Junior problems

J265. Let a, b, c be real numbers such that

$$5(a+b+c)-2(ab+bc+ca)=9.$$

Prove that any two of the equalities

$$|3a-4b|=|5c-6|, \qquad |3b-4c|=|5a-6|, \qquad |3c-4a|=|5b-6|$$

imply the third.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt , San Jose ,California, USA
Since

$$(3a-4b)^2-(5c-6)^2+(3b-4c)^2-(5a-6)^2+(3c-4a)^2-(5b-6)^2=$$

$$=60\,(a+b+c)-24\,(ab+ac+bc)-108=12\,(5(a+b+c)-2(ab+bc+ca)-9)=0$$

then from any two equalities, let it be $\begin{cases}|3a-4b|=|5c-6|\\|3b-4c|=|5a-6|\end{cases} \iff \begin{cases}(3a-4b)^2=(5c-6)^2\\(3b-4c)^2=(5a-6)^2\end{cases}$

immediately yields

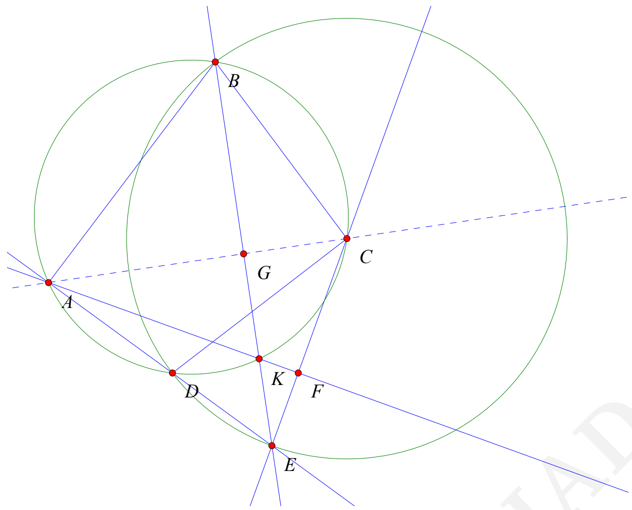
$$(3c-4a)^2-(5b-6)^2=0 \iff |3c-4a|=|5b-6|\,.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Aaron Doman, Pleasant Hill, CA, USA; David Stoner, South Aiken High School, Aiken, South Carolina, USA; Amedeo Sgueglia, University of Padua, Italy; Muhammad Thoriq; YoungSoo Kwon; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Prithwijit De, HBCSE, Mumbai, India; Radouan Boukharfane, Polytechnique de Montreal, Canada; Alessandro Ventullo, Milan, Italy; Polyahedra, Polk State College, FL, USA; Hoang Nguyen Viet, Hanoi, Vietnam; SHS Problem Solving Group, Tashkent, Uzbekistan; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Leonardo Boulay, Università di Roma "Tor Vergata", Roma, Italy.

J266. Let $ABCD$ be a cyclic quadrilateral such that $AB > AD$ and $BC = CD$. The circle of center C and radius CD intersects again the line AD in E. The line BE intersects again the circumcircle of the quadrilateral in K. Prove that AK is perpendicular to CE.

Proposed by Mircea Becheanu, University of Bucharest, Romania

Solution by Polyahedra, Polk State College, USA



Let F and G be the intersections of AK with CE and AC with BE, respectively. Since $\angle BAC = \angle CAD$, the reflection across AC interchanges the lines AB and AD, while preserving the circle through B, D, E. Since $AB \neq AD$, we must then have $AB = AE$. Therefore, $AG \perp BE$. Now $\angle CAK = \angle CBK = \angle CEK$, so $\triangle GAK \sim \triangle FEK$. Thus $KF \perp FE$.

Also solved by Hoang Nguyen Viet, Hanoi, Vietnam; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Radouan Boukharfane, Polytechnique de Montreal, Canada; David Stoner, South Aiken High School, Aiken, South Carolina, USA; SHS Problem Solving Group, Tashkent, Uzbekistan; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

J267. Solve the system of equations

$$\begin{cases} x^5+x-1=\left(y^3+y^2-1\right)z \\ y^5+y-1=\left(z^3+z^2-1\right)x \\ z^5+z-1=\left(x^3+x^2-1\right)y \end{cases}$$

where x, y, z are real numbers such that $x^3+y^3+z^3 \geq 3$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

$x^5+x-1=(x^3+x^2-1)(x^2-x+1)$ and

$$\prod_{\text{cyc}}(x^3+x^2-1)\prod_{\text{cyc}}(x^2-x+1)=xyz\prod_{\text{cyc}}(x^3+x^2-1)$$

Of course $xyz \neq 0$ and x^2-x+1 never annihilates. If

$$\prod_{\text{cyc}}(x^3+x^2-1)=0$$

we would have $x^3=1-x^2$ and $x^3+y^3+z^3=3-(x^2+y^2+z^2)<3$ thus the product never annihilates and we come to

$$\prod_{\text{cyc}}(x^2-x+1)=xyz \iff \prod_{\text{cyc}}(x-1+\frac{1}{x})=1$$

Since

$$x+\frac{1}{x}-1 \geq 2-1,\ x>0, \qquad x+\frac{1}{x}-1 \leq -2-1=-3,\ x<0$$

we can only have $x=y=z=1$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; YoungSoo Kwon; Arkady Alt , San Jose ,California, USA; Aaron Doman, Pleasant Hill, CA, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada; Polyahedra, Polk State College, FL, USA; SHS Problem Solving Group, Tashkent, Uzbekistan.

J268. Consider a convex m-gon $B_1 \dots B_m$ lying inside a convex n-gon $A_1 \dots A_n$. Their vertices define $m+n$ points in the plane. Prove that if $m+n \geq k^2-k+1$, then we can find a convex $(k+1)$-gon among these vertices that contains no other points inside it.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain
We consider two distinct cases:

Case 1: $m \geq k+1$. In this case, it suffices to pick any $k+1$ vertices out of $B_1, B_2, \dots, B_m$, and they clearly form a convex $(k+1)$-gon with no other vertices of the set inside it. The conclusion follows in this case.

Case 2: $m \leq k$. In this case, given any side of the m-gon B_iB_{i+1} for $i = 1, 2, \dots, m$ and where $B_{m+1} = B_1$, we will say that A_j *improves* B_iB_{i+1} iff A_j is on the opposite half-plane than the m-gon with respect to line B_iB_{i+1}. If A_j is on line B_iB_{i+1}, we will not say that A_j improves B_iB_{i+1}. Note that since lines $B_{i-1}B_i$ and B_iB_{i+1} meet inside the n-gon, each vertex of the n-gon improves at least one side of the m-gon, possibly more. Since $n \geq k^2-k+1-m \geq k(k-2)+1$, and there are at most k sides of the m-gon, then by Dirichlet's pigeonhole principle, there is at least one side of the m-gon improved by at least $k-2+1 = k-1$ vertices of the n-gon. Taking then B_i, B_{i+1}, and $k-1$ out of the at least $k-1$ vertices of the n-gon who improve B_iB_{i+1}, we find that they clearly are the vertices of a convex polygon, which contains in its interior no vertex of the m-gon (all of them are on line B_iB_{i+1}, or on the other side with respect to it), and no vertex of the n-gon (since it is clearly contained completely inside the n-gon). The conclusion follows also in this case.

Also solved by Polyahedra, Polk State College, FL, USA.

J269. Solve in positive integers the equation

$$\left(x^2-y^2\right)^2-6\min(x,y)=2013.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Clearly, $x\neq y$. Suppose without loss of generality that $x<y$. Then,

$$2013+6x=(x-y)^2(x+y)^2>(x+y)^2>4x^2,$$

which gives $4x^2-6x-2013<0$. Hence, $0<x<23$. Moreover,

$$(x^2-y^2)^2=3(671+2x),$$

therefore $671+2x$ must be divisible by 3 since the left member is a perfect square. This implies that $x=3k+2$ for some $k\in\mathbb{N}$, so

$$(x^2-y^2)^2=9(225+2k),$$

and $225+2k$ must be a perfect square. If $k=0$ it's obvious. The least positive integer such that $225+2k$ is a square is $k=32$, but for this value we get $x=98>23$. Therefore $k=0$, $x=2$ and $(x^2-y^2)^2=2025=45^2$ which gives $y^2-x^2=45$, i.e. $y=7$. By symmetry, $x=7,y=2$ is another solution of the equation. So, all the positive integer solutions of the given equation are $(2,7),(7,2)$.

Also solved by SHS Problem Solving Group, Tashkent, Uzbekistan; Polyahedra, Polk State College, FL, USA; David Stoner, South Aiken High School, Aiken, South Carolina, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Antonio Trusiani, Università di Roma "Tor Vergata", Roma, Italy.

J270. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a+2b+5c}+\frac{1}{b+2c+5a}+\frac{1}{c+2a+5b}\leq\frac{9}{8}\frac{a+b+c}{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2}.$$

Proposed by Tran Bach Hai, Bucharest, Romania

First solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Multiplying both sides by $\frac{8}{9}(a+b+c)$ and subtracting 1 from both sides, we obtain, after rearranging terms, the equivalent inequality

$$\sum_{\text{cyc}}\frac{(5a+41b+58c)(a-b)^2}{90(a^3+b^3+c^3)+423(a^2b+b^2c+c^2a)+531(ab^2+bc^2+ca^2)+1476abc}\leq$$

$$\leq\sum_{\text{cyc}}\left(1+\frac{2c^2}{\left(\sqrt{a}+\sqrt{b}\right)^2}\right)\frac{(a-b)^2}{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2}.$$

Now, it follows that the first factor in the RHS is larger than 1, while by the AM-QM inequality, the denominator in the second factor in the RHS is at most $3(ab+bc+ca)$, or it suffices to show that

$$(ab+bc+ca)(5a+41b+58c)\leq$$

$$\leq 30(a^3+b^3+c^3)+141(a^2b+b^2c+c^2a)+177(ab^2+bc^2+ca^2)+492abc,$$

clearly true. Since this last inequality is strict, equality in the originally proposed inequality holds iff $a-b, b-c, c-a$ are all zero, ie iff $a=b=c$.

Second solution by David Stoner, South Aiken High School, Aiken, South Carolina, USA

We will prove that:

$$(*) \qquad \frac{1}{a+2b+5c} \leq \frac{\frac{39a}{64}+\frac{30b}{64}+\frac{3c}{64}}{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2}$$

After which summing two analagous inequalities gives the desired result.

To prove this, note that by Cauchy-Schwarz:

$$(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2 \leq [(a+\frac{5b}{3})+(\frac{b}{3}+\frac{7c}{3})+(\frac{8c}{3})][\frac{3ab}{3a+5b}+\frac{3bc}{b+7c}+\frac{3a}{8}]$$

$$= [a+2b+5c][\frac{3ab}{3a+5b}+\frac{3bc}{b+7c}+\frac{3a}{8}]$$

Note that by the AM-HM inequality:

$$\frac{3ab}{3a+5b} = \frac{3}{8}[\frac{8}{\frac{1}{b}+\frac{1}{b}+\frac{1}{b}+\frac{1}{b}+\frac{1}{b}+\frac{1}{a}+\frac{1}{a}+\frac{1}{a}}] \leq \frac{3}{8}[\frac{b+b+b+a+a+a+a+a}{8}] = \frac{9b}{64}+\frac{15a}{64}$$

$$\frac{3bc}{b+7b} = \frac{3}{8}[\frac{8}{\frac{1}{c}+\frac{1}{b}+\frac{1}{b}+\frac{1}{b}+\frac{1}{b}+\frac{1}{b}+\frac{1}{b}+\frac{1}{b}}] \leq \frac{3}{8}[\frac{c+b+b+b+b+b+b+b}{8}] = \frac{3c}{64}+\frac{21b}{64}$$

This gives:

$$(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2 \leq$$

$$[a+2b+5c][\frac{3ab}{3a+5b}+\frac{3bc}{b+7c}+\frac{3a}{8}] \leq$$

$$[a+2b+5c][(\frac{9b}{64}+\frac{15a}{64})+(\frac{3c}{64}+\frac{21b}{64})+\frac{3a}{8}] =$$

$$[a+2b+5c][\frac{39a}{64}+\frac{30b}{64}+\frac{3c}{64}]$$

So $(*)$ is true, and we are done.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Sayan Das, Kolkata, India; SHS Problem Solving Group, Tashkent, Uzbekistan; Polyahedra, Polk State College, FL, USA.

S265. Find all pairs (m, n) of positive integers such that $m^2 + 5n$ and $n^2 + 5m$ are both perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by José Hernández Santiago, México

If $m < n$ then

$$n^2 < n^2 + 5m < n^2 + 5n < (n+3)^2.$$

Therefore, either $n^2 + 5m = (n+1)^2$ or $n^2 + 5m = (n+2)^2$. In the first case, we infer that $n = 5k + 2$ for some $k \in \mathbb{N}$. Then, $m^2 + 5n = (2k+1)^2 + 5(5k+2)$. The hypothesis that this number is a perfect square and the inequalities

$$(2k+4)^2 < (2k+1)^2 + 5(5k+2) = 4k^2 + 29k + 11 < (2k+8)^2$$

imply in turn that $k \in \{5, 38\}$. If $k = 5$ then $m = 11$ and $n = 27$; if $k = 38$ then $m = 77$ and $n = 192$. None of the associated pairs (m, n) satisfies that $m^2 + 5n$ and $n^2 + 5m$ are perfect squares. In the second case, we infer that $n = 5k - 1$ for some $k \in \mathbb{N} \setminus \{1\}$. Then, $m^2 + 5n = (4k)^2 + 5(5k-1)$. The hypothesis that this number is a perfect square and the inequalities

$$16k^2 < (4k)^2 + 5(5k-1) = 16k^2 + 25k - 5 < (4k+4)^2$$

imply in turn that $k = 14$, $m = 56$, and $n = 69$. The associated pair (m, n) doesn't satisfy that $m^2 + 5n$ and $n^2 + 5m$ are perfect squares either.

If $m = n$ then $m^2 < m^2 + 5n = m^2 + 5m < (m+3)^2$. Hence, $m^2 + 5m = (m+1)^2$ or $m^2 + 5m = (m+2)^2$. The former equation does not have solutions in $\mathbb{N}$ and the latter implies that $m = 4 = n$. The associated pair (m, n) is $(4, 4)$, which clearly satisfies that $m^2 + 5n$ and $n^2 + 5m$ are both perfect squares.

Finally, since the existence of a pair (m, n) with $m > n$ such that $m^2 + 5n$ and $n^2 + 5m$ are both perfect squares would contradict the conclusion we reached in the case $m < n$, we conclude that there is only one pair $(m, n) = (4, 4)$ that satisfies the constraints in question.

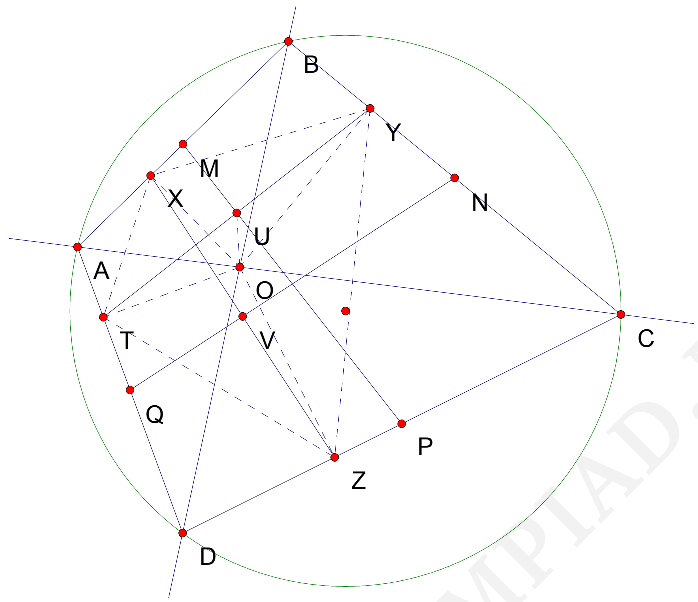
$\square$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; YoungSoo Kwon; Ioan Viorel Codreanu, Satulung, Maramures, Romania; SHS Problem Solving Group, Tashkent, Uzbekistan; Li Zhou, Polk State College, Winter Haven, FL, USA; Alessandro Ventullo, Milan, Italy; Leonardo Boulay, Università di Roma "Tor Vergata", Roma, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Radouan Boukharfane, Polytechnique de Montreal, Canada.

S266. Let $ABCD$ be a cyclic quadrilateral, $O = AC \cap BD$, M, N, P, Q be the midpoints of AB, BC, CD and DA, respectively, and X, Y, Z, T be the projections of O on AB, BC, CD and DA, respectively. Let $U = MP \cap YT$ and $V = NQ \cap XZ$. Prove that U, O, V are collinear.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA



Let $T = kA + (1-k)D$ for some $k \in (0,1)$. Since $\triangle AOD \sim \triangle BOC$, $Y = kB + (1-k)C$ as well. Then

$$kM + (1-k)P = \frac{k}{2}(A+B) + \frac{1-k}{2}(D+C) = \frac{1}{2}(T+Y).$$

Hence, U is the midpoint of TY. Likewise, V is the midpoint of XZ. Note next that $\angle TXO = \angle TAO = \angle YBO = \angle YXO$, etc., so O is the center of the incircle of $TXYZ$. Hence, $TX + YZ = XY + ZT$. Let $[\cdot]$ denotes area. Then

$$[TOX] + [YOZ] = [XOY] + [ZOT] = \frac{1}{2}[TXYZ].$$

On the other hand, we also have

$$[TUX] + [YUZ] = \frac{1}{2}[TYX] + \frac{1}{2}[YTZ] = \frac{1}{2}[TXYZ].$$

Thus,

$$[TOU] - [XOU] = [TOX] - [TUX] = [YUZ] - [YOZ] = [YOU] - [ZOU].$$

Since $[TOU] = [YOU]$, we get $[XOU] = [ZOU]$, which implies that V' is the midpoint of XZ, where $V' = UO \cap XZ$. Hence, $V' = V$, that is, U, O, V are collinear.

Also solved by SHS Problem Solving Group, Tashkent, Uzbekistan.

S267. Find all primes p, q, r such that $7p^3 - q^3 = r^6$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Suppose that $r = 2$. Then, $7p^3 = (q+4)(q^2-4q+16)$. Observe that both p and q are odd primes. Since

$$q^2 - 4q + 16 = (q+4)(q-8) + 48,$$

$\gcd(q+4, q^2-4q+16)|48$. Moreover, both factors are odd numbers, so $\gcd(q+4, q^2-4q+16) \in \{1, 3\}$. If $\gcd(q+4, q^2-4q+16) = 3$, then $p = 3$ by Unique Factorization. Substituting these values into the original equation we get $q = 5$. If $\gcd(q+4, q^2-4q+16) = 1$, since both factors are greater than 1, we get $p \neq 7$ and

$$\begin{aligned} q+4 &= 7 \\ q^2-4q+16 &= p^3 \end{aligned} \qquad \begin{aligned} q+4 &= p^3 \\ q^2-4q+16 &= 7. \end{aligned}$$

It's easy to see that both systems of equations have no solution. Now, suppose that $r > 2$. Then, exactly one between p and q is 2 and the other is odd. Suppose that $p = 2$. Then,

$$56 = (q + r^2)(q^2 - qr^2 + r^4).$$

Moreover both factors are greater than 1, $q + r^2$ is even and $q^2 - qr^2 + r^4$ is odd, so the only possibility is

$$\begin{aligned} q + r^2 &= 8 \\ q^2 - qr^2 + r^4 &= 7 \end{aligned}$$

and clearly the first equation has no solution in odd primes. Now, suppose that $q = 2$. Then,

$$7p^3 = (r^2+2)(r^4 - 2r^2 + 4).$$

Since

$$r^4 - 2r^2 + 4 = (r^2+2)(r^2-4) + 12,$$

then $\gcd(r^2+2, r^4-2r^2+4)|12$, but both factors are odd, so $\gcd(r^2+2, r^4-2r^2+4) \in \{1, 3\}$. If $\gcd(r^2+2, r^4-2r^2+4) = 3$, then $p = 3$ by Unique Factorization, but there is no solution for $p = 3, q = 2$. If $\gcd(r^2+2, r^4-2r^2+4) = 1$, since both factors are greater than 1, we get $p \neq 7$ and

$$\begin{aligned} r^2+2 &= 7 \\ r^4-2r^2+4 &= p^3 \end{aligned} \qquad \begin{aligned} r^2+2 &= p^3 \\ r^4-2r^2+4 &= 7. \end{aligned}$$

It's easy to see that both systems of equations have no solution. Therefore, the only primes which satisfy the given equation are $p = 3, q = 5, r = 2$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; YoungSoo Kwon; Cyril Letrouit, Lycee Jean-Baptiste Say, Paris, France; Li Zhou, Polk State College, Winter Haven, FL, USA; Albert Stadler, Switzerland; Radouan Boukharfane, Polytechnique de Montreal, Canada; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; SHS Problem Solving Group, Tashkent, Uzbekistan; Georgios Batzolis, Mandoulides High School, Thessaloniki, Greece; Anna Song, Lycée Henri IV, Paris, France.

S268. Let C be a circle with center O and let W be a point in its interior. From W we draw $2k$ rays such that the angle between any two adjacent rays is equal to $\frac{\pi}{k}$. These rays intersect the circumference of the circle C in points $A_1, \ldots, A_{2k}$. Prove that the centroid of $A_1 \ldots A_{2k}$ is the midpoint of OW.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Choose a cartesian coordinate system in the plane of C such that $O \equiv (0,0)$, $W \equiv (0,2d)$, and $C \equiv x^2+y^2=R^2$, where $R > 2d \geq 0$. The result is trivially true if $d=0$, since $A_1 \ldots A_{2k}$ would be a regular $2k$-gon inscribed in C. Otherwise, let α be the smallest positive angle such that one of the rays through W forms an angle α with the positive horizontal semiaxis. It follows that the vertices of the $2k$-gon are the intersection of C with lines $y = m_j x + 2d$, where $m_j = \tan\left(\alpha + \frac{j\pi}{k}\right)$ for $j = 0, 1, \ldots, k-1$. Note that each one of these lines intersects C at two points, whose respective x-coordinates are the roots of equation

$$R^2 = x^2 + (m_j x + 2d)^2 = \left(1 + {m_j}^2\right) x^2 + 4m_j dx + 4d^2,$$

or using Cardano-Vieta relations, both x coordinates add up to

$$-\frac{4dm_j}{1+{m_j}^2} = -2d \sin\left(2\alpha + \frac{2j\pi}{k}\right).$$

Since the equation describing the corresponding line through W is linear, the y coordinates of the intersection points of C with the line add up to

$$4d - 2m_j d \sin\left(2\alpha + \frac{2j\pi}{k}\right) = 4d - 4d\sin^2\left(\alpha + \frac{j\pi}{k}\right) = 2d + 2d\cos\left(2\alpha + \frac{2j\pi}{k}\right).$$

Now,

$$\sum_{j=0}^{k-1} \sin\left(2\alpha + \frac{2j\pi}{k}\right) = \sum_{j=0}^{k-1} \cos\left(2\alpha + \frac{2j\pi}{k}\right) = 0$$

is equivalent to the existence of a regular k-gon with unit sidelengths, one of whose sides forms an angle of 2α with the horizontal axis, or both these relations are true. It follows that the sum of all coordinates of the vertices of the $2k$-gon is $(0, 2kd)$, or its centroid has coordinates $(0, d)$, clearly the half sum of the coordinates of O and W. The conclusion follows.

Also solved by Li Zhou, Polk State College, Winter Haven, FL, USA.

S269. Find all integers n for which the equation $(n^2-1)x^2-y^2=2$ is solvable in integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Radouan Boukharfane, Polytechnique de Montreal, Canada

If (x,y) is a solution to the equation $(n^2-1)x^2-y^2=2$, then $(x,y+nx)$ is a solution to $x^2+y^2=2nxy$. We'll work with the later equation because of it's simplicity. If $x=y$, we get a solution $x=y=\pm1$ and $n=pm2$. The aim of the following demonstration is to show that those solutions of n are the unique that give a solution. To find a contradiction, let's suppose that $n>2$ and suppose there exists a solution (x,y) such that $x>y>0$. Among all the possible solutions $(r,s)=(x,y)$ we take the one such that $r+s$ is minimal. Therefore r is a root of the equation $t^2-2nst+s^2+2$, the other roots is a fortiori integer and verify $r'=\frac{s^2+2}{r}$, as $r>s$ so $r^2>s^2+2$ in a way that $r'<r$. Let's show now that $r'\neq s$, if on the contrary we have $r'=s$, we must get $r'=s=\pm1$ and so $r=3$ which leads to $n=2$, contradiction. We have found therefore a new couple of solution (r',s) such that $r'\neq s>0$ and $r'+s<r+s$, contradiction because $r+s$ is supposed to be minimal. We used here the prinicipal of Vieta Jumping.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Li Zhou, Polk State College, Winter Haven, FL, USA.

S270. Complex numbers z_1, z_2, z_3 satisfy $|z_1| = |z_2| = |z_3| = 1$. If ${z_1}^k + {z_2}^k + {z_3}^k$ is an integer for $k \in \{1, 2, 3\}$, prove that ${z_1}^{12} = {z_2}^{12} = {z_3}^{12}$.

Proposed by Mihai Piticari and Sorin Radulescu, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

We will show that ${z_1}^{12} = {z_2}^{12} = {z_3}^{12} = 1$, which we will refer to as the improved result. Note that

$$z_1 z_2 z_3 = \frac{2\left({z_1}^3 + {z_2}^3 + {z_3}^3\right) + (z_1 + z_2 + z_3)^3 - 3(z_1 + z_2 + z_3)\left({z_1}^2 + {z_2}^2 + {z_3}^2\right)}{6},$$

or with the conditions given in the problem statement, $z_1 z_2 z_3$ is rational, while at the same time $|z_1 z_2 z_3| = 1$, ie $z_1 z_2 z_3$ must be either 1 or -1. Since simultaneous substitution of all the z_i's by $-z_i$ leaves the problem unchanged, we can assume wlog that $z_1 z_2 z_3 = 1$. Therefore, wlog angles $\alpha, \beta \in (-\pi, \pi]$ exist such that $z_1 = e^{i\alpha}$, $z_2 = e^{i\beta}$, where wlog , we have $z_3 = e^{-i(\alpha+\beta)}$. But $z_1 + z_2 + z_3$ being an integer requires that $\sin\alpha + \sin\beta - \sin(\alpha+\beta) = 0$ for this sum to be real, or it is necessary that

$$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\alpha+\beta}{2} = 0,$$

for either $\alpha = \pi$, or $\beta = \pi$, or if $\alpha, \beta \neq \pi$, then $\alpha + \beta = 0$. Therefore, at least one of the z_i's is either 1 or -1.

Assume now wlog that $z_3 = 1$, and $z_1 z_2 z_3 = z_1 z_2 = \pm 1$. We may define wlog $z_1 = e^{i\alpha}$. If $z_1 z_2 = -1$, then $z_2 = -e^{-i\alpha}$, and $z_1 + z_2 = 2i\sin\alpha$ must be zero in order to be an integer, or α is an integral multiple of π, for (z_1, z_2) a permutation of $(-1, 1)$, and the improved result is clearly true. If $z_1 z_2 = 1$, then $z_2 = e^{-i\alpha}$, or $z_1 + z_2 = 2\cos\alpha$ must be an integer. Since $\cos\alpha \in [-1, 1]$, we may have $\cos(\alpha) \in \left\{0, \pm\frac{1}{2}, \pm 1\right\}$, for $\alpha \in \{0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}\}$, and in all these cases the improved result is clearly true again. The conclusion follows.

Also solved by Aaron Doman, Pleasant Hill, CA, USA; Moubinool Omarjee Lycée Henri IV , Paris, France; Jovan Jovanovic; Li Zhou, Polk State College, Winter Haven, FL, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada.

U265. Let $a > 1$ be a real number and let $f : [1, a] \to \mathbb{R}$ be twice differentiable. Prove that if the map $x \mapsto xf(x)$ is increasing, then

$$f\left(\sqrt{a}\right) \le \frac{1}{\ln a} \int_1^a \frac{f(t)}{t} dt.$$

Proposed by Marcel Chirita, Bucharest, Romania

Solution by Florin Stanescu, Serban Cioculescu School, Gaesti, Romania

We have $f(\sqrt{(a)}) \ln a = \int_1^a (f(\sqrt{x}) \ln x))' \, dx = \int_1^a \left(\frac{f'(\sqrt{x}) \ln x}{2\sqrt{x}} + \frac{f(\sqrt{x})}{x} \right) dx$.

On the pther hand, for $x \ge 1$, we have

$$f(x) - f(\sqrt{x}) = \int_{\sqrt{x}}^{x} f'(x)dx = \int_{\sqrt{x}}^{x} tf'(t) \cdot \frac{1}{t} dx \ge$$

$$\ge \sqrt{x} f'(\sqrt{x}) \cdot \int_{\sqrt{x}}^{x} \frac{1}{t} dt = \sqrt{x} f'(\sqrt{x}) \cdot \frac{\ln x}{2}$$

because function $xf'(x)$ is increasing, obtain

$$f(x) - f(\sqrt{x}) \ge \sqrt{x} f'(\sqrt{x}) \cdot \frac{\ln x}{2} \Rightarrow \frac{f(x)}{x} \ge$$

$$\ge \frac{f(\sqrt{x})}{x} + f'(\sqrt{x}) \cdot \frac{\ln x}{2\sqrt{x}} \Rightarrow f(\sqrt{a}) \ln a =$$

$$= \int_1^a \left(\frac{f'(\sqrt{x} \ln x)}{2\sqrt{x}} + \frac{f(\sqrt{x})}{x} \right) dx \le \int_1^a \frac{f(x)}{x} dx$$

and the conclusion follows.

Also solved by Radouan Boukharfane, Polytechnique de Montreal, Canada; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Li Zhou, Polk State College, Winter Haven, FL, USA.

U266. Let $A, B \in M_n(\mathbb{R})$ be symmetric positive definite matrices. Prove that

$$tr[(A^2 + AB^2A)^{-1}] \geq tr[(A^2 + BA^2B)^{-1}]$$

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Radouan Boukharfane, Polytechnique de Montreal, Canada

First we note that $(A + iAB)(A - iBA) = A^2 + AB^2A$. Otherwise the reverse product is equal to:

$$(A - iBA)(A + iAB) = \underbrace{A^2 + BA^2B}_{M} - i\underbrace{(BA^2 - A^2B)}_{N}$$

We note that the self-adjoint complex matrix $M - iN$ is positive definite because it's simply the product of an invertible operator and its adjoint. Therefore the given inequality is equivalent to

$\operatorname{Tr}\left(A^2 + AB^2A\right)^{-1} = \operatorname{Tr}\left(M - iN\right)^{-1} \geq \operatorname{Tr}\left(A^2 + BA^2B\right)^{-1} = \operatorname{Tr}\left(M\right)^{-1} \qquad (1)$

We note also that $(A - iAB)(A + iBA) = A^2 + AB^2A$, the reverse product is equal to:

$$(A + iBA)(A - iAB) = \underbrace{A^2 + BA^2B}_{M} + i\underbrace{(BA^2 - A^2B)}_{N}$$

The given inequality can be rewriten as

$$\operatorname{Tr}\left(A^2 + AB^2A\right)^{-1} = \operatorname{Tr}\left(M + iN\right)^{-1} \geq$$
$$\geq \operatorname{Tr}\left(A^2 + BA^2B\right)^{-1} = \operatorname{Tr}\left(M\right)^{-1} \qquad (2)$$

By summing up (1) and (2) we see that, because of the monotony of the function $M \to \operatorname{Tr}(M)$, we need only to show that

$$(M - iN)^{-1} + (M + iN)^{-1} \geq 2M^{-1}$$

We multiply both side by $N^{\frac{1}{2}}$, the last inequality can be reduced to

$$\left(I - i\underbrace{N^{-\frac{1}{2}}MN^{\frac{1}{2}}}_{P}\right)^{-1} + \left(I + i\underbrace{N^{-\frac{1}{2}}MN^{\frac{1}{2}}}_{P}\right)^{-1} \geq 2I$$

Noting again that both matrices on the left are positive definite. Diagonalizing the self-adjoint operator iP, we see that the last inequality is equivalent to prove

$$(1 + p)^{-1} + (1 - p)^{-1} = \frac{2}{1 - p^2} \geq 2$$

for $p \in (-1, 1)$ which is obviously true.

Also solved by Moubinool Omarjee , Lycée Henri IV , Paris, France.

U267. A continuous map $f:[0,1]\to\left[-\frac{1}{3},\frac{2}{3}\right]$ is onto and satisfies $\int_0^1 f(x)dx=0$. Prove that

$$\int_0^1 f(x)^3dx\le\frac{1}{9}.$$

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

First solution by Daniele Di Tullio, Università di Roma "Tor Vergata", Roma, Italy
For $-1/3\le y\le 2/3$,

$$\frac{4y^3}{3}=y^3+\frac{y^3}{3}\le y^3+\frac{y^2}{3}=\left(y+\frac{1}{3}\right)y^2\le\left(y+\frac{1}{3}\right)\frac{4}{9}=\frac{4y}{9}+\frac{4}{27}.$$

which implies that for $x\in[0,1]$,

$$\frac{4f(x)^3}{3}\le\frac{4f(x)}{9}+\frac{4}{27}.$$

Now we integrate with respect to x and we obtain

$$\frac{4}{3}\int_0^1 f(x)^3\,dx\le\frac{4}{9}\int_0^1 f(x)\,dx+\frac{4}{27}=\frac{4}{27},$$

which simplifies to

$$\int_0^1 f(x)^3\,dx\le\frac{1}{9}.$$

Second solution by G.R.A.20 Problem Solving Group, Roma, Italy.
We note that for $y\le 2/3$,

$$27y^3-9y-2=(3y-2)(3y+1)^2\le 0$$

which implies that for $x\in[0,1]$,

$$f(x)^3\le\frac{f(x)}{3}+\frac{2}{27}.$$

Now we integrate with respect to x and we obtain

$$\int_0^1 f(x)^3\,dx\le\frac{1}{3}\int_0^1 f(x)\,dx+\frac{2}{27}=\frac{2}{27}<\frac{1}{9}.$$

So the inequality holds with the smaller constant $2/27$ and it suffices to assume that $f:[0,1]\to(-\infty,2/3]$.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Arkady Alt , San Jose ,California, USA; Li Zhou, Polk State College, Winter Haven, FL, USA; Moubinool Omarjee Lycée Henri IV , Paris, France; Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Harun Immanuel, ITS Surabaya.

U268. Evaluate

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=2}^{n^2-1}\left\{\frac{n}{\sqrt{k}}\right\},$$

where $\{x\}$ is the fractional part of x.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Define

$$S(n)=\sum_{k=1}^{n^2}\frac{n}{\sqrt{k}},\qquad\qquad s(n)=\sum_{k=1}^{n^2}\left\lfloor\frac{n}{\sqrt{k}}\right\rfloor,$$

where $\lfloor x\rfloor$ is the greatest integer lower than or equal to x. Since $\frac{n}{\sqrt{k}}$ is clearly an integer for $k=1$ and $k=n$, the proposed problem clearly asks for $\lim_{n\to\infty}\frac{S(n)-s(n)}{n^2}$.

Note that, since $\frac{1}{\sqrt{x}}$ is a strictly decreasing function for all $x\geq 1$, we have

$$\int_x^{x+1}\frac{dx}{\sqrt{x}}<\frac{1}{\sqrt{x}}<\int_{x-1}^{x}\frac{dx}{\sqrt{x}},$$

or since

$$\sum_{k=1}^{n^2-1}\frac{1}{\sqrt{k}}>I(n)=\int_1^{n^2}\frac{dx}{\sqrt{x}}=2\sqrt{x}\Big|_1^{n^2}=2(n-1)>\sum_{k=2}^{n^2}\frac{1}{\sqrt{k}},$$

we have

$$\frac{1}{n}+2(n-1)<\frac{1}{n}+\sum_{k=1}^{n^2-1}\frac{1}{\sqrt{k}}=\frac{S(n)}{n}=1+\sum_{k=2}^{n^2}\frac{1}{\sqrt{k}}<1+2(n-1).$$

It follows that $2n^2-2n+1<S(n)<2n^2-n$, or $\lim_{n\to\infty}\frac{S(n)}{n^2}=2$. It therefore remains only to find $\lim_{n\to\infty}\frac{s(n)}{n^2}$.

Note that, for each $u\in\{1,2,\dots,n\}$, and for all $1\leq k\leq\frac{n^2}{u^2}$, we have $\frac{n}{\sqrt{k}}\geq u$, whereas if $k>\frac{n^2}{u^2}$, then $\frac{n}{\sqrt{k}}<u$. Since when $k\in\{1,2,\dots,n^2\}$, $\left\lfloor\frac{n}{\sqrt{k}}\right\rfloor$ takes exactly all integral values between 1 and n (both inclusive), it follows that

$$s(n)=\sum_{u=1}^{n}\left\lfloor\frac{n^2}{u^2}\right\rfloor,$$

since for each k that contributes exactly v to $s(n)$ through $\frac{n}{\sqrt{k}}$, we are counting said k exactly v times, one for each $u\in\{1,2,\dots,v\}$. Now, it is a well known result that

$$\sum_{u=1}^{\infty}\frac{1}{u^2}=\frac{\pi^2}{6},$$

whereas since $f(x)=\frac{1}{x^2}$ is a strictly decreasing function,

$$\sum_{u=n+1}^{\infty}\frac{1}{u^2}<\int_n^{\infty}\frac{dx}{x^2}=-\frac{1}{x}\Big|_n^{\infty}=\frac{1}{n},$$

or

$$\sum_{u=1}^{n}\frac{1}{u^2}=\frac{\pi^2}{6}-O\left(\frac{1}{n}\right).$$

Finally, since $\left\lfloor \frac{n^2}{u^2} \right\rfloor \geq \frac{n^2}{u^2} - 1$, we have

$$\frac{\pi^2}{6} - O\left(\frac{1}{n}\right) = \sum_{u=1}^{n} \frac{n^2}{u^2} > \frac{s(n)}{n^2} > \sum_{u=1}^{n} \left(\frac{1}{u^2} - \frac{1}{n^2}\right) = \frac{\pi^2}{6} - O\left(\frac{1}{n}\right) - \frac{1}{n},$$

or $\lim_{n\to\infty} \frac{s(n)}{n^2} = \frac{\pi^2}{6}$. We conclude that

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=2}^{n^2-1} \left\{ \frac{n}{\sqrt{k}} \right\} = 2 - \frac{\pi^2}{6} = \frac{12-\pi^2}{6}.$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Arkady Alt , San Jose ,California, USA; Li Zhou, Polk State College, Winter Haven, FL, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada; Albert Stadler, Switzerland; Konstantinos Tsouvalas, University of Athens, Athens, Greece; Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U269. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be positive real numbers with $a_i > b_i$ for $i = 1, \ldots, k$. If $\Delta_i = a_i - b_i$, prove that

$$\prod_{i=1}^{k} a_i - \prod_{i=1}^{k} b_i \geq k \sqrt[k]{\Delta_1 \cdots \Delta_k} \left(\prod_{i=1}^{k} a_i\right)^{\frac{k-1}{2k}} \left(\prod_{i=1}^{k} b_i\right)^{\frac{k-1}{2k}}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland

Using the Jensen inequality for concave function $f(x) = \ln(e^{kx} - 1)$ (since $f''(x) = \frac{-k^2 \cdot e^{kx}}{(e^{kx}-1)^2} < 0$) we obtain inequality:

$$f\left(\sum_{i=1}^{k} \frac{1}{k} x_i\right) \geq \sum_{i=1}^{k} \frac{1}{k} f(x_i) \qquad \Leftrightarrow \qquad \ln(e^{\sum_{i=1}^{k} x_i} - 1) \geq \sum_{i=1}^{k} \frac{1}{k} \ln(e^{kx_i} - 1)$$

or equivalently, by expotentiating:

$$\prod_{i=1}^{k} e^{x_i} - 1 \geq \sqrt[k]{\prod_{i=1}^{k} (e^{kx_i} - 1)}$$

By substituting $x_i = \ln \frac{a_i}{b_i}$ and then mulitplying both sides by $\prod_{i=1}^{k} b_i$ we get:

$$\prod_{i=1}^{k} \frac{a_i}{b_i} - 1 \geq \sqrt[k]{\prod_{i=1}^{k} \left(\left(\frac{a_i}{b_i}\right)^k - 1\right)}$$

$$\prod_{i=1}^{k} a_i - \prod_{i=1}^{k} b_i \geq \sqrt[k]{\prod_{i=1}^{k} (a_i^k - b_i^k)}$$

Using the AM-GM inequality we get:

$$\frac{a_i^{k-1} + a_i^{k-2} b_i + \ldots + a_i b_i^{k-2} + b_i^{k-1}}{k} \geq \sqrt[k]{a_i^{k-1} \cdot a_i^{k-2} b_i \cdot \ldots \cdot a_i b_i^{k-2} \cdot b_i^{k-1}} = a_i^{(k-1)/2} b_i^{(k-1)/2}$$

Therefore:

$$\prod_{i=1}^{k} a_i - \prod_{i=1}^{k} b_i \geq \sqrt[k]{\prod_{i=1}^{k} (a_i - b_i) \cdot (a_i^{k-1} + a_i^{k-2} b_i + \ldots + a_i b_i^{k-2} + b_i^{k-1})} \geq k^k \cdot \sqrt[k]{\prod_{i=1}^{k} (a_i - b_i)} \cdot \left(\prod_{i=1}^{k} a_i\right)^{\frac{k-1}{2k}} \cdot \left(\prod_{i=1}^{k} b_i\right)^{\frac{k-1}{2k}}$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Li Zhou, Polk State College, Winter Haven, FL, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U270. Let x_1 and x_2 be positive real numbers and define, for $n \geq 2$

$$x_{n+1} = \sqrt[n]{x_1} + \sqrt[n]{x_2} + \cdots + \sqrt[n]{x_n}.$$

Find $\lim_{n\to\infty} \frac{x_n - n}{\ln n}$.

Proposed by Gabriel Dospinescu, Ecole Normale Supérieure, Lyon, France

Solution by Jedrzej Garnek, University of Adam Mickiewicz, Poznan, Poland

Let $c = \max(1, x_1, x_2, x_3^2/4)$ and $d = \min(1, x_1, x_2, x_3^2/4)$. We'll prove inductively that

$$\sqrt[n!]{d} \cdot \left(n + \frac{\log(n-1)!}{n}\right) \leq x_{n+1} \leq n \cdot \sqrt[n]{n-1} \cdot \sqrt[n]{c}$$

for $n \geq 2$. Both inequalities are true for $n = 2$.

Using the AM-GM inequality, the inequality $\sqrt[k-1]{k-2} \leq 2$ (which follows easily from $2^x \geq 1 + x$) and the induction hypothesis (IH):

$$\begin{aligned} x_{n+1} &= \sum_{k=1}^{n} \sqrt[n]{x_k} = \sqrt[n]{x_1} + \sqrt[n]{x_2} + \sum_{k=3}^{n} \sqrt[n]{x_k} \overset{(IH)}{\leq} \sqrt[n]{c} + \sqrt[n]{c} + \sum_{k=3}^{n} \sqrt[n]{(k-1) \cdot \sqrt[k-1]{k-2} \cdot \sqrt[k-1]{c}} \\ &\leq \sqrt[n]{c} \cdot \left(1 + 1 + \sum_{k=3}^{n} \sqrt[n]{(k-1) \cdot \sqrt[k-1]{k-2}}\right) \overset{AM-GM}{\leq} \sqrt[n]{c} \cdot n \cdot \sqrt[n]{\frac{1^n + 1^n + \sum_{k=3}^{n}(k-1) \cdot \sqrt[k-1]{k-2}}{n}} \\ &\leq \sqrt[n]{c} n \cdot \sqrt[n]{\frac{1^n + 1^n + \sum_{k=3}^{n}(k-1) \cdot 2}{n}} = n \sqrt[n]{n-1} \cdot \sqrt[n]{c} \end{aligned}$$

To prove the second inequality, we will use the inequality $\sqrt[n]{x} = e^{\frac{\ln x}{n}} \geq 1 + \frac{\ln x}{n}$ and the induction hypothesis (IH):

$$\begin{aligned} x_{n+1} &= \sum_{k=1}^{n} \sqrt[n]{x_k} = \sqrt[n]{x_1} + \sqrt[n]{x_2} + \sum_{k=3}^{n} \sqrt[n]{x_k} \overset{(IH)}{\geq} \sqrt[n]{d} + \sqrt[n]{d} + \sum_{k=3}^{n} \sqrt[n]{\sqrt[(k-1)!]{d} \cdot \left((k-1) + \frac{\log(k-2)!}{k-1}\right)} \\ &\geq \sqrt[n]{d} + \sqrt[n]{d} + \sum_{k=3}^{n} \sqrt[n]{\sqrt[(k-1)!]{d} \cdot (k-1)} \geq \sqrt[n!]{d} \cdot \left(1 + 1 + \sum_{k=3}^{n} \sqrt[n]{k-1}\right) \\ &\geq \sqrt[n!]{d} \cdot \left(1 + 1 + \sum_{k=3}^{n} \left(1 + \frac{\log(k-1)}{n}\right)\right) = \sqrt[n!]{d} \cdot \left(n + \frac{\log(n-1)!}{n}\right) \end{aligned}$$

Thus we have:

$$\frac{\sqrt[n!]{d} \cdot \left(n + \frac{\log(n-1)!}{n}\right) - (n+1)}{\ln(n+1)} \leq \frac{x_{n+1} - (n+1)}{\ln(n+1)} \leq \frac{n \cdot \sqrt[n]{n-1} \cdot \sqrt[n]{c} - (n+1)}{\ln(n+1)}$$

Using Stirling formula and the limit $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$:

$$\begin{aligned} \lim_{n\to\infty} \quad & \frac{\sqrt[n!]{d} \cdot \left(n + \frac{\log(n-1)!}{n}\right) - (n+1)}{\ln(n+1)} = \lim_{n\to\infty} \left(\frac{n \cdot (\sqrt[n!]{d} - 1)}{\ln(n+1)} + \frac{\sqrt[n!]{d} \log(n-1)!}{n \ln(n+1)}\right) = \\ &= \lim_{n\to\infty} \left(\frac{n \cdot \frac{\ln d}{n!}}{\ln(n+1)} \cdot \frac{\exp(\frac{\ln d}{n!}) - 1}{\frac{\ln d}{n!}} + \frac{\sqrt[n!]{d} \log \sqrt[n]{(n-1)!}}{\ln(n+1)}\right) = \\ &= \lim_{n\to\infty} \left(\frac{n \cdot \frac{\ln d}{n!}}{\ln(n+1)} + \frac{\sqrt[n!]{d} \log(\sqrt[n]{(2\pi(n-1))^{1/2} \cdot ((n-1)/e)^{n-1}}) \cdot \delta_n}{\ln(n+1)}\right) = 0 + 1 \end{aligned}$$

(note that $\lim_{n\to\infty} \delta_n = 1$) and moreover:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n \cdot \sqrt[n]{n-1} \cdot \sqrt[n]{c} - n}{\ln(n+1)} &= \lim_{n\to\infty} \frac{n \cdot (\exp(\ln(c(n-1))/n) - 1)}{\ln(n+1)} \\
&= \lim_{n\to\infty} \frac{n \cdot \ln(c(n-1))/n) - 1)}{\ln(n+1)} \cdot \frac{\exp(\ln(c(n-1))/n) - 1}{\ln(c(n-1))/n} \\
&= \lim_{n\to\infty} \frac{n \cdot \ln(c(n-1))/n) - 1)}{\ln(n+1)} = 1
\end{aligned}$$

Thus, by squeeze theorem, the desired limit equals 1.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada; Albert Stadler, Switzerland; Moubinool Omarjee Lycée Henri IV , Paris, France; Konstantinos Tsouvalas, University of Athens, Athens, Greece.

O265. Solve in nonnegative real numbers the system of equations

$$\begin{cases} (x+1)(y+1)(z+1)=5 \\ \left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2-\min(x,y,z)=6. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Since we can exchange any two of x, y, z without altering the problem, we may assume wlog that $x \geq y \geq z$, or

$$x+y=-xy-1+\frac{5}{z+1},$$

$$x+y=2z+6-2\sqrt{z^2+6z}-2\sqrt{xy},$$

and subtracting both equations we find

$$(\sqrt{xy}-1)^2=2\sqrt{z^2+6z}+\frac{5}{z+1}-2z-6.$$

The RHS must clearly be non-negative, yielding

$$z^2+6z \geq \left(z+3-\frac{5}{2(z+1)}\right)^2=z^2+6z+9-\frac{5(z+3)}{z+1}+\frac{25}{4(z+1)^2},$$

or $0 \geq 16z^2-8z+1=(4z-1)^2$. We conclude that necessarily $z=\frac{1}{4}$, and consequently $\sqrt{xy}=1$, or $xy=1$, yielding finally $x+y=-1-1+\frac{5}{1+\frac{1}{4}}=2$, for $x=y=1$. There are no other solutions, hence restoring generality, the only solutions are $(x,y,z)=\left(1,1,\frac{1}{4}\right)$ and all its permutations.

Also solved by Li Zhou, Polk State College, Winter Haven, FL, USA; Radouan Boukharfane, Polytechnique de Montreal, Canada.

O266. Let $a, b, c \geq 1$ be real numbers such that $a+b+c=6$. Prove that

$$\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right) \leq 216.$$

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

Solution by David Stoner, South Aiken High School, Aiken, South Carolina, USA

Assume WLOG that $a \geq b \geq c$. This means that $6 = a+b+c \geq c+c+c$, so $c \leq 2 \rightarrow a+b \geq 4$. Now:
Lemma: $(a^2+2)(b^2+2) \leq ((\frac{a+b}{2})^2+2)^2$
Proof: This is equivalent with:

$$a^2b^2+2a^2+2b^2 \leq \frac{(a+b)^4}{16}+(a+b)^2$$

$$\Leftrightarrow 16(a-b)^2 \leq (a+b)^4-16a^2b^2$$

$$\Leftrightarrow 16(a-b)^2 \leq (a^2-b^2)^2+4ab(a-b)^2$$

$$\Leftrightarrow 16(a-b)^2 \leq (a-b)^2[(a+b)^2+4ab]$$

Which is true because $(a+b)^2 \geq 4^2 = 16$. Now, let $\frac{a+b}{2} = x$. We have:

$$(a^2+2)(b^2+2)(c^2+2) \leq (x^2+2)^2(c^2+2) = (x^2+2)^2((6-2x)^2+2)$$

Since $c \geq 1$, we have $2x+c = 6 \rightarrow x \leq \frac{5}{2}$. Also, $2x = a+b \geq 4$. Thus, $x \in [2, \frac{5}{2}]$. We wish to show that:

$$(x^2+2)^2((6-2x)^2+2)-216 \leq 0$$

$$\Leftrightarrow f(x) = -64-96x+168x^2-96x^3+54x^4-24x^5+4x^6 \leq 0$$

Note that $f'(x) = 12(x^2+2)(x-2)(x^2-3x+1)$, which only has one zero $x=2$ in $[2, \frac{5}{2}]$. So, since f is continuous and everywhere diffrentiable, we need only check $x=2, x=\frac{5}{2}$. The latter is strictly true, and the former gives the only case of equality $x=2$. This corresponds to $(a,b,c)=2$ in the original equation. The inequality is proven, and so we're done.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; Cyril Letrouit, Lycee Jean-Baptiste Say, Paris, France; Arkady Alt , San Jose ,California, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Li Zhou, Polk State College, Winter Haven, FL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

O267. Find all primes p, q, r such that

$$\frac{p^{2q}+q^{2p}}{p^3-pq+q^3}=r.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

If p, q are both odd primes, then the numerator in the LHS is even, the denominator is odd, and r must be an even prime, hence $r = 2$. Since $p, q \geq 2$, it follows that

$$p^4+q^4 \leq p^{2q}+q^{2p}=2p^3-2pq+2q^3<p^4+q^4,$$

absurd. Hence p, q cannot be both odd. If both are even, then both are equal to 2, and the LHS is $\frac{2^4+2^4}{2^3-2^2+2^3}=\frac{32}{12}$, which is not even an integer. Therefore, exactly one of p, q is an even prime, hence the numerator and denominator in the LHS are both odd, and r is an odd prime. Because of the symmetry between p and q, wlog $q = 2$, and the problem is equivalent to finding all pairs of primes (p, r) such that

$$\frac{p^4+2^{2p}}{p^3-2p+8}=r.$$

If $p = 5$, we have

$$r=\frac{625+1024}{125-10+8}=\frac{1649}{123},$$

absurd since 123 is a multiple of 3 but 1649 is not. Therefore, since $p \neq 5$, we have $p^4 \equiv 1 \pmod 5$ by Fermat's little theorem, whereas $2^{2p} = 4^p \equiv (-1)^p \equiv -1 \pmod 5$ because p is odd. Substitution of $p = 1, 2, 3, 4$ in $p^3 - 2p + 8$ yields remainders $2, 2, 4, 4$ when dividing by 5, or the denominator can never be a multiple of 5. Since the numerator is a multiple of 5, and r is prime, then $r = 5$, while p must satisfy

$$p^4+2^{2p}=5p^3-10p+40.$$

If $p \geq 5$, clearly the RHS is less than $5p^3 \leq p^4$, absurd, hence $p = 3$ is the only possible solution. Substitution yields that indeed p satisfies the required condition. Restoring generality allows us to write all possible solutions:

$$(p,q,r)=(2,3,5), \qquad (p,q,r)=(3,2,5).$$

Also solved by Albert Stadler, Switzerland; Radouan Boukharfane, Polytechnique de Montreal, Canada; Li Zhou, Polk State College, Winter Haven, FL, USA; David Stoner, South Aiken High School, Aiken, South Carolina, USA.

O268. Let $a_1, \ldots, a_{2n+1}$ be real numbers that add up to 0. Consider function $f(x) = \sum_{i=1}^{2n+1} |a_i - x|$. Let y be the point at which $f(x)$ attains its minimum. For $n \geq 1$, prove that

$$y \leq \frac{1}{2n+1} \sum_{i=1}^{2n+1} |a_i| .$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

We can assume that $a_1 \leq a_2 \leq \ldots \leq a_{2n} \leq a_{2n+1}$. Since the graph the given function is piecewise branch, then number of branches of graph of the function

$$f(x) = \sum_{i=1}^{2n+1} |a_i - x|$$

is $(2n+2)$ and on the each branches slopes are $-(2k+1), -(2k-1), \ldots, -1, 1, 3, \ldots, (2k+1)$ (but increasing order). We know $y = a_{n+1}, \sum_{i=1}^{2n+1} a_i = 0$ hence exist m such that

$$a_1 \leq a_2 \leq \ldots \leq a_m \leq 0 \leq a_{m+1} \leq \ldots \leq a_{2n+1}$$

. If $a_{n+1} \leq 0$ then we are done.

Let $a_{n+1} \geq 0$. Then $m \leq n$ and

$$-(a_1 + a_2 + \ldots + a_m) = a_{m+1} + \ldots + a_{n+1} + \ldots + a_{2n+1}.$$

Thus

$$\begin{aligned}\sum_{i=1}^{2n+1} |a_i| =& -(a_1 + a_2 + \ldots + a_m) + (a_{m+1} + \ldots + a_{n+1} + \ldots + a_{2n+1}) \\ =& 2(a_{m+1} + \ldots + a_{n+1} + \ldots + a_{2n+1}) \geq 2(a_{n+1} + a_{n+2} + \ldots + a_{2n+1}) \\ \geq& 2\underbrace{(a_{n+1} + a_{n+1} + \ldots + a_{n+1})}_{n+1} = 2(n+1)a_{n+1} = y.\end{aligned}$$

Consequently

$$y \leq \frac{1}{2(n+1)} \sum_{i=1}^{2n+1} |a_i|$$

Q.E.D.

Also solved by Li Zhou, Polk State College, Winter Haven, FL, USA; Arkady Alt , San Jose ,California, USA; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

O269. Let ABC be a triangle with circumcircle Γ and nine-point circle γ. Let X be a point on Γ and let Y, Z be on Γ so that the midpoints of segments XY and XZ are on γ.

a) Prove that the midpoint of YZ is on γ.

b) Find the locus of the symmedian point of triangle XYZ, as X moves along Γ.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Denote by O the circumcenter of ABC, by N the nine-point center of ABC, by U, V, W the respective midpoints of YZ, ZX, XY, and let γ' the circle with diameter OX. Clearly, circle γ' has diameter R, while the nine-point circle γ is well-known to have diameter R too. Since γ' is the result of scaling Γ with center X and scaling ratio 2, V, W are on γ', and since they are also on γ by hypothesis, U, V are the two intersection points of γ and γ'. Note that, since γ and γ' have the same radius, they are the result of reflecting each other over UV, or over the midpoint of VW. This means that the symmetric of X with respect to the midpoint of VW, which is clearly the midpoint U of YZ, is on γ. Part a) follows.

By part a), all triangles XYZ have the same circumcenter O and the same nine-point center N. Since N is the midpoint of OH, where H is the orthocenter, then all triangles XYZ also have the same orthocenter H. Denote by x, y, z the respective lengths of YZ, ZX, XY. It is relatively well known (or easily found by computing the power of the orthocenter with respect to the circumcircle) that

$$OH = \sqrt{9R^2 - (x^2 + y^2 + z^2)} =$$

$$= \frac{\sqrt{x^6 + y^6 + z^6 - x^4y^2 - x^4z^2 - y^4z^2 - y^4x^2 - z^4x^2 - z^4y^2 + 3x^2y^2z^2}}{4S},$$

where S is the area of XYZ. Relations expressing KO, KH as a function of the sidelengths and area of XYZ are also relatively well known, and can be found for example under the entry for the symmedian point in http://mathworld.wolfram.com. Using these relations and after some algebra, we can find

$$KO^2 = R^2 - \frac{48R^2S^2}{(x^2 + y^2 + z^2)^2},$$

$$KH^2 = \frac{8(3R^2 - OH^2)S^2}{(x^2 + y^2 + z^2)^2} - \frac{R^2 - OH^2}{2}.$$

Consider now a cartesian coordinate system (α, β) such that $O \equiv (0,0)$, $H \equiv (OH, 0)$. It follows that K satisfies

$$\alpha^2 + \beta^2 = KO^2, \qquad (\alpha - OH)^2 + \beta^2 = KH^2,$$

yielding

$$\alpha = \frac{OH^2 + KO^2 - KH^2}{2OH}, \qquad \beta^2 = KH^2 - (\alpha - OH)^2.$$

Substitution of the previous expressions for KO, KH produces

$$\alpha = \frac{3R^2 + OH^2}{4OH} - \frac{4S^2}{(9R^2 - OH^2)OH},$$

Note that

$$\left(\alpha - \frac{6R^2 \cdot OH}{9R^2 - OH^2}\right)^2 + \beta^2 = \frac{\left(27R^4 - 18R^2OH^2 - OH^4 - 16S^2\right)^2}{16(9R^2 - OH^2)^2OH^2} +$$

$$+KH^2 - \frac{\left(27R^4 - 30R^2OH^2 + 3OH^4 - 16S^2\right)^2}{16(9R^2 - OH^2)^2OH^2} =$$

$$= \frac{\left(27R^4 - 24R^2OH^2 + OH^4\right)\left(3R^2 - OH^2\right)}{2(9R^2 - OH^2)^2} - \frac{R^2 - OH^2}{2} = \left(\frac{2R \cdot OH^2}{9R^2 - OH^2}\right)^2,$$

or clearly the locus of the symmedian point is a circle, with center on ray OH at a distance $\frac{6R^2 \cdot OH}{9R^2 - OH^2}$ from O, and with radius $\frac{2R \cdot OH^2}{9R^2 - OH^2}$.

O270. No solutions hav yet been received.