Junior problems

J289. Let a be a real number such that $0 \le a < 1$. Prove that

$$\left\lfloor a\left(1+\left\lfloor\frac{1}{1-a}\right\rfloor\right)\right\rfloor+1=\left\lfloor\frac{1}{1-a}\right\rfloor.$$

Proposed by Arkady Alt, San Jose, California, USA

J290. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$\sqrt[3]{13a^3+14b^3}+\sqrt[3]{13b^3+14c^3}+\sqrt[3]{13c^3+14a^3}\ge 3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J291. Let ABC be a triangle such that $\angle BCA = 2\angle ABC$ and let P be a point in its interior such that $PA = AC$ and $PB = PC$. Evaluate the ratio of areas of triangles PAB and PAC.

Proposed by Panagiote Ligouras, Noci, Italy

J292. Find the least real number k such that for every positive real numbers x, y, z, the following inequality holds:

$$\prod_{cyc}(2xy+yz+zx)\le k(x+y+z)^6.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

J293. Find all positive integers x, y, z such that

$$(x+y^2+z^2)^2-8xyz=1.$$

Proposed by Aaron Doman, University of California, Berkeley , USA

J294. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$1\le(a^2-a+1)(b^2-b+1)(c^2-c+1)\le 7.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Senior problems

S289. Let x, y, z be positive real numbers such that $x \leq 4$, $y \leq 9$ and $x + y + z = 49$. Prove that

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} \geq 1.$$

Proposed by Marius Stanean, Zalau, Romania

S290. Prove that there is no integer n for which

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} = \left(\frac{4}{5}\right)^2.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S291. Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that

$$(2a^2 - 3ab + 2b^2)(2b^2 - 3bc + 2c^2)(2c^2 - 3ca + 2a^2) \geq \frac{5}{3}(a^2 + b^2 + c^2) - 4.$$

Proposed by Titu Andreescu, USA and Marius Stanean, Romania

S292. Given triangle ABC, prove that there exists X on the side BC such that the inradii of triangles AXB and AXC are equal and find a ruler and compass construction.

Proposed by Cosmin Pohoata, Princeton University, USA

S293. Let a, b, c be distinct real numbers and let n be a positive integer. Find all nonzero complex numbers z such that

$$az^n + b\overline{z} + \frac{c}{z} = bz^n + c\overline{z} + \frac{a}{z} = cz^n + a\overline{z} + \frac{b}{z}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S294. Let $s(n)$ be the sum of digits of $n^2 + 1$. Define the sequence $(a_n)_{n \geq 0}$ by $a_{n+1} = s(a_n)$, with a_0 an arbitrary positive integer. Prove that there is n_0 such that $a_{n+3} = a_n$ for all $n \geq n_0$.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

Undergraduate problems

U289. Let $a \geq 1$ be such that $(\lfloor a^n \rfloor)^{\frac{1}{n}} \in \mathbb{Z}$ for all sufficiently large integers n. Prove that $a \in \mathbb{Z}$.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

U290. Prove that there are infinitely many consecutive triples of primes (p_{n-1}, p_n, p_{n+1}) such that $\frac{1}{2}(p_{n+1} + p_{n-1}) \leq p_n$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U291. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function and let $\mathcal{S}$ be the set of all increasing maps $\varphi : \mathbb{R} \to \mathbb{R}$. Prove that there is a unique function g in $\mathcal{S}$ satisfying the conditions:

a) $f(x) \leq g(x)$, for all $x \in \mathbb{R}$.

b) If $h \in \mathcal{S}$ and $f(x) \leq h(x)$ for all $x \in \mathbb{R}$, then $g(x) \leq h(x)$ for all $x \in \mathbb{R}$.

Proposed by Marius Cavachi, Constanta, Romania

U292. Let r be a positive real number. Evaluate

$$\int_0^{\pi/2} \frac{1}{1 + \cot^r x} dx.$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

U293. Let $f : (0, \infty) \to \mathbb{R}$ be a bounded continuous function and let $\alpha \in [0, 1)$. Suppose there exist real numbers $a_0, \ldots, a_k$, with $k \geq 2$, so that $\sum_{p=0}^{k} a_p = 0$ and

$$\lim_{x \to \infty} x^\alpha \left| \sum_{p=0}^{k} a_p f(x + p) \right| = \alpha.$$

Prove that $\alpha = 0$.

Proposed by Marcel Chirita, Bucharest, Romania

U294. Let $p_1, p_2, \ldots, p_n$ be pairwise distinct prime numbers. Prove that

$$\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}) = \mathbb{Q}(\sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_n}).$$

Proposed by Marius Cavachi, Constanta, Romania

Olympiad problems

O289. Let a, b, x, y be positive real numbers such that $x^2 - x + 1 = a^2$, $y^2 + y + 1 = b^2$, and $(2x - 1)(2y + 1) = 2ab + 3$. Prove that $x + y = ab$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O290. Let Ω_1 and Ω_2 be the two circles in the plane of triangle ABC. Let α_1, α_2 be the circles through A that are tangent to both Ω_1 and Ω_2. Similarly, define β_1, β_2 for B and γ_1, γ_2 for C. Let A_1 be the second intersection of circles α_1 and α_2. Similarly, define B_1 and C_1. Prove that the lines AA_1, BB_1, CC_1 are concurrent.

Proposed by Cosmin Pohoata, Princeton University, USA

O291. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{\sqrt{4a^2+ab+4b^2}} + \frac{b^2}{\sqrt{4b^2+bc+4c^2}} + \frac{c^2}{\sqrt{4c^2+ca+4a^2}} \geq \frac{a+b+c}{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O292. For each positive integer n let

$$T_n = \sum_{k=1}^{n} \frac{1}{k \cdot 2^k}.$$

Find all prime numbers p for which

$$\sum_{k=1}^{p-2} \frac{T_k}{k+1} \equiv 0 \pmod{p}.$$

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Lyon

O293. Let x, y, z be positive real numbers and let $t^2 = \frac{xyz}{\max(x,y,z)}$. Prove that

$$4(x^3 + y^3 + z^3 + xyz)^2 \geq (x^2 + y^2 + z^2 + t^2)^3.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O294. Let ABC be a triangle with orthocenter H and let D, E, F be the feet of the altitudes from A, B and C. Let X, Y, Z be the reflections of D, E, F across EF, FD, and DE, respecitvely. Prove that the circumcircles of triangles HAX, HBY, HCZ share a common point, other than H.

Proposed by Cosmin Pohoata, Princeton University, USA