $(x : y : z)$ with $x + y + z = 0$ are called infinite points. The isotomic conjugate of $P = (x : y : z)$ is an infinite point if and only if $xy + yz + zx = 0$. This is the Steiner circum-ellipse which has center at the centroid G of triangle ABC. Another fruitful way is to view an infinite point as the difference $Q - P$ of the absolute barycentric coordinates of two points P and Q. As such, it represents the vector $\overrightarrow{PQ}$.

4. The basic results

The segment joining P to its isotomic conjugate is represented by the infinite point

$$\begin{aligned} PP' =& \frac{yzA + zxB + xyC}{xy + yz + zx} - \frac{xA + yB + zC}{x + y + z} \\ =& \frac{(y+z)(yz - x^2)A + (z+x)(zx - y^2)B + (x+y)(xy - z^2)C}{(x+y+z)(xy+yz+zx)}. \end{aligned} \tag{3}$$

This is parallel to the line BC if it is a multiple of the infinte point of BC, namely, $-B + C$. This is the case if and only if

$$(y + z)(x^2 - yz) = 0. \tag{4}$$

The equation $y + z = 0$ represents the line through A parallel to BC. It is clear that this line is invariant under isotomic conjugation. Every finite point on this line has coordinates $(x : 1 : -1)$ for a nonzero x. Its isotomic conjugate is the point $(\frac{1}{x} : 1 : -1)$ on the same line. On the other hand, the equation $x^2 - yz = 0$ represent an ellipse homothetic to the Steiner circum-ellipse. It passes through $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$, $G = (1 : 1 : 1)$, and $(-1 : 1 : 1)$. It is tangent to AB and AC at B and C respectively. It is obtained by translating the Steiner circum-ellipse along the vector $\overrightarrow{AG}$. We summarize this in the following theorem.

Theorem 1. *Let P be a finite point. The line joining P to its isotomic conjugate if parallel to BC if and only if P lies on the line through A parallel to BC or the ellipse through the centroid tangent to AB and AC at B and C respectively. In the latter case, the isotomic conjugate P' is the second intersection of the ellipse with the line through P parallel to BC.*

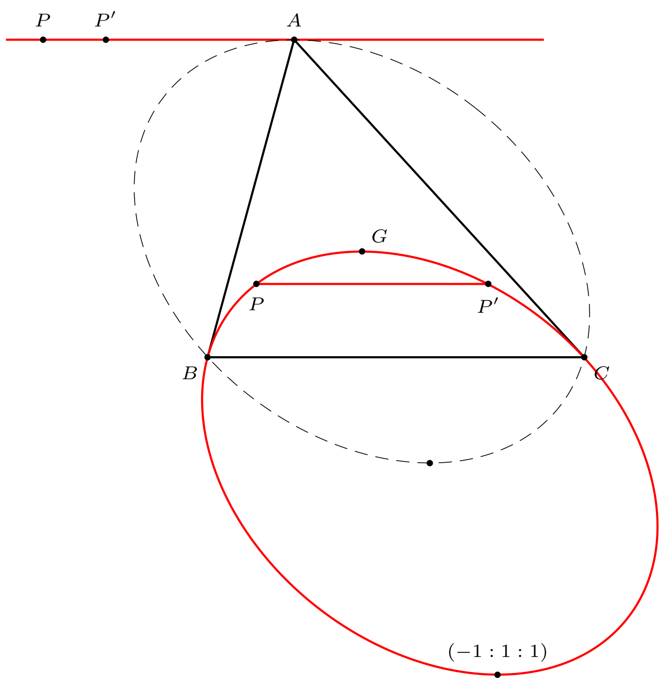

Figure 4

Now we consider the possibility for PP' not only to be parallel to BC, but also equal to one half of its length. This means that the vector PP' is $\pm\frac{1}{2}(C - B)$. If P is a finite point on the parallel to BC through A, we write $P = (x : 1 : -1)$, $x \neq 0$. From (3), we have $PP' = \frac{(1-x^2)(-B+C)}{x} = \frac{1}{2}(-B + C)$ if and only if $x = \frac{-1\pm\sqrt{17}}{4}$. These give the first two pairs of isotomic conjugates listed in Theorem 2 below.

By Theorem 1, P may also lie on the ellipse $x^2 - yz = 0$. It is convenient to use a parametrization

$$x = \mu, \quad y = \mu^2, \quad z = 1. \tag{5}$$

Setting the coefficient of C in (3) to $\frac{1}{2}$, simplifying, we obtain

$$\frac{\mu^2-\mu-3}{2(\mu^2+\mu+1)}=0.$$

The only possibilities are $\mu=\frac{1}{2}\left(1\pm\sqrt{13}\right)$. These give the last two pairs in Theorem 2 below.

Theorem 2. *There are four pairs of isotomic conjugates P, P' for which the segment PP' is parallel to BC and has half of its length.*

i	P_i	P'_i
1	$(\sqrt{17}-1:4:-4)$	$(\sqrt{17}+1:4:-4)$
2	$(\sqrt{17}+1:-4:4)$	$(\sqrt{17}-1:-4:4)$
3	$(\sqrt{13}+1:\ \sqrt{13}+7:\ 2)$	$(\sqrt{13}+1:\ 2:\ \sqrt{13}+7)$
4	$(-(\sqrt{13}-1):\ 7-\sqrt{13}:\ 2)$	$(-(\sqrt{13}-1):\ 2:\ 7-\sqrt{13})$

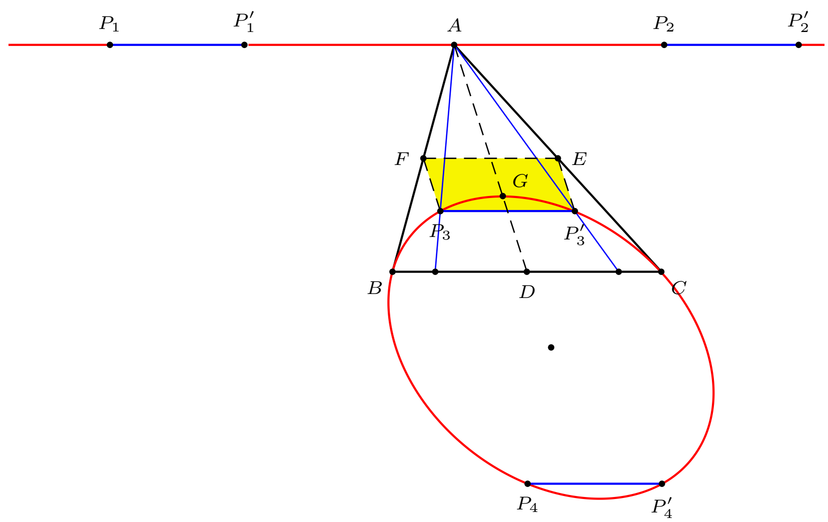

Figure 5

Among these four pairs, only the pair $(P_3,\ P'_3)$ are interior points. The segments FP_3 and EP'_3 are parallel to the median AD, and $P_3P'_3EF$ is a parallelogram with $FP_3=EP'_3=\frac{(5-\sqrt{13})m_a}{6}$.

5. Triangles with specific PP' parallel to BC

We examine the condition under which the line joining a pair of isotomic conjugates is parallel to C. We shall exclude the trivial case of equilateral triangles.

5.1. *The incenter.* Since the incenter has coordinates $(a : b : c)$, if II' is parallel to BC, we must have, according to (5), $a^2 - bc = 0$. Therefore, the triangle is self-altitude. See §2.4. It is, however, not possible to have II' equal to half of the side BC, since the coordinates of P_3 in Theorem 2 do not satisfy the triangle inequality.

5.2. *The symmedian and Brocard points.* Likewise, for the symmedian point K, the line KK' is parallel to BC if and only if $a^4 = b^2c^2$, or $a^2 = bc$. In other words, the triangle is self-altitude again. In fact, the following statements are equivalent.

(1) $a^2 = bc$.
(2) K is on the ellipse $x^2 - yz = 0$; KK' is parallel to BC.
(3) Ω_+ is on the ellipse $z^2 - xy = 0$; $\Omega_+\Omega_+'$ is parallel to CA.
(4) Ω_- is on the ellipse $y^2 - zx = 0$; $\Omega_-\Omega_-'$ is parallel to BA.

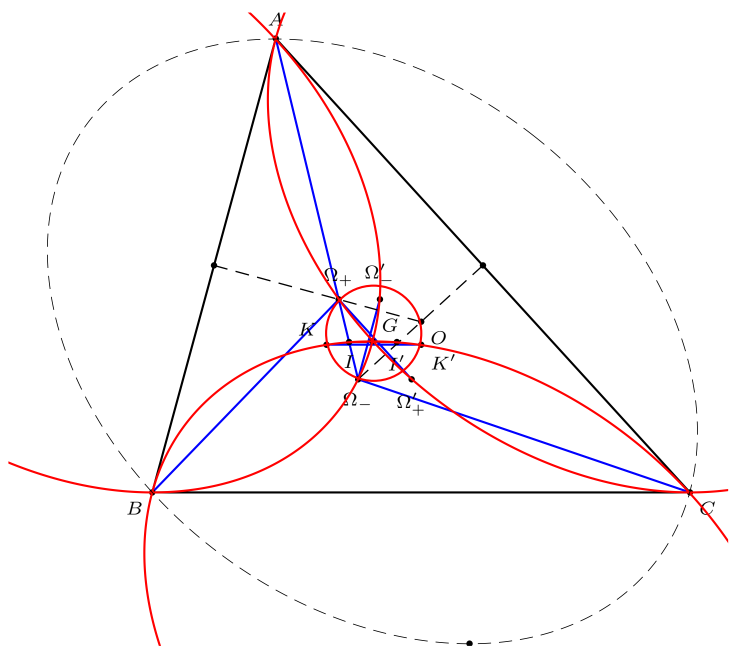

Figure 6

The self-altitude triangle with sides

$$a : b : c = \sqrt{2(1+\sqrt{13})} : 1 + \sqrt{13} : 2$$

has $KK' = \frac{1}{2}BC$.

5.3. *The circumcenter.* Unlike the incenter, the circumcenter may be outside the triangle. If O lies on the line $y + z = 0$, then $b\cos B + c\cos C = 0$. From this we deduce $\cos(B - C) = 0$, and $|B - C| = \pm\frac{\pi}{2}$. (This also follows from [2] by noting that the nine-point center lies on BC).

The homogeneous barycentric coordinates of the circumcenter are proportional to the sides of the orthic triangle (the pedal triangle of the orthocenter). To construct such a triangle, we take a self-altitude triangle $A'B'C'$ with incenter I_0, and construct the perpendiculars to $I'A'$, $I'B'$, $I'C'$ at A', B', C' respectively. These bound a triangle ABC whose orthocenter is I_0. Its circumcenter O is such that OO' is parallel to BC.

5.4. *The orthocenter.* The orthocenter has barycentric coordinates $(\tan A : \tan B : \tan C)$. If the triangle is acute, the condition $\tan^2 A = \tan B \tan C$ is equivalent to $A + \omega = \frac{\pi}{2}$ according to (1).

5.5. *The Gergonne and Nagel points.* The line joining the Gergonne and Nagel points is parallel to BC if and only if $(b+c-a)^2 = (c+a-b)(a+b-c)$. This is equivalent to (2). Hence, we have a characterization of such a triangle: the extension of the median m_a intersects the minor arc BC at a point P such that $AP = BP + CP$.

Since the Gergonne and Nagel points are interior points, there is a triangle (up to similarity) with ΓN parallel to BC and half in length. From

$$b+c-a : c+a-b : a+b-c = \sqrt{13}+1 : 2 : \sqrt{13}+7,$$

we obtain

$$a : b : c = \sqrt{13}+9 : 2\sqrt{13}+8 : \sqrt{13}+3 = 3\sqrt{13}-7 : \sqrt{13}+1 : 2.$$

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On the Intercepts of the OI-Line

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Abstract. We prove a new property of the intercepts of the line joining the circumcenter and the incenter on the sidelines of a triangle.

Given a triangle ABC with circumcenter O and incenter I, consider the intouch triangle XYZ. Let X' be the reflection of X in YZ, and similarly define Y' and Z'.

Theorem 1. *The intersections of AX' with BC, BY' with CA, and CZ' with AB are all on the line OI.*

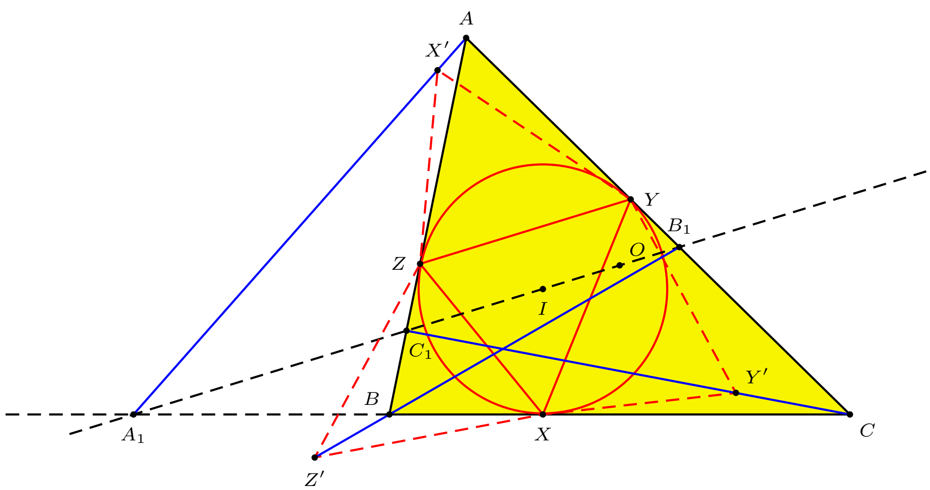

Figure 1.

Lemma 2. *The orthocenter H' of the intouch triangle lies on the line OI.*

Proof. Let $I_1I_2I_3$ be the excentral triangle. The lines YZ and I_2I_3 are parallel because both are perpendicular to AI. Similarly, $ZX//I_3I_1$ and $XY//I_1I_2$. See Figure 2. Hence, the excentral triangle and the intouch triangle are homothetic and their Euler lines are parallel. Now, I and O are the orthocenter and nine-point center of the excentral triangle. On the other hand, I is the circumcenter of the intouch triangle. Therefore, the line OI is their common Euler line, contains the orthocenter H' of XYZ. □

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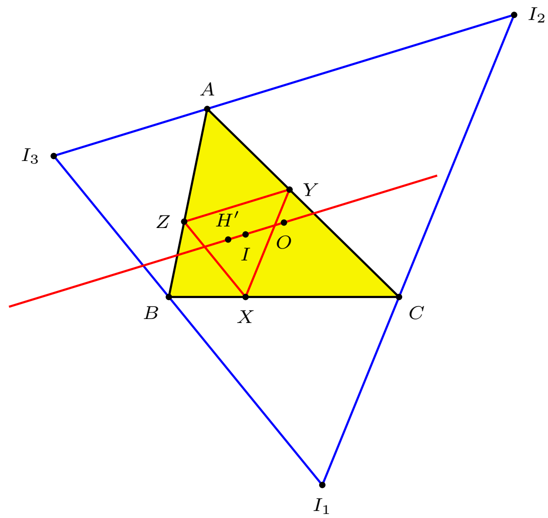

Figure 2.

Proof of Theorem 1. To prove that the intersection point A_1 of OI and AX' lies BC it is sufficient to show that $\frac{X'H'}{H'X} = \frac{AI}{IA_2}$, where A_2 is the foot of the bisector AI. See Figure 3.

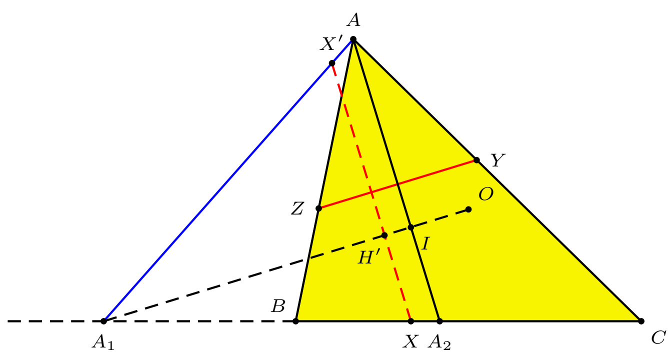

Figure 3.

It is known that

$$\frac{AI}{IA_2} = \frac{CA + AB}{BC} = \frac{\sin B + \sin C}{\sin A}.$$

For any acute triangle, $AH = 2R\cos A$. The angles of the intouch triangle are

$$X = \frac{B+C}{2}, \quad Y = \frac{C+A}{2}, \quad Z = \frac{A+B}{2}.$$

It is clear that triangle XYZ is always acute, and

$$XH' = 2r\cos X = 2r\cos\frac{B+C}{2} = 2r\sin\frac{A}{2},$$

where r is the inradius of triangle ABC.

$$\begin{aligned}\frac{X'H'}{H'X} &= \frac{X'X - H'X}{H'X} = \frac{X'X \cdot YZ}{H'X \cdot YZ} - 1\\ &= \frac{2 \cdot \text{area of } XYZ}{H'X \cdot YZ} - 1\\ &= \frac{2r^2(\sin 2X + \sin 2Y + \sin 2Z)}{2r \sin X \cdot 2r \cos X} - 1\\ &= \frac{\sin 2Y + \sin 2Z}{\sin 2X} = \frac{\sin B + \sin C}{\sin A}.\end{aligned}$$

This completes the proof of Theorem 1.

Similar results hold for the extouch triangle. In part it is in [1]. The following corollaries are clear.

Corollary 3. *The line joining A_1 to the projection of X on YZ passes through the midpoint of the bisector of angle A.*

Proof. In Figure 3, $X'X$ is parallel to the bisector of angle A and its midpoint is the projection of X on YZ. □

Corollary 4. *The OI-line is parallel to BC if and only if the projection of X on YZ lies on the line joining the midpoints of AB and AC.*

Corollary 5. *Let XYZ be the tangential triangle of ABC, X' the reflection of X in BC. If A_1 is the intersection of the Euler line and XX', then AA_1 is tangent to the circumcircle.*

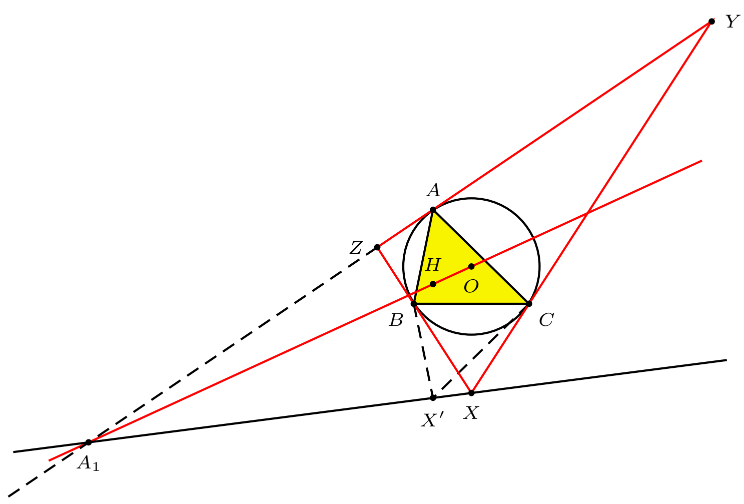

Figure 4.

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On the Schiffler center

Charles Thas

Abstract. Suppose that ABC is a triangle in the Euclidean plane and I its incenter. Then the Euler lines of ABC, IBC, ICA, and IAB concur at a point S, the Schiffler center of ABC. In the main theorem of this paper we give a projective generalization of this result and in the final part, we construct Schiffler-like points and a lot of other related centers. Other results in connection with the Schiffler center can be found in the articles [1] and [3].

1. Introduction

We recall some formulas and tools of projective geometry, which will be used in §2. Although we focus our attention on the real projective plane, it will be convenient to work in the complex projective plane $\mathcal{P}$.

1.1. Suppose that (x_1, x_2) are projective coordinates on a complex projective line and that two pairs of points are given as follows: P_1 and P_2 by the quadratic equation

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = 0 \tag{1}$$

and Q_1 and Q_2 by

$$a'x_1^2 + 2b'x_1x_2 + c'x_2^2 = 0. \tag{2}$$

Then the cross-ratio $(P_1P_2Q_1Q_2)$ equals -1 iff

$$ac' - 2bb' + a'c = 0. \tag{3}$$

Proof. Put $t = \frac{x_1}{x_2}$ and assume that t_1, t_2 (t'_1, t'_2 respectively) are the solutions of (1) ((2) respectively), divided by x_2^2. Then $(t_1\, t_2\, t'_1\, t'_2) = -1$ is equivalent to $2(t_1t_2 + t'_1t'_2) = (t_1 + t_2)(t'_1 + t'_2)$ or $2(\frac{c}{a} + \frac{c'}{a'}) = (-\frac{2b}{a})(-\frac{2b'}{a'})$, which gives (3). □

1.2.1. Consider a triangle ABC in the complex projective plane $\mathcal{P}$ and assume that ℓ is a line in $\mathcal{P}$, not through A, B, or C. Put $AB \cap \ell = M'_C, BC \cap \ell = M'_A$, and $CA \cap \ell = M'_B$ and determine the points M_C, M_A, and M_B by $(ABM'_CM_C) = (BCM'_AM_A) = (CAM'_BM_B) = -1$, then AM_A, BM_B, and CM_C concur at a point Z, the so-called trilinear pole of ℓ with regard to ABC.

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Proof. If $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$, and ℓ is the unit line $x_1 + x_2 + x_3 = 0$, then $M'_C = (1,-1,0)$, $M'_A = (0,1,-1)$, $M'_B = (1,0,-1)$, and $M_C = (1,1,0)$, $M_A = (0,1,1), M_B = (1,0,1)$, and Z is the unit point $(1,1,1)$. □

1.2.2. The trilinear pole Z_C of the unit-line ℓ with regard to ABQ, where $A = (1,0,0), B = (0,1,0)$, and $Q = (\mathcal{A},\mathcal{B},\mathcal{C})$, has coordinates $(2\mathcal{A}+\mathcal{B}+\mathcal{C}, \mathcal{A}+2\mathcal{B}+\mathcal{C}, \mathcal{C})$.

Proof. The point Z_C is the intersection of the line QM_C and BM_{QA}, with $M_C = (1,1,0)$, and M_{QA} the point of QA, such that $(Q\ A\ M_{QA}\ M'_{QA}) = -1$, with $M'_{QA} = QA \cap \ell$. We find for M_{QA} the coordinates $(2\mathcal{A}+\mathcal{B}+\mathcal{C}, \mathcal{B}, \mathcal{C})$, and a straightforward calculation completes the proof. □

1.3. Consider a non-degenerate conic $\mathcal{C}$ in the complex projective plane $\mathcal{P}$, and two points A, Q, not on $\mathcal{C}$, whose polar lines with respect to $\mathcal{C}$, intersect $\mathcal{C}$ at T_1, T_2, and I_1, I_2 respectively. Then Q lies on one of the lines ℓ_1, ℓ_2 through A which are determined by $(AT_1\ AT_2\ \ell_1\ \ell_2) = (AI_1\ AI_2\ \ell_1\ \ell_2) = -1$.

Proof. This follows immediately from the fact that the pole of the line AQ with respect to $\mathcal{C}$ is the point $T_1T_2 \cap I_1I_2$. □

1.4. For any triangle ABC of $\mathcal{P}$ and line ℓ not through a vertex, the Desargues-Sturm involution theorem ([7, p.341], [8, p.63]) provides a one-to-one correspondence between the involutions on ℓ and the points P in $\mathcal{P}$ that lie neither on ℓ nor on a side of the triangle. Specifically, the conics of the pencil $\mathcal{B}(A,B,C,P)$ intersect ℓ in pairs of points that are interchanged by an involution with fixed points I and J. Conversely, P is the fourth intersection point of the conics through A, B, and C that are tangent to ℓ at I and J. The point P can easily be constructed from A, B, C, I, and J as the point of intersection of the lines AA', and BB', where A' is the harmonic conjugate of $BC \cap \ell$ with respect to I and J, and B' is the harmonic conjugate of $AC \cap \ell$ with respect to I and J.

1.5. Denote the pencil of conics through the four points A_1, A_2, A_3, and A_4 by $\mathcal{B}(A_1, A_2, A_3, A_4)$, and assume that ℓ is a line not through A_i, $i = 1, \ldots, 4$. Put $M'_{12} = A_1A_2 \cap \ell$, and let M_{12} be the harmonic conjugate of M'_{12} with respect to A_1 and A_2, and define the points M_{23}, M_{34}, M_{13}, M_{14}, and M_{24} likewise. Let X, Y, and Z be the points $A_1A_2 \cap A_3A_4$, $A_2A_3 \cap A_1A_4$, and $A_1A_3 \cap A_2A_4$ respectively. Finally, let I and J be the tangent points with ℓ of the two conics of the pencil which are tangent at ℓ. Then the eleven points M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34}, X, Y, Z, I, and J belong to a conic ([8, p.109]).

Proof. We prove that this conic is the locus $\mathcal{C}$ of the poles of the line ℓ with regard to the conics of the pencil $\mathcal{B}(A_1, A_2, A_3, A_4)$. But first, let us prove that this locus is indeed a conic: if we represent the pencil by $F_1 + tF_2 = 0$, where $F_1 = 0$ and $F_2 = 0$ are two conics of the pencil, the equation of the locus is obtained by eliminating t from two linear equations which represent the polar lines of two points of ℓ, which gives a quadratic equation. Then, call A'_3 the point which is the harmonic conjugate of A_3 with respect to $M_{12}A_3 \cap \ell$ and M_{12}, and consider the conic of the pencil through A'_3: the pole of ℓ with respect to this conic clearly is M_{12}, which means that M_{12}, and thus also M_{ij}, is a point of the locus. Next, X, Y and Z are points of the locus, since they are singular points of the three degenerate conics of the pencil. And finally, I and J belong to the locus, because they are the poles of ℓ with regard to the two conics of the pencil which are tangent to ℓ. □

1.6. Consider again a triangle ABC in $\mathcal{P}$, and a point P not on a side of ABC. The *Ceva triangle* of P is the triangle with vertices $AP \cap BC$, $BP \cap CA$, and $CP \cap AB$. Example: with the notation of §1.2.1 the Ceva triangle of Z is $M_A M_B M_C$.

Next, assume that I and J are any two (different) points, not on a side of ABC, on a line ℓ, not through a vertex, and that P is the point which corresponds (according to 1.4) to the involution on ℓ with fixed points I and J. Let $H'_A H'_B H'_C$ be the Ceva triangle of P, let A' (B', and C' respectively) be the harmonic conjugate of $PA \cap \ell$ ($PB \cap \ell$, and $PC \cap \ell$ respectively) with respect to A and P (B and P, and C and P, respectively), and let $M_A M_B M_C$ be the Ceva triangle of the trilinear pole Z of ℓ with regard to ABC. Then there is a conic through I, J, and the triples $H'_A H'_B H'_C$, $A'B'C'$, and $M_A M_B M_C$. This conic is known as the *eleven-point conic* of ABC with regard to I and J ([7, pp.342–343]).

Proof. Apply 1.5 to the pencil $\mathcal{B}(A, B, C, P)$. □

2. The main theorem

Theorem. *Let ABC be a triangle in the complex projective plane $\mathcal{P}$, ℓ be a line not through a vertex, and I and J be any two (different) points of ℓ not on a side of the triangle. Choose $\mathcal{C}$ to be one of the four conics through I and J that are tangent to the sides of triangle ABC, and define Q to be the pole of ℓ with respect to $\mathcal{C}$. If Z, Z_A, Z_B, and Z_C are the trilinear poles of ℓ with respect to the triangles ABC, QBC, QCA, and QAB respectively, while P, P_A, P_B, and P_C respectively, are the points determined by these triangles and the involution on ℓ whose fixed points are I and J (see 1.4), then the lines PZ, $P_A Z_A$, $P_B Z_B$, and $P_C Z_C$ concur at a point S_P.*

Proof. We choose our projective coordinate system in $\mathcal{P}$ as follows : $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$, and ℓ is the unit line with equation $x_1 + x_2 + x_3 = 0$. The point P has coordinates (α, β, γ).

Two degenerate conics of the pencil $\mathcal{B}(A, B, C, P)$ are (CP, AB) and (BP, CA), which intersect ℓ at the points $(-\alpha, -\beta, \alpha + \beta)$, $(1, -1, 0)$ and $(-\alpha, \alpha + \gamma, -\gamma)$, $(1, 0, -1))$ respectively. Joining these points to A, we find the lines $(\alpha + \beta)x_2 + \beta x_3 = 0$, $x_3 = 0$ and $\gamma x_2 + (\alpha + \gamma)x_3 = 0$, $x_2 = 0$, or as quadratic equations $(\alpha + \beta)x_2 x_3 + \beta x_3^2 = 0$ and $\gamma x_2^2 + (\alpha + \gamma)x_2 x_3 = 0$ respectively. Therefore, the lines AI and AJ are given by $kx_2^2 + 2lx_2 x_3 + mx_3^2 = 0$ whereby k, l, and m are solution of (see 1.1):

$$\begin{cases} \beta k - (\alpha + \beta)l = 0 \\ -(\alpha + \gamma)l + \gamma m = 0, \end{cases}$$

and thus $(k,l,m) = (\gamma(\alpha+\beta), \beta\gamma, \beta(\alpha+\gamma))$. Next, the lines through A which form together with AI, AJ and with AB, AC an harmonic quadruple, are determined by $px_2^2 + 2qx_2x_3 + rx_3^2 = 0$ with p, q, r solutions of (see again 1.1)

$$\begin{cases} \beta(\alpha+\gamma)p - 2\beta\gamma q + \gamma(\alpha+\beta)r = 0 \\ q = 0, \end{cases}$$

and thus these lines are given by $\gamma(\alpha+\beta)x_2^2 - \beta(\alpha+\gamma)x_3^2 = 0$. In the same way, we find the quadratic equation of the two lines through B (C, respectively) which form together with BI, BJ and with BC, BA (with CI, CJ and with CA, CB respectively) an harmonic quadruple : $\alpha(\beta+\gamma)x_3^2 - \gamma(\beta+\alpha)x_1^2 = 0$ ($\beta(\gamma+\alpha)x_1^2 - \alpha(\gamma+\beta)x_2^2 = 0$ respectively). The intersection points of these three pairs of lines through A, B, and C are the poles Q_1, Q_2, Q_3, Q_4 of ℓ with respect to the four conics through I and J that are tangent to the sides of triangle ABC (see 1.3) and their coordinates are $Q_1(\mathcal{A},\mathcal{B},\mathcal{C})$, $Q_2(-\mathcal{A},\mathcal{B},\mathcal{C})$, $Q_3(\mathcal{A},-\mathcal{B},\mathcal{C})$, and $Q_4(\mathcal{A},\mathcal{B},-\mathcal{C})$, where

$$\mathcal{A} = \sqrt{\alpha(\beta+\gamma)}, \quad \mathcal{B} = \sqrt{\beta(\gamma+\alpha)}, \quad \mathcal{C} = \sqrt{\gamma(\alpha+\beta)}.$$

For now, let us choose for Q the point $Q_1(\mathcal{A},\mathcal{B},\mathcal{C})$.

The coordinates of the points Z, Z_A, Z_B, and Z_C are $(1,1,1)$, $(\mathcal{A}, \mathcal{A}+2\mathcal{B}+\mathcal{C}, \mathcal{A}+\mathcal{B}+2\mathcal{C})$, $(2\mathcal{A}+\mathcal{B}+\mathcal{C}, \mathcal{B}, \mathcal{A}+\mathcal{B}+2\mathcal{C})$, and $(2\mathcal{A}+\mathcal{B}+\mathcal{C}, \mathcal{A}+2\mathcal{B}+\mathcal{C}, \mathcal{C})$ (see 1.2.2).

Now, in connection with the point P_C, remark that $(AP_C \cap \ell\ (QB \cap \ell)\ I\ J) = -1$. But $(Q_2Q_4 \cap \ell\ (Q_1Q_3 \cap \ell)\ I\ J) = -1$ and $Q_2Q_4 = Q_2B$, $Q_1Q_3 = Q_1B$, so that $AP_C \cap \ell = Q_2B \cap \ell$, and since Q_2B has equation $\mathcal{C}x_1 + \mathcal{A}x_3 = 0$, the point $AP_C \cap \ell$ has coordinates $(\mathcal{A}, \mathcal{C}-\mathcal{A}, -\mathcal{C})$ and the line AP_C has equation $\mathcal{C}x_2 + (\mathcal{C}-\mathcal{A})x_3 = 0$. In the same way, we find the equation of the line BP_C: $\mathcal{C}x_1 + (\mathcal{C}-\mathcal{B})x_3 = 0$, and the common point of these two lines is the point P_C with coordinates $(\mathcal{B}-\mathcal{C}, \mathcal{A}-\mathcal{C}, \mathcal{C})$.

Finally, the line P_CZ_C has equation :

$$\mathcal{C}(\mathcal{B}+\mathcal{C})x_1 - \mathcal{C}(A+\mathcal{C})x_2 + (\mathcal{A}^2 - \mathcal{B}^2)x_3 = 0,$$

and cyclic permutation gives us the equations of P_AZ_A and P_BZ_B.

Now, P_AZ_A, P_BZ_B, and P_CZ_C are concurrent if the determinant

$$\begin{vmatrix} \mathcal{B}^2 - \mathcal{C}^2 & \mathcal{A}(\mathcal{C}+\mathcal{A}) & -\mathcal{A}(\mathcal{B}+\mathcal{A}) \\ -\mathcal{B}(\mathcal{C}+\mathcal{B}) & \mathcal{C}^2 - \mathcal{A}^2 & \mathcal{B}(\mathcal{A}+\mathcal{B}) \\ \mathcal{C}(\mathcal{B}+\mathcal{C}) & -\mathcal{C}(\mathcal{A}+\mathcal{C}) & \mathcal{A}^2 - \mathcal{B}^2 \end{vmatrix}$$

is zero, which is obviously the case, since the sum of the rows gives us three times zero. Then, the line PZ has equation $(\beta-\gamma)x_1 + (\gamma-\alpha)x_2 + (\alpha-\beta)x_3 = 0$. But $\mathcal{A}^2 = \alpha(\beta+\gamma)$, $\mathcal{B}^2 = \beta(\gamma+\alpha)$, and $\mathcal{C}^2 = \gamma(\alpha+\beta)$, so that $(\mathcal{B}^2 - \mathcal{C}^2)(-\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2) = 2\alpha\beta\gamma(\beta-\gamma)$, and PZ has also the following equation

$$(\mathcal{B}^2 - \mathcal{C}^2)(-\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2)x_1 + (\mathcal{C}^2 - \mathcal{A}^2)(\mathcal{A}^2 - \mathcal{B}^2 + \mathcal{C}^2)x_2$$
$$+(\mathcal{A}^2 - \mathcal{B}^2)(\mathcal{A}^2 + \mathcal{B}^2 - \mathcal{C}^2)x_3 = 0.$$

For PZ, P_AZ_A, and P_BZ_B to be concurrent, the following determinant must vanish :

$$\begin{aligned}&\begin{vmatrix} (\mathcal{B}^2-\mathcal{C}^2)(-\mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2) & (\mathcal{C}^2-\mathcal{A}^2)(\mathcal{A}^2-\mathcal{B}^2+\mathcal{C}^2) & (\mathcal{A}^2-\mathcal{B}^2)(\mathcal{A}^2+\mathcal{B}^2-\mathcal{C}^2) \\ \mathcal{B}^2-\mathcal{C}^2 & \mathcal{A}(\mathcal{C}+\mathcal{A}) & -\mathcal{A}(\mathcal{B}+\mathcal{A}) \\ -\mathcal{B}(\mathcal{C}+\mathcal{B}) & \mathcal{C}^2-\mathcal{A}^2 & \mathcal{B}(\mathcal{A}+\mathcal{B}) \end{vmatrix} \\ =&(\mathcal{B}+\mathcal{C})(\mathcal{C}+\mathcal{A})(\mathcal{A}+\mathcal{B})(\mathcal{A}(\mathcal{B}-\mathcal{C})(-\mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2)(-\mathcal{A}+\mathcal{B}+\mathcal{C}) \\ &+\mathcal{B}(\mathcal{C}-\mathcal{A})(\mathcal{A}^2-\mathcal{B}^2+\mathcal{C}^2)(\mathcal{A}-\mathcal{B}+\mathcal{C})+\mathcal{C}(\mathcal{A}-\mathcal{B})(\mathcal{A}^2+\mathcal{B}^2-\mathcal{C}^2)(\mathcal{A}+\mathcal{B}-\mathcal{C})) \\ =&0.\end{aligned}$$

We may conclude that PZ, P_AZ_A, P_BZ_B, and P_CZ_C are concurrent. This completes the proof. □

Remarks. (1) If Q is chosen as the point Q_2 (Q_3, or Q_4, respectively), then $\mathcal{A}$ ($\mathcal{B}$, or $\mathcal{C}$ respectively) must be replaced by $-\mathcal{A}$ ($-\mathcal{B}$, or $-\mathcal{C}$ respectively) in the foregoing proof.

(2) The coordinates of the common point S_P of the lines PZ, P_AZ_A, P_BZ_B, and P_CZ_C are $(\mathcal{A}\frac{-\mathcal{A}+\mathcal{B}+\mathcal{C}}{\mathcal{B}+\mathcal{C}}, \mathcal{B}\frac{\mathcal{A}-\mathcal{B}+\mathcal{C}}{\mathcal{C}+\mathcal{A}}, \mathcal{C}\frac{\mathcal{A}+\mathcal{B}-\mathcal{C}}{\mathcal{A}+\mathcal{B}})$.

(3) Of course, when we work in the real (complexified) projective plane $\mathcal{P}$ with a real triangle ABC, a real line ℓ and a real point P, the points Q and S_P, are not always real. That depends on the values of α, β, and γ and thus on the position of the point P in the plane. For instance, in example 5.5 of §5, the points Q and S_P will be imaginary.

(4) The conic through A, B, C, and through the points I, J on ℓ has equation

$$\alpha(\beta+\gamma)x_2x_3+\beta(\gamma+\alpha)x_3x_1+\gamma(\alpha+\beta)x_1x_2=0$$

or

$$\mathcal{A}^2x_2x_3+\mathcal{B}^2x_3x_1+\mathcal{C}^2x_1x_2=0.$$

Indeed, eliminating x_1 from this equation and from $x_1+x_2+x_3=0$, gives us $\gamma(\alpha+\beta)x_2^2+2\gamma\beta x_2x_3+\beta(\gamma+\alpha)x_3^2=0$, which determines the lines AI and AJ (see the proof of the theorem).
The pole of the line ℓ with respect to this conic is the point $Y(\beta+\gamma, \gamma+\alpha, \alpha+\beta)$, which clearly is a point of the line PZ. We denote this conic by (Y).

(5) The locus of the poles of the line ℓ with respect to the conics of the pencil $\mathcal{B}(A, B, C, P)$ is the conic with equation

$$\beta\gamma x_1^2+\gamma\alpha x_2^2+\alpha\beta x_3^2-\alpha(\gamma+\beta)x_2x_3-\beta(\alpha+\gamma)x_3x_1-\gamma(\beta+\alpha)x_1x_2=0.$$

It is the eleven-point conic of triangle ABC with regard to I and J (see 1.6): it is the conic through the points $M_A(0,1,1)$, $M_B(1,0,1)$, $M_C(1,1,0)$, $AP\cap BC=H'_A(0,\beta,\gamma)$, $BP\cap CA=H'_B(\alpha,0,\gamma)$, $CP\cap AB=H'_C(\alpha,\beta,0)$, $A'(2\alpha+\beta+\gamma,\beta,\gamma), B'(\alpha,\alpha+2\beta+\gamma,\gamma), C'(\alpha,\beta,\alpha+\beta+2\gamma), I$, and J. The pole of the line ℓ with regard to this conic is the point $Y'(2\alpha+\beta+\gamma, \alpha+2\beta+\gamma, \alpha+\beta+2\gamma)$, which is also a point of the line PZ. We denote this conic by (Y').

Here is an alternative formulation of the main theorem.

Theorem. *Let ABC be a triangle in the complex projective plane $\mathcal{P}$, ℓ be a line not through a vertex, and I and J be any two (different) points of ℓ not on a side* *of the triangle. Denote by Q the pole of ℓ with respect to one of the four conics through I and J that are tangent to the sides of the triangle. If Y, Y_A, Y_B, and Y_C are the poles of ℓ with respect to the conics determined by I, J, and the triples ABC, QBC, QCA, and QAB respectively, while Y', Y'_A, Y'_B, and Y'_C are the respective poles with respect to their eleven-point conics with regard to I and J, then YY', $Y_AY'_A$, $Y_BY'_B$, and $Y_CY'_C$ concur at a point S.*

3. The Euclidean case

In this section we give applications of the main theorem in the Euclidean plane Π. Throughout the following sections, we only consider a general real triangle ABC in Π, i.e., the side-lengths a, b, and c are distinct and the triangle has no right angle.

Corollary 1. *Let ABC be a triangle in Π and assume that ℓ is the line at infinity of Π. Suppose that P coincides with the orthocenter H of ABC; then the conics of the pencil $\mathcal{B}(A, B, C, H)$ are rectangular hyperbolas and the involution on ℓ, determined by H (see 1.4), becomes the absolute (or orthogonal) involution with fixed points the cyclic points (or circle points) J and J' of Π. The four conics through J, J' and tangent to the sidelines of ABC are now the incircle and the excircles of ABC, and the points $Q = Q_1$, Q_2, Q_3, Q_4 become the incenter I, and the excenters I_A (the line II_A contains A), I_B, and I_C, respectively.*

Next, the points Z, Z_A, Z_B, and Z_C, are the centroids of ABC, IBC, ICA, and of IAB respectively. Finally, P_A, P_B, P_C are the orthocenters H_A, H_B, H_C of IBC, ICA, and IAB respectively. Then the lines HZ, H_AZ_A, H_BZ_B, and H_CZ_C concur at a point S_H.

Remark that HZ, H_AZ_A, H_BZ_B, and H_CZ_C are the Euler lines of the triangles ABC, IBC, ICA, and IAB, respectively. The point of concurrence of these Euler lines is known as the Schiffler point S ([9]), but we prefer in this paper the notation S_H, since it results from setting $P = H$.

In connection with Remarks 4 and 5 of the foregoing section, and again working with ℓ as the line at infinity and J, J' the cyclic points, the conic (Y) becomes the circumcircle (O) of ABC, (Y') becomes its nine-point circle (O'), and OO' is the Euler line.

In connection with Remark 5, we recall that the locus of the centers of the rectangular hyperbolas through A, B, C (and H) is the nine-point circle (O') of ABC and that, for each point U of the circumcircle (O), the midpoint of HU is a point of (O') (and O' is the midpoint of HO on the Euler line).

The main theorem allows us to generalize the foregoing corollary as follows:

Corollary 2. *Let ABC be a triangle and let ℓ be the line at infinity in Π. Choose a general point P (*i.e.*, not on a sideline of ABC, not on ℓ and different from the centroid of ABC) and call J, J' the tangent points on ℓ of the two conics of the pencil $\mathcal{B}(A, B, C, P)$ which are tangent to ℓ (these are the centers of the parabolas through A, B, C and P). Denote by Q the center of one of the four conics through J and J', which are tangent at the sidelines of ABC. Next, Z is the centroid* *of ABC and Z_A, Z_B, Z_C are the centroids of the triangles QBC, QCA, QAB respectively. Finally, P_A (P_B, and P_C respectively) is the fourth common point of the two parabolas through Q, B, C (through Q, C, A, and through Q, A, B respectively) and tangent to ℓ at J and J'. Then the lines PZ, P_AZ_A, P_BZ_B, and P_CZ_C concur at a point S_P.*

4. The use of trilinear coordinates

From now on, we work with trilinear coordinates (x_1, x_2, x_3) with respect to the real triangle ABC in the Euclidean plane Π ([2, 5]): A, B, C, and the incenter I of ABC, have coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ respectively. The line at infinity ℓ has equation $ax_1 + bx_2 + cx_3 = 0$, where a, b, c are the side-lengths of ABC. The orthocenter H, the centroid Z, the circumcenter O, and the center of the nine-point circle O', have trilinear coordinates $\left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right)$, $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $(\cos A, \cos B, \cos C)$, and $(bc(a^2b^2 + a^2c^2 - (b^2 - c^2)^2), ca(b^2c^2 + b^2a^2 - (c^2-a^2)^2), ab(c^2a^2 + c^2b^2 - (a^2-b^2)^2))$ respectively. The equations of the circumcircle (O) and the nine-point circle (O') are $ax_2x_3 + bx_3x_1 + cx_1x_2 = 0$ and $x_1^2 \sin 2A + x_2^2 \sin 2B + x_3^2 \sin 2C - 2x_2x_3 \sin A - 2x_3x_1 \sin B - 2x_1x_2 \sin C = 0$.

The Schiffler point $S = S_H$ (the common point of the Euler lines of ABC, IBC, ICA, and IAB) has trilinear coordinates $\left(\frac{-a+b+c}{b+c}, \frac{a-b+c}{c+a}, \frac{a+b-c}{a+b}\right)$.

If T is a point of Π, not on a sideline of ABC, reflect the line AT about the line AI, and reflect BT and CT about the corresponding bisectors BI and CI. The three reflections concur in the isogonal conjugate T^{-1} of T, and T^{-1} has trilinear coordinates (t_2t_3, t_3t_1, t_1t_2) or $\left(\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}\right)$ if T has trilinear coordinates (t_1, t_2, t_3). Examples: the circumcenter O is the isogonal conjugate of the orthocenter H, and the centroid Z is the isogonal conjugate of the Lemoine point (or symmedian point) $K(a, b, c)$.

Let us now interpret the main theorem (or Corollary 2) in the Euclidean case using trilinear coordinates, with $\ell : ax_1 + bx_2 + cx_3 = 0$ as line at infinity and with $P(\alpha, \beta, \gamma)$ a general point of Π. In fact, the only thing that we have to do, is to replace in the proof of the main theorem the equation $x_1 + x_2 + x_3 = 0$ of ℓ, by $ax_1 + bx_2 + cx_3 = 0$, and a straightforward calculation gives us the following trilinear coordinates for the point Q: $(\sqrt{bc\alpha(b\beta + c\gamma)}, \sqrt{ca\beta(c\gamma + a\alpha)}, \sqrt{ab\gamma(a\alpha + b\beta)}) = (\mathcal{A}, \mathcal{B}, \mathcal{C})$. Next, the points Z, Z_A, Z_B, and Z_C are the centroids of ABC, QBC, QCA and QAB with trilinear coordinates $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $(bc\mathcal{A}, c(a\mathcal{A}+2b\mathcal{B}+c\mathcal{C}), b(a\mathcal{A}+b\mathcal{B}+2c\mathcal{C}))$, $(c(2a\mathcal{A}+b\mathcal{B}+c\mathcal{C}), ca\mathcal{B}, a(a\mathcal{A}+b\mathcal{B}+2c\mathcal{C}))$, $(b(2a\mathcal{A}+b\mathcal{B}+c\mathcal{C}), a(a\mathcal{A}+2b\mathcal{B}+c\mathcal{C}), ba\mathcal{C})$, respectively. Now, for the points P_A, P_B, P_C, again after a straightforward calculation, we find the coordinates: $P_A(bc\mathcal{A}, c(c\mathcal{C} - a\mathcal{A}), b(b\mathcal{B} - a\mathcal{A}))$, $P_B(c(c\mathcal{C} - b\mathcal{B}), ca\mathcal{B}, a(a\mathcal{A} - b\mathcal{B}))$ and $P_C(b(b\mathcal{B} - c\mathcal{C}), a(a\mathcal{A} - c\mathcal{C}), ab\mathcal{C})$.

And finally, we find the trilinear coordinates of the point S_P, corresponding to Q:

$$\left(\frac{\mathcal{A}(-a\mathcal{A} + b\mathcal{B} + c\mathcal{C})}{b\mathcal{B} + c\mathcal{C}}, \frac{\mathcal{B}(a\mathcal{A} - b\mathcal{B} + c\mathcal{C})}{c\mathcal{C} + a\mathcal{A}}, \frac{\mathcal{C}(a\mathcal{A} + b\mathcal{B} - c\mathcal{C})}{a\mathcal{A} + b\mathcal{B}}\right).$$

Remark that we find for the case $P(\alpha,\beta,\gamma) = H(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C})$:

$$\begin{aligned}\mathcal{A} &= \sqrt{bc\alpha(b\beta+c\gamma)} = \sqrt{\tfrac{bc}{\cos A}(\tfrac{b}{\cos B}+\tfrac{c}{\cos C})} = \sqrt{\tfrac{bc(b\cos C + c\cos B)}{\cos A\cos B\cos C}} \\ &= \sqrt{\tfrac{abc}{\cos A\cos B\cos C}} = \mathcal{B} = \mathcal{C}\end{aligned}$$

and $Q(\mathcal{A},\mathcal{B},\mathcal{C}) = I(1,1,1)$, while since $\mathcal{A} = \mathcal{B} = \mathcal{C}$, we get for S_H the coordinates $\left(\frac{-a+b+c}{b+c}, \frac{a-b+c}{c+a}, \frac{a+b-c}{a+b}\right)$, which gives us the Schiffler point S.

Let us also calculate the trilinear coordinates of the points Y and Y', defined above as the centers of the conic (Y) through A, B, C, J and J', and of the conic (Y') through the midpoints of the sides of ABC and through J, J' (or the eleven-point conic of ABC with regard to J and J'; remark that J and J' are the cyclic points only when $P = H$):

(Y) has equation $\alpha(b\beta+c\gamma)x_2x_3 + \beta(c\gamma+a\alpha)x_3x_1 + \gamma(a\alpha+b\beta)x_1x_2 = 0$ and center $Y(bc(b\beta+c\gamma), ca(c\gamma+a\alpha), ab(a\alpha+b\beta))$,

(Y') has equation $a\beta\gamma x_1^2 + b\gamma\alpha x_2^2 + c\alpha\beta x_3^2 - \alpha(\gamma c + b\beta)x_2x_3 - \beta(a\alpha + c\gamma)x_3x_1 - \gamma(b\beta + a\alpha)x_1x_2 = 0$ and center $Y'(bc(2a\alpha+b\beta+c\gamma), ca(a\alpha+2b\beta+c\gamma), ab(\alpha+b\beta+2c\gamma))$.

Remark that $Q = \sqrt{P * Y}$, with the notation $\sqrt{(x_1,x_2,x_3)*(y_1,y_2,y_3)} = (\sqrt{x_1y_1}, \sqrt{x_2y_2}, \sqrt{x_3y_3})$.
Recall that the coordinate transformation between trilinear coordinates (x_1,x_2,x_3) with regard to $\triangle ABC$ and trilinear coordinates (x'_1,x'_2,x'_3) with regard to the medial triangle $M_AM_BM_C$, is given by ([5, p.207]):

$$\begin{pmatrix} ax_1 \\ bx_2 \\ cx_3 \end{pmatrix} = \begin{pmatrix} 0 & b & c \\ a & 0 & c \\ a & b & 0 \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}.$$

Now, this gives for (x_1,x_2,x_3) the coordinates of the point Y, if (x'_1,x'_2,x'_3) are the coordinates (α,β,γ) of P and it gives for (x_1,x_2,x_3) the coordinates of Y' if (x'_1,x'_2,x'_3) are the coordinates of Y. Moreover, $\triangle ABC$ and its medial triangle are homothetic. As a corollary, we have that if P (Y, respectively) is *triangle center* $X(k)$ for $\triangle ABC$ (for the definition of triangle center, see [5, p.46]), then Y (Y' respectively) is center $X(k)$ for $\triangle M_AM_BM_C$.

5. Applications

In this section we choose $P(\alpha,\beta,\gamma)$ as a triangle center of the triangle ABC and calculate the coordinates of the corresponding points Y, Y', Q and S_P (sometimes Y' and S_P are not given).

Remark that P must be different from the centroid Z of ABC. The triangle centers are taken from Kimberling's list : $X(1)$, $X(2)$, $\ldots$, $X(2445)$ (list until 29 March 2004, see [6]). When we found the points Y, Y', Q or S_P in this list, we give the number $X(\cdots)$ and if possible, the name of the center. But, without doubt, we overlooked some centers and more points Y, Y', Q, S_P than indicated will occur in Kimberling's list. Several times, only the first trilinear coordinate is given: the second and the third are obtained by cyclic permutations.

5.1. The first example is of course:
$P(\alpha,\beta,\gamma)=H(\frac{1}{\cos A},\frac{1}{\cos B},\frac{1}{\cos C})=X(4)$ (orthocenter),
$Y=O(\cos A,\cos B,\cos C)=X(3)$ (circumcenter),
$Y'=O'(bc(a^2b^2+a^2c^2-(b^2-c^2)^2),\cdots,\cdots)=X(5)$ (nine-point center),
$Q=I(1,1,1)=X(1)$ (incenter), and
$S_H=S\left(\frac{-a+b+c}{b+c},\cdots,\cdots\right)=X(21)$ (Schiffler point).

5.2. $P(\alpha,\beta,\gamma)=I(1,1,1)=X(1)$,
$Y=(\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c})=X(10)$ (Spieker point = incenter of the medial triangle $M_AM_BM_C$),
$Y'=(\frac{2a+b+c}{a},\cdots,\cdots)=X(1125)$ (Spieker point of the medial triangle),
$Q=(\sqrt{bc(b+c)},\cdots,\cdots)$, and
$S_I=(\sqrt{bc(b+c)}\frac{-a\sqrt{bc(b+c)}+b\sqrt{ca(c+a)}+c\sqrt{ab(a+b)}}{b\sqrt{ca(c+a)}+c\sqrt{ab(a+b)}},\cdots,\cdots)$.

5.3. $P(\alpha,\beta,\gamma)=K(a,b,c)=X(6)$ (Lemoine point),
$Y=(\frac{b^2+c^2}{a},\cdots,\cdots)=X(141)$ = Lemoine point of medial triangle,
$Y'=(\frac{2a^2+b^2+c^2}{a},\cdots,\cdots)$,
$Q=(\sqrt{b^2+c^2},\cdots,\cdots)$, and
$S_K=(\sqrt{b^2+c^2}\frac{-a\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}+c\sqrt{a^2+b^2}}{b\sqrt{c^2+a^2}+c\sqrt{a^2+b^2}},\cdots,\cdots)$.

5.4. $P(\alpha,\beta,\gamma)=(\frac{1}{a(-a+b+c)},\cdots,\cdots)=X(7)$ (Gergonne point),
$Y=(-a+b+c,a-b+c,a+b-c)=X(9)$ (Mittenpunkt = Lemoine point of the excentral triangle $I_AI_BI_C=$ Gergonne point of medial triangle),
$Y'=(bc(a(b+c)-(b-c)^2),\cdots,\cdots)=X(142)$ (Mittenpunkt of medial triangle),
$Q=(\frac{1}{\sqrt{a}},\frac{1}{\sqrt{b}},\frac{1}{\sqrt{c}})=X(366)$, and
$S_{X(7)}=(\frac{1}{\sqrt{a}}\frac{-\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{b}+\sqrt{c}},\cdots,\cdots)$.

5.5. $P(\alpha,\beta,\gamma)=(\frac{1}{b-c},\frac{1}{c-a},\frac{1}{a-b})=X(100)$,
$Y=(bc(b-c)^2(-a+b+c),\cdots,\cdots)=X(11)$ (Feuerbach point = $X(100)$ of medial triangle),
$Y'=(bc((a-b)^2(a+b-c)+(c-a)^2(a-b+c)),\cdots,\cdots)$ (Feuerbach point of medial triangle), and $Q=(\sqrt{bc(b-c)(-a+b+c)},\cdots,\cdots)$.

In the foregoing examples, the coordinates of the point S_P are mostly rather complicated. Another method is to start with the coordinates of the point Q: if (k,l,m) are the trilinear coordinates of Q, then a short calculation shows that it corresponds with the point $P(\frac{1}{a(-a^2k^2+b^2l^2+c^2m^2)},\cdots,\cdots)$ and S_P becomes the point $(k\frac{-ak+bl+cm}{bl+cm},\cdots,\cdots)$. Finally, the coordinates of Y and Y' are $(ak^2(-a^2k^2+b^2l^2+c^2m^2),\cdots,\cdots)$, and $(bc(a^2k^2(b^2l^2+c^2m^2)-(b^2l^2-c^2m^2)^2),\cdots,\cdots)$, respectively. Here are some examples.

5.6. $Q(k,l,m) = K(a,b,c) = X(6)$(Lemoine point),
$P = (\frac{1}{a(-a^4+b^4+c^4)}, \cdots, \cdots) = X(66) = X(22)^{-1}$ ($X(22)$ is the Exeter point),
$Y = (a^3(-a^4+b^4+c^4), \cdots, \cdots) = X(206)$ ($X(66)$ of medial triangle),
$Y' = (bc(a^4(b^4+c^4) - (b^4-c^4)^2), \cdots, \cdots)$ ($X(206)$ of medial triangle), and
$S_{X(66)} = (\frac{a(-a^2+b^2+c^2)}{b^2+c^2}, \cdots, \cdots) = (\frac{\cos A}{b^2+c^2}, \cdots, \cdots) = X(1176)$.

5.7. $Q(k,l,m) = H(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}) = X(4)$ (orthocenter),
$P = (\frac{1}{a(-\frac{a^2}{\cos^2 A}+\frac{b^2}{\cos^2 B}+\frac{c^2}{\cos^2 C})}, \cdots, \cdots)$,
$Y = (\frac{a}{\cos^2 A}(-\frac{a^2}{\cos^2 A} + \frac{b^2}{\cos^2 B} + \frac{c^2}{\cos^2 C}), \cdots, \cdots)$, and
$S_P = (\frac{\cos A - \cos B \cos C}{\cos^2 A}, \cdots, \cdots)$.

5.8. $Q(k,l,m) = (\frac{b+c}{a}, \cdots, \cdots) = X(10)$ (Spieker point),
$P = (\frac{1}{a(-(b+c)^2+(c+a)^2+(a+b)^2)}, \cdots, \cdots) = X(596)$,
$Y = \left(\frac{(b+c)^2}{a}(-(b+c)^2 + (c+a)^2 + (a+b)^2), \cdots, \cdots\right)$ ($X(596)$ of medial triangle), and
$S_P = (\frac{b+c}{2a+b+c}, \frac{c+a}{a+2b+c}, \frac{a+b}{a+b+2c})$.

We also can start with the coordinates of the point $Y(p,q,r)$, then
$P = (\frac{-ap+bq+cr}{a}, \cdots, \cdots)$,
$Y'(bc(bq+cr), \cdots, \cdots)$, and
$Q = \sqrt{P * Y} = (\sqrt{\frac{p(-ap+bq+cr)}{a}}, \cdots, \cdots)$. Here are some examples.

5.9. $Y(p,q,r) = I(1,1,1) = X(1)$, $P = (\frac{-a+b+c}{a}, \frac{a-b+c}{b}, \frac{a+b-c}{c}) = X(8)$ (Nagel point),
$Y' = \left(\frac{b+c}{a}, \cdots, \cdots\right) = X(10)$ (Spieker point = incenter of medial triangle),
$Q = \left(\sqrt{\frac{-a+b+c}{a}}, \cdots, \cdots\right) = X(188)$, and
$S_P = (\mathcal{A}\frac{-a\mathcal{A}+b\mathcal{B}+c\mathcal{C}}{b\mathcal{B}+c\mathcal{C}}, \cdots, \cdots)$ with $Q(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

5.10. $Y = K(a,b,c) = X(6)$ (Lemoine point),
$P = (\frac{-a^2+b^2+c^2}{a}, \cdots, \cdots) = (\frac{cosA}{a^2}, \cdots, \cdots) = X(69)$,
$Y' = (\frac{b^2+c^2}{a}, \cdots, \cdots) = X(141)$ (Lemoine point of medial triangle), and
$Q = (\sqrt{-a^2+b^2+c^2}, \cdots, \cdots)$.

5.11. $Y = (\frac{2a+b+c}{a}, \cdots, \cdots) = X(1125)$ (Spieker point of medial triangle),
$P = (\frac{b+c}{a}, \cdots, \cdots) = X(10)$ (Spieker point),
$Y' = (\frac{2a+3b+3c}{a}, \cdots, \cdots)$ ($X(1125)$ of medial triangle), and
$Q = (bc\sqrt{(b+c)(2a+b+c)}, \cdots, \cdots)$.

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The Vertex-Midpoint-Centroid Triangles

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Abstract. This paper explores six triangles that have a vertex, a midpoint of a side, and the centroid of the base triangle ABC as vertices. They have many interesting properties and here we study how they monitor the shape of ABC. Our results show that certain geometric properties of these six triangles are equivalent to ABC being either equilateral or isosceles.

Let A', B', C' be midpoints of the sides BC, CA, AB of the triangle ABC and let G be its centroid (*i.e.*, the intersection of medians AA', BB', CC'). Let G_a^-, G_a^+, G_b^-, G_b^+, G_c^-, G_c^+ be triangles BGA', CGA', CGB', AGB', AGC', BGC' (see Figure 1).

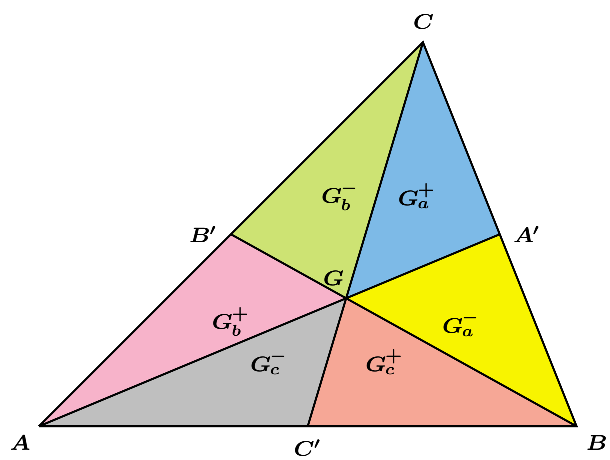

Figure 1. Six vertex–midpoint–centroid triangles of ABC.

This set of six triangles associated to the triangle ABC is a special case of the cevasix configuration (see [5] and [7]) when the chosen point is the centroid G. It has the following peculiar property (see [1]).

Theorem 1. *The triangle ABC is equilateral if and only if any three of the triangles from the set $\sigma_G = \{G_a^-, G_a^+, G_b^-, G_b^+, G_c^-, G_c^+\}$ have the same either perimeter or inradius.*

In this paper we wish to show several similar results. The idea is to replace perimeter and inradius with other geometric notions (like k-perimeter and Brocard angle) and to use various central points (like the circumcenter and the orthocenter – see [4]) of these six triangles.

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Let a, b, c be lengths of sides of the base triangle ABC. For a real number k, the sum $p_k = p_k(ABC) = a^k + b^k + c^k$ is called the k-*perimeter* of ABC. Of course, the 1-perimeter $p_1(ABC)$ is just the perimeter $p(ABC)$. The above theorem suggests the following problem.

Problem. *Find the set Ω of all real numbers k such that the following is true: The triangle ABC is equilateral if and only if any three of the triangles from σ_G have the same k-perimeter.*

Our first goal is to show that the set Ω contains some values of k besides the value $k = 1$. We start with $k = 2$ and $k = 4$.

Theorem 2. *The triangle ABC is equilateral if and only if any three of the triangles in σ_G have the same either 2-perimeter or 4-perimeter.*

Proof for $k = 2$. We shall position the triangle ABC in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex A is the origin with coordinates $(0, 0)$, the vertex B is on the x-axis and has coordinates $(r(f + g), 0)$, and the vertex C has coordinates $\left(\frac{rg(f^2-1)}{fg-1}, \frac{2rfg}{fg-1}\right)$. The three parameters r, f, and g are the inradius and the cotangents of half of angles at vertices A and B. Without loss of generality, we can assume that both f and g are larger than 1 (*i.e.*, that angles A and B are acute).

Nice features of this placement are that many important points of the triangle have rational functions in f, g, and r as coordinates and that we can easily switch from f, g, and r to side lengths a, b, and c and back with substitutions

$$a = \frac{rf\left(g^2+1\right)}{fg-1}, \quad b = \frac{rg\left(f^2+1\right)}{fg-1}, \quad c = r\,(f+g)\,,$$
$$f = \frac{(b+c)^2-a^2}{4\Delta}, \quad g = \frac{(a+c)^2-b^2}{4\Delta}, \quad r = \frac{2\Delta}{a+b+c},$$

where the area Δ is $\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a-b+c)(a+b-c)}$.

There are 20 ways in which we can choose 3 triangles from the set σ_G. The following three cases are important because all other cases are similar to one of these.

Case 1: (G_a^-, G_a^+, G_b^-). When we compute the 2-perimeters $p_2(G_a^-)$, $p_2(G_a^+)$, and $p_2(G_b^-)$ and convert to lengths of sides we get

$$p_2(G_a^-) - p_2(G_a^+) = \frac{(c-b)(c+b)}{3},$$
$$p_2(G_a^-) - p_2(G_b^-) = \frac{a^2}{6} - \frac{b^2}{2} + \frac{c^2}{3}.$$

Both of these differences are by assumption zero. From the first we get $b = c$ and when we substitute this into the second the conclusion is $\frac{(a-c)(a+c)}{6} = 0$. Hence, $b = c = a$ so that ABC is equilateral.

Case 2: (G_a^-, G_a^+, G_b^+). Now we have

$$p_2(G_a^-) - p_2(G_a^+) = \frac{(c-b)(c+b)}{3},$$
$$p_2(G_a^-) - p_2(G_b^+) = \frac{(a-b)(a+b)}{2},$$

which makes the conclusion easy.

Case 3: (G_a^-, G_b^-, G_c^-). This time we have

$$p_2(G_a^-) - p_2(G_b^-) = \frac{a^2}{6} - \frac{b^2}{2} + \frac{c^2}{3},$$
$$p_2(G_a^-) - p_2(G_c^-) = \frac{a^2}{2} - \frac{b^2}{3} - \frac{c^2}{6}.$$

The only solution of this linear system in a^2 and b^2 is $a^2 = c^2$ and $b^2 = c^2$. Thus the triangle ABC is equilateral because the lengths of sides are positive. □

Recall that the Brocard angle ω of the triangle ABC satisfies the relation

$$\cot \omega = \frac{p_2(ABC)}{4\Delta}.$$

Since all triangles in σ_G have the same area, from Theorem 2 we get the following corollary.

Corollary 3. *The triangle ABC is equilateral if and only if any three of the triangles in σ_G have the same Brocard angle.*

On the other hand, when we put $k = -2$ then for $a = \sqrt{-5 + 3\sqrt{3}}$ and $b = c = 1$ we find that the triangles G_a^-, G_a^+, and G_b^- have the same (-2)-perimeter while ABC is not equilateral. In other words the value -2 is not in Ω.

The following result answers the final question in [1]. It shows that some pairs of triangles from the set σ_G could be used to detect if ABC is isosceles. Let τ denote the set whose elements are pairs (G_a^-, G_a^+) (G_a^-, G_b^+), (G_a^-, G_c^+), (G_a^+, G_b^-), (G_a^+, G_c^-), (G_b^-, G_b^+), (G_b^-, G_c^+), (G_b^+, G_c^-), (G_c^-, G_c^+).

Theorem 4. *The triangle ABC is isosceles if and only if triangles from some element of τ have the same perimeter.*

Proof. This time there are only two representative cases.

Case 1: (G_a^-, G_a^+). By assumption,

$$p(G_a^-) - p(G_a^+) = \frac{\sqrt{2a^2 - b^2 + 2c^2}}{3} - \frac{\sqrt{2a^2 + 2b^2 - c^2}}{3} = 0.$$

When we move the second term to the right then take the square of both sides and move everything back to the left we obtain $\frac{(c-b)(c+b)}{3} = 0$. Hence, $b = c$ and ABC is isosceles.

Case 2: (G_a^-, G_b^+). This time our assumption is

$$p(G_a^-) - p(G_b^+) = \frac{a-b}{2} + \frac{\sqrt{2a^2 - b^2 + 2c^2}}{6} - \frac{\sqrt{2c^2 + 2b^2 - a^2}}{6} = 0.$$

When we move the third term to the right then take the square of both sides and move the right hand side back to the left and bring the only term with the square root to the right we obtain

$$\frac{2a^2 - 3ab + b^2}{6} = \frac{(b-a)\sqrt{2a^2 - b^2 + 2c^2}}{6}.$$

In order to eliminate the square root, we take the square of both sides and move the right hand side to the left to get $\frac{(a-b)^2(a-b-c)(a-b+c)}{18} = 0$. Hence, $a = b$ and the triangle ABC is again isosceles. □

Remark. The above theorem is true also when the perimeter is replaced with the 2-perimeter and the 4-perimeter. It is not true for $k = -2$ but it holds for any $k \neq 0$ when only pairs (G_a^-, G_a^+), (G_b^-, G_b^+), (G_c^-, G_c^+) are considered.

We continue with results that use various central points (see [4], [5, 6]) (like the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the symmedian or the Grebe-Lemoine point, and the Longchamps point) of the triangles from the set σ_G and try to detect when ABC is either equilateral or isosceles.

Recall that triangles ABC and XYZ are *homologic* provided lines AX, BY, and CZ are concurrent. The point in which they concur is their homology *center* and the line containing intersections of pairs of lines (BC, YZ), (CA, ZX), and (AB, XY) is their homology *axis*. Instead of homologic, homology center, and homology axis many authors use the terms *perspective*, *perspector*, and *perspectrix*.

The triangles ABC and XYZ are *orthologic* when the perpendiculars at vertices of ABC onto the corresponding sides of XYZ are concurrent. The point of concurrence is $[ABC, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto corresponding sides of ABC are concurrent at a point $[XYZ, ABC]$.

By replacing in the above definition perpendiculars with parallels we get the analogous notion of *paralogic* triangles and two centers of paralogy $\langle ABC, XYZ\rangle$ and $\langle XYZ, ABC\rangle$.

The triangle ABC is paralogic to its first Brocard triangle $A_bB_bC_b$ which has the orthogonal projections of the symmedian point K onto the perpendicular bisectors of sides as vertices (see [2] and [3]).

Theorem 5. *The centroids $G_{G_a^-}$, $G_{G_a^+}$, $G_{G_b^-}$, $G_{G_b^+}$, $G_{G_c^-}$, $G_{G_c^+}$ of the triangles from σ_G lie on the image of the Steiner ellipse of ABC under the homothety $\mathsf{h}(G, \frac{\sqrt{7}}{6})$. This ellipse is a circle if and only if ABC is equilateral. The triangles $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $G_{G_a^+}G_{G_b^+}G_{G_c^+}$ are both homologic and paralogic to triangles $A_bB_bC_b$, $B_bC_bA_b$ and $C_bA_bB_b$ and they share with ABC the centroid and the Brocard angle and both have $\frac{7}{36}$ of the area of ABC. They are directly similar to each other or to ABC if and only if ABC is an equilateral triangle. They are orthologic to either $A_bB_bC_b$, $B_bC_bA_b$ or $C_bA_bB_b$ if and only if ABC is an equilateral triangle.*

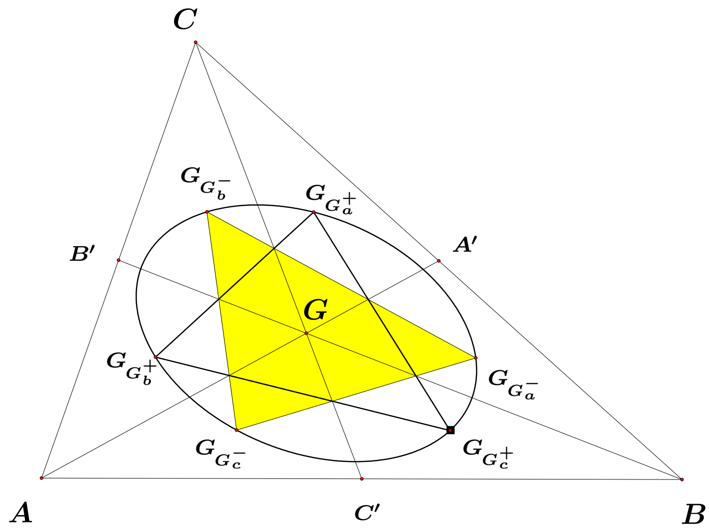

Figure 2. The ellipse containing vertices of $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $G_{G_a^+}G_{G_b^+}G_{G_c^+}$.

Proof. We look for the conic through five of the centroids and check that the the sixth centroid lies on it. The trilinear coordinates of $G_{G_a^-}$ are $\frac{2}{a} : \frac{11}{b} : \frac{5}{c}$ while those of other centroids are similar. It follows that they all lie on the ellipse with the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0,$$

where

$$\begin{aligned}&a_{11} = 432\Delta^2, \quad a_{12} = 108\Delta(a-b)(a+b),\\ &a_{22} = 27(a^4 + b^4 + 3c^4 - 2a^2b^2),\\ &a_{13} = -216\Delta^2 c, \quad a_{23} = -54\Delta c(a^2 - b^2 + c^2), \quad a_{33} = 116\Delta^2c^2.\end{aligned}$$

Since $D_0 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \frac{3c^4}{16\Delta^2} > 0$, and $\frac{A_0}{I_0} = \frac{-7c^4}{72(a^2+b^2+c^2)} < 0$ with $I_0 = a_{11} + a_{22}$, and $A_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$ it follows that this is an ellipse whose center is G. It will be a circle provided either $I_0^2 = 4D_0$ or $a_{11} = a_{22}$ and $a_{12} = 0$. This happens if and only if ABC is equilateral.

The precise identification of this ellipse is now easy. We take a point (p,q) which is on the Steiner ellipse of ABC (with the equation $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0$ in trilinear coordinates) and denote its image under $\mathsf{h}(G, \frac{\sqrt{7}}{6})$ by (x,y). By eliminating p and q we check that this image satisfies the above equation (of the common Steiner ellipse of $G_{G_a^-}G_{G_b^-}G_{G_c^-}$ and $G_{G_a^+}G_{G_b^+}G_{G_c^+}$).

Since the trilinear coordinates of A_b are $abc : c^3 : b^3$, the line $A_b G_{G_a^-}$ has the equation

$$a(11b^2 - 5c^2)x + b(5a^2 - 2b^2)y + c(11a^2 - 2c^2)z = 0.$$

The lines $B_b G_{G_b^-}$ and $C_b G_{G_c^-}$ have similar equations. The determinant of the coefficients of these three lines is equal to zero so that we conclude that the triangles $G_{G_a^-} G_{G_b^-} G_{G_c^-}$ and $A_b B_b C_b$ are homologic. The other claims about homologies and paralogies are proved in a similar way. We note that $\langle G_{G_a^-} G_{G_b^-} G_{G_c^-}, A_b B_b C_b \rangle$ is on the (above) Steiner ellipse of $G_{G_a^-} G_{G_b^-} G_{G_c^-}$ while $\langle A_b B_b C_b, G_{G_a^-} G_{G_b^-} G_{G_c^-} \rangle$ is on the Steiner ellipse of $A_b B_b C_b$. The other centers behave accordingly.

When we substitute the coordinates of the six centroids into the conditions

$$x_1(v_2 - v_3) + x_2(v_3 - v_1) + x_3(v_1 - v_2) - u_1(y_2 - y_3) - u_2(y_3 - y_1) - u_3(y_1 - y_2) = 0,$$
$$x_1(u_2 - u_3) + x_2(u_3 - u_1) + x_3(u_1 - u_2) - y_1(v_2 - v_3) - y_2(v_3 - v_1) - y_3(v_1 - v_2) = 0,$$

for triangles with vertices at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (u_1, v_1), (u_2, v_2), (u_3, v_3) to be directly similar and convert to the side lengths, we get

$$\frac{4\Delta(a-b)(a+b+c)}{9c^2} = 0 \quad \text{and} \quad \frac{h(1,1,2,1,1,2)}{9c^2} = 0,$$

where

$$h(u, v, w, x, y, z) = ub^2c^2 + vc^2a^2 + wa^2b^2 - xa^4 - yb^4 - zc^4.$$

The first relation implies $a = b$, which gives $h(1,1,2,1,1,2) = 2c^2(c-b)(c+b)$. Therefore, $b = c$ so that ABC is an equilateral triangle.

Substituting the coordinates of $G_{G_a^-}$, $G_{G_b^-}$, $G_{G_c^-}$, A_b, B_b, C_b into the left hand side of the condition

$$x_1(u_2 - u_3) + x_2(u_3 - u_1) + x_3(u_1 - u_2) + y_1(v_2 - v_3) + y_2(v_3 - v_1) + y_3(v_1 - v_2) = 0,$$

for triangles with vertices at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (u_1, v_1), (u_2, v_2), (u_3, v_3) to be orthologic, we obtain

$$\frac{-h(1,1,1,1,1,1)}{3p_2(ABC)} = \frac{(b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2}{6p_2(ABC)}$$

so that the triangles $G_{G_a^-} G_{G_b^-} G_{G_c^-}$ and $A_b B_b C_b$ are orthologic if and only if ABC is equilateral.

The remaining statements are proved similarly or by substitution of coordinates into well-known formulas for the area, the centroid, and the Brocard angle. □

Let m_a, m_b, m_c be lengths of medians of the triangle ABC. The following result is for the most part already proved in [7]. The center of the circle is given in [6] as $X(1153)$.

Theorem 6. *The circumcenters $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$, $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ of the triangles from σ_G lie on the circle whose center O_G is a central point with the first*

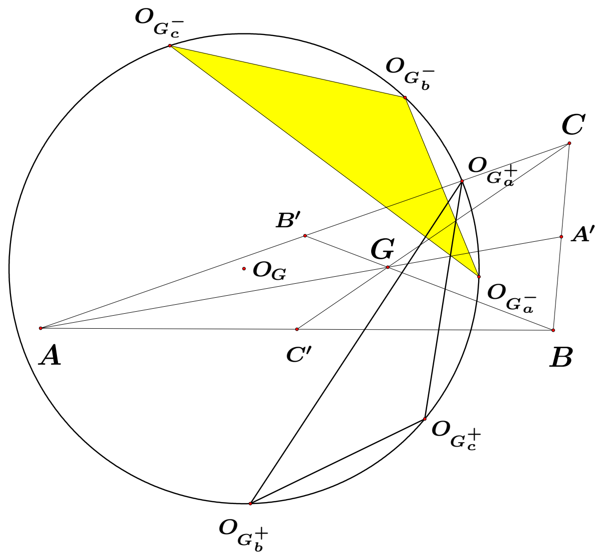

Figure 3. The vertices of $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_a^+}O_{G_b^+}O_{G_c^+}$ are on a circle.

trilinear coordinate

$$\frac{10a^4 - 13a^2(b^2+c^2) + 4b^4 + 4c^4 - 10b^2c^2}{a}$$

and whose radius is

$$\frac{m_a m_b m_c \sqrt{2(a^4+b^4+c^4) - 5(b^2c^2+c^2a^2+a^2b^2)}}{72\Delta}.$$

Also, $|O_G G| = \frac{m_a m_b m_c \sqrt{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2}}{72\sqrt{2}\Delta}$.

Proof. The proof is conceptually simple but technically involved so that we shall only outline how it could be done on a computer. In order to find points $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$, $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ we use the circumcenter function and evaluate it in vertices of the triangles from σ_G. Applying it again in points $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$ we obtain the point O_G. The remaining points $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ are at the same distance from it as the vertex $O_{G_a^-}$ is. The remaining tasks are standard (they involve only the distance function and the conversion to the side lengths). □

The last sentence in Theorem 6 implies the following corollary.

Corollary 7. *The triangle ABC is equilateral if and only if the circumcenters of any three of the triangles in σ_G have the same distance from the centroid G.*

Let P, Q and R denote vertices of similar isosceles triangles BCP, CAQ and ABR.

Theorem 8. *(1) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_b^+}O_{G_c^+}O_{G_a^+}$ are congruent. They are orthologic to BCA and CAB, respectively.*

(2) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_a^+}O_{G_b^+}O_{G_c^+}$ are orthologic to QRP and RPQ if and only if ABC is an equilateral triangle.

(3) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are orthologic if and only if the lengths of sides of ABC satisfy $h(7,7,7,4,4,4)=0$.

*(4) The line joining the centroids of triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ will go through the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the Longchamps point, or the Bevan point of ABC (*i.e., *$X(2)$, $X(3)$, $X(4)$, $X(5)$, $X(20)$, or $X(40)$ in* [6]*) if and only if it is an equilateral triangle.*

*(5) The line joining the symmedian points of $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ goes through the centroid of ABC. It will go through the centroid of its orthic triangle (*i.e., *$X(51)$ in* [6]*) if and only if ABC is an equilateral triangle.*

(6) The centroids of triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ have the same distance from $X(2)$, $X(3)$, $X(4)$, $X(5)$, $X(6)$, $X(20)$, $X(39)$, $X(40)$, or $X(98)$ if and only if ABC is an isosceles triangle.

Proof. (1) The points $O_{G_a^-}$ and $O_{G_a^+}$ have trilinear coordinates

$$a(5c^2-a^2-b^2) : \frac{2h(3,3,5,2,2,1)}{b} : \frac{h(6,1,3,1,2,4)}{c},$$

$$a(5b^2-a^2-c^2) : \frac{h(6,3,1,1,4,2)}{b} : \frac{2h(3,5,3,2,1,2)}{c},$$

while the trilinears of the points $O_{G_b^-}$, $O_{G_c^-}$, $O_{G_b^+}$, $O_{G_c^+}$ are their cyclic permutations. We can show easily that $|O_{G_b^-}O_{G_c^-}|^2 - |O_{G_c^+}O_{G_a^+}|^2 = 0$, $|O_{G_c^-}O_{G_a^-}|^2 - |O_{G_a^+}O_{G_b^+}|^2 = 0$, and $|O_{G_a^-}O_{G_b^-}|^2 - |O_{G_b^+}O_{G_c^+}|^2 = 0$, so that $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_b^+}O_{G_c^+}O_{G_a^+}$ are indeed congruent.

Substituting the coordinates of $O_{G_a^-}$, $O_{G_b^-}$, $O_{G_c^-}$, B, C, A into the left hand side of the above condition for triangles to be orthologic we conclude that it holds. The same is true for the triangles $O_{G_a^+}O_{G_b^+}O_{G_c^+}$ and CAB.

(2) The point P has the trilinear coordinates

$$2ka : \frac{k(a^2+b^2-c^2)+2\Delta}{b} : \frac{k(a^2-b^2+c^2)+2\Delta}{c}$$

for some real number $k \neq 0$. The coordinates of Q and R are analogous. It follows that the triangles $O_{G_c^-}O_{G_a^-}O_{G_b^-}$ and QRP are orthologic provided

$$\frac{h(1,1,1,1,1,1)k}{8\Delta} = 0,$$

i.e., if and only if ABC is equilateral.

(3) The triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are orthologic provided $\frac{p_2(ABC)h(7,7,7,4,4,4)}{384\Delta^2} = 0$. The triangle with lengths of sides $4, 4, 3\sqrt{2}+\sqrt{10}$ satisfies this condition.

(4) for $X(40)$. The first trilinear coordinates of the centroids of the triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are

$$\frac{3a^4-(2b^2+7c^2)a^2+b^4-3b^2c^2+2c^4}{a}$$

and

$$\frac{3a^4-(7b^2+2c^2)a^2+2b^4-3b^2c^2+c^4}{a}.$$

The line joining these centroids will go through $X(40)$ with the first trilinear coordinate $a^3+(b+c)a^2-(b+c)^2a-(b+c)(b-c)^2$ provided

$$\frac{(a^2+b^2+c^2-bc-ca-ab)(3bc+3ca+3ab+a^2+b^2+c^2)}{96\Delta}=0.$$

Since $a^2+b^2+c^2-bc-ca-ab=\frac{1}{2}\left((b-c)^2+(c-a)^2+(a-b)^2\right)$ it follows that this will happen if and only if ABC is equilateral.

(5) The first trilinear coordinates of the symmedian points of $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ are

$$\frac{2a^6-(b^2+3c^2)a^4+(3b^4-12b^2c^2-7c^4)a^2+2c^2(b^2-c^2)(b^2-2c^2)}{a}$$

and

$$\frac{2a^6-(3b^2+c^2)a^4-(7b^4+12b^2c^2-3c^4)a^2+2b^2(b^2-c^2)(2b^2-c^2)}{a}.$$

The line joining these symmedian points will go through $X(51)$ with the first trilinear coordinate $a\left((b^2+c^2)a^2-(b^2-c^2)^2\right)$ provided

$$\frac{2\Delta h(1,1,1,0,0,0)h(1,1,1,1,1,1)}{9a^2b^2c^2(a^2+b^2+c^2)}=0.$$

Since $h(1,1,1,1,1,1)=\frac{1}{2}\left((b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2\right)$ we see that this will happen if and only if ABC is equilateral. The trilinear coordinates $\frac{1}{a}:\frac{1}{b}:\frac{1}{c}$ of the centroid G satisfy the equation of this line.

(6) for $X(40)$. Using the information from the proof of (4), we see that the difference of squares of distances from $X(40)$ to the centroids of the triangles $O_{G_a^-}O_{G_b^-}O_{G_c^-}$ and $O_{G_c^+}O_{G_a^+}O_{G_b^+}$ is $\frac{(b-c)(c-a)(a-b)M}{192\Delta^2}$, where

$$M=2(a^3+b^3+c^3)+5(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)+18abc$$

is clearly positive. Hence, these distances are equal if and only if ABC is isosceles. □

With points $O_{G_a^-}$, $O_{G_a^+}$, $O_{G_b^-}$, $O_{G_b^+}$, $O_{G_c^-}$, $O_{G_c^+}$ we can also detect if ABC is isosceles as follows.

Theorem 9. *(1) The relation $b=c$ holds in ABC if and only if $O_{G_a^-}$ is on BG and/or $O_{G_a^+}$ is on CG.*

(2) The relation $c=a$ holds in ABC if and only if $O_{G_b^-}$ is on CG and/or $O_{G_b^+}$ is on AG.

(3) The relation $a = b$ holds in ABC if and only if $O_{G_c^-}$ is on AG and/or $O_{G_a^+}$ is on BG.

Proof. (1) for $O_{G_a^-}$. Since the trilinear coordinates of $O_{G_a^-}$, G and B are

$$a(5c^2 - a^2 - b^2) : \frac{2h(3,3,5,2,2,1)}{b} : \frac{h(6,1,3,1,2,4)}{c},$$

$\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$ and $(0 : 1 : 0)$, it follows that these points are collinear if and only if $\frac{m_b^2(b-c)(b+c)}{72\Delta} = 0$. □

For the following result I am grateful to an anonymous referee. It refers to the point T on the Euler line which divides the segment joining the circumcenter with the centroid in ratio k for some real number $k \neq -1$. Notice that for $k = 0, -\frac{3}{4}, -\frac{3}{2}, -3$ the point T will be the circumcenter, the Longchamps point, the orthocenter, and the center of the nine-point circle, respectively.

Theorem 10. *The triangles $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are directly similar to each other or to ABC if and only if ABC is equilateral.*

Proof. For $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$.

The point $T_{G_a^-}$ has $\frac{p_1}{a} : \frac{p_2}{b} : \frac{p_3}{c}$ as trilinear coordinates, where

$$\begin{aligned}
p_1 &= 3a^2(a^2 + b^2 - 5c^2) - 32\Delta^2 k,\\
p_2 &= 12a^4 - 6(5b^2 + 3c^2)a^2 + 6(b^2 - c^2)(2b^2 - c^2) - 176\Delta^2 k,\\
p_3 &= 12a^4 - 6(3b^2 + 5c^2)a^2 + 6(b^2 - c^2)(b^2 - 2c^2) - 176\Delta^2 k.
\end{aligned}$$

Applying the method of the proof of Theorem 4 we see that $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are directly similar if and only if

$$\frac{(a^2 - b^2)M}{288\Delta c^2(k+1)^2} = 0 \quad \text{and} \quad \frac{h(1,1,2,1,1,2)M}{1152S^2c^2(k+1)^2} = 0,$$

where $M = 128\Delta^2k^2 + 240\Delta^2k + h(15,15,15,6,6,6)$. The discriminant

$$-48\Delta^2 h(10,10,10,-11,-11,-11)$$

of the trinomial M is negative so that M is always positive. Hence, from the first condition it follows that $a = b$. Then the factor $h(1,1,2,1,1,2)$ in the second condition is $2c^2(c-b)(c+b)$ so that $b = c$ and ABC is equilateral. The converse is easy because for $a = b = c$ the left hand sides of both conditions are equal to zero.

For $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and ABC. The two conditions are

$$\begin{aligned}
&32\Delta^2(a^2 - b^2)k - a^6 + (4b^2 + 3c^2)a^4\\
&- (5b^4 + 2b^2c^2 + c^4)a^2 - 3b^4c^2 + 2b^2c^4 + 2b^6 + c^6 = 0
\end{aligned}$$

and

$$h(2,2,4,2,2,4)k + h(1,2,3,1,2,3) = 0.$$

When $a \neq b$, we can solve the first equation for k and substitute it into the second to obtain $\frac{c^4(a^2+b^2+c^2)h(1,1,1,1,1,1)}{8\Delta^2(a^2-b^2)} = 0$. This implies that $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and ABC are directly similar if and only if ABC is equilateral because the first condition is $c^2\,(b-c)\,(b+c)\left(c^2+2b^2\right) = 0$ for $a = b$. $\square$

Theorem 11. *(1) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are orthologic to ABC if and only if $k = -\frac{3}{2}$.*

(2) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are orthologic to $A_bB_bC_b$ if and only if either ABC is equilateral or $k = -\frac{3}{4}$.

(3) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ are paralogic to either $A_bB_bC_b$, $B_bC_bA_b$ or $C_bA_bB_b$ if and only if ABC is equilateral.

(4) $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ is orthologic to $B_bC_bA_b$ if and only if either ABC is equilateral or $k = -\frac{3}{2}$ and to $C_bA_bB_b$ if and only if ABC is equilateral.

(5) $T_{G_a^+}T_{G_b^+}T_{G_c^+}$ is orthologic to $B_bC_bA_b$ if and only if ABC is equilateral and to $C_bA_bB_b$ if and only if either ABC is equilateral or $k = -\frac{3}{2}$.

Proof. All parts have similar proofs. For example, in the first, we find that the triangles $T_{G_a^-}T_{G_b^-}T_{G_c^-}$ and ABC are orthologic if and only if $-\frac{(a^2+b^2+c^2)(2k+3)}{12(k+1)} = 0$. $\square$

The orthocenters $H_{G_a^-}$, $H_{G_a^+}$, $H_{G_b^-}$, $H_{G_b^+}$, $H_{G_c^-}$, $H_{G_c^+}$ of the triangles from σ_G also monitor the shape of the triangle ABC.

Theorem 12. *The triangles $H_{G_a^-}H_{G_b^-}H_{G_c^-}$ and $H_{G_a^+}H_{G_b^+}H_{G_c^+}$ are orthologic if and only if ABC is an equilateral triangle.*

Proof. Substituting the coordinates of $H_{G_a^-}$, $H_{G_b^-}$, $H_{G_c^-}$, $H_{G_a^+}$, $H_{G_b^+}$, $H_{G_c^+}$ into the condition for triangles to be orthologic (see the proof of Theorem 6), we obtain

$$\frac{(a^2+b^2+c^2)[(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2]}{192\Delta^2} = 0.$$

Hence, $a = b = c$ and the triangle ABC is equilateral. $\square$

Remark. Note that the triangles $H_{G_a^-}H_{G_b^-}H_{G_c^-}$ and $H_{G_a^+}H_{G_b^+}H_{G_c^+}$ have the same Brocard angle and both have the area equal to one fourth of the area of ABC.

The centers $F_{G_a^-}$, $F_{G_a^+}$, $F_{G_b^-}$, $F_{G_b^+}$, $F_{G_c^-}$, $F_{G_c^+}$ of the nine point circles of the triangles from σ_G allow the following analogous result.

Theorem 13. *The triangles $F_{G_a^-}F_{G_b^-}F_{G_c^-}$ and $F_{G_a^+}F_{G_b^+}F_{G_c^+}$ have the same Brocard angle and area. The triangle ABC is equilateral if and only if this area is $\frac{3}{16}$ of the area of ABC.*

Proof. Recall the formula $\frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$ for the area of the triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3). Since

$$\frac{3}{16}|ABC| - |F_{G_a^-}F_{G_b^-}F_{G_c^-}| = \frac{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2}{1536\Delta},$$

the second claim is true. The proof of the first are also substitutions of coordinates into well-known formulas. □

The symmedian points $K_{G_a^-}$, $K_{G_a^+}$, $K_{G_b^-}$, $K_{G_b^+}$, $K_{G_c^-}$, $K_{G_c^+}$ of the triangles from σ_G play the similar role.

Theorem 14. *The triangles $K_{G_a^-}K_{G_b^-}K_{G_c^-}$ and $K_{G_a^+}K_{G_b^+}K_{G_c^+}$ have the area equal to $\frac{7}{64}$ of the area of ABC if and only if ABC is an equilateral triangle.*

Proof. The difference $|K_{G_a^-}K_{G_b^-}K_{G_c^-}| - \frac{7}{64}|ABC|$ is equal to

$$\frac{3\Delta T}{64(5b^2+8c^2-a^2)(5c^2+8a^2-b^2)(5a^2+8b^2-c^2)},$$

where

$$T = 40(a^6+b^6+c^6)+231(b^4c^2+c^4a^2+a^4b^2)-147(b^2c^4+c^2a^4+a^2b^4)-372a^2b^2c^2.$$

We shall argue that T is equal to zero if and only if $a = b = c$. We can assume that $a \le b \le c$, $a = \sqrt{d}$, $b = \sqrt{(1+h)d}$, $c = \sqrt{(1+h+k)d}$ for some positive real numbers d, h and k. In new variables $\frac{T}{d^3}$ is

$$164h^3 + (204+57k)h^2 + 3k(68-9k)h + 4k^2(51+10k).$$

The quadratic part has the discriminant $-3k^2(41616+30056k+2797k^2)$. Thus T is always positive except when $h = k = 0$ which proves our claim. □

Theorem 15. *The triangles $K_{G_a^-}K_{G_b^-}K_{G_c^-}$ and $K_{G_a^+}K_{G_b^+}K_{G_c^+}$ have the same area if and only if the triangle ABC is isosceles.*

Proof. The difference $|K_{G_a^-}K_{G_b^-}K_{G_c^-}| - |K_{G_a^+}K_{G_b^+}K_{G_c^+}|$ is equal to

$$\frac{81\Delta(b-c)(b+c)(c-a)(c+a)(a-b)(a+b)T}{2t(-1,8,5)t(-1,5,8)t(8,-1,5)t(5,-1,8)t(8,5,-1)t(5,8,-1)},$$

where $t(u,v,w) = ua^2+vb^2+wc^2$ and

$$T = 10(a^6+b^6+c^6)-105(b^4c^2+c^4a^2+a^4b^2+b^2c^4+c^2a^4+a^2b^4)-156a^2b^2c^2.$$

We shall now argue that T is always negative. Without loss of generality we can assume that $a \le b \le c$ and that

$$a = \sqrt{d}, \quad b = \sqrt{(1+h)d}, \quad c = \sqrt{(1+h+k)d},$$

for some positive real numbers d, h and k. Since $a + b > c$ it follows that

$$k < 1 + 2\sqrt{h+1} \le h+3$$

because $\sqrt{h+1} = \sqrt{1\cdot(h+1)} \le \frac{1+(h+1)}{2}$. In new variables,

$$-\frac{T}{d^3} = 190h^3+(285k+936)h^2+(1512+936k+75k^2)h-10k^3+180k^2+756k+756.$$

For $k \le h$ it is obvious that the above polynomial is positive since $190h^3-10k^3 > 0$. On the other hand, when $k \in (h, h+3)$, then k can be represented as $(1-w)h + w(h+3)$ for some $w \in (0,1)$. The above polynomial for this k is

$$540h^3+(2052+1215w)h^2+(3888w+405w^2+2268)h-270w^3+1620w^2+2268w+756.$$

But, the free coefficient of this polynomial for w between 0 and 1 is positive. Thus T is always negative which proves our claim. $\square$

The Longchamps points (*i.e.*, the reflections of the orthocenters in the circumcenters) $L_{G_a^-}$, $L_{G_a^+}$, $L_{G_b^-}$, $L_{G_b^+}$, $L_{G_c^-}$, $L_{G_c^+}$ of the triangles from σ_G offer the following result.

Theorem 16. *The triangles $L_{G_a^-}L_{G_b^-}L_{G_c^-}$ and $L_{G_a^+}L_{G_b^+}L_{G_c^+}$ have the same areas and Brocard angles. This area is equal to $\frac{3}{4}$ of the area of ABC and/or this Brocard angle is equal to the Brocard angle of ABC if and only if ABC is an equilateral triangle.*

Proof. The common area is $\frac{h(10,10,10,1,1,1)}{112\Delta}$ while the tangent of the common Brocard angle is $\frac{h(10,10,10,1,1,1)}{4\Delta p_2(ABC)h(2,2,2,-7,-7,-7)}$. It follows that the difference

$$\frac{3}{4}|ABC| - |L_{G_a^-}L_{G_b^-}L_{G_c^-}| = \frac{h(1,1,1,1,1,1)}{24\Delta}$$

while the difference of tangents of the Brocard angles of the triangles $L_{G_a^-}L_{G_b^-}L_{G_c^-}$ and ABC is $\frac{32\Delta h(1,1,1,1,1,1)}{p_2(ABC)h(2,2,2,-7,-7,-7)}$. From here the conclusions are easy because $h(1,1,1,1,1,1) = \frac{1}{2}\left((b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2\right)$. $\square$

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Minimal Chords in Angular Regions

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Abstract. We use synthetic geometry to show that in an angular region minimal chords having a prescribed direction form a ray which is constructible with ruler and compass.

Let P be a fixed point inside a circle of center O. It is well-known that among the chords containing P one of minimal length is perpendicular to the diameter through P, if $P \neq O$, or is any diameter, if $P = O$. Consequently, such a chord is always constructible with ruler and compass.

When it comes to geometrically constructing minimal chords through given points in convex regions the circle is in some sense a singular case. Indeed, as shown in [1] this task is impossible even in the case of the conics. However, in general *it is* possible to construct all the points inside a convex region which support minimal chords parallel to a given direction. We proved this in [1, 2] by analytical means, with special emphasis on the conics.

The purpose of this note is to prove the same thing for angular regions, via essentially a purely geometrical argument.

To this end let $\angle AOB$ be an angle of vertex O and sides $\overrightarrow{OA}$, $\overrightarrow{OB}$, such that O, A, and B are not colinear, and let P be a point inside the angle. By definition, a *chord* in this angle is a straight segment $\overline{MN}$ such that $M \in \overrightarrow{OA}$ and $N \in \overrightarrow{OB}$. A continuity argument makes clear that among the chords containing P there is at least one of minimal length, that is a minimal chord through P in the given angle.

Problem. *Given a direction in the plane of $\angle AOB$, construct with ruler and compass the geometric locus of all the points inside the angle which support minimal chords parallel to that direction.*

In order to solve this problem we need the following

Lemma. *Inside $\angle AOB$ consider the chord $\overline{MN}$, $M \in \overrightarrow{OA}$, $N \in \overrightarrow{OB}$, such that $\angle OMN$ and $\angle ONM$ are acute angles. If P is the foot of the perpendicular on $\overline{MN}$ through the point Q diametrically opposite O on the circle circumscribed about $\triangle OMN$, then $\overline{MN}$ is the unique minimal chord through P inside $\angle AOB$. P is seen to be the unique point inside $\overline{MN}$ such that $\overline{ML} \cong \overline{NP}$, where L is*

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the foot of the perpendicular from O on $\overline{MN}$. Moreover, any point on the ray $\overrightarrow{OP}$ supports an unique minimal chord, parallel to $\overline{MN}$.

Proof. Clearly, Q is an interior point to $\angle AOB$, situated on the other side of the line $\overleftrightarrow{MN}$ with respect to O, and $\overline{MQ} \perp \overrightarrow{OA}$ and $\overline{NQ} \perp \overrightarrow{OB}$. Since $\angle OMN$ and $\angle ONM$ are acute angles, and $\angle QMN$ and $\angle QNM$ are acute angles too, as complements of acute angles , the points P and L described in the statement of the Lemma are interior points to the segment $\overline{MN}$. (See Figure 1).

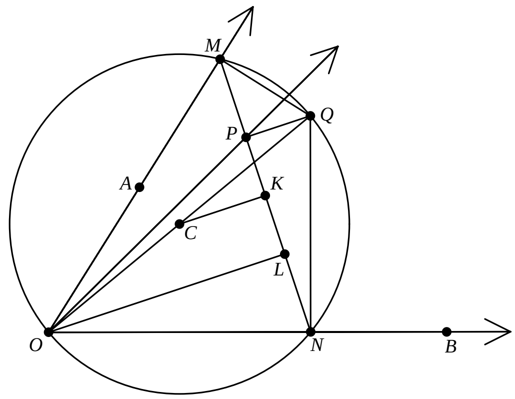

Figure 1

Let us prove first that $\overline{MN}$ is a minimal chord through P in $\angle AOB$. Let $\overline{M'N'}$, $M' \in \overrightarrow{OA}$, $N' \in \overrightarrow{OB}$, $P \in \overline{M'N'}$, be another chord through P (See Figure 2).

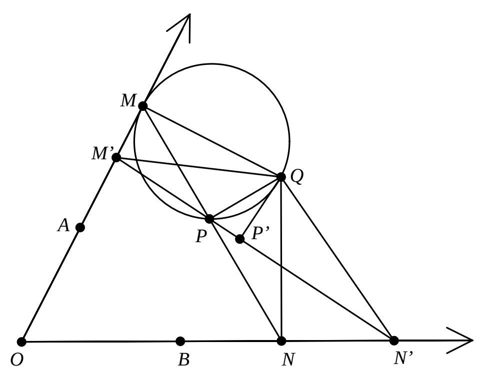

Figure 2

Notice now that the following angle inequalities hold:

$$\angle QM'P < \angle QMP, \ \ \angle QN'P < \angle QNP \tag{1}$$

Indeed, since the circle circumscribed about $\triangle MPQ$ is tangent to the ray $\overrightarrow{OA}$ at M, the point M' is located outside this circle. Now $\angle QMP$ and $\angle QM'P$ are precisely the angles the segment $\overline{PQ}$ is seen from M, respectively M'. Since M belongs to the circle circumscribed about $\triangle MPQ$ and M' is outside this circle, the inequality $\angle QM'P < \angle QMP$ becomes obvious. The other inequality (1) can be proved in a similar fashion.

The inequalities (1) prove that $\angle QM'N'$ and $\angle QN'M'$ are acute angles too, thus the foot P' of the perpendicular from Q on the line $\overleftrightarrow{M'N'}$ belongs to the interior of the segment $\overline{M'N'}$.

Notice now that

$$MQ < M'Q, \;\; NQ < N'Q, \;\; P'Q < PQ.$$

The above inequalities are obvious since in a right triangle a leg is shorter than the hypothenuse. Consequently, the Pythagorean Theorem yields

$$MP = \sqrt{MQ^2 - PQ^2} < \sqrt{M'Q^2 - P'Q^2} = M'P',$$

and similarly, $NP < N'P'$. In conclusion,

$$MN = MP + NP < M'P' + N'P' = M'N',$$

and so $\overline{MN}$ is indeed the unique minimal chord through P in $\angle AOB$.

The perpendicular line on $\overline{MN}$ through the center C of the circle circumscribed about the quadrilateral $OMQN$ intersects $\overline{MN}$ at its midpoint K (See Figure 1). Clearly, $\overline{KP} \cong \overline{KL}$, and so $\overline{ML} \cong \overline{NP}$ as stated.

Finally, the fact that any point on the ray $\overrightarrow{OP}$ supports an unique minimal chord parallel to $\overline{MN}$ is an immediate consequence of standard properties of similar triangles in the context of what was proved above. □

To $\angle AOB$ we associate now another angle, $\angle A'OB'$, according to the following recipe:

$a)$ If $\angle AOB$ is *acute* then $\angle A'OB'$ is obtained by rotating $\angle AOB$ counterclockwise 90° around O.

$b)$ If $\angle AOB$ is *not acute* (so it is either right or obtuse) then $\angle A'OB'$ is the supplementary angle to $\angle AOB$ along the line $\overleftrightarrow{OB}$ (See Figure 3).

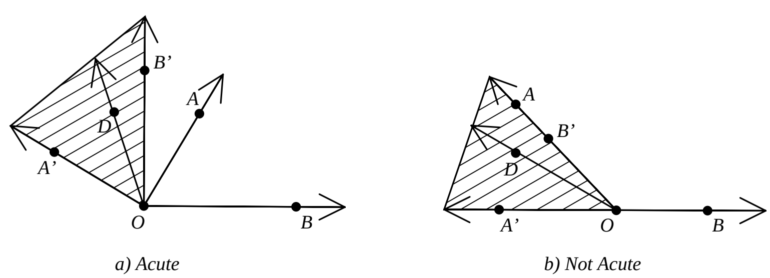

Figure 3

Definition. A ray $\overrightarrow{OD}$ is called an *admissible direction* for $\angle AOB$ if D is a point interior to $\angle A'OB'$.

It is easy to see that $\overrightarrow{OD}$ is an admissible direction for $\angle AOB$ if and only if any parallel line to $\overrightarrow{OD}$ through a point interior to $\angle AOB$ determines a chord $\overline{MN}$ such that $\angle OMN$ and $\angle ONM$ are acute angles.

Theorem. *Any point P inside $\angle AOB$ supports an unique minimal chord, parallel to an admissible direction. The geometric locus of all the points inside $\angle AOB$ which support minimal chords parallel to a given admissible direction can be constructed with ruler and compass as follows:*

$i)$ *Construct first the line $\overleftrightarrow{OL}$ perpendicular to the admissible direction, the point L being interior to $\angle AOB$.*

$ii)$ *Construct next the perpendicular through L to the line $\overleftrightarrow{OL}$, which intersects $\overrightarrow{OA}$ at M and $\overrightarrow{OB}$ at N.*

$iii)$ *Inside the segment $\overline{MN}$ construct the point P such that $\overline{NP} \cong \overline{ML}$.*

$iv)$ *Finally, construct the ray $\overrightarrow{OP}$, which is the desired geometric locus.*

Using the Lemma, an alternative construction can be provided by using the circle circumscribed about $\triangle OMN$, where the point M is chosen arbitrarily on $\overrightarrow{OA}$ and $N \in \overrightarrow{OB}$ is such that $\overline{MN}$ is parallel to the given admissible direction.

Proof. Let P be a fixed point inside $\angle AOB$. The proof splits naturally into two cases, according to $\angle AOB$ being acute or not.

$a)$ $\angle AOB$ *is acute*. Let $\overline{M_1N_1}$ be the perpendicular segment through P to $\overrightarrow{OA}$, $M_1 \in \overrightarrow{OA}$, $N_1 \in \overrightarrow{OB}$ and let $\overline{M_2N_2}$ be the perpendicular segment through P to $\overrightarrow{OB}$, $M_2 \in \overrightarrow{OA}$, $N_2 \in \overrightarrow{OB}$. Define now a function $f : \overline{M_1M_2} \longrightarrow \mathbf{R}$, by

$$f(M) = ML - NP, \quad M \in \overline{M_1M_2}, \tag{2}$$

where N is the intersection point of the line $\overleftrightarrow{MP}$ with $\overrightarrow{OB}$, and L is the foot of the perpendicular from O to the segment $\overline{MN}$ (See Figure 4).

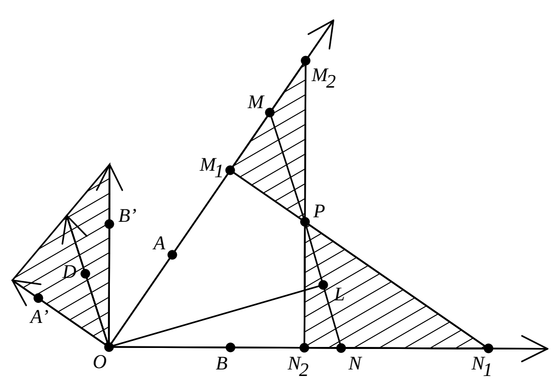

Figure 4

Clearly, this is a continuous function and $f(M_1) = -N_1P < 0$ and $f(M_2) = M_2P > 0$. By the intermediate value property there is some point $M \in \overline{M_1M_2}$ such that $f(M) = 0$, or equivalently $\overline{NP} \cong \overline{ML}$. According to the above Lemma, for this point M the chord $\overline{MN}$ is the unique minimal chord through P. It is also obvious that $\overline{MN}$ is parallel to an admissible direction.

$b)$ $\angle AOB$ *is not acute*. The proof in this case is a variant of that given at $a)$. Let M_0 be the point where the parallel line through P to $\overrightarrow{OB}$ intersects the ray $\overrightarrow{OA}$. Without loss of generality we can assume that M_0 is located between O and A. Defining now the function $f : \overrightarrow{M_0A} \longrightarrow \mathbf{R}$ by the same formula (2), we see that for points M close to M_0, $f(M)$ takes negative values and for points M far away on $\overrightarrow{M_0A}$, $f(M)$ takes positive values. One more time, the intermediate value property and the above Lemma guarantee the existence of an unique minimal chord through P, which is also parallel to an admissible direction.

Given now an admissible direction, the previous Lemma justifies the construction of the desired geometric locus as indicated in the statement of the theorem if we can prove that this locus does not contain points outside the ray $\overrightarrow{OP}$ described at $iv)$. Indeed this is the case since if there were other points then the equation $\overline{NP} \cong \overline{ML}$ would not hold. However, we have just proved that this equation is necessary for minimal chords. $\square$

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Three Pairs of Congruent Circles in a Circle

Li C. Tien

Abstract. Consider a closed chain of three pairs of congruent circles of radii a, b, c. The circle tangent internally to each of the 6 circles has radius $R = a + b + c$ if and only if there is a pair of congruent circles whose centers are on a diameter of the enclosing circle. Non-neighboring circles in the chain may overlap. Conditions for nonoverlapping are established. There can be a "central circle" tangent to four of the circles in the chain.

1. Introduction

Consider a closed chain of three pairs of congruent circles of radii a, b, c, as shown in Figure 1. Each of the circles is tangent internally to the enclosing circle (O) of radius R and tangent externally to its two neighboring circles.

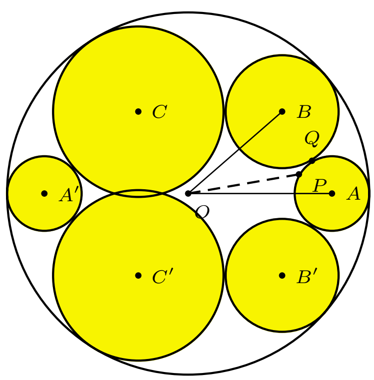

Figure 1A: $(abcacb)$

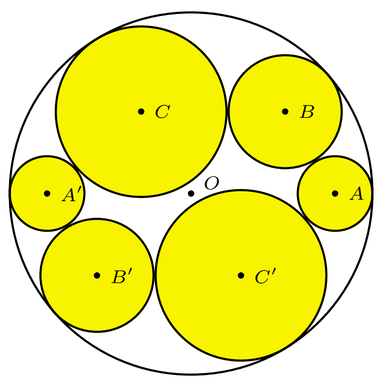

Figure 1B: $(abcabc)$

The essentially distinct arrangements, depending on the number of pairs of congruent neighboring circles, are

(A): $(aabcbc)$	(B): $(aacbbc)$
(C): $(aabbcc)$	(D): $(aaaabb)$
(E): $(abcabc)$, $(abcacb)$	(F): $(aabaab)$, $(aaabab)$
(G): $(aaaaaa)$	

Figures 1A and 1B illustrate the pattern (E). Patterns (D) and (F) have $c = a$. In pattern (G), $b = c = a$.

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According to [1, 3], in 1877 Sakuma proved $R = a + b + c$ for patterns (E). Hiroshi Okumura [1] published a much simpler proof. Unaware of this, Tien [4] rediscovered the theorem in 1995 and published a similar, simple proof. It is easy to see by symmetry that in each of the patterns (E), (F), (G), there is a pair of congruent circles with centers on a diameter of the enclosing circle. Let us call such a pair a *diametral* pair. Here is a stronger theorem:

Theorem 1. *In a closed chain of three pairs of congruent circles of radii a, b, c tangent internally to a circle of radius R, $R = a + b + c$ if and only if the closed chain contains a diametral pair of circles.*

In Figure 1, two non-neighboring circles intersect. The proof for $R = a + b + c$ does not forbid such an intersection. Sections 4 and 5 are about avoiding intersecting circles and about adding a "central" circle.

2. Preliminaries

In Figure 1, the enclosing circle (O) of radius R centers at O and the circles (A), (B), (C) of radii a, b, c, center at A, B, C, respectively. The circles (A'), (B'), (C') are also of radii a, b, c respectively.

Suppose two circles (A) and (B) of radii a and b are tangent externally each other, and each tangent internally to a circle $O(R)$. We denote the magnitude of angle AOB by θ_{ab}. See Figure 2A. This clearly depends on R. If $a < \frac{R}{2}$, then we can also speak of θ_{aa}. Note that the center O is outside each circle of radius a.

Lemma 2. (a) *If* $a < \frac{R}{2}$, $\sin\frac{\theta_{aa}}{2} = \frac{a}{R-a}$. (See Figure 2A).
(b) $\cos\theta_{bc} = \frac{(R-b)^2+(R-c)^2-(b+c)^2}{2(R-b)(R-c)}$. (See Figure 2B).

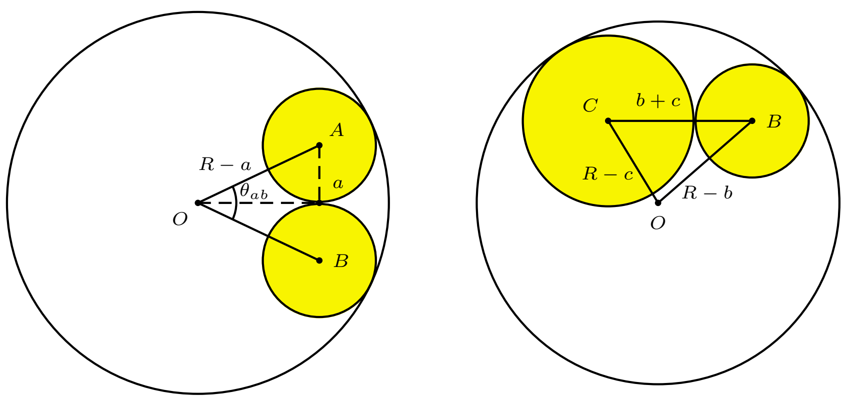

Figure 2A Figure 2B

Proof. These are clear from Figures 2A and 2B. □

Lemma 3. *If a and b are unequal and each $< \frac{R}{2}$, then $\theta_{aa} + \theta_{bb} > 2\theta_{ab}$.*

Proof. In Figure 1A, consider angle AOP, where P is a point on the circle (A). The angle AOP is maximum when line OP is tangent to the circle (A). This maximum is $\frac{\theta_{aa}}{2} \geq \angle AOQ$, where Q is the point of tangency of (A) and (B). Similarly, $\frac{\theta_{bb}}{2} \geq \angle BOQ$, and the result follows. $\square$

Corollary 4. *If a, b, c are not the same, then* $\theta_{aa} + \theta_{bb} + \theta_{cc} > \theta_{ab} + \theta_{bc} + \theta_{ca}$.

Proof. Write

$$\theta_{aa} + \theta_{bb} + \theta_{cc} = \frac{\theta_{aa} + \theta_{bb}}{2} + \frac{\theta_{bb} + \theta_{cc}}{2} + \frac{\theta_{cc} + \theta_{aa}}{2}$$

and apply Lemma 3. $\square$

3. Proof of Theorem 1

Sakuma, Okumura [1] and Tien [4] have proved the sufficiency part of the theorem. We need only the necessity part. This means showing that for distinct a, b, c in patterns (A) through (D) which do not have a diametral pair of circles, the assumption of $R = a + b + c$ causes contradictions. In patterns (E) with a pair of diametral circles and $R = a + b + c$, the sum of the angles around the center O of the enclosing circle is $2(\theta_{ab} + \theta_{bc} + \theta_{ca}) = 2\pi$, that is,

$$\theta_{ab} + \theta_{bc} + \theta_{ca} = \pi.$$

Pattern (A): $(aabcbc)$. The sum of the angles around O is

$$\begin{aligned}\theta_{aa} + \theta_{ab} + \theta_{bc} + \theta_{cb} + \theta_{bc} + \theta_{ca} =& \theta_{ab} + \theta_{bc} + \theta_{ca} + (\theta_{aa} + 2\theta_{bc}) \\ =& \pi + (\theta_{aa} + 2\theta_{bc}).\end{aligned}$$

This is 2π if and only if $(\theta_{aa} + 2\theta_{bc}) = \pi$, or $\frac{\pi}{2} - \frac{\theta_{aa}}{2} = \theta_{bc}$. The cosines of these angles, Lemma 2 and the assumption $R = a + b + c$ lead to

$$\frac{a}{b+c} = \frac{a^2 + ab + ac - bc}{(a+b)(a+c)},$$

which gives

$$(a-b)(a-c)(a+b+c) = 0,$$

an impossibility, if a, b, c are distinct.

Pattern (B): $(aacbbc)$. If $a > \frac{R}{2}$ or $b > \frac{R}{2}$, then the neighboring tangent circles of radii a or b, respectively, cannot fit inside the enclosing circle of radius $R = a + b + c$. For this equation to hold, it must be that $a \leq \frac{R}{2}$ and $b \leq \frac{R}{2}$. Then, O is outside $A(a)$ and $B(b)$. The sum of the angles around O exceeds 2π, by Lemma 3:

$$\begin{aligned}& \theta_{aa} + \theta_{ac} + \theta_{cb} + \theta_{bb} + \theta_{bc} + \theta_{ca} \\ =& (\theta_{aa} + \theta_{bb}) + 2(\theta_{bc} + \theta_{ca}) \\ >& 2(\theta_{ab} + \theta_{bc} + \theta_{ca}) \\ =& 2\pi.\end{aligned}$$

Patterns (C) and (D): $(aabbcc)$ *and* $(aaaabb)$. For $R = a + b + c$ to hold, O must be outside $A(a)$, $B(b)$, $C(c)$. Again, the sum of the angles around O exceeds 2π. For pattern (C),

$$\begin{aligned}
&\theta_{aa} + \theta_{ab} + \theta_{bb} + \theta_{bc} + \theta_{cc} + \theta_{ca} \\
=&(\theta_{aa} + \theta_{bb} + \theta_{cc}) + (\theta_{ab} + \theta_{bc} + \theta_{ca}) \\
>&(\theta_{ab} + \theta_{bc} + \theta_{ca}) + (\theta_{ab} + \theta_{bc} + \theta_{ca}) \\
=&2\pi.
\end{aligned}$$

Here, the inequality follows from Corollary 4 for a, b, c, not all the same.

For pattern (D) with $c = a$, the inequality remains true. This completes the proof of Theorem 1.

Remark. A narrower version of Theorem 1 treats a, b, c as variables, instead of any particular lengths. The proof for this version is simple. We see that when no pair of the enclosed circles is diametral, at least one pair has its two circles next to each other. Let these two be point circles and let the other four circles be of the same radius. Then the six circles become three equal tangent circles tangentially enclosed in a circle. In this special case $R = a + b + c = 0 + a + a$ is false. Then, a, b, c cannot be variables.

4. Nonoverlapping arrangements

Patterns (A) through (G) are adaptable to hands-on activities of trying to fit chains of three pairs of congruent circles into an enclosing circle of a fixed radius R. Most of the essential patterns have inessential variations. Assuming $a \le b \le c$, patterns (E) have four variations:

$$\begin{array}{ll}
\mathrm{E}_1: & (abcabc) \\
\mathrm{E}_2: & (cabcba) \\
\mathrm{E}_3: & (abcacb) \\
\mathrm{E}_4: & (bcabac)
\end{array}$$

For hands-on activities, it is desirable to find the conditions for the enclosed circles in patterns (E) not to overlap. We find the bounds of the ratio $\frac{a}{R}$ in these patterns.

4.1. *Patterns* E_1 *and* E_2. The largest circles (C) and $C')$ are diametral. For a nonoverlapping arrangement, Clearly, $a \le \frac{1}{3}R$ and $c \le \frac{1}{2}R$.

In Figure 3, a circle of radius b' is tangent externally to the two diametral circles of radii c, and internally to the enclosing circle of radius R. From

$$(b' + c)^2 = (R - b')^2 + (R - c)^2,$$

we have $b' = \frac{R(R-c)}{R+c}$. It follows that in a nonoverlapping patterns E_1 and E_2, with $\frac{1}{3}R \le c \le \frac{1}{2}R$, we have

$$b + c \le b' + c = \frac{R^2 + c^2}{R + c} \le \frac{5}{6}R.$$

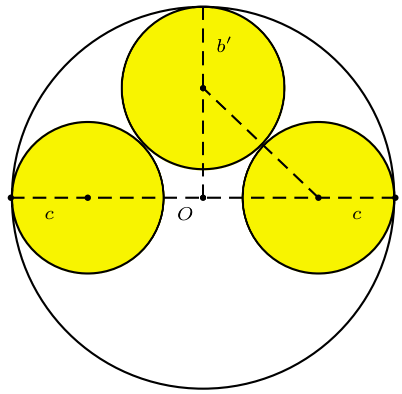

Figure 3

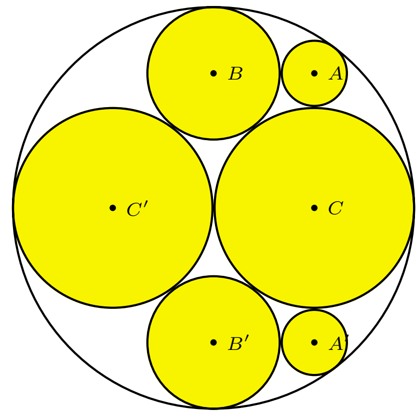

Figure 4

From this, $a \geq \frac{1}{6}R$. Figure 4 shows a nonoverlapping arrangement with $a = \frac{1}{6}R$, $b = \frac{1}{3}R$, $c = \frac{1}{2}R$. It is clear that for every a satisfying $\frac{1}{6}R \leq a \leq \frac{1}{3}R$, there are nonoverlapping patterns E_1 and E_2 (with $a \leq b \leq c$).

4.2. *Patterns* E_3 *and* E_4. In these cases the largest circles (C) and (C') are not diametral.

Lemma 5. *If three circles of radii x, z, z are tangent externally to each other, and are each tangent internally to a circle of radius R, then*

$$z = \frac{4Rx(R-x)}{(R+x)^2}.$$

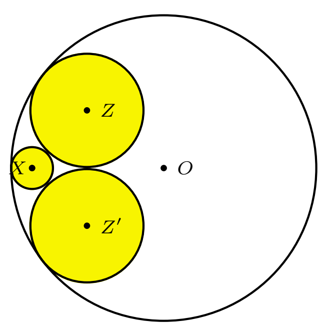

Figure 5

Proof. By the Descartes circle theorem [2], we have

$$2\left(\frac{1}{R^2} + \frac{1}{x^2} + \frac{2}{z^2}\right) = \left(-\frac{1}{R} + \frac{1}{x} + \frac{2}{z}\right)^2,$$

from which the result follows. □

Theorem 6. *For a given R, a nonoverlapping arrangement of pattern* $E_3(abcacb)$ *or* $E_4(bcabac)$ *with $a \le b \le c$ and $a+b+c=R$ exists if $\gamma R \le a \le \frac{1}{3}R$, where*

$$\gamma = \frac{1+\sqrt[3]{19+12\sqrt{87}}+\sqrt[3]{19-12\sqrt{87}}}{6} \approx 0.25805587\cdots .$$

Proof. For $b=a$ and the largest $c=R-2a$ for a nonoverlapping arrangement $E_3(abcacb)$, Lemma 5 gives

$$\frac{4Ra(R-a)}{(R+a)^2}-(R-2a)=\frac{f(\frac{a}{R})R^3}{(R+a)^2}=0,$$

where $f(x)=2x^3-x^2+4x-1$. It has a unique real root γ given above.

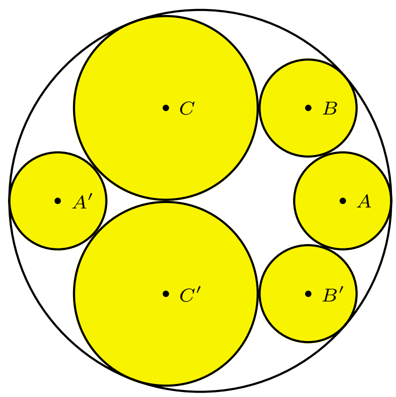

Figure 6

Figure 6 shows a nonoverlapping arrangement E_3 with $a=b=\gamma R$, and $c=(1-2\gamma)R$. For $\gamma R \le a \le \frac{1}{3}R$, from the figure we see that (C) and (C') and the other circles cannot overlap in arrangements of patterns $E_3(abcacb)$ and $E_4(bcabac)$. □

Corollary 7. *The sufficient condition $\gamma R \le a \le \frac{1}{3}R$ also applies to patterns* E_1 *and* E_2.

Outside the range $\gamma R \le a \le \frac{R}{3}$, patterns $E_3(abcacb)$ and $E_4(bcabac)$ still can have nonoverlapping circles. Both of the patterns involve Figure 5 and $z=\frac{4Rx(R-x)}{(R+x)^2}$, with $z=c$, $x=a$ or b, and $a \le b \le c$.

The equation gives the smallest $x=a=(3-2\sqrt{2})R \approx 0.1715\cdots R$ corresponding to the largest $b=c=(\sqrt{2}-1)R \approx 0.4142\cdots R$ and the largest $x=b=\frac{R}{3}$ corresponding to the largest $c=\frac{R}{2}$. Thus, the nonoverlapping conditions are $(3-2\sqrt{2})R \le x \le \frac{R}{3}$ and $c \le \frac{4Rx(R-x)}{R+x)^2}$.

For $x \ge \frac{R}{3}$, circles (Z) and (Z') overlap with (X'), which is diametral with (X). Now Figure 3 and the associated $b'=\frac{R(R-c)}{R+c}$ are relevant. With b' replaced by c and c by b, the equation becomes $c=\frac{R(R-b)}{R+b}$. By this equation, when b varies from $\frac{R}{3}$ to $(\sqrt{2}-1)R$, $c \geq b$ varies from $\frac{R}{2}$ to $(\sqrt{2}-1)R$. Thus, the nonoverlapping conditions are $\frac{R}{3} \leq b \leq (\sqrt{2}-1)R$ and $c \leq \frac{R(R-b)}{R+b}$. The case of $b > (\sqrt{2}-1)R$ makes $b > c$ and the largest pair of circles diametral, already covered in §4.1.

5. The central circle and avoiding intersecting circles

Obviously, pattern (G) $(aaaaaa)$ admits a "central" circle tangent to all 6 circles of radii a. In patterns (F) $(aabaab)$, (aaabab), we can add a central circle tangent to the four circles of radius a. Figure 7 shows the less obvious central circle for $(abcacb)$ of pattern (E).

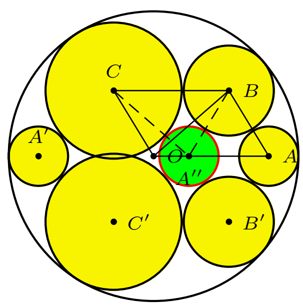

Figure 7

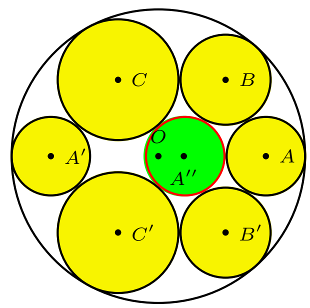

Figure 8

Theorem 8. *Consider a closed chain of pattern $(abcacb)$. There is a "central" circle of radius a tangent to the four circles of radii b and c. This circle does not overlap with the circle $A(a)$ if*

$$a \leq \frac{b(b+c)}{2c},$$

where $b \leq c$.

Proof. In Figure 7, the pattern of the chain tells that $R = a + b + c$. The central circle centered at A'' has radius a is tangent to $B(b)$, $B'(b)$, $C(c)$, $C'(c)$ because triangles $A''BC$ and OBC are mirror images of each other. When $b < c$, $A''(a)$ is closer to $A(a)$ than $A'(a)$. If $A''(a)$ and $A(a)$ are tangent to each other, then $AB^2 - a^2 = OB^2 - (OA-a)^2$. Now, $AB = a+b$ and $OB = a+c$, $OA = b+c$. This simplifies into $a = \frac{b(b+c)}{2c}$. If $a < \frac{b(b+c)}{2c}$, the circles $A(a)$ and $A''(a)$ are separate. □

Figure 8 shows an arrangement $(abcacb)$ with a central circle touching 5 inner circles except (A').

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The Intouch Triangle and the OI-line

Eric Danneels

Abstract. We prove some interesting results relating the intouch triangle and the OI line of a triangle. We also give some interesting properties of the triangle center X_{57}, the homothetic center of the intouch and excentral triangles.

1. Introduction

L. Emelyanov [4] has recently given an interesting relation between the OI-line and the triangle of reflections of the intouch triangle. Here, O and I are respectively the circumcenter and incenter of the triangle. Given triangle ABC with intouch triangle XYZ, let X_2, Y_2, Z_2 be the reflections of X, Y, Z in their respective opposite sides YZ, ZX, XY. Then the lines AX_2, BY_2, CZ_2 intersect BC, CA, AB at the intercepts of the OI-line.

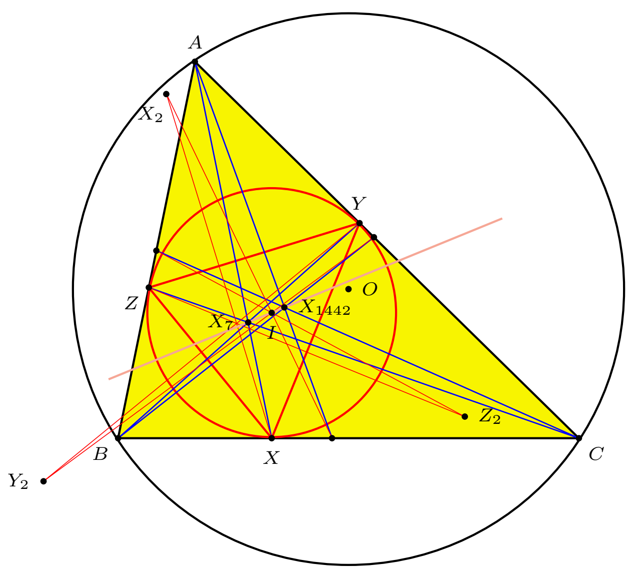

Figure 1.

Emelyanov [3] also noted that the intercepts of the points $IX_2 \cap BC$, $IY_2 \cap CA$, $IZ_2 \cap AB$ form a triangle perspective with ABC. See Figure 1. According to [7], this perspector is the point

$$X_{1442} = \left(\frac{a(b^2+bc+c^2-a^2)}{s-a} : \frac{b(c^2+ca+a^2-b^2)}{s-b} : \frac{c(a^2+ab+b^2-c^2)}{s-c} \right)$$

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on the Soddy line joining the incenter and the Gergonne point.

In this paper we generalize these results. We work with barycentric coordinates with reference to triangle ABC.

2. The triangle center X_{57}

Let a, b, c be the lengths of the sides BC, CA, AB of triangle ABC, and $s = \frac{1}{2}(a+b+c)$ the semiperimeter. The intouch triangle XYZ and the excentral triangle (with the excenters as vertices) are clearly homothetic, since their corresponding sides are perpendicular to the same angle bisector of triangle ABC. These triangles are respectively the cevian triangle of the Gergonne point $\left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right)$ and the anticevian triangle of the incenter $(a : b : c)$, their homothetic center has coordinates

$$\begin{aligned} &(a(-a(s-a)+b(s-b)+c(s-c)) : \cdots : \cdots) \\ =&(2a(s-b)(s-c) : \cdots : \cdots) \\ =&\left(\frac{a}{s-a} : \cdots : \cdots\right). \end{aligned}$$

This is the triangle center X_{57} in [6], defined as the isogonal conjugate of the Mittenpunkt $X_9 = (a(s-a) : b(s-b) : c(s-c))$. This is a point on the OI-line since the two triangles in question have circumcenters I and X_{40} (the reflection of I in O), [1]

We give some interesting properties of the triangle X_{57}.

Since ABC is the orthic triangle of the excentral triangle, it is homothetic to the orthic triangle $X_1Y_1Z_1$ of XYZ with the same homothetic center X_{57}. See Figure 2.

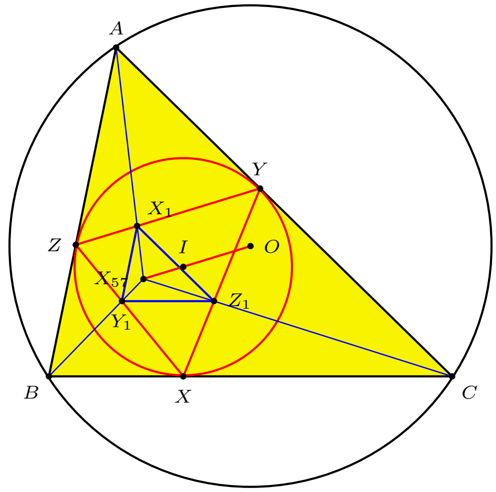

Figure 2.

[1] The circumcircle of ABC is the nine-point circle of the excentral triangle.

Let DEF be the circumcevian triangle of the incenter I, and D', E', F' the antipodes of D, E, F in the circumcircle. In other words, D and D' are the midpoints of the two arcs BC, D' on the arc containing the vertex A; similarly for the other two pairs. Clearly,

$$D = \left(\frac{a^2}{-(b+c)} : \frac{b^2}{b} : \frac{c^2}{c}\right) = (-a^2 : b(b+c) : c(b+c)).$$

Similarly,

$$E = (a(c+a) : -b^2 : c(c+a)) \quad \text{and} \quad F = (a(a+b) : b(a+b) : -c^2).$$

To compute the coordinates of D', E', F', we make use of the following formula.

Lemma 1. *Let $P = (a^2vw : b^2wu : c^2uv)$ be a point on the circumcircle (so that $u+v+w=0$). For a point $Q = (x : y : z)$ different from P and not lying on the circumcircle, the line PQ intersects the circumcircle again at the point $(a^2vw + tx : b^2wu + ty : c^2uv + tz)$, where*

$$t = \frac{b^2c^2u^2x + c^2a^2v^2y + a^2b^2w^2z}{a^2yz + b^2zx + c^2xy}. \tag{1}$$

Proof. Entering the coordinates

$$(\mathbb{X}, \mathbb{Y}, \mathbb{Z}) = (a^2vw + tx : b^2wu + ty : c^2uv + tz)$$

into the equation of the circumcircle

$$a^2\mathbb{Y}\mathbb{Z} + b^2\mathbb{Z}\mathbb{X} + c^2\mathbb{X}\mathbb{Y} = 0,$$

we obtain

$$\begin{aligned}
&(a^2yz + b^2zx + c^2xy)t^2 \\
&+(b^2c^2u(v+w)x + c^2a^2v(w+u)y + a^2b^2w(u+v)z)t \\
&+a^2b^2c^2uvw(u+v+w) = 0.
\end{aligned}$$

Since $u+v+w=0$, this gives $t=0$ or the value given in (1) above. $\square$

Let $M = (0:1:1)$ be the midpoint of BC. Applying Lemma 1 to D and M, we obtain

$$D' = (-a^2 : b(b-c) : c(c-b)).$$

Similarly,

$$E' = (a(a-c) : -b^2 : c(c-a) \quad \text{and} \quad F' = (a(a-b) : b(b-a) : -c^2).$$

Applying Lemma 1 to D' and $X = (0 : a+b-c : c+a-b)$, (likewise to E' and Y, and to F' and Z), we obtain the points

$$\begin{aligned}
X' &= \left(\frac{-a^2}{a(b+c)-(b-c)^2} : \frac{b}{c+a-b} : \frac{c}{a+b-c}\right), \\
Y' &= \left(\frac{a}{b+c-a} : \frac{-b^2}{b(c+a)-(c-a)^2} : \frac{c}{a+b-c}\right), \\
Z' &= \left(\frac{a}{b+c-a} : \frac{b}{c+a-b} : \frac{-c^2}{c(a+b)-(a-b)^2}\right).
\end{aligned}$$

These are clearly the vertices of the circumcevian triangle of X_{57}. We summarize this in the following proposition.

Proposition 2. *If X' (respectively Y', Z') are the second intersections of $D'X$ (respectively $E'Y$, $F'Z$) and the circumcircle, then $X'Y'Z'$ is the circumcevian triangle of X_{57}.*

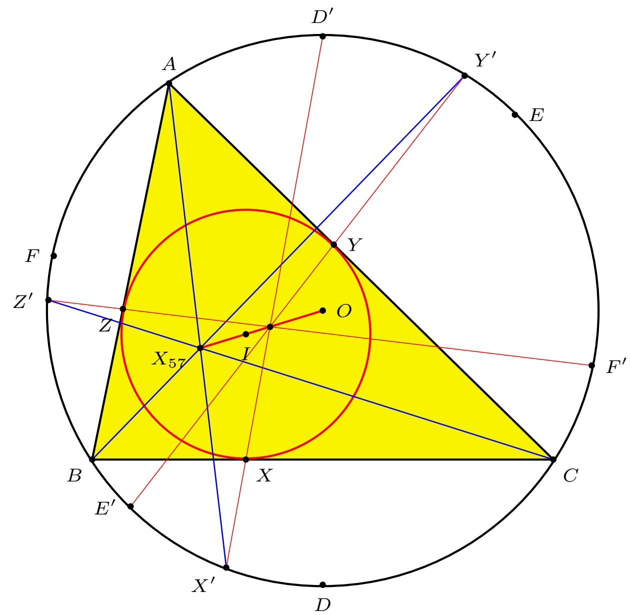

Figure 3.

Remark. The lines $D'X$, $E'Y$, $F'Z$ intersect at X_{55}, the internal center of similitude of the circumcircle and the incircle.

Proposition 3. *Let X'', Y'', Z'' be the second intersections of the circumcircle with the lines DX, EY, FZ respectively. The lines AX'', BY'', CZ'' bound the anticevian triangle of X_{57}.*

Proof. By Lemma 1, these are the points

$$X'' = \left(\frac{a^2}{s-a} : \frac{b(b-c)}{s-b} : \frac{c(c-b)}{s-c}\right),$$

$$Y'' = \left(\frac{a(a-c)}{s-a} : \frac{b^2}{s-b} : \frac{c(c-a)}{s-c}\right),$$

$$Z'' = \left(\frac{a(a-b)}{s-a} : \frac{b(b-a)}{s-b} : \frac{c^2}{s-c}\right).$$

The lines AX'', BY'', CZ'' have equations

$$\begin{array}{ccccccc} & & \frac{s-b}{b}y & + & \frac{s-c}{c}z & = & 0, \\ \frac{s-a}{a}x & & & + & \frac{s-c}{c}z & = & 0, \\ \frac{s-a}{a}x & + & \frac{s-b}{b}y & & & = & 0. \end{array}$$

They clearly bound the anticevian triangle of X_{57}. See Figure 4. $\square$

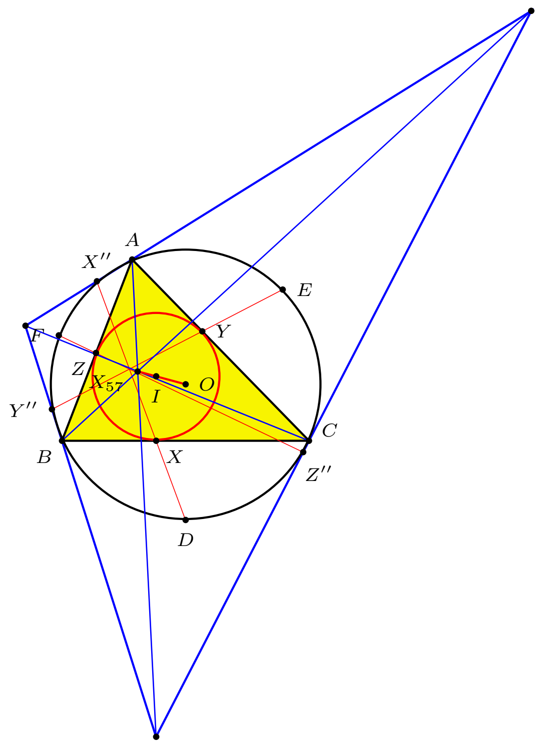

Figure 4.

Remark. The lines DX, EY, FZ intersect at X_{56}, the external center of similitude of the circumcircle and incircle.

Proposition 4. *X_{57} is the perspector of the triangle bounded by the polars of A, B, C with respect to the circle through the excenters.*

Proof. As is easily verified, the equation of the circumcircle of the excentral triangle is

$$a^2yz + b^2zx + c^2xy + (x+y+z)(bcx + cay + abz) = 0.$$

The polars are the lines

$$\begin{array}{ccccccc} \frac{x}{s} & + & \frac{y}{b} & + & \frac{z}{c} & = & 0, \\ \frac{x}{a} & + & \frac{y}{s} & + & \frac{z}{c} & = & 0, \\ \frac{x}{a} & + & \frac{y}{b} & + & \frac{z}{s} & = & 0. \end{array}$$

They bound a triangle with vertices

$$\left(-\frac{a(s^2-bc)}{s(s-b)(s-c)} : \frac{b}{s-b} : \frac{c}{s-c}\right),$$
$$\left(\frac{a}{s-a} : -\frac{b(s^2-ca)}{s(s-c)(s-a)} : \frac{c}{s-c}\right),$$
$$\left(\frac{a}{s-a} : \frac{s}{s-b} : -\frac{c(s^2-ab)}{s(s-a)(s-b)}\right).$$

This clearly has perspector X_{57}. □

Proposition 5. *X_{57} is the perspector of the reflections of the Gergonne point in the intouch triangle.*

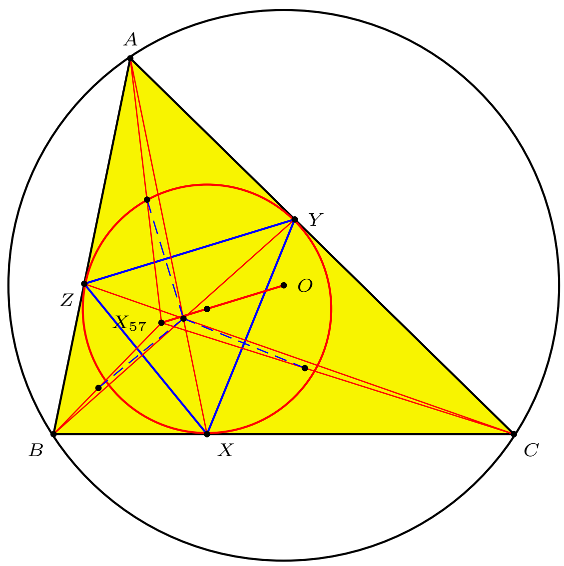

Figure 5.

More generally, the reflection triangle of $P = (u : v : w)$ in the cevian triangle of P is perspective with ABC at

$$\left(u\left(-\frac{a^2}{u^2}+\frac{b^2}{v^2}+\frac{c^2}{w^2}+\frac{b^2+c^2-a^2}{vw}\right) : \cdots : \cdots\right).$$

See [2]. For example, if P is the incenter, this perspector is the point

$$X_{35} = (a^2(b^2+bc+c^2-a^2) : b^2(c^2+ca+a^2-b^2) : c^2(a^2+ab+b^2-c^2))$$

which divides the segment OI in the ratio $OX_{35} : X_{35}I = R : 2r$.

Finally, we also mention from [5] that X_{57} is the orthocorrespondent of the incenter. This means that the trilinear polar of X_{57}, namely, the line

$$\frac{s-a}{a}x + \frac{s-b}{b}y + \frac{s-c}{c}z = 0$$

intersects the sidelines BC, CA, AB at X, Y, Z respectively such that $IX \perp IA$, $IY \perp IB$, and $IZ \perp IC$.

3. A locus of perspectors

As an extension of the result of [4], we consider, for a real number t, the triangle $X_tY_tZ_t$ with X_t, Y_t, Z_t dividing the segments XX_1, YY_1, ZZ_1 in the ratio

$$XX_t : X_tX_1 = YY_t : Y_tY_1 = ZZ_t : Z_tZ_1 = t : 1 - t.$$

Proposition 6. *The triangle $X_tY_tZ_t$ is perspective with ABC. The locus of the perspector is the Soddy line joining the incenter to the Gergonne point.*

Proof. We compute the coordinates of X_t, Y_t, Z_t. It is well known that $BX = s - b$, $XC = s - c$, *etc.*, so that, in absolute barycentric coordinates,

$$X = \frac{(s-c)B + (s-b)C}{a}, \qquad Y = \frac{(s-a)C + (s-c)A}{b}, \qquad Z = \frac{(s-b)A + (s-a)B}{c}.$$

Since the intouch triangle XYZ has (acute) angles $\frac{B+C}{2}$, $\frac{C+A}{2}$, and $\frac{A+B}{2}$ at X, Y, Z respectively, the pedal X_1 of X on YZ divides the segment in the ratio

$$YX_1 : X_1Z = \cot\frac{C+A}{2} : \cot\frac{A+B}{2} = \tan\frac{B}{2} : \tan\frac{C}{2} = s - c : s - b.$$

Similarly, Y_1 and Z_1 divide ZX and XY in the ratios

$$ZY_1 : Y_1X = s - a : s - c, \qquad XZ_1 : Z_1Y = s - b : s - a.$$

In absolute barycentric coordinates,

$$\begin{aligned} X_1 =& \frac{(s-b)Y + (s-c)Z}{a} \\ =& \frac{(b+c)(s-b)(s-c)A + b(s-c)(s-a)B + c(s-a)(s-b)C}{abc}. \end{aligned}$$

It follows that

$$\begin{aligned} X_t =& (1-t)X + tX_1 \\ =& \frac{t(b+c)(s-b)(s-c)A + b(s-c)(c - t(s-b))B + c(s-b)(b - t(s-c))C}{abc}. \end{aligned}$$

In homogeneous barycentric coordinates, this is

$$X_t = (t(b+c)(s-b)(s-c) : b(s-c)(c - t(s-b)) : c(s-b)(b - t(s-c))).$$

The line IX_t has equation

$$bc(b-c)(s-a)x + c(s-b)(ab - 2s(s-c)t)y - b(s-c)(ca - 2s(s-b)t)z = 0.$$

The line IX_t intersects BC at the point

$$\begin{aligned} X_t' =& (0 : b(s-c)(ca - 2s(s-b)t) : c(s-b)(ab - 2s(s-c)t)) \\ =& \left(0 : \frac{b(ca - 2s(s-b)t)}{s-b} : \frac{c(ab - 2s(s-c)t)}{s-c}\right). \end{aligned}$$

Similarly, the lines IY_t and IZ_t intersect CA and AB respectively at

$$Y_t' = \left(\frac{a(bc-2s(s-a)t)}{s-a} : 0 : \frac{c(ab-2s(s-c)t)}{s-c}\right),$$

$$Z_t' = \left(\frac{a(bc-2s(s-a)t)}{s-a} : \frac{b(ca-2s(s-b)t)}{s-b} : 0\right).$$

The triangle $X_t'Y_t'Z_t'$ is perspective with ABC at the point

$$\left(\frac{a(bc-2s(s-a)t)}{s-a} : \frac{b(ca-2s(s-b)t)}{s-b} : \frac{c(ab-2s(s-c)t)}{s-c}\right).$$

As t varies, this perspector traverses a straight line. Since the perspector is the Gergonne point for $t=0$ and the incenter for $t=\infty$, this line is the Soddy line joining these two points. □

The Soddy line has equation

$$(b-c)(s-a)^2x + (c-a)(s-b)^2y + (a-b)(s-c)^2z = 0.$$

Here are some triangle centers on the Soddy line, with the corresponding values of t. The symbol r_a stands for the radius of the A-excircle.

t	perspector	first barycentric coordinate
1	X_{77}	$\frac{a(b^2+c^2-a^2)}{s-a}$
2	X_{1442}	$\frac{a(b^2+bc+c^2-a^2)}{s-a}$
$\frac{1}{2}$	X_{269}	$\frac{a}{(s-a)^2}$
$\frac{R}{s}$	X_{481}	$2r_a - a$
$\frac{-R}{s}$	X_{482}	$2r_a + a$
$\frac{2R}{s}$	X_{175}	$r_a - a$
$\frac{-2R}{s}$	X_{176}	$r_a + a$
$\frac{3R}{2s}$	X_{1372}	$4r_a - 3a$
$\frac{-3R}{2s}$	X_{1371}	$4r_a + 3a$
$\frac{R}{2s}$	X_{1374}	$4r_a - a$
$\frac{-R}{2s}$	X_{1373}	$4r_a + a$

The infinite point of the Soddy point is the point

$$X_{516} = (2a^3-(b+c)(a^2+(b-c)^2) : 2b^3-(c+a)(b^2+(c-a)^2) : 2c^3-(a+b)(c^2+(a-b)^2)).$$

It corresponds to $t = \frac{R(4R+r)}{s^2}$. The deLongchamps point X_{20} also lies on the Soddy line. It corresponds to $t = \frac{2R(2R+r)}{s^2}$.

4. Emelyanov's first problem

From the coordinates of X_t, we easily find the intersections

$$A_t = AX_t \cap BC, \quad B_t = BX_t \cap CA, \quad C_t = CX_t \cap AB.$$

These are

$$\begin{aligned}A_t =&(0 : b(s-c)(c-(s-b)t) : c(s-b)(b-(s-c)t),\\ B_t =&(a(s-c)(c-(s-a)t) : 0 : c(s-a)(a-(s-c)t),\\ C_t =&(a(s-b)(b-(s-a)t) : b(s-a)(a-(s-b)t) : 0).\end{aligned} \tag{2}$$

They are collinear if and only if

$$\begin{aligned}&(a-(s-b)t)(b-(s-c)t)(c-(s-a)t)\\ +&(a-(s-c)t)(b-(s-a)t)(c-(s-b)t) = 0.\end{aligned} \tag{3}$$

Since this is a cubic equation in t, there are three values of t for which A_t, B_t, C_t are collinear. One of these is $t = 2$ according to [4]. The other two roots are given by

$$abc - abct + 2(s-a)(s-b)(s-c)t^2 = 0. \tag{4}$$

Since $abc = 4Rrs$ and $(s-a)(s-b)(s-c) = r^2s$, where R and r are respectively the circumradius and inradius, this becomes

$$2R - 2Rt + rt^2 = 0. \tag{5}$$

From this,

$$t = \frac{R \pm \sqrt{R^2 - 2Rr}}{r} = \frac{R \pm d}{r},$$

where d is the distance between O and I.

We identify the lines corresponding to these two values of t.

Proposition 7. *Corresponding to the two roots of (4), the lines containing A_t, B_t, C_t are the tangents to the incircle perpendicular to the OI-line.*

Lemma 8. *Consider a triangle ABC with intouch triangle XYZ, and a line $\mathcal{L}$ intersecting the sides BC, CA, AB at A', B', C' respectively. The line $\mathcal{L}$ is tangent to the incircle if and only if one of the following conditions holds.*

(1) *The intersection $BB' \cap CC'$ lies on the line YZ.*
(2) *The intersection $CC' \cap AA'$ lies on the line ZX.*
(3) *The intersection $AA' \cap BB'$ lies on the line XY.*

Proof. Let $A'B'$ be a tangent to the incircle. By Brianchon's theorem applied to the circumscribed hexagon $AYB'A'XB$ it immediately follows that AA', YX and $B'B$ are concurrent.

Now suppose AA', YX and $B'B$ are concurrent. Consider the tangent through A' (different from BC) to the incircle. Let B'' be the intersection of this tangent with AC. It follows from the preceeding that AA', YX and $B''B$ are concurrent. Therefore B'' must coincide with B'. This means that $A'B'$ is a tangent to the incircle. $\square$

5. Proof of Proposition 7

The lines BB_t and CC_t intersect at the point

$$A'' = \left(\frac{a}{s-a}(b-(s-a)t)(c-(s-a)t)\right.$$
$$: \frac{b}{s-b}(c-(s-a)t)(a-(s-b)t)$$
$$\left. : \frac{c}{s-c}(a-(s-c)t)(b-(s-a)t)\right).$$

This point lies on the line $YZ: \ -(s-a)x+(s-b)y+(s-c)z=0$ if and only if

$$-a(b-(s-a)t)(c-(s-a)t)$$
$$+b(c-(s-a)t)(a-(s-b)t)$$
$$+c(a-(s-c)t)(b-(s-a)t)=0.$$

This reduces to equation (4) above. By Lemma 8, these two lines are tangent to the incircle. We claim that these are the tangents perpendicular to the line OI. From the coordinates given in (2), the equation of the line B_tC_t is

$$-\frac{(s-a)(a-(s-b)t)(a-(s-c)t)}{a}x$$
$$+\frac{(s-b)(a-(s-c)t)(b-(s-a)t)}{b}y$$
$$+\frac{(s-c)(a-(s-b)t)(c-(s-a)t)}{c}z=0.$$

According to [6], lines perpendicular to OI have infinite point

$$X_{513} = (a(b-c) : b(c-a) : c(a-b)).$$

The line B_tC_t contains the infinite point X_{513} if and only if the same equation (4) holds. This shows that the two lines in question are indeed the tangents to the incircle perpendicular to the OI-line.

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A Theorem on Orthology Centers

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Abstract. We prove that if two triangles are orthologic, their orthology centers have the same barycentric coordinates with respect to the two triangles. For a point P with cevian triangle $A'B'C'$, we also study the orthology centers of the triangle of circumcenters of $PB'C'$, $PC'A'$, and $PA'B'$.

1. The barycentric coordinates of orthology centers

Let $A'B'C'$ be the cevian triangle of P with respect to a given triangle ABC. Denote by O_a, O_b, O_c the circumcenters of triangles $PB'C'$, $PC'A'$, $PA'B'$ respectively. Since O_bO_c, O_cO_a, and O_aO_b are perpendicular to AP, BP, CP, the triangles $O_aO_bO_c$ and ABC are orthologic at P. It follows that the perpendiculars from O_a, O_b, O_c to the sidelines BC, CA, AB are concurrent at a point Q. See Figure 1. We noted that the barycentric coordinates of Q with respect to triangle $O_aO_bO_c$ are the same as those of P with respect to triangle ABC. Alexey A. Zaslasky [7] pointed out that our original proof [3] generalizes to an arbitrary pair of orthologic triangles.

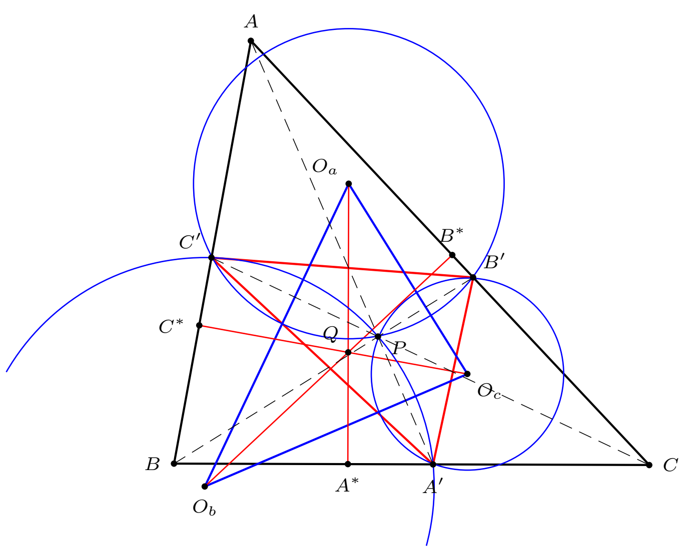

Figure 1

Theorem 1. *If triangles ABC and $A'B'C'$ are orthologic with centers P, P' then the barycentric coordinates of P with respect to ABC are equal to the barycentric coordinates of P' with respect to $A'B'C'$.*

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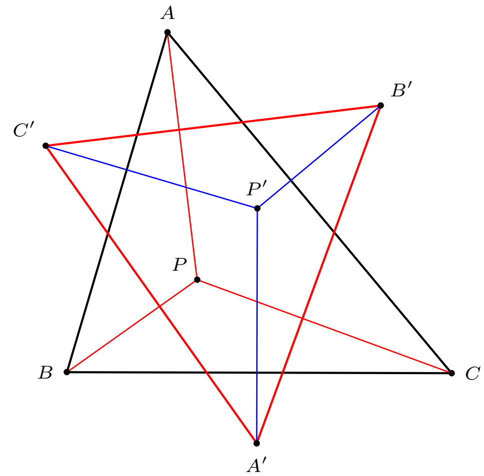

Figure 2

Proof. Since $A'P'$, $B'P'$, $C'P'$ are perpendicular to BC, CA, AB respectively, we have

$$\sin B'P'C' = \sin A, \quad \sin P'B'C' = \sin PAC, \quad \sin P'C'B' = \sin PAB.$$

Applying the law of sines to various triangles, we have

$$\begin{aligned}\frac{b}{P'B'} : \frac{c}{P'C'} =& \frac{1}{c \sin P'C'B'} : \frac{1}{b \sin P'B'C'} \\ =& \frac{1}{c \sin PAB} : \frac{1}{b \sin PAC} \\ =& \frac{1}{AP \cdot c \sin PAB} : \frac{1}{AP \cdot b \sin PAC} \\ =& \frac{1}{\text{area}(PAB)} : \frac{1}{\text{area}(PAC)} \\ =& \text{area}(PCA) : \text{area}(PAB).\end{aligned}$$

Similarly, $\frac{a}{P'A'} : \frac{b}{P'B'} = \text{area}(PBC) : \text{area}(PCA)$. It follows that the barycentric coordinates of P' with respect to triangle $A'B'C'$ are

$$\begin{aligned}&\text{area}(P'B'C') : \text{area}(P'C'A') : \text{area}(P'A'B') \\ =&(P'B')(P'C') \sin A : (P'C')(P'A') \sin B : (P'A')(P'B') \sin C \\ =& \frac{a}{P'A'} : \frac{b}{P'B'} : \frac{c}{P'C'} \\ =& \text{area}(PBC) : \text{area}(PCA) : \text{area}(PAB),\end{aligned}$$

the same as the barycentric coordinates of P with respect to triangle ABC. □

This property means that if P is the centroid of ABC then P' is also the centroid of $A'B'C'$.

2. The orthology center of $O_aO_bO_c$

We compute explicitly the coordinates (with respect to triangle ABC) of the orthology center Q of the triangle of circumcenters $O_aO_bO_c$. See Figure 3. Let $P = (x : y : z)$ and $Q = (u : v : w)$ in homogeneous barycentric coordinates. then $BC' = \frac{cx}{x+y}$, $CB' = \frac{bx}{x+z}$. In the notations of John H. Conway, the pedal A^* of O_a on BC has homogeneous barycentric coordinates $(0 : uS_C + a^2v : uS_B + a^2w)$. See, for example, [6, pp.32, 49].

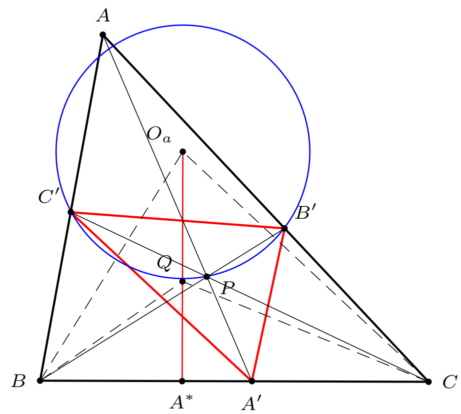

Figure 3

Note that $BA^* = \frac{uS_B+a^2w}{(u+v+w)a}$ and $A^*C = \frac{uS_C+a^2v}{(u+v+w)a}$. Also, by Stewart's theorem,

$$BB'^2 = \frac{c^2x^2 + a^2z^2 + (c^2 + a^2 - b^2)xz}{(x+z)^2},$$
$$CC'^2 = \frac{b^2x^2 + a^2y^2 + (a^2 + b^2 - c^2)xy}{(x+y)^2}.$$

Hence, if ρ is the circumradius of $PB'C'$, then

$$\begin{aligned}
&a(BA^* - A^*C)\\
=&(BA^* + A^*C)(BA^* - A^*C)\\
=&(BA^*)^2 - (A^*C)^2\\
=&(O_aB)^2 - (O_aA^*)^2 - (O_aC)^2 + (O_aA^*)^2\\
=&(O_aB)^2 - \rho^2 - (O_aC)^2 + \rho^2\\
=&BP \cdot BB' - CP \cdot CC'\\
=&\frac{c^2x^2 + a^2z^2 + (c^2 + a^2 - b^2)xz}{(x+z)(x+y+z)} - \frac{b^2x^2 + a^2y^2 + (a^2 + b^2 - c^2)xy}{(x+y)(x+y+z)}\\
=&-\frac{a^2(y-z)(x+y)(x+z) + b^2x(x+y)(x+2z) - c^2x(x+z)(x+2y)}{(x+y)(x+z)(x+y+z)}
\end{aligned}$$

since the powers of B and C with respect to the circle of $PB'C'$ are $BB' \cdot BP = (O_aB)^2 - \rho^2$ and $CC' \cdot CP = (O_aC)^2 - \rho^2$ respectively. In other words,

$$\begin{aligned}&\frac{(c^2-b^2)u - a^2(v-w)}{u+v+w}\\ &= -\frac{a^2(y-z)(x+y)(x+z) + b^2x(x+y)(x+2z) - c^2x(x+z)(x+2y)}{(x+y)(x+z)(x+y+z)},\end{aligned}$$

or

$$\begin{aligned}&(a^2(y-z)(x+y)(x+z) - b^2(x+y)(xy+yz+z^2) + c^2(x+z)(y^2+xz+yz))u\\ &-(a^2(x+y)(x+z)(x+2z) - b^2x(x+y)(x+2z) + c^2x(x+z)(x+2y))v\\ &+(a^2(x+y)(x+z)(x+2y) + b^2x(x+y)(x+2z) - c^2x(x+z)(x+2y))w = 0.\end{aligned}$$

By replacing x, y, z by y, z, x and u, v, w by v,w, u, we obtain another linear relation in u, v, w. From these we have $u : v : w$ given by

$$\begin{aligned}u =&(x^2-z^2)y^2S_{BB} + (x^2-y^2)z^2S_{CC} - x(2x+y)(x+z)(y+z)S_{AB}\\ &- x(2x+z)(x+y)(y+z)S_{CA} - 2(x+y)(x+z)(xy+yz+zx)S_{BC}.\end{aligned}$$

and v obtained from u by replacing x, y, z, S_A, S_B, S_C by v, w, u, S_B, S_C, S_A respectively, and w from v by the same replacements.

3. Examples

3.1. *The centroid.* For $P = G$,

$$\begin{aligned}O_a =&(5S_A(S_B+S_C) + 2(S_{BB} + 5S_{BC} + S_{CC})\\ &: 3S_{AB} + 4S_{AC} + S_{BC} - 2S_{CC}\\ &: 3S_{AC} + 4S_{AB} + S_{BC} - 2S_{BB}).\end{aligned}$$

Similarly, we write down the coordinates of O_b and O_c. The perpendiculars from O_a to BC, from O_b to CA, and from O_c to AB have equations

$$\begin{array}{rcrcrcl}(S_B-S_C)x & - & (3S_B+S_C)y & + & (S_B+3S_C)z & = & 0,\\ (S_C+3S_A)x & + & (S_C-S_A)y & - & (3S_C+S_A)z & = & 0,\\ -(3S_A+S_B)x & + & (S_A+3S_B)y & + & (S_A-S_B)z & = & 0.\end{array}$$

These three lines intersect at the nine-point center

$$X_5 = (S_{CA} + S_{AB} + 2S_{BC} : S_{AB} + S_{BC} + 2S_{CA} : S_{BC} + S_{CA} + 2S_{AB}),$$

which is the orthology center of $O_aO_bO_c$.

3.2. *The orthocenter.* If P is the orthocenter, the circumcenters O_a, O_b, O_c are simply the midpoints of the segments AP, BP, CP respectively. In this case, $Q = H$.

3.3. *The Steiner point.* If P is the Steiner point $\left(\frac{1}{S_B-S_C}:\frac{1}{S_C-S_A}:\frac{1}{S_A-S_B}\right)$, the perpendiculars from the circumcenters to the sidelines are

$$\begin{array}{ccccccc} (S_B-S_C)x & - & S_Cy & + & S_Bz & = & 0,\\ S_Cx & + & (S_C-S_A)y & - & S_Az & = & 0,\\ -S_Bx & + & S_Ay & + & (S_A-S_B)z & = & 0.\end{array}$$

These lines intersect at the deLongchamps point

$$X_{20}=(S_{CA}+S_{AB}-S_{BC}:S_{AB}+S_{BC}-S_{CA}:S_{BC}+S_{CA}-S_{AB}).$$

3.4. X_{671}. The point $P=X_{671}=\left(\frac{1}{S_B+S_C-2S_A}:\frac{1}{S_C+S_A-2S_B}:\frac{1}{S_A+S_B-2S_C}\right)$ is the antipode of the Steiner point on the Steiner circum-ellipse. It is also on the Kiepert hyperbola, with Kiepert parameter $-\operatorname{arccot}(\frac{1}{3}\cot\omega)$, where ω is the Brocard angle. In this case, the circumcenters are on the altitudes. This means that $Q=H$.

3.5. *An antipodal pair on the circumcircle.* The point X_{925} is the second intersection of the circumcircle with the line joining the deLongchamps point X_{20} to X_{74}, the isogonal conjugate of the Euler infinity point. It has coordinates

$$\left(\frac{1}{(S_B-S_C)(S^2-S_{AA})}:\frac{1}{(S_C-S_A)(S^2-S_{BB})}:\frac{1}{(S_A-S_B)(S^2-S_{CC})}\right).$$

For $P=X_{925}$, the orthology Q of $O_aO_bO_c$ is the point X_{68}, [1] which lies on the same line joining X_{20} to X_{74}.

The antipode of X_{925} is the point

$$X_{1300}=\left(\frac{1}{S_A((S_{AA}-S_{BC})(S_B+S_C)-S_A(S_B-S_C)^2)}:\cdots:\cdots\right).$$

It is the second intersection of the circumcircle with the line joining the orthocenter to the Euler reflection point [2] $X_{110}=\left(\frac{S_B+S_C}{S_B-S_C}:\frac{S_C+S_A}{S_C-S_A}:\frac{S_A+S_B}{S_A-S_B}\right)$. For $P=X_{1300}$, the orthology center Q of $O_aO_bO_c$ has first barycentric coordinate

$$\frac{S_{AA}(S_{BB}+S_{CC})(S_A(S_B+S_C)-(S_{BB}+S_{CC}))+S_{BC}(S_B-S_C)^2(S_{AA}-2S_A(S_B+S_C)-S_{BC}))}{S_A((S_B+S_C)(S_{AA}-S_{BC})-S_A(S_B-S_C)^2)}.$$

In this case, $O_aO_bO_c$ is also perspective to ABC at

$$X_{254}=\left(\frac{1}{S_A((S_{AA}-S_{BC})(S_B+S_C)-S_A(S_{BB}+S_{CC}))}:\cdots:\cdots\right).$$

By a theorem of Mitrea and Mitrea [5], this perspector lies on the line PQ.

[1] X_{68} is the perspector of the reflections of the orthic triangle in the nine-point center.

[2] The Euler reflection point is the intersection of the reflections of the Euler lines in the sidelines of triangle ABC.

3.6. More generally, for a generic point P on the circumcircle with coordinates $\left(\frac{S_B+S_C}{(S_A+t)(S_B-S_C)} : \cdots : \cdots\right)$, the center of orthology of $O_aO_bO_c$ is the point

$$\left(\frac{(S_B+S_C)(F(S_A,S_B,S_C)+G(S_A,S_B,S_C)t)}{S_A+t} : \cdots : \cdots\right),$$

where

$$\begin{aligned} F(S_A,S_B,S_C) =& S_{AA}(S_{BB}+S_{CC})(S_A+S_B+S_C)+S_{AABC}(S_B+S_C) \\ &- S_{BB}S_{CC}(2S_A+S_B+S_C), \\ G(S_A,S_B,S_C) =& 2(S_{AA}(S_{BB}+S_{BC}+S_{CC})-S_{BB}S_{CC}). \end{aligned}$$

Proposition 2. *If P lies on the circumcircle, the line joining P to Q always passes through the deLongchamps point X_{20}.*

Proof. The equation of the line PQ is

$$\begin{aligned} \sum_{\text{cyclic}} &(S_B-S_C)(S_A+t)(S_A^3(S_B-S_C)^2 \\ &+ (S_B+S_C+2t)(S_{AA}(S_{BB}-S_{BC}+S_{CC})-S_{BB}S_{CC})x = 0. \end{aligned}$$

□

3.7. *Some further examples.* We conclude with a few more examples of P with relative simple coordinates for Q, the orthology center of $O_aO_bO_c$.

P	first barycentric coordinate of Q
X_7	$4a^3+a^2(b+c)-2a(b-c)^2-3(b+c)(b-c)^2$
X_8	$4a^4-5a^3(b+c)-a^2(b^2-10bc+c^2)+5a(b-c)^2(b+c)-3(b^2-c^2)^2$
X_{69}	$3a^6-4a^4(b^2+c^2)+a^2(3b^4+2b^2c^2+3c^4)-2(b^2-c^2)^2(b^2+c^2)$
X_{80}	$\frac{4a^3-3a^2(b+c)-2a(2b^2-5bc+2c^2)+3(b-c)^2(b+c)}{(b^2+c^2-a^2-bc)}$

In each of the cases $P = X_7$ and X_{80}, the triangle $O_aO_bO_c$ is also perspective to ABC at the incenter.

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A Grand Tour of Pedals of Conics

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Abstract. We describe the pedal curves of conics and some of their relations to origami folding axioms. There are nine basic types of pedals depending on the location of the pedal point with respect to the conic. We illustrate the different pedals in our tour.

1. Introduction

The main 'axiom' of mathematical origami allows one to create a fold-line by sliding or folding point F onto a line L so that another point S is also folded onto yet another line M. One can regard this complicated axiom as making possible the folding of the common tangents to the parabola κ with focus F and directrix L and the parabola with focus S and directrix M. Since two parabolas have at most four common tangents in the projective plane and one of them is the line at infinity there are at most three folds in the Euclidean plane which will accomplish this origami operation. In the field theory associated to origami this operation yields construction methods for solving cubic equations, [1]. Hull has shown how to do the 'impossible' trisection of an angle using this folding, by a method due to Abe, [2]. In fact the trisection of Abe is quite similar to a classical method using Maclaurin's trisectrix, [3]. The trisectrix is one of the pedals along the tour.

One can simplify this origami folding operation into smaller steps: first fold S to the point P by reflection across the tangent of the parabola κ. The locus of points P for all the tangents of κ is a curve; finally, this locus is intersected with the line M. This 'origami locus' of points P is a cubic curve since intersecting with M will generally give three possible solutions. Since reflection of S across a line is just the double of the perpendicular projection S' of S onto L, this 'origami' locus is the scale by a factor of 2 of the locus of S', also known as the pedal curve of the parabola, [3]. As a generalization we shall investigate the pedal curves of an arbitrary conic; this pedal curve is generally a quartic curve.

Pedal of a conic. The points S' of the pedal curve lie on the lines through S at the places where the tangents to the curve are perpendicular to these lines. Suppose that S is at the origin. The line through the origin perpendicular to $\alpha x + \beta y + \gamma = 0$ is the line $\beta x - \alpha y = 0$; these meet when $x = -\frac{\alpha\gamma}{\alpha^2+\beta^2}$, $y = -\frac{\beta\gamma}{\alpha^2+\beta^2}$. This suggests

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using the inversion transform (at the origin), the map given by $x \to \frac{x}{x^2+y^2},\ y \to \frac{y}{x^2+y^2}$.

A conic has the homogenous quadratic equation $F(x,y,z) = 0$ which can also be given by the matrix equation $F(x,y,z) = (x,y,z)A(x,y,z)^t = 0$ for a 3 by 3 symmetric matrix A. It is well-known that the dual curve of tangent lines to a conic is also a conic having homogeneous equation $F'(x,y,z) = 0$ obtained from the adjoint matrix A' of A. Thus the pedal curve has the (inhomogeneous) equation obtained by applying the inversion transform to $F'(x,y,-z) = 0$, evaluated at $z = 1$, [4].

The polar line of a point T is the line through the points U and V on the conic where the tangents from T meet the conic. It is important to realize the polar line of a point with respect to the conic κ having equation $F = 0$ can be expressed in terms of the matrix A. In terms of equations, if T has (projective) coordinates (u,v,w) then the dual line has the equation $(x,y,z)A(u,v,w)^t = 0$. For example, when S is placed at the origin the dual line is $(x,y,z)A(0,0,1)^t = 0$.

2. Equation of a pedal of a conic

Let S be at the origin. Suppose the (non-degenerate) conic equation is $F(x,y,z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$. Applying the inversion to the adjoint equation gives after a bit of algebra the relatively simple equation

$$G = \Delta(x^2+y^2)^2 + (x^2+y^2)((4cd-2be)x + (4ae-2bd)y) + G_2 = 0$$

where $\Delta = 4ac - b^2$ is the discriminant of the conic; $\Delta = 0$ iff the conic is a parabola. In the case of a parabola, the pedal curve has a cubic equation. The origin is a singular point having as singular tangent lines the linear factors of the degree two term $G_2 = (4cf - e^2)x^2 + (2ed - 4bf)xy + (4af - d^2)y^2 = 0$.

3. Variety of pedals

Fix a (non-empty) real conic κ in the plane and a point S. There are two points U and V on the conic with tangents τ_U and τ_V meeting at S; the corresponding pedal point for each of these tangents is S. Thus S is a double point. The type of singularity or double point at S is either a node, cusp or acnode depending on whether or not the two tangents are real and distinct, real and equal or complex conjugates.

The perpendicular lines at S to τ_U and τ_V are the singular tangents. To see this notice that the dual line to $S = (0,0)$ is $(x,y,z)A(0,0,1)^t = 0$ or equivalently $dx + ey + 2f = 0$. This line meets the conic at the points U, V which are on the tangents from S. Determining the perpendiculars through the origin S to these tangents, and multiplying the two linear factors yields after a tedious calculation precisely the second degree terms G_2 of G.

The variety of pedals depending on the type of conic and the type of singularity, are displayed in Figures 1-9, along with their associated conics, the singular point S, the singular tangents, dual line and its intersections with the conic (whenever possible).

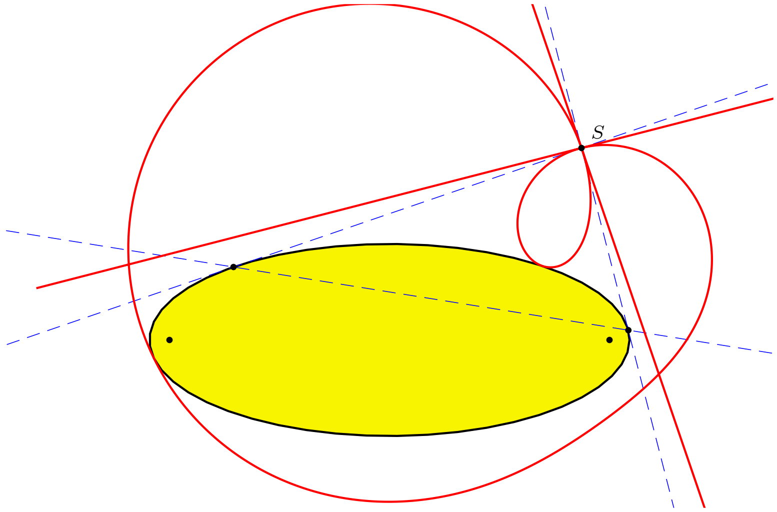

Figure 1. Elliptic node

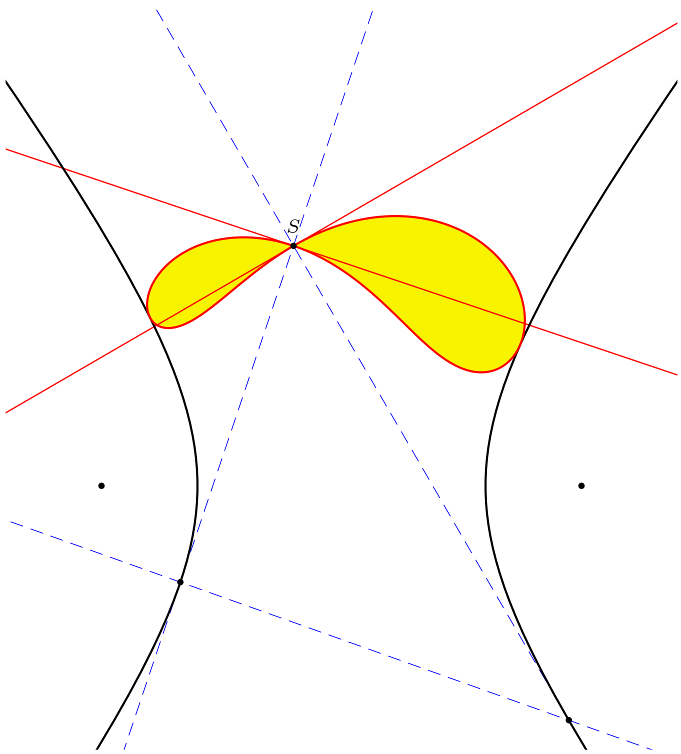

Figure 2. Hyperbolic node

Proposition 1. *The pedal of the real conic κ has a node, cusp or acnode depending on whether S is outside, on, or inside κ.*

Proof. By the calculation of the second degree terms of G, the singular tangents at the point S of the pedal are the perpendiculars to the two tangents from S to the conic κ. Thus the type of node depends on the position of S with respect to the conic since that determines how G_2 factors over the reals. □

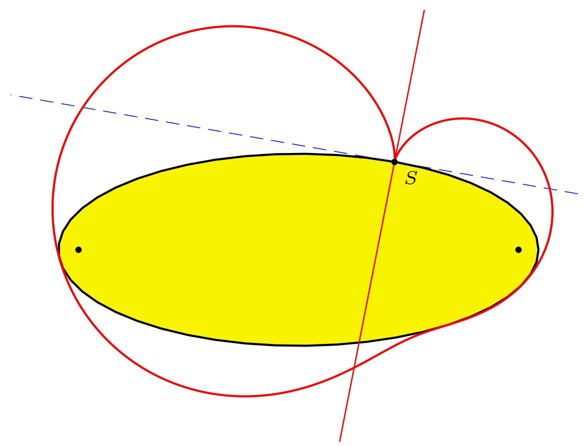

Figure 3. Elliptic cusp

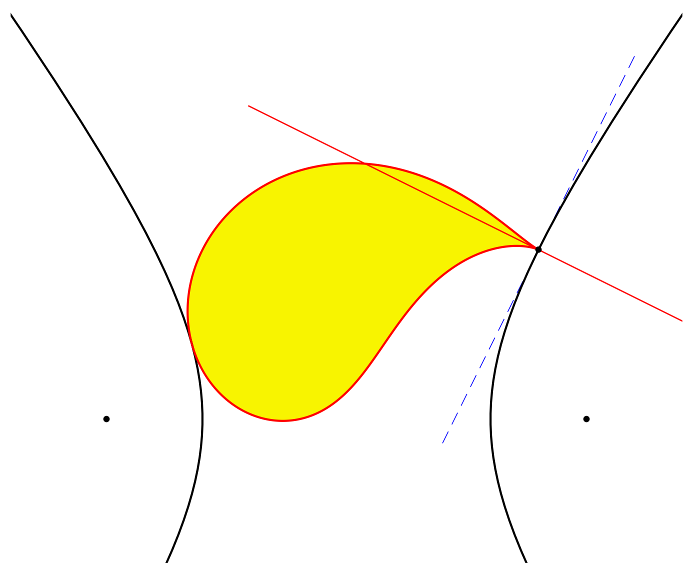

Figure 4. Hyperbolic cusp

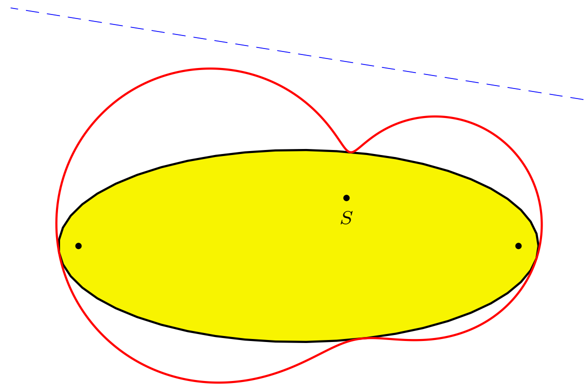

Figure 5. Elliptic acnode

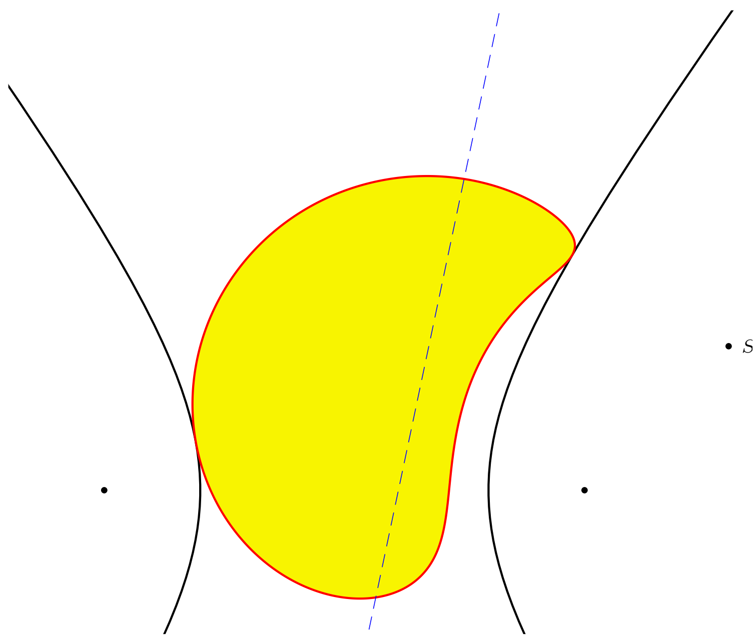

Figure 6. Hyperbolic acnode

4. Bicircular quartics

A quartic curve having circular double points is called bicircular.

Proposition 2. *A real quartic curve has the equation* $G = A(x^2 + y^2)^2 + (x^2 + y^2)(Bx + Cy) + Dx^2 + Exy + Fy^2 = 0$ *for* $A \neq 0$ *iff it is bicircular with double point at the origin. Thus the pedal of an ellipse or hyperbola is a bicircular quartic with a double point at* S*.*

Proof. A quartic has a double point at the origin iff there are no terms of degree less than 2 in the (inhomogeneous) equation $G = 0$. There are double points at the circular points iff $G(x,y,z)$ vanishes to second order when evaluated at the circular points; hence iff the gradient of G is zero at the circular points. Since $\frac{\partial G}{\partial z} = 2zG_2 + G_3$; this vanishes at the circular points iff G_3 is divisible by x^2+y^2. Also G vanishes at the circular points iff G_4 is divisible by x^2+y^2. Thus the homogeneous equation for the quartic is $G = (x^2+y^2)(ux^2+vxy+wy^2) + z(x^2+y^2)(Bx+Cy) + z^2G_2 = 0$. Finally $\frac{\partial G}{\partial x}$ or equivalently $\frac{\partial G}{\partial y}$ will also vanish at the circular points iff $ux^2+vxy+wy^2$ is divisible by x^2+y^2. Hence a bircular quartic with a double point at the origin has the equation as specified in the proposition and conversely.

The conclusion for the pedal follows immediately from the equation given in Section 2. □

We now show that any real bicircular quartic having a third double point can be realized as the pedal of a conic.

Proposition 3. *A bicircular quartic is the pedal of an ellipse or hyperbola.*

Proof. Using the equation for the pedal of a conic as in Section 2 we consider the system of equations $A = 4ac-b^2$, $B = 4cd-2be$, $C = 4ae-2bdy$, $D = 4cf-e^2$, $E = 2ed-4bf$, $F = 4af-d^2$. One can easily see that this is equivalent to a (symmetric) matrix equation $Y = X'$ where X' is the adjoint of X; we want to solve for X given Y. In our case here, Y involves the variables $A, B, \dots$ and X involves $a, b, \dots$ Certainly $det(Y) = det(X)^2$. Then we can solve using adjoints, $X = Y'$ iff the quadratic form $Q = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$ has positive determinant. However changing G to $-G$ changes the sign of this determinant so we can represent all these quartics by pedals. □

The type of singularity of a bicircular quartic with double point at S is determined from Proposition 1 and the previous Proposition. The type of singularity of the circular double points is determined by the low order terms of G when expanded at the circular points; since the circular point is complex it is nodal in general; a circular point is cuspidal when $BC = 8AE$ and $C^2 - B^2 = 16A(D-F)$ and then in fact both circular points are cusps.

5. Pedal of parabolas

In the case that the conic is a parabola ($\Delta = 0$) the pedal equation simplifies to a cubic equation. This pedal cubic is singular and circular.

Proposition 4. *A singular circular cubic with singularity at the origin has an equation $G = (x^2+y^2)(Bx+Cy) + Dx^2 + Exy + Fy^2 = 0$ and conversely. This is the pedal of a parabola.*

Proof. The cubic is singular at the origin iff there are no terms of degree less than two; the curve is circular iff the cubic terms vanish at the circular points iff x^2+y^2 is a factor of the cubic terms.

The pedal of a parabola having $\Delta = 4ac - b^2 = 0$, means the cubic equation is $G = (x^2+y^2)((4cd-2be)x + (4ae-2bd)y) + (4cf-e^2)x^2 + (2ed-4bf)xy +$

$(4af - d^2)y^2 = 0$. Solving the system of equations as in Proposition 3 we have a simpler system since $A = 0$ but similar methods give the desired result. $\square$

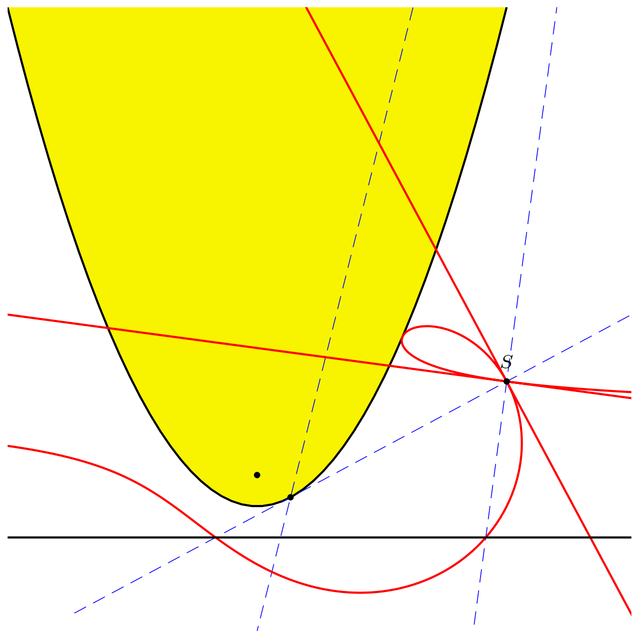

Figure 7. Parabolic node

6. Tangency of pedal and conic at their intersections

The pedal of a conic κ meets that conic at the places T iff the normal line to κ at that point passes through S. Thus the intersection occurs iff the line ST is a normal to the curve.

It follows from the fact that the conic and its pedal have a resultant which is a square (a horrendous calculation) that the pedal is tangent at all of its intersections with the conic. From Bezout's theorem, the conic and pedal have eight intersections (counted with multiplicity) and since each is a tangency there are at most four actual incidences just as expected from the figures.

Alternatively we can use elementary properties of a arbitrary curve $C(t)$ with unit speed parameterizations having tangent τ and normal η to see that when S is at the origin, the pedal $P(t)$ has a parametrization $P(t) = C(t) \cdot \eta(t)\eta(t)$ and tangent $P'(t) = -k(t)(C(t){\cdot}\tau(t)\eta(t){+}C(t){\cdot}\eta(t)\tau(t))$ where $k(t)$ is the curvature. Thus the tangent to P is parallel to τ iff $C(t) \cdot \tau(t) = 0$ iff $C(t)$ is parallel to the normal $\eta(t)$ iff the normal passes through S.

7. Linear families of pedals

Because of the importance of a parabola in the origami axioms, we illustrate in Figure 10 a family of origami curves. Recall that the origami curve is the pedal of

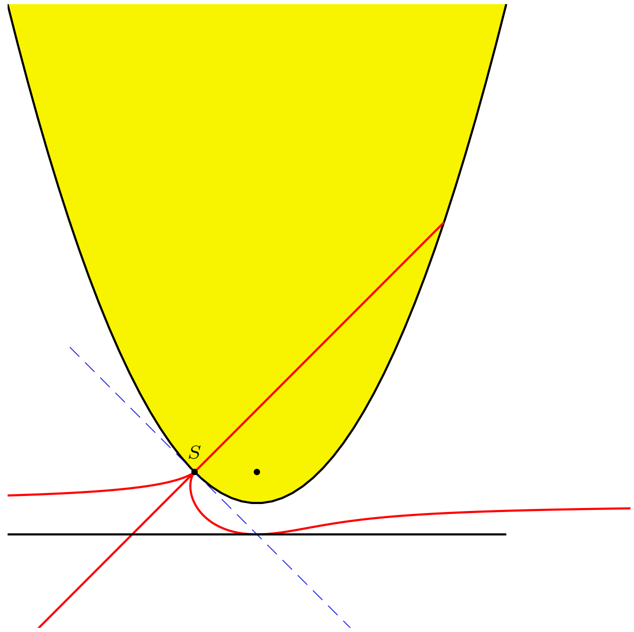

Figure 8. Parabolic cusp

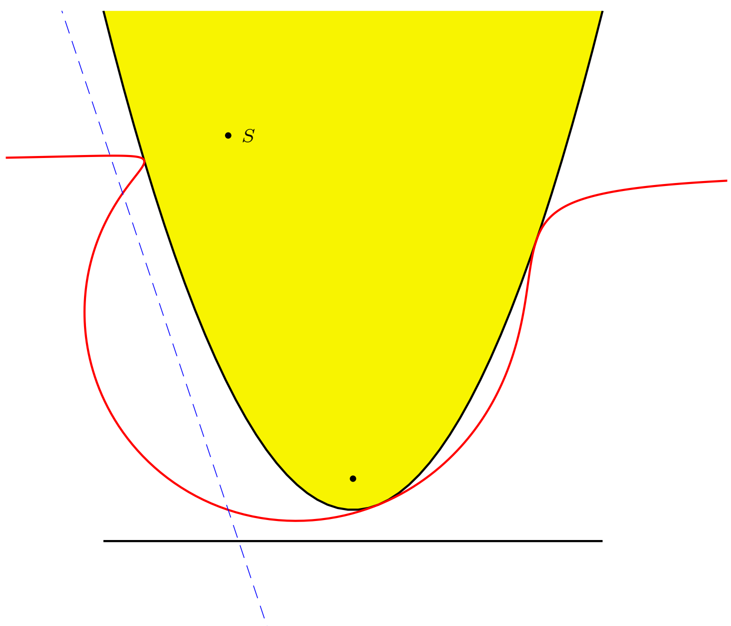

Figure 9. Parabolic acnode

a parabola scaled by 2 from the singular point S. The origami curves determined by a fixed parabola and S varying on a line parallel to the directrix are all tangent to a fixed circle of radius equal to the distance from S to the directrix. In case S varies on the directrix, then all the curves pass through the focus F.

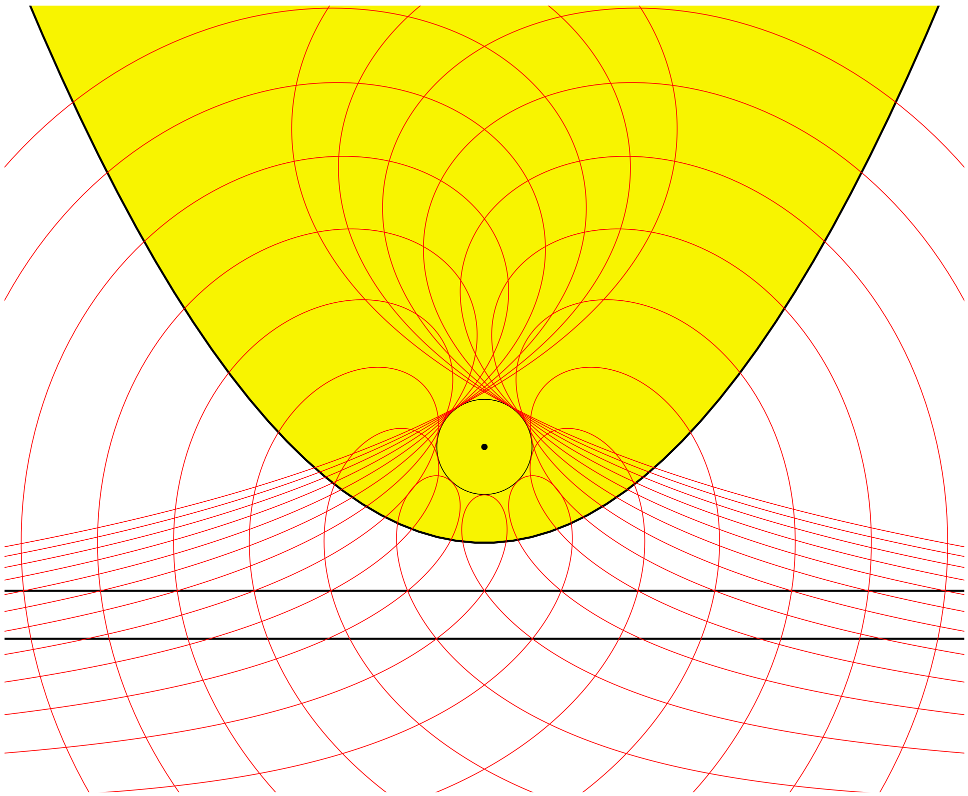

Figure 10. One parameter family of origami curves

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Garfunkel's Inequality

Nguyen Minh Ha and Nikolaos Dergiades

Abstract. Let I be the incenter of triangle ABC and U, V, W the intersections of the segments IA, IB, IC with the incircle. If the centroid G is inside the incircle, and D, E, F the intersections of the segments GA, GB, GC with the incircle. Jack Garfunkel [1] asked for a proof that the perimeter of UVW is not greater than that of DEF. This problem is hitherto unsolved. We give a proof in this note.

Consider a triangle ABC with centroid G lying inside its incircle (I). Let the segments AG, BG, CG, AI, BI, CI intersect the incircle at D, E, F, U, V, W respectively. Garfunkel posed the inequality $\partial(UVW) \le \partial(DEF)$ as Problem 648(b) of *Crux Mathematicorum* [1, 2].[1] Here, $\partial(\cdot)$ denotes the perimeter of a triangle. The problem is hitherto unresolved. In this note we give a proof of this inequality. We adopt standard notations: a, b, c, are the sidelengths of triangle ABC, s the semiperimeter and r the inradius.

Lemma 1. *If the centroid G of the triangle ABC is inside the incircle (I), then*

$$a^2 < 4bc, \quad b^2 < 4ca, \quad c^2 < 4ab.$$

Proof. Because G is inside (I), we have $\overrightarrow{IG}^2 \le r^2$, $(\overrightarrow{AG} - \overrightarrow{AI})^2 \le r^2$, $\overrightarrow{AG}^2 + \overrightarrow{AI}^2 - 2\overrightarrow{AG} \cdot \overrightarrow{AI} \le r^2$. This inequality is equivalent to the following

$$\overrightarrow{AG}^2 + (\overrightarrow{AI}^2 - r^2) - \frac{2}{3}(\overrightarrow{AB} + \overrightarrow{AC}) \cdot \overrightarrow{AI} \le 0$$

$$\frac{2(b^2+c^2) - a^2}{9} + (s-a)^2 - \frac{2(b+c)(s-a)}{3} \le 0$$

$$8(b^2+c^2) - 4a^2 + 9(b+c-a)^2 - 12(b+c)(b+c-a) \le 0$$

$$3(b+c-a)^2 + 2(b-c)^2 \le 2(4bc - a^2)$$

which implies $a^2 < 4bc$ and similarly $b^2 < 4ac$, $c^2 < 4ab$. □

Let the external bisectors of triangle UVW bound the triangle PQR, and intersect the incircle of ABC at U', V', W' respectively.

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[1]Problem 648(a) asked for a proof of $\partial(XYZ) \le \partial(UVW)$, XYZ being the intouch triangle. See Figure 1. A proof by Garfunkel was given in [1].

Lemma 2. *If the centroid G of ABC is inside the incircle, then the points D, E, F are on the minor arcs UU', VV', WW' respectively.*

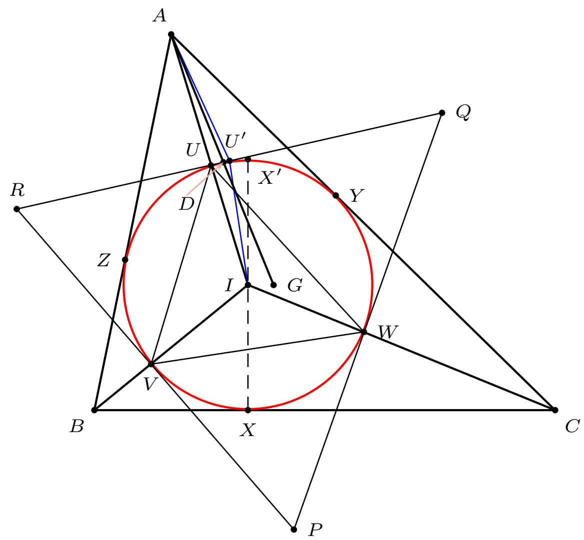

Figure 1

Proof. If $b = c$ then obviously U, D and U' are the same point.

Assume without loss of generality $b > c$. We set for brevity $\varphi = \frac{A}{2}$, $\theta = \frac{B-C}{4}$. Note that U' is the midpoint of the arc VUW. We have

$$\angle UIU' = \frac{1}{2}(\angle UIW - \angle UIV) = \frac{1}{2}\left(90^\circ + \frac{B}{2} - 90^\circ - \frac{C}{2}\right) = \theta.$$

Let X' be the antipode of the touch point X of the incircle with BC. Since $\angle UIV = \angle X'IW$, the point U' is the mid point of the arc UX'. We have

$$\begin{aligned}
\overrightarrow{AU'} =& \overrightarrow{AI} + \overrightarrow{IU'} = \overrightarrow{AI} + \frac{1}{2\cos\theta}\left(\overrightarrow{IU} + \overrightarrow{IX'}\right)\\
=& \overrightarrow{AI} + \frac{1}{2\cos\theta}\left(\sin\varphi\overrightarrow{IA} - \overrightarrow{IA} - \overrightarrow{AX}\right)\\
=& \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\overrightarrow{AI} - \frac{1}{2\cos\theta}\overrightarrow{AX}\\
=& \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\left(\frac{b}{2s}\overrightarrow{AB} + \frac{c}{2s}\overrightarrow{AC}\right)\\
&- \frac{1}{2\cos\theta}\left(\frac{s-c}{a}\overrightarrow{AB} + \frac{s-b}{a}\overrightarrow{AC}\right)\\
=& \left(\left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{b}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-c}{a}\right)\overrightarrow{AB}\\
&+ \left(\left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{c}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-b}{a}\right)\overrightarrow{AC}.
\end{aligned}$$

Since $b > c$, the centroid G lies inside the angle $\angle IAC$. To prove that D lies on the minor arc UU' it is sufficient to prove that the coefficient of $\overrightarrow{AC}$ is greater than that of $\overrightarrow{AB}$ in the above expression of $\overrightarrow{AU'}$. We need, therefore, to prove the inequality

$$\left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{c}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-b}{a} > \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{b}{2s} - \frac{1}{2\cos\theta}\cdot\frac{s-c}{a}.$$

Factoring and grouping common terms, the inequality is equivalent to

$$\begin{gathered}\frac{1}{2\cos\theta}\cdot\frac{b-c}{a} - \left(1 - \frac{\sin\varphi - 1}{2\cos\theta}\right)\frac{b-c}{2s} > 0\\ \frac{b-c}{4s\cos\theta}\left(\frac{b+c}{a} - 2\cos\theta + \sin\varphi\right) > 0\\ (b+c+a\sin\varphi)^2 > 4a^2\cos^2\theta. \qquad (1)\end{gathered}$$

Using the well-known identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, and $a\cos 2\theta = (b+c)\sin\varphi$ by the law of sines, inequality (1) can be written in the form

$$\begin{gathered}(b+c+a\sin\varphi)^2 > 2a^2 + 2a(b+c)\sin\varphi\\ (b+c)^2 - a^2 > a^2 - a^2\sin^2\varphi\\ 2bc + 2bc\cos A > a^2\cos^2\varphi\\ 4bc\cos^2(A/2) > a^2\cos^2\varphi\\ 4bc > a^2.\end{gathered}$$

This inequality holds by Lemma 1 since G is inside the incircle. This shows that D is on the minor arc UU'. The same reasoning also shows that E and F are on the minor arcs VV', WW' respectively. □

Theorem (Garfunkel's inequality)**.** *If the centroid G lies inside the incircle, then* $\partial(UVW) \leq \partial(DEF)$.

Proof. By Lemma 2, the points D, E, F lie on the minor arcs UU', VV', WW' respectively. Let X'' be the intersection point of DE and QR, Y'' be the intersection point of EF and RP, and Z'' be the intersection point of FD and PQ. Note that X'', Y'', Z'' belong to the segments DE, EF, FD respectively. See Figure 2. It follows that

$$\begin{aligned}\partial(DEF) &= DE + EF + FD\\ &= DX'' + X''E + EY'' + Y''F + FZ'' + Z''D\\ &= (EX'' + EY'') + (FY'' + FZ'') + (DZ'' + DX'')\\ &\geq X''Y'' + Y''Z'' + Z''X''\\ &= \partial(X''Y''Z'').\end{aligned}$$

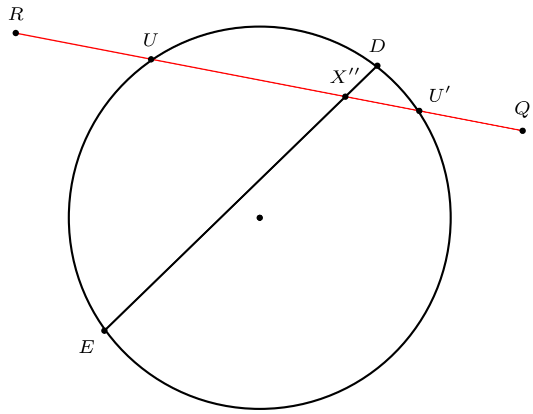

Figure 2

Therefore, $\partial(DEF) \geq \partial(X''Y''Z'')$. On the other hand, triangle PQR is acute and triangle UVW is its orthic triangle. See Figure 1. By Fagnano's theorem, we have $\partial(X''Y''Z'') \geq \partial(UVW)$. It follows that $\partial(DEF) \geq \partial(UVW)$. The equality holds if and only if triangle ABC is equilateral. $\square$

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On Some Actions of D_3 on a Triangle

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Abstract. The action of the dihedral group D_3 on the equilateral triangle is generalized to various actions on general triangles.

1. Introduction

The equilateral triangle admits in a natural way the action of the dihedral group D_3. The elements f of the group act as reflexions (order 2: $f^2 = 1$) or as rotations (order 3: $f^3 = 1$). If we relax the property of f from being isometry, we can define similar actions on an arbitrary triangle. In fact there are infinitely many actions of D_3 on an arbitrary triangle, described by the following setting.

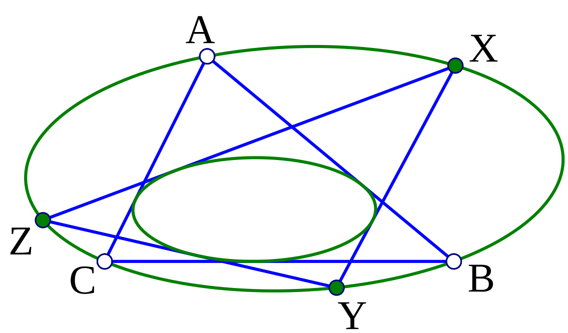

Figure 1. Projectivity preserving a conic

It is well known that given six points A, A', B, B', C, C' on a conic c, there is a unique projectivity preserving c and mapping A to A', B to B' and C to C'. Taking A', B', C' to be permutations of the set A, B, C we see that there is a group G of projectivities that permute the vertices of the triangle $t = (ABC)$ and preserve the conic c. It is not difficult to see that G is naturally isomorphic to the group of symmetries of the equilateral triangle. Thus from the algebraic point of view, the group action contains no significant information. But from the geometric point of view the situation is quite interesting. For example, fixing such a group, we can consider generalized rotations i.e. $f \in G$ of order three $f^3 = 1$, which applied to a point $X \in c$ generate an *orbital triangle* $X, Y = f(X), Z = f(f(X))$. All these orbital triangles envelope a second conic which is also invariant under the group G. For definitions, general facts on triangles, transformations and especially projectivities I refer to [1]. For special conics, circumscribed on a triangle, this setting unifies several dispersed properties and presents them under a new light.

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I shall illustrate this aspect by applying the above method to two special cases. Then I shall discuss an exceptional, similar setting, which results by replacing the circumconic with the circumcircle of the triangle and the projectivities by Moebius transformations. The first case will be that of the exterior Steiner ellipse of the triangle.

2. Steiner dihedral group of a triangle

We start with a triangle $t = (ABC)$ and its exterior Steiner ellipse. Then we consider the projectivities that preserve this conic and permute the vertices of the triangle. First I shall state the facts. The group, which I call the *Steiner dihedral group* of the triangle, comprises two kinds of maps: involutions, that resemble to reflections, and cyclic permutations of the vertices that resemble to rotations.

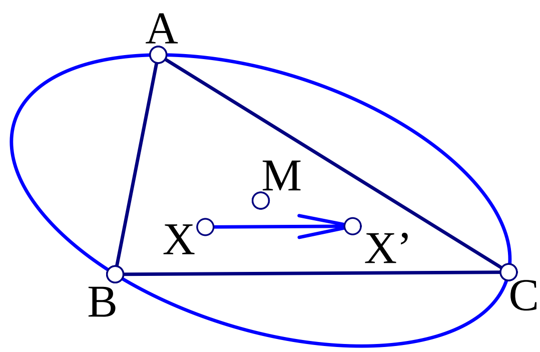

Figure 2. Isotomic conjugation

The involutions are related to the sides of the triangle and coincide with the isotomic conjugations with respect to the corresponding medians: Side a of the triangle defines an involution on the conic: $I_a(X) = Y$, where XY is parallel to the side a and bisected by the median to a. I_a has the median to a as its line of fixed points, which coincides with the conjugate diameter of a relative to the conic. The corresponding isolated fixed point (Fregier point of the involution) is the point at infinity of line a. Analogous definitions and properties have the involutions I_b, I_c.

More important seems to be the projectivity $f = I_b \circ I_a$, of order three $f^3 = 1$, that preserves the conic and cycles the vertices of the triangle. I call it the *isotomic rotation*.

As is the case with every projectivity f, preserving a conic, for all points X on c, the lines $[X, f(X)]$ envelope another conic, which in this case is the inner Steiner ellipse. By the same argument all *orbital* triangles i.e. triangles of the form $t' = (X, f(X), f(f(X)))$, are circumscribed on the inner Steiner ellipse. More precisely the following statements are valid and easy to prove:

(1) The centroid G of the triangle is the fixed point of f.

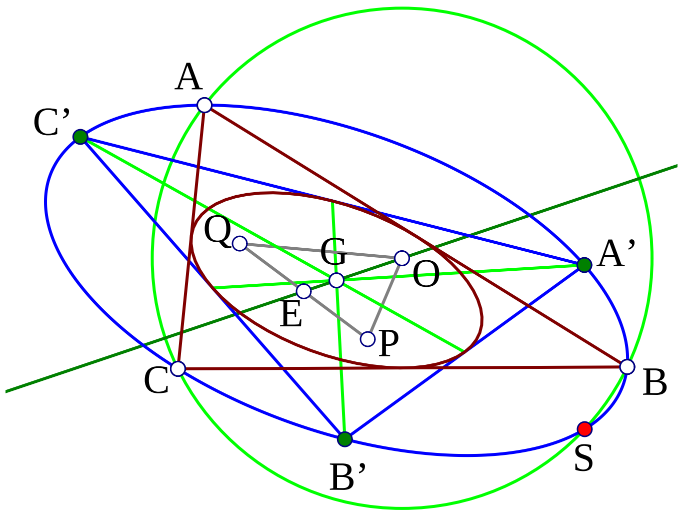

Figure 3. Isotomic rotation

(2) Every point X of the plane defines an *orbital* triangle

$$s = (X, f(X), f(f(X))),$$

which has G for its centroid.

(3) The orbital triangles s, as above, which have X on the external Steiner ellipse, are all circumscribed to the inner Steiner ellipse. They are precisely the only triangles that have these two ellipses as their external/internal Steiner ellipses.

(4) The inner and outer Steiner ellipses generate a family of homothetic conics, with homothety center the centroid G of the triangle. For every point X of the plane the orbital triangle s generated by X has the corresponding conics-family-member c, passing through X, as its outer Steiner ellipse. Besides, for all points X on c, the corresponding orbital triangles circumscribe another conics-family-member c', which is the inner Steiner ellipse of all these triangles.

(5) For a fixed orbital triangle $t = (ABC)$, the orbit of its circumcenter O, defines a triangle $u = (OPQ)$, whose median through O is the Euler line of the initial triangle t. The middle E of PQ is the center of the Euler circle of t.

(6) The trilinear coordinates of points $P = f(O)$ and $Q = f(P)$ are respectively:

$$P = \left(\frac{\sin 2C}{\sin A}, \frac{\sin 2A}{\sin B}, \frac{\sin 2B}{\sin C}\right),$$

$$Q = \left(\frac{\sin 2B}{\sin A}, \frac{\sin 2C}{\sin B}, \frac{\sin 2A}{\sin C}\right).$$

Deferring for a later moment the proofs, I shall pass now to the analogous group, of projectivities, which results by replacing the external ellipse with the circumcircle of the triangle. For a reason that will be made evident shortly I call the corresponding group the *Lemoine* dihedral group of the triangle.

3. Lemoine dihedral group of a triangle

We start with a triangle $t = (ABC)$ and its circumcircle c. Then we consider the projectivities that preserve c and permute the vertices of the triangle. There are again two kinds of such maps. Involutions, and maps of order three.

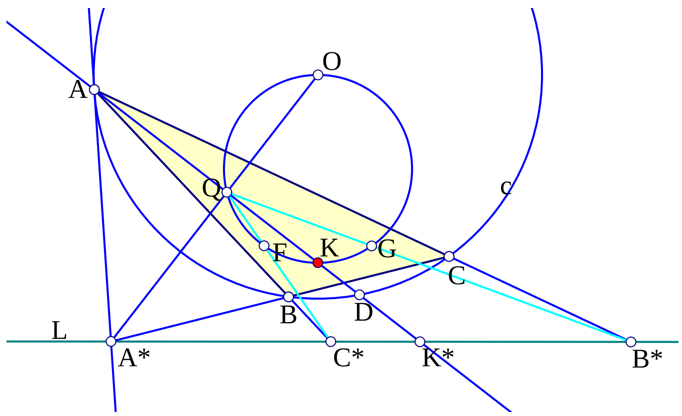

Figure 4. Lemoine reflexion

Side a of the triangle defines a projective involution $I_a(X) = X'$, by the properties $I_a(A) = A$ and $I_a(B) = C, I_a(C) = B$. Its line of fixed points, is the symmedian line AD. The corresponding isolated fixed point (Fregier point) is the pole A^* of the symmedian with respect to the circumcircle, which lies on the Lemoine axis L of the triangle. In the figure above, K is the symmedian point and Q is the projection of the circumcenter on the symmedian AD (is a vertex of the second Brocard triangle of t). From the invariance of cross-ratio and the fact that I_a maps L to itself, follows that $(C^*B^*K^*A^*) = 1$, hence the symmedian bisects the angle B^*QC^*. Joining Q with B^*, C^* we find the intersections F, G of these lines with the Brocard circle (with diameter OK). Below (in §6) we show that F, G coincide with the Brocard points of the triangle.

I_a could be called the *Lemoine reflexion* (on the symmedian through A). Analogous is the definition and the properties of the involutions I_b and I_c, corresponding to the other sides of the triangle.

More important seems to be the projectivity $f = I_b \circ I_a$, of order three $f^3 = 1$, which preserves the circumcircle and cycles the vertices of the triangle. I call it the *Lemoine rotation*.

As before, for all points X on c, the lines $[X, f(X)]$ envelope another conic, which in this case is the Brocard ellipse c' of the triangle t. By the same argument all *orbital* triangles i.e. triangles of the form $t' = (X, Y = f(X), Z = f(f(X)))$,

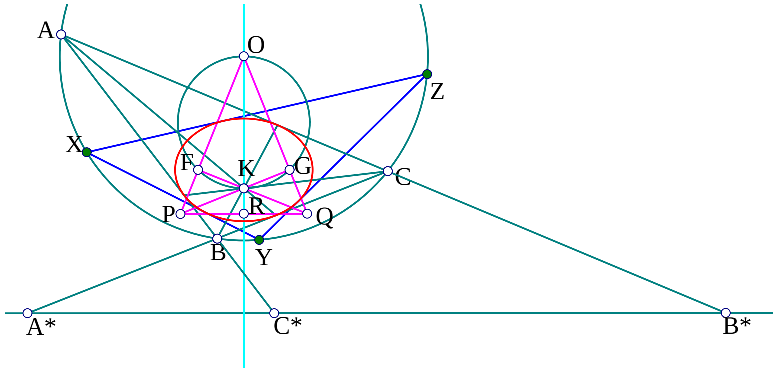

Figure 5. Lemoine rotation

are circumscribed on the Brocard ellipse. More precisely the following statements are valid and easy to prove:

(1) f leaves invariant each member of the family of conics generated by the circumcircle and the Brocard ellipse of t. In particular the Lemoine axis of t remains invariant under f, and permutes points A^{*}, B^{*}, C^{*}.

(2) The symmedian (or Lemoine) point K of the triangle is the fixed point of f.

(3) Every point X of the circle c defines an *orbital* triangle

$$s = (X, f(X), f(f(X))),$$

which has K as symmedian point.

(4) The orbital triangles s, as above, which have X on c, are all circumscribed to the Brocard ellipse c'. They are precisely the only triangles that have c and c' as circumcircle and Brocard ellipse, respectively.

(5) For a fixed orbital triangle $t = (ABC)$, the orbit of its circumcenter O, defines a triangle $u = (OPQ)$, whose median through O is the Brocard axis of the initial triangle t.

(6) The triangle u is isosceles and symmetric on the Brocard axis. The feet G, F of the altitudes of u from P and Q, respectively, coincide with the Brocard points of t.

(7) The triangles u, $u' = (PRF)$ and $u'' = (QRG)$ are similar. The similarity ratio of the two last to the first is equal to the sine of the Brocard angle.

Deferring once again the proofs at the end (§6), I shall pass to a third group, using now inversions instead of projectivities. For a reason that will be made evident shortly I call the corresponding group the *Brocard* dihedral group of the triangle.

4. Brocard dihedral group of a triangle

Once again we start with a triangle $t = (ABC)$ and its circumcircle c. Then we consider the Moebius transformations that permute the vertices of t. It is true that through such maps the sides are not mapped to sides. We do not have proper maps of the triangle's set of points onto itself, but we have a group that permutes its vertices, is isomorphic to D_3 and, as we will see, has intimate relations with the previous one and the geometry of the triangle.

Everything is based on the well known fact that a Moebius transformation is uniquely defined by prescribing three points and their images. Thus, fixing a vertex, A say, of the triangle and permuting the other two, we get a Moebius involution, I_a say. Analogously are defined the other two involutions I_b and I_c. I call them the *Brocard reflexions* of the triangle. Two of them generate the whole group. By the well known property of Moebius transformations, we know that all of them preserve the circumcircle c.

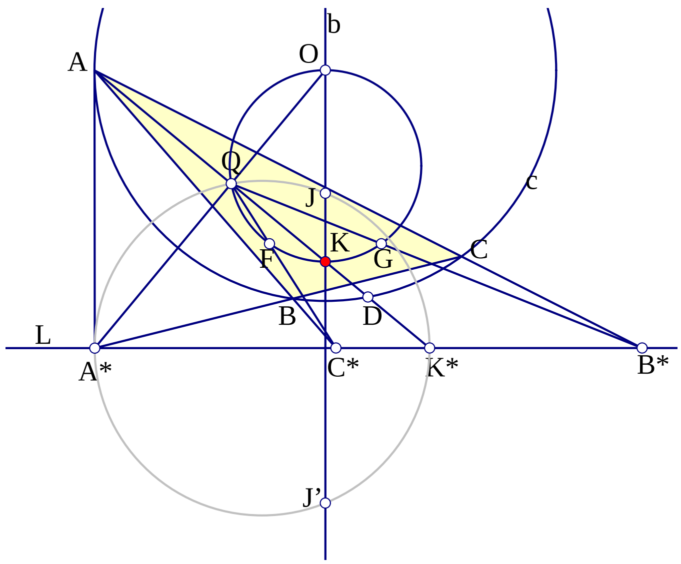

Figure 6. Brocard reflexion

I cite some properties of I_a that are easy to prove:

(1) On the points of the circumcircle the Brocard reflexion I_a coincides with the corresponding Lemoine reflexion.
(2) I_a leaves invariant each member of the bundle of circles through its fixed points A and D (D being the intersection of the symmedian from A with the circumcircle).
(3) I_a leaves invariant each member of the bundle of circles that is orthogonal to the previous one (i.e. the circles which are orthogonal to the symmedian AD and the circumcircle).
(4) In particular I_a leaves invariant the symmedian from A and maps the symmedian point K to the intersection K^* of the Lemoine axis with that symmedian.
(5) I_a permutes the circles of the bundle generated by the circumcircle and the Lemoine axis of t. The same happens with the orthogonal bundle to the previous one, which is the bundle generated by the Apollonian circles of t.

(6) I_a interchanges the circumcenter O with the pole A^* of the symmedian at A. It maps also the Brocard axis b onto the circle through the isodynamic points and A^*.
(7) All the circles through O, Q are mapped by I_a to lines through A^*. In particular the Brocard circle is mapped to the Lemoine axis.
(8) The line AB is mapped by I_a to the circle through A, C, tangent to this line at A.
(9) I_a maps the Brocard points F, G to the intersection points B^*, C^* of the sides AC and AB with the Lemoine axis respectively.

We pass now to the Moebius transformation that recycles the vertices of the triangle $t = (ABC)$. It is the product of two Brocard reflexions $f = I_b \circ I_a$. It is of order three: $f^3 = 1$ and I call it the *Brocard rotation*. The geometric properties of this transformation are related to the so called *characteristic parallelogram* of it. This is generally defined, for every Moebius transformation (may be degenarated), as the parallelogram whose vertices are the two fixed points and the poles of f and of its inverse f^{-1}. A short discussion of this parallelogram will be found in §8. Here are the main properties of our Brocard Rotation.

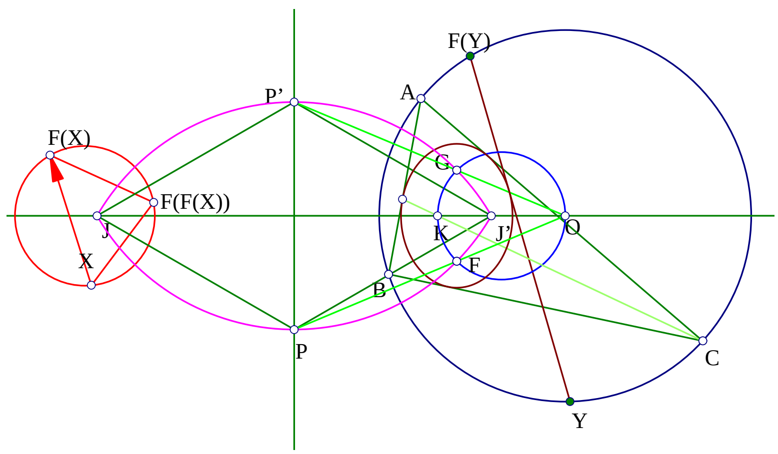

Figure 7. Brocard rotation

(10) On the points of the circumcircle c of t the Brocard Rotation coincides with the corresponding Lemoine rotation.
(11) The characteristic parallelogram of f is a rhombus with two angles of measure $\pi/3$. The vertices at these angles are the fixed points of f. They also coincide with the isodynamic points of the triangle. The other vertices of the parallelogram (angles $2\pi/3$) coincide with the inverses of the Brocard points with respect to the circumcircle.
(12) f leaves invariant every circle of the bundle of circles, generated by the circumcircle of t and its Brocard circle (circle throuch circumcenter and Brocard points).

(13) All circles of the bundle, which is orthogonal to the previous, pass through the isodynamic points J, J' of t. Each circle c of this bundle is mapped to a circle c' of the same bundle, which makes an angle of $\pi/6$ with c. In particular the Apollonian circles of the triangle are cyclically permuted by f.

(14) Every point X of the plane defines an *orbital* triangle

$$s = (X, f(X), f(f(X))),$$

which shares with t the same isodynamic points J, J', hence Brocard and Lemoine axes. Conversely, every triangle whose isodynamic points are J and J' is an orbital triangle of f.

(15) The Brocard points of all the above orbital triangles s fill the two $\pi/3$-angled arcs JPJ' and $JP'J'$ on the two circles with centers at the poles P, P' of f, joining the isodynamic points J and J'.

(16) The orbital triangles s, as above, which have X on the circumcircle of t, are all circumscribed to the Brocard ellipse c' of t. They are precisely the only triangles that have c and c' as their circumcircle and Brocard ellipse, respectively.

(17) The other two points of the orbital triangle of the circumcenter O, are the two Brocard points of t.

(18) The second Brocard triangle $A_2B_2C_2$ is an orbital triangle of f.

5. Proofs on Steiner

A convenient method to define the two Steiner ellipses of a triangle, is to use a projectivity F, that maps the vertices of an equilateral triangle $t' = (A'B'C')$ onto the vertices of an arbitrary triangle $t = (ABC)$ and the center of t' onto the centroid of t. As is well known, prescribing four points and their images, uniquely determines a projectivity of the plane. Thus the previous conditions uniquely determine F (up to permutation of vertices). Let a', b' be the circumcircle and incircle, correspondingly of t'. Their images $a = F(a')$ and $b = F(b')$ are correspondingly the exterior and interior Steiner ellipses of t.

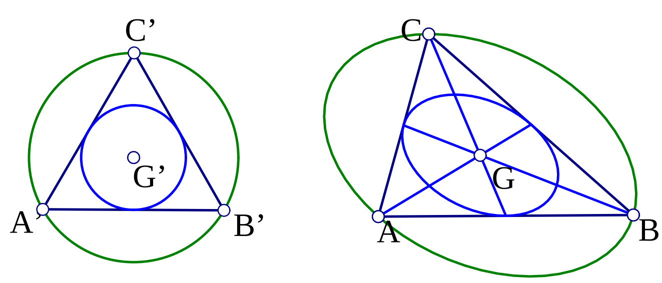

Figure 8. Creating the two Steiner ellipses of a triangle

From the general properties of projectivities result the main properties of Steiner's ellipses of the triangle t:

(1) From the invariance of cross-ratio, and the fact that F preserves the middles of the sides, follows that F preserves also the line at infinity. Thus, the images of circles are ellipses.
(2) The same reason implies, that the tangent to the outer ellipse at the vertex is parallel to the opposite side of the triangle.
(3) The same reason implies, that the centers of the two ellipses coincide with G and the ellipses are homothetic with ratio 2, with respect to that point.
(4) The invariance of cross-ratio implies also, that the *Steiner involution*, defined as the projectivity that fixes A and permutes B, C, coincides (on points of the conic) with the conjugation $X \mapsto Y$, where XY is parallel to a. It leaves the line at infinity fixed and coincides with the isotomic conjugation with respect to the median from A. The median being a conjugate direction to a with respect to the conic.
(5) The Fregier point of the involution I_a is the point at infinity of line $a = BC$ and the line of fixed points of I_a is the median from A.

The *isotomic rotation* is the projectivity $f = I_b \circ I_a$. One sees immediately that it has order three: $f^3 = 1$, that preserves the conic and cycles the vertices of the triangle. Besides it fixes the centroid G and cycles the middles of the sides. All the statements of §2, about orbital triangles, follow immediately from the previous facts and the property of f, to be conjugate, via F, to a rotation by $2\pi/3$ about G. For the statement on the particular orbital triangle of the circumcenter O of t, it suffices to do an easy calculation with trilinears. Actually the Euler line passes also through the symmetric O' of O with respect to G, which is one of the intersection points of the two conics of the figure below.

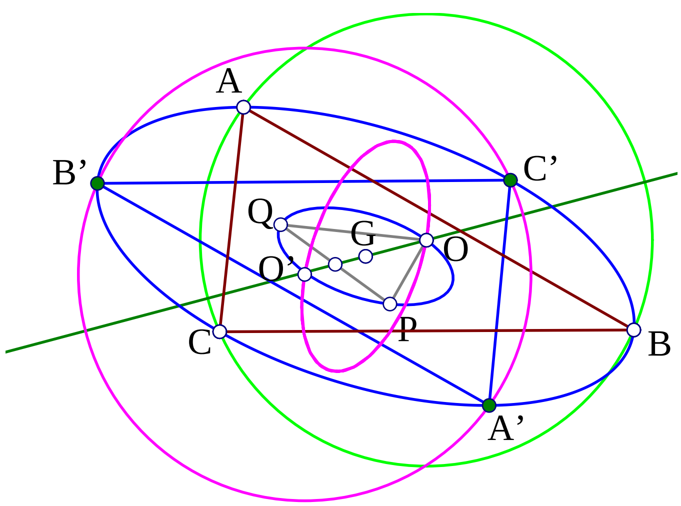

Figure 9. Circumcenters of orbital triangles

One of the conics is the member of the conics-family passing through O. The other ellipse has the same axes with the previous one and is the locus of the circumcenters of orbital triangles $u = (X, f(X), f^2(X))$, for X on the outer Steiner ellipse. O' is the circumcenter of the triangle $t' = (A'B'C')$ which is symmetric to t with respect to G.

6. Proofs on Lemoine

A convenient method to define the Brocard ellipse of a triangle, is to use a projectivity F, that maps the vertices of an equilateral triangle $t' = (A'B'C')$ onto the vertices of an arbitrary triangle $t = (ABC)$ and the center of t' onto the symmedian point of t. These conditions uniquely determine F (up to permutation of vertices).

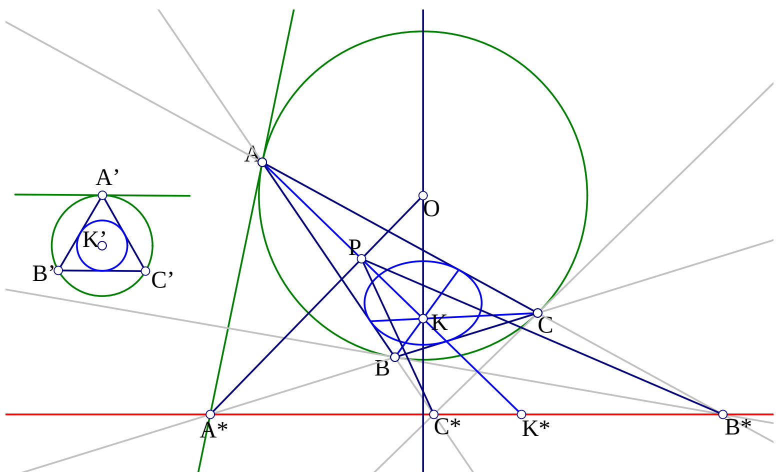

Figure 10. Creating the Brocard ellipse of a triangle

F maps the incircle of t' to the Brocard ellipse of t and the circumcircle of t' to the circumcircle of t. To see the later, notice that F preserves the cross ratio of a bundle of four lines through a point. Now the tangent of t' at A', the two sides $A'B'$, $A'C'$ and the median of t' from A' form a harmonic bundle. The same is true for the tangent of t at A the two sides AB, AC and the symmedian from A. Thus F maps the tangent of t' at A' to the tangent of t at A, and analogous properties hold for the other vertices. This forces the circumcircle of t to coincide with the image, under F, of the circumcircle of t'. The other statement, on the Brocard ellipse, follows from the fact, that this ellipse is characterized as the unique conic tangent to the sides of the triangle at the traces of the symmedians from the opposite vertices. The main properties of the *Lemoine reflexion* I_a result from the fact that it is conjugate, via F, to the reflexion of t' with respect to its median from A'. Thus the line of fixed points of I_a coincides with the symmedian from A. The intersection point A^* of the line BC with the tangent at A is the image, via F, of the point at infinity of the line $B'C'$. Analogous properties hold for the points

B^* and C^*. Since these points are known to be on the Lemoine axis, this implies that the line at infinity is mapped, via F, to the Lemoine axis of the triangle. All the lines through A^* remain invariant under I_a, hence this point coincides with the Fregier point of the involution.

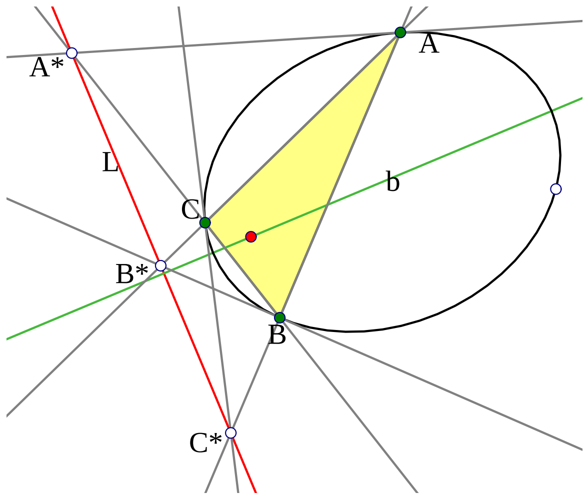

Figure 11. Orbital triangles

The *Lemoine rotation* is the projectivity $f = I_b \circ I_a$, of order three $f^3 = 1$, that preserves the circumcircle and cycles the vertices of the triangle. Besides it fixes the symmedian point K of the triangle and cycles the symmedians. f is conjugate, via F, to a rotation by $2\pi/3$ about K'. f leaves invariant the family of conics generated by the circumcircle and the Brocard ellipse. This family is the image, under F, of the bundle of concentric circles about K'. In particular the line at infinity is mapped onto the Lemoine axis of t, which is also invariant under f. The conics of the family, left invariant by f, are all symmetric with respect to the Brocard diameter b. Besides all orbital triangles $s = (A = X, B = f(X), C = f(f(X)))$ of f have the property shown in the above figure.

In this figure the point A^* is the intersection point of BC and the tangent at A of the conic-family member passing through A. Analogously are defined , B^* and C^*. The three points lie on the Lemoine axis L of t and are cyclically permuted by f. The proof is a repetition of the argument on harmonic bundles at the beginning of the paragraph. This has though a nice consequence. First, if A is on the Brocard diameter b of t, which is the symmetry axis of all the conics of the invariant family, then the coresponding orbital triangle s is symmetric. Besides the lines AB and AC pass through two fixed points C^* and B^* of L respectively. In fact, in that case, the tangent at A meets L at its point at infinity. Consequently the corresponding BC is parallel to L and s is isosceles. In addition, since f cycles the corresponding points A^*, B^*, C^*, the two last points are the image of the point at infinity of L, under f and its image respectively. Thus they are independent of the position of A on b.

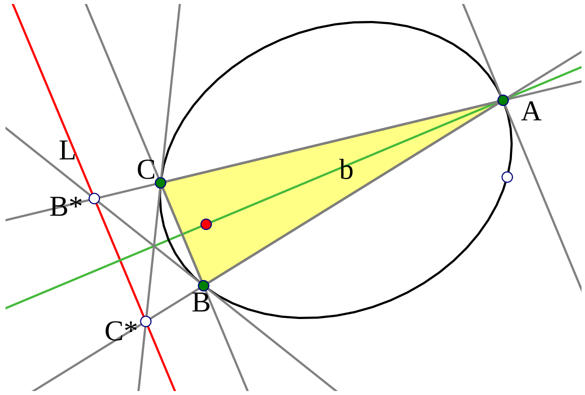

Figure 12. The orbit of the point at infinity of L

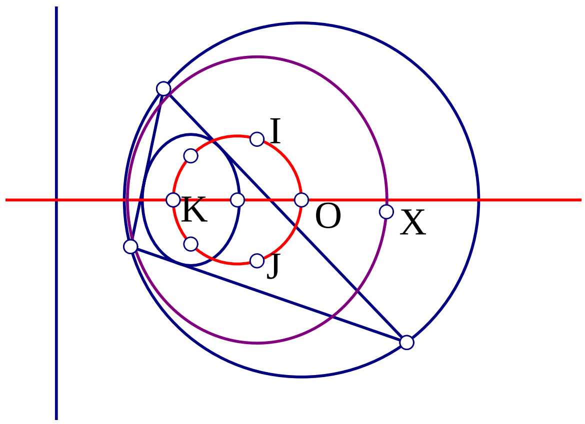

Figure 13. Focal points of the conics

Below B^*, C^* will be identified with the inverses of the Brocard points of t with respect to the circumcircle. Notice that the Brocard points of t are the focal points of the Brocard ellipse and they lie on the Brocard circle with diameter OK. It is well known, that in general the focal points of a family of conics lie on certain cubics. For a reference, see our paper with Apostolos Thoma [2], where we investigated such cubics from a geometric point of view. In the present case the family consists of conics that are symmetric with respect to the Brocard axis and the cubic must be reducible and equal to the product of a circle and a line. In fact a calculation shows that the cubic is the union of the Brocard cirlce and the Brocard axis. All points X inside the circumcircle of t define family members whose focal points are on the Brocard circle. All points X outside the circumcircle of t define family members whose focal points are on the Brocard axis. For X varying on b there are two positions, where the legs of the orbital isosceli contain the foci of the corresponding conic-member through X. One of these points is the center O of the circumcircle. Notice that the family of conics is generated also from the

Lemoine axis (squared) and the circumcircle. This representation makes simpler the computations of a proof of the last statements of §3, on the orbital triangle of the circumcenter. Another geometric proof of this fact may be derived from the arguments of the two next paragraphs.

7. Proofs on Brocard

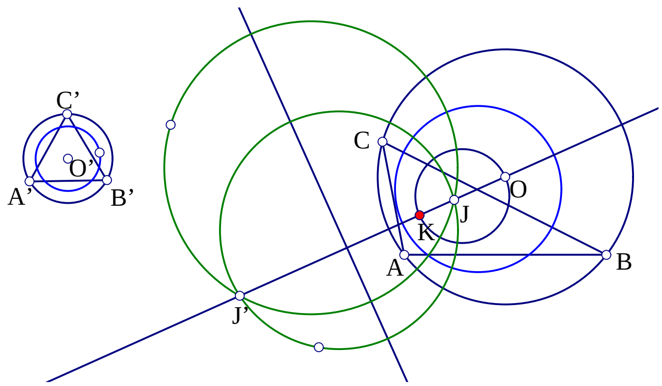

Figure 14. The isodynamic bundles of the triangle

In contrast to projectivities that need four, Moebius transformations are determined completely by three pairs of points. Imitating the procedures of the previous paragraphs, we define the Moebius transformation F that sends the vertices of an equilateral triangle $t' = (A'B'C')$ to the vertices of an arbitrary triangle $t = (ABC)$. Since Moebius transformations, preserve the set of circles and lines, the circumcircle of t' is mapped on the circumcircle of t. Moreover the bundle of concentric circles to the circumcircle of t' maps to the bundle Σ of circles generated by the circumcircle of t and its Lemoine axis. Below I call Σ the *Brocard bundle* of t. This is a hyperbolic bundle with focal (or limiting) points coinciding with the isodynamic points J, J' of t. Since F is conformal it maps the lines from O' to the circle bundle that is orthogonal to the previous one. All circles of this bundle pass through the isodynamic points. All these facts result immediately from the fact that the altitudes of t' map onto the corresponding Apollonian circles of t. This in turn follows from the invariance of the complex cross ratio, by considering the cross ratio of the vertices $(ABCD) = (A'B'C'D') = 1$. D on the circumcircle is uniquely determined by this condition and coincides with the trace of the symmedian from A. The conformality of Moebius transforms implies also that the Apollonian circles meet at J at angles equal to $\pi/3$. Below I call the bundle Σ' of circles through J, J' the *Apollonian bundle* of t. Now to the proofs of the statements in §4.

The first statement (1) is a general fact on Moebius transformations preserving a circle c. Given three pairs of points on c, there is a unique Moebius f and a unique projectivity f' preserving c and corresponding the points of the pairs. f and f' coincide on points $X \in c$. In fact, taking cross ratios $(ABCX)$ in complex or by projecting the points on a line, from a fixed point, $Z \in c$ say, gives the same result. The same is true for the images $(A'B'C'X')$ under both transformations, thus the images of X under f and f' coincide.

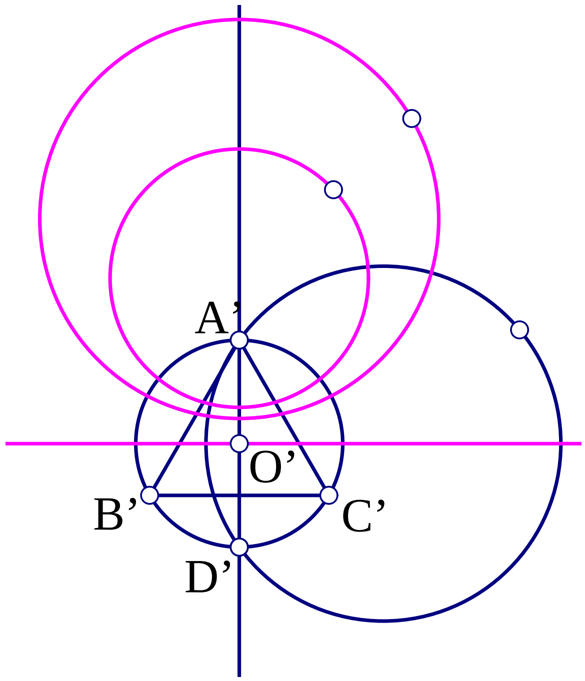

Figure 15. The I_a' invariant bundles

The next two statements (2,3) follow immediately from the fact that I_a is conjugate, via F, to the Moebius transformation I_a' fixing A', D' and mapping B' to C'. A short calculation shows that I_a' preserves the circles passing through A', D' and also preserves the circles of the orthogonal bundle to the previous one. These two I_a'-invariant bundles, map under F to the corresponding I_a-invariant bundles of the statements. The previous argument shows also that the bundle of concentric circles at O' is permuted by I_a', consequently the same is true for the bundle of lines through O'. But these two bundles map under F to the main bundles of our configuration, the Brocard Σ and the Apollonian Σ' correspondingly. This proves also statement (4).

Next statement (5) follows from the invariance of cross ratio, along the I_a-invariant symmedian from A, and the fact that the Lemoine axis is the polar of the symmedian point with respect to the circumcircle. A consequence of this, taking into account that I_a permutes the Brocard bundle, is that the Brocard circle of t maps via I_a to the Lemoine axis.

From the previous considerations, on the Brocard and Apollonian bundles, follows that I_a does the following: (a) It interchanges O, P, (b) sends Q (the projection of the circumcenter on the symmedian) at the point at infinity, (c) maps the circles with center at Q to circles with the same property, (d) maps the lines e

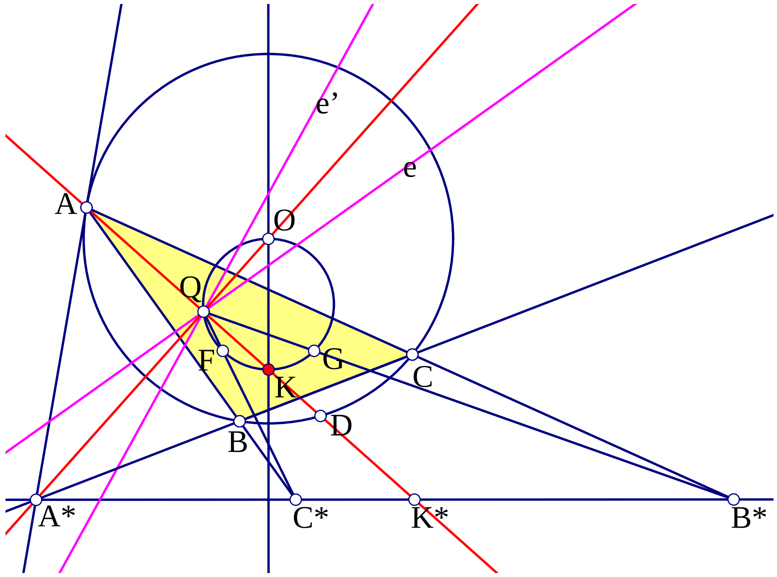

Figure 16. I_a on Brocard points

through Q to their symetrics e' with respect to PQ (or the symmedian at A). As a consequence I_a maps the line QB^* to the line QC^* and points G, F onto C^*, B^* correspondingly. Consider now the image of line AB via I_a. By the properties just described, points A, B, C^* are mapped onto A, C, G correspondingly. Also the point at infinity is mapped onto Q, thus the line maps to a circle r passing through the points (A, Q, C, G). It is trivial to show that the circle through the points (A, Q, C) is tangent to line AB at A. This identifies G with one of the two Brocard points of t. Statements (6-10) follow immediately from the previous remarks. Before to proceed to the proofs of the remaining statements of §4, let us review some facts about the characteristic parallelograms of Moebius transformations.

8. Characteristic parallelogram

For proofs of properties of Moebius transformations and their characteristic parallelogram I refer to Schwerdtfenger [3]. The characteristic parallelogram of a Moebius transformation f has one pair of opposite vertices coinciding with the fixed points of f, the other pair of vertices coinciding with the poles of f and f^{-1} respectively. The parallelogram can be degenerated or have infinite sides. It characterizes completely f, when we know which vertices are the fixed points and which are the poles. In the image below F, F' are the fixed points of f, P is its pole and P' is the pole of f^{-1}. Triangles zFP, $Fz'P'$ and $zz'F'$ are similar in that orientation. This defines the recipe by which we construct geometrically $z' = f(z)$.

Moebius transformations f permute the bundle Σ of circles which pass through their fixed points F, F'. Each circle a of Σ is mapped to a circle a' of the same

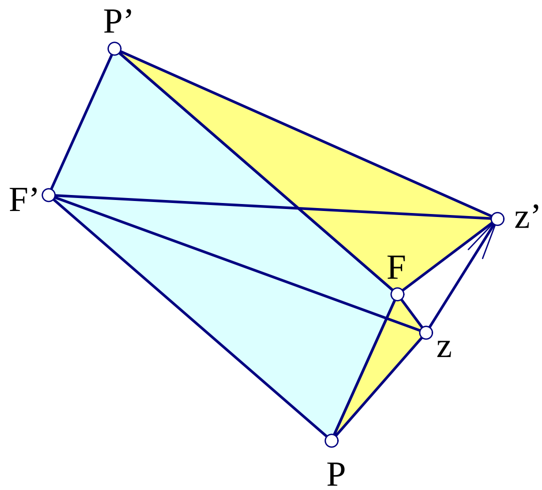

Figure 17. Building the image $z' = f(z)$

bundle, such that the angle at F is the same with the angle of the characteristic parallelogram at the pole P. In some sense the circles of Σ are *rotated* about the fixed points of f. The picture is complemented by the bundle Σ', which is orthogonal to the previous one. This is also permuted by f.

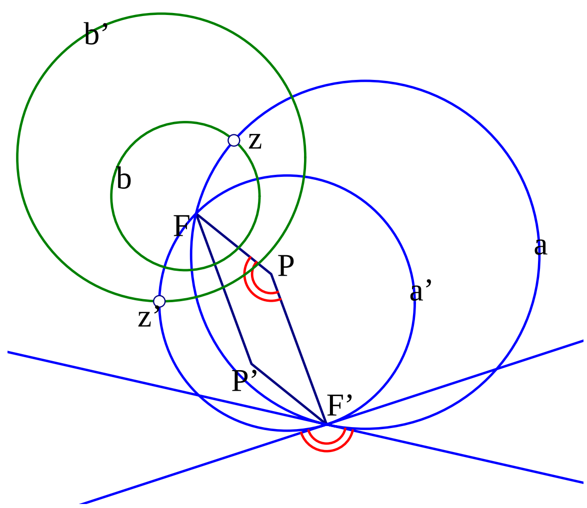

Figure 18. Characteristic bundles of a Moebius transformation

The *elliptic* Moebius transformations are characterized by their property to leave invariant a circle. The circle then belongs to the bundle Σ', whose all members remain also invariant by f. In fact, in that case f is conjugate to a rotation, and by this conjugation the two bundles correspond to the set of concentric circles about the rotation-center (Σ') and the set of lines through the rotation-center (Σ). In addition the parallelogram is then a rhombus.

Now to the proofs of the properties of Brocard rotations f of §4, preserving the notations introduced there. Since these transformations preserve the circumcircle of the triangle t, they are elliptic. Since they are conjugate, via the map F, to Rotations by $2\pi/3$, their characteristic parallelogram is a rhombus with an angle (at the pole) equal to $2\pi/3$. From the properties of F we know that the fixed points of f coincide with the isodynamic points of the triangle and the Apollonian circles are members of the bundle Σ, permuted by f. The Lemoine axis, being axis of symmetry of the isodynamic points, contains the other vertices of the rhombus. The other bundle Σ', of circles left invariant by f, coincides with the bundle generated by the circumcircle and the Lemoine axis. Later bundle contains the Brocard circle. The statement on orbital triangles follows from the corresponding property of Lemoine rotations, since the two maps coincide on the circumcircle.

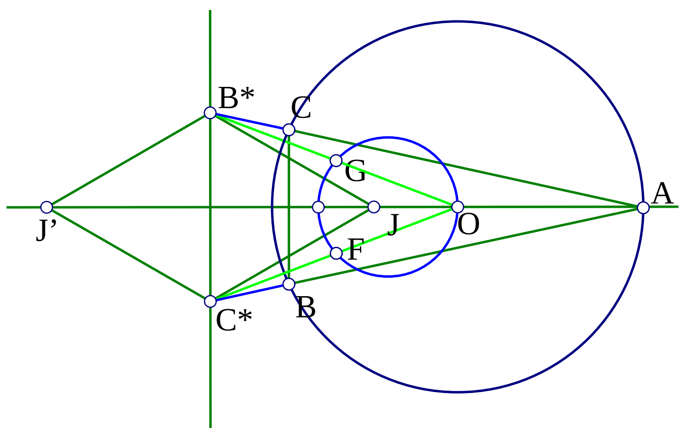

Figure 19. Projections of Brocard points on Lemoine axis

The fact that the circumcenter O, together with the two Brocard points F, G build an orbital triangle of f, follows now easily from the fact that $f = I_b \circ I_a$. In fact, from our discussion, on Brocard reflexions, we know that I_a maps the circumcenter onto A^*, the intersection of side $a = BC$ with the Lemoine axis. Then I_b, as shown there, maps A^* to one Brocard point. A similar argument proves that applying again f we get the other Brocard point. Analogously one proves that the second Brocard triangle is also an orbital triangle of f. All the statements (10-19) follow from the previous remarks.

Especially the statement about the fact that P, P' are the projections, from the circumcenter O, of the Brocard points, on the Lemoine axis, follows also easily from our arguments. In fact, the equibrocardian isosceles triangle $t = (ABC)$ of the previous picture, is also an orbital triangle of the corresponding Lemoine rotation. From there we know that its legs pass through the fixed points B^*, C^*. These points are identified as the images of the point at infinity of the Lemoine axis under the Lemoine Rotation. But this rotation coincides also with the Brocard rotation on that axis. This identifies P, P' with the other vertices of the characteristic parallelogram.

9. Remarks

(1) For every point P of the triangle's plane (e.g. some triangle center), one can define a projectivity F analogous to the one used in the two examples and establishing the conjugacy of the group G with the dihedral D_3. The projectivity F is required to map the vertices of the equilateral triangle to the vertices of the arbitrary triangle t. In addition, it is required to map the center P' of the equilateral to the selected point P. These conditions completely determine F and there are several phenomena, generalizing the previous examples. The bundle of circles centered at P' maps to a family Σ of conics. One of these conics, $c \in \Sigma$, circumcscribes t, one other being inscribed and touching the triangle's side at the feet of the cevians from P. One can define analogously the action of D_3, preserving c and permutting the vertices of the triangle. The properties of this action, reflect naturally properties of the point P with respect to triangle t. The action leaves invariant the whole family Σ.

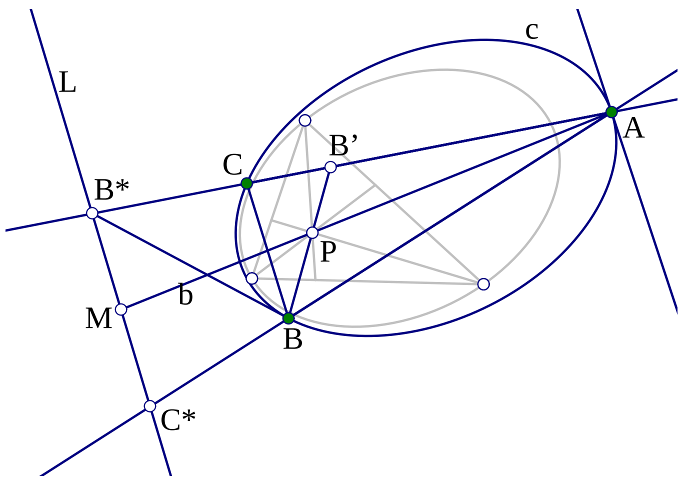

Figure 20. The limit points of the conics-family

Also, using essentially the same arguments as in the examples, one can show, that the line at infinity maps via F to the *trilinear polar* of P. The trilinear polar being then a singular member (double line) L of Σ. Besides all orbital triangles $t = (ABC)$ which have a side, BC say, parallel to this line, have the other two sides passing through two fixed points C^*, B^* of L, whereas the tangent to the member-conic c circumscribing the triangle at the other point A of the triangle is also parallel to L. The line $b = PA$, passes through the middle M of B^*C^* and is the conjugate direction to L, with respect to every conic of the family. In this case also the corresponding projective rotation f recycles points B^*, C^* and the point

at infinity of line L.
(2) The data L, P and the location of points B^*, C^* on L uniquely determine the invariant family of conics Σ and the related orbital triangles. In fact, once B^*, C^* are known, the line MP, where M is the middle of B^*, C^*, is conjugate to the direction of L, with respect to all the conics of Σ. A point A on this line can be determined, so that a special orbital triangle ABC can be constructed from the previous data. In fact, point B' on AB^* satisfies the condition that the four points $(ACB'B^*) = 1$, form a harmonic ratio. A triangle ABC is immediately constructed, so that BB^* and BB' are its bisectors and BC is parallel to L. Consequently the projectivity F can be defined, and from this the whole family is also constructed.

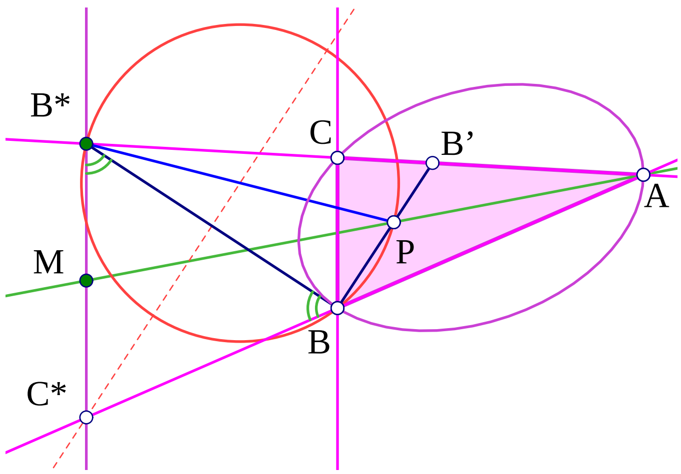

Figure 21. Special orbital triangle determined from B^*, C^*, P

(3) The previous considerations give a nice description of the set of triangles having a given line L and a given point $P \notin L$ as their trilinear polar with respect to P. They are orbital triangles of actions of the previous kind and they fall into families. Each family is characterized by the location of its limit points B^*, C^* on L.
(4) An easy calculation shows that the focal points of the members of Σ describe a singular cubic, self-intersecting at P. Besides the asymptotic line of this cubic coincides with b. When P is the Symmedian-point, the corresponding cubic coincides with the reducible one, consting of the Brocard circle and the Brocard line.
(5) Inscribed conics and corresponding actions of D_3, permutting their contact points with the sides of the triangle, could be also considered. They offer though nothing new, since they are equivalent to actions of the previous kind.
(6) In all the above groups of projectivities, the rotations are identical to the projectivities fixing the point P and cycling the vertices. One could start from such a projectivity and show the existence and invariance of the resepective family of conics. I prefer however the variant with the circumconics which introduces them

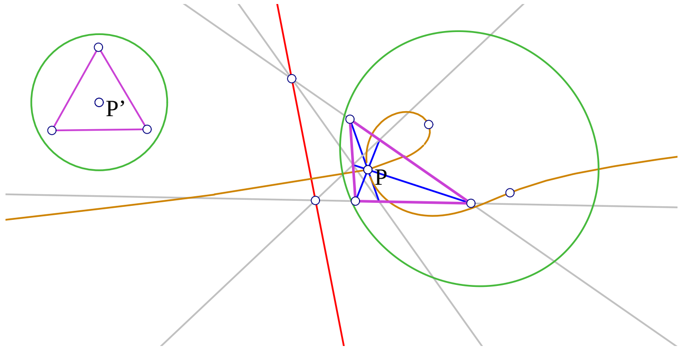

Figure 22. The focal cubic of the invariant family

into the play right from the beginning.
(7) The Brocard action is a singularium. It does not fit completely into the framework of circumconics and projectivities. As we have seen however, it has a close relationship to the Lemoine dihedral group. On Brocard Geometry there is an alternative exposition by John Conway [4], described in a letter to Hyacinthos .
(8) Finally a comment on the many figures used. They are produced with *EucliDraw*. This is a program, developed at the University of Crete, that does quickly the job of drawing interesting figures. It has many tools that do complicated jobs, reflecting the fact that it uses a conceptual granularity a bit wider than the very basic axioms. I am quite involved in its development and hope that other geometers will find it interesting, since it does quickly its job (sometimes even correctly), and new tools are continuously added. The program can be downloaded and tested from *www.euclidraw.com*.

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Generalized Mandart Conics

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Abstract. We consider interesting conics associated with the configuration of three points on the perpendiculars from a point P to the sidelines of a given triangle ABC, all equidistant from P. This generalizes the work of H. Mandart in 1894.

1. Mandart triangles

Let ABC be a given triangle and $A'B'C'$ its medial triangle. Denote by Δ, R, r the area, the circumradius, the inradius of ABC. For any $t \in \mathbb{R} \cup \{\infty\}$, consider the points P_a, P_b, P_c on the perpendicular bisectors of BC, CA, AB such that the signed distances verify $A'P_a = B'P_b = C'P_c = t$ with the following convention: for $t > 0$, P_a lies in the half-plane bounded by BC which does not contain A. We call $\mathbf{T}_t = P_aP_bP_c$ the t-*Mandart triangle* with respect to ABC. H. Mandart has studied in detail these triangles and associated conics ([5, 6]). We begin a modernized review with supplementary results, and identify the triangle centers in the notations of [4]. In the second part of this paper, we generalize the Mandart triangles and conics.

The vertices of the Mandart triangle $\mathbf{T}_t$, in homogeneous barycentric coordinates, are

$$\begin{aligned}
P_a =& -ta^2 : a\Delta + tS_C : a\Delta + tS_B,\\
P_b =& b\Delta + tS_C : -tb^2 : b\Delta + tS_A,\\
P_c =& c\Delta + tS_B : c\Delta + tS_A : -tc^2,
\end{aligned}$$

where

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

Proposition 1 ([6, §2]). *The points P_a, P_b, P_c are collinear if and only if $t^2+Rt+\frac{1}{2}Rr = 0$,* i.e.,

$$t = \frac{R \pm \sqrt{R^2-2Rr}}{2} = \frac{R \pm OI}{2}.$$

The two lines containing those collinear points are the parallels at X_{10} (Spieker center) to the asymptotes of the Feuerbach hyperbola.

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In other words, there are exactly two sets of collinear points on the perpendicular bisectors of ABC situated at the same (signed) distance from the sidelines of ABC. See Figure 1.

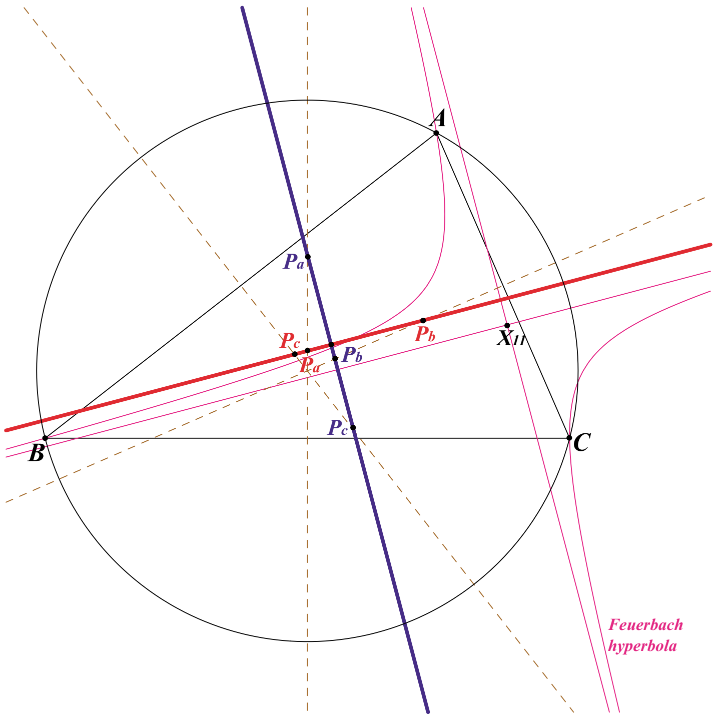

Figure 1. Collinear P_a, P_b, P_c

Proposition 2. *The triangles ABC and $P_aP_bP_c$ are perspective if and only if*
(1) $t = 0$: $P_aP_bP_c$ is the medial triangle, or
(2) $t = -r$: P_a, P_b, P_c are the projections of the incenter $I = X_1$ on the perpendicular bisectors.

In the latter case, P_a, P_b, P_c obviously lie on the circle with diameter IO. The two triangles are indirectly similar and their perspector is X_8 (Nagel point).

Remark. For any t, the triangle $Q_aQ_bQ_c$ bounded by the parallels at P_a, P_b, P_c to the sidelines BC, CA, AB is homothetic at I (incenter) to ABC.

Proposition 3. *The Mandart triangle $\mathbf{T}_t$ and the medial triangle $A'B'C'$ have the same area if and only if either :*
(1) $t = 0$: $\mathbf{T}_t$ is the medial triangle,
(2) $t = -R$,
(3) t is solution of: $t^2 + Rt + Rr = 0$.

This equation has two distinct (real) solutions when $R > 4r$, hence there are three Mandart triangles, distinct of $A'B'C'$, having the same area as $A'B'C'$. See Figure 2. In the very particular situation $R = 4r$, the equation gives the unique

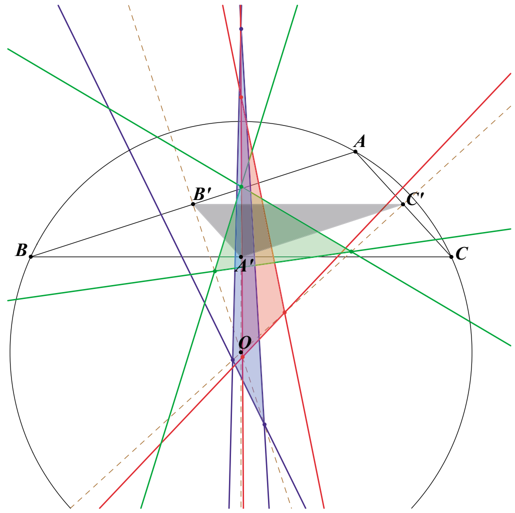

Figure 2. Three equal area triangles when $R > 4r$

solution $t = -2r = -\frac{R}{2}$ and we find only two such triangles. See Figure 3.

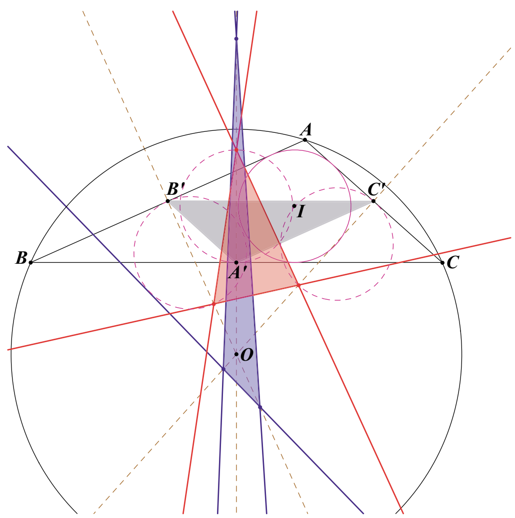

Figure 3. Only two equal area triangles when $R = 4r$

Proposition 4 ([5, §1])**.** *As t varies, the line P_bP_c envelopes a parabola $\mathcal{P}_a$.*

The parabola $\mathcal{P}_a$ is tangent to the perpendicular bisectors of AB and AC, to the line $B'C'$ and to the two lines met in proposition 1 above. Its focus F_a is the projection of O on the bisector AI. Its directrix ℓ_a is the bisector $A'X_{10}$ of the medial triangle. See Figure 4.

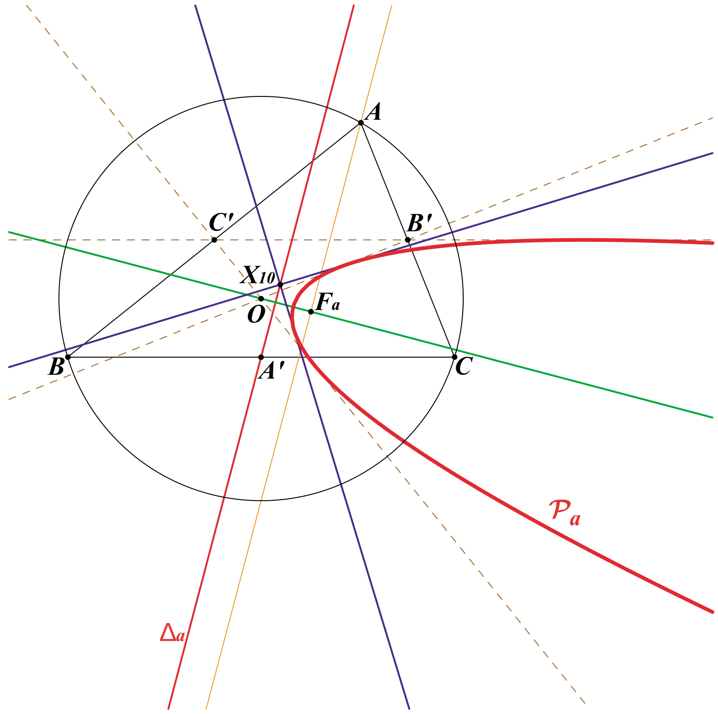

Figure 4. The parabola $\mathcal{P}_a$

Similarly, the lines P_cP_a and P_aP_b envelope parabolas $\mathcal{P}_b$ and $\mathcal{P}_c$ respectively. From this, we note the following.
(i) The foci of $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ lie on the circle with diameter OI.
(ii) The directrices concur at X_{10}.
(iii) The axes concur at O.
(iv) The contacts of the lines P_bP_c, P_cP_a, P_aP_b with $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ respectively are collinear. See Figure 5.

These three parabolas are generally not in the same pencil of conics since their jacobian is the union of the perpendicular at O to the line IX_{10} and the circle centered at X_{10} having the same radius as the Fuhrmann circle: the polar lines of any point on this circle in the parabolas concur on the line and conversely.

2. Mandart conics

Proposition 5 ([6, §7]). *The Mandart triangle* $\mathbf{T}_t$ *and the medial triangle are perspective at* O. *As* t *varies, the perspectrix envelopes the parabola* $\mathcal{P}_M$ *with focus* X_{124} *and directrix* X_3X_{10}.

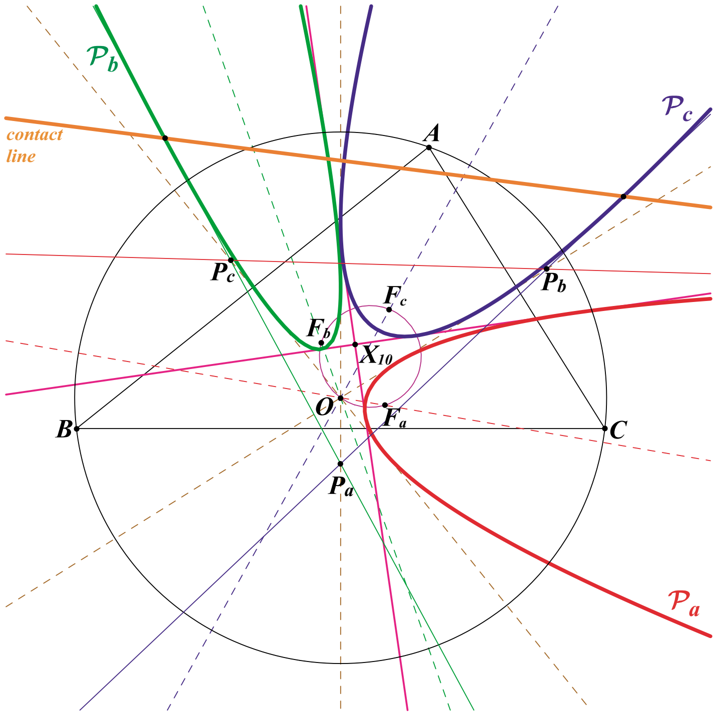

Figure 5. The three parabolas $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$

We call $\mathcal{P}_M$ the *Mandart parabola*. It has equation

$$\sum_{\text{cyclic}} \frac{x^2}{(b-c)(b+c-a)} = 0.$$

Triangle ABC is clearly self-polar with respect to $\mathcal{P}_M$. The directrix is the line X_3X_{10} and the focus is X_{124}. $\mathcal{P}_M$ is inscribed in the medial triangle with perspector

$$X_{1146} = ((b-c)^2(b+c-a)^2 : \cdots : \cdots),$$

the center of the circum-hyperbola passing through G and X_8 with respect to this triangle. The contacts of $\mathcal{P}_M$ with the sidelines of the medial triangle lie on the perpendiculars dropped from A, B, C to the directrix X_3X_{10}. $\mathcal{P}_M$ is the complement of the inscribed parabola with focus X_{109} and directrix the line IH. See Figure 6.

Proposition 6 ([5, 2, p.551])**.** *The Mandart triangle* $\mathbf{T}_t$ *and* ABC *are orthologic. The perpendiculars from* A, B, C *to the corresponding sidelines of* $P_aP_bP_c$ *are concurrent at*

$$Q_t = \left(\frac{a}{aS_A + 4\Delta t} : \cdots : \cdots\right).$$

As t *varies, the locus of* Q_t *is the Feuerbach hyperbola.*

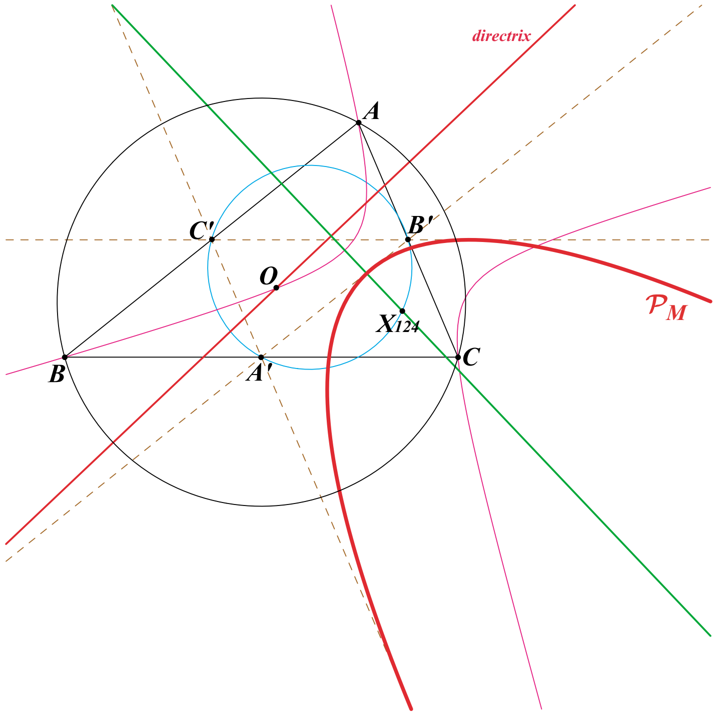

Figure 6. The Mandart parabola

Remark. The triangles $A'B'C'$ and $\mathbf{T}_t$ are also orthologic at Q'_t, the complement of Q_t.

Denote by $A_1B_1C_1$ the extouch triangle (see [3, p.158, §6.9]), *i.e.*, the cevian triangle of X_8 (Nagel point) or equivalently the pedal triangle of X_{40} (reflection of I in O). The circumcircle $\mathcal{C}_M$ of $A_1B_1C_1$ is called *Mandart circle*. $\mathcal{C}_M$ is therefore the pedal circle of X_{40} and X_{84} (isogonal conjugate of X_{40}), the cevian circumcircle of X_{189} (cyclocevian conjugate of X_8). $\mathcal{C}_M$ contains the Feuerbach point X_{11}. Its center is X_{1158}, intersection of the lines X_1X_{104} and X_8X_{40}. The second intersection with the incircle is X_{1364} and the second intersection with the nine-point circle is the complement of X_{934}. See Figure 7. The *Mandart ellipse* $\mathcal{E}_M$ (see [6, §§3,4]) is the inscribed ellipse with center X_9 (Mittenpunkt) and perspector X_8. It contains A_1, B_1, C_1, X_{11} and its axes are parallel to the asymptotes of the Feuerbach hyperbola. See Figure 7.

The equation of $\mathcal{E}_M$ is:

$$\sum_{\text{cyclic}} (c+a-b)^2(a+b-c)^2x^2 - 2(b+c-a)^2(c+a-b)(a+b-c)yz = 0$$

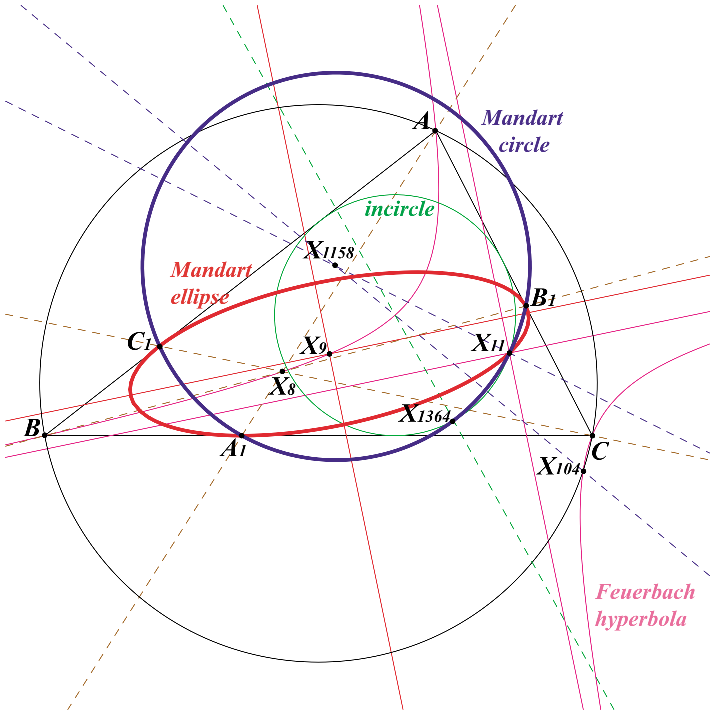

Figure 7. The Mandart circle and the Mandart ellipse

From this, we see that $\mathcal{C}_M$ is the Joachimsthal circle of X_{40} with respect to $\mathcal{E}_M$: the four normals drawn from X_{40} to $\mathcal{E}_M$ pass through A_1, B_1, C_1 and

$$F' = ((b+c-a)((b-c)^2 + a(b+c-2a))^2 : \cdots : \cdots),$$

the reflection X_{11} in X_9. [1]

The radical axis of $\mathcal{C}_M$ and the nine-point circle is the tangent at X_{11} to $\mathcal{E}_M$ and also the polar line of G in $\mathcal{P}_M$. The projection of X_9 on this tangent is the point X_{1364} we met above. Hence, $\mathcal{C}_M$, the nine-point circle and the circle with diameter X_9X_{11} belong to the same pencil of (coaxal) circles ([6, §§8,9]).

The radical axis of $\mathcal{C}_M$ and the incircle is the polar line of X_{10} in $\mathcal{P}_M$.

Proposition 7. [6, §§1,2] *The Mandart triangle $\mathbf{T}_t$ and the extouch triangle are orthologic. The perpendiculars drawn from A_1, B_1, C_1 to the corresponding sidelines of $\mathbf{T}_t = P_aP_bP_c$ are concurrent at S. As t varies, the locus of S is the rectangular hyperbola $\mathcal{H}_M$ passing through the traces of X_8 and $X_{190} = \left(\frac{1}{b-c} : \frac{1}{c-a} : \frac{1}{a-b}\right)$*

We call $\mathcal{H}_M$ the *Mandart hyperbola*. It has equation

$$\sum_{\text{cyclic}} (b-c)\left[(c+a-b)(a+b-c)x^2 + (b+c-a)^2 yz\right] = 0$$

[1] This point is not in the current edition of [4].

and contains the triangle centers X_8, X_9, X_{40}, X_{72}, X_{144}, X_{1145}, F', and F'' antipode of X_{11} on $\mathcal{C}_M$. Its asymptotes are parallel to those of the Feuerbach hyperbola. $\mathcal{H}_M$ is the Apollonian hyperbola of X_{40} with respect to $\mathcal{E}_M$. See Figure 8.

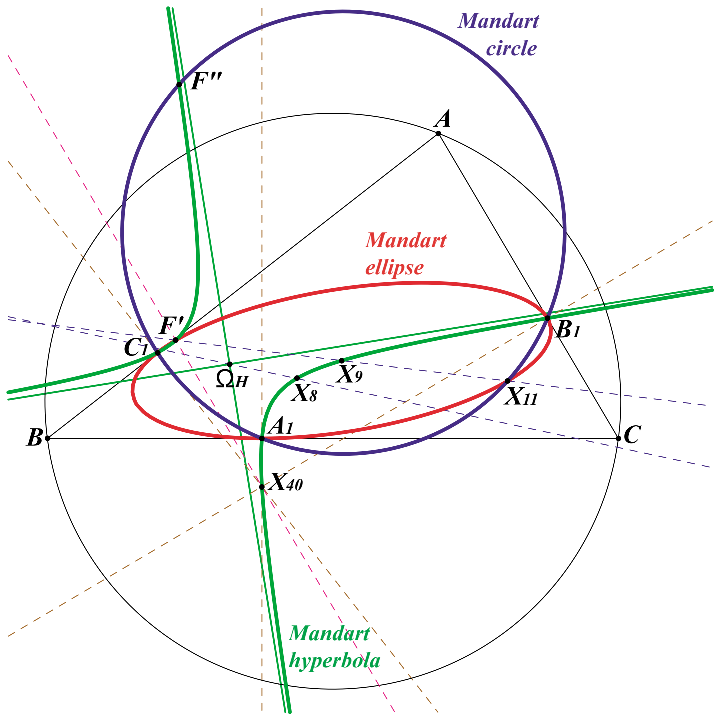

Figure 8. The Mandart hyperbola

3. Locus of some triangle centers in the Mandart triangles

We now examine the locus of some triangle centers of $\mathbf{T}_t = P_aP_bP_c$ when t varies. We shall consider the centroid, circumcenter, orthocenter, and Lemoine point.

Proposition 8. *The locus of the centroid of $\mathbf{T}_t$ is the parallel at G to the line OI.*

Proposition 9. *The locus of the circumcenter of $\mathbf{T}_t$ is the rectangular hyperbola passing through X_1, X_5, X_{10}, X_{21} (Schiffler point) and X_{1385}.* [2]

The equation of the hyperbola is

$$\sum_{\text{cyclic}} (b-c)\left[bc(b+c)x^2 + a(b^2+c^2-a^2+3bc)yz\right] = 0.$$

[2] X_{1385} is the midpoint of OI.

It has center X_{1125} (midpoint of IX_{10}) and asymptotes parallel to those of the Feuerbach hyperbola.

The locus of the orthocenter of $\mathbf{T}_t$ is a nodal cubic with node X_{10} passing through O, X_{1385}, meeting the line at infinity at X_{517} and the infinite points of the Feuerbach hyperbola. The line through the orthocenters of the t-Mandart triangle and the $(-t)$-Mandart triangle passes through a fixed point.

The locus of the Lemoine point of $\mathbf{T}_t$ is another nodal cubic with node X_{10}.

4. Generalized Mandart conics

Most of the results above can be generalized when X_8 is replaced by any point M on the Lucas cubic, the isotomic cubic with pivot X_{69}. The cevian triangle of such a point M is the pedal triangle of a point N on the Darboux cubic, the isogonal cubic with pivot the de Longchamps point X_{20}. [3]

For example, with $M = X_8$, we find $N = X_{40}$ and $M' = X_1 = I$.

Denote by $M_aM_bM_c$ the cevian triangle of M (on the Lucas cubic) and the pedal triangle of N (on the Darboux cubic). N^* is the isogonal conjugate of N also on the Darboux cubic. We now consider

– γ_M, inscribed conic in ABC with perspector M and center ω_M, which is the complement of the isotomic conjugate of M. It lies on the Thomson cubic and on the line KM' ($K = X_6$ is the Lemoine point),

– Γ_M, circumcircle of $M_aM_bM_c$ with center Ω_M, midpoint of NN^*. Γ_M is obviously the pedal circle of N and N^* and also the cevian circle of M°, cyclocevian conjugate of M (see [3, p.226, §8.12]). M° is a point on the Lucas cubic since this cubic is invariant under cyclocevian conjugation.

Since γ_M and Γ_M have already three points in common, they must have a fourth (always real) common point Z. Finally, denote by Z' the reflection of Z in ω_M. See Figure 9.

Table 1 gives examples for several known centers M on the Lucas cubic.[4] Those marked with $*$ are indicated in Table 2; those marked with ? are too complicated to give here.

Table 1

M	X_8	X_2	X_4	X_7	X_{20}	X_{69}	X_{189}	X_{253}	X_{329}	X_{1032}	X_{1034}
N	X_{40}	X_3	X_4	X_1	X_{1498}	X_{20}	X_{84}	X_{64}	X_{1490}	$*$	$*$
M'	X_1	X_2	X_3	X_9	X_4	X_6	X_{223}	X_{1249}	X_{57}	$*$	$*$
N^*	X_{84}	X_4	X_3	X_1	$*$	X_{64}	X_{40}	X_{20}	$*$	X_{1498}	X_{1490}
M°	X_{189}	X_4	X_2	X_7	X_{1032}	X_{253}	X_8	X_{69}	X_{1034}	X_{20}	X_{329}
ω_M	X_9	X_2	X_6	X_1	X_{1249}	X_3	X_{57}	X_4	X_{223}	X_{1073}	X_{282}
Ω_M	X_{1158}	X_5	X_5	X_1	?	?	X_{1158}	?	?	?	?
Z	X_{11}	X_{115}	X_{125}	X_{11}	X_{122}	X_{125}	$*$	X_{122}	$*$	?	$*$
Z'	$*$	$*$	$*$	X_{1317}	$*$	$*$	$*$	$*$	$*$	?	$*$

[3] It is also known that the complement of M is a point M' on the the Thomson cubic, the isogonal cubic with pivot $G = X_2$, the centroid.

[4] Two isotomic conjugates on the Lucas cubic are associated to the same point Z on the nine-point circle.

Table 2

Triangle center	First barycentric coordinate
$Z'(X_8)$	$(b+c-a)(2a^2-a(b+c)-(b-c)^2)^2$
$Z'(X_2)$	$(2a^2-b^2-c^2)^2$
$Z'(X_4)$	$\frac{(2a^2-b^2-c^2)^2}{S_A}$
$Z'(X_{20})$	$((3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2)$ $\cdot(2a^8-a^6(b^2+c^2)-5a^4(b^2-c^2)^2+5a^2(b^2-c^2)^2(b^2+c^2)$ $-(b^2-c^2)^2(b^4+6b^2c^2+c^4))^2$
$Z'(X_{69})$	$S_A(2a^4-a^2(b^2+c^2)-(b^2-c^2)^2)^2$
$Z(X_{189})$	$(b-c)^2(b+c-a)^2(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2$
$Z'(X_{189})$	$\frac{(2a^2-a(b+c)-(b-c)^2)^2}{a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2}$
$Z'(X_{253})$	$\frac{(2a^4-a^2(b^2+c^2)-(b^2-c^2)^2)^2}{3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2}$
$Z(X_{329})$	$(b-c)^2(b+c-a)^2(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2$
$Z'(X_{329})$	$(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)$ $\cdot(2a^5-a^4(b+c)-4a^3(b-c)^2+2a^2(b-c)^2(b+c)$ $+2a(b-c)^2(b^2+c^2)-(b-c)^2(b+c)^3)^2$
$N^*(X_{20})$	$1/(a^8-4a^6(b^2+c^2)+2a^4(3b^4-2b^2c^2+3c^4)$ $-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4))$
$N^*(X_{329})$	$a/(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b+c)(b-c)^2(b^2+c^2)+(b-c)^2(b+c)^4)$
$N(X_{1032})$	$1/(a^8-4a^6(b^2+c^2)+2a^4(3b^4-2b^2c^2+3c^4)$ $-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4))$
$M'(X_{1032})$	$(a^2(a^8-4a^6(b^2+c^2)+2a^4(3b^4-2b^2c^2+3c^4)$ $-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4))/$
	$(3a^4-2a^2(b^2+c^2)-(b^2-c^2)^2)$
$N(X_{1034})$	$a/(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$
	$-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)$
$M'(X_{1034})$	$a(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)/$ $(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)$
$Z(X_{1034})$	$(b-c)^2(b+c-a)(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)^2$ $\cdot(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4))$
$Z'(X_{1034})$	$(b+c-a)(2a^5-a^4(b+c)-4a^3(b-c)^2+2a^2(b-c)^2(b+c)$ $+2a(b-c)^2(b^2+c^2)-(b^2-c^2)^3)^2)/(a^6-2a^5(b+c)-a^4(b+c)^2$ $+4a^3(b+c)(b^2-bc+c^2)-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)$ $+(b-c)^2(b+c)^4)$
$M'(X_{1034})$	$a(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)/$ $(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)$
$Z(X_{1034})$	$(b-c)^2(b+c-a)(a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2)^2$ $\cdot(a^6-2a^5(b+c)-a^4(b+c)^2+4a^3(b+c)(b^2-bc+c^2)$ $-a^2(b^2-c^2)^2-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4))$
$Z'(X_{1034})$	$(b+c-a)(2a^5-a^4(b+c)-4a^3(b-c)^2+2a^2(b-c)^2(b+c)$ $+2a(b-c)^2(b^2+c^2)-(b^2-c^2)^3)^2)/(a^6-2a^5(b+c)-a^4(b+c)^2$ $+4a^3(b+c)(b^2-bc+c^2)-a^2(b^2-c^2)^2$ $-2a(b-c)^2(b+c)(b^2+c^2)+(b-c)^2(b+c)^4)$

Proposition 10. *Z is a point on the nine-point circle and Z' is the foot of the fourth normal drawn from N to γ_M.*

Proof. The lines NM_a, NM_b, NM_c are indeed already three such normals hence Γ_M is the Joachimsthal circle of N with respect to γ_M. This yields that Γ_M must pass through the reflection in ω_M of the foot of the fourth normal. See Figure 9. □

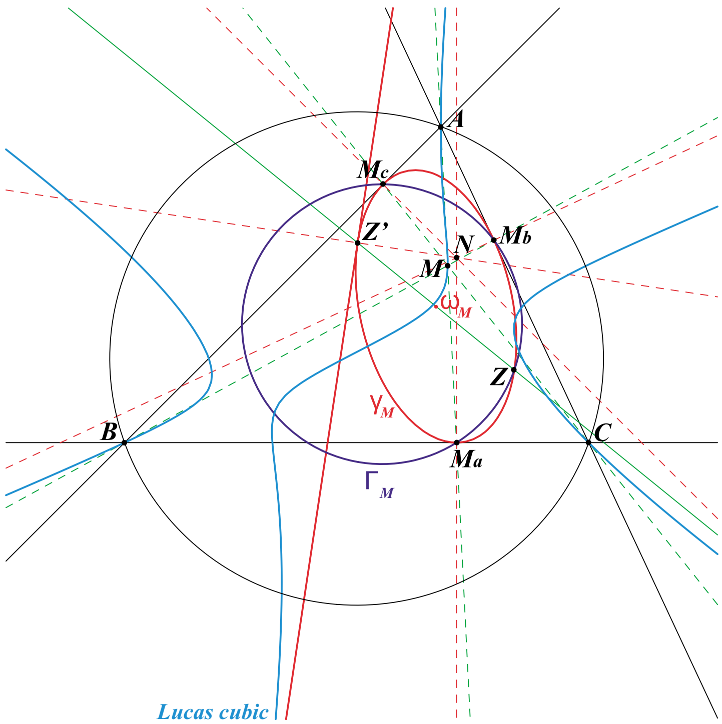

Figure 9. The generalized Mandart circle and conic

Remark. Z also lies on the cevian circumcircle of $M^\#$ isotomic conjugate of M and on the inscribed conic with perspector $M^\#$ and center M'.

Proposition 11. *The points M_a, M_b, M_c, M, N, ω_M and Z' lie on a same rectangular hyperbola whose asymptotes are parallel to the axes of γ_M.*

Proof. This hyperbola is the Apollonian hyperbola of N with respect to γ_M. □

Proposition 12. *The rectangular hyperbola passing through A, B, C, H and M is centered at Z. It also contains M', N^*, ω_M and $M^\#$. Its asymptotes are also parallel to the axes of γ_M.*

Remark. This hyperbola is the isogonal transform of the line ON and the isotomic transform of the line $X_{69}M$.

5. Generalized Mandart triangles

We now replace the circumcenter O by any finite point $P = (u : v : w)$ not lying on one sideline of ABC and we still call $A'B'C'$ its pedal triangle. For $t \in \mathbb{R} \cup \{\infty\}$, consider P_a, P_b, P_c defined as follows: draw three parallels to BC, CA, AB at the (signed) distance t with the conventions at the beginning of the paper. P_a, P_b, P_c are the projections of P on these parallels. See Figure 10.

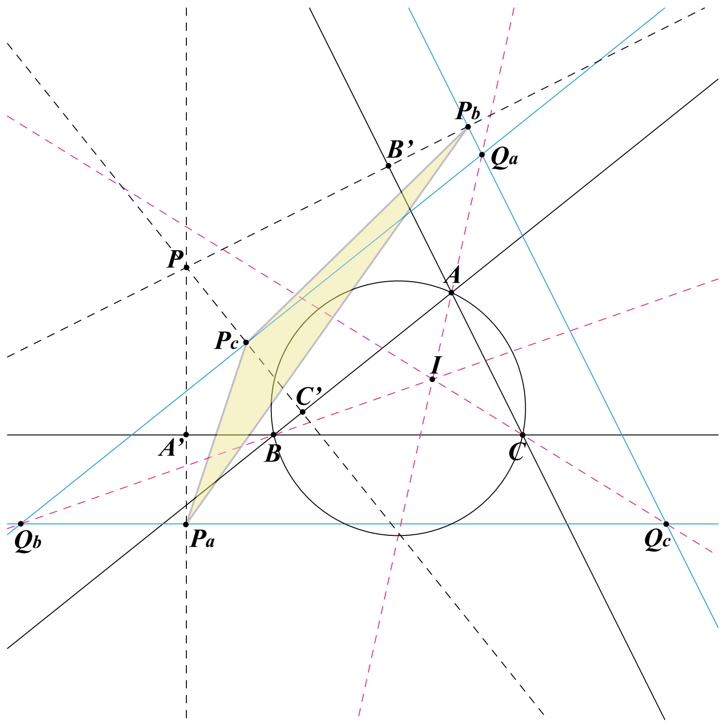

Figure 10. Generalized Mandart triangle

In homogeneous barycentric coordinates, these are the points

$$\begin{aligned}
P_a =& -a^3t : 2\Delta \cdot \frac{S_C u + a^2 v}{u+v+w} + taS_C : 2\Delta \cdot \frac{S_B u + a^2 w}{u+v+w} + taS_B,\\
P_b =& 2\Delta \cdot \frac{S_C v + b^2 u}{u+v+w} + tbS_C : -b^3t : 2\Delta \cdot \frac{S_A v + b^2 w}{u+v+w} + tbS_A,\\
P_c =& 2\Delta \cdot \frac{S_B w + c^2 u}{u+v+w} + tcS_B : 2\Delta \cdot \frac{S_A w + c^2 v}{u+v+w} + tcS_A : -c^3t.
\end{aligned}$$

The triangle $\mathbf{T}_t(P) = P_aP_bP_c$ is called $t-$Mandart triangle of P.

Proposition 13. *For any P distinct from the incenter I, there are always two sets of collinear points P_a, P_b, P_c. The two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ containing the points are* *parallel to the asymptotes of the hyperbola which is the isogonal conjugate of the parallel to IP at X_{40}*[5]*. They meet at the point :*

$$(a((b+c)bcu + cS_C v + bS_B w) : \cdots : \cdots).$$

They are perpendicular if and only if P lies on OI.

Proof. P_a, P_b, P_c are collinear if and only if t is solution of the equation :

$$abc(a+b+c)t^2 + 2\Delta\,\Phi_1(u,v,w)\,t + 4\Delta^2\,\Phi_2(u,v,w) = 0 \qquad (1)$$

where

$$\Phi_1(u,v,w) = \sum_{\text{cyclic}} bc(b+c)u \quad \text{and} \quad \Phi_2(u,v,w) = \sum_{\text{cyclic}} a^2vw.$$

We notice that $\Phi_1(u,v,w) = 0$ if and only if P lies on the polar line of I in the circumcircle and $\Phi_2(u,v,w) = 0$ if and only if P lies on the circumcircle.

The discriminant of (1) is non-negative for all P and null if and only if $P = I$. In this latter case, the points P_a, P_b, P_c are "collinear" if and only if they all coincide with I.

Considering now $P \neq I$, (1) always has two (real) solutions. □

Figure 11 shows the case $P = H$ with two (non-perpendicular) lines secant at X_{65} orthocenter of the intouch triangle.

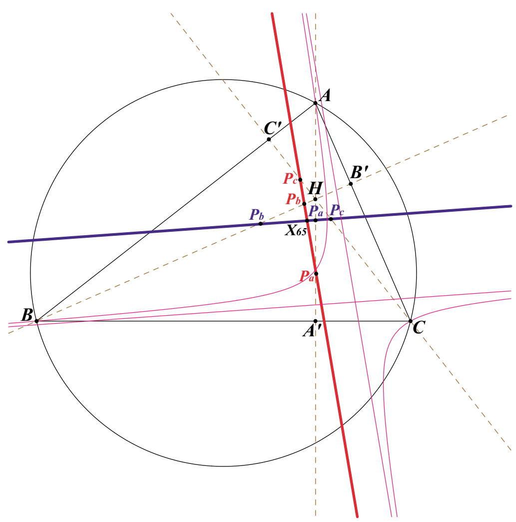

Figure 11. Collinear P_a, P_b, P_c with $P = H$

Figure 12 shows the case $P = X_{40}$ with two perpendicular lines secant at X_8 and parallel to the asymptotes of the Feuerbach hyperbola.

When P is a point on the circumcircle, equation (1) has a solution $t = 0$ and one of the two lines, say $\mathcal{L}_1$, is the Simson line of P: the triangle $A'B'C'$ degenerates

[5] X_{40} is the reflection of I in O.

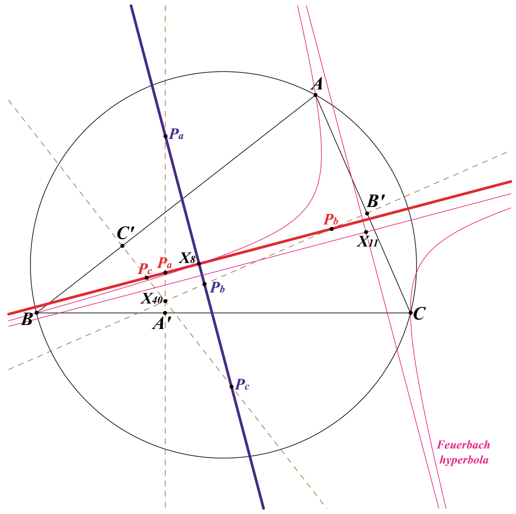

Figure 12. Collinear P_a, P_b, P_c with $P = X_{40}$

into this Simson line. $\mathcal{L}_1$ and $\mathcal{L}_2$ meet on the ellipse centered at X_{10} passing through X_{11}, the midpoints of ABC and the feet of the cevians of X_8. This ellipse is the complement of the circum-ellipse centered at I and has equation :

$$\sum_{\text{cyclic}} (a+b-c)(a-b+c)x^2 - 2a(b+c-a)yz = 0.$$

Figure 13 shows the case $P = X_{104}$ with two lines secant at X_{11}, one of them being the Simson line of X_{104}.

Following equation (1) again, we observe that, when P lies on the polar line of I in the circumcircle, we find to opposite values for t: the two corresponding points P_a are symmetric with respect to the sideline BC, P_b and P_c similarly. The most interesting situation is obtained with $P = X_{36}$ (inversive image of I in the circumcircle) since we find two perpendicular lines $\mathcal{L}_1$ and $\mathcal{L}_2$, parallel to the asymptotes of the Feuerbach hyperbola, intersecting at the midpoint of $X_{36}X_{80}$[6]. See Figure 14.

Construction of $\mathcal{L}_1$ and $\mathcal{L}_2$: the line IP[7] meets the circumcircle at S_1 and S_2. The parallels at P to OS_1 and OS_2 meet OI at T_1 and T_2. The homotheties with center I which map O to T_1 and T_2 also map the triangle ABC to the triangles $A_1B_1C_1$ and $A_2B_2C_2$. The perpendiculars PA', PB', PC' at P to the sidelines of ABC meet the corresponding sidelines of $A_1B_1C_1$ and $A_2B_2C_2$ at the requested points.

[6] X_{80} is the isogonal conjugate of X_{36}.

[7] We suppose $I \neq P$.

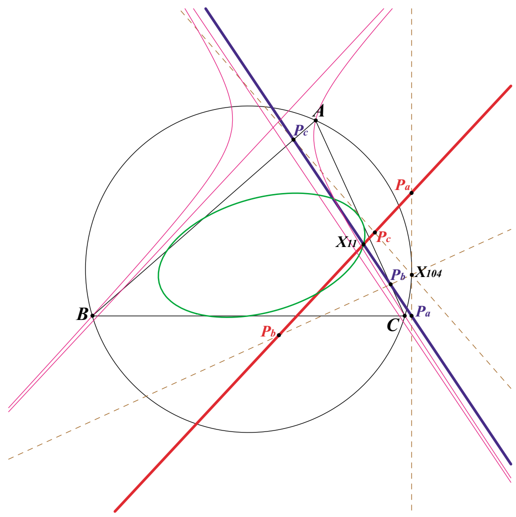

Figure 13. Collinear P_a, P_b, P_c with $P = X_{104}$

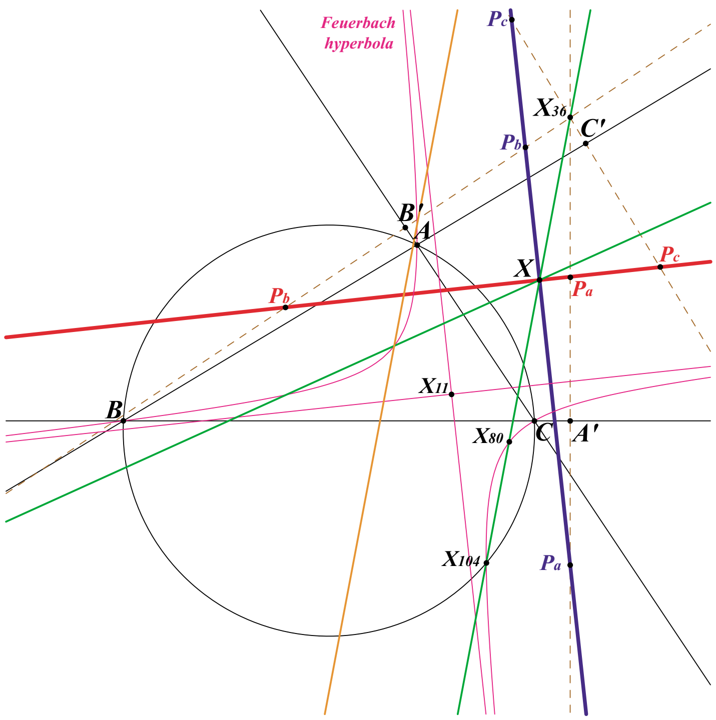

Figure 14. Collinear P_a, P_b, P_c with $P = X_{36}$

Proposition 14. *The triangles ABC and $P_aP_bP_c$ are perspective if and only if k is solution of :*

$$\Psi_2(u,v,w)\,t^2 + \Psi_1(u,v,w)\,t + \Psi_0(u,v,w) = 0 \tag{2}$$

where :

$$\begin{aligned}
\Psi_2(u,v,w) =& -\frac{1}{2}abc(a+b+c)(u+v+w)^2 \sum_{\text{cyclic}} (b-c)(b+c-a)S_A u,\\
\Psi_1(u,v,w) =& \frac{1}{2}(a+b+c)(u+v+w)\Delta \sum_{\text{cyclic}} \left(-2bc(b-c)(b+c-a)S_A u^2\right.\\
&\left. + a^2(b-c)(a+b+c)(b+c-a)^2 vw\right),\\
\Psi_0(u,v,w) =& \Delta^2 \sum_{\text{cyclic}} (3a^4 - 2a^2(b^2+c^2) - (b^2-c^2)^2)u(c^2v^2 - b^2w^2).
\end{aligned}$$

Remarks. (1) $\Psi_2(u,v,w) = 0$ if and only if P lies on the line IH.

(2) $\Psi_1(u,v,w) = 0$ if and only if P lies on the hyperbola passing through I, H, X_{500}, X_{573}, X_{1742} [8] and having the same asymptotic directions as the isogonal transform of the line $X_{40}X_{758}$, *i.e.*, the reflection in O of the line X_1X_{21}.

(3) $\Psi_0(u,v,w) = 0$ if and only if P lies on the Darboux cubic. See Figure 15.

The equation (2) is clearly realized for all t if and only if $P = I$ or $P = H$: all $t-$Mandart triangles of I and H are perspective to ABC. Furthermore, if $P = H$ the perspector is always H, and if $P = I$ the perspector lies on the Feuerbach hyperbola. In the sequel, we exclude those two points and see that there are at most two real numbers t_1 and t_2 for which t_1- and t_2-Mandart triangles of P are perspective to ABC. Let us denote by R_1 and R_2 the (not always real) corresponding perspectors.

We explain the construction of these two perspectors with the help of several lemmas.

Lemma 15. *For a given P and a corresponding Mandart triangle $\mathbf{T}_t(P) = P_aP_bP_c$, the locus of $R_a = BP_b \cap CP_c$, when t varies, is a conic γ_a.*

Proof. The correspondence on the pencils of lines with poles B and C mapping the lines BP_b and CP_c is clearly an involution. Hence, the common point of the two lines must lie on a conic. □

This conic γ_a obviously contains B, C, H, $S_a = BB' \cap CC'$ and two other points B_1 on AB, C_1 on AC defined as follows. Reflect $AB \cap PB'$ in the bisector AI to get a point B_2 on AC. The parallel to AB at B_2 meets PC' at B_3. B_1 is the intersection of AB and CB_3. The point C_1 on AC is constructed similarly. See Figure 16.

Lemma 16. *The three conics γ_a, γ_b, γ_c have three points in common: H and the (not always real) sought perspectors R_1 and R_2. Their jacobian must degenerate*

[8] $X_{500} = X_1X_{30} \cap X_3X_6$, $X_{573} = X_4X_9 \cap X_3X_6$ and $X_{1742} = X_1X_7 \cap X_3X_{238}$.

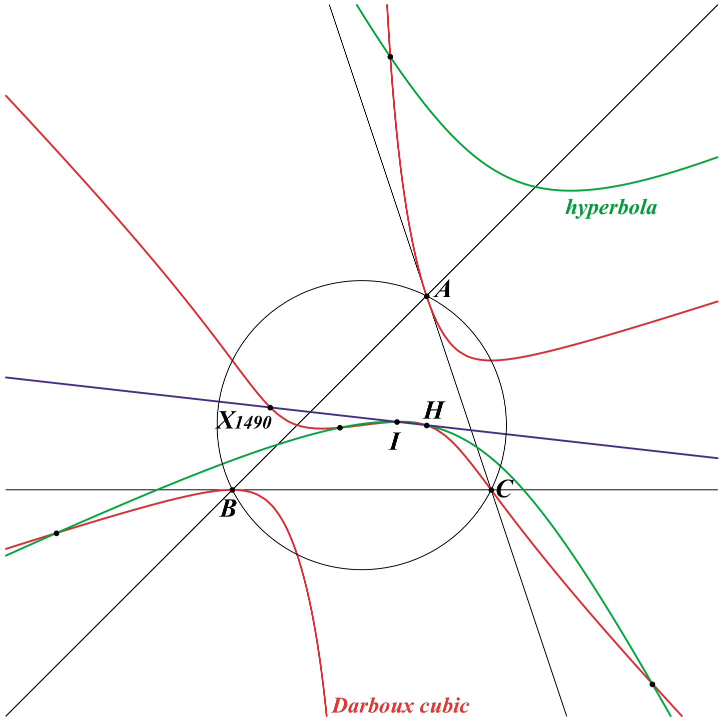

Figure 15. Proposition 14

into three lines, one always real $\mathcal{L}_P$ containing R_1 and R_2, two other passing through H.

Lemma 17. *$\mathcal{L}_P$ contains the Nagel point X_8. In other words, X_8, R_1 and R_2 are always collinear.*

With $P = (u : v : w)$, $\mathcal{L}_P$ has equation :

$$\sum_{\text{cyclic}} \frac{a(cv - bw)}{b + c - a}\, x = 0$$

$\mathcal{L}_P$ is the trilinear polar of the isotomic conjugate of point T, where T is the barycentric product of X_{57} and the isotomic conjugate of the trilinear pole of the line PI. The construction of R_1 and R_2 is now possible in the most general case with one of the conics and $\mathcal{L}_P$. Nevertheless, in three specific situations already mentioned, the construction simplifies as we see in the three following corollaries.

Corollary 18. *When P lies on IH, there is only one* (*always real*) *Mandart triangle $\mathbf{T}_t(P)$ perspective to ABC. The perspector R is the intersection of the lines HX_8 and PX_{78}.*

Proof. This is obvious since equation (2) is at most of the first degree when P lies on IH. □

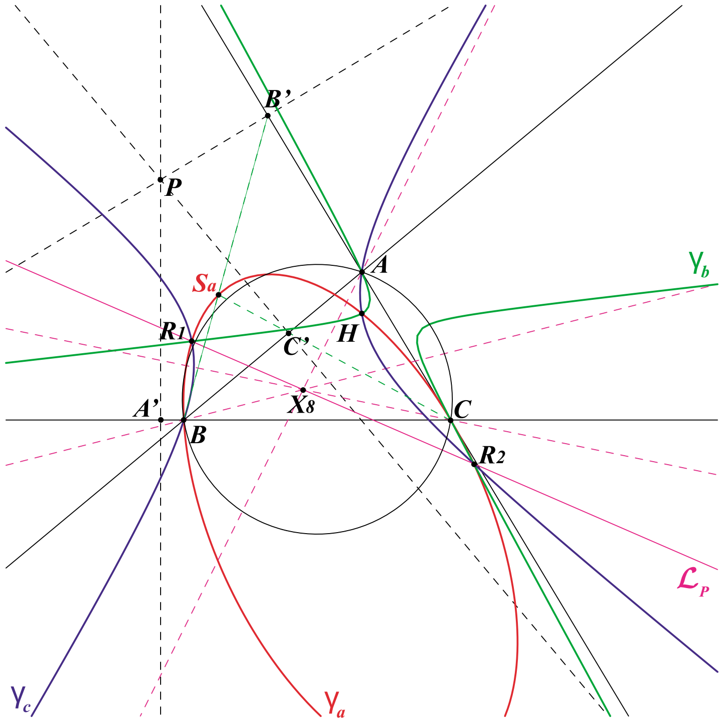

Figure 16. The three conics γ_a, γ_b, γ_c and the perspectors R_1, R_2

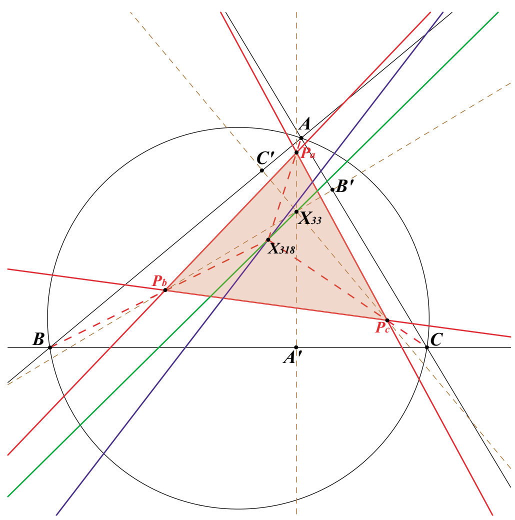

Figure 17. Only one triangle $P_aP_bP_c$ perspective to ABC when P lies on IH

In Figure 17, we have taken $P = X_{33}$ and $R = X_{318}$.

Remark. The line IH meets the Darboux cubic again at X_{1490}. The corresponding Mandart triangle $\mathbf{T}_t(P)$ is the pedal triangle of X_{1490} which is also the cevian triangle of X_{329}.

Corollary 19. *When P (different from I and H) lies on the conic seen above, there are two (not always real) Mandart triangles $\mathbf{T}_t(P)$ perspective to ABC obtained for two opposite values t_1 and t_2. The vertices of the triangles are therefore two by two symmetric in the sidelines of ABC.*

In the figure 18, we have taken $P = X_{500}$ (orthocenter of the incentral triangle).

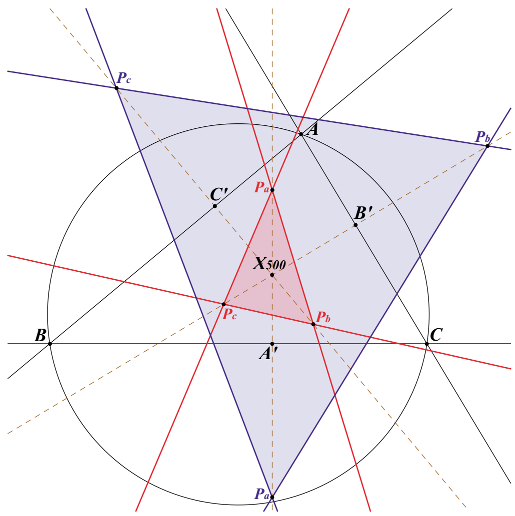

Figure 18. Two triangles $P_aP_bP_c$ perspective with ABC having vertices symmetric in the sidelines of ABC

Corollary 20. *When P (different from I, H, X_{1490}) lies on the Darboux cubic, there are two (always real) Mandart triangles $\mathbf{T}_t(P)$ perspective to ABC, one of them being the pedal triangle of P with a perspector on the Lucas cubic.*

Since one perspector, say R_1, is known, the construction of the other is simple: it is the "second" intersection of the line X_8R_1 with the conic $BCHS_aR_1$.

Table 3 gives P (on the Darboux cubic), the corresponding perspectors R_1 (on the Lucas cubic) and R_2.

Table 3

P	X_1	X_3	X_4	X_{20}	X_{40}	X_{64}	X_{84}	X_{1498}
R_1	X_7	X_2	X_4	X_{69}	X_8	X_{253}	X_{189}	X_{20}
R_2		X_8	X_4	X_{388}	X_{10}	$*$	X_{515}	$*$

Table 4

Triangle center	First barycentric coordinate
$R_2(X_{64})$	$\frac{a^8-4a^6(b+c)^2+2a^4(b+c)^2(3b^2-4bc+3c^2)-4a^2(b^2-c^2)^2(b^2+c^2)+(b-c)^2(b+c)^6}{b+c-a}$
$R_2(X_{1498})$	$\frac{a^4-2a^2(b+c)^2+(b^2-c^2)^2)}{a^3+a^2(b+c)-a(b+c)^2-(b+c)(b-c)^2}$

In Figure 19, we have taken $P = X_{40}$ (reflection of I in O).

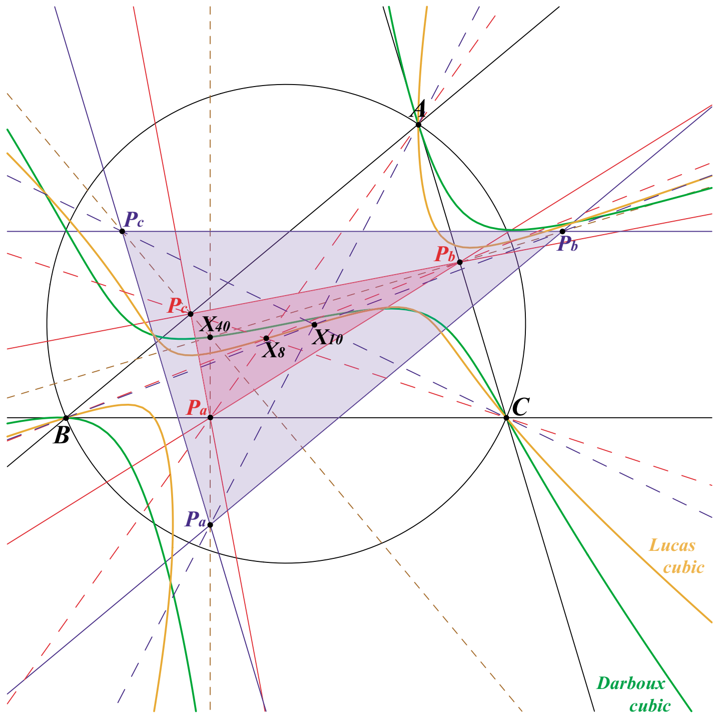

Figure 19. Two triangles $P_aP_bP_c$ perspective with ABC when $P = X_{40}$

Proposition 21. *The triangles $A'B'C'$ and $P_aP_bP_c$ have the same area if and only if*
(1) $t = 0$, or
(2) $t = -\frac{bc(b+c)u+ca(c+a)v+ab(a+b)w}{2R(a+b+c)(u+v+w)}$, [9]
(3) t is a solution of a quadratic equation [10] *whose discriminant has the same sign of*

$$f(u, v, w) = \sum_{\text{cyclic}} b^2c^2(b+c)^2u^2 + 2a^2bc(bc - 3a(a+b+c))vw.$$

---

[9] This can be interpreted as $t = -\frac{d(P)}{d(O)} \cdot R$, where $d(X)$ denotes the distance from X to the polar line of I in the circumcircle.

[10] $abc(a+b+c)(u+v+w)^2t^2 + 2\Delta(u+v+w)\left(\sum_{\text{cyclic}} bc(b+c)u\right)t + 8\Delta^2(a^2vw + b^2wu + c^2uv) = 0.$

The equation $f(x, y, z) = 0$ represents an ellipse $\mathcal{E}$ centered at X_{35} [11] whose axes are parallel and perpendicular to the line OI. See Figure 20.

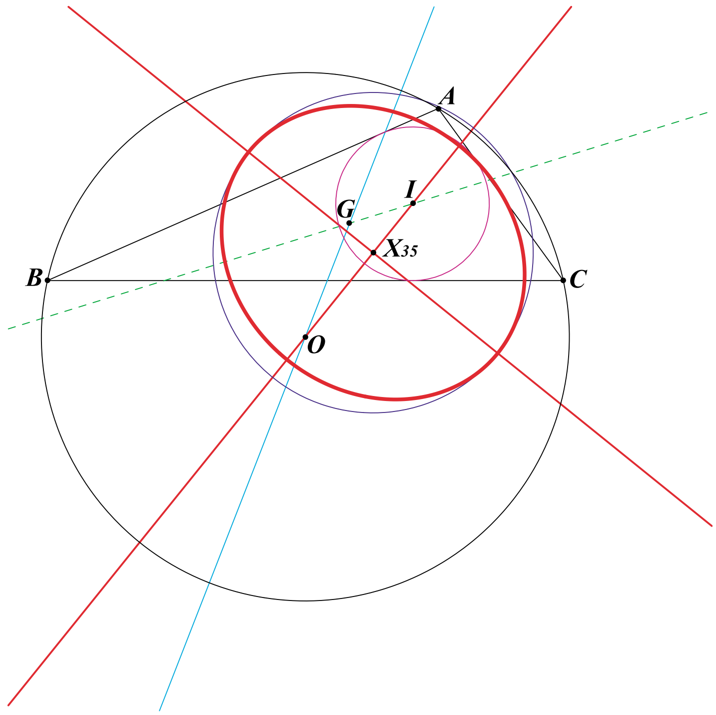

Figure 20. The "critical" ellipse $\mathcal{E}$

According to the position of P with respect to this ellipse, it is possible to have other triangles solution of the problem. More precisely, if P is
– inside $\mathcal{E}$, there is no other triangle,
– outside $\mathcal{E}$, there are two other (distinct) triangles,
– on $\mathcal{E}$, there is only one other triangle.

Proposition 22. *As t varies, each line P_bP_c, P_cP_a, P_aP_b still envelopes a parabola.*

Denote these parabolas by $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ respectively. $\mathcal{P}_a$ has focus the projection F_a of P on AI and directrix ℓ_a parallel to AI at E_a such that $\overrightarrow{PE_a} = \cos A\ \overrightarrow{PF_a}$. Note that the direction of the directrix (and the axis) is independent of P. $\mathcal{P}_a$ is still tangent to the lines PB', PC', $B'C'$.

In this more general case, the directrices ℓ_a, ℓ_b, ℓ_c are not necessarily concurrent. This happens if and only if P lies on the line OI and, then, their common point lies on IG.

Proposition 23. *The Mandart triangle $\mathbf{T}_t(P)$ and the pedal triangle of P are perspective at P. As t varies, the envelope of their perspectrix is a parabola.*

[11] Let I'_a be the inverse-in-circumcircle of the excenter I_a, and define I'_b and I'_c similarly. The triangles ABC and $I'_aI'_bI'_c$ are perspective at X_{35} which is a point on the line OI.

The directrix of this parabola is parallel to the line IP^*. It is still inscribed in the pedal triangle $A'B'C'$ of P and is tangent to the two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ met in proposition 13.

Remark. Unlike the case $P = X_8$, ABC is not necessary self polar with respect to this Mandart parabola.

Proposition 24. *The Mandart triangle $\mathbf{T}_t(P)$ and ABC are orthologic. The perpendiculars from A, B, C to the corresponding sidelines of $P_aP_bP_c$ are concurrent at $Q = \left(\frac{a^2}{at+2\Delta u} : \cdots : \cdots\right)$. As t varies, the locus of Q is generally the circumconic which is the isogonal transform of the line IP.*

This conic has equation

$$\sum_{\text{cyclic}} a^2(cv - bw)yz = 0.$$

It is tangent at I to IP, and is a rectangular hyperbola if and only if P lies on the line OI ($P \neq I$). When $P = I$, the triangles are homothetic at I and the perpendiculars concur at I.

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Another Proof of Fagnano's Inequality

Nguyen Minh Ha

Abstract. We prove Fagnano's inequality using the scalar product of vectors.

In 1775, I. F. Fagnano, an Italian mathematician, proposed the following extremum problem.

Problem (Fagnano)**.** *In a given acute-angled triangle ABC, inscribe a triangle XYZ whose perimeter is as small as possible.*

Fagnano himself gave a solution to this problem using calculus. The second proof given in [1] repeatedly using reflections and the mirror property of the orthic triangle was due to L. Fejér. While H. A. Schwarz gave another proof in which reflection was also used, we give another proof by using the scalar product of two vectors.

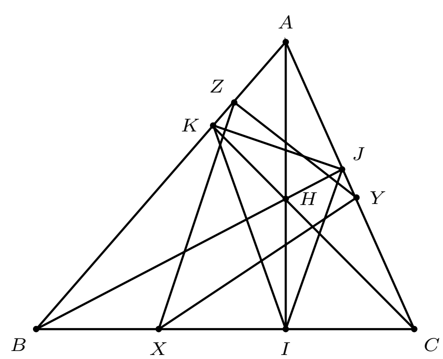

Figure 1

Let AI, BJ and CK be the altitudes of triangle ABC and H its orthocenter. Suppose that X, Y, Z are arbitrary points on the lines BC, CA and AB respectively. See Figure 1. We have

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$$\begin{aligned}
&YZ+ZX+XY\\
=&\frac{YZ\cdot JK}{JK}+\frac{ZX\cdot KI}{KI}+\frac{XY\cdot IJ}{IJ}\\
\geqslant&\frac{\overrightarrow{YZ}\cdot\overrightarrow{JK}}{JK}+\frac{\overrightarrow{ZX}\cdot\overrightarrow{KI}}{KI}+\frac{\overrightarrow{XY}\cdot\overrightarrow{IJ}}{IJ}\\
=&\frac{(\overrightarrow{YJ}+\overrightarrow{JK}+\overrightarrow{KZ})\cdot\overrightarrow{JK}}{JK}+\frac{(\overrightarrow{ZK}+\overrightarrow{KI}+\overrightarrow{IX})\cdot\overrightarrow{KI}}{KI}+\frac{(\overrightarrow{XI}+\overrightarrow{IJ}+\overrightarrow{JY})\cdot\overrightarrow{IJ}}{IJ}\\
=&JK+KI+IJ+\overrightarrow{XI}\cdot\Big(\frac{\overrightarrow{IJ}}{IJ}+\frac{\overrightarrow{IK}}{IK}\Big)+\overrightarrow{YJ}\cdot\Big(\frac{\overrightarrow{JK}}{JK}+\frac{\overrightarrow{JI}}{JI}\Big)+\overrightarrow{ZK}\cdot\Big(\frac{\overrightarrow{KI}}{KI}+\frac{\overrightarrow{KJ}}{KJ}\Big).
\end{aligned}$$

Since triangle ABC is acute-angled, its altitudes bisect the internal angles of its orthic triangle IJK. It follows that the vectors

$$\frac{\overrightarrow{IJ}}{IJ}+\frac{\overrightarrow{IK}}{IK},\qquad \frac{\overrightarrow{JK}}{JK}+\frac{\overrightarrow{JI}}{JI},\qquad \frac{\overrightarrow{KI}}{KI}+\frac{\overrightarrow{KJ}}{KJ}$$

are respectively perpendicular to the vectors $\overrightarrow{XI},\ \overrightarrow{YJ},\ \overrightarrow{ZK}$. It follows that

$$YZ+ZX+XY\geqslant JK+KI+IJ. \tag{1}$$

If the equality in (1) occurs, then the vectors $\overrightarrow{YZ},\overrightarrow{ZX},\overrightarrow{XY}$ point in the same directions of the vectors $\overrightarrow{JK},\overrightarrow{KI},\overrightarrow{IJ}$ respectively. Hence there exist positive numbers α, β and γ such that

$$\overrightarrow{YZ}=\alpha\overrightarrow{JK},\quad \overrightarrow{ZX}=\beta\overrightarrow{KI},\quad \overrightarrow{XY}=\gamma\overrightarrow{IJ}.$$

Now we have $\alpha\overrightarrow{JK}+\beta\overrightarrow{KI}+\gamma\overrightarrow{IJ}=\overrightarrow{0}$. It follows from this and the equality $\overrightarrow{JK}+\overrightarrow{KI}+\overrightarrow{IJ}=\overrightarrow{0}$ that $\alpha=\beta=\gamma$. Consequently,

$$\overrightarrow{YZ}=\alpha\overrightarrow{JK},\ \ \overrightarrow{ZX}=\alpha\overrightarrow{KI},\ \ \overrightarrow{XY}=\alpha\overrightarrow{IJ},$$

which implies that

$$YZ=\alpha JK,\ \ ZX=\alpha KI,\ \ XY=\alpha IJ,$$

and

$$YZ+ZX+XY=\alpha(JK+KI+IJ).$$

Note that the equality in (1) occurs, we have $\alpha=\beta=\gamma=1$. Then $\overrightarrow{YZ}=\overrightarrow{JK},\ \overrightarrow{ZX}=\overrightarrow{KI},\ \overrightarrow{XY}=\overrightarrow{IJ}$, which means that X,Y,Z respectively coincides with I,J,K.

Conversely, if X, Y, Z coincide with I, J, K respectively, then equality sign occurs in (1).

In conclusion, the triangle XYZ has the smallest possible perimeter when X, Y, Z coincide with I, J, K respectively.

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Further Inequalities of Erdős-Mordell Type

Walther Janous

To the memory of Murray S. Klamkin

Abstract. We extend the recent generalization of the famous Erdős-Mordell inequality by Dar and Gueron in the *American Mathematical Monthly*.

1. Introduction

In the recent note [1] the following generalization of the famous Erdős - Mordell inequality has been established. (For a proof of the original inequality see for instance [2]). For a triangle $A_1A_2A_3$, we denote by a_i the length of the side opposite to A_i, $i = 1, 2, 3$. Let P be an interior point. Denote the distances of P from the vertices A_i by R_i and from the sides opposite A_i by r_i. For positive real numbers λ_1, λ_2, λ_3,

$$\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 \geq 2\sqrt{\lambda_1\lambda_2\lambda_3}\left(\frac{r_1}{\sqrt{\lambda_1}} + \frac{r_2}{\sqrt{\lambda_2}} + \frac{r_3}{\sqrt{\lambda_3}}\right), \tag{1}$$

This inequality appears in [3, p.318, Theorem 15] without proof and with an incorrect characterization for equality. In [3, Chapter XI] and [4, Chapter 12], there are quoted very many extensions and variations of the original Erdős - Mordell inequality. It is the goal of this note to prove a further generalization containing the results of [1] and to apply it to specific points in a triangle, resulting in new inequalities for several elements of triangles.

2. The inequalities

Let λ_1, λ_2, λ_3 and t denote positive real numbers, with $0 < t \leq 1$.

Theorem 1.

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \geq 2^t\sqrt{\lambda_1\lambda_2\lambda_3}\left(\frac{r_1^t}{\sqrt{\lambda_1}} + \frac{r_2^t}{\sqrt{\lambda_2}} + \frac{r_3^t}{\sqrt{\lambda_3}}\right). \tag{2}$$

Equality holds if and only if $\lambda_1 : \lambda_2 : \lambda_3 = a_1^{2t} : a_2^{2t} : a_3^{2t}$ *and* P *is the circumcenter of triangle* $A_1A_2A_3$.

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Proof. As for instance in [1] we have

$$R_1 \geq \frac{a_3}{a_1} r_2 + \frac{a_2}{a_1} r_3, \qquad R_2 \geq \frac{a_1}{a_2} r_3 + \frac{a_3}{a_2} r_1, \qquad R_3 \geq \frac{a_2}{a_3} r_1 + \frac{a_1}{a_3} r_2.$$

Using the power means inequality we obtain (for $0 < t < 1$)

$$R_1^t \geq 2^t \left(\frac{\frac{a_3}{a_1} r_2 + \frac{a_2}{a_1} r_3}{2} \right)^t \geq 2^t \cdot \frac{\left(\frac{a_3}{a_1}\right)^t r_2^t + \left(\frac{a_2}{a_1}\right)^t r_3^t}{2}$$

and two similar inequalities. Applying several times the elementary estimation $x + \frac{1}{x} \geq 2$ for $x > 0$ we obtain

$$\begin{aligned} &\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \\ \geq &2^t \left(\frac{\left(\frac{a_3}{a_2}\right)^t \lambda_2 + \left(\frac{a_2}{a_3}\right)^t \lambda_3}{2} r_1^t + \frac{\left(\frac{a_1}{a_3}\right)^t \lambda_3 + \left(\frac{a_3}{a_1}\right)^t \lambda_1}{2} r_2^t + \frac{\left(\frac{a_2}{a_1}\right)^t \lambda_1 + \left(\frac{a_1}{a_2}\right)^t \lambda_2}{2} r_3^t \right) \\ \geq &2^t \left(\sqrt{\lambda_2 \lambda_3} r_1^t + \sqrt{\lambda_3 \lambda_1} r_2^t + \sqrt{\lambda_1 \lambda_2} r_3^t \right) \end{aligned}$$

as claimed. The conditions of equality are derived as in [1].

□

In view of the obvious inequality $(x+y)^t > x^t + y^t$ for $x, y > 0$, we have the following theorem.

Theorem 2. *For* $t > 1$,

$$\lambda_1 R_1^t + \lambda_2 R_2^t + \lambda_3 R_3^t \geq 2\sqrt{\lambda_1 \lambda_2 \lambda_3} \left(\frac{r_1^t}{\sqrt{\lambda_1}} + \frac{r_2^t}{\sqrt{\lambda_2}} + \frac{r_3^t}{\sqrt{\lambda_3}} \right). \tag{3}$$

As a consequence of Theorem 1 we get

Theorem 3.

$$\textstyle\sum_{i=1}^3 \frac{\lambda_i}{r_i^t} \geq 2^t \sqrt{\lambda_1 \lambda_2 \lambda_3} \sum_{i=1}^3 \frac{1}{\sqrt{\lambda_i} R_i^t}, \tag{4}$$

$$\textstyle\frac{\lambda_i}{R_i^t} \geq \frac{2^t \sqrt{\lambda_1 \lambda_2 \lambda_3}}{(R_1 R_2 R_3)^t \sum_{i=1}^3} \sum_{i=1}^3 \frac{(R_i r_i)^t}{\sqrt{\lambda_i}}, \tag{5}$$

$$\textstyle\sum_{i=1}^3 \lambda_i (R_i r_i)^t \geq 2^t \sqrt{\lambda_1 \lambda_2 \lambda_3} (r_1 r_2 r_3)^t \sum_{i=1}^3 \frac{1}{\sqrt{\lambda_i} r_i^t}, \tag{6}$$

$$\textstyle\sum_{i=1}^3 \lambda_i r_i^t \geq 2^t \sqrt{\lambda_1 \lambda_2 \lambda_3} (r_1 r_2 r_3)^t \sum_{i=1}^3 \frac{1}{\sqrt{\lambda_i} (R_i r_i)^t}, \tag{7}$$

$$\textstyle\sum_{i=1}^3 \frac{\lambda_i}{(R_i r_i)^t} \geq \frac{2^t \sqrt{\lambda_1 \lambda_2 \lambda_3}}{(R_1 R_2 R_3)^t} \sum_{i=1}^3 \frac{R_i^t}{\sqrt{\lambda_i}}. \tag{8}$$

The proofs of these inequalities follow from Theorem 1 upon application of transformations such as
(i) inversion with respect to the circle $\mathcal{C}(P, \sqrt{R_1 R_2 R_3})$ resulting in $R_i \mapsto \frac{R_1 R_2 R_3}{R_i}$ and $r_i \mapsto R_i r_i$ for $i = 1, 2, 3$,
(ii) reciprocation of $A_1 A_2 A_3$ yielding $R_i \mapsto \frac{r_1 r_2 r_3}{r_i}$ and $r_i \mapsto \frac{r_1 r_2 r_3}{R_i}$ for $i = 1, 2, 3$, and

(iii) isogonal conjugation.
For the details consult [3, pp. 293 - 295].

Remarks. (1) From (5) and (6) the following inequality is easily derived.

$$(R_1R_2R_3)^t \sum_{i=1}^{3} \frac{\lambda_i}{R_i^t} \geq 4^t \sqrt[4]{\lambda_1\lambda_2\lambda_3}(r_1r_2r_3)^t \sum_{i=1}^{3} \frac{\sqrt[4]{\lambda_i}}{r_i^t}. \tag{9}$$

whereas (7) and (8) lead to the "converse" of (9), *i.e.*,

$$\frac{1}{(r_1r_2r_3)^t} \sum_{i=1}^{3} \lambda_i r_i^t \geq \frac{4^t \sqrt[4]{\lambda_1\lambda_2\lambda_3}}{(R_1R_2R_3)^t} \sum_{i=1}^{3} \sqrt[4]{\lambda_i} R_i^t. \tag{10}$$

(2) We leave it as an exercise to the reader to derive an analogue of Theorem 2. It should be noted that the above inequalities include very many results of [3, 4] as special cases.

3. Applications to special triangle points

In this section we show that the theorems above, when specialized to suitably chosen interior points P, imply an abundance of new interesting triangle inequalities.

3.1. Let P be the incenter I of $A_1A_2A_3$. Then $r_1 = r_2 = r_3 = r$, the inradius of $A_1A_2A_3$, and $R_i = A_iI = r \csc \frac{A_i}{2}$, $i = 1, 2, 3$. Thus, from (8), we obtain, upon recalling that

$$\sin \frac{A_1}{2} \sin \frac{A_2}{2} \sin \frac{A_3}{2} = \frac{r}{4R},$$

the following inequality for $0 < t \leq 1$:

$$\sum_{i=1}^{3} \lambda_i \sin^t \frac{A_i}{2} \geq \sqrt{\lambda_1\lambda_2\lambda_3} \left(\frac{r}{2R}\right)^t \sum_{i=1}^{3} \frac{1}{\sqrt{\lambda_i}} \csc^t \frac{A_i}{2}. \tag{11}$$

3.2. Let P be the centroid G of $A_1A_2A_3$. Then $R_i = A_iG = \frac{2}{3}m_i$, and $r_i = \frac{h_i}{3}$, where, for $i = 1, 2, 3$, m_i and h_i denote respectively the median and altitude emanating from vertex A_i. Therefore, as an example, (4) becomes, for $0 < t \leq 1$,

$$\sum_{i=1}^{3} \frac{\lambda_i}{h_i^t} \geq \sqrt{\lambda_1\lambda_2\lambda_3} \sum_{i=1}^{3} \frac{1}{\sqrt{\lambda_i} m_i^t}. \tag{12}$$

If we put $\lambda_i = h_i^t$, $i = 1, 2, 3$, then

$$\left(\frac{\sqrt{h_2h_3}}{m_1}\right)^t + \left(\frac{\sqrt{h_3h_1}}{m_2}\right)^t + \left(\frac{\sqrt{h_1h_2}}{m_3}\right)^t \leq 3. \tag{13}$$

This inequality should be compared with the following one by Klamkin and Meir in [3, p. 215]:

$$\frac{\overline{h_1}}{m_1} + \frac{\overline{h_2}}{m_2} + \frac{\overline{h_3}}{m_3} \leq 3,$$

where $(\overline{h_1},\ \overline{h_2},\ \overline{h_3})$ is any permutation of (h_1, h_2, h_3).

Via the median - duality transforming an arbitrary triangle $A_1A_2A_3$ into one formed by its medians ([3, pp.109 - 111]), inequality (13) becomes

$$\left(\frac{h_1}{\sqrt{m_2m_3}}\right)^t + \left(\frac{h_2}{\sqrt{m_3m_1}}\right)^t + \left(\frac{h_3}{\sqrt{m_1m_2}}\right)^t \leq 3. \tag{14}$$

Finally, in (12), we put $\lambda_i = \frac{1}{a_i^t}$ for $i = 1, 2, 3$. A short calculation gives

$$3\left(\frac{R}{F}\right)^{\frac{t}{2}} \geq \sum_{i=1}^{3}\left(\frac{\sqrt{a_i}}{m_i}\right)^t. \tag{15}$$

Here, we make use of the identity $a_1a_2a_3 = 4RF$, where F denotes the area of $A_1A_2A_3$.

The median - dual of this inequality in turn reads

$$\sum_{i=1}^{3}\left(\frac{\sqrt{m_i}}{a_i}\right)^t \leq 3\left(\frac{\sqrt{m_1m_2m_3}}{2F}\right)^t. \tag{16}$$

Of course, if in (12) had we put $\lambda_i = \frac{\mu_i}{a_i^t}$ with $\mu_i > 0$, $i = 1, 2, 3$, we would obtain an even more general but less elegant inequality.

Remarks. (1) Clearly, many further inequalities could be deduced by the methods of this section. We leave this as an exercise to the reader.

(2) As the right hand side of inequality (1) indeed reads $2(\sqrt{\lambda_2\lambda_3}r_1 + \sqrt{\lambda_3\lambda_1}r_2 + \sqrt{\lambda_1\lambda_2}r_3)$, it is enough to assume λ_1, λ_2, λ_2 nonnegative throughout this note.

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Inscribed Squares

Floor van Lamoen

Abstract. We give simple constructions of various squares inscribed in a triangle, and some relations among these squares.

1. Inscribed squares

Given a triangle ABC, an inscribed square is one whose vertices are on the sidelines of ABC. Two of the vertices of an inscribed square must fall on a sideline. There are two kinds of inscribed squares.

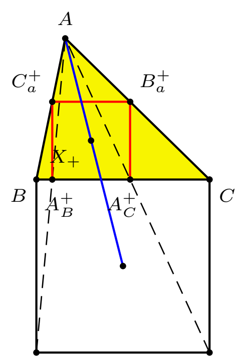

Figure 1A

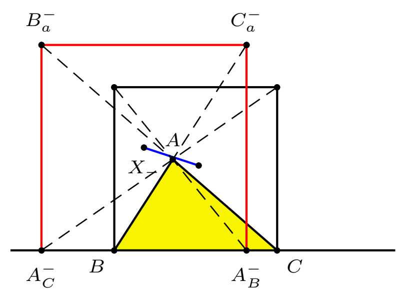

Figure 1B

1.1. *Inscribed squares of type I.* The Inscribed squares with two adjacent vertices on a sideline of ABC can be constructed easily from a homothety of a square erected on the side of ABC. Consider the two squares erected on the side BC. Their centers are the points with homogeneous barycentric coordinates $(-a^2 : S_C + \varepsilon S : S_B + \varepsilon S)$ for $\varepsilon = \pm 1$. Here, we use standard notations in triangle geometry. See, for example, [4, §1]. By applying the homothety $\mathsf{h}(A, \frac{\varepsilon S}{a^2+\varepsilon S})$, we obtain an inscribed square $\mathsf{Sq}^\varepsilon(A) = A_B^\varepsilon A_C^\varepsilon B_a^\varepsilon C_a^\varepsilon$ with center

$$\begin{aligned} X_\varepsilon =& \mathsf{h}(A, \frac{\varepsilon S}{a^2 + \varepsilon S})(-a^2 : S_C + \varepsilon S : S_B + \varepsilon S) \\ =& (a^2 : S_C + \varepsilon S : S_B + \varepsilon S), \end{aligned}$$

and two vertices (A_B^ε and A_C^ε) on the sideline BC. See Figure 1. Similarly there are the inscribed squares $\mathsf{Sq}^\varepsilon(B)$ and $\mathsf{Sq}^\varepsilon(C)$.

We give the coordinates of the centers and vertices of these squares in Table 1 below.

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Table 1. Centers and vertices of inscribed squares of type I

$\mathsf{Sq}^{\varepsilon}(A)$	$\mathsf{Sq}^{\varepsilon}(B)$	$\mathsf{Sq}^{\varepsilon}(C)$
$X_\varepsilon = (a^2 : S_C + \varepsilon S : S_B + \varepsilon S)$	$Y_\varepsilon = (S_C + \varepsilon S : b^2 : S_A + \varepsilon S)$	$Z_\varepsilon = (S_B + \varepsilon S : S_A + \varepsilon S : c^2)$
$A_B^\varepsilon = (0 : S_C + \varepsilon S : S_B)$ $A_C^\varepsilon = (0 : S_C : S_B + \varepsilon S)$	$A_b^\varepsilon = (0 : b^2 : \varepsilon S)$	$A_c^\varepsilon = (0 : \varepsilon S : c^2)$
$B_a^\varepsilon = (a^2 : 0 : \varepsilon S),$	$B_C^\varepsilon = (S_C : 0 : S_A + \varepsilon S)$ $B_A^\varepsilon = (S_C + \varepsilon S : 0 : S_A)$	$B_c^\varepsilon = (\varepsilon S : 0 : c^2)$
$C_a^\varepsilon = (a^2 : \varepsilon S : 0)$	$C_b^\varepsilon = (\varepsilon S : b^2 : 0)$	$C_A^\varepsilon = (S_B + \varepsilon S : S_A : 0)$ $C_B^\varepsilon = (S_B : S_A + \varepsilon S : 0)$

Proposition 1. *The triangle $X_\varepsilon Y_\varepsilon Z_\varepsilon$ and ABC perspective at the Vecten point*

$$V_\varepsilon = \left(\frac{1}{S_A + \varepsilon S} : \frac{1}{S_B + \varepsilon S} : \frac{1}{S_C + \varepsilon S}\right).$$

For V_+ and V_- are respectively X_{485} and X_{486} of [3].

1.2. *Inscribed squares of type II.* Another type of inscribed squares has two opposite vertices on a sideline of ABC. There are three such squares $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\mathrm{d}}(B)$, $\mathsf{Sq}^{\mathrm{d}}(C)$. The square $\mathsf{Sq}^{\mathrm{d}}(A)$ has two opposite vertices on the sideline BC. Its center X can be found as follows. The perpendicular at X to BC intersects CA and AB at B_a and C_a such that $B_aX + C_aX = 0$. If $X = (0 : v : w)$, it is easy to see that

$$B_aX = CX \cdot \tan C = \frac{av}{S_C(v+w)},$$
$$C_aX = BX \cdot \tan B = \frac{aw}{S_B(v+w)}.$$

It follows that $B_aX + C_aX = 0$ if and only if $v : w = -S_C : S_B$, and the center of $\mathsf{Sq}^{\mathrm{d}}(A)$ is the point $X = (0 : -S_C : S_B)$ on the line BC. The vertices can be easily determined, as given in Table 2 below.

Table 2. Centers and vertices of inscribed squares of type II

$\mathsf{Sq}^{\mathrm{d}}(A)$	$\mathsf{Sq}^{\mathrm{d}}(B)$	$\mathsf{Sq}^{\mathrm{d}}(C)$
$X = (0 : -S_C : S_B)$	$Y = (S_C : 0 : -S_A)$	$Z = (-S_B : S_A : 0)$
$A_+ = (0 : -S_C - S : S_B + S)$ $A_- = (0 : -S_C + S : S_B - S)$	$A_b = (0 : -b^2 : 2S_A)$	$A_c = (0 : 2S_A : -c^2)$
$B_a = (-a^2 : 0 : 2S_B)$	$B_+ = (S_C + S : 0 : -S_A - S)$ $B_- = (S_C - S : 0 : -S_A + S)$	$B_c = (2S_B : 0 : -c^2)$
$C_a = (-a^2 : 2S_C : 0)$	$C_b = (2S_C : -b^2 : 0)$	$C_+ = (-S_B - S : S_A + S : 0)$ $C_- = (-S_B + S : S_A - S : 0)$

2. Some collinearity relations

Proposition 2. (a) *The centers X, Y, Z are the intercepts of the orthic axis with the sidelines of triangle ABC.*
(b) *For $\varepsilon = \pm 1$, the points A_ε, B_ε and C_ε are collinear. The line containing them is parallel to the orthic axis.*

Proof. The line containing the points A_ε, B_ε and C_ε has equation

$$(S_A + \varepsilon S)x + (S_B + \varepsilon S)y + (S_C + \varepsilon S)z = 0.$$

See Figure 2. □

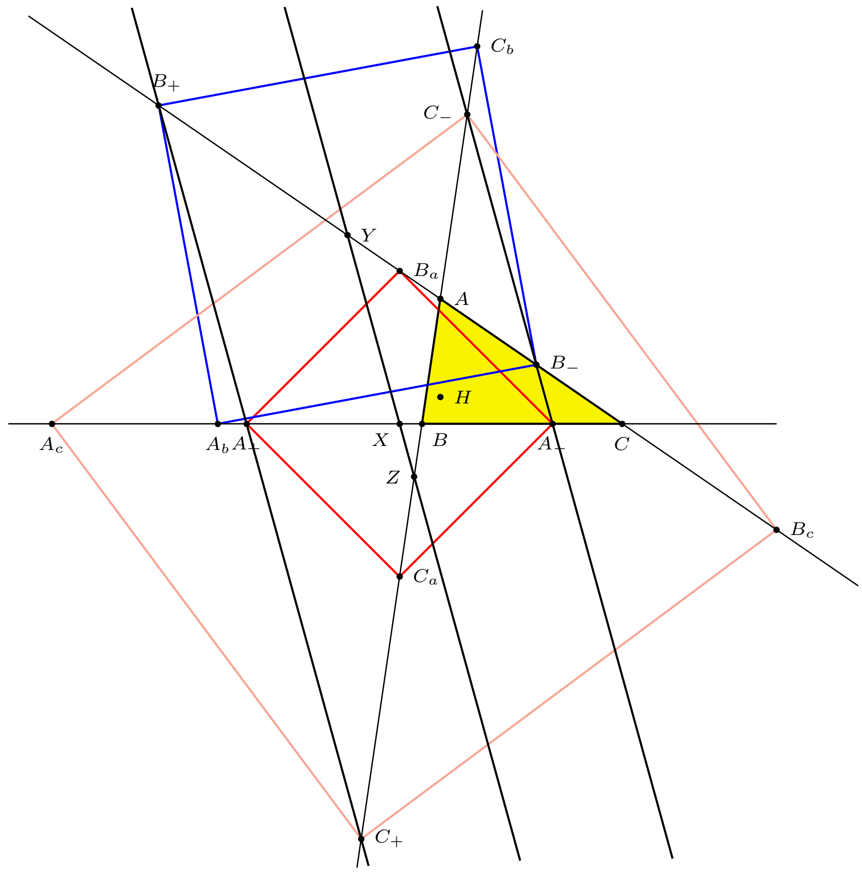

Figure 2

Proposition 3. (a) *The centers* X, Y_ε, Z_ε *of the squares* $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\varepsilon}(B)$, $\mathsf{Sq}^{\varepsilon}(C)$ *are collinear.*
(b) *The line* $B_C^\varepsilon C_B^\varepsilon$ *passes through the center* X *of* $\mathsf{Sq}^{\mathrm{d}}(A)$.
(c) *The line* $B_A^\varepsilon C_A^\varepsilon$ *passes through the point* A_ε.

Proof. (a) The line joining Y_ε and Z_ε has equation

$$-\varepsilon Sx + S_B y + S_C z = 0$$

as is easily verified. This line clearly contains $X = (0 : -S_C : S_B)$.

(b) The line $B_C^\varepsilon C_B^\varepsilon$ has equation

$$-(S_A + \varepsilon S)x + S_B y + S_C z = 0.$$

It clearly passes through X.

(c) The line $B_A^\varepsilon C_A^\varepsilon$ has equation

$$-S_A x + (S_B + \varepsilon S)y + (S_C + \varepsilon S)z = 0.$$

It contains the point $A_\varepsilon = (0 : -S_C - \varepsilon S : S_B + \varepsilon S)$. See Figure 3 for $\varepsilon = 1$. $\square$

Remark. For $\varepsilon = \pm 1$, the lines in (b) and (c) above are parallel.

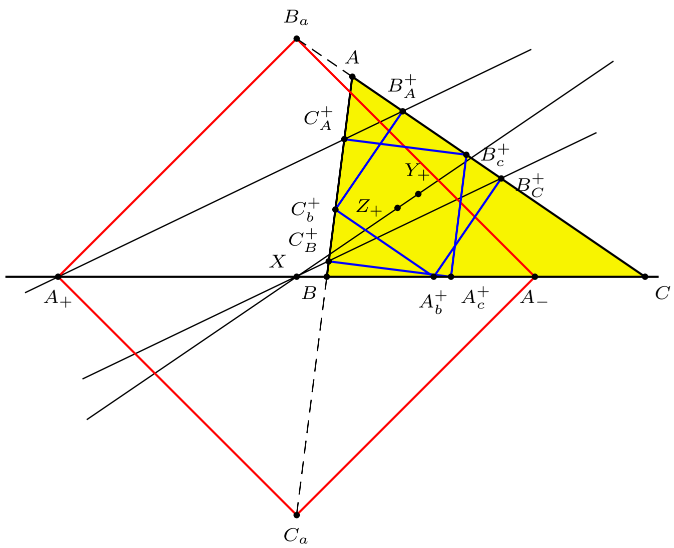

Figure 3

Let $T_A := B_A^+ C_A^+ \cap B_A^- C_A^- = (S_B - S_C : S_A : -S_A)$. The lines AT_A and BC are parallel. The three points T_A, T_B, T_C are collinear. The line connecting them has equation

$$S_A(S_B + S_C - S_A)x + S_B(S_C + S_A - S_B)y + S_C(S_A + S_B - S_C)z = 0.$$

Each of the squares of type II has a diagonal perpendicular to a sideline of triangle ABC. These diagonals clearly bound a triangle perspective to ABC with perspectrix the orthic axis. By [1] we know that the perspector lies on the circumcircle. Specifically, it is X_{74}, the Miquel perspector of the orthic axis.

The lines $B_A^\varepsilon C_A^\varepsilon$, $C_B^\varepsilon A_B^\varepsilon$, $A_C^\varepsilon B_C^\varepsilon$ bound a triangle perspective with ABC at the Kiepert perspector

$$K(\varepsilon \cdot \arctan 2) = \left(\frac{1}{2S_A + \varepsilon S} : \frac{1}{2S_B + \varepsilon S} : \frac{1}{2S_C + \varepsilon S}\right).$$

For $\varepsilon = +1$ and -1 respectively, these are X_{1131} and X_{1132} of [3]. The same perspector is found for the triangle bounded by the lines $B_C^\varepsilon C_B^\varepsilon$, $A_C^\varepsilon C_A^\varepsilon$, $A_B^\varepsilon B_A^\varepsilon$.

3. Inscribed squares and Miquel's theorem

We first recall Miquel's theorem.

Theorem 4 (Miquel)**.** *Let $A_1B_1C_1$ be a triangle inscribed in triangle ABC. There is a* pivot point *P such that $A_1B_1C_1$ is the image of the pedal triangle of P after a rotation about P followed by a homothety with center P. All inscribed triangles directly similar to $A_1B_1C_1$ have the same pivot point.*

A corollary of this theorem is for instance given in [2, Problem 8(ii), p.245].

Corollary 5. *Let X be a point defined with respect to the pedal triangle $A_PB_PC_P$ triangle of P. The images of X after the pivoting as in Miquel's theorem lie on a line.*

Proof. Let $A_2B_2C_2$ be the image of $A_PB_PC_P$ after pivoting, and let Y be the image of X. Clearly triangles PA_PA_2, PB_PB_2, PC_PC_2, and PXY are similar right triangles. This shows that Y lies on the line through X perpendicular to XP. □

Miquel's pivot theorem and Corollary 5 together give an easy explanation of Proposition 3(c). See Figure 4.

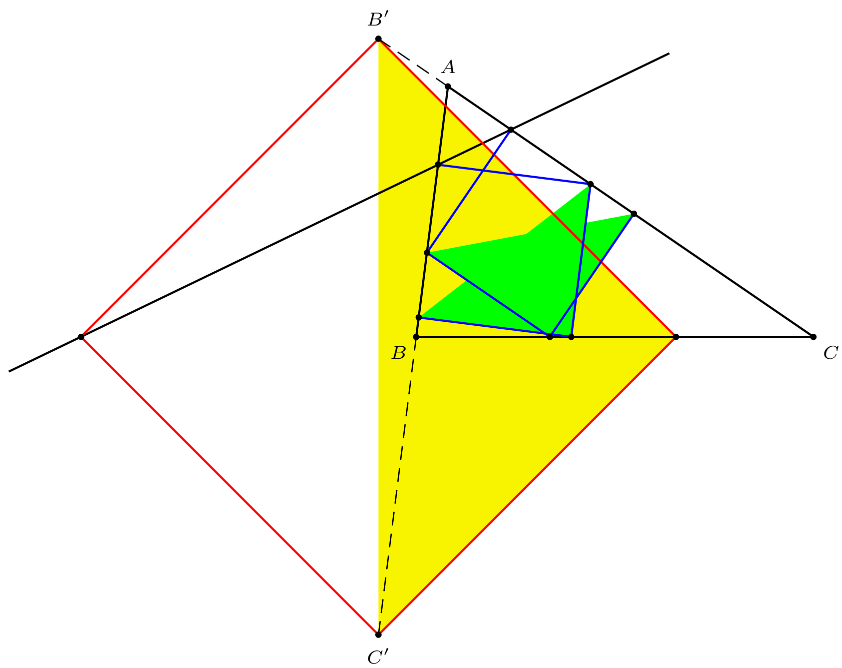

Figure 4

We have already seen that the centers of the inscribed squares of type II lie on the orthic axis. By Proposition 3(a), these centers are the intersections of the corresponding sides of the triangles $X_+Y_+Z_+$ and $X_-Y_-Z_-$ of the inscribed squares of type I. This means that the triangles $X_+Y_+Z_+$ and $X_-Y_-Z_-$ are perspective. The perspector is symmedian point $K = (a^2 : b^2 : c^2)$.

4. Squares with vertices on four given lines

Let us consider a fourth line in the plane of ABC. With the help of the inscribed squares of type I, we can construct two sets of three squares inscribing a fourline $\{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$, depending on the line containing the vertex opposite to that on d. Let ABC be the triangle bounded by the lines a, b, c. For $\varepsilon = \pm 1$, there is a square $\mathsf{Sq}^{\varepsilon}(\mathsf{a}) := A_{\mathsf{a}}^{\varepsilon} B_{\mathsf{a}}^{\varepsilon} D_{\mathsf{a}}^{\varepsilon} C_{\mathsf{a}}^{\varepsilon}$ with a pair of opposite vertices on a and d. The vertex on d is simply $D_{\mathsf{a}}^{\varepsilon} = B_A^{\varepsilon} C_A^{\varepsilon} \cap \mathsf{d}$. See the solution of Problem 55(a) of [5, p.146]. The other vertices of the square are determined by the same division ratio (of $B_A^{\varepsilon} C_A^{\varepsilon}$ by $D_{\mathsf{a}}^{\varepsilon}$):

$$B_A^{\varepsilon} C_A^{\varepsilon} : C_A^{\varepsilon} D_{\mathsf{a}}^{\varepsilon} = A_b^{\varepsilon} A_c^{\varepsilon} : A_c^{\varepsilon} A_{\mathsf{a}}^{\varepsilon} = B_C^{\varepsilon} B_c^{\varepsilon} : B_c^{\varepsilon} B_{\mathsf{a}}^{\varepsilon} = C_b^{\varepsilon} C_B^{\varepsilon} : C_B^{\varepsilon} C_{\mathsf{a}}^{\varepsilon}.$$

See Figure 5 for $\varepsilon = +1$. In fact, if $D_{\mathsf{a}}^{\varepsilon} = (S_C + \varepsilon S, 0, S_A) + t(S_B + \varepsilon S, S_A, 0)$, then

$$\begin{aligned} A_{\mathsf{a}}^{\varepsilon} =& (0, b^2, \varepsilon S) + t(0, \varepsilon S, c^2), \\ B_{\mathsf{a}}^{\varepsilon} =& (S_C, 0, S_A + \varepsilon S) + t(\varepsilon S, 0, c^2), \\ C_{\mathsf{a}}^{\varepsilon} =& (\varepsilon S, b^2, 0) + t(S_B, S_A + \varepsilon S, 0), \end{aligned}$$

and the center of the square is the point

$$X_{\mathsf{a}}^{\varepsilon} = (S_C + \varepsilon S, b^2, S_A + \varepsilon S) + t(S_B + \varepsilon S, S_A + \varepsilon S, c^2).$$

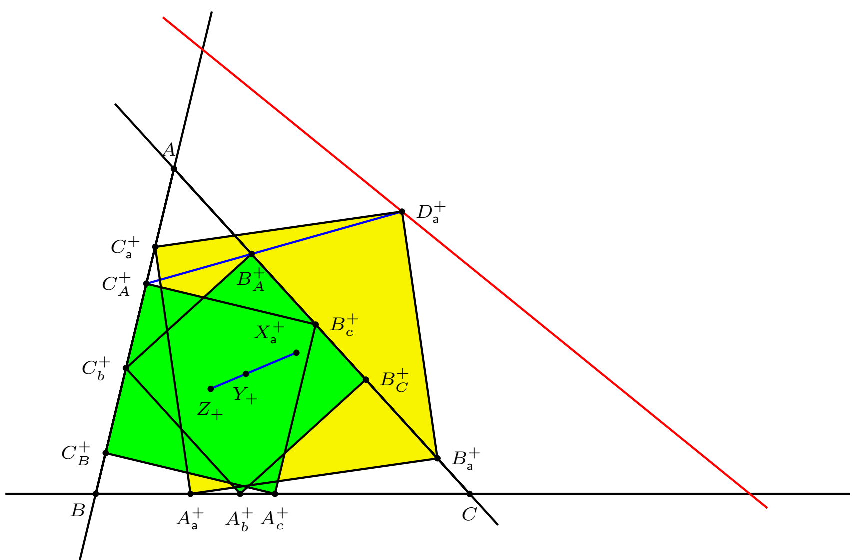

Figure 5

It is now clear that the position of $A_{\mathsf{a}}^{\varepsilon}$ relative to A_b^{ε} and A_c^{ε} fixes $D_{\mathsf{a}}^{\varepsilon}$ as well, even if we do not have a given line d. Similarly we $D_{\mathsf{b}}^{\varepsilon}$ and $D_{\mathsf{c}}^{\varepsilon}$ are fixed by $B_{\mathsf{b}}^{\varepsilon}$ and $C_{\mathsf{c}}^{\varepsilon}$ respectively. We may thus take $A_{\mathsf{a}}^{\varepsilon}$, $B_{\mathsf{b}}^{\varepsilon}$ and $C_{\mathsf{c}}^{\varepsilon}$ to be the traces of a point $P = (u : v : w)$ and see if the corresponding $D_{\mathsf{a}}^{\varepsilon}$, $D_{\mathsf{b}}^{\varepsilon}$ and $D_{\mathsf{c}}^{\varepsilon}$ are collinear. A simple calculation gives

$$\begin{aligned}
D_{\mathsf{a}}^{\varepsilon} =&((S_B-\varepsilon S)v+(S_C-\varepsilon S)w : \varepsilon Sv - b^2 w : \varepsilon Sw - c^2 v),\\
D_{\mathsf{b}}^{\varepsilon} =&(\varepsilon Su - a^2 w : (S_C-\varepsilon S)w+(S_A-\varepsilon S)u : \varepsilon Sw - c^2 u),\\
D_{\mathsf{c}}^{\varepsilon} =&(\varepsilon Su - a^2 v : \varepsilon Sv - b^2 u : (S_A-\varepsilon S)u+(S_B-\varepsilon S)v).
\end{aligned}$$

Also, the centers of the squares $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\mathrm{d}}(B)$, $\mathsf{Sq}^{\mathrm{d}}(C)$ are the points

$$\begin{aligned}
X_{\mathsf{a}}^{\varepsilon} =&(-(S_B-\varepsilon S)v-(S_C-\varepsilon S)w : (S_A-\varepsilon S)v+b^2 w : c^2 v+(S_A-\varepsilon S)w),\\
Y_{\mathsf{b}}^{\varepsilon} =&(a^2 w+(S_B-\varepsilon S)u : -(S_C-\varepsilon S)w-(S_A-\varepsilon S)u : (S_B-\varepsilon S)w+c^2 u),\\
Z_{\mathsf{c}}^{\varepsilon} =&((S_C-\varepsilon S)u+a^2 v : b^2 u+(S_C-\varepsilon S)v : -(S_A-\varepsilon S)u-(S_B-\varepsilon S)v).
\end{aligned}$$

Proposition 6. *Let $A_{\mathsf{a}}^{\varepsilon}$, $B_{\mathsf{b}}^{\varepsilon}$ and $C_{\mathsf{c}}^{\varepsilon}$ be the traces of a point $P=(u:v:w)$.* (a) *The three points $D_{\mathsf{a}}^{\varepsilon}$, $D_{\mathsf{b}}^{\varepsilon}$ and $D_{\mathsf{c}}^{\varepsilon}$ are collinear if and only if P lies on the circumcubic*

$$\begin{aligned}
&4a^2b^2c^2uvw+S^2\sum_{\text{cyclic}}u((2S_A+S_B)v^2+(2S_A+S_B)w^2)\\
=&\varepsilon S\left(2S^2uvw+\sum_{\text{cyclic}}u((2c^2a^2-S_{AB})v^2+(2a^2b^2-S_{CA})w^2)\right).
\end{aligned}$$

(b) *The centers of the squares $\mathsf{Sq}^{\mathrm{d}}(A)$, $\mathsf{Sq}^{\mathrm{d}}(B)$, $\mathsf{Sq}^{\mathrm{d}}(C)$ are collinear if and only if*

$$\begin{aligned}
&2a^2b^2c^2uvw+S^2\sum_{\text{cyclic}}u(c^2v^2+b^2w^2)\\
=&\varepsilon S\left(2S^2uvw+\sum_{\text{cyclic}}a^2u(c^2v^2+b^2w^2)\right).
\end{aligned}$$

Remarks. (1) The locus of P for which $D_{\mathsf{a}}^{\varepsilon}D_{\mathsf{b}}^{\varepsilon}D_{\mathsf{c}}^{\varepsilon}$ and ABC are perspective is the isogonal cubic with pivot $(a^2+\varepsilon S : b^2+\varepsilon S : c^2+\varepsilon S)$.

(2) The locus of P for which $X_{\mathsf{a}}^{\varepsilon}Y_{\mathsf{b}}^{\varepsilon}Z_{\mathsf{c}}^{\varepsilon}$ and ABC are perspective is the isogonal cubic with pivot H. Here are some examples of the perspectors for P on the cubic.

Table 3. Perspectors of $X_{\mathsf{a}}^{\varepsilon}Y_{\mathsf{b}}^{\varepsilon}Z_{\mathsf{c}}^{\varepsilon}$ for $\varepsilon=\pm 1$

P	$\varepsilon=+1$	$\varepsilon=-1$
I	I	I
O	$X_{372}=(a^2(S_A-S):\cdots:\cdots)$	$X_{371}=(a^2(S_A+S):\cdots:\cdots)$
H	$X_{486}=\left(\frac{1}{S_A-S}:\cdots:\cdots\right)$	$X_{485}=\left(\frac{1}{S_A+S}:\cdots:\cdots\right)$
X_{485}	G	$(a^2+S:\cdots:\cdots)$
X_{486}	$(a^2-S:\cdots:\cdots)$	G
X_{487}	$\left(\frac{1}{b^2c^2+S_{BC}-(S_A+S_B+S_C)S}:\cdots:\cdots\right)$	$\left(\frac{S_A-a^2}{S_A-S}:\cdots:\cdots\right)$
X_{488}	$\left(\frac{S_A-a^2}{S_A+S}:\cdots:\cdots\right)$	$\left(\frac{1}{b^2c^2+S_{BC}+(S_A+S_B+S_C)S}:\cdots:\cdots\right)$

(3) In comparison with Proposition 6 (a), if instead of traces, we take A_a^ε, B_b^ε and C_c^ε to be the *pedals* of a point P on the sidelines of ABC, then the locus of P for which D_a^ε, D_b^ε and D_c^ε are collinear turns out to be a conic, though with equation too complicated to record here.

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On the Existence of Triangles with Given Lengths of One Side and Two Adjacent Angle Bisectors

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Abstract. We give a necessary and sufficient condition for the existence of a triangle with given lengths of one side and the two adjacent angle bisectors.

1. Introduction

It is known that given three lengths ℓ_1, ℓ_2, ℓ_3, there is always a triangle whose three internal angle bisectors have lengths ℓ_1, ℓ_2, ℓ_3. See [1]. In this note we consider the question of existence and uniqueness of a triangle with given lengths of one side and the bisectors of the two angles adjacent to it. Recall that in a triangle ABC with sidelengths a, b, c, the bisector of angle A (with opposite side a) has length

$$\ell = \frac{2bc}{b+c}\cos\frac{A}{2} = \sqrt{bc\left(1-\frac{a^2}{(b+c)^2}\right)}. \tag{1}$$

We shall prove the following theorem.

Theorem 1. *Given a, ℓ_1, $\ell_2 > 0$, there is a unique triangle ABC with $BC = a$, and the lengths of the bisectors of angles B, C equal to ℓ_1 and ℓ_2 if and only if*

$$\sqrt{\ell_1^2+\ell_2^2} < 2a < \ell_1+\ell_2+\sqrt{\ell_1^2-\ell_1\ell_2+\ell_2^2}.$$

2. Uniqueness

First we prove that if such a triangle exists, then it is unique.

Denote the sidelengths of the triangle by a, x, y. If the angle bisectors on the sides x and y have lengths ℓ_1 and ℓ_2 respectively, then from (1) above,

$$y =(a+x)\sqrt{1-\frac{t_2}{x}}, \tag{2}$$

$$x =(a+y)\sqrt{1-\frac{t_1}{y}}, \tag{3}$$

where $t_1 = \frac{\ell_1^2}{a}$, $t_2 = \frac{\ell_2^2}{a}$, $(t_1 < y, t_2 < x)$.

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Let $t > 0$. We consider the function $y : (t, \infty) \to (0, \infty)$ defined by

$$y(x) = (a + x)\sqrt{1 - \frac{t}{x}}.$$

Obviously, y is a continuous function on the interval (t, ∞). It is increasing and has an oblique asymptote $y = x + a - \frac{t}{2}$. It is easy to check that $y'' < 0$ in (t, ∞), so that y is a *convex* funcion and its graph is below its oblique asymptote. See Figure 1.

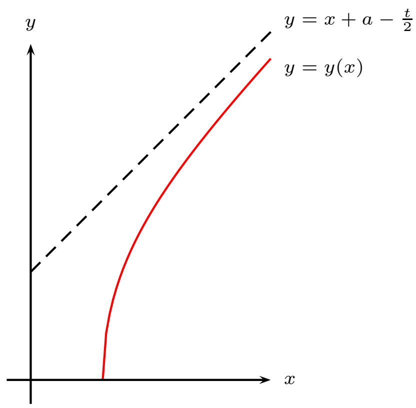

Figure 1

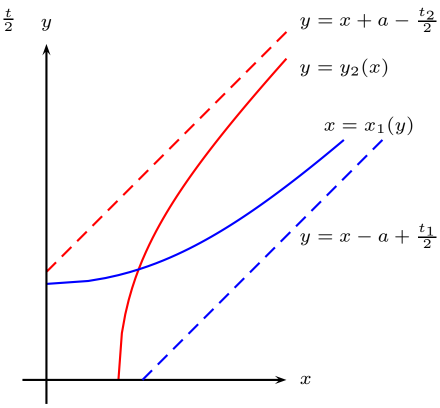

Figure 2

Now consider the system of equations

$$y = (a + x)\sqrt{1 - \frac{t}{x}}, \tag{4}$$

$$x = (a + y)\sqrt{1 - \frac{t}{y}}. \tag{5}$$

It is obvious that if a pair (x, y) satisfies (4), the pair (y, x) satisfies (5), and conversely. These equations therefore define inverse functions, and (5) defines a *concave* function $(0, \infty) \to (t, \infty)$ with an oblique asymptote $y = x - a + \frac{t}{2}$.

Applying to functions $y = y_2(x)$ and $x = x_1(y)$ defined by (2) and (3) respectively, we conclude that the system of equations (2), (3) cannot have more than one solution. See Figure 2.

Proposition 2. *If the side and the bisectors of the adjacent angles of triangle are respectively equal to the side and the bisectors of the adjacent angles of another triangle, then the triangles are congruent.*

Corollary 3 (Steiner-Lehmus theorem)**.** *If a triangle has two equal bisectors, then it is an isosceles triangle.*

Indeed, if the bisectors of the angles A and C of triangle ABC are equal, then triangle ABC is congruent to CBA, and so $AB = CB$.

3. Existence

Now we consider the question of existence of a triangle with given a, ℓ_1 and ℓ_2.

First of all note that in order for the system of equations (2), (3) to have a solution, it is necessary that $x + a - \frac{t_2}{2} > x - a + \frac{t_1}{2}$. Geometrically, this means that the asymptote of (2) is above that of (3). Thus, $2a > \frac{t_1+t_2}{2} = \frac{\ell_1^2+\ell_2^2}{2a}$, and

$$2a > \sqrt{\ell_1^2 + \ell_2^2}. \tag{6}$$

For the three lengths a, x, y to satisfy the triangle inequality, note that from (2) and (3), we have $y < a + x$ and $x < a + y$. If $x > a$ or $y > a$, then clearly $x + y > a$. We shall therefore restrict to $x < a$ and $y < a$.

Let BC be a given segment of length a. Consider a point Y in the plane such that the bisector of angle B of triangle YBC has a given length ℓ_1. It is easy to see from (1) that the length of BY is given by

$$y = \frac{a\ell_1}{2a\cos\frac{\theta}{2} - \ell_1} \quad \text{if } \angle CBY = \theta. \tag{7}$$

Let $\alpha = 2\arccos\frac{\ell_1}{2a}$. (7) defines a monotonic increasing function $y = y(\theta)$: $(0, \alpha) \to \left(\frac{a\ell_1}{2a-\ell_1}, \infty\right)$. It is easy to check that for $\theta \in (0, \alpha)$,

$$y > \frac{a\ell_1}{2a - \ell_1} > y\cos\theta.$$

The locus of Y is a continuous curve ξ_1 beginning at (but not including) a point M on the interval BC with $BM = \frac{a\ell_1}{2a-\ell_1}$. It has an oblique asymptote which forms an angle α with the line BC. See Figure 3. Since we are interested only in the case $y < a$, we may assume $a > \ell_1$. The angle α exceeds $\frac{2\pi}{3}$.

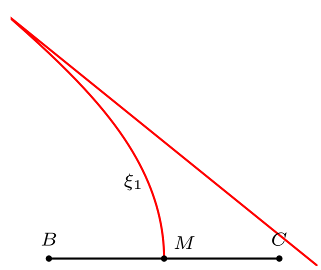

Figure 3

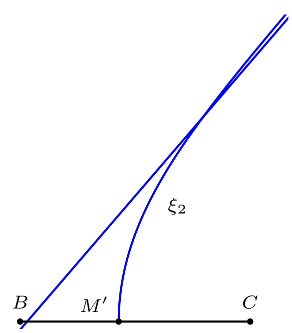

Figure 4

Consider now the locus of point Z such that the bisector of angle C of triangle ZBC has length $\ell_2 < a$. The same reasoning shows that this is a curve ξ_2 beginning at (but not including) a point M' on BC such that $M'C = \frac{a\ell_2}{2a-\ell_2}$, which has an oblique asymptote making an angle $2\arccos\frac{\ell_2}{2a}$ with CB. Again, this angle exceeds $\frac{2\pi}{3}$. See Figure 4.

The two curves ξ_1 and ξ_2 intersect if and only if $BM > BM'$, *i.e.*, $BM + M'C > a$. This gives

$$\frac{\ell_1}{2a-\ell_1}+\frac{\ell_2}{2a-\ell_2}>1.$$

Simplifying, we have $4a^2 - 4a(\ell_1+\ell_2)+3\ell_1\ell_2 < 0$, or

$$\ell_1+\ell_2-\sqrt{\ell_1^2-\ell_1\ell_2+\ell_2^2}<2a<\ell_1+\ell_2+\sqrt{\ell_1^2-\ell_1\ell_2+\ell_2^2}.$$

Since $a > \ell_1, \ell_2$, the first inequality always holds. Comparison with (6) now completes the proof of Theorem 1.

In particular, for the existence of an isosceles triangle with base a and bisectors of the equal angles of length ℓ, it is necessary and sufficient that $\frac{\sqrt{2}}{2} < \frac{a}{\ell} < \frac{3}{2}$.

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A Purely Synthetic Proof of the Droz-Farny Line Theorem

Jean-Louis Ayme

Abstract. We present a purely synthetic proof of the theorem on the Droz-Farny line, and a brief biographical note on Arnold Droz-Farny.

1. The Droz-Farny line theorem

In 1899, Arnold Droz-Farny published without proof the following remarkable theorem.

Theorem 1 (Droz-Farny [2]). *If two perpendicular straight lines are drawn through the orthocenter of a triangle, they intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.*

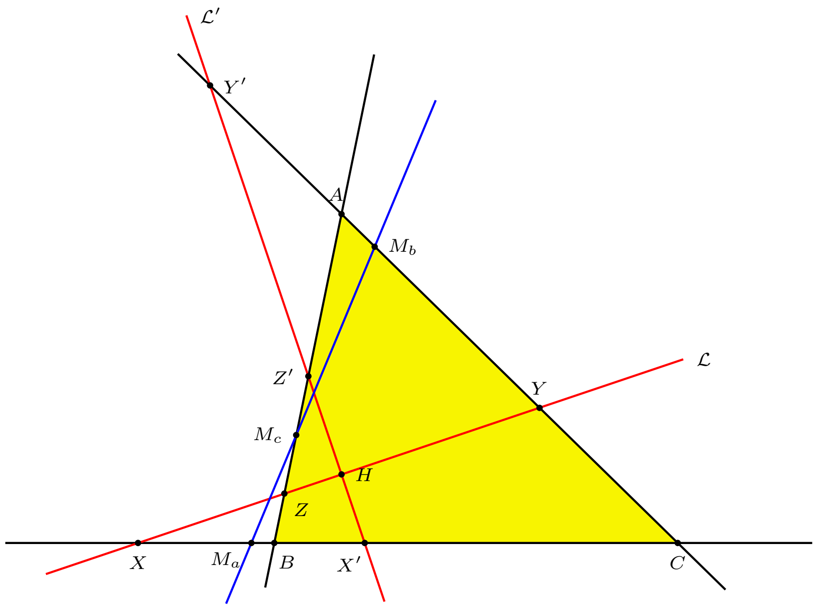

Figure 1.

Figure 1 illustrates the Droz-Farny line theorem. The perpendicular lines $\mathcal{L}$ and $\mathcal{L}'$ through the orthocenter H of triangle ABC intersect the sidelines BC at X, X', CA at Y, Y', and AB at Z, Z' respectively. The midpoints M_a, M_b, M_c of the segments XX', YY', ZZ' are collinear.

It is not known if Droz-Farny himself has given a proof. The Droz-Farny line theorem was presented again without any proof in 1995 by Ross Honsberger [9,

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p.72]. It also appeared in 1986 as Problem II 206 of [16, pp.111,311-313] without references but with an analytic proof. This "remarkable theorem", as it was named by Honsberger, has been the subject of many recent messages in the Hyacinthos group. If Nick Reingold [15] proposes a projective proof of it, he does not yet show that the considered circles intersect on the circumcircle. Darij Grinberg taking up an elegant idea of Floor van Lamoen presents a first trigonometric proof of this "rather difficult theorem" [5, 12, 3] which is based on the pivot theorem and applied on degenerated triangles. Grinberg also offers a second trigonometric proof, which starts from a generalization of the Droz-Farny's theorem simplifying by the way the one of Nicolaos Dergiades and gives a demonstration based on the law of sines [6]. Milorad Stevanović [17] presents a vector proof. Recently, Grinberg [8] picks up an idea in a newsgroup on the internet and proposes a proof using inversion and a second proof using angle chasing. In this note, we present a purely synthetic proof.

2. Three basic theorems

Theorem 2 (Carnot[1, p.101])**.** *The segment of an altitude from the orthocenter to the side equals its extension from the side to the circumcircle.*

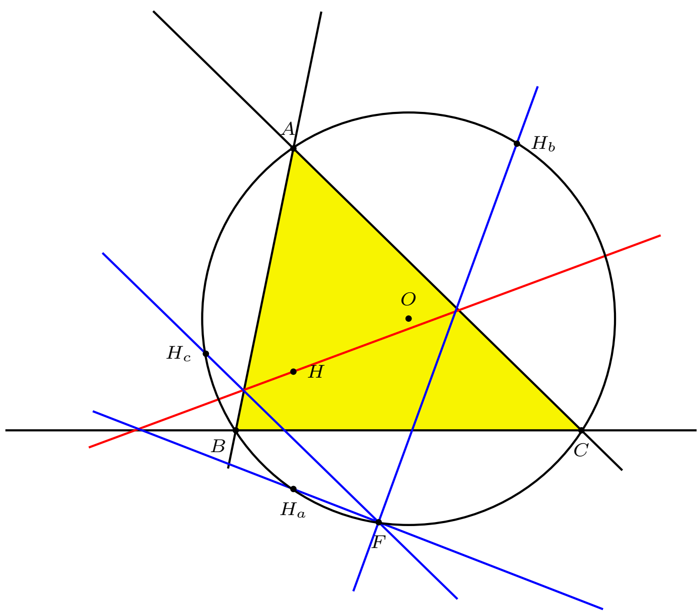

Figure 2.

Theorem 3. *Let $\mathcal{L}$ be a line through the orthocenter of a triangle ABC. The reflections of $\mathcal{L}$ in the sidelines of ABC are concurrent at a point on the circumcircle.*

See [11, p.99] or [10, §333].

Theorem 4 (Miquel's pivot theorem [13])**.** *If a point is marked on each side of a triangle, and through each vertex of the triangle and the marked points on the adjacent sides a circle is drawn, these three circles meet at a point.*

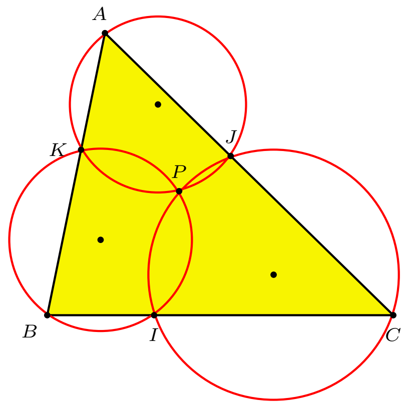

Figure 3.

See also [10, §184, p.131]. This result stays true in the case of tangency of lines or of two circles. Very few geometers contemporary to Miquel had realised that this result was going to become the spring of a large number of theorem.

3. A synthetic proof of Theorem 1

The right triangle case of the Droz-Farny theorem being trivial, we assume triangle ABC not containing a right angle. Let $\mathcal{C}$ be the circumcircle of ABC.

Let $\mathcal{C}_a$ (respectively $\mathcal{C}_b$, $\mathcal{C}_c$) be the circumcircle of triangle HXX' (respectively HYY', HZZ'), and H_a (respectively H_b, H_c) be the symmetric point of H in the line BC (respectively CA, AB). The circles $\mathcal{C}_a$, $\mathcal{C}_b$ and $\mathcal{C}_c$ have centers M_a, M_b and M_c respectively.

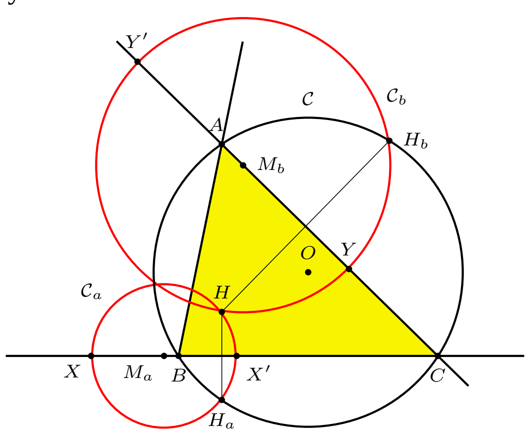

Figure 4.

According to Theorem 2, H_a is on the circle $\mathcal{C}$. XX' being a diameter of the circle $\mathcal{C}_a$, H_a is on the circle. Consequently, H_a is an intersection of $\mathcal{C}$ and $\mathcal{C}_a$, and the perpendicular to BC through H. In the same way, H_b is an intersection of $\mathcal{C}$ and $\mathcal{C}_b$, and the perpendicular to CA through H. See Figure 4.

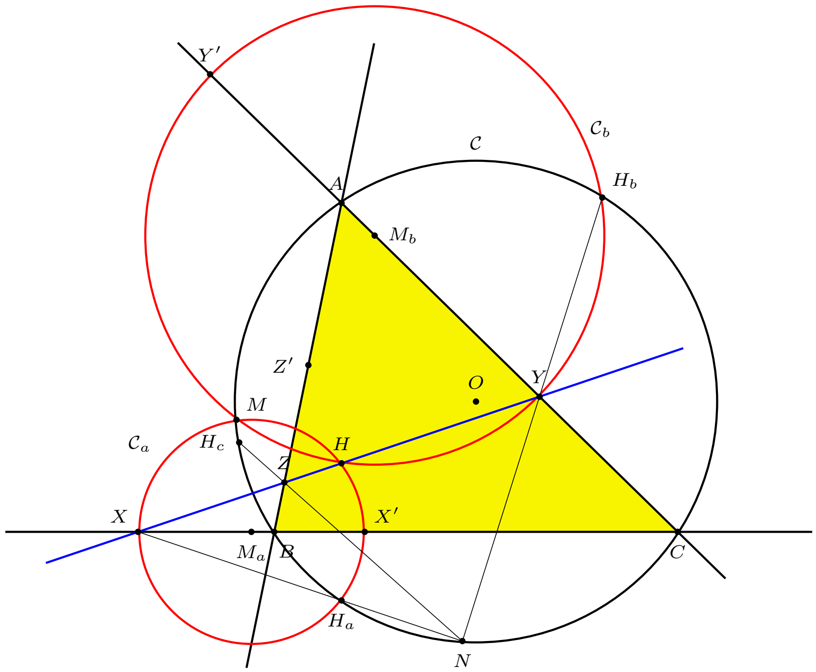

Figure 5.

Consider the point H_c, the symmetric of H in the line AB. According to Theorem 2, H_a is on the circle $\mathcal{C}$. Applying Theorem 3 to the line XYZ through H, we conclude that the lines H_aX, H_bY and H_cZ intersect at a point N on the circle $\mathcal{C}$. See Figure 5.

Applying Theorem 4 to the triangle XNY with the points H_a, H_b and H (on the lines XN, NY and YX respectively), we conclude that the circles $\mathcal{C}$, $\mathcal{C}_a$, and $\mathcal{C}_b$ pass through a common point M.

Mutatis mutandis, we show that the circles $\mathcal{C}$, $\mathcal{C}_b$, and $\mathcal{C}_c$ also pass through the same point M.

The circle $\mathcal{C}_a$, $\mathcal{C}_b$, and $\mathcal{C}_c$, all passing through H and M, are coaxial. Their centers are collinear. This completes the proof of Theorem 1.

4. A biographical note on Arnold Droz-Farny

Arnold Droz, son of Edouard and Louise Droz, was born in La Chaux-de-Fonds (Switzerland) on February 12, 1856. After his studies in the canton of Neufchatel, he went to Munich (Germany) where he attended lectures given by Felix Klein, but he finally preferred geometry. In 1880, he started teaching physics and mathematics in the school of Porrentruy (near Basel) where he stayed until 1908. He is known for having written four books between 1897 and 1909, two of them about geometry. He also published in the *Journal de Mathématiques Élementaires et*

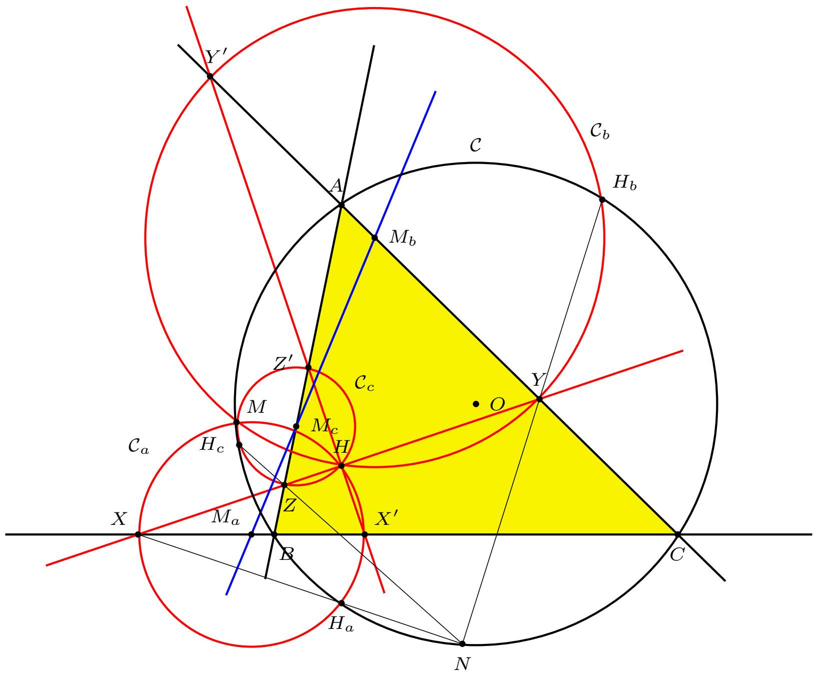

Figure 6.

Spéciales (1894, 1895), and in *L'intermédiaire des Mathématiciens* and in the *Educational Times* (1899) as well as in *Mathesis* (1901). As he was very sociable, he liked to be in contact with other geometers likes the Italian Virginio Retali and the Spanish Juan Jacobo Duran Loriga. In his free time, he liked to climb little mountains and to watch horse races. He was married to Lina Farny who was born also in La Chaux-de-Fonds. He died in Porrentruy on January 14, 1912 after having suffered from a long illness. See [4, 14].

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A Projective Generalization of the Droz-Farny Line Theorem

Jean-Pierre Ehrmann and Floor van Lamoen

Dedicated to the fifth anniversary of
the Hyacinthos group on triangle geometry

Abstract. We give a projective generalization of the Droz-Farny line theorem.

Ayme [1] has given a simple, purely synthetic proof of the following theorem by Droz-Farny.

Theorem 1 (Droz-Farny [1])**.** *If two perpendicular straight lines are drawn through the orthocenter of a triangle. they intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.*

In this note we give and prove a projective generalization. We begin with a simple observation. Given triangle ABC and a point S, the perpendiculars to AS, BS, CS at A, B, C respectively concur if and only if S lies on the circumcircle of ABC. In this case, their common point is the antipode of S on the circumcircle.

Now, consider 5 points A, B, C, I, I' lying on a conic $\mathcal{E}$ and a point S not lying on the line II'. Using a projective transformation mapping the circular points at infinity to I and I', we obtain the following.

Proposition 2. *The polar lines of S with respect to the pairs of lines (AI, AI'), (BI, BI'), (CI, CI') concur if and only if S lies on $\mathcal{E}$. In this case, their common point lies on $\mathcal{E}$ and on the line joining S to the pole of II' with respect to $\mathcal{E}$.*

The dual form of this proposition is the following.

Theorem 3. *Let ℓ and ℓ' be two lines intersecting at P, tangent to the same inscribed conic $\mathcal{E}$, and d be a line not passing through P. Let X, Y, Z (respectively X', Y', Z'; X_{d}, Y_{d}, Z_{d}) be the intersections of ℓ (respectively ℓ', d) with the sidelines BC, CA, AB. If X'_{d} is the harmonic conjugate of X_{d} with respect to (X, X'), and similarly for Y'_{d} and Z'_{d}, then X'_{d}, Y'_{d}, Z'_{d} lie on a same line d' if and only if d touches $\mathcal{E}$. In this case, d' touches $\mathcal{E}$ and the intersection of d and d' lies on the polar of P with respect to $\mathcal{E}$.*

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An equivalent condition is that A, B, C and the vertices of the triangle with sidelines ℓ, ℓ', d lie on a same conic.

More generally, consider points X'_{d}, Y'_{d} and Z'_{d} such that the cross ratios

$$(X, X', X_{\mathsf{d}}, X'_{\mathsf{d}}) = (Y, Y', Y_{\mathsf{d}}, Y'_{\mathsf{d}}) = (Z, Z', Z_{\mathsf{d}}, Z'_{\mathsf{d}}).$$

These points X'_{d}, Y'_{d}, Z'_{d} lie on a line d' if and only if d is tangent to $\mathcal{E}$. This follows easily from the dual of Steiner's theorem and its converse: two points P, Q lie on a conic through four given points A, B, C, D if and only if the cross ratios

$$(PA, PB, PC, PD) = (QA, QB, QC, QD).$$

If in Theorem 3 we take for d the line at infinity, we obtain the following.

Corollary 4. *The midpoints of XX', YY', ZZ' lie on a same line d' if and only if ℓ and ℓ' touch the same inscribed parabola. In this case, if ℓ and ℓ touch the parabola at M and M', d' is the tangent to the parabola parallel to MM'.*

An equivalent condition is that the circumhyperbola through the infinite points of ℓ and ℓ' passes through P.

We shall say that (ℓ, ℓ') is a pair of DF-lines if it satisfies the conditions of Corollary 4 above.

Now, if ℓ and ℓ' are perpendicular, we get immediately:

(a) if $P = H$, then (ℓ, ℓ') is a pair of DF-lines because H lies on any rectangular circumhyperbola, or, equivalently, on the directrix of any inscribed parabola. This is the Droz-Farny line theorem (Theorem 1 above).

(b) if $P \neq H$, then (ℓ, ℓ') is a pair of DF-lines if and only if they are the tangents from P to the inscribed parabola with directrix HP, or, equivalently, they are the parallels at P to the asymptotes of the rectangular circumhyperbola through P.

Remarks. (1) The focus of the inscribed parabola touching ℓ is the Miquel point F of the complete quadrilateral formed by AB, BC, CA, ℓ, and the directrix is the Steiner line of F. See [3].

(2) If the circle through F and with center P intersects the directrix at M, M', the tangents from P to the parabola are the perpendicular bisectors of FM and FM'.

(3) The tripoles of tangents to an inscribed parabola are collinear in a line through G.

(4) Let A_ℓ, B_ℓ, C_ℓ be the intercepts of ℓ on the sides of ABC. Let A_{r}, B_{r}, C_{r} be the reflections of these intercepts through the midpoints of the corresponding sides. Then A_{r}, B_{r}, and C_{r} are collinear on the "isotomic conjugate" of ℓ. Clearly, the isotomic conjugates of lines from a pencil are tangents to an inscribed conic and vice versa. In the case of inscribed parabolas, as above, the isotomic conjugates of the tangents are a pencil of parallel lines. It is trivial that lines dividing in equal ratios the intercepted segments by two parallel lines are again parallel. So, by isotomic conjugation of lines this holds for tangents to a parabola as well.

These remarks lead to a number of simple constructions of pairs of DF-lines satisfying a given condition.

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The Twin Circles of Archimedes in a Skewed Arbelos

Hiroshi Okumura and Masayuki Watanabe

Abstract. Any area surrounded by three mutually touching circles is called a skewed arbelos. The twin circles of Archimedes in the ordinary arbelos can be generalized to the skewed arbelos. The existence of several pairs of twin circles, under certain conditions, is demonstrated.

1. Introduction

Let O be an arbitrary point on the segment AB in the plane and α, β and γ the semicircles on the same side of the diameters AO, BO and AB, respectively. The area surrounded by the three semicircles is called an arbelos or a shoemaker's knife (see Figure 1). The common internal tangent of α and β divides the arbelos into two curvilinear triangles and the incircles of these triangles are congruent. They are called the twin circles of Archimedes or Archimedean twin circles. The authors of [3] pose the following question: Is it possible to find any interesting properties of a "skewed arbelos", in which the centers of the three circles α, β and γ are not collinear (see Figure 2), without resorting to trigonometry? In this article, we show several interesting properties of the skewed arbelos, one of them being the existence, in certain situations, of up to four pairs of twin circles. This property is a generalization of the existence of the twin circles of Archimedes in the ordinary arbelos.

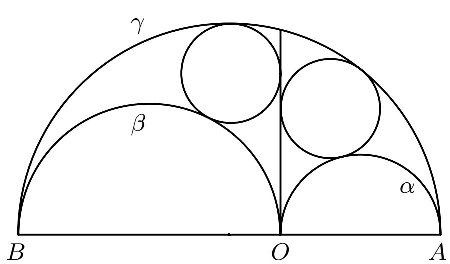

Figure 1.

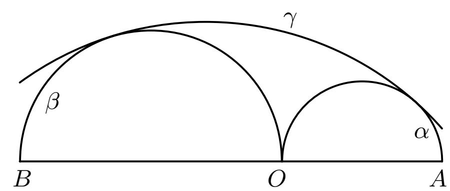

Figure 2.

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2. The skewed arbelos

Throughout this paper, α and β are circles with centers $(a, 0)$ and $(0, -b)$ for positive real numbers a and b, touching externally at the origin O, and γ is another circle touching α and β at points different from O. We do not exclude the case, when γ touches α and β externally or when γ is one of the common external tangents of α and β. There are always two different areas surrounded by α, β and γ (if γ touches α and β externally, we still consider the exterior infinite area to be surrounded by these three circles). We select one of these areas in the following way (see Figure 3): If γ touches α and β externally from above, we choose the finite area, if γ touches α and β internally, we choose the upper area, and if γ touches α and β externally from below, we choose the infinite area. We call this area the *skewed arbelos* formed by the circles α, β and γ.

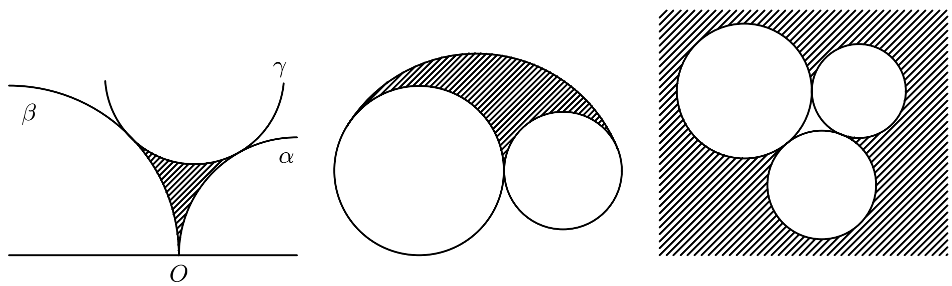


Figure 3.

Now we define four sets of tangent circles (or four chains of circles). If we include the lines parallel to the y-axis (circles of infinite radius) among the circles touching the y-axis, there are always two different circles touching γ, α and the y-axis, which do not pass through the tangency point of α and γ. We label the one inside of the skewed arbelos as α_0^+ and the other one as α_0^-. The circles β_0^+ and β_0^- touching γ, β and the y-axis are defined similarly (see Figure 4). There are also two circles touching α, α_0^+ and the y-axis, one intersecting γ and the other not. We label the former as α_{-1}^+ and the latter as α_1^+. The circles $\alpha_2^+, \alpha_3^+, \cdots$ can be defined inductively in the following way: Assuming the circles α_{i-1}^+ and α_i^+ are defined, α_{i+1}^+ is the circles touching α, α_i^+ and the y-axis and different from α_{i-1}^+. The circles $\alpha_{-2}^+, \alpha_{-3}^+, \cdots$ are defined similarly. Now the entire chain of circles

$$\{\cdots, \alpha_{-2}^+, \alpha_{-1}^+, \alpha_0^+, \alpha_1^+, \alpha_2^+, \cdots\}$$

is defined. The other three chains of circles

$$\{\cdots, \alpha_{-2}^-, \alpha_{-1}^-, \alpha_0^-, \alpha_1^-, \alpha_2^-, \cdots\},$$
$$\{\cdots, \beta_{-2}^+, \beta_{-1}^+, \beta_0^+, \beta_1^+, \beta_2^+, \cdots\},$$
$$\{\cdots, \beta_{-2}^-, \beta_{-1}^-, \beta_0^-, \beta_1^-, \beta_2^-, \cdots\},$$

where α_{-1}^-, β_{-1}^+ and β_{-1}^- intersect γ, are defined similarly. If α_i^+, α_i^-, β_i^+ and β_i^- are proper circles, there radii are denoted by a_i^+, a_i^-, b_i^+ and b_i^-, respectively. If, for example, α_i^+ is a line parallel to the y-axis, we consider the reciprocal value of its radius to be zero, even though we cannot define the radius a_i^+ itself.

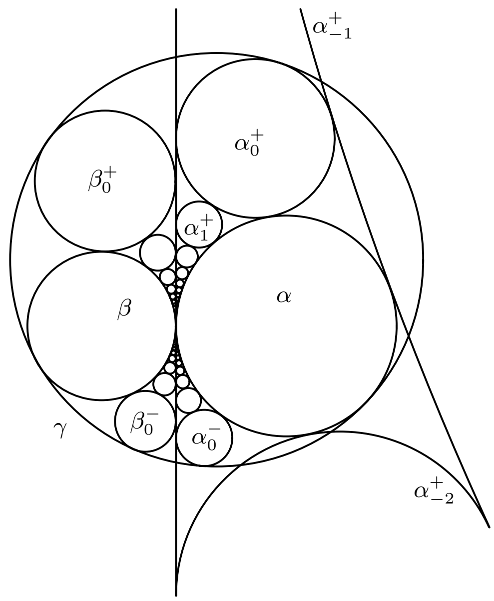


Figure 4.

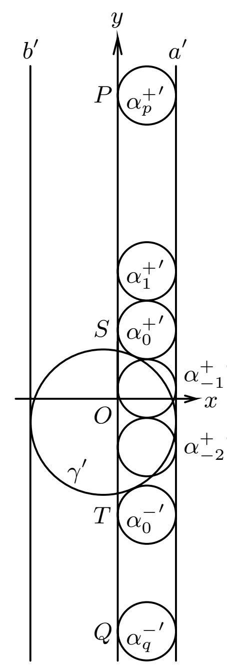


Figure 5.

If α_k^+ is a proper circle and the centers of α_k^+ and α_i^+ lie on the same side of the x-axis for all proper circles α_i^+ ($i > k$), we define $\sigma(\alpha_k^+) = 1$, otherwise we define $\sigma(\alpha_k^+) = -1$. If α_k^+ is a line parallel to the y-axis, we define $\sigma(\alpha_k^+) = 1$. The numbers $\sigma(\alpha_k^-)$, $\sigma(\beta_k^+)$, $\sigma(\beta_k^-)$ are defined similarly. If γ touches α and β internally, $\sigma(\alpha_0^+) = \sigma(\alpha_0^-) = 1$ and consequently, $\sigma(\alpha_i^+) = \sigma(\alpha_i^-) = 1$ for all non-negative integers i. Let s_i and t_j be the y-coordinates of the tangency points of the circles α_i^+ and α_j^- with the y-axis. If α_i^+ (or α_j^-) is a line, we consider $s_i = 0$ (or $t_j = 0$). We define $\sigma(\alpha_i^+, \alpha_j^-) = 1$, when $s_i t_j > 0$ and $s_i \leq t_j$, or when $s_i t_j \leq 0$ and $s_i \geq t_j$, otherwise $\sigma(\alpha_i^+, \alpha_j^-) = -1$. The number $\sigma(\beta_i^+, \beta_j^-)$ is defined similarly. If the centers of the three circles α, β and γ are collinear, we get an ordinary arbelos. In this case, the radii of the twin circles, which we denote as r_{A}, are equal to $ab/(a + b)$.

Theorem 1. *For any integers p and q,*

$$\sigma(\alpha_p^+, \alpha_q^-)\left(\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_q^-)}{\sqrt{a_q^-}}\right) = \left|\frac{2}{\sqrt{r_{\mathrm{A}}}} + \frac{p+q}{\sqrt{a}}\right|$$

and for given circles α and β, the value on the right side does not depend on the circle γ.

Proof. Let p and q be arbitrary integers. We invert the figure in the circle with center O and radius $k = 2\sqrt{ab}$, and label the images of all circles with a prime (see Figure 5). The circles ${\alpha_0^+}'$ and ${\beta_0^+}'$ always lie above the circles ${\alpha_0^-}'$ and ${\beta_0^-}'$ respectively. $\sigma(\alpha_p^+) = 1$ (resp. $\sigma(\alpha_q^-) = 1$) is equivalent to the fact that the center of ${\alpha_p^+}'$ (resp. ${\alpha_q^-}'$) lies in the region $y \geq 0$ (resp. $y \leq 0$) and $\sigma(\alpha_p^+, \alpha_q^-) = 1$ is equivalent to the fact that the y-coordinate of the center of ${\alpha_p^+}'$ is greater than or equal to the y-coordinate of the center of ${\alpha_q^-}'$. Since α' is a line parallel to the y-axis, the circles ${\alpha_p^+}'$ and ${\alpha_q^-}'$ are congruent, and we denote their common radius as a'. Similarly, we denote the common radius of the circles ${\beta_p^+}'$ and ${\beta_q^-}'$ as b'. Let us assume that ${\alpha_0^+}'$, ${\alpha_0^-}'$, ${\alpha_p^+}'$ and ${\alpha_q^-}'$ touch the y-axis at the points S, T, P and Q. If α_p^+ is a proper circle, the inversion center O is also the center of homothety of the circles α_p^+ and ${\alpha_p^+}'$ with homothety coefficient equal to the square of the radius of the inversion circle (i.e., to the power of inversion) divided by the power $O({\alpha_p^+}')$ of the point O to the inverted circle ${\alpha_p^+}'$: $k^2/O({\alpha_p^+}')$. Hence, the radius of α_p^+ can be expressed as $a_p^+ = k^2 a'/O({\alpha_p^+}')$ [5, p. 50]. The reciprocal value of this radius is then $1/a_p^+ = |OP|^2/(4aba')$. The last equation holds even if α_p^+ is a line parallel to the y-axis. Similarly, the reciprocal value of the radius of the circle α_q^- is equal to $1/a_q^- = |OQ|^2/(4aba')$. The segment length of the common external tangent of the externally touching circles γ', ${\alpha_0^+}'$, or γ', ${\alpha_0^-}'$ between the tangency points is equal to $|ST|/2 = 2\sqrt{(a'+b')a'}$. Consequently,

$$\sigma(\alpha_p^+, \alpha_q^-)\left(\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_q^-)}{\sqrt{a_q^-}}\right) = \sigma(\alpha_p^+, \alpha_q^-)\left(\frac{\sigma(\alpha_p^+)|OP| + \sigma(\alpha_q^-)|OQ|}{2\sqrt{ab}\sqrt{a'}}\right)$$

$$= \frac{|PQ|}{2\sqrt{ab}\sqrt{a'}} = \frac{||ST| + 2pa' + 2qa'|}{2\sqrt{ab}\sqrt{a'}} = \frac{\left|4\sqrt{(a'+b')a'} + 2(p+q)a'\right|}{2\sqrt{ab}\sqrt{a'}}.$$

Since $4aa' = 4bb' = 4ab$ by the definition of inversion, we get $a' = b$ and $b' = a$, and we finally obtain

$$\sigma(\alpha_p^+, \alpha_q^-)\left(\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_q^-)}{\sqrt{a_q^-}}\right) = \left|2\sqrt{\frac{1}{a} + \frac{1}{b}} + \frac{p+q}{\sqrt{a}}\right|.$$

The proof of the theorem is now complete. □

We can get a similar expression for the radii of the circles β_r^+ and β_s^- for any integers s and r. According to the proof of Theorem 1, the circles α_p^+ and α_q^- coincide if and only if $P = Q$ and this is also equivalent to

$$\sqrt{1+\frac{a}{b}} = -\frac{p+q}{2}.$$

Hence, we obtain the following corollary:

Corollary 2. *The two chains* $\{\cdots, \alpha_{-2}^+, \alpha_{-1}^+, \alpha_0^+, \alpha_1^+, \alpha_2^+, \cdots\}$ *and* $\{\cdots, \alpha_{-2}^-, \alpha_{-1}^-, \alpha_0^-, \alpha_1^-, \alpha_2^-, \cdots\}$ *coincide if and only if there is an integer* n *such that*

$$\frac{a}{b} = \frac{n^2}{4} - 1.$$

In this event, $\alpha_p^+ = \alpha_{-|n|-p}^-$ *for any integer* p. *For given circles* α *and* β*, this property does not depend on the circle* γ *.*

From the inverted skewed arbelos (see Figure 5), it is easy to see that the circles α_p^+, α_p^-, β_q^+ and β_q^- have two common tangent circles for any integers p and q. The line passing through the center $O_{\gamma'}$ of the circle γ' and perpendicular to the y-axis is also perpendicular to the lines α' and β' and to the circle γ'. Let δ be the circle, which is inverted into this line. Since inversion preserves angles between circles or lines, the circle δ is centered on the y-axis and perpendicular to the circles α, β and γ. Consequently, the inversion in δ with positive power leaves the y-axis and these circles in place and exchanges α_p^+, α_p^- and β_q^+ and β_q^-, respectively. Since the inversion center is also the center of homothety of a circle and its image (external, if the inversion center is outside of the circle, and internal in the opposite case), the external center of similitude of the circles α_p^+ and α_p^- is the same point on the y-axis (the center of the circle δ) for any integer p. This point is also the external center of similitude of β_q^+ and β_q^- for any integer q.

Since $\sigma(\alpha_p^+, \alpha_{-p}^-) = \sigma(\beta_q^+, \beta_{-q}^-) = 1$ for any integers p and q, we get the following corollary:

Corollary 3. *For any integers* p *and* q*,*

$$\frac{\sigma(\alpha_p^+)}{\sqrt{a_p^+}} + \frac{\sigma(\alpha_{-p}^-)}{\sqrt{a_{-p}^-}} = \frac{\sigma(\beta_q^+)}{\sqrt{b_q^+}} + \frac{\sigma(\beta_{-q}^-)}{\sqrt{b_{-q}^-}} = \frac{2}{\sqrt{r_{\mathrm{A}}}}$$

and for given circles α *and* β*, the constant value on the right side does not depend on the circle* γ*.*

Corollary 4. *If* γ *touches* α *and* β *internally,*

$$\frac{1}{\sqrt{a_0^+}} + \frac{1}{\sqrt{a_0^-}} = \frac{1}{\sqrt{b_0^+}} + \frac{1}{\sqrt{b_0^-}} = \frac{2}{\sqrt{r_{\mathrm{A}}}}$$

and for given circles α *and* β*, the constant value on the right side does not depend on the circle* γ*.*

From the last corollary, it is obvious that Theorem 1 is a generalization of the existence of the twin circles of Archimedes in the ordinary arbelos.

3. The n-th twin circles of Archimedes (symmetrical case)

In this section, we demonstrate that in certain situations, a skewed arbelos also has a twin circle property, which is a generalization of the twin circles of Archimedes in an ordinary arbelos. We use the same notations as in the previous section. If one circle of the set $\{\alpha_n^+, \alpha_{-n}^-, \alpha_{-n}^+, \alpha_n^-\}$ is congruent to one circle from the set $\{\beta_n^+, \beta_{-n}^-, \beta_{-n}^+, \beta_n^-\}$ for some integer n, the congruent pair is called *a pair of the n-th twin circles of Archimedes*. The twin circles of Archimedes in the ordinary arbelos are represented by one pair of the 0-th twin circles.

If the circles α, β and γ form an ordinary arbelos, the intersection of γ with the y-axis in the region $y > 0$ has the coordinates $(0, 2\sqrt{ab})$. For a real number z, the point $(0, 2\sqrt{ab}/z)$ is denoted by V_z and we consider V_0 to be the point at infinity on the y-axis. We show that $V_{n\pm1}$ are closely related to some pairs of the n-th twin circles of Archimedes. There are also other points on the y-axis, related to pairs of the n-th twin circles of Archimedes. For a real number z, consider the following points with the y-coordinates

$$\begin{aligned}
W_z^{++}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{z\left(\sqrt{a}+\sqrt{b}\right)+2\sqrt{a+b}}, \\
W_z^{--}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{z\left(\sqrt{a}+\sqrt{b}\right)-2\sqrt{a+b}}, \\
W_z^{+-}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{z\left(\sqrt{a}-\sqrt{b}\right)+2\sqrt{a+b}}, \\
W_z^{-+}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{z\left(\sqrt{a}-\sqrt{b}\right)-2\sqrt{a+b}}.
\end{aligned}$$

Reflecting the points V_z, W_z^{++} and W_z^{+-} in the x-axis, we get the points V_{-z}, W_{-z}^{--} and W_{-z}^{-+}. Since $\sqrt{2} \le 2\sqrt{a+b}/\left(\sqrt{a}+\sqrt{b}\right) < 2$, W_n^{++} and W_n^{--} cannot be the point at infinity on the y-axis for any integer n, but it can happen that each of W_n^{+-} and W_n^{-+} is identical with the point at infinity for some a, b and integer n. If the circle γ passes, for example, through both V_{n+1} and V_{n-1}, we say that γ passes through $V_{n\pm1}$.

Theorem 5. *Let n be an integer and $a \neq b$.*
(i) $1/a_n^+ = 1/b_n^+$ *if and only if the circle γ passes through $V_{n\pm1}$ or $W_{n\pm1}^{++}$. If γ passes through $V_{n\pm1}$,*

$$\frac{1}{a_n^+} = \frac{1}{b_n^+} = \left(n\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2 \tag{1}$$

and if γ passes through $W_{n\pm1}^{++}$,

$$\frac{1}{a_n^+} = \frac{1}{b_n^+} = \left(\left(n\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)^2. \tag{2}$$

(ii) $1/a^-_{-n} = 1/b^-_{-n}$ *if and only if the circle* γ *passes through* $V_{n\pm1}$ *or* $W^{--}_{n\pm1}$*. If* γ *passes through* $V_{n\pm1}$*,*

$$\frac{1}{a^-_{-n}} = \frac{1}{b^-_{-n}} = \left(-n\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2$$

and if γ *passes through* $W^{--}_{n\pm1}$*,*

$$\frac{1}{a^-_{-n}} = \frac{1}{b^-_{-n}} = \left(\left(-n\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)^2.$$

(iii) $1/a^+_{-n} = 1/b^-_{n}$ *if and only if the circle* γ *passes through* $V_{n\pm1}$ *or* $W^{+-}_{n\pm1}$*. If* γ *passes through* $V_{n\pm1}$*,*

$$\frac{1}{a^+_{-n}} = \frac{1}{b^-_{n}} = \left(-n\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2$$

and if γ *passes through* $W^{+-}_{n\pm1}$*,*

$$\frac{1}{a^+_{-n}} = \frac{1}{b^-_{n}} = \left(\left(-n\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)\right)^2.$$

(iv) $1/a^-_{n} = 1/b^+_{-n}$ *if and only if the circle* γ *passes through* $V_{n\pm1}$ *or* $W^{-+}_{n\pm1}$*. If* γ *passes through* $V_{n\pm1}$*,*

$$\frac{1}{a^-_{n}} = \frac{1}{b^+_{-n}} = \left(n\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2$$

and if γ *passes through* $W^{-+}_{n\pm1}$*,*

$$\frac{1}{a^-_{n}} = \frac{1}{b^+_{-n}} = \left(\left(n\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)\right)^2.$$

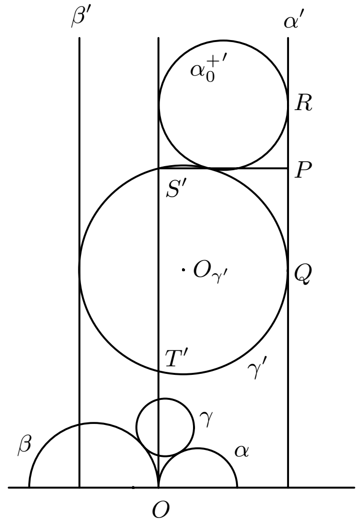


Figure 6.

Proof. Let S and T be the intersections of γ and the y-axis, where S lies on the arc or the line forming the boundary of the skewed arbelos. We denote the y-coordinates of S and T by s and t. If the circle γ touches α and β internally, $t < 0 < s$, otherwise $s < t$. We invert the figure in the circle centered at O and with radius $2\sqrt{ab}$ as in the proof of Theorem 1 (see Figure 6), label the images of all circles and points with a prime and denote the radii of $\alpha_n^{+\prime}$ and $\beta_n^{+\prime}$ by a' and b'. Then we obtain $a' = b$ and $b' = a$. Let the line parallel to the x-axis and passing through S' intersect the line α' at the point P. Let γ' and $\alpha_0^{+\prime}$ touch α' at the points Q and R, respectively, and let $O_{\gamma'}$ be the center of the circle γ'. From the right triangle formed by the lines $O_{\gamma'}S'$, $S'P$ and the line through $O_{\gamma'}$ parallel to the y-axis, we get $|PQ| = 2\sqrt{a'b'}$. The segment length of the common external tangent of the touching circles γ', $\alpha_0^{+\prime}$ between the tangency points is equal to $|QR| = 2\sqrt{(a'+b')a'}$. Hence, the reciprocal radius of α_n^+ is equal to

$$\begin{aligned}\frac{1}{a_n^+} &= \frac{O(\alpha_n^{+\prime})}{4aba'} = \frac{(s' - |PQ| + |QR| + 2na')^2}{4aba'} \\ &= \frac{(s' - 2\sqrt{a'b'} + 2\sqrt{(a'+b')a'} + 2na')^2}{4aba'} \\ &= \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb)^2}{4ab^2},\end{aligned}$$

where s' is the y-coordinate of the point S' and $O(\alpha_n^{+\prime})$ is the power of the point O to the inverted circle $\alpha_n^{+\prime}$. Therefore, $1/a_n^+ = 1/b_n^+$ is equivalent to

$$\frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb)^2}{4ab^2} = \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)a} + 2na)^2}{4a^2b}.$$

This quadratic equation for s' has two roots:

$$s' = 2(n+1)\sqrt{ab}. \tag{3}$$

and

$$s' = -2(n-1)\sqrt{ab} - \frac{4\sqrt{ab(a+b)}}{\sqrt{a}+\sqrt{b}}. \tag{4}$$

Since $ss' = 4ab$, these are equivalent to

$$s = \frac{2\sqrt{ab}}{n+1}$$

and

$$s = \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{(n-1)\left(\sqrt{a}+\sqrt{b}\right)+2\sqrt{a+b}}.$$

Hence, $1/a_n^+ = 1/b_n^+$ is equivalent to $S = V_{n+1}$ or $S = W_{n-1}^{++}$. If $S = V_{n+1}$, then

$$t' = s' - 2|PQ| = 2(n-1)\sqrt{ab},$$

where t' is the y-coordinate of the point T'. Hence,

$$t = \frac{4ab}{t'} = \frac{2\sqrt{ab}}{n-1},$$

and we obtain $T = V_{n-1}$. Similarly, $S = W_{n-1}^{++}$ implies $T = W_{n+1}^{++}$. Assume now that the circle γ passes through $V_{n\pm1}$. If $S = V_{n-1}$ and $T = V_{n+1}$, we would have

$$s' - t' = \frac{4ab}{s} - \frac{4ab}{t} = -4\sqrt{ab} < 0,$$

which contradicts to the fact $s' > t'$. Therefore, $S = V_{n+1}$ and s' is given by equation (3). Consequently, we arrive to equation (1):

$$\frac{1}{a_n^+} = \frac{\left(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb\right)^2}{4ab^2} = \left(n\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)+\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2.$$

If γ passes through $W_{n\pm1}^{++}$, $S = W_{n-1}^{++}$. For if $S = W_{n+1}^{++}$, we would again have

$$s' - t' = \frac{4ab}{s} - \frac{4ab}{t} = -4\sqrt{ab} < 0,$$

which is a contradiction. Using equation (4), we arrive to equation (2):

$$\begin{aligned}\frac{1}{a_n^+} =& \frac{\left(-2n\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb - \frac{4\sqrt{ab(a+b)}}{\sqrt{a}+\sqrt{b}}\right)^2}{4ab^2} \\ =& \left(\left(n\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)+\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)^2.\end{aligned}$$

Cases (ii), (iii) and (iv) can be proved similarly as case (i). The reciprocal radii $1/a^-_{-n}$, $1/a^+_{-n}$ and $1/a^-_n$ are equal to

$$\frac{1}{a^-_{-n}} = \frac{(s' - |PQ| - |QR| + 2na')^2}{4aba'} = \frac{(s' - 2\sqrt{ab} - 2\sqrt{(a+b)b} + 2nb)^2}{4ab^2},$$

$$\frac{1}{a^+_{-n}} = \frac{(s' - |PQ| + |QR| - 2na')^2}{4aba'} = \frac{(s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} - 2nb)^2}{4ab^2},$$

$$\frac{1}{a^-_{n}} = \frac{(s' - |PQ| - |QR| - 2na')^2}{4aba'} = \frac{(s' - 2\sqrt{ab} - 2\sqrt{(a+b)b} - 2nb)^2}{4ab^2}.$$

One root of the quadratic equations corresponding to cases (ii), (iii) and (iv) is always given by equation (3) and the other roots are

$$s' = -2(n-1)\sqrt{ab} + \frac{4\sqrt{ab(a+b)}}{\sqrt{a}+\sqrt{b}}, \tag{5}$$

$$s' = -2(n-1)\sqrt{ab} - \frac{4\sqrt{ab(a+b)}}{\sqrt{a}-\sqrt{b}}, \tag{6}$$

$$s' = -2(n-1)\sqrt{ab} + \frac{4\sqrt{ab(a+b)}}{\sqrt{a}-\sqrt{b}}. \tag{7}$$

$\square$

If the circle γ passes through the point $V_{n\pm1}$, we label the arbelos as $(V_{n\pm1})$. The arbeloi $(W^{++}_{n\pm1})$, $(W^{--}_{n\pm1})$, $(W^{+-}_{n\pm1})$ and $(W^{-+}_{n\pm1})$ are defined similarly. Reflecting the arbeloi $(V_{n\pm1})$, $(W^{++}_{n\pm1})$, $(W^{+-}_{n\pm1})$ in the x-axis yields the arbeloi $(V_{-n\pm1})$, $(W^{--}_{-n\pm1})$, $(W^{-+}_{-n\pm1})$, respectively. Equation (3) is obtained, when the signs of the expressions $s' - 2\sqrt{ab} + 2\sqrt{(a+b)b} + 2nb$ and $s' - 2\sqrt{ab} + 2\sqrt{(a+b)a} + 2na$ are the same. This implies that in $(V_{n\pm1})$, the centers of the circles α^+_n and β^+_n lie on the same side of the x-axis. On the other hand, equation (4) is obtained, when the signs of these expressions are different from each other. Consequently, in $(W^{++}_{n\pm1})$, the centers of α^+_n and β^+_n lie on the opposite sides of the x-axis. Similarly, we can find, on which sides of the x-axis lie the centers of the n-th twin circles of Archimedes in the remaining arbeloi. These results are arranged in Table 1.

	$(V_{n\pm1})$	$(W^{++}_{n\pm1})$	$(W^{--}_{n\pm1})$	$(W^{+-}_{n\pm1})$	$(W^{-+}_{n\pm1})$
same side	α^+_n, β^+_n $\alpha^-_{-n}, \beta^-_{-n}$			α^+_{-n}, β^-_n	α^-_n, β^+_{-n}
opposite side	α^+_{-n}, β^-_n α^-_n, β^+_{-n}	α^+_n, β^+_n	$\alpha^-_{-n}, \beta^-_{-n}$		

Table 1.

According to Theorem 5, there are four different pairs of the n-th twin circles of Archimedes in $(V_{n\pm1})$, for any non-zero integer n (see Figure 9). In this case, γ touches α and β externally from below for $n \le -1$, internally for $n = 0$, externally from above for $n \ge 1$. The twin circles of Archimedes in the ordinary arbelos $(V_{0\pm1})$ and their radii are obtained for $n = 0$. Figures 7 and 8 show the other pairs of the 0-th twin circles of Archimedes in the arbeloi $(W^{++}_{0\pm1})$ and $(W^{+-}_{0\pm1})$. The 0-th twin circles of Archimedes in $(W^{--}_{0\pm1})$ and $(W^{-+}_{0\pm1})$ are obtained by reflecting these figures in the x-axis and exchanging all plus and minus signs in the notation.

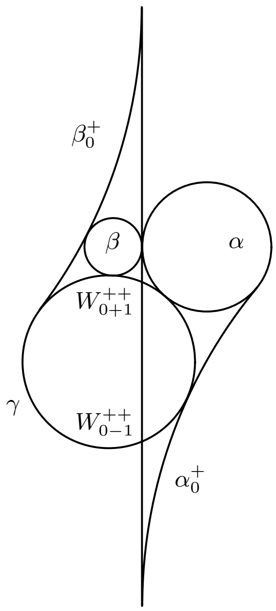


Figure 7. $a_0^+ = b_0^+$ for $(W^{++}_{0\pm1})$

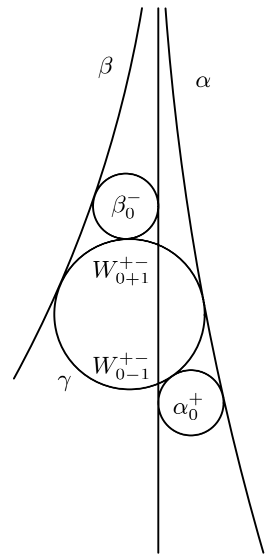


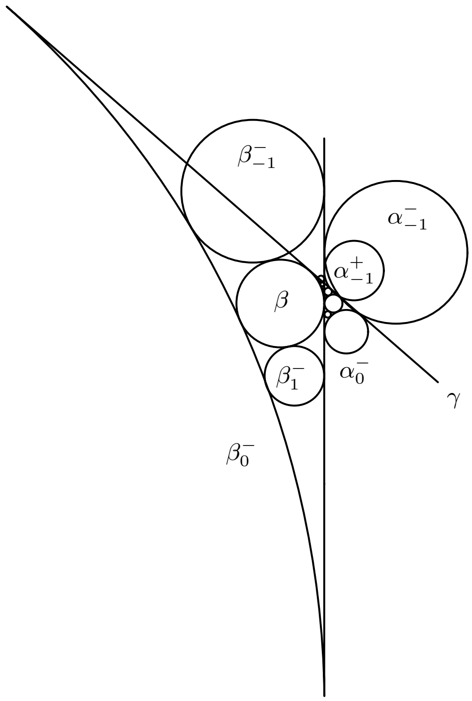
Figure 8. $a_0^+ = b_0^-$ for $(W^{+-}_{0\pm1})$

If γ is the common external tangent of α and β touching these circles from above, it passes through $V_{1\pm1}$, because this tangent bisects the segment OV_1 [2]. Hence, we get the following corollary (see Figure 9):

Corollary 6. *If γ is the common external tangent of α and β, touching these circles from above, then* (i) $a_1^+ = b_1^+$, (ii) $a_{-1}^- = b_{-1}^-$, (iii) $a_{-1}^+ = b_1^-$, (iv) $a_1^- = b_{-1}^+$, *and*

$$\text{(v)}\ \frac{1}{\sqrt{a_1^+}} = \frac{1}{\sqrt{a_1^-}} + \frac{1}{\sqrt{a_{-1}^+}} + \frac{1}{\sqrt{a_{-1}^-}} = \frac{1}{\sqrt{b_1^+}} = \frac{1}{\sqrt{b_1^-}} + \frac{1}{\sqrt{b_{-1}^+}} + \frac{1}{\sqrt{b_{-1}^-}}.$$

Proof. Since $1/\sqrt{a}$, $1/\sqrt{b}$, $1/\sqrt{r_A}$ satisfy the triangle inequality, relation (v) immediately follows from Theorem 5. □



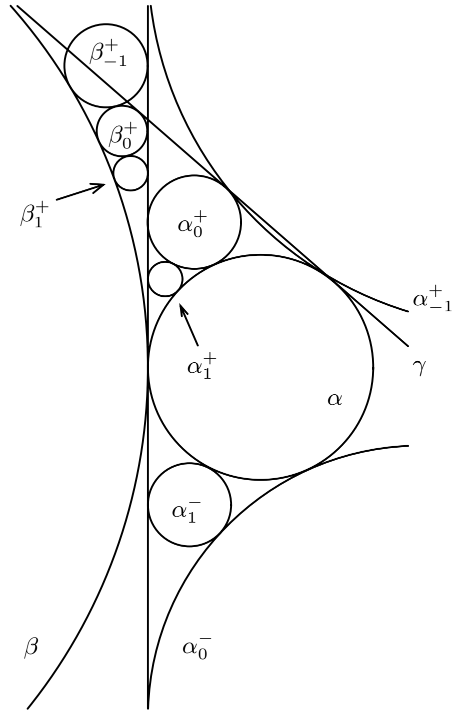


Figure 9. $a_{-1}^+ = b_1^-$, $a_{-1}^- = b_{-1}^-$ for $(V_{1\pm1})$ Magnified, $a_1^+ = b_1^+$, $a_1^- = b_{-1}^+$

Theorem 7. *Any circle touching α and β at points different from O passes through $V_{z\pm1}$ for some real number z. The proper circle touching α and β at points different from O and passing through $V_{z\pm1}$ for a real number $z \neq \pm1$ can be given by the equation*

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{8}$$

and conversely. The common external tangents of α and β can be expressed by the equations

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0, \tag{9}$$

which are obtained from equation (8) by approaching z to ±1.

Proof. We again invert the circles α, β and γ in the circle centered at O and with radius $2\sqrt{ab}$ as in the proofs of Theorems 1 and 5 and use the same notation. The circle γ is then carried into the circle γ' with radius $c' = a + b$, because $a' = b$ and $b' = a$. The intersection of the skewed arbelos boundary and the y-axis can be expressed as V_{z+1} for some real number z. Let t be the y-coordinate of the other intersection of γ and the y-axis. These intersections are carried into the intersections of γ' and the y-axis with the y-coordinates $s' = 4ab/s = 2(z+1)\sqrt{ab}$ and $t' = s' - 4\sqrt{ab} = 2(z-1)\sqrt{ab}$ (see the proof of Theorem 5), leading to $t = 4ab/t' = 2\sqrt{ab}/(z-1)$. Hence, the other intersection of γ and the y-axis is identical with the point V_{z-1}. Assume that γ is a proper circle passing through $V_{z\pm1}$ for a real number $z \neq \pm1$ and let (x_0, y_0) be the coordinates of the center of γ. Obviously, $y_0' = (s' + t')/2 = 2z\sqrt{ab}$ and $x_0' = (2a' - 2b')/2 = b - a$, where (x_0', y_0') are the coordinates of the center of γ'. The inversion center at the coordinate origin O is also the center of homothety of the circles γ and γ', with homothety coefficient equal to $h = 4ab/O(\gamma')$. Since $O(\gamma') = s't' = 4(z^2-1)ab$, this homothety coefficient is equal to $h = 1/(z-1)^2$. Hence, $x_0 = x_0'h = (b-a)/(z^2-1)$, $y_0 = y_0'h = 2z\sqrt{ab}/(z^2-1)$ and the radius of the circle γ is $c = c'h = (a+b)/|z^2-1|$, which leads to equation (8). The converse follows from the fact that (8) determines a circle touching α and β at points different from O and passing through V_{z+1} at the skewed arbelos boundary and this circle is then expressed by (8) again as we have already demonstrated. If $z \to \pm1$ and we neglect the terms quadratic in $z^2 - 1$ in (8), the remaining factors $z^2 - 1$ cancel out and we arrive to equation (9). □

4. Relationship of two skewed arbeloi

In this section, we analyze further properties of the skewed arbeloi $(V_{n\pm1})$, $(W^{++}_{n\pm1})$, $(W^{--}_{n\pm1})$, $(W^{+-}_{n\pm1})$ and $(W^{-+}_{n\pm1})$ for an arbitrary integer n and also consider properties of the circle orthogonal to α and β. We assume that the circles α and β are fixed. For these arbeloi, the circles formerly denoted by α^+_m for an integer m are now labeled explicitly as $\alpha^+_{n,m}$ and their radii as $a^+_{n,m}$. Similarly, we relabel the circles formerly denoted by α^-_m, β^+_m and β^-_m and their radii. The circle passing through $V_{z\pm1}$ and touching α and β at points different from O is denoted by γ_z for a real number z. If γ_z is a proper circle, it is expressed by (8), and the circle γ_n forms $(V_{n\pm1})$ with α and β. Reflecting the arbeloi $(V_{n\pm1})$, $(W^{++}_{n\pm1})$ and $(W^{+-}_{n\pm1})$ in the x-axis yields the arbeloi $(V_{-n\pm1})$, $(W^{--}_{-n\pm1})$, $(W^{-+}_{-n\pm1})$, respectively. Therefore $1/a^{\pm}_{n,m} = 1/a^{\mp}_{-n,m}$ and $1/b^{\pm}_{n,m} = 1/b^{\mp}_{-n,m}$ in the arbelos pairs $(V_{n\pm1})$ and $(V_{-n\pm1})$; $(W^{++}_{n\pm1})$ and $(W^{--}_{-n\pm1})$; $(W^{+-}_{n\pm1})$ and $(W^{-+}_{-n\pm1})$, but this is trivial.

Since the y-coordinates of the points $V_{n\pm1}$, $W^{++}_{n\pm1}$ and $W^{--}_{n\pm1}$ are symmetrical in a and b, the radii $b^{\pm}_{n,m}$ can be obtained from $a^{\pm}_{n,m}$ by replacing a with b and b with a in the arbeloi $(V_{n\pm1})$, $(W^{++}_{n\pm1})$ and $(W^{--}_{n\pm1})$. On the other hand, the y-coordinates of the points $W^{+-}_{n\pm1}$ and $W^{-+}_{n\pm1}$ are not symmetrical in a and b. Hence, we cannot draw the same conclusion for the arbeloi $(W^{+-}_{n\pm1})$ and $(W^{-+}_{n\pm1})$. Using the same notations as in the proof of Theorem 5, from equation (3) for the arbelos $(V_{n\pm1})$, we get

$$\frac{1}{a^{\pm}_{n,m}} = \frac{\left(s' - 2\sqrt{ab} \pm 2\sqrt{(a+b)b} \pm 2mb\right)^2}{4ab^2} = \left(\frac{n}{\sqrt{b}} \pm \frac{m}{\sqrt{a}} \pm \frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2.$$

Using equation (4) for the arbelos $(W^{++}_{n\pm1})$,

$$\frac{1}{a^{+}_{n,m}} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} + \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\frac{1}{\sqrt{r_{\mathrm{A}}}}\right)^2,$$

$$\frac{1}{a^-_{n,m}} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} + \frac{3\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2.$$

Using equation (5) for the arbelos $(W^{--}_{n\pm1})$,

$$\frac{1}{a^+_{n,m}} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} - \frac{3\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2,$$

$$\frac{1}{a^-_{n,m}} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} - \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2.$$

Using equation (6) for the arbelos $(W^{+-}_{n\pm1})$,

$$\frac{1}{a^+_{n,m}} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2,$$

$$\frac{1}{a^-_{n,m}} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} + \frac{3\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2,$$

$$\frac{1}{b^+_{n,m}} = \left(\frac{n}{\sqrt{a}} - \frac{m}{\sqrt{b}} - \frac{3\sqrt{b}-\sqrt{a}}{\sqrt{b}-\sqrt{a}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2,$$

$$\frac{1}{b^-_{n,m}} = \left(\frac{n}{\sqrt{a}} + \frac{m}{\sqrt{b}} - \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2.$$

Using equation (7) for the arbelos $(W^{-+}_{n\pm1})$,

$$\frac{1}{a^+_{n,m}} = \left(\frac{n}{\sqrt{b}} - \frac{m}{\sqrt{a}} - \frac{3\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2,$$

$$\frac{1}{a^-_{n,m}} = \left(\frac{n}{\sqrt{b}} + \frac{m}{\sqrt{a}} - \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2,$$

$$\frac{1}{b^+_{n,m}} = \left(\frac{n}{\sqrt{a}} - \frac{m}{\sqrt{b}} + \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2,$$

$$\frac{1}{b^-_{n,m}} = \left(\frac{n}{\sqrt{a}} + \frac{m}{\sqrt{b}} + \frac{3\sqrt{b}-\sqrt{a}}{\sqrt{b}-\sqrt{a}}\frac{1}{\sqrt{r_\mathrm{A}}}\right)^2.$$

By comparing the above equations, we obtain the following theorem (see Figure 10):

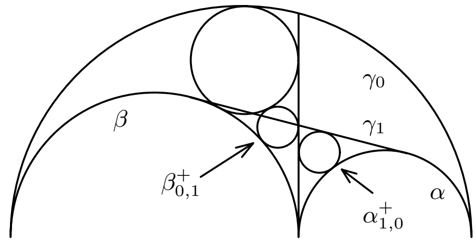


Figure 10. $a^+_{1,0} = b^+_{0,1}$ for $(V_{1\pm1})$ and $(V_{0\pm1})$

Theorem 8. *Let n and m be integers.*
(i) *For $(V_{n\pm1})$ and $(V_{m\pm1})$, we have $1/a^+_{n,m} = 1/b^+_{m,n}$, $1/b^+_{n,m} = 1/a^+_{m,n}$, $1/a^-_{n,-m} = 1/b^-_{m,-n}$, and $1/b^-_{n,-m} = 1/a^-_{m,-n}$.*
(ii) *For $(W^{++}_{n\pm1})$ and $(W^{++}_{m\pm1})$, we have $1/a^+_{n,m} = 1/b^+_{m,n}$ and $1/b^+_{n,m} = 1/a^+_{m,n}$.*
(iii) *For $(W^{--}_{n\pm1})$ and $(W^{--}_{m\pm1})$, we have $1/a^-_{n,-m} = 1/b^-_{m,-n}$ and $1/b^-_{n,-m} = 1/a^-_{m,-n}$.*
(iv) *For $(W^{+-}_{n\pm1})$ and $(W^{+-}_{m\pm1})$, we have $1/a^+_{n,-m} = 1/b^-_{m,n}$.*
(v) *For $(W^{-+}_{n\pm1})$ and $(W^{-+}_{m\pm1})$, we have $1/a^-_{n,m} = 1/b^+_{m,-n}$.*
(vi) *For $(W^{--}_{n\pm1})$ and $(W^{++}_{m\pm1})$, we have $1/a^-_{n,m} = 1/b^+_{m,-n}$ and $1/b^-_{n,m} = 1/a^+_{m,-n}$.*
(vii) *For $(W^{+-}_{n\pm1})$ and $(W^{-+}_{m\pm1})$, we have $1/a^+_{n,m} = 1/b^+_{m,n}$ and $1/b^-_{n,-m} = 1/a^-_{m,-n}$.*

For different real numbers z and w, $\zeta^\alpha_{z,w}$ is the circle touching α, γ_z and γ_w and passing through neither the tangency point of α and γ_z nor the tangency point of α and γ_w and different from β. Similarly the circle $\zeta^\beta_{z,w}$ is defined. In the figure formed by $(V_{0\pm1})$ and $(V_{1\pm1})$, two other congruent pairs of inscribed circles can be found (see Figure 11).

Theorem 9. *The circle inscribed in the curvilinear triangle formed by γ_0, the y-axis, and one of the twin circles of Archimedes touching β is congruent to $\zeta^\alpha_{0,1}$.*

To prove this theorem, we use the following result of the old Japanese geometry [7] (see Figure 12):

Lemma 10. *Assume that the circle C with radius r is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t. If two externally touching circles C_1 and C_2 with radii r_1 and r_2 also touch the chord t and the other arc of the circle C internally, then h, r, r_1 and r_2 are related as*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}.$$

Proof. The centers of C_1 and C_2 can be on the opposite sides of the normal dropped on t from the center of C or on the same side of this normal. From the right triangles formed by the centers of C and C_i $(i = 1, 2)$, the line parallel to t through the center of C, and the normal dropped on t from the center of C_i, we have

$$|\sqrt{(r-r_1)^2 - (h+r_1-r)^2} \pm \sqrt{(r-r_2)^2 - (h+r_2-r)^2}| = 2\sqrt{r_1 r_2},$$

where we used the fact that the segment length of the common external tangent of C_1 and C_2 between the tangency points is equal to $2\sqrt{r_1 r_2}$. The formula of the lemma follows from this equation. □

Now we can prove Theorem 9. The distance between the common external tangent of α and β and the midpoint of the minor arc of the circle γ_0 formed by this tangent is $2r_{\rm A}$ [2]. According to Lemma 10, the radii of the two inscribed circles are the root of the same quadratic equation

$$\frac{1}{r} + \frac{1}{a} + \frac{a+b}{ab} = 2\sqrt{\frac{(a+b)^2}{a^2br}}.$$

From Figure 11, it is obvious that one root of this quadratic equation is equal to b. The other root is then $a^2b/(a+2b)^2$. □

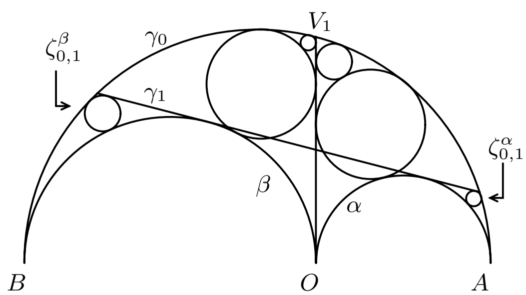


Figure 11. Two small congruent pairs

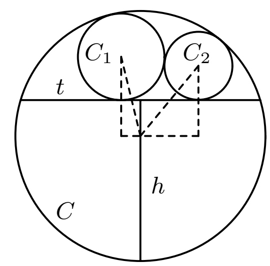


Figure 12.

Now we consider circles orthogonal to α and β. Let $t = (a+b)/\sqrt{ab}$ and let ϵ_z be the circle with a diameter OV_z for a real number z, where we consider ϵ_0 is identical with the x-axis. The mapping $\gamma_z \to \epsilon_z$ gives a one to one correspondence between the circles touching α and β at points different from O and the circles orthogonal to α and β. The circle ϵ_1 intersects α and γ_1 perpendicularly at their tangency point and the line segment AV_1 also passes through this point [2].

Theorem 11. *Let z and w be real numbers.*
(i) *The circle ϵ_z intersects α and γ_z perpendicularly at their tangency point and the line segment AV_z also passes through this point.*
(ii) *Let $w \neq 0$. The circle ϵ_z is orthogonal to any circle touching γ_{z-w} and γ_{z+w}. In particular ϵ_z intersects α and $\zeta^{\alpha}_{z-w,z+w}$ perpendicularly at their tangency point. If the two circles γ_{z-w} and γ_{z+w} intersect, ϵ_z also passes through their intersection.*
(iii) *The two circles γ_z and γ_w touch if and only if $z - w = \pm t$. The circle ϵ_z touches $\gamma_{z-t/2}$ and $\gamma_{z+t/2}$ at their tangency point.*
(iv) *The reciprocal radius of ϵ_{zt} is $|z|/r_{\rm A}$.*

Proof. We once again invert the circles in the circle centered at O and with radius $2\sqrt{ab}$ as in the proofs of Theorems 1, 5 and 7 and use the same notation.

The circle γ_z is then carried into the circle $\gamma_z{}'$ touching α' at a point with the y-coordinate $2z\sqrt{ab}$ as shown in the proof of Theorem 7 and ϵ_z is carried into the line $\epsilon_z{}'$: $y = 2z\sqrt{ab}$. This implies that ϵ_z intersects α and γ_z at their tangency point perpendicularly. The last part of (i) follows from the fact that the three points A', the tangency point of α' and $\gamma_z{}'$ and $V_z{}'$ lie on a circle passing through O in this order. (ii) follows from the fact that the two circles $\gamma_{z-w}{}'$ and $\gamma_{z+w}{}'$ are symmetrical in the line $\epsilon_z{}'$. The two circles $\gamma_z{}'$ and $\gamma_w{}'$ touch if and only if $2z\sqrt{ab} - 2w\sqrt{ab} = \pm 2(a+b)$ and this is equivalent to $z - w = \pm t$. This gives the first half part of (iii). The remaining part of (iii) and (iv) are now obvious. □

The circle $\zeta^{\alpha}_{z-w,z+w}$ touches α at a fixed point for any non-zero real number w, which is the intersection of α and ϵ_z by (ii) of the theorem. For any chain of circles touching α and β, the reciprocals of the radii of their associated circles orthogonal to α and β and the circles in this chain form a geometric progression by the first half part of (iii) and (iv) of the theorem, where we assume that the radius of the associated circle touching the x-axis from below has minus sign. In particular, starting with the ordinary arbelos, we get the chain of circles

$$\{\cdots, \gamma_{-2t}, \gamma_{-t}, \gamma_0, \gamma_t, \gamma_{2t}, \cdots\}$$

and the reciprocal radius of the circle ϵ_{nt} associated with γ_{nt} in this chain is n/r_{A}. In the case $n = 1$, we get the well-known fact that the circle orthogonal to α, β and the inscribed circle of the ordinary arbelos is congruent to the twin circles of Archimedes in the ordinary arbelos [1]. Now let us consider some other special cases of Theorem 11. In Figure 11, the circle with center V_1 passing through O, i.e., $\epsilon_{1/2}$, intersects α and $\zeta^{\alpha}_{0,1}$ (also β and $\zeta^{\beta}_{0,1}$) perpendicularly at their tangency point and also intersects γ_0 and γ_1 at their intersections. These results are obtained by letting $z = w = 1/2$ in (ii). The circle $\epsilon_{(n+1/2)t}$ with radius $r_{\mathrm{A}}/(n+\frac{1}{2})$ touches γ_{nt} and $\gamma_{(n+1)t}$ at their tangency point by (iii) and (iv). In particular the circle $\epsilon_{t/2}$, which is double the size of the twin circles of Archimedes in the ordinary arbelos, intersects α and $\zeta^{\alpha}_{0,t}$ (also β and $\zeta^{\beta}_{0,t}$) perpendicularly at their tangency point and also touches γ_0 and γ_t at their tangency point (see Figure 13).

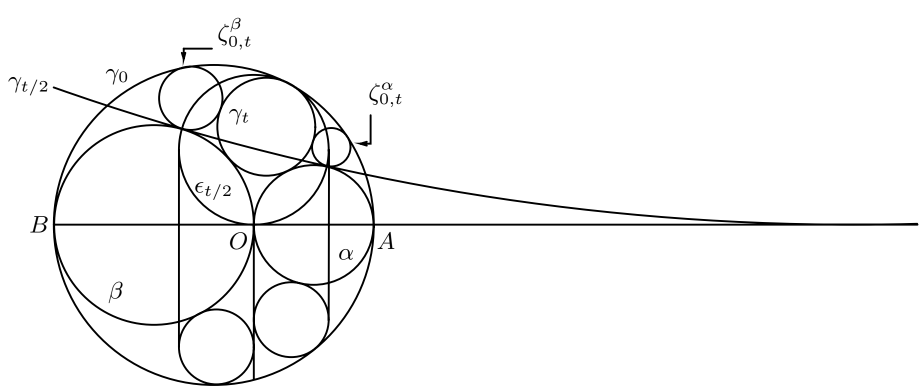


Figure 13.

There is a tangent between $\epsilon_{t/2}$ and each of the twin circles of Archimedes in the ordinary arbelos which is parallel to the y-axis. In order to avoid the overlapping circles, reflected twin circles of Archimedes in the x-axis are drawn in Figure 13. From (8) we can see that the circle $\gamma_{t/2}$ (also $\gamma_{-t/2}$) touches the x-axis.

5. The n-th twin circles of Archimedes (asymmetrical case)

To investigate further possibilities of the existence of pairs of the n-th twin circles of Archimedes, we define several other points on the y-axis, which are also related to some of those pairs. Consider the following points on the y-axis with given y-coordinates:

$$\begin{aligned}
X_{n,+}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)+\left(\sqrt{a}-\sqrt{b}\right)},\\
X_{n,-}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)-\left(\sqrt{a}-\sqrt{b}\right)};\\
Y_{n,+}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)+\left(\sqrt{a}+\sqrt{b}\right)},\\
Y_{n,-}: &\quad \frac{-2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)-\left(\sqrt{a}+\sqrt{b}\right)}.
\end{aligned}$$

Also,

$$\begin{aligned}
Z_{n,+}^{++}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)+\left(\sqrt{a}+\sqrt{b}\right)-2\sqrt{a+b}},\\
Z_{n,-}^{++}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)-\left(\sqrt{a}+\sqrt{b}\right)-2\sqrt{a+b}},\\
Z_{n,+}^{--}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)+\left(\sqrt{a}+\sqrt{b}\right)+2\sqrt{a+b}},\\
Z_{n,-}^{--}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{n\left(\sqrt{a}-\sqrt{b}\right)-\left(\sqrt{a}+\sqrt{b}\right)+2\sqrt{a+b}},\\
Z_{n,+}^{+-}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)+\left(\sqrt{a}-\sqrt{b}\right)-2\sqrt{a+b}},\\
Z_{n,-}^{+-}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)-\left(\sqrt{a}-\sqrt{b}\right)-2\sqrt{a+b}},\\
Z_{n,+}^{-+}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)+\left(\sqrt{a}-\sqrt{b}\right)+2\sqrt{a+b}},\\
Z_{n,-}^{-+}: &\quad \frac{2\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)}{n\left(\sqrt{a}+\sqrt{b}\right)-\left(\sqrt{a}-\sqrt{b}\right)+2\sqrt{a+b}}.
\end{aligned}$$

Reflecting the points $X_{n,+}$, $X_{n,-}$, $Y_{n,+}$, $Y_{n,-}$, $Z_{n,+}^{++}$, $Z_{n,-}^{++}$, $Z_{n,+}^{+-}$ and $Z_{n,-}^{+-}$ in the x-axis, we get the points $X_{-n,-}$, $X_{-n,+}$, $Y_{-n,-}$, $Y_{-n,+}$, $Z_{-n,-}^{--}$, $Z_{-n,+}^{--}$, $Z_{-n,-}^{-+}$ and $Z_{-n,+}^{-+}$, respectively. Since $-1 < (\sqrt{a}-\sqrt{b})/(\sqrt{a}+\sqrt{b}) < 1$, $X_{n,+}$ and $X_{n,-}$ cannot be the point at infinity on the y-axis for any integer n, if $a \neq b$. However, any of the other points can be identical with the point at infinity for some a and b and integer n. The proof of the next theorem is similar to the proof of Theorem 5.

Theorem 12. *Let n be an arbitrary integer and $a \neq b$.*

(i) $1/a_n^+ = 1/b_{-n}^+$ *if and only if the circle* γ *passes through* $X_{n,\pm}$ *or* $Z_{n,\pm}^{++}$. *If* γ *passes through* $X_{n,\pm}$,

$$\frac{1}{a_n^+} = \frac{1}{b_{-n}^+} = \left(n\frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} - 1\right)^2 \frac{1}{r_{\mathrm{A}}}$$

and if γ *passes through* $Z_{n,\pm}^{++}$,

$$\frac{1}{a_n^+} = \frac{1}{b_{-n}^+} = \left(\left(n\frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} - 1\right)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)^2 \frac{1}{r_{\mathrm{A}}}.$$

(ii) $1/a_{-n}^- = 1/b_n^-$ *if and only if the circle* γ *passes through* $X_{n,\pm}$ *or* $Z_{n,\pm}^{--}$. *If* γ *passes through* $X_{n,\pm}$,

$$\frac{1}{a_{-n}^-} = \frac{1}{b_n^-} = \left(n\frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} + 1\right)^2 \frac{1}{r_{\mathrm{A}}}$$

and if γ *passes through* $Z_{n,\pm}^{--}$,

$$\frac{1}{a_{-n}^-} = \frac{1}{b_n^-} = \left(\left(n\frac{\sqrt{a+b}}{\sqrt{a}-\sqrt{b}} + 1\right)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)^2 \frac{1}{r_{\mathrm{A}}}.$$

(iii) $1/a_{-n}^+ = 1/b_{-n}^-$ *if and only if the circle* γ *passes through* $Y_{n,\pm}$ *or* $Z_{n,\pm}^{+-}$. *If* γ *passes through* $Y_{n,\pm}$,

$$\frac{1}{a_{-n}^+} = \frac{1}{b_{-n}^-} = \left(n\frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - 1\right)^2 \frac{1}{r_{\mathrm{A}}}$$

and if γ *passes through* $Z_{n,\pm}^{+-}$,

$$\frac{1}{a_{-n}^+} = \frac{1}{b_{-n}^-} = \left(\left(n\frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - 1\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)\right)^2 \frac{1}{r_{\mathrm{A}}}.$$

(iv) $1/a_n^- = 1/b_n^+$ *if and only if the circle* γ *passes through* $Y_{n,\pm}$ *or* $Z_{n,\pm}^{-+}$. *If* γ *passes through* $Y_{n,\pm}$,

$$\frac{1}{a_n^-} = \frac{1}{b_n^+} = \left(n\frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} + 1\right)^2 \frac{1}{r_{\mathrm{A}}}$$

and if γ *passes through* $Z_{n,\pm}^{-+}$,

$$\frac{1}{a_n^-} = \frac{1}{b_n^+} = \left(\left(n\frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} + 1\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)\right)^2 \frac{1}{r_{\mathrm{A}}}.$$

Each of the propositions (i), (ii), (iii) and (iv) in Theorems 5 and 11 asserts the existence of two different pairs of the n-th twin circles of Archimedes in two different arbeloi, but the ratio of their radii is independent of n and the circle γ and always equal to $\left((\sqrt{a}+\sqrt{b})/(\sqrt{a}-\sqrt{b})\right)^{\pm 2}$.

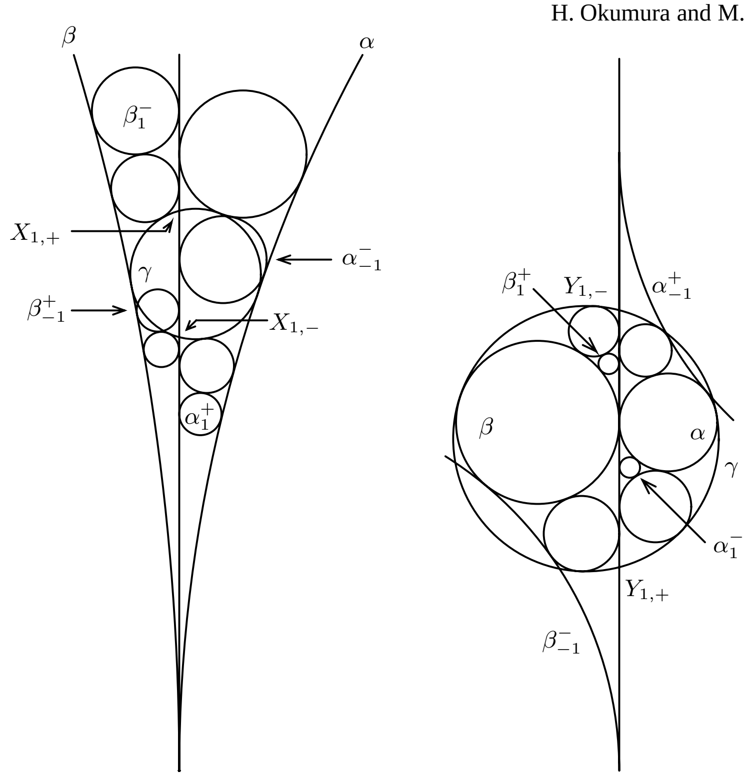


Figure 14. $a_1^+ = b_{-1}^+$, $a_{-1}^- = b_1^-$ for $(X_{1,\pm})$ $\quad$ $a_{-1}^+ = b_{-1}^-$, $a_1^- = b_1^+$ for $(Y_{1,\pm})$

If the circle γ passes through the points $X_{n,\pm}$, we label the arbelos as $(X_{n,\pm})$. The arbeloi $(Y_{n,\pm})$, $(Z_{n,\pm}^{++})$, $(Z_{n,\pm}^{--})$, $(Z_{n,\pm}^{+-})$ and $(Z_{n,\pm}^{-+})$ are defined similarly. Reflecting $(X_{n,\pm})$, $(Y_{n,\pm})$, $(Z_{n,\pm}^{++})$ and $(Z_{n,\pm}^{+-})$ in the x-axis, we get $(X_{-n,\pm})$, $(Y_{-n,\pm})$, $(Z_{-n,\pm}^{--})$ and $(Z_{-n,\pm}^{-+})$, respectively. Table 2 shows, on which sides of the x-axis lie the centers of the n-th twin circles of Archimedes in these arbeloi. According to Theorem 12, there are two pairs of the n-th twin circles of Archimedes in the arbeloi $(X_{n,\pm})$ and $(Y_{n,\pm})$ (see Figure 14).

	$(X_{n,\pm})$	$(Y_{n,\pm})$	$(Z_{n,\pm}^{++})$	$(Z_{n,\pm}^{--})$	$(Z_{n,\pm}^{+-})$	$(Z_{n,\pm}^{-+})$
same side	α_n^+, β_{-n}^+ α_{-n}^-, β_n^-				$\alpha_{-n}^+, \beta_{-n}^-$	α_n^-, β_n^+
opposite side		$\alpha_{-n}^+, \beta_{-n}^-$ α_n^-, β_n^+	α_n^+, β_{-n}^+	α_{-n}^-, β_n^-		

Table 2.

6. Another twin circle property

We demonstrate the existence of another pair of twin circles in the case, when the circle γ and the line joining the centers of α and β intersect. This pair of twin circles is a generalization of the circles W_6 and W_7 in [4]. A related result can be seen in [6]. We start by proving the following lemma:

Lemma 13. *Let A_0B_0 be the diameter of the circle γ parallel to the x-axis and intersecting the y-axis at the point O'. Let $a_0 = |A_0O'|$ and $b_0 = |B_0O'|$, where A_0 and B_0 lie on the same sides of the y-axis as the circles α and β, respectively. If γ touches α and β internally, $a/b = a_0/b_0$ and if γ touches α and β externally, $a/b = b_0/a_0$.*

Proof. Assume that γ touches α and β internally and $a < b$ (see Figure 15). Let O_α, O_β and O_γ be the centers of α, β and γ and F the foot of the normal dropped from O_γ to the x-axis. By Pythagorean theorem we get

$$|O_\gamma O_\alpha|^2 - |O_\alpha F|^2 = |O_\gamma O_\beta|^2 - |O_\beta F|^2.$$

Substituting $|O_\gamma O_\alpha| = (a_0 + b_0)/2 - a$, $|O_\gamma O_\beta| = (a_0 + b_0)/2 - b$, $|O_\alpha F| = a + |O_\gamma O'|$, $|O_\beta F| = b - |O_\gamma O'|$ and $|O_\gamma O'| = (a_0 + b_0)/2 - a_0$, we obtain $a/b = a_0/b_0$. The case, when γ touches α and β externally, can be proved in a similar way. $\square$

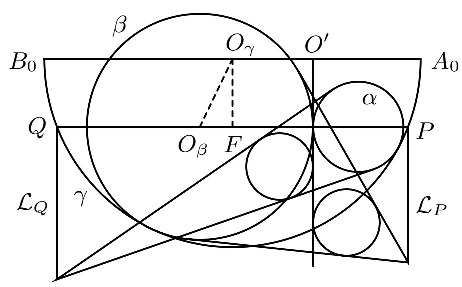


Figure 15.

Theorem 14. *Let AO and BO be the diameters of the circles α and β on the x-axis. Let P and Q be the intersections of the circle γ with the x-axis, choosing P and Q so that A, P, Q, B follow in this order on the x-axis, if we regard it as a circle of infinite radius closed through the point at infinity. Let $\mathcal{L}_P$ and $\mathcal{L}_Q$ be the lines through P and Q perpendicular to the x-axis. The circle touching the y-axis from the side opposite to β and the tangents to β from an arbitrary point on $\mathcal{L}_P$ is congruent to the circle touching the y-axis from the side opposite to α and the tangents to α from an arbitrary point on $\mathcal{L}_Q$.*

Proof. We use the same notation as in Lemma 13 and its proof. Assume that γ touches α and β internally and $a < b$. According to Lemma 13, there is a real number k, such that $a = a_0k$ and $b = b_0k$. Hence,

$$|O_\gamma F|^2 =((a_0 + b_0)/2 - b_0k)^2 - (b_0k - d)^2,$$
$$|QF|^2 =|O_\gamma Q|^2 - |O_\gamma F|^2 = 2a_0b_0k + d^2,$$

where $d = |OF| = (b_0 - a_0)/2$. Let r_b be the radius of the circle touching the y-axis from the side opposite to α and the common external tangents of α from an arbitrary point on $\mathcal{L}_Q$. Similarly, let r_a be the radius of the circle touching the y-axis from the side opposite to β and the common external tangent of β from an arbitrary point on $\mathcal{L}_P$. From the similarity of the circle with radius r_b and the circle α, we have

$$\frac{\sqrt{d^2 + 2a_0b_0k} + d - r_b}{r_b} = \frac{\sqrt{d^2 + 2a_0b_0k} + d + a_0k}{a_0k},$$

$$\frac{1}{r_b} = \frac{1}{a_0k} + \frac{\sqrt{d^2 + 2a_0b_0k} - d}{a_0b_0k}.$$

Similarly we obtain

$$\frac{1}{r_a} = \frac{1}{b_0k} + \frac{\sqrt{d^2 + 2a_0b_0k} + d}{a_0b_0k}.$$

But we can easily show that $1/r_a - 1/r_b = 0$ or $r_a = r_b$. The case, when γ touches α and β externally, can be proved in a similar way. □

Theorem 14 holds even in the case, when γ is one of the common external tangents of the circles α and β, if we consider γ to intersect the x-axis at the point at infinity. In this case, if $a < b$, these twin circles are congruent to α. If γ touches α and β internally, the minimum radii of these twin circles are equal to r_{A}, which is the case of the ordinary arbelos. If γ touches α and β externally, the radii of the twin circles are maximum in the case, when γ touches the x-axis. Let r be the maximum radius of the twin circles, c the radius of γ and d the distance of the tangency point of γ with the x-axis from the origin O and assume $a < b$. In this case

$$c^2 = (c + a)^2 - (d - a)^2 = (c + b)^2 - (d + b)^2.$$

Eliminating c and solving this equation for d, we get $d = 4ab/(b - a)$. From the similarity of the circle α and the corresponding twin circle, $(d-a)/a = (d+r)/r$, which implies $r = 2r_{\mathrm{A}}$. Consequently, we obtain that if $a < b$, $r_{\mathrm{A}} < a < 2r_{\mathrm{A}}$, and the the common radii of the twin circles take the minimum value r_{A} for the ordinary arbelos, a when γ is one of the common external tangents of α and β, and the maximum value $2r_{\mathrm{A}}$ when γ touches the x-axis. Since the circle γ touching the x-axis is identical with $\gamma_{\pm t/2}$ as mentioned at the end of §4, there is one more circle congruent to the twin circles in the last case, which is the circle $\epsilon_{\pm t/2}$ associated to $\gamma_{\pm t/2}$ by (iv) of Theorem 11 (see Figure 13).

7. Conclusion

We have demonstrated several interesting properties of the skewed arbelos, which could not have been found by consider the ordinary one. Since we confined our discussion largely to a generalization of the twin circles of Archimedes, it appears to be worth the effort to investigate other topics related to the skewed arbelos. We conclude our paper by proposing a problem. Let α, β and γ be three circles forming a skewed arbelos, i.e., γ is given by equations (8) or (9), and let δ be a circle touching α and β at their tangency point O and intersecting γ. The circle δ divides the skewed arbelos into two curvilinear triangles. Find (or construct) the circle δ, such that the incircles of the two curvilinear triangles are congruent (see Figure 16).

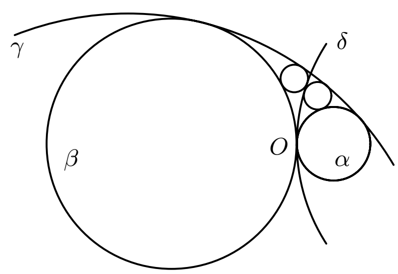


Figure 16.

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A Generalization of the Kiepert Hyperbola

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Abstract. Consider an arbitrary point P in the plane of triangle ABC with cevian triangle $A_1B_1C_1$. Erecting similar isosceles triangles on the segments BA_1, CA_1, CB_1, AB_1, AC_1, BC_1, we get six apices. If the apices of the two isosceles triangles with bases BA_1 and CA_1 are connected by a line, and the two similar lines for B_1 and C_1 are drawn, then these three lines form a new triangle, which is perspective to triangle ABC. For fixed P and varying base angle of the isosceles triangles, the perspector draws a hyperbola. Some properties of this hyperbola are studied in the paper.

1. Introduction

We consider the following configuration. Let P be a point in the plane of a triangle ABC, and AA_1, BB_1 and CC_1 be the three cevians of P. For an arbitrary nonzero angle φ satisfying $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, we erect two isosceles triangles BA_bA_1 and CA_cA_1 with the bases BA_1 and A_1C and base angle φ, both externally to triangle ABC if $\varphi > 0$, and internally otherwise. The same construction also gives the points B_c, B_a, C_a, C_b, with isosceles triangles all with base angle φ. This configuration depends on triangle ABC, the point P and $\varphi \in \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right) \setminus \{0\}$.

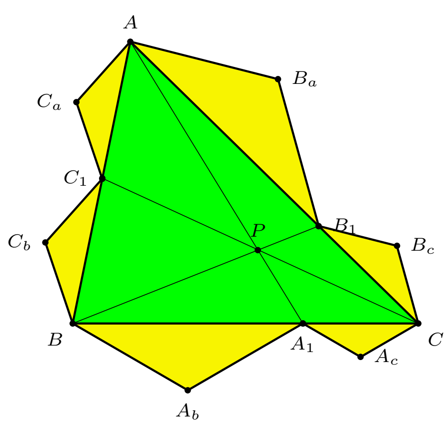

Figure 1.

We study an interesting locus problem associated with this configuration.

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2. The coordinates of the vertices

In this paper we work with homogeneous barycentric coordinates, and make use of John H. Conway's notations. See [1] for some basic properties of the Conway symbols. We begin by calculating the barycentric coordinates of the apices of our isosceles triangles. Let $(u : v : w)$ be the homogeneous barycentric coordinates of the point P.

Proposition 1. *The apices of the isosceles triangles on BA_1 and A_1C are the points*

$$A_b = (-a^2 w : 2S_\varphi v + (S_C + S_\varphi)w : (S_B + S_\varphi)w), \tag{1}$$

$$A_c = (-a^2 v : (S_C + S_\varphi)v : 2S_\varphi w + (S_B + S_\varphi)v). \tag{2}$$

Proof. Let A_φ be the apex of the isosceles triangle with base BC and base angle φ. It is well known that the point A_φ has the coordinates $(-a^2 : S_C + S_\varphi : S_B + S_\varphi)$. The line $A_b A_1$ is parallel to the line $A_\varphi C$; hence, using directed segments, we have $\frac{BA_b}{A_b A_\varphi} = \frac{BA_1}{A_1 C} = \frac{w}{v}$, so that (identifying every point with the vector to the point from an arbitrarily chosen origin),

$$A_b = \frac{vB + wA_\varphi}{v + w} = \left(-\frac{a^2 w}{2S_\varphi} : \frac{(S_C + S_\varphi)w}{2S_\varphi} + v : \frac{(S_B + S_\varphi)w}{2S_\varphi}\right).$$

Here, we have used the fact that the sum of the coordinates of the point A_φ is $-a^2 + (S_B + S_C) + 2S_\varphi = 2S_\varphi$.

This yields the coordinates of A_b given in (1) above. Similarly, A_c is as given in (2). The four remaining apices can be computed readily. □

Let $\mathcal{L}_a$ be the line joining the apices A_b and A_c. It is routine to compute the barycentric equation of the line $\mathcal{L}_a$.

Proposition 2. *The equation of the line $\mathcal{L}_a$ is*

$$(S_B v^2 + S_C w^2 + S_\varphi (v + w)^2)x + a^2 w^2 y + a^2 v^2 z = 0. \tag{3}$$

Proof. For $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$, the equation of the line joining A_b and A_c is

$$\begin{vmatrix} x & y & z \\ -a^2 w & 2S_\varphi v + (S_C + S_\varphi)w & (S_B + S_\varphi)w \\ -a^2 v & (S_C + S_\varphi)v & 2S_\varphi w + (S_B + S_\varphi)v \end{vmatrix} = 0.$$

This simplifies into (3) above. □

Similarly, we define $\mathcal{L}_b$ and $\mathcal{L}_c$. Their equations can be easily written down:

$$b^2 w^2 x + (S_C w^2 + S_A u^2 + S_\varphi (w + u)^2)y + b^2 u^2 z = 0, \tag{4}$$

$$c^2 v^2 x + c^2 u^2 y + (S_A u^2 + S_B v^2 + S_\varphi (u + v)^2)z = 0. \tag{5}$$

3. The triangle formed by the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$

Consider the triangle bounded by the lines $\mathcal{L}_a$, $\mathcal{L}_b$ and $\mathcal{L}_c$. This has vertices

$$A_2 = \mathcal{L}_b \cap \mathcal{L}_c, \qquad B_2 = \mathcal{L}_c \cap \mathcal{L}_a, \qquad C_2 = \mathcal{L}_a \cap \mathcal{L}_b.$$

Theorem 3. *The triangle bounded by the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ is perspective with ABC. Their axis of perspectivity is the trilinear polar of the barycentric square of the point P.*

Proof. Let $A_0 = BC \cap \mathcal{L}_a$, $B_0 = CA \cap \mathcal{L}_b$, and $C_0 = AB \cap \mathcal{L}_c$. In homogeneous barycentric coordinates, these are the points

$$A_0 = (0 : -v^2 : w^2), \quad B_0 = (u^2 : 0 : -w^2), \quad C_0 = (-u^2 : v^2 : 0)$$

respectively, and are all on the line

$$\frac{x}{u^2} + \frac{y}{v^2} + \frac{z}{w^2} = 0. \tag{6}$$

It follows from the Desargues theorem that ABC and the triangle bounded by the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are perspective. Note that the axis of perspectivity (6) is the trilinear polar of the point $(u^2 : v^2 : w^2)$, the barycentric square of P.[1] It is independent of φ. $\square$

The perspector of the triangles, however, varies with φ. We work out its coordinates explicitly. The vertices of the triangle in question are

$$A_2 = \mathcal{L}_b \cap \mathcal{L}_c, \qquad B_2 = \mathcal{L}_c \cap \mathcal{L}_a, \qquad C_2 = \mathcal{L}_a \cap \mathcal{L}_b.$$

From (4) and (5), the line joining A_2 to A has equation

$$\begin{aligned} &c^2(v^2(S_C w^2 + S_A u^2 + 2S_\varphi (w+u)^2) - b^2 w^2 u^2) y \\ -\, &b^2(w^2(S_A u^2 + S_B v^2 + 2S_\varphi (u+v)^2) - c^2 u^2 v^2) z = 0. \end{aligned}$$

Similarly, by writing down the equations of the lines BB_2 and CC_2, we easily find the perspector of the triangles ABC and $A_2B_2C_2$.

Theorem 4. *For any point P and any angle φ, the perspector of the triangles ABC and $A_2B_2C_2$ is the point*

$$K_P(\varphi) = \left(\frac{a^2}{-S_B v^2(w^2-u^2) + S_C w^2(u^2-v^2) + u^2(v+w)^2 S_\varphi} : \cdots : \cdots \right) \tag{7}$$

Strictly speaking, the perspector $K_P(\varphi)$ is not defined in the cases $\varphi = 0$ and $\varphi = \frac{\pi}{2}$. However, in these two cases we can define the perspectors as the limits of the perspector when the angle approaches 0 and $\frac{\pi}{2}$, respectively. The coordinates of these limiting perspectors can be obtained from (7) by substituting $\varphi = 0$ and $\frac{\pi}{2}$ respectively.

---

[1] If we take the harmonic conjugates A_0', B_0' and C_0' of the points A_0, B_0, C_0 with respect to the sides BC, CA, AB respectively, then the lines AA_0', BB_0' and CC_0' concur at the trilinear pole of the line $A_0B_0C_0$, which is the barycentric square of P. This gives an interesting construction of the barycentric square of a point. For another construction, see [2].

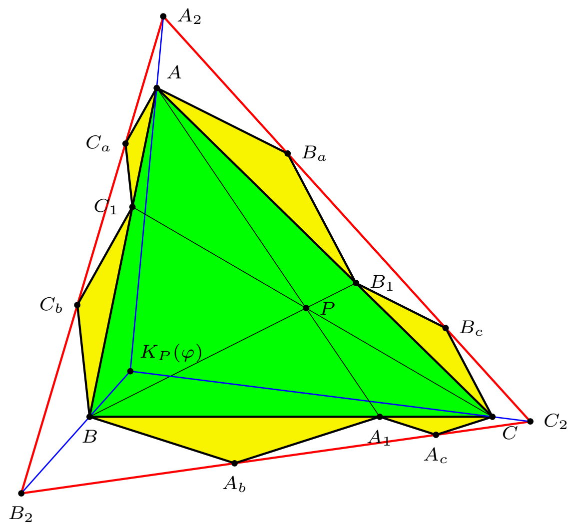

Figure 2.

$$K_P(0) = \left(\frac{a^2}{u^2(v+w)^2} : \frac{b^2}{v^2(w+u)^2} : \frac{c^2}{w^2(u+v)^2}\right),$$
$$K_P\left(\frac{\pi}{2}\right) = \left(\frac{a^2}{S_B\left(\frac{1}{w^2}-\frac{1}{u^2}\right) - S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right)} : \cdots : \cdots\right).$$

4. The locus of the perspector

From the coordinates of the perspector $K_P(\varphi)$ given in (7), it is clear that the point lies on the isogonal conjugate of the line joining the points

$$\begin{aligned} P_1 =&(u^2(v+w)^2 : v^2(w+u)^2 : w^2(u+v)^2), \\ P_2 =&(-S_B v^2(w^2-u^2) + S_C w^2(u^2-v^2) : \cdots : \cdots) \\ =&\left(S_B\left(\frac{1}{w^2}-\frac{1}{u^2}\right) - S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right) : \cdots : \cdots\right). \end{aligned}$$

Obviously P_1 is an interior point of triangle ABC. It is more interesting to note that P_2 is an infinite point, evidently of the line

$$S_A\left(\frac{1}{v^2}-\frac{1}{w^2}\right)x + S_B\left(\frac{1}{w^2}-\frac{1}{u^2}\right)y + S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right)z = 0. \qquad (8)$$

Note that $\left(\frac{1}{v^2}-\frac{1}{w^2} : \frac{1}{w^2}-\frac{1}{u^2} : \frac{1}{u^2}-\frac{1}{v^2}\right)$ is also an infinite point, of the line (6). From (8), these two lines are orthogonal. See [3, p.52].

Theorem 5. *Let $P = (u : v : w)$. The locus $\mathcal{K}_P$ is the isogonal conjugate of the line through the point $(u^2(v+w)^2 : v^2(w+u)^2 : w^2(u+v)^2)$ perpendicular to the trilinear polar of $(u^2 : v^2 : w^2)$.*

If P is not the centroid[2] and if this line does not pass through any of the vertices of ABC or its antimedial triangle, then the locus $\mathcal{K}_P$ is a circum-hyperbola,[3] which is rectangular if and only if P lies on the quintic

$$\begin{aligned}&a^2v^2w^2(v-w)+b^2w^2u^2(w-u)+c^2u^2v^2(u-v)\\=&uvw(u+v+w)((b^2-c^2)u+(c^2-a^2)v+(a^2-b^2)w).\end{aligned} \tag{9}$$

We shall study the degenerate case in §6 below.

5. Special cases

5.1. *The orthocenter.* If $P=H$, the orthocenter, $P_1=(a^4:b^4:c^4)$ and the trilinear polar of $\left(\frac{1}{S_{AA}}:\frac{1}{S_{BB}}:\frac{1}{S_{CC}}\right)$ is the line $S_{AA}x+S_{BB}y+S_{CC}z=0$. The perpendicular from P_1 to this line is the line

$$\frac{b^2-c^2}{a^2}x+\frac{c^2-a^2}{b^2}y+\frac{a^2-b^2}{c^2}z=0,$$

which is clearly the Brocard axis OK. The locus $\mathcal{K}_H$ is therefore the Kiepert hyperbola $\mathcal{K}$. A typical point on $\mathcal{K}$ is the Kiepert perspector

$$K(\theta)=\left(\frac{1}{S_A+S_\theta}:\frac{1}{S_B+S_\theta}:\frac{1}{S_C+S_\theta}\right)$$

which is the perspector of the triangle of apices of isosceles triangles of base angles θ erected on the sides of triangle ABC.

Theorem 6. $K_H(\varphi)=K(\theta)$ *if and only if*

$$\cot\varphi(\cot\omega+\cot\theta)+\cot\theta\cot\omega+1=0, \tag{10}$$

where ω is the Brocard angle of triangle ABC.

Proof. From (7),

$$K_H(\varphi)=\left(\frac{1}{S_{BC}-S_{AA}+a^2S_\varphi}:\cdots:\cdots\right).$$

This is the same as $K(\theta)$ if and only if

$$\begin{aligned}&((S_{CA}-S_{BB}+b^2S_\varphi)(S_{AB}-S_{CC}+c^2S_\varphi),\cdots,\cdots)\\=&k((S_B+S_\theta)(S_C+S_\theta),\cdots,\cdots)\end{aligned}$$

for some k. These conditions are satisfied if and only if

$$k=(S_A+S_B+S_C+S_\varphi)^2,$$

and

$$S_\theta S_\varphi+(S_A+S_B+S_C)(S_\theta+S_\varphi)+S^2=0.$$

This latter condition translates into (10) above. □

[2] $K_G(\varphi)=K$, the symmedian point, for every φ.

[3] In fact, being the isogonal conjugate of the line P_1P_2, this is a circumconic. Since the line P_1P_2 intersects the circumcircle of triangle ABC (as P_1 is an interior point), it is a circum-hyperbola. The isogonal conjugate of the point P_2 is the fourth point of intersection of the circumscribed hyperbola with the circumcircle of triangle ABC.

Note that the relation (10) is symmetric in φ, ω, and θ. From this we obtain the following interesting corollary.

Corollary 7. *$K_H(\varphi) = K(\theta)$ if and only if $K_H(\theta) = K(\varphi)$.*

Here are some examples of corresponding φ and θ.

φ	$\frac{\pi}{4}$	$-\frac{\pi}{4}$	ω	$-\omega$
θ	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\arctan(\sin 2\omega)$	0

5.2. *The incenter*. If $P = I$, the incenter, we have

$$P_1 = (a^2(b+c)^2 : b^2(c+a)^2 : c^2(a+b)^2) = X_{1500}.$$

The point P_2 is the infinite point of the perpendicular to the Lemoine axis, namely,

$$X_{511} = (a^2(a^2(b^2+c^2) - (b^4+c^4)) : \cdots : \cdots),$$

the same as the case $P = H$. The hyperbola $\mathcal{K}_I$ is the circum-hyperbola through X_{98} and

$$X_{1509} = \left(\frac{1}{(b+c)^2} : \frac{1}{(c+a)^2} : \frac{1}{(a+b)^2}\right).$$

The center of the hyperbola is the point

$$((b-c)^2 f(a,b,c)g(a,b,c) : (c-a)^2 f(b,c,a)g(b,c,a) : (a-b)^2 f(c,a,b)g(c,a,b)),$$

where

$$\begin{aligned} f(a,b,c) =& a^5 - a^3(b^2+bc+c^2) - a^2(b+c)(b^2+c^2) - abc(b+c)^2 - b^2c^2(b+c), \\ g(a,b,c) =& a^5 - a^3(2b^2+bc+2c^2) - a^2(b+c)(2b^2+bc+2c^2) \\ & - a(b^4 - b^3c - 2b^2c^2 - bc^3 + c^4) - bc(b+c)^3. \end{aligned}$$

5.3. *The Gergonne point.* If P is the Gergonne point, P_1 is the symmedian point K and the infinite point of the perpendicular to the trilinear polar of

$$X_{279} = \left(\frac{1}{(b+c-a)^2} : \frac{1}{(c+a-b)^2} : \frac{1}{(a+b-c)^2}\right)$$

is

$$X_{517} = (a(a^2(b+c) - 2abc - (b+c)(b-c)^2) : \cdots : \cdots).$$

The hyperbola passes through the centroid and X_{104} and has center

$$(a^2(b-c)^2(a^3 - a^2(b+c) - a(b-c)^2 + (b+c)(b^2+c^2))^2 : \cdots : \cdots).$$

5.4. $P = X_{671}$. The point $X_{671} = \left(\frac{1}{2a^2-b^2-c^2} : \frac{1}{2b^2-c^2-a^2} : \frac{1}{2c^2-a^2-b^2}\right)$ lies on the quintic (9). It is the reflection of the centroid in the Kiepert center X_{115}. If $P = X_{671}$, the locus $\mathcal{K}_P$ is the rectangular hyperbola whose center is the point

$$((b^2-c^2)^2(2a^2-b^2-c^2)(a^4-b^4+b^2c^2-c^4) : \cdots : \cdots)$$

on the nine-point circle.

5.5. *P on a sideline.* If P is a point on a sideline of triangle ABC, say, BC, then $P_1 = (0 : 1 : 1)$ is the midpoint of BC, and $P_2 = (-a^2 : S_C : S_B)$ is the infinite point of the A-altitude. It follows that P_1P_2 is the perpendicular bisector of BC. Its isogonal conjugate is the circum-hyperbola whose center is the midpoint of BC. It also passes through the antipode of A in the circumcircle.

6. The degenerate case

The locus $\mathcal{K}_P$ is a circum-hyperbola if and only if the line P_1P_2 does not contain a vertex of the triangle. The equation of the line P_1P_2 is of the form

$$U(P)x + V(P)y + W(P)z = 0,$$

where

$$\begin{aligned} U(P) = & \, v^2(w+u)^2(w^2(u^2S_A + v^2S_B) - c^2u^2v^2) \\ & - w^2(u+v)^2(v^2(u^2S_A + w^2S_C) - b^2u^2w^2), \end{aligned}$$

and $V(P)$ and $W(P)$ are obtained from $U(P)$ by cyclic permutations of (u, v, w), (a, b, c) and (S_A, S_B, S_C). The locus of the perspectors $\mathcal{K}_P$ is degenerate (*i.e.*, it is not a hyperbola) if and only if at least one of the three coefficients $U(P)$, $V(P)$ and $W(P)$ in the equation of the line P_1P_2 is zero, *i.e.*, if the point P lies on at least one of the three curves of 8 degree defined by the equations $U(P) = 0$, $V(P) = 0$ and $W(P) = 0$. Each of these three curves contains the vertices of the triangle ABC, its centroid G, and also the vertices of the antimedial triangle $G_aG_bG_c$. Moreover, for any two of these three curves, the only real common points are these 7 points just listed. We conclude with the following observations.

- If P is one of the vertices A, B and C, then the locus $\mathcal{K}_P$ is not defined. It is possibly an isolated singularity of one or more of the curves $U(P) = 0$, $V(P) = 0$, and $W(P) = 0$.
- If P is a vertex of the antimedial triangle, then the locus $\mathcal{K}_P$ is the corresponding sideline of the triangle ABC. For example, $\mathcal{K}_{G_a} = BC$.
- If $P = G$, the centroid of triangle ABC, then $\mathcal{K}_P$ consists of one single point, the symmedian point K of triangle ABC.
- In all other degenerate cases, the hyperbola degenerates into a pair of lines, one of them being a sideline of the triangle, while the other one passes through the opposite vertex (but does not coincide with a sideline).

  If we put

  $$(u, v, w) = \left(\frac{1}{y+z-x}, \frac{1}{z+x-y}, \frac{1}{x+y-z}\right),$$

  the equation $U(P) = 0$ defines the quartic curve

  $$x = \frac{yz(S_By^2 - S_Cz^2)}{S_A(y-z)(y^2+z^2) + S_By^3 - S_Cz^3}$$

  with respect to the antimedial triangle of ABC. Figure 3 shows an example of these curves in which the vertex B is an isolated singularity of the curves $U(P) = 0$ and $W(P) = 0$.

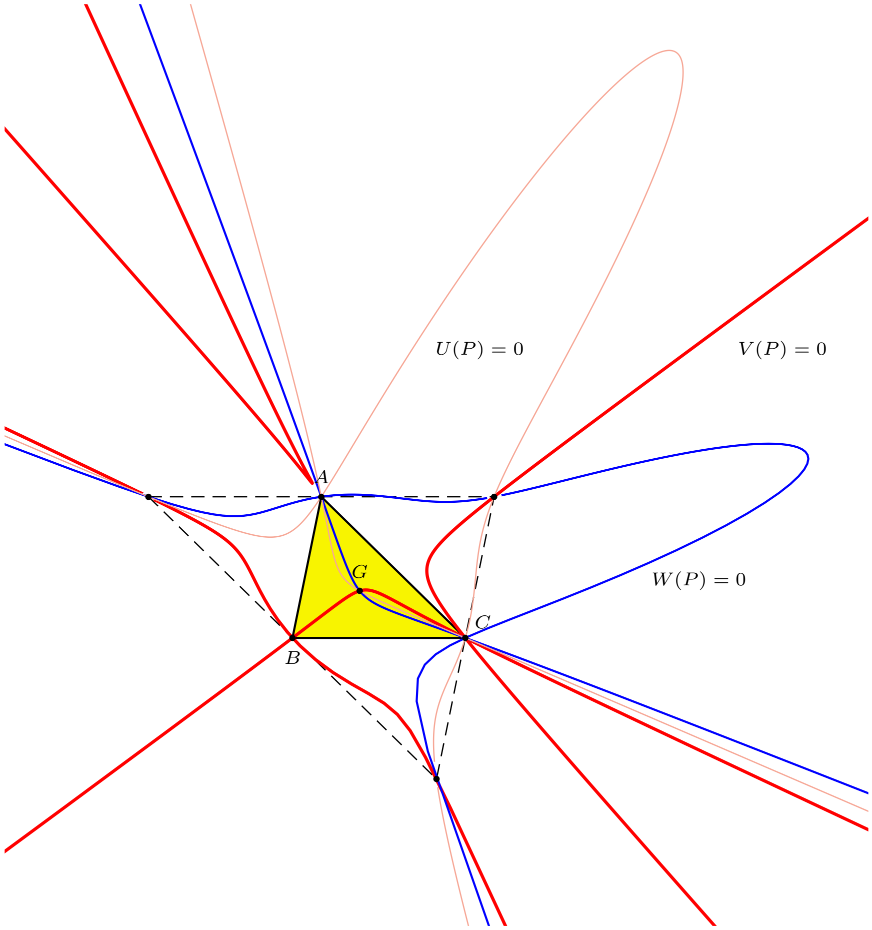

Figure 3.

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