Mathematical Excalibur

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Olympiad Corner

Below are the First Round problems of the 26th Iranian Math Olympiad.

Problem 1. In how many ways can one choose $n-3$ diagonals of a regular n-gon, so that no two have an intersection strictly inside the n-gon, and no three form a triangle?

Problem 2. Let ABC be a triangle. Let I_a be the center of its A-excircle. Assume that the A-excircle touches AB and AC in B' and C', respectively. Let I_aB and I_aC intersect $B'C'$ in P and Q, respectively. Let M be the intersection of CP and BQ. Prove that the distance between M and the line BC is equal to the inradius of $\triangle ABC$.

Problem 3. Let a, b, c and d be real numbers, and at least one of c or d is not zero. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(x)=\frac{ax+b}{cx+d}.$$

Assume that $f(x)\neq x$ for every $x\in\mathbb{R}$. Prove that there exists at least one p such that $f^{1387}(p)=p$, then for every x, for which $f^{1387}(x)$ is defined, we have $f^{1387}(x)=x$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***July 10, 2010***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Primitive Roots Modulo Primes

Kin Y. Li

The well-known Fermat's little theorem asserts that if p is a prime number and x is an integer not divisible by p, then

$$x^{p-1}\equiv 1 (\text{mod } p).$$

For positive integer $n>1$ and integer x, if there exists a least positive integer d such that $x^d\equiv 1$ (mod n), then we say d is the <u>order</u> of x (mod n). We denote this by $ord_n(x)=d$. It is natural to ask for a prime p, if there exists x such that $ord_p(x)=p-1$. Such x is called a <u>primitive root (mod p)</u>. Indeed, we have the following

<u>Theorem.</u> For every prime number p, there exists a primitive root (mod p). (We will comment on the proof at the end of the article.)

As a consequence, if x is a primitive root (mod p), then $1, x, x^2, \ldots, x^{p-2}$ (mod p) are distinct and they form a permutation of $1, 2, \ldots, p-1$ (mod p). This is useful in solving some problems in math competitions. The following are some examples. (Below, we will use the common notation $a|b$ to denote a is a divisor of b.)

<u>Example 1.</u> (*2009 Hungary-Israel Math Competition*) Let $p\ge 2$ be a prime number. Determine all positive integers k such that $S_k=1^k+2^k+\cdots+(p-1)^k$ is divisible by p.

<u>Solution.</u> Let x be a primitive root (mod p). Then

$$S_k\equiv 1+x^k+\cdots+x^{(p-2)k} (\text{mod } p).$$

If $p-1\,|\,k$, then $S_k\equiv 1+\cdots+1=p-1$ (mod p). If $p-1\nmid k$, then since $x^k\not\equiv 1$ (mod p) and $x^{(p-1)k}\equiv 1$(mod p), we have

$$S_k\equiv\frac{x^{(p-1)k}-1}{x^k-1}\equiv 0(\text{mod } p).$$

Therefore, all the k's that satisfy the requirement are precisely those integers that are not divisible by $p-1$.

<u>Example 2.</u> Prove that if p is a prime number, then $(p-1)!\equiv -1$ (mod p). This is *<u>Wilson's theorem</u>*.

<u>Solution.</u> The case $p=2$ is easy. For $p>2$, let x be a primitive root (mod p). Then

$$(p-1)!\equiv x^1x^2\cdots x^{p-1}=x^{(p-1)p/2} (\text{mod } p).$$

By the property of x, $w=x^{(p-1)/2}$ satisfies $w\not\equiv 1$(mod p) and $w^2\equiv 1$(mod p). So $w\equiv -1$(mod p). Then

$$(p-1)!\equiv x^{(p-1)p/2}=w^p=-1 (\text{mod } p).$$

<u>Example 3.</u> (1993 *Chinese IMO Team Selection Test*) For every prime number $p\ge 3$, define

$$F(p)=\sum_{k=1}^{(p-1)/2}k^{120},\quad f(p)=\frac{1}{2}-\left\{\frac{F(p)}{p}\right\},$$

where $\{x\}=x-[x]$ is the fractional part of x. Find the value of $f(p)$.

<u>Solution.</u> Let x be a primitive root (mod p). If $p-1\nmid 120$, then $x^{120}\not\equiv 1$(mod p) and $x^{120(p-1)}\equiv 1$(mod p). So

$$F(p)\equiv\frac{1}{2}\sum_{i=1}^{p-1}x^{120i}$$

$$=\frac{x^{120}(x^{120(p-1)}-1)}{2(x^{120}-1)}\equiv 0(\text{mod } p).$$

Then $f(p)=1/2$.

If $p-1\,|\,120$, then $p\in\{3, 5, 7, 11, 13, 31, 41, 61\}$ and $x^{120}\equiv 1$(mod p). So

$$F(p)\equiv\frac{1}{2}\sum_{i=1}^{p-1}x^{120i}=\frac{p-1}{2}(\text{mod } p).$$

Then

$$f(p)=\frac{1}{2}-\frac{p-1}{2p}=\frac{1}{2p}.$$

<u>Example 4.</u> If a and b are nonnegative integers such that $2^a\equiv 2^b$ (mod 101), then prove that $a\equiv b$ (mod 100).

Solution. We first check 2 is a primitive root of (mod 101). If d is the least positive integer such that $2^d \equiv 1$ (mod 101), then dividing 100 by d, we get $100 = qd + r$ for some integers q, r, where $0 \le r < d$. By Fermat's little theorem,

$$1 \equiv 2^{100} = (2^d)^q 2^r \equiv 2^r \pmod{101},$$

which implies the remainder $r = 0$. So $d|100$.

Assume $d < 100$. Then $d|50$ or $d|20$, which implies 2^{20} or $2^{50} \equiv 1$ (mod 101). But $2^{10} = 1024 \equiv 14$(mod 101) implies $2^{20} \equiv 14^2 \equiv -6$ (mod 101) and $2^{50} \equiv 14(-6)^2 \equiv -1$ (mod 101). So $d = 100$.

Finally, $2^a \equiv 2^b$ (mod 101) implies $2^{|a-b|} \equiv 1$ (mod 101). Then as above, dividing $|a\text{-}b|$ by 100, we will see the remainder is 0. Therefore, $a \equiv b$ (mod 100).

Comments: The division argument in the solution above shows if $ord_n(x) = d$, then $x^k \equiv 1$ (mod n) if and only if $d \mid k$. This is useful.

Example 5. (1994 *Putnam Exam*) For any integer a, set

$$n_a = 101a - 100\times 2^a.$$

Show that for $0 \le a,b,c,d \le 99$,

$$n_a + n_b \equiv n_c + n_d \pmod{10100}$$

implies $\{a,b\} = \{c,d\}$.

Solution. Since 100 and 101 are relatively prime, $n_a+n_b \equiv n_c+n_d$ (mod 10100) is equivalent to

$$n_a + n_b \equiv n_c + n_d \pmod{100}$$

and

$$n_a + n_b \equiv n_c + n_d \pmod{101}.$$

As $n_a \equiv a$ (mod 100) and $n_a \equiv 2^a$ (mod 101). These can be simplified to

$$a + b \equiv c + d \pmod{100} \quad (*)$$

and

$$2^a + 2^b \equiv 2^c + 2^d \pmod{101}.$$

Using $2^{100} \equiv 1$(mod 101) and (*), we get

$$2^a 2^b = 2^{a+b} \equiv 2^{c+d} = 2^c 2^d \pmod{101}.$$

Since $2^b \equiv 2^c + 2^d - 2^a$ (mod 101), we get $2^a(2^c + 2^d - 2^a) \equiv 2^c 2^d$ (mod 101). This can be rearranged as

$$(2^a - 2^c)(2^a - 2^d) \equiv 0 \pmod{101}.$$

Then $2^a \equiv 2^c$ (mod 101) or $2^a \equiv 2^d$ (mod 101). By the last example, we get $a \equiv c$ or d (mod 100). Finally, using $a+b \equiv c+d$ (mod 100), we get $\{a,b\} = \{c,d\}$.

Example 6. Find all two digit numbers n (i.e. $n = 10a + b$, where $a, b \in \{0,1,\ldots,9\}$ and $a \ne 0$) such that for all integers k, we have $n \mid k^a - k^b$.

Solution. Clearly, $n = 11, 22, \ldots, 99$ work. Suppose n is such an integer with $a \ne b$. Let p be a prime divisor of n. Let x be a primitive root (mod p). Then $p \mid x^a - x^b$, which implies $x^{|a-b|} \equiv 1$(mod p). By the comment at the end of example 4, we have $p-1 \mid |a\text{-}b| \le 9$. Hence, $p = 2, 3, 5$ or 7.

If $p = 7 \mid n$, then $6 \mid |a\text{-}b|$ implies $n = 28$. Now $k^2 \equiv k^8$ (mod 4) and (mod 7) hold by property of (mod 4) and Fermat's little theorem respectively. So $n = 28$ works.

Similarly the $p = 5$ case will lead to $n = 15$ or 40. Checking shows $n = 15$ works. The $p = 3$ case will lead to $n = 24$ or 48. Checking shows $n = 48$ works. The $p = 2$ case will lead to $n = 16$, 32 or 64, but checking shows none of them works. Therefore, the only answers are 11, 22, …, 99, 28, 15, 48.

Example 7. Let p be an odd prime number. Determine all functions $f: \mathbb{Z} \to \mathbb{Z}$ such that for all $m,n \in \mathbb{Z}$,

(i) if $m \equiv n$ (mod p), then $f(m) = f(n)$ and
(ii) $f(mn) = f(m)f(n)$.

Solution. For such functions, taking $m = n = 0$, we have $f(0) = f(0)^2$, so $f(0) = 0$ or 1. If $f(0) = 1$, then taking $m = 0$, we have $1 = f(0) = f(0)\,f(n) = f(n)$ for all $n \in \mathbb{Z}$, which is clearly a solution.

If $f(0) = 0$, then $n \equiv 0$ (mod p) implies $f(n) = 0$. For $n \not\equiv 0$ (mod p), let x be a primitive root (mod p). Then $n \equiv x^k$(mod p) for some $k \in \{1,2,\ldots,p-1\}$. So $f(n) = f(x^k) = f(x)^k$. By Fermat's little theorem, $x^p \equiv x$ (mod p). This implies $f(x)^p = f(x)$. So $f(x) = 0$, 1 or -1. If $f(x) = 0$, then $f(n) = 0$ for all $n \in \mathbb{Z}$. If $f(x) = 1$, then $f(n) = 1$ for all $n \not\equiv 0$ (mod p). If $f(x) = -1$, then for n congruent to a nonzero square number (mod p), $f(n) = 1$, otherwise $f(n) = -1$.

After seeing how primitive roots can solve problem, it is time to examine the proof of the theorem more closely. We will divide the proofs into a few observations.

For a polynomial $f(x)$ of degree n with coefficients in (mod p), the congruence

$$f(x) \equiv 0 \pmod{p}$$

has at most n solutions (mod p). This can be proved by doing induction on n and imitating the proof for real coefficient polynomials having at most n roots.

If $d|p-1$, then $x^d - 1 \equiv 0$(mod p) has exactly d solutions (mod p). To see this, let $n = (p-1)/d$, then

$$x^{p-1} - 1 = (x^d - 1)(x^{(n-1)d} + x^{(n-2)d} + \cdots + 1).$$

Since $x^{p-1} - 1 \equiv 0$ (mod p) has $p-1$ solutions by Fermat's little theorem, so if $x^d - 1 \equiv 0$ (mod p) has less than d solutions, then

$$(x^d - 1)(x^{(n-1)d} + x^{(n-2)d} + \cdots + 1) \equiv 0 \pmod{p}$$

would have less than $d + (n-1)d = p-1$ solutions, which is a contradiction.

Suppose the prime factorization of $p-1$ is $p_1^{e_1} \cdots p_k^{e_k}$, where p_i's are distinct primes and $e_i \ge 1$. For $i = 1, 2, \ldots, k$, let $m_i = p_i^{e_i}$. Using the observation in the last paragraph, we see there exist $m_i - m_i/p_i > 1$ solutions x_i of equation $x^{m_i} - 1 \equiv 0$ (mod p), which are not solutions of $x^{m_i/p_i} - 1 \equiv 0$ (mod p). It follows that the least positive integer d such that $x_i^d - 1 \equiv 0$ (mod p) is $m_i = p_i^{e_i}$. That means x_i has order $m_i = p_i^{e_i}$ in (mod p).

Let r be the order of $x_i x_j$ in (mod p). By the comment at the end of example 4, we have $r \mid p_i^{e_i} p_j^{e_j}$. Now

$$x_j^{rd} \equiv (x_i^d)^r x_j^{rd} = (x_i x_j)^{rd} \equiv 1 (\text{mod} p),$$

which by the comment again, we get $p_j^{e_j} \mid rd$. Since $p_j^{e_j}$ and $d = p_i^{e_i}$ are relatively prime, we get $p_j^{e_j} \mid r$. Interchanging the roles of p_i and p_j, we also get $p_j^{e_j} \mid r$. So $p_i^{e_i} p_j^{e_j} \mid r$. Then $r = p_i^{e_i} p_j^{e_j}$. So $x = x_1 x_2 \cdots x_k$ will have order $p_1^{e_1} \cdots p_k^{e_k} = p-1$, which implies x is a primitive root (mod p).

For $n > 1$, Euler's theorem asserts that if x and n are relatively prime integers, then $x^{\varphi(n)} \equiv 1$(mod n), where $\varphi(n)$ is the number of positive integers among $1,2,\ldots,n$ that are relatively prime to n. Similarly, we can define x to be a primitive root (mod n) if and only if the least positive integer d satisfying $x^d \equiv 1$(mod n) is $\varphi(n)$. For the inquisitive mind who wants to know for which n, there exists primitive roots (mod n), the answers are $n = 2, 4, p^k$ and $2p^k$, where p is an odd prime. This is much harder to prove. The important thing is for such a primitive root x (mod n), the numbers x^i (mod n) for $i = 1$ to $\varphi(n)$ is a permutation of the $\varphi(n)$ numbers among $1,2,\ldots,n$ that are relatively prime to n.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***July 10, 2010.***

Problem 346. Let k be a positive integer. Divide $3k$ pebbles into five piles (with possibly unequal number of pebbles). Operate on the five piles by selecting three of them and removing one pebble from each of the three piles. If it is possible to remove all pebbles after k operations, then we say it is a *harmonious ending*.

Determine a necessary and sufficient condition for a harmonious ending to exist in terms of the number k and the distribution of pebbles in the five piles.

(*Source*: 2008 *Zhejiang Province High School Math Competition*)

Problem 347. $P(x)$ is a polynomial of degree n such that for all $w\in\{1, 2, 2^2, \ldots, 2^n\}$, we have $P(w) = 1/w$.

Determine $P(0)$ with proof.

Problem 348. In $\triangle ABC$, we have $\angle BAC = 90°$ and $AB < AC$. Let D be the foot of the perpendicular from A to side BC. Let I_1 and I_2 be the incenters of $\triangle ABD$ and $\triangle ACD$ respectively. The circumcircle of $\triangle AI_1I_2$ (with center O) intersects sides AB and AC at E and F respectively. Let M be the intersection of lines EF and BC.

Prove that I_1 or I_2 is the incenter of the $\triangle ODM$, while the other one is an excenter of $\triangle ODM$.

(*Source:* 2008 *Jiangxi Province Math Competition*)

Problem 349. Let $a_1, a_2, \ldots, a_n$ be rational numbers such that for every positive integer m,

$$a_1^m + a_2^m + \cdots + a_n^m$$

is an integer. Prove that $a_1, a_2, \ldots, a_n$ are integers.

Problem 350. Prove that there exists a positive constant c such that for all positive integer n and all real numbers $a_1, a_2, \ldots, a_n$, if

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_n),$$

then

$$\max_{x\in[0,2]}|P(x)| \le c^n \max_{x\in[0,1]}|P(x)|.$$

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Solutions

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Problem 341. Show that there exists an infinite set S of points in the 3-dimensional space such that every plane contains at least one, but not infinitely many points of S.

Solution. **Emanuele NATALE and Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Consider the curve $\sigma : \mathbb{R}\to\mathbb{R}^3$ defined by $\sigma(x) = (x, x^3, x^5)$. Let S be the graph of σ. If $ax+by+cz=d$ is the equation of a plane in $\mathbb{R}^3$, then the intersection of the plane and the curve is determined by the equation

$$ax + bx^3 + cx^5 = d,$$

which has at least one and at most five solutions.

Other commended solvers: **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **D. Kipp JOHNSON** (Valley Catholic School, Beaverton, Oregon, USA) and **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

Problem 342. Let $f(x)=a_nx^n+\cdots+a_1x+p$ be a polynomial with coefficients in the integers and degree $n>1$, where p is a prime number and

$$|a_n|+|a_{n-1}|+\cdots+|a_1| < p.$$

Then prove that $f(x)$ is not the product of two polynomials with coefficients in the integers and degrees less than n.

Solution. **The 6B Mathematics Group** (Carmel Alison Lam Foundation Secondary School), **CHUNG Ping Ngai** (La Salle College, Form 6), **LEE Kai Seng** (HKUST), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy), **Pedro Henrique O. PANTOJA** (University of Lisbon, Portugal).

Let w be a root of $f(x)$ in $\mathbb{C}$. Assume $|w|\le 1$. Using $a_nw^n+\cdots+a_1w+p=0$ and the triangle inequality, we have

$$p = \left|\sum_{i=1}^n a_i w^i\right| \le \sum_{i=1}^n |a_i|\,|w|^i \le \sum_{i=1}^n |a_i|,$$

which contradicts the given inequality. So all roots of $f(x)$ have absolute values greater than 1.

Assume $f(x)$ is the product of two integral coefficient polynomials $g(x)$ and $h(x)$ with degrees less than n. Let b and c be the nonzero coefficients of the highest degree terms of $g(x)$ and $h(x)$ respectively. Then $|b|$ and $|c| \ge 1$. By Vieta's theorem, $|g(0)/b|$ and $|h(0)/c|$ are the products of the absolute values of their roots respectively. Since their roots are also roots of $f(x)$, we have $|g(0)/b| > 1$ and $|h(0)/c| > 1$. Now $p = |f(0)| = |g(0)h(0)|$, but $g(0)$, $h(0)$ are integers and $|g(0)| > |b| \ge 1$ and $|h(0)| > |c| \ge 1$, which contradicts p is prime.

Problem 343. Determine all ordered pairs (a,b) of positive integers such that $a\ne b$, $b^2+a=p^m$ (where p is a prime number, m is a positive integer) and a^2+b is divisible by b^2+a.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

For such (a,b),

$$\frac{a^2+b}{a+b^2} = a - b^2 + \frac{b^4+b}{a+b^2}$$

implies $p^m = a + b^2 \mid b^4 + b = b(b^3+1)$. From $a \ne b$, we get $b < 1+b < a+b^2$. As $\gcd(b, b^3+1) = 1$, so p^m divides $b^3+1 = (b+1)(b^2-b+1)$.

Next, by the Euclidean algorithm, we have $\gcd(b+1,b^2-b+1) = \gcd(b+1,3) \mid 3$.

Assume we have $\gcd(b+1,b^2-b+1)=1$. Then $b^2+a=p^m$ divides only one of $b+1$ or b^2-b+1. However, both $b+1, b^2-b+1 < b^2+a=p^m$. Hence, $b+1$ and b^2-b+1 must be divisible by p. Then the assumption is false and

$$p = \gcd(b+1,b^2-b+1) = 3. \qquad (*)$$

If $m = 1$, then $b^2+a = 3$ has no solution. If $m = 2$, then $b^2+a = 9$ yields $(a,b) = (5,2)$.

For $m \ge 3$, by (*), one of $b+1$ or b^2-b+1 is divisible by 3, while the other one is divisible by 3^{m-1}. Since

$$b+1 < \sqrt{b^2+a}+1 = 3^{m/2}+1 < 3^{m-1},$$

so $3^{m-1} \mid b^2-b+1$. Since $m \ge 3$, we have $b^2 - b + 1 \equiv 0 \pmod 9$. Checking $b \equiv -4, -3, -2, -1, 0, 1, 2, 3, 4 \pmod 9$ shows there cannot be any solution.

Problem 344. $ABCD$ is a cyclic quadrilateral. Let M, N be midpoints of diagonals AC, BD respectively. Lines BA, CD intersect at E and lines AD, BC intersect at F. Prove that

$$\left|\frac{BD}{AC}-\frac{AC}{BD}\right|=\frac{2MN}{EF}.$$

***Solution* 1. LEE Kai Seng** (HKUST).

Without loss of generality, let the circumcircle of $ABCD$ be the unit circle in the complex plane. We have

$M=(A+C)/2$ and $N=(B+D)/2$.

The equations of lines AB and CD are

$$Z+AB\overline{Z}=A+B$$

and

$$Z+CD\overline{Z}=C+D$$

respectively. Solving for Z, we get

$$E=Z=\frac{\overline{A}+\overline{B}-\overline{C}-\overline{D}}{\overline{AB}-\overline{CD}}.$$

Similarly,

$$F=\frac{\overline{A}-\overline{B}-\overline{C}+\overline{D}}{\overline{AD}-\overline{BC}}.$$

In terms of A, B, C, D, we have

$2MN=|A+C-B-D|$,

$$EF=\left|\overline{E}-\overline{F}\right|$$
$$=\left|\frac{A+B-C-D}{AB-CD}-\frac{A-B-C+D}{AD-BC}\right|$$
$$=\left|\frac{(B-D)(C-A)(A+C-B-D)}{(AB-CD)(AD-BC)}\right|.$$

The left and right hand sides of the equation become

$$\left|\frac{BD}{AC}-\frac{AC}{BD}\right|=\left|\frac{|B-D|^2-|A-C|^2}{(A-C)(B-D)}\right|,$$

$$\frac{2MN}{EF}=\left|\frac{(AB-CD)(AD-BC)}{(B-D)(C-A)}\right|.$$

It suffices to show the numerators of the right sides are equal. We have

$$\left||B-D|^2-|A-C|^2\right|$$
$$=\left|(B-D)(\overline{B}-\overline{D})-(A-C)(\overline{A}-\overline{C})\right|$$
$$=\left|A\overline{C}+C\overline{A}-B\overline{D}-D\overline{B}\right|$$

and

$$\left|(AB-CD)(AD-BC)\right|$$
$$=\left|(AB-CD)(\overline{AD}-\overline{BC})\right|$$
$$=\left|B\overline{D}-C\overline{A}-A\overline{C}+D\overline{B}\right|.$$

Comments: For complex method of solving geometry problems, please see <u>*Math Excalibur*</u>, *vol.* 9, *no*. 1.

***Solution* 2. CHUNG Ping Ngai** (La Salle College, Form 6).

Without loss of generality, let $AC > BD$. Since $\angle EAC=\angle EDB$ and $\angle AEC=\angle DEB$, we get $\Delta AEC\sim\Delta DEB$. Then

$$\frac{AE}{DE}=\frac{AC}{DB}=\frac{AM}{DN}=\frac{MC}{DB}$$

and $\angle ECA=\angle EBD$. So $\Delta AEM\sim\Delta DEN$ and $\Delta CEM\sim\Delta BEN$. Similarly, we have $\Delta AFC\sim\Delta BFD$, $\Delta AFM\sim\Delta BFN$ and $\Delta CFM\sim\Delta DFN$. Then

$$\frac{EN}{EM}=\frac{DE}{AE}=\frac{BD}{AC}=\frac{FB}{FA}=\frac{FN}{FM}. \quad (*)$$

Define Q so that $QENF$ is a parallelogram. Let $P=MQ\cap EF$. Then

$\angle EQF=\angle FNE=180°-\angle ENB-\angle FND$
$=180°-\angle EMC-\angle FMC=180°-\angle EMF.$

Hence, M, E, Q, F are concyclic. Then $\angle MEQ=180°-\angle MFQ$.

By (1), $EN\times FM=EM\times FN$. Then

$[EMQ]=½\, EM\times FN\sin\angle MEQ$
$=½\, EN\times FM\sin\angle MFQ=[FMQ]$,

where $[XYZ]$ denotes the area of ΔXYZ. Then $EP=FP$, which implies M, N, P, Q are collinear. Due to M, E, Q, F concyclic, so $\Delta PEM\sim\Delta PQF$ and $\Delta PEQ\sim\Delta PMF$. Then

$$\frac{EM}{EN}=\frac{EM}{QF}=\frac{PM}{PF},\ \frac{FN}{FM}=\frac{QE}{FM}=\frac{QP}{PF}=\frac{NP}{PF}.$$

Using these relations, we have

$$\frac{AC}{BD}-\frac{BD}{AC}=\frac{EM}{EN}-\frac{FN}{FM}$$

$$=\frac{MP}{PF}-\frac{NP}{PF}=\frac{MN}{EF/2},$$

which is the desired equation.

Problem 345. Let $a_1, a_2, a_3, \cdots$ be a sequence of integers such that there are infinitely many positive terms and also infinitely many negative terms. For every positive integer n, the remainders of $a_1, a_2, \cdots, a_n$ upon divisions by n are all distinct. Prove that every integer appears exactly one time in the sequence.

***Solution.* CHUNG Ping Ngai** (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Emanuele NATALE and Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Assume there are $i>j$ such that $a_i=a_j$. Then for $n>i$, $a_i\equiv a_j \pmod n$, which is a contradiction. So any number appears at most once.

Next, for every positive integer n, let $S_n=\{a_1, a_2, \ldots, a_n\}$, max $S_n=a_v$ and min $S_n=a_w$. If $k=a_v-a_w\ge n$, then $k\ge n\ge v, w$ and $a_v\equiv a_w \pmod k$, contradicting the given fact. So

$$\max S_n-\min S_n=a_v-a_w\le n-1.$$

Now $S_n\subseteq[\min S_n, \max S_n]$ and both contain n integers. So the n numbers in S_n are the n consecutive integers from min S_n to max S_n.

Now for every integer m, since there are infinitely many positive terms and also infinitely many negative terms, there exists a_p and a_q such that $a_p<m<a_q$. Let $r>\max\{p,q\}$, then m is in S_r. Therefore, every integer appears exactly one time in the sequence.

Comment: An example of such a sequence is $0, 1, -1, 2, -2, 3, -3, \ldots$.

Olympiad Corner

(continued from page 1)

Problem 4. Let $a\in\mathbb{N}$ be such that for every $n\in\mathbb{N}$, $4(a^n+1)$ is a perfect cube. Show that $a=1$.

Problem 5. We want to choose some phone numbers for a new city. The phone numbers should consist of exactly ten digits, and 0 is not allowed as a digit in them. To make sure that different phone numbers are not confused with each other, we want every two phone numbers to either be different in at least two places or have digits separated by at least 2 units, in at least one of the ten places.

What is the maximum number of phone numbers that can be chosen, satisfying the constraints? In how many ways can one choose this amount of phone numbers?

Problem 6. Let ABC be a triangle and H be the foot of the altitude drawn from A. Let T, T' be the feet of the perpendicular lines drawn from H onto AB, AC, respectively. Let O be the circumcenter of ΔABC, and assume that $AC=2OT$. Prove that $AB=2OT'$.

Mathematical Excalibur

Volume 15, Number 2 *July - September, 2010*

Olympiad Corner

Below are the problems used in the selection of the Indian team for IMO-2010.

Problem 1. Is there a positive integer n, which is a multiple of 103, such that $2^{2n+1} \equiv 2 \pmod{n}$?

Problem 2. Let a, b, c be integers such that b is even. Suppose the equation $x^3+ax^2+bx+c=0$ has roots α, β, γ such that $\alpha^2 = \beta+\gamma$. Prove that α is an integer and $\beta \neq \gamma$.

Problem 3. Let ABC be a triangle in which $BC < AC$. Let M be the midpoint of AB; AP be the altitude from A on to BC; and BQ be the altitude from B on to AC. Suppose QP produced meet AB (extended) in T. If H is the orthocenter of ABC, prove that TH is perpendicular to CM.

Problem 4. Let $ABCD$ be a cyclic quadrilateral and let E be the point of intersection of its diagonals AC and BD. Suppose AD and BC meet in F. Let the midpoints of AB and CD be G and H respectively. If Γ is the circumcircle of triangle EGH, prove that FE is tangent to Γ.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 20, 2010***.

For individual subscription for the next five issues for the 10-11 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Lagrange Interpolation Formula

Kin Y. Li

Let n be a positive integer. If we are given two collections of n+1 real (or complex) numbers $w_0, w_1, \ldots, w_n$ and $c_0, c_1, \ldots, c_n$ with the w_k's distinct, then there *exists* a *unique* polynomial $P(x)$ of degree at most n satisfying $P(w_k) = c_k$ for $k = 0,1,\ldots,n$. The uniqueness is clear since if $Q(x)$ is also such a polynomial, then $P(x)-Q(x)$ would be a polynomial of degree at most n and have roots at the n+1 numbers $w_0, w_1, \ldots, w_n$, which leads to $P(x)-Q(x)$ be the zero polynomial.

Now, to exhibit such a polynomial, we define $f_0(x)=(x-w_1)(x-w_2)\cdots(x-w_n)$ and similarly for i from 1 to n, define

$f_i(x)=(x-w_0)\cdots(x-w_{i-1})(x-w_{i+1})\cdots(x-w_n)$.

Observe that $f_i(w_k) = 0$ if and only if $i \neq k$. Using this, we see

$$P(x) = \sum_{i=0}^{n} c_i \frac{f_i(x)}{f_i(w_i)}$$

satisfies $P(w_k) = c_k$ for $k = 0,1,\ldots,n$. This is the famous *Lagrange interpolation formula*.

Below we will present some examples of using this formula to solve math problems.

Example 1. (*Romanian Proposal to* 1981 *IMO*) Let P be a polynomial of degree n satisfying for $k = 0,1,\ldots,n$,

$$P(k) = \binom{n+1}{k}^{-1}.$$

Determine $P(n+1)$.

Solution. For $k = 0,1,\ldots,n$, let $w_k=k$ and

$$c_k = \binom{n+1}{k}^{-1} = \frac{k!(n+1-k)!}{(n+1)!}.$$

Define $f_0, f_1, \ldots, f_n$ as above. We get

$$f_k(k) = (-1)^{n-k}k!(n-k)!$$

and

$$f_k(n+1) = \frac{(n+1)!}{(n+1-k)}.$$

By the Lagrange interpolation formula,

$$P(n+1) = \sum_{k=0}^{n} c_k \frac{f_k(n+1)}{f_k(k)} = \sum_{k=0}^{n} (-1)^{n-k},$$

which is 0 if n is odd and 1 if n is even.

Example 2. (*Vietnamese Proposal to* 1977 *IMO*) Suppose $x_0, x_1, \ldots, x_n$ are integers and $x_0 > x_1 > \cdots > x_n$. Prove that one of the numbers $|P(x_0)|, |P(x_1)|, \ldots, |P(x_n)|$ is at least $n!/2^n$, where $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ is a polynomial with real coefficients.

Solution. Define $f_0, f_1, \ldots, f_n$ using $x_0, x_1, \ldots, x_n$. By the Lagrange interpolation formula, we have

$$P(x) = \sum_{i=0}^{n} P(x_i) \frac{f_i(x)}{f_i(x_i)},$$

since both sides are polynomials of degrees at most n and are equal at $x_0, x_1, \ldots, x_n$. Comparing coefficients of x^n, we get

$$1 = \sum_{i=0}^{n} \frac{P(x_i)}{f_i(x_i)}.$$

Since $x_0, x_1, \ldots, x_n$ are strictly decreasing integers, we have

$$|f_i(x_i)| = \prod_{j=0}^{i-1} |x_j - x_i| \prod_{j=i+1}^{n} |x_j - x_i|$$
$$\geq i!(n-i)! = \frac{1}{n!}\binom{n}{i}.$$

Let the maximum of $|P(x_0)|, |P(x_1)|, \ldots, |P(x_n)|$ be $|P(x_k)|$. By the triangle inequality, we have

$$1 \leq \sum_{i=0}^{n} \frac{|P(x_i)|}{|f_i(x_i)|} \leq \frac{|P(x_k)|}{n!} \sum_{i=0}^{n} \binom{n}{i} = \frac{2^n |P(x_k)|}{n!}.$$

Then $|P(x_k)| \geq n!/2^n$.

Example 3. Let P be a point on the plane of ΔABC. Prove that

$$\frac{PA}{BC} + \frac{PB}{CA} + \frac{PC}{AB} \geq \sqrt{3}.$$

Solution. We may take the plane of ΔABC to be the complex plane and let P, A, B, C be corresponded to the complex numbers w, w_1, w_2, w_3 respectively. Then $PA=|w-w_1|$, $BC=|w_2-w_3|$, etc.

Now the only polynomial $P(x)$ of degree at most 2 that equals 1 at w_1, w_2, w_3 is the constant polynomial $P(x)\equiv 1$. So, expressing $P(x)$ by the Lagrange interpolation formula, we have

$$\frac{(x-w_1)(x-w_2)}{(w_3-w_1)(w_3-w_2)}+\frac{(x-w_2)(x-w_3)}{(w_1-w_2)(w_1-w_3)}+\frac{(x-w_3)(x-w_1)}{(w_2-w_3)(w_2-w_1)}\equiv 1.$$

Next, setting $x = w$ and applying the triangle inequality, we get

$$\frac{PA}{BC}\frac{PB}{CA}+\frac{PB}{CA}\frac{PC}{AB}+\frac{PC}{AB}\frac{PA}{BC}\ge 1. \quad (*)$$

The inequality $(r+s+t)^2\ge 3(rs+st+tw)$, after subtracting the two sides, reduces to $[(r-s)^2+(s-t)^2+(t-r)^2]/2\ge 0$, which is true. Setting $r=PA/BC$, $s=PB/CA$ and $t=PC/AB$, we get

$$\left(\frac{PA}{BC}+\frac{PB}{CA}+\frac{PC}{AB}\right)^2\ge 3\left(\frac{PA}{BC}\frac{PB}{CA}+\frac{PB}{CA}\frac{PC}{AB}+\frac{PC}{AB}\frac{PA}{BC}\right).$$

Taking square roots of both sides and applying (*), we get the desired inequality.

Example 4. (2002 *USAMO*) Prove that any *monic* polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Solution. Suppose $F(x)$ is a monic real polynomial. Choose real $y_1, y_2, \ldots, y_n$ such that for odd i, $y_i<\min\{0,2F(i)\}$ and for even i, $y_i>\max\{0,2F(i)\}$.

By the Lagrange interpolation formula, there is a polynomial of degree less than n such that $P(i)=y_i$ for $i=1,2,\ldots,n$. Let

$$G(x) = P(x)+(x-1)(x-2)\cdots(x-n)$$

and

$$H(x) = 2F(x)-G(x).$$

Then $G(x)$ and $H(x)$ are monic real polynomials of degree n and their average is $F(x)$.

As $y_1, y_3, y_5, \ldots<0$ and $y_2, y_4, y_6, \cdots>0$, $G(i)=y_i$ and $G(i+1)=y_{i+1}$ have opposite signs (hence $G(x)$ has a root in $[i,i+1]$) for $i=1,2,\ldots,n-1$. So $G(x)$ has at least $n-1$ real roots. The other root must also be real since non-real roots come in conjugate pair. Therefore, all roots of $G(x)$ are real.

Similarly, for odd i, $G(i) = y_i < 2F(i)$ implies $H(i)=2F(i)-G(i)>0$ and for even i, $G(i)=y_i>2F(i)$ implies $H(i)=2F(i)-G(i)<0$. These imply $H(x)$ has n real roots by reasoning similar to $G(x)$.

Example 5. Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ be real numbers such that $b_i-a_j\neq 0$ for i,j=1,2,3,4. Suppose there is a unique set of numbers X_1, X_2, X_3, X_4 such that

$$\frac{X_1}{b_1-a_1}+\frac{X_2}{b_1-a_2}+\frac{X_3}{b_1-a_3}+\frac{X_4}{b_1-a_4}=1,$$

$$\frac{X_1}{b_2-a_1}+\frac{X_2}{b_2-a_2}+\frac{X_3}{b_2-a_3}+\frac{X_4}{b_2-a_4}=1,$$

$$\frac{X_1}{b_3-a_1}+\frac{X_2}{b_3-a_2}+\frac{X_3}{b_3-a_3}+\frac{X_4}{b_3-a_4}=1,$$

$$\frac{X_1}{b_4-a_1}+\frac{X_2}{b_4-a_2}+\frac{X_3}{b_4-a_3}+\frac{X_4}{b_4-a_4}=1.$$

Determine $X_1+X_2+X_3+X_4$ in terms of the a_i's and b_i's.

Solution. Let

$$P(x)=\prod_{i=1}^4(x-a_i)-\prod_{i=1}^4(x-b_i).$$

Then the coefficient of x^3 in $P(x)$ is

$$\sum_{i=1}^4 b_i-\sum_{i=1}^4 a_i.$$

Define f_1, f_2, f_3, f_4 using a_1, a_2, a_3, a_4 as above to get the Lagrange interpolation formula

$$P(x)=\sum_{i=i}^4 P(a_i)\frac{f_i(x)}{f_i(a_i)}.$$

Since the coefficient of x^3 in $f_i(x)$ is 1, the coefficient of x^3 in $P(x)$ is also

$$\sum_{i=1}^4\frac{P(a_i)}{f_i(a_i)}.$$

Next, observe that $P(b_j)/f_i(b_j) = b_j - a_i$, which are the denominators of the four given equations! For j = 1,2,3,4, setting $x = b_j$ in the interpolation formula and dividing both sides by $P(b_j)$, we get

$$1=\sum_{i=i}^4\frac{P(a_i)}{P(b_j)}\frac{f_i(b_j)}{f_i(a_i)}=\sum_{i=1}^4\frac{P(a_i)/f_i(a_i)}{b_j-a_i}.$$

Comparing with the given equations, by uniqueness, we get $X_i=P(a_i)/f_i(a_i)$ for i = 1,2,3,4. So

$$\sum_{i=1}^4 X_i=\sum_{i=1}^4\frac{P(a_i)}{f_i(a_i)}=\sum_{i=1}^4 b_i-\sum_{i=}^4 a_i.$$

Comment: This example is inspired by problem 15 of the 1984 American Invitational Mathematics Examination.

Example 6. (*Italian Proposal to* 1997 *IMO*) Let p be a prime number and let $P(x)$ be a polynomial of degree d with integer coefficients such that:

(i) $P(0) = 0$, $P(1) = 1$;
(ii) for every positive integer n, the remainder of the division of $P(n)$ by p is either 0 or 1.

Prove that $d\ge p-1$.

Solution. By (i) and (ii), we see

$P(0)+P(1)+\cdots+P(p-1)\equiv k \pmod p$ (#)

for some $k\in\{1,2,\ldots,p-1\}$.

Assume $d\le p-2$. Then $P(x)$ will be uniquely determined by the values $P(0)$, $P(1)$, …, $P(p-2)$. Define $f_0, f_1, \ldots, f_{p-2}$ using 0, 1, …, $p-2$ as above to get the Lagrange interpolation formula

$$P(x)=\sum_{k=0}^{p-2}P(k)\frac{f_k(x)}{f_k(k)}.$$

As in example (1), we have

$$f_k(k) = (-1)^{p-2-k}k!(p-2-k)!,$$

$$f_k(p-1)=\frac{(p-1)!}{p-1-k}$$

and so

$$P(p-1)=\sum_{k=0}^{p-2}P(k)(-1)^{p-k}\binom{p-1}{k}.$$

Next, we claim that

$$\binom{p-1}{k}\equiv(-1)^k \pmod p \quad for \quad 0\le k\le p-2.$$

This is true for $k = 0$. Now for $0 < i < p$,

$$\binom{p}{i}=\frac{p!}{i!(p-i)!}\equiv 0 \pmod p$$

because p divides $p!$, but not $i!(p-i)!$. If the claim is true for k, then

$$\binom{p-1}{k+1}=\binom{p}{k+1}-\binom{p-1}{k}\equiv(-1)^{k+1} \pmod p$$

and the induction step follows. Finally the claim yields

$$P(p-1)\equiv(-1)^p\sum_{k=0}^{p-2}P(k) \pmod p.$$

So $P(0)+P(1)+\cdots+P(p-1)\equiv 0 \pmod p$, a contradiction to (#) above.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 20, 2010.***

Problem 351. Let S be a unit sphere with center O. Can there be three arcs on S such that each is a 300˚ arc on some circle with O as center and no two of the arcs intersect?

Problem 352. (*Proposed by Pedro Henrique O. PANTOJA*, *University of Lisbon, Portugal*) Let a, b, c be real numbers that are at least 1. Prove that

$$\frac{a^2bc}{\sqrt{bc}+1}+\frac{b^2ca}{\sqrt{ca}+1}+\frac{c^2ab}{\sqrt{ab}+1}\ge\frac{3}{2}.$$

Problem 353. Determine all pairs (x, y) of integers such that $x^5-y^2=4$.

Problem 354. For 20 boxers, find the least number n such that there exists a schedule of n matches between pairs of them so that for every three boxers, two of them will face each other in one of the matches.

Problem 355. In a plane, there are two *similar* convex quadrilaterals $ABCD$ and $AB_1C_1D_1$ such that C, D are inside $AB_1C_1D_1$ and B is outside $AB_1C_1D_1$ Prove that if lines BB_1, CC_1 and DD_1 concur, then $ABCD$ is cyclic. Is the converse also true?

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Solutions

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Problem 346. Let k be a positive integer. Divide $3k$ pebbles into five piles (with possibly unequal number of pebbles). Operate on the five piles by selecting three of them and removing one pebble from each of the three piles. If it is possible to remove all pebbles after k operations, then we say it is a *harmonious ending*.

Determine a necessary and sufficient condition for a harmonious ending to exist in terms of the number k and the distribution of pebbles in the five piles.

(*Source*: 2008 *Zhejiang Province High School Math Competition*)

Solution. **CHOW Tseung Man** (True Light Girl's College), **CHUNG Ping Ngai** (MIT Year 1), **HUNG Ka Kin Kenneth** (CalTech Year 1).

The necessary and sufficient condition is every pile has at most k pebbles in the beginning.

The necessity is clear. If there is a pile with more than k pebbles in the beginning, then in each of the k operations, we can only remove at most 1 pebble from that pile, hence we cannot empty the pile after k operations.

For the sufficiency, we will prove by induction. In the case k=1, three pebbles are distributed with each pebble to a different pile. So we can finish in one operation. Suppose the cases less than k are true. For case k, since $3k$ pebbles are distributed. So at most 3 piles have k pebbles. In the first operation, we remove one pebble from each of the three piles with the maximum numbers of pebbles. This will take us to a case less than k. We are done by the inductive assumption.

Problem 347. $P(x)$ is a polynomial of degree n such that for all $w\in\{1, 2, 2^2, \ldots, 2^n\}$, we have $P(w) = 1/w$.

Determine $P(0)$ with proof.

***Solution* 1.** **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy). **William CHAN Wai-lam** (Carmel Alison Lam Foundation Secondary School) and **Thien Nguyen** (Nguyen Van Thien Luong High School, Dong Nai Province, Vietnam).

Let $Q(x) = xP(x)-1 = a(x-1)(x-2)\cdots(x-2^n)$. For $x\neq 1, 2, 2^2, \ldots, 2^n$,

$$\frac{Q'(x)}{Q(x)}=\frac{1}{x-1}+\frac{1}{x-2}+\cdots+\frac{1}{x-2^n}.$$

Since $Q(0)= -1$ and $Q'(x)=P(x)+xP'(x)$,

$$P(0)=Q'(0)=-\frac{Q'(0)}{Q(0)}=\sum_{k=0}^{n}\frac{1}{2^k}=2-\frac{1}{2^n}.$$

***Solution* 2.** **CHUNG Ping Ngai** (MIT Year 1), **HUNG Ka Kin Kenneth** (CalTech Year 1), **Abby LEE** (SKH Lam Woo Memorial Secondary School, Form 5) and **WONG Kam Wing** (HKUST, Physics, Year 2).

Let $Q(x) = xP(x)-1 = a(x-1)(x-2)\cdots(x-2^n)$. Now $Q(0) = -1 = a(-1)^{n+1}2^s$, where $s = 1+2+\cdots+n$. So $a = (-1)^n\, 2^{-s}$. Then $P(0)$ is the coefficient of x in $Q(x)$, which is

$$a(-1)^n(2^s+2^{s-1}+\cdots+2^{s-n})=\sum_{k=0}^{n}\frac{1}{2^k}=2-\frac{1}{2^n}.$$

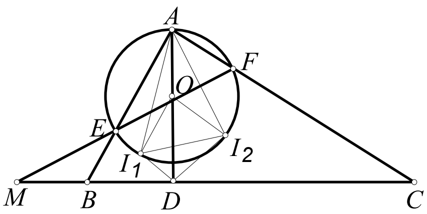
Other commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil),

Problem 348. In ΔABC, we have $\angle BAC$ = 90˚ and $AB < AC$. Let D be the foot of the perpendicular from A to side BC. Let I_1 and I_2 be the incenters of ΔABD and ΔACD respectively. The circumcircle of ΔAI_1I_2 (with center O) intersects sides AB and AC at E and F respectively. Let M be the intersection of lines EF and BC.

Prove that I_1 or I_2 is the incenter of the ΔODM, while the other one is an excenter of ΔODM.

(*Source:* 2008 *Jiangxi Province Math Competition*)

Solution. **CHOW Tseung Man** (True Light Girl's College).



We *claim* EF intersects AD at O. Since $\angle EAF$=90˚, EF is a diameter through O. Next we will show O is on AD.

Since AI_1, AI_2 bisect $\angle BAD$, $\angle CAD$ respectively, we get $\angle I_1AI_2$=45˚. Then $\angle I_1OI_2$=90˚. Since OI_1=OI_2, $\angle OI_1I_2$=45˚. Also, DI_1, DI_2 bisect $\angle BDA$, $\angle CDA$ respectively implies $\angle I_1DI_2$=90˚. Then D, I_1, O, I_2 are concyclic. So

$$\angle ODI_2=\angle OI_1I_2=45^\circ=\angle ADI_2.$$

Then O is on AD and the claim is true.

Since $\angle EOI_1 = 2\angle EAI_1 = 2\angle DAI_1 = \angle DOI_1$ and I_1 is on the angle bisector of $\angle ODM$, we see I_1 is the incenter of ΔODM. Similarly, replacing E by F and I_1 by I_2 in the last sentence, we see I_2 is an excenter of ΔODM.

Other commended solvers: **CHUNG Ping Ngai** (MIT Year 1), **HUNG Ka Kin Kenneth** (CalTech Year 1) and **Abby LEE** (SKH Lam Woo Memorial Secondary School, Form 5).

Problem 349. Let $a_1, a_2, \ldots, a_n$ be rational numbers such that for every positive integer m,

$$a_1^m+a_2^m+\cdots+a_n^m$$

is an integer. Prove that $a_1, a_2, \ldots, a_n$ are integers.

Solution. **CHUNG Ping Ngai** (MIT Year 1) and **HUNG Ka Kin Kenneth** (CalTech Year 1).

We may first remove all the integers among $a_1, a_2, \ldots, a_n$ since their m-th powers are integers, so the rest of $a_1, a_2, \ldots, a_n$ will still have the same property. Hence, without loss of generality, we may assume all $a_1, a_2, \ldots, a_n$ are rational numbers and not integers. First write every a_i in simplest term. Let Q be their least common denominator and for all $1 \le i \le n$, let $a_i = k_i/Q$. Take a prime factor p of Q. Then p is not a prime factor of one of the k_i's. So one of the remainders r_i when k_i is divided by p is nonzero! Since $k_i \equiv r_i \pmod p$, so for every positive integer m,

$$\sum_{i=1}^{n} r_i^m \equiv \sum_{i=1}^{n} k_i^m = \left(\sum_{i=1}^{n} a_i^m\right) Q^m \equiv 0 \pmod{p^m}.$$

This implies $p^m \le \sum_{i=1}^{n} r_i^m$. Since $r_i < p$,

$$1 \le \lim_{m\to\infty} \frac{1}{p^m} \sum_{i=1}^{n} r_i^m = \lim_{m\to\infty} \sum_{i=1}^{n} \left(\frac{r_i}{p}\right)^m = 0,$$

which is a contradiction.

Comments: In the above solution, it does not need all positive integers m, just an infinite sequence of positive integers m with the given property will be sufficient.

Problem 350. Prove that there exists a positive constant c such that for all positive integer n and all real numbers $a_1, a_2, \ldots, a_n$, if

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_n),$$

then

$$\max_{x\in[0,2]} |P(x)| \le c^n \max_{x\in[0,1]} |P(x)|.$$

(*Ed.*-Both solutions below show the conclusion holds for any polynomial!)

Solution 1. **LEE Kai Seng.**

Let S be the maximum of $|P(x)|$ for all $x \in [0,1]$. For $i = 0,1,2,\ldots,n$, let $b_i = i/n$ and

$$f_i(x) = (x - b_0)\cdots(x - b_{i-1})(x - b_{i+1})\cdots(x - b_n).$$

By the Lagrange interpolation formula, for all real x,

$$P(x) = \sum_{i=0}^{n} P(b_i) \frac{f_i(x)}{f_i(b_i)}.$$

For every $w \in [0,2]$, $|w - b_k| \le |2 - b_k|$ for all $k = 0,1,2,\ldots,n$. So

$$|f_i(w)| \le |f_i(2)| = \prod_{i=0}^{n} \left(2 - \frac{i}{n}\right)$$

$$= \frac{2n(2n-1)(2n-2)\cdots(n+1)}{n^n}$$

$$= \frac{(2n)!}{n!\,n^n}.$$

Also, $|P(b_i)| \le S$ and

$$|f_i(b_i)| = \frac{i!(n-i)!}{n^n}.$$

By the triangle inequality,

$$|P(w)| \le \sum_{i=0}^{n} |P(b_i)| \frac{|f_i(w)|}{|f_i(b_i)|} \le S \sum_{i=0}^{n} \binom{2n}{i}\binom{2n-i}{n}.$$

Finally,

$$\sum_{i=0}^{n} \binom{2n}{i}\binom{2n-i}{n} \le \sum_{i=0}^{n} \binom{2n}{i}\binom{2n}{n} = 2^{2n}\binom{2n}{n} \le 2^{4n}.$$

Then

$$\max_{w\in[0,2]} |P(w)| \le 2^{4n} S = 16^n \max_{x\in[0,1]} |P(x)|.$$

Solution 2. **G.R.A.20 Problem Solving Group** (Roma, Italy).

For a bounded closed interval I and polynomial $f(x)$, let $\|f\|_I$ denote the maximum of $|f(x)|$ for all x in I. The Chebyschev polynomial of order n is defined by $T_0(x) = 1$, $T_1(x) = x$ and

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \text{ for } n \ge 2.$$

(*Ed.*-By induction, we can obtain

$$T_n(x) = 2^n x^n + c_{n-1}x^{n-1} + \cdots + c_0$$

and $T_n(\cos\theta) = \cos n\theta$. So $T_n(\cos(\pi k/n)) = (-1)^k$, which implies all n roots of $T_n(x)$ are in $(-1,1)$ as it changes sign n times.)

It is known that for any polynomial $Q(x)$ with degree at most $n>0$ and all $t \notin [-1,1]$,

$$|Q(t)| \le \|Q\|_{[-1,1]}\, |T_n(t)|. \qquad (!)$$

To see this, we may assume $\|Q\|_{[-1,1]} = 1$ by dividing $Q(x)$ by such maximum. Assume $x_0 \notin [-1,1]$ and $|Q(x_0)| > |T_n(x_0)|$. Let

$$a = T(x_0)/Q(x_0) \text{ and } R(x) = aQ(x) - T_n(x).$$

For $k = 0, 1, 2, \cdots, n$, since $T_n(\cos(\pi k/n)) = (-1)^k$ and $|a|<1$, we see $R(\cos(\pi k/n))$ is positive or negative depending on whether k is odd or even. (In particular, $R(x) \not\equiv 0$.) By continuity, $R(x)$ has $n+1$ distinct roots on $[-1,1] \cup \{x_0\}$, which contradicts the degree of $R(x)$ is at most n.

Next, for the problem, we claim that for every $t \in [1,2]$, we have $|P(t)| \le 6^n \|P\|_{[0,1]}$.

(*Ed.*-Observe that the change of variable $t = (s+1)/2$ is a bijection between $s \in [-1,1]$ and $t \in [0,1]$. It is also a bijection between $s \in [1,3]$ and $t \in [1,2]$.) By letting $Q(s) = P((s+1)/2)$, the claim is equivalent to proving that for every $s \in [1,3]$, we have $|Q(s)| \le 6^n \|Q\|_{[-1,1]}$. By (!) above, it suffices to show that $|T_n(s)| \le 6^n$ for every $s \in [1,3]$.

Clearly, $|T_0(s)| = 1 = 6^0$. For $n=1$ and $s \in [1,3]$, $|T_1(s)| = s \le 3 < 6$. Next, since the largest root of T_n is less than 1, we see all $T_n(s) > 0$ for all $s \in [1,3]$. Suppose cases $n-2$ and $n-1$ are true. Then for all $s \in [1,3]$, we have $2sT_{n-1}(s)$, $T_{n-2}(s) > 0$ and so

$$\begin{aligned} |T_n(s)| &= |2sT_{n-1}(s) - T_{n-2}(s)| \\ &\le \max(2sT_{n-1}(s), T_{n-2}(s)) \\ &\le \max(6\cdot 6^{n-1}, 6^{n-2}) = 6^n. \end{aligned}$$

This finishes everything.

Olympiad Corner

(continued from page 1)

Problem 5. Let $A = (a_{jk})$ be a 10×10 array of positive real numbers such that the sum of the numbers in each row as well as in each column is 1. Show that there exist $j<k$ and $l<m$ such that

$$a_{jl}a_{km} + a_{jm}a_{kl} \ge \frac{1}{50}.$$

Problem 6. Let ABC be a triangle. Let AD, BE, CF be cevians such that $\angle BAD = \angle CBE = \angle ACF$. Suppose these cevians concur at a point Ω. (Such a point exists for each triangle and it is called a Brocard point.) Prove that

$$\frac{A\Omega^2}{BC^2} + \frac{B\Omega^2}{CA^2} + \frac{C\Omega^2}{AB^2} \ge 1.$$

(*Ed.*-A cevian is a line segment which joins a vertex of a triangle to a point on the opposite side or its extension.)

Problem 7. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y) + xy = f(x)f(y)$$

for all reals x,y.

Problem 8. Prove that there are infinitely many positive integers m for which there exist consecutive odd positive integers p_m, q_m $(=p_m+2)$ such that the pairs (p_m, q_m) are all distinct and

$$p_m^2 + p_m q_m + q_m^2,\quad p_m^2 + mp_m q_m + q_m^2$$

are both perfect squares.

Mathematical Excalibur

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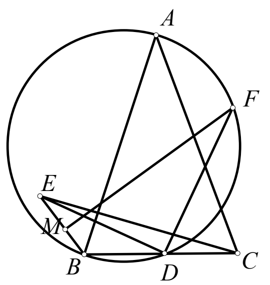
Olympiad Corner

Below are the problems of the 2010 Chinese Girls' Math Olympiad, which was held on August 10-11, 2010.

Problem 1. Let n be an integer greater than two, and let $A_1, A_2, \ldots, A_{2n}$ be pairwise disjoint nonempty subsets of $\{1,2,\ldots,n\}$. Determine the maximum value of $\sum_{i=1}^{2n}\frac{|A_i\cap A_{i+1}|}{|A_i|\cdot|A_{i+1}|}$. (Here we set $A_{2n+1}=A_1$. For a set X, let $|X|$ denote the number of elements in X.)

Problem 2. In $\triangle ABC$, $AB=AC$. Point D is the midpoint of side BC. Point E lies outside $\triangle ABC$ such that $CE\perp AB$ and $BE=BD$. Let M be the midpoint of segment BE. Point F lies on the minor arc AD of the circumcircle of $\triangle ABD$ such that $MF\perp BE$. Prove that $ED\perp FD$.



(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 14, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO Shortlisted Problems

Kin Y. Li

Every year, before the IMO begins, a problem selection committee collects problem proposals from many nations. Then it prepares a short list of problems for the leaders to consider when the leaders meet at the IMO site. The following were some of the interesting shortlisted problems in past years that were not chosen. Perhaps some of the ideas may reappear in later proposals in coming years.

Example 1. (*1985 IMO Proposal by Israel*) For which integer $n \geq 3$ does there exist a regular n-gon in the plane such that all its vertices have integer coordinates in a rectangular coordinate system?

Solution. Let A_i have coordinates (x_i,y_i), where x_i, y_i are integers for $i=1,2,\cdots,n$. In the case $n = 3$, if $A_1A_2A_3$ is equilateral, then on one hand, its area is

$$\frac{\sqrt{3}}{4}A_1A_2^2=\frac{\sqrt{3}}{4}\left((x_1-x_2)^2+(y_1-y_2)^2\right),$$

which is irrational. On the other hand, its area is also

$$\frac{\left|\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\right|}{2}=\pm\frac{1}{2}\begin{vmatrix}x_2-x_1 & y_2-y_1\\ x_3-x_1 & y_3-y_1\end{vmatrix},$$

which is rational. Hence, the case $n = 3$ leads to contradiction. The case $n = 4$ is true by taking (0,0),(0,1),(1,1) and (1,0). The case $n = 6$ is false since $A_1A_3A_5$ would be equilateral.

For the other cases, suppose $A_1A_2\cdots A_n$ is such a regular n-gon with minimal side length. For $i=1,2,\cdots,n$, define point B_i so that $A_iA_{i+1}A_{i+2}B_i$ is a parallelogram (where $A_{n+1}=A_1$ and $A_{n+2}=A_2$). Since $A_{i+1}A_{i+2}$ is parallel to A_iA_{i+3} (where $A_{n+3}=A_3$) and $A_{i+1}A_{i+2} < A_iA_{i+3}$, we see B_i is between A_i and A_{i+3} on the segment A_iA_{i+3}. In particular, B_i is inside $A_1A_2\cdots A_n$.

Next the coordinates of B_i are $(x_{i+2}-x_{i+1}+x_i,\ \ y_{i+2}-y_{i+1}+y_i)$, both of which are integers.

Using A_iA_{i+3} is parallel to $A_{i+1}A_{i+2}$, by subtracting coordinates, we can see $B_i \neq B_{i+1}$ and B_iB_{i+1} is parallel to $A_{i+1}A_{i+2}$. By symmetry, $B_1B_2\cdots B_n$ is a regular n-gon inside $A_1A_2\cdots A_n$. Hence, the side length of $B_1B_2\cdots B_n$ is less than the side length of $A_1A_2\cdots A_n$. This contradicts the side length of $A_1A_2\cdots A_n$ is supposed to be minimal. Therefore, n=4 is the only possible case.

Example 2. (*1987 IMO Proposal by Yugoslavia*) Prove that for every natural number k ($k \geq 2$) there exists an irrational number r such that for every natural number m,

$$[r^m] \equiv -1 \pmod{k}.$$

(Here $[x]$ denotes the greatest integer less than or equal to x.)

(*Comment:* The congruence equation is equivalent to $[r^m]+1$ is divisible by k. Since $[r^m] \leq r^m < [r^m]+1$, we want to add a small amount $\delta \in (0,1]$ to r^m to make it an integer divisible by k. If we can get $\delta = s^m$ for some $s\in(0,1)$, then some algebra may lead to a solution.)

Solution. If I have a quadratic equation

$$f(x) = x^2-akx+bk = 0$$

with a, b integers and irrational roots r and s such that $s\in(0,1)$, then $r+s=ak\equiv 0 \pmod{k}$ and $rs=bk\equiv 0 \pmod{k}$. Using

$$r^{m+1}+s^{m+1}=(r+s)(r^m+s^m)-rs(r^{m-1}+s^{m-1}),$$

by induction on m, we see r^m+s^m is also an integer as cases m=0,1 are clear. So

$$[r^m]+1= r^m+s^m \equiv (r+s)^m \equiv 0 \pmod{k}.$$

Finally, to get such a quadratic, we compute the discriminant $\Delta= a^2k^2-4bk$. By taking $a = 2$ and $b = 1$, we have

$$(2k-2)^2 < \Delta= 4k^2-4k < (2k-1)^2.$$

This leads to roots r, s irrational and

$$\frac{1}{2} < s = \frac{2k-\sqrt{\Delta}}{2} < 1.$$

In the next example, we will need to compute the exponent e of a prime number p such that p^e is the largest power of p dividing $n!$. The formula is

$$e=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots.$$

Basically, since $n!=1\times2\times\cdots\times n$, we first factor out p from numbers between 1 to n that are divisible by p (this gives $[n/p]$ factors of p), then we factor out another p from numbers between 1 to n that are divisible by p^2 (this give $[n/p^2]$ more factors of p) and so on.

Example 3. (*1983 and 1991 IMO Proposal by USSR*) Let a_n be the last nonzero digit (from left to right) in the decimal representation of $n!$. Prove that the sequence $a_1, a_2, a_3, \ldots$ is not periodic after a finite number of terms (equivalently $0.a_1a_2a_3\ldots$ is irrational).

Solution. Assume beginning with the term a_M, the sequence becomes periodic with period t. Then for $m\ge M$, we have $a_{m+t}=a_m$. To get a contradiction, we will do it in steps.

Step 1. For every positive integer k, $(10^k)!=(10^k-1)!\times10^k$ implies

$$a_{10^k}=a_{10^k-1}.$$

Step 2. We can get integers $k>m\ge M$ such that 10^k-10^m is a multiple of t as follow. We factor t into the form 2^r5^sw, where w is an integer relatively prime to 10. By Euler's theorem, $10^{\varphi(w)}-1$ is a number divisible by w. Choose $m=\max\{M,r,s\}$ and $k=m+\varphi(w)$. Then $10^k-10^m=2^m5^m(10^{\varphi(w)}-1)$ is a multiple of t, say $10^k-10^m=ct$ for some integer c.

Step 3. Let $n=10^k-1+ct$. By periodicity, we have

$$a_n=a_{10^k-1}=a_{10^k}=a_{n+1}.$$

Let $a_n=d$, that is the last nonzero digit of $n!$ is d. Since $(n+1)!=(n+1)\times n!$ and the last nonzero digit of $n+1=2\times10^k-10^m$ is 9, we see $a_{n+1}=a_n$ implies the units digit of $9d$ is d. Checking d=1 to 9, we see only $d=5$ is possible. So $n!$ ends in 50…0.

Step 4. By step 3, we see the prime factorization of $n!$ is of the form 2^r5^sw with w relatively prime to 10 and $s\ge r+1>r$. However,

$$r=\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]+\left[\frac{n}{2^3}\right]+\cdots$$
$$>\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\left[\frac{n}{5^3}\right]+\cdots=s.$$

This is a contradiction and we are done.

Example 4. (*2001 IMO Proposal by Great Britain*) Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that

$$AP\cdot AG+BP\cdot BG+CP\cdot CG$$

is minimum, and express this minimum value in terms of the side lengths of ABC.

Solution. (*Due to the late Professor Murray Klamkin*) Use a vector system with the origin taken to be the centroid of ABC. Denoting the vector from the origin to the point X by $\boldsymbol{X}$, we have

$$\begin{aligned}&AP\cdot AG+BP\cdot BG+CP\cdot CG\\&=|\boldsymbol{A}-\boldsymbol{P}||\boldsymbol{A}|+|\boldsymbol{B}-\boldsymbol{P}||\boldsymbol{B}|+|\boldsymbol{C}-\boldsymbol{P}||\boldsymbol{C}|\\&\ge|(\boldsymbol{A}-\boldsymbol{P})\cdot\boldsymbol{A}|+|(\boldsymbol{B}-\boldsymbol{P})\cdot\boldsymbol{B}|+|(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{C}|\\&=|\boldsymbol{A}|^2+|\boldsymbol{B}|^2+|\boldsymbol{C}|^2\quad(\text{since } \boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}=\boldsymbol{0})\\&=(BC^2+CA^2+AB^2)/3.\end{aligned}$$

Equality holds if and only if

$$|\boldsymbol{A}-\boldsymbol{P}||\boldsymbol{A}|=|(\boldsymbol{A}-\boldsymbol{P})\cdot\boldsymbol{A}|,$$
$$|\boldsymbol{B}-\boldsymbol{P}||\boldsymbol{B}|=|(\boldsymbol{B}-\boldsymbol{P})\cdot\boldsymbol{B}|$$

and $\quad|\boldsymbol{C}-\boldsymbol{P}||\boldsymbol{C}|=|(\boldsymbol{C}-\boldsymbol{P})\cdot\boldsymbol{C}|,$

which is equivalent to P is on the lines GA, GB and GC, i.e. $P=G$.

The next example is a proof of a theorem of Fermat. It is (the contrapositive of) an infinite descent argument that Fermat might have used.

Example 5. (*1978 IMO Proposal by France*) Prove that for any positive integers x, y, z with $xy-z^2=1$ one can find nonnegative integers a, b, c, d such that $x=a^2+b^2$, $y=c^2+d^2$ and $z=ac+bd$. Set $z=(2n)!$ to deduce that for any prime number $p=4n+1$, p can be represented as the sum of squares of two integers.

Solution. We will prove the first statement by induction on z. If z=1, then $(x,y)=(1,2)$ or $(2,1)$ and we take $(a,b,c,d)=(0,1,1,1)$ or $(1,1,0,1)$ respectively.

Next for integer $w>1$, suppose cases $z=1$ to $w-1$ are true. Let positive integers u,v,w satisfy $uv-w^2=1$ with $w>1$. Note $u=v$ leads to w=0, which is absurd. Also $u=w$ leads to w=1, again absurd. Due to symmetry in u, v, we may assume $u<v$. Let $x=u$, $y=u+v-2w$ and $z=w-u$. Since

$$uv=w^2+1>w^2=uv-1>u^2-1,$$

so $y\ge2(uv)^{1/2}-2w>0$ and $z=w-u>0$. Next we can check $xy-z^2=uv-w^2=1$. By inductive hypothesis, we have

$$x=a^2+b^2,\ y=c^2+d^2,\ z=ac+bd.$$

So $u=x=a^2+b^2$, $w=x+z=a^2+b^2+ac+bd=a(a+c)+b(b+d)$ and $v=y-u+2w=(a+c)^2+(b+d)^2$. This completes the proof of the first statement.

For the second statement, we have

$$\begin{aligned}z^2&=(2n)!(2n)(2n-1)\cdots1\\&=(2n)!(p-(2n+1))\cdots(p-4n)\\&\equiv(-1)^{2n}(4n)!=(p-1)!\equiv-1\pmod p,\end{aligned}$$

where the last congruence is by Wilson's theorem. This implies z^2+1 is a multiple of p, i.e. $z^2+1=py$ for some positive integer y. By the first statement, we see $p=a^2+b^2$ for some positive integers a and b.

Example 6. (*1997 IMO Proposal by Russia*) An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

Solution. Let a be the first term and d be the common difference. We will prove by induction on d. If d=1, then the terms are consecutive integers, hence the result is true. Next, suppose d>1. Let $r=\gcd(a,d)$ and $h=d/r$, then $\gcd(a/r,h)$=1. We have two cases.

Case 1: $\gcd(r,h)=1$. Then $\gcd(a,h)$=1. Since there exist x^2 and y^3 in the progression, so x^2 and $y^3\equiv a \pmod d$. Since h divides d, x^2 and $y^3\equiv a\pmod h$. From $\gcd(a,h)$=1, we get $\gcd(y,h)$=1. Then there exists an integer t such that $ty\equiv x\pmod h$. So

$$t^6a^2\equiv t^6y^6\equiv x^6\equiv a^3 \pmod h.$$

Since $\gcd(a,h)$=1, we may cancel a^2 to get $t^6\equiv a\pmod h$.

Since $\gcd(r,h)$=1, there exists an integer k such that $kh\equiv-t\pmod r$. Then we have $(t+kh)^6\equiv0\equiv a\pmod r$ and also $(t+kh)^6\equiv a\pmod h$. Since $\gcd(r,h)$=1 and $rh=d$, we get $(t+kh)^6\equiv a\pmod d$. Hence, $(t+kh)^6$ is in the progression.

Case 2: $\gcd(r,h)>1$. Let p be a prime dividing $\gcd(r,h)$. Then p divides r, which divides a and d. Let p^m be the greatest power of p dividing a and p^n be the greatest power of p dividing d. Since $d=rh$, p divides h and $\gcd(a,d)=r$, we see $n>m\ge1$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***January 14, 2011.***

Problem 356. A and B alternately color points on an initially colorless plane as follow. A plays first. When A takes his turn, he will choose a point not yet colored and paint it red. When B takes his turn, he will choose 2010 points not yet colored and paint them blue. When the plane contains three red points that are the vertices of an equilateral triangle, then A wins. Following the rules of the game, can B stop A from winning?

Problem 357. Prove that for every positive integer n, there do not exist four integers a, b, c, d such that $ad=bc$ and $n^2 < a < b < c < d < (n+1)^2$.

Problem 358. $ABCD$ is a cyclic quadrilateral with AC intersects BD at P. Let E, F, G, H be the feet of perpendiculars from P to sides AB, BC, CD, DA respectively. Prove that lines EH, BD, FG are concurrent or are parallel.

Problem 359. (*Due to Michel BATAILLE*) Determine (with proof) all real numbers x,y,z such that $x+y+z \ge 3$ and

$$x^3+y^3+z^3+x^4+y^4+z^4 \le 2(x^2+y^2+z^2).$$

Problem 360. (*Due to Terence ZHU, Affiliated High School of Southern China Normal University*) Let n be a positive integer. We call a set S of at least n distinct positive integers a <u>*n-divisible*</u> set if among every n elements of S, there always exist two of them, one is divisible by the other.

Determine the least integer m (in terms of n) such that every n-divisible set S with m elements contains n integers, one of them is divisible by all the remaining $n-1$ integers.

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Solutions

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Problem 351. Let S be a unit sphere with center O. Can there be three arcs on S such that each is a 300° arc on some circle with O as center and no two of the arcs intersect?

Solution. **Andy LOO** (St. Paul's Co-ed College).

The answer is no. Assume there exist three such arcs l_1, l_2 and l_3. For k=1,2,3, let C_k be the unit circle with center O that l_k is on. Since l_k is a 300° arc on C_k, <u>every point P on C_k is on l_k or its reflection point with respect to O is on l_k</u>. Let P_{ij} and P_{ji} be the intersection points of C_i and C_j. (Since P_{ij} and P_{ji} are reflection points with respect to O, if P_{ij} does not lie on both l_i and l_j, then P_{ji} will be on l_i and l_j, contradiction.) So we may let P_{ij} be the point on l_i and not on l_j and P_{ji} be the point on l_j and not on l_i.

Now P_{21} and P_{31} are on C_1 and outside of l_1, so $\angle P_{21}OP_{31} < 60°$. Hence the length of arcs $P_{21}P_{31}$ and $P_{12}P_{13}$ are equal and are less than $\pi/3$ (and similarly for $\angle P_{32}OP_{12}$, $\angle P_{13}OP_{23}$ and their arcs). Denote the distance (i.e. the length of shortest path) between P and Q on S by $d(P,Q)$. We have

$$\begin{aligned}\pi &= d(P_{12},P_{21})\\ &\le d(P_{12},P_{32})+d(P_{32},P_{31})+d(P_{31},P_{21})\\ &< \pi/3+\pi/3+\pi/3=\pi,\end{aligned}$$

which is absurd.

Other commended solvers: **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

Problem 352. (*Proposed by Pedro Henrique O. PANTOJA, University of Lisbon, Portugal*) Let a, b, c be real numbers that are at least 1. Prove that

$$\frac{a^2bc}{\sqrt{bc}+1}+\frac{b^2ca}{\sqrt{ca}+1}+\frac{c^2ab}{\sqrt{ab}+1} \ge \frac{3}{2}.$$

Solution. **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

From $a^2\sqrt{bc} \ge \sqrt{bc} \ge 1$, we get

$$\sum_{cyclic}\frac{a^2bc}{\sqrt{bc}+1} \ge \sum_{cyclic}\frac{a^2bc}{2\sqrt{bc}} = \sum_{cyclic}\frac{a^2\sqrt{bc}}{2} \ge \frac{3}{2}.$$

Moreover, we will prove the stronger fact: if $a, b, c > 0$ and $abc \ge 1$, then the inequality still holds. From $k = abc \ge 1$, we get

$$\frac{a^2bc}{\sqrt{bc}+1} = \frac{ka^{3/2}}{\sqrt{k}+a^{1/2}} \ge \frac{a^{3/2}}{1+a^{1/2}}, \quad (*)$$

where the inequality can be checked by cross-multiplication. For $x > 0$, define

$$f(x) = \frac{x^{3/2}}{1+x^{1/2}} - \frac{5}{8}\ln x.$$

Its derivative is

$$f'(x) = \frac{(\sqrt{x}-1)(8x^{3/2}+20x+15x^{1/2}+5)}{8x(\sqrt{x}+1)^2}.$$

This shows $f(1)=1/2$ is the minimum value of f, since $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. Then by (*),

$$\sum_{cyclic}\frac{a^2bc}{\sqrt{bc}+1} \ge \sum_{cyclic}\frac{a^{3/2}}{1+a^{1/2}} \ge \frac{3}{2}+\frac{5}{8}\ln abc \ge \frac{3}{2}.$$

Other commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil), **CHAN Chiu Yuen Oscar** (Wah Yan College Hong Kong), **Ozgur KIRCAK** (Jahja Kemal College, Skopje, Macedonia), **LAM Lai Him** (HKUST Math UG Year 2), **Andy LOO** (St. Paul's Co-ed College), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Salem MALIKIĆ** (Student, University of Sarajevo, Bosnia and Herzegovina), **NG Chau Lok** (HKUST Math UG Year 1), **Thien NGUYEN** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (GT(Ellen Yeung) College), **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Karatapanis SAVVAS** (3rd Senior High School of Rhoades, Greece), **TRAN Trong Hoang Tuan John** (Bac Lieu Specialized Secondary School, Vietnam), **WONG Chi Man** (CUHK Info Engg Grad), **WONG Sze Nga** (Diocesan Girls' School), **WONG Tat Yuen Simon** and **POON Lok Wing** (Carmel Divine Grace Foundation Secondary School) and **Simon YAU.**

Problem 353. Determine all pairs (x, y) of integers such that $x^5-y^2=4$.

Solution. **Ozgur KIRCAK** (Jahja Kemal College, Skopje, Macedonia), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy), **Anderson TORRES** (São Paulo, Brazil) and **Ghaleo TSOI Kwok-Wing** (University of Warwick, Year 1).

Let x, y take on values −5 to 5. We see $x^5 \equiv 0$, 1 or 10 (mod 11), but $y^2+4 \equiv 2$, 4, 5, 7, 8 or 9 (mod 11). Therefore, there can be no solution.

Other commended solvers: **Andy LOO** (St. Paul's Co-ed College).

Problem 354. For 20 boxers, find the least number n such that there exists a schedule of n matches between pairs of them so that for every three boxers, two of them will face each other in one of the matches.

Solution. **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College) and **Andy LOO** (St. Paul's Co-ed College).

Among the boxers, let A be a boxer that will be in the least number of matches, say m matches. For the 19−m boxers that do not have a match with A, each pair of them with A form a triple. Since A doesn't play them, every one of these $(19-m)(18-m)/2$ pairs must play each other in a match by the required condition.

For the m boxers that have a match with A, each of them (by the minimal condition on A) has at least m matches. Since each of these matches may be counted at most twice, we get at least $(m+1)m/2$ more matches. So

$$n \geq \frac{(19-m)(18-m)}{2} + \frac{(m+1)m}{2}$$
$$= (m-9)^2 + 90 \geq 90.$$

Finally, $n = 90$ is possible by dividing the 20 boxers into two groups of 10 boxers and in each group, every pair is scheduled a match. This gives a total of 90 matches.

Other commended solvers: **WONG Sze Nga** (Diocesan Girls' School).

Problem 355. In a plane, there are two *similar* convex quadrilaterals $ABCD$ and $AB_1C_1D_1$ such that C, D are inside $AB_1C_1D_1$ and B is outside $AB_1C_1D_1$ Prove that if lines BB_1, CC_1 and DD_1 concur, then $ABCD$ is cyclic. Is the converse also true?

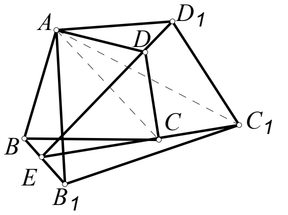
Solution. **CHAN Chiu Yuen Oscar** (Wah Yan College Hong Kong) and **LEE Shing Chi** (SKH Lam Woo Memorial Secondary School).

Since $ABCD$ and $AB_1C_1D_1$ are similar, we have

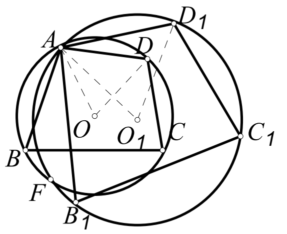
$$\frac{AB}{AB_1} = \frac{AC}{AC_1} = \frac{AD}{AD_1}. \qquad (1)$$

Also, $\triangle ABC$ and $\triangle AB_1C_1$ are similar. Then $\angle BAC = \angle B_1AC_1$. Subtracting $\angle B_1AC$ from both sides, we get $\angle BAB_1 = \angle CAC_1$. Similarly, $\angle CAC_1 = \angle DAD_1$. Along with (1), these give us $\triangle BAB_1$, $\triangle CAC_1$ and $\triangle DAD_1$ are similar. So

$$\angle AB_1B = \angle AC_1C = \angle AD_1D. \qquad (2)$$



Now if lines BB_1, CC_1 and DD_1 concur at E, then (2) can be restated as $\angle AB_1E = \angle AC_1E = \angle AD_1E$. These imply A, B_1, C_1, D_1, E are concyclic. So $AB_1C_1D_1$ is cyclic. Then by similarity, $ABCD$ is cyclic.



For the converse, suppose $ABCD$ is cyclic, then $AB_1C_1D_1$ is cyclic by similarity. Let the two circumcircles intersect at A and F. Let O be the circumcenter of $ABCD$ and O_1 be the circumcenter of $AB_1C_1D_1$. It follows $\triangle AOD$ and $\triangle AO_1D_1$ are similar. Hence $\angle AOD = \angle AO_1D_1$. From this we get

$$\angle AFD = \frac{1}{2}\angle AOD = \frac{1}{2}\angle AOD_1 = \angle AFD_1.$$

This implies line DD_1 passes through F. Similarly, lines BB_1 and CC_1 pass through F. Therefore, lines BB_1, CC_1 and DD_1 concur.

Other commended solvers: **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

Olympiad Corner

(continued from page 1)

Problem 3. Prove that for every given positive integer n, there exists a prime p and an integer m such that

(a) $p \equiv 5 \pmod 6$;
(b) $p \nmid n$;
(c) $n \equiv m^3 \pmod p$.

Problem 4. Let $x_1, x_2, \ldots, x_n$ (with $n \geq 2$) be real numbers such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

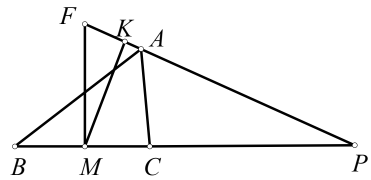
Prove that

$$\sum_{k=1}^{n}\left(1 - \frac{k}{\sum_{i=1}^{n} i x_i^2}\right)^2 \frac{x_k^2}{k} \leq \left(\frac{n-1}{n+1}\right)^2 \sum_{k=1}^{n} \frac{x_k^2}{k}.$$

Determine when equality holds.

Problem 5. Let $f(x)$ and $g(x)$ be strictly increasing linear functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x)$ is an integer if and only if $g(x)$ is an integer. Prove that for any real number x, $f(x) - g(x)$ is an integer.

Problem 6. In acute $\triangle ABC$, $AB > AC$. Let M be the midpoint of side BC. The exterior angle bisector of $\angle BAC$ meets ray BC at P. Points K and F lie on line PA such that $MF \perp BC$ and $MK \perp PA$. Prove that $BC^2 = 4PF \cdot AK$.



Problem 7. Let n be an integer greater than or equal to 3. For a permutation $p = (x_1, x_2, \ldots, x_n)$ of $(1,2,\ldots,n)$, we say x_j *lies between* x_i *and* x_k if $i < j < k$. (For example, in the permutation (1,3,2,4), 3 lies between 1 and 4, and 4 does not lie between 1 and 2.) Set $S = \{p_1, p_2, \ldots, p_m\}$ consists of (distinct) permutations p_i of $(1,2,\ldots,n)$. Suppose that among every three distinct numbers in $\{1,2,\ldots,n\}$, one of these numbers does not lie between the other two numbers in every permutation $p_i \in S$. Determine the maximum value of m.

Problem 8. Determine the least odd number $a > 5$ satisfying the following conditions: There are positive integers m_1, m_2, n_1, n_2 such that $a = m_1^2 + n_1^2$, $a^2 = m_2^2 + n_2^2$ and $m_1 - n_1 = m_2 - n_2$.

IMO Shortlisted Problems

(continued from page 2)

Then p^m divides a and d, hence all terms a, $a+d$, $a+2d$, ⋯ of the progression. In particular, p^m divides x^2 and y^3. Hence, m is a multiple of 6.

Consider the arithmetic progression obtained by dividing all terms of a, $a+d$, $a+2d$,⋯ by p^6. All terms are positive integers, the common difference is $d/p^6 < d$ and also contains $(x/p^3)^2$ and $(y/p^2)^3$. By induction hypothesis, this progression contains a sixth power j^6. Then $(pj)^6$ is a sixth power in a, $a+d$, $a+2d$,⋯ and we are done.

Mathematical Excalibur

Volume 15, Number 4 *January 2011*

Olympiad Corner

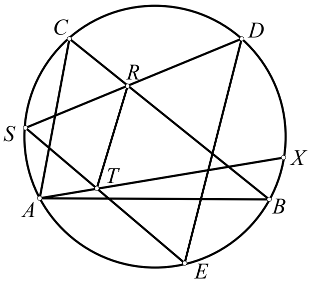
Below are the problems of the 2011 Chinese Math Olympiad, which was held on January 2011.

Problem 1. Let $a_1,a_2,\ldots,a_n$ ($n\geq 3$) be real numbers. Prove that

$$\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i a_{i+1} \leq \left[\frac{n}{2}\right](M-m)^2,$$

where $a_{n+1}=a_1$, $M=\max\limits_{1\leq i\leq n} a_i$, $m=\min\limits_{1\leq i\leq n} a_i$, $[x]$ denotes the greatest integer not exceeding x.

Problem 2. In the figure, D is the midpoint of the arc BC on the circumcircle Γ of triangle ABC. Point X is on arc BD. E is the midpoint of arc AX. S is a point on arc AC. Lines SD and BC intersect at point R. Lines SE and AX intersect at point T. Prove that if $RT \parallel DE$, then the incenter of triangle ABC is on line RT.



(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Klamkin's Inequality

Kin Y. Li

In 1971 Professor Murray Klamkin established the following

Theorem. For any real numbers x,y,z, integer n and angles α,β,γ of any triangle, we have

$$x^2+y^2+z^2 \geq (-1)^{n+1}2(yz\cos n\alpha + zx\cos n\beta + xy\cos n\gamma).$$

Equality holds if and only if

$$\frac{x}{\sin n\alpha}=\frac{y}{\sin n\beta}=\frac{z}{\sin n\gamma}.$$

The proof follows immediately from expanding

$$\left(x+(-1)^n(y\cos n\gamma + z\cos n\beta)\right)^2 + (y\sin n\gamma - z\sin n\beta)^2 \geq 0.$$

There are many nice inequalities that we can obtain from this inequality. The following are some examples (see references [1] and [2] for more).

Example 1. For angles α,β,γ of any triangle, if n is an odd integer, then

$$\cos n\alpha + \cos n\beta + \cos n\gamma \leq 3/2.$$

If n is an even integer, then

$$\cos n\alpha + \cos n\beta + \cos n\gamma \geq -3/2.$$

(This is just the case $x=y=z=1$.)

Example 2. For angles α,β,γ of any triangle,

$$\sqrt{3}\cos\alpha + 2\cos\beta + 2\sqrt{3}\cos\gamma \leq 4.$$

(This is just the case $n=1$, $x=\sin 90^\circ$, $y=\sin 60^\circ$, $z=\sin 30^\circ$.)

There are many symmetric inequalities in α,β,γ, which can be proved by standard identities or methods. However, if we encounter *asymmetric* inequality like the one in example 2, it may be puzzling in coming up with a proof.

Example 3. Let a,b,c be sides of a triangle with area Δ. If r,s,t are any real numbers, then prove that

$$\left(\frac{ar+bs+ct}{4\Delta}\right)^2 \geq \frac{st}{bc}+\frac{tr}{ca}+\frac{rs}{ab}.$$

Solution. Let α,β,γ be the angles of the triangle. We first observe that

$$4\Delta^2 = a^2b^2\sin^2\gamma = b^2c^2\sin^2\alpha = c^2a^2\sin^2\beta$$

and $\cos 2\theta = 1-2\sin^2\theta$. So we can try to set $n=2$, $x=ar$, $y=bs$, $z=ct$. Indeed, after applying Klamkin's inequality, we get the result.

Example 4. Let a,b,c be sides of a triangle with area Δ. Prove that

$$\left(\frac{a^2+b^2+c^2}{4\Delta}\right)^2 \geq \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}.$$

Comment: It may seem that we can use example 3 by setting $r=a$, $s=b$, $t=c$, but unfortunately

$$\frac{st}{bc}+\frac{tr}{ca}+\frac{rs}{ab}=3 \geq \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}$$

holds only when $a=b=c$ by the AM-GM inequality.

Solution. To solve this one, we bring in the circumradius R of the triangle. We recall that $2\Delta = bc\sin\alpha$ and by extended sine law, $2R=a/(\sin\alpha)$. So $4\Delta R=abc$. Now we set $r=bcx$, $s=cay$ and $t=abz$. Then the inequality in example 3 becomes

$$(x+y+z)^2R^2 \geq yza^2 + zxb^2 + xyc^2. \quad (*)$$

Next, we set $yz=1/b^2$, $zx=1/c^2$, $xy=1/a^2$, from which we can solve for x,y,z to get

$$x=\frac{b}{ac}=\frac{b^2}{4\Delta R},\quad y=\frac{c}{4\Delta R},\quad z=\frac{a}{4\Delta R}.$$

Then (*) becomes

$$\left(\frac{a^2+b^2+c^2}{4\Delta}\right)^2 \geq \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}.$$

Example 5. (*1998 Korean Math Olympiad*) Postive real numbers a,b,c satisfy $a+b+c=abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\le\frac{3}{2}$$

and determine when equality holds.

Solution. Let $a = \tan u$, $b = \tan v$ and $c = \tan w$, where $u,v,w > 0$. As $a+b+c=abc$,

$\tan u+\tan v+\tan w = \tan u \tan v \tan w$,

which can be written as

$$-\tan u=\frac{\tan v+\tan w}{1-\tan v\tan w}=\tan(v+w).$$

This implies $u+v+w=n\pi$ for some odd positive integer n. Let $\alpha = u/n$, $\beta = v/n$ and $\gamma = w/n$. Taking $x = y = z = 1$ in Klamkin's inequality (as in example 1), we have

$$\cos n\alpha+\cos n\beta+\cos n\gamma\le 3/2,$$

which is the desired inequality. Equality holds if and only if $a = b = c = \sqrt{3}$.

For the next two examples, we will introduce the following

Fact: Three positive real numbers x,y,z satisfy the equation

$$x^2+y^2+z^2+xyz = 4 \qquad (**)$$

if and only if there exists an acute triangle with angles α,β,γ such that

$$x = 2\cos\alpha,\ y = 2\cos\beta,\ z = 2\cos\gamma.$$

Proof. If $x,y,z > 0$ and $x^2+y^2+z^2+xyz = 4$, then x^2, y^2, $z^2 < 4$. So $0 < x, y, z < 2$. Hence, there are positive $\alpha,\beta,\gamma < \pi/2$ such that

$$x = 2\cos\alpha,\ y = 2\cos\beta \text{ and } z = 2\cos\gamma.$$

Substituting these into (**) and simplifying, we get $\cos\gamma = -\cos(\alpha+\beta)$, which implies $\alpha+\beta+\gamma = \pi$. We can get the converse by using trigonometric identities.

Example 6. (*1995 IMO Shortlisted Problem*) Let a,b,c be positive real numbers. Determine all positive real numbers x,y,z satisfying the system of equations

$$x+y+z = a+b+c,$$

$$4xyz-(a^2x+b^2y+c^2z) = abc.$$

Solution. We can rewrite the second equation as

$$\left(\frac{a}{\sqrt{yz}}\right)^2+\left(\frac{b}{\sqrt{zx}}\right)^2+\left(\frac{c}{\sqrt{xy}}\right)^2+\frac{abc}{xyz}=4.$$

By the fact, there exists an acute triangle with angles α,β,γ such that

$$\frac{a}{\sqrt{yz}}=2\cos\alpha,\frac{b}{\sqrt{zx}}=2\cos\beta,\frac{c}{\sqrt{xy}}=2\cos\gamma.$$

Then the first equation becomes

$$x+y+z=2(\sqrt{yz}\cos\alpha+\sqrt{zx}\cos\beta+\sqrt{xy}\cos\gamma).$$

This is the equality case of Klamkin's inequality. So

$$\frac{\sqrt{x}}{\sin\alpha}=\frac{\sqrt{y}}{\sin\beta}=\frac{\sqrt{z}}{\sin\gamma}.$$

As $\gamma+\beta = \pi-\alpha$, so $\sin(\gamma+\beta)/\sin\alpha=1$. Then

$$\frac{b}{2x}+\frac{c}{2x}=\frac{\sqrt{z}}{\sqrt{x}}\cos\beta+\frac{\sqrt{y}}{\sqrt{x}}\cos\gamma$$

$$=\frac{\sin\gamma\cos\beta+\sin\beta\cos\gamma}{\sin\alpha}=1.$$

So $x = (b+c)/2$. Similarly, $y = (c+a)/2$ and $z = (a+b)/2$.

Example 7. (*2007 IMO Chinese Team Training Test*) Positive real numbers u,v,w satisfy the equation $u+v+w+\sqrt{uvw}=4$.

Prove that

$$\sqrt{\frac{vw}{u}}+\sqrt{\frac{uw}{v}}+\sqrt{\frac{uv}{w}}\ge u+v+w.$$

Solution. By the fact, there exists an acute triangle with angles α,β,γ such that

$$\sqrt{u}=2\cos\alpha,\sqrt{v}=2\cos\beta,\sqrt{w}=2\cos\gamma.$$

The desired inequality becomes

$$\frac{2\cos\beta\cos\gamma}{\cos\alpha}+\frac{2\cos\gamma\cos\alpha}{\cos\beta}+\frac{2\cos\alpha\cos\beta}{\cos\gamma}$$

$$\ge 4(\cos^2\alpha+\cos^2\beta+\cos^2\gamma).$$

Comparing with Klamkin's inequality, all we have to do is to take $n = 1$ and

$$x=\sqrt{\frac{2\cos\beta\cos\gamma}{\cos\alpha}},\qquad y=\sqrt{\frac{2\cos\gamma\cos\alpha}{\cos\beta}}$$

$$z=\sqrt{\frac{2\cos\alpha\cos\beta}{\cos\gamma}}.$$

Example 8. (*1988 IMO Shortlisted Problem*) Let n be an integer greater than 1. For $i=1,2,\ldots,n$, $\alpha_i > 0$, $\beta_i > 0$ and

$$\sum_{i=1}^n\alpha_i=\sum_{i=1}^n\beta_i=\pi.$$

Prove that $\displaystyle\sum_{i=1}^n\frac{\cos\beta_i}{\sin\alpha_i}\le\sum_{i=1}^n\cot\alpha_i.$

Solution. For $n = 2$, we have equality

$$\frac{\cos\beta_1}{\sin\alpha_1}+\frac{\cos\beta_2}{\sin\alpha_2}=\frac{\cos\beta_1}{\sin\alpha_1}-\frac{\cos\beta_1}{\sin\alpha_1}$$

$$=0=\cot\alpha_1+\cot\alpha_2.$$

For $n = 3$, α_1, α_2, α_3 are angles of a triangle, say with opposite sides a,b,c. Let Δ be the area of the triangle. Now $2\Delta = bc\sin\alpha_1 = ca\sin\alpha_2 = ab\sin\alpha_3$. Combining with the cosine law, we get

$$\cot\alpha_1=\frac{\cos\alpha_1}{\sin\alpha_1}=\frac{b^2+c^2-a^2}{4\Delta}$$

and similarly for $\cot\alpha_2$ and $\cot\alpha_3$. By Klamkin's inequality,

$$\sum_{i=1}^n\frac{4\Delta\cos\beta_i}{\sin\alpha_i}=2(bc\cos\beta_1+ca\cos\beta_2+ab\cos\beta_3)$$

$$\le a^2+b^2+c^2=\sum_{i=1}^3 4\Delta\cot\alpha_i.$$

Cancelling 4Δ, we will finish the case $n = 3$. For the case $n > 3$, suppose the case $n-1$ is true. We have

$$\sum_{i=1}^n\frac{\cos\beta_i}{\sin\alpha_i}=\left[\frac{\cos\beta_1}{\sin\alpha_1}+\frac{\cos\beta_2}{\sin\alpha_2}-\frac{\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\alpha_2)}\right]$$

$$+\left[\sum_{i=3}^n\frac{\cos\beta_i}{\sin\alpha_i}+\frac{\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\alpha_2)}\right]$$

$$=\left[\frac{\cos\beta_1}{\sin\alpha_1}+\frac{\cos\beta_2}{\sin\alpha_2}+\frac{\cos(\pi-(\beta_1+\beta_2))}{\sin(\pi-(\alpha_1+\alpha_2))}\right]$$

$$+\left[\sum_{i=3}^n\frac{\cos\beta_i}{\sin\alpha_i}+\frac{\cos(\beta_1+\beta_2)}{\sin(\alpha_1+\alpha_2)}\right]$$

$$\le\left[\cot\alpha_1+\cot\alpha_2+\cot(\pi-(\alpha_1+\alpha_2))\right]$$

$$+\left[\sum_{i=3}^n\cot\alpha_i+\cot(\alpha_1+\alpha_2)\right]$$

$$=\sum_{i=1}^n\cot\alpha_i.$$

This finishes the induction.

References

[1] M.S.Klamkin, "*Asymetric Triangle Inequalities*," Publ.Elektrotehn. Fak. Ser. Mat. Fiz. Univ. Beograd, No. 357-380 (1971) pp. 33-44.

[2] Zhu Hua-Wei, From Mathematical Competitions to Competition Mathematics, Science Press, 2009 (in Chinese).

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 28, 2011.***

Problem 361. Among all real numbers a and b satisfying the property that the equation $x^4+ax^3+bx^2+ax+1=0$ has a real root, determine the minimum possible value of a^2+b^2 with proof.

Problem 362. Determine all positive rational numbers x,y,z such that

$$x+y+z,\ \ xyz,\ \ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

are integers.

Problem 363. Extend side CB of triangle ABC beyond B to a point D such that $DB=AB$. Let M be the midpoint of side AC. Let the bisector of $\angle ABC$ intersect line DM at P. Prove that $\angle BAP=\angle ACB$.

Problem 364. Eleven robbers own a treasure box. What is the least number of locks they can put on the box so that there is a way to distribute the keys of the locks to the eleven robbers with no five of them can open all the locks, but every six of them can open all the locks? The robbers agree to make enough duplicate keys of the locks for this plan to work.

Problem 365. For nonnegative real numbers a,b,c satisfying $ab+bc+ca=1$, prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{1}{a+b+c}\ge 2.$$

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Solutions

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Problem 356. A and B alternately color points on an initially colorless plane as follow. A plays first. When A takes his turn, he will choose a point not yet colored and paint it red. When B takes his turn, he will choose 2010 points not yet colored and paint them blue. When the plane contains three red points that are the vertices of an equilateral triangle, then A wins. Following the rules of the game, can B stop A from winning?

Solution. **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Anna PUN Ying** (HKU Math) and **The 7B Mathematics Group** (Carmel Alison Lam Foundation Secondary School).

The answer is negative. In the first $2n$ moves, A can color n red points <u>on a line</u>, while B can color $2010n$ blue points. For each pair of the n red points A colored, there are two points (on the perpendicular bisector of the pair) that can be chosen as vertices for making equilateral triangles with the pair. When $n>2011$, we have

$$2\binom{n}{2}=n(n-1)>2010n.$$

Then B cannot stop A from winning.

Other commended solvers: **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Andy LOO** (St. Paul's Co-ed College),**Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy) and **Lorenzo PASCALI** (Università di Roma "La Sapienza", Roma, Italy), **WONG Sze Nga** (Diocesan Girls' School).

Problem 357. Prove that for every positive integer n, there do not exist four integers a, b, c, d such that $ad=bc$ and $n^2<a<b<c<d<(n+1)^2$.

Solution. **U. BATZORIG** (National University of Mongolia) and **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College).

We first prove a useful

<u>*Fact (Four Number Theorem)*</u>: Let a,b,c,d be positive integers with $ad=bc$, then there exists positive integers p,q,r,s such that $a=pq,\ b=qr,\ c=ps,\ d=rs.$

To see this, let $p=\gcd(a,c)$, then $p|a$ and $p|c$. So $q=a/p$ and $s=c/p$ are positive integers. Now $p=\gcd(a,c)$ implies $\gcd(q,s)=1$. From $ad=bc$, we get $qd=sb$. Then $s|d$. So $r=d/s$ is a positive integer and $a=pq,\ b=qr,\ c=ps,\ d=rs.$

For the problem, assume a,b,c,d exist as required. Applying the fact, since $d>b>a$, we get $s>q$ and $r>p$. Then $s\ge q+1$, $r\ge p+1$ and we get

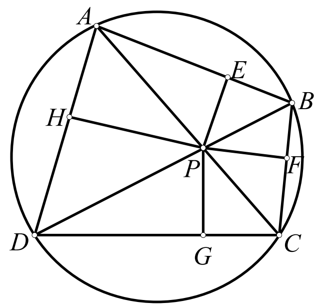
$$d=rs\ge(p+1)(q+1)\ge(\sqrt{pq}+1)^2$$
$$=(\sqrt{a}+1)^2>(n+1)^2,$$

a contradiction.

Other commended solvers: **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Anna PUN Ying** (HKU Math), **The 7B Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **WONG Sze Nga** (Diocesan Girls' School).

Problem 358. $ABCD$ is a cyclic quadrilateral with AC intersects BD at P. Let E, F, G, H be the feet of perpendiculars from P to sides AB, BC, CD, DA respectively. Prove that lines EH, BD, FG are concurrent or are parallel.

Solution. **U. BATZORIG** (National University of Mongolia), **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Abby LEE Shing Chi** (SKH Lam Woo Memorial Secondary School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Anna PUN Ying** (HKU Math), **Anderson TORRES** (São Paulo, Brazil) and **WONG Sze Nga** (Diocesan Girls' School).



Since $ABCD$ is cyclic, $\angle BAC=\angle CDB$ and $\angle ABD=\angle DCA$, which imply ΔAPB and ΔDPC are similar. As E and G are feet of perpendiculars from P to these triangles (and similarity implies the corresponding segments of triangles are proportional), we get $AE/EB=DG/GC$. Similarly, we get $AH/HD=BF/FC$.

If $EH \parallel BD$, then $AE/EB = AH/HD$, which is equivalent to $DG/GC=BF/FC$, and hence $FG \parallel BD$.

Otherwise, lines EH and BD intersect at some point I. By Menelaus theorem and its converse, we have

$$\frac{\overrightarrow{AE}}{\overrightarrow{EB}}\cdot\frac{\overrightarrow{BI}}{\overrightarrow{ID}}\cdot\frac{\overrightarrow{DH}}{\overrightarrow{HA}}=-1,$$

which is equivalent to

$$\frac{\overline{BI}}{\overline{ID}} \cdot \frac{\overline{DG}}{\overline{GC}} \cdot \frac{\overline{CF}}{\overline{FB}} = -1,$$

and lines BD and FG also intersect at I.

Other commended solvers: **Lorenzo PASCALI** (Università di Roma "La Sapienza", Roma, Italy).

Problem 359. (*Due to Michel BATAILLE*) Determine (with proof) all real numbers x,y,z such that $x+y+z \ge 3$ and

$$x^3+y^3+z^3+x^4+y^4+z^4 \le 2(x^2+y^2+z^2).$$

Solution. **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy) and **Terence ZHU** (Affilated High School of South China Normal University).

Let x,y,z be real numbers satisfying the conditions. For all real w, $w^2+3w+3 \ge (w+3/2)^2$ implies $(w^2+3w+3)(w-1)^2 \ge 0$. Expanding, we get (*) $w^4+w^3-2w^2 \ge 3w-3$. Applying (*) to $w=x,y,z$ and adding, then using the conditions on x,y,z, we get

$$0 \ge x^3+y^3+z^3+x^4+y^4+z^4-2(x^2+y^2+z^2)$$
$$\ge 3(x+y+z)-9 \ge 0.$$

Thus, for such x,y,z, we must have equalities in the (*) inequality for x,y,z. So $x = y = z = 1$ is the only solution.

Comments: For the idea behind this solution, we refer the readers to the article on the tangent line method (see *Math Excalibur*, *vol.* 10, *no.* 5, *page* 1). For those who do not know this method, we provide the

Proposer's solution. Suppose (x,y,z) is a solution. Let $s=x+y+z$ and $S=x^2y+y^2z+z^2x+xy^2+yz^2+zx^2$. By expansion, we have $s(x^2+y^2+z^2)-S= x^3+y^3+z^3$. Hence,

$$s(x^2+y^2+z^2)-S+x^4+y^4+z^4 \le 2(x^2+y^2+z^2),$$

which is equivalent to

$$(s-2)(x^2+y^2+z^2)+x^4+y^4+z^4 \le S. \quad (*)$$

Since S is the dot product of the vectors $v=(x^2,y^2,z^2,x,y,z)$ and $w=(y,z,x,y^2,z^2,x^2)$, by the Cauchy Schwarz inequality,

$$S \le x^2+y^2+z^2+x^4+y^4+z^4. \quad (**)$$

Combining (*) and (**), we conclude $(s-3)(x^2+y^2+z^2) \le 0$. Since $s \ge 3$, we get $s=3$ and (*) and (**) are equalities. Hence, vectors v and w are scalar multiple of each other. Since x,y,z are not all zeros, simple algebra yields $x=y=z=1$. This is the only solution.

Comments: Some solvers overlooked the possibility that x or y or z may be negative in applying the Cauchy Schwarz inequality!

Other commended solvers: **U. BATZORIG** (National University of Mongolia) and **Shaarvdorj** (11th High School of UB, Mongolia), **King's College Problem Solving Team (Angus CHUNG, Raymond LO, Benjamin LUI)**, **Thien NGUYEN** (Luong The Vinh High School, Dong Nai, Vietnam), **Anna PUN Ying** (HKU Math), **The 7B Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **WONG Sze Nga** (Diocesan Girls' School).

Problem 360. (*Due to Terence ZHU, Affiliated High School of Southern China Normal University*) Let n be a positive integer. We call a set S of at least n distinct positive integers a *n-divisible* set if among every n elements of S, there always exist two of them, one is divisible by the other.

Determine the least integer m (in terms of n) such that every n-divisible set S with m elements contains n integers, one of them is divisible by all the remaining $n-1$ integers.

Solution. **Anna PUN Ying** (HKU Math) and the proposer independently.

The smallest m is $(n-1)^2+1$. First choose distinct prime numbers $p_1, p_2, \ldots, p_{n-1}$. For i from 1 to $n-1$, let

$$A_i = \{p_i, p_i^2, \ldots, p_i^{n-1}\}$$

and let A be any nonempty subset of their union. Then A is n-divisible because among every n of the elements, by the pigeonhole principle, two of them will be in the same A_i, then one is divisible by the other. However, among n elements, two of them will also be in different A_i's and neither one is divisible by the other. So $m \le (n-1)^2$ will not work.

If $m \ge (n-1)^2+1$ and S is a n-divisible set with m elements, then let k_1 be the largest element in S and let B_1 be the subset of S consisted of all the divisors of k_1 in S. Let k_2 be the largest element in S and not in B_1. Let B_2 be the subset of S consisted of all the divisors of k_2 in S and not in B_1. Repeat this to get a partition of S.

Assume there are at least n of these B_i set. For i from 1 to n, let j_i be the largest element in B_i. However, by the definition of the B_i sets, $\{j_1, j_2, \ldots, j_n\}$ contradicts the n-divisiblity of S. So there are at most $n-1$ B_i's.

Since $m \ge (n-1)^2+1$, one of the B_i must have at least n elements. Then for S, we can choose n elements from this B_i with k_i included so that k_i is divisible by all the remaining $n-1$ integers. Therefore, the least m is $(n-1)^2+1$.

Other commended solvers: **WONG Sze Nga** (Diocesan Girls' School).

Olympiad Corner

(continued from page 1)

Problem 3. Let A be a finite set of real numbers. $A_1, A_2, \ldots, A_n$ are nonempty subsets of A satisfying the following conditions:

(1) the sum of all elements in A is 0;
(2) for every $x_i \in A_i$ ($i=1,2,\ldots,n$), we have $x_1+x_2+\cdots+x_n > 0$.

Prove that there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that

$$\left|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}\right| < \frac{k}{n}|A|.$$

Here $|X|$ denotes the number of elements in the finite set X.

Problem 4. Let n be a positive integer, set $S = \{1,2,\ldots,n\}$. For nonempty finite sets A and B of real numbers, find the minimum of $|A \triangle S|+|B \triangle S|+|C \triangle S|$, where $C = A+B = \{a+b \mid a \in A, b \in B\}$, $X \triangle Y = \{x \mid x$ belongs to exactly one of X or $Y\}$, $|X|$ denotes the number of elements in the finite set X.

Problem 5. Let $n \ge 4$ be a given integer. For nonnegative real numbers $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ satisfying $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n > 0$, find the maximum of

$$\frac{\sum_{i=1}^{n} a_i(a_i+b_i)}{\sum_{i=1}^{n} b_i(a_i+b_i)}.$$

Problem 6. Prove that for every given positive integers m,n, there exist infinitely many pairs of coprime positive integers a,b such that

$$a+b \mid am^a+bn^b.$$

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2011 Canadian Math Olympiad, which was held on March 23, 2011.

Problem 1. Consider 70-digit numbers n, with the property that each of the digits 1, 2, 3, …, 7 appears in the decimal expansion of n ten times (and 8, 9 and 0 do not appear). Show that no number of this form can divide another number of this form.

Problem 2. Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.

Problem 3. Amy has divided a square up into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 29, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Harmonic Series (I)

Leung Tat-Wing

A series of the form

$$\frac{1}{m}+\frac{1}{m+d}+\frac{1}{m+2d}+\cdots,$$

where m, d are numbers such that the denominators are never zero, is called a *harmonic series*. For example, the series

$$H(n)=H(1,n)=1+\frac{1}{2}+\ldots+\frac{1}{n}$$

is a harmonic series, or more generally

$$H(m,n)=\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n}$$

is also a harmonic series. Below we always assume $1 \le m < n$. There are many interesting properties concerning this kind of series.

Example 1: $H(1,n)$ is unbounded, i.e. for any positive number A, we can find n big enough, so that $H(1,n) \ge A$.

Solution For any positive integer r, note

$$\frac{1}{r+1}+\frac{1}{r+2}+\cdots+\frac{1}{2r}\ge\frac{1}{2},$$

which can be proved by induction. Hence we can take enough pieces of these fractions to make $H(1,n)$ as large as possible.

Example 2: $H(m,n)$ is never an integer.

Solution (i) For the special case $m = 1$, let s be such that $2^s \le n < 2^{s+1}$. We then multiply $H(1,n)$ by $2^{s-1}Q$, where Q is the product of all odd integers in $[1, n]$. All terms in $H(1,n)$ will become an integer except the term 2^s will become an integer divided by 2 (a half integer). This implies $H(1,n)$ is not an integer.

(ii) Alternatively, for the case $m = 1$, let p be the greatest prime number not exceeding n. By Bertrand's postulate there is a prime q with $p < q < 2p$. Therefore we have $n < 2p$. If $H(1,n)$ is an integer, then

$$n!H(n)=\sum_{i=1}^{n}\frac{n!}{i}$$

is an integer divisible by p. However the term $n!/p$ (an addend) is not divisible by p but all other addends are.

(iii) We deal with the case $m > 1$. Suppose $2^\alpha \mid k$ but $2^{\alpha+1}$ does not divide k (write this as $2^\alpha \parallel k$), then we call α the "parity order" of k. Now observe 2^α, $3\cdot2^\alpha$, $5\cdot2^\alpha$, ⋯ all have the same parity order. Between these numbers, there are $2\cdot2^\alpha$, $4\cdot2^\alpha$, $6\cdot2^\alpha$, ⋯, all have greater parity orders. Hence, between any two numbers of the same parity order, there is one with greater parity order. This implies among m, m+1, …, n, there is a unique integer with the greatest parity order, say q of parity order μ. Now multiply

$$\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n}$$

by $2^\mu L$, where L is the product of all odd integers in $[m, n]$. Then $2^\mu L\cdot H(m,n)$ is an odd number. Hence

$$H(m,n)=\frac{2r+1}{2^\mu L}=\frac{q}{p},$$

where p is even and q is odd and so is not an integer.

Example 3 (APMO 1997): Given that

$$S=1+\frac{1}{1+\frac{1}{3}}+\frac{1}{1+\frac{1}{3}+\frac{1}{6}}+\ldots+\frac{1}{1+\frac{1}{3}+\frac{1}{6}+\ldots+\frac{1}{1993006}},$$

where the denominators contains partial sum of the sequence of reciprocals of triangular numbers. Prove that $S > 1001$.

Solution Let T_n be the nth triangular number. Then $T_n=n(n+1)/2$ and hence

$$\frac{1}{T_1}+\frac{1}{T_2}+\ldots+\frac{1}{T_n}=\frac{2}{1\cdot2}+\frac{2}{2\cdot3}+\ldots+\frac{2}{n(n+1)}$$

$$=2\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}\right)=2\left(1-\frac{1}{n+1}\right)=\frac{2n}{n+1}.$$

Since 1993006=1996·1997/2, we get

$$S=\frac{1}{2}\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{1997}{1996}\right)$$

$$>\frac{1}{2}\left(1996+1+\frac{1}{2}+\left(\frac{1}{3}+\cdots+\frac{1}{1024}\right)\right).$$

Hence, S > (1996+6)/2=1001 using example 1 that $H(r+1,2r) \ge 1/2$ for r = 2, 4, 8, 16, 32, 64, 128, 256, 512.

Congruence relations of harmonic series are of some interest. First, let us look at an example.

Example 4 (IMO 1979): Let p, q be natural numbers such that

$$\frac{p}{q}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{1318}+\frac{1}{1319}.$$

Prove that p is divisible by 1979.

Solution We will prove the famous Catalan identity (due to N. Botez (1872) and later used by Catalan):

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

It is proved as follows:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

Thus

$$\frac{p}{q}=\frac{1}{660}+\frac{1}{661}+\ldots+\frac{1}{1318}+\frac{1}{1319}$$
$$=\frac{1}{2}\left(\frac{1}{660}+\frac{1}{1319}+\frac{1}{661}+\frac{1}{1318}+\ldots+\frac{1}{1319}+\frac{1}{660}\right)$$
$$=\frac{1}{2}\left(\frac{1979}{660\cdot1319}+\frac{1979}{661\cdot1318}+\ldots+\frac{1979}{1319\cdot660}\right)$$
$$=1979\cdot\frac{A}{B},$$

where B is the product of some positive integers less than 1319. However, 1979 is prime, hence $1979|\,p$.

For another proof using congruence relations, observe that if $(k,1979)=1$, then by Fermat's little theorem, $k^{1978}\equiv 1 \pmod{1979}$. Hence, we can consider $1/k\equiv k^{1977} \pmod{1979}$. Then

$$\sum_{k=1}^{1319}(-1)^{k-1}\frac{1}{k}\equiv\sum_{k=1}^{1319}(-1)^{k-1}k^{1977}$$

$$=\sum_{k=1}^{1319}k^{1977}-2\sum_{k=1}^{659}(2k)^{1977}$$

$$=\sum_{k=1}^{1319}k^{1977}-2\cdot2^{1977}\sum_{k=1}^{659}k^{1977}$$

$$\equiv\sum_{k=1}^{1319}k^{1977}-\sum_{k=1}^{659}k^{1977}$$

$$=\sum_{k=660}^{1319}k^{1977}=\sum_{k=660}^{989}(k^{1977}+(1979-k)^{1977})$$

$$\equiv\sum_{k=660}^{989}(k^{1977}+(-k)^{1977})=0 \pmod{1979}.$$

Note that $1/k \pmod p$ (as well as many fraction mod p) makes sense if $k\not\equiv 0 \pmod p$. Also, as a generalization, we have

Example 5: If $H(m,n)=q/p$ and $m+n$ is an odd prime number, then $m+n\mid q$.

Solution Note that $H(m,n)$ has an even number of terms and it equals

$$\sum_{j=0}^{(n-m-1)/2}\left(\frac{1}{m+j}+\frac{1}{n-j}\right)$$

$$=\sum_{j=0}^{(n-m-1)/2}\frac{m+n}{(m+j)(n-j)}=(m+n)\frac{s}{r}.$$

where $\gcd(s,r)=1$. Since $m+n$ is prime, $\gcd(r;m+n)=1$. Then $q/p=(m+n)s/r$ and $m+n\mid q$.

The Catalan identity is also used in the following example.

Example 6 (Rom Math Magazine, July 1998): Let

$$A=\frac{1}{1\cdot2}+\frac{1}{3\cdot4}+\ldots+\frac{1}{2011\cdot2012}$$

and

$$B=\frac{1}{1007\cdot2012}+\frac{1}{1008\cdot2011}+\ldots+\frac{1}{2012\cdot1007}.$$

Evaluate A/B.

Solution

$$A=\sum_{k=1}^{1006}\frac{1}{(2k-1)2k}=\sum_{k=1}^{1006}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)$$

$$=\frac{1}{1007}+\frac{1}{1008}+\cdots+\frac{1}{2012}$$

$$=\frac{1}{2}\left(\frac{1}{1007}+\frac{1}{2012}+\frac{1}{1008}+\frac{1}{2011}+\cdots+\frac{1}{2012}+\frac{1}{1007}\right)$$

$$=\frac{1}{2}\left(\frac{3019}{1007\cdot2012}+\frac{3019}{1008\cdot2011}+\cdots+\frac{3019}{2012\cdot1007}\right)$$

$$=\frac{3019B}{2}.$$

Hence $\dfrac{A}{B}=\dfrac{3019}{2}$.

Example 7: Given any proper fraction m/n, where m, n are positive integers satisfying $0<m<n$, then prove it is the sum of fractions of the form

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_k},$$

where $x_1, x_2, \ldots, x_k$ are distinct positive integers.

Solution We use the "greedy method". Let x_1 be the positive integer such that

$$\frac{1}{x_1}\le\frac{m}{n}<\frac{1}{x_1-1},$$

i.e. x_1 is the least integer greater than or equal to n/m. If $1/x_1=m/n$, then the problem is done. Otherwise

$$\frac{m}{n}-\frac{1}{x_1}=\frac{mx_1-n}{nx_1}=\frac{m_1}{nx_1},$$

where $m_1=mx_1-n<m$ (due to $m/n<1/(x_1-1)$) and obviously $nx_1>n$. Let x_2 be another positive integer such that

$$\frac{1}{x_2}\le\frac{m_1}{nx_1}<\frac{1}{x_2-1}.$$

The procedure can be repeated until $m>m_1>m_2>\cdots>m_k>0$ and

$$\frac{m}{n}=\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_k},$$

where $1\le k\le m$. (Note: writing

$$\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)},$$

we observe actually there are infinitely many ways of writing any proper fractions as sum of fractions of this kind. These fractions are called *unit fractions* or *Egyptian fractions*.)

Example 8: Remove those terms in

$$1+\frac{1}{2}+\cdots+\frac{1}{n}+\cdots$$

such that its denominator in decimal expansion contains the digit "9", then prove that the sequence is bounded.

Solution The integers without the digit 9 in the interval $[10^{m-1}, 10^m-1]$ are m-digit numbers. The first digit from the left cannot be the digits "0" and "9", (8 choices), the other digits cannot contain "9", hence nine choices 0, 1, 2, 3, 4, 5, 6, 7 and 8. Altogether there are $8\cdot9^{m-1}$ such integers. The sum of their reciprocals is less than

$$\frac{8\cdot9^{m-1}}{10^{m-1}}=8\left(\frac{9}{10}\right)^{m-1}.$$

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 29, 2011.***

Problem 366. Let n be a positive integer in base 10. For i =1,2,…,9, let $a(i)$ be the number of digits of n that equal i. Prove that

$$2^{a(1)}3^{a(2)}\cdots 9^{a(8)}10^{a(9)} \le n+1$$

and determine all equality cases.

Problem 367. For n = 1,2,3,…, let x_n and y_n be positive real numbers such that

$$x_{n+2} = x_n + x_{n+1}^2$$

and

$$y_{n+2} = y_n^2 + y_{n+1}.$$

If x_1, x_2, y_1, y_2 are all greater than 1, then prove that there exists a positive integer N such that for all $n > N$, we have $x_n > y_n$.

Problem 368. Let C be a circle, A_1, A_2, …, A_n be distinct points inside C and B_1, B_2, …, B_n be distinct points on C such that no two of the segments A_1B_1, A_2B_2,…, A_nB_n intersect. A grasshopper can jump from A_r to A_s if the line segment A_rA_s does not intersect any line segment A_tB_t ($t\neq r,s$). Prove that after a certain number of jumps, the grasshopper can jump from any A_u to any A_v.

Problem 369. ABC is a triangle with $BC > CA > AB$. D is a point on side BC and E is a point on ray BA beyond A so that $BD=BE=CA$. Let P be a point on side AC such that E, B, D, P are concyclic. Let Q be the intersection point of ray BP and the circumcircle of $\triangle ABC$ different from B. Prove that $AQ+CQ=BP$.

Problem 370. On the coordinate plane, at every lattice point (x,y) (these are points where x, y are integers), there is a light. At time t = 0, exactly one light is turned on. For n = 1, 2, 3, …, at time $t = n$, every light at a lattice point is turned on if it is at a distance 2005 from a light that was turned on at time $t = n - 1$. Prove that every light at a lattice point will eventually be turned on at some time.

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Solutions

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Problem 361. Among all real numbers a and b satisfying the property that the equation $x^4+ax^3+bx^2+ax+1=0$ has a real root, determine the minimum possible value of a^2+b^2 with proof.

Solution. **U. BATZORIG** (National University of Mongolia) and **Evangelos MOUROUKOS** (Agrinio, Greece).

Consider all a,b such that the equation has x as a real root. The equation implies $x \neq 0$. Using the Cauchy-Schwarz inequality (<u>or</u> looking at the equation as the line $(x^3+x)a + x^2b + (x^4+1) = 0$ in the (a,b)-plane and computing its distance from the origin), as

$$(a^2+2b^2+a^2)\left(x^6+\frac{x^4}{2}+x^2\right)$$

$$\ge (ax^3+bx^2+ax)^2 = (x^4+1)^2,$$

we get $a^2+b^2 \ge \dfrac{(x^4+1)^2}{2x^6+x^4+2x^2}$ with equality if and only if $x = \pm 1$ (at which both sides are 4/5). For $x = 1$, $(a,b) = (-4/5, -2/5)$. For $x = -1$, $(a,b) = (-2/5, 4/5)$. Finally,

$$\frac{(x^4+1)^2}{2x^6+x^4+2x^2} \ge \frac{4}{5}$$

by calculus or rewriting it as

$$5(x^4+1)^2 - 4(2x^6+x^4+2x^2)$$
$$= (x^2-1)^2(5x^4+2x^2+5) \ge 0.$$

So the minimum of $a^2 + b^2$ is 4/5.

Other commended solvers: **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Raymond LO** (King's College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Anna PUN Ying** (HKU Math), **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Problem 362. Determine all positive rational numbers x,y,z such that

$$x+y+z,\ \ xyz,\ \ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

are integers.

Solution. **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, Raymond LO** (King's College), **Anna PUN Ying** (HKU Math) and **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School).

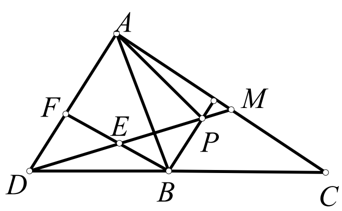
Let $A = x + y + z$, $B = xyz$ and $C = 1/x + 1/y + 1/z$, then A, B, C are integers. Since $xy + yz + zx = BC$, so x,y,z are the roots of the equation $t^3 - At^2 + BCt - B = 0$. Since the coefficients are integers and the coefficient of t^3 is 1, by Gauss lemma or the rational root theorem, the roots x, y, z are integers.

Since they are positive, without loss of generality, we may assume $z \ge y \ge x \ge 1$. Now $1 \le 1/x + 1/y + 1/z \le 3/x$ lead to x=1, 2 or 3. For x = 1, $1/y + 1/z$ = 1 or 2, which yields (y,z) = (1,1) or (2,2). For x = 2, $1/y + 1/z = 1/2$, which yields (y,z) = (3,6) or (4,4). For x = 3, $1/y + 1/z = 2/3$, which yields (y,z) = (3,3). So the solutions are (x,y,z) = (1,1,1), (1,2,2), (2,3,6), (2,4,4), (3,3,3) and permutations of coordinates.

Other commended solvers: **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Problem 363. Extend side CB of triangle ABC beyond B to a point D such that $DB=AB$. Let M be the midpoint of side AC. Let the bisector of $\angle ABC$ intersect line DM at P. Prove that $\angle BAP = \angle ACB$.

Solution. **Raymond LO** (King's College).



Construct line $BF \parallel$ line CA with F on line AD. Let DM intersect BF at E.

Since $BD=AB$, we get $\angle BDF = \angle BAF = \frac{1}{2}\angle ABC = \angle ABP = \angle CBP$. Then line $FD \parallel$ line PB. Hence, $\triangle DFE$ is similar to $\triangle PBE$.

Since $BF \parallel CA$ and M is the midpoint of

AC, so E is the midpoint of FB, i.e. $FE=BE$. Then $\triangle DFE$ is congruent to $\triangle PBE$. Hence, $FD=PB$.

This along with $DB = BA$ and $\angle BDF = \angle ABP$ imply $\triangle BDF$ is congruent to $\triangle ABP$. Therefore, $\angle BAP = \angle DBF = \angle ACB$.

Other commended solvers: **U. BATZORIG** (National University of Mongolia), **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, Abby LEE Shing Chi** (SKH Lam Woo Memorial Secondary School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Anna PUN Ying** (HKU Math), **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School), **Ercole SUPPA** (Liceo Scientifico Statale E.Einstein, Teramo, Italy) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Problem 364. Eleven robbers own a treasure box. What is the least number of locks they can put on the box so that there is a way to distribute the keys of the locks to the eleven robbers with no five of them can open all the locks, but every six of them can open all the locks? The robbers agree to make enough duplicate keys of the locks for this plan to work.

Solution. **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Raymond LO** (King's College), **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy), **Anna PUN Ying** (HKU Math), **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Let n be the least number of locks required. If for every group of 5 robbers, we put a new lock on the box and give a key to each of 6 other robbers only, then the plan works. Thus

$$n \le \binom{11}{5} = 462.$$

Conversely, in the case when there are n locks, for every group G of 5 robbers, there exists a lock $L(G)$, which they do not have the key, but the other 6 robbers all have keys to $L(G)$. Assume there exist $G \ne G'$ such that $L(G)=L(G')$. Then there is a robber in G and not in G'. Since G is one of the 6 robbers not in G', he has a key to $L(G')$, which is $L(G)$, contradiction. So $G \ne G'$ implies $L(G) \ne L(G')$. Then the number of locks is at least as many groups of 5 robbers. So

$n \ge \binom{11}{5} = 462.$ Therefore, $n = 462$.

Problem 365. For nonnegative real numbers a,b,c satisfying $ab+bc+ca = 1$, prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{1}{a+b+c} \ge 2.$$

Solution. **CHAN Long Tin** (Diocesan Boys' School) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Since $a, b, c \ge 0$ and $ab+bc+ca = 1$, none of the denominators can be zero. Multiplying both sides by $a+b+c$, we need to show

$$\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}+2 \ge 2(a+b+c).$$

This follows from using the Cauchy-Schwarz inequality and expanding $(c+a+b-2)^2 \ge 0$ as shown below

$$2\left(\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\right)$$

$$=((a+b)c+(b+c)a+(c+a)b)\left(\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\right)$$

$$\ge (c+a+b)^2$$

$$\ge 4(a+b+c)-4.$$

Other commended solvers: **Andrea FANCHINI** (Cantu, Italy), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Anna PUN Ying** (HKU Math) and **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School).

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) If the total area of the white rectangles equals the total area of the red rectangles, what is the smallest possible value of x?

Problem 4. Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a which is divisible by 2011. (For instance, if $a = 153204$, then 15, 532, and 0 are all contiguous substrings of a. Note that 0 is divisible by 2011.)

Problem 5. Let d be a positive integer. Show that for every integer S there exists an integer $n > 0$ and a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, where for any k, $\varepsilon_k = 1$ or $\varepsilon_k = -1$, such that

$$S = \varepsilon_1(1+d)^2 + \varepsilon_2(1+2d)^2 + \varepsilon_3(1+3d)^2 + \cdots + \varepsilon_n(1+nd)^2.$$

Harmonic Series (I)

(continued from page 2)

The sum of reciprocals of all such numbers is therefore less than

$$\sum_{m=0}^{\infty} 8\left(\frac{9}{10}\right)^m = \frac{8}{1-\frac{9}{10}} = 80.$$

<u>**Example 9**</u>: Let $m > 1$ be a positive integer. Show that $1/m$ is the sum of consecutive terms in the sequence

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)}.$$

<u>**Solution**</u> Since

$$\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1},$$

the problem is reduced to finding integers a and b such that

$$\frac{1}{m} = \frac{1}{a} - \frac{1}{b} \quad (*).$$

One obvious solution is $a = m-1$ and $b = m(m-1)$. To find other solutions of (*), we note that $1/a > 1/m$, so $m > a$.

Let $a = m-c$, then $b = (m^2/c)-m$. For each c satisfying $c \mid m^2$ and $1 \le c \le m$, there exists one and only one pair of a and b satisfying (*), and because $a < b$, the representation is unique. Let $d(n)$ count the number of factors of n. Now consider all factors of m^2 except m, there are $d(m^2)-1$ of them. If c is one of them, then exactly one of c or m^2/c will be less than m. Hence the number of solutions of (*) is $[d(m^2)-1]/2$.