Concurrency, Coliniarity, and Cyclicity using Homotheties

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Abstract

This article intends to reveal new interesting points of concurrency, colliniarity or cyclicity in the geometry of triangle using the homothety and the beautiful Monge D'Alembert Theorem. The focus is on some of the external and internal centers of homothety for the circumcenter, incircle, excircles, mixtilinear incircles or the Euler circle.

1 Preliminaries

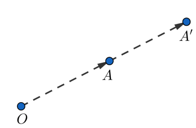
Definition 1.1. *Let be O a fixed point in the plane and k a fixed real number. The transformation of the plane that maps the point X into the point X' such that*

$$\overrightarrow{OX'} = k\,\overrightarrow{OX}$$

is called the homothety of center O and factor k and will be denoted by $\boldsymbol{H}_{O,k}$.

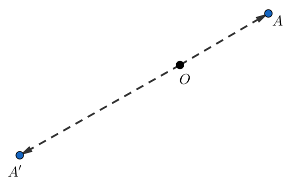
Shortly, $\mathbf{H}_{O,k}(X) = X' \iff \overrightarrow{OX'} = k \cdot \overrightarrow{OX} \quad (\star)$

The point X' is called the *homothetic* or the *image* of the point X.
The point X is called the *pre-image* of the point X' through the homothety $\mathbf{H}_{O,k}$.
When there is no confusion, the homothety $\mathbf{H}_{O,k}$ will be denoted simply by $\mathbf{H}$.
*Note $k = 0$ implies $\overrightarrow{OX'} = 0$, so $X' = 0$ (the banal homothety, any point is mapped into 0)
$k = 1$ implies $\overrightarrow{OX'} = \overrightarrow{OX}$, so $X = X'$ (the identical homothety, any point is mapped into itself)
$k = -1$, $\overrightarrow{OX'} = -\overrightarrow{OX}$, so O is the midpoint of the segment XX' (the symmetry with respect to O)



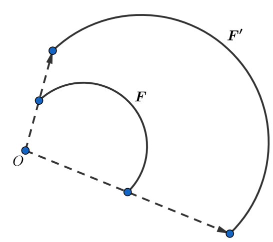
Example 1 $\mathbf{H}_{O,2}(A) = A' \iff \overrightarrow{OA'} = 2\,\overrightarrow{OA}$

Example 2 $\mathbf{H}_{O,-2}(A) = A' \iff \overrightarrow{OA'} = -2\,\overrightarrow{OA}$



Definition 1.2. *The homothetic of the figure **F** through the homothety $\boldsymbol{H}_{O,k}$ is defined as being the figure **F'** of the homothetics of all the points of the figure **F**.*

Example 3 $\boldsymbol{H}_{O,2}(\boldsymbol{F}) = \boldsymbol{F'}$



2 The Basic Properties of the Homothety

In the next sections, the following definitions and properties of the homothety will be considered.

Definition 2.1. *Let be l a fixed line. Direction of the line l is defined by all the lines in the plane that are parallel to the line l and the line l itself. Consequently, the lines l and l' have the same direction iff the lines are parallel or coincident.*

Theorem 2.1. *Let be $\boldsymbol{H}_{O,k}$ the homothety of center O and factor $k, k \neq 0$. Then the following properties hold:*

(i) $\boldsymbol{H}_{O,k}$ preserves the shape of the figure, meaning it maps a line into a line, a segment into a segment, a circle into a circle, etc.

(ii) $\boldsymbol{H}_{O,k}$ *preserves the direction of the lines, meaning it maps a line* l *into a line* l' *that has the same direction as the line* l*. More specifically, if the point* O *is not on the line* l*, then the image line* l' *is parallel with the line* l *but if the point* O *is on the line* l*, then the image line* l' *is the initial line* l *itself.*

(iii) $\boldsymbol{H}_{O,k}$ *preserves the parallelism of the lines, meaning it maps two parallel lines into two parallel lines.*

(iv) $\boldsymbol{H}_{O,k}$ *preserves the perpendicularity of the lines, meaning it maps two perpendicular lines into two perpendicular lines.*

(v) $\boldsymbol{H}_{O,k}$ *preserves the concurrency of the lines, meaning it maps the concurrent lines* a *and* b *into the concurrent lines* a' *and* b'*, where the lines* a', b' *are the images of the lines* a, b*.*

(vi) $\boldsymbol{H}_{O,k}$ *preserves the points of intersection of the lines, meaning it maps the intersection point* P *of the lines* a *and* b *into the intersection point* P' *of the lines* a' *and* b'*, where* a', b' *are the images of the lines* a, b*.*

(vii) $\boldsymbol{H}_{O,k}$ *preserves the measure of the angle, meaning it maps an angle into a congruent one.*

(viii) $\boldsymbol{H}_{O,k}$ *preserves the congruence of the figures, meaning it maps two congruent figures into two congruent figures.*

(ix) $\boldsymbol{H}_{O,k}$ *preserves the similarity of the figures, meaning it maps two similar figures into two similar figures.*

(x) $\boldsymbol{H}_{O,k}$ *preserves the ratios, meaning it maps two segments that have the ratio* r *into two segments that have the same ratio* r*. In particular,* $\boldsymbol{H}_{O,k}$ *preserves the midpoint of an segment, meaning it maps the midpoint of a segment into the midpoint of its image.*

(xi) $\boldsymbol{H}_{O,k}$ *preserves the important points of the triangle (orthocenter, centroid, circumcenter, incenter, etc), meaning it maps the orthocenter of a triangle into the orthocenter of its image, the centroid of a triangle into the centroid of its image, etc.*

(xii) $\boldsymbol{H}_{O,k}$ *preserves the tangency, meaning it maps the line* l *that is tangent to a figure* ***F*** *into its image* l' *that is also a line tangent to the figure* ***F'****, the image of the figure* ***F****.*

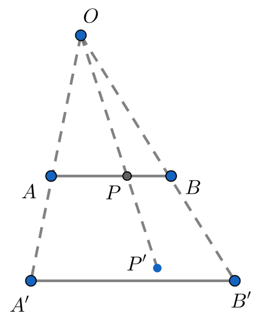
Proof. (i) When $k = 1$, any point of the line l is mapped into itself so the line l is mapped into itself. The conclusion is obviously valid in this case.

When $k \neq 1$, two cases will be considered.

(i) The first case is when the line l does not passing through the center of homothety O.

Consider A, B two fixed points of the line l. The points A, B are mapped through the homothety $\mathbf{H}_{O,k}$ into the points A', B'. From the relationship $(\star)$ it follows: $\overrightarrow{OA'} = k \cdot \overrightarrow{OA}$ and $\overrightarrow{OB'} = k \cdot \overrightarrow{OB}$. It follows immediately $A'B' \parallel AB$ or $A'B' \parallel l \quad (1)$.

Denote l' the line $A'B'$. The goal is to prove that the line l is mapped into the line l' i.e. $\mathbf{H}_{O,k}(l) = l' \quad (2)$.



To prove (2) it is necessary to prove $\mathbf{H}_{O,k}(l) \subseteq l'$ (that any point of the line l is mapped into a point of the line l') and then conversely, $l' \subseteq \mathbf{H}_{O,k}(l)$ (that any point of the line l' is the image of a point of the line l).

($\mathbf{H}_{O,k}(l) \subseteq l'$) Let be P an arbitrary point of the line l and P' its image through the homothety $\mathbf{H}_{O,k}$. It follows analogously $A'P' \parallel AP$ or $A'P' \parallel l \quad (3)$. From (1) and (3) it results A', B', P' collinear so the point P' is on the line l'.

($l' \subseteq \mathbf{H}_{O,k}(l)$) Let be P' an arbitrary point of the line l'. Define P s.t.$\overrightarrow{OP} = \dfrac{1}{k} \cdot \overrightarrow{OP'}$. (Note that this is possible because $k \neq 0$ from the hypothesis of Theorem 2.1). Similarly with the proof above, it follows $AP \parallel A'P'$ or equivalently $AP \parallel l'$ so the point P is on the line l. In conclusion, (2) is proved.

(ii) The second case is when the line l is passing through the center of homothety O. In this case, any point A of the line l is mapped into a point A' of the line l through the homothety $\mathbf{H}_{O,k}$. It results immediately that $\mathbf{H}_{O,k}\ (l) = l'$ holds.

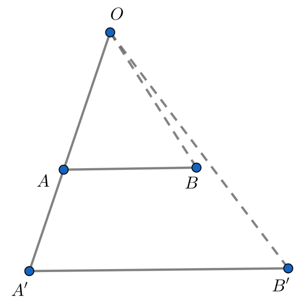
The other statements of the theorem can be proved similarly. $\square$

Proposition 2.1. *Let be $\boldsymbol{H}$ the homothety of center O and factor $k \neq 0$ and the segments AB and $A'B'$.*

(i) if $\boldsymbol{H}(A) = A'$ and $\overrightarrow{A'B'} = k\,\overrightarrow{AB}$, then $\boldsymbol{H}(B) = B' \quad (*)$

(ii) if $\boldsymbol{H}(AB) = A'B'$ and $\boldsymbol{H}(A) = A'$, then $\boldsymbol{H}(B) = B' \quad (\diamond)$

Proof. (i) Consider $k > 1$ to justify the figure used for the proof (the other cases: $0 < k < 1$, $k < -1$, $-1 < k < 0, k = 1, k = -1$ are similar).



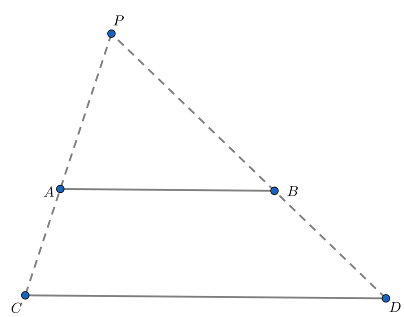
Adding the equations $\overrightarrow{OA'} = k\,\overrightarrow{OA}$ and $\overrightarrow{A'B'} = k\,\overrightarrow{AB}$, it follows $\overrightarrow{OB'} = k\,\overline{OB}$. According to $(\star)$ it results $\mathbf{H}(B) = B'$.

The (ii) statement is equivalent with (i).

$\square$

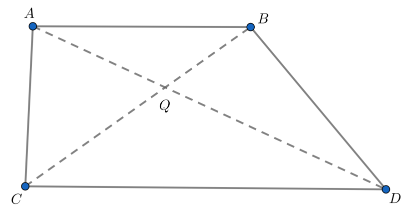
Proposition 2.2. *Let be AB and CD two parallel segments with $AB < CD$ s.t. the line AB separates the points P, C. The point Q is the intersection of the segments AD and BC, and the point P is the intersection of the lines AC and BD. There are two homotheties that maps the segment AB into the segment CD:*

(i) the (external) homothety $\boldsymbol{H}_e$ of center P and factor $\dfrac{CD}{AB}$



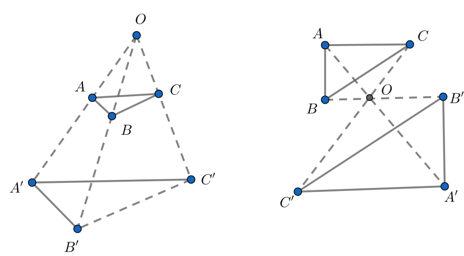
It holds $\boldsymbol{H}_e(AB) = CD$ with $\boldsymbol{H}_e(A) = C$, $\boldsymbol{H}_e(B) = D$

(ii) the (internal) homothety $\boldsymbol{H}_i$ of center Q and factor $-\dfrac{CD}{AB}$



It holds $\boldsymbol{H}_i(AB) = CD$ *with* $\boldsymbol{H}_i(A) = D$, $\boldsymbol{H}_i(B) = C$

(iii) if the $\triangle ABC$ *and* $\triangle A'B'C'$ *have the sides parallel, i.e.* $AB \parallel A'B'$, $AC \parallel A'C'$, $BC \parallel B'C'$, *then the lines* AA', BB' *and* CC' *are concurrent.*



Proof. (i) From $AB \parallel CD$ it follows $\dfrac{OC}{OA} = \dfrac{CD}{AB} = \dfrac{OD}{OB}$ that can be written equivalently as $\overrightarrow{OC} = \dfrac{CD}{AB} \cdot \overrightarrow{OA}$. Choosing $k = \dfrac{CD}{AB}$, it results from $(\star)$: $\mathbf{H}_e(A) = C$.Note that $k \neq -1, 0, 1$ from hypotheses. Analogously, $\mathbf{H}_e(B) = D$. The last two relationships and the Theorem 2.1 (1) lead to $\mathbf{H}_e(AB) = A'B'$

(ii) it can be proved analogously.

(iii) it follows from (i) (if the triangles have the same orientation) or (ii) (if the triangles have an opposite orientation) applied for the pairs of the sides.

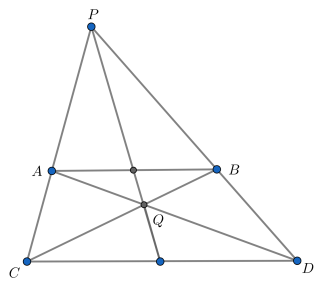
□

Definition 2.2. *In the hypotheses of the Proposition 2.1, the following centers of homothety can be defined:*

(i) the external center of homothety of the segments AB and CD: $P = exh(AB, CD)$.

(ii) the internal center of homothety of the segments AB and CD: $P = inh(AB, CD)$.

Proposition 2.3. *In a trapezoid, the following 4 points are collinear:*
- *the point of the intersection of the sides that are not parallel*
- *the point of the intersection of the diagonals*
- *the midpoints of the bases*

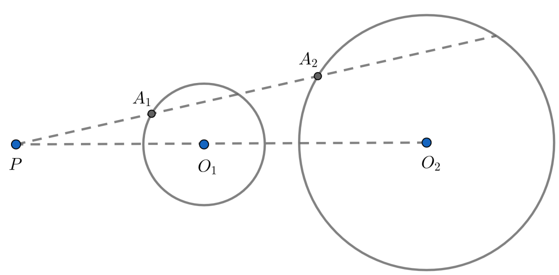


Proof. Immediately from the Proposition 2.1 and Theorem 2.1 (10). *Note: The points $exh(AB, CD)$ and $inh(AB, CD)$ are harmonic conjugates with respect to the segment that joins the midpoints of the trapezoid bases. □

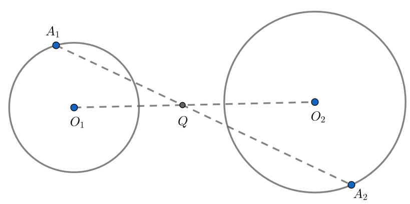
Proposition 2.4. *Let be* $\boldsymbol{C}_1(O_1, R_1)$ *and* $\boldsymbol{C}_2(O_2, R_2)$ *two circles of centers* O_1, O_2 *and radii* R_1, R_2, $R_1 < R_2$.

There are two homotheties that map the circle $\boldsymbol{C}_1$ *into the circle* $\boldsymbol{C}_2$*:*

(i) the (external) homothety of center P (P lies on the line O_1O_2 but not on the segment O_1O_2 and ratio $\dfrac{R_2}{R_1}$*; the position of P is given by* $\overrightarrow{PO_2} = \dfrac{R_2}{R_1}\overrightarrow{PO_1}$



(ii) the (internal) homothety of center Q (Q lies on the segment O_1O_2) and ratio $-\dfrac{R_2}{R_1}$; the position of Q is given by $\overrightarrow{PO_2} = -\dfrac{R_2}{R_1}\overrightarrow{PO_1}$



Proof. (i) From $\overrightarrow{PO_2} = \dfrac{R_2}{R_1}\overrightarrow{PO_1}$ and denoting $\dfrac{R_2}{R_1} = k$, using ($\star$) it results $\mathbf{H}_{P,k}(O_1) = O_2$. Then, an arbitrary point A_1 of the circle $\mathbf{C}_1$ it is mapped through $\mathbf{H}_{P,k}$ into A_2 and from Theorem 2.1 (i) it results $\mathbf{H}_{P,k}$ $(\mathbf{C}_1) = \mathbf{C}_2$.

□

Definition 2.3. *In the hypotheses of the Proposition 2.2, the following centers of homothety can be defined:*

(i) the external center of homothety of the circles $\boldsymbol{C}_1$ (O_1, R_1), $\boldsymbol{C}_2$ (O_2, R_2): $P = exh(\boldsymbol{C}_1,\ \boldsymbol{C}_2)$.

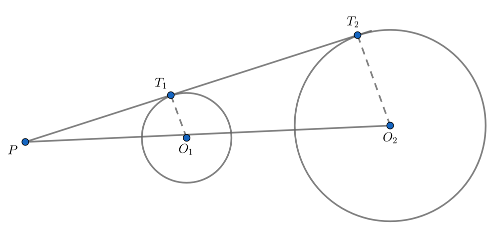
(ii) the internal center of homothety of the circles $\boldsymbol{C}_1$ $(O_1,\ R_1)$, $\boldsymbol{C}_2$ $(O_2,\ R_2)$: $Q = inh(\ \boldsymbol{C}1,\ \boldsymbol{C}_2)$

Note: 1.The points $exh(\mathbf{C}_1, \mathbf{C}_2)$ and $inh(\mathbf{C}_1, \mathbf{C}_2)$ are harmonic conjugates with respect to the segment O_1O_2

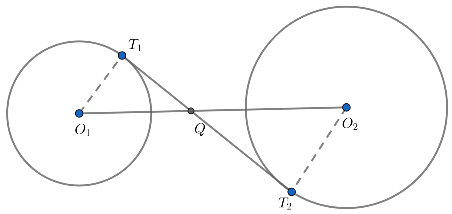
2. O_1A_1 and O_2A_2 are parallel radii of the circles $\mathbf{C}_1$ and $\mathbf{C}_2$ iff A_1 it is mapped into A_2 through the external/internal homothety (it depends on the orientation of the radii) that maps the circle $\mathbf{C}_1$ into $\mathbf{C}_2$.

Proposition 2.5. *Let be $\boldsymbol{C}_1(O_1, R_1)$ and $\boldsymbol{C}_2(O_2, R_2)$ two circles.*

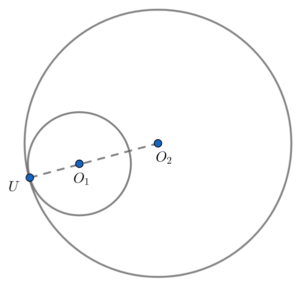
(i) If the circles share a common external tangent T_1T_2, then the point $P = exh(\boldsymbol{C}_1,\ \boldsymbol{C}_2)$ is the intersection of the tangent line T_1T_2 with the center line O_1O_2



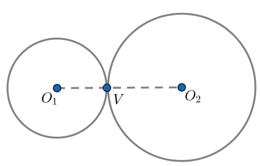
(ii) If the circles share a common internal tangent T_1T_2, then the point $P = exh(\boldsymbol{C}_1, \boldsymbol{C}_2)$ is the intersection of the tangent line T_1T_2 with the center line O_1O_2



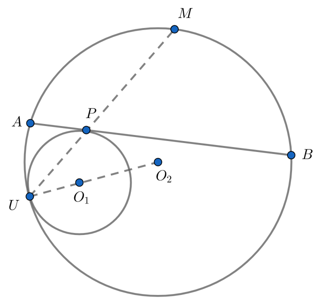
(iii) If the circles are internally tangent at U, then $U = exh(\boldsymbol{C}_1,\ \boldsymbol{C}_2)$



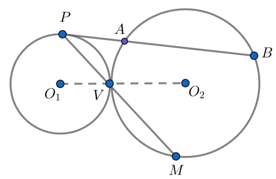
(iv) If the circles are externally tangent at V, then $V = inh(\boldsymbol{C}_1,\ \boldsymbol{C}_2)$



(v) If the circles are internally tangent at V, P is a point of the circle $\boldsymbol{C}_1$, AB is the tangent line in P to the circle $\boldsymbol{C}_1$ (A, B are on the circle $\boldsymbol{C}_2$), then the line VP intersects the circle $\boldsymbol{C}_2$ in the midpoint of the arc AB.



(vi) If the circles are externally tangent at U, P is a point of the circle $\boldsymbol{C}_1$, AB is the tangent line in P to the the circle $\boldsymbol{C}_2$ (A, B are on the circle $\boldsymbol{C}_2$), then the line UP intersects the circle $\boldsymbol{C}_2$ in the midpoint of the arc AB.



Proof. (i) Consider $\mathbf{H}$ the homothety of center P that maps $\mathbf{C}_1$ into $\mathbf{C}_2$. Then $\mathbf{H}(O_1) = O_2$, $O_1T_1 \parallel O_2T_2$ (because both of them are perpendicular on the common tangent T_1T_2) and $\dfrac{\overrightarrow{O_1T_1}}{\overrightarrow{O_2T_2}} = \dfrac{R_1}{R_2}$. From $(*)$ (Proposition 2.1, (i)) it results $\mathbf{H}(T_1) = T_2$ and from there it results further using $(\star)$ that P, T_1, T_2 are collinear. (ii)- (iv) analogously

(v) Consider $\mathbf{H}$ the homothety of center P that maps $\mathbf{C_1}$ into $\mathbf{C_2}$. Then $\mathbf{H}(O_1) = O_2$ and $\mathbf{H}(P) = M$ implies $\mathbf{H}(O_1P) = O_2M$, so $O_1P \parallel O_2M$. Due to the fact that $O_1P \perp AB$ it follows $O_2M \perp AB$. Because the line O_2M contains the diameter in $\mathbf{C}_2$, it results that the point M is the midpoint of the arc AB (according to the theorem that states that the diameter perpendicular to the chord halves the arcs determined by the chord). (vi) analogously $\square$

Proposition 2.6. *(i) Any homothety $\boldsymbol{H}_{O,k}$ with $k \neq O$ has an inverse that is also an homothety and*

$$\boldsymbol{H}_{O,k}^{-1} = \boldsymbol{H}_{O,k^{-1}}$$

(ii) if $\boldsymbol{H}_{O,k}$ and $\boldsymbol{H}_{Q,p}$ are two homotheties with $k, p \neq 1$, then $\boldsymbol{H}_{O,k} \circ \boldsymbol{H}_{Q,p}$ is also a homothety and $\boldsymbol{H}_{O,k} \circ \boldsymbol{H}_{Q,p} = \boldsymbol{H}_{R,kp}$ where the center R is on the line OQ s.t. $\overrightarrow{RQ} = k\,(p-1)\,\overrightarrow{QO}$

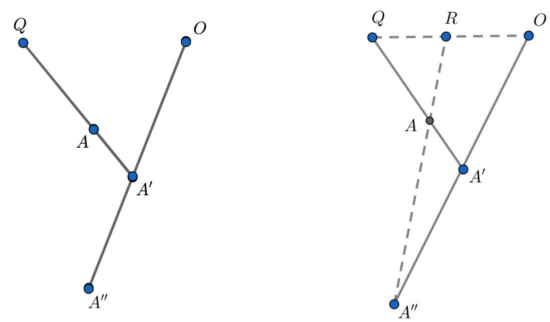
Proof. (i) Let be A an arbitrary point in the plane. From the definition of homothety ($\star$), the following holds: $\mathbf{H}_{O,k}(A) = A' \iff$

$\overrightarrow{OA'} = k\,\overrightarrow{OA}$. Because $k \neq 0$, the last relationship is equivalent with $\overrightarrow{OA} = \dfrac{1}{k}\,\overrightarrow{OA'}$ and again from ($\star$) this is equivalent with $\mathbf{H}_{O,k^{-1}}(A') = A$. The last equation shows that $\mathbf{H}_{O,k}$ has an inverse and $\mathbf{H}^{-1}_{O,k} = \mathbf{H}_{O,k^{-1}}$

(ii) To simplify the writing, denote $\mathbf{H}_{O,k} = \mathbf{H}_k$ and $\mathbf{H}_{Q,p} = \mathbf{H}_p$. For any point A, the homothety $\mathbf{H}_k \circ \mathbf{H}_p$ acts as below:

$$(\mathbf{H}_k \circ \mathbf{H}_p)(A) = (\mathbf{H}_k(\mathbf{H}_p(A))) = \mathbf{H}_k(A') = A''$$

where $A' = \mathbf{H}_p(A)$.



So $\mathbf{H}_p$ maps A into A' and $\mathbf{H}_k$ maps A' into A''. To prove that the transformation that maps A into A" is a homothety it is needed (according to Definition 1.1 and ($\star$)) to find a fixed point R that stands for the center and a fixed number r that stands for the factor of the presumptive homothety that maps A into A", so $\overrightarrow{RA''} = r\,\overrightarrow{RA'}$. Denote the intersection point of AA'' and OQ by R. The Menelaus's Theorem in $\triangle AA'A$" for the secant QRO provides:

$$\frac{\overrightarrow{QA'}}{\overrightarrow{QA}} \cdot \frac{\overrightarrow{RA}}{\overrightarrow{RA''}} \cdot \frac{\overrightarrow{OA''}}{\overrightarrow{OA'}} = -1$$

or equivalently $p \cdot \dfrac{\overrightarrow{RA}}{\overrightarrow{RA"}} \cdot k = -1$ or $\overrightarrow{RA''} = kp\,\overrightarrow{RA}$ (4). The position of the point R on the line OQ can be found analogously by applying Menelaus's Theorem in the $\triangle QA'O$ for the secant A"AR: $\overrightarrow{RQ} = \frac{k-1}{k(p-1)}\,\overrightarrow{RO}$ (5). This prove that the position of the point R is fixed and not dependant of the position of the arbitrary point A. The equation (4) reveals also that the factor kp is also constant because k, p does not depend on the position of A, the factors k, p being given initially in the problem. Now the Definition 1.1 and ($\star$) can be applied (because R is a fixed point in the plane and kp is a fixed number). It results that A is mapped into A" through the homothety of center R and factor kp i.e. $\mathbf{H}_{R,kp}(A) = A''$ for any point A.

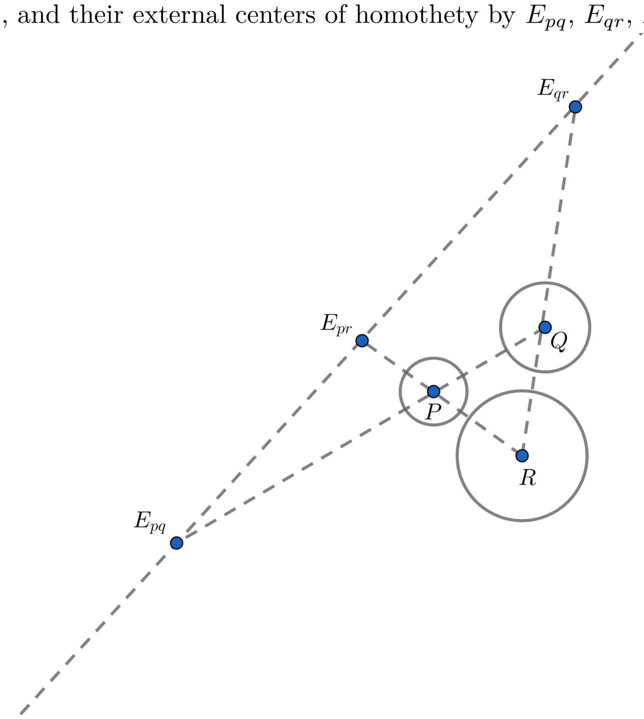
□

Theorem 2.2 (Monge-D'Alembert)**.** *Let be three circles in the plane of different radii. All the three pairs of these circles are providing three external centers of homothety and three internal center of homothety. The following hold:*

(i) all the external centers of homothety are collinear

(ii) two of the internal centers of homothety and the third external center of homothety provided by the all three pairs of circles are collinear.

Proof. (i) WLOG $p < q < r$. Denote the circles by **P**, **Q**, **R**, their centers by P, Q, R, their radii by p, q, r, and their external centers of homothety by E_{pq}, E_{qr}, E_{pr}.



Consider $\mathbf{H}_{pq}$ the external homothety that maps the circle **P** into the circle **Q** (the center is E_{pq} and the factor is $\dfrac{q}{p}$ due to Proposition 2.4)). Analogously define the external homotheties $\mathbf{H}_{qr}$ and $\mathbf{H}_{pr}$. Basically, $\mathbf{H}_{pq}$ maps **P** into **Q** and $\mathbf{H}_{qr}$ maps **Q** into **R**, so **P** is mapped into **R** through the homothety $\mathbf{H}_{qr} \circ \mathbf{H}_{pq}$. Due to the proof of the Proposition 2.6 (i), $\mathbf{H}_{qr} \circ \mathbf{H}_{pq}$ has the center on the line PR, outside the segment PR (from (5) and $\frac{q}{p}$, $\frac{r}{q}$ are greater than 1). (6).
Also, the external homothety $\mathbf{H}_{pr}$ maps the circle **P** into the circle **R** (5).
From (6), (7), and again Proposition 2.4 (i), it follows that these homotheties are identical and their centers are collinear. □

3 Conciclicity

Notations 3.1.

Consider $\triangle ABC$. The following notations will be used for $\triangle ABC$:

- $\angle A, \angle B, \angle C$ for the angles or the measure of the angles of the $\triangle ABC$

- O, I, G, H for the circumcenter, incenter, centroid, and orthocenter

- **O** for the circumcircle, **I** for the incircle, generally $\mathbf{X}(Y, r)$ for the circle centered at Y with the radius r

- A', B', C' for the feet of the altitudes from A, B, C in $\triangle ABC$, and A”, B”, C” for the intersections of the altitudes with the circumcircle of the $\triangle ABC$

- D, E, F for the tangency points of the incircle with the sides AB, BC, CA, and D', E', F' for the feet of the angle bisectors on the sides AB, BC, CA

- M, N, P for the midpoints of the sides AB, BC, CA

- M_A, N_A, P_A the midpoints of the circumcircle's arcs BC, CA, AB

- A_1, B_1, C_1 the pedal points of A, B, C in the circumcircle (* A_1 is the *pedal point* of the point A in the circumcircle if the segment AA' is a diameter in the circumcircle)

- A_2, B_2, C_2 for the midpoints of the segments HA, HB, HC

- $\angle XYZ$ or $\angle Y$ for the angle or measure of the angle XYZ when there is no confusion

Definition 3.1. *Consider triangle ABC. The lines Ax and Ay are isogonal conjugates iff they are symmetric over the angle bisector of A. The points P and Q are isogonal conjugates conjugates iff the pairs of lines $(AP, AQ), (BP, BQ), (CP, CQ)$ are isogonal conjugates.*

Proposition 3.1. *Consider triangle ABC and Notations 3.1. The following holds:*

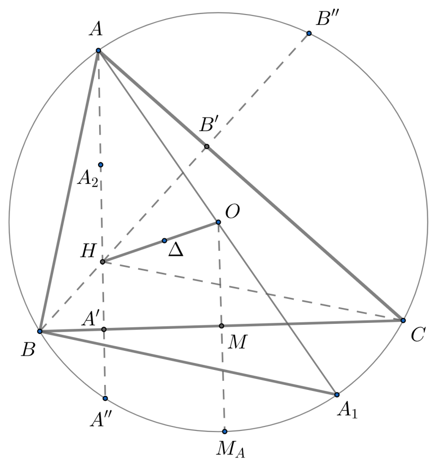
(i) the points H and O are isogonal conjugates

(ii) the quadrilateral $BHCA_1$ is a parallelogram and the quadrilateral $BHCA$” is a kite

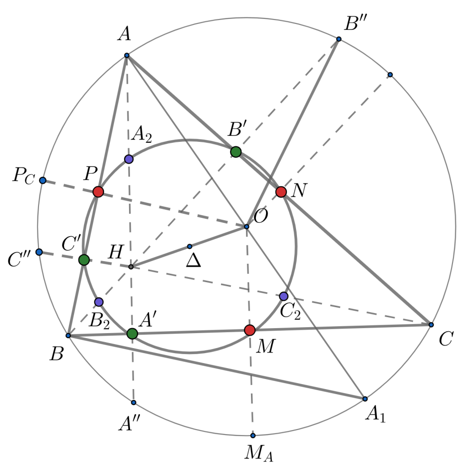
(iii) the points $A', B', C', M, N, P,\ A_2, B_2, C_2$ are concyclic and they are situated on the circle $\boldsymbol{E}(\Delta, \frac{R}{2})$, where Δ is the midpoint of the segment HG (the Euler circle)

(iv) the midpoints H_A, H_B, H_C of the segments HM_A, H_MB, H_MC are on the Euler circle $\boldsymbol{E}$.

Proof. (i) $\triangle BAA'$ and $\triangle CAA_1$ are right angle triangles (AA' altitude, AA_1 diameter) and $\angle B = \angle AA_1C$ (both subtend the arc AC). It follows that $\angle BAA' = \angle CAA_1$, hence the lines AA' and AA_1 are isogonal conjugate. Analogously for BB', BB_1 and hence the conclusion.



(ii) From $A_1C \perp AC$ and $BH \perp AC$ it follows $A_1C \parallel BH$. Analogously, $A_1B \parallel CH$, so $BHCA_1$ parallelogram. $\triangle BB'C$ and $\triangle AA'C$ are right angle triangles sharing $\angle C$ in common, so $\angle B'BC = \angle A''AC$. Because $\angle A''AC = A''BC$ (subtending the same arc $A''C$) it follows $\angle B'BC = \angle A''BC$. Then BA' is angle bisector and altitude in $\triangle HBA''$ and this implies $\triangle HBA''$ isosceles. Analogously, $\triangle HCA''$ is isosceles and that's why $BHCA''$ is a kite.



(iii) $BHCA_1$ parallelogram implies M midpoint of segment HA_1, $BHCA''$ kite implies A' midpoint of the segment HA''. Denote $\mathbf{H}$ the homothety with center H and factor $\dfrac{1}{2}$. The following hold:

$$\mathbf{H}(A'') = A', \quad \mathbf{H}(A_1) = M, \quad \mathbf{H}(A) = A_2 \quad (8)$$

These three points A'', A_1, A lie on the circumcircle $\mathbf{O}$. Denote by $\mathbf{E}$ the image of the circumcircle $\mathbf{O}$ through the homothety $\mathbf{H}$, so $\mathbf{H}(\mathbf{O}) = \mathbf{E}$. It follows that the points A', M, A_2 lie on the circle $\mathbf{E}$. Analogously, the points A', M, A_2 and A', M, A_2 lie on the circle $\mathbf{E}$ (Euler's circle or the 9-points' circle). The homothety $\mathbf{H}$ provides also the location of the center Δ of the Euler circle $\mathbf{E}$ (the midpoint of the segment HO) and its radius $\dfrac{R}{2}$.
*Note From AA_1 diameter in the circumcircle $\mathbf{O}$ and A_2, Δ, M midpoints of the segments HA, HO, HA_1, using the same homothety $\mathbf{H}$ it follows that A_2M is diameter in the Euler circle $\mathbf{E}$.

(iv) it follows immediately from the $\mathbf{H}(M_A) = M'_A$, from the $\mathbf{H}(\mathbf{O}) = \mathbf{E}$ proved at (iii) and Theorem 1. The analogously hold for M_B and M_C.

*Note As a conclusion, the 9-points Euler circle of $\triangle ABC$ contains at least 12 relevant points of the $\triangle ABC$

□

Proposition 3.2. *Consider $\triangle ABC$ and the Notations 3.1.The following holds:*

(i) the points H, O, G are collinear and $\overrightarrow{HG} = 2\overrightarrow{GO}$ (Euler line)

(ii) the segments A_2M, B_2N, C_2P are diameters in the Euler circle $\mathbf{E}$ of $\triangle ABC$

(iii) H is the orthocenter of $\triangle A_2, B_2, C_2$

(iv) A_2H_A, B_2H_B, C_2H_C are the angle bisectors of $\triangle A_2B_2C_2$, where H_A, H_B, H_C are the midpoints of the segments HM_A, HM_B, HM_C

Proof. (i) The segments AM and HO are medians in $\triangle AHA_1$ and because AM is median in $\triangle ABC$ it follows that the centroid G of $\triangle ABC$ is also the centroid for $\triangle AHA_1$. Therefore $\overrightarrow{HG} = 2\overrightarrow{GO}$

(ii) it follows from the points A_2, A', M lying on the Euler circle $\mathbf{E}$ and $\angle A_2A'M = 90^0$

(iii) it follows from B_2C_2 midline in $\triangle HBC$ and the similar statements for A_2B_2 and A_2C_2

(iv) it follows from M_A midpoint of the arc $A''A_1$ in the circumcircle $\mathbf{O}$ and the homothety $\mathbf{H}$ of center H and factor $\dfrac{1}{2}$ that maps M_A into H_A, the midpoint of the arc $A'M$ in the Euler circle $\mathbf{E}$ (because $\mathbf{H}(A'') = A', \mathbf{H}(A_1) = M, \mathbf{H}(\mathbf{O}) = \mathbf{E}$ and theorem 2.1)

□

Proposition 3.3. *Consider $\triangle ABC$ and the Notations 3.1.If $\overline{A}, \overline{B}, \overline{C}, \overline{H}$ are the reflections of A, B, C, H over M, N, P, O respectively, then the following holds for $\triangle \overline{ABC}$:*

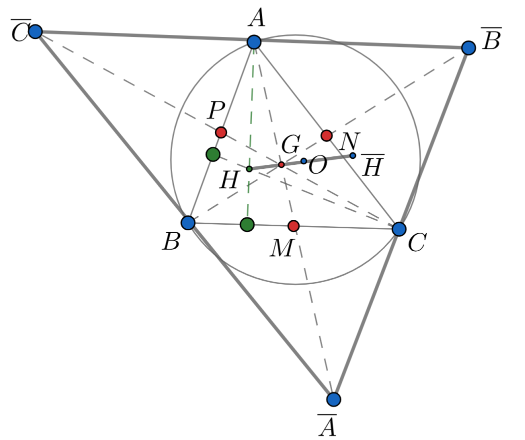
(i) G is the centroid

(i) H is the circumcenter and $2R$ is the circumradius

(ii) $\overline{H}$ is the orthocenter

(iii) O is the center of the Euler circle of the $\triangle \overline{ABC}$ and R is its radius.

Proof. (i) The segments BC and $A\overline{A}$ has the same midpoint M so $AB\overline{A}C$ is parallelogram. Analogously $BC\overline{B}A$ and $CA\overline{C}B$ are parallelograms. In consequence $AB = C\overline{A} = C\overline{B}$, so C is the midpoint of the segment $\overline{A}\ \overline{B}$. Analogously A is the midpoint of $\overline{B}\ \overline{C}$ and B is the midpoint of $\overline{A}$ $\overline{C}$. In consequence, $\triangle ABC$ and $\triangle\overline{ABC}$ have the same centroid G.



If $\mathbf{G}$ is the homothety of center G and factor -2, then

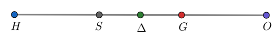
$$\mathbf{G}(A) = \overline{A}, \mathbf{G}(B) = \overline{B}, \mathbf{G}(C) = \overline{C}, \mathbf{G}(\mathbf{O}) = \mathbf{K}, \mathbf{G}(H) = \overline{H}$$

where $\mathbf{O}$ is the circumcircle of $\triangle ABC$ centered in O with the circumradius R and $\mathbf{K}$ is the circle centered in $\mathbf{H}$ with radius $2R$.

These makes $\mathbf{G}(\triangle ABC) = \triangle\overline{ABC}$ and using Theorem 2.1 (ix) it follows that the homothety $\mathbf{G}$ preserves the orthocenter (H maps into $\overline{H}$), the circumcenter and circumcircle (the point O maps into the point H and the circle $\mathbf{O}$ maps into the circle $\mathbf{K}$ through the homothety $\mathbf{G}$).This justify also the radius $2R$ for the circumcircle of $\triangle\overline{ABC}$ centered at H.

Denote by S the midpoint of the segment HG. From $\overrightarrow{HG} = 2\overrightarrow{GO}$ it follows S is the midpoint of HG and because the Euler center Δ of the $\triangle ABC$ is the midpoint of the segment HO, it follows that the point Δ is the midpoint of the segment SG. Finally, $\overrightarrow{GO} = -2\overrightarrow{\Delta G}$ makes O the image of the point Δ through the homothety $\mathbf{G}$, i.e. $\Delta = O$.

In consequence, the Euler circle $\mathbf{E}$ of $\triangle ABC$ centered at Δ and having the radius $\dfrac{R}{2}$ will be mapped through the homothety $\mathbf{G}$ into the Euler circle of the $\triangle\overline{ABC}$ centered at O and having the radius R.



□

4 Concurrency and Colliniarity

Theorem 4.1. *Consider $\triangle ABC$, the Notations 3.1, and the excircle $\boldsymbol{I_A}$ centered in I_A with the radius r_A.*

If T_{AB}, T_{BC} are the tangency points of the excircle $\boldsymbol{I_A}$ to the sides AB, AC, then the following holds:

(i) the lines $T_{AC}M_B, T_{AB}M_C$ and OI_A are concurrent.

Denote by S_A the concurrency point from (i). Similarly define the points S_B, S_C.

(ii) the lines AS_A, BS_B, CS_C are concurrent.

Denote by V the concurrency point from (ii).

(iii) the lines $D'S_A, E'S_B, F'S_C$ are concurrent.

Denote by W the concurrency point from (iii).

(iv) the points O, I, V, W are collinear and V, W are harmonic conjugates with respect to the segment OI.

Proof. (i) The line AB is the common tangent of the incircle $\mathbf{I}$ and the excircle $\mathbf{I}_A$ and because A, I, I_A are collinear, it results from Proposition 2.5 (i) that $A = exh(\mathbf{I}, \mathbf{I}_A)$. (9)

The circumcircle O and the excircle I_A have the radius OM_B and I_AT_{AC} parallel (both are perpendicular on AC), so $\overrightarrow{OM_B} = \dfrac{R}{r_A}\overrightarrow{I_AT_{AC}}$.

Consider $\mathbf{X}$ the external homothety that maps the circumcircle O into excircle I_A, so $\mathbf{X}(\mathbf{O}) = \mathbf{I}_A$.

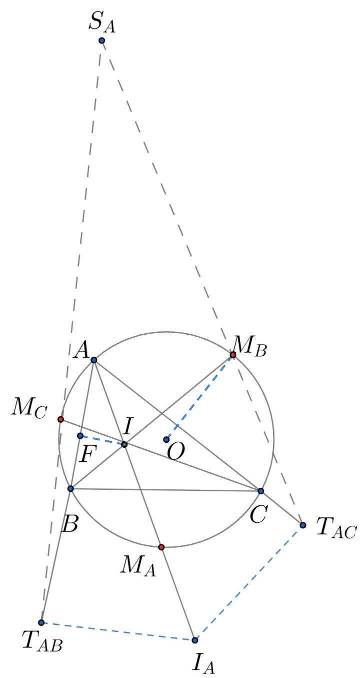
According to Proposition 2.1 (i) ($\star$) it follows that $\mathbf{X}(M_B) = T_{AC}$ and $\mathbf{X}(M_C) = T_{AB}$. From the last three relationships involving the homothety $\mathbf{X}$ it results that the lines $T_{AC}M_B, T_{AB}M_C$ and OI_A are concurrent in the $exh(\mathbf{O}, \mathbf{I}_A)$ that is S_A.

(ii) From the proof of (i) it follows that $S_A = exh(\mathbf{O}, \mathbf{I}_A)$. (10)

Denote by U the external center of homothety of the incenter and circumcenter of $\triangle ABC$, so $U = exh(\mathbf{I}, \mathbf{O})$ (11).

From (9), (10), (11), and Theorem 2.2 (Monge-D'Alembert) it results that U belongs to the line AS_A. Analogously can be proved that U belongs to the lines BS_B and CS_C.

Therefore, the lines AS_A, BS_B and CS_C are concurrent at U. Using the notations from Proposition 4.1 (ii) it follows that the points U and V are coincident, so $V = exh(\mathbf{I}, \mathbf{O})$. (12)

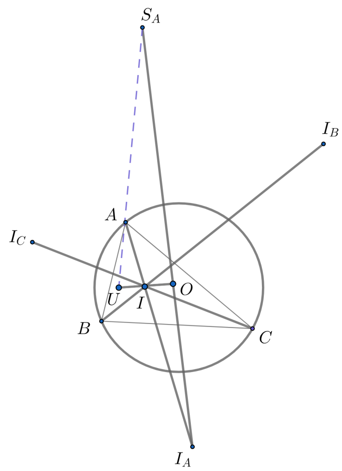


(iii) The point M'_A is the pedal point of M_A (M_A being the midpoint of the arc BC) in the circumcircle $\mathbf{O}$ (see Notations 3.1), so OM'_A is perpendicular on the side BC.

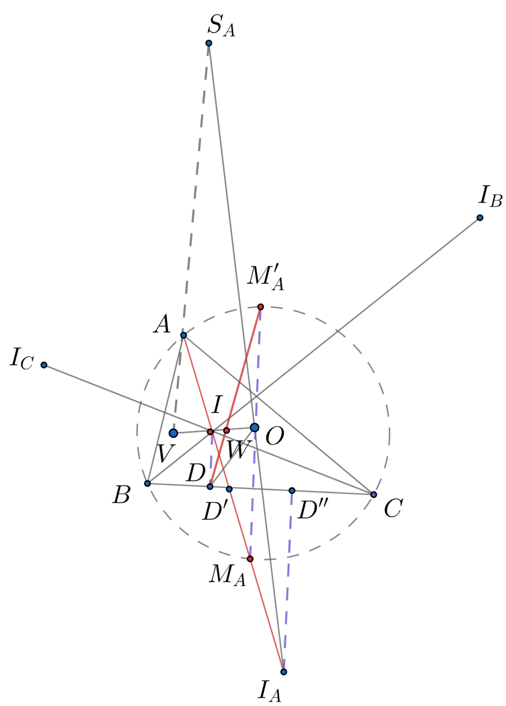
The internal center of homothety of the circumcenter $\mathbf{O}$ and $\mathbf{I}$ is at the intersection of the lines IO and DM'_A according to Proposition 2.4 (ii) ($ID \parallel OM'_A$, both of them being perpendicular on the side BC). Denote $Z = inh(\mathbf{O}, \mathbf{I})$. (13).

Proposition 2.4 can be used analogously to find the location of the internal center of homothety of the incenter $\mathbf{I}$ and the excircle $\mathbf{I}_A$, i. e. at the intersection of the lines II_A and DD''(D'' being the point of tangency of the excircle $\mathbf{I}_A$ to the side BC.

In consequence, ID, I_AD'' are both perpendicular on the side BC hence $ID \parallel I_AD''$ and Proposition 2.4 (ii) can be applied). But the line DD'' is the line BC, so II_A intersects BC at the point D' (the foot of the angle bisector AI), hence $D' = inh(\mathbf{I}, \mathbf{I}_A)$. (14)



From (10), (13), (14), and Theorem 2.2 (Monge-D'Alembert) it results that Z belongs to the line $D'S_A$. Analogously, Z belongs to the lines $E'S_B$ and $F'S_C$.
In other words, the lines the lines $D'S_A, E'S_B, F'S_C$ are concurrent at Z. Using the notations from Proposition 4.1 (iii) it follows that Z and W are coincident, so $W = inh(\mathbf{O}, \mathbf{I})$. (15)

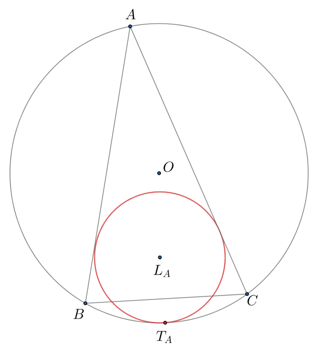


(iv) In (ii) was proved that O, I, V are collinear and in (iii) was proved that O, I, W are collinear, so all the four points O, I, V, W are collinear. A closer look at (12) and (15) emphasizes that the points V and W are the external and internal centers of homothety of the circumcircle $\mathbf{O}$ and incircle $\mathbf{I}$ of $\triangle ABC$, therefore V and W are points harmonic conjugates for the segment OI. $\square$

Theorem 4.2. *Let be $\triangle ABC$. There is an unique circle internally tangent to the circumcircle of $\triangle ABC$ and to the sides AB and AC (see proof in [2]).*

Definition 4.1. *The circle from the Theorem 4.2 is called the $A-mixtilinearincircle$ of the $\triangle ABC$.*

Notations 4.1. *The $A-mixtilinearincircle$ of the $\triangle ABC$ will be denoted by $\boldsymbol{L}_A$, its center by L_A, its radius by p_A, and the tangency point with the circumcircle $\boldsymbol{O}$ of $\triangle ABC$ by T_A.*



Proposition 4.1. *Let be $\triangle ABC$ and T_A, T_B, T_C the tangency points of the mixtilinear incircles of the vertices A, B, C to the circumcircle of $\triangle ABC$. Then the lines AT_A, BT_B, CT_C are concurrent.*

Proof. The $A-$mixtilinear incircle and the incircle $\mathbf{I}$ of $\triangle ABC$ have the sides AB, AC as common external tangents. From Proposition 2.4 (ii) it follows that A is the external center of homothety of these circles, so $A = exh(\mathbf{L}_A, \mathbf{I}) \quad (16)$.
Proposition 2.4 (i) provides T_A as the external center of homothety for the $A-$mixtilinear incircle and the circumcircle $\mathbf{O}$ of $\triangle ABC$, so $T_A = exh(\mathbf{O}, \mathbf{L}_A) \quad (17)$
In Theorem 4.1 was proved (9), i.e. $V = exh(\mathbf{I}, \mathbf{O})$.

Finally, (9), (16), (17), and the Theorem 2.2 (Monge-D'Alembert) imply the collinearity of the points A, T_A, V, so the line AT_A passes through the point V. Analogously, the lines BT_B and CT_C pass through the point V and hence it follows the concurrency of the lines AT_A, BT_B, CT_C.

□

References

[1] Andreescu, T. and Enescu, B., Mathematical Olympiad treasures, Birkhäuser, Boston (2006).

[2] Baca, J., On Mixtilinear Incircles, Mathemathical Reflections 2, Texas, USA (2020)

[3] Nguyen, K.L., Salazar, J.C., On Mixtilinear Incircles and Excircles, Forum Geometricorum Volume 6 (2006) 1–16