Junior problems

J283. Let a, b, c be positive real numbers. Prove that

$$\frac{2a+1}{b+c}+\frac{2b+1}{c+a}+\frac{2c+1}{a+b}\geq 3+\frac{9}{2(a+b+c)}.$$

Proposed by Zarif Ibragimov , SamSU, Samarkand, Uzbekistan

J284. Find the greatest integer that cannot be written as a sum of distinct prime numbers.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J285. Let a, b, c be the sidelengths of a triangle. Prove that

$$8<\frac{(a+b+c)(2ab+2bc+2ca-a^2-b^2-c^2)}{abc}\leq 9.$$

Proposed by Adithya Ganesh, Plano, USA

J286. Let $ABCD$ be a square inscribed in a circle. If P is a point on the arc AB, find the maximum of the expression

$$\frac{PC\cdot PD}{PA\cdot PB}.$$

Proposed by Panagiote Ligouras, Noci, Italy

J287. Let n be a positive integer and let $a_1, a_2 \ldots, a_n$ be real numbers in the interval $\left(0, \frac{1}{n}\right)$. Prove that

$$\log_{1-a_1}(1-na_2)+\log_{1-a_2}(1-na_3)+\cdots+\log_{1-a_n}(1-na_1)\geq n^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J288. Four points are given in the plane such that no three of them are collinear. These four points form six segments. Prove that if five of their midpoints lie on a single circle, then the sixth midpoint lies on this circle too.

Proposed by Michal Rolinek and Josef Tkadlec, Czech Republic

Senior problems

S283. Let a, b, c be positive real numbers greater than or equal to 1, such that

$$5(a^2-4a+5)(b^2-4b+5)(c^2-4c+5) \leq a+b+c-1.$$

Prove that

$$(a^2+1)(b^2+1)(c^2+1) \geq (a+b+c-1)^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S284. Let I be the incenter of triangle ABC and let D, E, F be the feet of the angle bisectors. The perpendicular dropped from D onto BC intersects the semicircle with diameter BC in A', where A and A' lie on the same side of line BC. Then angle bisector of $\angle BA'C$ intersects BC at D'. Denote by $V = BE \cap FD'$ and $W = CF \cap ED'$. Prove that VW is parallel to BC.

Proposed by Ercole Suppa, Teramo, Italy

S285. Let be a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\frac{a}{b^2+c^2+2} + \frac{b}{c^2+a^2+2} + \frac{c}{a^2+b^2+2} \leq \frac{1}{8}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

S286. Let ABC be a triangle and let P be a point in its interior. Lines AP, BP, CP intersect BC, CA, AB at A_1, B_1, C_1, respectively. Prove that the circumcenters of triangles APB_1, APC_1, BPC_1, BPA_1 are concyclic if and only if lines A_1B_1 and AB are parallel.

Proposed by Mher Mnatsakanyan, Armenia

S287. Let m and n be positive integers such that $\gcd(m, n) = 1$. Prove that

$$\sum_{k=1}^{m} \phi\left(\left\lfloor \frac{nk}{m} \right\rfloor\right) \left\lfloor \frac{m}{k} \right\rfloor = \sum_{k=1}^{n} \phi\left(\left\lfloor \frac{mk}{n} \right\rfloor\right) \left\lfloor \frac{n}{k} \right\rfloor,$$

where ϕ is the Euler totient function.

Proposed by Marius Cavachi, Romania

S288. Consider triangle ABC with vertices A, B, C in the counterclockwise order. On the sides AB and AC construct outwards rectangles $ABXY$ and $CAZT$ in the clockwise order. Let D be the second point of intersection of the circumcircle of triangle AYZ and ω, the circumcircle of triangle ABC. Denote by $U = DY \cap \omega$ and $V = DZ \cap \omega$ and let W be the midpoint of UV. Prove that $AW \perp YT$.

Proposed by Marius Stanean, Zalau, Romania

Undergraduate problems

U283. Let a, b, c, d be nonnegative real numbers such that $a+b+c+d=2$. Prove that for any positive integer n,

$$\frac{a^n+b^n+c^n+d^n+2^{3-n}}{3} \geq \left(\frac{2-a}{3}\right)^n + \left(\frac{2-b}{3}\right)^n + \left(\frac{2-c}{3}\right)^n + \left(\frac{2-d}{3}\right)^n.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U284. Let $a_n = \left\{\sqrt{n^2+1}\right\}$ be the sequence of real numbers, where $\{x\}$ denotes the fractional part of x. Find $\lim_{n\to\infty} na_n$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U285. Let $(a_n)_{n>0}$ be the sequence defined by $a_n = \dfrac{e^{\frac{1}{n+1}}}{n+1} + \dfrac{e^{\frac{1}{n+2}}}{n+2} + \cdots + \dfrac{e^{\frac{1}{2n}}}{2n}$. Prove that sequence $(a_n)_{n>0}$ is decreasing and find its limit.

Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

U286. Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers such that

$$(x_{n+1}-x_n)(x_{n+1}x_n-1) \leq 0$$

for all n and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$. Prove that $(x_n)_{n\geq 1}$ is convergent.

Proposed by Mihai Piticari, Campulung Moldovenesc, Romania

U287. Let $u, v : [0+\infty) \to (0, +\infty)$ be differentiable functions such that $u' = v^v$ and $v' = u^u$. Prove that $\lim_{x\to\infty}(u(x)-v(x)) = 0$.

Proposed by Aaron Doman, University of California, Berkeley , USA

U288. Let $f : [a, b] \to \mathbb{R}$ be an integrable function which has an antiderivative and satisfies the property: for each interval $(m, n) \subset [a, b]$ there is an interval (m', n') contained in (m, n) such that $f(x) \geq 0$ for all x in (m', n'). Prove that $f(x) \geq 0$ for all x in $[a, b]$.

Proposed by Mihai Piticari and Sorin Radulescu, Romania

Olympiad problems

O283. Prove that for all positive real numbers $x_1, x_2, \ldots, x_n$ the following inequality holds:

$$\sum_{i=1}^{n} \frac{x_i^3}{x_1^2 + \cdots + x_{i-1}^2 + x_{i+1}^2 + \cdots + x_n^2} \geq \frac{x_1 + \cdots + x_n}{n-1}.$$

Proposed by Mircea Becheanu, Bucharest, Romania

O284. Consider a convex hexagon $ABCDEF$ with $AB||DE$, $BC||EF$, $CD||FA$. The distance between the lines AB and DE is equal to the distance between the lines BC and EF and is equal to the distance between the lines CD and FA. Prove that $AD + BE + CF$ does not exceed the perimeter of the hexagon $ABCDEF$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O285. Let a_n be the sequence of integers defined as $a_1 = 1$ and $a_{n+1} = 2^n(2^{a_n} - 1)$ for $n \geq 1$. Prove that $n!$ divides a_n.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O286. Let ABC be a triangle with orthocenter H. Let HM be the median and HS be the symmedian in triangle BHC. Denote by P the orthogonal projection of A onto HS. Prove that the circumcircle of triangle MPS is tangent to the circumcircle of triangle ABC.

Proposed by Marius Stanean, Zalau, Romania

O287. We are given a 6×6 table with 36 unit cells. A 2×2 square with 4 unit cells is called a *block*. A set of blocks covers the table if each cell of the table is covered by at least one cell of one block in the set. Blocks can overlap with each other. Find the largest integer n such that there is a cover of the table with n blocks, but if we remove any block from the covering, the remaining set of blocks will no longer cover the table.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O288. Let $ABCD$ be a square situated in the plane $\mathcal{P}$. Find the minimum and the maximum of the function $f : \mathcal{P} \to \mathbb{R}$ defined by

$$f(P) = \frac{PA + PB}{PC + PD}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania