J319. Let $0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1$ such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\frac{a_1}{a_2 - a_0} + \frac{a_2}{a_3 - a_1} + \cdots + \frac{a_n}{a_{n+1} - a_{n-1}} \geq \frac{1}{a_n}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

The expression on the left-hand-side can be rewritten as

$$\frac{a_1^2}{a_1a_2 - a_0a_1} + \frac{a_2^2}{a_2a_3 - a_1a_2} + \cdots + \frac{a_n^2}{a_na_{n+1} - a_{n-1}a_n}$$

Now applying the Cauchy-Schwartz Inequality on the expression, we have

$$\frac{a_1^2}{a_1a_2 - a_0a_1} + \frac{a_2^2}{a_2a_3 - a_1a_2} + \cdots + \frac{a_n^2}{a_na_{n+1} - a_{n-1}a_n}$$
$$\geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{a_1a_2 - a_0a_1 + a_2a_3 - a_1a_2 + \cdots + a_na_{n+1} - a_{n-1}a_n} = \frac{1}{a_na_{n+1} - a_0a_1} = \frac{1}{a_n}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Alok Kumar, Delhi, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arber Igrishta, Eqrem Qabej, Vushtrri, Kosovo; Arkady Alt, San Jose, California, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Daniel Văcaru, Pitești, Romania; David E. Manes, Oneonta, NY, USA; Ilyes Hamdi, Lycée Voltaire, Doha, Qatar; Farrukh Mukhammadiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Shatlyk Mamedov, Dashoguz , Turkmenistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; William Kang, Bergen County Academies, Hackensack, NJ, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Cody Johnson, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Michael Tang, Edina High School, MN, USA; Yong Xi Wang,East China Institute Of Technology, China; Yooree Ha, Ponte Vedra High School, Ponte Vedra, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

J320. Find all positive integers n for which $2014^n + 11^n$ is a perfect square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Prithwijit De, HBCSE, Mumbai, India

Let $p(n) = 2014^n + 11^n$ for $n \geq 1$. Then $p(1) = 2025 = 45^2$. Also $p(n)$ is odd for all n. We will show that for no other value of n is $p(n)$ a perfect square. For n even note that the last digit of $p(n)$ is 7. Thus it cannot be a perfect square.

For n odd and $n > 1$ note that $p(n)$ leaves 3 as remainder when divided by 8. But the square of any odd integer leaves 1 as remainder when divided by 8. Thus $p(n)$ cannot be a square for odd positive integers n greater than 1.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis ,Athens College (HAEF), Nea Penteli; William Kang, Bergen County Academies, Hackensack, NJ, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Cody Johnson, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Michael Tang, Edina High School, MN, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, Delhi, India; Arber Avdullahu, Mehmet Akif College, Kosovo; Arkady Alt, San Jose, California, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; David E. Manes, Oneonta, NY, USA; Ilyes Hamdi, Lycée Voltaire, Doha, Qatar; Jean Heibig, Paris, France; Farrukh Mukhammadiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; Paul Revenant,Lycée Champollion,Grenoble,France; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Yooree Ha, Ponte Vedra High School, Ponte Vedra, FL, USA.

J321. Let x, y, z be positive real numbers such that $xyz(x+y+z) = 3$. Prove that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{54}{(x+y+z)^2} \geq 9.$$

Proposed by Marius Stânean, Zalau, Romania

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy
Rewriting the inequality yields to

$$\frac{(xy)^2 + (yz)^2 + (zx)^2}{x^2y^2z^2} + \frac{54}{(x+y+z)^2} \geq 9$$

Define now the new variables

$$x + y + z = 3u, \quad xy + uyz + zx = 3v^2, \quad xyz = w^3$$

and trivial AGM yields $u \geq v \geq w$. We also use the well known inequality

$$xy + yz + zx \geq \sqrt{3(xyz)(x+y+z)} \iff 3v^2 \geq 3$$

that is $v \geq 1$.

The inequality becomes

$$w^3u = 1 \implies \frac{9v^4 - 6uw^3}{w^6} + \frac{9}{u^2} \geq 9$$

that is

$$f(u) \doteq (3v^4 - 2)u^2 + \frac{2}{u^2} - 3 \geq 0,$$

$$f'(u) = 2u(3v^4 - 2) - \frac{4}{u^3} = 0 \iff u = u_0(v) = \sqrt[4]{\frac{2}{3v^2 - 2}}$$

If $v \leq 2/\sqrt{3}$ then $u_0(v) \leq 1$, and this implies $f(u) \geq f(1)$.

$$f(1) = (3v^4 - 2)u^2 + \frac{2}{u^2} - 3 \geq 3v^4 - 2 + 2 - 3 \geq 3 \cdot 1 - 2 + 2 - 3 = 0$$

and this part of the proof is complete. Now let $v > 2/\sqrt{3}$.

$$f(u) = (3v^4 - 2)u^2 + \frac{2}{u^2} - 3 > (4 - 2)u^2 + \frac{2}{u^2} - 3 \geq 4 - 3 = 1$$

and also this part is complete.

Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis ,Athens College (HAEF), Nea Penteli; Timothy Chon, Horace Mann School, Bronx, NY, USA; Cody Johnson, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.

J322. Let ABC be a triangle with centroid G. The parallel lines through a point P situated in the plane of the triangle to the medians AA', BB', CC' intersect lines BC, CA, AB at A_1, B_1, C_1, respectively. Prove that

$$A'A_1 + B'B_1 + C'C_1 \geq \frac{3}{2}PG.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

Solution by Marius Stânean, Zalau, Romania

Let $(x : y : z)$ be the barycentric coordinates of P with respect to triangle ABC so that the three (signed) areas $[PBC]$, $[PCA]$, and $[PAB]$ are in the ratio $x : y : z$. We have

$$\frac{PA_1}{AA'} = \frac{[PAB]}{[ABC]} = x \Longrightarrow$$

$$\overrightarrow{A'A_1} = \overrightarrow{PA_1} - \overrightarrow{PG} - \overrightarrow{GA'} = 3x \cdot \overrightarrow{GA'} - \overrightarrow{PG} - \overrightarrow{GA'} =$$

$$(3x-1)\left(-\frac{1}{3}\cdot\overrightarrow{A} + \left(\frac{1}{2}-\frac{1}{3}\right)\cdot\overrightarrow{B} + \left(\frac{1}{2}-\frac{1}{3}\right)\cdot\overrightarrow{C}\right) - \overrightarrow{PG} =$$

$$\frac{3x-1}{6}\cdot\left(-2\cdot\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}\right) - \overrightarrow{PG}$$

Analog obtain

$$\overrightarrow{B'B_1} = \frac{3y-1}{6}\cdot\left(\overrightarrow{A} - 2\cdot\overrightarrow{B} + \overrightarrow{C}\right) - \overrightarrow{PG}$$

$$\overrightarrow{C'C_1} = \frac{3z-1}{6}\cdot\left(\overrightarrow{A} + \overrightarrow{B} - 2\cdot\overrightarrow{C}\right) - \overrightarrow{PG}$$

Therefore

$$\overrightarrow{A'A_1} + \overrightarrow{B'B_1} + \overrightarrow{C'C_1} = \frac{1-3x}{2}\cdot\overrightarrow{A} + \frac{1-3y}{2}\cdot\overrightarrow{B} + \frac{1-3z}{2}\cdot\overrightarrow{C} - 3\cdot\overrightarrow{PG} =$$

$$\frac{3}{2}\left(\left(\frac{1}{3}-x\right)\cdot\overrightarrow{A} + \left(\frac{1}{3}-y\right)\cdot\overrightarrow{B} + \left(\frac{1}{3}-z\right)\cdot\overrightarrow{C}\right) - 3\cdot\overrightarrow{PG} =$$

$$\frac{3}{2}\cdot\overrightarrow{PG} - 3\cdot\overrightarrow{PG} = -\frac{3}{2}\cdot\overrightarrow{PG}.$$

Considering this we have

$$\frac{3}{2}PG = \left|-\frac{3}{2}\cdot\overrightarrow{PG}\right| = \left|\overrightarrow{A'A_1} + \overrightarrow{B'B_1} + \overrightarrow{C'C_1}\right| \leq$$

$$\left|\overrightarrow{A'A_1}\right| + \left|\overrightarrow{B'B_1}\right| + \left|\overrightarrow{C'C_1}\right| = A'A_1 + B'B_1 + C'C_1.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Seung Hwan An, Taft School, Watertown, CT, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA.

J323. In triangle ABC,

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=\frac{\sqrt{5}-1}{2}.$$

Prove that $\max(A,B,C)>144^\circ$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Henry Ricardo, New York Math Circle

WLOG, assume that $\max\{A,B,C\}=A\leq 162^\circ$. Then $\sin A\geq\sin 162^\circ=\sin 18^\circ=\frac{\sqrt{5}-1}{4}$, $18^\circ\leq B+C<180^\circ$, and $\sin(B+C)\geq\frac{\sqrt{5}-1}{4}$.

Since $\sin B+\sin C>\sin(B+C)$ for $0<B,C<180^\circ$, we have

$$\begin{aligned}\frac{\sqrt{5}-1}{2} &= \sin A+\sin B+\sin C \;>\; \sin A+\sin(B+C)\\ &> \frac{\sqrt{5}-1}{4}+\frac{\sqrt{5}-1}{4}=\frac{\sqrt{5}-1}{2}.\end{aligned}$$

This contradiction establishes that $\max\{A,B,C\}>162^\circ$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Cody Johnson, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Arkady Alt, San Jose, California, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Yooree Ha, Ponte Vedra High School, Ponte Vedra, FL, USA.

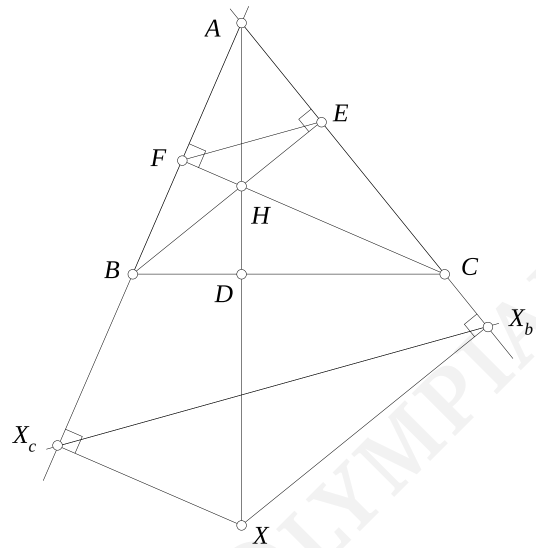
J324. Let ABC be a triangle and let X, Y , Z be the reflections of A, B, C in the opposite sides. Let X_b, X_c be the orthogonal projections of X on AC, AB, Y_c, Y_a the orthogonal projections of Y on BA, BC, and Z_a, Z_b the orthogonal projections of Z on CB, CA, respectively. Prove that X_b, X_c, Y_c, Y_a, Z_a, Z_b are concyclic.

Proposed by Cosmin Pohoata, Columbia University, USA

Solution by Ercole Suppa, Teramo, Italy

We first prove the following claims:

Claim 1. The lines X_bX_c, Y_aY_c, Z_aZ_b are antiparallel to BC, AC, AB respectively.



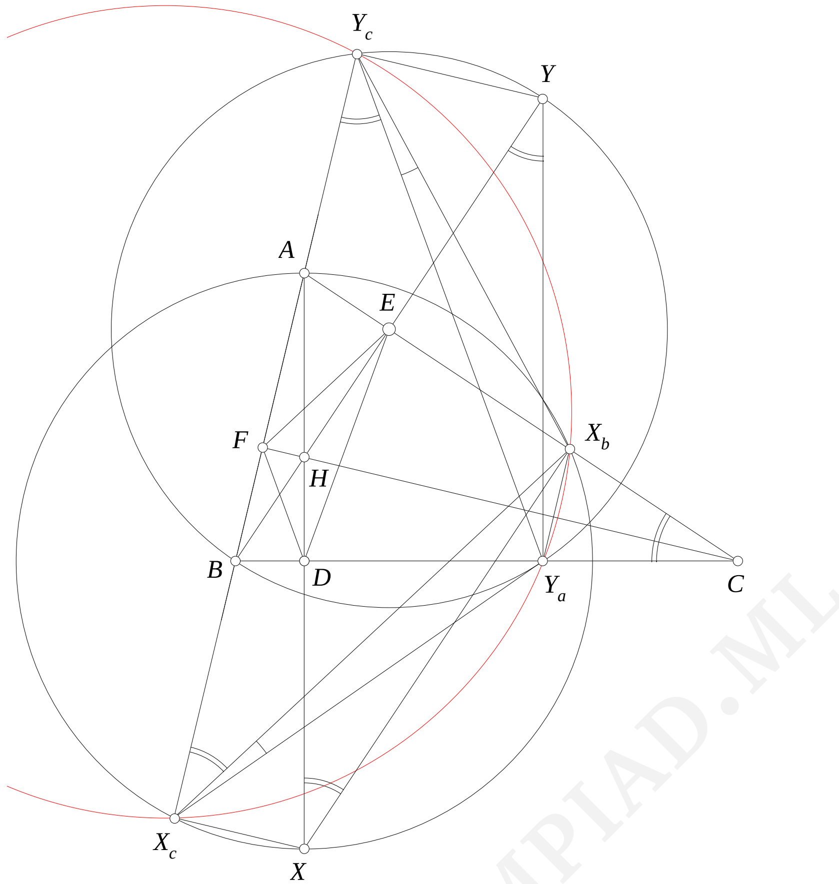
Proof of Claim 1. From the similar triangles $\triangle AHE \sim \triangle AXX_b$ and $\triangle AHF \sim \triangle AXX_c$ we get

$$AE : AX_b = AH : AX, \quad AF : AX_c = AH : AX \quad \Rightarrow \quad AE : AX_b = AF : AX_c$$

Therefore $\triangle AFE$ and $\triangle AX_cX_b$ are homotetic, so $X_bX_c \parallel EF$, i.e. X_bX_c is antiparallel to BC. A similar reasoning show that Y_aY_c is antiparallel to AC and Z_aZ_b is antiparallel to AB, as claimed. ■

Claim 2. The lines X_bY_a, Y_cZ_b, X_cZ_a are parallel to AB, BC, AC respectively.

Proof of Claim 2. Let $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$ and let D, E, F be the orthogonal projections of A, B, C on BC, CA, AB respectively, as shown in figure.



From the cyclic quadrilaterals BY_aYY_c and AX_cXX_b we have

$$\begin{aligned} Y_aY_c = BY \cdot \sin\beta = 2 \cdot AB \cdot \sin\alpha \sin\beta = AX \sin\alpha = X_bX_c \\ \angle X_cY_cY_a = \angle BYY_a = \gamma = \angle AXX_b = \angle Y_cX_cX_b \end{aligned} \tag{1}$$

whence it follows that $\triangle X_cY_cX_b$ and $\triangle X_cY_cY_a$ are congruent (SAS). Therefore

$$\angle X_cX_bY_c = \angle X_cY_aY_c$$

and this implies that X_c, X_b, Y_c, Y_a are concyclic, so

$$\angle Y_aY_cX_b = \angle X_bX_cY_a \tag{2}$$

Adding (1) and (2) gives

$$\angle X_cY_cX_b = \angle X_cY_cY_a + \angle Y_aY_cX_b = \angle Y_cX_cX_b + \angle X_bX_cY_a = \angle Y_cX_cY_a$$

Therefore $X_cY_aX_bY_c$ is an isosceles trapezoid. Thus $X_cY_c \parallel Y_aX_b$, i.e. $AB \parallel X_bY_a$. We can argue similarly to show that $BC \parallel Y_cZ_b$ and $AC \parallel X_cZ_a$ and the claim follows. ■

Returning to the original problem it suffices to notice that the above claims tell us that $X_cX_bY_aY_cZ_bZ_a$ is a Tucker hexagon, so X_b, X_c, Y_c, Y_a, Z_a, Z_b are concyclic, which is what we wanted to prove.

Also solved by Andrea Fanchini, Cantú, Italy; Mehtaab Sawhney, USA; Misiakos Panagiotis ,Athens College (HAEF), Nea Penteli; Cody Johnson, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania.

S319. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that for any positive real number t,

$$\left(at^2+bt+c\right)\left(bt^2+ct+a\right)\left(ct^2+at+b\right)\geq t^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the proposer
By Hölder's inequality:

$$(at^2+bt+c)(bt^2+ct+a)(ct^2+at+b)=(at^2+bt+c)(a+bt^2+ct)(at+b+ct^2)\geq(at+bt+ct)^3=t^3.$$

Second solution by Daniel Lasaosa, Pamplona, Spain
After some algebra, the inequality rewrites as

$$abc\left(t^6+1\right)+\left(a^2b+b^2c+c^2a\right)\left(t^5+2t^2\right)+\left(ab^2+bc^2+ca^2\right)\left(2t^4+t\right)+$$

$$+\left(a^3+b^3+c^3+4abc\right)t^3\geq t^3.$$

Now, by the AM-GM inequality, we have $t^6+1\geq 2t^3$, $t^5+2t^2\geq 3t^3$ and $2t^4+t=3t^3$, with equality iff $t=1$. It therefore remains only to prove that

$$2abc+3\left(a^2b+b^2c+c^2a\right)+3\left(ab^2+bc^2+ca^2\right)+\left(a^3+b^3+c^3+4abc\right)\geq 1,$$

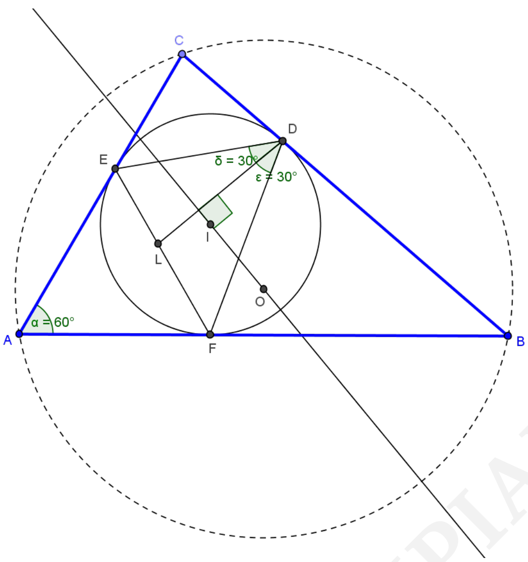
clearly true and with equality always because the LHS is nothing other than $(a+b+c)^3=1^3$. The conclusion follows, equality holds iff $t=1$.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Seung Hwan An, Taft School, Watertown, CT, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis ,Athens College (HAEF), Nea Penteli; William Kang, Bergen County Academies, Hackensack, NJ, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Yong Xi Wang,East China Institute Of Technology, China; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, Delhi, India; Arkady Alt, San Jose, California, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Marin Chirciu, Pitești, Romania; Farrukh Mukhammadiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Samin Riasat, University of Waterloo, ON, Canada; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

S320. Let ABC be a triangle with circumcenter O and incenter I. Let D, E, F be the tangency points of the incircle with BC, CA, AB, respectively. Prove that line OI is perpendicular to angle bisector of $\angle EDF$ if and only if $\angle BAC = 60°$.

Proposed by Marius Stânean, Zalau, Romania

Solution by Andrea Fanchini, Cantú, Italy



Using barycentric coordinates and the Conway's notation, if $\angle BAC = 60°$, then

$$a^2 = b^2 + c^2 - bc, \quad S_A = S_{60°} = \frac{bc}{2}, \quad S_B = \frac{2c^2 - bc}{2}, \quad S_C = \frac{2b^2 - bc}{2}, \quad S = \frac{\sqrt{3}}{2}bc, \tag{1}$$

Now line DE have equation

$$\begin{vmatrix} 0 & s-c & s-b \\ s-c & 0 & s-a \\ x & y & z \end{vmatrix} = 0 \quad \Rightarrow \quad DE \equiv (s-a)x + (s-b)y - (s-c)z = 0$$

and line DF have equation

$$\begin{vmatrix} 0 & s-c & s-b \\ s-b & s-a & 0 \\ x & y & z \end{vmatrix} = 0 \quad \Rightarrow \quad DF \equiv (s-a)x - (s-b)y + (s-c)z = 0$$

Now, the formula to calculate the angle between two lines is

$$S_\theta = S\cot\theta = \frac{S_A(q_1 - r_1)(q_2 - r_2) + S_B(r_1 - p_1)(r_2 - p_2) + S_C(p_1 - q_1)(p_2 - q_2)}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

so applying this to the $\angle EDF$ between the lines DE and DF we have

$$S_{EDF} = \frac{-a^2S_A - b(a-c)S_B + c(b-a)S_C}{}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ s-a & s-b & c-s \\ s-a & b-s & s-c \end{vmatrix}$$

and keeping in mind the (1), we have that

$$S_{EDF} = \frac{bc}{2} = S_{60^\circ}$$

so $\angle EDF = 60^\circ$.
Now we know that if θ is the oriented angle between the line $px+qy+rz=0$ and a line d, the coordinates of the infinite point of this line are

$$(pa^2 + q(S_\theta - S_C) - r(S_\theta + S_B) : qb^2 + r(S_\theta - S_A) - p(S_\theta + S_C) : rc^2 + p(S_\theta - S_B) - q(S_\theta + S_A))$$

in our case, the angle between line $DE \equiv (s-a)x + (s-b)y - (s-c)z = 0$ and the angle bisector DL is 30°, so the coordinates of the infinite point of DL are

$$\begin{gathered} DL_\infty(a^2(s-a) + (s-b)(\sqrt{3}S - S_C) + (s-c)(\sqrt{3}S + S_B) : \\ b^2(s-b) - (s-c)(\sqrt{3}S - S_A) - (s-a)(\sqrt{3}S + S_C) : \\ -c^2(s-c) + (s-a)(\sqrt{3}S - S_B) - (s-b)(\sqrt{3}S + S_A)) \end{gathered}$$

and using the (1) we obtain

$$DL_\infty(a(a-b+2c) : b(a-b-c) : c(2b-2a-c)).$$

Line OI have equation

$$\begin{vmatrix} a & b & c \\ a^2 S_A & b^2 S_B & c^2 S_C \\ x & y & z \end{vmatrix} = 0 \quad \Rightarrow \quad OI \equiv bc(cS_C - bS_B)x - ac(cS_C - aS_A)y + ab(bS_B - aS_A)z = 0$$

so the infinite point of line OI using the (1) is

$$OI_\infty(a(ab - 4bc + ac + b^2 + c^2) : b(-ab - 2bc + 2ac - b^2 + 2c^2) : c(2ab - 2bc - ac + 2b^2 - c^2))$$

Now we know that two lines with infinite points $(f : g : h)$ and $(f' : g' : h')$ are perpendicular to each other if and only if

$$S_A f f' + S_B g g' + S_C h h' = 0$$

and this is the case of our two infinite points DL_∞ and OI_∞, so line OI is perpendicular to angle bisector of $\angle EDF$ and also the contrary is immediately proved.

Also solved by Daniel Lasaosa, Pamplona, Spain; Mehtaab Sawhney, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Misiakos Panagiotis ,Athens College (HAEF), Nea Penteli; Ercole Suppa, Teramo, Italy; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania.

S321. Let x be a real number such that $x^m(x+1)$ and $x^n(x+1)$ are rational for some relatively prime positive integers m and n. Prove that x is rational.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Yujin Kim, Stony Brook School, Stony Brook, NY, USA

Suppose $m > n$ and denote $m - n = q > 0$. One can assume $x \neq 0, -1$.

$$\frac{x^m(x+1)}{x^n(x+1)} \in \mathbb{Q} \to x^2 \in \mathbb{Q}$$

$$\frac{[x^n(x+1)]^m}{[x^m(x+1)]^n} \in \mathbb{Q} \to \frac{x^{mn}(x+1)^m}{x^{mn}(x+1)^n} \in \mathbb{Q} \to (x+1)^q \in \mathbb{Q}$$

We will show that if x^2, $(x+1)^2$ are both rational and x is real.Thus x is rational.

Consider two polynomials: $R(z) = z^2 - x^2$ and $S(z) = (z+1)^2 - (x+1)^2$

Note that both R and S have rational coefficients and $R(x) = S(x) = 0$. Hence R and S have x as a common root. Suppose that $x \notin \mathbb{Q}$. Denote by $m(z)$ the minimal polynomial of x that has the smallest degree polynomial with rational coefficients which has x as a root. Such polynomial exists since $R(x) = S(x) = 0$. Since x is irrational, it follows that degree $m \geq 2$.

On the other hand, both $R(z)$ and $S(z)$ are multiples of $m(z)$. This means that R and S have some other common root besides x. Denote such a root by y. We have $y \neq x$. Note that y could be a complex unreal number.

We have $R(y) = S(y) = 0 \to y^\Sigma = x^\Sigma$ and $(1+y)^\Sigma = (1+x)^\Sigma$.

Consequently, $|y|^\Sigma = |x|^\Sigma$ and $|1+y|^\Sigma = |1+x|^\Sigma \to$

$$\begin{cases} |y| = |x| & \text{Recall that } x \in \mathbb{R} \\ |1+y| = |1+x| & \text{Denote } y = a + ib \end{cases}$$

Then, $|a+ib| = |x| \to \sqrt{a^2+b^2} = |x| \to a^2 + b^2 = x^2$ Also $|1+a+ib| = |1+x| \to \sqrt{(1+a)^2+b^2} = |1+x| \to (1+a)^2 + b^2 = (1+x)^2 \to 1 + 2a + a^2 + b^2 = 1 + 2x + x^2 \to a = x$ and since $a^2 + b^2 = x^2$ this implies $b = 0$.

But then $y = a + ib = x + i \cdot 0 = x$, contradiction. The conclusion follows.

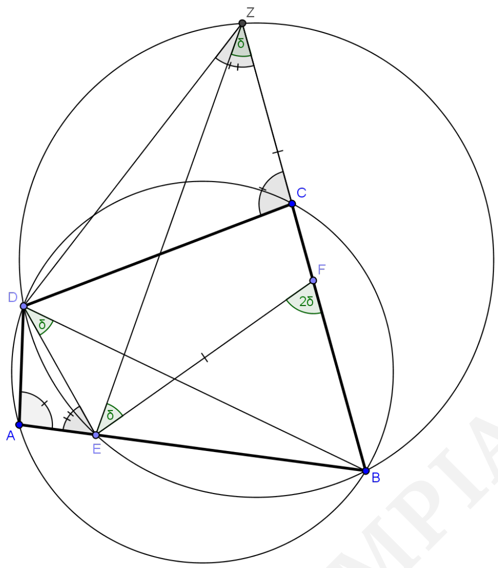
Also solved by Seung Hwan An, Taft School, Watertown, CT, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Timothy Chon, Horace Mann School, Bronx, NY, USA; Yooree Ha, Ponte Vedra High School, Ponte Vedra, FL, USA.

S322. Let $ABCD$ be a cyclic quadrilateral. Points E and F lie on the sides AB and BC, respectively, such that $\angle BFE = 2\angle BDE$. Prove that

$$\frac{EF}{AE} = \frac{FC}{AE} + \frac{CD}{AD}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Andrea Fanchini, Cantú, Italy



Let Z be the intersection point between BC and the circumcircle of $\triangle DEB$.
Then $\angle BZE = \angle BDE = \delta$
Now if we consider $\triangle EFZ$, we have

$$\angle FEZ + \angle EZF + \angle ZFE = \pi \quad \Rightarrow \quad \angle FEZ = \pi - \delta - (\pi - 2\delta) = \delta$$

Therefore $\triangle EFZ$ is isosceles and then $EF = FZ$.

Now being $ABCD$ a cyclic quadrilateral

$$\angle BAD + \angle BCD = \pi \quad \Rightarrow \quad \angle DCZ = \angle BAD$$

Furthermore being $BEDZ$ a cyclic quadrilateral

$$\angle BED + \angle BZD = \pi \quad \Rightarrow \quad \angle AED = \angle BZD$$

So $\triangle DCZ$ and $\triangle DAE$ are similar, that is

$$\frac{CZ}{CD} = \frac{AE}{AD}$$

but $CZ = FZ - FC$ or also, being $EF = FZ$, we have that $CZ = EF - FC$.Therefore

$$\frac{EF - FC}{CD} = \frac{AE}{AD} \quad \Rightarrow \quad \frac{EF}{AE} = \frac{FC}{AE} + \frac{CD}{AD}$$

so the equality is proved.

Also solved by Yujin Kim, Stony Brook School, Stony Brook, NY, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.

S323. Solve in positive integers the equation

$$x + y + (x - y)^2 = xy.$$

Proposed by Neculai Stanciu and Titu Zvonaru, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Denote $s = x + y$, $d = x - y$, and note that the proposed equation rewrites as

$$s^2 - d^2 = 4xy = 4x + 4y + 4(x - y)^2 = 4s + 4d^2, \qquad (s - 2)^2 - 5d^2 = 4.$$

This is clearly a Pell-like equation $c^2 - 5d^2 = 4$, where $s - 2 = c$ and d are integers of the same parity iff x, y are integers, and all of whose infinite solutions (c_n, d_n) may be found through the recurrent relations

$$c_{n+2} = 3c_{n+1} - c_n, \qquad d_{n+2} = 3d_{n+1} - d_n, \qquad n \geq 0,$$

with initial conditions $(c_0, d_0) = (2, 0)$ and $(c_1, d_1) = (3, 1)$. Therefore, all solutions are of the form

$$s_n = \left(\frac{\sqrt{5}+1}{2}\right)^{2n} + \left(\frac{\sqrt{5}-1}{2}\right)^{2n} + 2, \qquad d_n = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{2n} - \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right)^{2n},$$

or equivalently,

$$x_n = \frac{s_n + d_n}{2} = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{2n+1} + \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right)^{2n+1} + 1,$$

$$y_n = \frac{s_n - d_n}{2} = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{2n-1} + \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right)^{2n-1} + 1,$$

where n can take any non-negative integer value. We can readily check that indeed the following relation holds:

$$x_n y_n - x_n - y_n = \frac{1}{5}\left(\left(\frac{\sqrt{5}+1}{2}\right)^{4n} + \left(\frac{\sqrt{5}-1}{2}\right)^{4n} - 2\right) = {d_n}^2 = (x_n - y_n)^2.$$

We conclude that all solutions of the proposed equation are those (x_n, y_n) already found, or the result of interchanging x_n, y_n by symmetry in the problem.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA; Mehtaab Sawhney, USA; Cody Johnson, USA; Jhiseung Daniel Hahn, Phillips Exeter Academy, Exeter, NH, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, Delhi, India; Arber Avdullahu, Mehmet Akif College, Kosovo; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; David E. Manes, Oneonta, NY, USA; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.

S324. Find all functions $f : S \to S$ satisfying

$$f(x)f(y) + f(x) + f(y) = f(xy) + f(x+y)$$

for all $x, y \in S$ when (i) $S = \mathbb{Z}$; (ii) $S = \mathbb{R}$.

Proposed by Prasanna Ramakrishnan, Port of Spain, Trinidad and Tobago

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

For the first part where $S = \mathbb{Z}$, putting $x = y = 0$ we obtain $f(0) = 0$. Then, putting $(x, y) = (1, -1)$, we have

$$f(1) + f(-1) + f(1)f(-1) = f(0) + f(-1) \Rightarrow f(1)[f(-1) + 1] = 0.$$

(a) $f(1) = 0$, then,

$$f(x-1) + f(1) + f(1)f(x-1) = f(x) + f(x-1) \Rightarrow \boxed{f(x) = 0}\ \forall x \in \mathbb{Z}$$

(b) $f(-1) = -1$, then,

$$\begin{gathered} f(2) + f(-1) + f(-1)f(2) = f(1) + f(-2) \Rightarrow f(1) + f(-2) = f(-1) \\ f(-2) + f(1) + f(1)f(-2) = f(-1) + f(-2) \Rightarrow f(-1) = f(1) + f(1)f(-2) \end{gathered}$$

Combining the above two results, $f(-2)[f(1) - 1] = 0$.
(i) $f(1) = 1$, then,

$$f(x) + f(1) + f(x)f(1) = f(x) + f(x+1) \Rightarrow f(x) + 1 = f(x+1)\ \forall x \in \mathbb{Z}$$

and due to induction, $\boxed{f(x) = x}$ $\forall x \in \mathbb{Z}$.

(ii) $f(-2) = 0$, then, $f(1) + f(-2) = f(-1) = -1 \Rightarrow f(1) = -1$ and so,

$$f(x) + f(1) + f(x)f(1) = f(x) + f(x+1) \Rightarrow f(x) + f(x+1) + 1 = 0$$

and replacing x by $x + 1$ in the above relation,

$$f(x+1) + f(x+2) + 1 = 0$$

and so,

$$\textbf{(1.0)} \cdots\ f(x) = f(x+2),$$

thus applying induction, we have $f(2n) = 0$ and $f(2n-1) = -1$ for all $n \in \mathbb{Z}$ or

$$\boxed{f(x) = \begin{cases} 0, \text{when } x \text{ is even} \\ -1, \text{when } x \text{ is odd} \end{cases}}$$

Now we come to the second part where $S = \mathbb{R}$. We have the same conditions as before except that the domain is the real set. So,
(a) $f(1) = 0$, then,

$$f(x-1) + f(1) + f(1)f(x-1) = f(x) + f(x-1) \Rightarrow \boxed{f(x) = 0}\ \forall x \in \mathbb{R}$$

(b) $f(-1) = -1$ and so, $f(-2)[f(1) - 1] = 0$.
(i) $f(1) = 1$, then,

$$f(x) + f(-1) + f(x)f(-1) = f(x-1) + f(-x) \Rightarrow f(x-1) = -f(-x) - 1.$$

And also, in

$$\textbf{(1.1)} \cdots \; f(x+1) = f(x)+1 \text{ (as it was in the previous part)},$$

replacing $x+1$ by x, we have $f(x) = f(x-1)+1 = -f(-x)$. Thus we conclude that $-f(x) = f(-x)$ $\forall x \in \mathbb{R}$. And replacing x by $x+1$ in **(1.1)**, $f(x+2) = f(x+1)+1 = f(x)+2$.
So, $f(x+1) = f(x)+1 \Rightarrow f(2) = f(1)+1 = 2$. Thus,

$$f(2)+f(x)+f(2)f(x) = f(2x)+f(x+2) \Rightarrow 2f(x) = f(2x) \;\forall x \in \mathbb{R}.$$

Now,

$$\begin{aligned} f(x+y)+f(x-y)+f(x+y)f(x-y) &= f(x^2-y^2)+f(2x) \\ f(x+y)+f(y-x)+f(x+y)f(y-x) &= f(y^2-x^2)+f(2y) \end{aligned}$$

adding up the above two relations and using $f(-a) = -f(a)$ $\forall a \in \mathbb{R}$, we have $f(x+y) = f(x)+f(y)$ implying $f(x)f(y) = f(xy)$, thus the function is both additive and multiplicative, implying that $\boxed{f(x) = x}$ $\forall x \in \mathbb{R}$ and $\boxed{f(x) = 0}$ $\forall x \in \mathbb{R}$. It is easy to see that both of them satisfy the functional equation.
(ii)

$$f(x) = \begin{cases} 0, \text{when } x \text{ is even } \forall x \in \mathbb{Z} \\ -1, \text{when } x \text{ is odd } \forall x \in \mathbb{Z} \end{cases}$$

So, according to the previous part we have the result **(1.0)**,

$$f(x+2) = f(x)$$

which here, holds true for all real x. So, $x = \dfrac{1}{2}$ yields

$$f\left(\frac{1}{2}\right) = f\left(\frac{5}{2}\right) \cdots \textbf{(1.2)}$$

Whereas, $f\left(\dfrac{1}{2}\right) + f(2) + f\left(\dfrac{1}{2}\right)f(2) = f(1) + f\left(\dfrac{5}{2}\right)$ implying that

$$f\left(\frac{1}{2}\right) = -1 + f\left(\frac{5}{2}\right)$$

contradicting **(1.2)**. Thus in the real domain, $f(-1) = -1$ and $f(-2) = 0$ cannot both be simultaneously true. Thus in summary, we have the functions as:

(1) When $S = \mathbb{Z}$,

$$\begin{aligned} f(x) &= 0 \\ f(x) &= x \end{aligned}$$

$$f(x) = \begin{cases} 0, \text{when } x \text{ is even} \\ -1, \text{when } x \text{ is odd} \end{cases}$$

(2) When $S = \mathbb{R}$.

$$\begin{aligned} f(x) &= 0 \\ f(x) &= x \end{aligned}$$

Also solved by Mehtaab Sawhney, USA; Misiakos Panagiotis ,Athens College (HAEF), Nea Penteli; Cody Johnson, USA; Arber Avdullahu, Mehmet Akif College, Kosovo; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.

U319. Let A, B, C be the measured (in radians) of the angles of a triangle with circumradius R and inradius r . Prove that

$$\frac{A}{B} + \frac{B}{C} + \frac{C}{A} \leq \frac{2R}{r} - 1$$

Proposed by Nermin Hodžić, Bosnia and Herzegovina and Salem Malikić, Canada

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

We employ two known inequalities. The first one is

$$\frac{1}{ABC} \leq \frac{27}{2\pi^3} \frac{R}{r}$$

and can be found in problem 3757 of Crux Mathematicorum, vol.39–6, 2013. The second one is a known inequality

$$a^2c + b^2a + c^2b \leq \frac{4}{27}(a+b+c)^3 - abc$$

given that $a, b, c \geq 0$.

Therefore, we have

$$\frac{A}{B} + \frac{B}{C} + \frac{C}{A} = \frac{A^2C + B^2A + C^2B}{ABC} \leq$$

$$\frac{1}{ABC}\left(\frac{4}{27}(A+B+C)^3 - ABC\right) =$$

$$= \frac{1}{ABC}\frac{4}{27}\pi^3 - 1 \leq \frac{27}{2\pi^3}\frac{R}{r}\frac{4}{27}\pi^3 - 1 = 2\frac{R}{r} - 1$$

Also solved by Dragoljub Milošević, Gornji Milanovac, Serbia; Cody Johnson, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U320. Evaluate

$$\sum_{n\geq 0} \frac{2^n}{2^{2^n}+1}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Daniel Lasaosa, Pamplona, Spain

Denote for all non-negative integer N,

$$S_N = \sum_{n=0}^{N} \frac{2^n}{2^{2^n}+1}.$$

We show that for all positive integer N, we have

$$S_N = 1 - \frac{2^{N+1}}{2^{2^{N+1}}-1}.$$

The result is clearly true for $N = 0$ and $N = 1$, since

$$S_0 = \frac{1}{3} = 1 - \frac{2}{3} = 1 - \frac{2^1}{2^{2^1}-1}, \qquad S_1 = S_0 + \frac{2}{5} = 1 - \frac{4}{15} = 1 - \frac{2^2}{2^{2^2}-1}.$$

If the result is true for $N-1$, then for N we have

$$S_N = S_{N-1} + \frac{2^N}{2^{2^N}+1} = 1 - 2^N \frac{\left(2^{2^N}+1\right) - \left(2^{2^N}-1\right)}{\left(2^{2^N}+1\right)\left(2^{2^N}-1\right)} = 1 - \frac{2^{N+1}}{2^{2\cdot 2^N}-1},$$

or the result is true by inducton for all non-negative integer N. It follows that

$$\sum_{n\geq 0} \frac{2^n}{2^{2^n}+1} = \lim_{N\to\infty} S_{N-1} = 1 - \lim_{N\to\infty} \frac{2^N}{2^{2^N}-1} = 1 - \lim_{x\to\infty} \frac{x}{2^x-1} = 1,$$

since as it is well known, the exponential function increases much more rapidly than the linear function, and where we have defined $x = 2^N$. The conclusion follows.

Also solved by Reiner Martin, Bad Soden-Neuenhain, Germany; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; David E. Manes, Oneonta, NY, USA; Arkady Alt, San Jose, California, USA; Alok Kumar, Delhi, India; Albert Stadler, Herrliberg, Switzerland; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Yong Xi Wang,East China Institute Of Technology, China; Cody Johnson, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Mehtaab Sawhney, USA.

U321. Consider the sequence of polynomials $(P_s)_{s\geq 1}$ defined by

$$P_{k+1}(x) = (x^a - 1)P_k^{'}(x) - (k+1)P_k(x), k = 1, 2, \ldots,$$

where $P_1(x) = x^{a-1}$ and a is an integer greater than 1.
1. Find the degree of P_k
2. Determine $P_k(0)$

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, Romania

Solution by the proposer

1. We have $\deg P_1 = a - 1$, and from the recursive relation we obtain $\deg P_2 = 2a - 2$. Now, a simple inductive argument shows that $\deg P_k = k(a-1)$.

2. Let us consider the function $f : (-1, 1) \to \mathbb{R}$, defined by $f(x) = \frac{1}{x^a - 1}$. Observe that

$$f'(x) = \frac{ax^{a-1}}{(x^a-1)^2} = a\frac{P_1(x)}{(x^a-1)^2}.$$

One can prove immediately by induction that

$$f^{(k)}(x) = a\frac{P_k(x)}{(x^a-1)^{k+1}},$$

where we use the recursive relation in the definition of the sequence. This implies

$$P_k(0) = \frac{(-1)^{k+1}}{a} f^{(k)}(0),$$

and we reduce the problem of finding the value of $P_k(0)$ to determine $f^{(k)}(0)$. In this respect, using the geometric series we have

$$f(x) = \frac{1}{x^a - 1} = -\sum_{s=0}^{\infty} x^{as},$$

and we obtain

$$f^{(k)}(0) = \begin{cases} -k! & \text{if } \ a|k \\ 0 & \text{otherwise} \end{cases}$$

Finally,

$$P_k(0) = \frac{(-1)^{k+1}}{a} f^{(k)}(0) = \begin{cases} \frac{(-1)^k k!}{a} & \text{if } \ a|k \\ 0 & \text{otherwise} \end{cases}$$

Also solved by Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India.

U322. Evaluate

$$\sum_{n=1}^{\infty}\frac{16n^2-12n+1}{n(4n-2)!}.$$

Proposed by Titu Andreescu, USA and Oleg Mushkarov, Bulgaria

First solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Note that

$$\begin{aligned}\frac{16n^2-12n+1}{n(4n-2)!} &= \frac{4n(4n-3)}{n(4n-2)!}+\frac{1}{n(4n-2)!}\\ &= 4\cdot\frac{4n-3}{(4n-2)!}+4\cdot\frac{4n-1}{(4n)!}\\ &= 4\left(\frac{1}{(4n-3)!}-\frac{1}{(4n-2)!}+\frac{1}{(4n-1)!}-\frac{1}{(4n)!}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{16n^2-12n+1}{n(4n-2)!} &= 4\sum_{n=1}^{\infty}\left(\frac{1}{(4n-3)!}-\frac{1}{(4n-2)!}+\frac{1}{(4n-1)!}-\frac{1}{(4n)!}\right)\\ &= 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n!}=4\left(1-\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\right)\\ &= 4\left(1-\frac{1}{e}\right).\end{aligned}$$

Second solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Write

$$16n^2-12n+1=(4n-2)(4n-3)+2(4n-2)-1,$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{16n^2-12n+1}{n(4n-2)!} &= \sum_{n=1}^{\infty}\frac{1}{n(4n-4)!}+2\sum_{n=1}^{\infty}\frac{1}{n(4n-3)!}-\sum_{n=1}^{\infty}\frac{1}{n(4n-2)!}\\ &= f''(1)+2f'(1)-f(1),\end{aligned}$$

where

$$f(x)=\sum_{n=1}^{\infty}\frac{x^{4n-2}}{n(4n-2)!}.$$

Next, consider the function

$$g(x)=\sum_{n=1}^{\infty}\frac{x^{4n-2}}{(4n-2)!},$$

and note that $g^{(4)}(x)-g(x)=0$ subject to the initial conditions $g(0)=0$, $g'(0)=0$, $g''(0)=1$, and $g'''(0)=0$. Thus,

$$g(x)=\frac{1}{2}(\cosh x-\cos x).$$

To obtain f, multiply g by x, integrate term-by-term, determine the constant of integration using the initial condition $f(0)=0$, multiply by 4, and divide by x^2. This yields

$$f(x)=\frac{2}{x^2}(2+x\sinh x-\cosh x-x\sin x-\cos x).$$

Now,

$$\begin{aligned}
f'(x) &= \frac{2}{x^2}(x\cosh x - x\cos x) - \frac{4}{x^3}(2 + x\sinh x - \cosh x - x\sin x - \cos x),\\
f''(x) &= \frac{2}{x^2}(x\sinh x + \cosh x + x\sin x - \cos x) - \frac{8}{x^3}(x\cosh x - x\cos x) + \\
&\qquad \frac{12}{x^4}(2 + x\sinh x - \cosh x - x\sin x - \cos x),
\end{aligned}$$

so that

$$\begin{aligned}
f(1) &= 2\left(2 - \frac{1}{e} - \sin 1 - \cos 1\right),\\
f'(1) &= 2(\cosh 1 - \cos 1) - 4\left(2 - \frac{1}{e} - \sin 1 - \cos 1\right),\\
f''(1) &= 2(e + \sin 1 - \cos 1) - 8(\cosh 1 - \cos 1) + 12\left(2 - \frac{1}{e} - \sin 1 - \cos 1\right),
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{16n^2 - 12n + 1}{n(4n-2)!} = f''(1) + 2f'(1) - f(1) = 4\left(1 - \frac{1}{e}\right).$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Arkady Alt, San Jose, California, USA; Albert Stadler, Herrliberg, Switzerland; Cemal Kadirov, Istanbul University, Turkey; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Cody Johnson, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Mehtaab Sawhney, USA; Chaeyeon Oh, Episcopal High School, Alexandra, VA, USA.

U323. Let X and Y be independent random variables following a uniform distribution

$$p_X(x) = \begin{cases} 1 & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that inequality $X^2 + Y^2 \geq 3XY$ is true?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that the proposed inequality can be written as

$$\left(Y - \frac{3+\sqrt{5}}{2}X\right)\left(Y - \frac{3-\sqrt{5}}{2}\right) \geq 0,$$

ie since X, Y take only non-negative values, we must have either $Y \geq \frac{3+\sqrt{5}}{2}X \geq \frac{3-\sqrt{5}}{2}X$ or $Y \leq \frac{3-\sqrt{5}}{2}X \leq \frac{3+\sqrt{5}}{2}X$. This defines inside the unit square with vertices $(X, Y) = (0,0), (0,1), (1,1), (1,0)$ two triangles, one with vertices $(X, Y) = (0,0), (1,0), \left(1, \frac{3-\sqrt{5}}{2}\right)$, and one with vertices $(X, Y) = (0,0), (0,1), \left(\frac{3-\sqrt{5}}{2}, 1\right)$, inside which or on whose boundaries the inequality is true, being false in the rest of the unit square. Since both variables X, Y are independent and uniform, the probability of (X, Y) falling inside or on the boundary of one of these triangles equals the sum $\frac{3-\sqrt{5}}{2}$ of the areas of both triangles, divided by the area 1 of the unit square. The probability is therefore

$$\frac{3-\sqrt{5}}{2}.$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Cody Johnson, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; William Kang, Bergen County Academies, Hackensack, NJ, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA.

U324. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that $f(1) = 0$. Prove that there is $c \in (0,1)$ such that $|f(c)| \leq |f'(c)|$

Proposed by Marius Cavachi, Constanta, Romania

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

We will prove that for all $M > 0$ there is $c \in (0,1)$ such that $M|f(c)| \leq |f'(c)|$.

If there is a $c \in (0,1)$ such that $f(c) = 0$ then the inequality is trivial. Otherwise by continuity f has constant sign and without loss of generality we may assume that $f(x) > 0$ in $(0,1)$. By the Mean Value Theorem, for all $t \in (1/2, 1)$ there is $c_t \in (1/2, t)$ such that

$$\frac{\ln(f(t)) - \ln(f(1/2))}{t - 1/2} = \frac{f'(c_t)}{f(c_t)}.$$

The condition $f(1) = 0$ implies that the LHS tends to $-\infty$ as $t \to 1^-$.
Hence for all $M > 0$ there is a c_t such that

$$\frac{f'(c_t)}{f(c_t)} \leq -M.$$

Therefore $f'(c_t) < 0$ and

$$\frac{|f'(c_t)|}{|f(c_t)|} = \frac{-f'(c_t)}{f(c_t)} \geq M.$$

Also solved by Reiner Martin, Bad Soden-Neuenhain, Germany; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Corneliu Mănescu- Avram, Transportation High School, Ploieşti, Romania; Arkady Alt, San Jose, California, USA; Cody Johnson, USA.

O319. Let $f(x)$ and $g(x)$ be arbitrary functions defined for all $x \in \mathbb{R}$. Prove that there is a function $h(x)$ such that $(f(x)+h(x))^{2014}+(g(x)+h(x))^{2014}$ is an even function for all $x \in \mathbb{R}$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

It is well-known that any function f may be written as $f(x) = f_e(x) + f_o(x)$ where f_e and $f_o(x)$ are respectively the even and odd part of function f. Also $f_e(x) = \dfrac{f(x)+f(-x)}{2}$ while $f_o(x) = \dfrac{f(x)-f(-x)}{2}$.

Let us take $h(x) = -f_o(x) - g_e(x)$. Then, $f(x)+h(x) = f_e(x) - g_e(x)$, and $g(x)+h(x) = -f_o(x)+g_o(x)$. Therefore,

$$\begin{aligned} F(x) &= (f(x)+h(x))^{2014} + (g(x)+h(x))^{2014} \\ &= (f_e(x)-g_e(x))^{2014} + (-f_o(x)+g_o(x))^{2014}. \end{aligned}$$

Then,

$$\begin{aligned} F(-x) &= (f_e(-x)-g_e(-x))^{2014} + (-f_o(-x)+g_o(-x))^{2014} \\ &= (f_e(x)-g_e(x))^{2014} + (f_o(x)-g_e(x))^{2014} \\ &= F(x). \end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Yassine Hamdi, Lycee du Parc, Lyon, France; George - Petru Scărlătescu, Pitești, Romania; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Mehtaab Sawhney, USA; Michael Tang, Edina High School, MN, USA.

O320. Let n be a positive integer and let $0 < y_i \le x_i < 1$ for $1 \le i \le n$. Prove that

$$\frac{1 - x_1 \dots x_n}{1 - y_1 \dots y_n} \le \frac{1 - x_1}{1 - y_1} + \dots + \frac{1 - x_n}{1 - y_n}.$$

Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Marius Stânean, Zalau, Romania

We can prove this inequality by mathematical induction. For $n = 1$ it's obvious. For $n = 2$ we have

$$\frac{1 - x_1 x_2}{1 - y_1 y_2} \le \frac{1 - x_1}{1 - y_1} + \frac{1 - x_2}{1 - y_2}.$$

Let $a = \frac{y_1}{x_1} \le 1$ and $b = \frac{y_2}{x_2} \le 1$, so the inequality becomes

$$\frac{1 - x_1 x_2}{1 - abx_1 x_2} \le \frac{1 - x_1}{1 - ax_1} + \frac{1 - x_2}{1 - bx_2} \iff$$

$$(1 - x_1)(1 - x_2) + ax_1 x_2(1 - x_1) + bx_1 x_2(1 - x_2) - 3abx_1 x_2 + abx_1^2 x_2^2 +$$
$$abx_1 x_2(x_1 + x_2) + a^2 b x_1^2 x_2 + ab^2 x_1 x_2^2 - ab(a + b)x_1^2 x_2^2 \ge 0$$

dividing this inequality with $x_1 x_2$ we obtain

$$\frac{(1 - x_1)(1 - x_2)}{x_1 x_2} + ab(1 - x_1)(1 - x_2) + 2ab(x_1 + x_2 - 2) +$$
$$a^2 x_1(1 - x_2) + ab^2 x_2(1 - x_1) + a(1 - x_1) + b(1 - x_2) \ge 0 \iff$$
$$\frac{(1 - x_1)(1 - x_2)}{x_1 x_2} + a(1 - x_1)(b^2 x_2 - 2b + 1) + b(1 - x_2)(a^2 x_1 - 2a + 1) + ab(1 - x_1)(1 - x_2) \ge 0 \iff$$
$$\frac{(1 - x_1)(1 - x_2)}{x_1 x_2} + ab(1 - x_1)(1 - x_2) + a(1 - x_1)(b - 1)^2 +$$
$$b(1 - x_2)(a - 1)^2 - ab(a + b)(1 - x_1)(1 - x_2) \ge 0 \iff$$
$$(1 - x_1)(1 - x_2)\left[\frac{1}{x_1 x_2} + ab - ab(a + b)\right] + a(1 - x_1)(b - 1)^2 + b(1 - x_2)(a - 1)^2 \ge 0$$

which is true because

$$\frac{1}{x_1 x_2} + ab - ab(a + b) \ge 1 + ab - a - b = (1 - a)(1 - b) \ge 0.$$

Suppose that the original inequality holds for $n - 1 \in \mathbb{N}$ and we want to prove it for n. Therefore we have

$$\frac{1 - x_1 \cdots x_{n-1}}{1 - y_1 \cdots y_{n-1}} \le \frac{1 - x_1}{1 - y_1} + \ldots + \frac{1 - x_{n-1}}{1 - y_{n-1}}$$

but $Y = y_1 y_2 \cdots y_{n-1} \le x_1 x_2 \cdots x_{n-1} = X < 1$ and then applying the inequality for X, Y, respectively x_n, y_n we obtain

$$\frac{1 - x_1 \cdots x_n}{1 - y_1 \cdots y_n} = \frac{1 - X \cdot x_n}{1 - Y \cdot y_n} \le \frac{1 - X}{1 - Y} + \frac{1 - x_n}{1 - y_n} \le$$
$$\frac{1 - x_1}{1 - y_1} + \ldots + \frac{1 - x_{n-1}}{1 - y_{n-1}} + \frac{1 - x_n}{1 - y_n}.$$

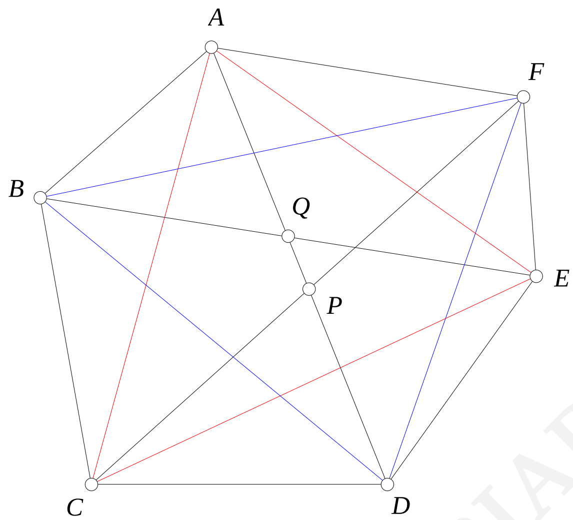
Also solved by Daniel Lasaosa, Pamplona, Spain; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Yassine Hamdi, Lycee du Parc, Lyon, France; Arkady Alt, San Jose, California, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; William Kang, Bergen County Academies, Hackensack, NJ, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Yujin Kim, Stony Brook School, Stony Brook, NY, USA; Mehtaab Sawhney, USA.

O321. Each of the diagonals AD, BE, CF of the convex hexagon $ABCDEF$ divides its area in half. Prove that

$$AB^2 + CD^2 + EF^2 = BC^2 + DE^2 + FA^2.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Ercole Suppa, Teramo, Italy



We begin by proving the following preliminary results:

Claim 1. We have $AC/DF = CE/FB = EA/BD$.

Proof of Claim 1. Let $P = AD \cap CF$. Observe that the areas of triangles $\triangle APF$ and $\triangle DPC$ are equal because

$$[APF] = [ADEF] - [DEFP] = [CDEF] - [DEFP] = [DPC]$$

Therefore $AP \cdot PF = CP \cdot PD$, or $AP/PD = CP/PF$.

Thus $\triangle APC \sim \triangle DPF$ (SAS) whence $\angle CAP = \angle FDP$ and $AC \parallel DF$.

In a similar way, we can prove that $BF \parallel CE$ and $AE \parallel BD$, so that $\triangle ACE \sim \triangle DFB$ and the Claim 1 follows. ■

Claim 2. The three diagonals AD, BE, CF are concurrent.

Proof of Claim 2. Let $P = AD \cap CF$, $Q = AD \cap BE$. As proved in Claim 1 we have $\triangle APC \sim \triangle DPF$, so

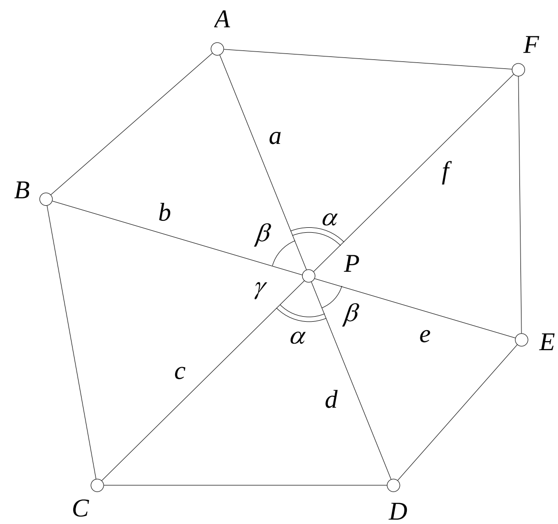
$$\frac{AP}{PD} = \frac{AC}{DF} \tag{1}$$

In a similar way, we can prove that

$$\frac{AQ}{QD} = \frac{EA}{BD} \tag{2}$$

From (1) and (2), taking into account of Claim 1, we have $AP/PD = AQ/QD$ so that $P = Q$. Hence, the three diagonals AD, BE, CF concur in P, as claimed. ■

Coming back to the proposed problem, note first that the pairs of triangles $\triangle FAP$ and $\triangle CDP$, $\triangle ABP$ and $\triangle DEP$, $\triangle BCP$ and $\triangle EFP$ are equivalent.



Therefore denoting by a, b, c, d, e, f the lengths of PA, PB, PC, PD, PE, PF respectively, we have

$$a \cdot f = c \cdot d, \quad a \cdot b = d \cdot e, \quad b \cdot c = e \cdot f \tag{3}$$

Finally, putting $\alpha = \angle FPA$, $\beta = \angle APB$, $\gamma = \angle BPC$, the cosinus law yields

$$AB^2 + CD^2 + EF^2 = a^2 + b^2 - 2ab\cos\beta + c^2 + d^2 - 2cd\cos\alpha + e^2 + f^2 - 2ef\cos\gamma$$
$$BC^2 + DE^2 + FA^2 = b^2 + c^2 - 2bc\cos\gamma + d^2 + e^2 - 2de\cos\beta + f^2 + a^2 - 2fa\cos\alpha$$

The above equalities and (3) gives

$$AB^2 + CD^2 + EF^2 = BC^2 + DE^2 + FA^2$$

which is precisely what we want to prove

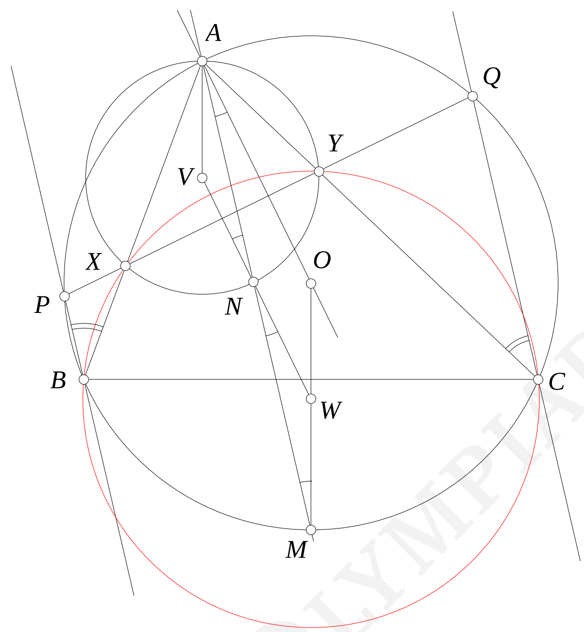
Also solved by Daniel Lasaosa, Pamplona, Spain; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Farrukh Mukhammadiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; William Kang, Bergen County Academies, Hackensack, NJ, USA; Yujin Kim, Stony Brook School, Stony Brook, NY, USA.

O322. Let ABC be a triangle with circumcircle Γ and let M be the midpoint of arc BC not containing A. Lines ℓ_b and ℓ_c passing through B and C, respectively, are parallel to AM and meet Γ at $P \neq B$ and $Q \neq C$. Line PQ intersects AB and AC at X and Y, respectively, and the circumcircle of AXY intersects AM again at N.

Prove that the perpendicular bisectors of BC, XY, and MN are concurrent.

Proposed by Prasanna Ramakrishnan, Port of Spain, Trinidad and Tobago

Solution by Ercole Suppa, Teramo, Italy



The parallelisms $BP \parallel AM$ and $CQ \parallel AM$ imply

$$\angle PBX = \frac{1}{2}\angle BAC = \angle QCY \tag{1}$$

The cyclic quadrilateral $BCQP$ yields

$$\angle PBC + \angle PQC = 180^\circ \tag{2}$$

From (1) and (2) we obtain

$$\begin{aligned} \angle XBC + \angle XYC &= 180^\circ - \angle PQC - \angle PBX + \angle XYC = \\ &= 180^\circ - \angle PQC - \angle QCY + \angle XYC = \\ &= \angle QYC + \angle XYC = 180^\circ \end{aligned}$$

so $BXYC$ is a cyclic quadrilateral.

Let V, W denote the centers of $\odot(AXY)$ and $\odot(BXYC)$ respectively. Observe that V, N, W are collinear since N is the midpoint of the arc XY and VW is the perpendicular bisector of XY.

Since $\angle PBX = \angle QCY$ we have $AP = AQ$, so A lies on the perpendicular bisector of PQ. Therefore OA is the perpendicular bisector of PQ and this implies that $OA \perp PQ$.

Now, from $VW \perp PQ$ and $OA \perp PQ$, it follows that $OA \parallel VW$.

Therefore we have

$$\angle WMN = \angle OAM = \angle ANV = \angle WNM \quad \Rightarrow \quad WN = WM$$

hence W belongs to the perpendicular bisector of MN.

Clearly W also belongs to the perpendicular bisectors of BC and XY so the proof is complete.

Also solved by Daniel Lasaosa, Pamplona, Spain; Andrea Fanchini, Cantú, Italy; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Farrukh Mukhammadiev, Academic Lyceum Nr1, Samarkand, Uzbekistan; Bodhisattwa Bhowmik, RKMV, Agartala, Tripura, India; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Mehtaab Sawhney, USA.

O323. Prove that the sequence $2^{2^1}+1, 2^{2^2}+1, \ldots, 2^{2^n}+1, \ldots$ and an arbitrary infinite increasing arithmetic sequence have either infinitely many terms in common or at most one term in common.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

Assume to the contrary that there exists an arbitrary infinite increasing arithmetic progression (or sequence) $\{a_n\}_{n\geq 1}$ with common difference equal to d, such that it has a terms in common with the sequence $\{F_n\}_{n\geq 1} = 2^{2^1}+1, 2^{2^2}+1, \cdots, 2^{2^n}+1, \cdots$, and $2 \leq a < \infty$. Then assume that the last two terms which are common to both the sequences, are $a_k = 2^{2^j}+1$ and $a_\ell = 2^{2^m}+1$ and $k < \ell \Leftrightarrow j < m$. Then, $d | a_\ell - a_k \Leftrightarrow d | 2^{2^j}(2^{2^m-2^j}-1)$. If $d = 2^b, b \leq 2^n$, then we have $d | 2^{2^m}(2^{2^{m+1}-2^m}-1) \Leftrightarrow d | F_{m+1} - F_m$ and thus there $\exists$ a u such that $a_{\ell+u} = F_{m+1}$. But this is a contradiction to the assumption that the sequences $\{a_n\}$ and $\{F_n\}$ share only a terms in common and $2 \leq a < \infty$. Thus if $\{a_n\}$ and $\{F_n\}$ share more than one term, then using the above method we can generate infinitely many common terms.

Next, let d have an odd factor say d_0. Then, $2^{2^m-2^j} \equiv 1 \pmod{d_0} \Rightarrow (2^{2^m-2^j})^{2^{m-j}} \equiv 1 \pmod{d_0}$. Thus $d_0 | 2^{2^m}(2^{2^{2m-j}-2^m}-1) \Leftrightarrow d | F_{2m-j} - F_m$. This shows the existence of a positive integer r such that $a_{\ell+r} = F_{2m-j}$. This again contradicts our assumption, thus there exist infinitely many common terms in $\{a_n\}$ and $\{F_n\}$ if they have more than one term in common, or they share at most one common term.

Note: An example of the arithmetic sequence which shares only one term with $\{F_n\}_{n\geq 1}$ is the sequence $5, 15, 25, 35, \cdots$ which shares only 5 as a common term and none else.

By the Claim, the two describe sequences have either zero, one, or infinitely many terms in common. The conclusion follows.

Note: We may easily construct infinite increasing arithmetic sequences with no terms in common by taking sequences of non-integers, or of even integers. We may also easily construct infinite increasing arithmetic sequences with exactly one term in common, by taking one of its terms in common and an irrational difference, or by appropriately choosing the starting point and the difference. For example, defining $s_m = 2^{2^1}+1+(m-1)2^5$ for $m = 1, 2, \ldots$, every term of this arithmetic sequence is congruent to 1 modulus $2^{2^1} = 4$, hence not of the form $2^{2^n}+1$, except for $n = 1$ when $m = 1$, which is the only common term, and there are no others.

Also solved by Reiner Martin, Bad Soden-Neuenhain, Germany; Mehtaab Sawhney, USA; Misiakos Panagiotis, Athens College (HAEF), Nea Penteli; Daniel Lasaosa, Pamplona, Spain; Samin Riasat, University of Waterloo, ON, Canada; Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan.

O324. Let a, b, c, d be nonnegative real numbers such that $a^3+b^3+c^3+d^3+abcd=5$. Prove that

$$abc+bcd+cda+dab-abcd\le 3$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Semchankau Aliaksei, Moscow Institute of Physics and Technology

We will prove more general problem: let $(a^3+b^3+c^3+d^3)t+abcd=5t^4, t>0$, prove that

$$(abc+bcd+cda+dab)t-abcd\le 3t^4$$

In required ineaquality $t=1$.

Let $r=abc+bcd+cda+dab, S_3=a^3+b^3+c^3+d^3, s=abcd$. Then we have $S_3t+s=5t^4$ and we have to prove that $rt-s\le 3t^4$. Obviously $t^4=\frac{S_3t+s}{5}$, so, our inequality transforms to

$$rt-s\le\frac{3}{5}(S_3t+s)\Leftrightarrow 5rt-5s\le 3S_3t+3s\Leftrightarrow(5r-S_3)t\le 8s$$

If $5r-3S_3\le 0$, then it is obvious, so we can conclude that $5r-3S_3>0$. Let's suppose, that inequality isn't hold (and after that we will get a contradiction). So, now we conclude that

$$(5r-3S_3)t>8s\Leftrightarrow t>\frac{8s}{5r-3S_3}.$$

Using it, we get:

$$S_3t+s=5t^4\Rightarrow$$

$$s=t(5t^3-S_3)>\frac{8s}{5r-3S_3}(5t^3-S_3)\Rightarrow$$

$$\Rightarrow 5r-3S_3>8(5t^3-S_3)\Rightarrow 5r+5S_3>40t^3\Leftrightarrow r+S_3>8t^3$$

We know, that $t>\frac{8s}{5r-3S_3}$, so $r+S_3>8(\frac{8s}{5r-3S_3})^3$, and it gives us that

$$(r+S_3)(5r-3S_3)^3>8^4s^3$$

$$(r+S_3)(5r-3S_3)=(3r-S_3)^2-(2r-2S_3)^2\le(3r-S_3)^2$$

So, we can get that $8^4s^3<(3r-S_3)^2(5r-3S_3)^2\Rightarrow 8^2s^{\frac{3}{3}}<(3r-S_3)(5r-3S_3)$.

$$(3r-S_3)(5r-3S_3)=(4r-2S_3)^2-(r-S_3)^2\le(4r-2S_3)^2,$$

so we get that

$$8s^{\frac{3}{4}}<4r-2S_3\Leftrightarrow 4s^{\frac{3}{4}}<2r-S_3\Leftrightarrow 4a^{\frac{3}{4}}b^{\frac{3}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}}\le 2(abc+bcd+cda+dab)-(a^3+b^3+c^3+d^3)$$

This is a contradiction because of the following lemma:

Lemma.

$$a^3+b^3+c^3+d^3+4a^{\frac{3}{4}}b^{\frac{3}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}} \ge 2(abc+bcd+cda+dab)$$

Proof

Let $f(a,b,c,d) = a^3+b^3+c^3+d^3+4a^{\frac{3}{4}}b^{\frac{3}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}} - 2(abc+bcd+cda+dab)$. Our goal is to prove that $f(a,b,c,d) \ge 0$. At first we will prove it in case $a=b, c=d$:

$$f(x,x,y,y) \ge 0 \Leftrightarrow 2(x^3-x^2y-xy^2+y^3) - 2xy(x-2\sqrt{xy}+y) \ge 0 \Leftrightarrow (\sqrt{x}+\sqrt{y})^2(x+y) \ge xy$$

which is obvious.

Now we will find, when inequality $f(a,b,c,d) \ge f(a,b,\sqrt{cd},\sqrt{cd})$ holds:

$$f(a,b,c,d) \ge f(a,b,\sqrt{cd},\sqrt{cd}) \Leftrightarrow (c\sqrt{c}-d\sqrt{d})^2 \ge 2ab(\sqrt{c}-\sqrt{d})^2 \Leftarrow$$

$$\Leftarrow (c+\sqrt{cd}+d)^2 \ge 2ab \Leftarrow (3\sqrt{cd})^2 \ge 2ab \Leftarrow 9cd \ge 2ab \Leftarrow 4cd \ge ab$$

So, if $4cd \ge ab$, then we can change c,d to x,x, where $x=\sqrt{cd}$. We will try to do it with our numbers. Lets rearrange a,b,c,d in such way that $a \ge b \ge c \ge d$.

Obviously, $4ab \ge cd$, so we can change a,b to x,x. Now we need to prove, that $f(x,x,c,d) \ge 0$. If $4cd \ge x^2$,then we can replace c,d to y,y and we are done.

So, we will consider the case, when $x^2 \ge 4cd$. Let's take a look at pairs x,c and x,d. If $4xc \ge xd$ and $4xd \ge xc$, then we can replace them to pairs l,l and r,r, $l=\sqrt{xc}, r=\sqrt{xd}$.

$4xc \ge xd$ obviously holds, so we have to check $4xd \ge xc$. If it is true, then we are done. Else, let's suppose that $4xd \le xc \Leftrightarrow 4d < c$.

$$f(x,x,c,d) \ge 0 \Leftrightarrow 2x^3+c^3+4x^{\frac{6}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}} \ge 2x^2(c+d)+4xcd$$

$4x^{\frac{6}{4}}c^{\frac{3}{4}}d^{\frac{3}{4}} \ge 4xcd$ - obvious. It remains to prove, that

$$2x^3+c^3+d^3 \ge 2x^2(c+d)$$

$2x^3+c^3+d^3 \ge 2x^3+c^3 = x^3+x^3+c^3 \ge 3x^2c$. $3x^2c \ge 2x^2(c+d) \Leftrightarrow 3c \ge 2c+2d \Leftrightarrow c \ge 2d$, but we already have $c \ge 4d$.

Also solved by Sardor Bozorboyev, Lyceum S.H.Sirojjidinov, Tashkent, Uzbekistan; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Arber Avdullahu, Mehmet Akif College, Kosovo; Adnan Ali, Student in A.E.C.S-4, Mumbai, India.