Junior problems

J247. Let a and b be distinct zeros of the polynomial $x^3 - 2x + c$. Prove that $a^2(2a^2 + 4ab + 3b^2) = 3$ if and only if $b^2(3a^2 + 4ab + 2b^2) = 5$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Francesco De Sclavis, Daniele Fakhoury and Andrea Fiacco, Università di Roma "Tor Vergata", Roma, Italy
We first note that

$$0 = (a^3 - 2a + c) - (b^3 - 2b + c) = (a - b)(a^2 + ab + b^2 - 2).$$

Since $a \neq b$, it follows that $a^2 + ab + b^2 = 2$. Now, let $x = a^2(2a^2 + 4ab + 3b^2)$ and $y = b^2(3a^2 + 4ab + 2b^2)$. We have that

$$x + y - 8 = 2(ab + 2)(a^2 + ab + b^2 - 2) = 0.$$

Hence, $x = 3$ if and only if $y = 8 - x = 5$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Corneliu Manescu-Avram, Transportation High School, Ploieti, Romania; Alessandro Ventullo, Milan, Italy; José Hernández Santiago, Oaxaca, México; Radouan Boukharfane, Polytechnique de Montreal, Canada; Arkady Alt, San Jose, California, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

J248. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{\{x\}^2}{\lfloor x \rfloor}$. Prove that $f(x+y) \leq f(x)+f(y)$, for any real numbers x and y.

Proposed by Sorin Radulescu, Bucharest, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote $a = \lfloor x \rfloor$, $b = \lfloor y \rfloor$, $u = \{x\}$ and $v = \{y\}$, where clearly a, b are positive integers, and $u, v \in [0, 1)$. Note that, either $\lfloor x + y \rfloor = a + b + 1$ and $\{x + y\} = u + v - 1$ when $u + v \geq 1$, or $\lfloor x + y \rfloor = a + b$ and $\{x + y\} = u + v$ when $u + v < 1$. In the first case, $f(x + y) = \frac{(u+v-1)^2}{a+b+1} < \frac{(u+v)^2}{a+b}$, or it suffices to show, for any positive integers a, b, and any reals $u, v \in [0, 1)$, that

$$\frac{(u+v)^2}{a+b} \leq \frac{u^2}{a} + \frac{v^2}{b},$$

equivalent after some algebra to $2abuv \leq b^2u^2 + a^2v^2$, which is clearly true by the AM-GM inequality, with equality iff $bu = av$. The conclusion follows, equality holds iff $x = a + \frac{k}{b}$, $y = b + \frac{k}{a}$, where a, b are positive integers, and k is a non-negative real such that $\frac{k}{a} + \frac{k}{b} < 1$, ie such that $k < \frac{ab}{a+b}$.

Also solved by Alessandro Ventullo, Milan, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Arkady Alt, San Jose, California, USA; Solving Group of Qafqaz University, Baku, Azerbaijan; Ioan Viorel Codreanu, Satulung, Maramures, Romania; G.R.A.20 Problem Solving Group, Roma, Italy; Salem Malikić, Simon Fraser University, Burnaby, BC, Canada.

J249. Find the least prime $p > 3$ that divides $3^q - 4^q + 1$ for all primes $q > 3$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy
Since $3^5 - 4^5 + 1 = -780 = -(2^2 \cdot 3 \cdot 5 \cdot 13)$, and $3^7 - 4^7 + 1$ is not divisible by 5, we only have to show that $p = 13$ divides $3^q - 4^q + 1$ for all primes $q > 3$. Since q is a prime, $q = 6k \pm 1$ for some $k \in \mathbb{N}^*$. Moreover, $3^{6k} \equiv 1 \pmod{13}$ and $4^{6k} \equiv 1 \pmod{13}$, then

$$\begin{aligned} 3^{6k+1} - 4^{6k+1} + 1 &\equiv 3 - 4 + 1 \equiv 0 \pmod{13} \\ 3^{6k-1} - 4^{6k-1} + 1 &\equiv 9 - 10 + 1 \equiv 0 \pmod{13}, \end{aligned}$$

which gives the conclusion.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Radouan Boukharfane, Polytechnique de Montreal, Canada; Corneliu Manescu-Avram, Transportation High School, Ploieti, Romania; Prithwijit De, HBCSE, Mumbai, India; Francesco De Sclavis, Emiliano Torti and Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy; Arkady Alt, San Jose ,California, USA; Salem Malikić, Simon Fraser University, Burnaby, BC, Canada; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong, China.

J250. Let ABC be a triangle with $\angle A \geq 120^\circ$ and let s be the semiperimeter of the triangle. Prove that

$$\sqrt{(s-b)(s-c)} \geq (3+\sqrt{6})(s-a).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Using the Law of Cosines with the given condition, we get that

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\angle A \\ &\geq b^2 + c^2 - 2bc\cos 120^\circ \\ &\geq b^2 + c^2 + bc, \end{aligned}$$

thus

$$\begin{aligned} 4a^2 &\geq 4b^2 + 4c^2 + 4bc \\ &\geq 3b^2 + 3c^2 + 6bc \\ &= 3(b+c)^2. \end{aligned}$$

Thus, $2a \geq \sqrt{3}(b+c)$. The last inequality can be written in its equivalent form,

$$\sqrt{3} \geq \sqrt{\frac{s-a}{s}} \cdot (3 + 2\sqrt{3}).$$

Since $\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \tan\frac{\angle A}{2} \geq \tan 60^\circ = \sqrt{3}$, we obtain

$$\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \geq \sqrt{\frac{s-a}{s}} \cdot (3+2\sqrt{3}), \text{ i.e } \sqrt{(s-b)(s-c)} \geq (3+2\sqrt{3})(s-a).$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy; Arkady Alt, San Jose ,California, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada.

J251. Let a, b, c be positive real numbers such that $a \geq b \geq c$ and $b^2 > ac$. Prove that

$$\frac{1}{a^2 - bc} + \frac{1}{b^2 - ca} + \frac{1}{c^2 - ab} > 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Giulia Giovannotti, Università di Roma "Tor Vergata", Roma, Italy
We note that

$$\frac{1}{a^2 - bc} + \frac{1}{b^2 - ca} + \frac{1}{c^2 - ab} = \frac{(ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca)}{(a^2 - bc)(b^2 - ca)(ab - c^2)}$$

so it suffices to prove that each factor is positive.
By hypothesis, $(b^2 - ca) > 0$ and $(ab + bc + ca) > 0$. Moreover $ac < b^2 \leq ab$ implies that $c < b \leq a$. Hence

$$(ab - c^2) > c^2 - c^2 = 0 \qquad \text{and} \qquad (a^2 - bc) \geq b^2 - bc = b(b - c) > 0.$$

Also, by the Cauchy-Schwarz inequality, we have that

$$a^2 + b^2 + c^2 = (a^2 + b^2 + c^2)^{1/2}(b^2 + c^2 + a^2)^{1/2} > ab + bc + ca$$

because equality holds if and only if $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$, whereas $b^2 > ac$.

Also solved by Arkady Alt, San Jose, California, USA; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong, China; Titouan Morvan, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo, Milan, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Radouan Boukharfane, Polytechnique de Montreal, Canada; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

J252. Let ABC be an acute triangle and let O_a be a point in its plane such that

$$|\angle BO_aC| = 2\alpha, \qquad |\angle CO_aA| = 180^\circ - \alpha, \qquad |\angle AO_aB| = 180^\circ - \alpha.$$

Similarly, define points O_b and O_c. Prove that the circumcircle of triangle $O_aO_bO_c$ passes through the circumcenter of triangle ABC.

Proposed by Michal Rolinek, Charles University, Czech Republic

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let A', B', C' be the respective midpoints of OA, OB, OC, and let D, E, F be the respective circumcenters of BOC, COA, AOB. Note that $\angle BOC = 2\angle A$, or O_a is on the circumcircle of BOC, or D is on the perpendicular bisector of OO_a. If moreover $\angle B = \angle C$, by symmetry O_a is on the perpendicular bisector of BC, hence $O_a = O$, and the proposed result is trivially true. We will assume wlog from now on that $\angle A > \angle B > \angle C$.

Since BO_aOC is a convex cyclic quadrilateral, then

$$\angle BO_aO = 180^\circ - \angle OCB = 90^\circ + \angle A,$$

and consequently

$$\angle OO_aA = 360^\circ - \angle AO_aB - \angle BO_aO = 90^\circ.$$

It follows that A' is the circumcenter of AOO_a, or A' is on the perpendicular bisector of OO_a, which clearly coincides with line DA'.

Now, since D is the circumcenter of BOC, which is isosceles at O, it follows that $DB' = DC'$. Moreover, D is on the perpendicular bisectors of OB, OC, which by similar reasoning are found to be DF, DE. It follows that, in triangle DEF, points A', B', C', respectively on sides EF, FD, DE, satisfy $DB' = DC'$, $EC' = EA'$, and $FA' = FB'$. Or, A', B', C' are the points where the incircle of DEF respectively touches sides EF, FD, DE. Therefore, the respective perpendicular bisectors DA', EB', FC' of OO_a, OO_b, OO_c concur at a point, which is the center of a circle through O, O_a, O_b, O_c. The conclusion follows.

Senior problems

S247. Prove that for any positive integers m and n, the number $8m^6 + 27m^3n^3 + 27n^6$ is composite.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by José Hernández Santiago, Oaxaca, México

Since

$$(2m^2 + 3n^2)^3 = 8m^6 + 36m^4n^2 + 54m^2n^4 + 27n^6,$$

it follows that

$$\begin{aligned} 8m^6 + 27m^3n^3 + 27n^6 &= (2m^2 + 3n^2)^3 - (36m^4n^2 + 54m^2n^4) + 27m^3n^3 \\ &= (2m^2 + 3n^2)^3 - 18m^2n^2(2m^2 + 3n^2) + 27m^3n^3. \end{aligned}$$

Then, letting $A := 2m^2 + 3n^2$ and $B := 3mn$, the expression in the previous line becomes

$$\begin{aligned} A^3 - 2AB^2 + B^3 &= A^3 - AB^2 - AB^2 + B^3 \\ &= A(A - B)(A + B) - B^2(A - B) \end{aligned}$$

Therefore, $2m^2 + 3n^2 - 3mn$ is always a divisor of $8m^6 + 27m^3n^3 + 27n^6$. In addition, the assumption that both m and n are positive integers allows us to ascertain at once that $2m^2 + 3n^2 - 3mn$ is a proper divisor of $8m^6 + 27m^3n^3 + 27n^6$. Indeed, $3mn > 0$ and whence

$$2m^2 + 3n^2 - 3mn < 2m^2 + 3n^2 < 8m^6 + 27m^3n^3 + 27n^6.$$

On the other hand,

$$2m^2 + 3n^2 - 3mn = (m - n)^2 + (m - n)^2 + n^2 + mn > 1.$$

So we indeed get $8m^6 + 27m^3n^3 + 27n^6$ is composite, as claimed.

Also solved by Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Radouan Boukharfane, Polytechnique de Montreal, Canada; Sayan Das, Kolkata, India; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong, China; Arkady Alt, San Jose ,California, USA; Albert Stadler, Switzerland; Vlad Petrache, Roxana Petrica and Daniel Vacaru, Colegiul Economic "Maria Teiuleanu, Pitesti, Romania; Cyril Letrouit, Lycee Jean-Baptiste Say, Paris, France; Prithwijit De, HBCSE, Mumbai, India;Salem Malikić, Simon Fraser University, Burnaby, BC, Canada.

S248. Let $\mathcal{C}(O,R)$ be a circle and let P be a point in its plane. Consider a pair of diametrically opposite points A and B lying on $\mathcal{C}$. Prove that while points A and B vary on the circumference of $\mathcal{C}$, the circumcircles of triangles ABP pass through another fixed point.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Prithwijit De, HBCSE, Mumbai, India

Let the equation of the circle C with respect to a given fixed rectangular coordinate system be $x^2+y^2-R^2=0$ and let the point P have coordinates (α,β). Let A and B be the points of intersection of the variable line $y-mx=0$ and C. Then the equation of a family of circles passing through A, B is given by

$$(x^2+y^2-R^2)+\lambda(y-mx)=0 \tag{1}$$

where λ is a real parameter. If these circles must pass through P then $x=\alpha, y=\beta$ must satisfy (1), whence

$$\lambda=\frac{\alpha^2+\beta^2-R^2}{m\alpha-\beta}.$$

Substituting this value of λ in (1) we can rewrite the equation of the family as

$$m\{\alpha(x^2+y^2)-(\alpha^2+\beta^2-R^2)x-\alpha R^2\}-\{\beta(x^2+y^2)-(\alpha^2+\beta^2-R^2)y-\beta R^2)\}=0. \tag{2}$$

But this is the equation of a family of circles passing through the points of intersection of two fixed circles whose equations are:

$$\alpha(x^2+y^2)-(\alpha^2+\beta^2-R^2)x-\alpha R^2=0; \tag{3}$$

$$\beta(x^2+y^2)-(\alpha^2+\beta^2-R^2)y-\beta R^2=0. \tag{4}$$

Note that the circles are distinct as they have different centres and different radii. One of the points of intersection is P, as $x=\alpha, y=\beta$ satisfy both equations, and the other one is the fixed point we seek.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Alessandro Ventullo, Milan, Italy; Cyril Letrouit, Lycee Jean-Baptiste Say, Paris, France; Arkady Alt, San Jose, California, USA; Mehdi Mikael Trense, Montpellier, France.

S249. Find the minimum of $2^x - 4^x + 6^x - 8^x - 9^x + 12^x$ where x is a positive real number.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Giulia Giovannotti, Università di Roma "Tor Vergata", Roma, Italy

Let $f(x) = 2^x - 4^x + 6^x - 8^x - 9^x + 12^x$ for $x \geq 0$. We note that

$$1 + f(x) = (3^x - 2^x - 1)(4^x - 3^x - 1) = g_2(x)g_3(x)$$

where $g_n(x) = (n+1)^x - n^x - 1$. Hence the minimum value of $f(x)$ for $x \geq 0$ is $-1 = f(1)$ as soon as we show that for $n \geq 1$, $g_n(x) > 0$ for any $x > 1$ and $g_n(x) < 0$ for any $0 \leq x < 1$. Indeed

$$g_n'(x) = (n+1)^x \ln(n+1) - n^x \ln(n) > 0 \text{ iff } \left(\frac{n+1}{n}\right)^x > \frac{\ln(n)}{\ln(n+1)},$$

which holds for all $x \geq 0$ because $\left(\frac{n+1}{n}\right)^x \geq 1$ and $\frac{\ln(n)}{\ln(n+1)} < 1$. Therefore g_n is strictly increasing in $[0, +\infty)$ and $g_n(1) = 0$.

Also solved by Albert Stadler, Switzerland; Mehdi Mikael Trense, Montpellier, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Arkady Alt, San Jose, California, USA; Alessandro Ventullo, Milan, Italy; Corneliu Manescu-Avram, Transportation High School, Ploieti, Romania; Salem Malikić, Simon Fraser University, Burnaby, BC, Canada; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada.

S250. Let Γ be a circle and ℓ be a line lying outside Γ. Let $K \in \ell$ and let AB and CD be chords of Γ passing through K. Let P and Q lie on Γ. Let PA, PB, PC, PD meet ℓ at X, Y, Z, T, respectively, and then let QX, QY, QZ, QT meet again Γ at R, S, U, V, respectively. Prove that RS and UV meet on ℓ.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Consider the hexagon $PAUQRC$. Since $PA \cap QR = X \in \ell$ and $PC \cap QU = Z \in \ell$, by Pascal's theorem it follows that $L = AU \cap CR \in \ell$.

Consider next hexagon $PBVQSD$. Since $PB \cap QS = Y \in \ell$ and $PD \cap QV = T \in \ell$, by Pascal's theorem it follows that $M = BV \cap DS \in \ell$.

Consider now hexagon $ABVCDS$. Since $AB \cap CD = K \in \ell$ and $BV \cap DS = M \in \ell$, by Pascal's theorem we have $N = AS \cap CV \in \ell$.

Consider finally hexagon $VUASRC$. Since $AU \cap CR = L \in \ell$ and $AS \cap CV = N \in \ell$, by Pascal's theorem, it follows that UV, RS meet on ℓ. The conclusion follows.

Also solved by Cyril Letrouit, Lycee Jean-Baptiste Say, Paris, France.

S251. Find all triples (x, y, z) of positive real numbers for which there is a positive real number t such that the following inequalities hold simultaneously:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + t \leq 4, \qquad x^2 + y^2 + z^2 + \frac{2}{t} \leq 5.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Using the AM-GM inequality, we deduce that

$$4 \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + t \geq 4 \cdot \sqrt[4]{\frac{t}{xyz}}. \tag{1}$$

and

$$5 \geq x^2 + y^2 + z^2 + \frac{2}{t} \geq 5 \sqrt[5]{\left(\frac{xyz}{t}\right)^2}. \tag{2}$$

Hence, we have $xyz = t$. Therefore,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz = 4$$

and

$$x^2 + y^2 + z^2 + \frac{2}{xyz} = 5.$$

From the equality case in the AM-GM inequality, we have $\frac{1}{x} = \frac{1}{y} = \frac{1}{z} = xyz = 1 \Rightarrow x = y = z = 1$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Arkady Alt, San Jose, California, USA; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong, China; Vlad Petrache, Roxana Petrica and Daniel Vacaru, Colegiul Economic "Maria Teiuleanu, Pitesti, Romania; Alessandro Ventullo, Milan, Italy; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Radouan Boukharfane, Polytechnique de Montreal, Canada; Nguyen Dang Qua, HUS, Hanoi, Vietnam; Cyril Letrouit, Lycee Jean-Baptiste Say, Paris, France; Salem Malikić, Simon Fraser University, Burnaby, BC, Canada.

S252. Let a, b, c be positive real numbers. Prove that

$$a(b-c)^2+b(c-a)^2+c(a-b)^2+2abc \geq 2\frac{\sqrt{3(a^4b^4+b^4c^4+c^4a^4)}}{a+b+c}.$$

Proposed by Pham Huu Duc, Australia and Cosmin Pohoata, Princeton University, USA

Solution by Arkady Alt, San Jose, California, USA
Since $\left(a^2b^2+b^2c^2+c^2a^2\right)^2 \geq 3\left(a^4b^4+b^4c^4+c^4a^4\right)$, it suffices that

$$a(b-c)^2+b(c-a)^2+c(a-b)^2+2abc \geq \frac{2\left(a^2b^2+b^2c^2+c^2a^2\right)}{a+b+c}.$$

To see this, write

$$\begin{aligned} & (a+b+c)\left(a(b-c)^2+b(c-a)^2+c(a-b)^2+2abc\right)-2\left(a^2b^2+b^2c^2+c^2a^2\right) \\ = \; & (a+b+c)^2(ab+bc+ca)-7abc(a+b+c)-2\Big((ab+bc+ca)^2-2abc(a+b+c)\Big) \\ = \; & (a+b+c)^2(ab+bc+ca)-2(ab+bc+ca)^2-3abc(a+b+c). \end{aligned}$$

The latter rewrites as

$$(ab+bc+ca)\Big((a+b+c)^2-3(ab+bc+ca)^2\Big)+\Big((ab+bc+ca)^2-3abc(a+b+c)\Big) \geq 0,$$

where the positivity holds because

$$(a+b+c)^2 \geq 3(ab+bc+ca)^2 \text{ and } (ab+bc+ca)^2 \geq 3abc(a+b+c).$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Michel Faleiros Martins, Instituto Tecnológico de Aeronáutica, Brazil; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada.

U247. Let a be a real number greater than 1. Evaluate

$$\frac{1}{a^2-a+1}-\frac{2a}{a^4-a^2+1}+\frac{4a^3}{a^8-a^4+1}-\frac{8a^7}{a^{16}-a^8+1}+\dots$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Let

$$S_n=\sum_{k=1}^{n}(-1)^{k-1}\frac{2^{k-1}a^{2^{k-1}-1}}{a^{2^k}-a^{2^{k-1}}+1}.$$

Since

$$\begin{aligned}
\frac{1}{a^2-a+1}-\frac{1}{a^2+a+1} &= \frac{2a}{a^4+a^2+1}\\
-\frac{2a}{a^4-a^2+1}+\frac{2a}{a^4+a^2+1} &= -\frac{4a^3}{a^8+a^4+1}\\
&\ \ \vdots\\
(-1)^{n-1}\frac{2^{n-1}a^{2^{n-1}-1}}{a^{2^n}-a^{2^{n-1}}+1}+(-1)^n\frac{2^{n-1}a^{2^{n-1}-1}}{a^{2^n}+a^{2^{n-1}}+1} &= (-1)^{n+1}\frac{2^n a^{2^n-1}}{a^{2^{n+1}}+a^{2^n}+1}.
\end{aligned}$$

summing up the two columns, we get

$$S_n-\frac{1}{a^2+a+1}=(-1)^{n+1}\frac{2^n a^{2^n-1}}{a^{2^{n+1}}+a^{2^n}+1}.$$

Now,

$$0\le\left|(-1)^{n+1}\frac{2^n a^{2^n-1}}{a^{2^{n+1}}+a^{2^n}+1}\right|\le\frac{2^n a^{2^n}}{a^{2^{n+1}}}=\frac{2^n}{a^{2^n}}$$

and $\dfrac{2^n}{a^{2^n}}\to 0$ as $n\to\infty$. So,

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{k-1}a^{2^{k-1}-1}}{a^{2^k}-a^{2^{k-1}}+1}=\lim_{n\to\infty}S_n=\frac{1}{a^2+a+1}.$$

Also solved by Moubinool Omarjee, Lycée Henri IV , Paris, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Konstantinos Tsouvalas, University of Athens, Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Radouan Boukharfane, Polytechnique de Montreal, Canada; Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland; Arkady Alt, San Jose, California, USA.

U248. Let A, S, X be matrices in $M_4(\mathbb{R})$ such that A is skew-symmetric, S is invertible, and $X = AS$. If $X^4 = O_4$, prove that $X^2 = O_4$.

Proposed by Dorin Andrica and Mihai Piticari, Romania

Solution by Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland

The matrix A is skew-symmetric and hence normal (i.e. $AA^* = A^*A$), and hence it's unitarily diagonalisable as a complex matrix. Hence in some base B of $\mathbb{C}^2$ matrix A has form:

$$A' = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

Let X', S' denote the matrices X, S in the base B. Let us notice that $0 = \det X^4 = (\det X)^4$ implies that $\det A = 0$ (because $\det S \neq 0$) and zero is an eigenvalue of A. Moreover all eigenvalues of a skew-symmetric matrix are pure imaginary and hence, because $\sum_{i=1}^4 \lambda_i = tr(A) \in \mathbb{R}$, zero must be an eigenvalue with even multiplicity.

Thus, without loss of generality we can assume that $\lambda_1 = \lambda_2 = 0$ and:

$$X' = A'S' = \begin{pmatrix} O_2 & O_2 \\ O_2 & diag(\lambda_3, \lambda_4) \end{pmatrix} S' = \begin{pmatrix} O_2 & O_2 \\ O_2 & D \end{pmatrix}$$

for some $D \in M_2(\mathbb{C})$, and because of $X'^4 = 0$, we have $D^4 = 0$. However, by the Cayley-Hamilton theorem the degree of minimal polynomial of D is at most 2. Hence $D^2 = O_2$ and $X'^2 = \begin{pmatrix} O_2 & O_2 \\ O_2 & D^2 \end{pmatrix} = O_4$, implying $X^2 = O_4$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy; Moubinool Omarjee, Lycée Henri IV , Paris, France; Konstantinos Tsouvalas, University of Athens, Athens, Greece.

U249. Let $(a_n)_{n\geq 1}$ be a decreasing sequence of positive numbers. Let

$$s_n = a_1 + a_2 + \cdots + a_n,$$

and

$$b_n = \frac{1}{a_{n+1}} - \frac{1}{a_n},$$

for all $n \geq 1$. Prove that if $(s_n)_{n\geq 1}$ is convergent, then $(b_n)_{n\geq 1}$ is unbounded.

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania

Solution by Konstantinos Tsouvalas, University of Athens, Athens, Greece

We begin with a preliminary result.

Lemma. We have: $\lim_{n\to\infty} na_n = 0$

Proof. Since $\sum_{k=1}^{\infty} a_k < \infty$, we obtain that $\lim_{n\to\infty} \sum_{k=n}^{\infty} a_k = 0$.
Because the sequence is decreasing we have

$$\sum_{k=n}^{\infty} a_k \geq \sum_{k=n}^{2n} a_k \geq na_{2n} \geq 0,$$

so $\lim_{n\to\infty} 2na_{2n} = 0$.
We also get

$$\sum_{k=n}^{2n+1} a_k \geqslant na_{2n+1} \geqslant 0, \text{ thus } \lim_{n\to\infty} (2n+1)\, a_{2n+1} = 0$$

because

$$a_n \to 0, \sum_{k=n}^{2n+1} a_k \to 0, n \to \infty$$

Now we assume that exists a positive real number M such that: $\left|\frac{1}{a_{k+1}} - \frac{1}{a_k}\right| \leq M$, for all $k \in M$. It follows that

$$\sum_{k=1}^{n} \left|\frac{1}{a_{k+1}} - \frac{1}{a_k}\right| \leq Mn, \text{ so } \left|\sum_{k=1}^{n} \left(\frac{1}{a_{k+1}} - \frac{1}{a_k}\right)\right| \leq Mn.$$

This yields $\left|\frac{1}{a_{n+1}} - \frac{1}{a_1}\right| \leq Mn, n \in \mathbb{N}$, which means that

$$\left|\frac{1}{(n+1)\, a_n} - \frac{1}{a_1\,(n+1)}\right| \leqslant \frac{Mn}{n+1} \leqslant 2M, \text{ for all } n \in \mathbb{N}.$$

This is however a contradiction because the sequence $\left(\frac{1}{na_n} - \frac{1}{na_1}\right)_{n\in\mathbb{N}}$ is not bounded ($\lim_{n\to\infty} na_n = 0$). Hence, the sequence: $\left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right)_{n\in\mathbb{N}}$ is not bounded.

Also solved by Moubinool Omarjee, Lycée Henri IV , Paris, France; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland; Arkady Alt, San Jose, California, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada; Alessandro Ventullo, Milan, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Robinson Higuita, Universidad de Antioquia, Colombia.

U250. Let f be a real valued function, continuous on an interval I, such that f has a continuous and nonnegative lateral derivative at any point in I. Prove that f is non-decreasing.

Proposed by Dan Marinescu and Mihai Piticari, Romania

Solution by G.R.A.20 Problem Solving Group, Roma, Italy

It suffices to show that for any $a < x_1 < x_2 < b$, and for any $\epsilon > 0$ then

$$f(x_2) - f(x_1) \geq (-\epsilon)(x_2 - x_1).$$

We consider the set

$$S = \{x \in [x_1, x_2] : f(x) - f(x_1) \geq (-\epsilon)(x - x_1)\}.$$

Then S is not empty because $x_1 \in S$ and, by the continuity of f, $s = \sup S \in S$. Let us prove that $s = x_2$. Indeed, if $s < x_2$ then, $f'_+(s) \geq 0$ implies that there is $s < t \leq x_2$ such that

$$f(t) - f(s) \geq (-\epsilon)(t - s).$$

Hence $t \in S$ because

$$f(t) - f(x_1) = f(t) - f(s) + f(s) - f(x_1) \geq (-\epsilon)(t - s) + (-\epsilon)(s - x_1) = (-\epsilon)(t - x_1)$$

and this is a contradiction with the definition of s.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

U251. Find all polynomials $p(x) = a_n x^n + \cdots + a_1 x + a_0$ in $\mathbb{Z}[x]$ such that for all distinct integers x and y the following condition is satisfied:

$$\frac{p(x) - p(y)}{x - y} = \frac{1}{n}\left(p'(x) + p'(y) + (n-2)\sqrt{p'(x)p'(y)}\right),$$

where $p'(x)$ is the derivative of $p(x)$.

Proposed by Dan Marinescu and Mihai Piticari, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

The property cannot apply to constant polynomials, since the LHS would be zero, and the RHS undefined as an indeterminancy of the form $\frac{0}{0}$.

For $n = 1$, we have $p(x) = a_1 x + a_0$, and consequently the LHS is a_1 for all x, y, while $p'(x) = a_1$, or the RHS is also a_1 for all distinct x, y. Therefore, all first degree polynomials in $\mathbb{Z}[x]$ satisfy the proposed condition (actually, all first degree polynomials in $\mathbb{R}[x]$ satisfy the condition for any two distinct reals x, y).

For $n = 2$, we have $p(x) = a_2 x^2 + a_1 x + a_0$, and consequently the LHS is $a_2(x + y) + a_1$ for all distinct x, y, while $p'(x) = 2a_2 x + a_1$, or the RHS is also $a_2(x + y) + a_1$, for all distinct x, y. Therefore, all second degree polynomials in $\mathbb{Z}[x]$ satisfy the proposed condition (actually, all second degree polynomials in $\mathbb{R}[x]$ satisfy the condition for any two distinct reals x, y).

Let $n \geq 3$, fix y, and take $x > |y|$ variable. Clearly,

$$n\frac{p(x) - p(y)}{x - y} - p'(x) - p'(y) = (na_n y + a_{n-1})x^{n-2} + O\left(x^{n-3}\right),$$

where Landau notation has been used, or taking an infinite, strictly increasing sequence of integers x, we have

$$\lim_{x\to\infty}(n-2)\sqrt{\frac{p'(y)p'(x)}{x^{2n-4}}} = \lim_{x\to\infty}\left(n\frac{p(x) - p(y)}{x^{n-2}(x - y)} - \frac{p'(x) + p'(y)}{x^{n-2}}\right) = na_n y + a_{n-1}.$$

Since this is true for as many integers y as we care to fix (as long as we take x sufficiently larger than $|y|$), we find that $p'(x)$ has necessarily degree $2n - 4$, or $2n - 4 = n - 1$, for $n = 3$, and no polynomial may meet the condition given in the problem statement for $n \geq 4$.

For $n = 3$, we have $p(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$, and $p'(x) = 3a_3 x^2 + 2a_2 x + a_1$, or the condition given in the problem statement becomes, after some algebra, equivalent to

$$3a_3 xy + a_2(x + y) + a_1 = \sqrt{(3a_3 x^2 + 2a_2 x + a_1)(3a_3 y^2 + 2a_2 y + a_1)}.$$

Squaring both sides yields the equivalent condition ${a_2}^2 = 3a_1 a_3$. Provided that this holds, the condition given in the problem statement is true for any real x, y. Since the problem asks specifically for $p(x) \in \mathbb{Z}[x]$, it is necessary that $a_1 a_3$ is three times a perfect square, and $a_2 = \pm\sqrt{3a_1 a_2}$.

We conclude that all first degree polynomials in $\mathbb{Z}[x]$, all second degree polynomials in $\mathbb{Z}[x]$, and all third degree polynomials $p(x) = a_3 x^3 \pm \sqrt{3a_1 a_3}x^2 + a_1 x + a_0$ such that a_0, a_1, a_3 are integers, and $a_1 a_3$ is three times a perfect square, are all polynomials that satisfy the requirements of the problem statement.

Also solved by Harun Immanuel, ITS Surabaya; Arkady Alt, San Jose, California, USA, Radouan Boukharfane, Polytechnique de Montreal, Canada.

U252. Find the number of automorphisms of the group of invertible residue classes mod n.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote by R_n the set of invertible residues modulus n. Clearly, all these residues must be prime with n, in other words, R_n has $\varphi(n)$ elements, where φ denotes Euler's totient function.

Any positive integer $n > 1$ can be written as $n = {p_1}^{a_1}{p_2}^{a_2} \dots {p_u}^{a_u}$, where $p_1, p_2, \dots, p_u$ are distinct primes and $a_1, a_2, \dots, a_u$ are positive integers. By the Chinese Remainder Theorem, we can associate biunivocally, to each invertible residue $r \in R_n$, each one of the possible sets of invertible residues $r_1 \in R_{{p_1}^{a_1}}, r_2 \in R_{{p_2}^{a_2}}, \dots, r_u \in R_{{p_u}^{a_u}}$. Clearly, if r, s are two invertible residues modulus n, to which we associate respectively the sets $(r_1, r_2, \dots, r_u), (s_1, s_2, \dots, s_u)$ of invertible residues modulus the ${p_i}^{a_i}$, the (clearly invertible) residue rs is associated to the set $(r_1s_1, r_2s_2, \dots, r_us_u)$. It therefore follows that each automorphism $f : R_n \to R_n$ can be decomposed into a set of automorphisms $f_i : R_{{p_i}^{a_i}} \to R_{{p_i}^{a_i}}$, so that if $s = f(r)$, then $s_i = f_i(r_i)$. Reciprocally, a set of automorphisms $f_i : R_{{p_i}^{a_i}} \to R_{{p_i}^{a_i}}$ generates an automorphism $f : R_n \to R_n$, so that if $f_i(r_i) = s_i$, then $f(r) = s$. Therefore, denoting by $A(n)$ the number of automorphisms modulus n, then $A(n) = A({p_1}^{a_1})A({p_2}^{a_2}) \dots A({p_u}^{a_u})$. It therefore suffices to find $A(p^a)$ for any prime p and any positive integer a.

Let p be any odd prime, and a any positive integer. It is well known that R_{p^a} is isomorph to the cyclic group of order $\varphi(p^a)$, where φ denotes Euler's totient function. More specifically, there exists at least one residue g (actually, there exist $\varphi(\varphi(n))$ of them), called primitive root, such that $g, g^2, g^3, \dots, g^{\varphi(n)} = 1$ are all distinct invertible residues modulus p^a. Then, the image $f(g) = r$ of g determines the image of any other invertible residue g^v, clearly equal to $f(g^v) = r^v$. The number of automorphisms in R_n is therefore equal to the number of values that $f(g)$ may take for a given primitive root g. Since $f(g)$ may take any of the $\varphi(p^a)$ values in R_n, $A(p^a) = \varphi(p^a)$ when p is an odd prime.

Let $p = 2$, and a a positive integer. If $a = 1$ then $R_{2^1} = \{1\}$, and there is clearly exactly $1 = \varphi(2^1)$ possible automorphism $f : R_{2^1} \to R_{2^1}$. If $a = 2$ then $R_{2^2} = \{1, 3\}$, where for any automorphism, $f(1) = 1$, and $f(3) \in \{1, 3\}$, or there are exactly $2 = \varphi(2^2)$ possible automorphisms $f : R_{2^2} \to R_{2^2}$. For $a \geq 3$, it is known that R_{2^a} is isomorph to the direct product of a cyclic subgroup of order 2, and a cyclic subgroup of order 2^{a-2}. Each one of the cyclic subgroups will have a primitive root g, where for the cyclic group of order 2, $g^2 = 1$, or $(f(g))^2 = 1$, ie $f(g) \in \{1, 2^{n-1} - 1, 2^{n-1} + 1, 2^n - 1\}$ for the primitive root g of the cyclic subgroup of order 2. Moreover, $f(g)$ can take any of the 2^{a-1} values in R_{2^a} for the primitive root g of the cyclic subgroup of order 2^{a-2}. Or there are $2^{a+1} = 4\varphi(2^a)$ possible automorphisms $f : R_{2^a} \to R_{2^a}$ for any $a \geq 3$.

We conclude that there are exactly $\varphi(n)$ distinct automorphisms if n is not a multiple of 8, and exactly $4\varphi(n)$ distinct automorphisms if n is a multiple of 8, where we have used that, just like $A(n)$, Euler's totient function $\varphi(n)$ is also multiplicative.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

O247. Solve in positive integers the equation

$$xy + yz + zx - 5\sqrt{x^2 + y^2 + z^2} = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy
Clearly, $xy + yz + zx \geq 3$. Rewriting the equation in the form

$$(xy + yz + zx - 1)^2 = 25(x^2 + y^2 + z^2),$$

we put $xy + yz + zx = 5t + 1, t > 0$, so that $x^2 + y^2 + z^2 = t^2$. Then

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = t^2 + 2(5t + 1) = (t + 5)^2 - 23,$$

which gives

$$(t + 5 - x - y - z)(t + 5 + x + y + z) = 23.$$

Since x, y, z, t are positive integers, it must be

$$\begin{cases} t + 5 - x - y - z &= 1 \\ t + 5 + x + y + z &= 23. \end{cases}$$

Summing up the two equations, we get $t = 7$. So, $x + y + z = 11$ and $xy + yz + zx = 36$. Suppose without loss of generality that $x \leq y \leq z$. Clearly, $x \leq 3$, so we have three cases.

(i) $x = 1$. Therefore, $y + z = 10, yz = 26$, so there are no solution.

(ii) $x = 2$. Therefore, $y + z = 9, yz = 18$, so $y = 3, z = 6$.

(iii) $x = 3$. Therefore, $y + z = 8, yz = 12$, so there are no solution.

In conclusion, we get the positive integer solutions

$$(2, 3, 6), (2, 6, 3), (3, 2, 6), (3, 6, 2), (6, 2, 3), (6, 3, 2).$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Radouan Boukharfane, Polytechnique de Montreal, Canada; Arkady Alt, San Jose, California, USA; Salem Malikić, Simon Fraser University, Burnaby, BC, Canada.

O248. What is the maximal number of elements that one can choose from the set $\{1, 2, \ldots, 31\}$ such that the sum of any two is not a perfect square?

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let T be a subset of $S = \{1, 2, \ldots, 31\}$ such that no two of its elements add up to a perfect square. Note that out of $A = \{6, 19, 30\}$, at most one can be in T, since the sum of any two of them is a perfect square. We can partition $S \backslash A$ into the following 14 pairs:

$$(31, 18), \quad (29, 20), \quad (28, 21), \quad (27, 22), \quad (26, 23), \quad (24, 25), \quad (17, 8),$$

$$(16, 9), \quad (15, 10), \quad (14, 11), \quad (13, 12), \quad (7, 2), \quad (5, 4), \quad (3, 1).$$

Note that each element of S appears exactly once, either in A, or in one of the previous pairs, and the sum of both elements of each one of the pairs is a perfect square. It follows that T can have no more than 15 elements, and this is only possible if there is exactly one element from one of the previous pairs, plus exactly one element of A. Assume that such a set T of 15 elements exists.

Assume further that $14 \in T$. Therefore, $2, 11, 22 \notin T$ because they add up to a perfect square with 14, or $7, 27 \in T$ because they are in the same pairs as $2, 22$. It follows that $9, 18, 29 \notin T$ because they add up to a perfect square with either 7 or 27, or $16, 31, 20 \in T$ because they are in the same pairs as $9, 18, 29$. But $20 + 16 = 36$ is a perfect square, contradiction, or $14 \notin T$.

Therefore, any set T with 15 elements must contain 11. Or it cannot contain $5, 14, 25$, who add up to a perfect square with 11. Thus, $4, 24 \in T$ because they are in the same pairs as $5, 25$, or $1, 12, 21 \notin T$ because they add up to a perfect square with either 4 or 24, hence $3, 13, 28 \in T$ becuase they are in the same pairs as $1, 12, 21$, respectively. But $13 + 3 = 16$ is a perfect square, contradiction.

Hence T can have at most 14 elements, and such a set T is, for example:

$$T = \{1, 5, 10, 12, 14, 16, 17, 18, 21, 23, 25, 27, 29, 30\}.$$

Also solved by G.R.A.20 Problem Solving Group, Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

O249. Find all triples (x, y, z) of positive integers such that

$$\frac{x}{y} + \frac{y}{z+1} + \frac{z}{x} = \frac{5}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Salem Malikić, Simon Fraser University, Burnaby, BC, Canada

We have $\frac{5}{2} + \frac{1}{x} = \frac{x}{y} + \frac{y}{z+1} + \frac{z+1}{x} \geq 3\sqrt[3]{\frac{x}{y} \cdot \frac{y}{z+1} \cdot \frac{z+1}{x}} = 3$ implying $\frac{1}{x} \geq \frac{1}{2}$ which is equivalent to $x \leq 2$. We thus distinguish 2 cases.

If $x = 2$, then equality must be achieved in above inequality, i.e. $\frac{x}{y} = \frac{y}{z+1} = \frac{z+1}{x}$, or equivalently $\frac{2}{y} = \frac{y}{z+1} = \frac{z+1}{2}$. It follows that $y(z+1) = 4$ so we have two candidates: $y = 2,\ z = 1$ and $y = 1,\ z = 3$. However, note that the second pair does not satisfy $\frac{2}{y} = \frac{y}{z+1}$; thus, we get only one solution in this case, namely $(x, y, z) = (2, 2, 1)$. in this case.

If $x = 1$, the equality becomes $\frac{1}{y} + \frac{y}{z+1} + z = \frac{5}{2}$, which implies $z \leq 2$. For $z = 1$, we get $\frac{1}{y} + \frac{y}{2} = \frac{3}{2}$, i.e. $y^2 - 3y + 2 = 0$. This gives two solutions $y = 1$ and $y = 2$ that give $(x, y, z) = \{(1, 1, 1), (1, 2, 1)\}$. For $z = 2$ we get $\frac{1}{y} + \frac{y}{3} = \frac{1}{2}$, which is equivalent to $2y^2 - 3y + 6 = 0$. However, this doesn't have positive integer solutions; so this case doesn't give us anything.

We conclude that all solutions of the equation are given by

$$(x, y, z) = \{(1, 1, 1), (1, 2, 1), (2, 2, 1)\}.$$

Also solved by Albert Stadler, Switzerland; Arkady Alt, San Jose, California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Cyril Letrouit, Lycee Jean-Baptiste Say, Paris, France; Vlad Petrache, Mihaela Petrica and Daniel Vacaru, Colegiul Economic "Maria Teiuleanu, Pitesti, Romania; Mehdi Mikael Trense, Montpellier, France; Kwan Chung Hang, Sir Ellis Kadoorie Secondary School, Hong Kong, China; Nguyen Dang Qua, HUS, Hanoi, Vietnam; Radouan Boukharfane, Polytechnique de Montreal, Canada; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Alessandro Ventullo, Milan, Italy; Ioan Viorel Codreanu, Satulung, Maramures, Romania.

O250. Given a triangle ABC, we define the A-mixtilinear excircle as the circle externally tangent to the circumcircle of ABC, and tangent to rays AB and AC. Find a formula for the radius of the A-mixtilinear excircle and give a ruler and compass construction for the A-mixtilinear excircle.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Solution by the proposer

Denote by J_a the mixtilinear excenter, and ρ_a the mixtilinear exradius. Since the mixtilinear excircle is tangent to rays AB, AC, a scaling exists with center A that transforms the incircle into the mixtilinear excircle, or $\frac{AJ_a}{\rho_a} = \frac{AI}{r}$, where r, I are the inradius and incenter of ABC. Applying the Cosine Law to triangles IAO, J_aAO, where O is the circumcenter of ABC, and since $\angle OAI = \angle OAJ_a$ because I, J_a are both clearly on the angle bisector of $\angle BAC$ and hence collinear with A, we find

$$\frac{AO^2 + AJ_a{}^2 - OJ_a{}^2}{2AO \cdot AJ_a} = \cos\angle OAJ_a = \cos\angle OAI = \frac{AO^2 + AI^2 - OI^2}{2AO \cdot AI},$$

where $AO = R$ is the circumradius of ABC, and it is well known that $OI^2 = R^2 - 2Rr$. Moreover, $OJ_a = R + \rho_a$ because the circumcircle and the mixtilinear excircle are externally tangent. Performing these substitutions and using the proportionality between mixtilinear excircle and incircle, we find

$$\rho_a = r\frac{AI^2 + 4Rr}{AI^2 - r^2}.$$

Now, $r = AI\sin\frac{A}{2}$, and as it is well known (or easily found using different expressions for the area of ABC), $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. It follows that

$$\rho_a = \frac{1}{\cos^2\frac{A}{2}} \cdot \frac{r}{\tan\frac{B}{2}\tan\frac{C}{2}} = \frac{1}{\cos^2\frac{A}{2}} \cdot \frac{(s-b)(s-c)}{r} = \frac{1}{\cos^2\frac{A}{2}} \cdot \frac{rs}{s-a},$$

where we have used that $\sin\frac{A}{2} = \cos\frac{B+C}{2}$, and Heron's formula for the area $S = rs$ of ABC. We will use this last expression of ρ_a to justify a ruler and compass construction for the mixtilinear excircle, remembering always that the incircle, A-excircle, and A-mixtilinear excircle are each one of them the result of scaling one of the others, with center A and an appropriately chosen scaling factor.

It is well known (or easily found by considering symmetries around the angle bisectors) that the incircle touches sides AB, AC at points which are at a distance $s - a$ from A, while the A-excircle touches rays AB, AC at points which are at a distance s from A. It follows that $\frac{rs}{s-a}$ is actually the A-exradius. Denote now by I_a the center of the A-excircle, by U the projection of I_a onto AB, and by V the point where AB intersects the perpendicular to AI_a drawn through I_a. Clearly, triangles AI_aU and AVI_a are similar, respectively rectangle at U, I_a, and with $\angle I_aAU = \angle VAI_a = \frac{A}{2}$, or

$$\frac{AV}{AU} = \frac{AI_a{}^2}{AU^2} = \frac{1}{\cos^2\frac{A}{2}}.$$

In other words, the circle that is tangent to rays AB, AC at a distance AV from A is the result of scaling the A-excircle, with center A, and scaling factor $\frac{1}{\cos^2\frac{A}{2}}$, hence the A-mixtilinear excircle. The construction of this A-mixtilinear excircle can be done as follows:

1) Draw the internal bisector of $\angle BAC$ and the external bisector of $\angle ABC$, they meet at I_a.

2) Draw a perpendicular to AI_a through I_a, it meets rays AB, AC at V, V'.

3) Draw perpendiculars to AB, AC, respectively through V, V', they meet at a point J_a on the angle bisector of $\angle BAC$.

4) Draw a circle with center J_a through V, V'. This is the A-mixtilinear excircle.

Also solved by Cyril Letrouit, Lycee Jean-Baptiste Say, Paris, France; Mehdi Mikael Trense, Montpellier, France.

O251. Let ABC be a triangle. Find the locus of points P in its plane, different from A, B, C, with the following property: if A', B', C' lie on the rays PA, PB, PC, respectively, such that triangles $A'B'C'$ and ABC are similar, then the triangles are homothetic.

Proposed by Josef Tkadlec, Charles University, Czech Republic

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Assume that $A'B'C'$ is similar to ABC but not homothetic. Note first that we may perform a homothety τ on $A'B'C'$ with center P such that the result of applying τ on A' is A, and the result of applying τ on $A'B'C'$ is a triangle $AB''C''$ which is similar to ABC but not homothetic, and such that B'', C'' are on rays PB, PC. Hence we may assume wlog that $A' = A$ in triangle $A'B'C'$.

Note that there are now two distinct cases: ABC can be obtained from $AB'C'$ after a rotation around A and a scaling with center A, or ABC can be obtained from $AB'C'$ after a reflection around the internal bisector of $\angle C'AB'$, a rotation around A and a scaling with center A. In the second case, and unless my long and tedious calculations are wrong (which I will not repeat here both for my sake and the reader's), any point P in the plane enables us to generate a triangle $AB'C'$ satisfying the conditions of the problem statement. We will therefore concentrate henceforth only in the first case.

Since ABC can be obtained from $AB'C'$ by a rotation around A and a scaling with center A, an angle $\delta = \angle PAB' - \angle PAB = \angle PAC' - \angle PAC$ (not necessarily positive) exists. Applying the Sine Law to triangles APB, APB', APC, APC', we find

$$\frac{\sin\angle PBA}{PA} = \frac{\sin\angle APB}{AB} = \frac{AB'}{AB}\cdot\frac{\sin\angle APB'}{AB'} = \frac{AB'}{AB}\cdot\frac{\sin\angle PB'A}{PA},$$

$$\frac{\sin\angle PCA}{PA} = \frac{\sin\angle APC}{AC} = \frac{AC'}{AC}\cdot\frac{\sin\angle APC'}{AC'} = \frac{AC'}{AC}\cdot\frac{\sin\angle PC'A}{PA},$$

and since $\frac{AB'}{AB} = \frac{AC'}{AC}$ is the scaling factor between $AB'C'$ and ABC, we conclude that

$$\frac{\sin\angle PCA}{\sin\angle PC'A} = \frac{\sin\angle PBA}{\sin\angle PB'A},$$

where moreover $\angle PB'A = \angle PBA - \delta$ and $\angle PC'A = \angle PCA - \delta$, or

$$\cos(\angle PCA - \angle PBA + \delta) = \cos(\angle PBA - \angle PCA + \delta).$$

If the angle on one side equals minus the angle on the other side, then $\delta = 0$, or $AB'C'$ and ABC are homothetic. Therefore, for $AB'C'$ not to be homothetic, we need the angles on both sides to be equal, or $\angle PCA = \angle PBA$, hence P is on the circumcircle of ABC. Hence the only possibility for ABC and $AB'C'$ not to be homothetic is when P is on the circumcircle of ABC. Reciprocally, if P is on the circumcircle of ABC, it can be shown analogously that for any angle δ, lines forming angle δ with AB, AC may be constructed, which intersect lines PB, PC at points B', C' such that $AB'C'$ is similar to ABC.

We therefore conclude that, if $A'B'C'$ can be brought onto ABC through a rotation and a scaling, then $A'B'C'$ and ABC are necessarily homothetic unless P is on the circumcircle of ABC, while if $A'B'C'$ can be brought onto ABC through a reflection with respect to a line, a rotation and a scaling, then $A'B'C'$ and ABC are not necessarily homothetic, regardless of the choice of P.

O252. Let there be an $N \times N$ grid of squares and two players A and B playing the following game. First, player A has to draw a line ℓ that needs to intersect the grid; then, B has to select a square of the grid that has been cut by ℓ and remove it from the grid; then, B has to draw a line intersecting the grid but which doesn't cut the previously removed square, and so on (A has to remove a square cut by the previous line and draw a new line intersecting the grid but not cutting the previously removed squares, etc). The loser is the the one who cannot draw any more lines. Is there a winning strategy for some player? If yes, find it.

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No solutions have been received yet.