Junior problems

J247. Let a and b be distinct zeros of the polynomial x^3-2x+c. Prove that $a^2(2a^2+4ab+3b^2)=3$ if and only if $b^2(3a^2+4ab+2b^2)=5$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J248. Let $f:[1,\infty)\to\mathbb{R}$ be defined by $f(x)=\frac{\{x\}^2}{\lfloor x\rfloor}$. Prove that $f(x+y)\le f(x)+f(y)$, for any real numbers x and y.

Proposed by Sorin Radulescu, Bucharest, Romania

J249. Find the least prime $p>3$ that divides 3^q-4^q+1 for all primes $q>3$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J250. Let ABC be a triangle with $\angle A\ge 120^\circ$ and let s be the semiperimeter of the triangle. Prove that

$$\sqrt{(s-b)(s-c)}\ge(3+\sqrt{6})(s-a).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J251. Let a, b, c be positive real numbers such that $a\ge b\ge c$ and $b^2>ac$. Prove that

$$\frac{1}{a^2-bc}+\frac{1}{b^2-ca}+\frac{1}{c^2-ab}>0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J252. Let ABC be an acute triangle and let O_a be a point in its plane such that

$$|\angle BO_aC|=2\alpha,\quad |\angle CO_aA|=180^\circ-\alpha,\quad |\angle AO_aB|=180^\circ-\alpha.$$

Similarly, define points O_b and O_c. Prove that the circumcircle of triangle $O_aO_bO_c$ passes through the circumcenter of triangle ABC.

Proposed by Michal Rolinek, Charles University, Czech Republic

Senior problems

S247. Prove that for any positive integers m and n, the number $8m^6 + 27m^3n^3 + 27n^6$ is composite.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S248. Let $\mathcal{C}(O, R)$ be a circle and let P be a point in its plane. Consider a pair of diametrically opposite points A and B lying on $\mathcal{C}$. Prove that while points A and B vary on the circumference of $\mathcal{C}$, the circumcircles of triangles ABP pass through another fixed point.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S249. Find the minimum of $2^x - 4^x + 6^x - 8^x - 9^x + 12^x$ where x is a positive real number.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S250. Let Γ be a circle and ℓ be a line lying outside Γ. Let $K \in \ell$ and let AB and CD be chords of Γ passing through K. Let P and Q lie on Γ. Let PA, PB, PC, PD meet ℓ at X, Y, Z, T, respectively, and then let QX, QY, QZ, QT meet again Γ at R, S, U, V, respectively. Prove that RS and UV meet on ℓ.

Proposed by Cosmin Pohoata, Princeton University, USA

S251. Find all triples (x, y, z) of positive real numbers for which there is a positive real number t such that the following inequalities hold simultaneously:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + t \leq 4, \quad x^2 + y^2 + z^2 + \frac{2}{t} \leq 5.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S252. Let a, b, c be positive real numbers. Prove that

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 + 2abc \geq \frac{2\sqrt{3(a^4b^4 + b^4c^4 + c^4a^4)}}{a+b+c}.$$

Proposed by Pham Huu Duc, Australia and Cosmin Pohoata, Princeton University, USA

Undergraduate problems

U247. Let a be a real number greater than 1. Evaluate

$$\frac{1}{a^2-a+1}-\frac{2a}{a^4-a^2+1}+\frac{4a^3}{a^8-a^4+1}-\frac{8a^7}{a^{16}-a^8+1}+\dots$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U248. Let A, S, X be matrices in $M_4(\mathbb{R})$ such that A is skew-symmetric, S is invertible, and $X = AS$. If $X^4 = O_4$, prove that $X^3 = O_4$.

Proposed by Dorin Andrica and Mihai Piticari, Romania

U249. Let $(a_n)_{n\geq 1}$be a decreasing sequence of positive numbers. Let

$$s_n = a_1 + a_2 + \ldots + a_n,$$

and

$$b_n = \frac{1}{a_{n+1}} - \frac{1}{a_n},$$

for all $n \geq 1$. Prove that if $(s_n)_{n\geq 1}$ is convergent, then $(b_n)_{n\geq 1}$ is unbounded.

Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Buzau, Romania

U250. Let f be a real valued function, continuous on an interval I, such that f has a continuous and nonnegative lateral derivative at any point in I. Prove that f is non-decreasing.

Proposed by Dan Marinescu and Mihai Piticari, Romania

U251. Find all polynomials $p(x) = a_n x^n + \cdots + a_1 x + a_0$ in $\mathbb{Z}[x]$ such that for all distinct integers x and y the following condition is satisfied:

$$\frac{p(x)-p(y)}{x-y} = \frac{1}{n}\left(p'(x)+p'(y)+(n-2)\sqrt{p'(x)p'(y)}\right),$$

where $p'(x)$ is the derivative of $p(x)$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U252. Find the number of automorphisms of the group of invertible residue classes mod n.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

Olympiad problems

O247. Solve in positive integers the equation

$$xy + yz + zx - 5\sqrt{x^2 + y^2 + z^2} = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O248. What is the maximal number of elements that one can choose from the set $\{1, 2, \ldots, 31\}$ such that the sum of any two is not a perfect square?

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O249. Find all triples (x, y, z) of positive integers such that

$$\frac{x}{y} + \frac{y}{z+1} + \frac{z}{x} = \frac{5}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O250. Given a triangle ABC, we define the A-mixtilinear excircle as the circle externally tangent to the circumcircle of ABC, and tangent to rays AB and AC. Find a formula for the radius of the A-mixtilinear excircle and give a ruler and compass construction for the A-mixtilinear excircle.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

O251. Let ABC be a triangle. Find the locus of points P in its plane, different from A, B, C, with the following property: if A', B', C' lie on the rays PA, PB, PC, respectively, such that triangles $A'B'C'$ and ABC are similar, then the triangles are homothetic.

Proposed by Josef Tkadlec, Charles University, Czech Republic

O252. Let there be an $N \times N$ grid of squares and two players A and B playing the following game. First, player A has to draw a line ℓ that needs to intersect the grid; then, B has to select a square of the grid that has been cut by ℓ and remove it from the grid; then, B has to draw a line intersecting the grid but which doesn't cut the previously removed square, and so on (A has to remove a square cut by the previous line and draw a new line intersecting the grid but not cutting the previously removed squares, etc). The loser is the the one who cannot draw any more lines. Is there a winning strategy for some player? If yes, find it.

Proposed by Cosmin Pohoata, Princeton University, USA