Junior problems

J331. Determine all positive integers n such that

$$n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n!}\right)$$

is divisible by n.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Henry Ricardo, New York Math Circle
We get divisibility by n for $n=1$ and for n a prime.

Clearly the given product is divisible by n when $n=1$. For $n \geq 2$ we have

$$\begin{aligned} n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n!}\right) &= n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)+(n-1)!+1 \\ &\equiv (n-1)!+1 \pmod{n} \equiv 0 \pmod{n} \Longleftrightarrow n \text{ is prime} \end{aligned}$$

by Wilson's theorem.

Also solved by Daniel Lasaosa, Pamplona, Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Haimoshri Das, South Point High School, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Alberto Espuny Díaz, CFIS, Universitat Politècnica de Catalunya, Barcelona, Spain; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; David E. Manes, Oneonta, NY, USA; Problem Solving Group, Department of Financial and Management Engineering, University of the Aegean, Greece; G. C. Greubel, Newport News, VA, USA; James Pierog, SMIC School, Shanghai, China; Robert Bosch, Archimedean Academy, FL, USA; Albert Stadler, Herrliberg, Switzerland; Suhas Vadan Gondi, KV-IISc High School, Bengaluru, India; Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania; Francesco Bonesi, University of East Anglia, UK; Polyahedra, Polk State College, FL, USA; Kwonil Kobe Ko, Cushing Academy, MA, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Dhruv Nevatia, Ramanujan Academy, Nashik, Maharashtra, India; Utamuratov Odilbek, Academic Lyceum Nr.2, Uzbekistan; Michael Tang, Edina High School, MN, USA; Mahmoud Ezzaki and Chakib Belgani; Louis Cahyadi, Cirebon, Indonesia; Daniel Jhiseung Hahn, Phillips Exeter Academy, Exeter, NH, USA; Théo Lenoir, Institut Saint-Lô, Agneaux, France; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria.

J332. Let $n \geq 3$ and $0 = a_0 < a_1 < \cdots < a_{n+1}$ such that $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n = a_na_{n+1}$. Prove that

$$\frac{1}{{a_3}^2 - {a_0}^2} + \frac{1}{{a_4}^2 - {a_1}^2} + \cdots + \frac{1}{{a_{n+1}}^2 - {a_{n-2}}^2} \geq \frac{1}{{a_{n-1}}^2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
The expression on the left-hand side can be rewritten as

$$\frac{a_1^2a_2^2}{a_1^2a_2^2a_3^2 - a_0^2a_1^2a_2^2} + \frac{a_2^2a_3^2}{a_2^2a_3^2a_4^2 - a_1^2a_2^2a_3^2} + \cdots + \frac{a_{n-1}^2a_n^2}{a_{n-1}^2a_n^2a_{n+1}^2 - a_{n-2}^2a_{n-1}^2a_n^2}.$$

Applying the Cauchy-Schwartz inequality then yields

$$\begin{aligned}
&\frac{a_1^2a_2^2}{a_1^2a_2^2a_3^2 - a_0^2a_1^2a_2^2} + \frac{a_2^2a_3^2}{a_2^2a_3^2a_4^2 - a_1^2a_2^2a_3^2} + \cdots + \frac{a_{n-1}^2a_n^2}{a_{n-1}^2a_n^2a_{n+1}^2 - a_{n-2}^2a_{n-1}^2a_n^2} \\
\geq\ & \frac{(a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n)^2}{a_1^2a_2^2a_3^2 - a_0^2a_1^2a_2^2 + a_2^2a_3^2a_4^2 - a_1^2a_2^2a_3^2 + \cdots + a_{n-1}^2a_n^2a_{n+1}^2 - a_{n-2}^2a_{n-1}^2a_n^2} \\
=\ & \frac{a_n^2a_{n+1}^2}{a_{n-1}^2a_n^2a_{n+1}^2 - a_0^2a_1^2a_2^2} = \frac{1}{a_{n-1}^2}.
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Kwonil Kobe Ko, Cushing Academy, MA, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Jhiseung Hahn, Phillips Exeter Academy, Exeter, NH, USA; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Adnan Ali, A.E.C.S-4, Mumbai, India; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

J333. Consider an equiangular hexagon $ABCDEF$. Prove that

$$AC^2 + CE^2 + EA^2 = BD^2 + DF^2 + FB^2.$$

Proposed by Nairi Sedrakyan, Armenia

Solution by Robert Bosch, Archimedean Academy, Florida, USA
Denote the sides of equiangular hexagon by a, b, c, d, e, f in clockwise order. More precisely,

$$AB = a,\ BC = b,\ CD = c,\ DE = d,\ EF = e,\ FA = f.$$

The following lemma is true.

$$a - d = e - b = c - f.$$

The proof may be found in the paper "*Equiangular polygons. An algebraic approach.* Titu Andreescu, Bogdan Enescu. Mathematical Reflections.

Now, by Cosines's law, it follows

$$\begin{aligned} AC^2 &= a^2 + b^2 - 2ab\cos\alpha, \\ CE^2 &= c^2 + d^2 - 2cd\cos\alpha, \\ EA^2 &= e^2 + f^2 - 2ef\cos\alpha, \\ BD^2 &= b^2 + c^2 - 2bc\cos\alpha, \\ DF^2 &= d^2 + e^2 - 2de\cos\alpha, \\ FB^2 &= f^2 + a^2 - 2fa\cos\alpha. \end{aligned}$$

Where, α is the common inner angle. By above relations,

$$AC^2 + CE^2 + EA^2 = BD^2 + DF^2 + FB^2 \Leftrightarrow ab + cd + ef = bc + de + fa.$$

By lemma:

$$\begin{aligned} 2(ab - de) &= e^2 + d^2 - a^2 - b^2, \\ 2(cd - fa) &= a^2 + f^2 - c^2 - d^2, \\ 2(ef - bc) &= b^2 + c^2 - e^2 - f^2. \end{aligned}$$

Adding, we get the conclusion.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Kwonil Kobe Ko, Cushing Academy, MA, USA; Dhruv Nevatia, Ramanujan Academy, Nashik, Maharashtra, India; Utamuratov Odilbek, Academic Lyceum Nr.2, Uzbekistan; Michael Tang, Edina High School, MN, USA; Daniel Jhiseung Hahn, Phillips Exeter Academy, Exeter, NH, USA; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; David E. Manes, Oneonta, NY, USA; Nicolas Huynh, Lycée Richelieu, Rueil-Malmaison, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

J334. Let ABC be a triangle with $\angle A \geq 60^\circ$ and let D and E be points on the sides AB and AC, respectively. Prove that

$$\frac{BC}{\min(BD, DE, EC)} \geq \sqrt{5 - 4\cos A}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by the author Suppose $BD = \min(BD, DE, EC)$. We can start moving segment BD closer to A keeping the distance BD fixed. Say this achieved for $B'D' = BD$, where B' and D' are new points. We draw a line through D' parallel to DE and a line through B' parallel to BC. Because $AB' < AB$, we have $D'E' < DE$ and $B'C' < BC$. Using this optimizing procedure can reach a configuration with either $BD = DE = \min(BD, DE, EC)$ or $BD = EC = \min(BD, DE, EC)$, because $\min(BD, DE, EC)$ stays fixed, while BC is decreasing.

Suppose $DE = \min(BD, DE, EC)$. Then we can can draw a line $B'C'$ closer to A and parallel to BC such that $B'D < BD$ and $C'E < CE$. Using this optmizing procedure, without loss of generality, we reach a configuration with $BD = DE = \min(BD, DE, EC)$.

Consider the configuration with $BD = EC = \min(BD, DE, EC)$. Denote by $u = AD$, $v = AE$, $x = BD = CE$ and $y = DE$, where $y \geq x$. Then

$$a^2 = (u+x)^2 + (v+x)^2 - 2(u+x)(v+x)\cos A, \quad y^2 = u^2 + v^2 - 2uv\cos A.$$

It follows that

$$a^2 = y^2 + 2x^2(1-\cos A) + 2(u+v)x(1-\cos A) \geq x^2(5 - 4\cos A),$$

because $y \geq x$ and $(u+v) \geq x$.

Consider the configuration with $BD = DE = \min(BD, DE, EC)$. We can attempt to move point C, while minimize EC and BC. This is possible if $\angle C < 90^\circ$. However, if $\angle C \geq 90^\circ$ it is not, because BC is not going to lessen. From here, we can reach two possible configurations:

- $BD = DE = EC$, which follows from the previously analyzed case.
- $BD = DE < CE$ and $\angle C \geq 90^\circ$.

The distance from D to BC is $x\sin B$ and the distance from E to BC is $y\sin(A+B)$. Because $y\sin(A+B) \geq x\sin B$, we get $\angle DEA$ is obtuse. Hence $DC \leq DA$ and $x \leq \frac{AB}{2}$ and also $x \leq y \leq AC$. Therefore $\min(BD, DE, EC) \leq \min\left(\frac{c}{2}, b\right)$.

$$a^2 = b^2 + c^2 - 2bc\cos A = 4\left(\frac{c}{2}\right)^2 + b^2 - 4b\left(\frac{c}{2}\right)\cos A \geq$$

$$\geq 3\left(\frac{c}{2}\right)^2 + 2b\left(\frac{c}{2}\right)(1 - 2\cos A) \geq (5 - 4\cos A)(\min(BD, DE, EC))^2,$$

as desired.

J335. Prove that for any $a, b > -1$,

$$\max\left\{(a+3)\left(b^2+3\right), (b+3)\left(a^2+3\right)\right\} \geq 2(a+b+2)^{\frac{3}{2}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Michael Tang, Edina High School, MN, USA

Since $a+1, b+1 > 0$, by Hölder's Inequality,

$$\begin{gathered}(1+1)((a+1)^3+2^3)(2^3+(b+1)^3) \geq (2(a+1)+2(b+1))^3 \\ 2((a+1)^3+8)((b+1)^3+8) \geq 8(a+b+2)^3 \\ ((a+1)^3+8)((b+1)^3+8) \geq 4(a+b+2)^3.\end{gathered}$$

But note that $(a+1)^3+8 = a^3+3a^2+3a+9 = (a+3)(a^2+3)$, so we get

$$(a+3)(a^2+3)(b+3)(b^2+3) \geq 4(a+b+2)^3.$$

Now let

$$X = \max\{(a+3)(b^2+3), (a^2+3)(b+3)\}.$$

Then $X \geq (a+3)(b^2+3)$ and $X \geq (a^2+3)(b+3)$, so

$$X^2 \geq (a+3)(a^2+3)(b+3)(b^2+3) \geq 4(a+b+2)^3.$$

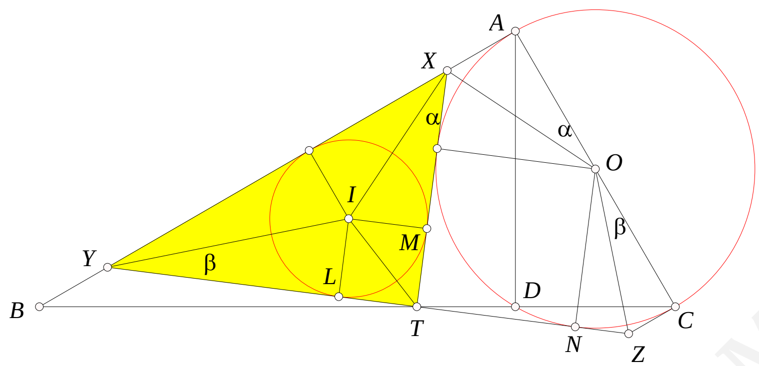
Thus $X \geq 2(a+b+2)^{3/2}$.

Also solved by Arkady Alt, San Jose, CA, USA; Polyahedra, Polk State College, USA; Kwonil Kobe Ko, Cushing Academy, MA, USA; Mahmoud Ezzaki and Chakib Belgani; Daniel Jhiseung Hahn, Phillips Exeter Academy, Exeter, NH, USA; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Haimoshri Das, South Point High School, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Inequaliter and Equaliter; Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania.

J336. Let ABC be a right triangle with altitude AD, and let T be an arbitrary point on segment BD. Lines through T tangent to the circumcircle of triangle ADC intersect line AB at X, Y, respectively. Prove that $AX = BY$.

Proposed by Josef Tkadlec, Charles University, Czech Republic

Solution by Polyahedra, Polk State College, USA



In the figure, $CZ \parallel AB$. Let $k = AX/CZ$ and let s be the semiperimeter of $\triangle TXY$. Then $LT = s - XY = AY - XY = AX = kNZ$, and $YL = XM \tan\alpha / \tan\beta = (s - YT)k = k(YN - YT) = kTN$. Hence,

$$\frac{BY}{CZ} = \frac{YT}{TZ} = \frac{YL + LT}{TN + NZ} = k,$$

thus $BY = AX$.

Also solved by George Gavrilopoulos, Nea Makri High School, Athens, Greece; Michael Tang, Edina High School, MN, USA; Arkady Alt, San Jose, CA, USA; Robert Bosch, Archimedean Academy, FL, USA; Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comănești, Romania.

S331. Find the minimum value of

$$E\left(x_1,\ldots,x_n\right)={x_1}^2+\cdots+{x_n}^2+x_1x_2+\cdots+x_1x_n+\cdots+x_{n-1}x_n+x_1+\cdots+x_n,$$

when $x_1,\ldots,x_n\in\mathbb{R}$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Romania

Solution by Daniel Lasaosa, Pamplona, Spain
Denote $s=x_1+x_2+\cdots+x_n$. We can then rewrite

$$E\left(x_1,\ldots,x_n\right)=\frac{s^2}{2}+\frac{\left(x_1+1\right)^2+\cdots+\left(x_n+1\right)^2}{2}-\frac{n}{2}\geq$$

$$\geq\frac{s^2}{2}+\frac{n}{2}\left(\frac{\left(x_1+1\right)+\cdots+\left(x_n+1\right)}{n}\right)^2-\frac{n}{2}=\frac{(n+1)s^2+2ns}{2n}=$$

$$=\frac{\left((n+1)s+n\right)^2}{2n(n+1)}-\frac{n}{2(n+1)}\geq-\frac{n}{2(n+1)}.$$

The first inequality is a consequence of the AM-QM inequality, or equality holds iff $x_1=\cdots=x_n$, whereas equality holds in the second equality iff $s=-\frac{n}{n+1}$. It follows that the minimum value of $E\left(x_1,\ldots,x_n\right)$ is $-\frac{n}{2(n+1)}$, obtained iff $x_1=\cdots=x_n=-\frac{1}{n+1}$.

An alternative (less *Olympic*) method arises from considering that the first and second derivatives of $E\left(x_1,\ldots,x_n\right)$ with respect to x_i for $i=1,\ldots,n$ are respectively $s+x_i+1$ and 2, whereas any mixed second derivative is 1 (resulting in a positive-definite Hessian matrix) or a local minimum is reached whenever $\frac{s}{n}=x_1=\cdots=x_n=-s-1$, resulting again in the minimum of $E\left(x_1,\ldots,x_n\right)$ occurring at $x_1=\cdots=x_n=-\frac{1}{n+1}$, and reaching the same value $-\frac{n}{2(n+1)}$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, USA; Kwonil Kobe Ko, Cushing Academy, MA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Michael Tang, Edina High School, MN, USA; Mahmoud Ezzaki and Chakib Belgani; Théo Lenoir, Institut Saint-Lô, Agneaux, France; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Adnan Ali, A.E.C.S-4, Mumbai, India; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Moubinool Omarjee Lycée Henri IV, Paris France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Francesco Bonesi, University of East Anglia, UK; Robert Bosch, Archimedean Academy, FL, USA.

S332. Prove that in any triangle with side lengths a, b, c and median lengths m_a, m_b, m_c,

$$4\left(m_a+m_b+m_c\right) \leq \sqrt{8a^2+(b+c)^2}+\sqrt{8b^2+(c+a)^2}+\sqrt{8c^2+(a+b)^2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

We have

$$\begin{aligned}
&(a-b)^2(a+b+c)(a+b-c) \geq 0\\
&\Leftrightarrow (a-b)^2((a+b)^2-c^2) \geq 0\\
&\Leftrightarrow (a^2-b^2)^2 \geq (ac-bc)^2\\
&\Leftrightarrow 4c^2ab \geq 2(ac)^2+2(bc)^2-2(a^2-b^2)^2\\
&\Leftrightarrow 4c^4+4c^2ab+(ab)^2 \geq 2(ac)^2+2(bc)^2+4c^4+(ab)^2-2(a^2-b^2)^2\\
&\Leftrightarrow (2c^2+ab)^2 \geq (2b^2+2c^2-a^2)(2c^2+2a^2-b^2)\\
&\Leftrightarrow 2(2c^2+ab) \geq 2\sqrt{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)}\\
&\Leftrightarrow 8c^2+a^2+b^2+2ab \geq 2\sqrt{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)}+4c^2+a^2+b^2\\
&\Leftrightarrow \sqrt{8c^2+(a+b)^2} \geq \sqrt{2c^2+2b^2-a^2}+\sqrt{2c^2+2a^2-b^2}.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\sqrt{8b^2+(c+a)^2} &\geq \sqrt{2c^2+2b^2-a^2}+\sqrt{2a^2+2b^2-c^2},\\
\sqrt{8a^2+(c+b)^2} &\geq \sqrt{2a^2+2b^2-c^2}+\sqrt{2c^2+2a^2-b^2}.
\end{aligned}$$

Adding these inequalities and using

$$\sqrt{2c^2+2b^2-a^2}=2m_a, \sqrt{2c^2+2a^2-b^2}=2m_b, \sqrt{2a^2+2b^2-c^2}=2m_c$$

we get the required inequality. Equality holds if and only if the triangle is equilateral.

Also solved by Daniel Lasaosa, Pamplona, Spain; José Hernández Santiago, México; Kwonil Kobe Ko, Cushing Academy, MA, USA; Louis Cahyadi, Cirebon, Indonesia; Inequaliter and Equaliter; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Arkady Alt, San Jose, CA, USA; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Rade Krenkov, SOU Goce Delcev, Valandovo, Macedonia; Robert Bosch, Archimedean Academy, FL, USA; Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania.

S333. Let $x > 1$ and let $(a_n)_{n\geq 1}$ be the sequence defined by $a_n = [x^n]$ for every positive integer n. Prove that if $(a_n)_{n\geq 1}$ is a geometric progression, then x is an integer.

Proposed by Marius Cavachi, Constanta, Romania

Solution by Robert Bosch, Archimedean Academy, Florida, USA

Denote by k the ratio of the geometric progression. We have $\lfloor x^n \rfloor = k^{n-1}\lfloor x \rfloor$, now we fix x and dividing by x^n both sides we get $\frac{\lfloor x^n \rfloor}{x^n} = \left(\frac{k}{x}\right)^{n-1}\frac{\lfloor x \rfloor}{x}$. Now we have three different cases, say $0 < \frac{k}{x} < 1$, $\frac{k}{x} = 1$ or $\frac{k}{x} > 1$. Note that $\frac{x^n - 1}{x^n} < \frac{\lfloor x^n \rfloor}{x^n} < 1$, thus $\lim_{n\to\infty} \frac{\lfloor x^n \rfloor}{x^n} = 1$. So, taking the limit when n tends to infinity we reach to a contradiction unless $\frac{k}{x} = 1$, thus getting $\lfloor x \rfloor = x$, which means x must be an integer.

Also solved by Daniel Lasaosa, Pamplona, Spain; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Alberto Espuny Díaz, CFIS, Universitat Politècnica de Catalunya, Barcelona, Spain; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Théo Lenoir, Institut Saint-Lô, Agneaux, France; Mahmoud Ezzaki and Chakib Belgani; Kwonil Kobe Ko, Cushing Academy, MA, USA.

S334. Let $a_0 \geq 0$ and $a_{n+1} = a_0 \cdot \ldots \cdot a_n + 4$ for $n \geq 0$. Prove that

$$a_n - \sqrt[4]{(a_{n+1}+1)\,(a_n{}^2+1) - 4} = 1$$

for all $n \geq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

We have

$$\begin{aligned}
a_{n+1} &= a_0 \cdot \ldots \cdot a_{n-1} a_n + 4 \\
\Rightarrow a_{n+1} &= (a_n - 4)a_n + 4 \\
\Rightarrow a_{n+1} + 1 &= a_n^2 - 4a_n + 5 \\
\Rightarrow (a_{n+1}+1)(a_n^2+1) - 4 &= (a_n^2 - 4a_n + 5)(a_n^2+1) - 4 \\
\Rightarrow (a_{n+1}+1)(a_n^2+1) - 4 &= a_n^4 - 4a_n^3 + 6a_n^2 - 4a_n + 1 \\
\Rightarrow (a_{n+1}+1)(a_n^2+1) - 4 &= (a_n - 1)^4 \\
\Rightarrow \sqrt{(a_{n+1}+1)(a_n^2+1) - 4} &= a_n - 1 \\
\Rightarrow a_n - \sqrt[4]{(a_{n+1}+1)(a_n^2+1) - 4} &= 4.
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Mahmoud Ezzaki and Chakib Belgani; Théo Lenoir, Institut Saint-Lô, Agneaux, France; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; David E. Manes, Oneonta, NY, USA; G. C. Greubel, Newport News, VA, USA; Haimoshri Das, South Point High School, India; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria; Henry Ricardo, New York Math Circle; Moubinool Omarjee Lycée Henri IV, Paris France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Rade Krenkov, SOU Goce Delcev, Valandovo, Macedonia; Robert Bosch, Archimedean Academy, FL, USA; Russelle Guadalupe, University of the Philippines - Diliman, Quezon City, Philippines; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania.

S335. Let ABC be a triangle. Let D be a point on the ray BA, which is not on the side AB, and let E be a point on the side AC, which is different from A And C. Let X be the reflection of D with respect to B and let Y be the reflection of E with respect to C. Suppose $4BC^2 + DE^2 = XY^2$. Prove that $BE \perp CD$ if and only if $\angle BAC = 90^\circ$.

Proposed by İlker Can Çiçek, Instanbul, Turkey

Solution by Andrea Fanchini, Cantú, Italy

We set $AD = d$ and $AE = e$, then points D and E have barycentric coordinates $D(c+d : -d : 0)$, $E(b-e : 0 : e)$.
So, points X and Y will have coordinates

$$X(-c-d : 2c+d : 0), \quad Y(e-b : 0 : 2b-e)$$

with the usual distance formula and the Conway's notation we have

$$DE^2 = \frac{S_A(ce+bd)^2 + S_B b^2 d^2 + S_C c^2 e^2}{b^2c^2} = \frac{bcd^2 + bce^2 + 2edS_A}{bc}$$

$$XY^2 = \frac{S_A((2b-e)(-c-d)+(b-e)(2c+d))^2 + S_B(b(2c+d))^2 + S_C(c(e-2b))^2}{b^2c^2} =$$

$$= \frac{bcd^2 + bce^2 + 4a^2bc + 2edS_A + 4bdS_B - 4ceS_C}{bc}$$

with the statement $4BC^2 + DE^2 = XY^2$, it follows that

$$4a^2 + \frac{bcd^2 + bce^2 + 2edS_A}{bc} = \frac{bcd^2 + bce^2 + 4a^2bc + 2edS_A + 4bdS_B - 4ceS_C}{bc}$$

developing and simplifying, we obtain that

$$bdS_B = ceS_C \quad (*)$$

I) $\angle BAC = 90^\circ \quad \Rightarrow \quad BE \perp CD$

If $\angle BAC = 90^\circ$ then $S_A = 0$ and $S_B = c^2, \quad S_C = b^2$. Then it follows that the $(*)$ relation it becomes

$$bdS_B = ceS_C \quad \Rightarrow \quad cd = be$$

the line BE have equation $ex + (e-b)z = 0$ and the infinite point is $BE_\infty(b-e : -b : e)$,

the line CD have equation $dx + (c+d)y = 0$ and the infinite perpendicular point is

$$CD_{\infty\perp}(da^2 - (c+d)S_C : b^2(c+d) - dS_C : -dS_B - (c+d)S_A)$$

but $S_A = 0, \quad S_B = c^2, \quad S_C = b^2$ and $cd = be$ so developing and simplifying

$$CD_{\infty\perp} \equiv (b-e : -b : e) = BE_\infty \quad \Rightarrow \quad BE \perp CD, \quad q.e.d.$$

II) $BE \perp CD \quad \Rightarrow \quad \angle BAC = 90^\circ$

now we know that two lines $p_i x + q_i y + r_i z = 0 (i = 1, 2)$ are perpendiculars if and only if

$$S_A(q_1 - r_1)(q_2 - r_2) + S_B(r_1 - p_1)(r_2 - p_2) + S_C(p_1 - q_1)(p_2 - q_2) = 0$$

so for the lines BE and CD we have

$$S_A(b - e)(c + d) + bdS_B - ceS_C = 0$$

but substituing the $(*)$ relation it follows that $S_A = 0$ and this means that $\angle BAC = 90°$, q.e.d.

Also solved by Daniel Lasaosa, Pamplona, Spain; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania.

S336. Let M be a point inside the triangle ABC and let K be a point symmetric to point M with respect to AC. Line BM intersects AC in point N and line BK intersects AC in point P. Prove that if $\angle AMP = \angle CMN$, then $\angle ABP = \angle CBN$.

Proposed by Nairi Sedrakyan, Armenia

Solution by Daniel Lasaosa, Pamplona, Spain

Consider the circumcircle Γ of BMK, which is clearly symmetric with respect to AC. By construction, MP is the symmetric of BK, and NK is the symmetric of BM, or they meet again at a point B' on Γ which is the symmetric of B with respect to AC. Let Q be the point where segment AC intersects Γ, or clearly $BQ = BQ'$, and

$$\angle PMQ = \angle QMB' = \angle BKQ = 180^\circ - \angle BMQ = \angle QMN,$$

or $\angle AMQ = \angle CMQ$. Now, applying the Sine Law to triangles AMQ and CMQ, and noting that $\sin\angle AQM = \sin\angle CQM$ because $\angle AQM + \angle CQM = 180^\circ$, we conclude that $\frac{AM}{AQ} = \frac{CM}{CQ}$, or M is on the Apollonius' circle defined by $\frac{AX}{CX} = \frac{AB}{CB}$ since the circle through every point M and Q passes also through B. We conclude that if $\angle AMP = \angle CMN$, then M is necessarily on the circle symmetric with respect to AC, through B, and through the intersection of AC and the internal bisector of angle B.

Since BQ is then the internal bisector of angle B, it follows that $\angle CBQ = \angle ABQ$, or it suffices to prove that $\angle MQB = \angle QBK$, clearly true since $MQ = QK$ by symmetry with respect to AC, and $BMQK$ is cyclic. The conclusion follows.

Also solved by Arkady Alt, San Jose, CA, USA; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Haimoshri Das, South Point High School, India.

U331. Find all positive integers $a > b \geq 2$ such that

$$a^b - a = b^a - b.$$

Proposed by Mircea Becheanu, Bucharest, Romania

Solution by Robert Bosch, Archimedean Academy, FL, USA
To start, consider $b = 2$; we have to solve in positive integers the equation $a^2 - a + 2 = 2^a$ with $a \geq 3$. The inequality $2^a > a^2$ for $a \geq 5$ is easily provable by induction. Hence by the above equation we obtain $a^2 - a + 2 > a^2$, which it holds only when $a < 2$. Then, we must consider the values $a = 3$ and $a = 4$, which leads to the solution $(a, b) = (3, 2)$. When $a > b > 2$, we consider the function $f(x) = \dfrac{\ln x}{x}$, which is decreasing when $x > e$ because the derivative $f'(x) = \dfrac{1 - \ln x}{x^2}$ is clearly negative. Now we have the following chain of equivalences

$$a > b \leftrightarrow \frac{\ln a}{a} < \frac{\ln b}{b} \leftrightarrow a^b < b^a \leftrightarrow a^b - b^a < 0 \leftrightarrow a - b < 0 \leftrightarrow a < b.$$

Contradiction. Hence the only solution is $(a, b) = (3, 2)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Francesco Bonesi, University of East Anglia, UK; David E. Manes, Oneonta, NY, USA; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Théo Lenoir, Institut Saint-Lô, Agneaux, France; Mahmoud Ezzaki and Chakib Belgani.

U332. Find $\inf_{(x,y)\in D}(x+1)(y+1)$, where $D = \{(x,y)|x,y \in \mathbb{R}^+, x \neq y, x^y = y^x\}$.

Proposed by Arkady Alt, San Jose, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note that $e^{y\ln x} = x^y = y^x = e^{x\ln y}$ iff $\frac{\ln x}{x} = \frac{\ln y}{y}$, since e^x is a strictly increasing function for all real x. Note also that

$$\frac{d}{dx}\left(\frac{\ln(x)}{x}\right) = \frac{1-\ln(x)}{x^2}$$

is negative iff $x > e$, positive iff $x < e$, and zero iff $x = e$, ie for any $x \neq y$ such that $x^y = y^x$, we must have either $x > e > y$ or $x < e < y$. We may therefore define $D^* = D \cup \{(x,y) = (e,e)\}$, and the problem is equivalent to finding, for $(x+1)(y+1)$, either its minimum in D^* if it exists at $(x,y) = (e,e)$ (since it will coincide by continuity of functions $(x+1)(y+1), x^y, y^x$ with the infimum in D), or otherwise its infimum in D.

Define $f(x,y) = xy$, and $g(x,y) = x\ln(y) - y\ln(x)$. Note that the extrema of $f(x,y)$ subject to condition $g(x,y) = 0$ may be found by Lagrange's multiplier method, or real constant λ exists such that

$$\lambda y = \ln(y) - \frac{y}{x}, \qquad \lambda xy = x\ln(y) - y = y\ln(x) - x,$$

and since $x\ln(y) = y\ln(x)$, we find that a local extremum occurs iff $x = y$, ie the only local extremum of xy in D^* occurs when $x = y = e$, with a value e^2. Moreover, the borders of D^* occur when $x \to 1$ and $y \to \infty$ or *vice versa*, for $xy \to \infty$, or the extremum of xy at $x = y = e$ is a minimum with value e^2. We conclude that, for $(x,y) \in D^*$, we have

$$(x+1)(y+1) = xy + x + y + 1 \geq xy + 2\sqrt{xy} + 1 \geq e^2 + 2e + 1 = (e+1)^2,$$

with equality iff $x = y = e$. By continuity of condition $x^y = y^x$ and of function $(x+1)(y+1)$, we conclude that

$$\inf_{(x,y)\in D}(x+1)(y+1) = (e+1)^2,$$

where $(x+1)(y+1)$ can get arbitrarily close to $(e+1)^2$ as $x > e$ gets arbitrarily close to e, with the corresponding value of $y < e$ getting arbitrarily close to e too, or *vice versa*.

Also solved by Francesco Bonesi, University of East Anglia, UK; Robert Bosch, Archimedean Academy, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U333. Evaluate

$$\prod_{n\geq 0}\left(1-\frac{2^{2^n}}{2^{2^{n+1}}+1}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

Let $x_n = 2^{2^n}$, then $x_{n+1} = x_n^2$, and

$$\begin{aligned}
\prod_{n\geq 0}\left(1-\frac{2^{2^n}}{2^{2^{n+1}}+1}\right) &= \prod_{n\geq 0}\left(1-\frac{1/x_n}{1/x_n^2+1}\right)\\
&= \prod_{n\geq 0}\frac{x_n^2-x_n+1}{1+x_n^2}\cdot\frac{1+x_n}{1+x_n}\cdot\frac{1-x_n}{1-x_n}\cdot\frac{1-x_n^3}{1-x_n^3}\\
&= \prod_{n\geq 0}\frac{(1-x_n^6)(1-x_n)}{(1-x_n^4)(1-x_n^3)}\\
&= \prod_{n\geq 0}\frac{1-x_{n+1}^3}{1-x_n^3}\cdot\prod_{n\geq 0}\frac{1-x_n}{1-x_{n+2}}\\
&= \frac{(1-x_0)(1-x_1)}{1-x_0^3} = \frac{(1-1/2)(1-1/4)}{1-1/8} = \frac{3}{7}.
\end{aligned}$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Robert Bosch, Archimedean Academy, FL, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Moubinool Omarjee Lycée Henri IV, Paris France; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Arkady Alt, San Jose, CA, USA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Michael Tang, Edina High School, MN, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kwonil Kobe Ko, Cushing Academy, MA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA.

U334. Prove that if $x \in \mathbb{R}$ with $|x| \geq e$, then $e^{|x|} \geq \left(\frac{e^2+x^2}{2e}\right)^e$, while the inequality is reversed if $|x| \leq e$.

Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Robert Bosch, Archimedean Academy, FL, USA

Taking natural logarithms on both sides and dividing by e, the inequality becomes

$$\left|\frac{x}{e}\right| > \ln\left(\left(\frac{x}{e}\right)^2 + 1\right) + 1 - \ln 2,$$

so, after the substitution $y = \dfrac{x}{e}$ we must show that

$$|y| \geq 1 \Rightarrow |y| > \ln(y^2+1) + 1 - \ln 2.$$

Now we consider two cases: when $y \geq 1$ and when $y \leq -1$. In the first one we consider the function $f(y) = \ln(y^2+1) - y + 1 - \ln 2$ with derivative $f'(y) = \dfrac{2y}{y^2+1} - 1 < 0$, hence $f(y)$ is decreasing, it follows that $f(y) \leq f(1) = 0$. Now, in the second one $g(y) = \ln(y^2+1) + y + 1 - \ln 2$ with first derivative $g'(y) = \dfrac{2y}{y^2+1} + 1 > 0$, thus is an increasing function, and then $g(y) \leq g(-1) = 0$. In a similar way the reversed inequality may be proved.

Also solved by Daniel Lasaosa, Pamplona, Spain; Francesco Bonesi, University of East Anglia, UK; Albert Stadler, Herrliberg, Switzerland; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Arkady Alt, San Jose, CA, USA; Théo Lenoir, Institut Saint-Lô, Agneaux, France; Mahmoud Ezzaki and Chakib Belgani; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kwonil Kobe Ko, Cushing Academy, MA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA.

U335. Let $p, a_1, ..., a_n$, $b_1, ..., b_n$ be positive real numbers. Prover that

$$a_1\left(\frac{a_1}{b_1}\right)^p + ... + a_n\left(\frac{a_1+...+a_n}{b_1+...+b_n}\right)^p < \left(\frac{p+1}{p}\right)^p\left(\frac{{a_1}^{p+1}}{{b_1}^p} + ... + \frac{{a_n}^{p+1}}{{b_n}^p}\right).$$

Proposed by Nairi Sedrakyan, Armenia

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

We need the weighted Hardy inequality, which is stated as: Let $a_1, \ldots, a_n$ and $\omega_1, \ldots, \omega_n$ be positive real numbers. If $p > 1$ then

$$\sum_{k=1}^{n}\omega_k\left(\frac{\omega_1 a_1+\cdots+\omega_k a_k}{\omega_1+\cdots+\omega_k}\right)^p < \left(\frac{p}{p-1}\right)^p\sum_{k=1}^{n}\omega_k a_k^p.$$

Replacing p, a_k, ω_k by $p+1, \frac{a_k}{b_k}, b_k$ for $k = 1, \ldots, n$, the inequality reduces to

$$\sum_{k=1}^{n} b_k\left(\frac{a_1+\cdots+a_k}{b_1+\cdots+b_k}\right)^{p+1} < \left(\frac{p+1}{p}\right)^{p+1}\sum_{k=1}^{n}\frac{a_k^{p+1}}{b_k^p}.$$

By Hölder's inequality and the above inequality we have

$$\begin{aligned}
\sum_{k=1}^{n} a_k\left(\frac{a_1+\cdots+a_k}{b_1+\cdots+b_k}\right)^p &= \sum_{k=1}^{n}\frac{a_k}{b_k^{\frac{p}{p+1}}}\left(b_k^{\frac{p}{p+1}}\left(\frac{a_1+\cdots+a_k}{b_1+\cdots+b_k}\right)^p\right)\\
&\le \left(\sum_{k=1}^{n}\frac{a_k^{p+1}}{b_k^p}\right)^{\frac{1}{p+1}}\left(\sum_{k=1}^{n} b_k\left(\frac{a_1+\cdots+a_k}{b_1+\cdots+b_k}\right)^{p+1}\right)^{\frac{p}{p+1}}\\
&< \left(\sum_{k=1}^{n}\frac{a_k^{p+1}}{b_k^p}\right)^{\frac{1}{p+1}}\left(\left(\frac{p+1}{p}\right)^{p+1}\sum_{k=1}^{n}\frac{a_k^{p+1}}{b_k^p}\right)^{\frac{p}{p+1}}\\
&= \left(\frac{p}{p+1}\right)^p\sum_{k=1}^{n}\frac{a_k^{p+1}}{b_k^p},
\end{aligned}$$

and the inequality is proved.

Also solved by Robert Bosch, Archimedean Academy, FL, USA.

U336. Find a closed form for the sum $E_n = \sum_{k=0}^{n} \binom{n}{2k+1} 3^k$. Deduce the value of constant c such that $0 < \lim_{n\to\infty} \frac{E_n}{c^n} < \infty$, and the limit itself.

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Solution by Haroun Meghaichi, University of Science and Technology Houari Boumediene, Algiers, Algeria

Let us consider the general case $E_n(x) = \sum_{k=0}^{n} \binom{n}{2k+1} x^k$ where $x > 0$. We have

$$E_n(x) = \frac{1}{\sqrt{x}} \sum_{k=0}^{n} \binom{n}{2k+1} \left(\sqrt{x}\right)^{2k+1} = \frac{1}{2\sqrt{x}} \left(\left(1+\sqrt{x}\right)^n + \left(1-\sqrt{x}\right)^n\right)$$

In this particluar case $x = 3$ then

$$E_n = E_n(3) = \frac{1}{2\sqrt{3}} \left(\left(1+\sqrt{3}\right)^n + \left(1-\sqrt{3}\right)^n\right)$$

Now, it is clear that $c = 1+\sqrt{3}$ (since $\left|1-\sqrt{3}\right| < 1$) and the limit is $\frac{1}{2\sqrt{3}}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kwonil Kobe Ko, Cushing Academy, MA, USA; Mahmoud Ezzaki and Chakib Belgani; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Byeong Yeon Ryu, The Hotchkiss School, Lakeville, CT, USA; David E. Manes, Oneonta, NY, USA; Henry Ricardo, New York Math Circle; Moubinool Omarjee Lycée Henri IV, Paris France; Robert Bosch, Archimedean Academy, FL, USA; Albert Stadler, Herrliberg, Switzerland.

O331. Let ABC be a triangle, let m_a, m_b, m_c be the lengths of its medians, and let p_0 be the semiperimeter of its orthic triangle. Prove that

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\leq\frac{3\sqrt{3}}{2p_0}$$

Proposed by Mircea Lascu, Zalau, Romania

Solution by Robert Bosch, Archimedean Academy, FL, USA

We start with some well-known formulas for orthic triangle. The area is

$$S_0=\frac{abc|\cos A\cos B\cos C|}{2R}$$

and the inradius

$$r_0=2R|\cos A\cos B\cos C|$$

thus the semiperimeter has the formula $2p_0=\dfrac{2S}{R}$ where S is the area of triangle ABC. Now using that $S=\dfrac{ah_a}{2}$ the original inequality can be rewritten in the following form

$$a\cdot\frac{h_a}{m_a}+b\cdot\frac{h_b}{m_b}+c\cdot\frac{h_c}{m_c}\leq 3\sqrt{3}R.$$

But, $\dfrac{h_a}{m_a}=\sin\alpha_A$ where α_A is the angle formed by median and side BC. Consequently, using the Law of Sines in ABC, the above rewrites as

$$\sin A\sin\alpha_A+\sin B\sin\alpha_B+\sin C\sin\alpha_C\leq\frac{3\sqrt{3}}{2}.$$

The above inequality is true because the left side is less than or equal $\sin A+\sin B+\sin C$, which is less than $\dfrac{3\sqrt{3}}{2}$ by Jensen's inequality.

Also solved by Sardor Bozorboyev, Lyceum S.H.Sirojjidinov,Tashkent,Uzbekistan; Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Arkady Alt, San Jose, CA, USA; Inequaliter and Equaliter; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kwonil Kobe Ko, Cushing Academy, MA, USA.

O332. Let ABC be a triangle. Prove that

$$\frac{\cos^4 A}{\sin^4 B+\sin^4 C}+\frac{\cos^4 B}{\sin^4 C+\sin^4 A}+\frac{\cos^4 C}{\sin^4 A+\sin^4 B}\geq \frac{1}{6}.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Robert Bosch, Archimedean Academy, FL, USA

By the Cauchy-Schwarz inequality we have that

$$\sum_{\text{cyc}}\frac{\cos^4 A}{\sin^4 B+\sin^4 C}\geq \frac{(\cos^2 A+\cos^2 B+\cos^2 C)^2}{2(\sin^4 A+\sin^4 B+\sin^4 C)}.$$

We next show that

$$\frac{(\cos^2 A+\cos^2 B+\cos^2 C)^2}{2(\sin^4 A+\sin^4 B+\sin^4 C)}\geq \frac{1}{6}.$$

The above rewrites as

$$3(\cos^2 A+\cos^2 B+\cos^2 C)^2\geq \sin^4 A+\sin^4 B+\sin^4 C,$$

which by Law of Sines may be transformed into

$$3\left(12R^2-(a^2+b^2+c^2)\right)^2\geq a^4+b^4+c^4,$$

where a,b,c are the sides of triangle ABC and R is the circumradius. Now the idea is to write the symmetric expressions $a^2+b^2+c^2$ and $a^4+b^4+c^4$ as functions of s,r,R, with s the semiperimeter and r the inradius. To do the latter, we use Newton's relations, getting

$$\begin{aligned} a^2+b^2+c^2 &= 2s^2-8rR-2r^2,\\ a^4+b^4+c^4 &= 2s^4-16s^2rR-12s^2r^2+32r^2R^2+16r^3R+2r^4. \end{aligned}$$

In particular, note that by Blundon's inequality $s^2\leq 4R^2+4rR+3r^2$ the following inequalities holds:

$$\begin{aligned} a^2+b^2+c^2 &\leq 8R^2+4r^2,\\ a^4+b^4+c^4 &\leq 32R^4-32r^3R-16r^4. \end{aligned}$$

Finally, we have to prove that

$$16R^4-96r^2R^2+64r^4+32r^3R\geq 0.$$

Dividing by $16R^4$ the above inequality can be rewritten as

$$1-6\left(\frac{r}{R}\right)^2+4\left(\frac{r}{R}\right)^4+2\left(\frac{r}{R}\right)^3\geq 0.$$

In other words, writing $y=\frac{r}{R}$, we have to show that polynomial $4y^4+2y^3-6y^2+1$ is nonnegative on the interval $\mathcal{I}=(0,1/2]$. This suffices due to Euler's inequality. But

$$4y^4+2y^3-6y^2+1=(2y-1)(2y^3+2y^2-2y-1).$$

So take $g(y)=2y^3+2y^2-2y-1$ and note that $g(y)<0$ on $\mathcal{I}$. Indeed,

$$\begin{aligned} g(-2)\cdot g(-1) &= -5<0,\\ g(-1)\cdot g(0) &= -1<0,\\ g(0.6)\cdot g(1) &= -1.048<0, \end{aligned}$$

the three real roots are located outside $\mathcal{I}$. Also, $g(0)=-1$ thus the cubic preserves the sign on $\mathcal{I}$.

Also solved by Arkady Alt, San Jose, CA, USA; Kwonil Kobe Ko, Cushing Academy, MA, USA.

O333. Let ABC be a scalene acute triangle and denote by O, I, H its circumcenter, incenter, and orthocenter, respectively. Prove that if the circumcircle of triangle OIH passes through one of the vertices of triangle ABC then it also passes through one other vertex.

Proposed by Josef Tkadlec, Charles University, Czech Republic

Solution by Adnan Ali, A.E.C.S-4, Mumbai, India

We prove the statement not only for acute-angled triangles, but for all scalene triangles. Without loss of generality, assume that $\angle CAB > \angle ABC > \angle BCA$. It is well-known that O, H are isogonal conjugates with respect to triangle ABC. So, let the circle pass through A. Thus, we know that $AOIH$ is cyclic. Hence $\angle IOH = \angle IAH$ and $\angle IHO = \angle IAO$. But O, H being isogonal conjugates w.r.t $\triangle ABC$, we have $\angle IAO = \angle IAH$. Thus, $\angle IHO = \angle IOH \Rightarrow IO = IH$. Now using the Sine Law for $\triangle IOC$ and $\triangle IHC$ and $IO = IH$, we have

$$\frac{CI}{\sin \angle COI} = \frac{IO}{\sin \angle IOC} = \frac{IH}{\sin \angle IHC} = \frac{CI}{\sin \angle CHI}$$

which implies that $\angle COI = \angle CHI$ or $\angle COI + \angle CHI = 180^\circ$. Similarly, we have either $\angle BOI = \angle BHI$ or $\angle BOI + \angle BHI = 180^\circ$. It is quite evident that all $AOIH, BOIH, COIH$ cannot be cyclic quadrilaterals. Thus, if $\angle BOI + \angle BHI = 180^\circ$, then $BOIH$ is cyclic, and that implies that $AHOB$ is cyclic, implying that $\angle AOB = \angle AHB \Rightarrow 2\angle ACB = 180^\circ - \angle ACB \Rightarrow \angle ACB = 60^\circ$, which is a contradiction, as the smallest angle of a scalene triangle is always less than 60°. Hence $\angle COI + \angle CHI = 180^\circ$ and $COIH$ is cyclic. Thus we have shown more than what was to be proved, i.e. when the circumcircle of OIH passes through one vertex, it must pass through the other vertex provided that the vertex is opposite to either the largest or the shortest side of the scalene triangle.

Also solved by Daniel Lasaosa, Pamplona, Spain; Mahmoud Ezzaki and Chakib Belgani; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Francisco Javier García Capitán, I.E.S Álvarez Cubero, Priego de Córdoba, Spain.

O334. Let a, b, c be positive real numbers such that $a, b, c \geq 1$. Prove that

$$\frac{1}{(a^3+1)^2}+\frac{1}{(b^3+1)^2}+\frac{1}{(c^3+1)^2} \geq \frac{3}{2\left(a^2b^2c^2+1\right)}.$$

Proposed by İlker Can Çiçek, Instanbul, Turkey

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

We will use the known inequality

$$x, y \geq 1 \implies \frac{1}{1+x}+\frac{1}{1+y} \geq \frac{2}{1+\sqrt{xy}} \tag{1}$$

By $x^2+y^2 \geq (x+y)^2/2 \geq 0$ we get

$$\frac{1}{2}\frac{1}{(1+a^3)^2}+\frac{1}{2}\frac{1}{(1+b^3)^2} \geq \frac{1}{4}\left(\frac{1}{1+a^3}+\frac{1}{1+b^3}\right)^2$$

and by (1)

$$\frac{1}{4}\left(\frac{1}{1+a^3}+\frac{1}{1+b^3}\right)^2 \geq \frac{1}{\left(1+(ab)^{\frac{3}{2}}\right)^2}$$

We obtain

$$\sum_{\text{cyc}}\frac{1}{(1+a^3)^2}-\frac{3}{2(a^2b^2c^2+1)} \geq \sum_{\text{cyc}}\frac{1}{(1+(ab)^{\frac{3}{2}})^2}-\frac{3}{2(a^2b^2c^2+1)} \doteq F(a,b,c)$$

The inequality is symmetric so we suppose $b \leq c \leq a$ and show that

$$F(a,b,c) \geq F(\sqrt{ab}, \sqrt{ab}, c)$$

namely

$$\frac{1}{(1+(ab)^{\frac{3}{2}})^2}+\frac{1}{(1+(bc)^{\frac{3}{2}})^2}+\frac{1}{(1+(ca)^{\frac{3}{2}})^2} \geq \frac{1}{(1+(ab)^{\frac{3}{2}})^2}+\frac{2}{(1+(\sqrt{ab}c)^{\frac{3}{2}})^2}$$

or

$$\frac{1}{(1+(bc)^{\frac{3}{2}})^2}+\frac{1}{(1+(ca)^{\frac{3}{2}})^2} \geq \frac{2}{(1+(\sqrt{ab}c)^{\frac{3}{2}})^2} \tag{2}$$

Moreover

$$\frac{1}{(1+(bc)^{\frac{3}{2}})^2}+\frac{1}{(1+(ca)^{\frac{3}{2}})^2}\geq\frac{1}{2}\left(\frac{1}{(1+(bc)^{\frac{3}{2}})}+\frac{1}{(1+(ca)^{\frac{3}{2}})}\right)^2$$

and a further use of (1) yields

$$\frac{1}{2}\left(\frac{1}{(1+(bc)^{\frac{3}{2}})}+\frac{1}{(1+(ca)^{\frac{3}{2}})}\right)^2\geq\frac{1}{2}\frac{4}{\left(1+\sqrt{(bc)^{\frac{3}{2}}(ca)^{\frac{3}{2}}}\right)^2}=\frac{2}{\left(1+c^{\frac{3}{2}}\sqrt{(ab)^{\frac{3}{2}}}\right)^2}$$

which is (2).

Standard theorems says that

$$F(a,b,c)\geq F(a,a,a)=0$$

and we are done.

Also solved by Arkady Alt, San Jose, CA, USA; Daniel Lasaosa, Pamplona, Spain; Robert Bosch, Archimedean Academy, FL, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Adnan Ali, A.E.C.S-4, Mumbai, India; Adithya Bhaskar, Atomic Energy Central School - 2 , Mumbai , India; Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Inequaliter and Equaliter; Mahmoud Ezzaki and Chakib Belgani.

O335. Determine all positive integers n such that $f_n(x,y,z) = x^{2n} + y^{2n} + z^{2n} - xy - yz - zx$ divides $g_n(x,y,z) = (x-y)^{5n} + (y-z)^{5n} + (z-x)^{5n}$, as polynomials in x, y, z with integer coefficients.

Proposed by Dorin Andrica, Babeş-Bolyai University, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

A necessary condition is that $f_n(u,v,w)$ divides $g_n(u,v,w)$ for any triple of integers (u,v,w). In particular, taking $(x,y,z) = (2,1,0)$, we find that $2^{2n} - 1$ must divide $2 + (-2)^{5n}$.

For even n, we have that $2\left(2^{5n-1}+1\right)$ must be a multiple of $(2^n-1)(2^n+1)$. Now,

$$2^{5n-1} + 1 = (2^n+1)\left(2^{4n-1} - 2^{3n-1} + 2^{2n-1} - 2^{n-1}\right) + 2^{n-1} + 1,$$

and clearly $2^n + 1 > 2^{n-1} + 1 > 0$ for any positive integer n, or no even n satisfies the condition given in the problem statement.

For odd n, we have that $2\left(2^{5n-1}-1\right)$ must be a multiple of $(2^n-1)(2^n+1)$. Now,

$$2^{5n-1} - 1 = (2^n+1)\left(2^{4n-1} - 2^{3n-1} + 2^{2n-1} - 2^{n-1}\right) + 2^{n-1} - 1,$$

and for any $n \geq 3$, we have $2^n + 1 > 2^{n-1} - 1 > 0$, or only $n = 1$ may satisfy the condition given in the problem statement.

For $n = 1$, we have

$$g_1(x,y,z) = \left(5x^2(z-y) + 5y^2(x-z) + 5z^2(y-x)\right) f_1(x),$$

and we conclude that $n = 1$ is the only solution to this problem.

Also solved by Albert Stadler, Herrliberg, Switzerland

O336. Let a, b, c be positive distinct real numbers and let u, v, w be positive real numbers such that $a+b+c=u+v+w$ and $(a^2-bc)r+(b^2-ac)s+(c^2-ab)t \geq 0$ for $(r,s,t)=(u,v,w),(r,s,t)=(v,w,u),(r,s,t)=(w,u,v)$. Prove that

$$x^a y^b z^c + x^b y^c z^a + x^c y^a z^b \geq x^u y^v z^w + x^w y^u z^v + x^v y^w z^u$$

for all nonnegative numbers x, y, z.

Proposed by Albert Stadler, Switzerland

Solution by Arkady Alt , San Jose ,California, USA

By the weighted AM-GM inequality,

$$px^a y^b z^c + qx^b y^c z^a + rx^c y^a z^b \geq (p+q+r)\left(x^{pa+qb+rc} y^{pb+qc+ra} z^{pc+qa+rb}\right)^{\frac{1}{p+q+r}},$$

for any positive reals p, q, r.

We now search for p, q, r such that $\begin{matrix} pa+qb+rc=u \\ pb+qc+ra=v \\ pc+qa+rb=w \end{matrix}$ or in matrix form $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix}=\begin{pmatrix} u \\ v \\ w \end{pmatrix}$.

Since $a+b+c=u+v+w$ then adding all equations we obtain

$$(p+q+r)(a+b+c)=u+v+w \iff p+q+r=1.$$

Since $\det\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}=\sum_{cyc} a\left(bc-a^2\right)=-\sum_{cyc} a\left(a^2-bc\right)=-(a+b+c)\left(a^2+b^2+c^2\right)$

and $\det\begin{pmatrix} u & b & c \\ v & c & a \\ w & a & b \end{pmatrix}=\sum_{cyc} u\left(bc-a^2\right)=-\sum_{cyc} u\left(a^2-bc\right)$, then $p=\dfrac{\sum_{cyc} u\left(a^2-bc\right)}{(a+b+c)\left(a^2+b^2+c^2\right)}$,

and cyclically

$$q=\frac{\sum_{cyc} v\left(a^2-bc\right)}{(a+b+c)\left(a^2+b^2+c^2\right)}, r=\frac{\sum_{cyc} w\left(a^2-bc\right)}{(a+b+c)\left(a^2+b^2+c^2\right)},$$

where, we get p, q, r all positive due condition

$$(a^2-bc)r+(b^2-ac)s+(c^2-ab)t \geq 0$$

for $(r,s,t)=(u,v,w)$, $(r,s,t)=(v,w,u)$, $(r,s,t)=(w,u,v)$.

For the obtained p, q, rwe have

$$px^a y^b z^c + qx^b y^c z^a + rx^c y^a z^b \geq x^u y^v z^w,$$

and similarly

$$qx^a y^b z^c + rx^b y^c z^a + px^c y^a z^b \geq x^w y^u z^v, rx^a y^b z^c + px^b y^c z^a + qx^c y^a z^b \geq x^v y^w z^u.$$

Adding these inequalities and taking in account that $p+q+r=1$ we finally obtain

$$x^a y^b z^c + x^b y^c z^a + x^c y^a z^b \geq x^u y^v z^w + x^w y^u z^v + x^v y^w z^u.$$