Junior Problems

J619. In triangle ABC,

$$AB^4 + BC^4 + CA^4 = 2AB^2 \cdot BC^2 + AB^2 \cdot CA^2 + 2BC^2 \cdot CA^2.$$

Find all possible values of $\angle A$.

Proposed by Adrian Andreescu, Dallas, USA

J620. Let ABC be a right triangle and let M be the midpoint of the hypotenuses BC. It is known that $AM^2 = AB \cdot AC$. Find the measure of angle ACB.

Proposed by Vasile Lupulescu, Târgu Jiu, România

J621. Let ABC be a triangle with $AB \neq AC$ and let I be its incenter. Let X be the midpoint of segment BC. Line XI intersects the altitude from A in Y. Prove that $AY = r$.

Proposed by Mihaela Berindeanu, Bucharest, România

J622. Let a, b, c be real numbers such that $a, b, c \in \left[\frac{1}{2}, 1\right]$. Prove that

$$\frac{a}{\sqrt{b}+\sqrt{c}} + \frac{b}{\sqrt{c}+\sqrt{a}} + \frac{c}{\sqrt{a}+\sqrt{b}} < 2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J623. Let m_a, m_b, m_c be the medians in a triangle ABC. Prove that

$$\frac{m_a^4}{m_b + m_c - m_a} + \frac{m_b^4}{m_c + m_a - m_b} + \frac{m_c^4}{m_a + m_b - m_c} \geq m_a^3 + m_b^3 + m_c^3.$$

Proposed by Mihaly Bencze, Braşov and Neculai Stanciu, Buzău, România

J624. In triangle ABC let M, N, P be the midpoints of BC, CA, AB, respectively, and let D, E, F be the feet of the altitudes on sides BC, CA, AB, respectively. Prove that

$$\frac{DM + EN + FP}{2} \geq \max\{m_a, m_b, m_c\} - \min\{m_a, m_b, m_c\}.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S619. Let $a, b, c \in [0,1]$, no two of which are zero. Prove that

$$\frac{ab+1}{a+b}+\frac{bc+1}{b+c}+\frac{ca+1}{c+a}\geq\frac{ab+bc+ca+3}{a+b+c}+1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S620. Let a, b, c, d be positive real numbers. Prove that

$$(abc+abd+acd+bcd)^2\geq 4abcd(ab+bc+cd+da).$$

Proposed by An Zhenping, Xianyang Normal University, China

S621. Find all positive integers n for which there are positive integers a, b and a non-degenerate triangle with side lengths n, 3^a, 5^b.

Proposed by Josef Tkadlec, Czech Republic

S622. Let ABC be a triangle inscribed in a circle Γ of center O. The tangents at A and C to Γ intersect each other in P. The line BP intersect Γ in Q and let S be the midpoint of BQ. Prove that $\angle ACQ = \angle BCS$.

Proposed by Mihaela Berindeanu, Bucharest, România

S623. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ $(n \geq 2)$ be positive real numbers satisfying

$$\frac{a_1}{b_1}\geq\frac{a_2}{b_2}\geq\ldots\geq\frac{a_n}{b_n}.$$

Prove that

$$\sqrt{a_1}+\sqrt{b_1+a_2}+\sqrt{b_2+a_3}+\ldots+\sqrt{b_{n-1}+a_n}+\sqrt{b_n}>$$
$$>\sqrt{a_1+b_1}+\sqrt{a_2+b_2}+\ldots+\sqrt{a_n+b_n}.$$

Proposed by Waldemar Pompe, Warsaw, Poland

S624. Prove that the following inequality holds for all positive real numbers a, b, c:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\sqrt{\frac{b+c}{2a}}+\sqrt{\frac{c+a}{2b}}+\sqrt{\frac{a+b}{2c}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Undergraduate Problems

U619. Find all polynomials $P(x)$ with real coefficients such that

$$P(x)(P(x)-2P(y))^2+(2P(x)-P(y))^2P(y)=P(xP(x))+P(yP(y)),$$

for all $x,y\in\mathbb{R}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U620. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}.$$

Proposed by Vasile Lupulescu, Târgu Jiu, România

U621. Let x,y,z be nonnegative real numbers such that $x+y+z=2$. Find the minimum of

$$\sqrt{4+2x^2}+\sqrt{54-36\sqrt{2}+4y^2}+\sqrt{8+2z^2}$$

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Roma, Italy

U622. Prove that in any acute triangle ABC,

$$\left(\frac{4S}{3R}\right)^4\geq\frac{3(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{a^2+b^2+c^2}.$$

Proposed by Marius Stănean, Zalău, România

U623. Find all positive real numbers a for which the sequence

$$x_n=\frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n^a}$$

converges and find its limits in those cases.

Proposed by Mircea Becheanu, Canada

U624. Let p be a prime number. For every positive integer n denote by $rad_p(n)$ the product of all prime divisors of n, except p. Let $f:\mathbb{N}\longrightarrow\mathbb{N}$ be a multiplicative function for which there is a nonzero integer c such that

$$rad_p(n)|f(n+1)-c.$$

Prove that $f(n)=n^r$, for some positive integer r.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Olympiad Problems

O619. Let l be a nonnegative integer. Prove that there are infinitely many positive integers $k \geq l$, for each of which there are infinitely many blocks of k consecutive positive integers such that every such block contains precisely l numbers that can be represented as the sum of two perfect squares of integers.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

O620. Prove that for any positive integer n there is at most one triplet of positive integers $a \leq b \leq c$ such that $(a+b)(b+c)(c+a)(a+b+c+n)$ is a power of a prime.

Proposed by Josef Tkadlec, Czech Republic and Ján Mazák, Slovakia

O621. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be sets of integers, with $a_1, \ldots, a_k$ positive and mutually distinct, and let ε be a positive real number. Prove that there are infinitely many positive integers n such that $(a_1 n + b_1) \cdots (a_k n + b_k)$ divides $\lfloor \varepsilon n \rfloor !$. (As usual, $\lfloor x \rfloor$ denotes the integer part of the real number x.)

Proposed by Titu Andreescu, USA and Marian Tetiva, România

O622. Determine all positive integers n for which the numbers $1, 2, \ldots, n$ can be written on a paper in such an order that for each $k = 1, 2, \ldots, n$ the sum of the first k numbers is a multiple of k.

Proposed by Josef Tkadlec, Czech Republic

O623. Prove that there is a positive integer n and a list of bases $b_1, b_2, \ldots, b_{2022}$ such that n is a $2023-$palyndrome in each of the bases $b_1, b_2, \ldots, b_{2022}$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O624. Let $ABCD$ be a convex quadrilateral with $\angle BCA = \angle DCA$. Let r_1 and r_2 be the inradii of triangles ABC and ACD, respectively. Let r_3 be the radius of a circle that passes through C and is tangent to rays AC and AB. Similarly, let r_4 be the radius of a circle that passes through C and is tangent to rays AC and AD. Prove that

$$\frac{1}{r_1} + \frac{1}{r_4} = \frac{1}{r_2} + \frac{1}{r_3}.$$

Proposed by Waldemar Pompe, Warsaw, Poland