Junior problems

J373. Let a, b, c be real numbers greater than -1. Prove that

$$(a^2+b^2+2)(b^2+c^2+2)(c^2+a^2+2) \geq (a+1)^2(b+1)^2(c+1)^2.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

Solution by Arkady Alt, San Jose, California, USA

From Cauchy-Schwarz, we know that

$$a^2+b^2+2 = a^2+1+b^2+1 \geq \frac{(a+1)^2}{2} + \frac{(b+1)^2}{2}.$$

By AM-GM, this is at least $(a+1)\cdot(b+1)$. Thus

$$\prod_{cyc}\left(a^2+b^2+2\right) \geq \prod_{cyc}(a+1)\cdot(b+1) = (a+1)^2(b+1)^2(c+1)^2.$$

Equality holds if and only if $a=b=c=1$.

Also solved by Vincelot Ravoson, France, Paris, Lycée Henri IV; Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Sushanth Sathish Kumar, Jeffery Trail Middle School, Irvine, CA, USA; Byeong Yeon (Jackie) Ryu, The Hotchkiss School, Lakeville, CT, USA; Stanescu Florin, Serban Cioculescu School, Gaesti, Romania; Rajarshi Kanta Ghosh, Kolkata, India; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Joachim Studnia, Lycee Condorcet, Paris, France; A.S. Arun Srinivaas, Chennai, India; WSA; Albert Stadler, Herrliberg, Switzerland; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Pamplona, Spain; Bazar Tumurkhan, National University of Mongolia; Sutanay Bhattacharya, Bishnupur High School, West Bengal, India; Paul Revenant, Lycee du Parc, Lyon, France; Arpon Basu, AECS-4, Mumbai, India; Stefan Petrevski, Pearson College UWC, Victoria, Canada; Alysson Espíndola de Sá Silveira, Fortaleza, Ceará, Brazil; Catalin Prajitura, Student, College at Brockport, SUNY, USA; Polyahedra, Polk State College, FL, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Joel Schlosberg, Bayside, NY, USA; Robert Bosch, Archimedean Academy, USA; Toshihiro Shimizu, Kawasaki, Japan.

J374. Let a, b, c be positive real numbers such that $a+b+c \geq 3$. Prove that

$$abc + 2 \geq \frac{9}{a^3+b^3+c^3}.$$

Proposed by Mehmet Berke, İsler, Denizli, Turkey

Solution by Robert Bosch, Archimedean Academy, USA

Since $a+b+c \geq 3$, by Cauchy-Schwarz we have that

$$\begin{aligned} a^2+b^2+c^2 &\geq \frac{(a+b+c)^2}{3} \geq 3, \text{ and} \\ a^3+b^3+c^3 &\geq \frac{(a^2+b^2+c^2)^2}{a+b+c} \geq \frac{(a+b+c)^3}{9} \geq 3. \end{aligned}$$

Now recall the third-degree Schur's inequality

$$a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(c+a)+c^2(a+b),$$

or equivalently

$$2(a^3+b^3+c^3)+3abc \geq (a+b+c)(a^2+b^2+c^2).$$

The inequality to be proved is

$$abc(a^3+b^3+c^3)+2(a^3+b^3+c^3) \geq 9,$$

or equivalently

$$abc(a^3+b^3+c^3-3)+2(a^3+b^3+c^3)+3abc \geq 9.$$

Note that

$$\begin{aligned} abc(a^3+b^3+c^3-3)+2(a^3+b^3+c^3)+3abc &\geq 2(a^3+b^3+c^3)+3abc, \\ &\geq (a+b+c)(a^2+b^2+c^2), \\ &\geq 9. \end{aligned}$$

Equality holds if and only if $a=b=c=1$.

Also solved by Vincelot Ravoson, France, Paris, Lycée Henri IV; WSA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; A.S. Arun Srinivaas, Chennai, India; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Sutanay Bhattacharya, Bishnupur High School, West Bengal, India; Polyahedra, Polk State College, FL, USA; Daniel Lasaosa, Pamplona, Spain; Arkady Alt, San Jose, CA, USA; Toshihiro Shimizu, Kawasaki, Japan.

J375. Solve in real numbers the equation

$$\sqrt[3]{x}+\sqrt[3]{y}=\frac{1}{2}+\sqrt{x+y+\frac{1}{4}}.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

Solution by Stefan Petrevski, Pearson College UWC, Victoria, Canada

Let $a=\sqrt[3]{x}$ and $b=\sqrt[3]{y}$. After transferring $\frac{1}{2}$ to the left-hand side and squaring both sides, we obtain that

$$a^3+b^3=a^2+b^2+2ab-a-b.$$

But this is equivalent to $(a+b)(a^2+b^2+1-a-b-ab)=0$, so one of the factors must be 0. However, if $a+b=0$, the left-hand side of the initial equation is 0, while the right-hand side is positive, a contradiction. Therefore, $a^2+b^2+1=a+b+ab$. This rearranges nicely to $(a-1)^2+(b-1)^2+(a-b)^2=0$, from where we easily see that the only solution is $(a,b)=(1,1)$. Thus $(x,y)=(1,1)$.

Also solved by Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Sushanth Sathish Kumar, Jeffery Trail Middle School, Irvine, CA, USA; Byeong Yeon (Jackie) Ryu, The Hotchkiss School, Lakeville, CT, USA; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Albert Stadler, Herrliberg, Switzerland; WSA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Polyahedra, Polk State College, FL, USA; Marissa Meehan, Student, College at Brockport, SUNY, USA; Arpon Basu, AECS-4 , Mumbai, India; Joel Schlosberg, Bayside, NY, USA; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Alessandro Ventullo, Milan, Italy; Daniel Lasaosa, Pamplona, Spain; Bazar Tumurkhan, National University of Mongolia; Sutanay Bhattacharya, Bishnupur High School, West Bengal, India; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Robert Bosch, Archimedean Academy, USA; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Arkady Alt, San Jose, California, USA; Toshihiro Shimizu, Kawasaki, Japan.

J376. Let α, β, γ be the angles of a triangle. Prove that

$$\frac{1}{5-4\cos\alpha}+\frac{1}{5-4\cos\beta}+\frac{1}{5-4\cos\gamma}\geq 1.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Solution by Polyahedra, Polk State College, FL, USA

Let $x=s-a$, $y=s-b$, and $z=s-c$. Then by the law of cosines,

$$\frac{1}{5-4\cos\alpha}=\frac{bc}{5bc-2(b^2+c^2-a^2)}=\frac{bc}{bc+8(s-b)(s-c)}=\frac{(z+x)(x+y)}{(z+x)(x+y)+8yz}.$$

Now by the AM-GM inequality,

$$(x+y+z)(z+x)(x+y)-x[(z+x)(x+y)+8yz]=(x+y)(y+z)(z+x)-8xyz\geq 0,$$

so $\dfrac{1}{5-4\cos\alpha}\geq\dfrac{x}{x+y+z}$. Adding this with the other two analogous inequalities gives the desired result. Equality holds if and only if $x=y=z$, or $\alpha=\beta=\gamma$.

Also solved by Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Sushanth Sathish Kumar, Jeffery Trail Middle School, Irvine, CA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Albert Stadler, Herrliberg, Switzerland; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Arpon Basu, AECS-4, Mumbai, India; Joel Schlosberg, Bayside, NY, USA; Daniel Lasaosa, Pamplona, Spain; Michel Faleiros Martins, Petrobras University, Brazil; Sutanay Bhattacharya, Bishnupur High School, West Bengal, India; Bazar Tumurkhan, National University of Mongolia; Robert Bosch, Archimedean Academy, USA; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; WSA; Toshihiro Shimizu, Kawasaki, Japan.

J377. Let ABC be a triangle with $\angle A \leq 90^\circ$. Prove that

$$\sin^2 \frac{A}{2} \leq \frac{m_a}{2R} \leq \cos^2 \frac{A}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Nermin Hodzic, Dobosnica, Bosnia and Herzegovina

Let d_a be the distance from the circumcenter of triangle ABC to the side a. Then

$$d_a = R\cos A.$$

Using the triangle inequality, we have

$$R - d_a \leq m_a \leq R + d_a \Leftrightarrow$$

$$R(1-\cos A) \leq m_a \leq R(1+\cos A) \Leftrightarrow$$

$$\frac{1-\cos A}{2} \leq \frac{m_a}{2R} \leq \frac{1+\cos A}{2} \Leftrightarrow$$

$$\sin^2 \frac{A}{2} \leq \frac{m_a}{2R} \leq \cos^2 \frac{A}{2}$$

Equality holds on the RHS if and only if $b=c$ or if $\angle A = \frac{\pi}{2}$. Equality holds on the LHS if only if $\angle A = \frac{\pi}{2}$.

Also solved by WSA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Robert Bosch, Archimedean Academy, USA; Sutanay Bhattacharya, Bishnupur High School, West Bengal, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Arpon Basu, AECS-4 , Mumbai, India; Daniel Lasaosa, Pamplona, Spain; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Polyahedra, Polk State College, FL, USA; Arkady Alt, San Jose, CA, USA; Toshihiro Shimizu, Kawasaki, Japan.

J378. Let P be a point in the interior of the triangle ABC such that $\angle BAP = 105^\circ$, and let D, E, F be the intersections of BP, CP, DE with the sides AC, AB, BC, respectively. Assume that the point B lies between C and F and that $\angle BAF = \angle CAP$. Find $\angle BAC$.

Proposed by Marius Stănean, Zalău, România

Solution by Polyahedra, Polk State College, FL, USA

Suppose that AP intersects BC at Q. By Ceva's and Menelaus's theorems,

$$\frac{AE}{EB} \cdot \frac{BQ}{QC} \cdot \frac{CD}{DA} = 1 = \frac{AE}{EB} \cdot \frac{FB}{FC} \cdot \frac{CD}{DA}.$$

Hence, $FB \cdot QC = BQ \cdot FC = FQ \cdot FC - FB(FQ + QC) = FQ \cdot BC - FB \cdot QC$, so $2FB \cdot QC = FQ \cdot BC$. Let $x = \angle BAF$. Then by the Law of Sines,

$$\frac{FB}{\sin x} = \frac{AB}{\sin F}, \quad \frac{QC}{\sin x} = \frac{AQ}{\sin C}, \quad \frac{FQ}{\sin(x + 105^\circ)} = \frac{AQ}{\sin F}, \quad \frac{BC}{\sin(x + 105^\circ)} = \frac{AB}{\sin C}.$$

Thus

$$\sqrt{2} \sin x = \sin(x + 105^\circ) = \frac{\sqrt{2} - \sqrt{6}}{4} \sin x + \frac{\sqrt{6} + \sqrt{2}}{4} \cos x,$$

that is, $\tan x = \frac{\sqrt{6}+\sqrt{2}}{3\sqrt{2}+\sqrt{6}} = \frac{1}{\sqrt{3}}$ and $x = 30^\circ$. Therefore, $\angle BAC = x + 105^\circ = 135^\circ$.

Also solved by Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; WSA; Joel Schlosberg, Bayside, NY, USA; Robert Bosch, Archimedean Academy, USA; Daniel Lasaosa, Pamplona, Spain; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Toshihiro Shimizu, Kawasaki, Japan.

Senior problems

S373. Let x, y, z be positive real numbers. Prove that

$$\sum_{\text{cyc}} \frac{1}{xy + 2z^2} \leq \frac{xy + yz + zx}{xyz(x + y + z)}.$$

Proposed by Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan

Solution by Daniel Lasaosa, Pamplona, Spain

Multiplying throughout by the product of the denominators and rearranging terms, the proposed inequality is equivalent to

$$\sum_{\text{cyc}} \left(4(y + z)^2 x^4 - 4x^4 yz - 3x^2 y^2 z^2\right)(y - z)^2 \geq 0.$$

Now, clearly

$$4(y + z)^2 x^4 - 4x^4 yz - 3x^2 y^2 z^2 = 4x^4 \left(y^2 + yz + z^2\right) - 3x^2 y^2 z^2 =$$
$$= (y - z)^2 + 3x^2 (xy + yz + zx)(xy - yz + zx),$$

or defining $a = yz$, $b = zx$ and $c = xy$, it suffices to show that

$$\sum_{\text{cyc}} (a + b - c)(a - b)^2 \geq 0,$$

which is in turn equivalent to

$$a^3 + b^3 + c^3 + 3abc \geq a^2 b + ab^2 + b^2 c + bc^2 + c^2 a + ca^2.$$

This is a well-known form of Schur's inequality, and since x, y, z are positive reals, so are a, b, c, and equality holds iff $a = b = c$, or iff $x = y = z$.

Also solved by Vincelot Ravoson, France, Paris, Lycée Henri IV; WSA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Albert Stadler, Herrliberg, Switzerland; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Sutanay Bhattacharya, Bishnupur High School, West Bengal, India; Michel Faleiros Martins, Petrobras University, Brazil; Arkady Alt, San Jose, California, USA; Angel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Toshihiro Shimizu, Kawasaki, Japan.

S374. Let a, b, c be positive real numbers. Prove that at least one of the numbers

$$\frac{a+b}{a+b-c}, \frac{b+c}{b+c-a}, \frac{c+a}{c+a-b}$$

is not in the interval $(1,2)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Sutanay Bhattacharya, Bishnupur High School, West Bengal, India

If at least one of the numbers $a+b-c, b+c-a, c+a-b$, is negative, then the corresponding fraction(s) will be negative, and we are done immediately. So let us assume that $a+b-c, b+c-a, c+a-b > 0$. Without loss of generality, assume $c = \max\{a, b, c\}$. Then

$$\begin{aligned}\frac{a+b}{a+b-c} - 2 &= \frac{2c-a-b}{a+b-c} \\ &= \frac{(c-a)+(c-b)}{a+b-c} \geq 0 \implies \frac{a+b}{a+b-c} \geq 2,\end{aligned}$$

so we are done.

Also solved by Vincelot Ravoson, France, Paris, Lycée Henri IV; WSA; Duy Quan Tran, University of Health Science, Ho Chi Minh City, Vietnam; Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Sushanth Sathish Kumar, Jeffery Trail Middle School, Irvine, CA, USA; Byeong Yeon (Jackie) Ryu, The Hotchkiss School, Lakeville, CT, USA; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Arkady Alt, San Jose, California, USA; Robert Bosch, Archimedean Academy, USA; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Tan Qi Huan, Universiti Sains Malaysia, Malaysia; Arpon Basu, AECS-4, Mumbai, India; Christine Izyk, Student, College at Brockport, SUNY; Catalin Prajitura, Student, College at Brockport, SUNY, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Li Zhou, Polk State College, Winter Haven, FL, USA; Ángel Plaza, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Daniel Lasaosa, Pamplona, Spain; Alessandro Ventullo, Milan, Italy; Toshihiro Shimizu, Kawasaki, Japan.

S375. Let a, b, c be nonnegative real numbers such that $ab + bc + ca = a + b + c > 0$. Prove that

$$a^2 + b^2 + c^2 + 5abc \geq 8.$$

Proposed by An Zhen-Ping, Xianyang Normal University, China

Solution by Li Zhou, Polk State College, USA

Let $k = ab + bc + ca = a + b + c$. Then $k^2 = (a + b + c)^2 \geq 3(ab + bc + ca) = 3k$. So $k \geq 3$. If $k > 4$, then $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = k(k-2) > 8$. Assume thus that $k \leq 4$. By the Cauchy-Schwarz inequality, $a^3 + b^3 + c^3 \geq \frac{(a^2+b^2+c^2)^2}{a+b+c} = k(k-2)^2$. Hence,

$$\begin{aligned} 6abc &= (a + b + c)^3 + 2(a^3 + b^3 + c^3) - 3(a + b + c)(a^2 + b^2 + c^2) \\ &\geq k^3 + 2k(k-2)^2 - 3k^2(k-2) = -2k^2 + 8k. \end{aligned}$$

Therefore,

$$3(a^2 + b^2 + c^2 + 5abc - 8) \geq 3k(k-2) - 5k^2 + 20k - 24 = 2(k-3)(4-k) \geq 0,$$

completing the proof.

Also solved by WSA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Michel Faleiros Martins, Petrobras University, Brazil; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Sutanay Bhattacharya, Bishnupur High School, West Bengal, India; Albert Stadler, Herrliberg, Switzerland; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Andrea Fanchini, Cantù, Italy; Nguyen Viet Hung, HSGS, Hanoi University of Science, Vietnam; Arkady Alt, San Jose, California, USA; Toshihiro Shimizu, Kawasaki, Japan.

S376. Solve in integers the equation $x^5 - 2xy + y^5 = 2016$.

Proposed by Adrian Andreescu, Dallas, TX, USA

Solution by Robert Bosch, Archimedean Academy, USA and Richard Stong, Rice University, USA

If $x = 0$ or $y = 0$, the equation is unsolvable because 2016 is not a fifth-power. If x and y are negative then the equation is unsolvable because the left side is negative. Suppose that $0 < x \leq y$. We have

$$2016 = x^5 - 2xy + y^5 \geq x^5 - x^2 + y^5 - y^2 \geq y^5 - y^2.$$

Hence $1 \leq y \leq 4$. So the possible pairs (x, y) to test are

$$(1,1); (1,2); (2,2); (1,3); (2,3); (3,3); (1,4); (2,4); (3,4); (4,4).$$

Obtaining the solution $(x, y) = (4, 4)$. Note that if $x = y$ the equation is $x^5 - x^2 - 1008 = 0$ or $(x-4)(x^4 + 4x^3 + 16x^2 + 63x + 252) = 0$, thus $x = y = 4$.

It only remains to consider when $x > 0$ and $y < 0$. In this case the equation becomes $x^5 + 2xz - z^5 = 2016$ where $z = -y > 0$. Denoting by $s = x - z$ and $p = xz$ the equation becomes $5sp^2 + (5s^3+2)p + (s^5 - 2016) = 0$. If $s < 0$, then the left side is negative. Suppose now $s \geq 0$.

If $s \geq 5$ then the left side is positive, so $s = 0, 1, 2, 3, 4$. For $s = 0$ we obtain $x^2 = 1008$, which is not a perfect square. For the other values consider the equation as a quadratic on p, thus the discriminant $\Delta(s) = 5s^6 + 20s^3 + 40320s + 4$ have to be a perfect square, but $\Delta(1) = 40349, \Delta(2) = 81124, \Delta(3) = 125149$ and $\Delta(4) = 183044$ are not.

Finally, the only solution to the original equation is $(x, y) = (4, 4)$.

Also solved by Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Joseph Currier; Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Michel Faleiros Martins, Petrobras University, Brazil; Li Zhou, Polk State College, Winter Haven, FL, USA; Toshihiro Shimizu, Kawasaki, Japan.

S377. If z is a complex number with $|z| \geq 1$, prove that

$$\frac{|2z-1|^5}{25\sqrt{5}} \geq \frac{|z-1|^4}{4}.$$

Proposed by Florin Stănescu, Găesti, România

Solution by Nermin Hodzic, Dobosnica, Bosnia and Herzegovina

Let $z = a + bi$. We have $|z| \geq 1 \Rightarrow a^2 + b^2 \geq 1$. Now we have

$$\frac{|2z-1|^5}{25\sqrt{5}} \geq \frac{|z-1|^4}{4} \Leftrightarrow$$

$$\frac{|2z-1|^{10}}{5^5} \geq \frac{|z-1|^8}{16} \Leftrightarrow$$

$$\left(\frac{(2a-1)^2+4b^2}{5}\right)^5 \geq \left(\frac{(a-1)^2+b^2}{2}\right)^4.$$

From $a^2 + b^2 \geq 1$, we have

$$a^2+b^2-a \geq \frac{a^2+b^2}{2} + \frac{1}{2} - a = \frac{(a-1)^2+b^2}{2} \geq 0$$

So it suffices to prove

$$\left(\frac{4(a^2+b^2-a)+1}{5}\right)^5 \geq \left(a^2+b^2-a\right)^4$$

Since $a^2 + b^2 - a \geq 0$, from AM-GM we have

$$\left(\frac{4(a^2+b^2-a)+1}{5}\right)^5 \geq \left(\frac{5\sqrt[5]{(a^2+b^2-a)^4}}{5}\right)^5 = (a^2+b^2-a)^4$$

Equality holds if and only if $a^2 + b^2 = 1$ and $a^2 + b^2 - a = 1$, which imply $z = \pm i$.

Also solved by Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Ji Eun Kim, Tabor Academy, Marion, MA, USA; Albert Stadler, Herrliberg, Switzerland; Byeong Yeon (Jackie) Ryu, The Hotchkiss School, Lakeville, CT, USA; Michel Faleiros Martins, Petrobras University, Brazil; Li Zhou, Polk State College, Winter Haven, FL, USA; Robert Bosch, Archimedean Academy, USA; Toshihiro Shimizu, Kawasaki, Japan.

S378. In a triangle, let m_a, m_b, m_c be the lengths of the medians, w_a, w_b, w_c be the lengths of the angle bisectors, and r and R be the inradius and circumradius, respectively. Prove that

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \le \left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}}\right)^2.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

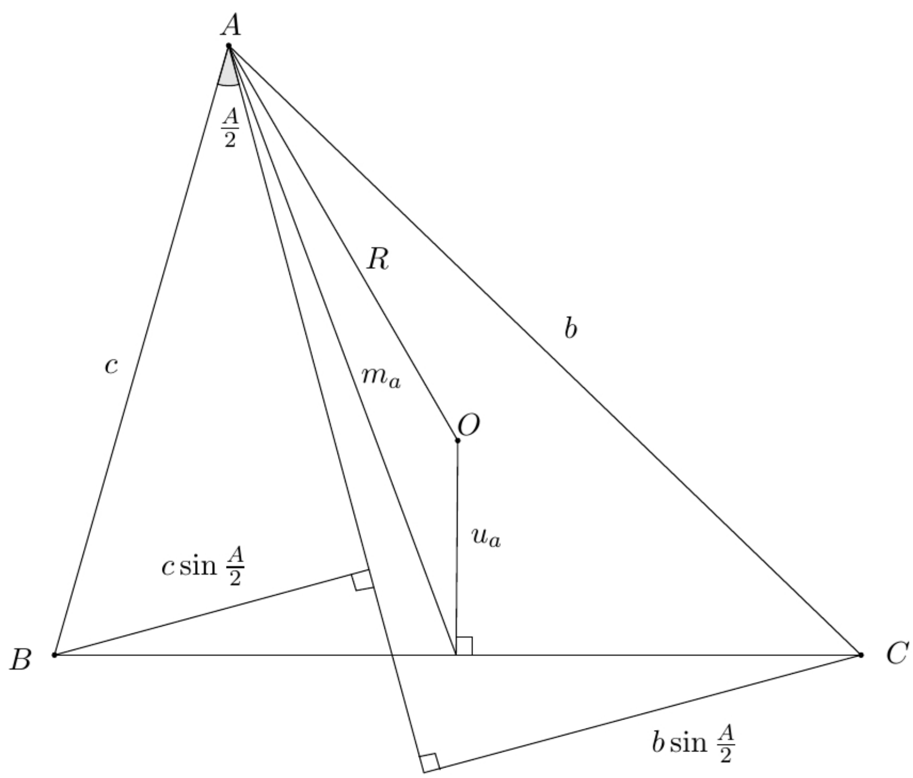
Solution by Michel Faleiros Martins, Petrobras University, Brazil

We will prove the following stronger statement

$$\frac{m_a}{w_a} + \frac{m_a}{w_a} + \frac{m_c}{w_c} \le 1 + \frac{R}{r}.$$

By the Figure we see quickly that

$$w_a c \sin\frac{A}{2} + w_a b \sin\frac{A}{2} = 2K \Rightarrow w_a = \frac{2K}{(b+c)\sin\frac{A}{2}}.$$



Using the similar expressions for w_b and w_c the inequality becomes

$$m_a(b+c)\sin\frac{A}{2} + m_b(c+a)\sin\frac{B}{2} + m_c(a+b)\sin\frac{C}{2} \le 2K\left(1 + \frac{R}{r}\right) = 2K + 2sR.$$

By the triangle inequality we obtain

$$m_a \le R + u_a \quad \text{and} \quad (b+c)\sin\frac{A}{2} \le a,$$

and the other similar expressions

$$m_b \le R + u_b, \quad (a+c)\sin\frac{B}{2} \le b \quad \text{and} \quad m_c \le R + u_c, \quad (a+b)\sin\frac{C}{2} \le c.$$

We conclude that

$$LHS \leq am_a + bm_b + cm_c \leq au_a + bu_b + cu_c + (a+b+c)R = 2K + 2sR.$$

Also solved by Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Dorina Mormocea, National College of Informatics, Piatra Neamt, Romania; Nermin Hodzic, Dobosnica, Bosnia and Herzegovina; Robert Bosch, Archimedean Academy, USA; Toshihiro Shimizu, Kawasaki, Japan.

Undergraduate problems

U373. Prove the following inequality holds for all positive integers $n \geq 2$,

$$\left(1+\frac{1}{1+2}\right)\left(1+\frac{1}{1+2+3}\right)\cdots\left(1+\frac{1}{1+2+\cdots+n}\right) < 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Albert Stadler, Herrliberg, Switzerland

$$\prod_{j=2}^{n}\left(1+\frac{1}{1+2+...+j}\right) = \prod_{j=2}^{n}\left(1+\frac{2}{j(j+1)}\right) = \exp\left(\sum_{j=2}^{n}\log\left(1+\frac{2}{j(j+1)}\right)\right) \leq \exp\left(2\sum_{j=2}^{n}\frac{1}{j(j+1)}\right) =$$

$$\exp\left(2\sum_{j=2}^{n}\left(\frac{1}{j}-\frac{1}{j+1}\right)\right) = \exp\left(1-\frac{2}{n+1}\right) \leq e < 3.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Sutanay Bhattacharya, Bishnupur High School, India; Vincelot Ravoson, Lycée Henri IV, Paris, France; Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Ji Eun Kim, Tabor Academy, MA, USA; Byeong Yeon Ryu, Hotchkiss School, Lakeville, CT, USA; Li Zhou, Polk State College, USA; Arpon Basu, AECS-4, Mumbai, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Michel Faleiros Martins, Petrobras University, Brazil; Robert Bosch, Archimedean Academy, USA; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Alessandro Ventullo, Milan, Italy; Zafar Ahmed, BARC, Mumbai, India and Timilan Mandal, SVNIT, Surat, India; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Toshihiro Shimizu, Kawasaki, Japan.

U374. Let p and q be complex numbers such that two of the zeros a, b, c of the polynomial $x^3 + 3px^2 + 3qx + 3pq = 0$ are equal. Evaluate $a^2b + b^2c + c^2a$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Assume WLOG that $a = b$. Then, $a^2b + b^2c + c^2a = b^3 + b^2c + bc^2$. By Viète's Formulas, we have

$$\begin{aligned} abc &= -3pq \\ ab + bc + ca &= 3q \\ a + b + c &= -3p. \end{aligned}$$

Since $a = b$, we have

$$\begin{aligned} b^2c &= -3pq \\ b^2 + 2bc &= 3q \\ 2b + c &= -3p. \end{aligned}$$

Multiplying side by side the last two equations, we get

$$(b^2 + 2bc)(2b + c) = -9pq.$$

Since $-9pq = 3(-3pq) = 3b^2c$, we get

$$(b^2 + 2bc)(2b + c) = 3b^2c,$$

i.e.

$$b^3 + b^2c + bc^2 = 0.$$

It follows that $a^2b + b^2c + c^2a = 0$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Ji Eun Kim, Tabor Academy, MA, USA; Li Zhou, Polk State College, USA; Arpon Basu, AECS-4, Mumbai, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Catalin Prajitura, College at Brockport, SUNY, NY, USA; Joel Schlosberg, Bayside, NY, USA; Michel Faleiros Martins, Petrobras University, Brazil; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece; Robert Bosch, Archimedean Academy, USA; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawasaki, Japan.

U375. Let

$$a_n = \sum_{k=1}^{n} \sqrt[k]{\frac{(k^2+1)^2}{k^4+k^2+1}}, \qquad n = 1, 2, 3, \ldots.$$

Determine $\lfloor a_n \rfloor$ and evaluate $\lim_{n\to\infty} \frac{a_n}{n}$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Albert Stadler, Herrliberg, Switzerland

We note that

$$1 \le \sqrt[k]{\frac{(k^2+1)^2}{k^4+k^2+1}} = \sqrt[k]{1+\frac{k^2}{k^4+k^2+1}} \le 1 + \frac{k}{k^4+k^2+1} = 1 + \frac{1}{2(k^2-k+1)} - \frac{1}{2(k^2+k+1)} =$$

$$= 1 + \frac{1}{2((k-1)^2+(k-1)+1)} - \frac{1}{2(k^2+k+1)}.$$

Therefore,

$$n \le \sum_{k=1}^{n} \sqrt[k]{\frac{(k^2+1)^2}{k^4+k^2+1}} \le n + \frac{1}{2} - \frac{1}{2(n^2+n+1)},$$

and thus,

$$\lfloor a_n \rfloor = n, \lim_{n\to\infty} \frac{a_n}{n}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Li Zhou, Polk State College, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Juan Manuel Sánchez Gallego, University of Antioquia, Medellín, Colombia; Arkady Alt, San Jose, CA, USA; Arpon Basu, AECS-4, Mumbai, India; Henry Ricardo, New York Math Circle; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Michel Faleiros Martins, Petrobras University, Brazil; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota ,Traian Vuia Technical College, Focsani, Romania; Robert Bosch, Archimedean Academy, USA; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Toshihiro Shimizu, Kawasaki, Japan.

U376. Evaluate

$$\lim_{n\to\infty}\left(1+\sin\frac{1}{n+1}\right)\left(1+\sin\frac{1}{n+2}\right)\cdots\left(1+\sin\frac{1}{n+n}\right).$$

Proposed by Marius Cavachi, Constanţa, România

Solution by Henry Ricardo, New York Math Circle

Letting $P(n)$ denote the given product, we have, since $\ln(1+x)=x+O(x^2)$ for x close to 0 and $\sin(1/(n+k))=O(1/n)$,

$$\begin{aligned}\ln P(n) &= \sum_{k=1}^{n}\ln\left(1+\sin\frac{1}{n+k}\right) = \sum_{k=1}^{n}\left(\sin\frac{1}{n+k}+O\left(\sin^2\frac{1}{n+k}\right)\right)\\ &= \sum_{k=1}^{n}\left(\sin\frac{1}{n+k}+O\left(\frac{1}{n^2}\right)\right)\\ &= \sum_{k=1}^{n}\sin\frac{1}{n+k} + O\left(\frac{1}{n}\right). \qquad (*)\end{aligned}$$

Since $\sin x = x+O(x^3)$ for small values of x, we see that

$$\sum_{k=1}^{n}\sin\frac{1}{n+k} = \sum_{k=1}^{n}\frac{1}{n+k}+\sum_{k=1}^{n}O\left(\frac{1}{n^3}\right) = \sum_{k=1}^{n}\frac{1}{n+k} + O\left(\frac{1}{n^2}\right).$$

Using the well-known result $\lim_{n\to\infty}\sum_{k=1}^{n}1/(n+k)=\ln 2$, we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sin\frac{1}{n+k} = \ln 2.$$

Finally, equation $(*)$ yields

$$\ln(\lim_{n\to\infty}P(n))=\lim_{n\to\infty}(\ln P(n))=\lim_{n\to\infty}\sum_{k=1}^{n}\sin\frac{1}{n+k} + \lim_{n\to\infty}O\left(\frac{1}{n}\right)=\ln 2,$$

so $\lim_{n\to\infty}P(n)=2$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Stanescu Florin, Serban Cioculescu School, Gaesti, Romania; Li Zhou, Polk State College, USA; Zafar Ahmed, BARC, Mumbai, India and Timilan Mandal, SVNIT, Surat, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, CA, USA; Juan Felipe Buitrago Velez, University of Antioquia, Colombia; Moubinool Omarjee, Lycée Henri IV, Paris; Michel Faleiros Martins, Petrobras University, Brazil; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Toshihiro Shimizu, Kawasaki, Japan.

U377. Let m and n be positive integers and let

$$f_k(x) = \underbrace{\sin(\sin(\dots(\sin x)\dots))}_{k \text{ times}}.$$

Evaluate

$$\lim_{x\to 0} \frac{f_m(x)}{f_n(x)}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

It is well known (or it can be easily proved by considering the Taylor expansion of $\sin x$ at $x = 0$) that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, and consequently $\lim_{x\to 0} \frac{x}{\sin x} = 1$. We may generalize this result into the following

Claim: For every positive integer k, we have

$$\lim_{x\to 0} \frac{f_k(x)}{x} = \lim_{x\to 0} \frac{x}{f_k(x)} = 1.$$

Proof: The initial result is clearly the Claim for $k = 1$. If the Claim is true for $k-1$, denote $y = f_{k-1}(x)$, or clearly $\lim_{x\to 0} y = 0$, and $f_k(x) = \sin(f_{k-1}(x))$, or

$$\frac{f_k(x)}{x} = \frac{\sin y}{y} \cdot \frac{f_{k-1}(x)}{x},$$

where the limit of both factors is 1 when $x \to 0$ by hypothesis of induction, and hence the limit of their product, and the limit of the inverse of their product, is also 1. The Claim follows.

If $n = m$, the expression whose limit is asked is clearly 1, and so is trivially its limit. Otherwise, if $m > n$, define $y = f_n(x)$ and $k = m - n$, or $f_m(x) = f_k(y)$, and

$$\lim_{x\to 0} \frac{f_m(x)}{f_n(x)} = \lim_{y\to 0} \frac{f_k(y)}{y} = 1,$$

and similarly when $m < n$ defining $y = f_m(x)$ and $k = n - m$. It follows that

$$\lim_{x\to 0} \frac{f_m(x)}{f_n(x)} = 1.$$

Also solved by Bhattacharya, Bishnupur High School, India; Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Byeong Yeon Ryu, Hotchkiss School, Lakeville, CT, USA; Li Zhou, Polk State College, USA; Zafar Ahmed, BARC, Mumbai, India and Timilan Mandal, SVNIT, Surat, India; Juan Manuel Sánchez Gallego, University of Antioquia, Medellín, Colombia; Adam Krause, College at Brockport, SUNY, NY, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, CA, USA; Henry Ricardo, New York Math Circle; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Robert Bosch, Archimedean Academy, USA; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawasaki, Japan.

U378. Let $f:[0,1]\to\mathbb{R}$ be a continuous function. Prove that

$$\frac{(-1)^{n-1}}{(n-1)!}\int_0^1 f(x)\ln^{n-1}x dx=\int_0^1\int_0^1\dots\int_0^1 f\left(x_1x_2\cdots x_n\right)dx_1dx_2\cdots dx_n.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Li Zhou, Polk State College, USA

For $n=1$, both sides become $\int_0^1 f(x)dx$. As an induction hypothesis, assume that the claim is true for some $n\geq 1$. Then integrating by parts we get

$$\frac{(-1)^n}{n!}\int_0^1 f(x)\ln^n x dx=I(1)-\lim_{x\to 0^+}I(x)+J,$$

where

$$I(x)=\frac{(-1)^n}{n!}\ln^n x\int_0^x f(t)dt,\quad J=\frac{(-1)^{n-1}}{(n-1)!}\int_0^1\frac{\ln^{n-1}x}{x}\int_0^x f(t)dtdx.$$

Now $I(1)=0$, and by L'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 0^+}I(x) &= \frac{(-1)^{n-1}}{n!}\lim_{x\to 0^+}\frac{xf(x)\ln^{n+1}x}{n}=\frac{(-1)^{n-2}(n+1)f(0)}{n}\lim_{x\to 0^+}\frac{x\ln^n x}{n!}\\ &= \frac{(-1)^{n-3}(n+1)f(0)}{n}\lim_{x\to 0^+}\frac{x\ln^{n-1}x}{(n-1)!}=\cdots\\ &= \frac{-(n+1)f(0)}{n}\lim_{x\to 0^+}x\ln x=\frac{(n+1)f(0)}{n}\lim_{x\to 0^+}x=0.\end{aligned}$$

Finally, using the substitution $t=xy$ and applying the induction hypothesis to $g(x)=f(yx)$, we get

$$J=\int_0^1\frac{(-1)^{n-1}}{(n-1)!}\int_0^1 f(yx)\ln^{n-1}xdxdy=\int_0^1\int_0^1\cdots\int_0^1 f(yx_1\cdots x_n)dx_1\cdots dx_ndy,$$

completing the induction.

Also solved by Daniel Lasaosa, Pamplona, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Juan Manuel Sánchez Gallego, University of Antioquia, Medellín, Colombia; Moubinool Omarjee, Lycée Henri IV, Paris, France; Michel Faleiros Martins, Petrobras University, Brazil; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Shohruh Ibragimov, National University of Uzbekistan, Tashkent, Uzbekistan; Toshihiro Shimizu, Kawasaki, Japan.

Olympiad problems

O373. Let $n \geq 3$ be a natural number. On a $n \times n$ table we perform the following operation: choose a $(n-1) \times (n-1)$ square and add or subtract 1 to all its entries. At the beginning all the entries in the table are 0. Is it possible after a finite number of operations to obtain all the numbers from 1 to n^2 in the table?

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Li Zhou, Polk State College, USA

It is possible if and only if $n = 6$. Let a, b, c, and d be the numbers of times when the $(n-1)\times(n-1)$ squares on the upper-left (UL), upper-right (UR), lower-left (LL), and lower-right (LR) are chosen, respectively. Then in modulo 2, the result will be: a, b, c, d appear in 1 cell each; $a+b$, $b+d$, $d+c$, $c+a$ appear in $n-2$ cells each; and $a+b+c+d$ appears in $(n-2)^2$ cells. Note that the set $\{1, 2, \ldots, n^2\}$ has $\lceil n^2/2 \rceil$ odd entries and $\lfloor n^2/2 \rfloor$ even entries.

If $a \equiv b \equiv c \equiv d \equiv 0, 1 \pmod 2$, then we have at least $4(n-2) + (n-2)^2$ even cells and at most 4 odd cells, and $4 - 4(n-2) - (n-2)^2 = 8 - n^2 < 0$.

If $a \equiv b \equiv c \equiv 0$ and $d \equiv 1 \pmod 2$, then we have $2n-1$ even cells and $(n-1)^2$ odd cells, and $(n-1)^2 - (2n-1) \notin \{0, 1\}$.

If $a \equiv b \equiv c \equiv 1$ and $d \equiv 0 \pmod 2$, then we have $2n-3$ even cells and $(n-1)^2 + 2$ odd cells, and $(n-1)^2 + 2 - (2n-3) \notin \{0, 1\}$.

If $a \equiv b \equiv 1$ and $c \equiv d \equiv 0 \pmod 2$, then there are $(n-2)^2$ more even cells than odd cells.

If $a \equiv d \equiv 1$ and $b \equiv c \equiv 0 \pmod 2$, then there are $(n-2)^2 + 2$ even cells and $4(n-2) + 2$ odd cells, and $4(n-2) + 2 - (n-2)^2 - 2 = (n-2)(6-n) \in \{0, 1\}$ if and only if $n = 6$.

Finally, we can see that the result is achievable for $n = 6$ by letting $a = d = 35$, $b = 2$, and $c = 4$. Also, the $b = 2$ times we choose UR, we use $1 + 1 = 2$ for its upper-right corner cell and $1 - 1 = 0$ for its other $(n-1)^2 - 1$ cells; the $c = 4$ times we choose LL, we use 4 for its lower-left corner cell and $2 - 2 = 0$ for its other $(n-1)^2 - 1$ cells; and the $d = 35$ times we choose LR, we use $18 - 17 = 1$ for its $(n-2)^2$ upper-left cells. Note that for any odd integer k, $1 \leq k \leq 35$, the system $x + y = 35$ and $x - y = k$ always has integer solutions with $0 \leq y < x \leq 35$, from which it is obvious that we can obtain all numbers from 1 to 36 in the table.

Also solved by Daniel Lasaosa, Pamplona, Spain; Sutanay Bhattacharya, Bishnupur High School, India; Arpon Basu, AECS-4, Mumbai, India; Joel Schlosberg, Bayside, NY, USA; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Toshihiro Shimizu, Kawasaki, Japan.

O374. Prove that in any triangle,

$$\max\left(|A-B|,|B-C|,|C-A|\right) \le \arccos\left(\frac{4r}{R}-1\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Robert Bosch, Archimedean Academy, USA

We can suppose without loss of generality that $A \le B \le C$. Hence

$$\max\left\{|A-B|,|B-C|,|C-A|\right\} = C-A.$$

So we need to prove the following inequality

$$C-A \le \arccos\left(\frac{4r}{R}-1\right),$$

or equivalently

$$\cos(C-A) \ge \frac{4r}{R}-1 = 4(\cos A+\cos B+\cos C)-5.$$

Note that

$$\begin{aligned}
\cos B &= \cos(180^\circ-(A+C)) = -\cos(A+C) = 2\cos^2\left(\frac{A+C}{2}\right)-1,\\
\cos A+\cos C &= 2\cos\left(\frac{A+C}{2}\right)\cos\left(\frac{C-A}{2}\right),\\
\cos(C-A) &= 2\cos^2\left(\frac{C-A}{2}\right)-1.
\end{aligned}$$

Finally the inequality to be proved becomes

$$\left[\cos\left(\frac{C-A}{2}\right)-2\cos\left(\frac{A+C}{2}\right)\right]^2 \ge 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Michel Faleiros Martins, Petrobras University, Brazil; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Li Zhou, Polk State College, USA; Toshihiro Shimizu, Kawasaki, Japan.

O375. Let a, b, c, d, e, f be real numbers such that $ad - bc = 1$ and $e, f \geq \frac{1}{2}$. Prove that

$$\sqrt{e^2(a^2+b^2+c^2+d^2)+e(ac+bd)}+\sqrt{f^2(a^2+b^2+c^2+d^2)-f(ac+bd)} \geq (e+f)\sqrt{2}$$

.

Proposed by Marius Stănean, Zalău, România

Solution by Michel Faleiros Martins, Petrobras University, Brazil
Using substitution

$$w = a + bi \quad \text{and} \quad z = d + ci$$

then

$$wz = (ad - bc) + (ac + bd)i = \rho(\cos\theta + i\sin\theta).$$

But

$$ad - bc = 1 \Rightarrow \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad 1 = \rho\cos\theta \Rightarrow \rho = \frac{1}{\cos\theta}, \quad ac + bd = \frac{\sin\theta}{\cos\theta}.$$

By the AM-GM inequality

$$a^2+b^2+c^2+d^2 \geq 2\sqrt{(a^2+b^2)(c^2+d^2)} = 2\sqrt{(ad-bc)^2+(ac+bd)^2} = 2\rho = \frac{2}{\cos\theta}.$$

It is sufficient to prove that

$$\sqrt{\frac{2e^2+e\sin\theta}{\cos\theta}}+\sqrt{\frac{2f^2-f\sin\theta}{\cos\theta}}-\sqrt{2}(e+f) \geq 0$$

or

$$\sqrt{2e^2+e\sin\theta}+\sqrt{2f^2-f\sin\theta}-\sqrt{2\cos\theta}(e+f) \geq 0 \quad (\star)$$

Let

$$\Omega_\theta(x) = \sqrt{2x^2+x\sin\theta}-\sqrt{2\cos\theta}\,x.$$

$$\Omega'_\theta(x) = \frac{4x+\sin\theta}{2\sqrt{2x^2+x\sin\theta}}-\sqrt{2\cos\theta}$$

For $x \geq \frac{1}{2}$ ($x = e$ or $x = f$), $x \geq \frac{\sin\theta}{4}$ and $x \geq -\frac{\sin\theta}{4}$.

$$\Omega'_\theta(x) \geq 0 \Leftrightarrow (4x+\sin\theta)^2 \geq 8\cos\theta(2x^2+x\sin\theta)$$

$$\Leftrightarrow (1-\cos\theta)(16x^2+8x\sin\theta+1+\cos\theta) \geq 0$$

$$\Leftrightarrow (4x+\sin\theta)^2+\cos\theta(1+\cos\theta) \geq 0.$$

The last inequality is true for any $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So $\Omega_\theta(x)$ is a increasing function and the same occurs for $\Omega_{-\theta}(x)$. Thus

$$LHS_\star = \Omega_\theta(e)+\Omega_{-\theta}(f) \geq \Omega_\theta\left(\frac{1}{2}\right)+\Omega_{-\theta}\left(\frac{1}{2}\right) = \sqrt{2}\left(\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}-2\sqrt{\cos\theta}\right)$$

$$\geq \sqrt{2}\left(2\sqrt{\sqrt{1-\sin^2\theta}}-2\sqrt{\cos\theta}\right) = 0$$

$$\therefore \Omega_\theta(e)+\Omega_{-\theta}(f) \geq 0.$$

Also solved by Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Toshihiro Shimizu, Kawasaki, Japan.

O376. Let $a_1, a_2, \ldots, a_{100}$ be a permutation of the numbers $1, 2, \ldots, 100$. Let $S_1 = a_1, S_2 = a_1 + a_2, \ldots, S_{100} = a_1 + a_2 + \cdots + a_{100}$. Find the maximum possible number of perfect squares among the numbers $S_1, S_2, \ldots, S_{100}$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Li Zhou, Polk State College, USA

We show that this maximum number is 60. First, $1+2+\cdots+100 = 5050 < 72^2$. Next, there are 71 changes of parity in the sequence $(0^2, 1^2, 2^2, \ldots, 71^2)$, and each change of parity requires adding at least one odd integer from the set $\{1, 3, \ldots, 99\}$. Note that removing one term from the sequence $(1^2, 2^2, \ldots, 71^2)$ eliminates at most two changes of parity. Thus at least 11 terms need to be removed to eliminate $71 - 50 = 21$ changes of parity. Hence, at most 60 squares are possible. Now for $1 \le i \le 50$, let $a_i = 2i - 1$, then $S_i = i^2$, achieving 50 squares. Also, let

$$\begin{aligned}
S_{53} &= S_{50} + 100 + 98 + 6 = 52^2, \\
S_{56} &= S_{53} + 96 + 94 + 22 = 54^2, \\
S_{59} &= S_{56} + 92 + 90 + 38 = 56^2, \\
S_{62} &= S_{59} + 88 + 86 + 54 = 58^2, \\
S_{65} &= S_{62} + 84 + 82 + 70 = 60^2, \\
S_{69} &= S_{65} + 80 + 78 + 76 + 10 = 62^2, \\
S_{74} &= S_{69} + 74 + 72 + 68 + 36 + 2 = 64^2, \\
S_{79} &= S_{74} + 66 + 64 + 62 + 60 + 8 = 66^2, \\
S_{85} &= S_{79} + 58 + 56 + 52 + 50 + 48 + 4 = 68^2, \\
S_{93} &= S_{85} + 46 + 44 + 42 + 40 + 34 + 32 + 26 + 12 = 70^2,
\end{aligned}$$

achieving 10 more squares.

Also solved by Daniel Lasaosa, Pamplona, Spain; Toshihiro Shimizu, Kawasaki, Japan.

O377. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_i b_i > 1$ for all $i \in \{1, 2, \ldots, n\}$. Denote

$$a = \frac{a_1 + a_2 + \cdots + a_n}{n} \qquad \text{and} \qquad b = \frac{b_1 + b_2 + \cdots + b_n}{n}.$$

Prove that

$$\frac{1}{\sqrt{a_1 b_1 - 1}} + \frac{1}{\sqrt{a_2 b_2 - 1}} + \cdots + \frac{1}{\sqrt{a_n b_n - 1}} \geq \frac{n}{\sqrt{ab - 1}}.$$

Proposed by Marius Stănean, Zalău, România

Solution by Daniel Lasaosa, Pamplona, Spain

It is well known that Jensen's inequality applies in multi-variable functions, as long as the Hessian matrix of the function is either positive definite (in which case the inequality holds as for strictly convex single-variable functions) or negative definite (in which case the inequality holds as for strictly concave single-variable functions). Define $f(x, y) = \frac{1}{\sqrt{xy-1}}$, or the Hessian matrix is

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \frac{1}{4\left(\sqrt{xy-1}\right)^5} \begin{pmatrix} 3y^2 & 3x - 2xy + 2 \\ 3y - 2xy + 2 & 3y^2 \end{pmatrix}.$$

Now, since the prefactor is positive since $xy > 1$, the trace has the same sign as $3x^2 + 3y^2$ and is therefore positive, and the determinant has the same sign as

$$9x^2y^2 - (3x - 2xy + 2)(3y - 2xy + 2) = 5x^2y^2 + 8xy - 4 + 6(x + y)(xy - 1),$$

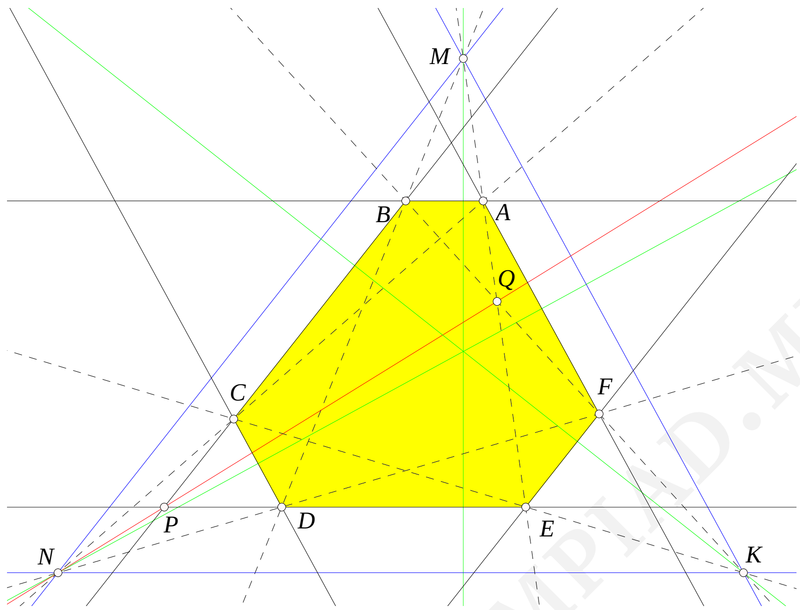
also clearly positive since $x + y > 0$ and $xy > 1$. Or both eigenvalues of the Hessian have positive sum and positive product, hence both are positive, and the Hessian is positive definite. Multi-variable Jensen's inequality therefore holds, and is equivalent to the proposed inequality, where equality holds iff all a_i's are equal and simultaneously all b_i's are equal.

Also solved by Rithvik Pasumarty, Wayzata High School, Plymouth, MN, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Joel Schlosberg, Bayside, NY, USA; Michel Faleiros Martins, Petrobras University, Brazil; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Robert Bosch, Archimedean Academy, USA; Li Zhou, Polk State College, USA; Toshihiro Shimizu, Kawasaki, Japan.

O378. Consider a convex hexagon $ABCDEF$ such that $AB \parallel DE$, $BC \parallel EF$, and $CD \parallel FA$. Let M, N, K be the intersections of lines BD and AE, AC and DF, CE and BF, respectively. Prove that the perpendiculars from M, N, K to the lines AB, CD, EF respectively, are concurrent.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Li Zhou, Polk State College, USA



Since the hexagon $ABCDEF$ has parallel opposite sides, it is well known that its six vertices lie on a conic. By Pascal's theorem, N, P, Q are collinear, where $P = CB \cap DE$ and $Q = EA \cap BF$. Now consider the hexagon $BANPEK$. The sides AN and EK concur with the diagonal BP (at C); the sides NP and KB concur with the diagonal AE (at Q). Thus, by Pappus' theorem, the sides BA and PE must also concur with the diagonal NK (at a point at infinity), that is, $NK \parallel AB$. Likewise, $KM \parallel CD$ and $MN \parallel EF$. Therefore, the perpendiculars from M, N, K to the lines AB, CD, EF respectively, are concurrent at the orthocenter of $\triangle MNK$.

Also solved by Michel Faleiros Martins, Petrobras University, Brazil; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Robert Bosch, Archimedean Academy, USA; Saturnino Campo Ruiz, Salamanca, Spain; Toshihiro Shimizu, Kawasaki, Japan.