Junior problems

J211. Let a, b, c be positive real numbers such that $a^3 + b^3 + c^3 = 1$. Prove that

$$\frac{1}{a^5(b^2+c^2)^2} + \frac{1}{b^5(c^2+a^2)^2} + \frac{1}{c^5(a^2+b^2)^2} \geq \frac{81}{4}.$$

Proposed by Titu Zvonaru, Comanesti, Romania

Solution proposed by G.R.A.20 Problem Solving Group, Roma, Italy.
By the Cauchy-Schwarz inequality

$$(a^3+b^3+c^3)\left(\frac{1}{a^5(b^2+c^2)^2} + \frac{1}{b^5(c^2+a^2)^2} + \frac{1}{c^5(a^2+b^2)^2}\right) \geq \left(\frac{1}{a(b^2+c^2)} + \frac{1}{b(c^2+a^2)} + \frac{1}{c(a^2+b^2)}\right)^2.$$

So, it suffices to show that

$$\frac{1}{a(b^2+c^2)} + \frac{1}{b(c^2+a^2)} + \frac{1}{c(a^2+b^2)} \geq \frac{9/2}{a^3+b^3+c^3},$$

that is

$$\frac{2(a^3+b^3+c^3)}{3} \geq \frac{3}{\frac{1}{a(b^2+c^2)} + \frac{1}{b(c^2+a^2)} + \frac{1}{c(a^2+b^2)}}.$$

Since the right-hand side is the harmonic mean of $a(b^2+c^2), b(c^2+a^2), c(a^2+b^2)$, it follows that it is less or equal to the arithmetic mean of the same numbers. So we have to show that

$$2(a^3+b^3+c^3) \geq a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)$$

which holds by the rearranging inequality.

Also solved by Albert Stadler, Switzerland; Harun Immanuel, ITS Surabaya; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Universidad Pública de Navarra,Spain; Pham Huy Hoang, Highschool for gifted Students, Hanoi University of Science, Hanoi, Vietnam

J212. Solve in real numbers the system of equations

$$
\begin{aligned}
(x-2y)(x-4z) &= 6\\
(y-2z)(y-4x) &= 10\\
(z-2x)(z-4y) &= -16.
\end{aligned}
$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria

Expanding the left hand side of these equations and adding them, we obtain $(x+y+z)^2=0$, so we must have $x=-y-z$. substituting this value of x in the first and the second equations we find that

$$
\begin{aligned}
(z+3y)(y+5z) &= 6\\
(y-2z)(5y+4z) &= 10
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
3(y^2-2)+5z^2+16yz &= 0\\
5(y^2-2)-8z^2-6yz &= 0
\end{aligned}
$$

Eliminating y^2-2, we conclude that $z(z+2y)=0$, and we distinguish two cases:

- If $z=-2y$, then from $5(y^2-2)-8z^2-6yz=0$ we conclude that $3y^2+2=0$ which is a contradiction, since we are looking for real solutions. Hence
- $z=0$, and consequently $y^2=2$. Recalling that $x=-y-z$, we obtain

$$(x,y,z)\in\left\{(\sqrt{2},-\sqrt{2},0),(-\sqrt{2},\sqrt{2},0)\right\},$$

and we check easily that these are effectively solutions to our system of equations.

Therefore the set of solutions to the considered system of equations is

$$\left\{(\sqrt{2},-\sqrt{2},0),(-\sqrt{2},\sqrt{2},0)\right\}.$$

Remark. If we want the complex solutions to the system as well, then from the first point above we obtain also

$$\left\{\left(i\sqrt{\frac{2}{3}},i\sqrt{\frac{2}{3}},-2i\sqrt{\frac{2}{3}}\right),\left(-i\sqrt{\frac{2}{3}},-i\sqrt{\frac{2}{3}},2i\sqrt{\frac{2}{3}}\right)\right\}.$$

Also solved by Matteo Elia, Francesco Bonesi, and Antonio Cirulli, Università di Roma "Tor Vergata", Roma, Italy; Albert Stadler, Switzerland; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Gabriel Alexander Chicas Reyes, El Salvador

J213. For any positive integer n, let $S(n)$ denote the sum of digits in its decimal representation. Prove that the set of all positive integers n such that n is not divisible by 10 and $S(n) > S(n^2+2012)$ is infinite.

Proposed by Preudtanan Sriwongleang, Ramkamhaeng University, Bangkok, Thailand

Solution by Alessandro Ventullo, Milan, Italy

Let $A = \{n \in \mathbb{N} : n \neq 10k \text{ and } S(n) > S(n^2+2012), k \in \mathbb{N}\}$. We prove that $A \supset \{5 \cdot 10^m - 1, m \in \mathbb{N} \setminus \{0\}\} = \{49, 499, \ldots, 4\underbrace{9\ldots9}_{m}, \ldots\}$, so that A must be infinite. Clearly, $49, 499 \in A$, since

$$49 \neq 10k, k \in \mathbb{N}, \qquad S(49) = 13 > 12 = S(4413)$$

and

$$499 \neq 10k, k \in \mathbb{N}, \qquad S(499) = 22 > 12 = S(251013).$$

Let $m > 2$. Then $5 \cdot 10^m - 1 \neq 10k, k \in \mathbb{N}$ and

$$(5 \cdot 10^m - 1)^2 + 2012 = 25 \cdot 10^{2m} - 10^{m+1} + 2013 = 24\underbrace{9\ldots9}_{m-1}\underbrace{0\ldots0}_{m+1}+2013,$$

from which

$$S((5 \cdot 10^m - 1)^2 + 2012) = 2 + 4 + 9(m-1) + 2 + 1 + 3 = 9m + 3.$$

Therefore, $S(5 \cdot 10^m - 1) = 9m + 4 > 9m + 3 = S((5 \cdot 10^m - 1)^2 + 2012)$, i.e.

$$5 \cdot 10^m - 1 \in A \qquad \forall m \in \mathbb{N} \setminus \{0\},$$

which gives the desired conclusion.

Also solved by Albert Stadler, Switzerland; Daniel Lasaosa, Universidad Publica de Navarra, Spain

J214. Let $a, b, c, d, e \in [1,2]$. Prove that

$$ab + bc + cd + de + ea \geq a^2 + b^2 + c^2 + d^2 - e^2$$

and find the values for which the equality occurs.

Proposed by Ion Dobrota, Romania and Adrian Zahariuc, Harvard University, USA

Solution by Francesco Bonesi, Università di Roma "Tor Vergata", Roma, Italy
Let

$$\begin{aligned} f(x_1, x_2, x_3, x_4, x_5) &= ((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + (x_4 - x_5)^2 + (x_5 - x_1)^2)/2 \\ &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_1x_5. \end{aligned}$$

For $1 \leq i \leq 5$, the quadratic function

$$[1,2] \ni x_i \to f(x_1, x_2, x_3, x_4, x_5) = x_i^2 + B_i x_i + C_i$$

attains its maximum value at $x_i = 1$ or at $x_i = 2$. Therefore, it is easy to verify that

$$\max_{[1,2]^5} f(x_1, x_2, x_3, x_4, x_5) = 2$$

and the maximum is attained at any point in the set

$$\begin{aligned} M = \{&(2,1,2,1,1), (1,2,1,2,1), (1,1,2,1,2), (2,1,1,2,1), (1,2,1,1,2), \\ &(1,2,1,2,2), (2,1,2,1,2), (2,2,1,2,1), (1,2,2,1,2), (2,1,2,2,1)\}. \end{aligned}$$

Hence,

$$f(a,b,c,d,e) - 2e^2 \leq \max_{[1,2]^5} f(a,b,c,d,e) - 2\min_{[1,2]} e^2 = 2 - 2 = 0$$

which is equivalent to the desired inequality. Moreover the equality holds if and only if $(a, b, c, d, 1) \in M$ that is:

$$(2,1,2,1,1), (1,2,1,2,1), (2,1,1,2,1), (2,2,1,2,1), (2,1,2,2,1).$$

Also solved by Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Albert Stadler, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Omran Kouba ,Higher Institute for Applied Sciences and Technology, Damascus, Syria; Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain

J215. Prove that for any prime $p > 3$, $\frac{p^6-7}{3} + 2p^2$ can be written as sum of two perfect cubes.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note that

$$\frac{p^6-7}{3} + 2p^2 = \left(\frac{2p^2+1}{3}\right)^3 + \left(\frac{p^2-4}{3}\right)^3,$$

where $\frac{2p^2+1}{3}$ and $\frac{p^2-4}{3}$ are integers, since $p^2 \equiv 1 \pmod 3$ for any p prime with 3, which clearly holds for any prime $p > 3$. The conclusion follows.

Also solved by Pham Huy Hoang,High School for Gifted students,Hanoi University of Science,Hanoi National University; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Omran Kouba ,Higher Institute for Applied Sciences and Technology, Damascus, Syria; Alessandro Ventullo, Milan, Italy; Albert Stadler, Switzerland

J216. Let ω be a circle and let M be a point outside it. Draw through M the lines ℓ_1, ℓ_2 and ℓ_3 intersecting ω and consider the intersections $\ell_1 \cap \omega = \{A_1, A_2\}$, $\ell_2 \cap \omega = \{B_1, B_2\}$, and $\ell_3 \cap \omega = \{C_1, C_2\}$. Denote $P = A_1B_2 \cap A_2B_1$, $Q = B_1C_2 \cap B_2C_1$ and R is one of the points of intersection between PQ and ω. Prove that MR is tangent at ω.

Proposed by Catalin Turcas, Warwick University, United Kingdom

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Claim: Let $ABCD$ be a convex cyclic quadrilateral inscribed in circle $\omega = C(O, \rho)$, such that its sides AB, CD meet at a point M (clearly outside ω). Let T, U be the points where lines through M are tangent to ω. Denote $P = AC \cap BD$, and if AD, BC intersect, $M' = AD \cap BC$. Then, $P, M' \in TU$.

Proof: We may assume WLOG that B, C are respectively inside segments MA, MD, otherwise we may rename the vertices of $ABCD$, exchanging A with B, and C with D, without altering the problem. Define $\omega' = C(O', \rho')$ as the circle with diameter OM. Clearly, $\omega \cap \omega' = \{T, U\}$, and TU is the radical axis of both circles. Let N be the midpoint of AB. Its distance from O is such that, by the median theorem, $ON^2 = \frac{OA^2+OB^2}{2} - \frac{AB^2}{4} = \rho^2 - \frac{AB^2}{4}$. Its distance from M is $MA - \frac{AB}{2} = MB + \frac{AB}{2}$, or since $AB = MA - MB$,

$$MN^2 = MA \cdot MB + (MA - MB)\frac{AB}{2} - \frac{AB^2}{4} = MA \cdot MB + \frac{AB^2}{4}.$$

Therefore, since $MA \cdot MB = 4\rho'^2 - \rho^2$ is the power of MN with respect to ω, then $MN^2 + ON^2 = 4\rho'^2 = OM^2$, and $N \in \omega'$. Denote now $X = TU \cap AB$, and $x = BX$. Therefore, the power of X with respect to ω and ω', which are equal since $x \in TU$, is

$$x(AB - x) = BX \cdot AX = MX \cdot NX = (MB + x)\left(\frac{AB}{2} - x\right),$$

$$x = \frac{AB \cdot MB}{MA + MB}.$$

Similarly, define $Y = TU \cdot CD$, and $y = CY$, or $y = \frac{CD \cdot MC}{MC+MD}$. Now,

$$\frac{MX}{XA} = \frac{MB + x}{AB - x} = \frac{2MB}{AB}, \qquad \frac{MX}{XB} = \frac{MB + x}{x} = \frac{2MA}{AB},$$

$$\frac{CY}{YM} = \frac{y}{MC + y} = \frac{CD}{2MD},$$

Denote h_A, h_C the lengths of the altitudes from A, C onto BD. Since triangles ABD and BCD share side BD and have the same circumradius ρ, and ΔADM is similar to ΔCBM, we conclude that

$$\frac{AP}{PC} = \frac{h_A}{h_C} = \frac{[ABD]}{[BCD]} = \frac{AB \cdot AD}{BC \cdot CD} = \frac{AB \cdot MD}{CD \cdot MB}.$$

By Menelaus' theorem, it follows that X, P, Y are collinear, or $P \in TU$. Applying the Sine Law to $\Delta M'BA$, $\Delta M'CD$, ΔABD and ΔBCD, we have

$$\frac{BM'}{M'A} = \frac{\sin \angle BAD}{\sin \angle ABC} = \frac{BD}{AC} = \frac{MD}{MA},$$

where we have also used that ΔMAC and ΔMDB are similar. Therefore,

$$\frac{BM'}{M'C} = \frac{M'A}{M'C} \cdot \frac{MD}{MA} = \frac{AB}{CD} \cdot \frac{MD}{MA},$$

where we have used that $\Delta M'AB$ and $\Delta M'CD$ are similar. We conclude again by Menelaus' theorem that M', X, Y are collinear, or $M' \in TU$. The claim follows.

Coming back to the proposed problem, define again by T, U the points where lines through M touch ω. Clearly, one of the quadrilaterals $A_1A_2B_1B_2$ or $A_1A_2B_2B_1$ is convex. Apply the claim to this quadrilateral, or TU passes through $A_1B_1 \cap A_2B_2$ and through $A_1B_2 \cap A_2B_1 = P$. Similarly, TU passes through Q, or $PQ = TU$. But then, either $R = T$ or $R = U$, and MR is tangent to ω. The conclusion follows.

Senior problems

S211. Let (a, b, c, d, e, f) be a 6-tuple of positive real numbers satisfying simultaneously the equations:

$$\begin{aligned} 2a^2 - 6b^2 - 7c^2 + 9d^2 &= -1, \\ 9a^2 + 7b^2 + 6c^2 + 2d^2 &= e, \\ 9a^2 - 7b^2 - 6c^2 + 2d^2 &= f, \\ 2a^2 + 6b^2 + 7c^2 + 9d^2 &= ef. \end{aligned}$$

Prove that $a^2 - b^2 - c^2 + d^2 = 0$ if and only if $7\frac{a}{b} = \frac{c}{d}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note that

$$0 = e \cdot f + (-1)ef = (9a^2 + 2d^2)^2 - (7b^2 + 6c^2)^2 + (2a^2 + 9d^2)^2 - (6b^2 + 7c^2) =$$

$$= 85(a^2 + b^2 + c^2 + d^2)(a^2 - b^2 - c^2 + d^2) - 98a^2d^2 + 2b^2c^2,$$

Hence $a^2 - b^2 - c^2 + d^2 = 0$ if and only if $98a^2d^2 = 2b^2c^2$, i.e. $7\frac{a}{b} = \frac{c}{d}$.

Also solved by Prithwijit De, HBCSE, Mumbai, India; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy, and Albert Stadler, Switzerland

S212. Consider a circle $\omega(I, r)$ and let Γ be a point inside it such that $I\Gamma = \ell$. Using only the straightedge and the compass, construct a triangle such that ω is its incircle and Γ its Gergonne point.

Note: The Gergonne point of a triangle is the intersection point of the lines determined by the vertices with the corresponding tangency points of the incircle with the opposite sides of the triangle.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let ABC be isosceles at A, let M be the midpoint of AC and E the point where the B-symmedian intersects AC, or $\angle ABM = \angle CBE$ and $\angle ABE = \angle CBM$. Applying the Sine Law to triangles ABM, BCE, ABE, BCM, we obtain

$$\frac{AM \sin \angle AMB}{AB} = \sin \angle ABM = \sin \angle CBE = \frac{CE \sin \angle CEB}{BC},$$
$$\frac{CM \sin \angle CMB}{BC} = \sin \angle CBM = \sin \angle ABE = \frac{AE \sin \angle AEB}{AB},$$

or $\frac{CE}{AE} = \frac{a^2}{c^2} = \frac{a^2}{b^2}$. Denote by D the point where the A-symmedian intersects BC. Since points B, K, E are collinear, by Menelaus' theorem we find

$$1 = \frac{AK}{KD} \cdot \frac{DB}{BC} \cdot \frac{CE}{EA} = \frac{AK}{KD} \cdot \frac{a^2}{2b^2}, \qquad \frac{AD}{AK} = 1 + \frac{KD}{AK} = \frac{a^2 + 2b^2}{2b^2},$$

and since denoting by A' the second point where AO intersects the circumcircle of ABC, we have that ABD and $AA'B$ are similar, we conclude that $AD = \frac{b^2}{2r}$, where r is the circumradius of ABC, or

$$AK = \frac{b^4}{r(a^2 + 2b^2)} = \frac{r(4b^2 - a^2)}{(a^2 + 2b^2)}, \qquad KO = AK - r = \frac{2r(b^2 - a^2)}{a^2 + 2b^2} = \ell.$$

It follows that

$$\frac{a}{2b} = \sqrt{\frac{2r - \ell}{4r + 2\ell}}.$$

Construct a segment XY of length $6r + \ell$, and find a point T such that $XT = 2r - \ell$, $YT = 4r + 2\ell$. Find also the midpoint U of XY. Choose a sufficiently small angle α, and draw the loci of points P, Q such that, on the same side of XT, $\angle XPT = \angle YQU = \alpha$; these loci are clearly circle arcs. The intesection of both arcs is a point Z such that T is the point where the Z-symmedian intersects XY in triangle XYZ, or $\frac{XZ}{YZ} = \sqrt{\frac{XT}{YT}} = \sqrt{\frac{2r-\ell}{4r+2\ell}} = \frac{a}{2b}$. Draw two circles, one with diameter YT, one with radius XT. One of their intersection points V satisfies $\frac{a}{2b} = \sin \angle YTV$.

In the incircle of $\omega(I, r)$, draw the diameter that passes through Γ, and choose A as its vertex such that O is inside AK. Construct angles $\angle YTV$ at both sides of AO, the lines forming these two angles with AO intersect again the incircle $\omega(I, r)$ at points B, C other than A. Triangle ABC, clearly isosceles at A by construction, is the such that $\omega(I, r)$ is it circumcircle, and Γ its symmedian point. Therefore, ABC is also the contact triangle of another triangle such that its incircle is $\omega(I, r)$, and its Gergonne point is Γ; draw tangents to $\omega(I, r)$ at A, B, C, these are the sides of the triangle whose construction is asked for in the problem statement.

Also solved by Antonio Cirulli, Università di Roma "Tor Vergata", Roma, Italy

S213. Let a, b, c be positive real numbers such that $a^2 \geq b^2 + bc + c^2$. Prove that

$$a > \min(b, c) + \frac{|b^2 - c^2|}{a}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Since we may interchange b, c without altering the problem, we may assume without loss of generality that $b \geq c$, or we need to prove that

$$a^2 > ac + b^2 - c^2.$$

Clearly $a^2 > b^2 - 2bc + c^2$, or $a > b - c$. If $a < 2c + b$, then

$$ac + b^2 - c^2 < 2c^2 + bc + b^2 - c^2 \leq a^2,$$

and the proposed inequality is true in this case. Otherwise, $a \geq 2c + b$, or

$$a^2 \geq a(b + c) + ac > b^2 - c^2 + ac,$$

and the proposed inequality is also true in this case. The conclusion follows.

Also solved by Matteo Elia, Università di Roma "Tor Vergata", Roma, Italy; Alessio Podda, Università di Roma "Tor Vergata", Roma, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo,Milan,Italy; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; Albert Stadler, Switzerland

S214. Let $x > y$ be positive rational numbers and R_0 a rectangle of dimensions $x \times y$. By a cut of R_0 we understand a dissection of the rectangle in two pieces: a square of dimensions $y \times y$ and a rectangle R_1 of dimensions $(x-y) \times y$. Similarly, R_2 is obtained from a cut of R_1, and so on. Prove that after finitely many cuts the sequence of rectangles $R_1, R_2, \dots R_k$ ends into the square R_k. Find k in terms of x, y and find the dimensions of R_k.

Proposed by Mircea Becheanu, University of Bucharest, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain Express $x = \frac{a}{b}$, $y = \frac{c}{d}$ in irreducible form, ie such that a, b are coprime positive integers, and so are c, d. Denote $M = \operatorname{lcm}(b,d)$, $D = \gcd(a,c)$, then the dimensions of R_k are $\frac{D}{M} \times \frac{D}{M}$; indeed, except for a scaling by a factor of M, the problem is equivalent to performing the cuts starting with a rectangle of dimensions $u \times v = \frac{Ma}{b} \times \frac{Mc}{d} = \frac{ad}{\gcd(b,d)} \times \frac{bc}{\gcd(b,d)}$, clearly integral. Note also that the process is identical to following Euclid's algorithm for finding $\gcd(u,v)$, since while $u > v$, we subtract v from u, until either $u = v$ when u is a multiple of v, or u has become $u - cv = r$, r being the nonzero remainder obtained when dividing u into v, then the process continues by taking $u' = v$ and $v' = r$, and so on. Now, $\frac{b}{\gcd(b,d)}$ is prime with a because a, b are coprime, and is prime with d because of the definition of $\gcd(b,d)$. Therefore the size of square R_k is

$$\gcd\left(\frac{ad}{\gcd(b,d)}, \frac{bc}{\gcd(b,d)}\right) = \gcd\left(a, \frac{bc}{\gcd(b,d)}\right) = \gcd(a,c) = D.$$

We now need to divide by M in order to undo the scaling that turned x, y into integers u, v, or R_k is indeed a square of side $\frac{D}{M}$.

Expressing the number of steps needed to find $\gcd(m,n)$ for arbitrary m, n is, as far as the author of this solution knows, an open problem. We may however define recursively $k(u,v)$, the number of steps needed to find R_k when the starting rectangle has integral dimensions $u \times v$, as:

$$k(u,v) = \begin{cases} \frac{u}{v} - 1 & \text{if } v \text{ divides } u, \\ c + k(v,r) & \text{if } u = cv + r \text{ for positive integers } c, r < v. \end{cases}$$

We just need to define u, v as earlier, ie $u = \frac{ad}{\gcd(b,d)}$, $v = \frac{bc}{\gcd(b,d)}$ where $x = \frac{a}{b}$ and $y = \frac{c}{d}$ are written in irreducible form.

Second solution by Alessio Podda ,Università di Roma "Tor Vergata", Roma, Italy
The pair (x,y) can be written as $(a/s, b/s)$, were a, b are positive integers and s is the least common multiple of the denominators of x, y (we assume that the fractions x, y are reduced to lowest terms).
The dissection described in the statement is equivalent to the Euclidean Algorithm applied to the couple (a,b). Therefore, after finitely many cuts, we will obtain a square R_k of dimensions $\frac{1}{s} \times \frac{1}{s}$. It is easy to verify that if

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_r}}}}$$

is the continued fraction expansion of $x/y = a/b$ with $a_0, a_1, a_2, \dots, a_r$ positive integers, then, at end of the dissection procedure, we get

$$k = a_0 + a_1 + a_2 + \cdots + a_r$$

S215. Let ABC be a given triangle and let ρ_A, ρ_B, ρ_C be the lines through the vertices A, B, C and parallel to the Euler line OH, where O and H are the circumcenter and orthocenter of ABC. Let X be the intersection of ρ_A with the sideline BC. The points Y, Z are defined analogously. If I_a, I_b, I_c are the corresponding excenters of triangle ABC, then the lines XI_a, YI_b, ZI_c are concurrent on the circumcircle of triangle $I_aI_bI_c$.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The trilinear coordinates α, β, γ of O, H are respectively

$$O \equiv (\cos A, \cos B, \cos C), \qquad H \equiv (\cos B \cos C, \cos C \cos A, \cos A \cos B).$$

Now, any parallel to OH has trillinear equation $u\alpha + v\beta + w\gamma$, where

$$0 = \begin{vmatrix} a & b & c \\ u & v & w \\ u_0 & v_0 & w_0 \end{vmatrix},$$

where $u_0\alpha + v_0\beta + w_0\gamma = 0$ is the equation in trilinear coordinates of OH. After some algebra, and since a line through A has $u = 0$, we find that ρ_A has trilinear equation

$$\frac{\beta}{\gamma} = \frac{b\cos^2 B - \cos C \cos A(c\cos C + a\cos A)}{c\cos^2 C - \cos A \cos B(a\cos A + b\cos B)} = \frac{\cos B - 2\cos C \cos A}{\cos C - 2\cos A \cos B},$$

where we have used that

$$a\cos A + b\cos B + c\cos C = \frac{8S^2}{abc} = \frac{2S}{R},$$
$$b\cos B + b\cos C \cos A = b\sin C \sin A = \frac{S}{R},$$

where S, R are area and circumradius of ABC, respectively. Define therefore

$$\alpha_0 = \cos A - 2\cos B \cos C, \qquad \beta_0 = \cos B - 2\cos C \cos A, \qquad \gamma_0 = \cos C - 2\cos A \cos B,$$

or ρ_A is defined by $\frac{\beta}{\gamma} = \frac{\beta_0}{\gamma_0}$. Now, any point on BC satisfies $\alpha = 0$, or $X \equiv (0, \beta_0, \gamma_0)$, whereas $I_a \equiv (-1, 1, 1)$, and line XI_a satisfies the trilinear equation

$$\alpha = \frac{\gamma_0\beta - \beta_0\gamma}{\beta_0 - \gamma_0}.$$

Now the circumcircle of $I_aI_bI_c$ has trilinear equation

$$(\alpha + \beta + \gamma)(a\alpha + b\beta + c\gamma) + (a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0,$$

or after some algebra, the point where XI_a meets this circumcircle has coordinates β, γ that satisfy

$$a\beta_0\gamma_0(\beta - \gamma)^2 + (\beta_0 - \gamma_0)(b\beta_0 + c\gamma_0)(\beta^2 - \gamma^2) = 0.$$

Now, if $\beta = \gamma$, then $\alpha = -1$, which corresponds to I_a. We may therefore divide by $\beta - \gamma$, and the second point where XI_a meets the circumcircle of $I_aI_bI_c$ satisfies

$$\frac{a\beta_0\gamma_0}{\beta_0 - \gamma_0}(\beta - \gamma) + (b\beta_0 + c\gamma_0)(\beta + \gamma) = 0.$$

But $a\alpha_0 + b\beta_0 + c\gamma_0 = 0$, and substituting this result we finally obtain

$$\frac{\beta}{\gamma} = \frac{\alpha_0\beta_0 + \beta_0\gamma_0 - \gamma_0\alpha_0}{\beta_0\gamma_0 + \gamma_0\alpha_0 - \alpha_0\beta_0}.$$

Note that, if indeed $\beta = \alpha_0\beta_0 + \beta_0\gamma_0 - \gamma_0\alpha_0$ and $\gamma = \beta_0\gamma_0 + \gamma_0\alpha_0 - \alpha_0\beta_0$, then

$$\alpha = \frac{\gamma_0\beta - \beta_0\gamma}{\beta_0 - \gamma_0} = \gamma_0\alpha_0 + \alpha_0\beta_0 - \beta_0\gamma_0,$$

and the point where XI_a meets the circumcircle of $I_aI_bI_c$ has trilinear coordinates

$$(\gamma_0\alpha_0 + \alpha_0\beta_0 - \beta_0\gamma_0, \alpha_0\beta_0 + \beta_0\gamma_0 - \gamma_0\alpha_0, \beta_0\gamma_0 + \gamma_0\alpha_0 - \alpha_0\beta_0)\,.$$

These coordinates are invariant under simultaneous cyclic permutation of A, B, C and the corresponding trilinear coordinates α, β, γ, or this point will also be where YI_b, ZI_c meets the circumcircle of $I_aI_bI_c$. The conclusion follows.

S216. Let p be a prime number. Prove that for each positive integer n the polynomial

$$P(X) = (X^p + 1^2)(X^p + 2^2)\dots(X^p + n^2) + 1$$

is irreducible in $\mathbb{Z}[X]$.

Cezar Lupu, University of Pittsburgh, USA, and and Tudorel Lupu, Decebal High School Constanta, Romania

Solution by Cosmin Pohoata, Princeton University, USA

Consider the polynomial $g(X) = (X + 1^2)(X + 2^2)\dots(X + n^2) + 1$. First, we claim that this polynomial is irreducible in $\mathbb{Z}[X]$. Indeed, assuming the contrary that $g(X) = A(X)B(X)$ for some polynomials A, B over $\mathbb{Z}$, we see that $1 = g(-1) = A(-k^2)B(-k^2)$, and so $A(-1) = \pm 1$ and $B(-1) = \pm 1$ for all $1 \le k \le n$, and furthermore $A(-k^2) = B(-k^2)$ for all these values. Consequently, $A - B$ is divisible by g and so it should be the zero polynomial (since A and B are assumed to have degree at most $n - 1$). It follows that $g = A^2$ and so $(n!)^2 + 1 = g(0)$ must be a perfect square, which is impossible. Hence, we conclude that g is indeed irreducible.

Now, the problem in question is a consequence of the following more general criterion that appeared as a question in a Romanian IMO Team Selection Test from 2003:

Lemma. Let g be a monic polynomial with integer coefficients and let p be a prime number. If g is irreducible in $\mathbb{Z}[X]$ and $\sqrt[p]{(-1)^{\deg g} g(0)}$ is irrational, then $g(X^p)$ is also irreducible in $\mathbb{Z}[X]$.

A proof for this useful irreducibility test can be seen in T. Andreescu, G. Dospinescu, *Problems from the Book*, XYZ Press, 2008, pp. 494.

U211. On the set $M = \mathbb{R} - \{3\}$ the following binary law is defined:

$$x * y = 3(xy - 3x - 3y) + m,$$

where $m \in \mathbb{R}$. Find all possible values of m such that $(M, *)$ is a group.

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Note first that if $(M, *)$ is a group, then $*$ is internal, or $x * y \neq 3$ for any x, y. Assume otherwise, hence

$$1 - \frac{m}{3} = xy - 3x - 3y = (x-3)(y-3) - 9, \qquad 10 - \frac{m}{3} = (x-3)(y-3).$$

Note that, if $10 - \frac{m}{3} \neq 0$, we may take $x = 4$ and $y = 13 - \frac{m}{3} \neq 3$, and $x, y \in M$ exist such that $x * y = 3$, or $(M, *)$ could never be a group. Thus $m = 30$, or

$$x * y = 3(x-3)(y-3) + 3.$$

Clearly, as long as $x, y \in M$, we have $(x-3)(y-3) \neq 0$, and $x * y \neq 3$, or $*$ is internal in M iff $m = 30$. Note also that x, y may be interchanged in the definition of the binary law without affecting its result, hence $*$ is commutative.

Given any $x, y, z \in M$, note that

$$(x * y) * z = 3((x * y) - 3)(z - 3) + 3 = 9(x-3)(y-3)(z-3) + 3,$$

$$x * (y * z) = 3(x-3)((y * z) - 3) + 3 = 9(x-3)(y-3)(z-3) + 3,$$

and $*$ is associative.

Given any $x \in M$, note that

$$x * \frac{10}{3} = \frac{10}{3} * x = 3(x-3)\left(\frac{10}{3} - 3\right) + 3 = (x-3) + 3 = x,$$

or $e = \frac{10}{3}$ is an identity in $(M, *)$. No other identity exists, since for any identity e, it must hold that

$$x * e - 3 = 3(x-3)(e-3) = x - 3, \qquad \text{or indeed,} \qquad e - 3 = \frac{1}{3}.$$

Finally, for any $x \in M$, take $y = 3 + \frac{1}{9(x-3)}$, clearly in M since it is real and different from 3, and

$$(x-3)(y-3) = \frac{1}{9}, \qquad \text{for} \qquad y * x = x * y = \frac{1}{3} + 3 = \frac{10}{3} = e,$$

and every element of M has an inverse in M with respect to $*$.

It follows that $(M, *)$ is a group iff $m = 30$, and it is abelian.

Also solved by Albert Stadler, Switzerland; Kannappan Sampath,Bangalore, India; Harun Immanuel, ITS Surabaya; Noah Taylor, Ohio State University, USA; Matteo Elia, Lorenzo Luzzi, Alessio Podda, and Francesco Bonesi, Università di Roma "Tor Vergata", Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo,Milan,Italy; Gabriel Alexander Chicas Reyes, El Salvador

U212. Let G be a finite abelian group such that G contains a subgroup $K \neq \{e\}$ with the property that $K \subset H$ for each subgroup H of G such that $H \neq \{e\}$. Prove that G is a cyclic group.

Proposed by Daniel Lopez Aguayo, Institute of Mathematics, UNAM, Morelia, Mexico

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let $k \in K$ be any element in K other than e, and let $p = o(k)$ be the order of k, ie the minimum positive integer such that $g^{o(g)} = e$, which clearly exists, and is 1 iff $g = e$. Clearly, $\{e, k, k^2, \ldots, k^{u-1}\}$ is a subgroup of G, or it contains K, whereas since this subgroup is generated by an element of K, it is a subgroup of K. Or $K = \{e, k, k^2, \ldots, k^{p-1}\}$ for each element $k \in K$ such that $k \neq e$. We conclude that $o(k) = p$ for any $k \in K$. Note that p must be prime, since if it had a proper divisor d, then $q = \frac{p}{d}$ such that $1 < d, q < p$, and $1 < o(k^d) \leq q < p$, contradiction. K is therefore a cyclic subgroup of G, generated by any one of its elements other than e, and of prime order p. Furthermore, assume that an element $g \in G$ is such that $g^p = e$; then $g \in K$, since $\{e, g, \ldots, g^{p-1}\}$ is a cyclic subgroup of G, then it contains K, but it hast the same number of elements as K, hence it must be K.

Assume that for some element $g \in G \backslash K$, $o(g)$ contains a prime divisor q that is prime with p, or $o(g) = qd$ for some positive integer d. Define $h = g^d$, clearly $o(g^d) = q$, and $\{e, h, h^2, \ldots, h^{q-1}\}$ is a subgroup of G, hence it contains K, or for some integer $1 < r < q$, $h^r \in K$, and $h^{rp} = e$. But since rp is prime with q, by Bezout's identity nonzero integers u, v of opposite sign exist such that $urp + vq = 1$. If $u > 0 > v$, $(h^{rp})^u = h \cdot (h^q)^{-v}$, and if $v > 0 > u$, $(h^q)^v = h \cdot (h^{rp})^{-u}$, or $h = e$ in both cases since $h^{rp} = h^q = e$, contradiction. Or every element $g \in G$ has order $o(g) = p^a$ for some non-negative integer a (and $a = 0$ iff $g = e$).

We finish the problem using the following

Lemma: Let G be a finite abelian group such that for any of its elements $g \neq e$, a positive integer u exists for which $o(g) = p^u$, and such that G contains a unique cyclic subgroup of order p. If $\max(o(g)) = U$ for any $g \in G$, then $|G| = p^U$, and hence G is cyclic.

Proof: By induction on U, if $U = 1$, then G is equal to its unique cyclic subgroup of order p, or $|G| = p$ and G is cyclic. If the result is true for U, then for a group such that $\max(o(g)) = U + 1$, define $G' = \{g^p : g \in G\}$, and the homomorphism $f : G \to G'$ by $f(g) = g^p$. Clearly $\ker(f) = K$; indeed, if $f(g) = g^p = e$, then $o(g)$ divides p, and $g \in K$, whereas if $g \in K$, then $g^p = e$ and $g \in \ker(f)$. Note therefore that G' is isomorph to $\frac{G}{K}$, but since G' is a subgroup of G, then it is a group such that $\max(o(g')) = U$ for any $g' \in G'$, and by hypothesis of induction, $|G'| = p^U$ and G' is cyclic, hence $|G| = |G'| \cdot |K| = p^U \cdot p = p^{U+1}$. Moreover, taking any element $g \in G$ of order $o(g) = p^{U+1}$, $G = \{e, g, g^2, \ldots, g^{p^U - 1}\}$ is cyclic. The conclusion follows.

Second solution by Lorenzo Luzzi and Alessio Podda, Università di Roma "Tor Vergata", Roma, Italy

Since G is abelian and finite, it is isomorphic to a direct sum of cyclic groups

$$G \simeq \bigoplus_{i}^{n} \mathbf{Z}_{l_i}$$

If G is not cyclic, then $n \geq 2$ and we can consider the non trivial subgroups H_1 and H_2 generated respectively by $(1, 0, \ldots, 0)$ and by $(0, 1, \ldots, 0)$. By hypothesis $K \subset H_1$ and $K \subset H_1$, hence

$$K \subset H_1 \cap H_2 = \{e\}$$

and we have a contradiction.

Also solved by Kannappan Sampath, Indian Statistical Institute, Bangalore, India; Harun Immanuel, ITS Surabaya; Alessandro Ventullo,Milan,Italy

U213. Let $x_0 \in (0, \pi)$ fixed. For $n \in \mathbb{N}$ we set $x_n = \sin x_{n-1}$. Show that

$$x_n = \frac{3^{1/2}}{n^{1/2}} - \frac{3^{3/2} \ln n}{10 n^{3/2}} + O(n^{-3/2}).$$

Proposed by Anastasios Kotronis, Athens, Greece

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

We have that $x_1 \in (0,1)$ and $x_n = \sin x_{n-1} < x_{n-1} < 1$ for $n > 1$. Hence x_n is strictly decreasing and tends to 0^+. Moreover

$$x_{n+1} = \sin(x_n) = x_n - \frac{x_n^3}{6} + \frac{x_n^5}{120} + O(x_n^7)$$

implies that

$$\frac{1}{x_{n+1}^2} = \frac{1}{x_n^2} + \frac{1}{3} + \frac{x_n^2}{15} + O(x_n^4).$$

Therefore, by Stoltz-Cesaro

$$\lim_{n\to\infty} \frac{\frac{1}{x_n^2}}{n} = \lim_{n\to\infty} \left(\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2} \right) = \frac{1}{3}$$

and

$$\lim_{n\to\infty} \frac{\frac{1}{x_n^2} - \frac{n}{3}}{\ln n} = \lim_{n\to\infty} n \left(\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2} - \frac{1}{3} \right) = \lim_{n\to\infty} \left(\frac{n x_n^2}{15} + O(1/n) \right) = \frac{3}{15} = \frac{1}{5}.$$

because $\ln(1 + 1/n) = 1/n + O(1/n^2)$ and $O(x_n) = O(1/\sqrt{n})$. Hence

$$\frac{1}{x_n^2} = \frac{n}{3} + \frac{\ln n}{5} + O(\ln n / n)$$

and finally we get

$$x_n = \frac{3^{1/2}}{n^{1/2}} \left(1 + \frac{3 \ln n}{5n} + O(\ln n / n^2) \right)^{-1/2} = \frac{3^{1/2}}{n^{1/2}} - \frac{3^{3/2} \ln n}{10 n^{3/2}} + O(\ln n / n^{5/2}).$$

Also solved by Albert Stadler, Switzerland; Moubinool Omarjee , Paris, France; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy

U214. Prove that

$$\lim_{n\to+\infty}\prod_{k=1}^{n}\frac{\cosh\left(k^2+k+\frac{1}{2}\right)+i\sinh\left(k+\frac{1}{2}\right)}{\cosh\left(k^2+k+\frac{1}{2}\right)-i\sinh\left(k+\frac{1}{2}\right)}=\frac{e^2-1+2ie}{e^2+1}.$$

Proposed by Moubinool Omarjee, Lycee Jean Murcat, Paris, France

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria

Note that

$$\begin{aligned}\frac{\cosh\left(k^2+k+\frac{1}{2}\right)+i\sinh\left(k+\frac{1}{2}\right)}{\cosh\left(k^2+k+\frac{1}{2}\right)-i\sinh\left(k+\frac{1}{2}\right)} &= \frac{e^{2k^2+2k+1}+1+i(e^{k^2+2k+1}-e^{k^2})}{e^{2k^2+2k+1}+1-i(e^{k^2+2k+1}-e^{k^2})}\\ &= \frac{(e^{k^2}+i)(e^{(k+1)^2}-i)}{(e^{(k+1)^2}+i)(e^{k^2}-i)}\\ &= \frac{e^{k^2}+i}{e^{(k+1)^2}+i}\cdot\frac{e^{(k+1)^2}-i}{e^{k^2}-i}.\end{aligned}$$

So, the considered product is telescoping, and

$$\prod_{k=1}^{n}\frac{\cosh\left(k^2+k+\frac{1}{2}\right)+i\sinh\left(k+\frac{1}{2}\right)}{\cosh\left(k^2+k+\frac{1}{2}\right)-i\sinh\left(k+\frac{1}{2}\right)}=\frac{e+i}{e^{(n+1)^2}+i}\cdot\frac{e^{(n+1)^2}-i}{e-i}=\frac{e+i}{e-i}\cdot\frac{1+ie^{-(n+1)^2}}{1-ie^{-(n+1)^2}}$$

Therefore,

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{\cosh\left(k^2+k+\frac{1}{2}\right)+i\sinh\left(k+\frac{1}{2}\right)}{\cosh\left(k^2+k+\frac{1}{2}\right)-i\sinh\left(k+\frac{1}{2}\right)}=\frac{e+i}{e-i}=\frac{(e+i)^2}{e^2+1}=\frac{e^2-1+2ie}{e^2+1},$$

which is the desired conclusion.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Albert Stadler, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

U215. Let $f : [-1,1] \to \mathbb{R}$ be a continuous function such that $\int_{-1}^{1} x^2 f(x)dx = 0$. Prove that

$$\int_{-1}^{1} f^2(x)dx \geq \frac{9}{8}\left(\int_{-1}^{1} f(x)dx\right)^2.$$

Proposed by Cezar Lupu, University of Pittsburgh, USA, and Tudorel Lupu, Decebal High School Constanta, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

For all $x \in [-1,1]$, let

$$E(x) = \frac{f(x)+f(-x)}{2}, \qquad O(x) = \frac{f(x)-f(-x)}{2}.$$

Clearly $E(x) = E(-x)$ and $O(x) = -O(-x)$, both functions are continuous, and $f(x) = E(x)+O(x)$. Now, since $O(x)$ is an odd function, and $E(x), x^2$ are even functions, we have

$$\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} E(x)dx, \qquad \int_{-1}^{1} x^2 f(x)dx = 2\int_{0}^{1} x^2 E(x)dx = 0,$$

$$\int_{-1}^{1} f^2(x)dx = 2\int_{0}^{1} E^2(x)dx + 2\int_{0}^{1} O^2(x)dx,$$

Also,

$$\int_{0}^{1}\left(1-\frac{5x^2}{3}\right)^2 dx = 1 - \frac{10}{3}\cdot\frac{1}{3} + \frac{25}{9}\cdot\frac{1}{5} = \frac{4}{9},$$

or by the Cauchy-Schwarz inequality,

$$\int_{0}^{1} E(x)dx = \int_{0}^{1}\left(1-\frac{5x^2}{3}\right)E(x)dx \leq \frac{2}{3}\sqrt{\int_{0}^{1} E^2(x)dx},$$

with equality iff $E(x) = K\left(1-\frac{5x^2}{3}\right)$ for some real constant K. Finally,

$$\frac{9}{8}\left(\int_{-1}^{1} f(x)dx\right)^2 = \frac{9}{2}\left(\int_{0}^{1} E(x)dx\right)^2 \leq 2\int_{0}^{1} E^2(x)dx =$$

$$= \int_{-1}^{1} f^2(x)dx - 2\int_{0}^{1} O^2(x)dx \leq \int_{-1}^{1} f^2(x)dx,$$

with equality in the second inequality iff $O(x) = 0$ since $O(x)$ must be a continuous real function whose square integrates to 0. The conclusion follows, equality is reached iff a real constant C exists such that $f(x) = C\left(3-5x^2\right)$.

Also solved by Harun Immanuel, ITS Surabaya; Albert Stadler, Switzerland; Perfetti Paolo, Dipartimento di Matematica, Universita degli studi di Tor Vergata Roma, Italy; G.R.A.20 Problem Solving Group, Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria;

U216. Let N be a positive integer and let Δ_{f_N} be the discriminant of the polynomial $f_N(x) = x^N - x - 1$. Prove that for any prime p dividing Δ_{f_N}, the reduction modulo p of $f_N(x)$ has one double root and $N-2$ simple roots.

Note. The discriminant Δ_f of a polynomial f of degree n is defined as

$$D_f = a_n^{2n-2} \prod_{i<j} (r_i - r_j)^2 ,$$

where a_n is the leading coefficient and $r_1, \ldots, r_n$ are the roots (counting multiplicity) of the polynomial in some splitting field.

Cosmin Pohoata, Princeton University, USA

Solution by the author

In its attempt to be something more, this problem turns out to be more or less obvious, as we shall see. Let p be a prime dividing Δ_{f_N}. This means f has an inseparable factor modulo p, and so it has a repeated root, say s, in some extension of $\mathbb{F}_p$, satisfying $s^N - s - 1 = 0$ and $Ns^{N-1} - 1 = 0$. From this it is easy to see that $s = \frac{N}{1-N}$. Also, s cannot be repeated 3 times because then additionally we must have $N(N-1) = 0$, and simple algebra on the other hand gives us that $N^N - (1-N)^{N-1} = 0$, whence $N = 0$ and $N - 1 = 0$ mod p; but $\gcd(N, N-1) = 1$, so we get a contradiction. Therefore s is the unique repeated root of $f(x)$ modulo p and so $x^N - x - 1$ has at precisely one inseparable quadratic factor modulo p for any prime p dividing Δ_{f_N}. This completes the proof.

The interesting question that lies behind it is whether the discriminant of f_N is squarefree for all N. Unfortunately, this proves to be wrong for $N = 259$, yet it would be interesting to find more about the behavior of $(\Delta_{f_N})_{N \geq 1}$.

Olympiad problems

O211. Prove that for any positive integer n the number $(2^n + 4^n)^2 + 4(6^n + 9^n + 12^n)$ has at least 9 positive divisors.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Lucas Mioranci, Special Program for International Competitions, Rio de Janeiro, Brazil

Note that $N := (2^n + 4^n)^2 + 4(6^n + 9^n + 12^n) = (2^n + 4^n + 2 \times 3^n)^2$, thus N is the perfect square of an even number. Now, if that even number is divisible by a prime number p different from 2, then N is divisible by 2^2p^2 which has 9 divisors, so we are done in this case. Otherwise, if N is a power of 2, then things are simple as well: if $2^n + 4^n + 2 \times 3^n \geq 2^4$ then N is divisible by 2^8 and that number has 9 divisors. But for $n \geq 2$ we have $2^n + 4^n + 2 \times 3^n \geq 4^n \geq 2^4$, so we are left to check what happens for $n = 1$, and in that case $N = (2^2 \times 3)^2$.

Also solved by Albert Stadler, Switzerland; Lorenzo Luzzi, Alessio Podda, and Francesco Bonesi, Università di Roma "Tor Vergata", Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo,Milan,Italy

O212. Let $f_0(i) = 1$ for all $i \in \mathbb{N}$ and let $f_k(n) = \sum_{i=1}^{n} f_{k-1}(i)$ for $k \geq 1$. Prove that

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} f_j(n-2j) = F_n,$$

where F_n is the n-th Fibonacci number.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Alessandro Ventullo, Milan,Italy

First we show that if $n, k \geq 1$, then

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}. \tag{1}$$

Indeed, by Pascal's Identity (which might be the official name for the fundamental binomial coefficient identity)

$$\begin{aligned}
\sum_{i=k}^{n} \binom{i}{k} &= \binom{k}{k+1} + \sum_{i=k}^{n} \binom{i}{k} \\
&= \binom{k+1}{k+1} + \sum_{i=k+1}^{n} \binom{i}{k} \\
&= \binom{k+2}{k+1} + \sum_{i=k+2}^{n} \binom{i}{k} \\
&= \vdots \\
&= \binom{n+1}{k+1}.
\end{aligned}$$

Now, we prove by induction on $k \geq 1$ that $f_k(n) = \binom{n+k-1}{k}$ for any $n \in \mathbb{N}$. For $k = 1$ we have $f_1(n) = \sum_{i=1}^{n} f_0(i) = n$, so the equality holds. Suppose that the equality is true for k. Then

$$f_{k+1}(n) = \sum_{i=1}^{n} f_k(i) = \sum_{i=1}^{n} \binom{i+k-1}{k} = \binom{n+k}{k+1}$$

by (1). Now, it is clear that

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} f_j(n-2j) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j-1}{j}.$$

So, we only have to prove that

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j-1}{j} = F_n,$$

where F_n is the n-th Fibonacci number. This can be done by induction on n. For $n = 1$ and $n = 2$ is obvious. Suppose that the equality holds for every m such that $1 \leq m \leq n$. Since $\lfloor \frac{n-1}{2} \rfloor + 1 = \lfloor \frac{n+1}{2} \rfloor$, we have

$$\begin{aligned}
F_{n-1}+F_n &= \sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-j-2}{j}+\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-j-1}{j}\\
&= \binom{n-1}{0}+\sum_{j=1}^{\lfloor\frac{n+1}{2}\rfloor}\left[\binom{n-j-1}{j-1}+\binom{n-j-1}{j}\right]\\
&= \binom{n}{0}+\sum_{j=1}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n-j}{j}\\
&= \sum_{j=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n-j}{j}=F_{n+1},
\end{aligned}$$

as we wanted to prove.

Also solved by Albert Stadler, Switzerland; G.R.A.20 Problem Solving Group, Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Gabriel Alexander Chicas Reyes, El Salvador

O213. Let n be a positive integer and let z be a complex number such that $z^{2^n-1}-1=0$. Evaluate

$$\prod_{k=0}^{n-1}\left(z^{2^k}+\frac{1}{z^{2^k}}-1\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia
Let

$$Z_n=\prod_{k=0}^{n-1}\left(z^{2^k}+\frac{1}{z^{2^k}}-1\right).$$

We have that

$$\left(z+\frac{1}{z}+1\right)Z_n=\left(z^2+\frac{1}{z^2}+1\right)\left(z^2+\frac{1}{z^2}-1\right)\cdots\left(z^{2^{n-1}}+\frac{1}{z^{2^{n-1}}}+1\right)=\left(z^{2^n}+\frac{1}{z^{2^n}}+1\right).$$

However, from the given condition we have that $z^{2^n}=z$. Finally,

$$\left(z+\frac{1}{z}+1\right)Z_n=\left(z+\frac{1}{z}+1\right).$$

Hence $Z_n=1$.

Also solved by Francesco Bonesi and Antonio Cirulli ,Università di Roma "Tor Vergata", Roma, Italy; Albert Stadler, Switzerland; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Alessandro Ventullo,Milan,Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Gabriel Alexander Chicas Reyes, El Salvador

O214. The vertices $A_1, A_2, \ldots, A_n$ of a regular polygon lie on a circle $\mathcal{C}$ of center O. Is it true that the map $P \to \sum_{k=1}^{n} \frac{1}{{PA_k}^4}$, defined on the set of points of the plane outside $\mathcal{C}$, is a rational function in OP?

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We may write WLOG that the coordinates of A_k, for $k = 1, 2, \ldots, n$, are

$$A_k \equiv \left(R\cos\left(\frac{\pi(2k-1)}{n}\right), R\sin\left(\frac{\pi(2k-1)}{n}\right) \right),$$

where R is the radius of $\mathcal{C}$, and the positive horizontal semiaxis is clearly the perpendicular bisector of side A_nA_1 of the polygon. Note that, for any point with coordinates $P \equiv (\rho, 0)$, where $\rho > R$ so that P is outside $\mathcal{C}$, we have

$$PA_k > PA_1 = PA_n > \frac{A_1A_n}{2} = 2R\sin\left(\frac{\pi}{n}\right),$$

or for all $\rho > R$,

$$\sum_{k=1}^{n} \frac{1}{{PA_k}^4} < \frac{1}{16R^4\sin^4\left(\frac{\pi}{n}\right)}.$$

Take now point Q such that, for some $\rho > R$ so that Q is outside $\mathcal{C}$, we have

$$Q \equiv \left(\rho\cos\left(\frac{\pi}{n}\right), \rho\sin\left(\frac{\pi}{n}\right) \right),$$

or clearly $QA_1 = \rho - R$, while $QA_k > 0$. Therefore,

$$\sum_{k=1}^{n} \frac{1}{{QA_k}^4} > \frac{1}{{QA_1}^4} = \frac{1}{(\rho - R)^4}.$$

Note that we may take $\rho - R$ positive but as small as we want, or this last expression has no upper bound, whereas for the same value of ρ, the corresponding value for P does have an upper bound.

We can conclude that if by *"a rational function in OP"* is meant *"a function that is rational in OP and depends on no other parameter"*, then the proposed statement is not true, since for the same value of $OP = OQ = \rho$, we obtain different values for the expression defined in the problem statement when we change the relative position of the point with respect to the vertices of the polygon. On the other hand, for any $P \equiv (\rho\cos\theta, \rho\sin\theta)$, we can readily find ${PA_k}^2 = \rho^2 + R^2 - 2R\rho\cos(\theta - \alpha_k)$, where $\alpha_k = \frac{\pi(2k-1)}{n}$, and clearly each one of the ${PA_k}^4$ is a function on α_k and $\rho = OP$, which is rational in OP for fixed α_k. We can also conclude that, if by *"a rational function in OP"* is meant *"a function that may depend on other parameters, but that is rational in OP when the rest of the parameters are kept constant"*, then the proposed statement is true.

Also solved by Albert Stadler, Switzerland

O215. Prove that there are no positive integers a, b, c, d that are consecutive terms of an arithmetic progression and also satisfy the condition that $ab+1$, $ac+1$, $ad+1$, $bc+1$, $bd+1$, $cd+1$ are all perfect squares.

Cosmin Pohoata, Princeton University, USA

No solutions have been received yet.

O216. Let $f \in \mathbb{Z}[X]$ be a monic polynomial of degree greater than 1. Suppose that $f(X^n)$ is a reducible polynomial in $\mathbb{Z}[X]$ for all $n \geq 2$. Does it follow that f is reducible in $\mathbb{Z}[X]$?

Gabriel Dospinescu, Ecole Polytechnique, France

Solution by the author We will prove that the answer is positive.

Assume that f is irreducible and let L be its splitting field (i.e. field generated by its roots).

Claim. Let p be a prime and let α be a root of f. Then α is a pth power in L.

Proof of Claim. Assuming the contrary, we will prove that $f(X^p)$ is irreducible in $\mathbf{Q}[X]$, contradicting the hypothesis. Let $K = \mathbf{Q}(\alpha)$. Clearly $K \subset L$, so α is not a pth power in K. It is standard that this implies the irreducibility of the polynomial $X^p - \alpha$ in $K[X]$. Take a root β of this polynomial. It is a root of $f(X^p)$ and since $\alpha = \beta^p$, we have $\mathbf{Q}(\beta) = K(\beta)$. Hence

$$[\mathbf{Q}(\beta) : \mathbf{Q}] = [K(\beta) : K] \cdot [K : \mathbf{Q}] = p \cdot \deg f,$$

the last equality being a consequence of the irreducibility of f and of the previous discussion. This shows that the minimal polynomial of β has degree $p \cdot \deg f$, which is the degree of $f(X^p)$. Hence $f(X^p)$ is irreducible and so the claim is proven.

Next, we show that any root α of f is a root of unity. Indeed, let O_L be the ring of integers in L. As f is monic, we have $\alpha \in O_L$. By looking at the prime decomposition of the ideal αO_L and by using the previous result, it is easy to see that α must be a unit in O_L (if a prime ideal divided α, the previous lemma says that its multiplicity in the decomposition of αO_L has infinitely many prime factors, which is absurd). By Dirichlet's unit theorem, the $\mathbf{Z}$-module $M = O_L^*$ is of the form $\mathbf{Z}^d \oplus \mathbf{Z}/k\mathbf{Z}$ for some k and d (we actually need the easy part of the theorem, since we don't need the value of d). The previous lemma says that α is in pM for all primes p. It is immediate to check that the image of α in $\mathbf{Z}^d$ is zero and so α is a torsion element. The result follows.