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Orthocorrespondence and Orthopivotal Cubics

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Abstract. We define and study a transformation in the triangle plane called the orthocorrespondence. This transformation leads to the consideration of a family of circular circumcubics containing the Neuberg cubic and several hitherto unknown ones.

1. The orthocorrespondence

Let P be a point in the plane of triangle ABC with barycentric coordinates $(u : v : w)$. The perpendicular lines at P to AP, BP, CP intersect BC, CA, AB respectively at P_a, P_b, P_c, which we call the *orthotraces* of P. These orthotraces lie on a line $\mathcal{L}_P$, which we call the *orthotransversal* of P.[1] We denote the trilinear pole of $\mathcal{L}_P$ by $P^{\perp}$, and call it the *orthocorrespondent* of P.

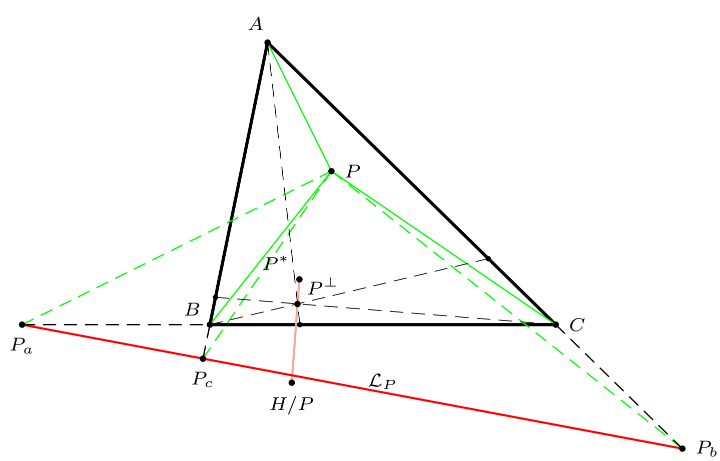

Figure 1. The orthotransversal and orthocorrespondent

In barycentric coordinates, [2]

$$P^{\perp} = (u(-uS_A + vS_B + wS_C) + a^2vw : \cdots : \cdots), \qquad (1)$$

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[1] The homography on the pencil of lines through P which swaps a line and its perpendicular at P is an involution. According to a Desargues theorem, the points are collinear.

[2] All coordinates in this paper are homogeneous barycentric coordinates. Often for triangle centers, we list only the first coordinate. The remaining two can be easily obtained by cyclically permuting a, b, c, and corresponding quantities. Thus, for example, in (1), the second and third coordinates are $v(-vS_B + wS_C + uS_A) + b^2wu$ and $w(-wS_C + uS_A + vS_B) + c^2uv$ respectively.

where, a, b, c are respectively the lengths of the sides BC, CA, AB of triangle ABC, and, in J.H. Conway's notations,

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2),\ S_B = \frac{1}{2}(c^2 + a^2 - b^2),\ S_C = \frac{1}{2}(a^2 + b^2 - c^2). \quad (2)$$

The mapping $\Phi : P \mapsto P^{\perp}$ is called the *orthocorrespondence* (with respect to triangle ABC).

Here are some examples. We adopt the notations of [5] for triangle centers, except for a few commonest ones. Triangle centers without an explicit identification as X_n are not in the current edition of [5].

(1) $I^{\perp} = X_{57}$, the isogonal conjugate of the Mittenpunkt X_9.
(2) $G^{\perp} = (b^2 + c^2 - 5a^2 : \cdots : \cdots)$ is the reflection of G about K, and the orthotransversal is perpendicular to GK.
(3) $H^{\perp} = G$.
(4) $O^{\perp} = (\cos 2A : \cos 2B : \cos 2C)$ on the line GK.
(5) More generally, the orthocorrespondent of the Euler line is the line GK. The orthotransversal envelopes the Kiepert parabola.
(6) $K^{\perp} = \left(a^2(b^4 + c^4 - a^4 - 4b^2c^2) : \cdots : \cdots\right)$ on the Euler line.
(7) $X_{15}^{\perp} = X_{62}$ and $X_{16}^{\perp} = X_{61}$.
(8) $X_{112}^{\perp} = X_{115}^{\perp} = X_{110}$.

See §2.3 for points on the circumcircle and the nine-point circle with orthocorrespondents having simple barycentric coordinates.

Remarks. (1) While the geometric definition above of $P^{\perp}$ is not valid when P is a vertex of triangle ABC, by (1) we extend the orthocorrespondence Φ to cover these points. Thus, $A^{\perp} = A$, $B^{\perp} = B$, and $C^{\perp} = C$.

(2) The orthocorrespondent of P is not defined if and only if the three coordinates of $P^{\perp}$ given in (1) are simultaneously zero. This is the case when P belongs to the three circles with diameters BC, CA, AB.[3] There are only two such points, namely, the circular points at infinity.

(3) We denote by P^* the isogonal conjugate of P and by H/P the cevian quotient of H and P.[4] It is known that

$$H/P = (u(-uS_A + vS_B + wS_C) : \cdots : \cdots).$$

This shows that $P^{\perp}$ lies on the line through P^* and H/P. In fact,

$$(H/P)P^{\perp} : (H/P)P^* = a^2vw + b^2wu + c^2uv : S_Au^2 + S_Bv^2 + S_Cw^2.$$

In [6], Jim Parish claimed that this line also contains the isogonal conjugate of P with respect to its anticevian triangle. We add that this point is in fact the harmonic conjugate of $P^{\perp}$ with respect to P^* and H/P. Note also that the line through P and H/P is perpendicular to the orthotransversal $\mathcal{L}_P$.

(4) The orthocorrespondent of any (real) point on the line at infinity $\mathcal{L}^{\infty}$ is G.

[3] See Proposition 2 below.

[4] H/P is the perspector of the cevian triangle of H (orthic triangle) and the anticevian triangle of P.

(5) A straightforward computation shows that the orthocorrespondence Φ has exactly five fixed points. These are the vertices A, B, C, and the two Fermat points X_{13}, X_{14}. Jim Parish [7] and Aad Goddijn [2] have given nice synthetic proofs of this in answering a question of Floor van Lamoen [3]. In other words, X_{13} and X_{14} are the only points whose orthotransversal and trilinear polar coincide.

Theorem 1. *The orthocorrespondent $P^\perp$ is a point at infinity if and only if P lies on the Monge (orthoptic) circle of the inscribed Steiner ellipse.*

Proof. From (1), $P^\perp$ is a point at infinity if and only if

$$\sum_{\text{cyclic}} S_A x^2 - 2a^2 yz = 0. \tag{3}$$

This is a circle in the pencil generated by the circumcircle and the nine-point circle, and is readily identified as the Monge circle of the inscribed Steiner ellipse.[5] □

It is obvious that $P^\perp$ is at infinity if and only if $\mathcal{L}_P$ is tangent to the inscribed Steiner ellipse. [6]

Proposition 2. *The orthocorrespondent $P^\perp$ lies on the sideline BC if and only if P lies on the circle Γ_{BC} with diameter BC. The perpendicular at P to AP intersects BC at the harmonic conjugate of $P^\perp$ with respect to B and C.*

Proof. $P^\perp$ lies on BC if and only if its first barycentric coordinate is 0, *i.e.*, if and only if $u(-uS_A + vS_B + wS_C) + a^2vw = 0$ which shows that P must lie on Γ_{BC}. □

2. Orthoassociates and the critical conic

2.1. *Orthoassociates and antiorthocorrespondents.*

Theorem 3. *Let Q be a finite point. There are exactly two points P_1 and P_2 (not necessarily real nor distinct) such that $Q = P_1^\perp = P_2^\perp$.*

Proof. Let Q be a finite point. The trilinear polar ℓ_Q of Q intersects the sidelines of triangle ABC at Q_a, Q_b, Q_c. The circles Γ_a, Γ_b, Γ_c with diameters AQ_a, BQ_b, CQ_c are in the same pencil of circles since their centers O_a, O_b, O_c are collinear (on the Newton line of the quadrilateral formed by the sidelines of ABC and ℓ_Q), and since they are all orthogonal to the polar circle. Thus, they have two points P_1 and P_2 in common. These points, if real, satisfy $P_1^\perp = Q = P_2^\perp$. [7] □

We call P_1 and P_2 the *antiorthocorrespondents* of Q and write $Q^\top = \{P_1, P_2\}$. We also say that P_1 and P_2 are *orthoassociates*, since they share the same orthocorrespondent and the same orthotransversal. Note that P_1 and P_2 are homologous

[5] The Monge (orthoptic) circle of a conic is the locus of points whose two tangents to the conic are perpendicular to each other. It has the same center of the conic. For the inscribed Steiner ellipse, the radius of the Monge circle is $\frac{\sqrt{2}}{6}\sqrt{a^2+b^2+c^2}$.

[6] The trilinear polar of a point at infinity is tangent to the in-Steiner ellipse since it is the in-conic with perspector G.

[7] P_1 and P_2 are not always real when ABC is obtuse angled, see §2.2 below.

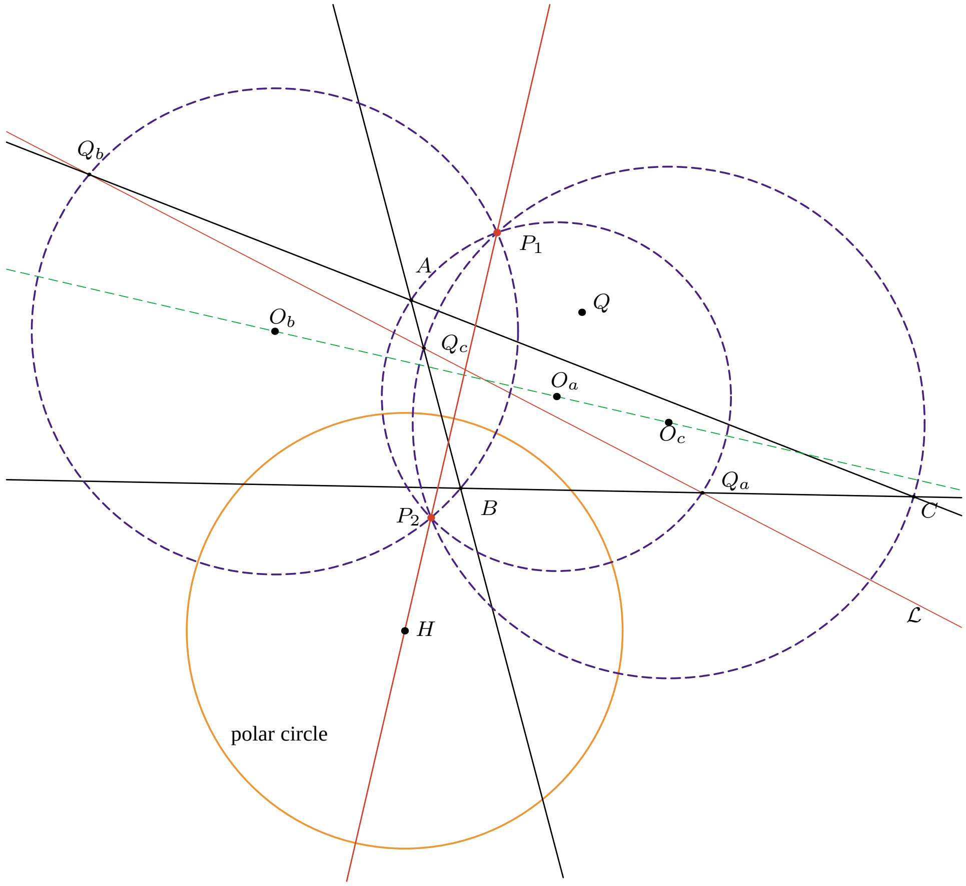

Figure 2. Antiorthocorrespondents

under the inversion ι_H with pole H which swaps the circumcircle and the nine-point circle.

Proposition 4. *The orthoassociate* $\overline{P}$ *of* $P(u : v : w)$ *has coordinates*

$$\left(\frac{S_Bv^2 + S_Cw^2 - S_Au(v+w)}{S_A} : \frac{S_Cw^2 + S_Au^2 - S_Bv(w+u)}{S_B} : \frac{S_Au^2 + S_Bv^2 - S_Cw(u+v)}{S_C}\right). \tag{4}$$

Let S denote *twice* of the area of triangle ABC. In terms of S_A, S_B, S_C in (2), we have

$$S^2 = S_AS_B + S_BS_C + S_CS_A.$$

Proposition 5. *Let*

$$K(u, v, w) = S^2(u + v + w)^2 - 4(a^2S_Avw + b^2S_Bwu + c^2S_Cuv).$$

The antiorthocorrespondents of $Q = (u : v : w)$ *are the points with barycentric coordinates*

$$((u-w)(u+v-w)S_B+(u-v)(u-v+w)S_C \pm \frac{\sqrt{K(u,v,w)}}{S}((u-w)S_B+(u-v)S_C) : \cdots : \cdots). \quad (5)$$

These are real points if and only if $K(u, v, w) \geq 0$*.*

2.2. *The critical conic* $\mathcal{C}$. Consider the *critical conic* $\mathcal{C}$ with equation

$$S^2(x+y+z)^2 - 4\sum_{\text{cyclic}} a^2 S_A yz = 0, \quad (6)$$

which is degenerate, real, imaginary according as triangle ABC is right-, obtuse-, or acute-angled. It has center the Lemoine point K, and the same infinite points as the circumconic

$$a^2 S_A yz + b^2 S_B zx + c^2 S_C xy = 0,$$

which is the isogonal conjugate of the orthic axis $S_A x + S_B y + S_C z = 0$, and has the same center K. This critical conic is a hyperbola when it is real. Clearly, if Q lies on the critical conic, its two real antiorthocorrespondents coincide.

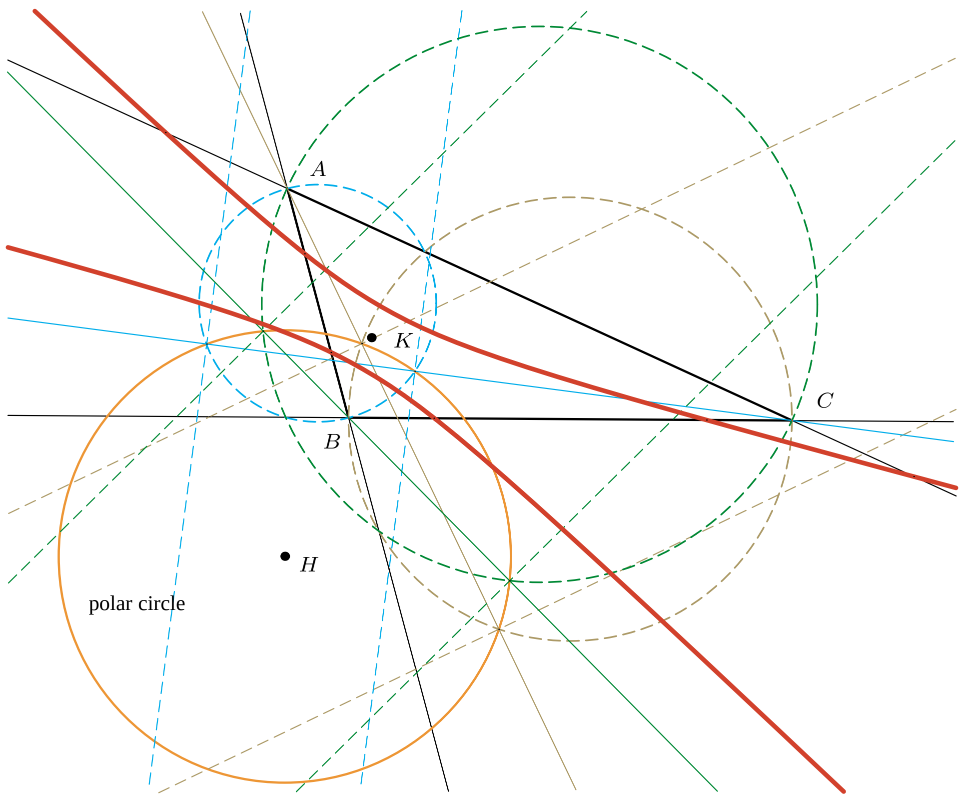

Figure 3. The critical conic

Proposition 6. *The antiorthocorrespondents of Q are real if and only if one of the following conditions holds.*
(1) Triangle ABC is acute-angled.
(2) Triangle ABC is obtuse-angled and Q lies in the component of the critical hyperbola not containing the center K.

Proposition 7. *The critical conic is the orthocorrespondent of the polar circle. When it is real, it intersects each sideline of ABC at two points symmetric about the corresponding midpoint. These points are the orthocorrespondents of the intersections of the polar circle and the circles Γ_{BC}, Γ_{CA}, Γ_{AB} with diameters BC, CA, AB.*

2.3. *Orthocorrespondent of the circumcircle.* Let P be a point on the circumcircle. Its orthotransversal passes through O, and $P^{\perp}$ lies on the circumconic centered at K. [8] The orthoassociate $\overline{P}$ lies on the nine-point circle. The table below shows several examples of such points. [9]

P	P^*	$\overline{P}$	$P^{\perp}$
X_{74}	X_{30}	X_{133}	$a^2S_A/((b^2-c^2)^2+a^2(2S_A-a^2))$
X_{98}	X_{511}	X_{132}	X_{287}
X_{99}	X_{512}	$(b^2-c^2)^2(S_A-a^2)/S_A$	$S_A/(b^2-c^2)$
X_{100}	X_{513}		$aS_A/(b-c)$
X_{101}	X_{514}		$a^2S_A/(b-c)$
X_{105}	X_{518}		$aS_A/(b^2+c^2-ab-ac)$
X_{106}	X_{519}		$a^2S_A/(b+c-2a)$
X_{107}	X_{520}	X_{125}	$X_{648}=X^*_{647}$
X_{108}	X_{521}	X_{11}	$X_{651}=X^*_{650}$
X_{109}	X_{522}		$a^2S_A/((b-c)(b+c-a))$
X_{110}	X_{523}	X_{136}	$a^2S_A/(b^2-c^2)$
X_{111}	X_{524}		$a^2S_A/(b^2+c^2-2a^2)=X^*_{468}$
X_{112}	X_{525}	X_{115}	$X_{110}=X^*_{523}$
X_{675}	X_{674}		$S_A/(b^3+c^3-a(b^2+c^2))$
X_{689}	X_{688}		$S_A/(a^2(b^4-c^4))$
X_{691}	X_{690}		$a^2S_A/((b^2-c^2)(b^2+c^2-2a^2))$
P_1	P_1^*	X_{114}	X^*_{230}

Remark. The coordinates of P_1 can be obtained from those of X_{230} by making use of the fact that X^*_{230} is the barycentric product of P_1 and X_{69}. Thus,

$$P_1 = \left(\frac{a^2}{S_A((b^2-c^2)^2-a^2(b^2+c^2-2a^2))} : \cdots : \cdots\right).$$

[8] If $P=(u:v:w)$ lies on the circumcircle, then $P^{\perp}=(uS_A:vS_B:wS_C)$ is the barycentric product of P and X_{69}. See [9]. The orthotransversal is the line $\frac{x}{uS_A}+\frac{y}{vS_B}+\frac{z}{wS_C}=0$ which contains O.

[9] The isogonal conjugates are trivially infinite points.

2.4. *The orthocorrespondent of a line.* The orthocorrespondent of a sideline, say BC, is the circumconic through G and its projection on the corresponding altitude. The orthoassociate is the circle with the segment AH as diameter.

Consider a line ℓ intersecting BC, CA, AB at X, Y, Z respectively. The orthocorrespondent $\ell^\perp$ of ℓ is a conic containing the centroid G (the orthocorrespondent of the infinite point of ℓ) and the points $X^\perp$, $Y^\perp$, $Z^\perp$. [10] A fifth point can be constructed as $P^\perp$, where P is the pedal of G on ℓ. [11] These five points entirely determine the conic. According to Proposition 2, $\ell^\perp$ meets BC at the orthocorrespondents of the points where ℓ intersects the circle Γ_{BC}. [12] It is also the orthocorrespondent of the circle through H which is the orthoassociate of ℓ.

If the line ℓ contains H, the conic $\ell^\perp$ degenerates into a double line containing G. If ℓ also contains $P = (u : v : w)$ other than H, then this line has equation

$$(S_B v - S_C w)x + (S_C w - S_A u)y + (S_A u - S_B v)z = 0.$$

This double line passes through the second intersection of ℓ with the Kiepert hyperbola. [13] It also contains the point $(uS_A : vS_B : wS_C)$. The two lines intersect at the point

$$\left(\frac{S_B - S_C}{S_B v - S_C w} : \frac{S_C - S_A}{S_C w - S_A u} : \frac{S_A - S_B}{S_A u - S_B v}\right).$$

The orthotransversals of points on ℓ envelope the inscribed parabola with directrix ℓ and focus the antipode (on the circumcircle) of the isogonal conjugate of the infinite point of ℓ.

2.5. *The antiorthocorrespondent of a line.* Let ℓ be the line with equation $lx + my + nz = 0$.
When ABC is acute angled, the antiorthocorrespondent $\ell^\top$ of ℓ is the circle centered at $\Omega_\ell = (m+n : n+l : l+m)$[14] and orthogonal to the polar circle. It has square radius

$$\frac{S_A(m+n)^2 + S_B(n+l)^2 + S_C(l+m)^2}{4(l+m+n)^2}$$

and equation

$$(x+y+z)\left(\sum_{\text{cyclic}} S_A l x\right) - (l+m+n)\left(\sum_{\text{cyclic}} a^2 yz\right) = 0.$$

When ABC is obtuse angled, $\ell^\top$ is only a part of this circle according to its position with respect to the critical hyperbola $\mathcal{C}$. This circle clearly degenerates

[10] These points can be easily constructed. For example, $X^\perp$ is the trilinear pole of the perpendicular at X to BC.

[11] $P^\perp$ is the antipode of G on the conic.

[12] These points can be real or imaginary, distinct or equal.

[13] In particular, the orthocorrespondent of the tangent at H to the Kiepert hyperbola, *i.e.*, the line HK, is the Euler line.

[14] Ω_ℓ is the complement of the isotomic conjugate of the trilinear pole of ℓ.

into the union of $\mathcal{L}^\infty$and a line through H when G lies on ℓ. This line is the directrix of the inscribed conic which is now a parabola.

Conversely, any circle centered at Ω (proper or degenerate) orthogonal to the polar circle is the orthoptic circle of the inscribed conic whose perspector P is the isotomic conjugate of the anticomplement of the center of the circle. The ortho-correspondent of this circle is the trilinear polar ℓ_P of P. The table below shows a selection of usual lines and inscribed conics.[15]

P	Ω	ℓ	inscribed conic
X_1	X_{37}	antiorthic axis	ellipse, center I
X_2	X_2	$\mathcal{L}^\infty$	Steiner in-ellipse
X_4	X_6	orthic axis	ellipse, center K
X_6	X_{39}	Lemoine axis	Brocard ellipse
X_7	X_1	Gergonne axis	incircle
X_8	X_9		Mandart ellipse
X_{13}	X_{396}		Simmons conic
X_{76}	X_{141}	de Longchamps axis	
X_{110}	X_{647}	Brocard axis	
X_{598}	X_{597}		Lemoine ellipse

2.6. *Orthocorrespondent and antiorthocorrespondent of a circle.* In general, the orthocorrespondent of a circle is a conic. More precisely, two orthoassociate circles share the same orthocorrespondent conic, or the part of it outside the critical conic $\mathcal{C}$ when ABC is obtuse-angled. For example, the circumcircle and the nine-point circle have the same orthocorrespondent which is the circumconic centered at K. The orthocorrespondent of each circle (and its orthoassociate) of the pencil generated by circumcircle and the nine-point circle is another conic also centered at K and homothetic of the previous one. The axis of these conics are the parallels at K to the asymptotes of the Kiepert hyperbola. The critical conic is one of them since the polar circle belongs to the pencil.

This conic degenerates into a double line (or part of it) if and only if the circle is orthogonal to the polar circle. If the radical axis of the circumcircle and this circle is $lx + my + nz = 0$, this double line has equation $\frac{l}{S_A}x + \frac{m}{S_B}y + \frac{n}{S_C}z = 0$. This is the trilinear polar of the barycentric product X_{69} and the trilinear pole of the radical axis.

The antiorthocorrespondent of a circle is in general a bicircular quartic.

[15]The conics in this table are entirely defined either by their center or their perspector in the table. See [1]. In fact, there are two Simmons conics (and not ellipses as Brocard and Lemoyne wrote) with perspectors (and foci) X_{13} and X_{14}.

3. Orthopivotal cubics

For a given a point P with barycentric coordinates $(u : v : w)$, the locus of point M such that P, M, $M^\perp$ are collinear is the cubic curve $\mathcal{O}(P)$:

$$\sum_{\text{cyclic}} x\left((c^2u - 2S_Bw)y^2 - (b^2u - 2S_Cv)z^2\right) = 0. \tag{7}$$

Equivalently, $\mathcal{O}(P)$ is the locus of the intersections of a line through P with the circle which is its antiorthocorrespondent. See §2.5. We shall say that $\mathcal{O}(P)$ is an *orthopivotal* cubic, and call P its *orthopivot*.

Equation (7) can be rewritten as

$$\sum_{\text{cyclic}} u\left(x(c^2y^2 - b^2z^2) + 2yz(S_By - S_Cz)\right) = 0. \tag{8}$$

Accordingly, we consider the cubic curves

$$\begin{array}{ll} \Sigma_a : & x(c^2y^2 - b^2z^2) + 2yz(S_By - S_Cz) = 0, \\ \Sigma_b : & y(a^2z^2 - c^2x^2) + 2zx(S_Cz - S_Ax) = 0, \\ \Sigma_c : & z(b^2x^2 - a^2y^2) + 2xy(S_Ax - S_By) = 0, \end{array} \tag{9}$$

and very loosely write (8) in the form

$$u\Sigma_a + v\Sigma_b + w\Sigma_c = 0. \tag{10}$$

We study the cubics Σ_a, Σ_b, Σ_c in §6.5 below, where we shall see that they are strophoids. We list some basic properties of the $\mathcal{O}(P)$.

Proposition 8. (1) *The orthopivotal cubic $\mathcal{O}(P)$ is a circular circumcubic*[16] *passing through the Fermat points, P, the infinite point of the line GP, and*

$$P' = \left(\frac{b^2 - c^2}{v - w} : \frac{c^2 - a^2}{w - u} : \frac{a^2 - b^2}{u - v}\right), \tag{11}$$

which is the second intersection of the line GP and the Kiepert hyperbola.[17]

(2) *The "third" intersection of $\mathcal{O}(P)$ and the Fermat line $X_{13}X_{14}$ is on the line PX_{110}.*

(3) *The tangent to $\mathcal{O}(P)$ at P is the line $PP^\perp$.*

(4) *$\mathcal{O}(P)$ intersects the sidelines BC, CA, AB at U, V, W respectively given by*

$$\begin{aligned} U =& (0 : 2S_Cu - a^2v : 2S_Bu - a^2w), \\ V =& (2S_Cv - b^2u : 0 : 2S_Av - b^2w), \\ W =& (2S_Bw - c^2u : 2S_Aw - c^2v : 0). \end{aligned}$$

(5) *$\mathcal{O}(P)$ also contains the (not always real) antiorthocorrespondents P_1 and P_2 of P.*

[16] This means that the cubic passes through the two circular points at infinity common to all circles, and the three vertices of the reference triangle.

[17] This is therefore the sixth intersection of $\mathcal{O}(P)$ with the Kiepert hyperbola.

Here is a simple construction of the intersection U in (4) above. If the parallel at G to BC intersects the altitude AH at H_a, then U is the intersection of PH_a and BC. [18]

4. Construction of $\mathcal{O}(P)$ and other points

Let the trilinear polar of P intersect the sidelines BC, CA, AB at X, Y, Z respectively. Denote by Γ_a, Γ_b, Γ_c the circles with diameters AX, BY, CZ and centers O_a, O_b, O_c. They are in the same pencil $\mathbb{F}$ whose radical axis is the perpendicular at H to the line $\mathcal{L}$ passing through O_a, O_b, O_c, and the points P_1 and P_2 seen above. [19]

For an arbitrary point M on $\mathcal{L}$, let Γ be the circle of $\mathbb{F}$ passing through M. The line $PM^{\perp}$ intersects Γ at two points N_1 and N_2 on $\mathcal{O}(P)$. From these we note the following.

(1) $\mathcal{O}(P)$ contains the second intersections A_2, B_2, C_2 of the lines AP, BP, CP with the circles Γ_a, Γ_b, Γ_c.
(2) The point P' in (11) lies on the radical axis of $\mathbb{F}$.
(3) The circle of $\mathbb{F}$ passing through P meets the line $PP^{\perp}$ at $\widetilde{P}$, tangential of P.
(4) The perpendicular bisector of N_1N_2 envelopes the parabola with focus F_P (see §5 below) and directrix the line GP. This parabola is tangent to $\mathcal{L}$ and to the two axes of the inscribed Steiner ellipse.

This yields another construction of $\mathcal{O}(P)$: a tangent to the parabola meets $\mathcal{L}$ at ω. The perpendicular at P to this tangent intersects the circle of $\mathbb{F}$ centered at ω at two points on $\mathcal{O}(P)$.

5. Singular focus and an involutive transformation

The singular focus of a circular cubic is the intersection of the two tangents to the curve at the circular points at infinity. When this singular focus lies on the curve, the cubic is said to be a focal cubic. The singular focus of $\mathcal{O}(P)$ is the point

$$F_P = \left(a^2(v^2+w^2-u^2-vw)+b^2u(u+v-2w)+c^2u(u+w-2v)\ :\cdots:\cdots\right).$$

If we denote by F_1 and F_2 the foci of the inscribed Steiner ellipse, then F_P is the inverse of the reflection of P in the line F_1F_2 with respect to the circle with diameter F_1F_2.

Consider the mapping $\Psi : P \mapsto F_P$ in the affine plane (without the centroid G) which transforms a point P into the singular focus F_P of $\mathcal{O}(P)$. This is clearly an involution: F_P is the singular focus of $\mathcal{O}(P)$ if and only if P is the singular focus of $\mathcal{O}(F_P)$. It has exactly two fixed points, *i.e.*, F_1 and F_2. [20]

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[18] H_a is the “third” intersection of AH with the Napoleon cubic, the isogonal cubic with pivot X_5.

[19] This line $\mathcal{L}$ is the trilinear polar of the isotomic conjugate of the anticomplement of P.

[20] The two cubics $\mathcal{O}(F_1)$ and $\mathcal{O}(F_2)$ are central focals with centers at F_1 and F_2 respectively, with inflexional tangents through K, sharing the same real asymptote F_1F_2.

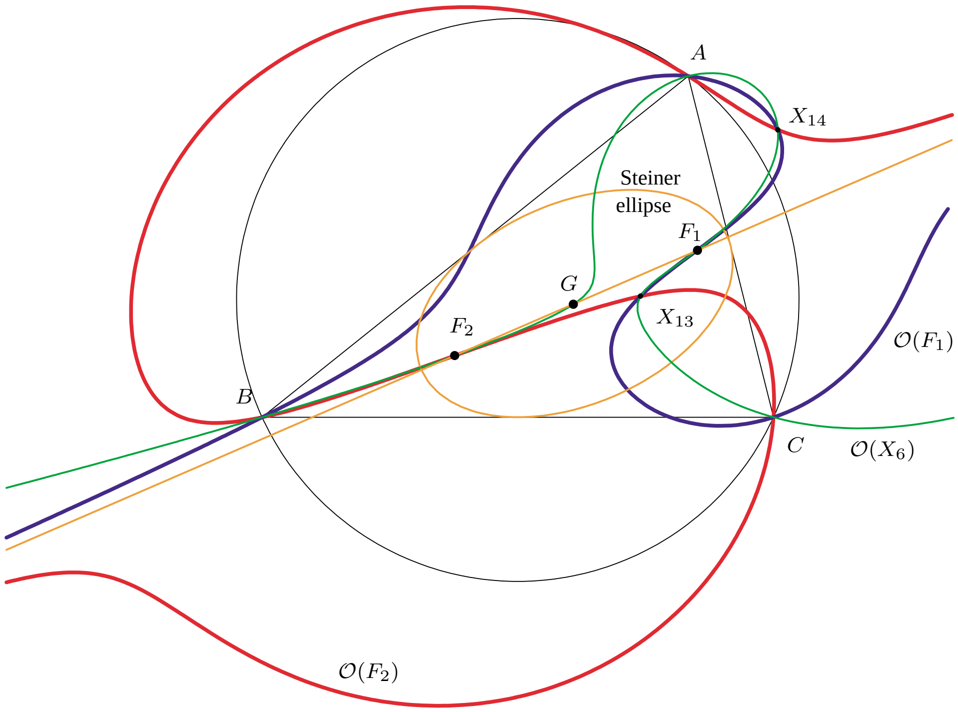

Figure 4. $\mathcal{O}(F_1)$ and $\mathcal{O}(F_2)$

The table below shows a selection of homologous points under Ψ, most of which we shall meet in the sequel. When P is at infinity, $F_P = G$, *i.e.*, all $\mathcal{O}(P)$ with orthopivot at infinity have G as singular focus.

P	X_1	X_3	X_4	X_6	X_{13}	X_{15}	X_{23}	X_{69}
F_P	X_{1054}	X_{110}	X_{125}	X_{111}	X_{14}	X_{16}	X_{182}	X_{216}

P	X_{100}	X_{184}	X_{187}	X_{352}	X_{616}	X_{617}	X_{621}	X_{622}
F_P	X_{1083}	X_{186}	X_{353}	X_{574}	X_{619}	X_{618}	X_{624}	X_{623}

The involutive transformation Ψ swaps

(1) the Euler line and the line through GX_{110}, [21]
(2) more generally, any line GP and its reflection in F_1F_2,
(3) the Brocard axis OK and the Parry circle.
(4) more generally, any line OP (which is not the Euler line) and the circle through G, X_{110}, and F_P,
(5) the circumcircle and the Brocard circle,
(6) more generally, any circle not through G and another circle not through G.

[21] The nine-point center is swapped into the anticomplement of X_{110}.

The involutive transformation Ψ leaves the second Brocard cubic $\mathcal{B}_2$ [22]

$$\sum_{\text{cyclic}} (b^2 - c^2)x(c^2y^2 + b^2z^2) = 0$$

globally invariant. See §6.4 below. More generally, Ψ leaves invariant the pencil of circular circumcubics through the vertices of the second Brocard triangle (they all pass through G). [23] There is another cubic from this pencil which is also globally invariant, namely,

$$(a^2b^2c^2 - 8S_AS_BS_C)xyz + \sum_{\text{cyclic}} (b^2 + c^2 - 2a^2)x(c^2S_Cy^2 + b^2S_Bz^2) = 0.$$

We call this cubic $\mathcal{B}_6$. It passes through X_3, X_{110}, and X_{525}.

If $\mathcal{O}(P)$ is nondegenerate, then its real asymptote is the homothetic image of the line GP under the homothety $\mathsf{h}(F_P, 2)$.

6. Special orthopivotal cubics

6.1. *Degenerate orthopivotal cubics.* There are only two situations where we find a degenerate $\mathcal{O}(P)$. A cubic can only degenerate into the union of a line and a conic. If the line is $\mathcal{L}^\infty$, we find only one such cubic. It is $\mathcal{O}(G)$, the union of $\mathcal{L}^\infty$ and the Kiepert hyperbola. If the line is not $\mathcal{L}^\infty$, there are ten different possibilities depending of the number of vertices of triangle ABC lying on the conic above which now must be a circle.

(1) $\mathcal{O}(X_{110})$ is the union of the circumcircle and the Fermat line. [24]
(2) $\mathcal{O}(P)$ is the union of one sideline of triangle ABC and the circle through the remaining vertex and the two Fermat points when P is the "third" intersection of an altitude of ABC with the Napoleon cubic. [25]
(3) $\mathcal{O}(P)$ is the union of a circle through two vertices of ABC and one Fermat point and a line through the remaining vertex and Fermat point when P is a vertex of one of the two Napoleon triangles. See [4, §6.31].

6.2. *Isocubics $\mathcal{O}(P)$.* We denote by $p\mathcal{K}$ a *pivotal* isocubic and by $n\mathcal{K}$ a *non-pivotal* isocubic. Consider an orthopivotal circumcubic $\mathcal{O}(P)$ intersecting the sidelines of triangle ABC at U, V, W respectively. The cubic $\mathcal{O}(P)$ is an isocubic in the two following cases.

---

[22] The second Brocard cubic $\mathcal{B}_2$ is the locus of foci of inscribed conics centered on the line GK. It is also the locus of M for which the line $MM^\perp$ contains the Lemoine point K.

[23] The inversive image of a circular cubic with respect to one of its points is another circular cubic through the same point. Here, Ψ swaps ABC and the second Brocard triangle $A_2B_2C_2$. Hence, each circular cubic through A, B, C, A_2, B_2, C_2 and G has an inversive image through the same points.

[24] X_{110} is the focus of the Kiepert parabola.

[25] The Napoleon cubic is the isogonal cubic with pivot X_5. These third intersections are the intersections of the altitudes with the parallel through G to the corresponding sidelines.

6.2.1. *Pivotal $\mathcal{O}(P)$.*

Proposition 9. *An orthopivotal cubic $\mathcal{O}(P)$ is a pivotal circumcubic $p\mathcal{K}$ if and only if the triangles ABC and UVW are perspective,* i.e., *if and only if P lies on the Napoleon cubic (isogonal $p\mathcal{K}$ with pivot X_5). In this case,*

(1) *the* pivot *Q of $\mathcal{O}(P)$ lies on the cubic $\mathcal{K}_n$:* [26] *it is the perspector of ABC and the (-2)-pedal triangle of P,* [27] *and lies on the line PX_5;*
(2) *the* pole *Ω of the isoconjugation lies on the cubic*

$$\mathcal{C}_o: \qquad \sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2(b^2z - c^2y) = 0.$$

The Ω-isoconjugate Q^* of Q lies on the Neuberg cubic and is the inverse in the circumcircle of the isogonal conjugate of Q. The Ω-isoconjugate P^* of P lies on $\mathcal{K}_n$ and is the third intersection with the line QX_5.

Here are several examples of such cubics.

(1) $\mathcal{O}(O) = \mathcal{O}(X_3)$ is the Neuberg cubic.
(2) $\mathcal{O}(X_5)$ is $\mathcal{K}_n$.
(3) $\mathcal{O}(I) = \mathcal{O}(X_1)$ has pivot $X_{80} = ((2S_C - ab)(2S_B - ac) : \cdots : \cdots)$, pole $(a(2S_C - ab)(2S_B - ac) : \cdots : \cdots)$, and singular focus
$$(a(2S_A + ab + ac - 3bc) : \cdots : \cdots).$$
(4) $\mathcal{O}(H) = \mathcal{O}(X_4)$ has pivot H, pole M_o the intersection of HK and the orthic axis, with coordinates
$$\left(\frac{a^2(b^2 + c^2 - 2a^2) + (b^2 - c^2)^2}{S_A} : \cdots : \cdots\right),$$
and singular focus X_{125}, center of the Jerabek hyperbola.

$\mathcal{O}(H)$ is a very remarkable cubic since every point on it has orthocorrespondent on the Kiepert hyperbola. It is invariant under the inversion with respect to the conjugated polar circle and is also invariant under the isogonal transformation with respect to the orthic triangle. It is an isogonal $p\mathcal{K}$ with pivot X_{30} with respect to this triangle.

6.2.2. *Non-pivotal $\mathcal{O}(P)$.*

Proposition 10. *An orthopivotal cubic $\mathcal{O}(P)$ is a non-pivotal circumcubic $n\mathcal{K}$ if and only if its "third" intersections with the sidelines* [28] *are collinear,* i.e., *if and only if P lies on the isogonal $n\mathcal{K}$ with root X_{30}:* [29]

$$\sum_{\text{cyclic}} \left((b^2 - c^2)^2 + a^2(b^2 + c^2 - 2a^2)\right) x(c^2y^2 + b^2z^2) + 2(8S_AS_BS_C - a^2b^2c^2)xyz = 0.$$

We give two examples of such cubics.

[26] $\mathcal{K}_n$ is the 2-cevian cubic associated with the Neuberg and the Napoleon cubics. See [8].

[27] For any non-zero real number t, the t-pedal triangle of P is the image of its pedal triangle under the homothety $\mathsf{h}(P, t)$.

[28] These are the points U, V, W in Proposition 8(4).

[29] This passes through G, K, X_{110}, and X_{523}.

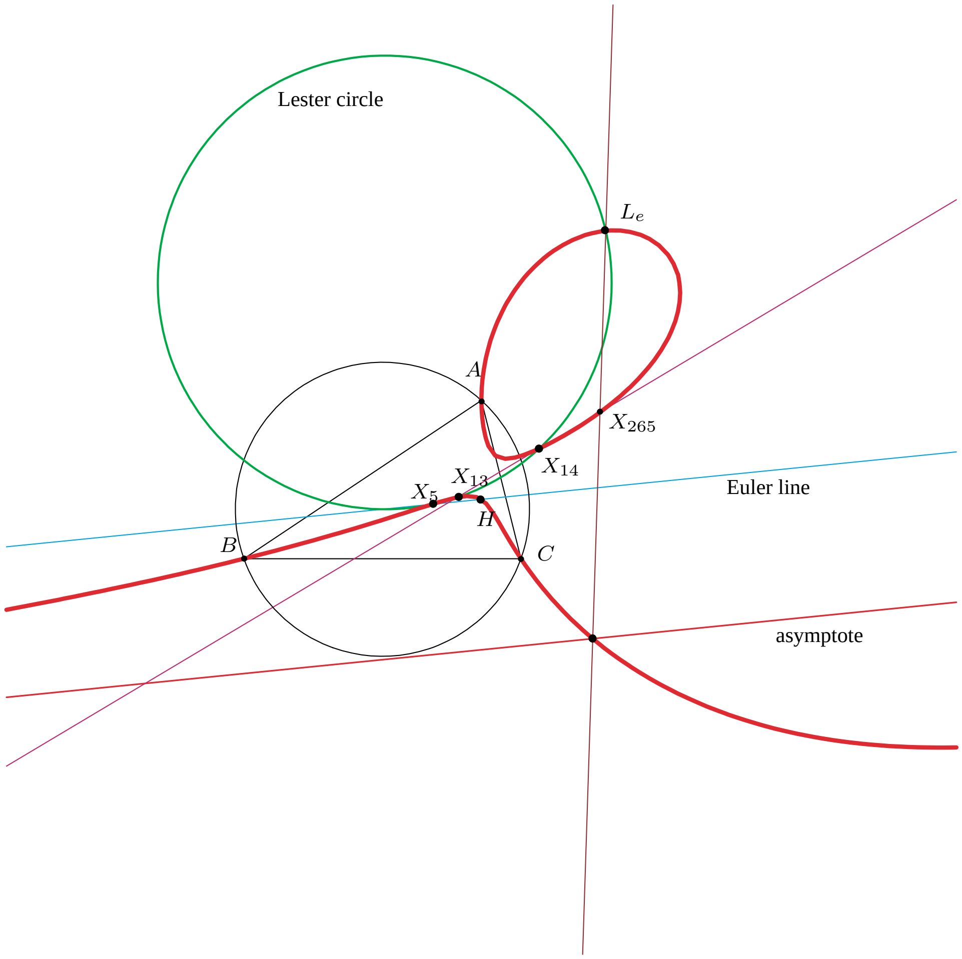

Figure 5. $\mathcal{K}_n$

(1) $\mathcal{O}(K) = \mathcal{O}(X_6)$ is the second Brocard cubic $\mathcal{B}_2$.
(2) $\mathcal{O}(X_{523})$ is a $n\mathcal{K}$ with pole and root both at the isogonal conjugate of X_{323}, and singular focus G: [30]

$$\sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2(y+z) = 0$$

6.3. *Isogonal* $\mathcal{O}(P)$. There are only two)$\mathcal{O}(P)$ which are isogonal cubics, one pivotal and one non-pivotal:

(i) $\mathcal{O}(X_3)$ is the Neuberg cubic (pivotal),
(ii) $\mathcal{O}(X_6)$ is $\mathcal{B}_2$ (nonpivotal).

[30] $\mathcal{O}(X_{523})$ meets the circumcircle at the Tixier point X_{476}.

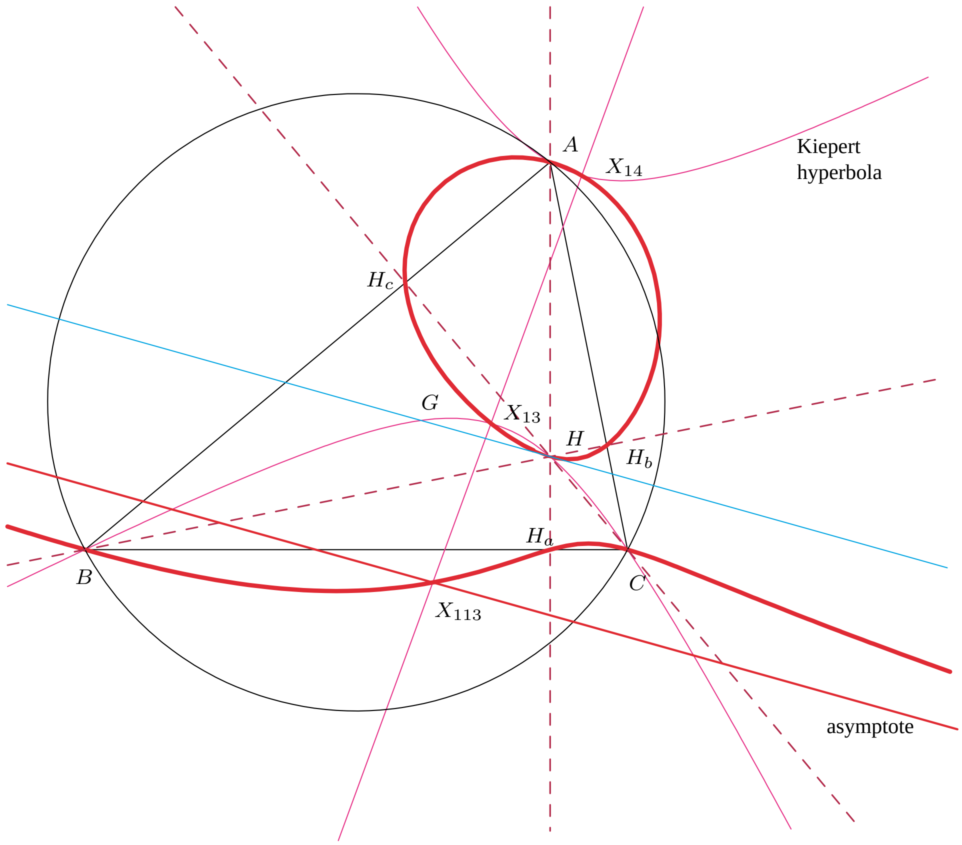

Figure 6. $\mathcal{O}(X_4)$

6.4. *Orthopivotal focals.* Recall that a focal is a circular cubic containing its own singular focus. [31]

Proposition 11. *An orthopivotal cubic $\mathcal{O}(P)$ is a focal if and only if P lies on $\mathcal{B}_2$.*

This is the case of $\mathcal{B}_2$ itself, which is an isogonal focal cubic passing through the following points: A, B, C, G, K, X_{13}, X_{14}, X_{15}, X_{16}, X_{111} (the singular focus), X_{368}, X_{524}, the vertices of the second Brocard triangle and their isogonal conjugates. All those points are orthopivots of orthopivotal focals. When the orthopivot is a fixed point of the orthocorrespondence, we shall see in §6.5 below that $\mathcal{O}(P)$ is a strophoid.

We have seen in §5 that F_1 and F_2 are invariant under Ψ. These two points lie on $\mathcal{B}_2$ (and also on the Thomson cubic). The singular focus of an orthopivotal focal $\mathcal{O}(P)$ always lies on $\mathcal{B}_2$; it is the "third" point of $\mathcal{B}_2$ and the line KP.

[31]Typically, a focal is the locus of foci of conics inscribed in a quadrilateral. The only focals having double points (nodes) are the strophoids.

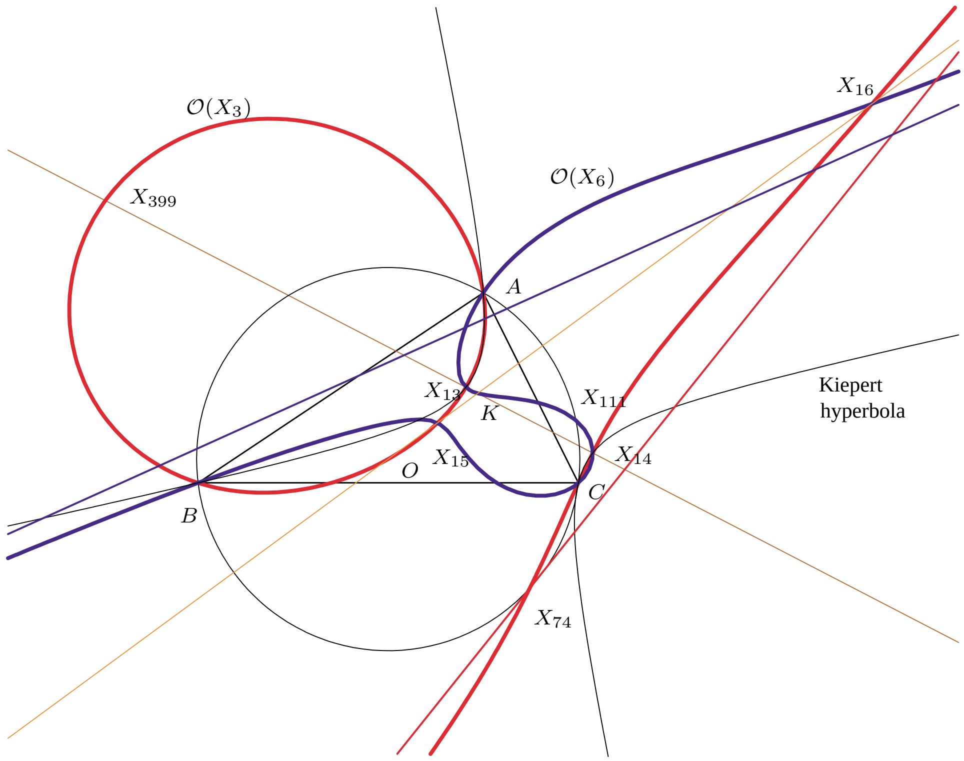

Figure 7. $\mathcal{O}(X_3)$ and $\mathcal{O}(X_6)$

One remarkable cubic is $\mathcal{O}(X_{524})$: it is another central cubic with center and singular focus at G and the line GK as real asymptote. This cubic passes through X_{67} and obviously the symmetrics of A, B, C, X_{13}, X_{14}, X_{67} about G. Its equation is

$$\sum_{\text{cyclic}} x\left(\left(b^2+c^4-a^4-c^2(a^2+2b^2-2c^2)\right)y^2-\left(b^4+c^4-a^4-b^2(a^2-2b^2+2c^2)\right)z^2\right)=0.$$

Another interesting cubic is $\mathcal{O}(X_{111})$ with K as singular focus. Its equation is

$$\sum_{\text{cyclic}} (b^2+c^2-2a^2)x^2\left(c^2(a^4-b^2c^2+3b^4-c^4-2a^2b^2)y-b^2(a^4-b^2c^2+3c^4-b^4-2a^2c^2)z\right)=0.$$

The sixth intersection with the Kiepert hyperbola is X_{671}, a point on the Steiner circumellipse and on the line through X_{99} and X_{111}.

6.5. *Orthopivotal strophoids.* It is easy to see that $\mathcal{O}(P)$ is a strophoid if and only if P is one of the five real fixed points of the orthocorrespondence, namely, A, B, C, X_{13}, X_{14}, the fixed point being the double point of the curve. This means that the mesh of orthopivotal cubics contains five strophoids denoted by $\mathcal{O}(A)$, $\mathcal{O}(B)$, $\mathcal{O}(C)$, $\mathcal{O}(X_{13})$, $\mathcal{O}(X_{14})$.

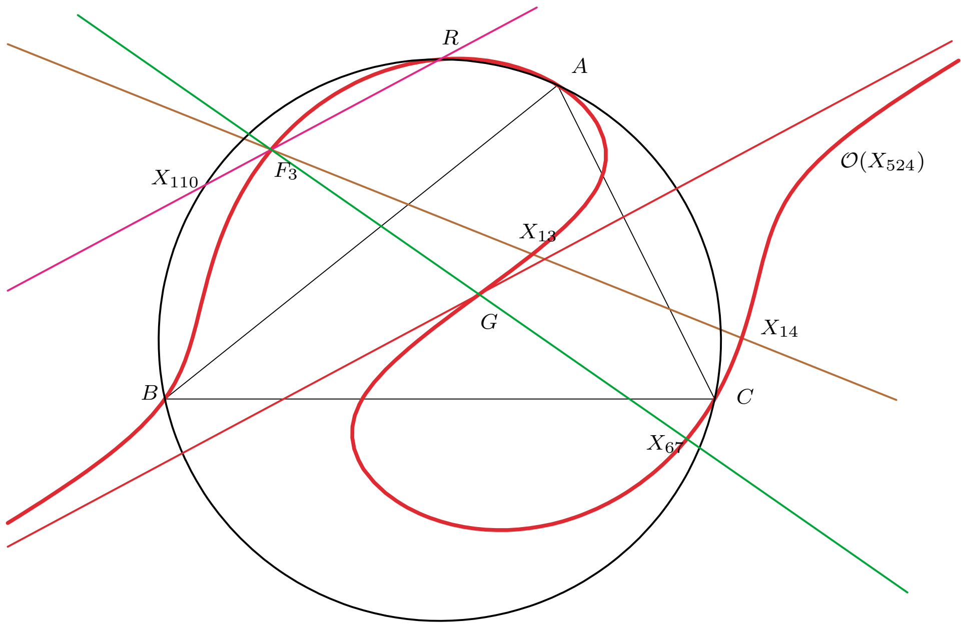

Figure 8. $\mathcal{O}(X_{524})$

6.5.1. *The strophoids* $\mathcal{O}(A)$, $\mathcal{O}(B)$, $\mathcal{O}(C)$. These are the cubics Σ_a, Σ_b, Σ_c with equations given in (9). It is enough to consider $\mathcal{O}(A) = \Sigma_a$. The bisectors of angle A are the tangents at the double point A. The singular focus is the corresponding vertex of the second Brocard triangle, namely, the point $A_2 = (2S_A : b^2 : c^2)$. [32] The real asymptote is parallel to the median AG, being the homothetic image of AG under $\mathsf{h}(A_2, 2)$.

Here are some interesting properties of $\mathcal{O}(A) = \Sigma_a$.

(1) Σ_a is the isogonal conjugate of the Apollonian A-circle

$$\mathcal{C}_A : \qquad a^2(b^2z^2 - c^2y^2) + 2x(b^2S_Bz - c^2S_Cy) = 0, \tag{12}$$

which passes through A and the two isodynamic points X_{15} and X_{16}.

(2) The isogonal conjugate of A_2 is the point $A_4 = (a^2 : 2S_A : 2S_A)$ on the Apollonian circle $\mathcal{C}_A$, which is the projection of H on AG. The isogonal conjugate of the antipode of A_4 on $\mathcal{C}_A$ is the intersection of Σ_a with its real asymptote. [33]

(3) $\mathcal{O}(A) = \Sigma_a$ is the pedal curve with respect to A of the parabola with focus at the second intersection of $\mathcal{C}_A$ and the circumcircle and with directrix the median AG.

[32]This is the projection of O on the symmedian AK, the tangent at A_2 being the reflection about OA_2 of the parallel at A_2 to AG.

[33]This isogonal conjugate is on the perpendicular at A to AK, and on the tangent at A_2 to Σ_a.

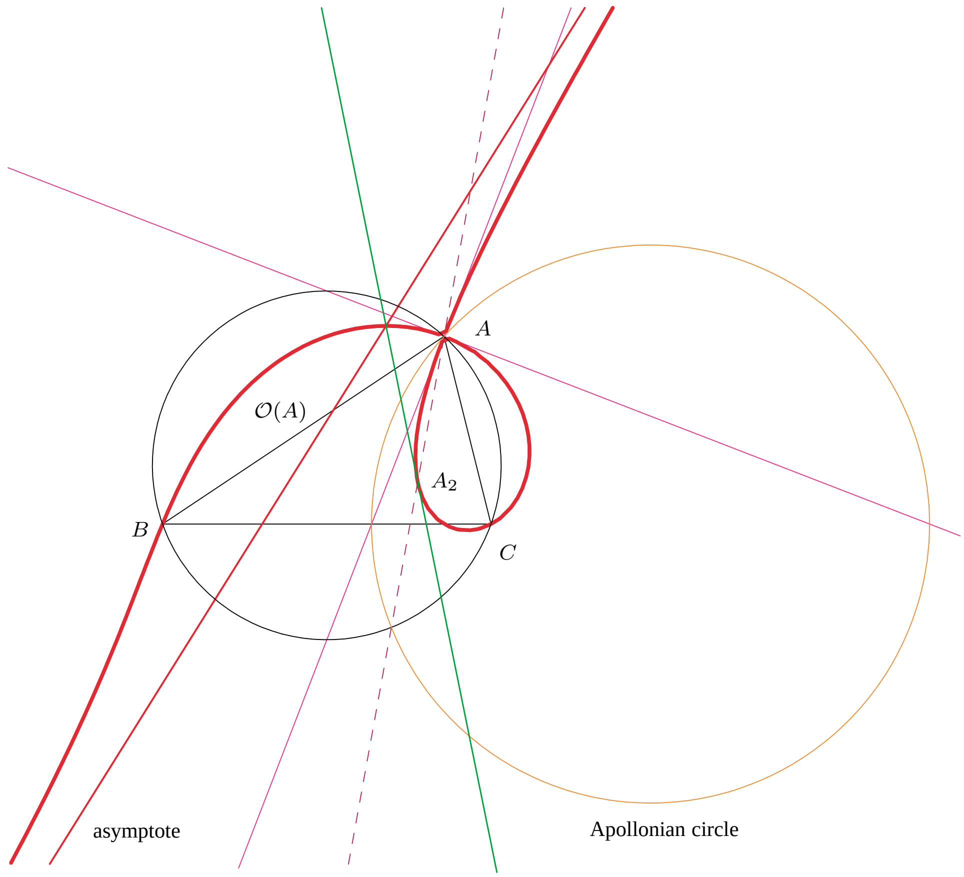

Figure 9. The strophoid $\mathcal{O}(A)$

6.5.2. *The strophoids $\mathcal{O}(X_{13})$ and $\mathcal{O}(X_{14})$.* The strophoid $\mathcal{O}(X_{13})$ has singular focus X_{14}, real asymptote the parallel at X_{99} to the line GX_{13}, [34] The circle centered at X_{14} passing through X_{13} intersects the parallel at X_{14} to GX_{13} at D_1 and D_2 which lie on the nodal tangents. The perpendicular at X_{14} to the Fermat line meets the bisectors of the nodal tangents at E_1 and E_2 which are the points where the tangents are parallel to the asymptote and therefore the centers of anallagmaty of the curve. [35]

$\mathcal{O}(X_{13})$ is the pedal curve with respect to X_{13} of the parabola with directrix the line GX_{13} and focus X'_{13}, the symmetric of X_{13} about X_{14}.

[34] The "third intersection" of this asymptote with the cubic lies on the perpendicular at X_{13} to the Fermat line. The intersection of the perpendicular at X_{13} to GX_{13} and the parallel at X_{14} to GX_{13} is another point on the curve.

[35] This means that E_1 and E_2 are the centers of two circles through X_{13} and the two inversions with respect to those circles leave $\mathcal{O}(X_{13})$ unchanged.

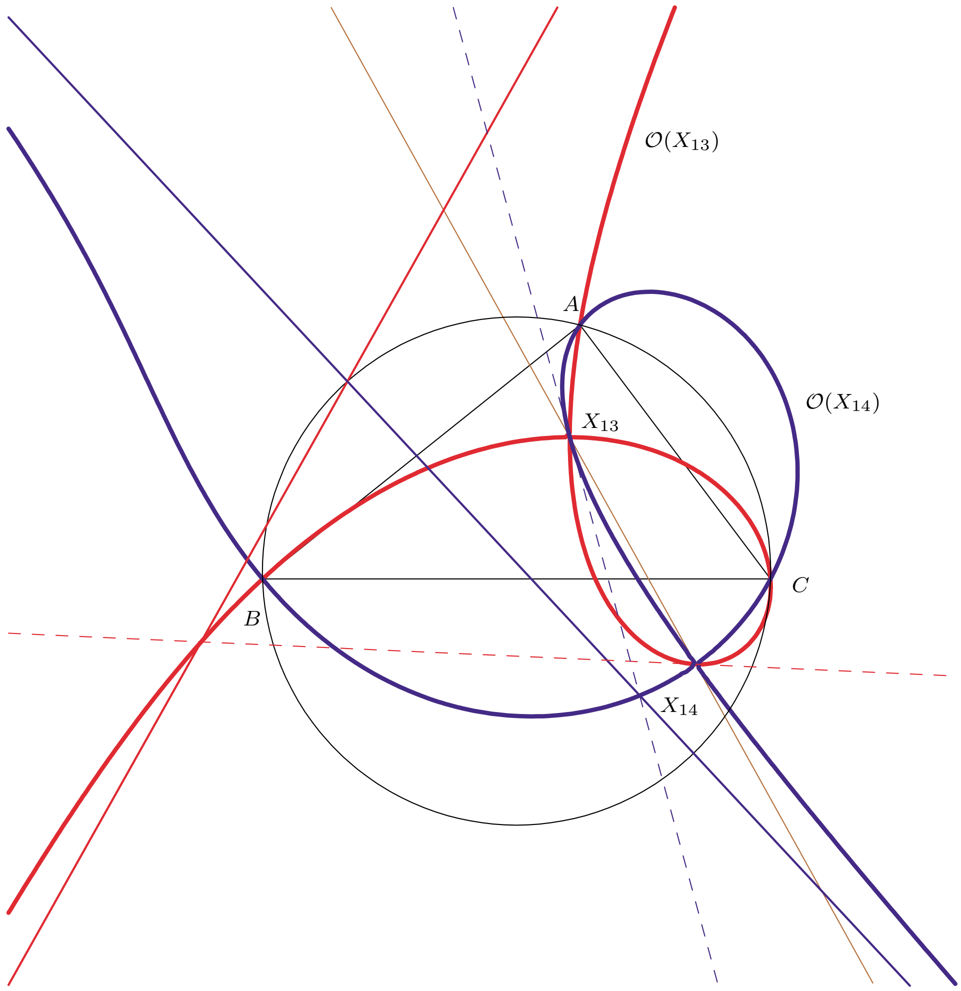

Figure 10. $\mathcal{O}(X_{13})$ and $\mathcal{O}(X_{14})$

The construction of $\mathcal{O}(X_{13})$ is easy to realize. Draw the parallel ℓ at X_{14} to GX_{13} and take a variable point M on it. The perpendicular at M to MX'_{13} and the parallel at X_{13} to MX'_{13} intersect at a point on the strophoid.

We can easily adapt all these to $\mathcal{O}(X_{14})$.

6.6. *Other remarkable* $\mathcal{O}(P)$. The following table gives a list triangle centers P with $\mathcal{O}(P)$ passing through the Fermat points X_{13}, X_{14}, and at least four more triangle centers of [5]. Some of them are already known and some others will be detailed in the next section. The very frequent appearance of X_{15}, X_{16} is explained in §7.3 below.

P	centers	P	centers
X_1	$X_{10,80,484,519,759}$	X_{182}	$X_{15,16,98,542}$
X_3	Neuberg cubic	X_{187}	$X_{15,16,598,843}$
X_5	$X_{4,30,79,80,265,621,622}$	X_{354}	$X_{1,105,484,518}$
X_6	$X_{2,15,16,111,368,524}$	X_{386}	$X_{10,15,16,519}$
X_{32}	$X_{15,16,83,729,754}$	X_{511}	$X_{15,16,262,842}$
X_{39}	$X_{15,16,76,538,755}$	X_{569}	$X_{15,16,96,539}$
X_{51}	$X_{61,62,250,262,511}$	X_{574}	$X_{15,16,543,671}$
X_{54}	$X_{3,96,265,539}$	X_{579}	$X_{15,16,226,527}$
X_{57}	$X_{1,226,484,527}$	X_{627}	$X_{17,532,617,618,622}$
X_{58}	$X_{15,16,106,540}$	X_{628}	$X_{18,533,616,619,621}$
X_{61}	$X_{15,16,18,533,618}$	X_{633}	$X_{18,533,617,623}$
X_{62}	$X_{15,16,17,532,619}$	X_{634}	$X_{17,532,616,624}$

7. Pencils of $\mathcal{O}(P)$

7.1. *Generalities.* The orthopivotal cubics with orthopivots on a given line ℓ form a pencil $\mathbb{F}_\ell$ generated by any two of them. Apart from the vertices, the Fermat points, and two circular points at infinity, all the cubics in the pencil pass through two fixed points depending on the line ℓ. Consequently, all the orthopivotal cubics passing through a given point Q have their orthopivots on the tangent at Q to $\mathcal{O}(Q)$, namely, the line $QQ^\perp$. They all pass through another point Q' on this line which is its second intersection with the circle which is its antiorthocorrespondent. For example, $\mathcal{O}(Q)$ passes through G, O, or H if and only if Q lies on GK, OX_{54}, or the Euler line respectively.

7.2. *Pencils with orthopivot on a line passing through G.* If ℓ contains the centroid G, every orthopivotal cubic in the pencil $\mathbb{F}_\ell$ passes through its infinite point and second intersection with the Kiepert hyperbola. As P traverses ℓ, the singular focus of $\mathcal{O}(P)$ traverses its reflection about F_1F_2 (see §5).

The most remarkable pencil is the one with ℓ the Euler line. In this case, the two fixed points are the infinite point X_{30} and the orthocenter H. In other words, all the cubics in this pencil have their asymptote parallel to the Euler line. In this pencil, we find the Neuberg cubic and $\mathcal{K}_n$. The singular focus traverses the line GX_{98}, X_{98} being the Tarry point.

Another worth noticing pencil is obtained when ℓ is the line GX_{98}. In this case, the two fixed points are the infinite point X_{542} and X_{98}. The singular focus traverses the Euler line. This pencil contains the two degenerate cubics $\mathcal{O}(G)$ and $\mathcal{O}(X_{110})$ seen in §6.1.

When ℓ is the line GK, the two fixed points are the infinite point X_{524} and the centroid G. The singular focus lies on the line GX_{99}, X_{99} being the Steiner point. This pencil contains $\mathcal{B}_2$ and the central cubic seen in §6.4.

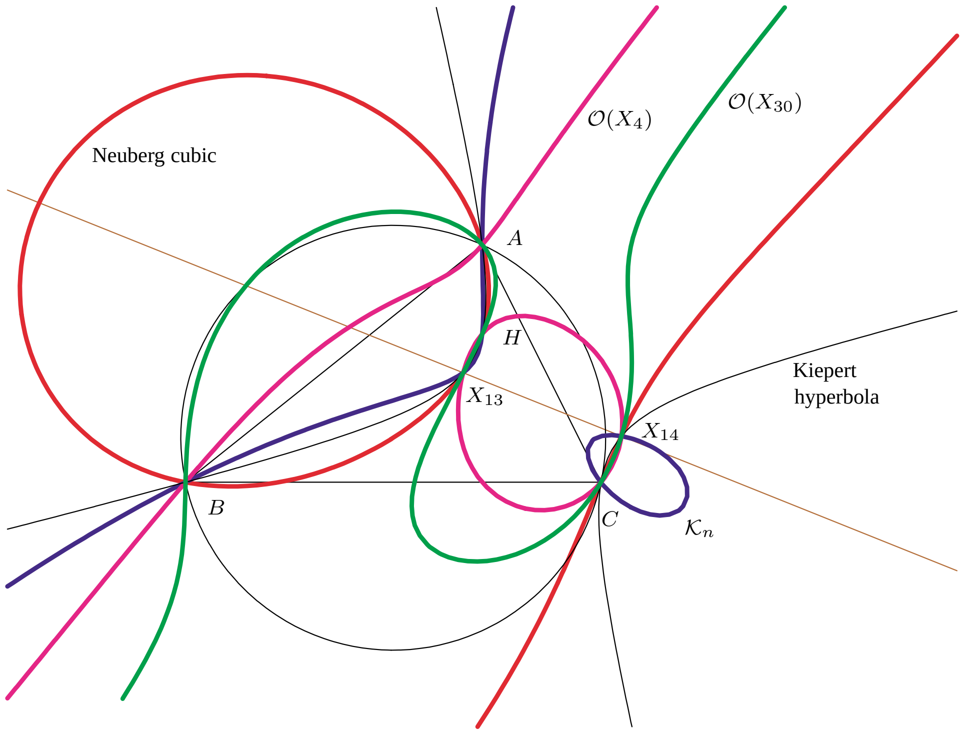

Figure 11. The Euler pencil

7.3. *Pencils with orthopivots on a line not passing through G.* If ℓ is a line not through G, the orthopivotal cubics in the pencil $\mathbb{F}_\ell$ pass through the two (not necessarily real nor distinct) intersections of ℓ with the circle which is its antiorthocorrespondent of. See §2.5 and §3. The singular focus lies on a circle through G, and the real asymptote envelopes a deltoid tangent to the line F_1F_2 and tritangent to the reflection of this circle about G.

According to §6.2.1, §6.2.2, §6.4, this pencil contains at least one, at most three $p\mathcal{K}$, $n\mathcal{K}$, focal(s) depending of the number of intersections of ℓ with the cubics met in those paragraphs respectively.

Consider, for example, the Brocard axis OK. We have seen in §6.3 that there are two and only two isogonal $\mathcal{O}(P)$, the Neuberg cubic and the second Brocard cubic $\mathcal{B}_2$ obtained when the orthopivots are O and K respectively. The two fixed points of the pencil are the isodynamic points.[36]

The singular focus lies on the Parry circle (see §5) and the asymptote envelopes a deltoid tritangent to the reflection of the Parry circle about G.

The pencil $\mathbb{F}_{OK}$ is invariant under isogonal conjugation, the isogonal conjugate of $\mathcal{O}(P)$ being $\mathcal{O}(Q)$, where Q is the harmonic conjugate of P with respect to

[36]The antiorthocorrespondent of the Brocard axis is a circle centered at X_{647}, the isogonal conjugate of the trilinear pole of the Euler line.

O and K. It is obvious that the Neuberg cubic and $\mathcal{B}_2$ are the only cubic which are "self-isogonal" and all the others correspond two by two. Since OK intersects the Napoleon cubic at O, X_{61} and X_{62}, there are only three $p\mathcal{K}$ in this pencil, the Neuberg cubic and $\mathcal{O}(X_{61})$, $\mathcal{O}(X_{62})$. [37]

$\mathcal{O}(X_{61})$ passes though X_{18}, X_{533}, X_{618}, and the isogonal conjugates of X_{532} and X_{619}.

$\mathcal{O}(X_{62})$ passes though X_{17}, X_{532}, X_{619}, and the isogonal conjugates of X_{533} and X_{618}. There are only three focals in the pencil $\mathbb{F}_{OK}$, namely, $\mathcal{B}_2$ and $\mathcal{O}(X_{15})$, $\mathcal{O}(X_{16})$ (with singular foci X_{16}, X_{15} respectively).

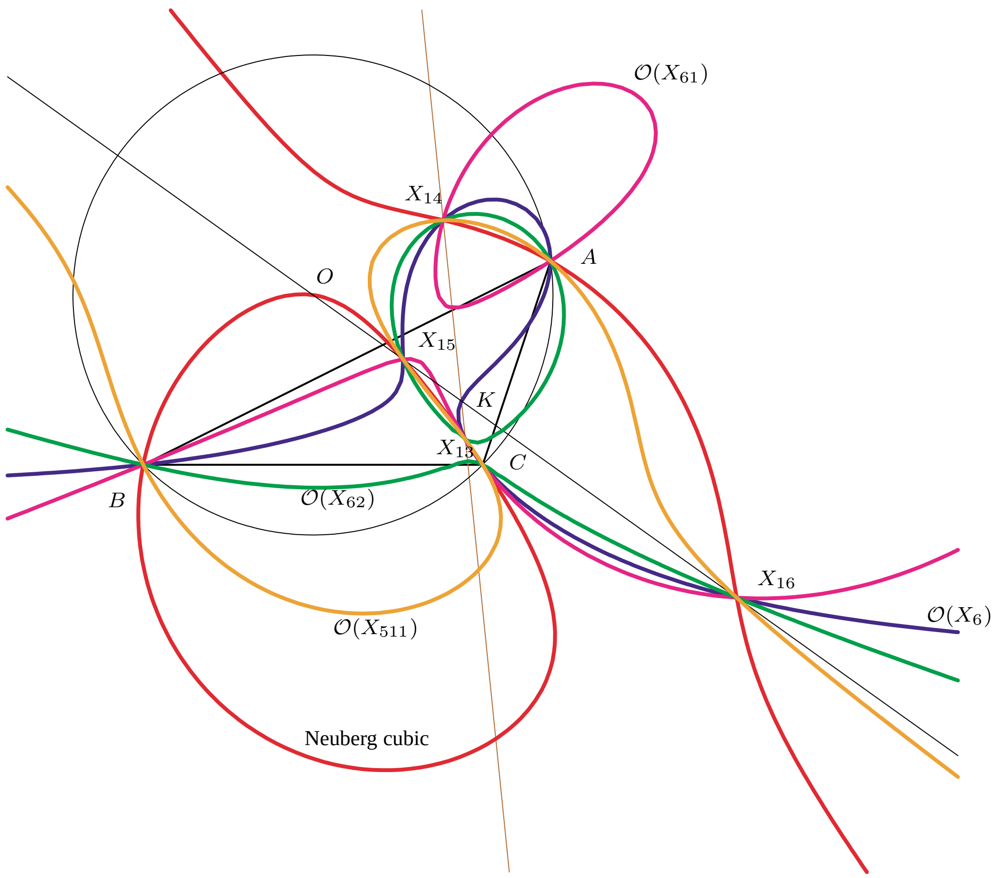

Figure 12. The Brocard pencil

An interesting situation is found when $P = X_{182}$, the midpoint of OK. Its harmonic conjugate with respect to OK is the infinite point $Q = X_{511}$. $\mathcal{O}(X_{511})$ passes through X_{262} which is its intersection with its real asymptote parallel at G

[37] $\mathcal{O}(X_{61})$ and $\mathcal{O}(X_{62})$ are isogonal conjugates of each other. Their pivots are X_{14} and X_{13} respectively and their poles are quite complicated and unknown in [5].

to OK. Its singular focus is G. The third intersection with the Fermat line is U_1 on $X_{23}X_{110}$ and the last intersection with the circumcircle is $X_{842} = X^*_{542}$.[38]

$\mathcal{O}(X_{182})$ is the isogonal conjugate of $\mathcal{O}(X_{511})$ and passes through X_{98}, X_{182}. Its singular focus is X_{23}, inverse of G in the circumcircle. Its real asymptote is parallel to the Fermat line at X_{323} and the intersection is the isogonal conjugate of U_1.

The following table gives several pairs of harmonic conjugates P and Q on OK. Each column gives two cubics $\mathcal{O}(P)$ and $\mathcal{O}(Q)$, each one being the isogonal conjugate of the other.

P	X_{32}	X_{50}	X_{52}	X_{58}	X_{187}	X_{216}	X_{284}	X_{371}	X_{389}	X_{500}
Q	X_{39}	X_{566}	X_{569}	X_{386}	X_{574}	X_{577}	X_{579}	X_{372}	X_{578}	X_{582}

8. A quintic and a quartic

We present a pair of interesting higher degree curves associated with the orthocorrespondence.

Theorem 12. *The locus of point P whose orthotransversal $\mathcal{L}_P$ and trilinear polar ℓ_P are parallel is the circular quintic*

$$\mathcal{Q}_1 : \qquad \sum_{\text{cyclic}} a^2y^2z^2(S_By - S_Cz) = 0.$$

Equivalently, $\mathcal{Q}_1$ is the locus of point P for which

(1) *the lines PP^* and ℓ_P (or $\mathcal{L}_P$) are perpendicular,*
(2) *P lies on the Euler line of the pedal triangle of P^*,*
(3) *P, P^*, H/P (and $P^\perp$) are collinear,*
(4) *P lies on $\mathcal{O}(P^*)$.*

Note that $\mathcal{L}_P$ and ℓ_P coincide when P is one of the Fermat points.[39]

Theorem 13. *The isogonal transform of the quintic $\mathcal{Q}_1$ is the circular quartic*

$$\mathcal{Q}_2 : \qquad \sum_{\text{cyclic}} a^4S_Ayz(c^2y^2 - b^2z^2) = 0,$$

which is also the locus of point P such that

(1) *the lines PP^* and ℓ_{P^*} (or $\mathcal{L}_{P^*}$) are perpendicular,*
(2) *P lies on the Euler line of its pedal triangle,*
(3) *P, P^*, H/P^* are collinear,*
(4) *P^* lies on $\mathcal{O}(P)$.*

These two curves $\mathcal{Q}_1$ and $\mathcal{Q}_2$ contain a large number of interesting points, which we enumerate below.

Proposition 14. *The quintic $\mathcal{Q}_1$ contains the 58 following points:*

[38]This is on $X_{23}X_{110}$ too. It is the reflection of the Tarry point X_{98} about the Euler line and the reflection of X_{74} about the Brocard line.

[39]See §1, Remark (5).

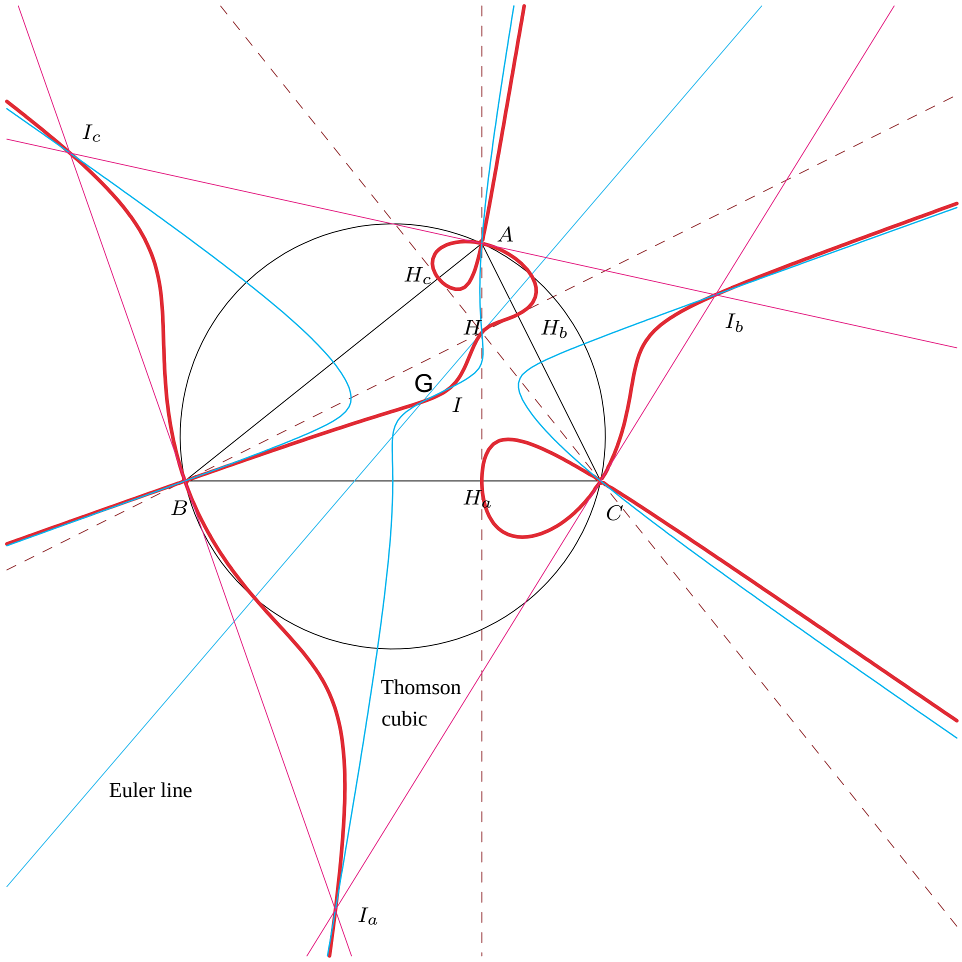

Figure 13. The quintic $\mathcal{Q}_1$

(1) *the vertices A, B, C, which are singular points with the bisectors as tangents,*
(2) *the circular points at infinity and the singular focus G,*[40]
(3) *the three infinite points of the Thomson cubic,*[41]
(4) *the in/excenters I, I_a, I_b, I_c, with tangents passing through O, and the isogonal conjugates of the intersections of these tangents with the trilinear polars of the corrresponding in/excenters,*
(5) *H, with tangent the Euler line,*

[40] The tangent at G passes through the isotomic conjugate of $G^{\perp}$, the point with coordinates $\left(\frac{1}{b^2+c^2-5a^2} : \cdots : \cdots\right)$.
[41] In other words, $\mathcal{Q}_1$ has three real asymptotes parallel to those of the Thomson cubic.

(6) *the six points where a circle with diameter a side of ABC intersects the corresponding median,* [42]
(7) *the feet of the altitudes, the tangents being the altitudes,*
(8) *the Fermat points X_{13} and X_{14},*
(9) *the points X_{1113} and X_{1114} where the Euler line meets the circumcircle,*
(10) *the perspectors of the 27 Morley triangles and ABC.* [43]

Proposition 15. *The quartic $\mathcal{Q}_2$ contains the 61 following points:*

(1) *the vertices A, B, C,* [44]
(2) *the circular points at infinity,* [45]
(3) *the three points where the Thomson cubic meets the circumcircle again,*
(4) *the in/excenters I, I_a, I_b, I_c, with tangents all passing through O, and the intersections of these tangents OI_x with the trilinear polars of the corresponding in/excenters,*
(5) *O and K,* [46]
(6) *the six points where a symmedian intersects a circle centered at the corresponding vertex of the tangential triangle passing through the remaining two vertices of ABC,* [47]
(7) *the six feet of bisectors,*
(8) *the isodynamic points X_{15} and X_{16}, with tangents passing through X_{23},*
(9) *the two infinite points of the Jerabek hyperbola,* [48]
(10) *the isogonal conjugates of the perspectors of the 27 Morley triangles and ABC.* [49]

We give a proof of (10). Let k_1, k_2, $k_3 = 0, \pm 1$, and consider

$$\varphi_1 = \frac{A + 2k_1\pi}{3}, \quad \varphi_2 = \frac{B + 2k_2\pi}{3}, \quad \varphi_3 = \frac{C + 2k_3\pi}{3}.$$

Denote by M one of the 27 points with barycentric coordinates

$$(a\cos\varphi_1 : b\cos\varphi_2 : c\cos\varphi_3).$$

[42] The two points on the median AG have coordinates

$$(2a : -a \pm \sqrt{2b^2 + 2c^2 - a^2} : -a \pm \sqrt{2b^2 + 2c^2 - a^2}).$$

[43] The existence of the these points was brought to my attention by Edward Brisse. In particular, X_{357}, the perspector of ABC and first Morley triangle.

[44] These are inflection points, with tangents passing through O.

[45] The singular focus is the inverse X_{23} of G in the circumcircle. This point is not on the curve $\mathcal{Q}_2$.

[46] Both tangents at O and K pass through the point $Z = (a^2 S_A(b^2 + c^2 - 2a^2) : \cdots : \cdots)$, the intersection of the trilinear polar of O with the orthotransversal of X_{110}. The tangent at O is also tangent to the Jerabek hyperbola and the orthocubic.

[47] The two points on the symmedian AK have coordinates $(-a^2 \pm a\sqrt{2b^2 + 2c^2 - a^2} : 2b^2 : 2c^2)$.

[48] The two real asymptotes of $\mathcal{Q}_2$ are parallel to those of the Jerabek hyperbola and meet at Z in footnote 46 above.

[49] In particular, the Morley-Yff center X_{358}.

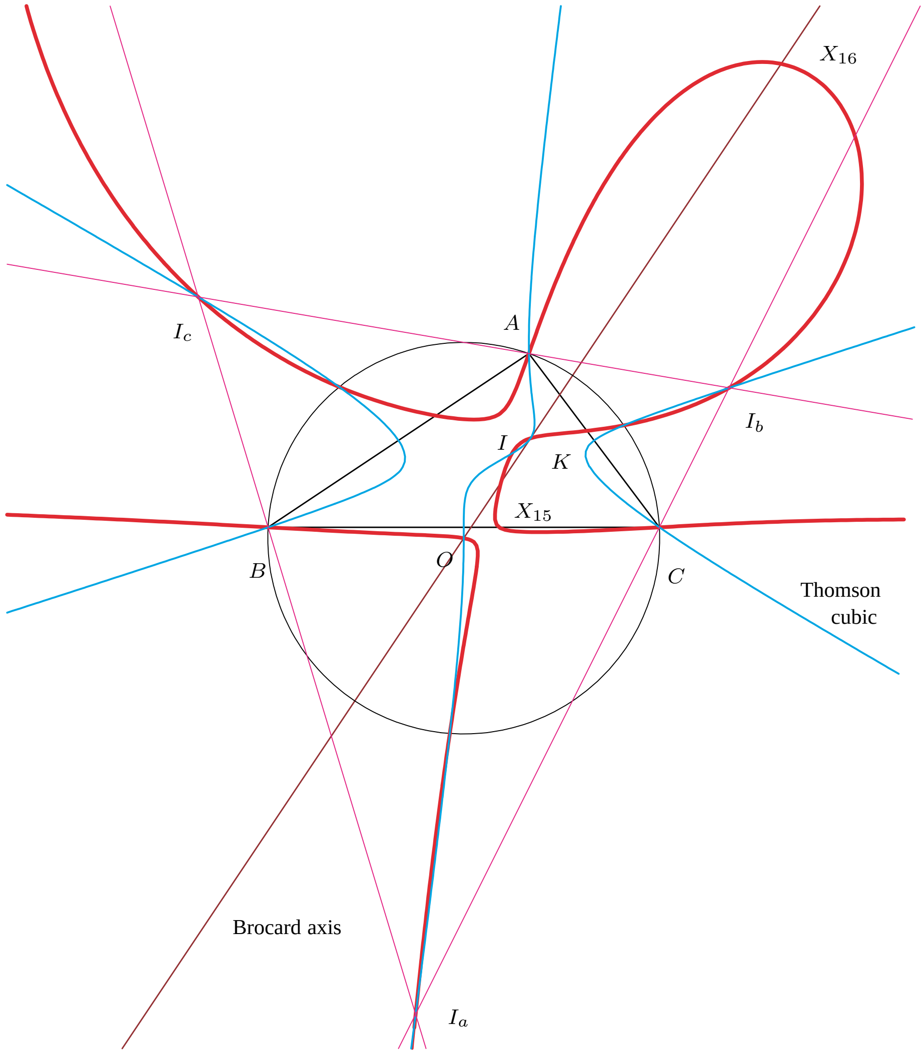

Figure 14. The quartic $\mathcal{Q}_2$

The isogonal conjugate of M is the perspector of ABC and one of the 27 Morley triangles. [50] We show that M lies on the quartic $\mathcal{Q}_2$. [51] Since $\cos A = \cos 3\varphi_1 = 4\cos^3\varphi_1 - 3\cos\varphi_1$, we have $\cos^3\varphi_1 = \frac{1}{4}(\cos A + 3\cos\varphi_1)$ and similar identities for $\cos^3\varphi_2$ and $\cos^3\varphi_3$. From this and the equation of $\mathcal{Q}_2$, we obtain

$$\sum_{\text{cyclic}} a^4 S_A b\cos\varphi_2\ c\cos\varphi_3\ (c^2b^2\cos^2\varphi_2 - b^2c^2\cos^2\varphi_3)$$

[50] For example, with $k_1 = k_2 = k_3 = 0$, $M^* = X_{357}$ and $M = X_{358}$.
[51] Consequently, M^* lies on the quintic $\mathcal{Q}_1$. See Proposition 14(10).

$$
\begin{aligned}
&= \sum_{\text{cyclic}} a^4b^3c^3S_A(\cos\varphi_3\ \cos^3\varphi_2 - \cos\varphi_2\ \cos^3\varphi_3) \\
&= \sum_{\text{cyclic}} \frac{1}{4}\, a^4b^3c^3S_A(\cos\varphi_3\cos B - \cos\varphi_2\cos C) \\
&= \sum_{\text{cyclic}} \frac{1}{4}\, a^4b^3c^3S_A\left(\frac{S_B}{ac}\cos\varphi_3 - \frac{S_C}{ab}\cos\varphi_2\right) \\
&= \frac{1}{4}\, a^3b^3c^3S_AS_BS_C \sum_{\text{cyclic}}\left(\frac{\cos\varphi_3}{c\,S_C} - \frac{\cos\varphi_2}{b\,S_B}\right) \\
&= 0.
\end{aligned}
$$

This completes the proof of (10).

Remark. $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are *strong* curves in the sense that they are invariant under extraversions: any point lying on one of them has its three extraversions also on the curve. [52]

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[52] The extraversions of a point are obtained by replacing one of a, b, c by its opposite. For example, the extraversions of the incenter I are the three excenters and I is said to be a *weak* point. On the contrary, K is said to be a ”strong” point.

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On the Procircumcenter and Related Points

Alexei Myakishev

Abstract. Given a triangle ABC, we solve the construction problem of a point P, together with points B_c, C_b on BC, C_a, A_c on CA, and A_b, B_a on AB such that PB_aC_a, A_bPC_b, and A_cB_cP are congruent triangles similar to ABC. There are altogether seven such triads. If these three congruent triangles are all oppositely similar to ABC, then P must be the procircumcenter, with trilinear coordinates $(a^2 \cos A : b^2 \cos B : c^2 \cos C)$. If at least one of the triangles in the triad is directly similar to ABC, then P is either a vertex or the midpoint of a side of the tangential triangle. We also determine the ratio of similarity in each case.

1. Introduction

Given a triangle ABC, we consider the construction of a point P, together with points B_c, C_b on BC, C_a, A_c on CA, and A_b, B_a on AB such that PB_aC_a, A_bPC_b, and A_cB_cP are congruent triangles similar to ABC. We first consider in §§2,3 the case when these triangles are all *oppositely* similar to ABC. See Figure 1. In §4, the possibilities when at least one of these congruent triangles is directly similar to ABC are considered. See, for example, Figure 2.

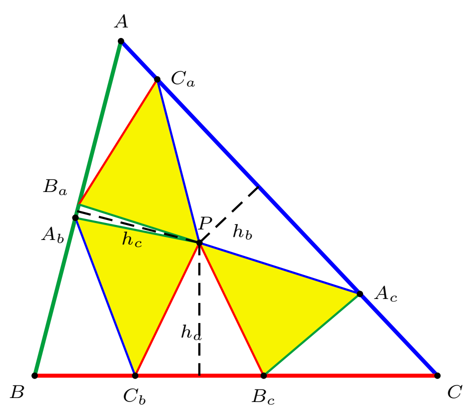

Figure 1

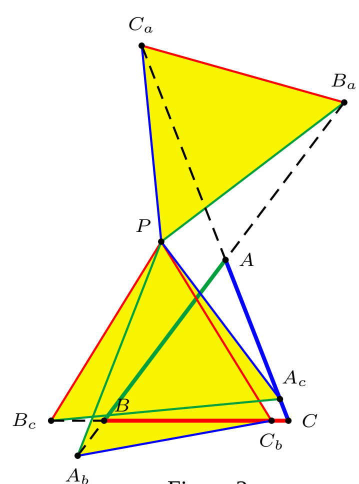

Figure 2

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The author thanks Jean-Pierre Ehrmann and Paul Yiu for their helps in the preparation of this paper.

2. The case of opposite similarity: construction of P

With reference to Figure 1, we try to find the trilinear coordinates of P. As usual, we denote the lengths of the sides opposite to angles A, B, C by a, b, c. Denote the *oriented* angles C_bPB_c by φ_a, A_cPC_a by φ_b, and B_aPA_b by φ_c. [1] Since $PC_b = PB_c$, $\angle PB_cC_b = \frac{1}{2}(\pi - \varphi_a)$. Since also $\angle PB_cA_c = B$, we have $\angle A_cB_cC = \frac{1}{2}(\pi + \varphi_a) - B$. For the same reason, $\angle B_cA_cC = \frac{1}{2}(\pi + \varphi_b) - A$. Considering the sum of the angles in triangle A_cB_cC, we have $\frac{1}{2}(\varphi_a + \varphi_b) = \pi - 2C$. Since $\varphi_a + \varphi_b + \varphi_c = \pi$, we have $\varphi_c = 4C - \pi$. Similarly, $\varphi_a = 4A - \pi$ and $\varphi_b = 4B - \pi$.

Let k be the ratio of similarity of the triangles PB_aC_a, A_bPC_b, and A_cB_cP with ABC, *i.e.*, $B_aC_a = PC_b = B_cP = k \cdot BC = ka$. The perpendicular distance from P to the line BC is

$$h_a = ka \cos \frac{\varphi_a}{2} = ka \cos \left(2A - \frac{\pi}{2}\right) = ka \sin 2A.$$

Similarly, the perpendicular distances from P to CA and AB are $h_b = kb \sin 2B$ and $h_c = kc \sin 2C$. It follows that P has trilinear coordinates,

$$(a \sin 2A : b \sin 2B : c \sin 2C) \sim (a^2 \cos A : b^2 \cos B : c^2 \cos C). \tag{1}$$

Note that we have found not only the trilinears of P, but also the angles of isosceles triangles PC_bB_c, PA_cC_a, PB_aA_b. It is therefore easy to construct the triangles by ruler and compass from P. Now, we easily identify P as the isogonal conjugate of the isotomic conjugate of the circumcenter O, which has trilinear coordinates $(\cos A : \cos B : \cos C)$. We denote this point by $\overline{O}$ and follow John H. Conway in calling it the *procircumcenter* of triangle ABC. We summarize the results in the following proposition.

Proposition 1. *Given a triangle ABC* not *satisfying (2), the point P for which there are congruent triangles PB_aC_a, A_bPC_b, and A_cB_cP oppositely similar to ABC (with B_c, C_b on BC, C_a, A_c on CA, and A_b, B_a on AB) is the procircumcenter $\overline{O}$. This is a finite point unless the given triangle satisfies*

$$a^4(b^2 + c^2 - a^2) + b^4(c^2 + a^2 - b^2) + c^4(a^2 + b^2 - c^2) = 0. \tag{2}$$

The procircumcenter $\overline{O}$ appears as X_{184} in [3], and is identified as the inverse of the Jerabek center X_{125} in the Brocard circle. A simple construction of $\overline{O}$ is made possible by the following property discovered by Fred Lang.

Proposition 2 (Lang [4])**.** *Let the perpendicular bisectors of BC, CA, AB intersect the other pairs of sides at B_1, C_1, C_2, A_2, A_3, B_3 respectively. The perpendicular bisectors of B_1C_1, C_2A_2 and A_3B_3 bound a triangle homothetic to ABC at the procircumcenter $\overline{O}$.*

---

[1] We regard the orientation of triangle ABC as positive. The oriented angles are defined modulo 2π.

3. The case of opposite similarity: ratio of similarity

We proceed to determine the ratio of similarity k. We shall make use of the following lemmas.

Lemma 3. *Let $\triangle$ denote the area of triangle ABC, and R its circumradius.*
(1) $\triangle = 2R^2 \sin A \sin B \sin C$;
(2) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$;
(3) $\sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \sin 2B \sin 2C$;
(4) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$.

Proof. (1) By the law of sines,

$$\triangle = \frac{1}{2} bc \sin A = \frac{1}{2}(2R \sin B)(2R \sin C) \sin A = 2R^2 \sin A \sin B \sin C.$$

For (2),

$$\begin{aligned}
&\sin 2A + \sin 2B + \sin 2C \\
=&2 \sin A \cos A + 2 \sin(B + C) \cos(B - C) \\
=&2 \sin A(-\cos(B + C) + \cos(B - C)) \\
=&4 \sin A \sin B \sin C.
\end{aligned}$$

The proof of (3) is similar. For (4),

$$\begin{aligned}
&\sin^2 A + \sin^2 B + \sin^2 C \\
=&\sin^2 A + 1 - \frac{1}{2}(\cos 2B + \cos 2C) \\
=&\sin^2 A + 1 - \cos(B + C) \cos(B - C) \\
=&2 - \cos^2 A + \cos A \cos(B - C) \\
=&2 + \cos A(\cos(B + C) + \cos(B - C)) \\
=&2 + 2 \cos A \cos B \cos C.
\end{aligned}$$

$\square$

Lemma 4. $a^2 + b^2 + c^2 = 9R^2 - OH^2$*, where R is the circumradius, and O, H are respectively the circumcenter and orthocenter of triangle ABC.*

This was originally due to Euler. An equivalent statement

$$a^2 + b^2 + c^2 = 9(R^2 - OG^2),$$

where G is the centroid of triangle ABC, can be found in [2, p.175].

Proposition 5 (Dergiades [1])**.** *The ratio of similarity of $\overline{O}B_aC_a$, $A_b\overline{O}C_b$, and $A_cB_c\overline{O}$ with ABC is*

$$k = \left| \frac{R^2}{3R^2 - OH^2} \right| .$$

Proof. Since $2\triangle = a \cdot h_a + b \cdot h_b + c \cdot h_c$, and $h_a = ka \sin 2A$, $h_b = kb \sin 2B$, and $h_c = kc \sin 2C$, the ratio of similarity is the absolute value of

$$\begin{aligned}
& \frac{2\triangle}{a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C} \\
= & \frac{4R^2 \sin A \sin B \sin C}{4R^2(\sin^2 A \sin 2A + \sin^2 B \sin 2B + \sin^2 C \sin 2C)} && \text{[Lemma 3(1)]} \\
= & \frac{2 \sin A \sin B \sin C}{(1 - \cos 2A) \sin 2A + (1 - \cos 2B) \sin 2B + (1 - \cos 2C) \sin 2C} \\
= & \frac{4 \sin A \sin B \sin C}{2(\sin 2A + \sin 2B + \sin 2C) - (\sin 4A + \sin 4B + \sin 4C)} \\
= & \frac{4 \sin A \sin B \sin C}{8 \sin A \sin B \sin C + 4 \sin 2A \sin 2B \sin 2C} && \text{[Lemma 3(2,3)]} \\
= & \frac{1}{2 + 8 \cos A \cos B \cos C} \\
= & \frac{1}{4(\sin^2 A + \sin^2 B + \sin^2 C) - 6} && \text{[Lemma 3(4)]} \\
= & \frac{R^2}{a^2 + b^2 + c^2 - 6R^2} \\
= & \frac{R^2}{3R^2 - OH^2}
\end{aligned}$$

by Lemma 4. □

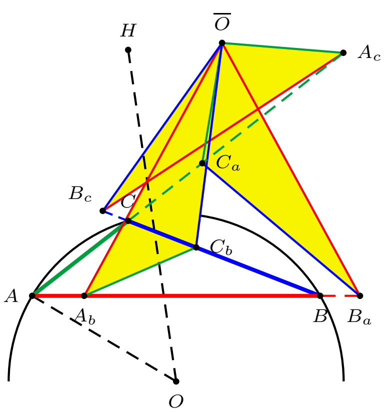

Figure 3: $OH = 2R$

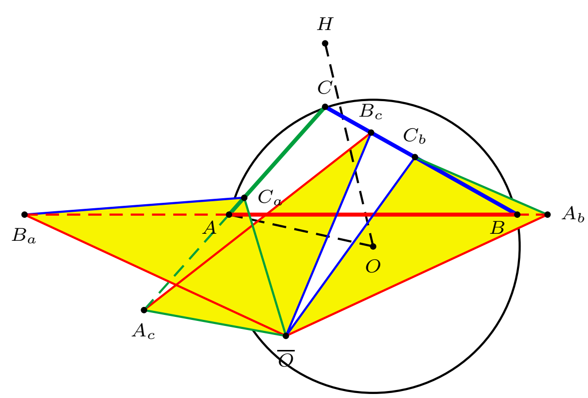

Figure 4: $OH = \sqrt{2}R$

From Proposition 5, we also infer that $\overline{O}$ is an infinite point if and only if $OH = \sqrt{3}R$. More interesting is that for triangles satisfying $OH = 2R$ or $\sqrt{2}R$, the congruent triangles in the triad are also congruent to the reference triangle ABC. See Figures 3 and 4. These are triangles satisfying

$$a^4(b^2 + c^2 - a^2) + b^4(c^2 + a^2 - b^2) + c^4(a^2 + b^2 - c^2) = \pm a^2b^2c^2.$$

4. Cases allowing direct similarity with ABC

As Jean-Pierre Ehrmann has pointed out, by considering all possible orientations of the triangles PB_aC_a, A_bPC_b, A_cB_cP, there are other points, apart from the procircumcenter $\overline{O}$, that yield triads of congruent triangles similar to ABC.

4.1. *Exactly one of the triangles oppositely similar to ABC.* Suppose, for example, that among the three congruent triangles, only PB_aC_a be oppositely similar to ABC, the other two, A_bPC_b and A_cB_cP being directly similar. We denote by P_a^+ the point P satisfying these conditions. Modifying the calculations in §2, we have

$$\varphi_a = \pi + 2A, \quad \varphi_b = \pi - 2A, \quad \varphi_c = \pi - 2A.$$

From these, we obtain the trilinears of P_a^+ as

$$(-a \sin A : b \sin A : c \sin A) = (-a : b : c).$$

It follows that P_a^+ is the A-vertex of the tangential triangle of ABC. See Figure 5.

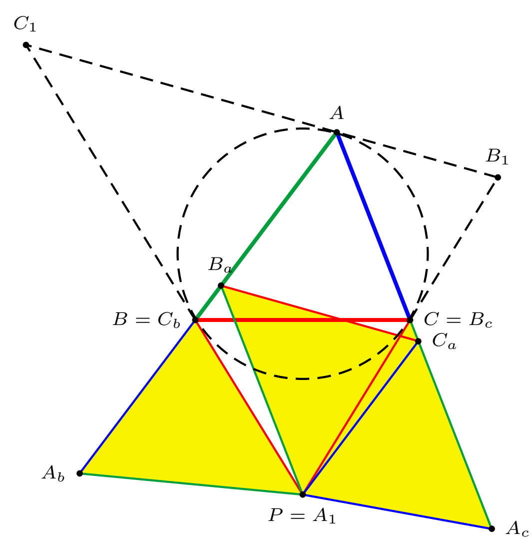

Figure 5

The ratio of similarity, by a calculation similar to that performed in §3, is $k = \left|\frac{1}{2\cos A}\right|$. This is equal to 1 only when $A = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. In these cases, the three triangles are congruent to ABC.

Clearly, there are two other triads of congruent triangles corresponding to the other two vertices of the tangential triangle.

4.2. *Exactly one of triangles directly similar to ABC.* Suppose, for example, that among the three congruent triangles, only PB_aC_a be directly similar to ABC, the other two, A_bPC_b and A_cB_cP being oppositely similar. We denote by P_a^- the point P satisfying these conditions. See Figure 6. In this case, we have

$$\varphi_a = 2A - \pi, \quad \varphi_b = \pi + 2B - 2C, \quad \varphi_c = \pi + 2C - 2B.$$

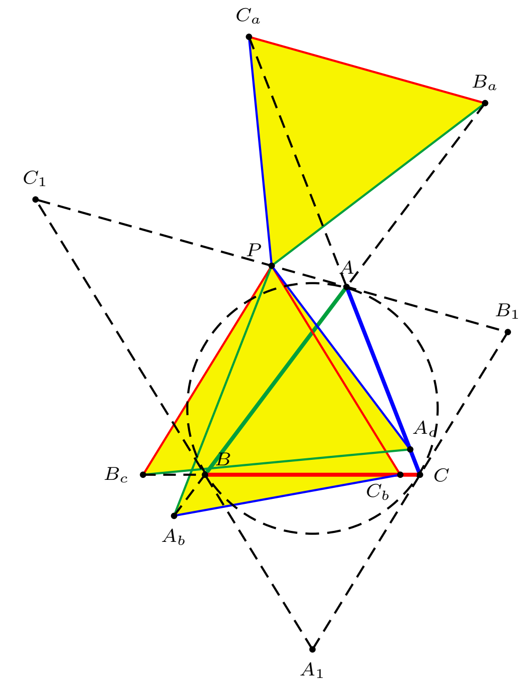

Figure 6

From these, we obtain the trilinears of P_a^- as

$$(-a\sin A : b\sin(B-C) : c\sin(C-B)) = (-a^3 : b(b^2-c^2) : c(c^2-b^2)).$$

It is easy to check that this is the midpoint of the side B_1C_1 of the tangential triangle of ABC. In this case, the ratio of similarity is $k = \left|\frac{1}{4\cos B\cos C}\right|$.

Clearly, there are two other triads of congruent triangles corresponding to the midpoints of the remaining two sides of the tangential triangle.

We conclude with the remark that it is not possible for all three of the congruent triangles to be directly similar to ABC, since this would require $\varphi_a = \varphi_b = \varphi_c = \pi$.

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Bicentric Pairs of Points and Related Triangle Centers

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Abstract. Bicentric pairs of points in the plane of triangle ABC occur in connection with three configurations: (1) cevian traces of a triangle center; (2) points of intersection of a central line and central circumconic; and (3) vertex-products of bicentric triangles. These bicentric pairs are formulated using trilinear coordinates. Various binary operations, when applied to bicentric pairs, yield triangle centers.

1. Introduction

Much of modern triangle geometry is carried out in in one or the other of two homogeneous coordinate systems: barycentric and trilinear. Definitions of triangle center, central line, and bicentric pair, given in [2] in terms of trilinears, carry over readily to barycentric definitions and representations. In this paper, we choose to work in trilinears, except as otherwise noted.

Definitions of *triangle center* (or simply *center*) and *bicentric pair* will now be briefly summarized. A triangle center is a point (as defined in [2] as a function of variables a, b, c that are sidelengths of a triangle) of the form

$$f(a, b, c) : f(b, c, a) : f(c, a, b),$$

where f is homogeneous in a, b, c, and

$$|f(a, c, b)| = |f(a, b, c)|. \tag{1}$$

If a point satisfies the other defining conditions but (1) fails, then the points

$$\begin{aligned} F_{ab} &:= f(a, b, c) : f(b, c, a) : f(c, a, b), \\ F_{ac} &:= f(a, c, b) : f(b, a, c) : f(c, b, a) \end{aligned} \tag{2}$$

are a *bicentric pair*. An example is the pair of Brocard points,

$$c/b : a/c : b/a \quad \text{and} \quad b/c : b/a : c/b.$$

Seven binary operations that carry bicentric pairs to centers are discussed in §§2, 3, along with three bicentric pairs associated with a center. In §4, bicentric pairs associated with cevian traces on the sidelines BC, CA, AB will be examined. §§6–10 examine points of intersection of a central line and central circumconic; these points are sometimes centers and sometimes bicentric pairs. §11 considers

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bicentric pairs associated with bicentric triangles. §5 supports §6, and §12 revisits two operations discussed in §3.

2. Products: trilinear and barycentric

Suppose $U = u : v : w$ and $X = x : y : z$ are points expressed in general homogeneous coordinates. Their product is defined by

$$U \cdot X = ux : vy : wz.$$

Thus, when coordinates are specified as trilinear or barycentric, we have here two distinct product operations, corresponding to constructions of barycentric product [8] and trilinear product [6]. Because we have chosen trilinears as the primary means of representation in this paper, it is desirable to write, for future reference, a formula for barycentric product in terms of trilinear coordinates. To that end, suppose $u : v : w$ and $x : y : z$ are trilinear representations, so that in barycentrics,

$$U = au : bv : cw \quad \text{and} \quad X = ax : by : cz.$$

Then the barycentric product is $a^2ux : b^2vy : c^2wz$, and we conclude as follows: the trilinear representation for the barycentric product of $U = u : v : w$ and $X = x : y : z$, these being trilinear representations, is given by

$$U \cdot_\flat X = aux : bvy : cwz.$$

3. Other centralizing operations

Given a bicentric pair, aside from their trilinear and barycentric products, various other binary operations applied to the pair yield a center. Consider the bicentric pair (2). In [2, p. 48], the points

$$F_{ab} \oplus F_{ac} := f_{ab} + f_{ac} : f_{bc} + f_{ba} : f_{ca} + f_{cb} \tag{3}$$

and

$$F_{ab} \ominus F_{ac} := f_{ab} - f_{ac} : f_{bc} - f_{ba} : f_{ca} - f_{cb} \tag{4}$$

are observed to be triangle centers. See §8 for a geometric discussion.

Next, suppose that the points F_{ab} and F_{ac} do not lie on the line at infinity, $\mathcal{L}^\infty$, and consider normalized trilinears, represented thus:

$$F'_{ab} = (k_{ab}f_{ab}, k_{ab}f_{bc}, k_{ab}f_{ca}), \quad F'_{ac} = (k_{ac}f_{ac}, k_{ac}f_{ba}, k_{ac}f_{cb}), \tag{5}$$

where

$$k_{ab} := \frac{2\sigma}{af_{ab} + bf_{bc} + cf_{ca}},\ k_{ac} := \frac{2\sigma}{af_{ac} + bf_{ba} + cf_{cb}},\ \sigma := \text{area}(\triangle ABC).$$

These representations give

$$F'_{ab} \oplus F'_{ac} = k_{ab}f_{ab} + k_{ac}f_{ac} : k_{ab}f_{bc} + k_{ac}f_{ba} : k_{ab}f_{ca} + k_{ac}f_{cb}, \tag{6}$$

which for many choices of $f(a, b, c)$ differs from (3). In any case, (6) gives the the midpoint of the bicentric pair (2), and the harmonic conjugate of this midpoint with respect to F_{ab} and F_{ac} is the point in which the line $F_{ab}F_{ac}$ meets $\mathcal{L}^\infty$.

We turn now to another centralizing operation on the pair (2). Their line is given by the equation

$$\begin{vmatrix} \alpha & \beta & \gamma \\ f_{ab} & f_{bc} & f_{ca} \\ f_{ac} & f_{ba} & f_{cb} \end{vmatrix} = 0$$

and is a central line. Its trilinear pole, P, and the isogonal conjugate of P, given by

$$f_{bc}f_{cb} - f_{ca}f_{ba} : f_{ca}f_{ac} - f_{ab}f_{cb} : f_{ab}f_{ba} - f_{bc}f_{ac},$$

are triangle centers.

If

$$X := x : y : z = f(a,b,c) : f(b,c,a) : f(c,a,b)$$

is a triangle center other than X_1, then the points

$$Y := y : z : x \quad \text{and} \quad Z := z : x : y$$

are clearly bicentric. The operations discussed in §§2,3, applied to $\{Y, Z\}$, yield the following centers:

- trilinear product $= X_1/X$ (the indexing of centers as X_i follows [3]);
- barycentric product $= X_6/X$ (here, "/" signifies trilinear division);
- $Y \oplus Z = y + z : z + x : x + y$;
- $Y \ominus Z = y - z : z - x : x - y$;
- midpoint $= m(a,b,c) : m(b,c,a) : m(c,a,b)$, where

$$m(a,b,c) = cy^2 + bz^2 + 2ayz + x(by + cz);$$

- $YZ \cap \mathcal{L}^\infty = n(a,b,c) : n(b,c,a) : n(c,a,b)$, where

$$n(a,b,c) = cy^2 - bz^2 + x(by - cz);$$

- (isogonal conjugate of trilinear pole of YZ)

$$\begin{aligned} &= x^2 - yz : y^2 - zx : z^2 - xy \\ &= (X_1\text{-Hirst inverse of } X). \end{aligned}$$

The points Z/Y and Y/Z are bicentric and readily yield the centers with first coordinates $x(y^2 + z^2)$, $x(y^2 - z^2)$, and $x^3 - y^2z^2/x$. One more way to make bicentric pairs from triangle centers will be mentioned: if $U = r : s : t$ and $X := x : y : z$ are centers, then ([2, p.49])

$$U \otimes X := sz : tx : ry, \qquad X \otimes U := ty : rz : sx$$

are a bicentric pair. For example, $(U \otimes X) \ominus (X \otimes U)$ has for trilinears the coefficients for line UX.

4. Bicentric pairs associated with cevian traces

Suppose P is a point in the plane of $\triangle ABC$ but not on one of the sidelines BC, CA, AB and not on $\mathcal{L}^\infty$. Let A', B', C' be the points in which the lines AP, BP, CP meet the sidelines BC, CA, AB, respectively. The points A', B', C' are the *cevian traces* of P. Letting $|XY|$ denote the directed length of a segment from a point X to a point Y, we recall a fundamental theorem of triangle geometry (often called Ceva's Theorem, but Hogendijk [1] concludes that it was stated and proved by an ancient king) as follows:

$$|BA'| \cdot |CB'| \cdot |AC'| = |A'C| \cdot |B'A| \cdot |C'B|.$$

(The theorem will not be invoked in the sequel.) We shall soon see that if P is a center, then the points

$$P_{BC} := |BA'| : |CB'| : |AC'| \quad \text{and} \quad P_{CB} := |A'C| : |B'A| : |C'B|$$

comprise a bicentric pair, except for $P = centroid$, in which case both points are the incenter. Let σ denote the area of $\triangle ABC$, and write $P = x : y : z$. Then the actual trilinear distances are given by

$$B = \left(0, \frac{2\sigma}{b}, 0\right) \quad \text{and} \quad A' = \left(0, \frac{2\sigma y}{by+cz}, \frac{2\sigma z}{by+cz}\right).$$

Substituting these into a distance formula (e.g. [2, p. 31]) and simplifying give

$$\begin{aligned} |BA'| &= \frac{z}{b(by+cz)}; \\ P_{BC} &= \frac{z}{b(by+cz)} : \frac{x}{c(cz+ax)} : \frac{y}{a(ax+by)}; \qquad (7) \\ P_{CB} &= \frac{y}{c(by+cz)} : \frac{z}{a(cz+ax)} : \frac{x}{b(ax+by)}. \qquad (8) \end{aligned}$$

So represented, it is clear that P_{BC} and P_{CB} comprise a bicentric pair if P is a center other that the centroid. Next, let

$$P'_{BC} = \frac{|BA'|}{|CA'|} : \frac{|CB'|}{|AB'|} : \frac{|AC'|}{|BC'|} \quad \text{and} \quad P'_{CB} = \frac{|CA'|}{|BA'|} : \frac{|AB'|}{|CB'|} : \frac{|BC'|}{|AC'|}.$$

Equation (7) implies

$$P'_{BC} = \frac{cz}{by} : \frac{ax}{cz} : \frac{by}{ax} \quad \text{and} \quad P'_{CB} = \frac{by}{cz} : \frac{cz}{ax} : \frac{ax}{by}. \qquad (9)$$

Thus, using "/" for trilinear quotient, or for barycentric quotient in case the coordinates in (7) and (8) are barycentrics, we have $P'_{BC} = P_{BC}/P_{CB}$ and $P'_{CB} = P_{CB}/P_{BC}$. The pair of isogonal conjugates in (9) generalize the previously mentioned Brocard points, represented by (9) when $P = X_1$.

As has been noted elsewhere, the trilinear (and hence barycentric) product of a bicentric pair is a triangle center. For both kinds of product, the representation is given by

$$P_{BC} \cdot P_{CB} = \frac{a}{x(by+cz)^2} : \frac{b}{y(cz+ax)^2} : \frac{c}{z(ax+by)^2}.$$

P	X_2	X_1	X_{75}	X_4	X_{69}	X_7	X_8
$P_{BC} \cdot P_{CB}$	X_{31}	X_{593}	X_{593}	X_{92}	X_{92}	X_{57}	X_{57}
$P_{BC} \cdot_{\flat} P_{CB}$	X_{32}	X_{849}	X_{849}	X_4	X_4	X_{56}	X_{56}

Table 1. Examples of trilinear and barycentric products

The line of a bicentric pair is clearly a central line. In particular, the line $P'_{BC}P'_{CB}$ is given by the equation

$$\left(\frac{a^2x^2}{bcyz} - \frac{bcyz}{a^2x^2}\right)\alpha + \left(\frac{b^2y^2}{cazx} - \frac{cazx}{b^2y^2}\right)\beta + \left(\frac{c^2z^2}{abxy} - \frac{abxy}{c^2z^2}\right)\gamma = 0.$$

This is the trilinear polar of the isogonal conjugate of the E-Hirst inverse of F, where $E = ax : by : cz$, and F is the isogonal conjugate of E. In other words, the point whose trilinears are the coefficients for the line $P'_{BC}P'_{CB}$ is the E-Hirst inverse of F.

The line $P_{BC}P_{CB}$ is given by $x'\alpha + y'\beta + z'\gamma = 0$, where

$$x' = bc(by + cz)(a^2x^2 - bcyz),$$

so that $P_{BC}P_{CB}$ is seen to be a certain product of centers if P is a center.

Regarding a euclidean construction for P_{BC}, it is easy to transfer distances for this purpose. Informally, we may describe P_{BC} and P'_{BC} as points constructed "by rotating through 90° the corresponding relative distances of the cevian traces from the vertices A, B, C".

5. The square of a line

Although this section does not involve bicentric pairs directly, the main result will make an appearance in §7, and it may also be of interest *per se*.

Suppose that $U_1 = u_1 : v_1 : w_1$ and $U_2 = u_2 : v_2 : w_2$ are distinct points on an arbitrary line L, represented in general homogeneous coordinates relative to $\triangle ABC$. For each point

$$X := u_1 + u_2t : v_1 + v_2t : w_1 + w_2t,$$

let

$$X^2 := (u_1 + u_2t)^2 : (v_1 + v_2t)^2 : (w_1 + w_2t)^2.$$

The locus of X^2 as t traverses the real number line is a conic section. Following the method in [4], we find an equation for this locus:

$$l^4\alpha^2 + m^4\beta^2 + n^4\gamma^2 - 2m^2n^2\beta\gamma - 2n^2l^2\gamma\alpha - 2l^2m^2\alpha\beta = 0, \qquad (10)$$

where l, m, n are coefficients for the line U_1U_2; that is,

$$l : m : n = v_1w_2 - w_1v_2 : w_1u_2 - u_1w_2 : u_1v_2 - v_1u_2.$$

Equation (10) represents an inscribed ellipse, which we denote by L^2. If the coordinates are trilinears, then the center of the ellipse is the point

$$bn^2 + cm^2 : cl^2 + an^2 : am^2 + bl^2.$$

6. (Line L)$\cap$(Circumconic Γ), two methods

Returning to general homogeneous coordinates, suppose that line L, given by $l\alpha + m\beta + n\gamma = 0$, meets circumconic Γ, given by $u/\alpha + v/\beta + w/\gamma = 0$. Let R and S denote the points of intersection, where $R = S$ if L is tangent to Γ. Substituting $-(m\beta + n\gamma)/l$ for α yields

$$mw\beta^2 + (mv + nw - lu)\beta\gamma + nv\gamma^2 = 0, \tag{11}$$

with discriminant

$$D := l^2u^2 + m^2v^2 + n^2w^2 - 2mnvw - 2nlwu - 2lmuv, \tag{12}$$

so that solutions of (11) are given by

$$\frac{\beta}{\gamma} = \frac{lu - mv - nw \pm \sqrt{D}}{2mw}. \tag{13}$$

Putting β and γ equal to the numerator and denominator, respectively, of the right-hand side (13), putting $\alpha = -(m\beta + n\gamma)/l$, and simplifying give for R and S the representation

$$x_1 : y_1 : z_1 = m(mv - lu - nw \mp \sqrt{D}) : l(lu - mv - nw \pm \sqrt{D}) : 2lmw. \tag{14}$$

Cyclically, we obtain two more representations for R and S:

$$x_2 : y_2 : z_2 = 2mnu : n(nw - mv - lu \mp \sqrt{D}) : m(mv - nw - lu \pm \sqrt{D}) \tag{15}$$

and

$$x_3 : y_3 : z_3 = n(nw - lu - mv \pm \sqrt{D}) : 2nlv : l(lu - nw - mv \mp \sqrt{D}). \tag{16}$$

Multiplying the equal points in (14)-(16) gives R^3 and S^3 as

$$x_1x_2x_3 : y_1y_2y_3 : z_1z_2z_3$$

in cyclic form. The first coordinates in this form are

$$2m^2n^2u(mv - lu - nw \mp \sqrt{D})(nw - lu - mv \pm \sqrt{D}),$$

and these yield

$$\text{(1st coordinate of } R^3) = m^2n^2u[l^2u^2 - (mv - nw - \sqrt{D})^2] \tag{17}$$

$$\text{(1st coordinate of } S^3) = m^2n^2u[l^2u^2 - (mv - nw + \sqrt{D})^2]. \tag{18}$$

The 2nd and 3rd coordinates are determined cyclically.

In general, products (as in §2) of points on Γ intercepted by a line are notable: multiplying the first coordinates shown in (17) and (18) gives

$$\text{(1st coordinate of } R^3 \cdot S^3) = l^2m^5n^5u^4vw,$$

so that

$$R \cdot S = mnu : nlv : lmw.$$

Thus, on writing $L = l : m : n$ and $U = u : v : w$, we have $R \cdot S = U/L$.

The above method for finding coordinates of R and S in symmetric form could be called the multiplicative method. There is also an additive method.[1] Adding the coordinates of the points in (14) gives

$$m(mv-lu-nw):l(lu-mv-nw):2lmw.$$

Do the same using (15) and (16), then add coordinates of all three resulting points, obtaining the point $U = u_1 : u_2 : u_3$, where

$$\begin{aligned}u_1 &= (lm+ln-2mn)u+(m-n)(nw-mv)\\ u_2 &= (mn+ml-2nl)v+(n-l)(lu-nw)\\ u_3 &= (nl+nm-2lm)w+(l-m)(mv-lu).\end{aligned}$$

Obviously, the point

$$V = v_1 : v_2 : v_3 = m-n : n-l : l-m$$

also lies on L, so that L is given parametrically by

$$u_1+tv_1 : u_2+tv_2 : u_3+tv_3. \tag{19}$$

Substituting into the equation for Γ gives

$$u(u_2+tv_2)(u_3+tv_3)+v(u_3+tv_3)(u_1+tv_1)+w(u_1+tv_1)(u_2+tv_2)=0.$$

The expression of the left side factors as

$$(t^2-D)F=0, \tag{20}$$

where

$$F = u(n-l)(l-m)+v(l-m)(m-n)+w(m-n)(n-l).$$

Equation (20) indicates two cases:

Case 1: $F=0$. Here, V lies on both L and Γ, and it is then easy to check that the point

$$W = mnu(n-l)(l-m) : nlv(l-m)(m-n) : lmw(m-n)(n-l)$$

also lies on both.

Case 2: $F \neq 0$. By (20), the points of intersection are

$$u_1 \pm v_1\sqrt{D} : u_2 \pm v_2\sqrt{D} : u_3 \pm v_3\sqrt{D}. \tag{21}$$

As an example to illustrate Case 1, take $u(a,b,c) = (b-c)^2$ and $l(a,b,c) = a$. Then $D = (b-c)^2(c-a)^2(a-b)^2$, and the points of intersection are $b-c : c-a : a-b$ and $(b-c)/a : (c-a)/b : (a-b)/c$.

[1] I thank the Jean-Pierre Ehrmann for describing this method and its application.

7. $L \cap \Gamma$ **when** $D = 0$

The points R and S are identical if and only if $D = 0$. In this case, if in equation (12) we regard either $l : m : n$ or $u : v : w$ as a variable $\alpha : \beta : \gamma$, then the resulting equation is that of a conic inscribed in $\triangle ABC$. In view of equation (10), we may also describe this locus in terms of squares of lines; to wit, if $u : v : w$ is the variable $\alpha : \beta : \gamma$, then the locus is the set of squares of points on the four lines indicated by the equations

$$\sqrt{|l|}\alpha \pm \sqrt{|m|}\beta \pm \sqrt{|n|}\gamma = 0.$$

Taking the coordinates to be trilinears, examples of centers $X_i = l : m : n$ and $X_j = u : v : w$ for which $D = 0$ are given in Table 2. It suffices to show results for $i \leq j$, since L and U are interchangeable in (12).

i	j
1	$244, 678$
2	$1015, 1017$
3	125
6	115
11	$55, 56, 181, 202, 203, 215$
31	$244, 1099, 1109, 1111$
44	44

Table 2. Examples for $D = 0$

8. $L \cap \Gamma$ **when** $D \neq 0$ **and** $l : m : n = u : v : w$

Returning to general homogeneous coordinates, suppose now that $l : m : n$ and $u : v : w$ are triangle centers for which $D \neq 0$. Then, sometimes, R and S are centers, and sometimes, a bicentric pair. We begin with the case $l : m : n = u : v : w$, for which (12) gives

$$D := (u + v + w)(u - v + w)(u + v - w)(u - v - w).$$

This factorization shows that if $u + v + w = 0$, then $D = 0$. We shall prove that converse also holds. Suppose $D = 0$ but $u + v + w \neq 0$. Then one of the other three factors must be 0, and by symmetry, they must all be 0, so that $u = v + w$, so that

$$\begin{aligned} u(a,b,c) &= v(a,b,c) + w(a,b,c) \\ u(a,b,c) &= u(b,c,a) + u(c,a,b) \\ u(b,c,a) &= u(c,a,b) + u(a,b,c). \end{aligned}$$

Applying the third equation to the second gives $u(a,b,c) = u(c,a,b) + u(a,b,c) + u(c,a,b)$, so that $u(a,b,c) = 0$, contrary to the hypothesis that U is a triangle center.

Writing the roots of (11) as r_2/r_3 and s_2/s_3, we find

$$\frac{r_2 s_2}{r_3 s_3} = \frac{(u^2 - v^2 - w^2 + \sqrt{D})(u^2 - v^2 - w^2 - \sqrt{D})}{4v^2 w^2} = 1,$$

which proves that R and S are a conjugate pair (isogonal conjugates in case the coordinates are trilinears). Of particular interest are cases for which these points are polynomial centers, as listed in Table 3, where, for convenience, we put

$$E := (b^2 - c^2)(c^2 - a^2)(a^2 - b^2).$$

u	$\sqrt{D}$	r_1	s_1
$a(b^2 - c^2)$	E	a	bc
$a(b^2 - c^2)(b^2 + c^2 - a^2)$	$16\sigma^2 E$	$\sec A$	$\cos A$
$a(b - c)(b + c - a)$	$4abc(b - c)(c - a)(a - b)$	$\cot(A/2)$	$\tan(A/2)$
$a^2(b^2 - c^2)(b^2 + c^2 - a^2)$	$4a^2b^2c^2E$	$\tan A$	$\cot A$
$bc(a^4 - b^2c^2)$	$(a^4 - b^2c^2)(b^4 - c^2a^2)(c^4 - a^2b^2)$	b/c	c/b

Table 3. Points $R = r_1 : r_2 : r_3$ and $S = s_1 : s_2 : s_3$ of intersection

In Table 3, the penultimate row indicates that for $u : v : w = X_{647}$, the Euler line meets the circumconic $u/\alpha + v/\beta + w/\gamma = 0$ in the points X_4 and X_3. The final row shows that R and S can be a bicentric pair.

9. $L \cap \Gamma$: Starting with Intersection Points

It is easy to check that a point R lies on Γ if and only if there exists a point $x : y : z$ for which

$$R = \frac{u}{by - cz} : \frac{v}{cz - ax} : \frac{w}{ax - by}.$$

From this representation, it follows that every line that meets Γ in distinct points

$$\frac{u}{by_i - cz_i} : \frac{v}{cz_i - ax_i} : \frac{w}{ax_i - by_i},\ i = 1, 2,$$

has the form

$$\frac{(by_1 - cz_1)(by_2 - cz_2)\alpha}{u} + \frac{(cz_1 - ax_1)(cz_2 - ax_2)\beta}{v} + \frac{(ax_1 - by_1)(ax_2 - by_2)\gamma}{w} = 0. \tag{22}$$

and conversely. In this case,

$$D = u^2v^2w^2 \begin{vmatrix} bc & ca & ab \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}^2,$$

indicating that $D = 0$ if and only if the points $x_i : y_i : z_i$ are collinear with the $bc : ca : ab$, which, in case the coordinates are trilinears, is the centroid of $\triangle ABC$.

Example 1. Let

$$x_1 : y_1 : z_1 = c/b : a/c : b/a \quad \text{and} \quad x_2 : y_2 : z_2 = b/c : c/a : a/b.$$

These are the 1st and 2nd Brocard points in case the coordinates are trilinears, but in any case, (22) represents the central line

$$\frac{\alpha}{ua^2(b^2 - c^2)} + \frac{\beta}{vb^2(c^2 - a^2)} + \frac{\gamma}{wc^2(a^2 - b^2)} = 0,$$

meeting Γ in the bicentric pair

$$\frac{u}{b^2(a^2-c^2)}:\frac{v}{c^2(b^2-a^2)}:\frac{w}{a^2(c^2-b^2)},\ \frac{u}{c^2(a^2-b^2)}:\frac{v}{a^2(b^2-c^2)}:\frac{w}{b^2(c^2-a^2)}.$$

Example 2. Let $X = x : y : z$ be a triangle center other than X_1, so that $y : z : x$ and $z : x : y$ are a bicentric pair. The points

$$\frac{u}{bz-cx}:\frac{v}{cx-ay}:\frac{w}{ay-bz},\quad \text{and}\quad \frac{u}{cy-bx}:\frac{v}{az-cy}:\frac{w}{bx-az}$$

are the bicentric pair in which the central line

$$vw(bx-cy)(cx-bz)\alpha+wu(cy-az)(ay-cx)\beta+uv(az-bx)(bz-ay)\gamma=0$$

meets Γ.

10. $L \cap \Gamma$: Euler Line and Circumcircle

Example 3. Using trilinears, the circumcircle is given by $u(a, b, c) = a$ and the Euler line by

$$l(a,b,c)=a(b^2-c^2)(b^2+c^2-a^2).$$

The discriminant $D = 4a^2b^2c^2d^2$, where

$$d=\sqrt{a^6+b^6+c^6+3a^2b^2c^2-b^2c^2(b^2+c^2)-c^2a^2(c^2+a^2)-a^2b^2(a^2+b^2)}.$$

Substitutions into (17) and (18) and simplification give the points of intersection, centers R and S, represented by 1st coordinates

$$\left\{\frac{[ca(a^2-c^2)\pm bd][ba(a^2-b^2)\pm cd]}{(b^2-c^2)^2(b^2+c^2-a^2)^2}\right\}^{1/3}.$$

11. Vertex-products of bicentric triangles

Suppose that $f(a, b, c) : g(b, c, a) : h(c, a, b)$ is a point, as defined in [2] We abbreviate this point as $f_{ab} : g_{bc} : h_{ca}$ and recall from [5, 7] that bicentric triangles are defined by the forms

$$\begin{pmatrix} f_{ab} & g_{bc} & h_{ca}\\ h_{ab} & f_{bc} & g_{ca}\\ g_{ab} & h_{bc} & f_{ca}\end{pmatrix}\ \text{and}\ \begin{pmatrix} f_{ac} & h_{ba} & g_{cb}\\ g_{ac} & f_{ba} & h_{cb}\\ g_{ac} & h_{ba} & f_{cb}\end{pmatrix}.$$

The vertices of the first of these two triangles are the rows of the first matrix, etc. We assume that $f_{ab}g_{ab}h_{ab} \neq 0$. Then the product of the three vertices, namely

$$f_{ab}g_{ab}h_{ab} : f_{bc}g_{bc}h_{bc} : f_{ca}g_{ca}h_{ca} \tag{23}$$

and the product of the vertices of the second triangle, namely

$$f_{ac}g_{ac}h_{ac} : f_{ba}g_{ba}h_{ba} : f_{cb}g_{cb}h_{cb} \tag{24}$$

clearly comprise a bicentric pair if they are distinct, and a triangle center otherwise.

Examples of bicentric pairs thus obtained will now be presented. An inductive method [6] of generating the non-circle-dependent objects of triangle geometry enumerates such objects in sets formally of size six. When the actual size is six, which means that no two of the six objects are identical, the objects form a pair of bicentric triangles. The least such pair for which $f_{ab}g_{ab}h_{ab} \neq 0$ are given by Objects 31-36:

$$\begin{pmatrix} b & c\cos B & -b\cos B \\ -c\cos C & c & a\cos C \\ b\cos A & -a\cos A & a \end{pmatrix} \text{ and } \begin{pmatrix} c & -c\cos C & b\cos C \\ c\cos A & a & -a\cos A \\ -b\cos B & a\cos B & b \end{pmatrix}.$$

In this example, the bicentric pair of points (23) and (24) are

$$\frac{b}{a\cos B} : \frac{c}{b\cos C} : \frac{a}{c\cos A} \quad \text{and} \quad \frac{c}{a\cos C} : \frac{a}{b\cos A} : \frac{b}{c\cos B},$$

and the product of these is the center $\cos A\csc^3 A : \cos B\csc^3 B : \cos C\csc^3 C$.

This example and others obtained successively from Generation 2 of the aforementioned enumeration are presented in Table 4. Column 1 tells the Object numbers in [5]; column 2, the A-vertex of the least Object; column 3, the first coordinate of point (23) after canceling a symmetric function of (a,b,c); and column 4, the first coordinate of the product of points (23) and (24) after canceling a symmetric function of (a,b,c). In Table 4, $\cos A, \cos B, \cos C$ are abbreviated as a_1, b_1, c_1, respectively.

Objects	$f_{ab} : g_{ab} : h_{ab}$	$[f_{ab}g_{ab}h_{ab}]$	$[f_{ab}f_{ac}g_{ab}g_{ac}h_{ab}h_{ac}]$
31-36	$b : cb_1 : -bb_1$	b/ab_1	a_1/a^3
37-42	$bc_1 : -ca_1 : ba_1$	bc_1/aa_1	$(aa_1)^{-3}$
43-48	$bb_1 : c : -b$	bb_1/a	$(a_1a^3)^{-1}$
49-54	$ab : -c^2 : bc$	b/c	1
58-63	$c + ba_1 : cc_1 : -bc_1$	$(ba_1 + c)/ac_1$	$a_1(ba_1 + c)(ca_1 + b)a^{-2}$
71-76	$-b_1^2 : c_1 : b_1$	b_1^2/a_1	a_1^{-4}
86-91	$c_1 - a_1b_1 : c_1^2 : b_1c_1$	$b_1(c_1 - a_1b_1)$	$[a_1(a_1 - b_1c_1)]^{-1}$
92-97	$a_1b_1 : 1 : -a_1$	b_1/c_1	1
98-103	$1 : -c_1 : c_1a_1$	b_1/c_1	1
104-109	$aa_1 : -c : ca_1$	a/cc_1	a^3a_1
110-115	$a : b : -ba_1$	ab_1/b	a^3/a_1
116-121	$c_1 - a_1b_1 : 1 : -a_1$	$b_1(c_1 - a_1b_1)$	$[a_1(a_1 - b_1c_1)]^{-1}$
122-127	$1 + a_1^2 : c_1 : -c_1a_1$	$b_1(1 + a_1^2)/c_1$	$(1 + a_1^2)^2$
128-133	$2a_1 : -b_1 : a_1b_1$	a_1	a_1^2

Table 4. Bicentric triangles, bicentric points, and central vertex-products

Table 4 includes examples of interest: (i) bicentric triangles for which (23) and (24) are identical and therefore represent a center; (ii) distinct pairs of bicentric triangles that yield the identical bicentric pairs of points; and (iii) cases in which the pair (23) and (24) are isogonal conjugates. Note that Objects 49-54 yield for (23) and (24) the 2nd Brocard point, $\Omega_2 = b/c : c/a : a/b$ and the 1st Brocard point, $\Omega_1 = c/b : a/c : b/a$.

12. Geometric discussion: $\oplus$ and $\ominus$

Equations (3) and (4) define operations $\oplus$ and $\ominus$ on pairs of bicentric points. Here, we shall consider the geometric meaning of these operations. First, note that one of the points in (2) lies on $\mathcal{L}^\infty$ if and only if the other lies on $\mathcal{L}^\infty$, since the transformation $(a,b,c) \to (a,c,b)$ carries each of the equations

$$af_{ab} + bf_{bc} + cf_{ca} = 0, \quad af_{ac} + bf_{ba} + cf_{cb} = 0$$

to the other. Accordingly, the discussion breaks into two cases.

Case 1: F_{ab} not on $\mathcal{L}^\infty$. Let k_{ab} and k_{ac} be the normalization factors given in §3. Then the actual directed trilinear distances of F_{ab} and F_{ac} (to the sidelines BC, CA, AB) are given by (5). The point F that separates the segment $F_{ab}F_{ac}$ into segments satisfying

$$\frac{|F_{ab}F|}{|FF_{ac}|} = \frac{k_{ab}}{k_{ac}},$$

where | | denotes directed length, is then

$$\frac{k_{ac}}{k_{ab}+k_{ac}}F'_{ab} + \frac{k_{ab}}{k_{ab}+k_{ac}}F'_{ac} = \frac{k_{ac}k_{ab}}{k_{ab}+k_{ac}}F_{ab} + \frac{k_{ab}k_{ac}}{k_{ab}+k_{ac}}F_{ac},$$

which, by homogeneity, equals $F_{ab} \oplus F_{ac}$. Similarly, the point "constructed" as

$$\frac{k_{ac}}{k_{ab}+k_{ac}}F'_{ab} - \frac{k_{ab}}{k_{ab}+k_{ac}}F'_{ac}$$

equals $F_{ab} \ominus F_{ac}$. These representations show that $F_{ab} \oplus F_{ac}$ and $F_{ab} \ominus F_{ac}$ are a harmonic conjugate pair with respect to F_{ab} and F_{ac}.

Case 2: F_{ab} on $\mathcal{L}^\infty$. In this case, the isogonal conjugates F_{ab}^{-1} and F_{ac}^{-1} lie on the circumcircle, so that Case 1 applies:

$$F_{ab}^{-1} \oplus F_{ac}^{-1} = \frac{f_{ab}+f_{ac}}{f_{ab}f_{ac}} : \frac{f_{bc}+f_{ba}}{f_{bc}f_{ba}} : \frac{f_{ca}+f_{cb}}{f_{ca}f_{cb}}.$$

Trilinear multiplication [6] by the center $F_{ab} \cdot F_{ac}$ gives

$$F_{ab} \oplus F_{ac} = (F_{ab}^{-1} \oplus F_{ac}^{-1}) \cdot F_{ab} \cdot F_{ac}.$$

In like manner, $F_{ab} \ominus F_{ac}$ is "constructed".

It is easy to prove that a pair P_{ab} and P_{ac} of bicentric points on $\mathcal{L}^\infty$ are necessarily given by

$$P_{ab} = bf_{ca} - cf_{bc} : cf_{ab} - af_{ca} : af_{bc} - bf_{ab}$$

for some bicentric pair as in (2). Consequently,

$$\begin{aligned} P_{ab} \oplus P_{ac} &= g(a,b,c) : g(b,c,a) : g(c,a,b), \\ P_{ab} \ominus P_{ac} &= h(a,b,c) : h(b,c,a) : h(c,a,b), \end{aligned}$$

where

$$\begin{aligned} g(a,b,c) &= b(f_{ca}+f_{cb}) - c(f_{bc}+f_{ba}), \\ h(a,b,c) &= b(f_{ca}-f_{cb}) + c(f_{ba}-f_{bc}). \end{aligned}$$

Example 4. We start with $f_{ab} = c/b$, so that F_{ab} and F_{ac} are the Brocard points, and P_{ab} and P_{ac} are given by 1st coordinates $a - c^2/a$ and $a - b^2/a$, respectively, yielding 1st coordinates $(2a^2 - b^2 - c^2)/a$ and $(b^2 - c^2)/a$ for $P_{ab} \oplus P_{ac}$ and $P_{ab} \ominus P_{ac}$. These points are the isogonal conjugates of X_{111} (the Parry point) and X_{110} (focus of the Kiepert parabola), respectively.

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Some Configurations of Triangle Centers

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Abstract. Many collections of triangle centers and symmetrically defined triangle points are vertices of configurations. This illustrates a high level of organization among the points and their collinearities. Some of the configurations illustrated are inscriptable in Neuberg's cubic curve and others arise from Monge's theorem.

1. Introduction

By a configuration $\mathcal{K}$ we shall mean a collection of p points and g lines with r points on each line and q lines meeting at each point. This implies the relationship $pq = gr$. We then say that $\mathcal{K}$ is a (p_q, g_r) configuration. The simplest configuration is a point with a line through it. Another example is the triangle configuration, $(3_2, 3_2)$ with $p = g = 3$ and $q = r = 2$. When $p = g$, $\mathcal{K}$ is called *self-dual*, and then we must also have $q = r$. The symbol for the configuration is now simplified to read (p_q). The smallest (n_3) self-dual configurations exist combinatorially, when the "lines" are considered as suitable triples of points (vertices), but they cannot be realized with lines in the Euclidean plane. Usually when configurations are presented graphically, the lines appear as segments to make the figure compact and easy to interpret. Only one (7_3) configuration exists, the Fano plane of projective geometry, and only one (8_3) configuration exists, the Möbius-Kantor configuration. Neither of these can be realized with straight line segments. For larger n, the symbol may not determine a configuration uniquely. The smallest (n_3) configurations consisting of line segments in the Euclidean plane are (9_3), and there are three of them, one of which is the familiar Pappus configuration [4, pp.94–170]. The number of distinct (n_3) configurations grows rapidly with n. For example, there are 228 different (12_3) configurations [11, p.40]. In the discussion here, we shall only be concerned with configurations lying in a plane.

While configurations have long been studied as combinatorial objects, it does not appear that in any examples the vertices have been identified with triangle-derived points. In recent years there has been a resurgence of interest in triangle geometry along with the recognition of many new special points defined in different very ways. Since each point is defined from original principles, it is somewhat surprising that so many of them are collinear in small sets. An even higher level of relationship among special points is seen when they can be incorporated into

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certain configurations of moderate size. Then the collinearities and their incidences are summarized in a tidy, symmetrical, and graphic way. Here we exhibit several configurations whose vertices are naturally defined by triangles and whose lines are collinearities among them. It happens that the general theory for the first three examples was worked out long ago, but then the configurations were not identified as consisting of familiar triangle points and their collinearities.

2. Some configurations inscriptable in a cubic

First let us set the notation for several triangles. Given a triangle $\mathbf{T}$ with vertices A, B, and C, let A^* be the reflection of vertex A in side BC, A_+ the apex of an equilateral triangle erected outward on BC, and A_- the apex of an equilateral triangle erected inward on BC. Similarly define the corresponding points for B and C. Denote the triangle with vertices A^*, B^*, C^* as $\mathbf{T}^*$ and similarly define the triangles $\mathbf{T}_+$ and $\mathbf{T}_-$. Using trilinear coordinates it is straightforward to verify that the four triangles above are pairwise in perspective to one another. The points of perspective are as follows.

	$\mathbf{T}$	$\mathbf{T}^*$	$\mathbf{T}_+$	$\mathbf{T}_-$
$\mathbf{T}$		H	F_+	F_-
$\mathbf{T}^*$	H		J_-	J_+
$\mathbf{T}_+$	F_+	J_-		O
$\mathbf{T}_-$	F_-	J_+	O	

Here, O and H are respectively the circumcenter and orthocenter, $F_\pm$ the isogonic (Fermat) points, and $J_\pm$ the isodynamic points. They are triangle centers as defined by Kimberling [5, 6, 7, 8], who gives their trilinear coordinates and discusses their geometric significance. See also the in §5. For a simple simultaneous construction of all these points, see Evans [2].

To assemble the configurations, we first need to identify certain sets of collinear points. Now it is advantageous to introduce a notation for collinearity. Write $\mathcal{L}(X, Y, Z, \dots)$ to denote the line containing X, Y, Z, ... The key to identifying configurations among all the previously mentioned points depends on the observation that A^* , B_+, and C_- are always collinear, so we may write $\mathcal{L}(A^*, B_+, C_-)$. One can easily verify this using trilinear coordinates. This is also true for any permutation of A, B, and C, so we have

(I): the 6 lines $\mathcal{L}(A^*, B_+, C_-)$, $\mathcal{L}(A^*, B_-, C_+)$, $\mathcal{L}(B^*, C_+, A_-)$, $\mathcal{L}(B^*, C_-, A_+)$, $\mathcal{L}(C^*, A_+, B_-)$, $\mathcal{L}(C^*, A_-, B_+)$.

They all occur in Figures 1, 2, and 3. In fact the nine points A_+, A_-, A^*, ... themselves form the vertices of a $(9_2, 6_3)$ configuration.

It is easy to see other collinearities, namely 3 from each pair of triangles in perspective. For example, triangles $\mathbf{T}_+$ and $\mathbf{T}_-$ are in perspective from O, so we have

(II): the 3 lines $\mathcal{L}(A_+, O, A_-)$, $\mathcal{L}(B_+, O, B_-)$ and $\mathcal{L}(C_+, O, C_-)$.

See Figure 2.

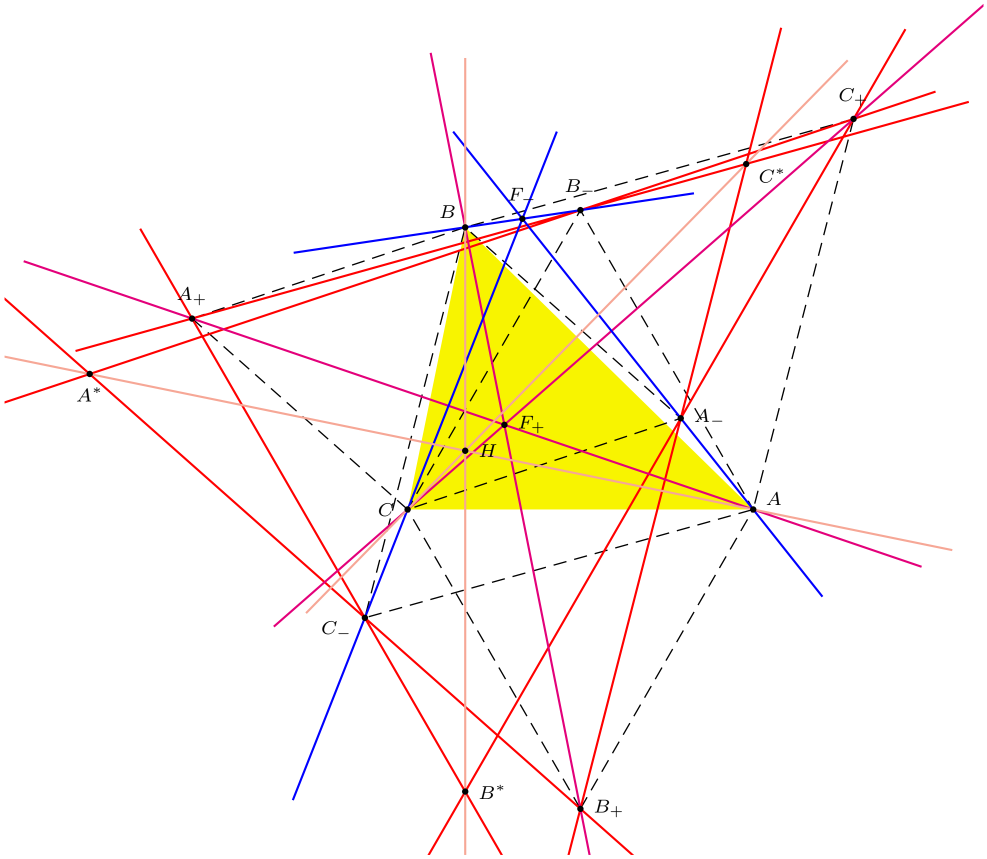

Figure 1. The Cremona-Richmond configuration

2.1. *The Cremona-Richmond configuration* (15_3). Consider the following sets of collinearities of three points:

(III): the 3 lines $\mathcal{L}(A, F_+, A_+)$, $\mathcal{L}(B, F_+, B_+)$ and $\mathcal{L}(C, F_+, C_+)$;
(IV): the 3 lines $\mathcal{L}(A, F_-, A_-)$, $\mathcal{L}(B, F_-, B_-)$ and $\mathcal{L}(C, F_-, C_-)$;
(V): the 3 lines $\mathcal{L}(A, H, A^*)$, $\mathcal{L}(B, H, B^*)$ and $\mathcal{L}(C, H, C^*)$.

The 15 points (A, B, C, A^*, B^*, C^*, $A_\pm$, $B_\pm$, $C_\pm$, H, $F_\pm$) and 15 lines in **(I), (III), (IV)**, and **(V)** form a figure which is called the Cremona-Richmond configuration [7]. See Figure 1. It has 3 lines meeting at each point with 3 points on each line, so it is self-dual with symbol (15_3). Inspection reveals that this configuration itself contains no triangles.

The reader may have noticed that the fifteen points in the configuration all lie on Neuberg's cubic curve, which is known to contain many triangle centers [7]. Recently a few papers, such as Pinkernell's [10] discussing Neuberg's cubic have

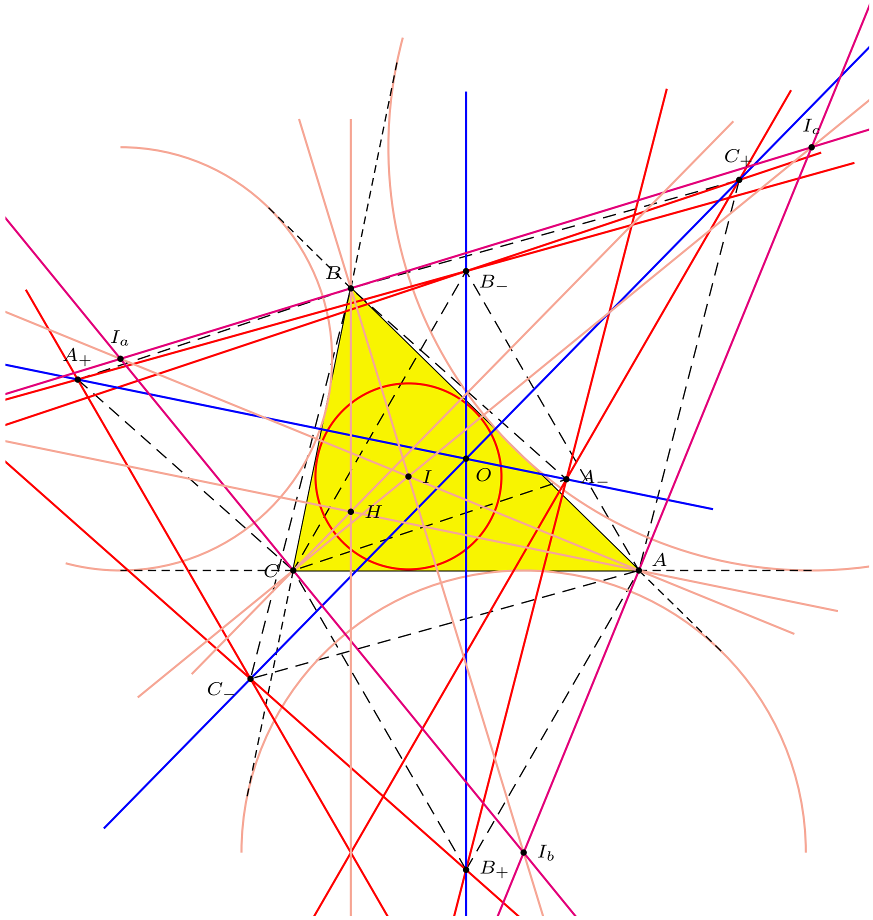

Figure 2

appeared, so we shall not elaborate on the curve itself. It has been known for a long time that many configurations are inscriptable in cubic curves, possibly first noticed by Schoenflies circa 1888 according to Feld [3]. However, it does not appear to be well-known that Neuberg's cubic in particular supports such configurations of familiar points. We shall exhibit two more configurations inscriptable in Neuberg's cubic.

2.2. *A* (18_3) *associated with the excentral triangle.* For another configuration, this one of the type (18_3), we employ the excentral triangle, that is, the triangle whose vertices are the excenters of $\mathbf{T}$. Denote the excenter opposite vertex A by I_a, etc., and denote the extriangle as $\mathbf{T}_{\mathrm{x}}$. Triangles $\mathbf{T}$ and $\mathbf{T}_{\mathrm{x}}$ are in perspective from the incenter, I. This introduces two more sets of collinearities involving the excenters:

(VI): the 3 lines $\mathcal{L}(A, I, I_a)$, $\mathcal{L}(B, I, I_b)$ and $\mathcal{L}(C, I, I_c)$;
(VII): the 3 lines $\mathcal{L}(I_b, A, I_c)$, $\mathcal{L}(I_c, B, I_a)$ and $\mathcal{L}(I_a, C, I_b)$.

The 18 lines of **(I), (II), (V), (VI), (VII)** and the 18 points A, B, C, I_a, I_b, I_c, A^*, B^*, C^*, $A_\pm$, $B_\pm$, $C_\pm$, O, H, and I form an (18_3) configuration. See Figure 2. There are enough points to suggest the outline of Neuberg's cubic, which is bipartite. The 10 points in the lower right portion of the figure lie on the ovoid portion of the curve. The 8 other points lie on the serpentine portion, which has an asymptote parallel to Euler's line (dashed). For other shapes of the basic triangle T, these points will not necessarily lie on the same components of the curve.

2.3. *A configuration* $(12_4, 16_3)$. Now we define two more sets of collinearities involving the isodynamic points:

(VIII): the 3 lines $\mathcal{L}(A^*, J_-, A_+)$, $\mathcal{L}(B^*, J_-, B_+)$ and $\mathcal{L}(C^*, J_-, C_+)$;
(IX): the 3 lines $\mathcal{L}(A^*, J_+, A_-)$, $\mathcal{L}(B^*, J_+, B_-)$ and $\mathcal{L}(C^*, J_+, C_-)$.

Among the centers of perspective we have defined so far, there is an additional collinearity, $\mathcal{L}(J_+, O, J_-)$, which is the Brocard axis. See Figure 3.

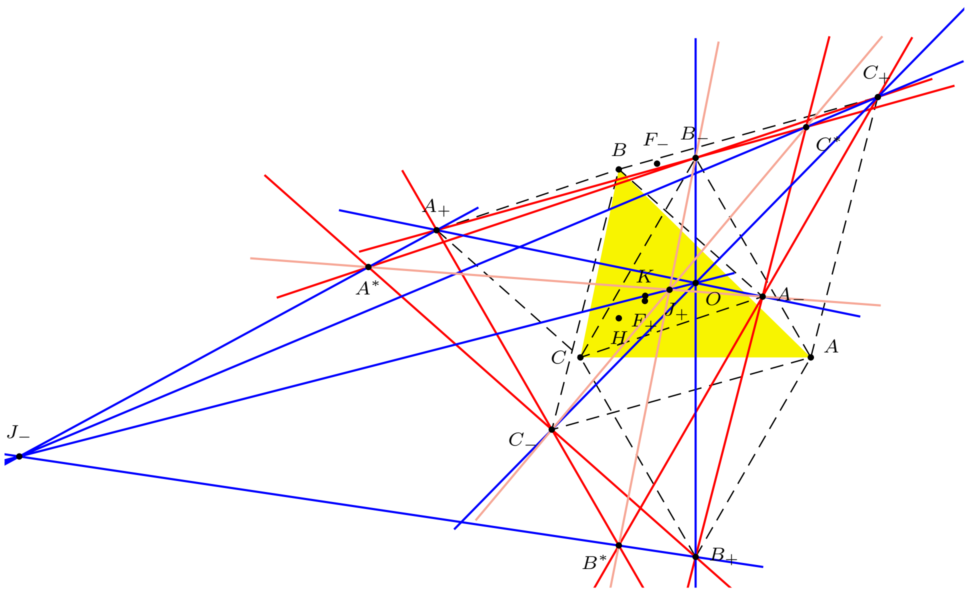

Figure 3

Using Weierstrass elliptic functions, Feld proved that within any bipartite cubic, a real configuration can be inscribed which has 12 points and 16 lines, with 4 lines meeting at each point and 3 points on each line [11], so that is, its symbol is $(12_4, 16_3)$. Now the Neuberg cubic of a non-equilateral triangle is bipartite, consisting of an ovoid portion and a serpentine portion whose asymptote is parallel to the Euler line of the triangle. Here one such inscriptable configuration consists of the following sets of lines: **(I), (II), (VIII), (IX)**, and the line, $\mathcal{L}(J_+, O, J_-)$. See Figure 3. The three triangles $\mathbf{T}_+$, $\mathbf{T}_-$, and $\mathbf{T}^*$ are pair-wise in perspective with collinear perspectors J_+, J_-, and O. The vertices of the basic triangle $\mathbf{T}$ are not in this configuration.

3. A Desargues configuration with triangle centers as vertices

There are so many collinearities involving triangle centers that we can also exhibit a Desargues (10_3) configuration with vertices consisting entirely of basic centers. Let K denote the symmedian (Lemoine's) point, N_p the center of the nine-point circle, G the centroid, N_+ the first Napoleon point, and N_- the second Napoleon point. Then the ten points F_+, F_-, J_+, J_-, N_+, N_-, K, G, H and N_p form the vertices of such a configuration. This is seen on noting that the triangles $F_-J_+N_+$ and $F_+J_-N_-$ are in perspective from K with the line of perspective $\mathcal{L}(G, N_p, H)$, which is Euler's line. See Figure 4. In a Desargues configuration any vertex may be chosen as the center of perspective of two suitable triangles. For simplicity we have chosen $\mathcal{K}$ in this example. Unlike the previous examples, Desargues configurations are not inscriptable in cubic curves [9].

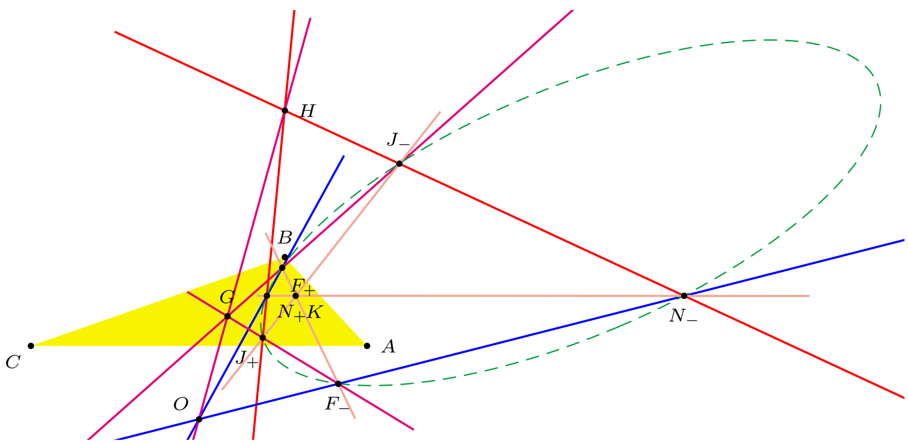

Figure 4

4. Configurations from Monge's theorem

Another way triangle centers form vertices of configurations arises from Monge's theorem [4, 11]. This theorem states that if we have three circles, then the 3 external centers of simitude (ecs) are collinear and that each external center of simitude is collinear with two of the internal centers of simitude (ics). These 4 collinearities form a $(4_3, 6_2)$ configuration, *i.e.*, a complete quadrilateral with the centers of similitude as vertices. This is best illustrated by an example. Suppose we have the circumcircle, the nine-point circle, and the incircle of a triangle. The ics of the circumcircle and the nine-point circle is the centroid, G, and their ecs is the orthocenter, H. The ics of the nine-point circle and the incircle is X_{12} in Kimberling's list and the ecs is Feuerbach's point, X_{11}. The ics of the circumcircle and the incircle is X_{55}, and the ecs is X_{56}. The lines of the configuration are then $\mathcal{L}(H, X_{56}, X_{11})$, $\mathcal{L}(G, X_{55}, X_{11})$, $\mathcal{L}(G, X_{56}, X_{12})$, and $\mathcal{L}(H, X_{55}, X_{12})$. This construction, of course, applies to any group of three circles related to the triangle. In the example given, the circles can be nested, so it may not be easy to see the centers of similitude. In such a case, the radii of the circles can be reduced in the same proportion to make the circles small enough that they do not overlap. The ecs's and ics's remain the same. The ecs of two such circles is the point where the two common external tangents meet, and the ics is the point where the two common internal tangents meet. When two of the circles have the same radii, their ics is the midpoint of the line joining their centers and their ecs is the point at infinity in the direction of the line joining their centers.

One may ask what happens when a fourth circle whose center is not collinear with any other two is also considered. Monge's theorem applies to each group of three circles. First it happens that the four lines containing only ecs's themselves form a $(6_2, 4_3)$ configuration. Second, when the twelve lines containing an ecs and two ics's are annexed, the result is a $(12_4, 16_3)$ configuration. This is a projection onto the plane of Reye's three-dimensional configuration, which arises from a three-dimensional analog of Monge's theorem for four spheres [4]. This is illustrated in Figure 5 with the vertices labelled with the points of Figure 3, which shows that these two $(12_4, 16_3)$ configurations are actually the same even though the representation in Figure 5 may not be inscriptable in a bipartite cubic. Evidently larger configurations arise by the same process when yet more circles are considered.

5. Final remarks

We have see that certain collections of collinear triangle points can be knitted together into highly symmetrical structures called configurations. Furthermore some relatively large configuratons such as the (18_3) shown above are inscriptable in low degree algebraic curves, in this case a cubic.

General information about configurations can be found in Hilbert and Cohn-Vossen [4]. Also we recommend Coxeter [1], which contains an extensive bibliography of related material pre-dating 1950.

The centers here appear in Kimberling [5, 6, 7, 8] as X_n for n below.

center	I	G	O	H	N_p	K	F_+	F_-	J_+	J_-	N_+	N_-
n	1	2	3	4	5	6	13	14	15	16	17	18

While not known by eponyms, X_{12}, X_{55}, and X_{56} are also geometrically significant in elementary ways [7, 8].

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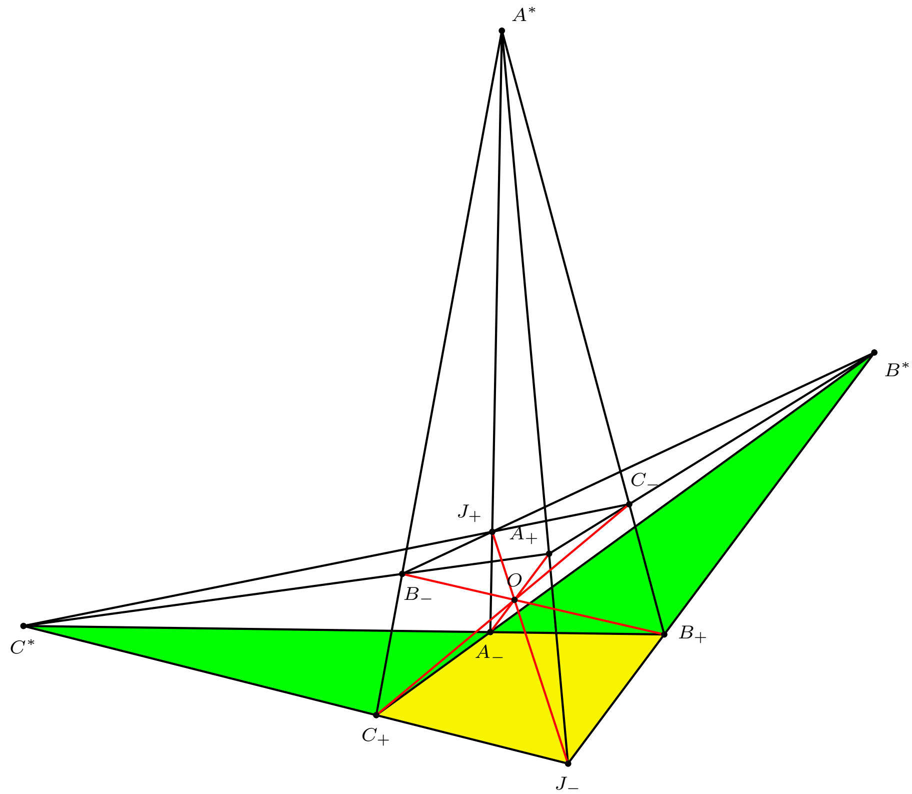

Figure 5

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On the Circumcenters of Cevasix Configurations

Alexei Myakishev and Peter Y. Woo

Abstract. We strengthen Floor van Lamoen's theorem that the 6 circumcenters of the cevasix configuration of the centroid of a triangle are concyclic by giving a proof which at the same time shows that the converse is also true with a minor qualification, *i.e.*, the circumcenters of the cevasix configuration of a point P are concyclic if and only if P is the centroid or the orthocenter of the triangle.

1. Introduction

Let P be a point in the plane of triangle ABC, with traces A', B', C' on the sidelines BC, CA, AB respectively. We assume that P does not lie on any of the sidelines. Triangle ABC is then divided by its cevians AA', BB', CC' into six triangles, giving rise to what Clark Kimberling [2, pp.257–260] called the *cevasix configuration* of P. See Figure 1. Floor van Lamoen has discovered that when P is the centroid of triangle ABC, the 6 circumcenters of the cevasix configuration are concyclic. See Figure 2. This was posed as a problem in the *American Mathematical Monthly* [3]. Solutions can be found in [3, 4]. In this note we study the converse.

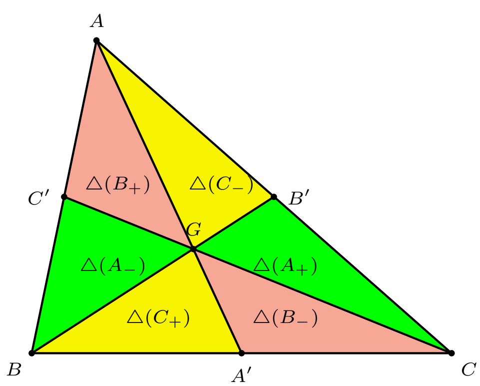

Figure 1

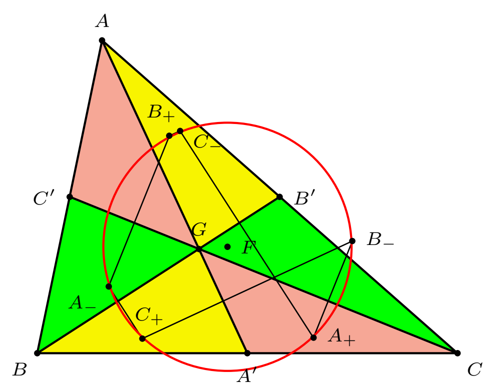

Figure 2

Theorem 1. *The circumcenters of the cevasix configuration of P are concyclic if and only if P is the centroid or the orthocenter of triangle ABC.*

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2. Preliminary results

We adopt the following notations.

Triangle	PCB'	$PC'B$	PAC'	$PA'C$	PBA'	$PB'A$
Notation	$\triangle(A_+)$	$\triangle(A_-)$	$\triangle(B_+)$	$\triangle(B_-)$	$\triangle(C_+)$	$\triangle(C_-)$
Circumcenter	A_+	A_-	B_+	B_-	C_+	C_-

It is easy to see that two of these triangle may possibly share a common circumcenter only when they share a common vertex of triangle ABC.

Lemma 2. *The circumcenters of triangles APB' and APC' coincide if and only if P lies on the reflection of the circumcircle in the line BC.*

Proof. Triangles APB' and APC' have the same circumcenter if and only if the four points A, B', P, C' are concyclic. In terms of directed angles, $\angle BPC = \angle B'PC' = \angle B'AC' = \angle CAB = -\angle BAC$. See, for example, [1, §§16–20]. It follows that the reflection of A in the line BC lies on the circumcircle of triangle PBC, and P lies on the reflection of the circumcircle in BC. The converse is clear. □

Thus, if $B_+ = C_-$ and $C_+ = A_-$, then necessarily P is the orthocenter H, and also $A_+ = B_-$. In this case, there are only three distinct circumcenters. They clearly lie on the nine-point circle of triangle ABC. We shall therefore assume $P \neq H$, so that there are at least five distinct points in the set $\{A_\pm, B_\pm, C_\pm\}$.

The next proposition appears in [2, p.259].

Proposition 3. *The 6 circumcenters of the cevasix configuration of P lie on a conic.*

Proof. We need only consider the case when these 6 circumcenters are all distinct. The circumcenters B_+ and C_- lie on the perpendicular bisector of the segment AP; similarly, B_- and C_+ lie on the perpendicular bisector of PA'. These two perpendicular bisectors are clearly parallel. This means that B_+C_- and B_-C_+ are parallel. Similarly, $C_+A_-//C_-A_+$ and $A_+B_-//A_-B_+$. The hexagon $A_+C_-B_+A_-C_+B_-$ has three pairs of parallel opposite sides. By the converse of Pascal's theorem, there is a conic passing through the six vertices of the hexagon. □

Proposition 4. *The vertices of a hexagon $A_+C_-B_+A_-C_+B_-$ with parallel opposite sides $B_+C_-//C_+B_-$, $C_+A_-//A_+C_-$, $A_+B_-//B_+A_-$ lie on a circle if and only if the main diagonals A_+A_-, B_+B_- and C_+C_- have equal lengths.*

Proof. If the vertices are concyclic, then $A_+C_-A_-C_+$ is an isosceles trapezoid, and $A_+A_- = C_+C_-$. Similarly, $C_+B_-C_-B_+$ is also an isosceles trapezoid, and $C_+C_- = B_+B_-$.

Conversely, consider the triangle XYZ bounded by the three diagonals A_+A_-, B_+B_- and C_+C_-. If these diagonals are equal in length, then the trapezoids $A_+C_-A_-C_+$, $C_+B_-C_-B_+$ and $B_+A_-B_-A_+$ are isosceles. From these we immediately conclude that the common perpendicular bisector of A_+C_- and A_-C_+

is the bisector of angle XYZ. Similarly, the common perpendicular bisector of B_+C_- and B_-C_+ is the bisector of angle X, and that of A_+B_- and A_-B_+ the bisector of angle Z. These three perpendicular bisectors clearly intersect at a point, the incenter of triangle XYZ, which is equidistant from the six vertices of the hexagon. $\square$

Proposition 5. *The vector sum* $\mathbf{AA'} + \mathbf{BB'} + \mathbf{CC'} = \mathbf{0}$ *if and only if* P *is the centroid.*

Proof. Suppose with reference to triangle ABC, the point P has *absolute* barycentric coordinates $uA + vB + wC$, where $u + v + w = 1$. Then,

$$A' = \frac{1}{v+w}(vB + wC), \quad B' = \frac{1}{w+u}(wC + uA), \quad C' = \frac{1}{u+v}(uA + vB).$$

From these,

$$\begin{aligned}&\mathbf{AA'} + \mathbf{BB'} + \mathbf{CC'}\\ =&(A' + B' + C') - (A + B + C)\\ =&\frac{u^2 - vw}{(w+u)(u+v)} \cdot A + \frac{v^2 - wu}{(u+v)(v+w)} \cdot B + \frac{w^2 - uv}{(v+w)(w+u)} \cdot C.\end{aligned}$$

This is zero if and only if

$$u^2 - vw = v^2 - wu = w^2 - uv = 0,$$

and $u = v = w = \frac{1}{3}$ since they are all real, and $u + v + w = 1$. $\square$

We denote by π_a, π_b, π_c the orthogonal projections on the lines AA', BB', CC' respectively.

Proposition 6.

$$\begin{aligned}\pi_b(\mathbf{A_+A_-}) &= -\tfrac{1}{2}\mathbf{BB'}, & \pi_c(\mathbf{A_+A_-}) &= \tfrac{1}{2}\mathbf{CC'},\\ \pi_c(\mathbf{B_+B_-}) &= -\tfrac{1}{2}\mathbf{CC'}, & \pi_a(\mathbf{B_+B_-}) &= \tfrac{1}{2}\mathbf{AA'},\\ \pi_a(\mathbf{C_+C_-}) &= -\tfrac{1}{2}\mathbf{AA'}, & \pi_b(\mathbf{C_+C_-}) &= \tfrac{1}{2}\mathbf{BB'}.\end{aligned} \tag{1}$$

Proof. The orthogonal projections of A_+ and A_- on the cevian BB' are respectively the midpoints of the segments PB' and BP. Therefore,

$$\pi_b(\mathbf{A_+A_-}) = \frac{B+P}{2} - \frac{P+B'}{2} = -\frac{B'-B}{2} = -\frac{1}{2}\mathbf{BB'}.$$

The others follow similarly. $\square$

3. Proof of Theorem 1

Sufficiency part. Let P be the centroid G of triangle ABC. By Proposition 4, it is enough to prove that the diagonals A_+A_-, B_+B_- and C_+C_- have equal lengths. By Proposition 5, we can construct a triangle $A^*B^*C^*$ whose sides as vectors $\mathbf{B^*C^*}$, $\mathbf{C^*A^*}$ and $\mathbf{A^*B^*}$ are equal to the medians $\mathbf{AA'}$, $\mathbf{BB'}$, $\mathbf{CC'}$ respectively.

Consider the vector $\mathbf{A}^*\mathbf{Q}$ equal to $\mathbf{A}_+\mathbf{A}_-$. By Proposition 6, the orthogonal projections of $\mathbf{A}_+\mathbf{A}_-$ on the two sides C^*A^* and A^*B^* are the midpoints of the sides. This means that Q is the circumcenter of triangle $A^*B^*C^*$, and the length of $\mathbf{A}_+\mathbf{A}_-$ is equal to the circumradius of triangle $A^*B^*C^*$. The same is true for the lengths of $\mathbf{B}_+\mathbf{B}_-$ and $\mathbf{C}_+\mathbf{C}_-$. The case $P = H$ is trivial.

Necessity part. Suppose the 6 circumcenters $A_\pm$, $B_\pm$, $C_\pm$ lie on a circle. By Proposition 3, the diagonals A_+A_-, B_+B_-, and C_+C_- have equal lengths. We show that $\mathbf{AA}' + \mathbf{BB}' + \mathbf{CC}' = 0$, so that P is the centroid of triangle ABC by Proposition 5. In terms of scalar products, we rewrite equation (1) as

$$\begin{aligned}
\mathbf{A}_+\mathbf{A}_- \cdot \mathbf{BB}' &= -\tfrac{1}{2}\mathbf{BB}' \cdot \mathbf{BB}', & \mathbf{A}_+\mathbf{A}_- \cdot \mathbf{CC}' &= \tfrac{1}{2}\mathbf{CC}' \cdot \mathbf{CC}',\\
\mathbf{B}_+\mathbf{B}_- \cdot \mathbf{CC}' &= -\tfrac{1}{2}\mathbf{CC}' \cdot \mathbf{CC}', & \mathbf{B}_+\mathbf{B}_- \cdot \mathbf{AA}' &= \tfrac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}',\\
\mathbf{C}_+\mathbf{C}_- \cdot \mathbf{AA}' &= -\tfrac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}', & \mathbf{C}_+\mathbf{C}_- \cdot \mathbf{BB}' &= \tfrac{1}{2}\mathbf{BB}' \cdot \mathbf{BB}'.
\end{aligned} \tag{2}$$

From these, $(\mathbf{B}_+\mathbf{B}_- + \mathbf{C}_+\mathbf{C}_-) \cdot \mathbf{AA}' = 0$, and $\mathbf{AA}'$ is orthogonal to $\mathbf{B}_+\mathbf{B}_- + \mathbf{C}_+\mathbf{C}_-$. Since $\mathbf{B}_+\mathbf{B}_-$, and $\mathbf{C}_+\mathbf{C}_-$ have equal lengths, $\mathbf{B}_+\mathbf{B}_- + \mathbf{C}_+\mathbf{C}_-$ and $\mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_-$ are orthogonal. We may therefore write $\mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_- = k\mathbf{AA}'$ for a scalar k. From (2) above,

$$\begin{aligned}
k\mathbf{AA}' \cdot \mathbf{AA}' &= (\mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_-) \cdot \mathbf{AA}'\\
&= \frac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}' + \frac{1}{2}\mathbf{AA}' \cdot \mathbf{AA}'\\
&= \mathbf{AA}' \cdot \mathbf{AA}'.
\end{aligned}$$

From this, $k = 1$ and we have

$$\mathbf{AA}' = \mathbf{B}_+\mathbf{B}_- - \mathbf{C}_+\mathbf{C}_-.$$

The same reasoning shows that

$$\begin{aligned}
\mathbf{BB}' &= \mathbf{C}_+\mathbf{C}_- - \mathbf{A}_+\mathbf{A}_-,\\
\mathbf{CC}' &= \mathbf{A}_+\mathbf{A}_- - \mathbf{B}_+\mathbf{B}_-.
\end{aligned}$$

Combining the three equations, we have

$$\mathbf{AA}' + \mathbf{BB}' + \mathbf{CC}' = \mathbf{0}.$$

It follows from Proposition 5 that P must be the centroid of triangle ABC.

4. An alternative proof of Theorem 1

We present another proof of Theorem 1 by considering an auxiliary hexagon. Let $\mathcal{L}_a$ and $\mathcal{L}'_a$ be the lines perpendicular to AA' at A and A' respectively; similarly, $\mathcal{L}_b$, $\mathcal{L}'_b$, and $\mathcal{L}_c$ and $\mathcal{L}'_c$. Consider the points

$$\begin{aligned}
X_+ &= \mathcal{L}_c \cap \mathcal{L}'_b, & X_- &= \mathcal{L}_b \cap \mathcal{L}'_c,\\
Y_+ &= \mathcal{L}_a \cap \mathcal{L}'_c, & Y_- &= \mathcal{L}_c \cap \mathcal{L}'_a,\\
Z_+ &= \mathcal{L}_b \cap \mathcal{L}'_a, & Z_- &= \mathcal{L}_a \cap \mathcal{L}'_b.
\end{aligned}$$

Note that the circumcenters $A_\pm$, $B_\pm$, $C_\pm$ are respectively the midpoints of $PX_\pm$, $PY_\pm$, $PZ_\pm$. Hence, the six circumcenters are concyclic if and only if $X_\pm$, $Y_\pm$, $Z_\pm$ are concyclic.

In Figure 3, let $\angle CPA' = \angle APC' = \alpha$. Since angles $PA'Y_-$ and PCY_- are both right angles, the four points P, A', C, Y_- are concyclic and $\angle Z_+Y_-X_+ = \angle A'Y_-X_+ = \angle A'PC = \alpha$. Similarly, $\angle CPB' = \angle BPC' = \angle Y_-X_+Z_-$, and we denote the common measure by β.

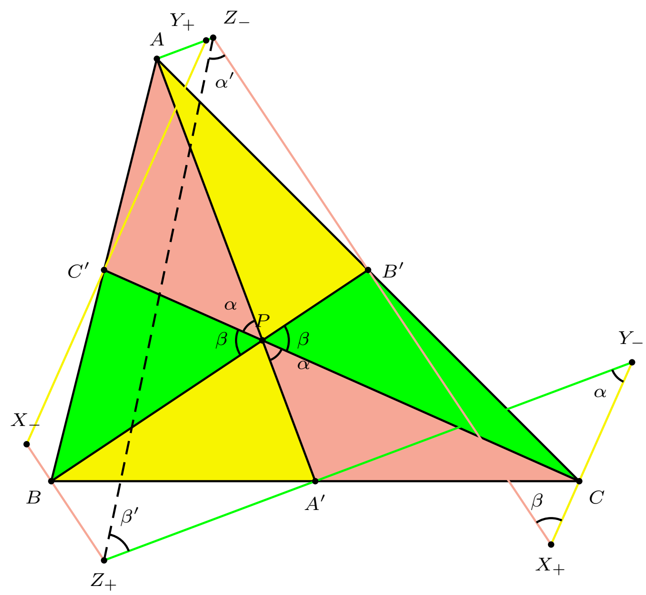

Figure 3

Lemma 7. *If the four points X_+, Y_-, Z_+, Z_- are concyclic, then P lies on the median through C.*

Proof. Let $x = \frac{AP}{AA'}$ and $y = \frac{BP}{BB'}$. If the four points X_+, Y_-, Z_+, Z_- are concyclic, then $\angle Z_+Z_-X_+ = \alpha$ and $\angle Y_-Z_+Z_- = \beta$. Now,

$$\frac{|BB'|}{|AA'|} = \frac{|Z_+Z_-| \cdot \sin\alpha'}{|Z_+Z_-| \cdot \sin\beta'} = \frac{\sin\alpha}{\sin\beta} = \frac{\frac{|AC'|}{|AP|}}{\frac{|BC'|}{|BP|}}.$$

It follows that

$$\frac{|BP|}{|BB'| \cdot |BC'|} = \frac{|AP|}{|AA'| \cdot |AC'|},$$

and, as a ratio of signed lengths,

$$\frac{BC'}{AC'} = -\frac{y}{x}. \tag{3}$$

Now applying Menelaus' theorem to triangle APC' with transversal $A'CB$, and triangle BGA' with transversal $B'CA$, we have

$$\frac{AA'}{A'P}\cdot\frac{PC}{CC'}\cdot\frac{C'B}{BA}=-1=\frac{BB'}{B'P}\cdot\frac{PC}{CC'}\cdot\frac{C'A}{AB}.$$

From this, $\frac{AA'}{A'P}\cdot BC'=\frac{BB'}{B'P}\cdot AC'$, or

$$\frac{BC'}{1-x}=-\frac{AC'}{1-y}. \tag{4}$$

Comparing (3) and (4), we have $\frac{1-x}{1-y}=\frac{y}{x}$, $(x-y)(x+y-1)=0$. Either $x=y$ or $x+y=1$. It is easy to eliminate the possibility $x+y=1$. If P has homogeneous barycentric coordinates $(u:v:w)$ with reference to triangle ABC, then $x=\frac{v+w}{u+v+w}$ and $y=\frac{w+u}{u+v+w}$. Thus, $x+y=1$ requires $w=0$ and P lies on the sideline AB, contrary to the assumption. It follows that $x=y$, and from (3), C' is the midpoint of AB, and P lies on the median through C. □

The necessity part of Theorem 1 is now an immediate corollary of Lemma 7.

5. Concluding remark

We conclude with a remark on triangles for which two of the circumcenters of the cevasix configuration of the centroid coincide. Clearly, $B_+=C_-$ if and only if A, B', G, C' are concyclic. Equivalently, the image of G under the homothety $\mathsf{h}(A,2)$ lies on the circumcircle of triangle ABC. This point has homogeneous barycentric coordinates $(-1:2:2)$. Since the circumcircle has equation

$$a^2yz+b^2zx+c^2xy=0,$$

we have $2a^2=b^2+c^2$. There are many interesting properties of such triangles. We simply mention that it is similar to its own triangle of medians. Specifically,

$$m_a=\frac{\sqrt{3}}{2}a,\quad m_b=\frac{\sqrt{3}}{2}c,\quad m_c=\frac{\sqrt{3}}{2}b.$$

Editor's endnote. John H. Conway [5] has located the center of the Van Lamoen circle (of the circumcenters of the cevasix configuration of the centroid) as

$$F=\mathsf{m}N+\frac{\cot^2\omega}{12}\cdot(G-K),$$

where $\mathsf{m}N$ is the medial Ninecenter,[1] G the centroid, K the symmedian point, and ω the Brocard angle of triangle ABC. In particular, the parallel through F to the symmedian line GK hits the Euler line in $\mathsf{m}N$. See Figure 4. The point F has homogeneous barycentric coordinates

$$(10a^4-13a^2(b^2+c^2)+(4b^4-10b^2c^2+4c^4):\cdots:\cdots).$$

This appears as X_{1153} of [6].

[1]This is the point which divides OH in the ratio $1:3$.

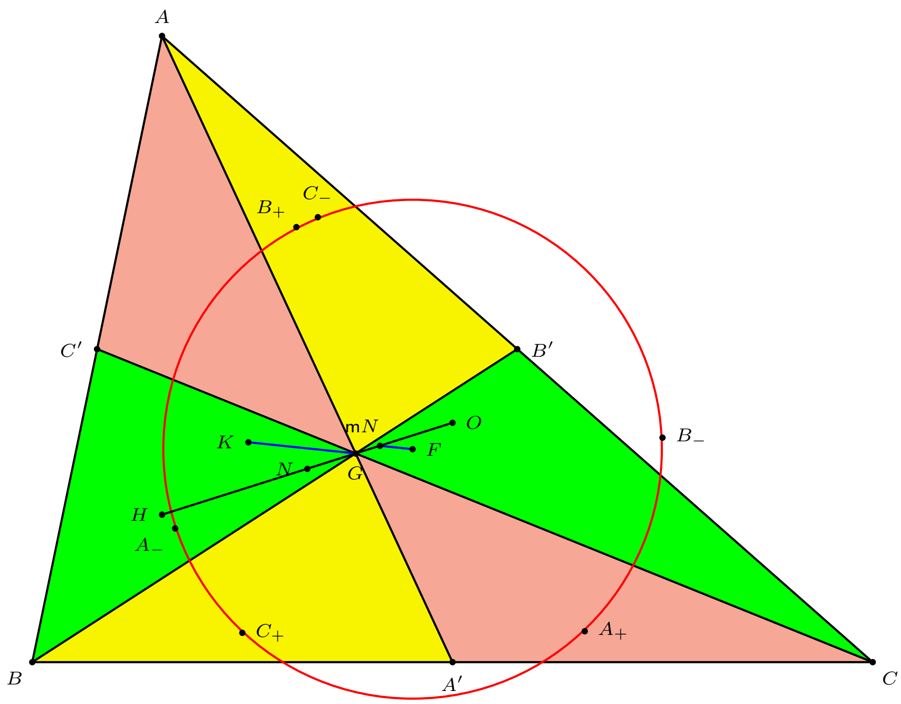

Figure 4

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Napoleon Triangles and Kiepert Perspectors

Two examples of the use of complex number coordinates

Floor van Lamoen

Abstract. In this paper we prove generalizations of the well known Napoleon Theorem and Kiepert Perspectors. We use complex numbers as coordinates to prove the generalizations, because this makes representation of isosceles triangles built on given segments very easy.

1. Introduction

In [1, XXVII] O. Bottema describes the famous (first) Fermat-Torricelli point of a triangle ABC. This point is found by attaching outwardly equilateral triangles to the sides of ABC. The new vertices form a triangle $A'B'C'$ that is perspective to ABC, that is, AA', BB' and CC' have a common point of concurrency, the perspector of ABC and $A'B'C'$. A lot can be said about this point, but for this paper we only need to know that the lines AA', BB' and CC' make angles of 60 degrees (see Figure 1), and that this is also the case when the equilateral triangles are pointed inwardly, which gives the second Fermat-Torricelli point.

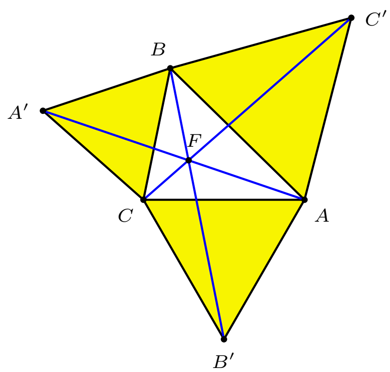

Figure 1

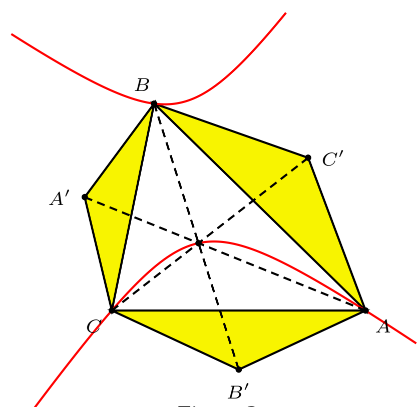

Figure 2

It is well known that to yield a perspector, the triangles attached to the sides of ABC do not need to be equilateral. For example they may be isosceles triangles with base angle ϕ, like Bottema tells us in [1, XI]. It was Ludwig Kiepert who studied these triangles - the perspectors with varying ϕ lie on a rectangular hyperbola

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named after Kiepert. See [4] for some further study on this hyperbola, and some references. See Figure 2. However, it is already sufficient for the lines AA', BB', CC' to concur when the attached triangles have oriented angles satisfying

$$\angle BAC' = \angle CAB', \qquad \angle ABC' = \angle CBA', \qquad \angle ACB = \angle BCA'.$$

When the attached triangles are equilateral, there is another nice geometric property: *the centroids of the triangles $A'BC$, $AB'C$ and ABC' form a triangle that is equilateral itself*, a fact that is known as Napoleon's Theorem. The triangles are referred to as the *first* and *second Napoleon triangles* (for the cases of outwardly and inwardly pointed attached triangles). See Figures 3a and 3b. The perspectors of these two triangles with ABC are called *first* and *second Napoleon points*. General informations on Napoleon triangles and Kiepert perspectors can be found in [2, 3, 5, 6].

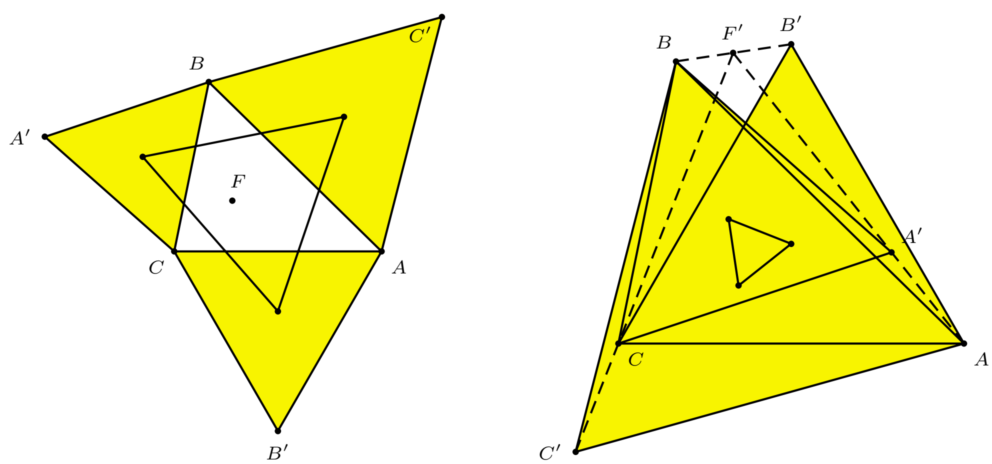

Figure 3a Figure 3b

2. The equation of a line in the complex plane

Complex coordinates are not that much different from the rectangular (x, y) - the two directions of the axes are now hidden in one complex number, that we call the *affix* of a point. Of course such an affix just exists of a real (x) and imaginary (y) part - the complex number $z = p + qi$ in fact resembles the point (p, q).

If $z = p + qi$, then the number $\overline{z} = p - qi$ is called complex conjugate of z. The combination of z and $\overline{z}$ is used to make formulas, since we do not have x and y anymore! A parametric formula for the line through the points a_1 and a_2 is $z = a_1 + t(a_2 - a_1)$, where t runs through the *real* numbers. The complex conjugate of this formula is $\overline{z} = \overline{a_1} + t(\overline{a_2} - \overline{a_1})$. Elimination of t from these two formulas gives the formula for the line through the points with affixes a_1 and a_2:

$$z(\overline{a_1 - a_2}) - \overline{z}(a_1 - a_2) + (a_1\overline{a_2} - \overline{a_1}a_2) = 0.$$

3. Isosceles triangle on a segment

Let the points A and B have affixes a and b. We shall find the affix of the point C for which ABC is an isosceles triangle with base angle ϕ and apex C. The midpoint of AB has affix $\frac{1}{2}(a+b)$. The distance from this midpoint to C is equal to $\frac{1}{2}|AB|\tan\phi$. With this we find the affix for C as

$$c = \frac{a+b}{2} + i\tan\phi\cdot\frac{b-a}{2} = \frac{1-i\tan\phi}{2}a + \frac{1+i\tan\phi}{2}b = \overline{\chi}a + \chi b$$

where $\chi = \frac{1}{2} + \frac{i}{2}\tan\phi$, so that $\chi + \overline{\chi} = 1$.

The special case that ABC is equilateral, yields for χ the sixth root of unity $\zeta = \frac{1}{2} + \frac{i}{2}\sqrt{3} = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. This number ζ is a sixth root of unity, because it satisfies

$$\zeta^6 = e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1.$$

It also satisfies the identities $\zeta^3 = -1$ and $\zeta\cdot\overline{\zeta} = \zeta + \overline{\zeta} = 1$. Depending on orientation one can find two vertices C that together with AB form an equilateral triangle, for which we have respectively $c = \zeta a + \overline{\zeta}b$ (negative orientation) and $c = \overline{\zeta}a + \zeta b$ (positive orientation). From this one easily derives

Proposition 1. *The complex numbers a, b and c are affixes of an equilateral triangle if and only if*

$$a + \zeta^2 b + \zeta^4 c = 0$$

for positive orientation or

$$a + \zeta^4 b + \zeta^2 c = 0$$

for negative orientation.

4. Napoleon triangles

We shall generalize Napoleon's Theorem, by extending the idea of the use of centroids. Napoleon triangles were indeed built in a triangle ABC by attaching to the sides of a triangle equilateral triangles, and taking the centroids of these. We now start with two triangles $A_kB_kC_k$ for $k = 1,\ 2$, and attach equilateral triangles to the connecting segments between the A's, the B's and the C's. This seems to be entirely different, but Napoleon's Theorem will be a special case by starting with triangles BCA and CAB.

So we start with two triangles $A_kB_kC_k$ for $k = 1,\ 2$ with affixes a_k, b_k and c_k for the vertices. The centroids Z_k have affixes $z_k = \frac{1}{3}(a_k + b_k + c_k)$. Now we attach positively orientated equilateral triangles to segments A_1A_2, B_1B_2 and C_1C_2 having A_{3+}, B_{3+}, C_{3+} as third vertices. In the same way we find A_{3-}, B_{3-}, C_{3-} from equilateral triangles with negative orientation. We find as affixes

$$a_{3+} = \zeta a_2 + \overline{\zeta}a_1$$

and

$$a_{3-} = \zeta a_1 + \overline{\zeta}a_2,$$

and similar expressions for b_{3+}, b_{3-}, c_{3+} and c_{3-}. The centroids Z_{3+} and Z_{3-} now have affixes $z_{3+} = \zeta z_2 + \overline{\zeta}z_1$ and $z_{3-} = \zeta z_1 + \overline{\zeta}z_2$ respectively, from which we

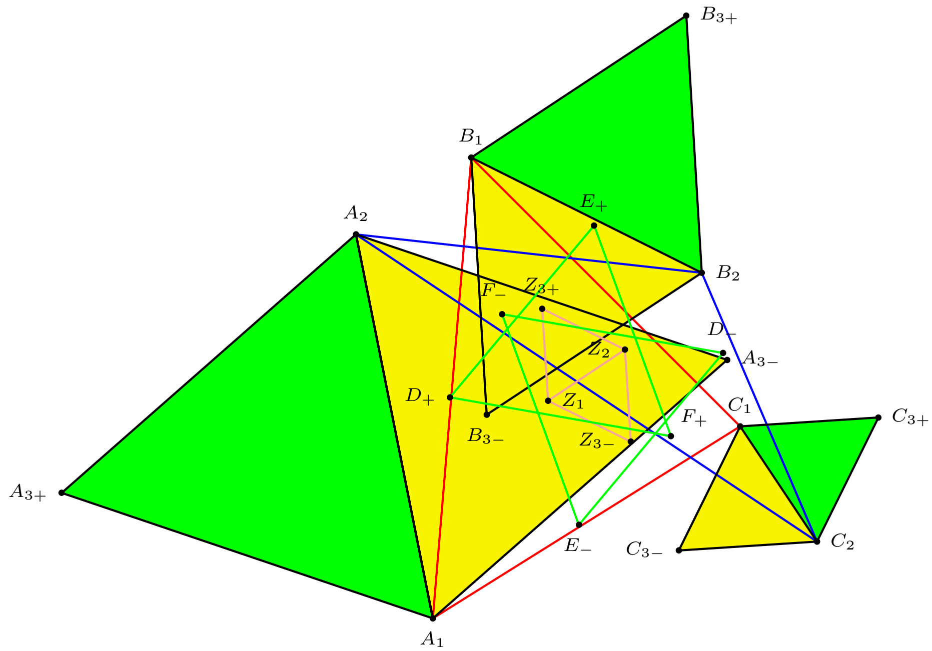

Figure 4

see that $Z_1Z_2Z_{3+}$ and $Z_1Z_2Z_{3-}$ are equilateral triangles of positive and negative orientation respectively.

We now work with the following centroids:
D_+, E_+ and F_+ of triangles $B_1C_2A_{3+}$, $C_1A_2B_{3+}$ and $A_1B_2C_{3+}$ respectively;
D_-, E_- and F_- of triangles $C_1B_2A_{3-}$, $A_1C_2B_{3-}$ and $B_1A_2C_{3-}$ respectively.

For these we claim

Theorem 2. *Given triangles $A_kB_kC_k$ and points Z_k for $k = 1,\ 2,\ 3+,\ 3-$ and $D_\pm E_\pm F_\pm$ as described above, triangles $D_+E_+F_+$ and $D_-E_-F_-$ are equilateral triangles of negative orientation, congruent and parallel, and their centroids coincide with the centroids of $Z_1Z_2Z_{3+}$ and $Z_1Z_2Z_{3-}$ respectively. (See Figure 4).*

Proof. To prove this we find the following affixes

$$d_+ = \tfrac{1}{3}(b_1 + c_2 + \zeta a_2 + \overline{\zeta} a_1), \qquad d_- = \tfrac{1}{3}(b_2 + c_1 + \zeta a_1 + \overline{\zeta} a_2),$$

$$e_+ = \tfrac{1}{3}(c_1 + a_2 + \zeta b_2 + \overline{\zeta} b_1), \qquad e_- = \tfrac{1}{3}(c_2 + a_1 + \zeta b_1 + \overline{\zeta} b_2),$$

$$f_+ = \tfrac{1}{3}(a_1 + b_2 + \zeta c_2 + \overline{\zeta} c_1), \qquad f_- = \tfrac{1}{3}(a_2 + b_1 + \zeta c_1 + \overline{\zeta} c_2).$$

Using Proposition 1 it is easy to show that $D_+E_+F_+$ and $D_-E_-F_-$ are equilateral triangles of negative orientation. For instance, the expression $d_+ + \zeta^4 e_+ + \zeta^2 f_+$ has as ‘coefficient’ of b_1 the number $\frac{1}{3}(1 + \zeta^4\overline{\zeta}) = 0$. We also find that

$$d_+ - e_+ = e_- - d_- = \overline{\zeta}(a_1 - a_2) + \zeta(b_1 - b_2) + (c_2 - c_1),$$

from which we see that D_+E_+ and D_-E_- are equal in length and directed oppositely. Finally it is easy to check that $\frac{1}{3}(d_+ + e_+ + f_+) = \frac{1}{3}(z_1 + z_2 + z_{3+})$ and $\frac{1}{3}(d_- + e_- + f_-) = \frac{1}{3}(z_1 + z_2 + z_{3-})$, and the theorem is proved. □

We can make a variation of Theorem 2 if in the creation of $D_\pm E_\pm F_\pm$ we interchange the roles of $A_{3+}B_{3+}C_{3+}$ and $A_{3-}B_{3-}C_{3-}$. The roles of Z_{3+} and Z_{3-} change as well, and the equilateral triangles found have positive orientation.

We note that if the centroids Z_1 and Z_2 coincide, then they coincide with Z_{3+} and Z_{3-}, so that $D_+E_-F_+D_-E_+F_-$ is a regular hexagon, of which the center coincides with Z_1 and Z_2.

Napoleon's Theorem is a special case. If we take $A_1B_1C_1 = BCA$ and $A_2B_2C_2 = CAB$, then $D_+E_+F_+$ is the second Napoleon Triangle, and indeed appears equilateral. We get as a bonus that $D_+E_-F_+D_-E_+F_-$ is a regular hexagon. Now D_- is the centroid of AAA_{3-}, that is, D_- is the point on AA_{3-} such that $AD_- : D_-A_{3-} = 1 : 2$. In similar ways we find E_- and F_-. Triangles ABC and $A_{3-}B_{3-}C_{3-}$ have the first point of Fermat-Torricelli F_1 as perspector, and the lines AA_{3-}, BB_{3-} and CC_{3-} make angles of 60 degrees. From this it is easy to see (congruent inscribed angles) that F_1 must be on the circumcircle of $D_-E_-F_-$ and thus also on the circumcircle of $D_+E_+F_+$. See Figure 5. In the same way, now using the variation of Theorem 2, we see that the second Fermat-Torricelli point lies on the circumcircle of the first Napoleon Triangle.

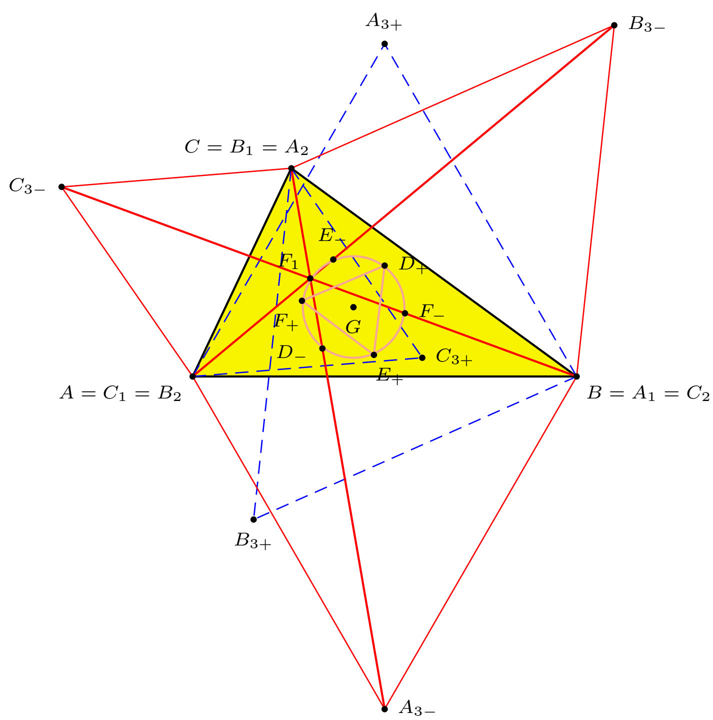

Figure 5

5. Kiepert perspectors

To generalize the Kiepert perspectors we start with two triangles as well. We label these ABC and $A'B'C'$ to distinguish from Theorem 2. These two triangles we take to be directly congruent (hence A corresponds to A', etc.) and of the same orientation. This means that the two triangles can be mapped to each other by a combination of a rotation and a translation (in fact one of both is sufficient). We now attach isosceles triangles to segments connecting ABC and $A'B'C'$. While we usually find Kiepert perspectors on a line, for example, from A to the apex of an isosceles triangle built on BC, now we start from the apex of an isosceles triangle on AA' and go to the apex of an isosceles triangle on on BC'. This gives the following theorem:

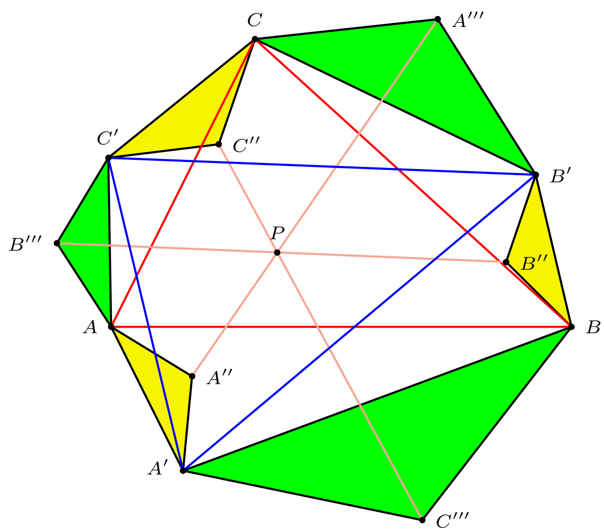

Figure 6

Theorem 3. *Given two directly congruent triangles ABC and $A'B'C'$ with the same orientation, attach to the segments AA', BB', CC', CB', AC' and BA' similar isosceles triangles with the same orientation and apexes A'', B'', C'', A''', B''' and C'''. The lines $A''A'''$, $B''B'''$ and $C''C'''$ are concurrent, so triangles $A''B''C''$ and $A'''B'''C'''$ are perspective. (See Figure 6).*

Proof. For the vertices A, B and C we take the affixes a, b and c. Because triangles ABC and $A'B'C'$ are directly congruent and of equal orientation, we can get $A'B'C'$ by applying on ABC a rotation about the origin, followed by a translation. This rotation about the origin can be represented by multiplication by a number τ on the unit circle, so that $\tau\overline{\tau} = 1$. The translation is represented by addition with a number σ. So the affixes of A', B' and C' are the numbers $\tau a + \sigma$, $\tau b + \sigma$ and $\tau c + \sigma$.

We take for the base angles of the isosceles triangle ϕ again, and we let $\chi = \frac{1}{2} + \frac{i}{2}\tan\phi$, so that the affix for A'' is $(\overline{\chi}+\chi\tau)a+\chi\sigma$. For A''' we find $\overline{\chi}c+\chi\tau b+\chi\sigma$. The equation of the line $A''A'''$ we can find after some calculations as

$$\begin{aligned}
&(\chi\overline{a}+\overline{\chi\tau a}-\chi\overline{c}-\overline{\chi\tau b})z-(\overline{\chi}a+\chi\tau a-\overline{\chi}c-\chi\tau b)\overline{z}\\
&+(\overline{\chi}+\chi\tau)a(\chi\overline{c}+\overline{\chi\tau b})+\chi\sigma(\chi\overline{c}+\overline{\chi\tau b})+\overline{\chi\sigma}(\overline{\chi}+\chi\tau)a\\
&-(\chi+\overline{\chi\tau})\overline{a}(\overline{\chi}c+\chi\tau b)-\overline{\chi\sigma}(\overline{\chi}c+\chi\tau b)-\chi\sigma(\chi+\overline{\chi\tau})\overline{a}\\
&=0.
\end{aligned}$$

In a similar fashion we find for $B''B'''$,

$$\begin{aligned}
&(\chi\overline{b}+\overline{\chi\tau b}-\chi\overline{a}-\overline{\chi\tau c})z-(\overline{\chi}b+\chi\tau b-\overline{\chi}a-\chi\tau c)\overline{z}\\
&+(\overline{\chi}+\chi\tau)b(\chi\overline{a}+\overline{\chi\tau c})+\chi\sigma(\chi\overline{a}+\overline{\chi\tau c})+\overline{\chi\sigma}(\overline{\chi}+\chi\tau)b\\
&-(\chi+\overline{\chi\tau})\overline{b}(\overline{\chi}a+\chi\tau c)-\overline{\chi\sigma}(\overline{\chi}a+\chi\tau c)-\chi\sigma(\chi+\overline{\chi\tau})\overline{b}\\
&=0,
\end{aligned}$$

and for $C''C'''$,

$$\begin{aligned}
&(\chi\overline{c}+\overline{\chi\tau c}-\chi\overline{b}-\overline{\chi\tau a})z-(\overline{\chi}c+\chi\tau c-\overline{\chi}b-\chi\tau a)\overline{z}\\
&+(\overline{\chi}+\chi\tau)c(\chi\overline{b}+\overline{\chi\tau a})+\chi\sigma(\chi\overline{b}+\overline{\chi\tau a})+\overline{\chi\sigma}(\overline{\chi}+\chi\tau)c\\
&-(\chi+\overline{\chi\tau})\overline{c}(\overline{\chi}b+\chi\tau a)-\overline{\chi\sigma}(\overline{\chi}b+\chi\tau a)-\chi\sigma(\chi+\overline{\chi\tau})\overline{c}\\
&=0.
\end{aligned}$$

We must do some more effort to see what happens if we add the three equations. Our effort is rewarded by noticing that the sum gives $0 = 0$. The three equations are dependent, so the lines are concurrent. This proves the theorem. □

We end with a question on the locus of the perspector for varying ϕ. It would have been nice if the perspector would, like in Kiepert's hyperbola, lie on an equilateral hyperbola. This, however, does not seem to be generally the case.

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On the Fermat Lines

Paul Yiu

Abstract. We study the triangle formed by three points each on a Fermat line of a given triangle, and at equal distances from the vertices. For two specific values of the common distance, the triangle degenerates into a line. The two resulting lines are the axes of the Steiner ellipse of the triangle.

1. The Fermat lines

This paper is on a variation of the theme of Bottema [2]. Bottema studied the triangles formed by three points each on an *altitude* of a given triangle, at equal distances from the respective vertices. See Figure 1. He obtained many interesting properties of this configuration. For example, these three points are collinear when the common distance is $R \pm d$, where R is the circumradius and d the distance between the circumcenter and the incenter of the reference triangle. The two lines containing the two sets of collinear points are perpendicular to each other at the incenter, and are parallel to the asymptotes of the Feuerbach hyperbola, the rectangular hyperbola through the vertices, the orthocenter, and the incenter. See Figure 2.

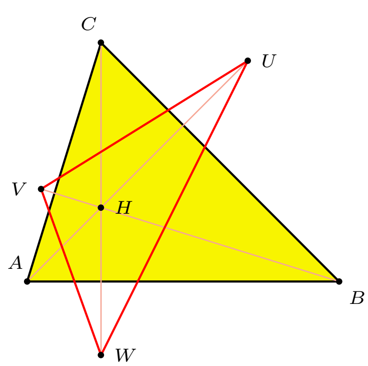

Figure 1

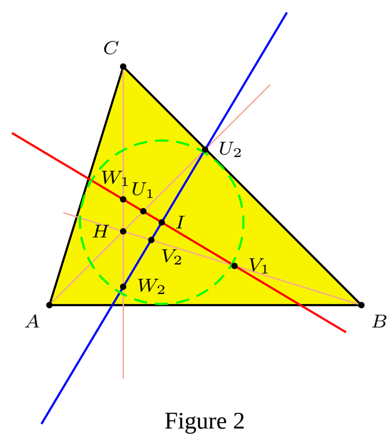

Figure 2

In this paper we consider the *Fermat lines*, which are the lines joining a vertex of the given triangle ABC to the apex of an equilateral triangle constructed on its opposite side. We label these triangles BCA_ϵ, CAB_ϵ, and ABC_ϵ, with $\epsilon = +1$

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This paper is an extended revision of its Dutch version, *Over de lijnen van Fermat*, Euclides, 77 (2002) nr. 4, 188–193. This issue of *Euclides* is a tribute to O. Bottema (1900 – 1992). The author thanks Floor van Lamoen for translation into Dutch, and the editors of *Euclides* for permission to publish the present English version.

for those erected externally, and $\epsilon = -1$ otherwise. There are 6 of such lines, AA_+, BB_+, CC_+, AA_-, BB_-, and CC_-. See Figure 3. The reason for choosing these lines is that, for $\epsilon = \pm 1$, the three segments AA_ϵ, BB_ϵ, and CC_ϵ have equal lengths τ_ϵ given by

$$\tau_\epsilon^2 = \frac{1}{2}(a^2 + b^2 + c^2) + \epsilon \cdot 2\sqrt{3}\triangle,$$

where a, b, c are the side lengths, and $\triangle$ the area of triangle ABC. See, for example, [1, XXVII.3].

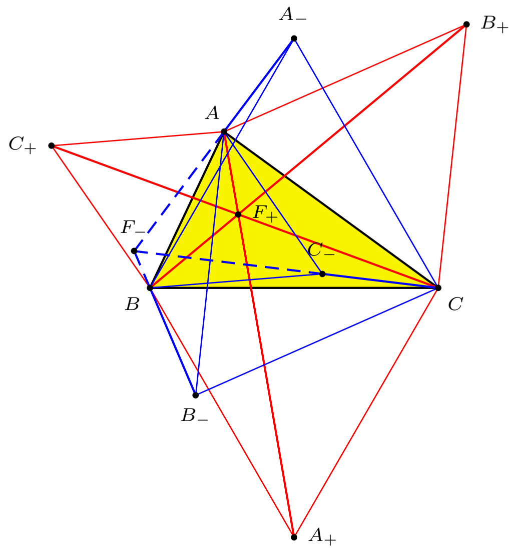

Figure 3

It is well known that the three Fermat lines AA_ϵ, BB_ϵ, and CC_ϵ intersect each other at the ϵ-Fermat point F_ϵ at 60° angles. The centers of the equilateral triangles BCA_ϵ, CAB_ϵ, and ABC_ϵ form the ϵ-Napoleon equilateral triangle. The circumcircle of the ϵ-Napoleon triangle has radius $\frac{\tau_\epsilon}{3}$ and passes through the $(-\epsilon)$-Fermat point. See, for example, [5].

2. The triangles $\mathcal{T}_\epsilon(t)$

We shall label points on the Fermat lines by their distances from the corresponding vertices of ABC, positive in the direction from the vertex to the Fermat point, negative otherwise. Thus, $A_+(t)$ is the unique point X on the positive Fermat line AF_+ such that $AX = t$. In particular,

$$A_\epsilon(\tau_\epsilon) = A_\epsilon, \qquad B_\epsilon(\tau_\epsilon) = B_\epsilon, \qquad C_\epsilon(\tau_\epsilon) = C_\epsilon.$$

We are mainly interested in the triangles $\mathcal{T}_\epsilon(t)$ whose vertices are $A_\epsilon(t)$, $B_\epsilon(t)$, $C_\epsilon(t)$, for various values of t. Here are some simple observations.

(1) The centroid of AA_+A_- is G. This is because the segments A_+A_- and BC have the same midpoint.

(2) The centers of the equilateral triangles BCA_+ and BCA_- trisect the segment A_+A_-. Therefore, the segment joining $A_\epsilon(\frac{\tau_\epsilon}{3})$ to the center of $BCA_{-\epsilon}$ is parallel to the Fermat line $AA_{-\epsilon}$ and has midpoint G.

(3) This means that $A_\epsilon(\frac{\tau_\epsilon}{3})$ is the reflection of the A-vertex of the $(-\epsilon)$-Napoleon triangle in the centroid G. See Figure 4, in which we label $A_+(\frac{\tau_+}{3})$ by X and $A_-(\frac{\tau_-}{3})$ by X' respectively.

This is the same for the other two points $B_\epsilon(\frac{\tau_\epsilon}{3})$ and $C_\epsilon(\frac{\tau_\epsilon}{3})$.

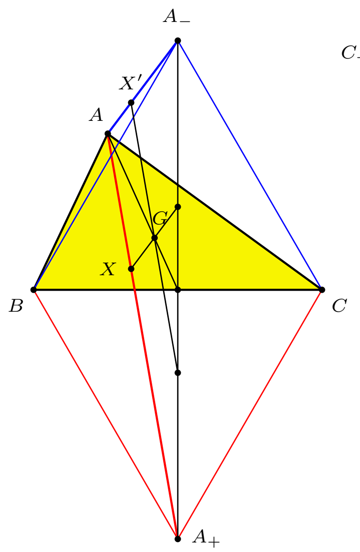

Figure 4

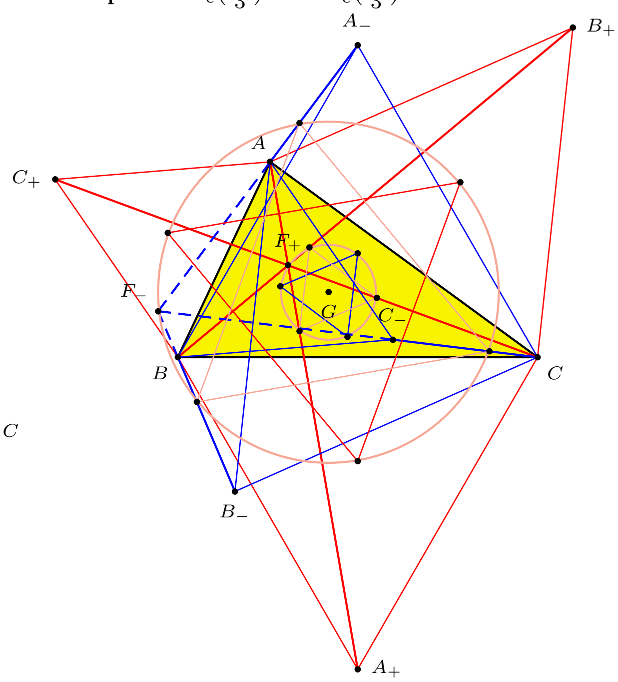

Figure 5

(4) It follows that the triangle $\mathcal{T}_\epsilon(\frac{\tau_\epsilon}{3})$ is the reflection of the $(-\epsilon)$-Napoleon triangle in G, and is therefore equilateral.

(5) The circle through the vertices of $\mathcal{T}_\epsilon(\frac{\tau_\epsilon}{3})$ and the $(-\epsilon)$-Napoleon triangle has radius $\frac{\tau_{-\epsilon}}{3}$ and also passes through the Fermat point F_ϵ. See Figure 5.

Since $GA_\epsilon(\frac{\tau_\epsilon}{3}) = \frac{\tau_{-\epsilon}}{3}$, (see Figure 4), the circle, center X, radius $\frac{\tau_{-\epsilon}}{3}$, passes through G. See Figure 6A. Likewise, the circle, center X', radius $\frac{\tau_\epsilon}{3}$ also passes through G. See Figure 6B. In these figures, we label

$$Y = A_+\left(\frac{\tau_+ - \tau_-}{3}\right), \qquad Z = A_+\left(\frac{\tau_+ + \tau_-}{3}\right),$$
$$Y' = A_-\left(\frac{\tau_- - \tau_+}{3}\right), \qquad Z' = A_-\left(\frac{\tau_+ + \tau_-}{3}\right).$$

It follows that GY and GZ are perpendicular to each other; so are GY' and GZ'.

(6) For $\epsilon = \pm 1$, the lines joining the centroid G to $A_\epsilon(\frac{\tau_\epsilon+\tau_{-\epsilon}}{3})$ and $A_\epsilon(\frac{\tau_\epsilon-\tau_{-\epsilon}}{3})$ are perpendicular to each other. Similarly, the lines joining G to $B_\epsilon(\frac{\tau_\epsilon+\tau_{-\epsilon}}{3})$ and $B_\epsilon(\frac{\tau_\epsilon-\tau_{-\epsilon}}{3})$ are perpendicular to each other; so are the lines joining G to $C_\epsilon(\frac{\tau_\epsilon+\tau_{-\epsilon}}{3})$ and $C_\epsilon(\frac{\tau_\epsilon-\tau_{-\epsilon}}{3})$.

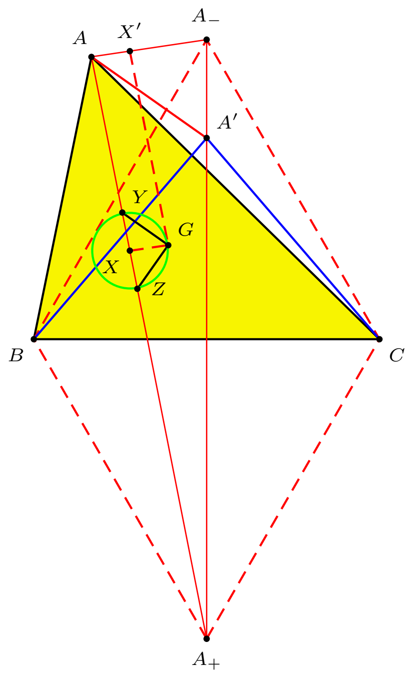

Figure 6A

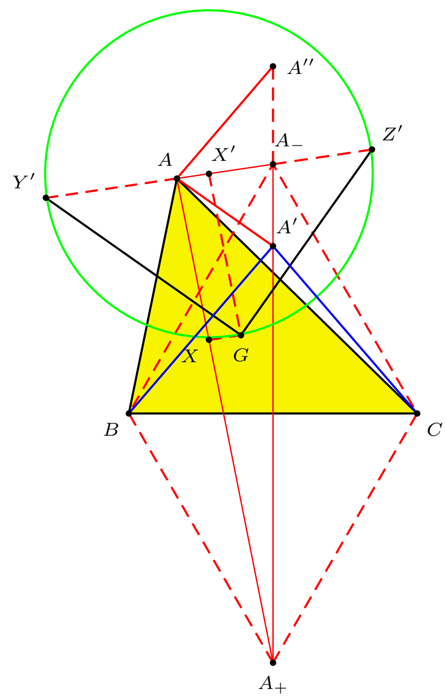

Figure 6B

In Figure 6A, since $\angle XGY = \angle XYG$ and $AXGX'$ is a parallelogram, the line GY is the bisector of angle XGX', and is parallel to the bisector of angle A_+AA_-. If the internal bisector of angle A_+AA_- intersects A_+A_- at A', then it is easy to see that A' is the apex of the isosceles triangle constructed *inwardly* on BC with base angle φ satisfying

$$\cot\varphi = \frac{\tau_+ + \tau_-}{\sqrt{3}(\tau_+ - \tau_-)}. \qquad (\dagger)$$

Similarly, in Figure 6B, the line GZ' is parallel to the external bisector of the same angle. We summarize these as follows.

(7) The lines joining $A_+(\frac{\tau_+-\tau_-}{3})$ to $A_-(\frac{\tau_--\tau_+}{3})$ and $A_+(\frac{\tau_++\tau_-}{3})$ to $A_-(\frac{\tau_++\tau_-}{3})$ are perpendicular at G, and are respectively parallel to the internal and external bisectors of angle A_+AA_-. Similarly, the two lines joining $B_+(\frac{\tau_+-\tau_-}{3})$ to $B_-(\frac{\tau_--\tau_+}{3})$ and $B_+(\frac{\tau_++\tau_-}{3})$ to $B_-(\frac{\tau_++\tau_-}{3})$ are perpendicular at G, being parallel to the internal and external bisectors of angle B_+BB_-; so are the lines joining

$C_+(\frac{\tau_+-\tau_-}{3})$ to $C_-(\frac{\tau_--\tau_+}{3})$, and $C_+(\frac{\tau_++\tau_-}{3})$ to $C_-(\frac{\tau_++\tau_-}{3})$, being parallel to the internal and external bisectors of angle C_+CC_-.

3. Collinearity

What is interesting is that these 3 pairs of perpendicular lines in (7) above form the same right angles at the centroid G. Specifically, the six points

$$A_+(\frac{\tau_++\tau_-}{3}), B_+(\frac{\tau_++\tau_-}{3}), C_+(\frac{\tau_++\tau_-}{3}), A_-(\frac{\tau_++\tau_-}{3}), B_-(\frac{\tau_++\tau_-}{3}), C_-(\frac{\tau_++\tau_-}{3})$$

are collinear with the centroid G on a line $\mathcal{L}_+$; so are the 6 points

$$A_+(\frac{\tau_+-\tau_-}{3}), B_+(\frac{\tau_+-\tau_-}{3}), C_+(\frac{\tau_+-\tau_-}{3}), A_-(\frac{\tau_--\tau_+}{3}), B_-(\frac{\tau_--\tau_+}{3}), C_-(\frac{\tau_--\tau_+}{3})$$

on a line $\mathcal{L}_-$ through G. See Figure 7. To justify this, we consider the triangle $\mathcal{T}_\epsilon(t) := A_\epsilon(t)B_\epsilon(t)C_\epsilon(t)$ for varying t.

(8) For $\epsilon = \pm 1$, the triangle $\mathcal{T}_\epsilon(t)$ degenerates into a line containing the centroid G if and only if $t = \frac{\tau_\epsilon+\delta\tau_{-\epsilon}}{3}$, $\delta = \pm 1$.

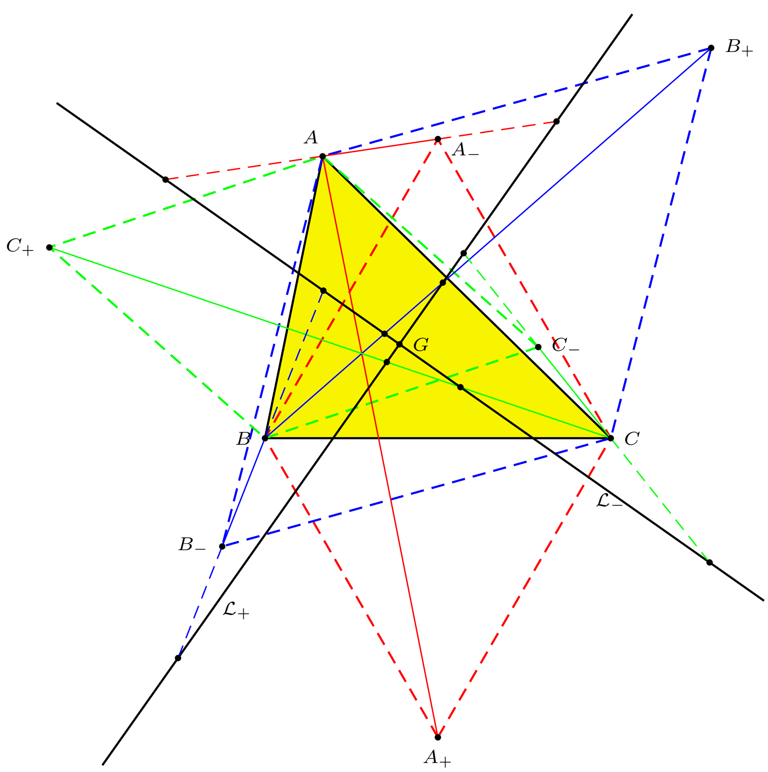

Figure 7

4. Barycentric coordinates

To prove (8) and to obtain further interesting geometric results, we make use of coordinates. Bottema has advocated the use of homogeneous barycentric coordinates. See [3, 6]. Let P be a point in the plane of triangle ABC. With reference to ABC, the homogeneous barycentric coordinates of P are the ratios of signed areas

$$(\triangle PBC : \triangle PCA : \triangle PAB).$$

The coordinates of the vertex A_+ of the equilateral triangle BCA_+, for example, are $(-\frac{\sqrt{3}}{4}a^2 : \frac{1}{2}ab\sin(C+60^\circ) : \frac{1}{2}ca\sin(B+60^\circ))$, which can be rewritten as

$$A_+ = (-2\sqrt{3}a^2 : \sqrt{3}(a^2+b^2-c^2)+4\triangle : \sqrt{3}(c^2+a^2-b^2)+4\triangle).$$

More generally, for $\epsilon = \pm 1$, the vertices of the equilateral triangles erected on the sides of triangle ABC are the points

$$\begin{aligned}
A_\epsilon =&(-2\sqrt{3}a^2 : \sqrt{3}(a^2+b^2-c^2)+4\epsilon\triangle : \sqrt{3}(c^2+a^2-b^2)+4\epsilon\triangle),\\
B_\epsilon =&(\sqrt{3}(a^2+b^2-c^2)+4\epsilon\triangle : -2\sqrt{3}b^2 : \sqrt{3}(b^2+c^2-a^2)+4\epsilon\triangle),\\
C_\epsilon =&(\sqrt{3}(c^2+a^2-b^2)+4\epsilon\triangle : \sqrt{3}(b^2+c^2-a^2)+4\epsilon\triangle : -2\sqrt{3}c^2).
\end{aligned}$$

Note that in each case, the coordinate sum is $8\epsilon\triangle$. From this we easily compute the coordinates of the centroid by simply adding the corresponding coordinates of the three vertices.

(9A) For $\epsilon = \pm 1$, triangles $A_\epsilon B_\epsilon C_\epsilon$ and ABC have the same centroid.

Sometimes it is convenient to work with *absolute* barycentric coordinates. For a finite point $P = (u : v : w)$, we obtain the absolute barycentric coordinates by normalizing its homogeneous barycentric coordinates, namely, by dividing by the coordinate sum. Thus,

$$P = \frac{1}{u+v+w}(uA+vB+wC),$$

provided $u+v+w$ is nonzero.

The absolute barycentric coordinates of the point $A_\epsilon(t)$ can be easily written down. For each value of t,

$$A_\epsilon(t) = \frac{1}{\tau_\epsilon}((\tau_\epsilon - t)A + t\cdot A_\epsilon),$$

and similarly for $B_\epsilon(t)$ and $C_\epsilon(t)$.

This, together with (9A), leads easily to the more general result.

(9B) For arbitrary t, the triangles $\mathcal{T}_\epsilon(t)$ and ABC have the same centroid.

5. Area of $\mathcal{T}_\epsilon(t)$

Let $X = (x_1 : x_2 : x_3)$, $Y = (y_1 : y_2 : y_3)$ and $Z = (z_1 : z_2 : z_3)$ be finite points with homogeneous coordinates with respect to triangle ABC. The *signed* area of the oriented triangle XYZ is

$$\frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}}{(x_1+x_2+x_3)(y_1+y_2+y_3)(z_1+z_2+z_3)} \cdot \triangle.$$

A proof of this elegant formula can be found in [1, VII] or [3]. A direct application of this formula yields the area of triangle $\mathcal{T}_\epsilon(t)$.

(10) The area of triangle $\mathcal{T}_\epsilon(t)$ is

$$\frac{3\sqrt{3}\epsilon}{4}\left(t - \frac{\tau_\epsilon + \tau_{-\epsilon}}{3}\right)\left(t - \frac{\tau_\epsilon - \tau_{-\epsilon}}{3}\right)\triangle.$$

Statement (8) follows immediately from this formula and (9B).

(11) $\mathcal{T}_\epsilon(t)$ has the same area as ABC if and only if $t = 0$ or $\frac{2\tau_\epsilon}{3}$. In fact, the two triangles $\mathcal{T}_+(\frac{2\tau_+}{3})$ and $\mathcal{T}_-(\frac{2\tau_-}{3})$ are symmetric with respect to the centroid. See Figures 8A and 8B.

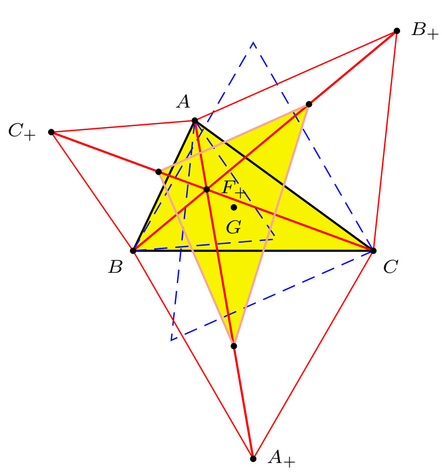

Figure 8A

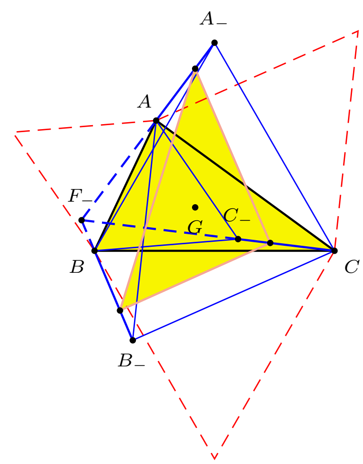

Figure 8B

6. Kiepert hyperbola and Steiner ellipse

The existence of the line $\mathcal{L}_-$ (see §3) shows that the internal bisectors of the angles A_+AA_-, B_+BB_-, and C_+CC_- are parallel. These bisectors contain the the apexes A', B', C' of isosceles triangles constructed inwardly on the sides with the same base angle given by (†). It is well known that $A'B'C'$ and ABC are perspective at a point on the Kiepert hyperbola, the rectangular circum-hyperbola through the orthocenter and the centroid. This perspector is necessarily an infinite point (of an asymptote of the hyperbola). In other words, the line $\mathcal{L}_-$ is parallel to an asymptote of this rectangular hyperbola.

(12) The lines $\mathcal{L}_\pm$ are the parallels through G to the asymptotes of the Kiepert hyperbola.

(13) It is also known that the asymptotes of the Kiepert hyperbola are parallel to the axes of the Steiner in-ellipse, (see [4]), the ellipse that touches the sides of triangle ABC at their midpoints, with center at the centroid G. See Figure 9.

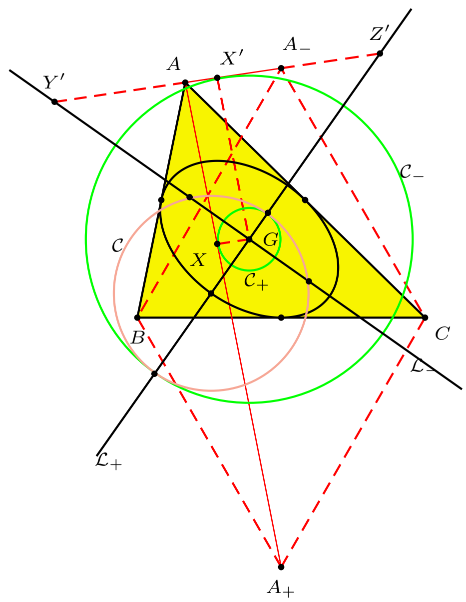

Figure 9

(14) The Steiner in-ellipse has major and minor axes of lengths $\frac{\tau_+ \pm \tau_-}{3}$. From this, we have the following construction of its foci. See Figure 9.

- Construct the concentric circles $\mathcal{C}_\pm$ at G through $A_\epsilon(\frac{\tau_\epsilon}{3})$.
- Construct a circle $\mathcal{C}$ with center on $\mathcal{L}_+$ tangent to the circles $\mathcal{C}_+$ internally and $\mathcal{C}_-$ externally. There are two such circles; any one of them will do.
- The intersections of the circle $\mathcal{C}$ with the line $\mathcal{L}_-$ are the foci of Steiner in-ellipse.

We conclude by recording the homogeneous barycentric coordinates of the two foci of the Steiner in-ellipse. Let

$$Q = a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2.$$

The line $\mathcal{L}_-$ containing the two foci has infinite point

$$\begin{aligned} I_-^\infty = ((b-c)(a(a+b+c)-(b^2+bc+c^2)-\sqrt{Q}),\\ (c-a)(b(a+b+c)-(c^2+ca+a^2)-\sqrt{Q}),\\ (a-b)(c(a+b+c)-(a^2+ab+b^2)-\sqrt{Q})). \end{aligned}$$

As a vector, this has square length $2\sqrt{Q}(f+g\sqrt{Q})$, where

$$f = \sum_{\text{cyclic}} a^6 - bc(b^4+c^4) + a^2bc(ab+ac-bc),$$

$$g = \sum_{\text{cyclic}} a^4 - bc(b^2+c^2-a^2).$$

Since the square distance from the centroid to each of the foci is $\frac{1}{9}\sqrt{Q}$, these two foci are the points

$$G \pm \frac{1}{3\sqrt{2(f+g\sqrt{Q})}} I_-^\infty.$$

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Triangle Centers Associated with the Malfatti Circles

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Abstract. Various formulae for the radii of the Malfatti circles of a triangle are presented. We also express the radii of the excircles in terms of the radii of the Malfatti circles, and give the coordinates of some interesting triangle centers associated with the Malfatti circles.

1. The radii of the Malfatti circles

The Malfatti circles of a triangle are the three circles inside the triangle, mutually tangent to each other, and each tangent to two sides of the triangle. See Figure 1. Given a triangle ABC, let a, b, c denote the lengths of the sides BC, CA, AB, s the semiperimeter, I the incenter, and r its inradius. The radii of the Malfatti circles of triangle ABC are given by

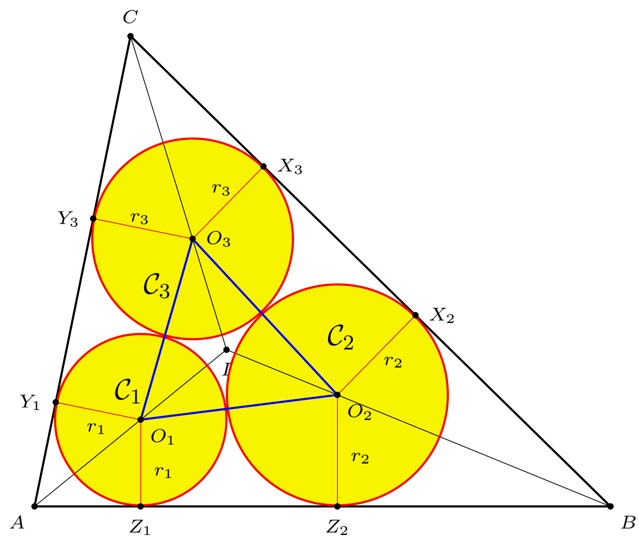

Figure 1

$$
\begin{aligned}
r_1 &= \frac{r}{2(s-a)}\left(s - r - (IB + IC - IA)\right), \\
r_2 &= \frac{r}{2(s-b)}\left(s - r - (IC + IA - IB)\right), \\
r_3 &= \frac{r}{2(s-c)}\left(s - r - (IA + IB - IC)\right).
\end{aligned}
\tag{1}
$$

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The author is grateful to the referee and the editor for their valuable comments and helps.

According to F.G.-M. [1, p.729], these results were given by Malfatti himself, and were published in [7] after his death. See also [6]. Another set of formulae give the same radii in terms of a, b, c and r:

$$\begin{aligned} r_1 =& \frac{(IB+r-(s-b))(IC+r-(s-c))}{2(IA+r-(s-a))}, \\ r_2 =& \frac{(IC+r-(s-c))(IA+r-(s-a))}{2(IB+r-(s-b))}, \\ r_3 =& \frac{(IA+r-(s-a))(IB+r-(s-b))}{2(IC+r-(s-c))}. \end{aligned} \tag{2}$$

These easily follow from (1) and the following formulae that express the radii r_1, r_2, r_3 in terms of r and trigonometric functions:

$$\begin{aligned} r_1 =& \frac{\left(1+\tan\frac{B}{4}\right)\left(1+\tan\frac{C}{4}\right)}{1+\tan\frac{A}{4}} \cdot \frac{r}{2}, \\ r_2 =& \frac{\left(1+\tan\frac{C}{4}\right)\left(1+\tan\frac{A}{4}\right)}{1+\tan\frac{B}{4}} \cdot \frac{r}{2}, \\ r_3 =& \frac{\left(1+\tan\frac{A}{4}\right)\left(1+\tan\frac{B}{4}\right)}{1+\tan\frac{C}{4}} \cdot \frac{r}{2}. \end{aligned} \tag{3}$$

These can be found in [10]. They can be used to obtain the following formula which is given in [2, pp.103–106]. See also [12].

$$\frac{2}{r} = \frac{1}{\sqrt{r_1 r_2}} + \frac{1}{\sqrt{r_2 r_3}} + \frac{1}{\sqrt{r_1 r_2}} - \sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}. \tag{4}$$

2. Exradii in terms of Malfatti radii

Antreas P. Hatzipolakis [3] asked for the exradii r_a, r_b, r_c of triangle ABC in terms of the Malfatti radii r_1, r_2, r_3 and the inradius r.

Proposition 1.

$$\begin{aligned} r_a - r_1 =& \frac{\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}\right)}, \\ r_b - r_2 =& \frac{\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}\right)}, \\ r_c - r_3 =& \frac{\frac{2}{r} - \frac{1}{\sqrt{r_1 r_2}}}{\left(\frac{2}{r} - \frac{1}{\sqrt{r_2 r_3}}\right)\left(\frac{2}{r} - \frac{1}{\sqrt{r_3 r_1}}\right)}. \end{aligned} \tag{5}$$

Proof. For convenience we write

$$t_1 := \tan\frac{A}{4}, \qquad t_2 := \tan\frac{B}{4}, \qquad t_3 := \tan\frac{C}{4}.$$

Note that from $\tan\left(\frac{A}{4}+\frac{B}{4}+\frac{C}{4}\right)=1$, we have

$$1 - t_1 - t_2 - t_3 - t_1t_2 - t_2t_3 - t_3t_1 + t_1t_2t_3 = 0. \tag{6}$$

From (3) we obtain

$$\begin{aligned}
\frac{2}{r} - \frac{1}{\sqrt{r_2r_3}} =& \frac{t_1}{1+t_1}\cdot\frac{2}{r},\\
\frac{2}{r} - \frac{1}{\sqrt{r_3r_1}} =& \frac{t_2}{1+t_2}\cdot\frac{2}{r},\\
\frac{2}{r} - \frac{1}{\sqrt{r_1r_2}} =& \frac{t_3}{1+t_3}\cdot\frac{2}{r}.
\end{aligned} \tag{7}$$

For the exradius r_a, we have

$$r_a = \frac{s}{s-a}\cdot r = \cot\frac{B}{2}\cot\frac{C}{2}\cdot r = \frac{(1-t_2^2)(1-t_3^2)}{4t_2t_3}\cdot r.$$

It follows that

$$\begin{aligned}
r_a - r_1 =& (1+t_2)(1+t_3)\cdot\frac{r}{2}\left(\frac{(1-t_2)(1-t_3)}{2t_2t_3} - \frac{1}{1+t_1}\right)\\
=& (1+t_2)(1+t_3)\cdot\frac{r}{2}\cdot\frac{(1+t_1)(1-t_2)(1-t_3)-2t_2t_3}{2t_2t_3(1+t_1)}\\
=& (1+t_2)(1+t_3)\cdot\frac{r}{2}\cdot\frac{2t_1}{2t_2t_3(1+t_1)} \qquad \text{(from (6))}\\
=& \frac{t_1}{1+t_1}\cdot\frac{1+t_2}{t_2}\cdot\frac{1+t_3}{t_3}\cdot\frac{r}{2}.
\end{aligned}$$

Now the result follows from (7). $\square$

Note that with the help of (4), the exradii r_a, r_b, r_c can be explicitly written in terms of the Malfatti radii r_1, r_2, r_3. We present another formula useful in the next sections in the organization of coordinates of triangle centers.

Proposition 2.

$$\frac{1}{r_1} - \frac{1}{r_a} = \frac{a}{rs}\cdot\frac{(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})}{1+\cos\frac{A}{2}}.$$

3. Triangle centers associated with the Malfatti circles

Let A' be the point of tangency of the Malfatti circles $\mathcal{C}_2$ and $\mathcal{C}_3$. Similarly define B' and C'. It is known ([4, p.97]) that triangle $A'B'C'$ is perspective with ABC at the *first Ajima-Malfatti point* X_{179}. See Figure 3. We work out the details here and construct a few more triangle centers associated with the Malfatti circles. In particular, we find two new triangle centers P_+ and P_- which divide the incenter I and the first Ajima-Malfatti point harmonically.

3.1. *The centers of the Malfatti circles.* We begin with the coordinates of the centers of the Malfatti circles.

Since O_1 divides the segment AI_a in the ratio $AO_1 : O_1I_a = r_1 : r_a - r_1$, we have $\frac{O_1}{r_1} = \left(\frac{1}{r_1} - \frac{1}{r_a}\right) A + \frac{1}{r_a} \cdot I_A$. With $r_a = \frac{rs}{s-a}$ we rewrite the absolute barycentric coordinates of O_1, along with those of O_2 and O_3, as follows.

$$\begin{aligned}
\frac{O_1}{r_1} &= \left(\frac{1}{r_1} - \frac{1}{r_a}\right) A + \frac{s-a}{rs} \cdot I_a, \\
\frac{O_2}{r_2} &= \left(\frac{1}{r_2} - \frac{1}{r_b}\right) B + \frac{s-b}{rs} \cdot I_b, \\
\frac{O_3}{r_3} &= \left(\frac{1}{r_3} - \frac{1}{r_c}\right) C + \frac{s-c}{rs} \cdot I_c.
\end{aligned} \tag{8}$$

From these expressions we have, in homogeneous barycentric coordinates,

$$\begin{aligned}
O_1 &= \left(2rs\left(\frac{1}{r_1} - \frac{1}{r_a}\right) - a : b : c\right), \\
O_2 &= \left(a : 2rs\left(\frac{1}{r_2} - \frac{1}{r_b}\right) - b : c\right), \\
O_3 &= \left(a : b : 2rs\left(\frac{1}{r_3} - \frac{1}{r_c}\right) - c\right).
\end{aligned}$$

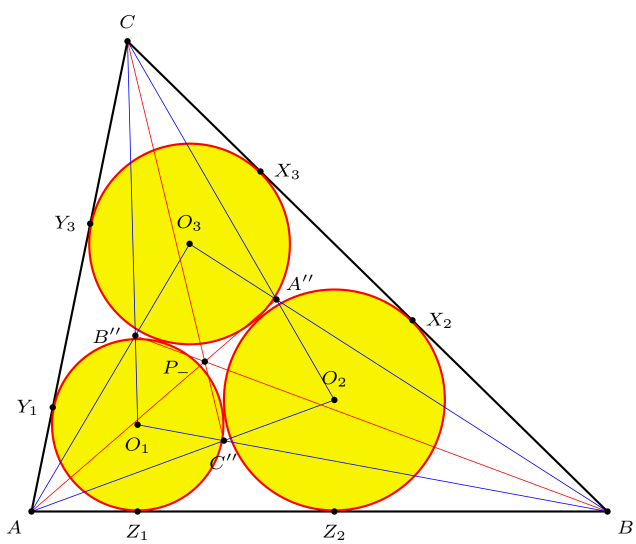

Figure 2

3.2. *The triangle center P_-.* It is clear that $O_1O_2O_3$ is perspective with ABC at the incenter $(a : b : c)$. However, it also follows that if we consider

$$A'' = BO_3 \cap CO_2, \quad B'' = CO_1 \cap AO_3, \quad C'' = AO_2 \cap BO_1,$$

then triangle $A''B''C''$ is perspective with ABC at

$$\begin{aligned} P_- &= \left(2rs\left(\frac{1}{r_1} - \frac{1}{r_a}\right) - a : 2rs\left(\frac{1}{r_2} - \frac{1}{r_b}\right) - b : 2rs\left(\frac{1}{r_3} - \frac{1}{r_c}\right) - c\right) \\ &= \left(\frac{1}{r_1} - \frac{1}{r_a} - \frac{a}{2rs} : \frac{1}{r_2} - \frac{1}{r_b} - \frac{b}{2rs} : \frac{1}{r_3} - \frac{1}{r_c} - \frac{c}{2rs}\right) \\ &= \left(a\left(\frac{(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})}{1+\cos\frac{A}{2}} - \frac{1}{2}\right) : \cdots : \cdots\right) \end{aligned} \tag{9}$$

by Proposition 2. See Figure 2.

Remark. The point P_- appears in [5] as the *first Malfatti-Rabinowitz point* X_{1142}.

3.3. *The first Ajima-Malfatti point.* For the points of tangency of the Malfatti circles, note that A' divides O_2O_3 in the ratio $O_2A' : A'O_3 = r_2 : r_3$. We have, in absolute barycentric coordinates,

$$\left(\frac{1}{r_2} + \frac{1}{r_3}\right) A' = \frac{O_2}{r_2} + \frac{O_3}{r_3} = \frac{a}{rs} \cdot A + \left(\frac{1}{r_2} - \frac{1}{r_b}\right) B + \left(\frac{1}{r_3} - \frac{1}{r_c}\right) C;$$

similarly for B' and C'. In homogeneous coordinates,

$$\begin{aligned} A' &= \left(\frac{a}{rs} : \frac{1}{r_2} - \frac{1}{r_b} : \frac{1}{r_3} - \frac{1}{r_c}\right), \\ B' &= \left(\frac{1}{r_1} - \frac{1}{r_a} : \frac{b}{rs} : \frac{1}{r_3} - \frac{1}{r_c}\right), \\ C' &= \left(\frac{1}{r_1} - \frac{1}{r_a} : \frac{1}{r_2} - \frac{1}{r_b} : \frac{c}{rs}\right). \end{aligned} \tag{10}$$

From these, it is clear that $A'B'C'$ is perspective with ABC at

$$\begin{aligned} P &= \left(\frac{1}{r_1} - \frac{1}{r_a} : \frac{1}{r_2} - \frac{1}{r_b} : \frac{1}{r_3} - \frac{1}{r_c}\right) \\ &= \left(\frac{a(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})}{1+\cos\frac{A}{2}} : \frac{b(1+\cos\frac{C}{2})(1+\cos\frac{A}{2})}{1+\cos\frac{B}{2}} \right. \\ &\qquad \left. : \frac{c(1+\cos\frac{A}{2})(1+\cos\frac{B}{2})}{1+\cos\frac{C}{2}}\right) \\ &= \left(\frac{a}{\left(1+\cos\frac{A}{2}\right)^2} : \frac{b}{\left(1+\cos\frac{B}{2}\right)^2} : \frac{c}{\left(1+\cos\frac{C}{2}\right)^2}\right) \end{aligned} \tag{11}$$

by Proposition 2. The point P appears as X_{179} in [4, p.97], with trilinear coordinates

$$\left(\sec^4\frac{A}{4} : \sec^4\frac{B}{4} : \sec^4\frac{C}{4}\right)$$

computed by Peter Yff, and is named the *first Ajima-Malfatti point*. See Figure 3.

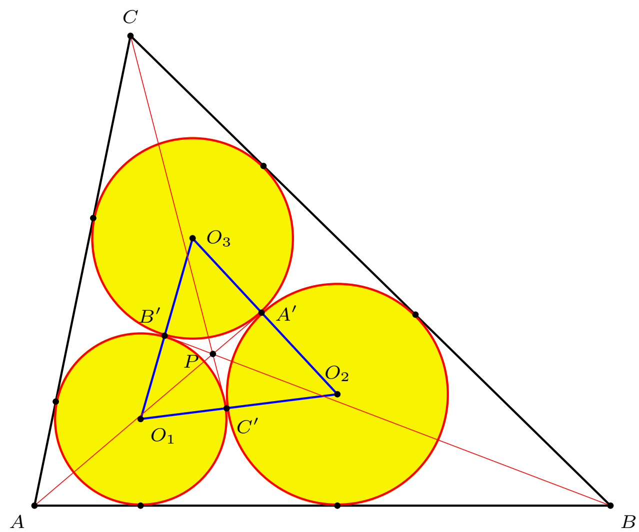

Figure 3

3.4. *The triangle center* P_+. Note that the circle through A', B', C' is orthogonal to the Malfatti circles. It is the radical circle of the Malfatti circles, and is the incircle of $O_1O_2O_3$. The lines O_1A', O_2B', O_3C' are concurrent at the Gergonne point of triangle $O_1O_2O_3$. See Figure 4. As such, this is the point P_+ given by

$$\begin{aligned}
&\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)P_+ = \frac{O_1}{r_1}+\frac{O_2}{r_2}+\frac{O_3}{r_3}\\
=&\left(\frac{1}{r_1}-\frac{1}{r_a}\right)A+\frac{I_a}{r_a}+\left(\frac{1}{r_2}-\frac{1}{r_b}\right)B+\frac{I_b}{r_b}+\left(\frac{1}{r_3}-\frac{1}{r_c}\right)C+\frac{I_c}{r_c}\\
=&\left(\frac{1}{r_1}-\frac{1}{r_a}\right)A+\left(\frac{1}{r_2}-\frac{1}{r_b}\right)B+\left(\frac{1}{r_3}-\frac{1}{r_c}\right)C+\frac{I}{r}\\
=&\left(\frac{1}{r_1}-\frac{1}{r_a}\right)A+\left(\frac{1}{r_2}-\frac{1}{r_b}\right)B+\left(\frac{1}{r_3}-\frac{1}{r_c}\right)C+\frac{1}{2rs}(aA+bB+cC)\\
=&\left(\frac{1}{r_1}-\frac{1}{r_a}+\frac{a}{2rs}\right)A+\left(\frac{1}{r_2}-\frac{1}{r_b}+\frac{b}{2rs}\right)B+\left(\frac{1}{r_3}-\frac{1}{r_c}+\frac{c}{2rs}\right)C.
\end{aligned}$$

It follows that in homogeneous coordinates,

$$P_+ = \left(\frac{1}{r_1} - \frac{1}{r_a} + \frac{a}{2rs} : \frac{1}{r_2} - \frac{1}{r_b} + \frac{b}{2rs} : \frac{1}{r_3} - \frac{1}{r_c} + \frac{c}{2rs}\right)$$
$$= \left(a\left(\frac{(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})}{1+\cos\frac{A}{2}} + \frac{1}{2}\right) : \cdots : \cdots\right) \qquad (12)$$

by Proposition 2.

Proposition 3. *The points P_+ and P_- divide the segment IP harmonically.*

Proof. This follows from their coordinates given in (12), (9), and (11). □

From the coordinates of P, P_+ and P_-, it is easy to see that P_+ and P_- divide the segment IP harmonically.

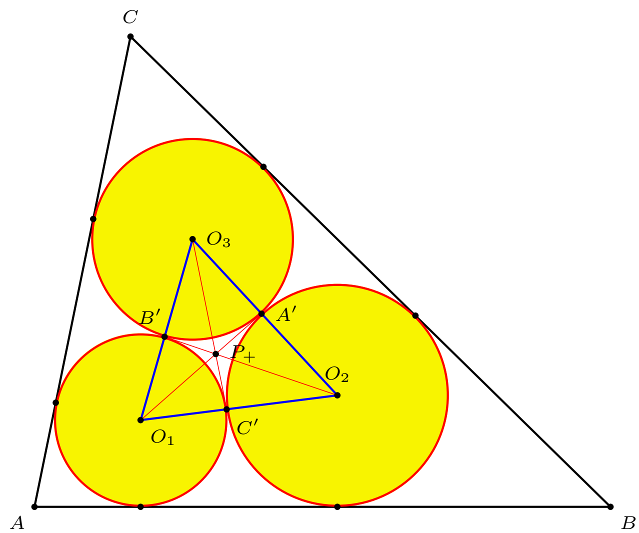

Figure 4

3.5. *The triangle center Q.* Let the Malfatti circle $\mathcal{C}_1$ touch the sides CA and AB at Y_1 and Z_1 respectively. Likewise, let $\mathcal{C}_2$ touch AB and BC at Z_2 and X_2, $\mathcal{C}_3$ touch BC and CA at X_3 and Y_3 respectively. Denote by X, Y, Z the midpoints of the segments X_2X_3, Y_3Y_1, Z_1Z_2 respectively. Stanley Rabinowitz [9] asked if the lines AX, BY, CZ are concurrent. We answer this in the affirmative.

Proposition 4. *The lines AX, BY, CZ are concurrent at a point Q with homogeneous barycentric coordinates*

$$\left(\tan\frac{A}{4} : \tan\frac{B}{4} : \tan\frac{C}{4}\right). \qquad (13)$$

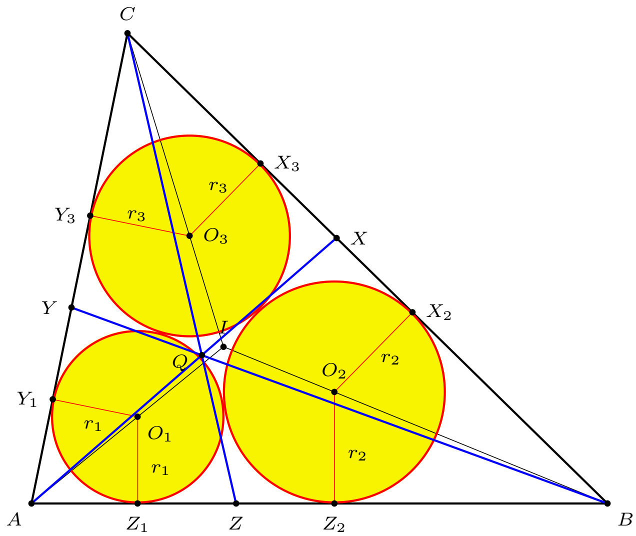

Figure 5

Proof. In Figure 5, we have

$$\begin{aligned}
BX =& \frac{1}{2}(a + BX_2 - X_3C) \\
=& \frac{1}{2}\left(a + \frac{r_2}{r}(s-b) - \frac{r_3}{r}(s-c)\right) \\
=& \frac{1}{2}(a + IB - IC) && \text{(from (1))} \\
=& \frac{1}{2}\left(2R\sin A + \frac{r}{\sin\frac{B}{2}} - \frac{r}{\sin\frac{C}{2}}\right) \\
=& 4R\sin\frac{A}{2}\cos\frac{B}{4}\sin\frac{C}{4}\cos\frac{B+C}{4}
\end{aligned}$$

by making use of the formula

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Similarly,

$$XC = \frac{1}{2}(a - BX_2 + X_3C) = 4R\sin\frac{A}{2}\sin\frac{B}{4}\cos\frac{C}{4}\cos\frac{B+C}{4}.$$

It follows that

$$\frac{BX}{XC} = \frac{\cos\frac{B}{4}\sin\frac{C}{4}}{\sin\frac{B}{4}\cos\frac{C}{4}} = \frac{\tan\frac{C}{4}}{\tan\frac{B}{4}}.$$

Likewise,

$$\frac{CY}{YA} = \frac{\tan\frac{A}{4}}{\tan\frac{C}{4}} \quad \text{and} \quad \frac{AZ}{ZB} = \frac{\tan\frac{B}{4}}{\tan\frac{A}{4}},$$

and it follows from Ceva's theorem that AX, BY, CZ are concurrent since

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

In fact, we can easily identify the homogeneous barycentric coordinates of the intersection Q as given in (13) above since those of X, Y, Z are

$$X = \left(0 : \tan\frac{B}{4} : \tan\frac{C}{4}\right),$$
$$Y = \left(\tan\frac{A}{4} : 0 : \tan\frac{C}{4}\right),$$
$$Z = \left(\tan\frac{A}{4} : \tan\frac{B}{4} : 0\right).$$

□

Remark. The coordinates of Q can also be written as

$$\left(\frac{\sin\frac{A}{2}}{1+\cos\frac{A}{2}} : \frac{\sin\frac{B}{2}}{1+\cos\frac{B}{2}} : \frac{\sin\frac{C}{2}}{1+\cos\frac{C}{2}}\right)$$

or

$$\left(\frac{a}{(1+\cos\frac{A}{2})\cos\frac{A}{2}} : \frac{b}{(1+\cos\frac{B}{2})\cos\frac{B}{2}} : \frac{c}{(1+\cos\frac{C}{2})\cos\frac{C}{2}}\right).$$

3.6. *The radical center of the Malfatti circles.* Note that the common tangent of $\mathcal{C}_2$ and $\mathcal{C}_3$ at A' passes through X. This means that $A'X$ is perpendicular to O_2O_3 at A'. This line therefore passes through the incenter I' of $O_1O_2O_3$. Now, the homogeneous coordinates of A' and X can be rewritten as

$$A' = \left(\frac{a}{(1+\cos\frac{A}{2})(1+\cos\frac{B}{2})(1+\cos\frac{C}{2})} : \frac{b}{(1+\cos\frac{B}{2})^2} : \frac{c}{(1+\cos\frac{C}{2})^2}\right),$$
$$X = \left(0 : \frac{b}{(1+\cos\frac{B}{2})\cos\frac{B}{2}} : \frac{c}{(1+\cos\frac{C}{2})\cos\frac{C}{2}}\right).$$

It is easy to verify that these two points lie on the line

$$\frac{(1+\cos\frac{A}{2})(\cos\frac{B}{2}-\cos\frac{C}{2})}{a}x - \frac{(1+\cos\frac{B}{2})\cos\frac{B}{2}}{b}y + \frac{(1+\cos\frac{C}{2})\cos\frac{C}{2}}{c}z = 0,$$

which also contains the point

$$\left(\frac{a}{1+\cos\frac{A}{2}} : \frac{b}{1+\cos\frac{B}{2}} : \frac{c}{1+\cos\frac{C}{2}}\right).$$

Similar calculations show that the latter point also lies on the lines $B'Y$ and $C'Z$. It is therefore the incenter I' of triangle $O_1O_2O_3$. See Figure 6. This point appears in [5] as X_{483} , the radical center of the Malfatti circles.

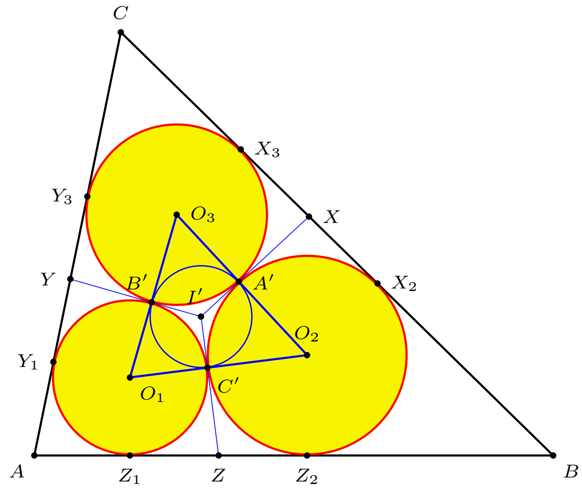

Figure 6

Remarks. (1) The line joining Q and I' has equation

$$\frac{(1+\cos\frac{A}{2})(\cos\frac{B}{2}-\cos\frac{C}{2})}{\sin\frac{A}{2}}x+\frac{(1+\cos\frac{B}{2})(\cos\frac{C}{2}-\cos\frac{A}{2})}{\sin\frac{B}{2}}y$$
$$+\frac{(1+\cos\frac{C}{2})(\cos\frac{A}{2}-\cos\frac{B}{2})}{\sin\frac{C}{2}}z=0.$$

This line clearly contains the point $\left(\sin\frac{A}{2}:\sin\frac{B}{2}:\sin\frac{C}{2}\right)$, which is the point X_{174}, the Yff center of congruence in [4, pp.94–95].

(2) According to [4], the triangle $A'B'C'$ in §3.3 is also perspective with the excentral triangle. This is because cevian triangles and anticevian triangles are always perspective. The perspector

$$\left(\frac{a\left((2+\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2})^2+\cos\frac{A}{2}(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}-(2+\cos\frac{A}{2})^2)\right)}{1+\cos\frac{A}{2}}:\cdots:\cdots\right)$$

is named the *second Ajima-Malfatti* point X_{180}. For the same reason, the triangle XYZ in §3.5 is also perspective with the excentral triangle. The perspector is the point

$$\left(a\left(-\cos\frac{A}{2}\left(1+\cos\frac{A}{2}\right)+\cos\frac{B}{2}\left(1+\cos\frac{B}{2}\right)+\cos\frac{C}{2}\left(1+\cos\frac{C}{2}\right)\right):\cdots:\cdots\right).$$

This point and the triangle center P_+ apparently do not appear in the current edition of [5].

Editor's endnote. The triangle center Q in §3.5 appears in [5] as the *second Malfatti-Rabinowitz point* X_{1143}. Its coordinates given by the present editor [13] were not correct owing to a mistake in a sign in the calculations. In the notations of [13], if

α, β, γ are such that

$$\sin^2\alpha = \frac{a}{s}, \quad \sin^2\beta = \frac{b}{s}, \quad \sin^2\gamma = \frac{c}{s},$$

and $\lambda = \frac{1}{2}(\alpha+\beta+\gamma)$, then the homogeneous barycentric coordinates of Q are

$$(\cot(\lambda-\alpha) : \cot(\lambda-\beta) : \cot(\lambda-\gamma)).$$

These are equivalent to those given in (13) in simpler form.

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The Lucas Circles and the Descartes Formula

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Abstract. We determine the radii of the three circles each tangent to the circumcircle of a given triangle at a vertex, and mutually tangent to each other externally. The calculations are then reversed to give the radii of the two Soddy circles associated with three circles tangent to each other externally.

1. The Lucas circles

Consider a triangle ABC with circumcircle $\mathcal{C}$. We set up a coordinate system with the circumcenter O at the origin and A, B, C represented by complex numbers of moduli R, the circumradius. If the lengths of the sides BC, CA, AB are a, b, c respectively, then

$$||A - B|| = c \quad \text{and} \quad \langle A, B\rangle = R^2 - \frac{c^2}{2}. \tag{1}$$

Analogous relations hold for the pairs B, C and C, A. Let $0 \leq \alpha < R$, and consider the circle $\mathcal{C}_A(\alpha)$ with center $\frac{R-\alpha}{R} \cdot A$ and radius α. This is internally tangent to the circumcircle at A, and is the image of $\mathcal{C}$ under the homothety $\mathsf{h}(A, \frac{\alpha}{R})$. See Figure 1. For real numbers β, γ satisfying $0 \leq \beta, \gamma < R$, we consider the circles $\mathcal{C}_B(\beta)$ and $\mathcal{C}_C(\gamma)$ analogously defined. Now, the circles $\mathcal{C}_A(\alpha)$ and $\mathcal{C}_B(\beta)$ are tangent externally if and only if

$$\left|\left|\frac{R-\alpha}{R}A - \frac{R-\beta}{R}B\right|\right| = \alpha + \beta.$$

This is equivalent, by a simple application of (1), to

$$c^2 = \frac{4\alpha\beta}{(R-\alpha)(R-\beta)}.$$

Therefore, the three circles $\mathcal{C}_A(\alpha)$, $\mathcal{C}_B(\beta)$ and $\mathcal{C}_C(\gamma)$ are tangent externally to each other if and only if

$$a^2 = \frac{4R^2\beta\gamma}{(R-\beta)(R-\gamma)}, \quad b^2 = \frac{4R^2\gamma\alpha}{(R-\gamma)(R-\alpha)}, \quad c^2 = \frac{4R^2\alpha\beta}{(R-\alpha)(R-\beta)}. \tag{2}$$

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These equations can be solved for the radii α, β, and γ in terms of a, b, c, and R. In fact, multiplying the equations in (2), we obtain

$$abc = \frac{8R^3\alpha\beta\gamma}{(R-\alpha)(R-\beta)(R-\gamma)}.$$

Consequently,

$$\frac{\alpha}{R-\alpha} = \frac{bc}{2Ra}, \quad \frac{\beta}{R-\beta} = \frac{ca}{2Rb}, \quad \frac{\gamma}{R-\gamma} = \frac{ab}{2Rc}.$$

From these, we obtain

$$\alpha = \frac{bc}{2Ra+bc}\cdot R, \quad \beta = \frac{ca}{2Rb+ca}\cdot R, \quad \gamma = \frac{ab}{2Rc+ab}\cdot R. \tag{3}$$

Denote by $\triangle$ the area of triangle ABC, and h_a, h_b, h_c its three altitudes. We have $2\triangle = a\cdot h_a = b\cdot h_b = c\cdot h_c$. Since $abc = 4R\triangle$, the expression for α in (3) can be rewritten as

$$\frac{\alpha}{R} = \frac{abc}{2Ra^2+abc} = \frac{4R\triangle}{2Ra^2+4R\triangle} = \frac{2\triangle}{a^2+2\triangle} = \frac{a\cdot h_a}{a^2+a\cdot h_a} = \frac{h_a}{a+h_a}.$$

Therefore, the homothety $\mathsf{h}(A, \frac{\alpha}{R})$ is the one that contracts the square on the side BC (externally) into the inscribed square on this side. See Figure 1. The same is true for the other two circles. The three circles $\mathcal{C}_A(\alpha)$, $\mathcal{C}_B(\beta)$, $\mathcal{C}_C(\gamma)$ are therefore the Lucas circles considered in [3]. See Figure 2.

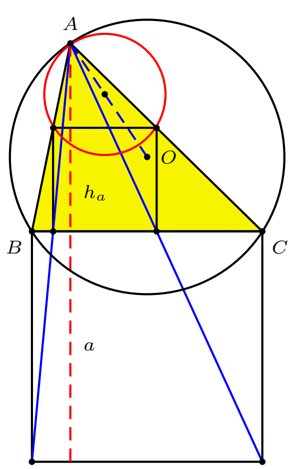

Figure 1

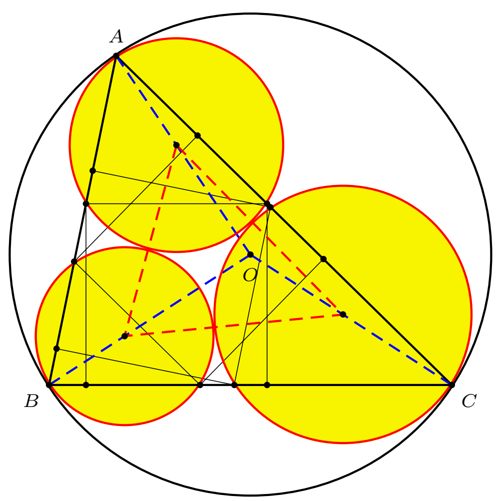

Figure 2

2. Another triad of circles

A simple modification of the above calculations shows that for positive numbers α', β', γ', the images of the circumcircle $\mathcal{C}$ under the homotheties $\mathsf{h}(A, -\frac{\alpha'}{R})$, $\mathsf{h}(B, -\frac{\beta'}{R})$ and $\mathsf{h}(C, -\frac{\gamma'}{R})$ (each tangent to $\mathcal{C}$ at a vertex) are tangent to each other if and only if

$$\alpha' = \frac{bc}{2Ra - bc} \cdot R, \quad \beta' = \frac{ca}{2Rb - ca} \cdot R, \quad \gamma' = \frac{ab}{2Rc - ab} \cdot R. \tag{4}$$

The tangencies are all external provided $2Ra - bc$, $2Rb - ca$ and $2Rc - ab$ are all positive. These quantities are essentially the excesses of the sides over the corresponding altitudes:

$$2Ra - bc = \frac{bc}{a}(a - h_a), \quad 2Rb - ca = \frac{ca}{b}(b - h_b), \quad 2Rc - ab = \frac{ab}{c}(c - h_c).$$

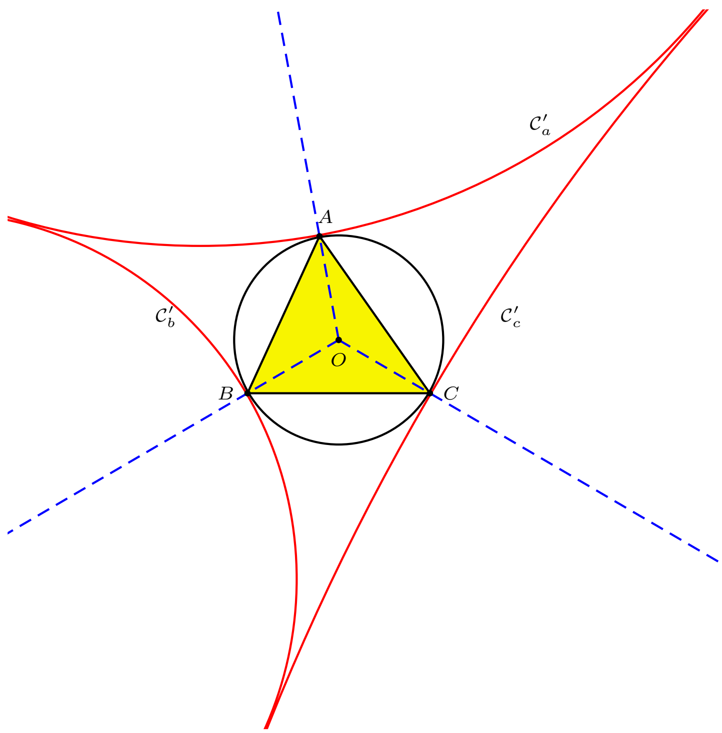

Figure 3

It may occur that one of them is negative. In that case, the tangencies with the corresponding circle are all internal. See Figure 4.

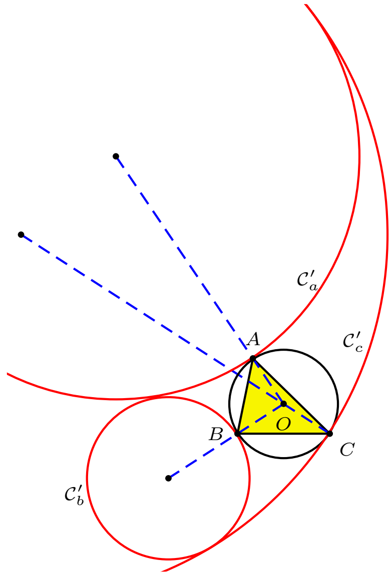

Figure 4

3. Inscribed squares

Consider the triad of circles in §2. The homothety $\mathsf{h}(A, -\frac{\alpha'}{R})$ transforms the square erected on BC on the same side of A into an inscribed square since $\frac{-\alpha'}{R} = \frac{-h_a}{a-h_a}$. See Figure 5.

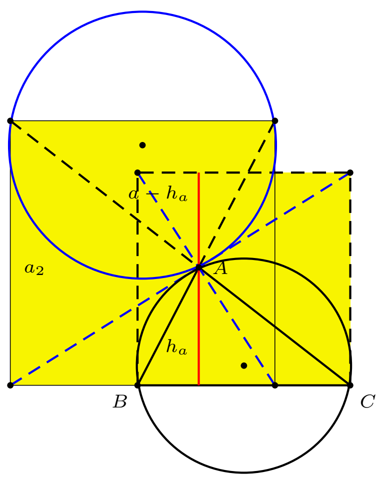

Figure 5

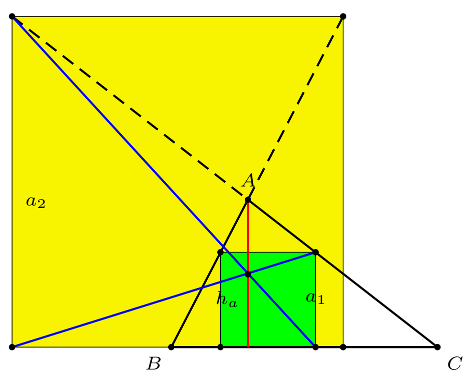

Figure 6

Denote by a_1 and a_2 the lengths of sides of the two inscribed squares on BC, under the homotheties $\mathsf{h}(A, \frac{\alpha}{R})$ and $\mathsf{h}(A, -\frac{\alpha'}{R})$ respectively, *i.e.*, $a_1 = \frac{\alpha}{R} \cdot a$ and

$a_2 = \frac{\alpha'}{R} \cdot a$. Making use of (3) and (4), we have

$$\frac{1}{a_1} + \frac{1}{a_2} = \left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right)\frac{R}{a} = \frac{4a}{bc} \cdot \frac{R}{a} = \frac{a}{\triangle} = \frac{2}{h_a}.$$

This means that the altitude h_a is the harmonic mean of the lengths of the sides of the two inscribed squares on the side BC. See Figure 6.

4. The Descartes formula

We reverse the calculations in §§1,2 to give a proof of the Descartes formula. See, [2, pp.90–92]. Given three circles of radii α, β, γ tangent to each other externally, we determine the radii of the two Soddy circles tangent to each of them. See, for example, [1, pp.13–16]. We first seek the radius R of the circle tangent *internally* to each of them, the *outer* Soddy circle. Regard, in equation (3), R, a, b, c as unknowns, and write $\triangle$ for the area of the unknown triangle ABC whose vertices are the points of tangency. Thus, by Heron's formula,

$$16\triangle^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4. \tag{5}$$

In terms of $\triangle$, (3) can be rewritten as

$$\alpha = \frac{2\triangle}{a^2 + 2\triangle} \cdot R, \qquad \beta = \frac{2\triangle}{b^2 + 2\triangle} \cdot R, \qquad \gamma = \frac{2\triangle}{c^2 + 2\triangle} \cdot R,$$

or

$$a^2 = \frac{2(R-\alpha)\triangle}{\alpha}, \qquad b^2 = \frac{2(R-\beta)\triangle}{\beta}, \qquad c^2 = \frac{2(R-\gamma)\triangle}{\gamma}. \tag{6}$$

Substituting these into (5) and simplifying, we obtain

$$\begin{aligned} &\alpha^2\beta^2\gamma^2 + 2\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)R \\ &+(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 - 2\alpha^2\beta\gamma - 2\alpha\beta^2\gamma - 2\alpha\beta\gamma^2)R^2 = 0. \end{aligned}$$

Dividing throughout by $\alpha^2\beta^2\gamma^2 \cdot R^2$, we have

$$\frac{1}{R^2} + 2\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)\frac{1}{R} + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\gamma\alpha}\right) = 0.$$

Since $R > \alpha, \beta, \gamma$, we have

$$\frac{1}{R} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - 2\sqrt{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}}.$$

This is positive if and only if

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\gamma\alpha} > 0. \tag{7}$$

This is the condition necessary and sufficient for the existence of a circle tangent *internally* to each of the given circles.

By reversing the calculations in §2, the radius of the circle tangent to the three given circles externally, the *inner* Soddy circle, is given by

$$\frac{1}{R'} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + 2\sqrt{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}}.$$

If condition (7) is not satisfied, both Soddy circles are tangent to each of the given circles externally.

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Similar Pedal and Cevian Triangles

Jean-Pierre Ehrmann

Abstract. The only point with similar pedal and cevian triangles, other than the orthocenter, is the isogonal conjugate of the Parry reflection point.

1. Introduction

We begin with notation. Let ABC be a triangle with sidelengths a, b, c, orthocenter H, and circumcenter O. Let K_A, K_B, K_C denote the vertices of the tangential triangle, O_A, O_B, O_C the reflections of O in A, B, C, and A_S, B_S, C_S the reflections of the vertices of A in BC, of B in CA, and of C in AB. Let

$M^* =$ isogonal conjugate of a point M;
$\overline{M} =$ inverse of M in the circumcircle;
$\measuredangle LL' =$ the measure, modulo π, of the directed angle of the lines L, L';
$S_A = bc\cos A = \frac{1}{2}(b^2+c^2-a^2)$, with S_B and S_C defined cyclically;
$x : y : z =$ barycentric coordinates relative to triangle ABC;
$\Gamma_A =$ circle with diameter K_AO_A, with circles Γ_B and Γ_C defined cyclically.

The circle Γ_A passes through the points B_S, C_S and is the locus of M such that $\measuredangle B_SMC_S = -2\measuredangle BAC$. An equation for Γ_A, in barycentrics, is

$$2S_A\left(a^2yz+b^2zx+c^2xy\right)+\left(b^2c^2x+2c^2S_Cy+2b^2S_Bz\right)(x+y+z)=0.$$

Consider a triangle $A'B'C'$, where A', B', C' lie respectively on the sidelines BC, CA, AB. The three circles $AB'C'$, $BC'A'$, $CA'B'$ meet in a point S called the Miquel point of $A'B'C'$. See [2, pp.131–135]. The point S (or $\overline{S}$) is the only point whose pedal triangle is directly (or indirectly) similar to $A'B'C'$.

The circles Γ_A, Γ_B, Γ_C have a common point T: the Parry reflection point, X_{399} in [3]; the three radical axes TA_S, TB_S, TC_S are the reflections with respect to a sideline of ABC of the parallel to the Euler line going through the opposite vertex. See [3, 4], and Figure 1. T lies on the circle $(O, 2R)$, on the Neuberg cubic, and is the antipode of O on the Stammler hyperbola. See [1].

2. Similar triangles

Let $A'B'C'$ be the cevian triangle of a point $P = p : q : r$.

Lemma 1. *The pedal and cevian triangles of P are directly (or indirectly) similar if and only if P (or $\overline{P}$) lies on the three circles $AB'C'$, $BC'A'$, $CA'B'$.*

Proof. This is an immediate consequence of the properties of the Miquel point above. □

Lemma 2. *A, B', C', P are concyclic if and only if P lies on the circle BCH.*

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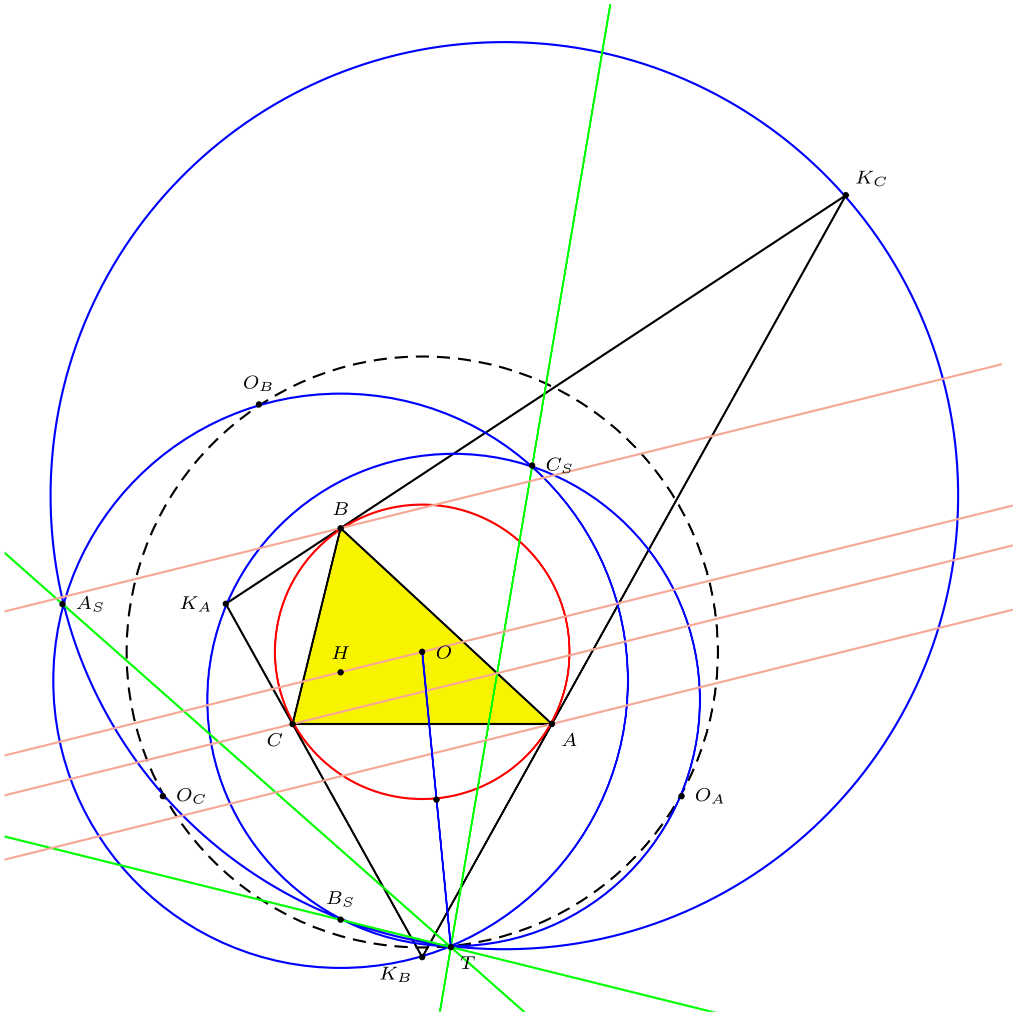

Figure 1

Proof. A, B', C' and P are concyclic $\Leftrightarrow \measuredangle B'PC' = \measuredangle B'AC' \Leftrightarrow \measuredangle BPC = \measuredangle BHC \Leftrightarrow P$ lies on the circle BCH. □

Proposition 3. *The pedal and cevian triangles of P are directly similar only in the trivial case of $P = H$.*

Proof. By Lemma 1, the pedal and cevian triangles of P are directly similar if and only if P lies on the three circles $AB'C'$, $BC'A'$, $CA'B'$. By Lemma 2, P lies on the three circles BCH, CAH, ABH. Hence, $P = H$. □

Lemma 4. A, B', C', $\overline{P}$ *are concyclic if and only if* P^* *lies on the circle* Γ_A.

Proof. If $P = p : q : r$, the circle Φ_A passing through A, B', C' is given by

$$a^2yz + b^2zx + c^2xy - p\,(x+y+z)\left(\frac{c^2}{p+q}y + \frac{b^2}{p+r}z\right) = 0,$$

and its inverse in the circumcircle is the circle $\overline{\Phi}_A$ given by

$$\begin{aligned} &(a^2(p^2-qr) + (b^2-c^2)p(q-r))(a^2yz + b^2zx + c^2xy) \\ - \quad & pa^2(x+y+z)(c^2(p+r)y + b^2(p+q)z) = 0. \end{aligned}$$

Since Φ_A contains $\overline{P}$, its inverse $\overline{\Phi}_A$ contains P. Changing (p, q, r) to (x, y, z) gives the locus of P satisfying $\overline{P} \in \Phi_A$. Then changing (x, y, z) to $\left(\frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z}\right)$ gives the locus $\widehat{\overline{\Phi}}_A$ of the point P^* such that $\overline{P} \in \Phi_A$. By examination, $\widehat{\overline{\Phi}}_A = \Gamma_A$. □

Proposition 5. *The pedal and cevian triangles of P are indirectly similar if and only if P is the isogonal conjugate of the Parry reflection point.*

Proof. By Lemma 1, the pedal and cevian triangles of P are indirectly similar if and only if $\overline{P}$ lies on the three circles $AB'C'$, $BC'A'$, $CA'B'$. By Lemma 4, P^* lies on each of the circles Γ_A, Γ_B, Γ_C. Hence, $P^* = T$, and $P = T^*$. □

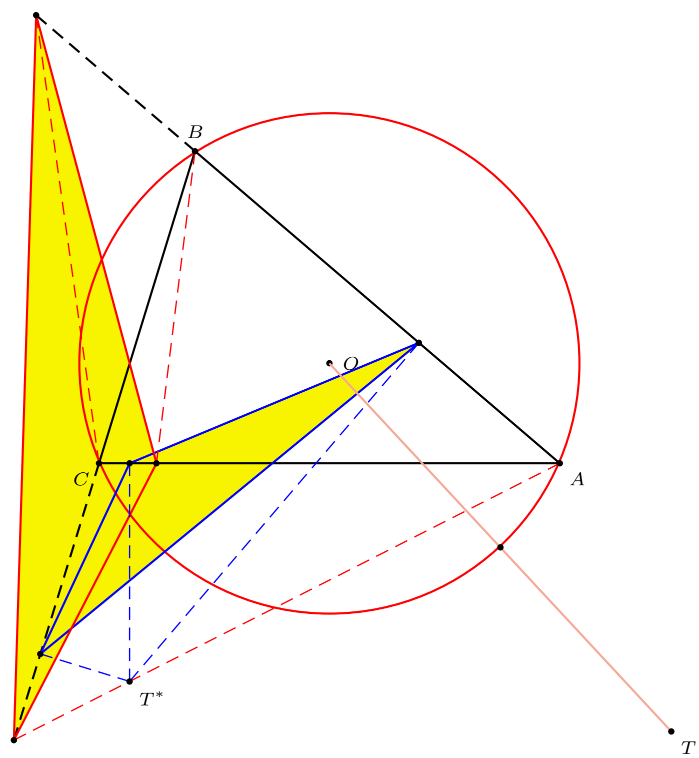

Figure 2

Remarks. (1) The isogonal conjugate of X_{399} is X_{1138} in [3]: this point lies on the Neuberg cubic.

(2) We can deduce Lemma 4 from the relation $\measuredangle B'\overline{P}C' - \measuredangle B_s P^* C_s = \measuredangle BAC$, which is true for every point P in the plane of ABC except the vertices A, B, C.

(3) As two indirectly similar triangles are orthologic and as the pedal and cevian triangles of P are orthologic if and only if P^* lies on the Stammler hyperbola, a point with indirectly similar cevian and pedal triangles must be the isogonal conjugate of a point of the Stammler hyperbola.

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On the Kosnita Point and the Reflection Triangle

Darij Grinberg

Abstract. The Kosnita point of a triangle is the isogonal conjugate of the nine-point center. We prove a few results relating the reflections of the vertices of a triangle in their opposite sides to triangle centers associated with the Kosnita point.

1. Introduction

By the Kosnita point of a triangle we mean the isogonal conjugate of its nine-point center. The name Kosnita point originated from J. Rigby [5].

Theorem 1 (Kosnita)**.** *Let ABC be a triangle with the circumcenter O, and X, Y, Z be the circumcenters of triangles BOC, COA, AOB. The lines AX, BY, CZ concur at the isogonal conjugate of the nine-point center.*

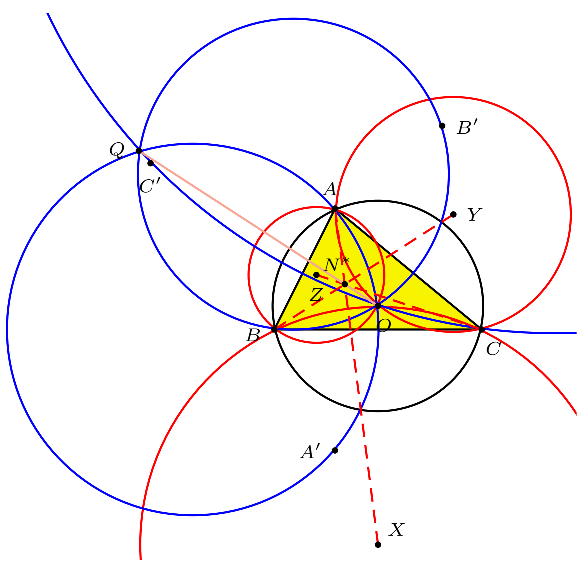

Figure 1

We denote the nine-point center by N and the Kosnita point by N^*. Note that N^* is an infinite point if and only if the nine-point center is on the circumcircle. We study this special case in §5 below. The points N and N^* appear in [3] as X_5 and X_{54} respectively. An old theorem of J. R. Musselman [4] relates the Kosnita

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point to the reflections A', B', C' of A, B, C in their opposite sides BC, CA, AB respectively.

Theorem 2 (Musselman)**.** *The circles AOA', BOB', COC' pass through the inversive image of the Kosnita point in the circumcircle of triangle ABC.*

This common point of the three circles is the triangle center X_{1157} in [3], which we denote by Q in Figure 1. The following theorem gives another triad of circles containing this point. It was obtained by Paul Yiu [7] by computations with barycentric coordinates. We give a synthetic proof in §2.

Theorem 3 (Yiu)**.** *The circles $AB'C'$, $BC'A'$, $CA'B'$ pass through the inversive image of the Kosnita point in the circumcircle.*

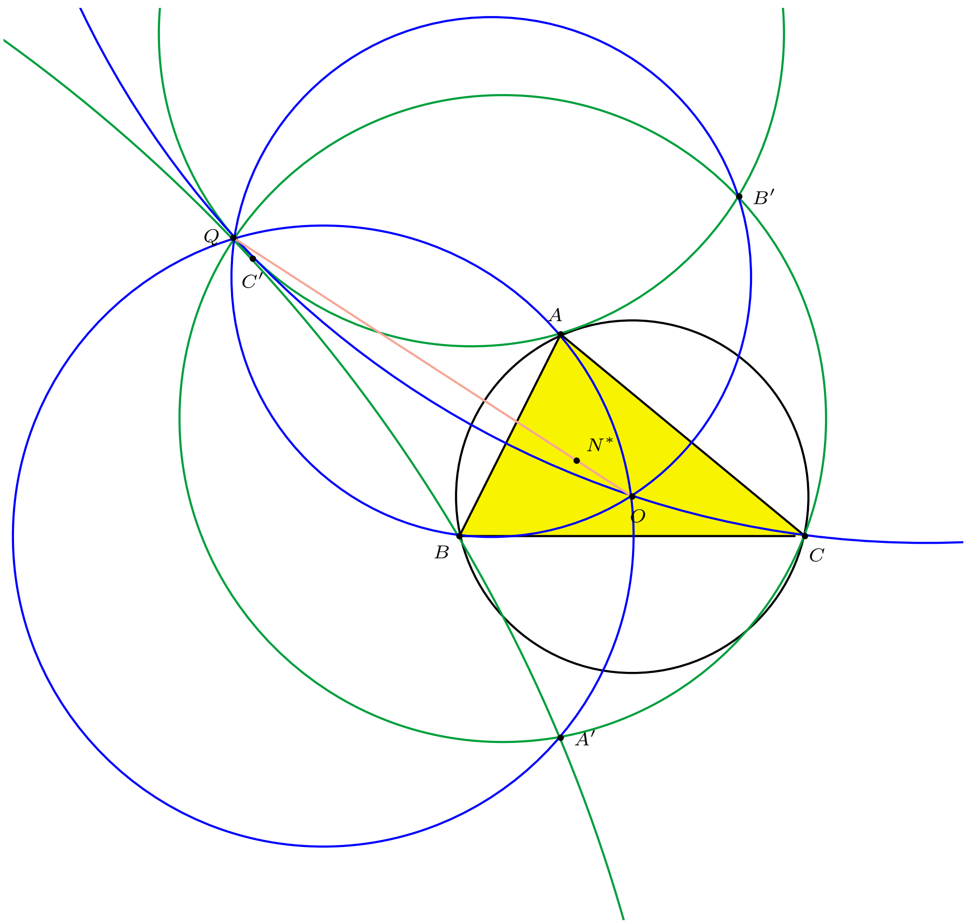

Figure 2

On the other hand, it is clear that the circles $A'BC$, $B'CA$, and $C'AB$ pass through the orthocenter of triangle ABC. It is natural to inquire about the circumcenter of the *reflection triangle* $A'B'C'$. A very simple answer is provided by the following characterization of $A'B'C'$ by G. Boutte [1].

Theorem 4 (Boutte)**.** *Let G be the centroid of ABC. The reflection triangle $A'B'C'$ is the image of the pedal triangle of the nine-point center N under the homothety* $\mathsf{h}(G, 4)$.

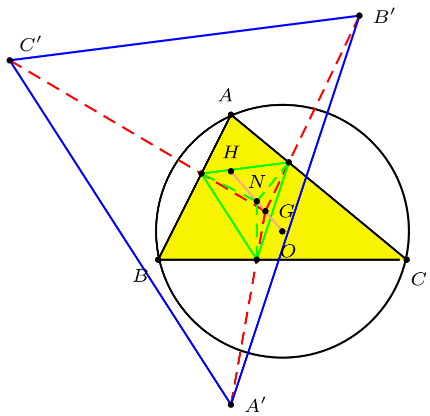

Figure 3

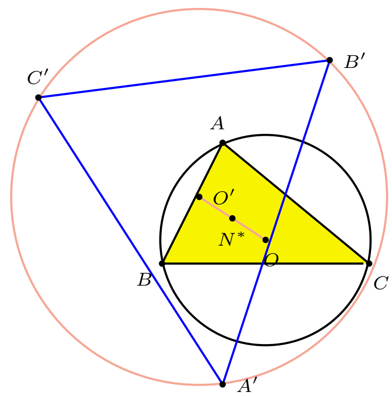

Figure 4

Corollary 5. *The circumcenter of the reflection triangle $A'B'C'$ is the reflection of the circumcenter in the Kosnita point.*

2. Proof of Theorem 3

Denote by Q the inverse of the Kosnita point N^* in the circumcircle. By Theorem 2, Q lies on the circles BOB' and COC'. So $\angle B'QO = \angle B'BO$ and $\angle C'QO = \angle C'CO$. Since $\angle B'QC' = \angle B'QO + \angle C'QO$, we get

$$\begin{aligned}\angle B'QC' =& \angle B'BO + \angle C'CO\\ =& \left(\angle CBB' - \angle CBO\right) + \left(\angle BCC' - \angle BCO\right)\\ =& \angle CBB' + \angle BCC' - (\angle CBO + \angle BCO)\\ =& \angle CBB' + \angle BCC' - (\pi - \angle BOC)\\ =& \angle CBB' + \angle BCC' - \pi + \angle BOC.\end{aligned}$$

But we have $\angle CBB' = \frac{\pi}{2} - C$ and $\angle BCC' = \frac{\pi}{2} - B$. Moreover, from the central angle theorem we get $\angle BOC = 2A$. Thus,

$$\begin{aligned}\angle B'QC' =& \left(\frac{\pi}{2} - C\right) + \left(\frac{\pi}{2} - B\right) - \pi + 2A\\ =& \pi - B - C - \pi + 2A = 2A - B - C\\ =& 3A - (A + B + C) = 3A - \pi,\end{aligned}$$

and consequently

$$\pi - \angle B'QC' = \pi - (3A - \pi) = 2\pi - 3A.$$

But on the other hand, $\angle BAC' = \angle BAC = A$ and $\angle CAB' = A$, so

$$\angle B'AC' = 2\pi - \left(\angle BAC' + \angle BAC + \angle CAB'\right) = 2\pi - (A + A + A) = 2\pi - 3A.$$

Consequently, $\angle B'AC' = \pi - \angle B'QC'$. Thus, Q lies on the circle $AB'C'$. Similar reasoning shows that Q also lies on the circles $BC'A'$ and $CA'B'$.

This completes the proof of Theorem 3.

Remark. In general, if a triangle ABC and three points A', B', C' are given, and the circles $A'BC$, $B'CA$, and $C'AB$ have a common point, then the circles $AB'C'$, $BC'A'$, and $CA'B'$ also have a common point. This can be proved with some elementary angle calculations. In our case, the common point of the circles $A'BC$, $B'CA$, and $C'AB$ is the orthocenter of ABC, and the common point of the circles $AB'C'$, $BC'A'$, and $CA'B'$ is Q.

3. Proof of Theorem 4

Let A_1, B_1, C_1 be the midpoints of BC, CA, AB, and A_2, B_2, C_2 the midpoints of B_1C_1, C_1A_1, A_1B_1. It is clear that $A_2B_2C_2$ is the image of ABC under the homothety $\mathsf{h}(G, \frac{1}{4})$. Denote by X the image of A' under this homothety. We show that this is the pedal of the nine-point center N on BC.

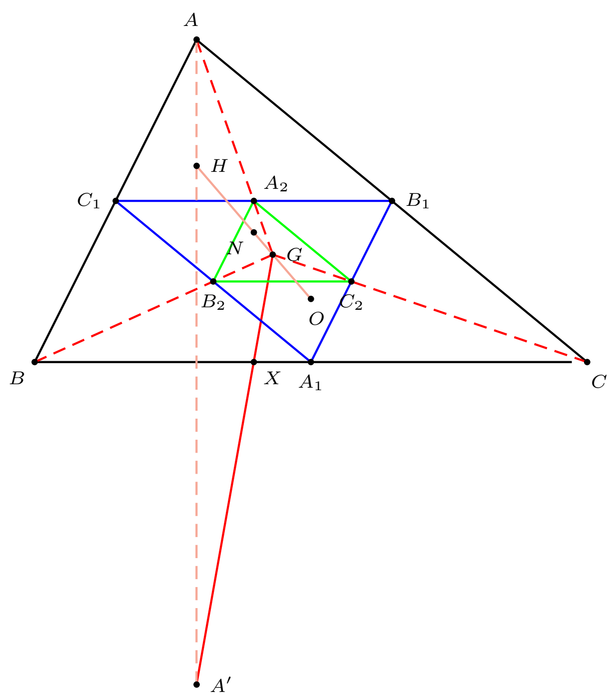

Figure 5

First, note that X, being the reflection of A_2 in B_2C_2, lies on BC. This is because A_2X is perpendicular to B_2C_2 and therefore to BC. The distance from

X to A_2 is twice of that from A_2 to B_2C_2. This is equal to the distance between the parallel lines B_2C_2 and BC.

The segment A_2X is clearly the perpendicular bisector of B_1C_1. It passes through the circumcenter of triangle $A_1B_1C_1$, which is the nine-point N of triangle ABC. It follows that X is the pedal of N on BC. For the same reasons, the images of B', C' under the same homothety $\mathsf{h}(G, \frac{1}{4})$ are the pedals of N on CA and AB respectively.

This completes the proof of Theorem 4.

4. Proof of Corollary 5

It is well known that the circumcenter of the pedal triangle of a point P is the midpoint of the segment PP^*, P^* being the isogonal conjugate of P. See, for example, [2, pp.155–156]. Applying this to the nine-point center N, we obtain the circumcenter of the reflection triangle $A'B'C'$ as the image of the midpoint of NN^* under the homothety $\mathsf{h}(G, 4)$. This is the point

$$\begin{aligned}
G + 4\left(\frac{N+N^*}{2} - G\right) =& 2(N+N^*) - 3G \\
=& 2N^* + 2N - 3G \\
=& 2N^* + (O+H) - (2\cdot O + H) \\
=& 2N^* - O,
\end{aligned}$$

the reflection of O in the Kosnita point N^*. Here, H is orthocenter, and we have made use of the well known facts that N is the midpoint of OH and G divides OH in the ratio $HG : GO = 2 : 1$.

This completes the proof of Corollary 5.

This point is the point X_{195} of [3]. Barry Wolk [6] has verified this theorem by computer calculations with barycentric coordinates.

5. Triangles with nine-point center on the circumcircle

Given a circle $O(R)$ and a point N on its circumference, let H be the reflection of O in N. For an arbitrary point P on the minor arc of the circle $N(\frac{R}{2})$ *inside* $O(R)$, let (i) A be the intersection of the *segment* HP with $O(R)$, (ii) the perpendicular to HP at P intersect $O(R)$ at B and C. Then triangle ABC has nine-point center N on its circumcircle $O(R)$. See Figure 6. This can be shown as follows. It is clear that $O(R)$ is the circumcircle of triangle ABC. Let M be the midpoint of BC so that OM is orthogonal to BC and parallel to PH. Thus, $OMPH$ is a (self-intersecting) trapezoid, and the line joining the midpoints of PM and OH is parallel to PH. Since the midpoint of OH is N and PH is orthogonal to BC, we conclude that N lies on the perpendicular bisector of PM. Consequently, $NM = NP = \frac{R}{2}$, and M lies on the circle $N(\frac{R}{2})$. This circle is the nine-point circle of triangle ABC, since it passes through the pedal P of A on BC and through the midpoint M of BC and has radius $\frac{R}{2}$.

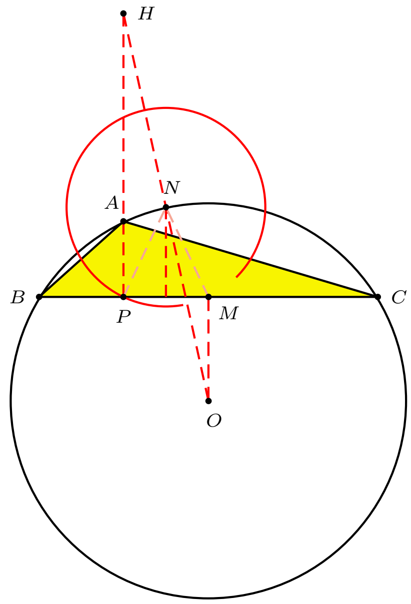

Figure 6

Remark. As P traverses the minor arc which the intersection of $N(\frac{R}{2})$ with the interior of $O(R)$, the line $\mathcal{L}$ passes through a fixed point, which is the reflection of O in H.

Theorem 6. *Suppose the nine-point center N of triangle ABC lies on the circumcircle.*

(1) *The reflection triangle $A'B'C'$ degenerates into a line $\mathcal{L}$.*
(2) *If X, Y, Z are the centers of the circles BOC, COA, AOB, the lines AX, BY, CZ are all perpendicular to $\mathcal{L}$.*
(3) *The circles AOA', BOB', COC' are mutually tangent at O. The line joining their centers is the parallel to $\mathcal{L}$ through O.*
(4) *The circles $AB'C'$, $BC'A'$, $CA'B'$ pass through O.*

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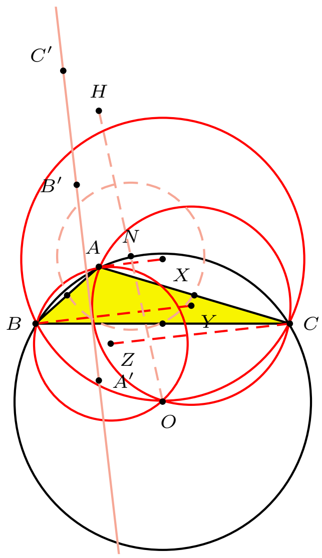

Figure 7

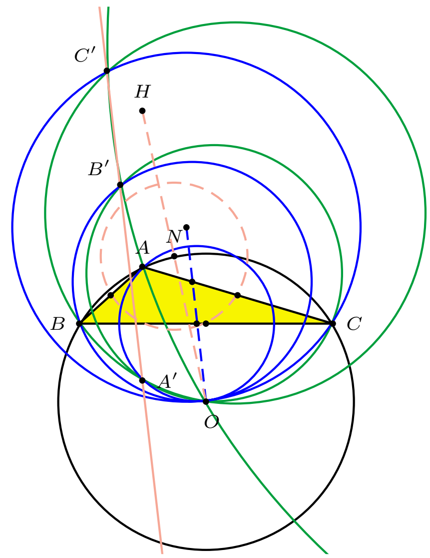

Figure 8

Added in proof: Bernard Gibert has kindly communincated the following results. Let A_1 be the intersection of the lines OA' and $B'C'$, and similarly define B_1 and C_1. Denote, as in §1, by Q be the inverse of the Kosnita point in the circumcircle.

Theorem 7 (Gibert)**.** *The lines AA_1, BB_1, CC_1 concur at the isogonal conjugate of Q.*

This is the point X_{1263} in [3]. The points A, B, C, A', B', C', O, Q, A_1, B_1, C_1 all lie on the Neuberg cubic of triangle ABC, which is the isogonal cubic with pivot the infinite point of the Euler line. This cubic is also the locus of all points whose reflections in the sides of triangle ABC form a triangle perspective to ABC. The point Q is the unique point whose triangle of reflections has perspector on the circumcircle. This perspector, called the Gibert point X_{1141} in [3], lies on the line joining the nine-point center to the Kosnita point.

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A Note on the Schiffler Point

Lev Emelyanov and Tatiana Emelyanova

Abstract. We prove two interesting properties of the Schiffler point.

1. Main results

The Schiffler point is the intersection of four Euler lines. Let I be the incenter of triangle ABC. The Schiffler point S is the point common to the Euler lines of triangles IBC, ICA, IAB, and ABC. See [1, p.70]. Not much is known about S. In this note, we prove two interesting properties of this point.

Theorem 1. *Let A and I_1 be the circumcenter and A-excenter of triangle ABC, and A_1 the intersection of OI_1 and BC. Similarly define B_1 and C_1. The lines AA_1, BB_1 and CC_1 concur at the Schiffler point S.*

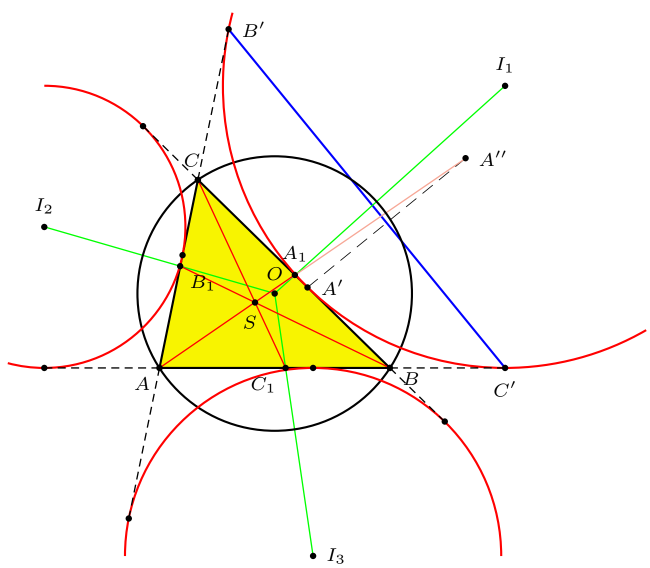

Figure 1

Theorem 2. *Let A', B', C' be the touch points of the A-excircle and BC, CA, AB respectively, and A'' the reflection of A' in $B'C'$. Similarly define B'' and C''. The lines AA'', BB'' and CC'' concur at the Schiffler point S.*

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We make use of trilinear coordinates with respect to triangle ABC. According to [1, p.70], the Schiffler point has coordinates

$$\left(\frac{1}{\cos B+\cos C}:\frac{1}{\cos C+\cos A}:\frac{1}{\cos A+\cos B}\right).$$

2. Proof of Theorem 1

We show that AA_1 passes through the Schiffler point S. Because

$$O=(\cos A:\cos B:\cos C)\quad\text{and}\quad I_1=(-1:1:1),$$

the line OI_1 is given by

$$(\cos B-\cos C)\alpha-(\cos C+\cos A)\beta+(\cos A+\cos B)\gamma=0.$$

The line BC is given by $\alpha=0$. Hence the intersection of OI_1 and BC is

$$A_1=(0:\cos A+\cos B:\cos A+\cos C).$$

The collinearity of A_1, S and A follows from

$$\begin{aligned}&\begin{vmatrix}1 & 0 & 0\\ 0 & \cos A+\cos B & \cos A+\cos C\\ \frac{1}{\cos B+\cos C} & \frac{1}{\cos C+\cos A} & \frac{1}{\cos A+\cos B}\end{vmatrix}\\ =&\begin{vmatrix}\cos A+\cos B & \cos A+\cos C\\ \frac{1}{\cos C+\cos A} & \frac{1}{\cos A+\cos B}\end{vmatrix}\\ =&0.\end{aligned}$$

This completes the proof of Theorem 1.

Remark. It is clear from the proof above that more generally, if P is a point with trilinear coordinates $(p:q:r)$, and A_1, B_1, C_1 the intersections of PI_a with BC, PI_2 with CA, PI_3 with AB, then the lines AA_1, BB_1, CC_1 intersect at a point with trilinear coordinates $\left(\frac{1}{q+r}:\frac{1}{r+p}:\frac{1}{p+q}\right)$. If P is the symmedian point, for example, this intersection is the point $X_{81}=\left(\frac{1}{b+c}:\frac{1}{c+a}:\frac{1}{a+b}\right)$.

3. Proof of Theorem 2

We deduce Theorem 2 as a consequence of the following two lemmas.

Lemma 3. *The line OI_1 is the Euler line of triangle $A'B'C'$.*

Proof. Triangle ABC is the tangential triangle of $A'B'C'$. It is known that the circumcenter of the tangential triangle lies on the Euler line. See, for example, [1, p.71]. It follows that OI_1 is the Euler line of triangle $A'B'C'$. □

Lemma 4. *Let A^* be the reflection of vertex A of triangle ABC with respect to BC, $A_1B_1C_1$ be the tangential triangle of ABC. Then the Euler line of ABC and line A_1A^* intersect line B_1C_1 in the same point.*

Proof. As is well known, the vertices of the tangential triangle are given by

$$A_1 = (-a : b : c), \quad B_1 = (a : -b : c), \quad C_1 = (a : b : -c).$$

The line B_1C_1 is given by $c\beta + b\gamma = 0$. According to [1, p.42], the Euler line of triangle ABC is given by

$$a(b^2-c^2)(b^2+c^2-a^2)\alpha+b(c^2-a^2)(c^2+a^2-b^2)\beta+c(a^2-b^2)(a^2+b^2-c^2)\gamma = 0.$$

Now, it is not difficult to see that

$$\begin{aligned} A^* =&(-1 : 2\cos C : 2\cos B) \\ =&(-abc : c(a^2+b^2-c^2) : b(c^2+a^2-b^2)). \end{aligned}$$

The equation of the line A^*A_1 is then

$$\begin{vmatrix} -abc & 2c(a^2+b^2-c^2) & 2b(c^2+a^2-b^2) \\ -a & b & c \\ \alpha & \beta & \gamma \end{vmatrix} = 0.$$

After simplification, this is

$$-(b^2-c^2)(b^2+c^2-a^2)\alpha + ab(a^2-b^2)\beta - ac(a^2-c^2)\gamma = 0.$$

Now, the lines B_1C_1, A^*A_1, and the Euler line are concurrent if the determinant

$$\begin{vmatrix} 0 & c & b \\ -(b^2-c^2)(b^2+c^2-a^2) & ab(a^2-b^2) & -ac(a^2-c^2) \\ a(b^2-c^2)(b^2+c^2-a^2) & b(c^2-a^2)(c^2+a^2-b^2) & c(a^2-b^2)(a^2+b^2-c^2) \end{vmatrix}$$

is zero. Factoring out $(b^2-c^2)(b^2+c^2-a^2)$, we have

$$\begin{aligned} &\begin{vmatrix} 0 & c & b \\ -1 & ab(a^2-b^2) & -ac(a^2-c^2) \\ a & b(c^2-a^2)(c^2+a^2-b^2) & c(a^2-b^2)(a^2+b^2-c^2) \end{vmatrix} \\ =&-c\begin{vmatrix} -1 & -ac(a^2-c^2) \\ a & c(a^2-b^2)(a^2+b^2-c^2) \end{vmatrix} + b\begin{vmatrix} -1 & ab(a^2-b^2) \\ a & b(c^2-a^2)(c^2+a^2-b^2) \end{vmatrix} \\ =&c^2((a^2-b^2)(a^2+b^2-c^2) - a^2(a^2-c^2)) \\ &-b^2((c^2-a^2)(c^2+a^2-b^2) + a^2(a^2-b^2)) \\ =&c^2 \cdot b^2(c^2-b^2) - b^2 \cdot c^2(c^2-b^2) \\ =&0. \end{aligned}$$

This confirms that the three lines are concurrent. □

To prove Theorem 2, it is enough to show that the line AA'' in Figure 1 contains S. Now, triangle $A'B'C'$ has tangential triangle ABC and Euler line OI_1 by Lemma 3. By Lemma 4, the lines OI_1, AA'' and BC are concurrent. This means that the line AA'' contains A_1. By Theorem 1, this line contains S.

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Harcourt's Theorem

Nikolaos Dergiades and Juan Carlos Salazar

Abstract. We give a proof of Harcourt's theorem that if the signed distances from the vertices of a triangle of sides a, b, c to a tangent of the incircle are a_1, b_1, c_1, then $aa_1 + bb_1 + cc_1$ is twice of the area of the triangle. We also show that there is a point on the circumconic with center I whose distances to the sidelines of ABC are precisely a_1, b_1, c_1. An application is given to the extangents triangle formed by the external common tangents of the excircles.

1. Harcourt's Theorem

The following interesting theorem appears in F. G.-M.[1, p.750] as Harcourt's theorem.

Theorem 1 (Harcourt)**.** *If the distances from the vertices A, B, C to a tangent to the incircle of triangle ABC are a_1, b_1, c_1 respectively, then the algebraic sum $aa_1 + bb_1 + cc_1$ is twice of the area of triangle ABC.*

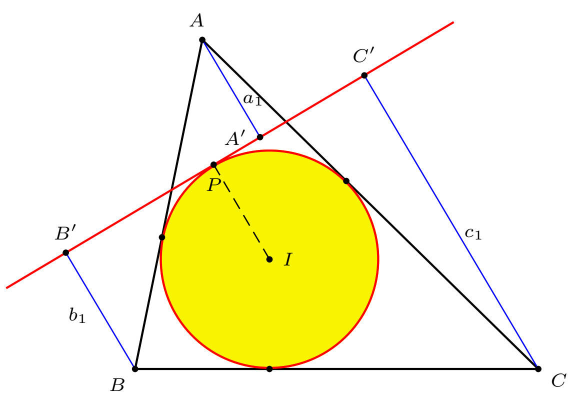

Figure 1

The distances are signed. Distances to a line from points on opposite sides are opposite in sign, while those from points on the same side have the same sign. For the tangent lines to the incircle, we stipulate that the distance from the incenter is positive. For example, in Figure 1, when the tangent line ℓ separates the vertex A from B and C, a_1 is negative while b_1 and c_1 are positive. With this sign convention, Harcourt's theorem states that

$$aa_1 + bb_1 + cc_1 = 2\triangle, \tag{1}$$

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where $\triangle$ is the area of triangle ABC.

We give a simple proof of Harcourt's theorem by making use of homogeneous barycentric coordinates with reference to triangle ABC. First, we establish a fundamental formula.

Proposition 2. *Let ℓ be a line passing through a point P with homogeneous barycentric coordinates $(x : y : z)$. If the signed distances from the vertices A, B, C to a line ℓ are d_1, d_2, d_3 respectively, then*

$$d_1x + d_2y + d_3z = 0. \tag{2}$$

Proof. It is enough to consider the case when ℓ separates A from B and C. We take d_1 as negative, and d_2, d_3 positive. See Figure 2. If A' is the trace of P on the side line BC, it is well known that

$$\frac{AP}{PA'} = \frac{x}{y+z}.$$

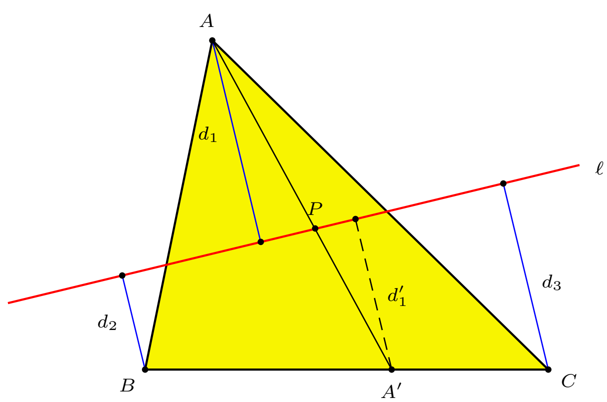

Figure 2

Since $\frac{BA'}{A'C} = \frac{z}{y}$, the distance from A' to ℓ is

$$d_1' = \frac{yd_2 + zd_3}{y+z}.$$

Since $\frac{-d_1}{d_1'} = \frac{AP}{PA'} = \frac{y+z}{x}$, the equation (2) follows. □

Proof of Harcourt's theorem. We apply Proposition 2 to the line ℓ through the incenter $I = (a : b : c)$ parallel to the tangent. The signed distances from A, B, C to ℓ are $d_1 = a_1 - r$, $d_2 = a_2 - r$, and $d_3 = a_3 - r$. From these,

$$\begin{aligned} aa_1 + bb_1 + cc_1 =& a(d_1 + r) + b(d_2 + r) + c(d_3 + r) \\ =& (ad_1 + bd_2 + cd_3) + (a + b + c)r \\ =& 2\triangle, \end{aligned}$$

since $ad_1 + bd_2 + cd_3 = 0$ by Proposition 2.

2. Harcourt's theorem for the excircles

Harcourt's theorem for the incircle and its proof above can be easily adapted to the excircles.

Theorem 3. *If the distances from the vertices A, B, C to a tangent to the A-excircle of triangle ABC are a_1, b_1, c_1 respectively, then $-aa_1 + bb_1 + cc_1 = 2\triangle$. Analogous statements hold for the B- and C-excircles.*

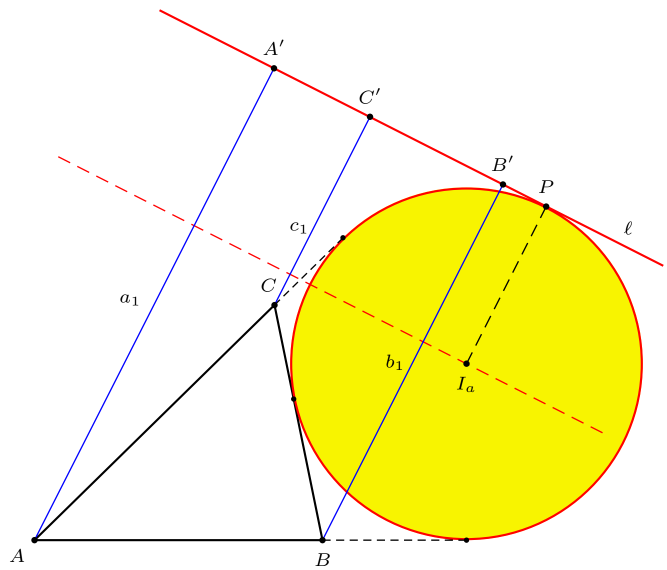

Figure 3

Proof. Apply Proposition 2 to the line ℓ through the excenter $I_a = (-a : b : c)$ parallel to the tangent. If the distances from A, B, C to ℓ are d_1, d_2, d_3 respectively, then

$$-ad_1 + bd_2 + cd_3 = 0.$$

Since $a_1 = d_1 + r_1$, $b_1 = d_2 + r_1$, $c_1 = d_3 + r_1$, where r_1 is the radius of the excircle, it easily follows that

$$\begin{aligned} -aa_1 + bb_1 + cc_1 =& -a(d_1 + r_1) + b(d_2 + r_1) + c(d_3 + r_1) \\ =&(-ad_1 + bd_2 + cd_3) + r_1(-a + b + c) \\ =&r_1(-a + b + c) \\ =&2\triangle. \end{aligned}$$

□

Consider the external common tangents of the excircles of triangle ABC. Let ℓ_a be the external common tangent of the B- and C-excircles. Denote by d_{a1}, d_{a2}, d_{a3} the distances from the A, B, C to this line. Clearly, $d_{a1} = h_a$, the altitude on BC. Similarly define ℓ_b, ℓ_c and the associated distances.

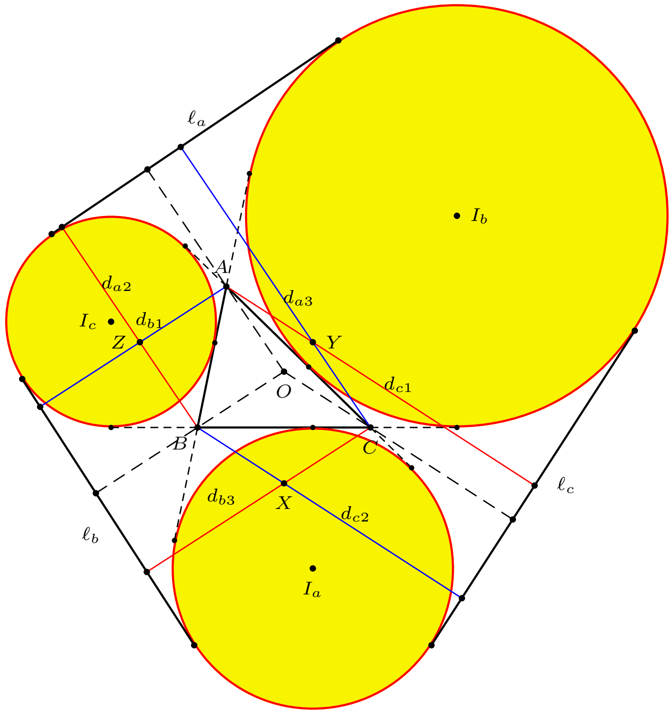

Figure 4

Theorem 4. $d_{a2}d_{b3}d_{c1} = d_{a3}d_{b1}d_{c2}$.

Proof. Applying Theorem 3 to the tangent ℓ_a of the B-excircle (respectively the C-excircle), we have

$$ad_{a1} - bd_{a2} + cd_{a3} = 2\triangle,$$
$$ad_{a1} + bd_{a2} - cd_{a3} = 2\triangle.$$

From these it is clear that $bd_{a2} = cd_{a3}$, and

$$\frac{d_{a2}}{d_{a3}} = \frac{c}{b}.$$

Similarly,

$$\frac{d_{b3}}{d_{b1}} = \frac{a}{c} \quad \text{and} \quad \frac{d_{c1}}{d_{c2}} = \frac{b}{a}.$$

Combining these three equations we have $d_{a2}d_{b3}d_{c1} = d_{a3}d_{b1}d_{c2}$. $\square$

It is clear that the perpendiculars from A to ℓ_a, being the reflection of the A-altitude, passes through the circumcenter; similarly for the perpendiculars from B to ℓ_b and from C to ℓ_c.

Let X be the intersection of the perpendiculars from B to ℓ_c and from C to ℓ_b. Note that OB and CX are parallel, so are OC and BX. Since $OB = OC$, it follows that $OBXC$ is a rhombus, and $BX = CX = R$, the circumradius of triangle ABC. It also follows that X is the reflection of O in the side BC. Similarly, if Y is the intersection of the perpendiculars from C to ℓ_a and from A to ℓ_c, and Z that of the perpendiculars from A to ℓ_b and from B to ℓ_a, then XYZ is the triangle of reflections of the circumcenter O. As such, it is oppositely congruent to ABC, and the center of homothety is the nine-point center of triangle ABC.

3. The circum-ellipse with center I

Consider a tangent $\mathcal{L}$ to the incircle at a point P. If the signed distances from the vertices A, B, C to $\mathcal{L}$ are a_1, b_1, c_1, then by Harcourt's theorem, there is a point $P^\#$ whose signed distances to the sides BC, CA, AB are precisely a_1, b_1, c_1. What is the locus of the point $P^\#$ as P traverses the incircle? By Proposition 2, the barycentric equation of $\mathcal{L}$ is

$$a_1x + b_1y + c_1z = 0.$$

This means that the point with homogeneous barycentric coordinates $(a_1 : b_1 : c_1)$ is a point on the dual conic of the incircle, which is the circumconic with equation

$$(s-a)yz + (s-b)zx + (s-c)xy = 0. \tag{3}$$

The point $P^\#$ in question has barycentric coordinates $(aa_1 : bb_1 : cc_1)$. Since (a_1, b_1, c_1) satisfies (3), if we put $(x, y, z) = (aa_1, bb_1, cc_1)$, then

$$a(s-a)yz + b(s-b)zx + c(s-c)xy = 0.$$

Thus, the locus of $P^\#$ is the circumconic with perspector $(a(s-a) : b(s-b) : c(s-c))$.[1] It is an ellipse, and its center is, surprisingly, the incenter I.[2] We denote this circum-ellipse by $\mathcal{C}_I$. See Figure 5.

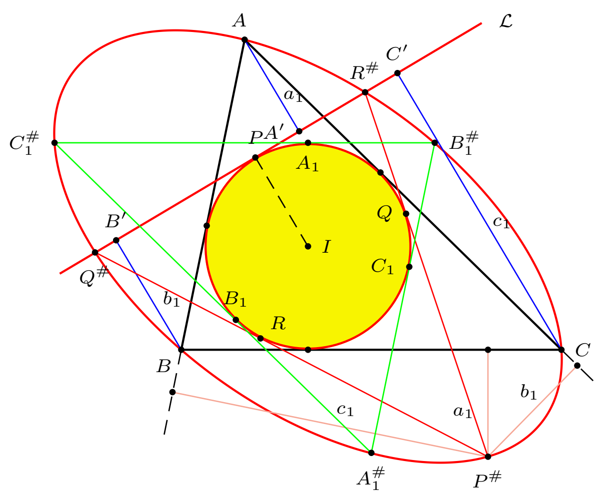

Figure 5

[1]This is the Mittenpunkt, the point X_9 in [4]. It can be constructed as the intersection of the lines joining the excenters to the midpoints of the corresponding sides of triangle ABC.

[2]In general, the center of the circumconic $pyz + qzx + rxy = 0$ is the point with homogeneous barycentric coordinates $(p(q + r - p) : q(r + p - q) : r(p + q - r))$.

Let A_1, B_1, C_1 be the antipodes of the points of tangency of the incircle with the sidelines. It is quite easy to see that $A_1^\#$, $B_1^\#$, $C_1^\#$ are the antipodes of A, B, C in the circum-ellipse $\mathcal{C}_I$. Note that $A_1^\# B_1^\# C_1^\#$ and ABC are oppositely congruent at I. It follows from Steiner's porism that if we denote the intersections of $\mathcal{L}$ and this ellipse by $Q^\#$ and $R^\#$, then the lines $P^\# Q^\#$ and $P^\# R^\#$ are tangent to the incircle at Q and R. This leads to the following construction of $P^\#$.

Construction. If the tangent to the incircle at P intersects the ellipse $\mathcal{C}_I$ at two points, the second tangents from these points to the incircle intersect at $P^\#$ on $\mathcal{C}_I$.

If the point of tangency P has coordinates $\left(\frac{u^2}{s-a} : \frac{v^2}{s-b} : \frac{w^2}{s-c}\right)$, with $u+v+w = 0$, then $P^\#$ is the point $\left(\frac{a(s-a)}{u} : \frac{b(s-b)}{v} : \frac{c(s-c)}{w}\right)$. In particular, if $\mathcal{L}$ is the common tangent of the incircle and the nine-point circle at the Feuerbach point, which has coordinates $((s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2)$, then $P^\#$ is the point $\left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$. This is X_{100} of [3, 4]. It is a point on the circumcircle, lying on the half line joining the Feuerbach point to the centroid of triangle ABC. See [3, Figure 3.12, p.82].

4. The extangents triangle

Consider the external common tangent ℓ_a of the excircles (I_b) and (I_c). Let d_{a1}, d_{a2}, d_{a3} be the distances from A, B, C to this line. We have shown that $\frac{d_{a2}}{d_{a3}} = \frac{c}{b}$. On the other hand, it is clear that $\frac{d_{a1}}{d_{a2}} = \frac{b}{b+c}$. See Figure 6. It follows that

$$d_{a1} : d_{a2} : d_{a3} = bc : c(b+c) : b(b+c).$$

By Proposition 2, the barycentric equation of ℓ_a is

$$bcx + c(b+c)y + b(b+c)z = 0.$$

Similarly, the equations of ℓ_b and ℓ_c are

$$\begin{aligned} c(c+a)x + cay + a(c+a)z =&0, \\ b(a+b)x + a(a+b)y + abz =&0. \end{aligned}$$

These three external common tangents bound a triangle called the *extangents triangle* in [3]. The vertices are the points [3]

$$\begin{aligned} A' =&(-a^2 s : b(c+a)(s-c) : c(a+b)(s-b)), \\ B' =&(a(b+c)(s-c) : -b^2 s : c(a+b)(s-a)), \\ C' =&(a(b+c)(s-b) : b(c+a)(s-a) : -c^2 s). \end{aligned}$$

Let I'_a be the incenter of the reflection of triangle ABC in A. It is clear that the distances from A and I'_a to ℓ_a are respectively h_a and r. Since A is the midpoint of II'_a, the distance from I to ℓ_a is $2h_a - r$.

[3] The *trilinear* coordinates of these vertices given in [3, p.162, §6.17] are not correct. The diagonal entries of the matrices should read $1 + \cos A$ etc. and $\frac{-a(a+b+c)}{(a-b+c)(a+b-c)}$ etc. respectively.

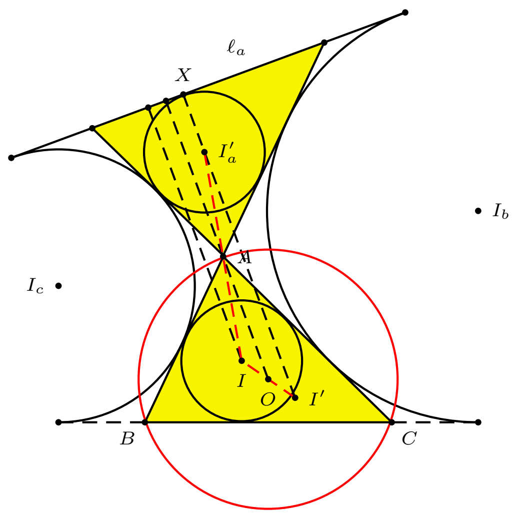

Figure 6

Now consider the reflection of I in O. We denote this point by I'. [4] Since the distances from I and O to ℓ_a are respectively $2h_a - r$ and $R + h_a$, it follows that the distance from I' to ℓ_a is $2(R + h_a) - (2h_a - r) = 2R + r$. For the same reason, the distances from I' to ℓ_b and ℓ_c are also $2R + r$. From this we deduce the following interesting facts about the extangents triangle.

Theorem 5. *The extangent triangle bounded by ℓ_a, ℓ_b, ℓ_c*

(1) *has incenter I' and inradius $2R + r$;*
(2) *is perspective with the excentral triangle at I';*
(3) *is homothetic to the tangential triangle at the internal center of similitude of the circumcircle and the incircle of triangle ABC, the ratio of the homothety being $\frac{2R+r}{R}$.*

Proof. It is enough to locate the homothetic center in (3). This is the point which divides $I'O$ in the ratio $2R + r : -R$, *i.e.*,

$$\frac{(2R + r)O - R(2O - I)}{R + r} = \frac{r \cdot O + R \cdot I}{R + r},$$

the internal center of similitude of the circumcircle and incircle of triangle ABC.[5] $\square$

Remarks. (1) The statement that the extangents triangle has inradius $2R + r$ can also be found in [2, Problem 2.5.4].

(2) Since the excentral triangle has circumcenter I' and circumradius $2R$, it follows that the excenters and the incenters of the reflections of triangle ABC in A, B, C are concyclic. It is well known that since ABC is the orthic triangle of the

[4]This point appears as X_{40} in [4].
[5]This point appears as X_{55} in [4].

excentral triangle, the circumcircle of ABC is the nine-point circle of the excentral triangle.

(3) If the incircle of the extangents triangle touches its sides at X, Y, Z respectively, [6] then triangle XYZ is homothetic to ABC, again at the internal center of similitude of the circumcircle and the incircle.

(4) More generally, the reflections of the traces of a point P in the respective sides of the excentral triangle are points on the sidelines of the extangents triangle. They form a triangle perspective with ABC at the isogonal conjugate of P. For example, the reflections of the points of tangency of the excircles (traces of the Nagel point $(s-a : s-b : s-c)$) form a triangle with perspector $\left(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}\right)$, the external center of similitude of the circumcircle and the incircle.[7]

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[6] These are the reflections of the traces of the Gergonne point in the respective sides of the excentral triangle.

[7] This point appears as X_{56} in [4].

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Isotomic Inscribed Triangles and Their Residuals

Mario Dalcín

Abstract. We prove some interesting results on inscribed triangles which are isotomic. For examples, we show that the triangles formed by the centroids (respectively orthocenters) of their residuals have equal areas, and those formed by the circumcenters are congruent.

1. Isotomic inscribed triangles

The starting point of this investigation was the interesting observation that if we consider the points of tangency of the sides of a triangle with its incircle and excircles, we have two triangles of equal areas.

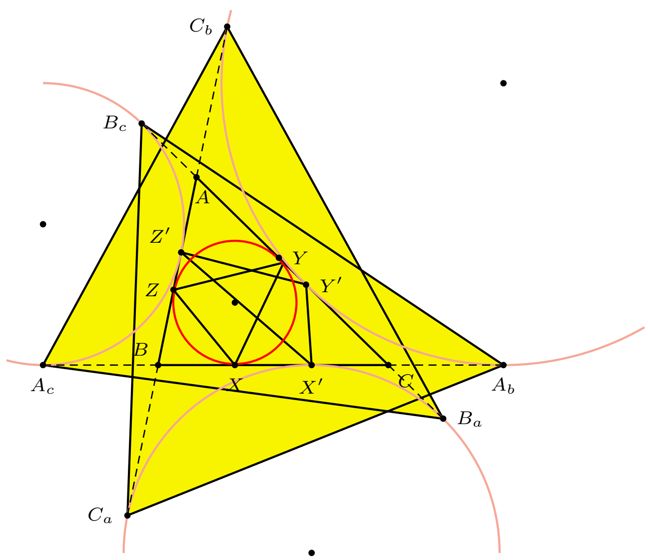

Figure 1

In Figure 1, X, Y, Z are the points of tangency of the incircle with the sides BC, CA, AB of triangle ABC, and X', Y', Z' those with the corresponding excircles. In [2], XYZ and $X'Y'Z'$ are called the intouch and extouch triangles of ABC respectively. That these two triangles have equal areas is best explained by the fact that each pair of points X, X'; Y, Y'; Z, Z' are isotomic on their respective sides, *i.e.*,

$$BX = X'C, \quad CY = Y'A, \quad AZ = Z'B. \tag{1}$$

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We shall say that XYZ and $X'Y'Z'$ are isotomic inscribed triangles. The following basic proposition follows from simple calculations with barycentric coordinates.

Proposition 1. *Isotomic inscribed triangles have equal areas.*

Proof. Let X, Y, Z be points on the sidelines BC, CA, AB dividing the sides in the ratios

$$BX : XC = x : 1-x, \quad CY : YA = y : 1-y, \quad AZ : ZB = z : 1-z.$$

In terms of barycentric coordinates with respect to ABC, we have

$$X = (1-x)B + xC, \quad Y = (1-y)C + yA, \quad Z = (1-z)A + zB. \tag{2}$$

The area of triangle XYZ, in terms of the area $\triangle$ of ABC, is

$$\begin{aligned}\triangle XYZ =& \begin{vmatrix} 0 & 1-x & x \\ y & 0 & 1-y \\ 1-z & z & 0 \end{vmatrix} \triangle \\ =&(1-(x+y+z)+(xy+yz+zx))\triangle \\ =&(xyz+(1-x)(1-y)(1-z))\triangle. \end{aligned} \tag{3}$$

See, for example, [4, Proposition 1]. If X', Y', Z' are points satisfying (1), then

$$BX' : X'C = 1-x : x, \quad CY' : Y'A = 1-y : y, \quad AZ' : Z'B = 1-z : z, \tag{4}$$

and

$$X' = xB + (1-x)C, \quad Y' = yC + (1-y)A, \quad Z' = zA + (1-z)B. \tag{5}$$

The area of triangle $X'Y'Z'$ can be obtained from (3) by replacing x, y, z by $1-x$, $1-y$, $1-z$ respectively. It is clear that this results in the same expression. This completes the proof of the proposition. □

Proposition 2. *The centroids of isotomic inscribed triangles are symmetric with respect to the centroid of the reference triangle.*

Proof. The expressions in (2) allow one to determine the centroid of triangle XYZ easily. This is the point

$$G_{XYZ} = \frac{1}{3}(X+Y+Z) = \frac{(1+y-z)A+(1+z-x)B+(1+x-y)C}{3}. \tag{6}$$

On the other hand, with the coordinates given in (5), the centroid of triangle $X'Y'Z'$ is

$$G_{X'Y'Z'} = \frac{1}{3}(X'+Y'+Z') = \frac{(1-y+z)A+(1-z+x)B+(1-x+y)C}{3}. \tag{7}$$

It follows easily that

$$\frac{1}{2}(G_{XYZ}+G_{X'Y'Z'}) = \frac{1}{3}(A+B+C) = G,$$

the centroid of triangle ABC. □

Corollary 3. *The intouch and extouch triangles have equal areas, and the midpoint of their centroids is the centroid of triangle ABC.*

Proof. These follow from the fact that the intouch triangle XYZ and the extouch triangle $X'Y'Z'$ are isotomic, as is clear from the following data, where a, b, c denote the lengths of the sides BC, CA, AB of triangle ABC, and $s = \frac{1}{2}(a+b+c)$.

$$\begin{array}{ll} BX = X'C = s - b, & BX' = XC = s - c, \\ CY = Y'A = s - c, & CY' = YA = s - a, \\ AZ = Z'B = s - a, & AZ' = ZB = s - b. \end{array}$$

□

In fact, we may take

$$x = \frac{s-b}{a}, \quad y = \frac{s-c}{b}, \quad z = \frac{s-a}{c},$$

and use (3) to obtain

$$\triangle XYZ = \triangle X'Y'Z' = \frac{2(s-a)(s-b)(s-c)}{abc}\triangle.$$

Let R and r denote respectively the circumradius and inradius of triangle ABC. Since $\triangle = rs$ and

$$R = \frac{abc}{4\triangle}, \qquad r^2 = \frac{(s-a)(s-b)(s-c)}{s},$$

we have

$$\triangle XYZ = \triangle X'Y'Z' = \frac{r}{2R} \cdot \triangle.$$

If we denote by A_b and A_c the points of tangency of the line BC with the B- and C-excircles, it is easy to see that A_b and A_c are isotomic points on BC. In fact,

$$BA_b = A_cC = s, \quad BA_c = A_bC = -(s-a).$$

Similarly, the other points of tangency B_c, B_a, C_a, C_b form pairs of isotomic points on the lines CA and AB respectively. See Figure 1.

Corollary 4. *The triangles $A_bB_cC_a$ and $A_cB_aC_b$ have equal areas. The centroids of these triangles are symmetric with respect to the centroid G of triangle ABC.*

These follow because $A_bB_cC_a$ and $A_cB_bC_a$ are isotomic inscribed triangles. Indeed,

$$\begin{aligned} BA_b : A_bC =& s : -(s-a) = 1 + \frac{s-a}{a} : -\frac{s-a}{a} = CA_c : A_cB, \\ CB_c : B_cA =& s : -(s-b) = 1 + \frac{s-b}{b} : -\frac{s-b}{b} = AB_a : B_aC, \\ AC_a : C_aB =& s : -(s-c) = 1 + \frac{s-c}{c} : -\frac{s-c}{c} = BC_b : C_bA. \end{aligned}$$

Furthermore, the centroids of the four triangles XYZ, $X'Y'Z'$, $A_bB_cC_a$ and $A_cB_aC_b$ form a parallelogram. See Figure 2.

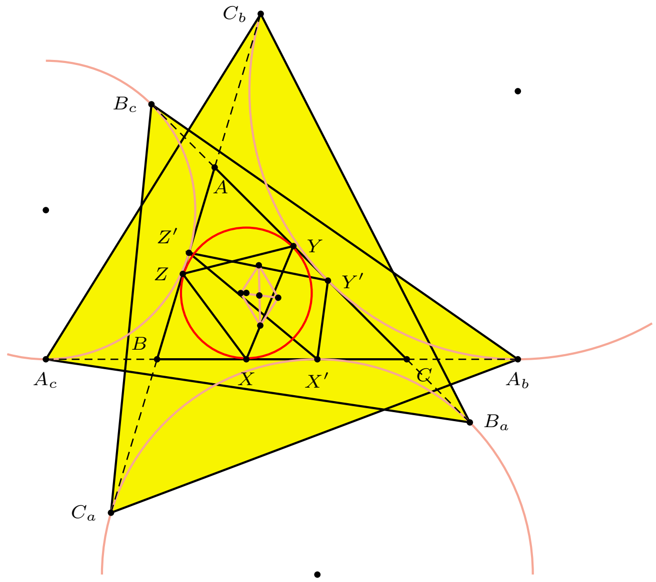

Figure 2

2. Triangles of residual centroids

For an inscribed triangle XYZ, we call the triangles AYZ, BZX, CXY its residuals. From (2, 5), we easily determine the centroids of these triangles.

$$\begin{aligned}
G_{AYZ} =& \frac{1}{3}((2+y-z)A + zB + (1-y)C),\\
G_{BZX} =& \frac{1}{3}((1-z)A + (2+z-x)B + xC),\\
G_{CXY} =& \frac{1}{3}(yA + (1-x)B + (2+x-y)C).
\end{aligned}$$

We call these the residual centroids of the inscribed triangle XYZ.

The following two propositions are very easily to established, by making the interchanges $(x, y, z) \leftrightarrow (1-x, 1-y, 1-z)$.

Proposition 5. *The triangles of residual centroids of isotomic inscribed triangles have equal areas.*

Proof. From the coordinates given above, we obtain the area of the triangle of residual centroids as

$$\begin{aligned}
&\frac{1}{27}\begin{vmatrix} 2+y-z & z & 1-y \\ 1-z & 2+z-x & x \\ y & 1-x & 2+x-y \end{vmatrix}\triangle\\
=&\frac{1}{9}(3-x-y-z+xy+yz+zx)\triangle\\
=&\frac{1}{9}(2+xyz+(1-x)(1-y)(1-z))\triangle
\end{aligned}$$

By effecting the interchanges $(x, y, z) \leftrightarrow (1-x, 1-y, 1-z)$, we obtain the area of the triangle of residual centroids of the isotomic inscribed triangle $X'Y'Z'$. This clearly remains unchanged. $\square$

Proposition 6. *Let XYZ and $X'Y'Z'$ be isotomic inscribed triangles of ABC. The centroids of the following five triangles are collinear:*

- *G of triangle ABC,*
- *G_{XYZ} and $G_{X'Y'Z'}$ of the inscribed triangles,*
- *$\widetilde{G}$ and $\widetilde{G'}$ of the triangles of their residual centroids.*

Furthermore,

$$G_{XYZ}\widetilde{G} : \widetilde{G}G : G\widetilde{G'} : \widetilde{G'}G_{X'Y'Z'} = 1 : 2 : 2 : 1.$$

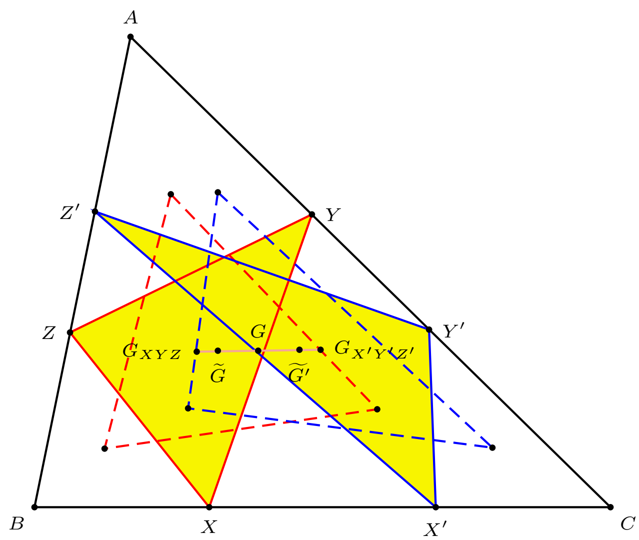

Figure 3

Proof. The centroid $\widetilde{G}$ is the point

$$\widetilde{G} = \frac{1}{9}((3 + 2y - 2z)A + (3 + 2z - 2x)B + (3 + 2x - 2y)C).$$

We obtain the centroid $\widetilde{G'}$ by interchanging $(x, y, z) \leftrightarrow (1-x, 1-y, 1-z)$. From these coordinates and those given in (6,7), the collinearity is clear, and it is easy to figure out the ratios of division. $\square$

3. Triangles of residual orthocenters

Proposition 7. *The triangles of residual orthocenters of isotomic inscribed triangles have equal areas.*

See Figure 4. This is an immediate corollary of the following proposition (see Figure 5), which in turn is a special case of a more general situation considered in Proposition 8 below.

Proposition 8. *An inscribed triangle and its triangle of residual orthocenters have equal areas.*

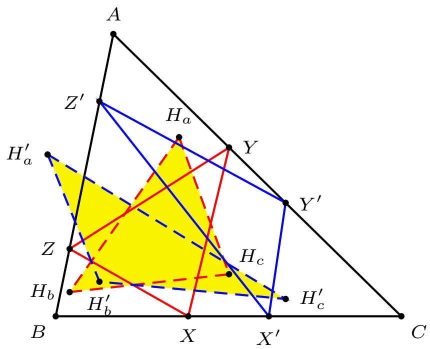

Figure 4

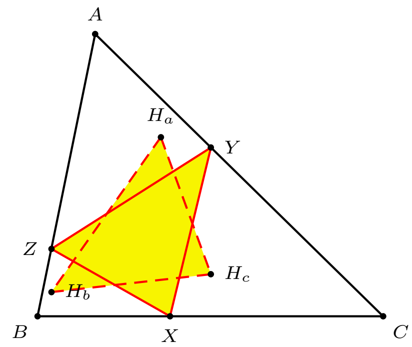

Figure 5

Proposition 9. *Given a triangle ABC, if pairs of parallel lines $\mathcal{L}_{1B}$, $\mathcal{L}_{1C}$ through B, C, $\mathcal{L}_{2C}$, $\mathcal{L}_{2A}$ through C, A, and $\mathcal{L}_{3A}$, $\mathcal{L}_{3B}$ through A, B are constructed, and if*

$$P_a = \mathcal{L}_{2C} \cap \mathcal{L}_{3B}, \qquad P_b = \mathcal{L}_{3A} \cap \mathcal{L}_{1C}, \qquad P_c = \mathcal{L}_{1B} \cap \mathcal{L}_{2A},$$

then the triangle $P_aP_bP_c$ has the same area as triangle ABC.

Proof. We write $Y = \mathcal{L}_{2C} \cap \mathcal{L}_{3A}$ and $Z = \mathcal{L}_{2A} \cap \mathcal{L}_{3B}$. Consider the parallelogram AZP_aY in Figure 6. If the points B and C divide the segments ZP_a and YP_a in the ratios

$$ZB : BP_a = v : 1 - v, \qquad YC : CP_a = w : 1 - w,$$

then it is easy to see that

$$\text{Area}(ABC) = \frac{1 + vw}{2} \cdot \text{Area}(AZP_aY). \tag{8}$$

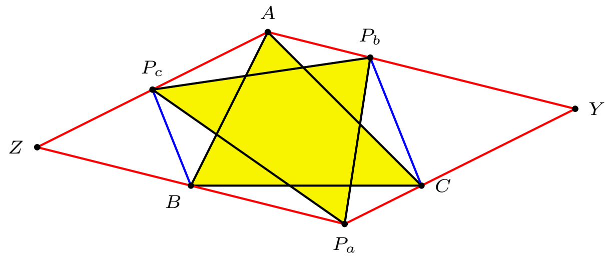

Figure 6

Now, P_b and P_c are points on AY and AZ such that BP_c and CP_b are parallel. If

$$YP_b : P_bA = v' : 1 - v', \qquad ZP_c : P_cA = w' : 1 - w',$$

then from the similarlity of triangles BZP_c and P_bYC, we have

$$ZB : ZP_c = YP_b : YC.$$

This means that $v : w' = v' : w$ and $v'w' = vw$. Now, in the same parallelogram AZP_aY, we have

$$\text{Area}(P_aP_bP_c) = \frac{1+v'w'}{2} \cdot \text{Area}(AZP_aY).$$

From this we conclude that $P_aP_bP_c$ and ABC have equal areas.

□

4. Triangles of residual circumcenters

Consider the circumcircles of the residuals of an inscribed triangle XYZ. By Miquel's theorem, the circles AYZ, BZX, and CXY have a common point. Furthermore, the centers O_a, O_b, O_c of these circles form a triangle similar to ABC. See, for example, [1, p.134]. We prove the following interesting theorem.

Theorem 10. *The triangles of residual circumcenters of the isotomic inscribed triangles are congruent.*

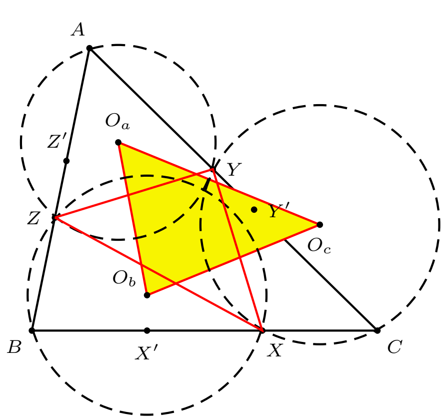

Figure 7A

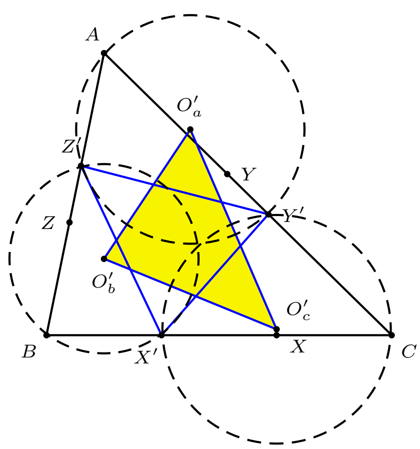

Figure 7B

We prove this theorem by calculations.

Lemma 11. *Let X, Y, Z be points on BC, CA, AB such that*

$$BX : XC = w : v, \quad CY : YA = u_c : w, \quad AZ : ZB = v : u_b.$$

The distance between the circumcenters O_b and O_c is the hypotenuse of a right triangle with one side $\frac{a}{2}$ and another side

$$\frac{(v-w)(u_b+v)(u_c+w)a^2 + (v+w)(w-u_c)(u_b+v)b^2 + (v+w)(w+u_c)(u_b-v)c^2}{8\triangle(u_b+v)(v+w)(w+u_c)} \cdot a. \tag{9}$$

Proof. The distance between O_b and O_c along the side BC is clearly $\frac{a}{2}$. We calculate their distance along the altitude on BC. The circumradius of BZX is clearly $R_b = \frac{ZX}{2\sin B}$. The distance of O_b above BC is

$$\begin{aligned}
R_b \cos BZX =& \frac{ZX\cos BZX}{2\sin B} = \frac{2BZ\cdot ZX\cos BZX}{4BZ\sin B} = \frac{BZ^2+ZX^2-BX^2}{4BZ\sin B}\\
=& \frac{BZ^2+BZ^2+BX^2-2BZ\cdot BX\cos B - BX^2}{4BZ\sin B}\\
=& \frac{BZ-BX\cos B}{2\sin B} = \frac{c(BZ-BX\cos B)}{4\triangle}\cdot a\\
=& \frac{c\left(\frac{u_b}{u_b+v}c - \frac{w}{v+w}a\cos B\right)}{4\triangle}\cdot a\\
=& \frac{u_b(v+w)2c^2 - w(u_b+v)(c^2+a^2-b^2)}{8\triangle(u_b+v)(v+w)}\cdot a\\
=& \frac{-(u_b+v)w(a^2-b^2)+(2u_bv+u_bw-vw)c^2}{8\triangle(u_b+v)(v+w)}\cdot a
\end{aligned}$$

By making the interchanges $b \leftrightarrow c$, $v \leftrightarrow w$, and $u_b \leftrightarrow u_c$, we obtain the distance of O_c above the same line as

$$\frac{-(u_c+w)v(a^2-c^2)+(2u_cw+u_cv-vw)b^2}{8\triangle(u_c+w)(v+w)}\cdot a.$$

The difference between these two is the expression given in (9) above. □

Consider now the isotomic inscribed triangle $X'Y'Z'$. We have

$$\begin{aligned}
BX' : X'C =& v : w,\\
CY' : Y'A =& w : u_c = \frac{vw}{u_c} : v,\\
AZ' : Z'B =& u_b : v = w : \frac{vw}{u_b}.
\end{aligned}$$

Let O_b' and O_c' be the circumcenters of $BZ'X'$ and $CX'Y'$. By making the following interchanges

$$v \leftrightarrow w, \qquad u_b \leftrightarrow \frac{vw}{u_b}, \qquad u_c \leftrightarrow \frac{vw}{u_c}$$

in (9), we obtain the distance between O_b' and O_c' along the altitude on BC as

$$\begin{aligned}
&\frac{(w-v)(\frac{vw}{u_b}+w)(\frac{vw}{u_c}+v)a^2+(v+w)(v-\frac{vw}{u_c})(\frac{vw}{u_b}+w)b^2+(v+w)(v+\frac{vw}{u_c})(\frac{vw}{u_b}-w)c^2}{8\triangle(\frac{vw}{u_b}+w)(v+w)(v+\frac{vw}{u_c})}\cdot a\\
=&\frac{(w-v)(v+u_b)(w+u_c)a^2+(v+w)(u_c-w)(v+u_b)b^2+(v+w)(w+u_c)(v-u_b)c^2}{8\triangle(v+u_b)(v+w)(u_c+w)}\cdot a.
\end{aligned}$$

Except for a reversal in sign, this is the same as (9).

From this we easily conclude that the segments O_bO_c and $O'_bO'_c$ are congruent. The same reasoning also yields the congruences of O_cO_a, $O'_cO'_a$, and of O_aO_b, $O'_aO'_b$. It follows that the triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are congruent. This completes the proof of Theorem 9.

5. Isotomic conjugates

Let XYZ be the cevian triangle of a point P, *i.e.*, X, Y, Z are respectively the intersections of the line pairs AP, BC; BP, CA; CP, AB. By the residual centroids ((respectively orthocenters, circumcenters) of P, we mean those of its cevian triangle. If we construct points X', Y', Z' satisfying (1), then the lines AX', BY', CZ' intersect at a point P' called the isotomic conjugate of P. If the point P has homogeneous barycentric coordinates $(x : y : z)$, then P' has homogeneous barycentric coordinates $\left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. All results in the preceding sections apply to the case when XYZ and $X'Y'Z'$ are the cevian triangles of two isotomic conjugates. In particular, in the case of residual circumcenters in §4 above, if XYZ is the cevian triangle of P with homogeneous barycentric coordinates $(u : v : w)$, then

$$BX : XC = w : v, \qquad CY : YA = u : w, \qquad AZ : ZB = v : u.$$

By putting $u_b = u_c = u$ in (9) we obtain a necessary and sufficient condition for the line O_bO_c to be parallel to BC, namely,

$$(v-w)(u+v)(u+w)a^2+(v+w)(w-u)(u+v)b^2+(v+w)(w+u)(u-v)c^2 = 0.$$

This can be reorganized into the form

$$(b^2+c^2-a^2)u(v^2-w^2)+(c^2+a^2-b^2)v(w^2-u^2)+(a^2+b^2-c^2)w(u^2-v^2) = 0.$$

This is the equation of the Lucas cubic, consisting of points P for which the line joining P to its isotomic conjugate P' passes through the orthocenter H. The symmetry of this equation leads to the following interesting theorem.

Theorem 12. *The triangle of residual circumcenters of P is homothetic to ABC if and only if P lies on the Lucas cubic.*

It is well known that the Lucas cubic is the locus of point P whose cevian triangle is also the pedal triangle of a point Q. In this case, the circumcircles of AYZ, BZX and CXY intersect at Q, and the circumcenters O_a, O_b, O_c are the midpoints of the segments AQ, BQ, CQ. The triangle $O_aO_bO_c$ is homothetic to ABC at Q.

For example, if P is the Gergonne point, then $O_aO_bO_c$ is homothetic to ABC at the incenter I. The isotomic conjugate of P is the Nagel point, and $O'_aO'_bO'_c$ is homothetic to ABC at the reflection of I in the circumcenter O.

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The M-Configuration of a Triangle

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Abstract. We give an easy construction of points A_a, B_a, C_a on the sides of a triangle ABC such that the figure M path $BC_aA_aB_aC$ consists of 4 segments of equal lengths. We study the configuration consisting of the three figures M of a triangle, and define an interesting mapping of triangle centers associated with such an M-configuration.

1. Introduction

Given a triangle ABC, we consider points A_a on the line BC, B_a on the half line CA, and C_a on the half line BA such that $BC_a = C_aA_a = A_aB_a = B_aC$. We shall refer to $BC_aA_aB_aC$ as M$_a$, because it looks like the letter M when triangle ABC is acute-angled. See Figures 1a. Figure 1b illustrates the case when the triangle is obtuse-angled. Similarly, we also have M$_b$ and M$_c$. The three figures M$_a$, M$_b$, M$_c$ constitute the M-configuration of triangle ABC. See Figure 2.

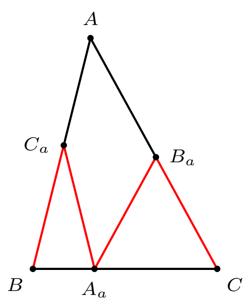

Figure 1a

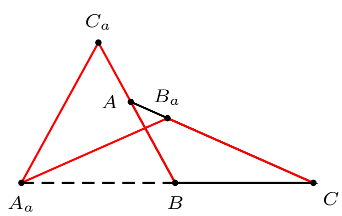

Figure 1b

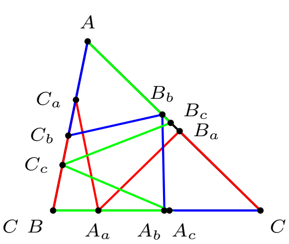

Figure 2

Proposition 1. *The lines AA_a, BB_a, CC_a concur at the point with homogeneous barycentric coordinates*

$$\left(\frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}\right).$$

Proof. Let l_a be the length of $BC_a = C_aA_a = A_aB_a = B_aC$. It is clear that the directed length $BA_a = 2l_a \cos B$ and $A_aC = 2l_a \cos C$, and $BA_a : A_aC = \cos B : \cos C$. For the same reason, $CB_b : B_bA = \cos C : \cos A$ and $AC_c : C_cB = \cos A : \cos B$. It follows by Ceva's theorem that the lines AA_a, BB_a, CC_a concur at the point with homogeneous barycentric coordinates given above.[1] □

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The author is grateful to the editor for his help in the preparation of this paper.

[1]This point appears in [3] as X_{92}.

Remark. Since $2l_a \cos B + 2l_a \cos C = a = 2R \sin A$, where R is the circumradius of triangle ABC,

$$l_a = \frac{a}{2(\cos B + \cos C)} = \frac{R \sin A}{\cos B + \cos C} = \frac{R \cos \frac{A}{2}}{\cos \frac{B-C}{2}}. \tag{1}$$

For later use, we record the absolute barycentric coordinates of A_a, B_a, C_a in terms of l_a:

$$\begin{aligned} A_a =& \frac{2l_a}{a}(\cos C \cdot B + \cos B \cdot C), \\ B_a =& \frac{1}{b}(l_a \cdot A + (b - l_a)C), \\ C_a =& \frac{1}{c}(l_a \cdot A + (c - l_a)B). \end{aligned} \tag{2}$$

2. Construction of $\mathbf{M}_a$

Proposition 2. *Let A' be the intersection of the bisector of angle A with the circumcircle of triangle ABC.*

(a) *A_a is the intersection of BC with the parallel to AA' through the orthocenter H.*

(b) *B_a (respectively C_a) is the intersection of CA (respectively BA) with the parallel to CA' (respectively BA') through the circumcenter O.*

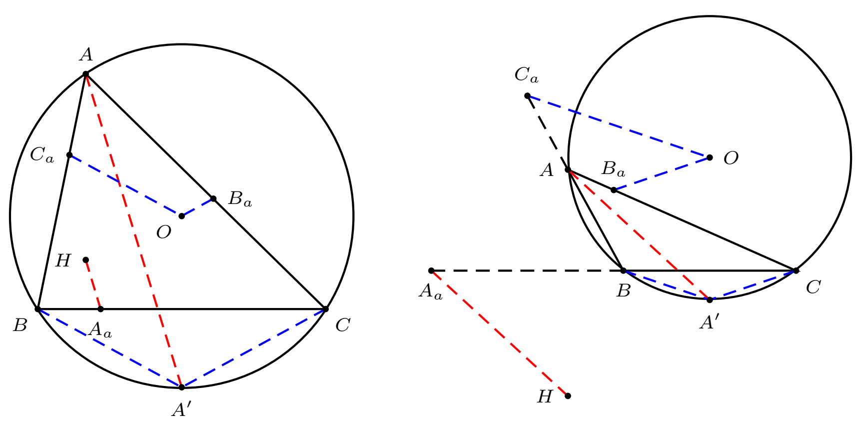

Figure 3a Figure 3b

Proof. (a) The line joining $A_a = (0 : \cos C : \cos B)$ to $H = \left(\frac{a}{\cos A} : \frac{b}{\cos B} : \frac{c}{\cos C}\right)$ has equation

$$\begin{vmatrix} 0 & \cos C & \cos B \\ \frac{a}{\cos A} & \frac{b}{\cos B} & \frac{c}{\cos C} \\ x & y & z \end{vmatrix} = 0.$$

This simplifies to

$$-(b-c)x\cos A + a(y\cos B - z\cos C) = 0.$$

It has infinite point

$$\begin{aligned}&(-a(\cos B+\cos C) : a\cos C-(b-c)\cos A : (b-c)\cos A + a\cos B)\\ =&(-a(\cos B+\cos C) : b(1-\cos A) : c(1-\cos A)).\end{aligned}$$

It is clear that this is the same as the infinite point $(-(b+c) : b : c)$, which is on the line joining A to the incenter.

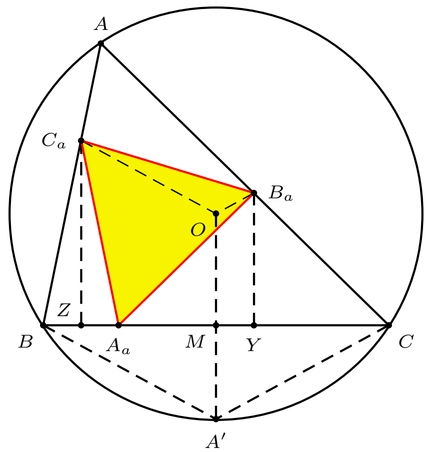

Figure 4

(b) Let M be the midpoint of BC, and Y, Z the pedals of B_a, C_a on BC. See Figure 4. We have

$$\begin{aligned}OM =&\frac{a}{2}\cot A = l_a(\cos B+\cos C)\cot A,\\ C_aZ =&l_a\sin B,\\ MZ =&\frac{a}{2} - l_a\cos B = l_a(\cos B+\cos C) - l_a\cos B = l_a\cos C.\end{aligned}$$

From this the acute angle between the line C_aO and BC has tangent ratio

$$\begin{aligned}\frac{C_aZ-OM}{MZ} =&\frac{\sin B-(\cos B+\cos C)\cot A}{\cos C}\\ =&\frac{\sin B\sin A-(\cos B+\cos C)\cos A}{\cos C\sin A}\\ =&\frac{-\cos(A+B)-\cos C\cos A}{\cos C\sin A} = \frac{\cos C(1-\cos A)}{\cos C\sin A}\\ =&\frac{1-\cos A}{\sin A} = \tan\frac{A}{2}.\end{aligned}$$

It follows that C_aO makes an angle $\frac{A}{2}$ with the line BC, and is parallel to BA'. The same reasoning shows that B_aO is parallel to CA'. □

3. Circumcenters in the M-configuration

Note that $\angle B_aA_aC_a = \angle A$. It is clear that the circumcircles of $B_aA_aC_a$ and B_aAC_a are congruent. The circumradius is

$$R_a = \frac{l_a}{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)} = \frac{l_a}{2\cos\frac{A}{2}} = \frac{R}{2\cos\frac{B-C}{2}} \qquad (3)$$

from (1).

Proposition 3. *The circumcircle of triangle AB_aC_a contains* (i) *the circumcenter O of triangle ABC,* (ii) *the orthocenter H_a of triangle $A_aB_aC_a$, and* (iii) *the midpoint of the arc BAC.*

Proof. (i) is an immediate corollary of Proposition 2(b) above.

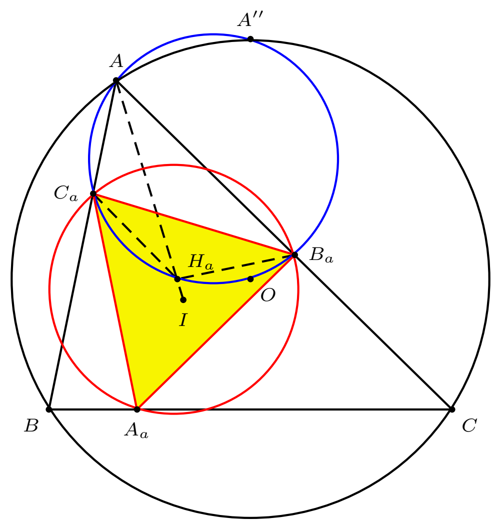

Figure 5

(ii) Let H_a be the orthocenter of triangle $A_aB_aC_a$. It is clear that

$$\angle B_aH_aC_a = \pi - \angle B_aA_aC_a = \pi - \angle BAC = \pi - \angle C_aAB_a.$$

It follows that H_a lies on the circumcircle of AB_aC_a. See Figure 5. Since the triangle $A_aB_aC_a$ is isosceles, $B_aH_a = C_aH_a$, and the point H_a lies on the bisector of angle A.

(iii) Let A'' be the midpoint of the arc BAC. By a simple calculation, $\angle AA''O = \frac{\pi}{2} - \frac{1}{2}|B - C|$. Also, $\angle AC_aO = \frac{\pi}{2} + \frac{1}{2}|B - C|$.[2] This shows that A'' also lies on the circle AB_aOC_a. □

The points B_a and C_a are therefore the intersections of the circle OAA'' with the sidelines AC and AB. This furnishes another simple construction of the figure M$_a$.

[2]This is $C + \frac{A}{2}$ if $C \geq B$ and $B + \frac{A}{2}$ otherwise.

Remarks. (1) If we take into consideration also the other figures M_b and M_c, we have three triangles AB_aC_a, BC_bA_b, CA_cB_c with their circumcircles intersecting at O.

(2) We also have three triangles A_aB_aCa, $A_bB_bC_b$, $A_cB_cC_c$ with their orthocenters forming a triangle perspective with ABC at the incenter I.

Proposition 4. *The circumcenter O_a of triangle $A_aB_aC_a$ is equidistant from O and H.*

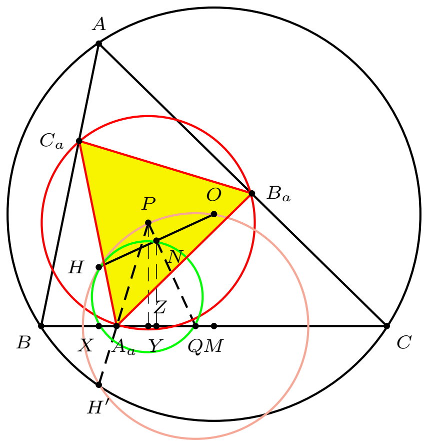

Figure 6

Proof. Construct the circle through O and H with center Q on the line BC. We prove that the midpoint P of the arc OH on the opposite side of Q is the circumcenter O_a of triangle $A_aB_aC_a$. See Figure 6. It will follow that O_a is equidistant from O and H. Let N be the midpoint of OH. Suppose the line PQ makes an angle φ with BC. Let X, Y, and M be the pedals of H, N, O on the line BC.

Since H, X, Q, N are concyclic, and the diameter of the circle containing them is $QH = \frac{NX}{\sin\varphi} = \frac{R}{2\sin\varphi}$. This is the radius of the circle OPH.

By symmetry, the circle OPH contains the reflection H' of H in the line BC.

$$\angle HH'P = \frac{1}{2}\angle HQP = \frac{1}{2}\angle HQN = \frac{1}{2}\angle HXN = \frac{1}{2}|B-C|.$$

Therefore, the angle between $H'P$ and BC is $\frac{\pi}{2} - \frac{1}{2}|B-C|$. It is obvious that the angle between A_aO_a and BC is the same. But from Proposition 2(a), the angle between HA_a and BC is the same too, so is the angle between the reflection $H'A_a$ and BC. From these we conclude that H', A_a, O_a and P are collinear. Now, let Z be the pedal of P on BC.

$$A_aP = \frac{PZ}{\cos\frac{1}{2}(B-C)} = \frac{QP\sin\varphi}{\cos\frac{1}{2}(B-C)} = \frac{R}{2\cos\frac{1}{2}(B-C)} = R_a.$$

Therefore, P is the circumcenter O_a of triangle $A_aB_aC_a$. □

Applying this to the other two figures M_b and M_c, we obtain the following remarkable theorem about the M-configuration of triangle ABC.

Theorem 5. *The circumcenters of triangles $A_aB_aC_a$, $A_bB_bC_b$, and $A_cB_cC_c$ are collinear. The line containing them is the perpendicular bisector of the segment OH.*

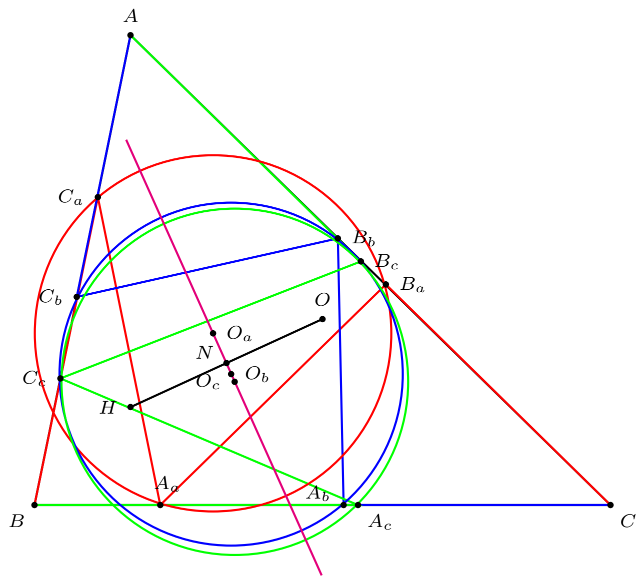

Figure 7

One can check without much effort that in homogeneous barycentric coordinates, the equation of this line is

$$\frac{\sin 3A}{\sin A}x + \frac{\sin 3B}{\sin B}y + \frac{\sin 3C}{\sin C}z = 0.$$

4. A central mapping

Let P be a triangle center in the sense of Kimberling [2, 3], given in homogeneous barycentric coordinates $(f(a,b,c) : f(b,c,a) : f(c,a,b))$ where $f = f_P$ satisfies $f(a,b,c) = f(a,c,b)$. If the reference triangle ABC is isosceles, say, with $AB = AC$, then P lies on the perpendicular bisector of BC and has coordinates of the form $(g_P : 1 : 1)$. The coordinate g depends only on the shape of the isosceles triangle, and we express it as a function of the *base* angle. We shall call $g = g_P$ the *isoscelized form* of the triangle center function f_P. Let P^* denote the isogonal conjugate of P.

Lemma 6. $g_{P^*}(B) = \frac{4\cos^2 B}{g_P(B)}$.

Proof. If $P = (g_P(B) : 1 : 1)$ for an isosceles triangle ABC with $B = C$, then

$$P^* = \left(\frac{\sin^2 A}{g_P(B)} : \sin^2 B : \sin^2 B\right) = \left(\frac{4\cos^2 B}{g_P(B)} : 1 : 1\right)$$

since $\sin^2 A = \sin^2(\pi - 2B) = \sin^2 2B = 4\sin^2 B \cos^2 B$. □

Here are some examples.

Center	f_P	g_P
centroid	1	1
incenter	a	$2\cos B$
circumcenter	$a^2(b^2+c^2-a^2)$	$-2\cos 2B$
orthocenter	$\frac{1}{b^2+c^2-a^2}$	$\frac{-2\cos^2 B}{\cos 2B}$
symmedian point	a^2	$4\cos^2 B$
Gergonne point	$\frac{1}{s-a}$	$\frac{\cos B}{1-\cos B}$
Nagel point	$s-a$	$\frac{1-\cos B}{\cos B}$
Mittenpunkt	$a(s-a)$	$2(1-\cos B)$
Spieker point	$b+c$	$\frac{2}{1+2\cos B}$
X_{55}	$a^2(s-a)$	$4\cos B(1-\cos B)$
X_{56}	$\frac{a^2}{s-a}$	$\frac{4\cos^3 B}{1-\cos B}$
X_{57}	$\frac{a}{s-a}$	$\frac{2\cos^2 B}{1-\cos B}$

Consider a triangle center given by a triangle center function with isoscelized form $g = g_P$. The triangle center of the isosceles triangle C_aBA_a is the point $P_{a,b}$ with coordinates $(g(B) : 1 : 1)$ relative to C_aBA_a. Making use of the absolute barycentric coordinates of A_a, B_a, C_a given in (2), it is easy to see that this is the point

$$P_{a,b} = \left(\frac{g(B)l_a}{c} : \frac{g(B)(c-l_a)}{c} + 1 + \frac{2l_a}{a}\cos C : \frac{2l_a}{a}\cos B\right).$$

The same triangle center of the isosceles triangle B_aA_aC is the point

$$P_{a,c} = \left(\frac{g(C)l_a}{b} : \frac{2l_a}{a}\cos C : \frac{g(C)(b-l_a)}{b} + \frac{2l_a}{a}\cos B + 1\right).$$

It is clear that the lines $BP_{a,b}$ and $CP_{a,c}$ intersect at the point

$$\begin{aligned} P_a &= \left(\frac{g(B)g(C)l_a^2}{bc} : \frac{2g(B)l_a^2\cos C}{ca} : \frac{2g(C)l_a^2\cos B}{ab}\right) \\ &= (ag(B)g(C) : 2bg(B)\cos C : 2cg(C)\cos B) \\ &= \left(\frac{ag(B)g(C)}{2\cos B\cos C} : \frac{bg(B)}{\cos B} : \frac{cg(C)}{\cos C}\right). \end{aligned}$$

Figure 8 illustrates the case of the Gergonne point.

In the M-configuration, we may also consider the same triangle center (given in isoscelized form g_P of the triangle center function) in the isosceles triangles . These are the point $P_{b,c}$, $P_{b,a}$, $P_{c,a}$, $P_{c,b}$. The pairs of lines $CP_{b,c}$, $AP_{b,a}$ intersecting at P_b and $AP_{c,a}$, $BP_{c,b}$ intersecting at P_c. The coordinates of P_b and P_c can be

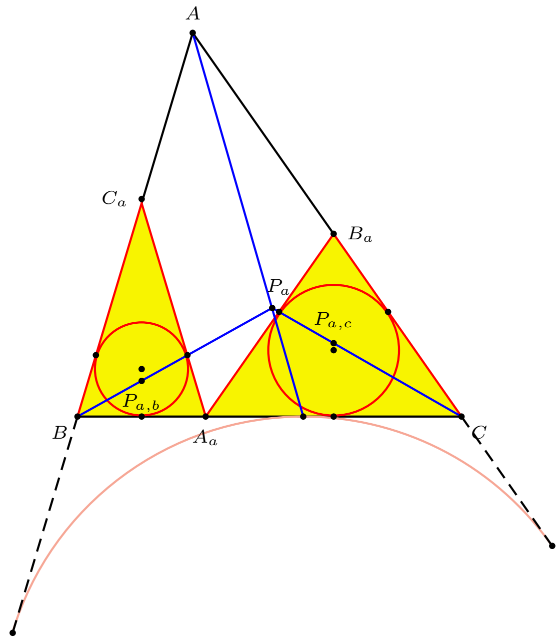

Figure 8

written down easily from those of P_a. From these coordinates, we easily conclude that that $P_aP_bP_c$ is perspective with triangle ABC at the point

$$\begin{aligned}\Phi(P) &= \left(\frac{ag_P(A)}{\cos A} : \frac{bg_P(B)}{\cos B} : \frac{cg_P(C)}{\cos C}\right)\\ &= (g_P(A)\tan A : g_P(B)\tan B : g_P(C)\tan C).\end{aligned}$$

Proposition 7. $\Phi(P^*) = \Phi(P)^*$.

Proof. We make use of Lemma 6.

$$\begin{aligned}\Phi(P^*) =&(g_{P^*}(A)\tan A : g_{P^*}(B)\tan B : g_{P^*}(C)\tan C)\\ =&\left(\frac{4\cos^2 A}{g_P(A)}\tan A : \frac{4\cos^2 B}{g_P(B)}\tan B : \frac{4\cos^2 C}{g_P(C)}\tan C\right)\\ =&\left(\frac{\sin^2 A}{g_P(A)\tan A} : \frac{\sin^2 B}{g_P(B)\tan B} : \frac{\sin^2 C}{g_P(C)\tan C}\right)\\ =&\Phi(P)^*.\end{aligned}$$

□

We conclude with some examples.

P	$\Phi(P)$	P^*	$\Phi(P^*) = \Phi(P)^*$
incenter	incenter		
centroid	orthocenter	symmedian point	circumcenter
circumcenter	X_{24}	orthocenter	X_{68}
Gergonne point	Nagel point	X_{55}	X_{56}
Nagel point	X_{1118}	X_{56}	$X_{1259} = X_{1118}^*$
Mittenpunkt	X_{34}	X_{57}	$X_{78} = X_{34}^*$

For the Spieker point, we have

$$\begin{aligned}\Phi(X_{10}) &= \left(\frac{\tan A}{1+2\cos A} : \frac{\tan B}{1+2\cos B} : \frac{\tan C}{1+2\cos C}\right)\\ &= \left(\frac{1}{a(b^2+c^2-a^2)(b^2+c^2-a^2+bc)} : \cdots : \cdots\right).\end{aligned}$$

This triangle center does not appear in the current edition of [3].

Remark. For $P = X_8$, the Nagel point, the point P_a has an alternative description. Antreas P. Hatzipolakis [1] considered the incircle of triangle ABC touching the sides CA and AB at Y and Z respectively, and constructed perpendiculars from Y, Z to BC intersecting the incircle again at Y' and Z'. See Figure 9. It happens that B, Z', $P_{a,b}$ are collinear; so are C, Y', $P_{a,c}$. Therefore, BZ' and CY' intersect at P_a. The coordinates of Y' and Z' are

$$\begin{aligned}Y' &= (a^2(b+c-a)(c+a-b) : (a^2+b^2-c^2)^2 : (b+c)^2(a+b-c)(c+a-b)),\\ Z' &= (a^2(b+c-a)(a+b-c) : (b+c)^2(c+a-b)(a+b-c) : (a^2-b^2+c^2)^2).\end{aligned}$$

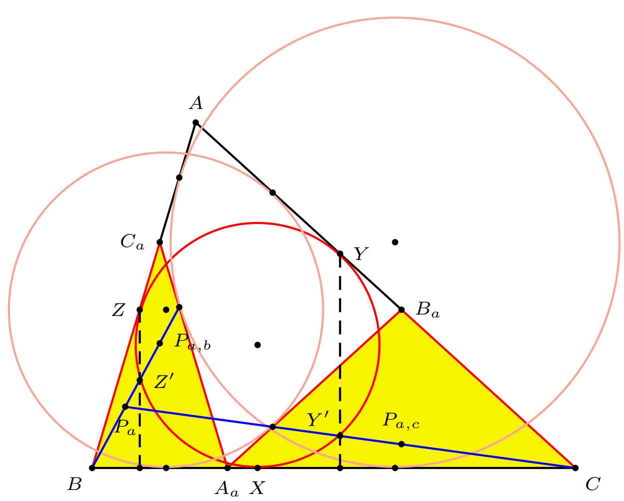

Figure 9

The lines BZ' and CY' intersect at

$$P_a = \left(a^2(b+c-a) : \frac{(a^2+b^2-c^2)^2}{c+a-b} : \frac{(a^2-b^2+c^2)^2}{a+b-c}\right)$$
$$= \left(\frac{a^2(b+c-a)}{(a^2-b^2+c^2)^2(a^2+b^2-c^2)^2} : \frac{1}{(c+a-b)(a^2-b^2+c^2)^2} : \frac{1}{(a+b-c)(a^2+b^2-c^2)^2}\right).$$

It was in this context that Hatzipolakis constructed the triangle center

$$X_{1118} = \left(\frac{1}{(b+c-a)(b^2+c^2-a^2)^2} : \cdots : \cdots\right).$$

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Rectangles Attached to Sides of a Triangle

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Abstract. We study the figure of a triangle with a rectangle attached to each side. In line with recent publications on special cases we find concurrencies and study homothetic triangles. Special attention is given to the cases in which the attached rectangles are similar, have equal areas and have equal perimeters, respectively.

1. Introduction

In recent publications [3, 4, 10, 11, 12] the configurations have been studied in which rectangles or squares are attached to the sides of a triangle. In these publications the rectangles are all similar. In this paper we study the more general case in which the attached rectangles are not necessarily similar. We consider a triangle ABC with attached rectangles BCA_cA_b, CAB_aB_c and ABC_bC_a. Let u be the length of CA_c, positive if A_c and A are on opposite sides of BC, otherwise negative. Similarly let v and w be the lengths of AB_a and BC_b. We describe the shapes of these rectangles by the ratios

$$U = \frac{a}{u}, \qquad V = \frac{b}{v}, \qquad W = \frac{c}{w}. \tag{1}$$

The vertices of these rectangles are [1]

$$\begin{aligned} A_b &= (-a^2 : S_C + SU : S_B), & A_c &= (-a^2 : S_C : S_B + SU), \\ B_a &= (S_C + SV : -b^2 : S_A), & B_c &= (S_C : -b^2 : S_A + SV), \\ C_a &= (S_B + SW : S_A : -c^2), & C_b &= (S_B : S_A + SW : -c^2). \end{aligned}$$

Consider the flank triangles AB_aC_a, A_bBC_b and A_cB_cC. With the same reasoning as in [10], or by a simple application of Ceva's theorem, we can see that the triangle $H_aH_bH_c$ of orthocenters of the flank triangles is perspective to ABC with perspector

$$P_1 = \left(\frac{a}{u} : \frac{b}{v} : \frac{c}{w}\right) = (U : V : W). \tag{2}$$

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[1] All coordinates in this note are homogeneous barycentric coordinates. We adopt J. H. Conway's notation by letting $S = 2\Delta$ denote twice the area of ABC, while $S_A = \frac{-a^2+b^2+c^2}{2} = S\cot A$, $S_B = S\cot B$, $S_C = S\cot C$, and generally $S_{XY} = S_XS_Y$.

See Figure 1. On the other hand, the triangle $O_aO_bO_c$ of circumcenters of the flank triangles is clearly homothetic to ABC, the homothetic center being the point

$$P_2 = (au : bv : cw) = \left(\frac{a^2}{U} : \frac{b^2}{V} : \frac{c^2}{W}\right). \tag{3}$$

Clearly, P_1 and P_2 are isogonal conjugates.

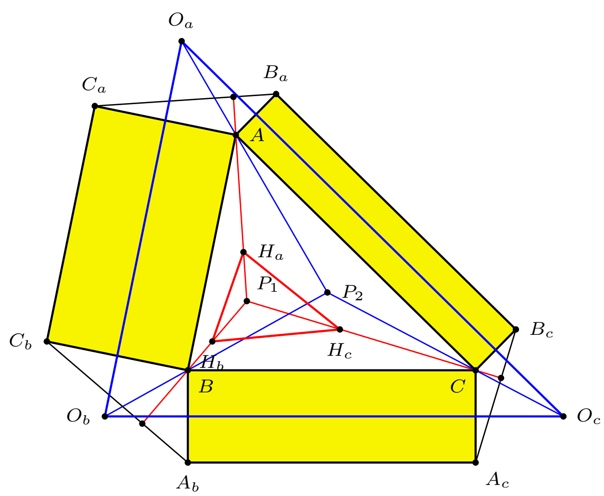

Figure 1

Now the perpendicular bisectors of B_aC_a, A_bC_b and A_cB_c pass through O_a, O_b and O_c respectively and are parallel to AP_1, BP_1 and CP_1 respectively. This shows that these perpendicular bisectors concur in a point P_3 on P_1P_2 satisfying

$$P_2P_1 : P_1P_3 = 2S : au + bv + cw,$$

where S is twice the area of ABC. See Figure 2. More explicitly,

$$\begin{aligned} P_3 =&(-a^2VW(V+W) + U^2(b^2W + c^2V) + 2SU^2VW \\ &: -b^2WU(W+U) + V^2(c^2U + a^2W) + 2SUV^2W) \\ &: -c^2UV(U+V) + W^2(a^2V + b^2U) + 2SUVW^2) \end{aligned} \tag{4}$$

This concurrency generalizes a similar result by Hoehn in [4], and was mentioned by L. Lagrangia [9]. It was also a question in the Bundeswettbewerb Mathematik Deutschland (German National Mathematics Competition) 1996, Second Round.

From the perspectivity of ABC and the orthocenters of the flank triangles, we see that ABC and the triangle $A'B'C'$ enclosed by the lines B_aC_a, A_bC_b and A_cB_c are orthologic. This means that the lines from the vertices of $A'B'C'$ to the corresponding sides of ABC are concurrent as well. The point of concurrency is the reflection of P_1 in O, *i.e.*,

$$P_4 = (-S_{BC}U + a^2 S_A(V+W) : \cdots : \cdots). \tag{5}$$

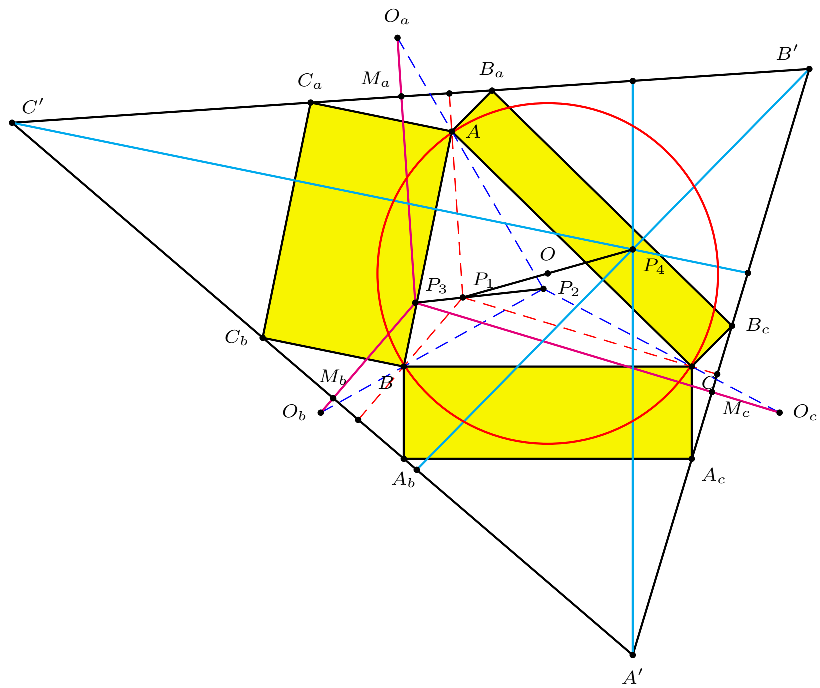

Figure 2

Remark. We record the coordinates of A'. Those of B' and C' can be written down accordingly.

$$\begin{aligned}A' =&(-(a^2S(U+V+W)+(a^2V+S_CU)(a^2W+S_BU))\\ &:S_CS(U+V+W)+(b^2U+S_CV)(a^2W+S_BU)\\ &:S_BS(U+V+W)+(a^2V+S_CU)(c^2U+S_BW)).\end{aligned}$$

2. Special cases

We are mainly interested in three special cases.

2.1. *The similarity case.* This is the case when the rectangles are similar, *i.e.*, $U = V = W = t$ for some t. In this case, $P_1 = G$, the centroid, and $P_2 = K$, the symmedian point. As t varies,

$$P_3 = (b^2+c^2-2a^2+2St : c^2+a^2-2b^2+2St : a^2+b^2-2c^2+2St)$$

traverses the line GK. The point P_4, being the reflection of G in O, is X_{376} in [7]. The triangle $M_aM_bM_c$ is clearly perspective with ABC at the orthocenter H. More interestingly, it is also perspective with the medial triangle at

$$((S_A+St)(a^2+2St) : (S_B+St)(b^2+2St) : (S_C+St)(c^2+2St)),$$

which is the complement of the Kiepert perspector

$$\left(\frac{1}{S_A+St}:\frac{1}{S_B+S_t}:\frac{1}{S_C+St}\right).$$

It follows that as t varies, this perspector traverses the Kiepert hyperbola of the medial triangle. See [8].

The case $t=1$ is the *Pythagorean* case, when the rectangles are squares erected externally. The perspector of $M_aM_bM_c$ and the medial triangle is the point

$$O_1=(2a^4-3a^2(b^2+c^2)+(b^2-c^2)^2-2(b^2+c^2)S:\cdots:\cdots),$$

which is the center of the circle through the centers of the squares. See Figure 3. This point appears as X_{641} in [7].

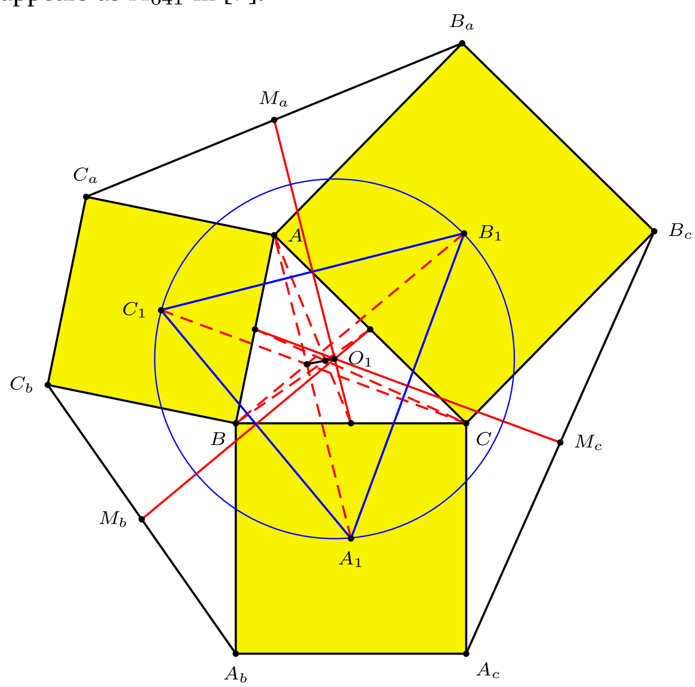

Figure 3

2.2. *The equiareal case.* When the rectangles have equal areas $\frac{T}{2}$, *i.e.*, $(U,V,W)=\left(\frac{2a^2}{T},\frac{2b^2}{T},\frac{2c^2}{T}\right)$, it is easy to see that $P_1=K$, $P_2=G$, and

$$\begin{aligned}P_4=&(a^2(-S_{BC}+S_A(b^2+c^2)):\cdots:\cdots)\\=&(a^2(a^4+2a^2(b^2+c^2)-(3b^4+2b^2c^2+3c^4)):\cdots:\cdots)\end{aligned}$$

is the reflection of K in O.[2] The *special equiareal case* is when $T=S$, the rectangles having the same area as triangle ABC. See Figure 4. In this case,

$$P_3=(6a^2-b^2-c^2:6b^2-c^2-a^2:6c^2-a^2-b^2).$$

[2]This point is not in the current edition of [7].

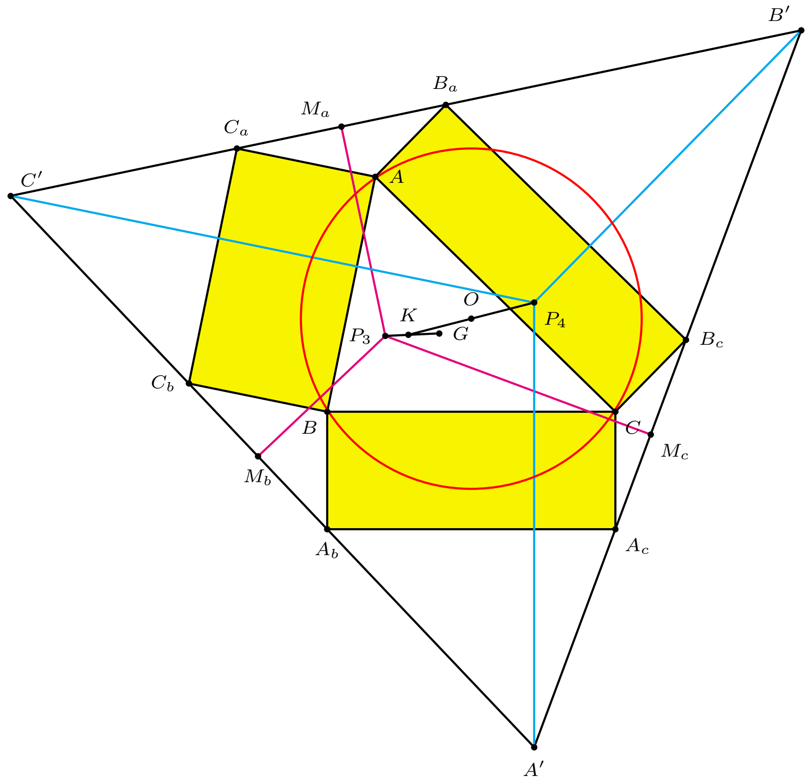

Figure 4

2.3. *The isoperimetric case.* This is the case when the rectangles have equal perimeters $2p$, *i.e.*, $(u, v, w) = (p - a, p - b, p - c)$. The *special isoperimetric* case is when $p = s$, the semiperimeter, the rectangles having the same perimeter as triangle ABC. In this case, $P_1 = X_{57}$, $P_2 = X_9$, the Mittenpunkt, and

$$\begin{aligned} P_3 =&(a(bc(2a^2 - a(b + c) - (b - c)^2) + 4(s - b)(s - c)S) : \cdots : \cdots),\\ P_4 =&(a(a^6 - 2a^5(b + c) - a^4(b^2 - 10bc + c^2) + 4a^3(b + c)(b^2 - bc + c^2)\\ &- a^2(b^4 + 8b^3c - 2b^2c^2 + 8c^3b + c^4) - 2a(b + c)(b - c)^2(b^2 + c^2)\\ &+ (b + c)^2(b - c)^4) : \cdots : \cdots). \end{aligned}$$

These points can be described in terms of division ratios as follows.[3]

$$\begin{aligned} P_3X_{57} : X_{57}X_9 =&4R + r : 2s,\\ P_4I : IX_{57} =&4R : r. \end{aligned}$$

3. A pair of homothetic triangles

Let A_1, B_1 and C_1 be the centers of the rectangles BCA_cA_b, CAB_aB_c and ABC_bC_a respectively, and $A_2B_2C_2$ the triangle bounded by the lines B_cC_b, C_aA_c and A_bB_a. Since, for instance, segments B_1C_1 and B_cC_b are homothetic through

[3]These points are not in the current edition of [7].

A, the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic. See Figure 5. Their homothetic center is the point

$$P_5 = \left(-a^2S_A(V+W) + U(S_B+SW)(S_C+SV) : \cdots : \cdots\right).$$

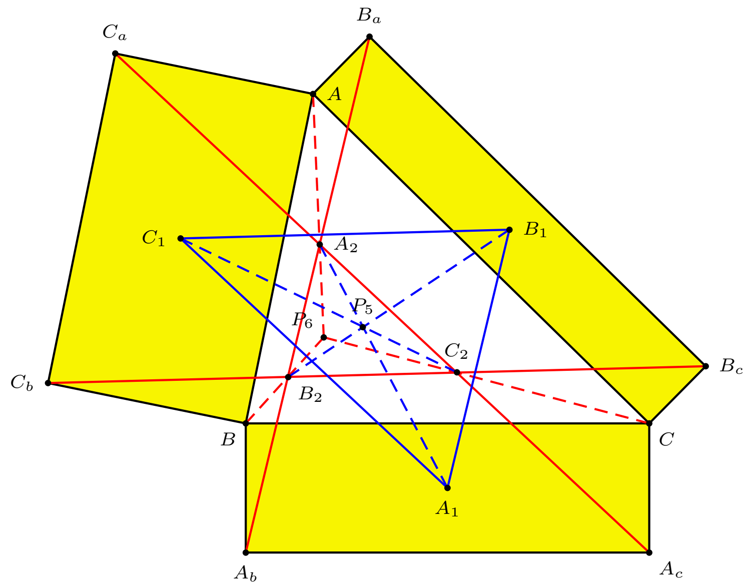

Figure 5

For the Pythagorean case with squares attached to triangles, *i.e.*, $U = V = W = 1$, Toshio Seimiya and Peter Woo [12] have proved the beautiful result that the areas Δ_1 and Δ_2 of $A_1B_1C_1$ and $A_2B_2C_2$ have geometric mean Δ. See Figure 5. We prove a more general result by computation using two fundamental area formulae.

Proposition 1. *For $i = 1, 2, 3$, let P_i be finite points with homogeneous barycentric coordinates $(x_i : y_i : z_i)$ with respect to triangle ABC. The oriented area of the triangle $P_1P_2P_3$ is*

$$\frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}{(x_1+y_1+z_1)(x_2+y_2+z_2)(x_3+y_3+z_3)} \cdot \Delta.$$

A proof of this proposition can be found in [1, 2].

Proposition 2. *For $i = 1, 2, 3$, let ℓ_i be a finite line with equation $p_ix+q_iy+r_iz = 0$. The oriented area of the triangle bounded by the three lines ℓ_1, ℓ_2, ℓ_3 is*

$$\frac{\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}^2}{D_1 \cdot D_2 \cdot D_3} \cdot \Delta,$$

where

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} p_1 & q_1 & r_1 \\ 1 & 1 & 1 \\ p_3 & q_3 & r_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ 1 & 1 & 1 \end{vmatrix}.$$

A proof of this proposition can be found in [5].

Theorem 3. $\frac{\Delta_1\Delta_2}{\Delta^2} = \frac{(U+V+W-UVW)^2}{4(UVW)^2}$.

Proof. The coordinates of A_1, B_1, C_1 are

$$\begin{aligned} A_1 =&(-a^2 : S_C + SU : S_B + SU), \\ B_1 =&(S_C + SV : -b^2 : S_A + SV), \\ C_1 =&(S_B + SW : S_A + SW : -c^2). \end{aligned}$$

By Proposition 1, the area of triangle $A_1B_1C_1$ is

$$\Delta_1 = \frac{S(U+V+W+UVW) + (a^2VW + b^2WU + c^2UV)}{4SUVW} \cdot \Delta. \tag{6}$$

The lines B_cC_b, C_aA_c, A_bB_a have equations

$$\begin{aligned} (S(1-VW) - S_A(V+W))x + (S+S_BV)y + (S+S_CW)z =&0, \\ (S+S_AU)x + (S(1-WU) - S_B(W+U))y + (S+S_CW)z =&0, \\ (S+S_AU)x + (S+S_BV)y + (S(1-UV) - S_C(U+V))z =&0. \end{aligned}$$

By Proposition 2, the area of the triangle bounded by these lines is

$$\Delta_2 = \frac{S(U+V+W-UVW)^2}{UVW(S(U+V+W+UVW) + (a^2VW+b^2WU+c^2UV))} \cdot \Delta. \tag{7}$$

From (6, 7), the result follows. □

Remarks. (1) The ratio of homothety is

$$\frac{-S(U+V+W-UVW)}{2(S(U+V+W+UVW) + (a^2VW+b^2WU+c^2UV))}.$$

(2) We record the coordinates of A_2 below. Those of B_2 and C_2 can be written down accordingly.

$$\begin{aligned} A_2 =&(-a^2((S+S_AU)(V+W) + SU(1-VW)) + (S_B+SW)(S_C+SV)U^2 \\ &: (S+S_AU)(SUV + S_C(U+V+W)) \\ &: (S+S_AU)(SUW + S_B(U+V+W))). \end{aligned}$$

From the coordinates of $A_2B_2C_2$ we see that this triangle is perspective to ABC at the point

$$P_6 = \left(\frac{1}{S_A(U+V+W)+SVW} : \cdots : \cdots\right).$$

4. Examples

4.1. *The similarity case.* If the rectangles are similar, $U = V = W = t$, then

$$P_6 = \left(\frac{1}{3S_A + St} : \frac{1}{3S_B + St} : \frac{1}{3S_C + St}\right)$$

traverses the Kiepert hyperbola. In the Pythagorean case, the homothetic center P_5 is the point

$$((S_B-S)(S_C-S)-4S_{BC} : (S_C-S)(S_A-S)-4S_{CA} : (S_A-S)(S_B-S)-4S_{AB}).$$

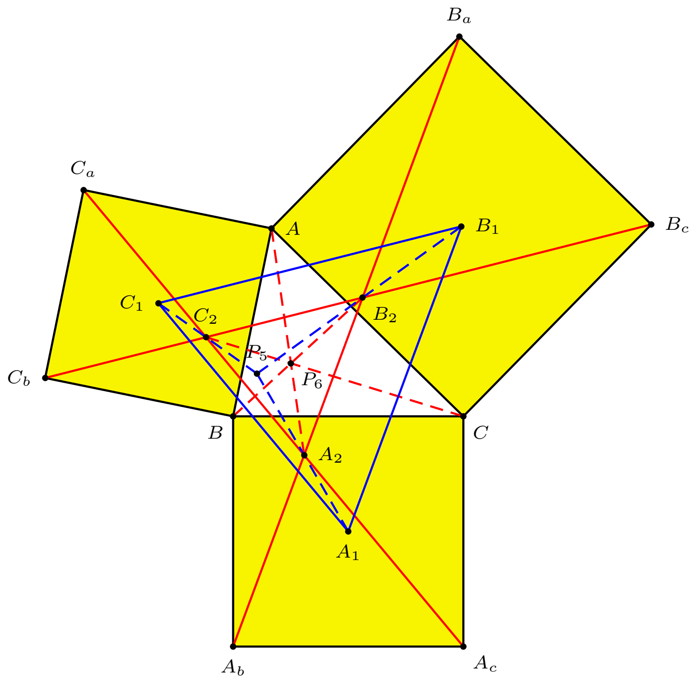

Figure 6

4.2. *The equiareal case.* For $(U, V, W) = (\frac{2a^2}{T}, \frac{2b^2}{T}, \frac{2c^2}{T})$, we have

$$P_6 = \left(\frac{1}{T(a^2 + b^2 + c^2)S_A + 2Sb^2c^2} : \cdots : \cdots\right).$$

This traverses the Jerabek hyperbola as T varies. When the rectangles have the same area as the triangle, the homothetic center P_5 is the point

$$(a^2((a^2 + 3b^2 + 3c^2)^2 - 4(4b^4 - b^2c^2 + 4c^4)) : \cdots : \cdots).$$

5. More homothetic triangles

Let $\mathcal{C}_A$, $\mathcal{C}_B$ and $\mathcal{C}_C$ be the circumcricles of rectangles BCA_cA_b, CAB_aB_c and ABC_bC_a respectively. See Figure 7. Since the circle $\mathcal{C}_A$ passes through B and C, its equation is of the form

$$a^2yz + b^2zx + c^2xy - px(x + y + z) = 0.$$

Since the same circle passes through A_b, we have $p = \frac{S_A U + S}{U} = S_A + \frac{S}{U}$. By the same method we derive the equations of the three circles:

$$a^2yz + b^2zx + c^2xy = (S_A + \frac{S}{U})x(x + y + z),$$
$$a^2yz + b^2zx + c^2xy = (S_B + \frac{S}{V})y(x + y + z),$$
$$a^2yz + b^2zx + c^2xy = (S_C + \frac{S}{W})z(x + y + z).$$

From these, the radical center of the three circles is the point

$$J = \left(\frac{1}{S_A + \frac{S}{U}} : \frac{1}{S_B + \frac{S}{V}} : \frac{1}{S_C + \frac{S}{W}}\right) = \left(\frac{U}{S_A U + S} : \frac{V}{S_B V + S} : \frac{W}{S_C W + S}\right).$$

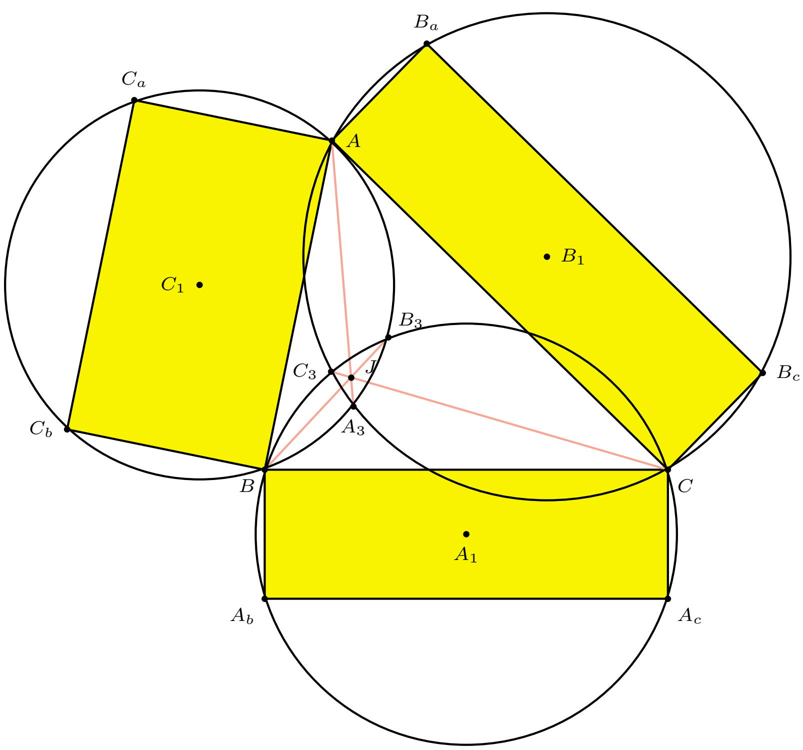

Figure 7

Note that the isogonal conjugate of J is the point

$$J^* = \left(a^2 S_A + S \cdot \frac{a^2}{U} : b^2 S_B + S \cdot \frac{b^2}{V} : c^2 S_C + S \cdot \frac{c^2}{W}\right).$$

It lies on the line joining O to P_2. In fact,

$$P_2 J^* : J^* O = 2S : au + bv + cw = P_2 P_1 : P_1 P_3.$$

The circles $\mathcal{C}_B$ and $\mathcal{C}_C$ meet at A and a second point A_3, which is the reflection of A in B_1C_1. See Figure 8. In homogeneous barycentric coordinates,

$$A_3 = \left(\frac{V+W}{S_A(V+W)-S(1-VW)} : \frac{V}{S_BV+S} : \frac{W}{S_CW+S}\right).$$

Similarly we have points B_3 and C_3. Clearly, the radical center J is the perspector of ABC and $A_3B_3C_3$.

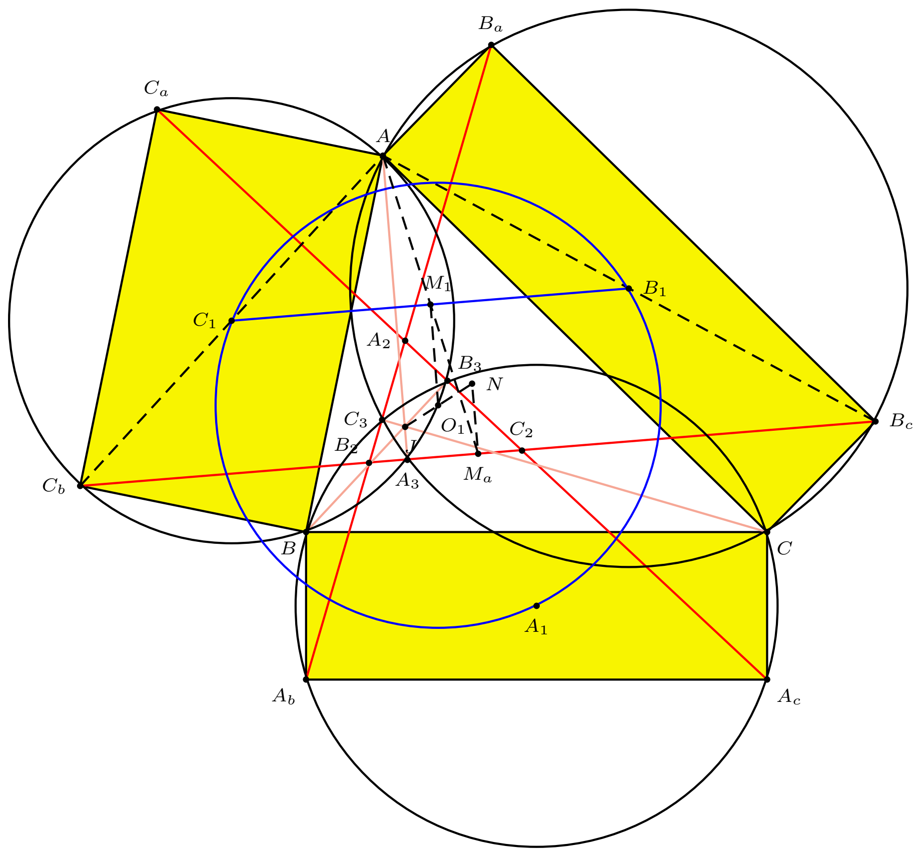

Figure 8

Proposition 4. *The triangles ABC and $A_2B_2C_2$ are orthologic. The perpendiculars from the vertices of one triangle to the corresponding lines of the other triangle concur at the point J.*

Proof. As C_1B_1 bisects AA_3, we see A_3 lies on B_cC_b and $AJ \perp B_cC_b$. Similarly, we have $BJ \perp C_aA_c$ and $CJ \perp A_bB_a$. The perpendiculars from A, B, C to the corresponding sides of $A_2B_2C_2$ concur at J.

On the other hand, the points B, C_3, B_3, C are concyclic and B_3C_3 is antiparallel to BC with respect to triangle JBC. The quadrilateral $JB_3A_2C_3$ is cyclic, with JA_2 as a diameter. It is known that every perpendicular to JA_2 is antiparallel to

B_3C_3 with respect to triangle JB_3C_3. Hence, $A_2J \perp BC$. Similarly, $B_2J \perp CA$ and $C_2J \perp AB$. $\square$

It is clear that the perpendiculars from A_3, B_3, C_3 to the corresponding sides of triangle $A_2B_2C_2$ intersect at J. Hence, the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are orthologic.

Proposition 5. *The perpendiculars from A_2, B_2, C_2 to the corresponding sides of $A_3B_3C_3$ meet at the reflection of J in the circumcenter O_3 of triangle $A_3B_3C_3$.*

Proof. Since triangle $A_3B_3C_3$ is the pedal triangle of J in $A_2B_2C_2$, and A_2J passes through the circumcenter of triangle $A_2B_3C_3$, the perpendicular from A_2 to B_3C_3 passes through the orthocenter of $A_2B_3C_3$ and is isogonal to A_2J in triangle $A_2B_2C_2$. This line therefore passes through the isogonal conjugate of J in $A_2B_2C_2$. We denote this point by $J^!$. Similarly, the perpendiculars from B_2, C_2 to the sides C_3A_3 and A_3B_3 pass through $J^!$. The circumcircle of $A_3B_3C_3$ is the pedal circle of J. Hence, its circumcenter O_3 is the midpoint of $JJ^!$. It follows that $J^!$ is the reflection of J in O_3. $\square$

Remark. The point J and the circumcenters O and O_3 of triangles ABC and $A_3B_3C_3$ are collinear. This is because $|JA \cdot JA_3| = |JB \cdot JB_3| = |JC \cdot JC_3|$, say, $= d^2$, and an inversion in the circle (J, d) transforms ABC into $A_3B_3C_3$ or its reflection in J.

Theorem 6. *The perpendicular bisectors of B_cC_b, C_aA_c, A_bB_a are concurrent at a point which is the reflection of J in the circumcenter O_1 of triangle $A_1B_1C_1$.*

Proof. Let M_1 and M_a be the midpoints of B_1C_1 and B_cC_b respectively. Note that M_1 is also the midpoint of AM_a. Also, let O_1 be the circumcenter of $A_1B_1C_1$, and the perpendicular bisector of B_cC_b meet JO_1 at N. See Figure 8. Consider the trapezium AM_aNJ. Since O_1M_1 is parallel to AJ, we conclude that O_1 is the midpoint of JN. Similarly the perpendicular bisectors of C_aA_c, A_bB_a pass through N, which is the reflection of J in O_1. $\square$

We record the coordinates of O_1:

$$\begin{aligned}
&((c^2U^2V - a^2VW(V+W) + b^2WU(W+U)\\
&\quad + UVW((S_A+3S_B)UV + (S_A+3S_C)UW))S\\
&\quad + c^2S_BU^2V^2 + b^2S_CU^2W^2 - a^4V^2W^2\\
&\quad + (S^2+S_{BC})U^2V^2W^2 + 4S^2U^2VW)\\
&: \cdots : \cdots)
\end{aligned}$$

In the Pythagorean case, the coordinates of O_1 are given in §2.1.

6. More triangles related to the attached rectangles

Write $U = \tan\alpha$, $V = \tan\beta$, and $W = \tan\gamma$ for angles α, β, γ in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$. The point A_4 for which the swing angles CBA_4 and BCA_4 are β and γ

respectively has coordinates

$$(-a^2 : S_C + S \cdot \cot\gamma : S_B + S \cdot \cot\beta) = \left(-a^2 : S_C + \frac{S}{W} : S_B + \frac{S}{V}\right).$$

It is clear that this point lies on the line AJ. See Figure 9. If B_4 and C_4 are analogously defined, the triangles $A_4B_4C_4$ and ABC are perspective at J.

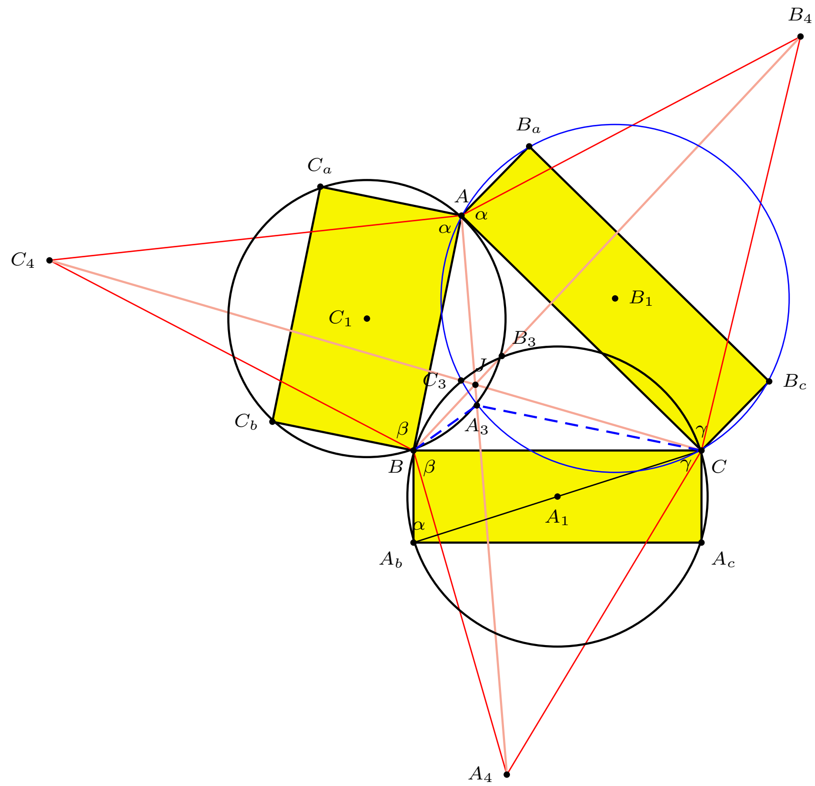

Figure 9

Note that A_3, B, A_4, C are concyclic since $\angle A_4BC = \beta = \angle AB_cV = \angle A_4A_3C$.

Let $d_1 = B_cC_b$, $d_2 = C_aA_c$, $d_3 = A_bB_a$, $d'_1 = AA_4$, $d'_2 = BB_4$, $d'_3 = CC_4$.

Proposition 7. *The ratios* $\frac{d_i}{d'_i}$, $i = 1, 2, 3$, *are independent of triangle* ABC. *More precisely,*

$$\frac{d_1}{d'_1} = \frac{1}{V} + \frac{1}{W}, \qquad \frac{d_2}{d'_2} = \frac{1}{W} + \frac{1}{U}, \qquad \frac{d_3}{d'_3} = \frac{1}{U} + \frac{1}{V}.$$

Proof. Since $AA_4 \perp C_bB_c$, the circumcircle of the cyclic quadrilateral A_3BA_4C meets C_bB_c besides A_3 at the antipode A_5 of A_4. See Figure 10. Let f, g, h denote, for vectors, the compositions of a rotation by $\frac{\pi}{2}$, and homotheties of ratios

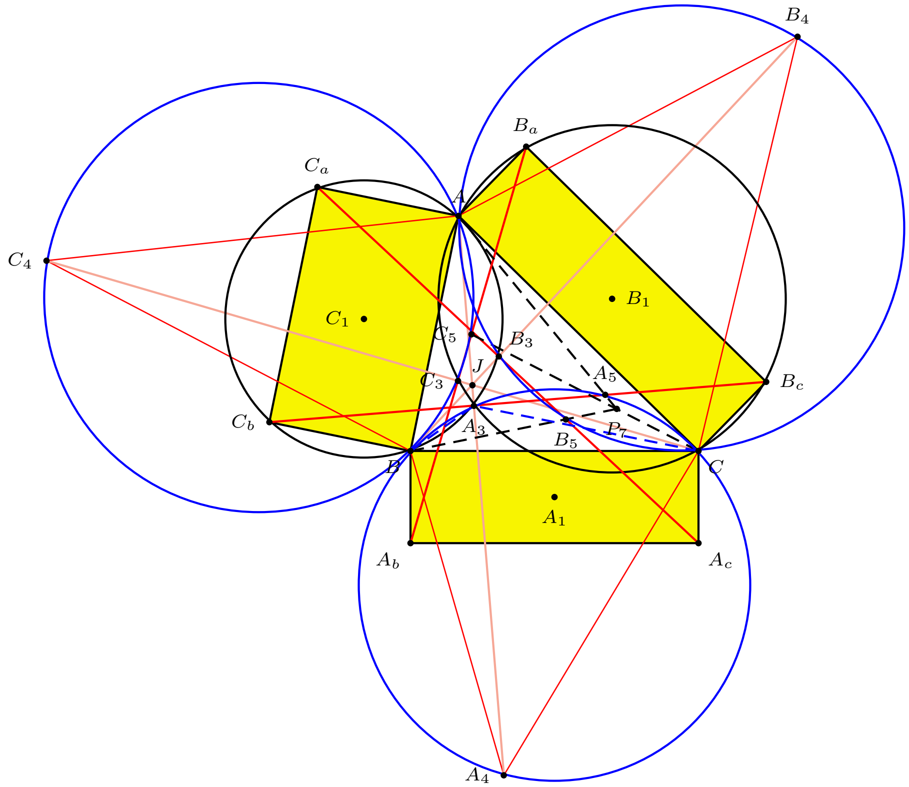

Figure 10

$\frac{1}{U}$, $\frac{1}{V}$, and $\frac{1}{W}$ respectively. Then

$$g(\overrightarrow{AA_4}) = g(\overrightarrow{AC}) + g(\overrightarrow{CA_4}) = \overrightarrow{CB_c} + \overrightarrow{A_5C} = \overrightarrow{A_5B_c},$$

and $\frac{A_5B_c}{AA_4} = \frac{1}{V}$. Similarly, $h(\overrightarrow{AA_4}) = \overrightarrow{C_bA_5}$, and $\frac{C_bA_5}{AA_4} = \frac{1}{W}$. It follows that $\frac{d_1}{d_1'} = \frac{1}{V} + \frac{1}{W}$. $\square$

The coordinates of A_5 can be seen immediately: Since A_4A_5 is a diameter of the circle (A_4BC), we see that $\angle BCA_5 = -\frac{\pi}{2} + \angle BCA_4$, and

$$A_5 = (-a^2 : S_C - SW : S_B - SV).$$

Similarly, we have the coordinates of B_5 and C_5. From these, it is clear that $A_5B_5C_5$ and ABC are perspective at

$$P_7 = \left(\frac{1}{S_A - SU} : \frac{1}{S_B - SV} : \frac{1}{S_C - SW}\right) = \left(\frac{1}{\cot A - U} : \frac{1}{\cot B - V} : \frac{1}{\cot C - W}\right).$$

For example, in the similarity case it is obvious from the above proof that the points A_5, B_5, C_5 are the midpoints of B_cC_b, C_aA_c, A_bB_a. Clearly in the Pythagorean case, the points A_4, B_4, C_4 coincide with A_1, B_1, C_1 respectively.

In this case, J is the Vecten point and from the above proof we have $d_1 = 2d'_1$, $d_2 = 2d'_2$, $d_3 = 2d'_3$ and $P_7 = X_{486}$.

7. Another interesting special case

If $\alpha + \beta + \gamma = \pi$, then $U + V + W = UVW$. From Theorem 3 we conclude that $\triangle_2 = 0$, and the points A_2, B_2, C_2, A_3, B_3, C_3 coincide with J, which now is the common point of the circumcircles of the three rectangles. Also, the points A_4, B_4, C_4 lie on the circles $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ respectively.

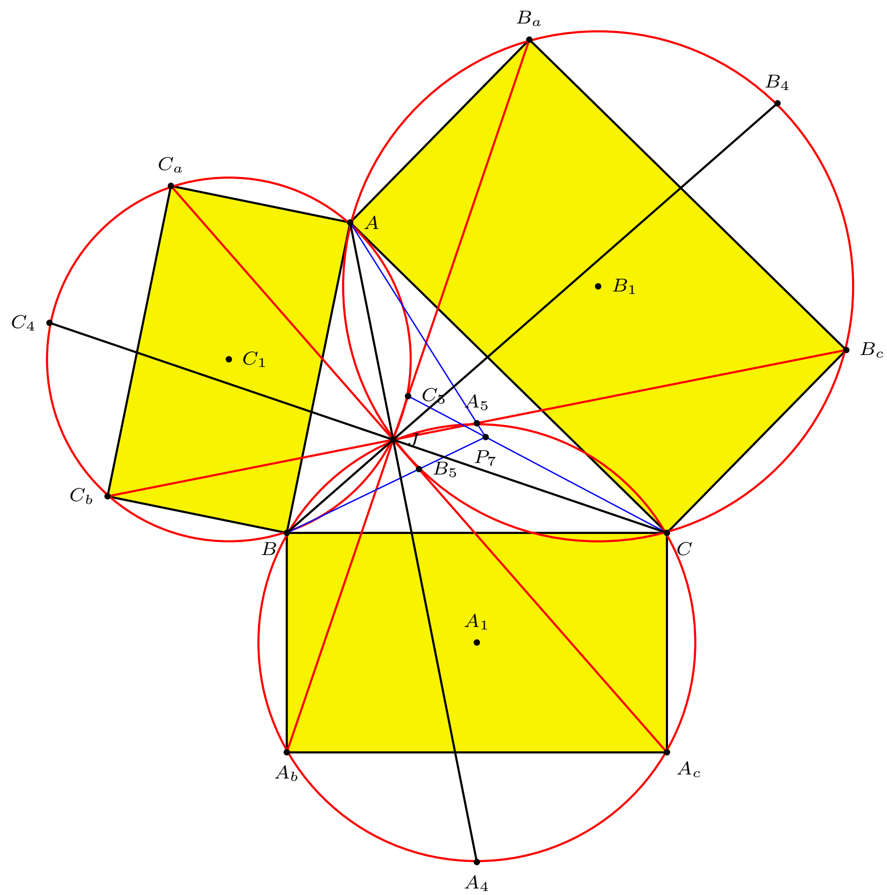

Figure 11

In Figure 11 we illustrate the case $\alpha = \beta = \gamma = \frac{\pi}{3}$. In this case, J is the Fermat point. The triangles BCA_4, CAB_4, ABC_4 are the Fermat equilateral triangles, and the angles of the lines AA_4, BB_4, CC_4, B_cC_b, C_aA_c, A_bB_a around J are $\frac{\pi}{6}$. The points A_5, B_5, C_5 are the mid points of B_cC_b, C_aA_c, A_bB_a. Also, $d'_1 = d'_2 = d'_3$, and $d_1 = d_2 = d_3 = \frac{2\sqrt{3}}{3}d'_1$. In this case, P_7 is the second Napoleon point, the point X_{18} in [7].

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A Generalization of the Lemoine Point

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Abstract. It is known that the Lemoine point K of a triangle in the Euclidean plane is the point of the plane where the sum of the squares of the distances d_1, d_2, and d_3 to the sides of the triangle takes its minimal value. There are several ways to generalize the Lemoine point. First, we can consider $n \geq 3$ lines u_1, ..., u_n instead of three in the Euclidean plane and search for the point which minimalizes the expression $d_1^2 + \cdots + d_n^2$, where d_i is the distance to the line u_i, $i = 1, \ldots, n$. Second, we can work in the Euclidean m-space R^m and consider n hyperplanes in R^m with $n \geq m + 1$. In this paper a combination of these two generalizations is presented.

1. Introduction

Let us start with a triangle $A_1A_2A_3$ in the Euclidean plane R^2 and suppose that its sides $a_1 = A_2A_3$, $a_2 = A_3A_1$, and $a_3 = A_1A_2$ have length l_1, l_2, and l_3, respectively. The easiest way to deal with the Lemoine point K of the triangle is to work with trilinear coordinates with regard to $A_1A_2A_3$ (also called normal coordinates). See [1, 5, 6]. These are homogeneous projective coordinates (x_1, x_2, x_3) such that A_1, A_2, A_3, and the incenter I of the triangle, have coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$, respectively. If (a_i^1, a_i^2) are the non-homogeneous coordinates (x, y) of the point A_i with respect to an orthonormal coordinate system in R^2, $i = 1, 2, 3$, then the relationship between homogeneous cartesian coordinates (x, y, z) and trilinear coordinates (x_1, x_2, x_3) is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_1a_1^1 & l_2a_2^1 & l_3a_3^1 \\ l_1a_1^2 & l_2a_2^2 & l_3a_3^2 \\ l_1 & l_2 & l_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

This follows from the fact that the position vector of the incenter I of $A_1A_2A_3$ is given by

$$\vec{r} = \frac{l_1\vec{r}_1 + l_2\vec{r}_2 + l_3\vec{r}_3}{l_1 + l_2 + l_3},$$

with $\vec{r}_i$ the position vector of A_i. Remark also that $z = 0$ corresponds with $l_1x_1 + l_2x_2 + l_3x_3 = 0$, which is the equation in trilinear coordinates of the line at infinity

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of R^2. If (x_1, x_2, x_3) are normal coordinates of any point P of R^2 with regard to $A_1A_2A_3$, then the so-called absolute normal coordinates of P are

$$(d_1, d_2, d_3) = \left(\frac{2Fx_1}{l_1x_1 + l_2x_2 + l_3x_3}, \frac{2Fx_2}{l_1x_1 + l_2x_2 + l_3x_3}, \frac{2Fx_3}{l_1x_1 + l_2x_2 + l_3x_3} \right),$$

where F is the area of $A_1A_2A_3$. It is well known that d_i is the relative distance from P to the side a_i of the triangle (d_i is positive or negative, according as P lies at the same side or opposite side as A_i, with regard to a_i).

Next, consider the locus of the points of R^2 for which $d_1^2 + d_2^2 + d_3^2 = k$, with k a given value. In trilinear coordinates this locus is given by

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - k(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0. \qquad (1)$$

For variable k, we get a pencil of homothetic ellipses (they all have the same points at infinity, the same asymptotes, the same center and the same axes), and the center of these ellipses is the Lemoine point K of the triangle $A_1A_2A_3$. A straightforward calculation gives that (l_1, l_2, l_3) are trilinear coordinates of K and the minimal value of $d_1^2 + d_2^2 + d_3^2$ reached at K is $\dfrac{4F^2}{l_1^2 + l_2^2 + l_3^2}$.

Remark also that K is the singular point of the degenerate ellipse of the pencil (1) corresponding with $k = \frac{1}{l_1^2+l_2^2+l_3^2}$ (set $\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = 0$).

More properties and constructions of the Lemoine point K can be found in [1]. And in [3] and [7] constructions for the axes of the ellipses (1) are given, while [7] contains a lot of generalizations.

Next, the foregoing can immediately be generalized to higher dimensions as follows. Consider in the Euclidean m-space R^m $(m \geq 2)$, $m+1$ hyperplanes not through a point and no two parallel; this determines an m-simplex with vertices $A_1, \ldots, A_{m+1}$. Let us denote the $(m-1)$-dimensional volume of the "face" a_i with vertices $A_1, \ldots, \hat{A}_i, \ldots, A_{m+1}$ by F_i, $i = 1, \ldots, m+1$. Then the position vector of the incenter I of $A_1A_2 \ldots A_{m+1}$ (= center of the hypersphere of R^m inscribed in $A_1 \ldots A_{m+1}$) is given by

$$\vec{r} = \frac{F_1\vec{r}_1 + F_2\vec{r}_2 + \cdots + F_{m+1}\vec{r}_{m+1}}{F_1 + F_2 + \cdots + F_{m+1}},$$

where $\vec{r}_i$ is the position vector of A_i, and normal coordinates $(x_1, \ldots, x_{m+1})$ with respect to $A_1 \ldots, A_{m+1}$ are homogeneous projective coordinates such that $A_1, \ldots, A_{m+1}$, and I, have coordinates $(1, 0, \ldots, 0)$, …, $(0, \ldots, 0, 1)$, and $(1, 1, \ldots, 1)$, respectively. If $(a_i^1, a_i^2, \ldots, a_i^m)$ are cartesian coordinates (with respect to an orthonormal coordinate system) of A_i, $i = 1, \ldots, m+1$, the coordinate transformation between homogeneous cartesian coordinates $(z_1, \ldots, z_{m+1})$ and normal coordinates $(x_1, \ldots, x_{m+1})$ with respect to $A_1 \ldots A_{m+1}$ is given by

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \\ z_{m+1} \end{pmatrix} = \begin{pmatrix} F_1 a_1^1 & F_2 a_2^1 & \dots & F_{m+1} a_{m+1}^1 \\ F_1 a_1^2 & F_2 a_2^2 & \dots & F_{m+1} a_{m+1}^2 \\ \vdots & \vdots & & \vdots \\ F_1 a_1^m & F_2 a_2^m & \dots & F_{m+1} a_{m+1}^m \\ F_1 & F_2 & \dots & F_{m+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ x_{m+1} \end{pmatrix}.$$

In normal coordinates the hyperplane at infinity of R^m has the equation $F_1 x_1 + \cdots + F_{m+1} x_{m+1} = 0$. Absolute normal coordinates of a point P of R^m with respect to $A_1, A_2, \dots, A_{m+1}$ are $d_i = \dfrac{mFx_i}{F_1 x_1 + \cdots + F_{m+1} x_{m+1}}$, $i = 1, \dots, m+1$, where F is the m-dimensional volume of $A_1 A_2 \dots A_{m+1}$ and d_i is the relative distance from P to the face a_i (d_i is positive or negative, according as P lies at the same side or at the opposite face as A_i, with regard to a_i). Remark that $F_1 d_1 + \cdots + F_{m+1} d_{m+1} = mF$.

The locus of the points of R^m for which $d_1^2 + \cdots + d_{m+1}^2 = k$ now determines a pencil of hyperquadrics (hyperellipsoids) with equation

$$x_1^2 + x_2^2 + \cdots + x_{m+1}^2 - k(F_1 x_1 + \cdots + F_{m+1} x_{m+1})^2 = 0 \qquad (2)$$

and all these (homothethic) hyperellipsoids have the same axes, the same points at infinity and the same center K, which we call the Lemoine point of $A_1 \dots A_{m+1}$ and which obviously has normal coordinates $(F_1, F_2, \dots, F_{m+1})$. The minimal value of $d_1^2 + \cdots + d_{m+1}^2$, reached at K is given by $\dfrac{m^2 F^2}{F_1^2 + \cdots + F_{m+1}^2}$. Remark that K is the singular point of the singular hyperquadric (hypercone) corresponding in the pencil (2) with the value $k = \frac{1}{F_1^2 + \cdots + F_{m+1}^2}$.

Remark. Some characterizations and constructions of the Lemoine point K of a triangle in the plane R^2 are no longer valid in higher dimensions. For instance, K is the perspective center of the triangle $A_1 A_2 A_3$ and the triangle $A_1' A_2' A_3'$ whose sides are the tangents of the circumscribed circle of $A_1 A_2 A_3$ at A_1, A_2, and A_3 (in trilinear coordinates the circumcircle has equation $l_1 x_2 x_3 + l_2 x_3 x_1 + l_3 x_1 x_2 = 0$). This construction is, in general, not correct in R^3 : a tetrahedron $A_1 A_2 A_3 A_4$ and its so called tangential tetrahedron, which is the tetrahedron $A_1' A_2' A_3' A_4'$ consisting of the tangent planes of the circumscribed sphere of $A_1 A_2 A_3 A_4$ at A_1, A_2, A_3, and A_4, are, in general, not perspective. If they are perspective, the tetrahedron is a special one, an *isodynamic* tetrahedron in which the three products of the three pairs of opposite edges are equal. The lines joining the vertices of an isodynamic tetrahedron to the Lemoine points of the respective opposite faces have a point in common and this common point is the perspective center of the isodynamic tetrahedron and its tangential tetrahedron (see [2]). It is not difficult to prove that this point of an isodynamic tetrahedron coincides with the Lemoine point K of the tetrahedron obtained with our definition of “Lemoine point”.

2. The main theorem

First we give some notations. Consider n hyperplanes, denoted by $u_1, \ldots, u_n$ in the Euclidean space R^m ($m \geq 2, n \geq m+1$), in general position (this means : no two are parallel and no $m+1$ are concurrent). The "figure" consisting of these n hyperplanes is called an n-hyperface (examples: for $m=2, n=3$ it determines a triangle in R^2, for $m=2$, $n=4$ it is an quadrilateral in R^2, and for $m=3$, $n=4$ it is a tetrahedron in R^3). The Lemoine point K of this n-hyperface is, by definition, the point of R^m for which the sum of the squares of the distances to the n hyperplanes $u_1, \ldots, u_n$ is minimal. The uniqueness of K follows from the proof of the next theorem.

Next, K^i is the Lemoine point of the $(n-1)$-hyperface $u_1u_2 \ldots \hat{u}_i \ldots u_n$, $i = 1, \ldots, n$. And $K^{rs} = K^{sr}$ is the Lemoine point of the $(n-2)$-hyperface $u_1u_2 \ldots \hat{u}_r \ldots \hat{u}_s \ldots u_n$, with $r, s = 1, \ldots, n$, $r \neq s$ (only defined if $n > m+1$).

Now, for an $(m+1)$-hyperface or m-simplex in R^m (a triangle in R^2, a tetrahedron in $R^3, \ldots$) we know the position (the normal coordinates) of the Lemoine point (see §1). The following theorem gives us a construction for the Lemoine point K of a general n-hyperface in R^m ($m \geq 2$ and $n > m+1$):

Theorem 1. *Working with an n-hyperface in R^m, we have, with the notations given above that $K^iK \cap u_j = K^jK^{ji} \cap u_j$, $i, j = 1, \ldots, n$ and $n > m+1$.*

Proof. In this proof, we work with cartesian coordinates $(x_1, \ldots, x_m)$ or homogeneous $(x_1, \ldots, x_{m+1})$ with respect to an orthonormal coordinate system in R^m. Suppose that the hyperplane u_r has equation $a_r^1x_1 + a_r^2x_2 + \cdots + a_r^m x_m + a_r^{m+1} = 0$, with $(a_r^1)^2 + (a_r^2)^2 + \cdots + (a_r^m)^2 = 1, r = 1, \ldots, n$. Then the Lemoine point K of the n-hyperface $u_1u_2 \ldots u_n$ is the center of the hyperquadrics of the pencil with equation

$$\mathcal{F}(x_1, \ldots, x_m) = \sum_{r=1}^{n} (a_r^1 x_1 + a_r^2 x_2 + \cdots + a_r^m x_m + a_r^{m+1})^2 - k = 0, \qquad (3)$$

where k is a parameter. Indeed, since the coordinates of K minimize the expression $\sum_{r=1}^{n} \left(a_r^1 x_r + \cdots + a_r^{m+1}\right)^2$, they are a (the) solution of $\frac{\partial \mathcal{F}}{\partial x_1} = \frac{\partial \mathcal{F}}{\partial x_2} = \cdots = \frac{\partial \mathcal{F}}{\partial x_m} = 0$. In homogeneous coordinates, (3) becomes

$$\mathcal{F}(x_1, \ldots, x_{m+1}) = \sum_{r=1}^{n} (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0. \qquad (4)$$

Next, the Lemoine point K^i of $u_1u_2 \ldots \hat{u}_i \ldots u_n$ is the center of the hyperquadrics of the pencil given by (we use the same notation k for the parameter)

$$\mathcal{F}^i(x_1, \ldots, x_{m+1}) = \sum_{\substack{r=1 \\ r \neq i}}^{n} (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0. \qquad (5)$$

The diameter of the hyperquadrics (5), conjugate with respect to the direction of the ith hyperplane u_i has the equations (consider the polar hyperplanes of the

$m-1$ points at infinity with coordinates $(a_i^2, -a_i^1, 0, \ldots, 0)$, $(a_i^3, 0, -a_i^1, 0, \ldots, 0)$, $(a_i^4, 0, 0, -a_i^1, 0, \ldots, 0)$, …, $(a_i^m, 0, \ldots, 0, -a_i^1, 0)$ of the hyperplane u_i):

$$\begin{cases} \sum_{r=1}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^2 - a_r^2 a_i^1) = 0, \\ \sum_{r=1}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^3 - a_r^3 a_i^1) = 0, \\ \quad\vdots \\ \sum_{r=1}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^m - a_r^m a_i^1) = 0. \end{cases} \tag{6}$$

But the first side of each of these equations becomes zero for $r = i$, and thus (6) gives us also the conjugate diameter with respect to the hyperplane u_i of the hyperquadrics of the pencil (5). It follows that (6) determines the line KK^i.

Next, the Lemoine point K^j is the center of the hyperquadrics of the pencil

$$\mathcal{F}^j(x_1, \ldots, x_{m+1}) = \sum_{\substack{r=1 \\ r\neq j}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0, \tag{7}$$

and K^{ji} is the center of the hyperquadrics:

$$\mathcal{F}^{ji}(x_1, \ldots, x_{m+1}) = \sum_{\substack{r=1 \\ r\neq j,i}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})^2 - k x_{m+1}^2 = 0. \tag{8}$$

The diameter of the hyperquadrics (7), conjugate with respect to the direction of u_i is given by

$$\begin{cases} \sum_{\substack{r=1 \\ r\neq j}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^2 - a_r^2 a_i^1) = 0 \\ \quad\vdots \\ \sum_{\substack{r=1 \\ r\neq j}}^n (a_r^1 x_1 + \cdots + a_r^{m+1} x_{m+1})(a_r^1 a_i^m - a_r^m a_i^1) = 0. \end{cases} \tag{9}$$

And this gives us also the diameter of the hyperquadrics (8) conjugate with regard to the direction of u_i; in other words, (9) determines the line $K^j K^{ji}$.
Finally, the coordinates of the point $K^i K \cap u_j$ are the solutions of the linear system

$$\begin{cases} (6) \\ a_j^1 x_1 + \cdots + a_j^{m+1} x_{m+1} = 0, \end{cases}$$

while the point $K^j K^{ji} \cap u_j$ is given by

$$\begin{cases} (9) \\ a_j^1 x_1 + \cdots + a_j^{m+1} x_{m+1} = 0. \end{cases}$$

It is obvious that this gives the same point and the proof is complete. □

3. Applications

3.1. Let us first consider the easiest example for trying out our construction: the case where $m = 2$ and $n = 4$, or four lines u_1, u_2, u_3, u_4 in general position (they form a quadrilateral) in R^2. Using orthonormal coordinates (x, y, z) in R^2, the homogeneous equation of u_r is $a_r x + b_r y + c_r z = 0$ with $a_r^2 + b_r^2 = 1$,

$r = 1,2,3,4$. Where lies the Lemoine point K of the quadrilateral $u_1u_2u_3u_4$? For instance K^1 is the Lemoine point of the triangle with sides (lines) u_2, u_3, u_4; K^2 of the triangle with sides u_1, u_3, u_4, and so on ... . We may assume that we can construct the Lemoine point of a triangle. But which point is, for instance, the point K^{12} : it is the Lemoine point of the 2-side u_3u_4, *i.e.*, it is the point $u_3 \cap u_4$.

Let us denote the six vertices of the quadrilateral as follows : $u_1 \cap u_2 = C, u_2 \cap u_3 = A, u_3 \cap u_4 = F, u_1 \cap u_4 = D, u_2 \cap u_4 = E$, and $u_1 \cap u_3 = B$, then $K^{12} = K^{21} = F$, $K^{23} = D, K^{34} = C, K^{14} = A, K^{24} = B$, and $K^{13} = E$. Now, from $K^iK \cap u_j = K^jK^{ji} \cap u_j$, we find, for instance for $i = 1$ and $j = 2$:

$$K^1K \cap u_2 = K^2K^{21} \cap u_2 = K^2F \cap u_2$$

and for $i = 2$ and $j = 1$: $K^2K \cap u_1 = K^1K^{12} \cap u_1 = K^1F \cap u_1$, with K^1 (K^2, resp.) the Lemoine point of the triangle AFE (of the triangle BFD, resp.). This allows us to construct the point K.
In particular, we can construct the diameters KK^1, KK^2, KK^3, and KK^4 of the ellipses of the pencil $\sum_{r=1}^{4}(a_rx + b_ry + c_rz)^2 = kz^2$, which are conjugate to the directions of the lines u_1, u_2, u_3, and u_4, respectively. In other words, we have four pairs of conjugate diameters of these ellipses : (KK^i, KI^i_∞), where I^i_∞ is the point at infinity of the line $u_i, i = 1, \ldots, 4$. From this, we can construct the axes of the conics of this bundle (in fact, two pairs of conjugate diameters are sufficient): consider any circle $\mathcal{C}$ through K and project the involution of conjugate diameters onto $\mathcal{C}$; if S is the center of this involution on $\mathcal{C}$ and if the diameter of $\mathcal{C}$ through S intersects $\mathcal{C}$ at the points S_1 and S_2, then KS_1 and KS_2 are the axes.

In the case of a triangle in R^2, constructions of the common axes of the ellipses determined by $d_1^2 + d_2^2 + d_3^2 = k$ with center the Lemoine point of the triangle, are given in [3] and [7]. In [3], J. Bilo proved that the axes are the perpendicular lines through K on the Simson lines of the common points of the Euler line and the circumscribed circle of the triangle. And in [7], we proved that these axes are the orthogonal lines through K which cut the sides of the triangle in pairs of points whose midpoints are three collinear points. Moreover [7] contains a lot of generalizations for pencils whose conics have any point P of the plane as common center and whose common axes are constructed in the same way.

3.2. In the case $m = 2$ and $n \geq 4$, we can construct the n diameters $KK^1, \ldots, KK^n$ of the ellipses $\sum_{r=1}^{n}(a_rx + b_ry + c_rz)^2 = kz^2$ which are conjugate to the directions of the n lines $u_1, \ldots, u_n$.

3.3. The easiest example in space is the case where $m = 3$ and $n = 5$, or five planes in R^3. Assume that the planes have equations $a_rx + b_ry + c_rz + d_ru = 0$, with $a_r^2 + b_r^2 + c_r^2 = 1, r = 1,2,\ldots,5$. We look for the Lemoine point K of the "5-plane" $u_1u_2u_3u_4u_5$ in R^3 and assume that we know the position of the Lemoine point of any tetrahedron in R^3 (we know its normal coordinates). The points $K^1, \ldots, K^5$ are the Lemoine points of the tetrahedra $u_2u_3u_4u_5, \ldots, u_1u_2u_3u_4$, respectively. And, for instance K^{12} is the Lemoine point of the "3-plane" $u_3u_4u_5$, *i.e.*, it is the common point of these three planes u_3, u_4, and u_5. Now, for instance from

$$K^1K \cap u_2 = K^2K^{21} \cap u_2 \quad \text{and} \quad K^2K \cap u_1 = K^1K^{12} \cap u_1,$$

we can construct the lines K^1K and K^2K, and thus the point K. In fact, we can construct the diameters $KK^1, \dots, KK^5$ conjugate to the plane directions of $u_1, \dots, u_5$, respectively, of the quadrics with center K of the pencil given by $d_1^2 + \cdots + d_5^2 = k$ or

$$\sum_{r=1}^{5}(a_r x + b_r y + c_r z + d_r u)^2 = ku^2.$$

Finally, the construction of the point K in the general case $n > m+1$, $m \geq 2$ is obvious.

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The Parasix Configuration and Orthocorrespondence

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Abstract. We introduce the parasix configuration, which consists of two congruent triangles. The conditions of these triangles to be orthologic with ABC or a circumcevian triangle, to form a cyclic hexagon, to be equilateral or to be degenerate reveal a relation with orthocorrespondence, as defined in [1].

1. The parasix configuration

Consider a triangle ABC of reference with finite points P and Q not on its sidelines. Clark Kimberling [2, §§9.7,8] has drawn attention to configurations defined by six triangles. As an example of such configurations we may create six triangles using the lines ℓ_a, ℓ_b and ℓ_c through Q parallel to sides a, b and c respectively. The triples of lines (ℓ_a, b, c), (a, ℓ_b, c) and (a, b, ℓ_c) bound three triangles which we refer to as the *great paratriple*. Figure 1a shows the A-triangle of the great paratriple. On the other hand, the triples (a, ℓ_b, ℓ_c), (ℓ_a, b, ℓ_c) and (ℓ_a, ℓ_b, c) bound three triangles which we refer to as the *small paratriple*. See Figure 1b.

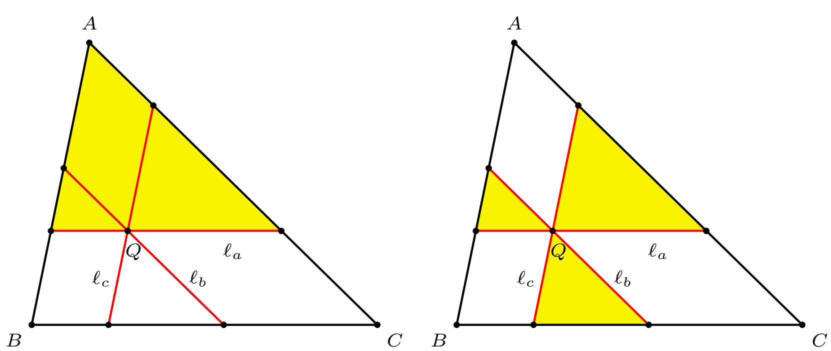

Figure 1a Figure 1b

Clearly these six triangles are all homothetic to ABC, and it is very easy to find the homothetic images of P in these triangles, A_g in the A-triangle bounded by (ℓ_a, b, c) in the great paratriple, and A_s in the A-triangle bounded by (a, ℓ_b, ℓ_c) in the small paratriple; similarly for B_g, C_g, B_s, C_s. These six points form the *parasix configuration of P with respect to Q*, or shortly $\mathsf{Parasix}(P, Q)$. See Figure 2. If in homogeneous barycentric coordinates with reference to ABC, $P = (u : v : w)$ and $Q = (f : g : h)$, then these are the points

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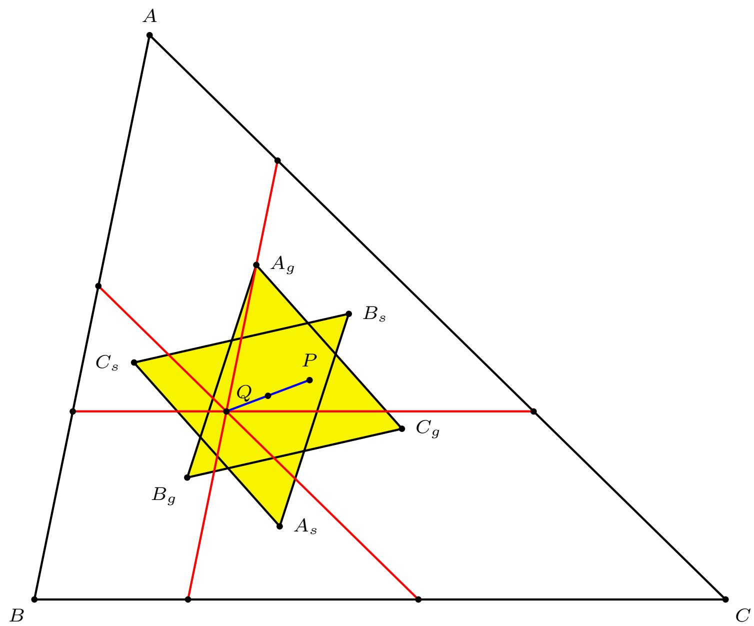

Figure 2. Parasix(P,Q)

$$
\begin{aligned}
A_g =&(u(f+g+h)+f(v+w):v(g+h):w(g+h)),\\
B_g =&(u(f+h):g(u+w)+v(f+g+h):w(f+h)),\\
C_g =&(u(f+g):v(f+g):h(u+v)+w(f+g+h));\\
A_s =&(uf:g(u+w)+v(f+g):h(u+v)+w(f+h)),\\
B_s =&(u(f+g)+f(v+w):vg:h(u+v)+w(g+h)),\\
C_s =&(u(f+h)+f(v+w):g(u+w)+v(g+h):wh).
\end{aligned} \tag{1}
$$

Proposition 1. (1) *Triangles $A_gB_gC_g$ and $A_sB_sC_s$ are symmetric about the midpoint of segment PQ.*
(2) *The six points of a parasix configuration lie on a central conic.*
(3) *The centroids of triangles $A_gB_gC_g$ and $A_sB_sC_s$ trisect the segment PQ.*

Proof. It is clear from the coordinates given above that the segments A_gA_s, B_gB_s, C_gC_s, PQ have a common midpoint

$$(f(u+v+w)+u(f+g+h):\cdots:\cdots).$$

The six points therefore lie on a conic with this common midpoint as center. For (3), it is enough to note that the centroids G_g and G_s of $A_gB_gC_g$ and $A_sB_sC_s$ are the points

$$
\begin{aligned}
G_g =&(2u(f+g+h)+f(u+v+w):\cdots:\cdots),\\
G_s =&(u(f+g+h)+2f(u+v+w):\cdots:\cdots).
\end{aligned}
$$

It follows that vectors $\overrightarrow{PG_g}=\frac{1}{3}\,\overrightarrow{PQ}$ and $\overrightarrow{PG_s}=\frac{2}{3}\,\overrightarrow{PQ}$. $\square$

While $\mathsf{Parasix}(P, Q)$ consists of the two triangles $A_gB_gC_g$ and $A_sB_sC_s$, we write $\widetilde{A}_g\widetilde{B}_g\widetilde{C}_g$ and $\widetilde{A}_s\widetilde{B}_s\widetilde{C}_s$ for the two corresponding triangles of $\mathsf{Parasix}(Q, P)$. From (1) we easily derive their coordinates by interchanging the roles of f, g, h, and u, v, w. Note that $\widetilde{G}_s = G_g$ and $\widetilde{G}_g = G_s$.

Let P_A and Q_A be the the points where AP and AQ meet BC respectively, and let $AP : PP_A = t_P : 1 - t_P$ while $AQ : QQ_A = t_Q : 1 - t_Q$. Then it is easy to see that

$$AA_g : A_gP_A = A\widetilde{A}_g : \widetilde{A}_gQ_A = t_Pt_Q : 1 - t_Pt_Q$$

so that the line $A_g\widetilde{A}_g$ is parallel to BC. By Proposition 1, $A_s\widetilde{A_s}$ is also parallel to BC.

Proposition 2. (a) *The lines $A_g\widetilde{A}_g$, $B_g\widetilde{B}_g$ and $C_g\widetilde{C}_g$ bound a triangle homothetic to ABC. The center of homothety is the point*

$$(f(u + v + w) + u(g + h) : g(u + v + w) + v(h + f) : h(u + v + w) + w(f + g)).$$

The ratio of homothety is

$$-\frac{fu + gv + hw}{(f + g + h)(u + v + w)}.$$

(b) *The lines $A_s\widetilde{A}_s$, $B_s\widetilde{B}_s$ and $C_s\widetilde{C}_s$ bound a triangle homothetic to ABC with center of homothety $(uf : vg : wh)$* [1] *The ratio of homothety is*

$$1 - \frac{fu + gv + hw}{(f + g + h)(u + v + w)}.$$

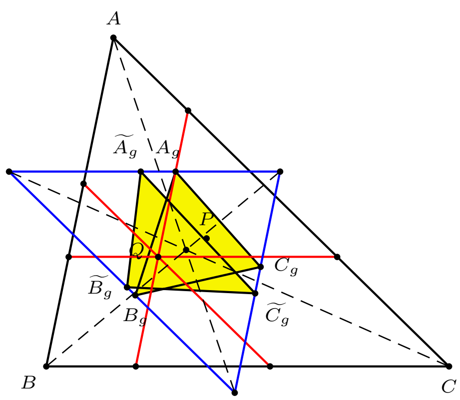

Figure 3a

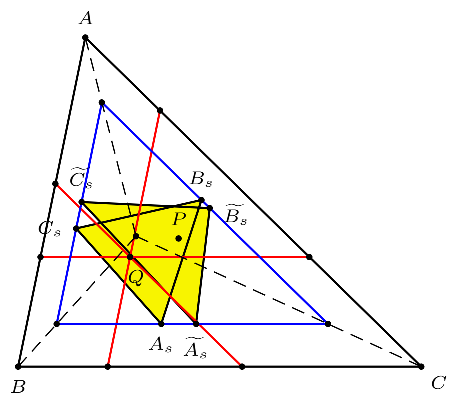

Figure 3b

[1] This point is called the barycentric product of P and Q. Another construction was given by P. Yiu in [4]. These homothetic centers are collinear with the midpoint of PQ.

2. Parasix loci

We present a few line and conic loci associated with parasix configurations. For $P = (u : v : w)$, we denote by

(i) $\mathcal{L}_P$ the trilinear polar of P, which has equation

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0;$$

(ii) $\mathcal{C}_P$ the circumconic with perspector P, which has equation

$$\frac{u}{x} + \frac{v}{y} + \frac{w}{z} = 0.$$

2.1. *Area of parasix triangles.* The parasix triangles $A_gB_gC_g$ and $A_sB_sC_s$ have a common area

$$\frac{ghu + hfv + fgw}{(f + g + h)^2(u + v + w)}. \tag{2}$$

Proposition 3. (a) *For a given Q, the locus of P for which the triangles $A_gB_gC_g$ and $A_sB_sC_s$ have a fixed (signed) area is a line parallel to $\mathcal{L}_P$.*

(b) *For a given P, the locus of Q for which the triangles $A_gB_gC_g$ and $A_sB_sC_s$ have a fixed (signed) area is a conic homothetic to $\mathcal{C}_P$ at its center.*

In particular, the parasix triangles degenerate into two parallel lines if and only if

$$\frac{u}{f} + \frac{v}{g} + \frac{w}{h} = 0. \tag{$*$}$$

This condition can be construed in two ways: $P \in \mathcal{L}_Q$, or equivalently, $P \in \mathcal{C}_P$. See §6.

2.2. *Perspectivity with the pedal triangle.*

Proposition 4. (a) *Given P, the locus of Q so that $A_sB_sC_s$ is perspective to the pedal triangle of Q is the line*[2]

$$\sum_{\text{cyclic}} S_A(S_Bv - S_Cw)(-uS_A + vS_B + wS_C)x = 0.$$

This line passes through the orthocenter H and the point

$$\left(\frac{1}{S_A(-uS_A + vS_B + wS_C)} : \cdots : \cdots\right),$$

which can be constructed as the perspector of ABC and the cevian triangle of P in the orthic triangle.

[2]Here we adopt J.H. Conway's notation by writing S for *twice* of the area of triangle ABC and

$$S_A = S{\cdot}\cot A = \frac{b^2 + c^2 - c^2}{2},\ S_B = S{\cdot}\cot B = \frac{c^2 + a^2 - b^2}{2},\ S_C = S{\cdot}\cot C = \frac{a^2 + b^2 - c^2}{2}.$$

These satisfy $S_{AB} + S_{BC} + S_{CA} = S^2$. The expressions S_{AB}, S_{BC}, S_{CA} stand for S_AS_B, S_BS_C, S_CS_A respectively.

2.3. *Parallelogy.* A triangle is said to be parallelogic to a second triangle if the lines through the vertices of the triangle parallel to the corresponding opposite sides of the second triangle are concurrent.

Proposition 5. (a) *Given $P = (u : v : w)$, the locus of Q for which ABC is parallelogic to $A_gB_gC_g$ (respectively $A_sB_sC_s$) is the line $(v+w)x+(w+u)y+(u+v)z = 0$, which can be constructed as the trilinear polar of the isotomic conjugate of the complement of P.*

(b) *Given $Q = (f : g : h)$, the locus of P for which ABC is parallelogic to $A_gB_gC_g$ (respectively $A_sB_sC_s$) is the line $(g+h)x+(h+f)y+(f+g)z = 0$, which can be constructed as the trilinear polar of the isotomic conjugate of the complement of Q.*

2.4. *Perspectivity with ABC.* Clearly $A_gB_gC_g$ is perspective to ABC at P. The perspectrix is the line $gh(g+h)x+fh(f+h)y+fg(f+g)z = 0$, parallel to the trilinear polar of Q. Given P, the locus of Q such that $A_sB_sC_s$ is perspective to ABC is the cubic

$$(v+w)x(wy^2-vz^2)+(u+w)y(uz^2-wx^2)+(u+v)z(vx^2-uy^2) = 0,$$

which is the isocubic with pivot $(v+w : w+u : u+v)$ and pole P. For $P = K$, the symmedian point, this is the isogonal cubic with pivot $X_{141} = (b^2+c^2 : c^2+a^2 : a^2+b^2)$.

3. Orthology

Some interesting loci associated with the orthology of triangles attracted our attention because of their connection with the orthocorrespondence defined in [1]. We recall that two triangles are orthologic if the perpendiculars from the vertices of one triangle to the opposite sides of the corresponding vertices of the other triangle are concurrent.

First, consider the locus of Q, given P, such that the triangles $A_gB_gC_g$ and $A_sB_sC_s$ are orthologic to ABC. We can find this locus by simple calculation since this is also the locus such that $A_gB_gC_g$ is perspective to the triangle of the infinite points of the altitudes, with coordinates

$$H_A^\infty = (-a^2, S_C, S_B), \quad H_B^\infty = (S_C, -b^2, S_A), \quad H_C^\infty = (S_B, S_A, -c^2).$$

The lines $A_gH_A^\infty$, $V_gH_B^\infty$ and $C_gH_C^\infty$ concur if and only if Q lies on the line

$$(S_Bv-S_Cw)x+(S_Cw-S_Au)y+(S_Au-S_Bv)z = 0, \tag{3}$$

which is the line through the centroid G and the orthocorrespondent of P, namely, the point [3]

$$P^\perp = (u(-S_Au+S_Bv+S_Cw)+a^2vw : \cdots : \cdots).$$

The line (3) is the orthocorrespondent of the line HP. See [1, §2.4].

[3] The lines perpendicular at P to AP, BP, CP intersect the respective sidelines at three collinear points. The orthocorrespondent of P is the trilinear pole $P^\perp$ of the line containing these three intersections.

For the second locus problem, we let Q be given, and ask for the locus of P such that the triangles $A_gB_gC_g$ and $A_sB_sC_s$ are orthologic to ABC. The computations are similar, and again we find a line as the locus:

$$S_A(g-h)x + S_B(h-f)y + S_C(f-g)z = 0.$$

This is the line through H, and the two anti-orthocorrespondents of Q. See [1, Figure 2]. It is the anti-orthocorrespondent of the line GQ.

Given P, for both $A_gB_gC_g$ and $\widetilde{A}_g\widetilde{B}_g\widetilde{C}_g$ to be orthologic to ABC, the point Q has to be the intersection of the line $GP^{\perp}$ ((3) above) and

$$S_A(v-w)x + S_B(w-u)y + S_C(u-v)z = 0,$$

the anti-orthocorrespondent of GP. This is the point

$$\tau(P) = \big(S_A(c^2-b^2)u^2 + (S_{AC}-S_{BB})uv - (S_{AB}-S_{CC})uw + a^2(c^2-b^2)vw \\ : \cdots : \cdots\big).$$

The point $\tau(P)$ is not well defined if all three coordinates of $\tau(P)$ are equal to zero, which is the case exactly when P is either K, the orthocenter H, or the centroid G. The pre-images of these points are lines: GH (the Euler line), GK, and HK for K, G and H respectively. Outside these lines the mapping $P \longmapsto \tau(P)$ is an involution. Note that P and $\tau(P)$ are collinear with the symmedian point K.

The fixed points of τ are the points of the Kiepert hyperbola

$$(b^2-c^2)yz + (c^2-a^2)xz + (a^2-b^2)xy = 0.$$

More precisely, the line joining $\tau(P)$ to H meets GP on the Kiepert hyperbola. Therefore we may characterize $\tau(P)$ as the intersection of the line PK with the polar of P in the Kiepert hyperbola. [4]

In the table below we give the first coordinates of some well known triangle centers and their images under τ. The indexing of triangle centers follows [3].

P	first coordinate	$\tau(P)$	first coordinate
X_1	a	X_9	$a(s-a)$
X_7	$(s-b)(s-c)$	X_{948}	$(s-b)(s-c)F$
X_8	$s-a$		$a^2+(b+c)^2$
X_{19}			aG
X_{34}			$a(s-b)(s-c)(a^2+(b+c)^2)$
X_{37}		X_{72}	$a(b+c)S_A$
X_{42}	$a^2(b+c)$	X_{71}	$a^2(b+c)S_A$
X_{57}	$a/(s-a)$	X_{223}	$a(s-b)(s-c)F$
X_{58}		X_{572}	a^2G

[4] This is also called the *Hirst inverse* of P with respect to K. See the glossary of [3].

Here,

$$F = a^3 + a^2(b+c) - a(b+c)^2 - (b+c)(b-c)^2,$$
$$G = a^3 + a^2(b+c) + a(b+c)^2 + (b+c)(b-c)^2,$$

We may also wonder, given P outside the circumcircle, for which Q are the $\mathsf{Parasix}(P,Q)$ triangles $A_gB_gC_g$ and $A_sB_sC_s$ orthologic to the circumcevian triangle of P. The A-vertex of the circumcevian triangle of P has coordinates

$$\left(-a^2yz : (b^2z+c^2y)y : (b^2z+c^2y)z\right).$$

Hence we find that the lines from the vertices of the circumcevian triangle of P perpendicular to the corresponding sides of $A_gB_gC_g$ concur if and only if

$$(uyz + vxz + wxy)L = 0, \tag{4}$$

where

$$L = \sum_{\text{cyclic}} (c^2v^2+2S_Avw+b^2w^2)((c^2S_Cv-b^2S_Bw)u^2+a^2((c^2v^2-b^2w^2)u+(S_Bv-S_Cw)vw))x.$$

The first factor in (4) represents the circumconic with perspector P, and when Q is on this conic, $\mathsf{Parasix}(P,Q)$ is degenerate, see §6 below. The second factor L yields the locus we are looking for, a line passing through $P^{\perp}$. [5]

A point X lies on the line $L = 0$ if and only if P lies on a bicircular circumquintic through the in- and excenters[6]. For the special case $X = G$ this quintic decomposes into $\mathcal{L}_\infty$ (with multiplicity 2) and the McCay cubic. [7] In other words, for any P on the McCay cubic, the circumcevian triangle of P is orthologic to the $\mathsf{Parasix}(P,Q)$ triangles if and only if Q lies on the line $GP^{\perp}$.

4. Concyclic $\mathsf{Parasix}(P,Q)$-hexagons

We may ask, given P, for which Q the parasix configuration yields a cyclic hexagon. This is equivalent to the circumcenter of $A_gB_gC_g$ being equal to the midpoint of segment PQ. Now the midpoint of PQ lies on the perpendicular bisector of B_gC_g if and only if Q lies on the line

$$-(w(S_Au+S_Bv-S_Cw)+c^2uv)y+(v(S_Au-S_Bv+S_Cw)v+b^2wu)z = 0,$$

which is indeed the cevian line $AP^{\perp}$. Remarkably, we find the same cevian line as locus for Q satisfying the condition that $B_gC_g \perp AP$.

Proposition 6. *The following statements are equivalent.*

(1) $\mathsf{Parasix}(P,Q)$ *yields a cyclic hexagon.*

---

[5] The line $L = 0$ is not defined when P is an in/excenter. This means that, for any Q, triangles $A_gB_gC_g$ and $A_sB_sC_s$ in $\mathsf{Parasix}(P,Q)$ are orthologic to the circumcevian triangle of P. This is not surprising since P is the orthocenter of its own circumcevian triangle. For $P = X_3$, $L = 0$ is the line GK, while for $P = X_{13}, X_{14}$, it is the parallel at P to the Euler line.

[6] This quintic has equation $Q_Ax + Q_By + Q_Cz = 0$ where Q_A represents the union of the circle center A, radius 0 and the Van Rees focal which is the isogonal pivotal cubic with pivot the infinite point of AH and singular focus A.

[7] The McCay cubic is the isogonal cubic with pivot O given by the equation $\sum_{\text{cyclic}} a^2S_Ax(c^2y^2-b^2z^2) = 0$.

(2) *$A_gB_gC_g$ and $A_sB_sC_s$ are homothetic to the antipedal triangle of P.*
(3) *Q is the orthocorrespondent of P.*

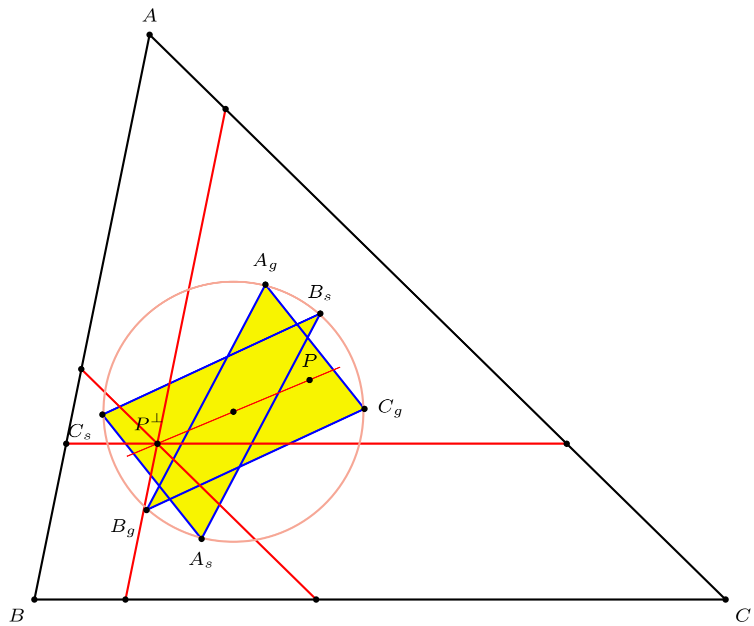

Figure 4

The center of the circle containing the 6 points is the midpoint of PQ.

The homothetic centers and the circumcenter of the cyclic hexagon are collinear.

A nice example is the circle around $\mathsf{Parasix}(H, G)$. It is homothetic to the circumcircle and nine point circle through H with factors $\frac{1}{3}$ and $\frac{2}{3}$ respectively. The center of the circle divides OH in the ratio $2 : 1$.[8] The antipedal triangle of H is clearly the anticomplementary triangle of ABC. The two homothetic centers divide the same segment in the ratios $5 : 2$ and $3 : 2$ respectively.[9] See Figure 5.

As noted in [1], $P = P^\perp$ only for the Fermat-Torricelli points X_{13} and X_{14}. The vertices of $\mathsf{parasix}(X_{13}, X_{13})$ and $\mathsf{Parasix}(X_{14}, X_{14})$ form regular hexagons. See Figure 6.

5. Equilateral triangles

The last example raises the question of finding, for given P, the points Q for which the triangles $A_gB_gC_g$ and $A_sB_sC_s$ are equilateral. We find that the A-median of $A_gB_gC_g$ is also an altitude in this triangle if and only if Q lies on the

[8]This is also the midpoint of GH, the center of the orthocentroidal circle, the point X_{381} in [3].

[9]These have homogeneous barycentric coordinates $(3a^4+2a^2(b^2+c^2)-5(b^2-c^2)^2 : \cdots : \cdots)$ and $(a^4-2a^2(b^2+c^2)+3(b^2-c^2)^2 : \cdots : \cdots)$ respectively. They are not in the current edition of [3].

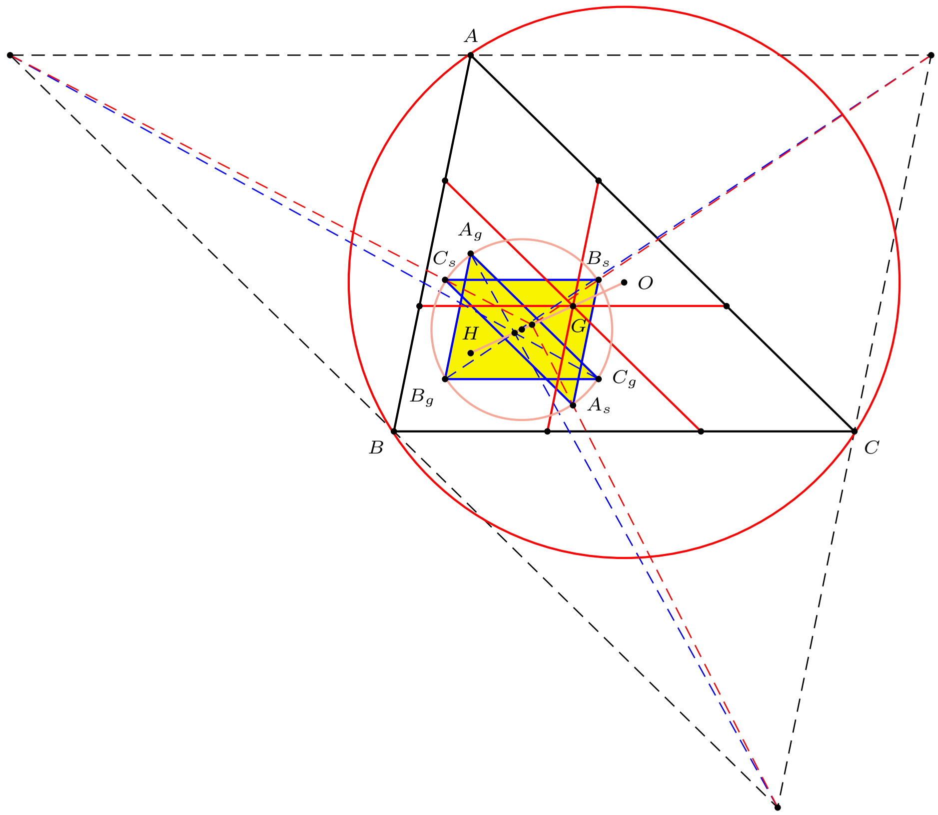

Figure 5. Parasix(H, G)

conic

$$\begin{aligned}&-2((S_Au+S_Bv-S_Cw)w+c^2uv)xy+2((S_Au-S_Bv+S_Cw)v+b^2uw)xz\\&-(c^2u^2+a^2w^2+2S_Buw)y^2+(b^2u^2+a^2v^2+2S_Cuv)z^2=0.\end{aligned}$$

We find an analogous conic for the B-median of $A_gB_gC_g$ to be an altitude. The two conics intersect in four points: two imaginary points and the points

$$Q_{1,2}=\left((-S_Au+S_Bv+S_Cw)u+a^2vw\pm\frac{1}{3}\sqrt{3}Su(u+v+w):\cdots:\cdots\right).$$

Proposition 7. *Given P, there are two (real) points Q for which triangles $A_gB_gC_g$ and $A_sB_sC_s$ are equilateral. These two points divide $PP^\perp$ harmonically.*

The points $Q_{1,2}$ from Proposition 7 can be constructed in the following way, using the fact that P, G_s, G_g and $P^\perp$ are collinear.

Start with a point G' on $PP^\perp$. We shall construct an equilateral triangle $A'B'C'$ with vertices on AP, BP and CP respectively and centroid at G'. This triangle must be homothetic to one of the equilateral triangles $A_gB_gC_g$ of Proposition 7 through P.

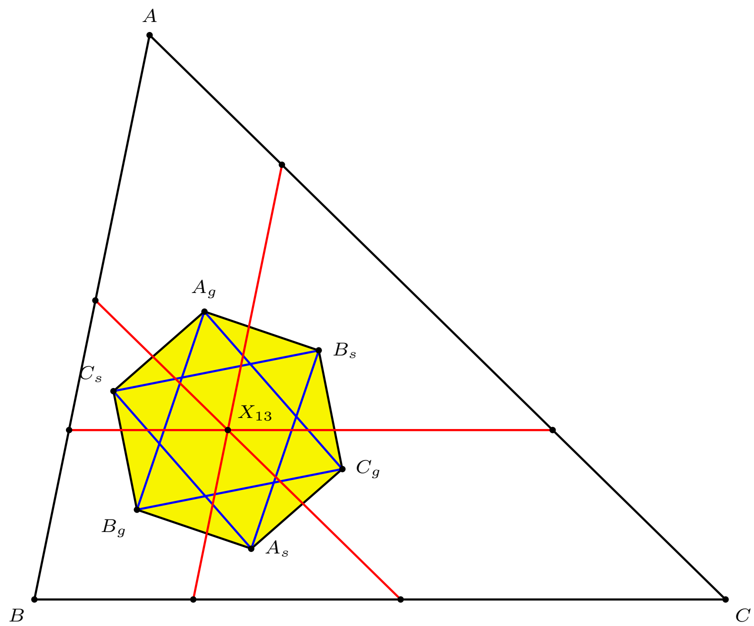

Figure 6. The parasix configuration $\mathsf{Parasix}(X_{13}, X_{13})$

Consider the rotation ρ about G' through $\pm\frac{2\pi}{3}$. The image of AP intersects BP in a point B'. Now let C' be the image of B' and A' the image of C'. Then $A'B'C'$ is equilateral, A' lies on AP, G' is the centroid and C' must lie on CP.

The homothety with center A that maps P to A' also maps BC to a line ℓ_a. Similarly we find ℓ_b and ℓ_c. These lines enclose a triangle $A''B''C''$ homothetic to ABC. We of course want to find the case for which $A''B''C''$ degenerates into one point, which is the Q we are looking for. Since all possible equilateral $A'B'C'$ of the same orientation are homothetic through P, the trianges $A''B''C''$ are all homothetic to ABC through the same point. So the homothety center of $A''B''C''$ and ABC is the point Q we are looking for.

6. Degenerate parasix triangles

We begin with a simple interesting fact.

Proposition 8. *Every line through P intersects the circumconic $\mathcal{C}_P$ at two real points.*

Proof. For the special case of the symmedian point K this is clear, since K is the interior of the circumcircle. Now, there is a homography φ fixing A, B, C and transforming $P = (u : v : w)$ into $K = (a^2 : b^2 : c^2)$. It is given by

$$\varphi(x : y : z) = \left(\frac{a^2}{u}x : \frac{b^2}{v}y : \frac{c^2}{w}z\right),$$

and is a projective transformation mapping $\mathcal{C}_P$ into the circumcircle and any line through P into a line through K. If ℓ is a line through P, then $\varphi(\ell)$ is a line through K, intersecting the circumcircle at two real points q_1 and q_2. The circumcircle and the circumconic $\mathcal{C}_P$ have a fourth real point Z in common, which is the trilinear pole of the line PK. For any point M on $\mathcal{C}_P$, the points Z, M, $\varphi(M)$ are collinear. The second intersections of the lines Zq_1 and Zq_2 are common points of ℓ and the circumconic $\mathcal{C}_P$. $\square$

In §2, we have seen that the parasix triangles are degenerate if and only if $P \in \mathcal{L}_Q$ or equivalently, $Q \in \mathcal{C}_P$. This means that for each line ℓ_P through P intersecting the circumconic $\mathcal{C}_P$ at Q_1 and Q_2, the triangles of $\mathsf{Parasix}(P, Q_i)$, $i = 1, 2$, are degenerate.

Theorem 9. *For $i = 1,\ 2$, the two lines containing the degenerate triangles of the parasix configuration $\mathsf{Parasix}(P, Q_i)$ are parallel to a tangent from P to the inscribed conic $\mathcal{C}_\ell$ with perspector the trilinear pole of ℓ_P. The two tangents for $i = 1$, 2 are perpendicular if and only if the line ℓ_P contains the orthocorrespondent $P^\perp$.*

For example, for $P = K$, the symmedian point, the circumconic $\mathcal{C}_P$ is the circumcircle. The orthocorrespondent is the point

$$K^\perp = (a^2(a^4 - b^4 + 4b^2c^2 - c^4) : \cdots : \cdots)$$

on the Euler line. The line ℓ joining K to this point has equation

$$\frac{(b^2-c^2)(b^2+c^2-2a^2)}{a^2}x + \frac{(c^2-a^2)(c^2+a^2-2b^2)}{b^2}y + \frac{(a^2-b^2)(a^2+b^2-2c^2)}{c^2}z = 0.$$

The inscribed conic $\mathcal{C}_\ell$ has center

$$(a^2(b^2-c^2)(a^4 - b^4 + b^2c^2 - c^4) : \cdots : \cdots).$$

The tangents from K to the conic $\mathcal{C}_\ell$ are the Brocard axis OK and its perpendicular at K.[10] The points of tangency are

$$\left(\frac{a^2(2a^2-b^2-c^2)}{b^2-c^2} : \frac{b^2(2b^2-c^2-a^2)}{c^2-a^2} : \frac{c^2(2c^2-a^2-b^2)}{a^2-b^2}\right)$$

on the Brocard axis and

$$\left(\frac{a^2(b^2-c^2)}{2a^2-b^2-c^2} : \frac{b^2(c^2-a^2)}{2b^2-c^2-a^2} : \frac{c^2(a^2-b^2)}{2c^2-a^2-b^2}\right)$$

on the perpendicular tangent. See Figure 7. The line ℓ intersects the circumcircle at the point

$$X_{110} = \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$$

and the Parry point

$$X_{111} = \left(\frac{a^2}{b^2+c^2-2a^2} : \frac{b^2}{c^2+a^2-2b^2} : \frac{c^2}{a^2+b^2-2c^2}\right).$$

The lines containing the degenerate triangles of $\mathsf{Parasix}(K, X_{110})$ are parallel to the Brocard axis, while those for $\mathsf{Parasix}(K, X_{111})$ are parallel to the tangent from K which is perpendicular to the Brocard axis.

[10] The infinite points of these lines are respectively X_{511} and X_{512}.

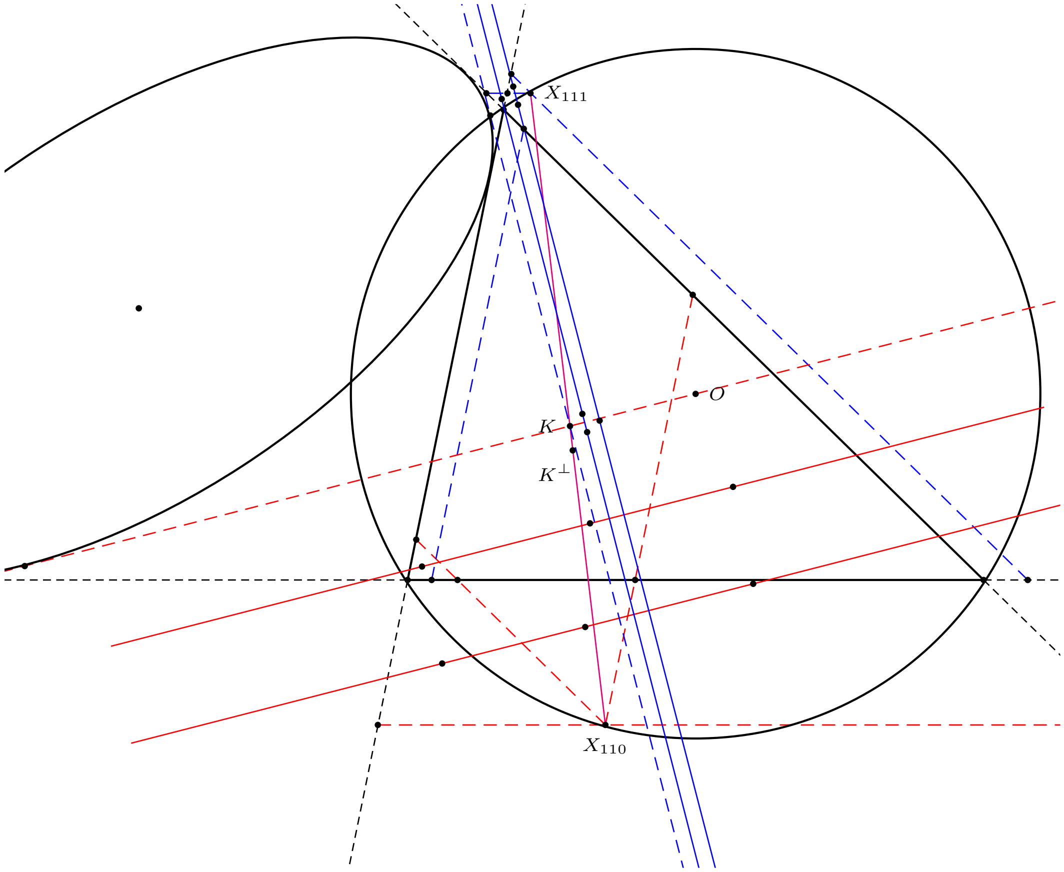

Figure 7. Degenerate $\mathsf{Parasix}(K, X_{110})$ and $\mathsf{Parasix}(K, X_{111})$

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A Tetrahedral Arrangement of Triangle Centers

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Abstract. We present a graphic scheme for indexing 25 collinearities of 17 triangle centers three at a time. The centers are used to label vertices and edges of nested polyhedra. Two new triangle centers are introduced to make this possible.

1. Introduction

Collinearities of triangle centers which are defined in apparently different ways has been of interest to geometers since it was first noticed that the orthocenter, centroid, and circumcenter are collinear, lying on Euler's line. Kimberling [3] lists a great many collinearites, including many more points on Euler's line. The object of this note is to present a three-dimensional graphical summary of 25 three-center collinearities involving 17 centers, in which the centers are represented as vertices and edge midpoints of nested polyhedra: a tetrahedron circumscribing an octahedron which then circumscribes a cubo-octahedron. Such a symmetric collection of collinearities may be a useful mnemonic. Probably the reason why this has not been recognized before is that two of the vertices of the tetrahedron represent previously undescribed centers. First we describe two new centers, which Kimberling lists as X_{1276} and X_{1277} in his *Encyclopedia of Triangle Centers* [3]. Then we describe the tetrahedron and work inward to the cubo-octahedron.

2. Perspectors and the excentral triangle

The excentral triangle, $\mathbf{T}_{\text{x}}$, of a triangle $\mathbf{T}$ is the triangle whose vertices are the excenters of $\mathbf{T}$. Let $\mathbf{T}_{+}$ be the triangle whose vertices are the apices of equilateral triangles erected outward on the sides of $\mathbf{T}$. Similarly let $\mathbf{T}_{-}$ be the triangle whose vertices are the apices of equilateral triangles erected inward on the sides of $\mathbf{T}$. It happens that $\mathbf{T}_{\text{x}}$ is in perspective from $\mathbf{T}_{+}$ from a point V_{+}, a previously undescribed triangle center now listed as X_{1276} in [3], and that $\mathbf{T}_{\text{x}}$ is also in perspective from $\mathbf{T}_{-}$ from another new center V_{-} listed as X_{1277} in [3]. See Figure 1.

For $\varepsilon = \pm 1$, the homogeneous trilinear coordinates of V_{ε} are

$$1 - v_a + v_b + v_c : 1 + v_a - v_b + v_c : 1 + v_a + v_b - v_c,$$

where $v_a = -\frac{2}{\sqrt{3}} \sin(A + \varepsilon \cdot 60^\circ)$ etc.

It is well known that $\mathbf{T}_{\text{x}}$ and $\mathbf{T}$ are in perspective from the incenter I. Define $\mathbf{T}^{*}$ as the triangle whose vertices are the reflections of the vertices of $\mathbf{T}$ in the opposite sides. Then $\mathbf{T}_{\text{x}}$ and $\mathbf{T}^{*}$ are in perspective from a point W listed as X_{484} in [3]. See Figure 2. The five triangles $\mathbf{T}$, $\mathbf{T}_{\text{x}}$, $\mathbf{T}_{+}$, $\mathbf{T}_{-}$, and $\mathbf{T}^{*}$ are pairwise in perspective, giving 10 perspectors. Denote the perspector of two triangles by enclosing the two triangles in brackets, so, for example $[\mathbf{T}_{\text{x}}, \mathbf{T}] = I$.

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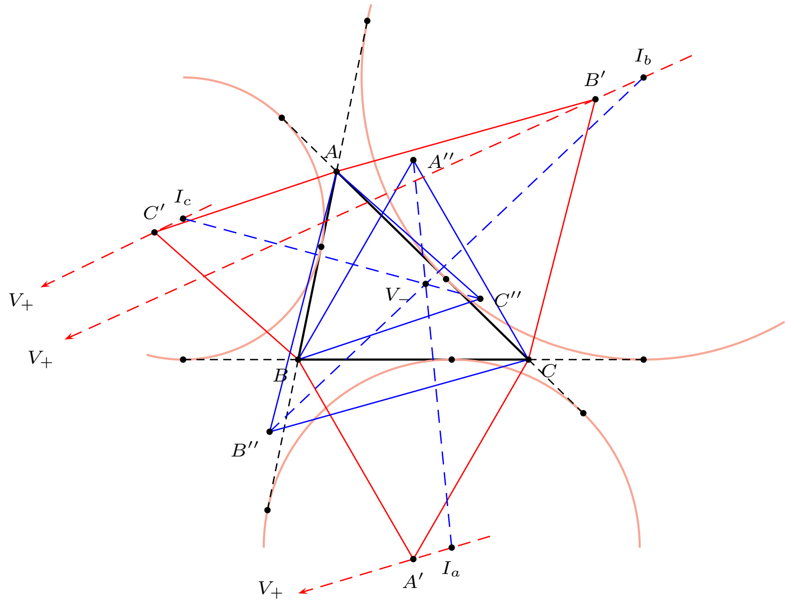

Figure 1

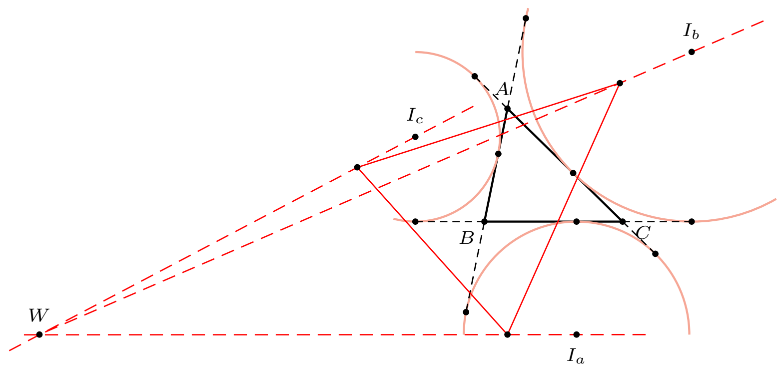

Figure 2

Here is a list of the 10 perspectors with their names and ETC numbers:

$[\mathbf{T}, \mathbf{T}_+]$	F_+	First Fermat point	X_{13}
$[\mathbf{T}, \mathbf{T}_-]$	F_-	Second Fermat point	X_{14}
$[\mathbf{T}, \mathbf{T}^*]$	H	Orthocenter	X_4
$[\mathbf{T}, \mathbf{T}_\mathrm{x}]$	I	Incenter	X_1
$[\mathbf{T}_+, \mathbf{T}_-]$	O	Circumcenter	X_3
$[\mathbf{T}_+, \mathbf{T}^*]$	J_-	Second isodynamic point	X_{16}
$[\mathbf{T}_-, \mathbf{T}^*]$	J_+	First isodynamic point	X_{15}
$[\mathbf{T}_\mathrm{x}, \mathbf{T}^*]$	W	First Evans perspector	X_{484}
$[\mathbf{T}_\mathrm{x}, \mathbf{T}_+]$	V_+	Second Evans perspector	X_{1276}
$[\mathbf{T}_\mathrm{x}, \mathbf{T}_-]$	V_-	Third Evans perspector	X_{1277}

3. Collinearities among the ten perspectors

As in [2], we shall write $\mathcal{L}(X, Y, Z, \dots)$ to denote the line containing X, Y, Z, ... The following collinearities may be easily verified:

$$\begin{array}{lll} \mathcal{L}(I, O, W), & \mathcal{L}(I, J_-, V_-), & \mathcal{L}(I, J_+, V_+), \\ \mathcal{L}(V_+, H, V_-), & \mathcal{L}(W, F_+, V_-), & \mathcal{L}(W, F-, V_+). \end{array}$$

What is remarkable is that all five triangles are involved in each collinearity, with $\mathbf{T}_\mathrm{x}$ used twice. For example, rewrite $\mathcal{L}(I, O, W)$ as

$$\mathcal{L}([\mathbf{T}, \mathbf{T}_\mathrm{x}], [\mathbf{T}_+, \mathbf{T}_-], [\mathbf{T}_\mathrm{x}, \mathbf{T}^*])$$

to see this. The six collinearites have been stated so that the first and third perspectors involve $\mathbf{T}_\mathrm{x}$, with the perspector of the remaining two triangles listed second. This lends itself to a graphical representation as a tetrahedron with vertices labelled with I, V_+, V_-, and W, and the edges labelled with the perspectors collinear with the vertices. See Figure 3. When these centers are actually constructed, they may not be in the order listed in these collinearities. For example, O is not necessarily between I and W. There is another collinearity which we do not use, however, namely, $\mathcal{L}(O, J_+, J_-)$, which is the Brocard axis. Triangle $\mathbf{T}_\mathrm{x}$ is not involved in any of the perspectors in this collinearity.

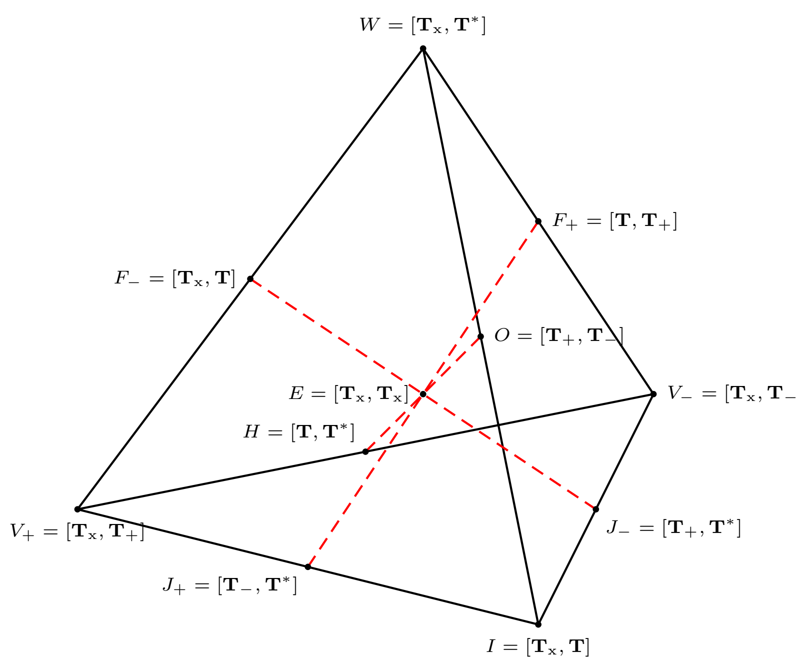

Figure 3

If we label each edge of the tetrahedron at its midpoint by the middle center listed in each of the collinearities above, then opposite edge midpoints are pairs of isogonal conjugates: H and O, J_+ and F_+, and J_- and F_-. Also the lines $\mathcal{L}(O, H)$, $\mathcal{L}(F_+, J_+)$, and $\mathcal{L}(F_-, J_-)$ are parallel to the Euler line, and may be interpreted as intersecting at the Euler infinity point E, listed as X_{30} in [3]. This adds three more collinearities to the tetrahedral scheme:

$$\mathcal{L}(O,E,H),\ \mathcal{L}(F_+,E,J_+),\ \mathcal{L}(F_-,E,J_-).$$

The five triangles $\mathbf{T}$, $\mathbf{T}_+$, $\mathbf{T}_-$, $\mathbf{T}^*$, and $\mathbf{T}_{\mathrm{x}}$ are all inscribed in Neuberg's cubic curve. Now consider a triangle $\mathbf{T}'_{\mathrm{x}}$ in perspective with $\mathbf{T}_{\mathrm{x}}$ and inscribed in the cubic with vertices very close to those of $\mathbf{T}_{\mathrm{x}}$ (the excenters of $\mathbf{T}$). The lines of perspective of $\mathbf{T}'_{\mathrm{x}}$ and $\mathbf{T}_{\mathrm{x}}$ approach the tangents to Neuberg's cubic at the vertices of $\mathbf{T}_{\mathrm{x}}$ as $\mathbf{T}'_{\mathrm{x}}$ approaches $\mathbf{T}_{\mathrm{x}}$. These tangents are known to be parallel to the Euler line and may be thought of as converging at the Euler point at infinity, $E = X_{30}$. So we can write $E = [\mathbf{T}_{\mathrm{x}}, \mathbf{T}_{\mathrm{x}}]$, interpreting this to mean that $\mathbf{T}_{\mathrm{x}}$ is in perspective from itself from E. I propose the term "ipseperspector" for such a point, from the Latin "ipse" for self. Note that the notion of ipseperspector is dependent on the curve circumscribing the triangle $\mathbf{T}$. A well-known example of an ipseperspector for a triangle curcumscribed in Neuberg's cubic is X_{74}, this being the point where the tangents to the curve at the vertices of $\mathbf{T}$ intersect.

4. Further nested polyhedra

We shall encounter other named centers, which are listed here for reference:

G	Centroid	X_2
K	Symmedian (Lemoine) point	X_6
N_+	First Napoleon point	X_{17}
N_-	Second Napoleon point	X_{18}
N_+^*	Isogonal conjugate of N_+	X_{61}
N_-^*	Isogonal conjugate of N_-	X_{62}

The six midpoints of the edges of the tetrahedron may be considered as the vertices of an inscribed octahedron. This leads to indexing more collinearities in the following way: label the midpoint of each edge of the octahedron by the point where the lines indexed by opposite edges meet. For example, opposite edges of the octahedron $\mathcal{L}(F_+,J_-)$ and $\mathcal{L}(F_-,J_+)$ meet at the centroid G. We can then write two 3-point collinearities as $\mathcal{L}(F_+,G,J_-)$ and $\mathcal{L}(F_-,G,J_+)$. Now the edges adjacent to both of these edges index the lines $\mathcal{L}(F_+,F_-)$ and $\mathcal{L}(J_+,J_-)$, which meet at the symmedian point K. This gives two more 3-point collinearities, $\mathcal{L}(F_+,K,F_-)$ and $\mathcal{L}(J_+,K,J_-)$. Note that G and K are isogonal conjugates. This pattern persists with the other pairs of opposite edges of the octahedron.

The intersections of other lines represented as opposite edges intersect at the Napoleon points and their isogonal conjugates. When we consider the four vertices O, F_-, H, and J_- of the octahedron, four more 3-point collinearities are indexed in the same manner: $\mathcal{L}(O,N_-^*,J_-)$, $\mathcal{L}(H,N_-^*,F_-)$, $\mathcal{L}(O,N_-,F_-)$, and $\mathcal{L}(H,N_-,J_-)$. Similarly, from vertices O, F_+ , H, and J_+, four more 3-point collinearites arise in the same indexing process: $\mathcal{L}(O,N_+^*,J_+)$, $\mathcal{L}(H,N_+^*,F_+)$, $\mathcal{L}(O,N_+,F_+)$, and $\mathcal{L}(H,N_+,J_+)$. So each of the twelve edges of the octahedron indexes a different 3-point collinearity.

Let us carry this indexing scheme further. Now consider the midpoints of the edges of the octahedron to be the vertices of a polyhedron inscribed in the octahedron. This third nested polyhedron is a cubo-octahedron: it has eight triangular faces, each of which is coplanar with a face of the octahedron, and six square faces. Yet again more 3-point collinearities are indexed, but this time by the triangular faces of the cubo-octahedron. It happens that the three vertices of each triangular face of the cubo-octahedron, which inherit their labels as edges of the octahedron, are collinear in the plane of the basic triangle $\mathbf{T}$. Opposite edges of the octahedron have the same point labelling their midpoints, so opposite triangular faces of the cubo-octahedron are labelled by the same three centers. This means that there are four instead of eight collinearities indexed by the triangular faces: $\mathcal{L}(G, N_+, N_-^*)$,$\mathcal{L}(G, N_-, N_+^*)$, $\mathcal{L}(K, N_+, N_-)$, and $\mathcal{L}(K, N_-^*, N_+^*)$. See Figure 4.

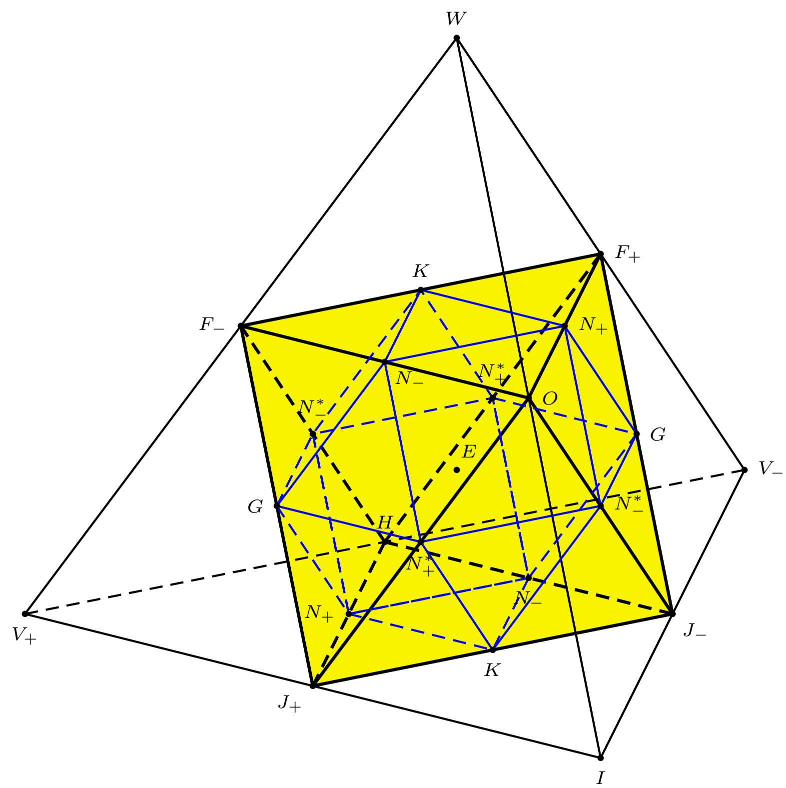

Figure 4

So we have 6 collinearites indexed by edges of the tetrahedron, 3 more by its diagonals, 12 by the inscribed octahedron, and 4 more by the further inscribed cubo-octahedron, for a total of 25.

5. Concluding remarks

In a sense, the location of each center entering into this graphical scheme places it in equal importance to the other centers in similar locations. So the four centers I, U, V, and W, which arose as perspectors with the excentral triangle are on one level. On the next level we may place the six centers O, H, J_+, J_-, F_+, and F_- which index the edges of the tetrahedron and the vertices of the inscribed octahedron. It is interesting that these six centers are the first to appear in the construction given by the author [1], and that the subsequent centers indexed by the midpoints of the edges of the octahedron arise as intersections of lines they determine. The Euler infinity point, E, is the only point at the third level of construction. Centers I, V_+, V_-, W, O, H, F_+ , J_+, F_-, J_-, and E all lie on Neuberg's cubic curve. The Euler line appears as the collinearity $\mathcal{L}(O, E, H)$, with no indication that G lies on the line. The Brocard axis appears four times as $\mathcal{L}(J_+, K, J_-)$, $\mathcal{L}(K, N_-^*, N_+^*)$, $\mathcal{L}(O, N_+^*, J_+)$, and $\mathcal{L}(O, N_-^*, J-)$, but the better-known collinearity $\mathcal{L}(O, J_+, J_-)$ does not.

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The Apollonius Circle and Related Triangle Centers

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Abstract. We give a simple construction of the Apollonius circle without directly invoking the excircles. This follows from a computation of the coordinates of the centers of similitude of the Apollonius circle with some basic circles associated with a triangle. We also find a circle orthogonal to the five circles, circumcircle, nine-point circle, excentral circle, radical circle of the excircles, and the Apollonius circle.

1. The Apollonius circle of a triangle

The Apollonius circle of a triangle is the circle tangent internally to each of the three excircles. Yiu [5] has given a construction of the Apollonius circle as the inversive image of the nine-point circle in the radical circle of the excircles, and the coordinates of its center Q. It is known that this radical circle has center the Spieker center S and radius $\rho = \frac{1}{2}\sqrt{r^2 + s^2}$. See, for example, [6, Theorem 4]. Ehrmann [1] found that this center can be constructed as the intersection of the Brocard axis and the line joining S to the nine-point center N. See Figure 1. A proof of this fact was given in [2], where Grinberg and Yiu showed that the Apollonius circle is a

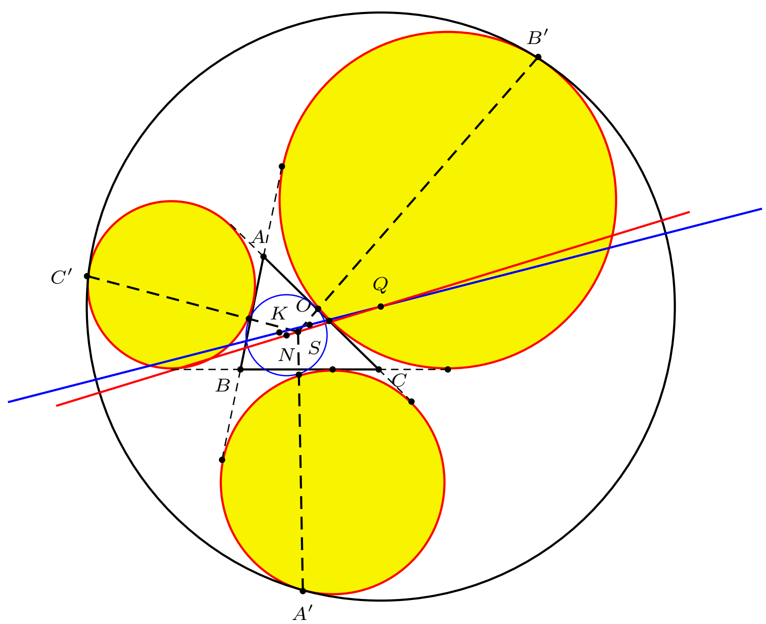

Figure 1

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Tucker circle. In this note we first verify these results by expressing the coordinates of Q in terms of R, r, and s, (the circumradius, inradius, and semiperimeter) of the triangle. By computing some homothetic centers of circles associated with the Apollonius circle, we find a simple construction of the Apollonius circle without directly invoking the excircles. See Figure 4.

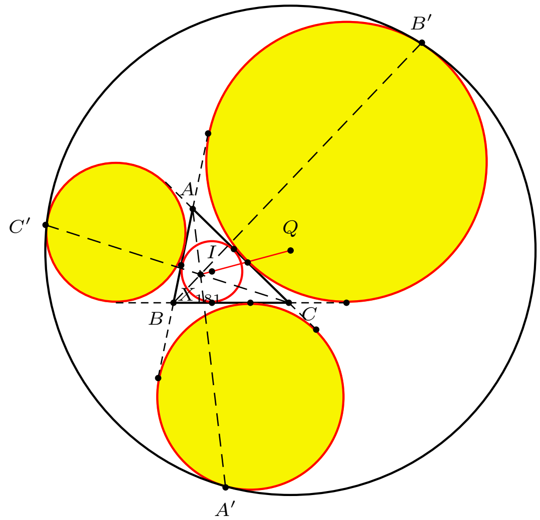

Figure 2

For triangle centers we shall adopt the notation of Kimberling's *Encyclopedia of Triangle Centers* [3], except for the most basic ones:

G	centroid	O	circumcenter
I	incenter	H	orthocenter
N	nine-point center	K	symmedian point
S	Spieker center	I'	reflection of I in O

We shall work with barycentric coordinates, absolute and homogeneous. It is known that if the Apollonius circle touches the three excircles respectively at A', B', C', then the lines AA', BB', CC' concur in the point [1]

$$X_{181} = \left(\frac{a^2(b+c)^2}{s-a} : \frac{b^2(c+a)^2}{s-b} : \frac{c^2(a+b)^2}{s-c}\right).$$

We shall make use of the following simple lemma.

Lemma 1. *Under inversion with respect to a circle, center P, radius ρ, the image of the circle center P', radius ρ', is the circle, radius $\left|\frac{\rho^2}{d^2-\rho'^2}\cdot\rho'\right|$ and center Q which divides the segment PP' in the ratio*

$$PQ : QP' = \rho^2 : d^2 - \rho^2 - \rho'^2,$$

[1] The trilinear coordinates of X_{181} were given by Peter Yff in 1992.

where d is the distance between P and P'. Thus,

$$Q = \frac{(d^2 - \rho^2 - \rho'^2)P + \rho^2 \cdot P'}{d^2 - \rho'^2}.$$

Theorem 2. *The Apollonius circle has center*

$$Q = \frac{1}{4Rr}\left((r^2 + 4Rr + s^2)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I\right)$$

and radius $\frac{r^2+s^2}{4r}$.

Proof. It is well known that the distance between O and I is given by

$$OI^2 = R^2 - 2Rr.$$

Since S and N divide the segments IG and OG in the ratio $3 : -1$,

$$SN^2 = \frac{R^2 - 2Rr}{4}.$$

Applying Lemma 1 with

$$\begin{array}{ll} P = S = \frac{1}{2}(3G - I) = \frac{1}{2}(2O + H - I), & P' = N = \frac{1}{2}(O + H), \\ \rho^2 = \frac{1}{4}(r^2 + s^2), & \rho'^2 = \frac{1}{4}R^2, \\ d^2 = SN^2 = \frac{1}{4}(R^2 - 2Rr), & \end{array}$$

we have

$$Q = \frac{1}{4Rr}\left((r^2 + 4Rr + s^2)O + 2Rr \cdot H - (r^2 + 2Rr + s^2)I\right).$$

The radius of the Apollonius circle is $\frac{r^2+s^2}{4r}$. □

The point Q appears in Kimberling's *Encyclopedia of Triangle Centers* [3] as

$$\begin{aligned} X_{970} =&(a^2(a^3(b+c)^2 + a^2(b+c)(b^2+c^2) - a(b^4 + 2b^3c + 2bc^3 + c^4) \\ &- (b+c)(b^4+c^4)) : \cdots : \cdots). \end{aligned}$$

We verify that it also lies on the Brocard axis.

Proposition 3.

$$\overrightarrow{OQ} = -\frac{s^2 - r^2 - 4Rr}{4Rr} \cdot \overrightarrow{OK}.$$

Proof. The oriented areas of the triangles KHI, OKI, and OHK are as follows.

$$\begin{aligned} \triangle(KHI) =& \frac{(a-b)(b-c)(c-a)f}{16(a^2+b^2+c^2) \cdot \triangle}, \\ \triangle(OKI) =& \frac{abc(a-b)(b-c)(c-a)}{8(a^2+b^2+c^2) \cdot \triangle}, \\ \triangle(OHK) =& \frac{-(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)}{8(a^2+b^2+c^2) \cdot \triangle}, \end{aligned}$$

where $\triangle$ is the area of triangle ABC and

$$\begin{aligned} f =& a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) + 2abc \\ =& 8rs(2R+r). \end{aligned}$$

Since $abc = 4Rrs$ and $(a+b)(b+c)(c+a) = 2s(r^2+2Rr+s^2)$, it follows that, with respect to OHI, the symmedian point K has homogeneous barycentric coordinates

$$\begin{aligned} & f : 2abc : -2(a+b)(b+c)(c+a) \\ =& 8rs(2R+r) : 8Rrs : -4s(r^2+2Rr+s^2) \\ =& 2r(2R+r) : 2Rr : -(r^2+2Rr+s^2). \end{aligned}$$

Therefore,

$$K = \frac{1}{4Rr+r^2-s^2}\left(2r(2R+r)O + 2Rr \cdot H - (r^2+2Rr+s^2)I\right),$$

and

$$\begin{aligned} \overrightarrow{OK} =& \frac{1}{4Rr+r^2-s^2}\left((r^2+s^2)O + 2Rr \cdot H - (r^2+2Rr+s^2)I\right) \\ =& -\frac{4Rr}{s^2-r^2-4Rr} \cdot \overrightarrow{OQ}. \end{aligned}$$

$\square$

2. Centers of similitude

We compute the coordinates of the centers of similitude of the Apollonius circle with several basic circles. Figure 3 below shows the Apollonius circle with the circumcircle, incircle, nine-point circle, excentral circle, and the radical circle (of the excircles). Recall that the excentral circle is the circle through the excenters of the triangle. It has center I' and radius $2R$.

Lemma 4. *Two circles with centers P, P', and radii ρ, ρ' respectively have internal center of similitude* $\dfrac{\rho' \cdot P + \rho \cdot P'}{\rho' + \rho}$ *and external center of similitude* $\dfrac{\rho' \cdot P - \rho \cdot P'}{\rho' - \rho}$.

Proposition 5. *The homogeneous barycentric coordinates (with respect to triangle ABC) of the centers of similitude of the Apollonius circle with the various circles are as follows.*

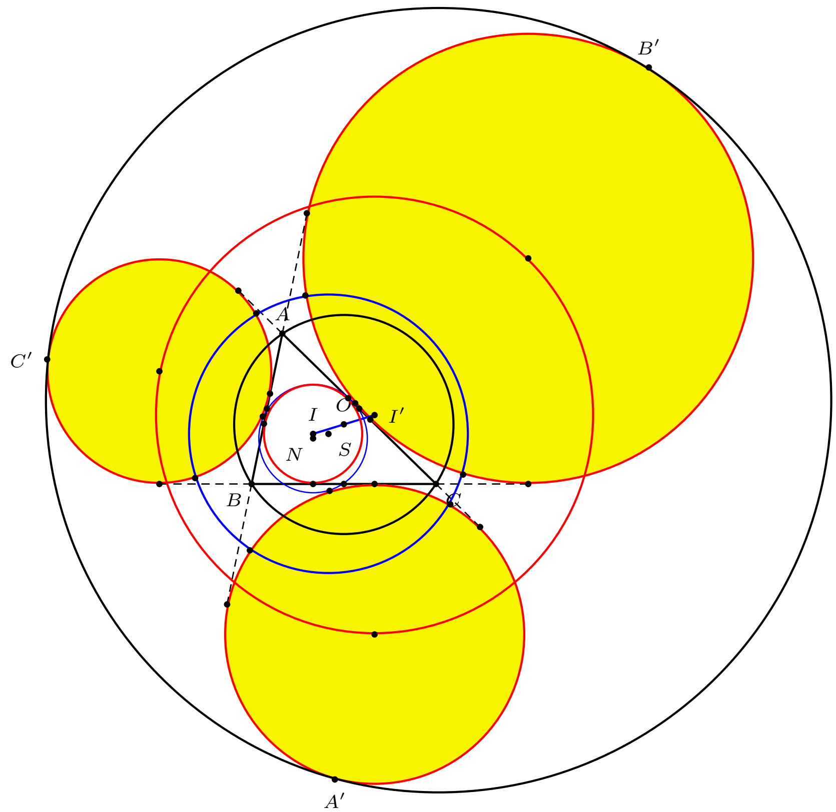

Figure 3

circumcircle	
internal X_{573}	$a^2(a^2(b+c)-abc-(b^3+c^3)) : \cdots : \cdots$
external X_{386}	$a^2(a(b+c)+b^2+bc+c^2) : \cdots : \cdots$
incircle	
internal X_{1682}	$a^2(s-a)(a(b+c)+b^2+c^2)^2 : \cdots : \cdots$
external X_{181}	$\frac{a^2(b+c)^2}{s-a} : \cdots : \cdots$
nine $-$ point circle	
internal S	$b+c : c+a : a+b$
external X_{2051}	$\frac{1}{a^3-a(b^2-bc+c^2)-bc(b+c)} : \cdots : \cdots$
excentral circle	
internal X_{1695}	$a \cdot F : \cdots : \cdots$
external X_{43}	$a(a(b+c)-bc) : \cdots : \cdots$

where

$$\begin{aligned} F =& a^5(b+c)+a^4(4b^2+7bc+4c^2)+2a^3(b+c)(b^2+c^2) \\ & -2a^2(2b^4+3b^3c+3bc^3+2c^4)-a(b+c)(3b^4+2b^2c^2+3c^4)-bc(b^2-c^2)^2. \end{aligned}$$

Proof. The homogenous barycentric coordinates (with respect to triangle OHI) of the centers of similitude of the Apollonius circle with the various circles are as follows.

circumcircle	
internal X_{573}	$2(r^2+2Rr+s^2) : 2Rr : -(r^2+2Rr+s^2)$
external X_{386}	$4Rr : 2Rr : -(r^2+2Rr+s^2)$
incircle	
internal X_{1682}	$-r(r^2+4Rr+s^2) : -2Rr^2 : r^3+Rr^2-(R-r)s^2$
external X_{181}	$-r(r^2+4Rr+s^2) : -2Rr^2 : r^3+3Rr^2+(R+r)s^2$
nine − point circle	
internal S	$2:1:-1$
external X_{2051}	$-4Rr : r^2-2Rr+s^2 : r^2+2Rr+s^2$
excentral circle	
internal X_{1695}	$4(r^2+2Rr+s^2) : 4Rr : -(3r^2+4Rr+3s^2)$
external X_{43}	$8Rr : 4Rr : -(r^2+4Rr+s^2)$

Using the relations

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} \quad \text{and} \quad R = \frac{abc}{4rs},$$

and the following coordinates of O, H, I (with equal coordinate sums),

$$\begin{aligned}
O =&(a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2)),\\
H =&((c^2+a^2-b^2)(a^2+b^2-c^2), (a^2+b^2-c^2)(b^2+c^2-a^2),\\
&(b^2+c^2-a^2)(c^2+a^2-b^2)),\\
I =&(b+c-a)(c+a-b)(a+b-c)(a,b,c),
\end{aligned}$$

these can be converted into those given in the proposition. □

Remarks. 1. $X_{386} = OK \cap IG$.
2. $X_{573} = OK \cap HI' = OK \cap X_{55}X_{181}$.
3. $X_{43} = IG \cap X_{57}X_{181}$.

From the observation that the Apollonius circle and the nine-point circle have S as internal center of similitude, we have an easy construction of the Apollonius circle without directly invoking the excircles.

Construct the center Q of Apollonius circle as the intersection of OK and NS. Let D be the midpoint of BC. Join ND and construct the parallel to ND through Q (the center of the Apollonius circle) to intersect DS at A', a point on the Apollonius circle, which can now be easily constructed. See Figure 4.

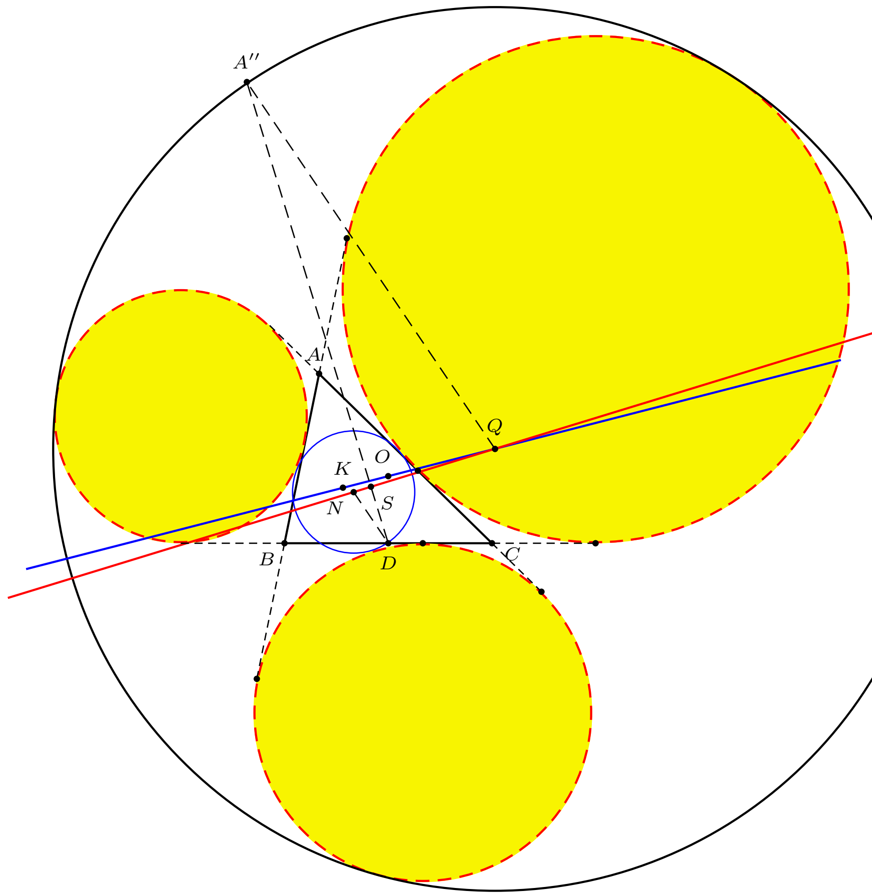

Figure 4

Proposition 6. *The center Q of the Apollonius circle lies on the each of the lines $X_{21}X_{51}$, $X_{40}X_{43}$ and $X_{411}X_{185}$. More precisely,*

$$X_{51}X_{21} : X_{21}Q = 2r : 3R,$$
$$X_{43}X_{40} : X_{43}Q = 8Rr : r^2 + s^2,$$
$$X_{185}X_{411} : X_{411}Q = 2r : R.$$

Remark. The Schiffler point X_{21} is the intersection of the Euler lines of the four triangles ABC, IBC, ICA and IAB. It divides OH in the ratio

$$OX_{21} : X_{21}H = R : 2(R + r).$$

The harmonic conjugate of X_{21} in OH is the triangle center

$$\begin{aligned} X_{411} =&(a(a^6 - a^5(b+c) - a^4(2b^2 + bc + 2c^2) + 2a^3(b+c)(b^2 - bc + c^2) \\ &+ a^2(b^2+c^2)^2 - a(b-c)^2(b+c)(b^2+c^2) + bc(b-c)^2(b+c)^2) \\ &: \cdots : \cdots). \end{aligned}$$

3. A circle orthogonal to 5 given ones

We write the equations of the circles encountered above in the form

$$a^2yz + b^2zx + c^2xy + (x+y+z)L_i = 0,$$

where L_i, $1 \le i \le 5$, are linear forms given below.

i	circle	L_i
1	circumcircle	0
2	nine − point circle	$-\frac{1}{4}((b^2+c^2-a^2)x + (c^2+a^2-b^2)y + (a^2+b^2-c^2)z)$
3	excentral circle	$bcx + cay + abz$
4	radical circle	$(s-b)(s-c)x + (s-c)(s-a)y + (s-a)(s-b)z$
5	Apollonius	$s\left((s+\frac{bc}{a})x + (s+\frac{ca}{b})y + (s+\frac{ab}{c})z\right)$

Remark. The equations of the Apollonius circle was computed in [2]. The equations of the other circles can be found, for example, in [6].

Proposition 7. *The four lines $L_i = 0$, $i = 2, 3, 4, 5$, are concurrent at the point*

$$X_{650} = (a(b-c)(s-a) : b(c-a)(s-b) : c(a-b)(s-c)).$$

It follows that this point is the radical center of the five circles above. From this we obtain a circle orthogonal to the five circles.

Theorem 8. *The circle*

$$a^2yz + b^2zx + c^2xy + (x+y+z)L = 0,$$

where

$$L = \frac{bc(b^2+c^2-a^2)}{2(c-a)(a-b)}x + \frac{ca(c^2+a^2-b^2)}{2(a-b)(b-c)}y + \frac{ab(a^2+b^2-c^2)}{2(b-c)(c-a)}z,$$

is orthogonal to the circumcircle, excentral circle, Apollonius circle, nine-point circle, and the radical circle of the excircles. It has center X_{650} and radius the square root of

$$\frac{abc \cdot G}{4(a-b)^2(b-c)^2(c-a)^2},$$

where

$$\begin{aligned} G =& abc(a^2+b^2+c^2) - a^4(b+c-a) - b^4(c+a-b) - c^4(a+b-c) \\ =& 16r^2s(r^2+5Rr+4R^2-s^2). \end{aligned}$$

This is an interesting result because among these five circles, only three are coaxal, namely, the Apollonius circle, the radical circle, and the nine-point circle.

Remark. X_{650} is also the perspector of the triangle formed by the intersections of the corresponding sides of the orthic and intouch triangles. It is the intersection of the trilinear polars of the Gergonne and Nagel points.

4. More centers of similitudes with the Apollonius circle

We record the coordinates of the centers of similitude of the Apollonius circle with the Spieker radical circle. These are

$$\begin{aligned}&(a^2(-a^3(b+c)^2 - a^2(b+c)(b^2+c^2) + a(b^4+2b^3c+2bc^3+c^4) + (b+c)(b^4+c^4))\\&\pm abc(b+c)\sqrt{(b+c-a)(c+a-b)(a+b-c)(a^2(b+c)+b^2(c+a)+c^2(a+b)+abc)}\\&:\cdots:\cdots)\end{aligned}$$

It turns out that the centers of similitude with the Spieker circle (the incircle of the medial triangle) and the Moses circle (the one tangent internally to the nine-point circle at the center of the Kiepert hyperbola) also have rational coordinates in a, b, c:

Spieker circle	
internal	$a(b+c-a)(a^2(b+c)^2 + a(b+c)(b^2+c^2) + 2b^2c^2)$
external	$a(a^4(b+c)^2 + a^3(b+c)(b^2+c^2) - a^2(b^4-4b^2c^2+c^4))$ $-a(b+c)(b^4-2b^3c-2b^2c^2-2bc^3+c^4) + 2b^2c^2(b+c)^2)$
Moses circle	
internal	$a^2(b+c)^2(a^3 - a(2b^2-bc+2c^2) - (b^3+c^3))$
external	$a^2(a^3(b+c)^2 + 2a^2(b+c)(b^2+c^2) - abc(b-c)^2$ $-(b-c)^2(b+c)(b^2+bc+c^2))$

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Two Triangle Centers Associated with the Excircles

Milorad R. Stevanović

Abstract. The triangle formed by the second intersections of the bisectors of a triangle and the respective excircles is perspective to each of the medial and intouch triangles. We identify the perspectors. In the former case, the perspector is closely related to the Yff center of congruence.

1. Introduction

In this note we construct two triangle centers associated with the excircles. Given a triangle ABC, let A' be the "second" intersection of the bisector of angle A with the A-excircle, which is outside the segment AI_a, I_a being the A-excenter. Similarly, define B' and C'.

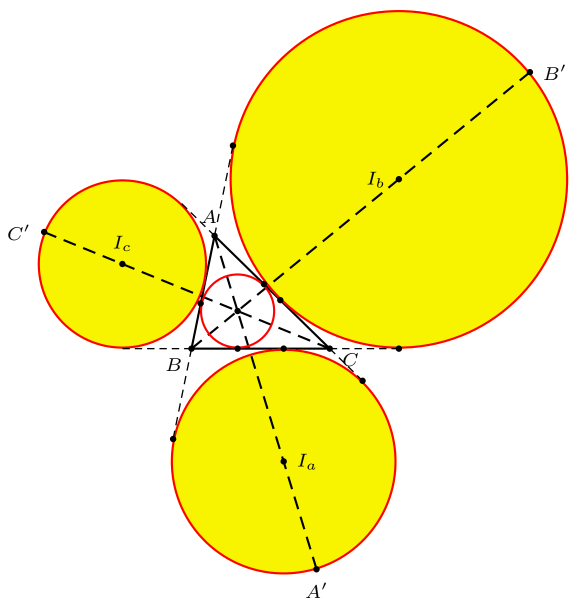

Figure 1

Theorem 1. *Triangle $A'B'C'$ is perspective with the medial triangle at the Yff center of congruence of the latter triangle, namely, the point P with homogeneous barycentric coordinates*

$$\left(\sin\frac{B}{2}+\sin\frac{C}{2}:\sin\frac{C}{2}+\sin\frac{A}{2}:\sin\frac{A}{2}+\sin\frac{B}{2}\right)$$

with respect to ABC.

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Theorem 2. *Triangle $A'B'C'$ is perspective with the intouch triangle at the point Q with homogeneous barycentric coordinates*

$$\left(\tan\frac{A}{2}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right):\tan\frac{B}{2}\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right):\tan\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\right).$$

Remark. These triangle centers now appear as X_{2090} and X_{2091} in [2].

2. Notations and preliminaries

We shall make use of the following notations. In a triangle ABC of sidelengths a, b, c, circumradius R, inradius r, and semiperimeter s, let

$$s_a=\sin\tfrac{A}{2},\quad s_b=\sin\tfrac{B}{2},\quad s_c=\sin\tfrac{C}{2};$$

$$c_a=\cos\tfrac{A}{2},\quad c_b=\cos\tfrac{B}{2},\quad c_c=\cos\tfrac{C}{2}.$$

The following formulae can be found, for example, in [1].

$$\begin{array}{lll} r=4Rs_as_bs_c, & s=4Rc_ac_bc_c; & \\ s-a=4Rc_as_bs_c, & s-b=4Rs_ac_bs_c, & s-c=4Rs_as_bc_c. \end{array}$$

2.1. *The medial triangle.* The medial triangle $A_1B_1C_1$ has vertices the midpoints of the sides BC, CA, AB of triangle ABC. From

$$\mathbf{A_1}=\frac{\mathbf{B}+\mathbf{C}}{2},\qquad \mathbf{B}_1=\frac{\mathbf{C}+\mathbf{A}}{2},\qquad \mathbf{C}_1=\frac{\mathbf{A}+\mathbf{B}}{2},$$

we have

$$\mathbf{A}=\mathbf{B}_1+\mathbf{C}_1-\mathbf{A}_1,\qquad \mathbf{B}=\mathbf{C}_1+\mathbf{A}_1-\mathbf{B}_1,\qquad \mathbf{C}=\mathbf{A}_1+\mathbf{B}_1-\mathbf{C}_1. \quad (1)$$

Lemma 3. *The barycentric coordinates of the excenters with respect to the medial triangle are*

$$\begin{aligned}
\mathbf{I}_a&=\frac{s\cdot\mathbf{A}_1-(s-c)\mathbf{B}_1-(s-b)\mathbf{C}_1}{s-a},\\
\mathbf{I}_b&=\frac{-(s-c)\mathbf{A}_1+s\cdot\mathbf{B}_1-(s-a)\mathbf{C}_1}{s-b},\\
\mathbf{I}_c&=\frac{-(s-b)\mathbf{A}_1-(s-a)\mathbf{B}_1+s\cdot\mathbf{C}_1}{s-c}.
\end{aligned}$$

Proof. It is enough to compute the coordinates of the excenter I_a:

$$\begin{aligned}
\mathbf{I}_a&=\frac{-a\cdot\mathbf{A}+b\cdot\mathbf{B}+c\cdot\mathbf{C}}{b+c-a}\\
&=\frac{-a(\mathbf{B}_1+\mathbf{C}_1-\mathbf{A}_1)+b(\mathbf{C}_1+\mathbf{A}_1-\mathbf{B}_1)+c(\mathbf{A}_1+\mathbf{B}_1-\mathbf{C}_1)}{b+c-a}\\
&=\frac{(a+b+c)\mathbf{A}_1-(a+b-c)\mathbf{B}_1-(c+a-b)\mathbf{C}_1}{b+c-a}\\
&=\frac{s\cdot\mathbf{A}_1-(s-c)\mathbf{B}_1-(s-b)\mathbf{C}_1}{s-a}.
\end{aligned}$$

□

2.2. *The intouch triangle.* The vertices of the intouch triangle are the points of tangency of the incircle with the sides. These are

$$\mathbf{X}=\frac{(s-c)\mathbf{B}+(s-b)\mathbf{C}}{a},\quad \mathbf{Y}=\frac{(s-c)\mathbf{A}+(s-a)\mathbf{C}}{b},\quad \mathbf{Z}=\frac{(s-b)\mathbf{A}+(s-a)\mathbf{B}}{c}.$$

Equivalently,

$$\begin{aligned}\mathbf{A}=&\frac{-a(s-a)\mathbf{X}+b(s-b)\mathbf{Y}+c(s-c)\mathbf{Z}}{2(s-b)(s-c)},\\ \mathbf{B}=&\frac{a(s-a)\mathbf{X}-b(s-b)\mathbf{Y}+c(s-c)\mathbf{Z}}{2(s-c)(s-a)},\\ \mathbf{C}=&\frac{a(s-a)\mathbf{X}+b(s-b)\mathbf{Y}-c(s-c)\mathbf{Z}}{2(s-a)(s-b)}.\end{aligned}\tag{2}$$

Lemma 4. *The barycentric coordinates of the excenters with respect to the intouch triangle are*

$$\begin{aligned}\mathbf{I}_a=&\frac{a(bc-(s-a)^2)\mathbf{X}-b(s-b)^2\mathbf{Y}-c(s-c)^2\mathbf{Z}}{2(s-a)(s-b)(s-c)},\\ \mathbf{I}_b=&\frac{-a(s-a)^2\mathbf{X}+b(ca-(s-b)^2)\mathbf{Y}-c(s-c)^2\mathbf{Z}}{2(s-a)(s-b)(s-c)},\\ \mathbf{I}_c=&\frac{-a(s-a)^2\mathbf{X}-b(s-b)^2\mathbf{Y}+c(ab-(s-c)^2)\mathbf{Z}}{2(s-a)(s-b)(s-c)}.\end{aligned}$$

3. Proof of Theorem 1

We compute the barycentric coordinates of A' with respect to the medial triangle. Note that A' divides AI_a externally in the ratio $AA':A'I_a=1+s_a:-s_a$. It follows that

$$\begin{aligned}\mathbf{A}'=&(1+s_a)\mathbf{I}_a-s_a\cdot\mathbf{A}\\ =&\frac{1+s_a}{s-a}(s\cdot\mathbf{A}_1-(s-c)\mathbf{B}_1-(s-b)\mathbf{C}_1)-s_a(\mathbf{B}_1+\mathbf{C}_1-\mathbf{A}_1).\end{aligned}$$

From this, the homogeneous barycentric coordinates of A' with respect to $A_1B_1C_1$ are

$$\begin{aligned}&(1+s_a)s+s_a(s-a):-(1+s_a)(s-c)-s_a(s-a)\\ &\qquad :-(1+s_a)(s-b)-s_a(s-a)\\ =&s+s_a(b+c):-((s-c)+s_ab):-((s-b)+s_ac)\\ =&4Rc_ac_bc_c+4Rs_a(s_bc_b+s_cc_c):-4R(s_as_bc_c+s_as_bc_b):-4R(s_ac_bs_c+s_as_cc_c)\\ =&-\frac{c_ac_bc_c+s_a(s_bc_b+s_cc_c)}{s_a(c_b+c_c)}:s_b:s_c.\end{aligned}$$

Similarly,

$$B' = \left(s_a : -\frac{c_a c_b c_c + s_b(s_c c_c + s_a c_a)}{s_b(c_c + c_a)} : s_c\right),$$
$$C' = \left(s_a : s_b : -\frac{c_a c_b c_c + s_c(s_a c_a + s_b c_b)}{s_c(c_a + c_b)}\right).$$

From these, it is clear that $A'B'C'$ and the medial triangle are perspective at the point with coordinates $(s_a : s_b : s_c)$ relative to $A_1B_1C_1$. This is clearly the Yff center of congruence of the medial triangle. See Figure 2. Its coordinates with respect to ABC are

$$(s_b + s_c : s_c + s_a : s_a + s_b).$$

This completes the proof of Theorem 1.

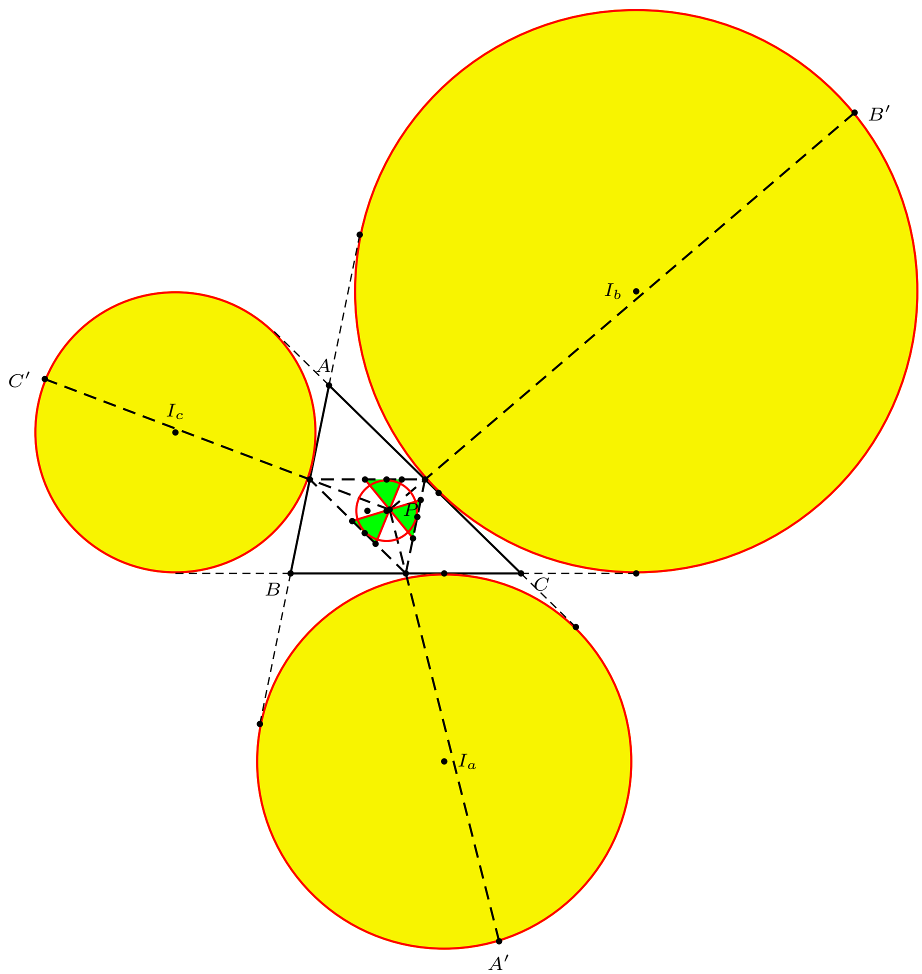

Figure 2

Remark. In triangle ABC, let A'', B'', C'' be the feet of the bisectors of angles BIC, CIA, AIB respectively on sides BC, CA, AB. Triangles $A''B''C''$ and ABC are perspective at the Yff center of congruence X_{174}, *i.e.*, if the perpendiculars from X_{174} to the bisectors of the angles of ABC intersect the sides of triangle ABC at X_b, X_c, Y_a, Y_c, Z_a, Z_b (see Figure 3), then the triangles $X_{174}X_bX_c$, $Y_aX_{174}Y_c$ and $Z_aZ_bX_{174}$ are congruent. See [3].

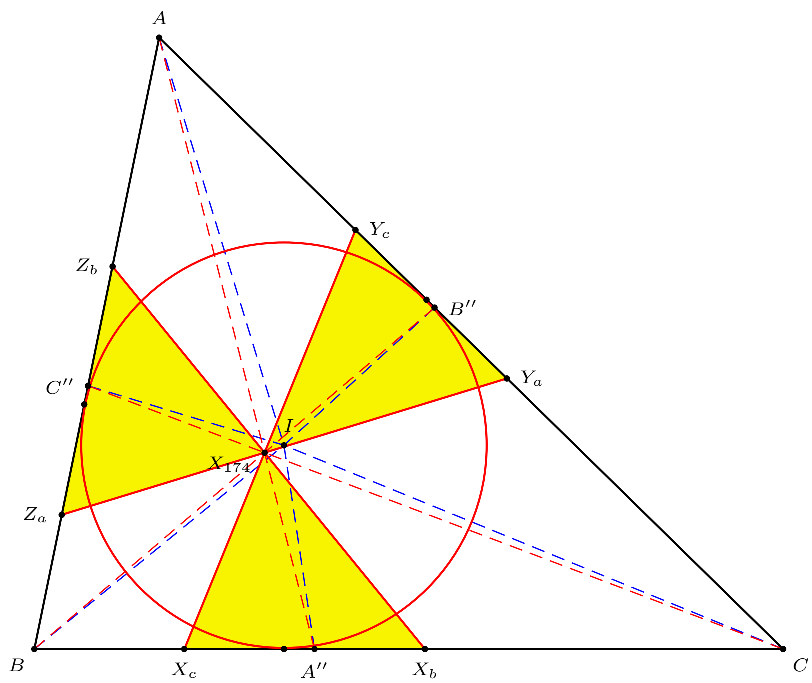

Figure 3

4. Proof of Theorem 2

Consider the coordinates of $\mathbf{A}' = (1+s_a)\mathbf{I}_a - s_a \cdot \mathbf{A}$ with respect to the intouch triangle XYZ. By Lemma 3, the Y-coordinate is

$$\begin{aligned}
&\frac{-(1+s_a)b(s-b)^2 - s_a b(s-a)(s-b)}{2(s-a)(s-b)(s-c)}\\
=&\frac{-b(s-b)((1+s_a)(s-b)+s_a(s-a))}{2(s-a)(s-b)(s-c)}\\
=&\frac{-b(s-b)(s-b+s_a\cdot c)}{2(s-a)(s-b)(s-c)}\\
=&\frac{-(c_b+c_c)}{2c_a c_b c_c}\cdot\frac{c_b^2}{s_b}.
\end{aligned}$$

Similarly for the Z-coordinate is $\frac{-(c_b+c_c)}{2c_ac_bc_c}\cdot\frac{c_c^2}{s_c}$. Therefore, $A'B'C'$ is perspective with XYZ at

$$Q=\left(\frac{c_a^2}{s_a}:\frac{c_b^2}{s_b}:\frac{c_c^2}{s_c}\right).$$

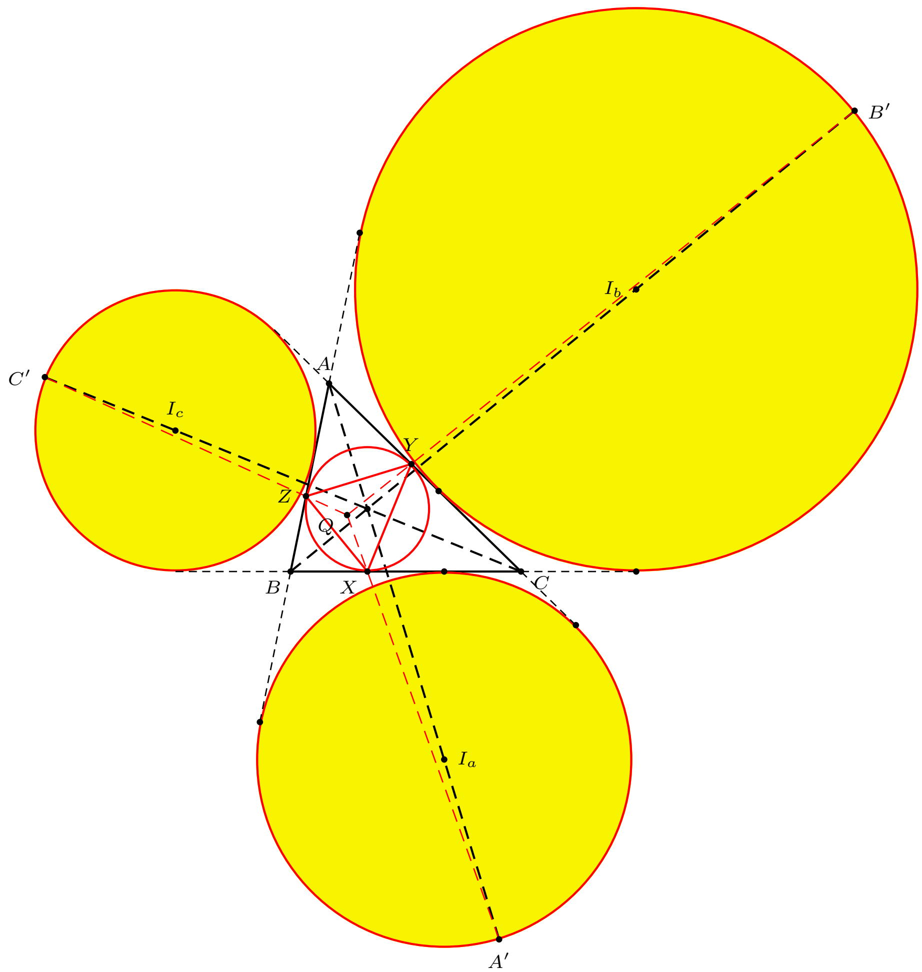

Figure 4

Note that the angles of the intouch triangles are $X=\frac{B+C}{2}$, $Y=\frac{C+A}{2}$, and $Z=\frac{A+B}{2}$. This means

$$s_a=\cos\frac{B+C}{2}=\cos X,\qquad c_a=\sin\frac{B+C}{2}=\sin X,$$

etc. It follows that Q has homogeneous barycentric coordinates

$$\left(\frac{\sin^2 X}{\cos X}:\frac{\sin^2 Y}{\cos Y}:\frac{\sin^2 Z}{\cos Z}\right)$$

and is the Clawson point of the intouch triangle XYZ. With respect to triangle ABC, this perspector Q has coordinates given by

$$\begin{aligned}
&\left(\frac{a(s-a)}{s_a}+\frac{b(s-b)}{s_b}+\frac{c(s-c)}{s_c}\right)\mathbf{Q}\\
=&\frac{a(s-a)\mathbf{X}}{s_a}+\frac{b(s-b)\mathbf{Y}}{s_b}+\frac{c(s-c)\mathbf{Z}}{s_c}\\
=&\frac{(s-b)(s-c)(s_b+s_c)}{s_bs_c}\mathbf{A}+\frac{(s-c)(s-a)(s_c+s_a)}{s_cs_a}\mathbf{B}+\frac{(s-a)(s-b)(s_a+s_b)}{s_as_b}\mathbf{C}\\
=&(4R)^2s_a^2c_bc_c(s_b+s_c)\mathbf{A}+(4R)^2s_b^2c_cc_a(s_c+s_a)\mathbf{B}+(4R)^2s_c^2c_ac_b(s_a+s_b)\mathbf{C}\\
=&(4R)^2c_ac_bc_c\left(\frac{s_a^2(s_b+s_c)}{c_a}\cdot\mathbf{A}+\frac{s_b^2(s_c+s_a)}{c_b}\cdot\mathbf{B}+\frac{s_c^2(s_a+s_b)}{c_c}\cdot\mathbf{C}\right).
\end{aligned}$$

Therefore, the homogeneous barycentric coordinates of Q with respect to ABC are

$$\begin{aligned}
&\left(\frac{s_a^2(s_b+s_c)}{c_a}:\frac{s_b^2(s_c+s_a)}{c_b}:\frac{s_c^2(s_a+s_b)}{c_c}\right)\\
=&\left(\tan\frac{A}{2}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right):\tan\frac{B}{2}\left(\csc\frac{C}{2}+\csc\frac{A}{2}\right):\tan\frac{C}{2}\left(\csc\frac{A}{2}+\csc\frac{B}{2}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 2.

Inasmuch as Q is the Clawson point of the intouch triangle, it is interesting to point out that the congruent isoscelizers point X_{173}, a point closely related to the Yff center of congruence X_{174} and with coordinates

$$(a(-c_a+c_b+c_c):b(c_a-c_b+c_c):c(c_a+c_b-c_c)),$$

is the Clawson point of the excentral triangle $I_aI_bI_c$ (which is homothetic to the intouch triangle at X_{57}). This fact was stated in an earlier edition of [2], and can be easily proved by the method of this paper.

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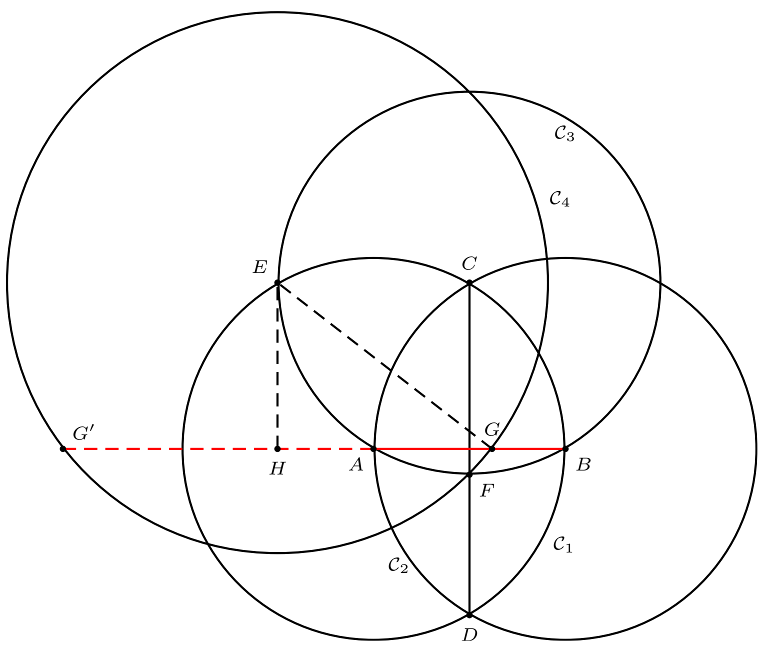
A 5-step Division of a Segment in the Golden Section

Kurt Hofstetter

Abstract. Using ruler and compass only in five steps, we divide a given segment in the golden section.

Inasmuch as we have given in [1] a construction of the golden section by drawing 5 circular arcs, we present here a very simple division of a given segment in the golden section, in 5 euclidean steps, using ruler and compass only. For two points P and Q, we denote by $P(Q)$ the circle with P as center and PQ as radius.



Construction. Given a segment AB, construct

(1) $\mathcal{C}_1 = A(B)$,
(2) $\mathcal{C}_2 = B(A)$, intersecting $\mathcal{C}_1$ at C and D,
(3) $\mathcal{C}_3 = C(A)$, intersecting $\mathcal{C}_1$ again at E,
(4) the segment CD to intersect $\mathcal{C}_3$ at F,
(5) $\mathcal{C}_4 = E(F)$ to intersect AB at G.

The point G divides the segment AB in the golden section.

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Proof. Suppose AB has unit length. Then $CD = \sqrt{3}$ and $EG = EF = \sqrt{2}$. Let H be the orthogonal projection of E on the line AB. Since $HA = \frac{1}{2}$, and $HG^2 = EG^2 - EH^2 = 2 - \frac{3}{4} = \frac{5}{4}$, we have $AG = HG - HA = \frac{1}{2}(\sqrt{5} - 1)$. This shows that G divides AB in the golden section. □

Remark. The other intersection G' of $\mathcal{C}_4$ and the line AB is such that $G'A : AB = \frac{1}{2}(\sqrt{5} + 1) : 1$.

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Circumcenters of Residual Triangles

Eckart Schmidt

Abstract. This paper is an extension of Mario Dalcín's work on isotomic inscribed triangles and their residuals [1]. Considering the circumcircles of residual triangles with respect to isotomic inscribed triangles there are two congruent triangles of circumcenters. We show that there is a rotation mapping these triangles to each other. The center and angle of rotation depend on the Miquel points. Furthermore we give an interesting generalization of Dalcin's definitive example.

1. Introduction

If X, Y, Z are points on the sides of a triangle ABC, there are three residual triangles AZY, BXZ, CYX. The circumcenters of these triangles form a triangle $O_aO_bO_c$ similar to the reference triangle ABC [2]. The circumcircles have a common point M by Miquel's theorem. The lines MX, MY, MZ and the corresponding side lines have the same angle of intersection $\mu = (AY, YM) = (BZ, ZM) = (CX, XM)$. The angles are directed angles measured between 0 and π.

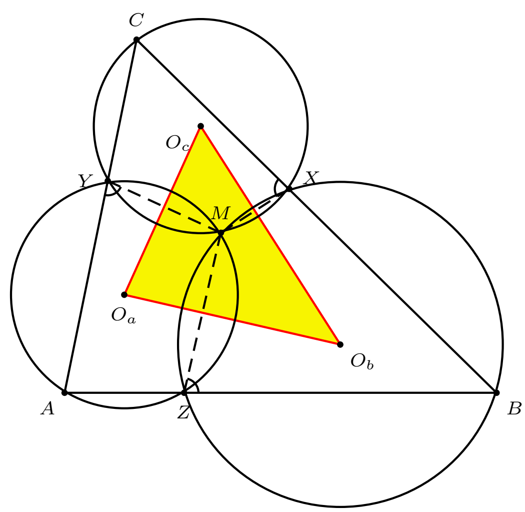

Figure 1

Dalcín considers isotomic inscribed triangles XYZ and $X'Y'Z'$. Here, X', Y', Z' are the reflections of X, Y, Z in the midpoints of the respective sides. The triangle XYZ may or may not be cevian. If it is the cevian triangle of a point P, then $X'Y'Z'$ is the cevian triangle of the isotomic conjugate of P. The

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corresponding Miquel point M' of X', Y', Z' has Miquel angle $\mu' = \pi - \mu$. The circumcircles of the residual triangles $AZ'Y'$, $BX'Z'$, $CY'X'$ give further points of intersection. The intersections A' of the circles AZY and $AZ'Y'$, B' of BXZ and $BX'Z'$, and C' of CYX and $CY'X'$ form a triangle $A'B'C'$ perspective to the reference triangle ABC with the center of perspectivity Q. See Figure 2. It can be shown that the points M, M', A', B', C', Q and the circumcenter O of the reference triangle lie on a circle with the diameter OQ.

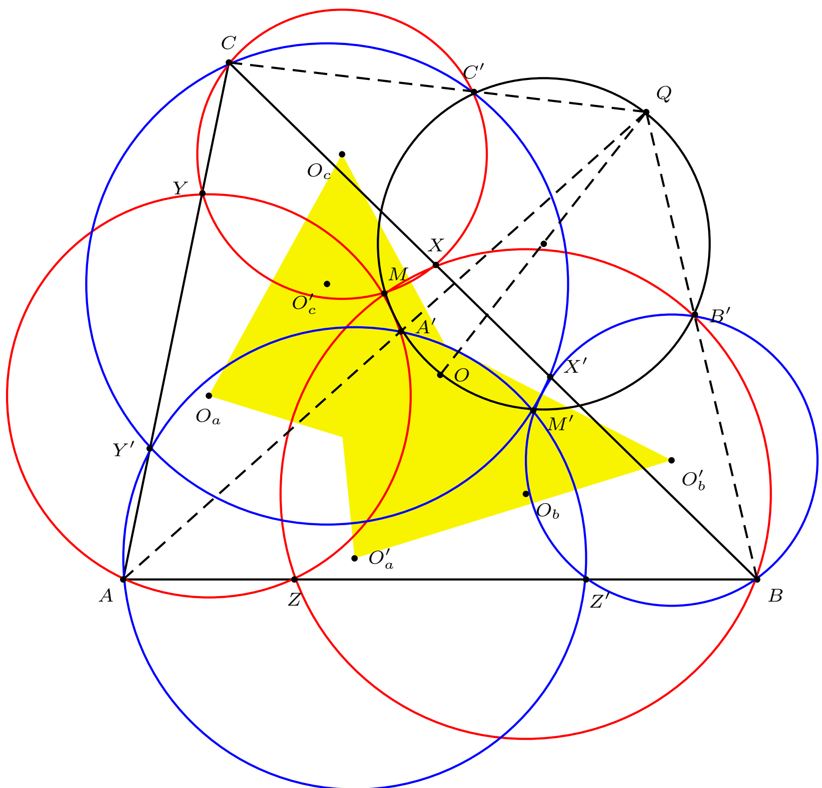

Figure 2

These results can be proved by analytical calculations. We make use of homogeneous barycentric coordinates. Let X, Y, Z divide the sides BC, CA, AB respectively in the ratios

$$BX : XC = x : 1, \qquad CY : YA = y : 1, \qquad AZ : ZB = z : 1.$$

These points have coordinates

$$\begin{aligned} X &= (0 : 1 : x), \quad & Y &= (y : 0 : 1), \quad & Z &= (1 : z : 0); \\ X' &= (0 : x : 1), \quad & Y' &= (1 : 0 : y), \quad & Z' &= (z : 1 : 0). \end{aligned}$$

The circumcenter, the Miquel points, and the center of perspectivity are the points

$$\begin{aligned}
O =&(a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)),\\
M =&(a^2x(1+y)(1+z) - b^2xy(1+x)(1+z) - c^2(1+x)(1+y) : \cdots : \cdots),\\
M' =&(a^2x(1+y)(1+z) - b^2(1+x)(1+z) - c^2xz(1+x)(1+y) : \cdots : \cdots),\\
Q =&\left(\frac{(1-x)a^2}{1+x} : \frac{(1-y)b^2}{1+y} : \frac{(1-z)c^2}{1+z}\right).
\end{aligned}$$

The Miquel angle μ is given by

$$\cot\mu = \frac{1-yz}{(1+y)(1+z)}\cot A + \frac{1-zx}{(1+z)(1+x)}\cot B + \frac{1-xy}{(1+x)(1+y)}\cot C.$$

For example, let X, Y, Z divide the sides in the same ratio k, *i.e.*, $x = y = z = k$, then we have

$$\begin{aligned}
M =&(a^2(-c^2+a^2k-b^2k^2) : b^2(-a^2+b^2k-c^2k^2) : c^2(-b^2+c^2k-a^2k^2)),\\
M' =&(a^2(-b^2+a^2k-c^2k^2) : b^2(-c^2+b^2k-a^2k^2) : c^2(-a^2+c^2k-b^2k^2)),\\
Q =&(a^2 : b^2 : c^2) = X_6(\text{ Lemoine point});\\
\cot\mu =&\frac{1-k}{1+k}\cot\omega,
\end{aligned}$$

where ω is the Brocard angle.

2. Two triangles of circumcenters

Considering the circumcenters of the residual triangles for XYZ and $X'Y'Z'$, Dalcín ([1, Theorem 10]) has shown that the triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are congruent. We show that there is a rotation mapping $O_aO_bO_c$ to $O'_aO'_bO'_c$. This rotation also maps the Miquel point M to the circumcenter O, and O to the other Miquel point M'. See Figure 3. The center of rotation is therefore the midpoint of OQ. This center of rotation is situated with respect to $O_aO_bO_c$ and $O'_aO'_bO'_c$ as the center of perspectivity with respect to the reference triangle ABC. The angle φ of rotation is given by

$$\varphi = \pi - 2\mu.$$

The similarity ratio of triangles $O_aO_bO_c$ and ABC is

$$\frac{1}{2\cos\frac{\varphi}{2}} = \frac{1}{2\sin\mu},$$

similarly for triangle $O'_aO'_bO'_c$.

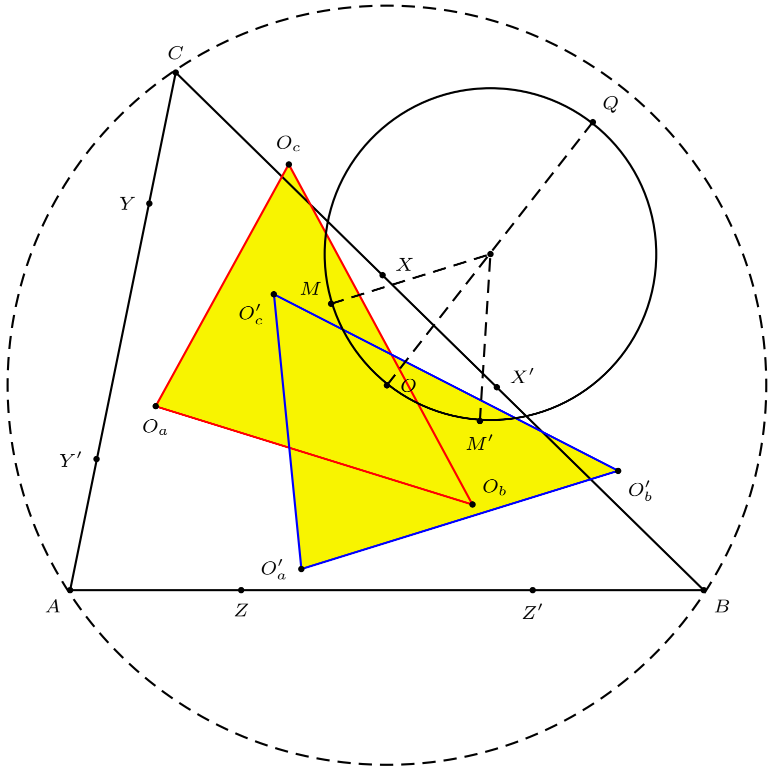

Figure 3

3. Dalcín's example

If we choose X, Y, Z as the points of tangency of the incircle with the sides, XYZ is the cevian triangle of the Gergonne point G_e and $X'Y'Z'$ is the cevian triangle of the Nagel point N_a. The Miquel point M is the incenter I and the Miquel point M' is the reflection of I in O, *i.e.*,

$$X_{40} = (a(a^3 - b^3 - c^3 + (a-b)(a-c)(b+c)) : \cdots : \cdots).$$

In this case, $O_aO_bO_c$ is homothetic to ABC at M, with factor $\frac{1}{2}$. This is also the case when XYZ is the cevian triangle of the Nagel point, with $M = X_{40}$.

Therefore, the circle described in §2, degenerates into a line. The center of perspectivity $Q(a(b-c) : b(c-a) : c(a-b))$ is a point of infinity. The triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are homothetic to the triangle ABC at the Miquel points M and M' with factor $\frac{1}{2}$. There is a parallel translation mapping $O_aO_bO_c$ to $O'_aO'_bO'_c$.

The fact that ABC is homothetic to $OaObOc$ with the factor $\frac{1}{2}$ does not only hold for the Gergonne and Nagel points. Here are further examples.

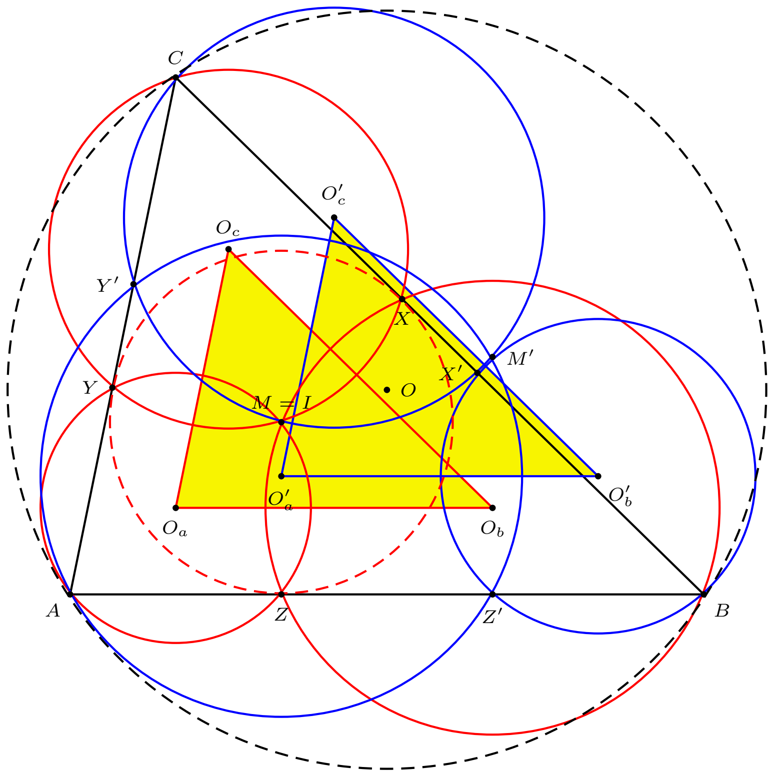

Figure 4

P	Homothetic center and Miquel point M
centroid G	circumcenter O
orthocenter H	H
X_{69}	X_{20}
X_{189}	X_{84}
X_{253}	X_{64}
X_{329}	X_{1490}

These points $P(u : v : w)$, whose cevian triangle is also the pedal triangle of the point M, lie on the Lucas cubic [1]

$$(b^2+c^2-a^2)u(v^2-w^2)+(c^2+a^2-b^2)v(w^2-u^2)+(a^2+b^2-c^2)w(u^2-v^2) = 0.$$

The points M lie on the Darboux cubic. [2] Isotomic points P and $P^\wedge$ on the Lucas cubic have corresponding points M and M' on the Darboux cubic symmetric with respect to the circumcenter. Isogonal points M and M^* on the Darboux cubic have

[1]The Lucas cubic is invariant under the isotomic conjugation and the isotomic conjugate X_{69} of the orthocenter is the pivot point.

[2]The Darboux cubic is invariant under the isogonal conjugation and the pivot point is the De-Longchamps point X_{20}, the reflection of the orthocenter in the circumcenter. It is symmetric with respect to the circumcenter.

corresponding points P and P' on the Lucas cubic with $P' = P^{\wedge * \wedge}$. Here, $()^*$ is the isogonal conjugation with respect to the anticomplementary triangle of ABC. The line PM and MM^* all correspond with the DeLongchamps point X_{20} and so the points P, $P^{\wedge * \wedge}$, M, M^* and X_{20} are collinear. For example, for $P = N_a$, the five points N_a, X_{189}, X_{40}, X_{84}, X_{20} are collinear.

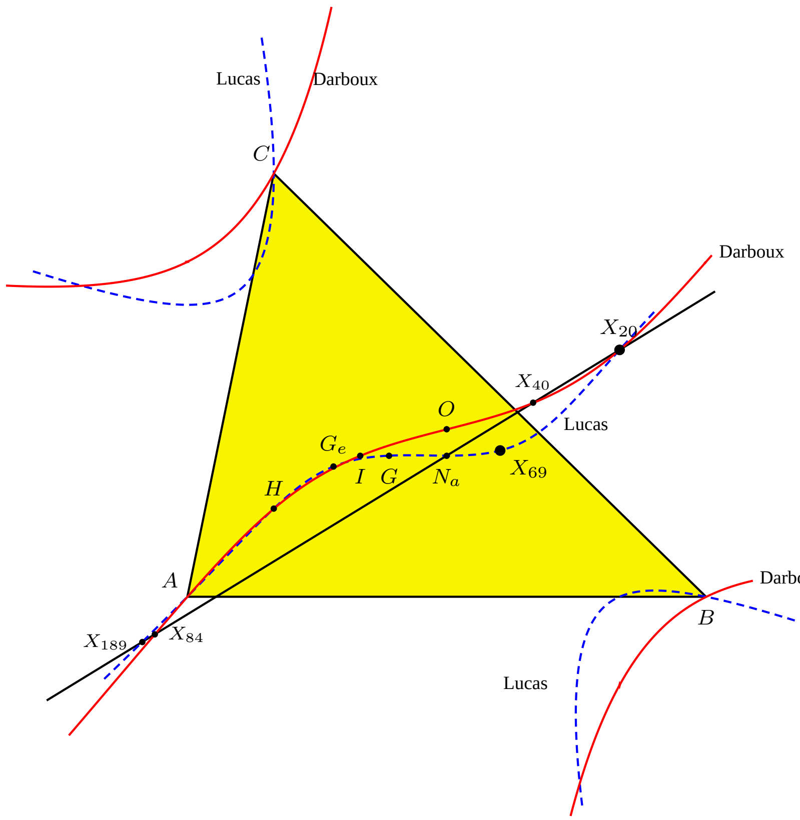

Figure 5. The Darboux and Lucas cubics

4. Further results

Dalcín's example can be extended. The cevian triangle of the Gergonne point G_e is the triangle of tangency of the incircle, the cevian triangle of the Nagel point N_a is the triangle of the inner points of tangency of the excircles. Consider the points of tangency of the excircles with the sidelines:

A−excircle	$B_a = (-a+b-c : 0 : a+b+c)$	with CA
	$C_a = (-a-b+c : a+b+c : 0)$	with AB
B−excircle	$A_b = (0 : a-b-c : a+b+c)$	with BC
	$C_b = (a+b+c : -a-b+c : 0)$	with AB
C−excircle	$A_c = (0 : a+b+c : a-b-c)$	with BC
	$B_c = (a+b+c : 0 : -a+b-c)$	with CA

The point pairs (A_b, A_c), (B_c, B_a) and (C_a, C_b) are symmetric with respect to the corresponding midpoints of the sides. If $XYZ = A_bB_cC_a$, then $X'Y'Z' = A_cB_aC_b$. See Figure 6.

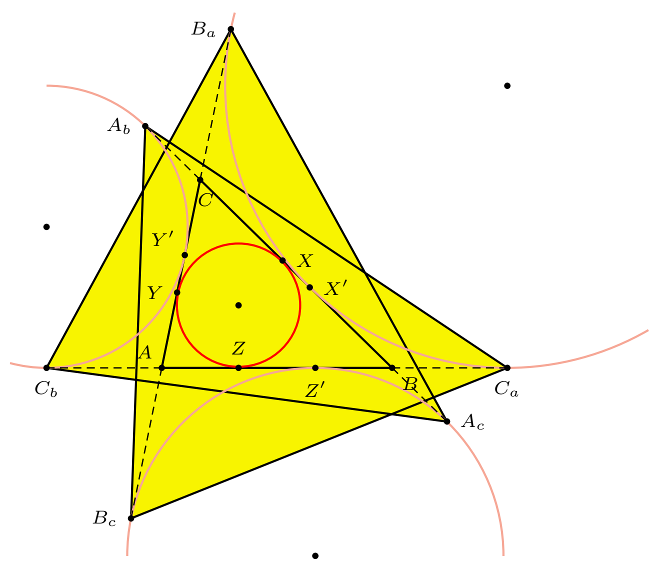

Figure 6

Consider the residual triangles of $A_bB_cC_a$ and those of $A_cB_aC_b$, with the circumcenters. The two congruent triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ have a common area

$$\frac{\triangle}{4} + \frac{(ab+bc+ca)^2}{16\triangle}.$$

The center of perspectivity is

$$Q = (a(b+c) : b(c+a) : c(a+b)) = X_{37}.$$

The center of rotation which maps $O_aO_bO_c$ to $O'_aO'_bO'_c$ is the midpoint of OQ. The point X_{37} of a triangle is the complement of the isotomic conjugate of the incenter. The center of rotation is the common point X_{37} of $O_aO_bO_c$ and $O'_aO'_bO'_c$. The angle of rotation is given by

$$\tan\frac{\varphi}{2} = \frac{ab+bc+ca}{2\triangle} = \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}.$$

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Circumrhombi

Floor van Lamoen

Abstract. We consider rhombi circumscribing a given triangle ABC in the sense that one vertex of the rhombus coincides with a vertex of ABC, while the sidelines of the rhombus opposite to this vertex pass through the two remaining vertices of ABC respectively. We construct some new triangle centers associated with these rhombi.

1. Introduction

In this paper we further study the rhombi circumscribing a given reference triangle ABC that the author defined in [4]. These rhombi circumscribe ABC in the sense that each of them shares one vertex with ABC, with its two opposite sides passing through the two remaining vertices of ABC. These rhombi will depend on a fixed angle ϕ and its complement $\overline{\phi} = \frac{\pi}{2} - \phi$. More precisely, for a given ϕ, the rhombus $\mathcal{R}_A(\phi) = AA_cA_aA_b$ will be such that $\angle A_bAA_c = 2\phi$, $B \in A_cA_a$ and $C \in A_bA_a$. Similarly there are rhombi $BB_aB_bB_c$ and $CC_bC_cC_a$.

In [4] it was shown that the vertices of the rhombi opposite to ABC form a triangle $A_aB_bC_c$ perspective to ABC, and that their perspector lies on the Kiepert hyperbola. We give another proof of this result (Theorem 3).

We denote by $\mathcal{K}(\phi) = A^\phi B^\phi C^\phi$ the Kiepert triangle formed by isosceles triangles built on the sides of ABC with base angles ϕ. When the isosceles triangles are constructed outwardly, $\phi > 0$. Otherwise, $\phi < 0$. These vertices have homogeneous barycentric coordinates [1]

$$\begin{aligned} A^\phi =&(-(S_B+S_C) : S_C+S_\phi : S_B+S_\phi),\\ B^\phi =&(S_C+S_\phi : -(S_C+S_A) : S_A+S_\phi),\\ C^\phi =&(S_B+S_\phi : S_A+S_\phi : -(S_A+S_B)). \end{aligned}$$

From these it is clear that $\mathcal{K}(\phi)$ is perspective with ABC at the point

$$K(\phi) = \left(\frac{1}{S_A+S_\phi} : \frac{1}{S_B+S_\phi} : \frac{1}{S_C+S_\phi}\right).$$

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[1]For the notations, see [5].

2. Circumrhombi to a triangle

Theorem 1. *Consider* $\triangle ABC$ *and* $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$. *There are unique rhombi* $\mathcal{R}_A(\phi) = AA_cA_aA_b$, $\mathcal{R}_B(\phi) = BB_aB_bB_c$ *and* $\mathcal{R}_C(\phi) = CC_bC_cC_a$ *with*

$$\angle A_bAA_c = \angle B_cBB_a = \angle C_aCC_b = 2\phi,$$

and $B \in A_cA_a$ *and* $C \in A_bA_a$. *Similarly there are rhombi* $C \in B_aB_b$, $A \in B_cB_b$, $A \in C_bC_c$, $B \in C_aC_c$.

Proof. It is enough to show the construction of $\mathcal{R}_A = \mathcal{R}_A(\phi)$.

Let B_r be the image of B after a rotation through $-2\overline{\phi}$ about A, and C_r the image of C after a rotation through $2\overline{\phi}$ about A. Then let $A_a = B_rC \cap C_rB$. Points $A_c \in C_rA_a$ and $A_b \in B_rA_a$ can be constructed in such a way that $AA_cA_aA_b$ is a parallelogram. Observe that $\triangle AC_rB \equiv \triangle ACB_r$, so that the perpendicular distances from A to lines B_rA_a and C_rA_a are equal. And $AA_cA_aA_b$ must be a rhombus. See Figure 1.

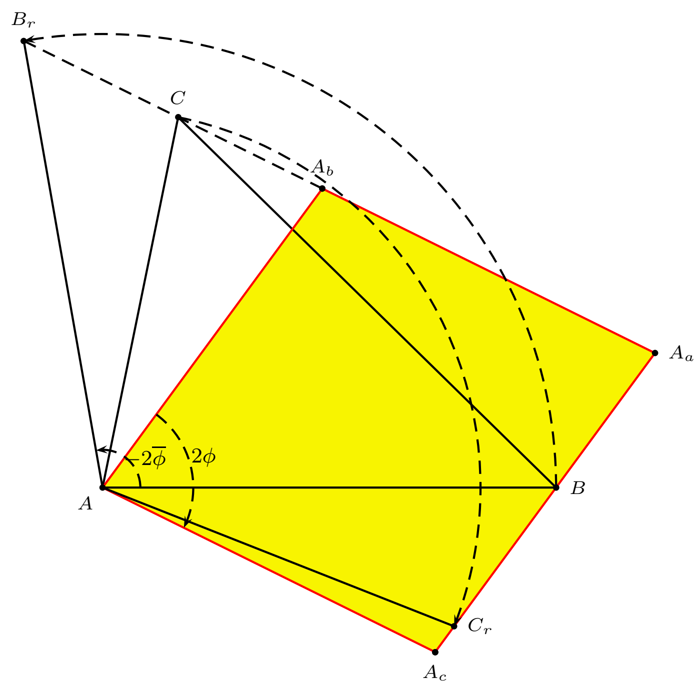

Figure 1

Note that line $B_rC = A_aA_b$ is the image of line $C_rB = A_aA_c$ after rotation through $2\overline{\phi}$ about A, so that the directed angle $\angle A_cA_aA_b = 2\phi$. It follows that $AA_cA_aA_b$ is the rhombus desired in the theorem.

It is easy to see that this is the unique rhombus fulfilling these requirements. When we rotate the complete figure of $\triangle ABC$ and rhombus $AA_cA_aA_b$ through $-2\overline{\phi}$ about A, and let B_r be the image of B again, we see immediately that $B_r \in A_aC$. In the same way we see that the image of C after rotation through $2\overline{\phi}$ about A must be on the line A_aB. $\square$

Consider $\mathcal{R}_A$ and $\mathcal{R}_B$. We note that $\angle AA_aB \equiv \phi \bmod \pi$ and also $\angle AB_bB \equiv \phi \bmod \pi$. This means that ABA_aB_b is cyclic. The center P of its circle should be the apex of the isosceles triangle built on AB such that $\angle APB = 2\phi$,[2] so that $P = C^{\overline{\phi}}$. This shows that $C^{\overline{\phi}}$ lies on the perpendicular bisectors of AA_a and BB_b, hence $A_bA_c \cap B_aB_c = C^{\overline{\phi}}$. See Figure 2.

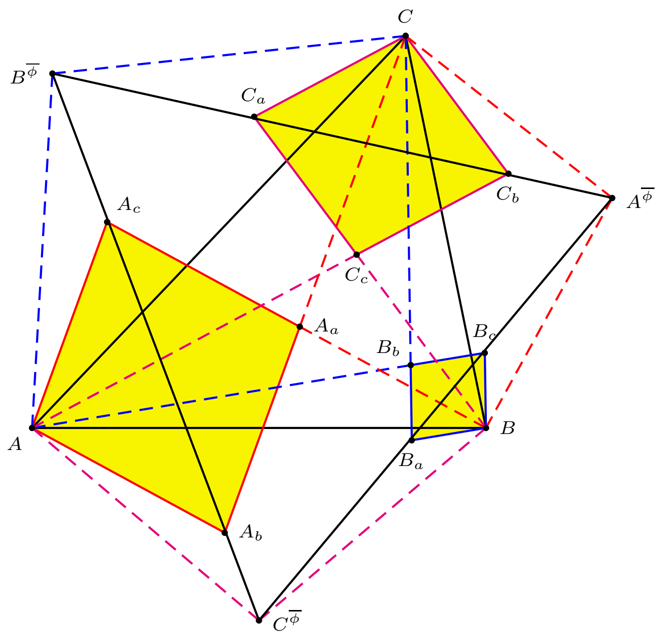

Figure 2

Theorem 2. *The diagonals A_bA_c, B_aB_c and C_aC_b of the circumrhombi $\mathcal{R}_A(\phi)$, $\mathcal{R}_B(\phi)$, $\mathcal{R}_C(\phi)$ bound the Kiepert triangle $\mathcal{K}(\overline{\phi})$.*

3. Radical center of a triad of circles

It is now interesting to further study the circles $A^{\overline{\phi}}(B)$, $B^{\overline{\phi}}(C)$ and $C^{\overline{\phi}}(A)$ with centers at the apices of $\mathcal{K}(\overline{\phi})$, passing through the vertices of ABC. Since the circle $A^{\overline{\phi}}(B)$ passes through B and C, it is represented by an equation of the form

$$a^2yz + b^2zx + c^2xy - kx(x+y+z) = 0.$$

Since it also passes through $A^{-\phi/2} = (-(S_B+S_C) : S_C - S_{\phi/2} : S_B - S_{\phi/2})$, we find

$$k = \frac{S_\phi^2 + 2S_AS_{\phi/2} - S^2}{2S_{\phi/2}} = S_A + S_\phi.$$

[2]Hence, when ϕ is negative, the apex is on the same side of AB as the vertex C.

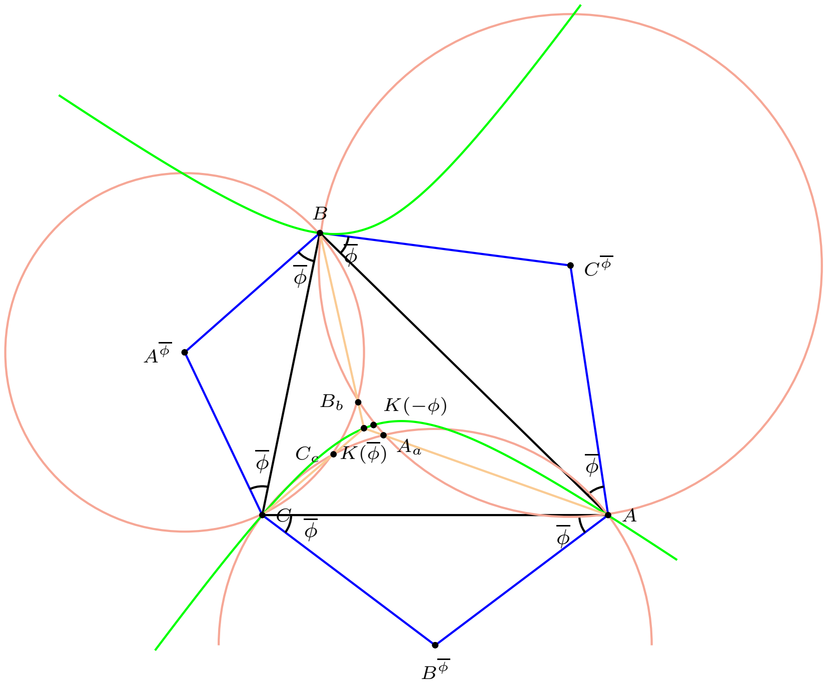

Figure 3

The equations of the three circles are thus

$$\begin{aligned}
a^2yz + b^2zx + c^2xy - (S_A + S_\phi)x(x+y+z) &= 0,\\
a^2yz + b^2zx + c^2xy - (S_B + S_\phi)y(x+y+z) &= 0,\\
a^2yz + b^2zx + c^2xy - (S_C + S_\phi)z(x+y+z) &= 0.
\end{aligned}$$

From this, it is clear that the radical center of the three circles is the point

$$K(\phi) = \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi}\right).$$

The intersections of the circles apart from A, B and C are the points

$$\begin{aligned}
A_a &= \left(\frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C + S_\phi}\right),\\
B_b &= \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C + S_\phi}\right),\\
C_c &= \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C - S_{2\phi}}\right).
\end{aligned} \tag{1}$$

Theorem 3. *The triangle $A_aB_bC_c$ is perspective to ABC and the perspector is $K(\phi)$.*

Remark. For $\phi = \pm\frac{\pi}{3}$, triangle $A_aB_bC_c$ degenerates into the Fermat point $K\left(\pm\frac{\pi}{3}\right)$.

The coordinates of the circumcenter of $A_aB_bC_c$ are too complicated to record here, even in the case of circumsquares. However, we prove the following interesting collinearity.

Theorem 4. *The circumcenters of triangles ABC and $A_aB_bC_c$ are collinear with $K(\phi)$.*

Proof. Since $P = K(\phi)$ is the radical center of $A^\phi(B)$, $B^\phi(C)$ and $C^\phi(A)$ we see that

$$\overline{PA} \cdot \overline{PA_a} = \overline{PB} \cdot \overline{PB_b} = \overline{PC} \cdot \overline{PC_c},$$

which product we will denote by Γ. When $\Gamma > 0$ then the inversion with center P and radius $\sqrt{\Gamma}$ maps A to A_a, B to B_b and C to C_c. Consequently the circumcircles of ABC and $A_aB_bC_c$ are inverses of each other, and the centers of these circles are collinear with the center of inversion.

When $\Gamma < 0$ then the inversion with center P and radius $\sqrt{-\Gamma}$ maps A, B and C to the reflections of A_a, B_b and C_c through P. And the collinearity follows in the same way as above.

When $\Gamma = 0$ the theorem is trivial. $\square$

4. Coordinates of the vertices of the circumrhombi

Along with the coordinates given in (1), we record those of the remaining vertices of the circumrhombi.

$$\begin{aligned}
A_b =&((b^2 + S\csc 2\phi)(S_B + S_\phi) : (S_A - S_{2\phi})(b^2 + S\csc 2\phi) : -(S_A - S_{2\phi})^2),\\
A_c =& \left((c^2 + S\csc 2\phi)(S_C + S_\phi) : -(S_A - S_{2\phi})^2 : (S_A - S_{2\phi})(c^2 + S\csc 2\phi)\right);\\
B_c =&(-(S_B - S_{2\phi})^2 : (c^2 + S\csc 2\phi)(S_C + S_\phi) : (S_B - S_{2\phi})(c^2 + S\csc 2\phi)),\\
B_a =& \left((S_B - S_{2\phi})(a^2 + S\csc 2\phi) : (a^2 + S\csc 2\phi)(S_A + S_\phi) : -(S_B - S_{2\phi})^2\right);\\
C_a =&((S_C - S_{2\phi})(a^2 + S\csc 2\phi) : -(S_C - S_{2\phi})^2 : (a^2 + S\csc 2\phi)(S_A + S_\phi)),\\
C_b =& \left(-(S_C - S_{2\phi})^2 : (S_C - S_{2\phi})(b^2 + S\csc 2\phi) : (b^2 + S\csc 2\phi)(S_B + S_\phi)\right).
\end{aligned} \tag{2}$$

5. The triangle $A'B'C'$

Let $A' = CC_a \cap BB_a$, $B' = CC_b \cap AA_b$ and $C' = AA_c \cap BB_c$. The coordinates of A', using (2), are

$$\begin{aligned}
A' &= \left(a^2 + S\csc 2\phi : -(S_C - S_{2\phi}) : -(S_B - S_{2\phi})\right)\\
&= \left(\frac{a^2 + S\csc 2\phi}{(S_B - S_{2\phi})(S_C - S_{2\phi})} : \frac{-1}{S_B - S_{2\phi}} : \frac{-1}{S_C - S_{2\phi}}\right);
\end{aligned}$$

Similarly for B' and C'. It is clear that $A'B'C'$ is perspective to ABC at $K(-2\phi)$. Note that in absolute barycentric coordinates,

$$\begin{aligned}
&S(\csc 2\phi + 2\cot 2\phi)A' \\
=&\left(a^2 + S\csc 2\phi, -(S_C - S_{2\phi}), -(S_B - S_{2\phi})\right) \\
=&(S_B + S_C, -(S_C + S_{\overline{\phi}}), -(S_B + S_{\overline{\phi}})) + S(\csc 2\phi, \cot 2\phi + \tan\phi, \cot 2\phi + \tan\phi) \\
=&(S_B + S_C, -(S_C + S_{\overline{\phi}}), -(S_B + S_{\overline{\phi}})) + S\csc 2\phi(1, 1, 1) \\
=&S(-2\tan\phi A^{\overline{\phi}} + 3\csc 2\phi G).
\end{aligned}$$

Now, $\frac{-2\tan\phi}{-2\tan\phi + 3\csc 2\phi} = \frac{4}{1-3\cot^2\phi}$. It follows that

$$A' = \mathsf{h}\left(G, \frac{4}{1-3\cot^2\phi}\right)(A^{\overline{\phi}}).$$

Similarly for B' and C'.

Proposition 5. *Triangles $A'B'C'$ and $\mathcal{K}(\overline{\phi})$ are homothetic at G.*

Corollary 6. *ABC is the Kiepert triangle $\mathcal{K}(-\phi)$ with respect to $A'B'C'$.*

See [5, Proposition 4].

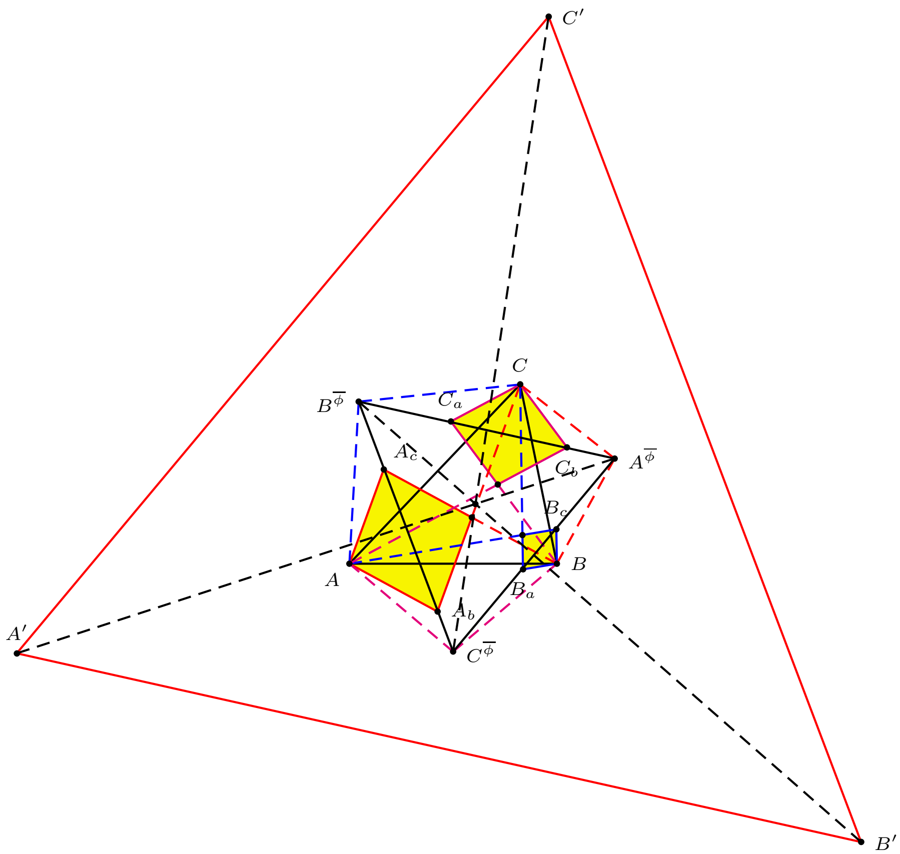

Figure 4

6. The desmic mates

Let XYZ be a triangle perspective with ABC at $P = (u : v : w)$. Its vertices have coordinates

$$X = (x : v : w), \qquad Y = (u : y : w), \qquad Z = (u : v : z),$$

for some x, y, z. The desmic mate of XYZ is the triangle with vertices $X' = BZ \cap CY$, $Y' = CX \cap AZ$, $Z' = AY \cap BX$. These have coordinates

$$X' = (u : y : z), \qquad Y' = (x : v : z), \qquad Z' = (x : y : w).$$

Lemma 7. *The triangle $X'Y'Z'$ is perspective to ABC at $(x : y : z)$ and to XYZ at $(u + x : v + y : w + z)$.*

See, for example, [1, §4].

The desmic mate of $A_aB_bC_a$ has vertices

$$\begin{aligned} A'_a &= \left(\frac{1}{S_A + S_\phi} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C - S_{2\phi}}\right), \\ B'_b &= \left(\frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B + S_\phi} : \frac{1}{S_C - S_{2\phi}}\right), \qquad (3) \\ C'_c &= \left(\frac{1}{S_A - S_{2\phi}} : \frac{1}{S_B - S_{2\phi}} : \frac{1}{S_C + S_\phi}\right). \end{aligned}$$

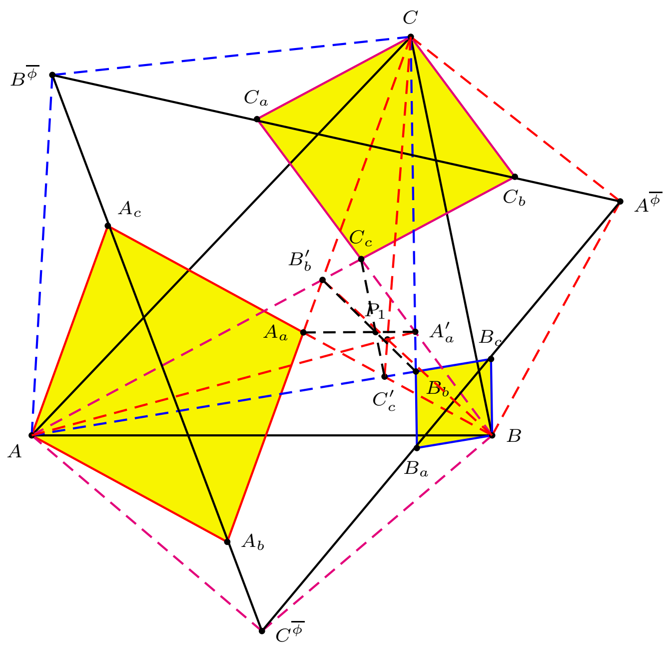

Figure 5

Proposition 8. *Triangle $A'_aB'_bC'_c$ is perspective to ABC at $K(-2\phi)$. It is also perspective to $A_aB_bC_c$ at*

$$P_1(\phi)=\left(\frac{2S_A+S\csc 2\phi}{(S_A+S_\phi)(S_A+S_{2\phi})}:\frac{2S_B+S\csc 2\phi}{(S_B+S_\phi)(S_B+S_{2\phi})}:\frac{2S_C+S\csc 2\phi}{(S_C+S_\phi)(S_C+S_{2\phi})}\right).$$

See Figure 5.

The desmic mate of $A'B'C'$ has vertices

$$\begin{aligned}A'' =&(-(S_B-S_{2\phi})(S_C-S_{2\phi}):(S_B-S_{2\phi})(b^2+S\csc 2\phi)\\&:(S_C-S_{2\phi})(c^2+S\csc 2\phi));\\B''=&((S_A-S_{2\phi})(a^2+S\csc 2\phi):-(S_C-S_{2\phi})(S_A-S_{2\phi})\\&:(S_C-S_{2\phi})(c^2+S\csc 2\phi)),\\C''=&((S_A-S_{2\phi})(a^2+S\csc 2\phi):(S_B-S_{2\phi})(b^2+S\csc 2\phi)\\&:-(S_A-S_{2\phi})(S_B-S_{2\phi})).\end{aligned}\tag{4}$$

Proposition 9. *Triangle $A''B''C''$ is perspective to*

(1) *ABC at*

$$P_2(\phi)=\left((S_A-S_{2\phi})(a^2+S\csc 2\phi):\cdots:\cdots\right),$$

(2) *$A'B'C'$ at*

$$P_3(\phi)=((a^2S_A-S_{BC})-S\csc 2\phi(S_A-S_\phi\cos 2\phi):\cdots:\cdots),$$

(3) *the dilated triangle*[3] *at*

$$P_4(\phi)=(S_B+S_C-S_A-S_{2\phi}:\cdots:\cdots).$$

Proof. (1) is clear from the coordinates given in (4). Since

$$\begin{aligned}&(a^2+S\csc 2\phi)(S_A-S\cot 2\phi)-(S_B-S\cot 2\phi)(S_C-S\cot 2\phi)\\=&(a^2S_A-S_{BC})+S^2\csc 2\phi\cot A-S\cot 2\phi(a^2+S\csc 2\phi-(S_B+S_C)+S\cot 2\phi)\\=&(a^2S_A-S_{BC})+S^2\csc 2\phi\cot A-S^2\cot 2\phi\cot\phi\\=&(a^2S_A-S_{BC})-S_AS\csc 2\phi+S_{2\phi}S_\phi\\=&(a^2S_A-S_{BC})-S\csc 2\phi(S_A-S_\phi\cos 2\phi),\end{aligned}$$

it follows from Lemma 7 that $A''B''C''$ is perspective to $A'B'C'$ at

$$\begin{aligned}&\left(\frac{a^2+S\csc 2\phi}{(S_B-S_{2\phi})(S_C-S_{2\phi})}-\frac{1}{S_A-S_{2\phi}}:\cdots:\cdots\right)\\=&((a^2S_A-S_{BC})-S\csc 2\phi(S_A-S_\phi\cos 2\phi):\cdots:\cdots).\end{aligned}$$

[3]This is also called the anticomplementary triangle, it is formed by the lines through the vertices of ABC, parallel to the corresponding opposite sides.

This proves (2). For (3), we rewrite the coordinates for A'' as

$$\begin{aligned}A'' =&(S_B - S_{2\phi})(S_C - S_{2\phi}) \cdot (-1, 1, 1)\\ &+ (S \csc(2\phi) + S_{2\phi} + S_A) \cdot (0, S_B - S_{2\phi}, S_C - S_{2\phi})\\ =&(S_B - S_{2\phi})(S_C - S_{2\phi}) \cdot (-1, 1, 1)\\ &+ (S_A + S_\phi) \cdot (0, S_B - S_{2\phi}, S_C - S_{2\phi})\end{aligned}$$

From this we see that A'' is on the line connecting the A-vertices of the dilated triangle and the cevian triangle of the isotomic conjugate of $K(-2\phi)$, namely, the point

$$K^\bullet(-2\phi) = (S_A - S_{2\phi} : S_B - S_{2\phi} : S_C - S_{2\phi}).$$

This shows that $A''B''C''$ is perspective to both triangles, and that the perspector is the *cevian quotient* $K^\bullet(-2\phi)/G$, [4] where G denotes the centroid. It is easy to see that this is the superior of $K^\bullet(-2\phi)$. Equivalently, it is $K^\bullet(-2\phi)$ of the dilated triangle, with coordinates

$$(S_B + S_C - S_A - S_{2\phi} : \cdots : \cdots).$$

□

We conclude with a table showing the triangle centers associated with the circumsquares, when $\phi = \pm\frac{\pi}{4}$.

k	$P_k(\frac{\pi}{4})$	$P_k(-\frac{\pi}{4})$
1	$K(\frac{\pi}{4})$	$K(-\frac{\pi}{4})$
2	circumcenter	circumcenter
3	de Longchamps point	de Longchamps point
4	X_{193}	X_{193}

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[4]The cevian quotient X/Y is the perspector of the cevian triangle of X and the precevian triangle of Y. This is the X-Ceva conjugate of Y in the terminology of [2].

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Sawayama and Thébault's theorem

Jean-Louis Ayme

Abstract. We present a purely synthetic proof of Thébault's theorem, known earlier to Y. Sawayama.

1. Introduction

In 1938 in a "Problems and Solutions" section of the Monthly [24], the famous French problemist Victor Thébault (1882-1960) proposed a problem about three circles with collinear centers (see Figure 1) to which he added a correct ratio and a relation which finally turned out to be wrong. The date of the first three metric

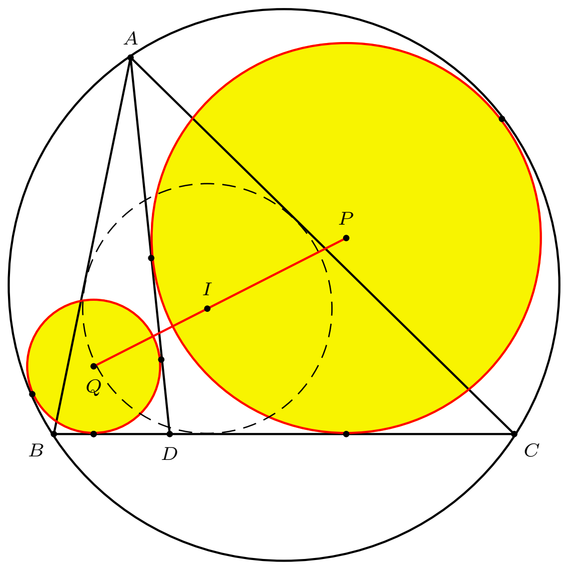

Figure 1

solutions [22] which appeared discreetly in 1973 in the Netherlands was more widely known in 1989 when the Canadian revue *Crux Mathematicorum* [27] published the simplified solution by Veldkamp who was one of the two first authors to prove the theorem in the Netherlands [26, 5, 6]. It was necessary to wait until the end of this same year when the Swiss R. Stark, a teacher of the Kantonsschule of Schaffhausen, published in the Helvetic revue *Elemente der Mathematik* [21] the first synthetic solution of a "more general problem" in which the one of Thébault's appeared as a particular case. This generalization, which gives a special importance to a rectangle known by J. Neuberg [15], citing [4], has been pointed out in 1983 by the editorial comment of the *Monthly* in an outline publication about the supposed

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first metric solution of the English K. B. Taylor [23] which amounted to 24 pages. In 1986, a much shorter proof [25], due to Gerhard Turnwald, appeared. In 2001, R. Shail considered in his analytic approach, a “more complete” problem [19] in which the one of Stark appeared as a particular case. This last generalization was studied again by S. Gueron [11] in a metric and less complete way. In 2003, the *Monthly* published the angular solution by B. J. English, received in 1975 and “lost in the mists of time” [7].

Thanks to *JSTOR*, the present author has discovered in an anciant edition of the *Monthly* [18] that the problem of Shail was proposed in 1905 by an instructor Y. Sawayama of the central military School of Tokyo, and geometrically resolved by himself, mixing the synthetic and metric approach. On this basis, we elaborate a new, purely synthetic proof of Sawayama-Thébault theorem which includes several theorems that can all be synthetically proved. The initial step of our approach refers to the beginning of the Sawayama’s proof and the end refers to Stark’s proof. Furthermore, our point of view leads easily to the Sawayama-Shail result.

2. A lemma

Lemma 1. *Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. Let P be the center of the circle $\mathcal{C}_1$ which touches DC, DA at E, F and the circumcircle $\mathcal{C}_2$ of ABC at K. Then the chord of contact EF passes through the incenter I of triangle ABC.*

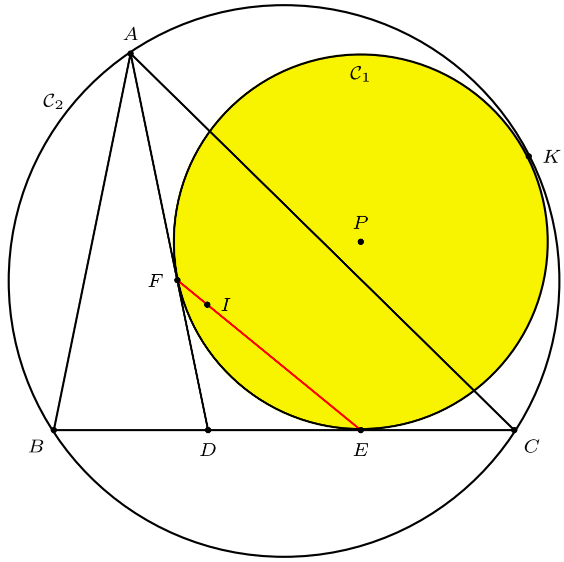

Figure 2

Proof. Let M, N be the points of intersection of KE, KF with $\mathcal{C}_2$, and J the point of intersection of AM and EF (see Figure 3). KE is the internal bisector of $\angle BKC$ [8, Théorème 119]. The point M being the midpoint of the arc BC which does not contain K, AM is the A-internal bisector of ABC and passes through I.

The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ being tangent at K, EF and MN are parallel.

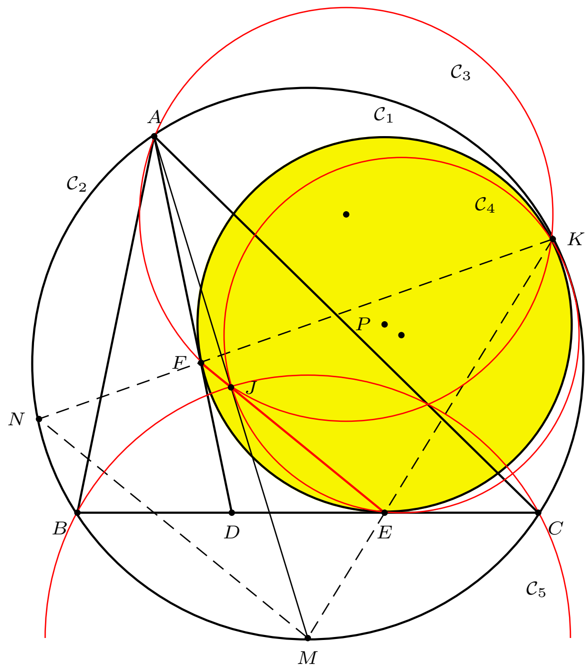

Figure 3

The circle $\mathcal{C}_2$, the basic points A and K, the lines MAJ and NKF, the parallels MN and JF, lead to a converse of Reim's theorem ([8, Théorème 124]). Therefore, the points A, K, F and J are concyclic. This can also be seen directly from the fact that angles FJA and FKA are congruent.

Miquel's pivot theorem [14, 9] applied to the triangle AFJ by considering F on AF, E on FJ, and J on AJ, shows that the circle $\mathcal{C}_4$ passing through E, J and K is tangent to AJ at J. The circle $\mathcal{C}_5$ with center M, passing through B, also passes through I ([2, Livre II, p.46, théorème XXI] and [12, p.185]). This circle being orthogonal to circle $\mathcal{C}_1$ [13, 20] is also orthogonal to circle $\mathcal{C}_4$ ([10, 1]) as KEM is the radical axis of circles $\mathcal{C}_1$ and $\mathcal{C}_4$. [1] Therefore, $MB = MJ$, and $J = I$. Conclusion: the chord of contact EF passes through the incenter I. □

Remark. When D is at B, this is the theorem of Nixon [16].

3. Sawayama-Thébault theorem

Theorem 2. *Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. I is the center of the incircle of triangle ABC. Let P be the center of the circle which touches DC, DA at E, F, and the circumcircle of ABC, and let Q be the center of a further circle which touches DB, DA in G, H and the circumcircle of ABC. Then P, I and Q are collinear.*

[1] From $\angle BKE = \angle MAC = \angle MBE$, we see that he circumcircle of BKE is tangent to BM at B. So circle $\mathcal{C}_5$ is orthogonal to this circumcircle and consequently also to $\mathcal{C}_1$ as M lies on their radical axis.

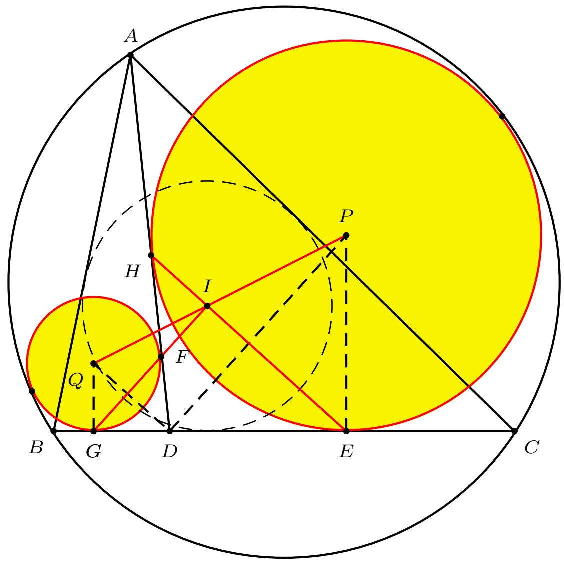

Figure 4

Proof. According to the hypothesis, $QG \perp BC$, $BC \perp PE$; so $QG//PE$. By Lemma 1, GH and EF pass through I. Triangles DHG and QGH being isosceles in D and Q respectively, DQ is

(1) the perpendicular bisector of GH,

(2) the D-internal angle bisector of triangle DHG.

Mutatis mutandis, DP is

(1) the perpendicular bisector of EF,

(2) the D-internal angle bisector of triangle DEF.

As the bisectors of two adjacent and supplementary angles are perpendicular, we have $DQ \perp DP$. Therefore, $GH//DP$ and $DQ//EF$. Conclusion: using the converse of Pappus's theorem ([17, Proposition 139] and [3, p.67]), applied to the hexagon $PEIGQDP$, the points P, I and Q are collinear. □

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Antiorthocorrespondents of Circumconics

Bernard Gibert

Abstract. This note is a complement to our previous paper [3]. We study how circumconics are transformed under antiorthocorrespondence. This leads us to consider a pencil of pivotal circular cubics which contains in particular the Neuberg cubic of the orthic triangle.

1. Introduction

This paper is a complement to our previous paper [3] on the orthocorrespondence. Recall that in the plane of a given triangle ABC, the orthocorrespondent of a point M is the point $M^\perp$ whose trilinear polar intersects the sidelines of ABC at the orthotraces of M. If $M = (p : q : r)$ in homogeneous barycentric coordinates, then [1]

$$M^\perp = (p(-pS_A + qS_B + rS_C) + a^2qr : \cdots : \cdots). \tag{1}$$

The antiorthocorrespondents of M consists of the two points M_1 and M_2, not necessarily real, for which $M_1^\perp = M = M_2^\perp$. We write $M^\top = \{M_1, M_2\}$, and say that M_1 and M_2 are orthoassociates. We shall make use of the following basic results.

Lemma 1. *Let $M = (p : q : r)$ and $M^\top = \{M_1, M_2\}$.*

(1) *The line M_1M_2 [2] has equation*

$$S_A(q-r)x + S_B(r-p)y + S_C(p-q)z = 0.$$

It always passes through the orthocenter H, and intersects the line GM at the point

$$\big((b^2-c^2)/(q-r) : \cdots : \cdots\big)$$

on the Kiepert hyperbola.

(2) *The perpendicular bisector ℓ_M of the segment M_1M_2 is the trilinear polar of the isotomic conjugate of the anticomplement of M*, i.e.,

$$(q+r-p)x + (r+p-q)y + (p+q-r)z = 0.$$

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[1]Throughout this paper, we use the same notations in [3]. All coordinates are barycentric coordinates with respect to the reference triangle ABC.

[2]M_1M_2 is the Steiner line of the isogonal conjugate of the infinite point of the trilinear polar of the isotomic conjugate of M.

We study how circumconics are transformed under antiorthocorrespondence. Let $P = (u : v : w)$ be a point not lying on the sidelines of ABC. Denote by Γ_P the circumconic with perspector P, namely,

$$\frac{u}{x} + \frac{v}{y} + \frac{w}{z} = 0.$$

This has center [3]

$$G/P = (u(-u+v+w) : v(-v+w+u) : w(u+v-w)),$$

and is the locus of trilinear poles of lines passing through P.

A point $(x : y : z)$ is the orthocorrespondent of a point on Γ_P if and only if

$$\sum_{\text{cyclic}} \frac{u}{x(-xS_A + yS_B + zS_C) + a^2yz} = 0. \qquad (2)$$

The antiorthocorrespondent of Γ_P is therefore in general a quartic $\mathcal{Q}_P$. It is easy to check that $\mathcal{Q}_P$ passes through the vertices of the orthic triangle and the pedal triangle of P. It is obviously invariant under orthoassociation, *i.e.*, inversion with respect to the polar circle. See [3, §2]. It is therefore a special case of anallagmatic fourth degree curve.

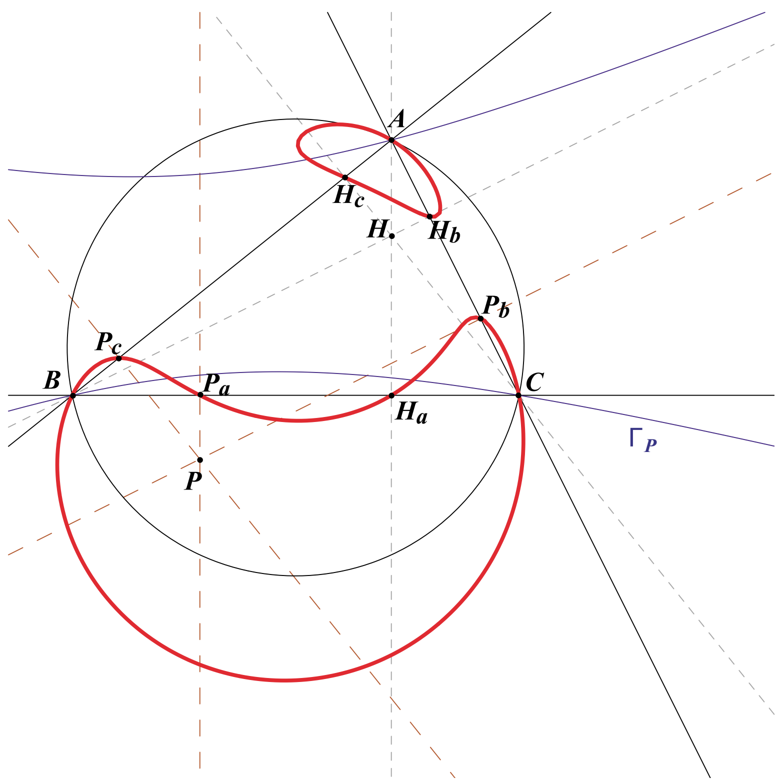

Figure 1. The bicircular circum-quartic $\mathcal{Q}_P$

[3]This is the perspector of the medial triangle and anticevian triangle of P.

The equation of $\mathcal{Q}_P$ can be rewritten as

$$(u+v+w)\mathcal{C}^2 - \left(\sum_{\text{cyclic}} (v+w)S_A\,x\right)\mathcal{LC} - \left(\sum_{\text{cyclic}} uS_BS_C\,yz\right)\mathcal{L}^2 = 0, \quad (3)$$

with

$$\mathcal{C} = a^2yz + b^2zx + c^2xy, \qquad \mathcal{L} = x+y+z.$$

From this it is clear that $\mathcal{Q}_P$ is a bicircular quartic if and only if $u+v+w \neq 0$; equivalently, Γ_P does not contain the centroid G. We shall study this case in §3 below, and the case $G \in \Gamma_P$ in §4.

2. The conic γ_P

A generic point on the conic Γ_P is

$$M = M(t) = \left(\frac{u}{(v-w)(u+t)} : \frac{v}{(w-u)(v+t)} : \frac{w}{(u-v)(w+t)}\right).$$

As M varies on the circumconic Γ_P, the perpendicular bisector ℓ_M of M_1M_2 envelopes the conic γ_P:

$$\sum((u+v+w)^2 - 4vw)x^2 - 2(u+v+w)(v+w-u)yz = 0.$$

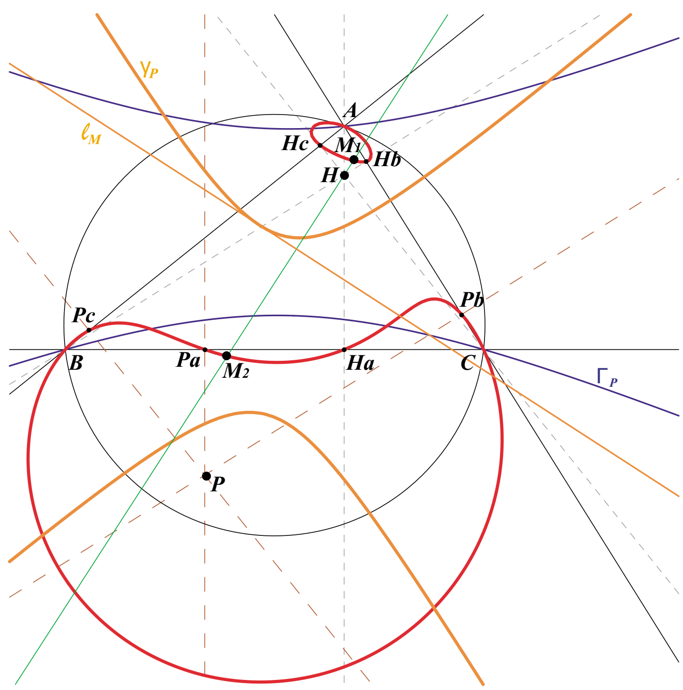

Figure 2. The conic γ_P

The point of tangency of γ_P and the perpendicular bisector of M_1M_2 is

$$T_M = (v(u-v)^2(w+t)^2 + w(u-w)^2(v+t)^2 : \cdots : \cdots).$$

The conic γ_P is called the déférente of Γ_P in [1]. It has center $\omega_P = (2u + v + w : \cdots : \cdots)$, and is homothetic to the circumconic with perspector $((v + w)^2 : (w + u)^2 : (u + v)^2)$. [4] It is therefore a circle when P is the Nagel point or one of its extraversions. This circle is the Spieker circle. We shall see in §3.5 below that $\mathcal{Q}_P$ is an oval of Descartes.

It is clear that γ_P is a parabola if and only if ω_P and therefore P are at infinity. In this case, Γ_P contains the centroid G. See §4 below.

3. Antiorthocorrespondent of a circumconic not containing the centroid

Throughout this section we assume P a finite point so that the circumconic Γ_P does not contain the centroid G.

Proposition 2. *Let ℓ be a line through G intersecting Γ_P at two points M and N. The antiorthocorrespondents of M and N are four collinear points on $\mathcal{Q}_P$.*

Proof. Let M_1, M_2 be the antiorthocorrespondents of M, and N_1, N_2 those of N. By Lemma 1, each of the lines M_1M_2 and N_1N_2 intersects ℓ at the same point on the Kiepert hyperbola. Since they both contain H, M_1M_2 and N_1N_2 are the same line. □

Corollary 3. *Let the medians of ABC meet Γ_P again at A_g, B_g, C_g. The antiorthocorrespondents of these points are the third and fourth intersections of $\mathcal{Q}_P$ with the altitudes of ABC.* [5]

Proof. The antiorthocorrespondents of A are A and H_a. □

In this case, the third and fourth points on AH are symmetric about the second tangent to γ_P which is parallel to BC. The first tangent is the perpendicular bisector of AH_a with contact $(v + w : v : w)$, the contact with this second tangent is $(u(v + w) : uw + (v + w)^2 : uv + (v + w)^2)$ while $A_g = (-u : v + w : v + w)$.

For distinct points P_1 and P_2, the circumconics Γ_{P_1} and Γ_{P_2} have a "fourth" common point T, which is the trilinear pole of the line P_1P_2. Let $T^\top = \{T_1, T_2\}$. The conics Γ_{P_1} and Γ_{P_2} generate a pencil $\mathcal{F}$ consisting of Γ_P for P on the line P_1P_2. The antiorthocorrespondent of every conic $\Gamma_P \in \mathcal{F}$ contains the following 16 points:
(i) the vertices of ABC and the orthic triangle $H_aH_bH_c$,
(ii) the circular points at infinity with multiplicity 4, [6]
(iii) the antiorthocorrespondents $T^\top = \{T_1, T_2\}$.

Proposition 4. *Apart from the circular points at infinity and the vertices of ABC and the orthic triangle, the common points of the quartics $\mathcal{Q}_{P_1}$ and $\mathcal{Q}_{P_2}$ are the antiorthocorrespondents of the trilinear pole of the line P_1P_2.*

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[4] It is inscribed in the medial triangle; its anticomplement is the circumconic with center the complement of P, with perspector the isotomic conjugate of P.

[5] They are not always real when ABC is obtuse angle.

[6] Think of $\mathcal{Q}_{P_1}$ as the union of two circles and $\mathcal{Q}_{P_2}$ likewise. These have at most 8 real finite points and the remaining 8 are the circular points at infinity, each counted with multiplicity 4.

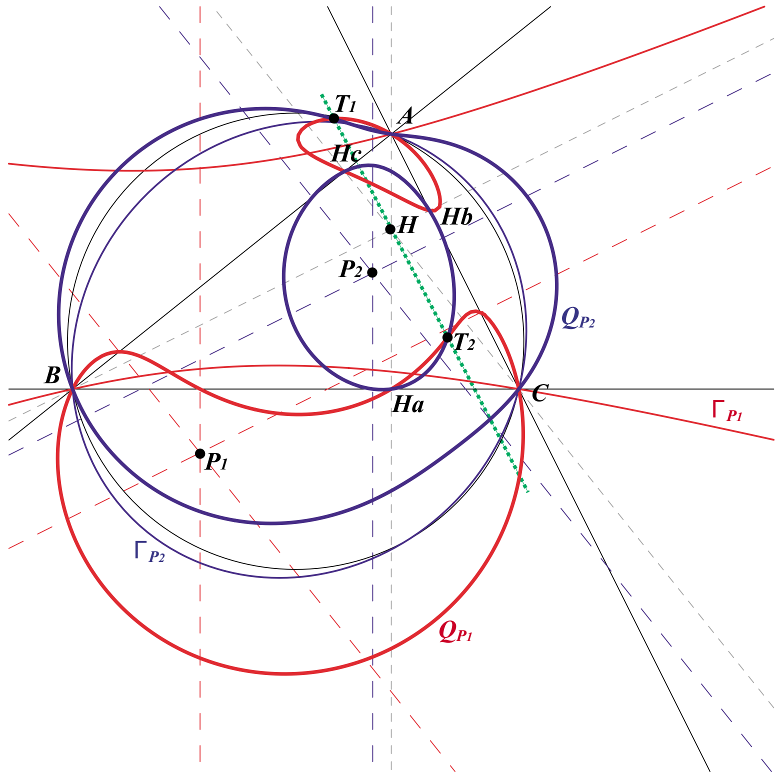

Figure 3. The bicircular quartics $\mathcal{Q}_{P_1}$ and $\mathcal{Q}_{P_2}$

Remarks. 1. T_1 and T_2 lie on the line through H which is the orthocorrespondent of the line GT. See [3, §2.4]. This line T_1T_2 is the directrix of the inscribed (in ABC) parabola tangent to the line P_1P_2.

2. The pencil $\mathcal{F}$ contains three degenerate conics $BC \cup AT$, $CA \cup BT$, and $AB \cup CT$. The antiorthocorrespondent of $BC \cup AT$, for example, degenerates into the circle with diameter BC and another circle through A, H_a, T_1 and T_2 (see [3, Proposition 2]).

3.1. *The points* S_1 *and* S_2. Since $\mathcal{Q}_P$ and the circumcircle have already seven common points, the vertices A, B, C, and the circular points at infinity, each of multiplicity 2, they must have an eighth common point, namely,

$$S_1 = \left(\frac{a^2}{\frac{v}{b^2S_B} - \frac{w}{c^2S_C}} : \cdots : \cdots \right), \tag{4}$$

which is the isogonal conjugate of the infinite point of the line

$$\frac{u}{a^2S_A}x + \frac{v}{b^2S_B}y + \frac{w}{c^2S_C}z = 0.$$

Similarly, $\mathcal{Q}_P$ and the nine-point circle also have a real eighth common point

$$S_2 = ((S_B(u-v+w) - S_C(u+v-w))(c^2S_Cv - b^2S_Bw) : \cdots : \cdots), \quad (5)$$

which is the inferior of

$$\left(\frac{a^2}{S_B(u-v+w) - S_C(u+v-w)} : \cdots : \cdots\right)$$

on the circumcircle.

We know that the orthocorrespondent of the circumcircle is the circum-ellipse Γ_O, with center K, the Lemoine point, ([3, §2.6]). If $P \neq O$, this ellipse meets Γ_P at A, B, C and a fourth point

$$S = S(P) = \left(\frac{1}{c^2S_Cv - b^2S_Bw} : \cdots : \cdots\right), \quad (6)$$

which is the trilinear pole of the line OP. The point S lies on the circumcircle if and only if P is on the Brocard axis OK.

Proposition 5. $S^\top = \{S_1, S_2\}$.

Corollary 6. *$S(P) = S(P')$ if and only if P, P' and O are collinear.*

Remark. When $P = O$ (circumcenter), Γ_P is the circum-ellipse with center K. In this case $\mathcal{Q}_P$ decomposes into the union of the circumcircle and the nine point circle.

3.2. *Bitangents.*

Proposition 7. *The points of tangency of the two bitangents to $\mathcal{Q}_P$ passing through H are the antiorthocorrespondents of the points where the polar line of G in Γ_P meets Γ_P.*

Proof. Consider a line ℓ_H through H which is supposed to be tangent to $\mathcal{Q}_P$ at two (orthoassociate) points M and N. The orthocorrespondents of M and N must lie on Γ_P and on the orthocorrespondent of ℓ_H which is a line through G. Since M and N are double points, the line through G must be tangent to Γ_P and MN is the polar of G in Γ_P. □

Remark. M and N are not necessarily real. If $M^\top = \{M_1, M_2\}$ and $N^\top = \{N_1, N_2\}$, the perpendicular bisectors of M_1M_2 and N_1N_2 are the asymptotes of γ_P.[7] The four points M_1, M_2, N_1, N_2 are concyclic and the circle passing through them is centered at ω_P.

Denote by H_1, H_2, H_3 the vertices of the triangle which is self polar in both the polar circle and γ_P. The orthocenter of this triangle is obviously H. For $i = 1, 2, 3$, let $\mathcal{C}_i$ be the circle centered at H_i orthogonal to the polar circle and Γ_i the circle centered at ω_P orthogonal to $\mathcal{C}_i$. The circle Γ_i intersects $\mathcal{Q}_P$ at the circular points at infinity (with multiplicity 2) and four other points two by two homologous in the inversion with respect to $\mathcal{C}_i$ which are the points of tangency of the (not

[7] The union of the line at infinity and a bitangent is a degenerate circle which is bitangent to $\mathcal{Q}_P$. Its center must be an infinite point of γ_P.

always real) bitangents drawn from H_i to $\mathcal{Q}_P$. The orthocorrespondent of Γ_i is a conic (see [3, §2.6]) intersecting Γ_P at four points whose antiorthocorrespondents are eight points, two by two orthoassociate. Four of them lie on Γ_i and are the required points of tangency. The remaining four are their orthoassociates and they lie on the circle which is the orthoassociate of Γ_i. Figure 4 below shows an example of $\mathcal{Q}_P$ with three pairs of real bitangents.

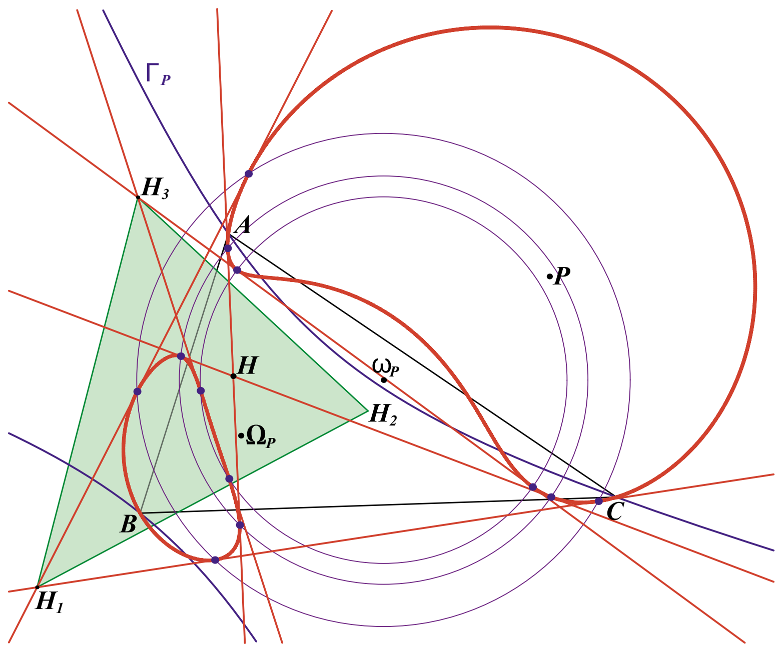

Figure 4. Bitangents to $\mathcal{Q}_P$

Proposition 8. *$\mathcal{Q}_P$ is tangent at H_a, H_b, H_c to BC, CA, AB if and only if $P = H$.*

3.3. *$\mathcal{Q}_P$ as an envelope of circles.*

Theorem 9. *The circle $\mathcal{C}_M$ centered at T_M passing through M_1 and M_2 is bitangent to $\mathcal{Q}_P$ at those points and orthogonal to the polar circle.*

This is a consequence of the following result from [1, tome 3, p.170]. A bicircular quartic is a special case of "plane cyclic curve". Such a curve always can be considered in four different ways as the envelope of circles centered on a conic (déférente) cutting orthogonally a fixed circle. Here the fixed circle is the polar circle with center H, and since M_1 and M_2 are anallagmatic (inverse in the polar circle) and collinear with H, there is a circle passing through M_1, M_2, centered on the déférente, which must be bitangent to the quartic.

Corollary 10. *$\mathcal{Q}_P$ is the envelope of circles $\mathcal{C}_M$, $M \in \Gamma_P$, centered on γ_P and orthogonal to the polar circle.*

Construction. It is easy to draw γ_P since we know its center ω_P. For m on γ_P, draw the tangent t_m at m to γ_P. The perpendicular at m to Hm meets the perpendicular bisector of AH_a at a point which is the center of a circle through A (and H_a). This circle intersects Hm at two points which lie on the circle centered at m and orthogonal to the polar circle. This circle intersects the perpendicular at H to t_m at two points of $\mathcal{Q}_P$.

Corollary 11. *The tangents at M_1 and M_2 to $\mathcal{Q}_P$ are the tangents to the circle $\mathcal{C}_M$ at these points.*

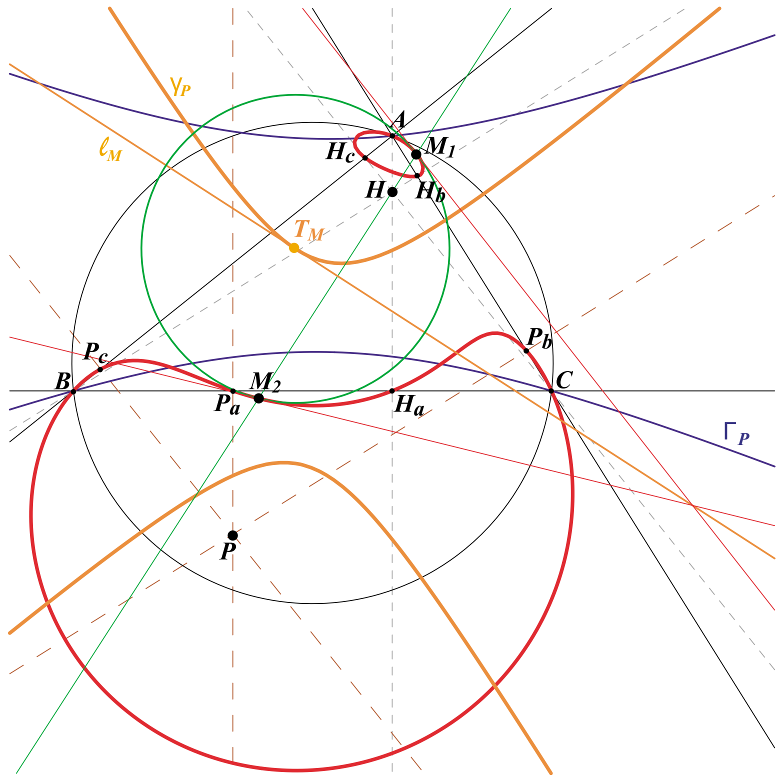

Figure 5. $\mathcal{Q}_P$ as an envelope of circles

3.4. *Inversions leaving $\mathcal{Q}_P$ invariant.*

Theorem 12. *$\mathcal{Q}_P$ is invariant under three other inversions whose poles are the vertices of the triangle which is self-polar in both the polar circle and γ_P.*

Proof. This is a consequence of [1, tome 3, p.172].

Construction: Consider the transformation ϕ which maps any point M of the plane to the intersection M' of the polars of M in both the polar circle and γ_P. Let Σ_a, Σ_b, Σ_c be the conics which are the images of the altitudes AH, BH, CH under ϕ. The conic Σ_a is entirely defined by the following five points:

(1) the point at infinity of BC.
(2) the point at infinity of the polar of H in γ_P.
(3) the foot on BC of the polar of A in γ_P.
(4) the intersection of the polar of H_a in γ_P with the parallel at A to BC.
(5) the pole of AH in γ_P.

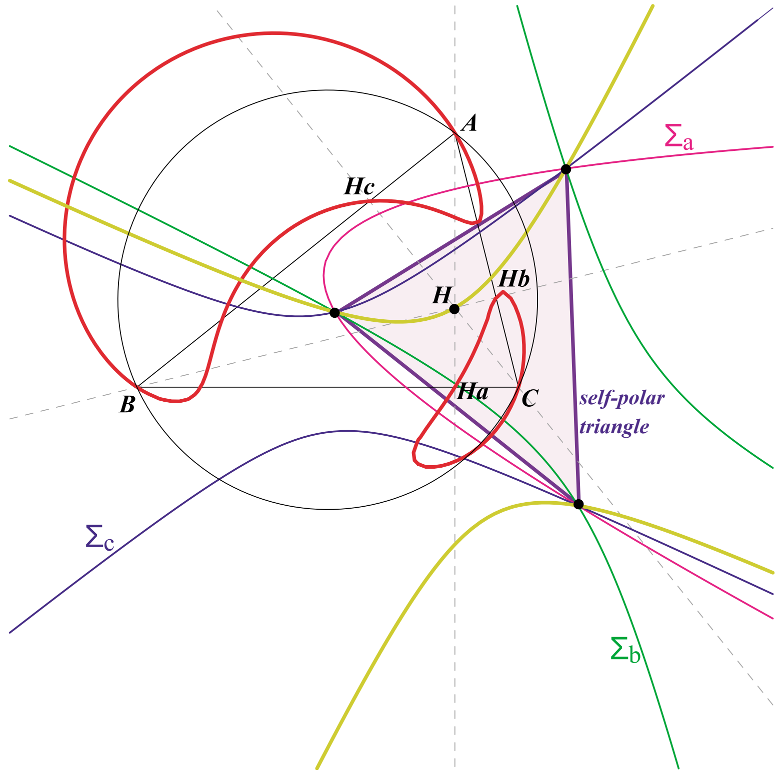

Figure 6. The conics Σ_a, Σ_b, Σ_c

Similarly, we define the conics Σ_b and Σ_c. These conics are in the same pencil and meet at four points: one of them is the point at infinity of the polar of H in γ_P and the three others are the required poles. The circles of inversion are centered at those points and are orthogonal to the polar circle. Their radical axes with the polar circle are the sidelines of the self-polar triangle. □

Another construction is possible : the transformation of the sidelines of triangle ABC under ϕ gives three other conics σ_a, σ_b, σ_c but not defining a pencil since the three lines are not now concurrent. σ_a passes through A, the two points where the trilinear polar of P^+ (anticomplement of P) meets AB and AC, the pole of the line BC in γ_P, the intersection of the parallel at A to BC with the polar of H_a in γ_P. See Figure 7.

Remark. The Jacobian of σ_a, σ_b, σ_c is a degenerate cubic consisting of the union of the sidelines of the self-polar triangle.

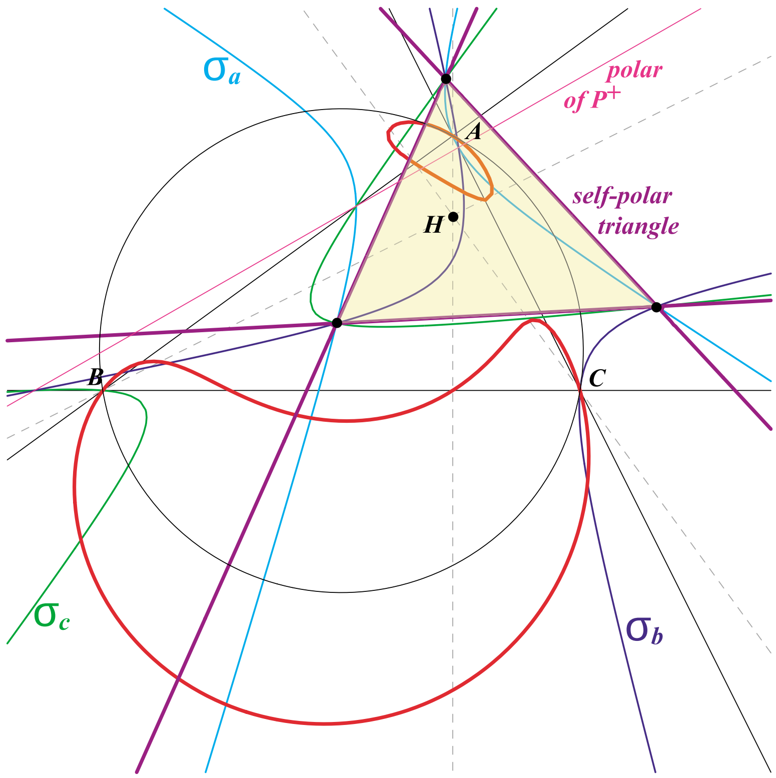

Figure 7. The conics σ_a, σ_b, σ_c

3.5. *Examples.* We provide some examples related to common centers of ABC.

P	S	S_1	S_2	Γ_P	Remark
H	X_{648}	X_{107}	X_{125}		see Figure 8
K	X_{110}	X_{112}	X_{115}	circumcircle	
G	X_{648}	X_{107}	S_{125}	Steiner circum $-$ ellipse	
X_{647}				Jerabek hyperbola	

Remarks. 1. For $P = H$, $\mathcal{Q}_P$ is tangent at H_a, H_b, H_c to the sidelines of ABC. See Figure 8.

2. $P = X_{647}$, the isogonal conjugate of the tripole of the Euler line: Γ_P is the Jerabek hyperbola.

3. $\mathcal{Q}_P$ has two axes of symmetry if and only if P is the point such that $\overrightarrow{OP} = 3\,\overrightarrow{OH}$ (this is a consequence of [1, tome 3, p.172, §15]. Those axes are the parallels at H to the asymptotes of the Kiepert hyperbola. See Figure 9.

4. When $P = X_8$ (Nagel point), γ_P is the incircle of the medial triangle (its center is X_{10} = Spieker center) and Γ_P the circum-conic centered at $\Omega_P = ((b + c - a)(b + c - 3a) : \cdots : \cdots)$. Since the déférente is a circle, $\mathcal{Q}_P$ is now an oval

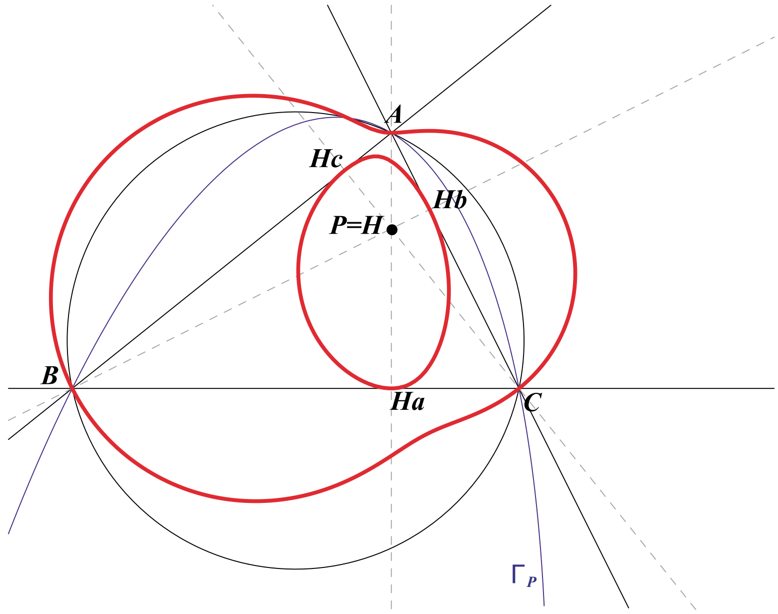

Figure 8. The quartic $\mathcal{Q}_H$

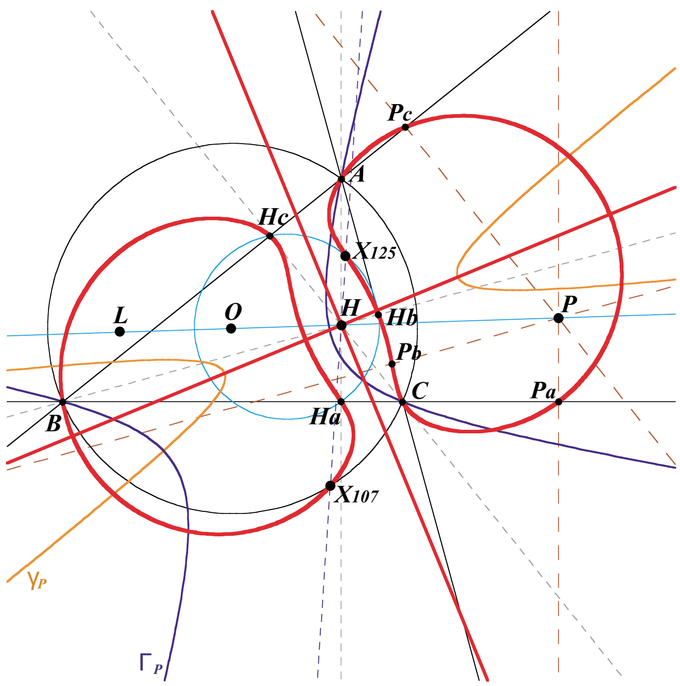

Figure 9. $\mathcal{Q}_P$ with two axes of symmetry

of Descartes (see [1, tome 1, p.8]) with axis the line HX_{10}. We obtain three more ovals of Descartes if X_8 is replaced by one of its extraversions. See Figure 10.

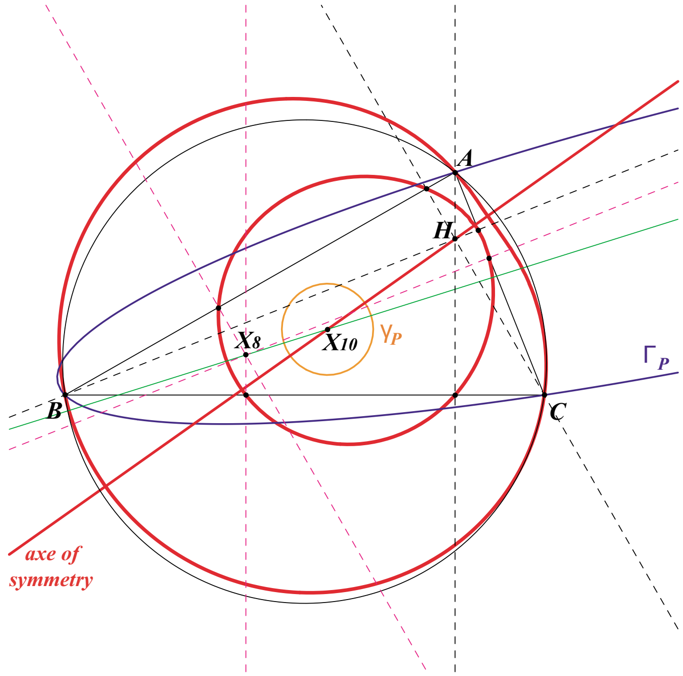

Figure 10. $\mathcal{Q}_P$ as an oval of Descartes

4. Antiorthocorrespondent of a circum-conic passing through G

We consider the case when the circumconic Γ_P contains the centroid G; equivalently, $P = (u : v : w)$ is an infinite point. In this case, Γ_P has center $(u^2 : v^2 : w^2)$ on the inscribed Steiner ellipse. The trilinear polar of points $Q \neq G$ on Γ_P are all parallel, and have infinite point P. It is clear from (3) that the curve $\mathcal{Q}_P$ decomposes into the union of the line at infinity $\mathcal{L}^\infty : x + y + z = 0$ and the cubic $\mathcal{K}_P$

$$\sum x(S_B(S_A u - S_B v)y^2 - S_C(S_C w - S_A u)z^2) = 0. \tag{7}$$

This is the pivotal isocubic p$\mathcal{K}(\Omega_P, H)$, with pivot H and pole

$$\Omega_P = \left(\frac{S_B v - S_C w}{S_A} : \frac{S_C w - S_A u}{S_B} : \frac{S_A u - S_B v}{S_C} \right).$$

Since the orthocorrespondent of the line at infinity is the centroid G, we shall simply say that the antiorthocorrespondent of Γ_P is the cubic $\mathcal{K}_P$. The orthocenter H is the only finite point whose orthocorrespondent is G. We know that $\mathcal{Q}_P$ has already the circular points (counted twice) on $\mathcal{L}^\infty$. This means that the cubic $\mathcal{K}_P$ is also a circular cubic. In fact, equation (7) can be rewritten as

$$\begin{aligned}(uS_Ax + vS_By + wS_Cz)&(a^2yz + b^2zx + c^2xy)\\ +(x + y + z)&(uS_{BC}yz + vS_{CA}zx + wS_{AB}xy) = 0.\end{aligned} \tag{8}$$

As P traverses $\mathcal{L}^\infty$, these cubics $\mathcal{K}_P$ form a pencil of circular pivotal isocubics since they all contain A, B, C, H, H_a, H_b, H_c and the circular points at infinity. The poles of these isocubics all lie on the orthic axis.

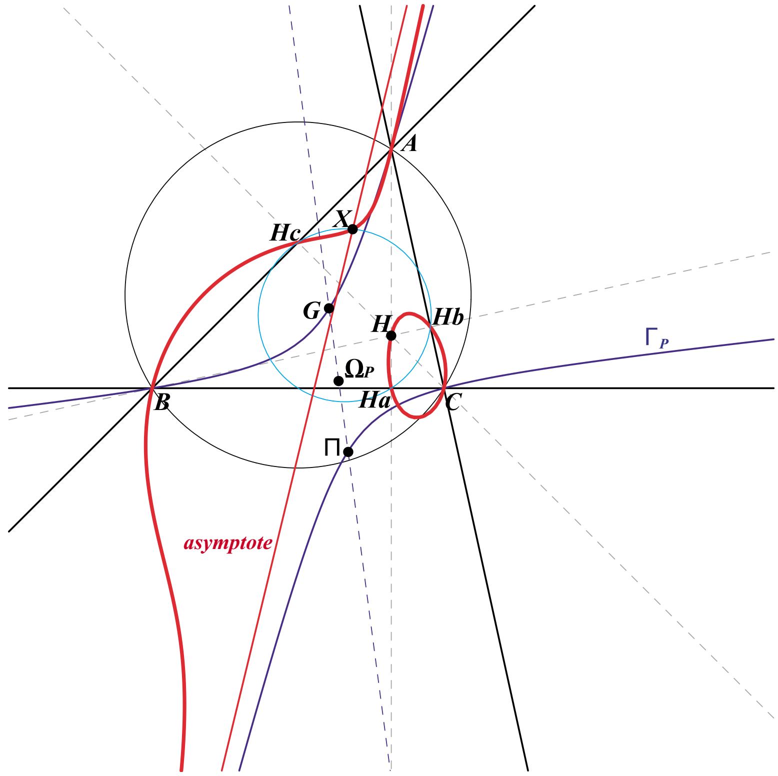

Figure 11. The circular pivotal cubic $\mathcal{K}_P$

4.1. *Properties of* $\mathcal{K}_P$.

(1) $\mathcal{K}_P$ is invariant under orthoassociation: the line through H and M on $\mathcal{K}_P$ meets $\mathcal{K}_P$ again at M' simultaneously the Ω_P-isoconjugate and orthoassociate of M. $\mathcal{K}_P$ is also invariant under the three inversions with poles A, B, C which swap H and H_a, H_b, H_c respectively. [8] See Figure 11.

(2) The real asymptote of $\mathcal{K}_P$ is the line ℓ_P

$$\frac{u}{S_Bv - S_Cw}x + \frac{v}{S_Cw - S_Au}y + \frac{w}{S_Au - S_Bv}z = 0. \tag{9}$$

It has infinite point

$$P' = (S_Bv - S_Cw : S_Cw - S_Au : S_Au - S_Bv),$$

[8] H, H_a, H_b, H_c are often called the centers of anallagmaty of the circular cubic.

and is parallel to the tangents at A, B, C, and H.[9] It is indeed the Simson line of the isogonal conjugate of P. It is therefore tangent to the Steiner deltoid.

(3) The tangents to $\mathcal{K}_P$ at H_a, H_b, H_c are the reflections of those at A, B, C, about the perpendicular bisectors of AH_a, BH_b, CH_c respectively.[10] They concur on the cubic at the point

$$X = \left(\frac{S_Bv - S_Cw}{u}\left(\frac{b^2S_B}{v} - \frac{c^2S_C}{w}\right) : \cdots : \cdots\right),$$

which is also the intersection of ℓ_P and the nine point circle. This is the inferior of the isogonal conjugate of P'. It is also the image of P^*, the isogonal conjugate of P, under the homothety $\mathsf{h}(H, \frac{1}{2})$.

(4) The antipode F of X on the nine point circle is the singular focus of $\mathcal{K}_P$:

$$F = (u(b^2v - c^2w) : v(c^2w - a^2u) : w(a^2u - b^2v)).$$

(5) The orthoassociate Y of X is the "last" intersection of $\mathcal{K}_P$ with the circumcircle, apart from the vertices and the circular points at infinity.

(6) The second intersection of the line XY with the circumcircle is $Z = P^*$.

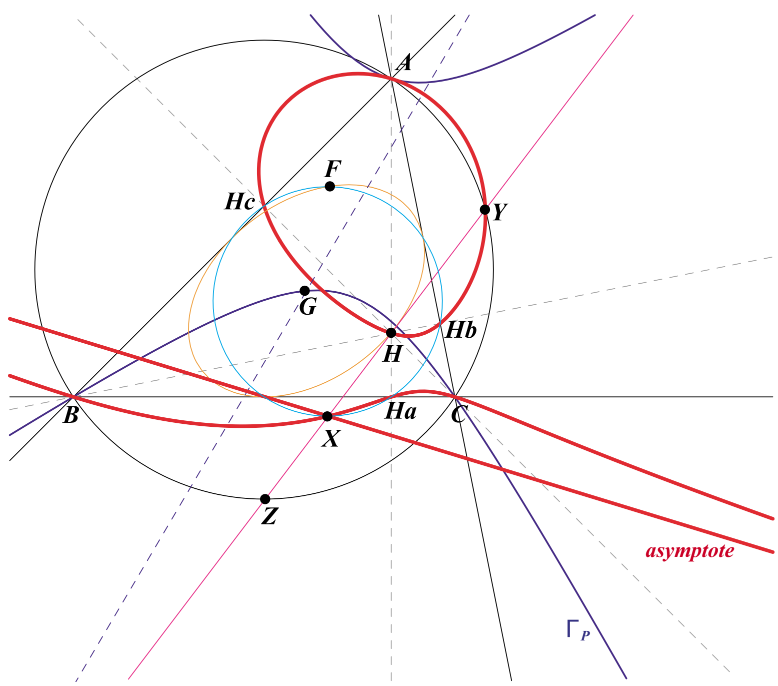

Figure 12. The points X, Y, Z and $\mathcal{K}_P$ for $P = X_{512}$

[9]The latter is the line $uS_Ax + vS_By + wS_Cz = 0$.

[10]These are the lines $S^2ux - (S_Bv - S_Cw)(S_By - S_Cz) = 0$ etc.

(7) $\mathcal{K}_P$ intersects the sidelines of the orthic triangle at three points lying on the cevian lines of Y in ABC.

(8) $\mathcal{K}_P$ is the envelope of circles centered on the parabola $\mathcal{P}_P$ (focus F, directrix the parallel at O to the Simson line of Z) and orthogonal to the polar circle. See Figure 13.

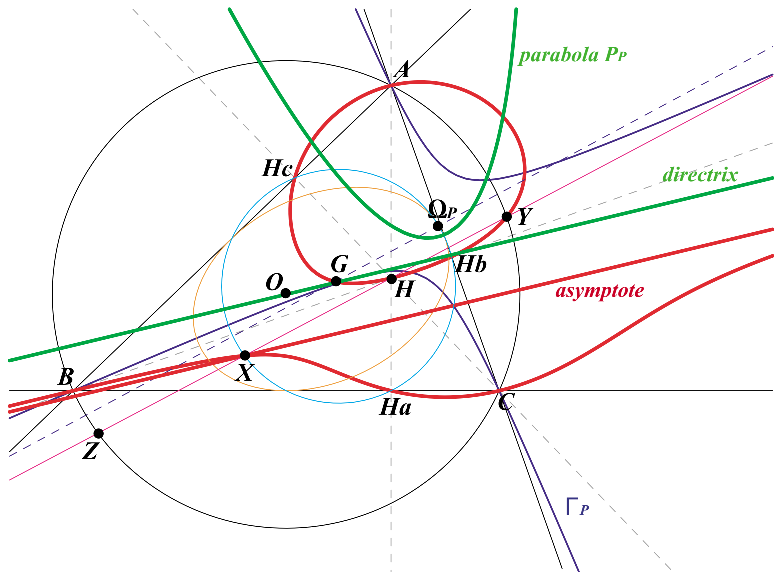

Figure 13. $\mathcal{K}_P$ and the parabola $\mathcal{P}_P$

(9) Γ_P meets the circumcircle again at

$$S = \left(\frac{1}{b^2v - c^2w} : \frac{1}{c^2w - a^2u} : \frac{1}{a^2u - b^2v}\right)$$

and the Steiner circum-ellipse again at

$$R = \left(\frac{1}{v - w} : \frac{1}{w - u} : \frac{1}{u - v}\right).$$

The antiorthocorrespondents of these two points S are four points on $\mathcal{K}_P$. They lie on a same circle orthogonal to the polar circle. See [3, §2.5] and Figure 14.

4.2. *$\mathcal{K}_P$ passing through a given point.* Since all the cubics form a pencil, there is a unique $\mathcal{K}_P$ passing through a given point Q which is not a base-point of the pencil. The circumconic Γ_P clearly contains G and $Q^\perp$, the orthocorrespondent of Q. It follows that P is the infinite point of the tripolar of $Q^\perp$.

Here is another construction of P. The circumconic through G and $Q^\perp$ intersects the Steiner circum-ellipse at a fourth point R. The midpoint M of GR is the center of Γ_P. The anticevian triangle of M is perspective to the medial triangle at P. The lines through their corresponding vertices are parallel to the tangents to

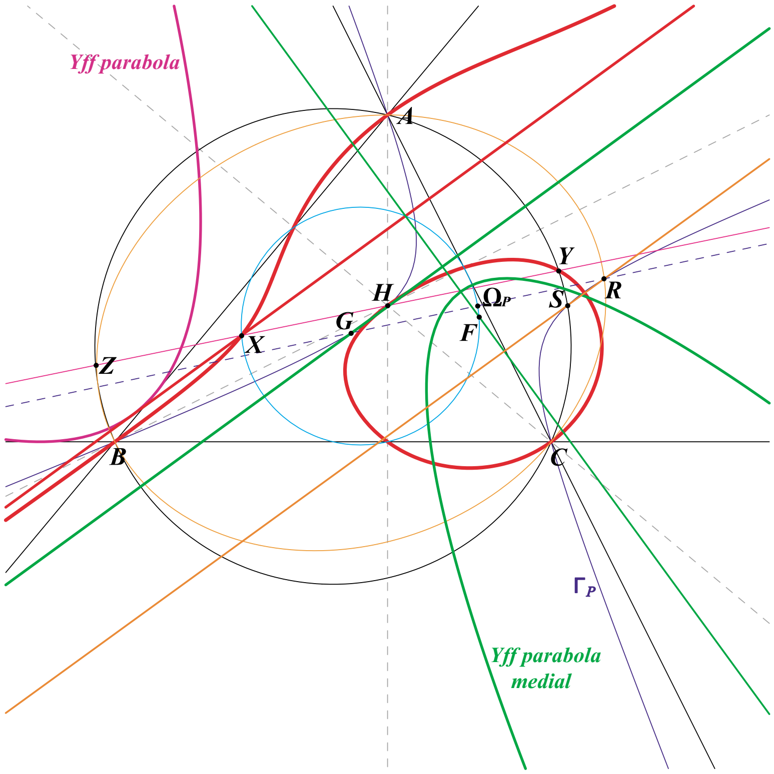

Figure 14. The points R, S and $\mathcal{K}_P$ for $P = X_{514}$

$\mathcal{K}_P$ at A, B, C. The point at infinity of these parallel lines is the point P for which $\mathcal{K}_P$ contains Q.

In particular, if Q is a point on the circumcircle, P is simply the isogonal conjugate of the second intersection of the line HQ with the circumcircle.

4.3. *Some examples and special cases.*

(1) The most remarkable circum-conic through G is probably the Kiepert rectangular hyperbola with perspector $P = X_{523}$, point at infinity of the orthic axis. Its antiorthocorrespondent is p$\mathcal{K}(X_{1990}, H)$, identified as the orthopivotal cubic $\mathcal{O}(H)$ in [3, §6.2.1]. See Figure 15.

(2) With $P =$ isogonal conjugate of X_{930} [11], $\mathcal{K}_P$ is the Neuberg cubic of the orthic triangle. We have $F = X_{137}$, $X = X_{128}$, $Y =$ isogonal conjugate of X_{539}, $Z = X_{930}$. The cubic contains X_5, X_{15}, X_{16}, X_{52}, X_{186}, X_{1154} (at infinity). See Figure 16.

(3) $\mathcal{K}_P$ degenerates when P is the point at infinity of one altitude. For example, with the altitude AH, $\mathcal{K}_P$ is the union of the sideline BC and the circle through A, H, H_b, H_c.

[11] $P = (a^2(b^2 - c^2)(4S_A^2 - 3b^2c^2) : \cdots : \cdots)$. The point X_{930} is the anticomplement of X_{137} which is X_{110} of the orthic triangle.

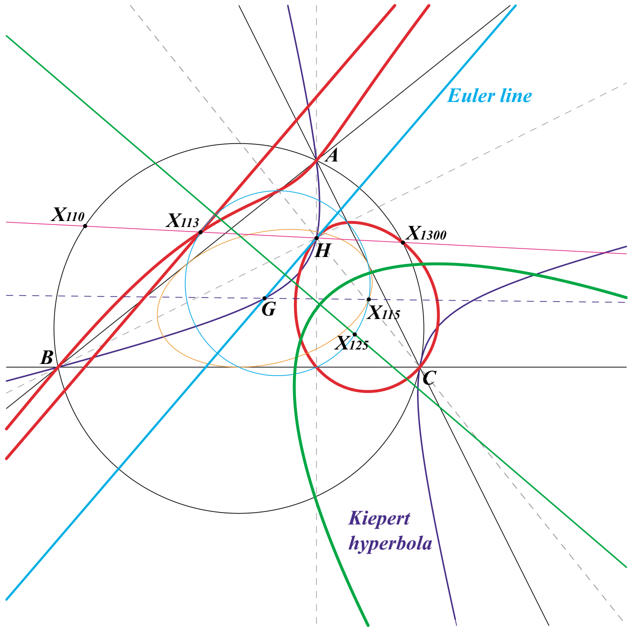

Figure 15. $\mathcal{O}(H)$ or $\mathcal{K}_P$ for $P = X_{523}$

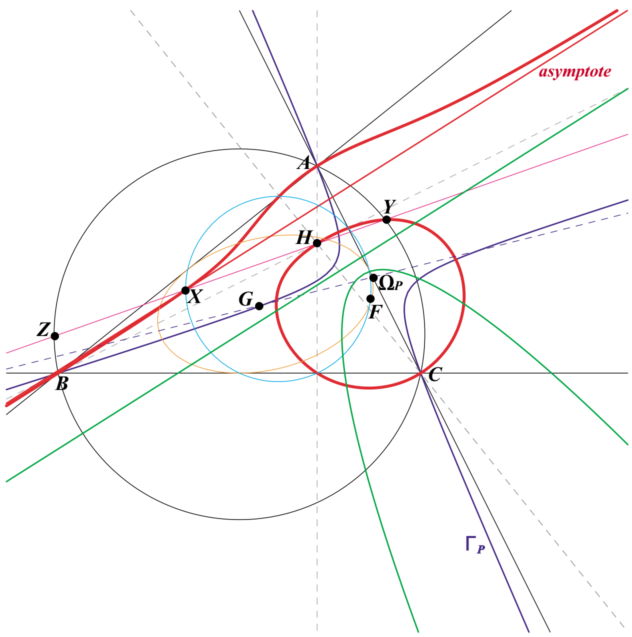

Figure 16. $\mathcal{K}_P$ as the Neuberg cubic of the orthic triangle

(4) $\mathcal{K}_P$ is a focal cubic if and only if P is the point at infinity of one tangent to the circumcircle at A, B, C. For example, with A, $\mathcal{K}_P$ is the focal cubic

denoted $\mathcal{K}_a$ with singular focus H_a and pole the intersection of the orthic axis with the symmedian AK. The tangents at A, B, C, H are parallel to the line OA. Γ_P is the isogonal conjugate of the line passing through K and the midpoint of BC. $\mathcal{P}_P$ is the parabola with focus H_a and directrix the line OA.

$\mathcal{K}_a$ is the locus of point M from which the segments BH_b, CH_c are seen under equal or supplementary angles. It is also the locus of contacts of tangents drawn from H_a to the circles centered on H_bH_c and orthogonal to the circle with diameter H_bH_c. See Figure 17.

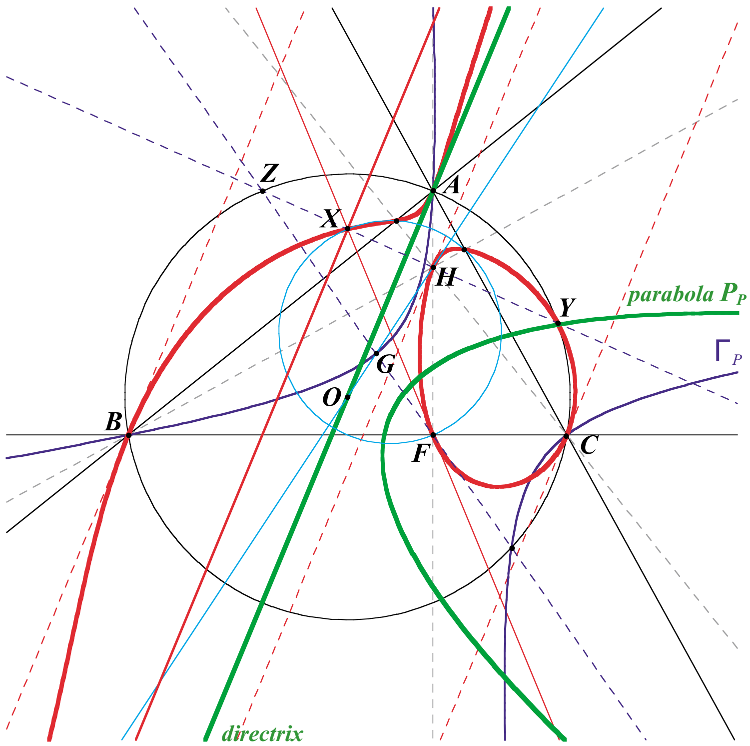

Figure 17. The focal cubic $\mathcal{K}_a$

4.4. *Conclusion.* We conclude with the following table showing the repartition of the points we met in the study above in some particular situations. Recall that P', X, Y always lie on $\mathcal{K}_P$, Y, Z, S on the circumcircle, X, F on the nine point circle, R on the Steiner circum-ellipse. When the point is not mentioned in [6], its first barycentric coordinate is given, as far as it is not too complicated. M^* denotes the isogonal conjugate of M, and $M^\#$ denotes the isotomic conjugate of M.

P	P'	X	Y	Z	F	S	R	Remark
X_{30}	X_{523}	X_{125}	X_{107}	X_{74}	X_{113}	X_{1302}	X_{648}	
X_{523}	X_{30}	X_{113}	X_{1300}	X_{110}	X_{125}	X_{98}	X_{671}	(1)
X_{514}	X_{516}	X_{118}	X_{917}	X_{101}	X_{116}	X_{675}	X_{903}	(2)
X_{511}	X_{512}	X_{115}	X_{112}	X_{98}	X_{114}	X_{110}	M_1	
X_{512}	X_{511}	X_{114}	M_2	X_{99}	X_{115}	X_{111}	$X^{\#}_{538}$	(3)
X_{513}	X_{517}	X_{119}	X_{915}	X_{100}	X_{11}	X_{105}	$X^{\#}_{536}$	(4)
X_{524}	X_{1499}	M_3	M_4	X_{111}	X_{126}	X_{99}	X_{99}	
X_{520}	X^{*}_{1294}	X_{133}	X_{74}	X_{107}	X_{122}	X_{1297}		
X_{525}	X_{1503}	X_{132}	X_{98}	X_{112}	X_{127}	$X^{\#}_{858}$	$X^{\#}_{30}$	
X^{*}_{930}	X_{1154}	X_{128}	X^{*}_{539}	X_{930}	X_{137}			
X_{515}	X_{522}	X_{124}	M_5	X_{102}	X_{117}			
X_{516}	X_{514}	X_{116}	M_6	X_{103}	X_{118}		M_7	

Remarks. (1) $\Omega_P = X_{115}$. Γ_P is the Kiepert hyperbola. $\mathcal{P}_P$ is the Kiepert parabola of the medial triangle with directrix the Euler line. See Figure 15.

(2) $\Omega_P = X_{1086}$. $\mathcal{P}_P$ is the Yff parabola of the medial triangle. See Figure 14.

(3) $\Omega_P = X_{1084}$. The directrix of $\mathcal{P}_P$ is the Brocard line.

(4) $\Omega_P = X_{1015}$. The directrix of $\mathcal{P}_P$ is the line OI.

The points $M_1, \dots, M_7$ are defined by their first barycentric coordinates as follows.

M_1	$1/[(b^2-c^2)(a^2S_A+b^2c^2)]$
M_2	$a^2/[S_A((b^2-c^2)^2-a^2(b^2+c^2-2a^2))]$
M_3	$(b^2-c^2)^2(b^2+c^2-5a^2)(b^4+c^4-a^4-4b^2c^2)$
M_4	$1/[S_A(b^2-c^2)(b^4+c^4-a^4-4b^2c^2)]$
M_5	$S_A(b-c)(b^3+c^3-a^2b-a^2c+abc)$
M_6	$1/[S_A(b-c)(b^2+c^2-ab-ac+bc)]$
M_7	$1/[(b-c)(3b^2+3c^2-a^2-2ab-2ac+2bc)]$

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