Mathematical Excalibur

Volume 20, Number 1 *July 2015 – October 2015*

Olympiad Corner

Below are the problems of the *2015 International Mathematical Olympiad* held in July 10-11, 2015.

Problem 1. We say that a finite set $\boldsymbol{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\boldsymbol{S}$, there is a point C in $\boldsymbol{S}$ such that $AC=BC$. We say that $\boldsymbol{S}$ is *center-free* if for any three different points A, B and C in $\boldsymbol{S}$, there is no point P in $\boldsymbol{S}$ such that $PA=PB=PC$.

(a) Show that for all integers $n\geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n\geq 3$ for which there exists a balanced center-free set consisting of n points.

Problem 2. Determine all triples (a,b,c) of positive integers such that each of the numbers

$$ab-c,\ \ bc-a,\ \ ca-b$$

is a power of 2.

(A power of 2 is an integer of the form 2^n, where n is a non-negative integer.)

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 27, 2015***.

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IMO 2015 – Problem Report

Law Ka Ho

IMO 2015 was held in Chiang Mai, Thailand from July 4 to 16. The examinations were held in the mornings of July 10 and 11 (contestants unable to adhere to this schedule with religious reasons were allowed to be quarantined in the day and sit the Day 2 paper after sunset). The Hong Kong team was consisted of the following students:

CHEUNG Wai Lam (Queen Elizabeth School, Form 5)
KWOK Man Yi (Baptist Lui Ming Choi College, Form 4)
LEE Shun Ming Samuel (CNEC Christian College, Form 4)
TUNG Kam Chuen (La Salle College, Form 6)
WU John Michael (Hong Kong International School, Form 4)
YU Hoi Wai (La Salle College, Form 4)

Cheung and Yu were in the IMO team last year, while the rest are first-timers.

Since Hong Kong will host IMO 2016, we sent a total of 14 observers in addition to the contestants, the leader and the deputy leader.

The following consists mainly of the discussions of the problems, marking schemes, performance etc., rather than of the solutions. The problems can be found from the Olympiad Corner in this issue. (Some readers may want to try the problems before reading this section.)

Problem 1. This is quite a standard question in combinatorial geometry. Clearly odd polygons would work for both (a) and (b). The construction for even n in (a) would take some effort, although there were a number of ways to get it done. In (b), the proof that even n does not work involves a standard double counting technique. The Hong Kong team did very well in this question, with five perfect scores plus a 6 out of 7.

This question allows partial progress to various degrees. One may complete the whole question. Those who didn't may just figure out the odd polygons, or in addition they could complete the rest of either part (a) or (b). This is better than an all-or-nothing problem. (The marking scheme does not require students to give any proof that their constructions are balanced and/or centre-free.)

Students raised quite a lot of queries on this question during the contest. The most popular question was whether the point C has to be unique. There were also questions like whether the points must be lattice points, and whether the points A, B, C could be collinear.

Problem 2. This looks like a typical number theory problem. The problem is easy to understand. However, all known solutions involve a heavy amount of considerations of different cases, and very limited number theory techniques were involved. It ended up more like an algebra problem, where one deals with the different algebraic expressions by inequality bounds and so on.

Although the known solutions were not particularly elegant, the answers turned out to be surprisingly nice. While most contestants would get (2,2,2) and (2,2,3) (and its permutations) by trial-and-error or whatever methods, there are two other sets of solutions (3,5,7) and (2,6,11) (and their permutations).

The problem was much more difficult than imagined. Very few students managed to get a complete solution, even among the strongest teams. Most of our team members obtained partial results on this one. The question also killed a lot of the contestants' time, leaving them with little time for the last problem of Day 1.

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During the problem selection, there were discussions of whether the note defining what a power of 2 is should be included. Some leaders felt that this destroyed the beauty and elegance of the paper. Some others insisted that it should be there because it would otherwise lead to heaps of questions as for whether 1 is a power of 2. Some even said that in their countries a power of 2 would mean 2 to the power 2 or above!

While discussing the marking scheme, it was decided that no penalty would be levied on students who forgot to list the permutations. In other words, one would not be penalized for saying that there are in total four solutions, namely, (2,2,2), (2,2,3), (3,5,7) and (2,6,11). I also asked for clarification whether points would be deducted for not checking the solutions satisfied the conditions of the problem. The answer was negative.

Problem 3. This is again a difficult question, even for members of some strong teams. Of course, as previously mentioned, most students spent a lot of time dealing with the different cases in Problem 2. So they simply did not have much time left for this one. This could be one of the important reasons for the general poor performance. However, our deputy leader pointed out there is a very simple solution using inversion. Interested readers may wish to try it out.

During problem selection, there had been discussions of whether there should be a note (as in Problem 2) explaining what orthocenter means. It was eventually decided that such note should not appear in the question paper. During the contest, when a question on the meaning of the orthocenter arrived, the leader of UK shouted *"Finally!"*.

Another issue is the possibility of having two different configurations. To avoid making students spend extra time working on the two cases, it was decided to fix one configuration, and so the phrase '*A, B, C, K and Q are all different, and lie on Γ in this order*' was added.

Our team obtained little in this question. Only two students managed to show that Q, M, H are collinear. According to the marking scheme, it is worth 1 point. One of the students, however, did not include much detail of the proof (after all, the question was not to prove that Q, M, H are collinear!), and the coordinator refused to award the point. This went into a long fight. The coordinators referred the case to the problem captain, then the chief coordinator. It turned out that there were many similar cases in which students mentioned the collinearity of the three points but were not accepted by the coordinators as a *proof*.

To prove that Q, M, H are collinear, one simple way is to show that Q, H, A' are collinear (where A' is the point on Γ that is diametrically opposite A), and that H, M, A' are collinear. The coordinators decided that the latter is well-known, but the former requires an explicit mention that $\angle AQH = \angle AQA' = 90^\circ$. To me, it is clear that proving the former is more trivial than the latter. If a student mentioned that A' is the antipodal point of A, then clearly (s)he knew that $\angle AQA'=90^\circ$ (it's the IMO!). Furthermore, $\angle AQH=90^\circ$ is given in the problem. What is the point of penalizing students who failed to copy this again? I didn't really see the consistency in accepting the latter as well-known but requiring such a detailed proof for the former. An urgent Jury Meeting was called to discuss this issue. The motion of sticking to the original marking scheme (i.e. to accept H, M, A' being collinear as well-known but to award 1 point only if $\angle AQH = \angle AQA' = 90^\circ$ is explicitly mentioned) was passed by a narrow margin.

The next day when we went on excursion, the Deputy Leader of Paraguay talked to me saying that many people thought that my speech was really to-the-point (by that time the deputy leaders had moved to the leaders' site and were allowed to sit in the Jury Meetings). But obviously more thought the opposite, as shown by the result of the vote!

Problem 4. This is the first problem of Day 2. It is a geometry problem, phrased carefully to make it as easy as possible. The order of the points was clearly given to ensure that only one configuration is possible. The statement to be proved was also rephrased from the original version so that the word *collinear* could be avoided.

Our team did not do well in this question. Only three students solved it. Another student showed that it suffices to prove $\angle AFK=\angle AGL$, which according to the marking scheme is worth 2 points. This sounds pretty much trivial, and the other two students would probably know it as well (only that they did not write it down because they did not find that useful).

In fact, there had been quite a lot of discussions on this point. Suppose a student showed $\angle AFK = \angle AGL$. How many points should that be worth? According to the original marking scheme, this would be worth 4 points; if a student added that *hence we are done*, that would make it 5; by writing *by symmetry we are done*, that would make it 6. (A perfect score would require some explanation on how symmetry leads to the result.) This led to strong opinion from the leaders. Eventually the (4,5,6) above was revised to (5,6,6).

Problem 5. This is the only question for which no student asked questions. This is interesting because in Problem 1 set notations were deliberately avoided, but in this question notation like $f\colon \mathbb{R} \to \mathbb{R}$ did not lead to any question, which to me is a bit of surprise.

By nature this problem is quite similar to Problem 2. Most students managed to make some partial progress, as one naturally starts by plugging in certain values of x and y into the functional equation, leading to some preliminary discoveries. However not many students obtained full solutions. We are glad everyone in our team got partial marks.

The solution to this problem depends heavily on fixed points, which in hindsight is reasonable considering that the expression $x+f(x+y)$ occurs on both sides. This also justifies starting the problem with setting $y=1$ as it would equate the terms $f(xy)$ and $yf(x)$ on the two sides of the equation. Completing the solution, on the other hand, is much more difficult, as there are too many equations and sometimes it is not clear what to put into which equation.

There were heated debates when discussing the marking scheme to this problem. As there were two functions satisfying the equation, most solutions could be divided into two parts (e.g. according to whether $f(0)=0$ or not). Each part would lead to one solution, and then one needs to check that the two solutions obtained, namely, $f(x)=x$ and $f(x)=2-x$, indeed satisfy the equation in the question. In the original proposal of the marking scheme, the coordinators said that they would accept students directly claiming that the former is a solution, while for the latter, it must be explicitly checked (expanding brackets and showing that the two sides are equal).

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 27, 2015.***

Problem 471. For $n \ge 2$, let $A_1, A_2, \ldots, A_n$ be positive integers such that $A_k \le k$ for $1 \le k \le n$. Prove that $A_1+A_2+\cdots+A_n$ is even if and only if there exists a way of selecting + or – signs such that

$$A_1 \pm A_2 \pm \cdots \pm A_n = 0.$$

Problem 472. There are $2n$ distinct points marked on a line, n of them are colored red and the other n points are colored blue. Prove that the sum of the distances of all pairs of points with same color is less than or equal to the sum of the distances of all pairs of points with different color.

Problem 473. Determine all functions f: $\mathbb{R} \to \mathbb{R}$ such that for all $x,y \in \mathbb{R}$,

$$f(x)\,f(y\,f(x)-1) = x^2 f(y) - f(x).$$

Problem 474. Quadrilateral $ABCD$ is convex and lines AB, CD are not parallel. Circle Γ passes through A, B and side CD is tangent to Γ at P. Circle L passes through C, D and side AB is tangent to L at Q. Circles Γ and L intersect at E and F. Prove that line EF bisects line segment PQ if and only if lines AD, BC are parallel.

Problem 475. Let a, b, n be integers greater than 1. If b^n-1 is a divisor of a, then prove that in base b, a has at least n digits not equal to zero.

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Solutions

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Problem 466. Let k be an integer greater than 1. If k+2 integers are chosen among 1,2,3,…,3k, then there exist two of these integers m,n such that $k<|m-n|<2k$.

Solution. **Adnan ALI** (A. E. C. S.-4, Mumbai, India), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania).

Let S be the set of the k+2 chosen integers and a be the smallest number in S. Subtracting $a-1$ from each element in S do not change the differences between the elements of S. So, without loss of generality, we can suppose $1 \in S$.

If S contains an element b such that $k+2 \le b \le 2k$, then take $m=b$ and $n=1$ to get $k<|m-n|=b-1<2k$. Otherwise, none of the numbers $k+2$, $k+3$,…, $2k$ belong to S. The k+1 numbers from $S\backslash\{1\}$ are then among the components of the k pairs $(2,2k+1)$, $(3,2k+2)$, … , $(k+1,3k)$. By the pigeonhole principle, there is a pair containing two numbers m, n from $S\backslash\{1\}$. Then we have $k<|m-n|=2k-1<2k$.

Other commended solvers: **Prithwijit DE** (HBCSE, Mumbai, India), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Simon YAU**.

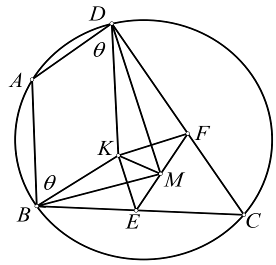
Problem 467. Let p be a prime number and q be a positive integer. Take any pq consecutive integers. Among these integers, remove all multiples of p. Let M be the product of the remaining integers. Determine the remainder when M is divided by p in terms of q.

Solution. **Adnan ALI** (A. E. C. S.-4, Mumbai, India), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Mark LAU Tin Wai**, **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

For r = 0, 1, 2, …, $p-1$, among the pq consecutive integers, there are q integers having remainders r when divided by p. Then $M \equiv 1^q 2^q \cdots (p-1)^q = (p-1)!^q \pmod p$. By Wilson's theorem, $(p-1)! \equiv -1 \pmod p$. So $M \equiv (-1)^q \pmod p$. Then the remainder when M is divided by p is 1 if q is even and is $p-1$ if q is odd.

Problem 468. Let $ABCD$ be a cyclic quadrilateral satisfying $BC>AD$ and $CD>AB$. E, F are points on chords BC, CD respectively and M is the midpoint of EF. If $BE=AD$ and $DF=AB$, then prove that $BM \perp DM$.

Solution. **George APOSTOLOPOULOS (**2 High School, Messolonghi, Greece), **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India) and **MANOLOUDIS Apostolis** (4 High School of Korydallos, Piraeus, Greece).



Let K be the point such that $ABKD$ is a parallelogram. Let $\theta = \angle ABK = \angle ADK$. Now $BE=AD=BK$, $DF=AB=DK$ and

$$\angle BKE = 90^\circ - \tfrac{1}{2}\angle KBE = 90^\circ - \tfrac{1}{2}(\angle ABC - \theta),$$
$$\angle DKF = 90^\circ - \tfrac{1}{2}\angle KDF = 90^\circ - \tfrac{1}{2}(\angle ADC - \theta),$$
$$\angle BKD = 180^\circ - \theta.$$

Adding these and using $\angle ABC + \angle ADC = 180^\circ$, we get $\angle BKE+\angle BKD+\angle DKF = 270^\circ$. Then $\angle EKF = 90^\circ$, i.e. $KF \perp KE$. So $ME = MK = MF$. Also $BE = BK$ and $DF = DK$. Then $BM \perp KE$ and $DM \perp KF$. So $BM \| KF$ and $DM \| KE$. So $BM \perp DM$.

Other commended solvers: **Prithwijit DE** (HBCSE, Mumbai, India), **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 469. Let m be an integer greater than 4. On the plane, if m points satisfy no three of them are collinear and every four of them are the vertices of a convex quadrilateral, then prove that all m of the points are the vertices of a m-sided convex polygon.

Solution. **Adnan ALI** (A. E. C. S.-4, Mumbai, India), **William FUNG**, **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let S be the set of the m points and C be the set of the vertices of the convex hull H of S. Then S contains C and C has at least 3 elements. Assume there is a point P in S and not in C. Let n be the number of elements in C. Since H is a convex polygon, H can be decomposed into $n-2$ triangles by selecting a vertex and connecting all other vertices to this vertex. Since no three points of S are collinear, P is in the interior of one of these triangles. This contradicts every four of them are the vertices of a convex quadrilateral. So $S=C$, $m=n$ and S is the set of the vertices of a m-sided convex polygon.

Problem 470. If a, b, $c > 0$, then prove that

$$\frac{a}{b(a^2+2b^2)}+\frac{b}{c(b^2+2c^2)}+\frac{c}{a(c^2+2a^2)}$$
$$\geq \frac{3}{ab+bc+ca}.$$

Solution. **Jon GLIMMS** and **Henry RICARDO** (New York Math Circle, New York, USA).

Let $x=1/a$, $y=1/b$ and $z=1/c$. Below all sums are cyclic in the order x,y,z. The desired inequality is the same as

$$\sum\frac{y^2}{z(2x^2+y^2)}\geq\frac{3}{x+y+z}.$$

By Cauchy's inequality, we have

$$\sum\frac{y^2}{z(2x^2+y^2)}\geq\frac{(x^2+y^2+z^2)^2}{\sum y^2z(2x^2+y^2)}.$$

It suffices to show

$$\frac{(x^2+y^2+z^2)^2}{\sum y^2z(2x^2+y^2)}\geq\frac{3}{x+y+z}.$$

Cross-multiplying and expanding, this is the same as

$$\sum(x^5+2x^3y^2+x^2y^3+xy^4)$$
$$\geq\sum(2x^4y+4x^2y^2z). \qquad (*)$$

By the AM-GM inequality, we have

(1) $\sum(x^5+x^3y^2)\geq\sum 2x^4y,$

(2) $\sum(x^2y^3+xy^4)=\sum(x^2y^3+yz^4)$
$\geq\sum 2y^2z^2x=\sum 2x^2y^2z.$

Next, (3) $\sum(x^3y^2+x^2y^3)\geq\sum 2x^2y^2z$ is the same as $\sum x\sum x^2y^2\geq 3xyz\sum xy$ after expansion. To get it, we have

$$\sum x\sum xy\geq\sum x\frac{\left(\sum xy\right)^2}{3}\geq 3xyz\sum xy$$

by Cauchy's inequality and the AM-GM inequality. Finally adding up (1), (2), (3), we get (*).

Other commended solvers: **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Olympiad Corner

(*Continued from page 1*)

Problem 3. Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90^\circ$, and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K and Q are all different, and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Problem 4. Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects segment BC at points D and E, such that B, D, E and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R}\to\mathbb{R}$ satisfying the equation

$$f(x+f(x+y))+f(xy)=x+f(x+y)+y\,f(x)$$

for all real numbers x and y.

Problem 6. The sequence a_1, a_2, … of integers satisfies the following conditions:

(i) $1\leq a_j\leq 2015$ for all $j\geq 1$;

(ii) $k+a_k\neq l+a_l$ for all $1\leq k\leq l$.

Prove that there exist two positive integers b and N such that

$$\left|\sum_{j=m+1}^{n}(a_j-b)\right|\leq 1007^2$$

for all integers m and n satisfying $n > m \geq N$.

IMO2015–Problem Report

(*Continued from page 2*)

This led to strong reactions from almost all the leaders, as the process of checking is indeed trivial, so an indication that the student is aware of the need of checking should be sufficient. This was eventually accepted by the coordinators.

Then the Canadian leader suggested that no mark should be deducted at all for omitting the checking. The UK leader said that he was surprised to hear such a suggestion as omitting the checking constitutes a logical error, but he would be happy to let this suggestion go to a vote. The Jury eventually voted against the suggestion. So in the end a student must somehow mention the checking (but need not actually show it) to get full mark for this question.

Interestingly, not checking that the solutions work would also constitute a logical error in Problem 2, but nobody made a suggestion to deduct points in that case. Also, while the coordinators first expected the checking to be explicitly carried out, in Problem 1 the coordinators did not even expect students to do anything to show that their constructed sets are balanced and center-free. It seems that such inconsistency between different problems is a common phenomenon.

<u>*Problem 6.*</u> Traditionally, Problem 6 is the most difficult problem of the IMO. This year's Problem 6 turned out to be not as difficult. Although only 11 out of the 577 contestants obtained perfect scores, the mean 0.355 for this question was one of the highest in recent years.

One of our team members solved this question. He mentioned that he got the idea by working on small cases first. So after all, this simple rule sometimes helps us solve not-so-simple problems!

At first sight the problem looks like one in mathematical analysis concerning the convergence of a sequence. One may even be tempted to try to prove that the sequence eventually becomes constant, which is not true.

There is an interesting interpretation of this problem (which is probably how this problem came up in the first place). At each second a ball is thrown upward, and the ball thrown at the i-th second will return to the ground after a_i seconds. So the condition $k+a_k\neq l+a_l$ for all $1\leq k\leq l$ means that no two balls shall return to the ground at the same time. The interested reader may follow this line to see whether a solution could be obtained more easily.

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Olympiad Corner

Below are the problems of the *32nd Balkan Mathematical Olympiad* held in May 5, 2015.

Problem 1. Let a, b and c be positive real numbers. Prove that

$$a^3b^6+b^3c^6+c^3a^6+3a^3b^3c^3 \geq abc(a^3b^3+b^3c^3+c^3a^3)+a^2b^2c^2(a^3+b^3+c^3).$$

Problem 2. Let ABC be a scalene triangle with incenter I and circumcircle (ω). The lines AI, BI, CI intersect (ω) for the second time at the point D, E, F, respectively. The line through I parallel to the sides BC, AC, AB intersect the lines EF, DF, DE at the points K, L, M, respectively. Prove that the points K, L, M are collinear.

Problem 3. A jury of 3366 film critics is judging the Oscars. Each critic makes a single vote for his favorite actor, and a single vote for his favorite actress. It turns out that for every integer $n\in\{1,2,\ldots,100\}$ there is an actor or actress who has been voted for exactly n times. Show that there are two critics who voted for the same actor and the same actress.

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Divisibility Problems

Kin Y. Li

Divisibility problems are common in many math competitions. Below we will look at some of these interesting problems. As usual, for integers a and b with $a\neq 0$, we will write $a \mid b$ to denote <u>b is divisible by a</u> (or in short <u>a divides b</u>).

In dividing b by a, we get a quotient q and a remainder r, we get $b/a=q+r/a$. Notice that b/a is an integer if and only if r/a is an integer. The following examples exploit this observation.

<u>*Example 1.*</u> (1999 *AIME*) Find the greatest positive integer n such that $(n-2)^2(n+1)/(2n-1)$ is an integer.

<u>*Solution.*</u> The numerator is n^3-3n^2+4. So

$$\frac{n^3-3n^2+4}{2n-1}=\frac{1}{2}n^2-\frac{5}{4}n-\frac{5}{8}+\frac{27/8}{2n-1}.$$

Multiplying by 8, we get

$$\frac{8(n^3-3n^2+4)}{2n-1}=4n^2-10n-5+\frac{27}{2n-1}.$$

Then $2n-1|27$. The greatest such n is 14.

<u>*Example 2.*</u> (*1998 IMO*) Determine all pairs (a,b) of positive integers such that ab^2+b+7 divides a^2b+a+b.

<u>*Solution.*</u> We can think of a as a variable and b as a constant, then do division of polynomials to get

$$\frac{a^2b+a+b}{ab^2+b+7}=\frac{1}{b}a-\frac{7a/b-b}{ab^2+b+7}.$$

Multiplying by $b(ab^2+b+7)$, we get

$$b(a^2b+a+b)=(ab^2+b+7)a-(7a-b^2).$$

If $ab^2+b+7 \mid a^2b+a+b$, then

$$ab^2+b+7 \mid (ab^2+b+7)a-b(a^2b+a+b) =7a-b^2. \quad (*)$$

<u>*Case 1* $(7a-b^2=0)$.</u> Then $7a=b^2$. So $7| b$. Then for some positive integer k, $b=7k$ and $a=7k^2$. We can check $(a,b)=(7k^2,7k)$ are indeed solutions.

<u>*Case 2* $(7a-b^2<0)$.</u> Then $7a<b^2$ and

$$ab^2+b+7 \leq |7a-b^2| = b^2-7a.$$

However, $b^2-7a < b^2 < ab^2+b+7$, which leads to a contradiction.

<u>*Case 3* $(7a-b^2>0)$.</u> Then $ab^2+b+7 \leq 7a-b^2$. If $b\geq 3$, then $ab^2+b+7 \geq 9a > 7a > 7a-b^2$, contradicting (*).

So $b = 1$ or 2. If $b = 1$, then (*) yields $a+8 \mid 7a-1 = 7(a+8)-57$. Hence, $a+8 \mid 57$, which leads to $a = 11$ or 49. Then we can check $(a,b) = (11,1)$ and $(49,1)$ are solutions. If $b=2$, then (*) yields $4a+9 \mid 7a-4$. Now

$$4a+9 \leq 7a-4 < 8a+18 = 2(4a+9).$$

So $4a+9 = 7a-4$, contradicting a is an integer.

<u>*Example 3.*</u> (*2003* IMO) Determine all pairs of positive integers (a,b) such that $a^2/(2ab^2-b^3+1)$ is a positive integer.

<u>*Solution.*</u> Suppose $a^2/(2ab^2-b^3+1)=k$ is a positive integer. Then $a^2-2kb^2a+kb^3-k = 0$. Multiplying by 4 and completing squares, we get

$$(2a-2kb^2)^2 = (2kb^2-b)^2 + (4k-b^2). \quad (**)$$

Let $M = 2a-2kb^2$ and $N = 2kb^2-b$.

<u>*Case 1* $(4k-b^2 = 0)$.</u> Then b is even and $M = \pm N$. If $M=-N$, then $b=2a$. If $M=N$, then $2a = 4kb^2-b = b^4-b$. We get $(a,b) = (b/2,b)$ or $((b^4-b)/2,b)$ with b even. These are easily checked to be solutions.

<u>*Case 2* $(4k-b^2 > 0)$.</u> Then $M^2 > N^2$ and $N = 2kb^2-b = b(2kb-1) \geq 1(2-1) = 1$. So $M^2 \geq (N+1)^2$. Hence, by (**)

$$\begin{aligned}4k-b^2 &= M^2 - N^2\\ &\geq (N+1)^2-N^2=2N+1\\ &= 4kb^2-2b+1,\end{aligned}$$

which implies $4k(b^2-1) + (b-1)^2 \leq 0$.

(continued on page 2)

Then $b = 1$, $k = a/2$ and $(a,b) = (2k,1)$ are easily checked to be solutions for all positive integer k.

Case 3 ($4k-b^2 < 0$). Then $M^2 \le (N-1)^2$. By (**),

$$\begin{aligned}4k-b^2 &= M^2 - N^2\\ &\le (N-1)^2-N^2=-2N+1\\ &= -4kb^2+2b+1.\end{aligned}$$

This implies

$$\begin{aligned}0 &\le (1-4k)b^2+2b+(1-4k)\\ &= (1-4k)\left(b+\frac{1}{1-4k}\right)^2+\frac{8k(2k-1)}{1-4k}<0,\end{aligned}$$

which is a contradiction.

Exercise 1. Find all positive integers n, a, and b such that

$$n^b-1 \mid n^a+1.$$

For divisibility problems involving exponential terms, like 2^n, often we will need to do modulo arithmetic and apply Fermat's little theorem. A useful fact is *if $m>n\ge 0$, then there exist integers s, t such that $\gcd(m,n) = ms+nt$.* (*Proof.* If $n=0$, then let $s=1$, $t=0$. Suppose it is true for all r with $0 \le r < n$. Then $m=qn+r$, where $q=[m/n]$. We have

$$\begin{aligned}\gcd(m,n) &= \gcd(m,r) = ms+rt\\ &= ms+(m-qn)t = m(s+t)+n(-qt).\ )\end{aligned}$$

So if $d = \gcd(m,n)$ and a^m, $a^n \equiv 1 \pmod k$, then $a^d \equiv 1 \pmod k$ by the fact.

Example 4. (*1972 Putnam Exam*) Show that if n is an integer greater than 1, then 2^n-1 is not divisible by n.

Solution. Assume there exists an integer $n > 1$ such that $n \mid 2^n-1$. Since 2^n-1 is odd, n must be odd. Let p be the least prime divisor of n. Then $p \mid 2^n-1$, which is the same as $2^n\equiv 1 \pmod p$. By Fermat's little theorem, $2^{p-1}\equiv 1 \pmod p$. Let $d = \gcd(n,p-1)$. Then $2^d \equiv 1 \pmod p$. By the definition of p, since $d \mid n$ and $d \le p-1 < p$, we get $d = 1$. Then $2 = 2^d \equiv 1 \pmod p$ lead to a contradiction.

Having seen the last example, here comes a hard problem that one needs to know the last example to get a start.

Example 5. (*1990 IMO*) Determine all integer $n>1$ such that $(2^n+1)/n^2$ is an integer.

Solution. Since 2^n+1 is odd, n must be odd. Let p be the least prime divisor of n. Then $p|2^n+1$, which implies $(2^n)^2\equiv(-1)^2 \equiv 1 \pmod p$. By Fermat's little theorem, $2^{p-1}\equiv 1 \pmod p$. Let $d = \gcd(2n,p-1)\ge 2$. Then $2^d\equiv 1 \pmod p$. By the definition of p, we get $\gcd(n,p-1)=1$. This gives $d=2$ and $4=2^d\equiv 1 \pmod p$ gives $p=3$. Then $n = 3^k m$ for some $k \ge 1$ and m satisfying $\gcd(3,m)=1$.

Using $x^3+1=(x+1)(x^2-x+1)$ for $x=2^m$, 2^{3m}, 2^{9m}, …, we have

$$2^n+1=(2^m+1)\prod_{j=0}^{k-1}(2^{2\cdot 3^j m}-2^{3^j m}+1).\quad (*)$$

For odd c, $2^c\equiv 2,-1,-4 \pmod 9$ implies $2^{2c}-2^c+1\equiv 3 \pmod 9$. From the binomial expansion, we see $2^m+1 = (3-1)^m+1 \equiv 3m \equiv 3$ or $6 \pmod 9$. So each of the factor on the right side of (*) is divisible by 3, but not by 9. So $2^n+1=3^{k+1}s$ for some integer s satisfying $\gcd(3,s)=1$. Now $n^2 = 3^{2k}m^2 \mid 2^n+1 = 3^{k+1}s$, which implies $k=1$ and $n=3m$.

Assume $m>1$. Let q be the least prime divisor of m. Now q is odd and $q>3$. Then $\gcd(m,q-1)=1$. Since $q \mid m \mid n$, we have $q^2 \mid n^2 \mid 2^n+1$. Then 2^{q-1} and $2^{2n}\equiv 1 \pmod q$ lead to $2^w\equiv 1 \pmod q$, where $w = \gcd(2n, q-1)$. Then $w|2n=6m$. Also, from $w|q-1$ and $\gcd(m,q-1) = 1$, we get $w|6$. Now $q>3$, $w=1,2,3,6$ and $2^w\equiv 1 \pmod q$ imply $q=7$. Then $7=q \mid 2^n+1$, but $2^n\equiv 1, 2, 4\not\equiv -1 \pmod 7$, contradiction. Therefore, $m=1$ and $n=3$. Indeed, $3^2=9 \mid 2^3+1$.

Exercise 2. (1999 *IMO*) Find all pairs of positive integers (x,p) such that p is prime, $x\le 2p$, and x^{p-1} divides $(p-1)^x+1$.

In the following examples, we will see there is a very clever trick in solving certain divisibility problems.

Example 6. (*1988 IMO*) Let a and b positive integers such that $ab+1$ divides a^2+b^2. Show that $(a^2+b^2)/(ab+1)$ is square of an integer.

Solution. Let $k = (a^2+b^2)/(ab+1)$. Assume there is a case k is an integer, but not a square. Among all such cases, consider the case when $\max\{a,b\}$ is least possible. Note $a=b$ implies $0 < k = 2a^2/(a^2+1) < 2$, which implies $k=1=1^2$. So in the least case, $a\ne b$, say $a>b$. Now $k = (a^2+b^2)/(ab+1)>0$ and it can be rewritten as $a^2-kba+b^2-k=0$. Note $k\ne b^2$ implies $a\ne 0$.

Other than a, let c be the second root of $x^2-kbx+b^2-k=0$. Then $k = (c^2+b^2)/(cb+1)$, $a+c=kb$ and $ac=b^2-k$. So $c=kb-a=(b^2-k)/a$ is an integer. Now $cb+1=(c^2+b^2)/k > 0$ and $c=(b^2-k)/a\ne 0$ imply c is a positive integer. Finally, $c = (b^2-k)/a < (a^2-k)/a < a$. Now $k = (c^2+b^2)/(cb+1)$ is an integer, not a square and $\max\{b,c\} < a = \max\{a,b\}$. This contradicts $\max\{a,b\}$ is the least.

Example 7. (*2007 IMO*) Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

Solution. We can consider a as variable and b as constant to do a division as in example 2, but a nicer way is as follows: from $(4a^2-1)b=a(4ab-1)+(a-b)$, we get

$$(4a^2-1)^2b^2 = J(4ab-1)+(a-b)^2,$$

where $J=a^2(4ab-1)+2a(a-b)$. Observe that $\gcd(b^2,4ab-1) =1$ (otherwise prime $p \mid \gcd(b^2,4ab-1)$ would imply $p|b$ and $p|4ab-(4ab-1)=1$). Hence,

$$4ab-1|(4a^2-1)^2 \Leftrightarrow 4ab-1|(a-b)^2.$$

Now $k=(a-b)^2/(4ab-1) > 0$ and it can be rewritten as $a^2-(4k+2)ba+b^2+k=0$.

Assume there exists (a,b) such that k is an integer and $a\ne b$, say $a>b$. Among all such cases, consider the case when $a+b$ is least possible.

Other than a, let c be the second root of $x^2-(4k+2)bx+b^2+k=0$. Then $k = (c-b)^2/(4cb-1)$, $a+c = (4k+2)b$ and $ac = b^2+k$. So $c = (4k+2)b-a = (b^2+k)/a$ is a positive integer. So (c,b) is another case k is an integer. Since $a+b$ is least possible, we would have $c \ge a > b$. Now $c = (b^2+k)/a \ge a$ leads to $k \ge a^2-b^2$. Then

$$(a-b)^2 = k(4ab-1) \ge (a^2-b^2)(4ab-1).$$

Canceling $a-b$ on both sides, we get

$$a-b \ge (a+b)(4ab-1) > a,$$

a contradiction.

The next example is short and cute.

Example 8. (*2005 IMO Shortlisted Problem*) Let a and b be positive integers such that a^n+n divides b^n+n for every positive integer n. Show that $a=b$.

Solution. Assume $a\ne b$. For $n = 1$, we have $a+1|b+1$ and so $a < b$. Let p be a prime greater than b. Then let $n = (a+1)(p-1)+1$. By Fermat's little theorem, $a^n=(a^{p-1})^{a+1}a\equiv a \pmod p$.

So $a^n+n \equiv a+n = (a+1)p \equiv 0 \pmod p$. Then $p \mid a^n+n \mid b^n+n$. By Fermat's little theorem,

$$0\equiv b^n+n=(b^{p-1})^{a+1}b+n\equiv b-a \pmod p,$$

which contradicts $0 < a < b < p$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***January 7, 2016.***

Problem 476. Let p be a prime number. Define sequence a_n by $a_0=0$, $a_1=1$ and $a_{k+2}=2a_{k+1}-pa_k$. If one of the terms of the sequence is −1, then determine all possible value of p.

Problem 477. In $\triangle ABC$, points D, E are on sides AC, AB respectively. Lines BD, CE intersect at a point P on the bisector of $\angle BAC$.

Prove that quadrilateral $ADPE$ has an inscribed circle if and only if $AB=AC$.

Problem 478. Let a and b be a pair of coprime positive integers of opposite parity. If a set S satisfies the following conditions:

(1) $a, b \in S$;
(2) if $x,y,z\in S$, then $x+y+z\in S$,

then prove that every positive integer greater than $2ab$ belongs to S.

Problem 479. Prove that there exists infinitely many positive integers k such that for every positive integer n, the number $k2^n+1$ is composite.

Problem 480. Let m, n be integers with $n > m > 0$. Prove that if $0 < x < \pi/2$, then

$$2|\sin^n x-\cos^n x| \le 3|\sin^m x-\cos^m x|.$$

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Solutions

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Problem 471. For $n \ge 2$, let $A_1, A_2, \ldots, A_n$ be positive integers such that $A_k\le k$ for $1 \le k \le n$. Prove that $A_1+A_2+\cdots+A_n$ is even if and only if there exists a way of selecting + or – signs such that

$$A_1 \pm A_2 \pm \cdots \pm A_n = 0.$$

Solution. **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

If $A_1 \pm A_2 \pm \cdots \pm A_n = 0$, then using $A_i \equiv \pm A_i \pmod 2$, we get $A_1+A_2+\cdots+A_n \equiv 0 \pmod 2$. Hence $A_1+A_2+\cdots+A_n$ is even.

Conversely, we will prove by induction that for t from n to 1 that there exists a way of selecting signs so that

$$0 \le S_t = \pm A_t \pm A_{t+1} \pm\cdots\pm A_n \le t.$$

The case $t=n$ is $0 < A_n \le n$. Suppose the case $t=k$ is true, that is

$$0 \le S_k = \pm A_k \pm A_{k+1} \pm\cdots\pm A_n \le k.$$

If $A_{k-1}\le S_k$, then let $S_{k-1}= -A_{k-1}+S_k$ and we have $0\le S_{k-1} = S_k - A_{k-1} \le k-1$. If $A_{k-1}>S_k$, then let $S_{k-1}=A_{k-1}-S_k$ (here $-S_k$ means reversing all the signs of S_k) and we have $0\le S_{k-1}\le A_{k-1}\le k-1$. This completes the induction.

The case $t=1$ gives us $0 \le \pm A_1\pm A_2\pm\cdots\pm A_n \le 1$. As $\pm A_1\pm A_2\pm\cdots\pm A_n$ is an even integer, $\pm A_1\pm A_2\pm\cdots\pm A_n=0$.

Problem 472. There are $2n$ distinct points marked on a line, n of them are colored red and the other n points are colored blue. Prove that the sum of the distances of all pairs of points with same color is less than or equal to the sum of the distances of all pairs of points with different color.

Solution. **Jon GLIMMS**, **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Raul A. SIMON** (Chile).

Let the points be on the real axis with red points having coordinates $x_1<x_2<\cdots<x_n$ and the blue points having coordinates $y_1<y_2<\cdots<y_n$. Let S_n denote the sum of distances of all pairs of points with same color and D_n denote the sum of distances of all pairs of points with different color. We will prove $S_i\le D_i$ for all i by induction. Now $S_1=0\le|x_1-y_1|=D_1$. Suppose $S_n\le D_n$. For case $n+1$,

$$S_{n+1}-S_n=\sum_{i=1}^{n}(x_{n+1}-x_i)+(y_{n+1}-y_i)$$

$$\le|x_{n+1}-y_{n+1}|+\sum_{i=1}^{n}|x_{n+1}-y_i|+|y_{n+1}-x_i|$$

$$=D_{n+1}-D_n.$$

Then $S_{n+1}-D_{n+1}\le S_n-D_n\le 0$. So $S_{n+1}\le D_{n+1}$.

Problem 473. Determine all functions f: $\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in\mathbb{R}$,

$$f(x)f(yf(x)-1)=x^2f(y)-f(x).$$

Solution. **Coco YAU** (Pui Ching Middle School).

The zero function is a solution. Suppose f is a solution that is not the zero function. Then there exists $a\in\mathbb{R}$ such that $f(a)\ne 0$. Denote the functional equation by (*). Setting $x=0$ in (*), we get

$$f(0)(f(yf(0)-1)+1)=0.$$

If $f(0)\ne 0$, then $f(yf(0)-1)=-1$. Since $\{yf(0)-1: y\in\mathbb{R}\}=\mathbb{R}$, we can see f is the constant function −1. Then (*) with $x=1$ yields $(-1)^2=-1^2+1$, which is a contradiction. So $f(0)=0$.

Now setting $x=a$, $y=0$ in (*), we can get

$$f(-1)=-1.$$

Also, if $f(b)=0$, then setting $x=b$ and $y=a$, we get $b=0$. Hence,

$$f(x)=0 \Leftrightarrow x=0.$$

Next by setting $x=y=1$ in (*), we get $f(1)f(f(1)-1)=0 \Leftrightarrow f(1)-1=0 \Leftrightarrow f(1)=1$.

Setting $x=1$ in (*), we get

$$f(y-1)=f(y)-1. \quad (1)$$

Applying (1) to $f(yf(x)-1)$ in (*), we can simplify (*) to

$$f(x)f(yf(x))=x^2f(y). \quad (2)$$

Setting $x=-1$ in (2), we get $-f(-y)=f(y)$. So f is an odd function.

Applying induction to (1), we get for $n=1,2,3,\ldots$,

$$f(y-n)=f(y)-n. \quad (3)$$

Setting $y=0$, this gives $f(-n)=-n$. As f is odd, we get $f(n)=n$ for all integers n. Setting $x=n$ in (2), we get

$$f(ny)=nf(y). \quad (4)$$

Setting $y=1/n$ and $y=1/m$ we get $1=nf(1/n)$ and $f(n/m)=nf(1/m)=n/m$. So $f(x)=x$ for all rational x.

Setting $y=1$ in (2), we get

$$f(x)f(f(x))=x^2. \quad (5)$$

Setting x,y to be $f(x)$ in (2), we also get

$$f(f(x))f(f(x)f(f(x)))=f(x)^2f(f(x)).$$

Cancelling $f(f(x))$ on both sides, we get

$$f(x)^2=f(f(x)f(f(x)))=f(x^2),$$

where the second equality follows from applying f to both sides of (5). Then we see $w>0$ implies $f(w)>0$.

For irrational $w>0$, assume $f(w)>w$. Take rational $q=n/m$ such that $m>0$ and $f(w)>q>w$. We have $m(q-w)>0$. So $f(n-mw)=f(m(q-w))>0$. As f is odd, using (4) and (3), we get

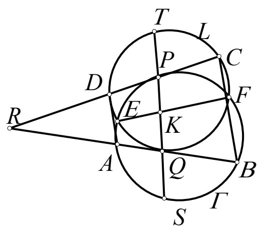
$mf(w)-n = f(mw)-n = f(mw-n) < 0$,

which contradicts $f(w) > q$. Similarly, $f(w) < w$ will lead to a contradiction. Therefore, $f(w)=w$ for all w and we can check (*) holds in this case.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 474. Quadrilateral $ABCD$ is convex and lines AB, CD are not parallel. Circle Γ passes through A, B and side CD is tangent to Γ at P. Circle L passes through C, D and side AB is tangent to L at Q. Circles Γ and L intersect at E and F. Prove that line EF bisects line segment PQ if and only if lines AD, BC are parallel.

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Let EF meet PQ at K. Extend PQ to meet Γ and L at S and T respectively. Let lines AB, CD meet at R. We have

$RP^2=RA\cdot RB$ and $RQ^2=RC\cdot RD$. (*)

By the intersecting chord theorem, we have $KP\cdot KS=KE\cdot KF=KQ\cdot KT$. Then $KP(KQ+QS) = KQ(KP+PT)$. Cancel $KP\cdot KQ$. We have

$$KP\cdot QS= KQ\cdot PT.$$

Then

$$\begin{aligned} & KP=KQ \\ \Leftrightarrow\ & QS=PT \\ \Leftrightarrow\ & PQ\cdot QS=QP\cdot PT \\ \Leftrightarrow\ & AQ\cdot QB=DP\cdot PC. \end{aligned}$$

Using $AQ=RQ-RA$, $QB=RB-RQ$, $DP=RP-RD$, $PC=RC-RP$ and (*), we get

$$\begin{aligned} & AQ\cdot QB = DP\cdot PC \\ \Leftrightarrow\ & RQ(RA+RB) = RP(RC+RD) \\ \Leftrightarrow\ & RC\cdot RD(RA+RB)^2 = RA\cdot RB(RC+RD)^2 \\ \Leftarrow\ & (RA\cdot RC - RB\cdot RD)^2 = 0 \\ \Leftrightarrow\ & \frac{RA}{RB}=\frac{RD}{RC} \\ \Leftrightarrow\ & AD \parallel BC. \end{aligned}$$

Problem 475. Let a, b, n be integers greater than 1. If b^n-1 is a divisor of a, then prove that in base b, a has at least n digits not equal to zero.

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Among all numbers that are multiples of b^n-1, suppose the least number of nonzero digits in base b of these numbers is s. Let A be one of these numbers with least digit sum, say

$$A = a_1 b^{n_1} + a_2 b^{n_2} + \cdots + a_s b^{n_s},$$

where $n_1 > n_2 > \cdots > n_s \geq 0$ and $1 \leq a_i < b$ for $i=1,2,\cdots,s$.

Assume there are i,j such that $1\leq i<j\leq s$ and $n_i \equiv n_j \equiv r \pmod n$ with $0\leq r\leq n-1$. Then consider

$$B = A - a_i b^{n_i} - a_j b^{n_j} + (a_i + a_j) b^{nn_1 + r}.$$

From $b^n \equiv 1 \pmod{b^n-1}$, we get $B \equiv 0 \pmod{b^n-1}$. If $a_i+a_j<b$, then the number of nonzero digits of B in base b is $s-1$, contradicting the choice of A. So we must have $b \leq a_i+a_j < 2b$. Let $a_i+a_j = b+q$, where $0\leq q<b$. Then

$$\begin{aligned} B = {} & b^{nn_1+r+1} + qb^{nn_1+r} + a_1 b^{n_1} + \cdots \\ & + a_{i-1} b^{n_{i-1}} + a_{i+1} b^{n_{i+1}} + \cdots \\ & + a_{j-1} b^{n_{j-1}} + a_{j+1} b^{n_{j+1}} + \cdots + a_s b^{n_s}. \end{aligned}$$

Then the digit sum of B is

$$\begin{aligned} & \sum_{k=1}^{s} a_k - (a_i + a_j) + 1 + q \\ = & \sum_{k=1}^{s} a_k + 1 - b \\ < & \sum_{k=1}^{s} a_k, \end{aligned}$$

which is the digit sum of A. This contradicts the choice of A. So n_1, n_2, …, n_s (mod n) are pairwise distinct. Then $s\leq n$.

Assume $s<n$. Then let $n_i \equiv r_i \pmod n$ with $0\leq r_i<n$ and consider

$$C = a_1 b^{r_1} + a_2 b^{r_2} + \cdots + a_s b^{r_s}.$$

Since $b^{n_i} \equiv b^{r_i} \pmod{b^n-1}$, so C is a multiple of b^n-1. Now $s<n$ implies

$$\begin{aligned} 0 < C \leq & (b-1)b + (b-1)b^2 \\ & + \cdots + (b-1)b^{n-1} \\ < & b^n - 1, \end{aligned}$$

contradiction. Therefore, $s = n$.

Other commended solvers: **Mark LAU Tin Wai** (Pui Ching Middle School) and **LEUNG Kit Yat** (St. Paul's College, Hong Kong).

Olympiad Corner

(Continued from page 1)

Problem 4. Prove that among any 20 consecutive positive integers there exists an integer d such that for each positive integer n we have the inequality

$$n\sqrt{d}\,\{n\sqrt{d}\} > \frac{5}{2}$$

where $\{x\}$ denotes the fractional part of the real number x. The fractional part of a real number x is x minus the greatest integer less than or equal to x.

Divisibility Problems

(Continued from page 2)

Solution of Exercise 1. Let $a=qb+r$ with $0 \leq r \leq b-1$. Then

$$\frac{n^a+1}{n^b-1} = n^r \sum_{j=0}^{q-1} n^{bj} + \frac{n^r+1}{n^b-1}.$$

So we need to find when $n^b-1 \mid n^r+1$. If $b=1$, then $r=0$ and we get $n=2,3$. If $b>1$, then $n>1$ and $n^b \geq 4$. For $n^b>4$, we have $0<n^r+1\leq n^{b-1}+1\leq n^b/2+1<n^b-1$, hence no solution. For $n^b\leq 4$, we have three cases, namely (n,b,a) = $(2,2,2k-1)$, $(3,1,k)$ and $(2,1,k)$, where $k=1,2,3,\ldots$.

Solution of Exercise 2. For $x<3$ or $p<3$, the solutions are $(x,p)=(2,2)$ and (1,prime). For x and $p\geq 3$, since p is odd, $(p-1)^x+1$ is odd, so x is odd. Let q be the least prime divisor of x, which must be odd. We have $q \mid x \mid x^{p-1} \mid (p-1)^x+1$. So $(p-1)^x \equiv -1 \pmod q$. By Fermat's little theorem, $(p-1)^{q-1} \equiv 1 \pmod q$. By the definition of q, we have $\gcd(x,q-1)=1$. Then there are integers a,b such that $ax=b(q-1)+1$ is odd. Then a is odd. Now

$$p-1 \equiv (p-1)^{b(q-1)+1} = (p-1)^{ax} \equiv -1 \pmod q$$

implies $q|p$. So $q=p$. Since x is odd, $p = q \mid x$ and the problem require the condition $x\leq 2p$, we must have $x=p$ for the cases $x,p \geq 3$. Observe that

$$p^{p-1} \mid (p-1)^p + 1 = p^2(mp+1)$$

for some m. Then $p-1\leq 2$. So $x=p=3$ is the only solution.

Mathematical Excalibur

Volume 20, Number 3 *January 2016 – February 2016*

Olympiad Corner

Below are the problems of the Second Round of the *32nd Iranian Math Olympiad*.

Problem 1. A local supermarket is responsible for the distribution of 100 supply boxes. Each box is ought to contain 10 kilograms of rice and 30 eggs. It is known that a total of 1000 kilograms of rice and 3000 eggs are in these boxes, but in some of them the amount of either item is more or less than the amount required. In each step, supermarket workers can choose two arbitrary boxes and transfer any amount of rice or any number of eggs between them. At least how many steps are required so that, starting from any arbitrary initial condition, after these steps the amount of rice and the number of eggs in all these boxes are equal?

Problem 2. Square $ABCD$ is given. Points N and P are selected on sides AB and AD, respectively, such that $PN = NC$, and point Q is selected on segment AN such that $\angle NCB = \angle QPN$. Prove that $\angle BCQ = \frac{1}{2}\angle PQA$.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 29, 2016***.

For individual subscription for the next five issues for the 15-16 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Coloring Problems

Kin Y. Li

In some math competitions, there are certain combinatorial problems that are about partitioning a board (or a set) into pieces like dominos. We will look at some of these interesting problems. Often clever ways of assigning color patterns to the squares of the board allow simple solutions. Below, a <u>*m*×*n* rectangle</u> will mean a *m*-by-*n* or a *n*-by-*m* rectangle.

<u>*Example 1.*</u> A 8×8 chessboard with the the northeast and southwest corner unit squares removed is given. Is it possible to partition such a board into thirty-one dominoes (where a domino is a 1×2 rectangle)?

<u>*Solution.*</u> For such a board, we can color the unit squares alternatively in black and white, say black is color 1 and white is color 2. Then we have the following pattern.

1	2	1	2	1	2	1	
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
	1	2	1	2	1	2	1

Each domino will cover two adjacent squares, one with color 1 and the other with color 2. If 31 dominoes can cover the board, there should be 31 squares with color 1 and 31 squares with color 2. However, in the board above there are 32 squares of color 1 and 30 squares of color 2. So the task is impossible.

<u>*Example 2.*</u> Eight 1×3 rectangles and one 1×1 square covered a 5×5 board. Prove that the 1×1 square must be over the center unit square of the board.

<u>*Solution.*</u> Let us paint the 25 unit squares of the 5×5 board with colors A, B and C as shown on the top of the next column.

A	B	C	A	B
B	C	A	B	C
C	A	B	C	A
A	B	C	A	B
B	C	A	B	C

There are 8 color A squares, 9 color B squares and 8 color C squares. Each 1×3 rectangle covers a color A, a color B and a color C square. So the 1×1 square piece must be over a color B square.

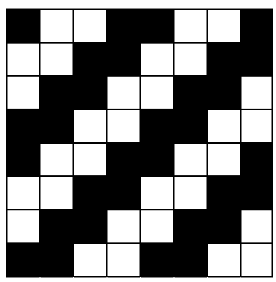
Next, we rotate the <u>*coloring*</u> of the board (not the board itself) clockwise 90˚ around the center unit square.

B	A	C	B	A
C	B	A	C	B
A	C	B	A	C
B	A	C	B	A
C	B	A	C	B

Then observe that the 1×1 square piece must still be over a color B square due to reasoning used in the top paragraph. However, the only color B square that remains color B after the 90˚ rotation is the center unit square. So the 1×1 square piece must be over the center unit square.

<u>*Example 3.*</u> Can a 8×8 board be covered by fifteen 1×4 rectangles and one 2×2 square without overlapping?

<u>*Solution.*</u> Consider the following coloring of the 8×8 board.



(continued on page 2)

In the coloring of the board, there are 32 white and 32 black squares respectively. By simple checking, we can see every 1×4 rectangle will cover 2 white and 2 black squares. The 2×2 square will cover *either* 1 black and 3 white squares *or* 3 black and 1 white squares. Assume the task is possible. Then the 16 pieces together should cover *either* 31 black and 33 white squares *or* 33 black and 31 white squares, which is a contradiction to the underlined statement above.

In coloring problems, other than assigning different colors to all the squares, sometimes assigning different numerical values for different types of squares can be useful in solving the problem. Below is one such example.

Example 4. Let m,n be integers greater than 2. Color every 1×1 square of a $m\times n$ board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then call this pair of squares a *distinct pair*. Let S be the number of distinct pairs in the $m\times n$ board. Prove that whether S is odd or even depends only on the 1×1 squares on the boundary of the board excluding the 4 corner 1×1 squares.

Solution. We first divide the 1×1 squares into three types. Type 1 squares are the four 1×1 squares at the corners of the board. Type 2 squares are the 1×1 squares on the boundary of the board, but not the type 1 squares. Type 3 squares are the remaining 1×1 squares.

Assign every white 1×1 square the value 1 and every black 1×1 square the value −1. Let the type 1 squares have values a, b, c, d respectively. Let the type 2 squares have values $x_1, x_2, \ldots, x_{2m+2n-8}$ and the type 3 squares have values $y_1, y_2, \ldots, y_{(m-2)(n-2)}$.

Next for every pair of 1×1 squares sharing a common edge, write the product of the values in the two squares on their common edge. Let H be the product of these values on all the common edges. For every type 1 square, it has two neighbor squares sharing a common edge with it. So the number in a type 1 square appears two times as factors in H. For every type 2 square, it has three neighbor squares sharing a common edge with it. So the number in a type 2 square appears three times as factors in H. For every type 3 square, it has four neighbor squares sharing a common edge with it. So the number in a type 3 square appears four times as factors in H. Hence,

$$\begin{aligned} H &= (abcd)^2(x_1x_2\cdots x_{2m+2n-8})^3(y_1y_2\cdots y_{(m-2)(n-2)})^4 \\ &= (x_1x_2\cdots x_{2m+2n-8})^3. \end{aligned}$$

If $x_1x_2\cdots x_{2m+2n-8}=1$, then $H=1$ and there are an even number of distinct pairs in the board. If $x_1x_2\cdots x_{2m+2n-8}=-1$, then $H=-1$ and there are an odd number of distinct pairs in the board. So whether S is even or odd is totally determined by the set of type 2 squares.

Next we will look at problems about coloring elements of some sets.

Example 5. There are 1004 distinct points on a plane. Connect each pair of these points and mark the midpoints of these line segments black. Prove that there are at least 2005 black points and there exists a set of 1004 distinct points generating exactly 2005 black midpoints of the line segments connecting pairs of them.

Solution. From 1004 distinct points, we can draw $k={}_{1004}C_2$ line segments connecting pairs of them. Among these, there exists a *longest* segment AB. Now the midpoints of the line segments joining A to the other 1003 points lie inside or on the circle center at A and radius ½AB. Similarly, the midpoints of the line segments joining B to the other 1003 points lie inside or on another circle center at B and radius ½AB. These two circles intersect only at the midpoint of AB. Then there are at least 2×1003−1=2005 black midpoints generated by the line segments.

To construct an example of a set of 1004 points generating exactly 2005 black midpoints, we can simply take 0, 2, 4, …, 2006 on the x-axis. Then the black midpoints generated are exactly the point at 1, 2, 3, …, 2005 of the x-axis..

Example 6. Find all ways of coloring all positive integers such that

(1) every positive integer is colored either black or white (but not both) and

(2) the sum of two numbers with distinct colors is always colored black and their product is always colored white.

Also, determine the color of the product of two white numbers.

Solution. Other than coloring all positive integers the same color, we have the following coloring satisfying conditions (1) and (2). We claim if m and n are white numbers, then mn is a white number. To see this, assume there are m, n both white, but mn is black. Let k be black. By (1), $m+k$ is black and $(m+k)n = mn+kn$ is white. On the other hand, kn is white and mn is black. So by (2), $mn+kn$ would also be black, which is a contradiction.

Next, let j be the *smallest* white positive integer. From (2) and the last paragraph, we see every sj is white, where s is any positive integer. We will prove every positive integer p that is not a multiple of j is black. Suppose $p=qj+r$, where q is a nonnegative integer and $0 < r < j$. Since j is the smallest white integer, so r is black. When $q=0$, $p=r$ is black. When $q \geq 1$, qj is white and so by (2), $p=qj+r$ is black.

Example 7. In the coordinate plane, a point (x,y) is called a *lattice point* if and only if x and y are integers. Suppose there is a convex pentagon $ABCDE$ whose vertices are lattice points and the lengths of its five sides are all integers. Prove that the perimeter of the pentagon $ABCDE$ is an even integer.

Solution. Let us color every lattice point of the coordinate plane either black or white. If $x+y$ is even, then color (x,y) white. If $x+y$ is odd, then color (x,y) black. Notice (x,y) is assigned a color different from its four *neighbors* $(x\pm1,y)$ and $(x,y\pm1)$.

Now for each of the five sides, say AB, of the pentagon $ABCDE$, let A be at (x_1, y_1) and B be at (x_2, y_2). Also let T_{AB} to be at (x_1,y_2). Then $\triangle ABT_{AB}$ is a right triangle with AB as the hypotenuse or it is a line segment (which we can consider as a degenerate right triangle).

Since each lattice point is assigned a color different from any one of its four neighbors, the polygonal path

$$AT_{AB}BT_{BC}CT_{CD}DT_{DE}ET_{EA}A$$

has even length. For positive integers a, b, c satisfying $a^2+b^2=c^2$, since $n^2\equiv n \pmod 2$, we get $a+b\equiv c \pmod 2$. It follows the perimeter of $ABCDE$ and the length of $AT_{AB}BT_{BC}CT_{CD}DT_{DE}ET_{EA}A$ are of the same parity. So the perimeter of $ABCDE$ is even.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 29, 2016.***

Problem 481. Let $S=\{1,2,\ldots,2016\}$. Determine the least positive integer n such that whenever there are n numbers in S satisfying every pair is relatively prime, then at least one of the n numbers is prime.

Problem 482. On $\triangle ABD$, C is a point on side BD with $C\neq B,D$. Let K_1 be the circumcircle of $\triangle ABC$. Line AD is tangent to K_1 at A. A circle K_2 passes through A and D and line BD is tangent to K_2 at D. Suppose K_1 and K_2 intersect at A and E with E inside $\triangle ACD$. Prove that $EB/EC=(AB/AC)^3$.

Problem 483. In the open interval $(0,1)$, n distinct rational numbers a_i/b_i $(i=1,2,\ldots,n)$ are chosen, where $n>1$ and a_i, b_i are positive integers. Prove that the sum of the b_i's is at least $(n/2)^{3/2}$.

Problem 484. In a multiple choice test, there are four problems. For each problem, there are choices *A, B* and *C*. For any three students who took the test, there exist a problem the three students selected distinct choices. Determine the maximum number of students who took the test.

Problem 485. Let m and n be integers such that $m>n>1$, $S=\{1,2,\ldots,m\}$ and $T=\{a_1, a_2, \ldots, a_n\}$ is a subset of S. It is known that every two numbers in T do not both divide any number in S. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{n}.$$

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Solutions

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Problem 476. Let p be a prime number. Define sequence a_n by $a_0=0$, $a_1=1$ and $a_{k+2}=2a_{k+1}-pa_k$. If one of the terms of the sequence is -1, then determine all possible value of p.

Solution. **Jon GLIMMS** and **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5).

Observe that $p\neq 2$ (otherwise beginning with a_2, the rest of the terms will be even, then -1 cannot appear). On one hand, using the recurrence relation, we get

$$a_{k+2}\equiv 2a_{k+1}\equiv\cdots\equiv 2^{k+1}a_1=2^{k+1} \pmod p.$$

If $a_m=-1$ for some $m\geq 2$, then letting $k=m-2$, we get

$$-1=a_m\equiv 2^{m-1} \pmod p. \qquad (*)$$

On the other hand, using the recurrence relation again, we also have

$$a_{k+2}\equiv 2a_{k+1}-a_k \pmod{p-1},$$

which implies $a_{k+2}-a_{k+1}\equiv a_{k+1}-a_k\equiv\cdots\equiv a_1-a_0=1 \pmod{p-1}$. Then

$$-1=a_m\equiv m+a_0=m \pmod{p-1},$$

which implies $p-1$ divides $m+1$. By Fermat's little theorem and (*), we get

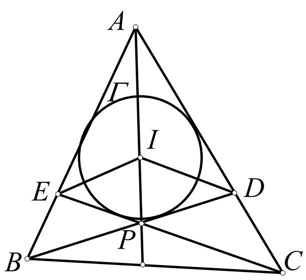
$$1\equiv 2^{m+1}\equiv 4\cdot 2^{m-1}\equiv -4 \pmod p.$$

Then $p=5$. Finally, if $p=5$, then $a_3=-1$.

Problem 477. In $\triangle ABC$, points *D, E* are on sides *AC, AB* respectively. Lines *BD* and *CE* intersect at a point *P* on the bisector of $\angle BAC$.

Prove that quadrilateral *ADPE* has an inscribed circle if and only if $AB=AC$.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece), **Jafet Alejandro Baca OBANDO** (IDEAS High School, Nicaragua) and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Suppose *ADPE* has an inscribed circle Γ. Since the center of Γ is on the bisector of $\angle BAC$, the center is on line *AP*. Similarly, *AP* also bisects $\angle DPE$, so $\angle APE=\angle APD$. It also follows that $\angle APB=\angle APC$, since $\angle EPB=\angle DPC$. By ASA, we get $\triangle APB\cong\triangle APC$ with *AP* common. Then $AB=AC$.

Conversely, if $AB=AC$, then $\triangle ABC$ is symmetric with respect to *AP*. Thus, lines *BP* and *CP* (hence also *D* and *E*) are symmetric with respect to *AP*. By symmetry, the bisectors of $\angle ADP$ and $\angle AEP$ meet at a point *I* on *AP*. Then the distances from *I* to lines *EA, EP, DP, DA* are the same. So *ADPE* has an inscribed circle with center *I*.

Other commended solvers: **Mark LAU Tin Wai** (Pui Ching Middle School), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 478. Let a and b be a pair of coprime positive integers of opposite parity. If a set S satisfies the following conditions:

(1) $a, b\in S$;
(2) if $x,y,z\in S$, then $x+y+z\in S$,

then prove that every positive integer greater than $2ab$ belongs to S.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

Without loss of generality, we assume that a is odd and b is even. Let $n>2ab$. Since a and b are coprime, the equation $ax\equiv n \pmod b$ has a solution satisfying $0\leq x<b$. Then $y=(n-ax)/b$ is a positive integer. Now

$$a=\frac{2ab-ab}{b}<\frac{n-ax}{b}=y\leq\frac{2ab}{b}=2a.$$

Let $x'=x+b$, $y'=y-a$, then x', y' are positive and $ax'+by'=n$. Observe $x+y$ and $x'+y'=x+y+b-a$ are of opposite parity. So we may assume $x+y$ is odd (otherwise take $x'+y'$). Then $x+y\geq 3$ and by (1) and (2),

$$n=a+\cdots+a+b+\cdots+b\in S,$$

where a appeared x times and b appeared y times.

Other commended solvers: **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5) and **Mark LAU Tin Wai** (Pui Ching Middle School).

Problem 479. Prove that there exists infinitely many positive integers k such that for every positive integer n, the number $k2^n+1$ is composite.

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5).

By the Chinese remainder theorem, there exist infinitely many positive integers k such that

$$k \equiv 1 \pmod{3},$$
$$k \equiv 1 \pmod{5},$$
$$k \equiv 3 \pmod{7},$$
$$k \equiv 10 \pmod{13},$$
$$k \equiv 1 \pmod{17},$$
$$k \equiv -1 \pmod{241}.$$

If $n \equiv 1 \pmod 2$, then $k2^n+1 \equiv 2+1 \equiv 0 \pmod 3$. Otherwise $2|n$. If $n \equiv 2 \pmod 4$, then $k2^n+1 \equiv 2^2+1 \equiv 0 \pmod 5$. Otherwise $4|n$. If $n \equiv 4 \pmod 8$, then $k2^n+1 \equiv 2^4+1 \equiv 0 \pmod{17}$. Otherwise $8|n$. Then we have three cases:

<u>*Case 1: $n \equiv 8 \pmod{24}$.*</u> By Fermat's little theorem, $2^{24} = (2^{12})^2 \equiv 1 \pmod{13}$. So $2^n = 2^{8+24m} \equiv 256 \equiv -4 \pmod{13}$ and $k2^n+1 \equiv 10(-4)+1 \equiv 0 \pmod{13}$.

<u>*Case 2: $n \equiv 16 \pmod{24}$.*</u> Since $2^{24} = (2^3)^8 \equiv 1 \pmod 7$, we have $2^n = 2^{16+24m} \equiv 2^{1+3(5+8m)} \equiv 2 \pmod 7$ and $k2^n+1 \equiv 3\cdot 2+1 \equiv 0 \pmod 7$.

<u>*Case 3: $n \equiv 0 \pmod{24}$.*</u> Since $2^{24} = (2^8)^3 \equiv 15^3 \equiv 225\cdot 15 \equiv -16\cdot 15 \equiv 1 \pmod{241}$. So $2^n = 2^{24m} \equiv 1 \pmod{241}$ and Then $k2^n+1 \equiv -1+1 = 0 \pmod{241}$.

Comment: We may wonder why modulo 3, 5, 7, 13, 17, 241 work. It may be that in dealing with $n \equiv 8, 16, 0 \pmod{24}$, we want $2^{24} \equiv 1 \pmod p$ for some useful primes p. Then we notice

$$2^{24}-1 = (2^3-1)(2^3+1)(2^6+1)(2^{12}+1) = 7\cdot 3^2\cdot 5\cdot 13\cdot 17\cdot 241.$$

Other commended solvers: **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Prishtina Math Gymnasium Problem Solving Group** (Republic of Kosova), **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 480. Let m, n be integers with $n > m > 0$. Prove that if $0 < x < \pi/2$, then

$$2|\sin^n x - \cos^n x| \le 3|\sin^m x - \cos^m x|.$$

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5).

If $x=\pi/4$, both sides are 0. Since the inequality for x and $\pi/2-x$ are the same, we only need to consider $0 < x < \pi/4$. Let $k \ge 0$. Define $a_k = \cos^k x - \sin^k x$. We have $a_k \ge 0$. For $k \ge 2$, we have

$$\begin{aligned} a_k &= (\cos^k x - \sin^k x)(\cos^2 x + \sin^2 x) \\ &= a_{k+2} + \sin^2 x \cos^2 x\, a_{k-2} \\ &\ge a_{k+2}. \end{aligned}$$

Let $m \ge 2$. For the case $n-m = 2,4,6,\ldots$, we have $3a_m \ge 2a_m \ge 2a_n$. Next, for the case $n-m = 1,3,5,\ldots$, observe that

$$(\cos x + \sin x)a_m = a_{m+1} + \sin x \cos x\, a_{m-1}.$$

Using this, we have

$$\begin{aligned} & 3a_m \ge 2a_{m+1} \\ \Leftrightarrow\ & 3a_m \ge 2[(\cos x + \sin x)a_m - \sin x \cos x\, a_{m-1}] \\ \Leftrightarrow\ & [3 - 2\sqrt{2}\sin(x + \tfrac{\pi}{4})]a_m \ge -2\sin x \cos x\, a_{m-1}, \end{aligned}$$

which is true as the left side is positive and the right side is negative. Then $3a_m \ge 2a_{m+1} \ge 2a_n$.

Finally, for the case $m=1$, we get $3a_1 \ge 2a_2$ from $3 > 2\sqrt{2} \ge 2(\cos x + \sin x) = 2a_2/a_1$. Then $3a_1 \ge 2a_2 \ge 2a_n$ for $n = 2, 4, 6, \ldots$. Also, we get $3a_1 \ge 2a_3$ from $3 \ge 2 + \sin 2x = 2a_3/a_1$. Then $3a_1 \ge 2a_3 \ge 2a_n$ for $n = 3, 5, 7, \ldots$.

Other commended solvers: **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Olympiad Corner

(Continued from page 1)

Problem 3. Let x, y and z be nonnegative real numbers. Knowing that $2(xy+yz+zx) = x^2+y^2+z^2$, prove

$$\frac{x+y+z}{3} \ge \sqrt[3]{2xyz}.$$

Problem 4. Find all of the solutions of the following equation in natural numbers:

$$n^{n^n} = m^m.$$

Problem 5. A non-empty set S of positive real numbers is called **powerful** if for any two distinct elements of it like a and b, at least one of the numbers a^b or b^a is an element of S.

a) Present an example of a powerful set having four elements.

b) Prove that a finite powerful set cannot have more than four elements.

Problem 6. In the **Majestic Mystery Club (MMC)**, members are divided into several groups, and groupings change by the end of each week in the following manner: in each group, a member is selected as king; all of the kings leave their respective groups and form a new group. If a group has only one member, that member goes to the new group and his former group is deleted. Suppose that MMC has n members and at the beginning all of them form a single group. Prove that there comes a week for which thereafter each group will have at most $1+\sqrt{2n}$ members.

Coloring Problems

(Continued from page 2)

<u>*Example 8.*</u> Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are divided into two groups, each having at least one number. Prove that there always exists a three term arithmetic progression (AP in short) in one of the two groups.

<u>*Solution.*</u> Assume no three term AP is in any of the two groups. Color numbers in one group red and the other group blue. Since 5/2>2, among 1, 3, 5, 7, 9, there exist three of them assigned the same color, say they are red. By assumption, they are not the terms of an AP. Below are the possibilities of these red numbers: {1,3,7}, {1,3,9}, {1,5,7}, {1,7,9}, {3,5,9} or {3,7,9}.

If 1,3,7 are red, then as 1,2,3 and 1,4,7 and 3,5,7 are AP, so 2, 4, 5 are blue. As 4,5,6 and 2,5,8 are AP, so 6, 8 are red. So 6,7,8 is a red AP, contradiction.

If 1,3,9 are red, then as 1,2,3 and 1,5,9 and 3,6,9 are AP, so 2, 5, 6 are blue. As 4,5,6 and 5,6,7 an AP, so 4,7 are red. Then 1,4,7 is a red AP, contradiction.

If 1,5,7 are red, then as 1,3,5 and 5,6,7 and 1,5,9 are AP, so 3,6,9 are blue. Then 3, 6, 9 is a blue AP, contradiction.

If 1,7,9 are red, then as 1,4,7 and 1,5,9 and 7,8,9 are AP, so 4, 5, 8 are blue. As 3,4,5 and 4,5,6 an AP, so 3,6 are red. Then 3,6,9 is a red AP, contradiction.

If 3,5,9 are red, then as 1,5,9 and 3,4,5 and 5,7,9 are AP, so 1,4,7 are blue. Then 1,4,7 a blue AP, contradiction.

If 3,7,9 are red, then as 3,5,7 and 3,6,9 and 7,8,9 are AP, so 5, 6, 8 are blue. As 2,5,8 and 4,5,6 are AP, so 2, 4 are red. So 2,3,4 is a red AP, contradiction.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the *28th Asian Pacific Math Olympiad*, which was held in March 2016.

Problem 1. We say that a triangle ABC is *great* if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A=90^\circ$ and $AB=AC$.

Problem 2. A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1}+2^{a_2}+\cdots+2^{a_{100}},$$

where $a_1, a_2, \ldots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

Problem 3. Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 14, 2016***.

For individual subscription for the next five issues for the 15-16 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Inequalities of Sequences

Kin Y. Li

There are many math competition problems on inequalities. While most symmetric inequalities can be solved by powerful facts like the Muirhead and Schur inequalities, there are not many tools for general inequalities involving sequences. Below we will first take a look at some relatively easy examples on inequalities of sequences.

Example 1. (1997 *Chinese Math Winter Camp*) Let $a_1, a_2, a_3, \ldots$ be a sequence of nonnegative numbers. If for all positive integers m and n, $a_{n+m} \le a_n+a_m$, then prove that

$$a_n \le ma_1+\left(\frac{n}{m}-1\right)a_m.$$

Solution. Let $n = mq + r$, where q, r are integers and $0 \le r < m$. We have

$$\begin{aligned} a_n &\le a_{mq}+a_r \le qa_m+a_r \\ &=\frac{n-r}{m}a_m+a_r \\ &=\left(\frac{n}{m}-1\right)a_m+\frac{m-r}{m}a_m+a_r \\ &\le\left(\frac{n}{m}-1\right)a_m+\frac{m-r}{m}ma_1+ra_1 \\ &=\left(\frac{n}{m}-1\right)a_m+ma_1. \end{aligned}$$

Example 2. (*IMO* 2014) Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \ge 1$ such that

$$a_n < \frac{a_0+a_1+\cdots+a_n}{n} \le a_{n+1}.$$

Solution. For $n = 1,2,3,\ldots$, define

$$d_n=(a_0+a_1+\cdots+a_n)-na_n.$$

We have

$$\begin{aligned} &na_{n+1}-(a_0+a_1+\cdots+a_n) \\ &=(n+1)a_{n+1}-(a_0+a_1+\cdots+a_{n+1}) \\ &=-d_{n+1}. \end{aligned}$$

In terms of d_i's, the required conclusion is the same as $d_n > 0 \ge d_{n+1}$ for some unique $n \ge 1$.

Now observe that $d_1 = (a_0+a_1) - a_1 > 0$. Also the d_i's are strictly decreasing as

$$\begin{aligned} &d_{n+1}-d_n \\ &=\sum_{i=1}^{n+1}a_i-(n+1)a_{n+1}-\sum_{i=1}^{n}a_i-na_n \\ &=n(a_n-a_{n+1})<0. \end{aligned}$$

Finally, from $d_1 > 0$, the d_i's are integers and strictly decreasing, there must be a first non-positive d_i. So $d_n > 0 \ge d_{n+1}$ for some unique $n \ge 1$.

Example 3. (1980 *Austrian-Polish Math Competition*) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers satisfying the inequality

$$|a_{k+m}-a_k-a_m| \le 1 \quad \text{for all } k, m.$$

Show that the following inequality holds for all positive integers k and m,

$$\left|\frac{a_k}{k}-\frac{a_m}{m}\right|<\frac{1}{k}+\frac{1}{m}.$$

Solution. Observe that multiplying by km, the desired inequality is the same as $|ma_k-ka_m| < m+k$. To get this, we will prove for a fixed m, $|a_{km}-ka_m| < k$ holds for all positive integer k by induction. The case $k = 1$ is $|a_m-a_m|=0<1$. Suppose the k-th case is true. Then

$$\begin{aligned} &|a_{(k+1)m}-(k+1)a_m| \\ &=|a_{km+m}-a_{km}-a_m+a_{km}-ka_m| \\ &\le|a_{km+m}-a_{km}-a_m|+|a_{km}-ka_m| \\ &\le 1+|a_{km}-ka_m|<1+k. \end{aligned}$$

This completes the inductive step. Now interchanging k and m, similarly we also have $|a_{km}-ma_k| < m$. Then

$$\begin{aligned} |ma_k-ka_m| &\le |a_{km}-ma_k|+|a_{km}-ka_m| \\ &\le m+k \end{aligned}$$

and we are done.

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Example 4. (2006 *IMO Shortlisted Problem*) The sequence of real numbers a_0, a_1, a_2, … is defined recursively by

$$a_0=-1,\ \sum_{k=0}^{n}\frac{a_{n-k}}{k+1}=0\quad for\ \ n\ge 1.$$

Show that $a_n>0$ for $n\ge 1$.

Solution. Setting n=1, we find a_1=1/2. For $n\ge 1$, reversing the order of the terms in the given sum, we have

$$\sum_{k=0}^{n}\frac{a_k}{n-k+1}=0\ \ and\ \ \sum_{k=0}^{n+1}\frac{a_k}{n-k+2}=0.$$

Suppose a_1 to a_n are positive. Then

$$0=(n+2)\sum_{k=0}^{n+1}\frac{a_k}{n-k+2}-(n+1)\sum_{k=0}^{n}\frac{a_k}{n-k+1}$$
$$=(n+2)a_{n+1}+\sum_{k=0}^{n}\left(\frac{n+2}{n-k+2}-\frac{n+1}{n-k+1}\right)a_k.$$

Notice the k=0 term in the last sum is 0. Solving for a_{n+1}, we get

$$a_{n+1}=\frac{1}{n+1}\sum_{k=1}^{n}\left(\frac{n+1}{n-k+1}-\frac{n+2}{n-k+2}\right)a_k$$
$$=\frac{1}{n+2}\sum_{k=1}^{n}\frac{k}{(n-k+1)(n-k+2)}a_k$$

is positive as a_1 to a_n are positive.

Next we will study certain examples that require more observation and possibly involve some calculations of limit of sequences.

Example 5. (1988 *IMO Shortlisted Problem*) Let a_1, a_2, a_3, … be a sequence of nonnegative real numbers such that

$$a_k-2a_{k+1}+a_{k+2}\ge 0\ \text{ and }\ \sum_{j=1}^{k}a_j\le 1$$

for all k = 1, 2, …. Prove that

$$0\le a_k-a_{k+1}<\frac{2}{k^2}$$

for all k = 1, 2, ….

Solution. We claim $0\le a_k-a_{k+1}$ for all k. (Otherwise assume for some k, we have $a_k-a_{k+1}<0$. From $a_k-2a_{k+1}+a_{k+2}\ge 0$, we get $a_{k+1}-a_{k+2}\le a_k-a_{k+1}<0$. It follows $a_k<a_{k+1}<a_{k+2}<\cdots$. Then

$$a_k+a_{k+1}+a_{k+2}+\cdots$$

diverges to infinity, which leads to a contradiction.)

Let $b_k=a_k-a_{k+1}$. Then for all positive integer k, we have $b_k\ge b_{k+1}\ge 0$. Now we have

$$b_k\sum_{i=1}^{k}i\le\sum_{i=1}^{k}ib_i=\sum_{i=1}^{k}ia_i-\sum_{i=1}^{k}ia_{i+1}$$
$$=\sum_{i=1}^{k}ia_i-\sum_{i=2}^{k+1}(i-1)a_i$$
$$=\sum_{i=1}^{k}a_i-ka_{k+1}\le\sum_{i=1}^{k}a_i\le 1.$$

Therefore,

$$b_k\le 1\Big/\sum_{i=1}^{k}i=\frac{2}{k(k+1)}<\frac{2}{k^2}.$$

Example 6. (1970 *IMO*) Let $1=a_0\le a_1\le a_2\le\cdots\le a_n\le\cdots$ be a sequence of real numbers. Consider the sequence defined by

$$b_n=\sum_{k=1}^{n}\left(1-\frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}}.$$

Prove that :

(a) For all positive integers n, $0\le b_n\le 2$.

(b) Given an arbitrary $0\le b<2$, there is a sequence a_0, a_1, …, a_n, … of the above type such that $b_n>b$ is true for infinitely many natural numbers n.

Solution. (a) For all k, we have

$$0\le\left(1-\frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}}$$
$$=\frac{(\sqrt{a_k}+\sqrt{a_{k-1}})(\sqrt{a_k}-\sqrt{a_{k-1}})}{a_k\sqrt{a_k}}$$
$$\le 2\frac{\sqrt{a_k}-\sqrt{a_{k-1}}}{\sqrt{a_ka_{k-1}}}$$
$$=2\left(\frac{1}{\sqrt{a_{k-1}}}-\frac{1}{\sqrt{a_k}}\right).$$

Then

$$0\le b_n\le 2\sum_{k=1}^{n}\left(\frac{1}{\sqrt{a_{k-1}}}-\frac{1}{\sqrt{a_k}}\right)$$
$$=2\left(1-\frac{1}{\sqrt{a_n}}\right)<2.$$

(b) Let $0<q<1$. Then $a_n=q^{-2n}$ for n = 0, 1, 2, … satisfy $1=a_0<a_1<a_2<\cdots$ and the sequence $b_n=q(1+q)(1-q^n)$ has

$$\lim_{n\to\infty}b_n=q(1+q).$$

For an arbitrary $0\le b<2$, take q satisfy

$$\frac{\sqrt{1+4b}-1}{2}<q<1.$$

Then $0<q<1$ and $q(1+q)>b$. So eventually the sequence b_n (on its way to $q(1+q)$) will be greater than b.

Example 7. (2006 *IMO Shortlisted Problem*) Prove the inequality

$$\sum_{1\le i<j\le n}\frac{a_ia_j}{a_i+a_j}\le\frac{n}{2(a_1+a_2+\cdots+a_n)}\sum_{1\le i<j\le n}a_ia_j$$

for positive real numbers a_1, a_2, …, a_n.

Solution. Let S be the sum of the n numbers. Let L and R be the left and the right expressions in the inequality. Observe that

$$\sum_{1\le i<j\le n}(a_i+a_j)=(n-1)\sum_{k=1}^{n}a_k=(n-1)S$$

and

$$L=\sum_{1\le i<j\le n}\frac{a_ia_j}{a_i+a_j}$$
$$=\sum_{1\le i<j\le n}\frac{1}{4}\left(a_i+a_j-\frac{(a_i-a_j)^2}{a_i+a_j}\right)$$
$$=\frac{n-1}{4}S-\frac{1}{4}\sum_{1\le i<j\le n}\frac{(a_i-a_j)^2}{a_i+a_j}.$$

Next we will write the expression R in two ways. On one hand, we have

$$R=\frac{n}{2S}\sum_{1\le i<j\le n}a_ia_j=\frac{n}{4S}\left(S^2-\sum_{i=1}^{n}a_i^2\right).$$

On the other hand,

$$R=\frac{n}{4S}\sum_{1\le i<j\le n}\left(a_i^2+a_j^2-(a_i-a_j)^2\right)$$
$$=\frac{n(n-1)}{4S}\sum_{i=1}^{n}a_i^2-\frac{n}{4S}\sum_{1\le i<j\le n}(a_i-a_j)^2.$$

Multiplying the first of these equations by $n-1$ and adding it to the second equation, then dividing the sum by n, we get

$$R=\frac{n-1}{4}S-\frac{1}{4}\sum_{1\le i<j\le n}\frac{(a_i-a_j)^2}{S}.$$

Comparing L and R and using $S\ge a_i+a_j$, we get $L\le R$.

Example 8. (1998 *IMO Longlisted Problem*) Let

$$a_n=[\sqrt{(1+n)^2+n^2}],\quad n=1,2,\ldots,$$

where $[x]$ denotes the integer part of x. Prove that

(a) there are infinitely many positive integers m such that $a_{m+1}-a_m>1$;

(b) there are infinitely many positive integers m such that $a_{m+1}-a_{m=1}=1$.

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Problem Corner

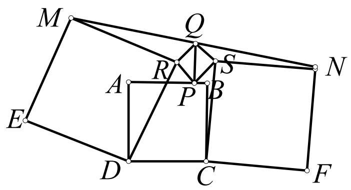
We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***August 14, 2016.***

Problem 486. Let a_0=1 and

$$a_n = \frac{\sqrt{1+a_{n-1}^2}-1}{a_{n-1}}.$$

for n=1,2,3,…. Prove that $2^{n+2}a_n > \pi$ for all positive integers n.

Problem 487. Let $ABCD$ and $PSQR$ be squares with point P on side AB and $AP>PB$. Let point Q be outside square $ABCD$ such that $AB\perp PQ$ and $AB=2PQ$. Let $DRME$ and $CSNF$ be squares as shown below. Prove Q is the midpoint of line segment MN.



Problem 488. Let $\mathbb{Q}$ denote the set of all rational numbers. Let $f: \mathbb{Q} \to \{0,1\}$ satisfy $f(0)=0$, $f(1)=1$ and the condition $f(x) = f(y)$ implies $f(x) = f((x+y)/2)$. Prove that if $x\geq 1$, then $f(x) = 1$.

Problem 489. Determine all prime numbers p such that there exist positive integers m and n satisfying $p=m^2+n^2$ and m^3+n^3-4 is divisible by p.

Problem 490. For a parallelogram $ABCD$, it is known that $\triangle ABD$ is acute and AD=1. Prove that the unit circles with centers A, B, C, D cover $ABCD$ if and only if

$$AB \leq \cos\angle BAD + \sqrt{3}\sin\angle BAD.$$

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Solutions

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Problem 481. Let $S=\{1,2,\ldots,2016\}$. Determine the least positive integer n such that whenever there are n numbers in S satisfying every pair is relatively prime, then at least one of the n numbers is prime.

Solution. **BOBOJONOVA Latofat** (academic lycuem S.H.Sirojiddinov, Tashkent, Uzbekistan), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5), **Toshihiro SHIMIZU** (Kawasaki, Japan),**WONG Yat.**

Let k_0=1 and k_i be the square of the i-th prime number. Then $k_{14}=43^2<2016$. Since the numbers $k_0,k_1,\ldots,k_{14}$ are in S and are pairwise coprime, so $n\geq 16$.

Next suppose $A=\{a_1,a_2,\ldots,a_{16}\}\subset S$ with no a_i prime and a_r, a_s are coprime for $r\neq s$.

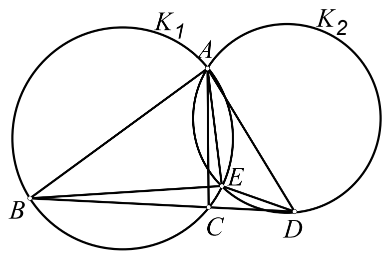
Then in case $1\notin A$, let p_i be the least prime divisor of a_i. We have $a_i\geq p_i^2$. As the a_i's are pairwise coprime, no two p_i's are the same. Now the 15th prime is 47. So the largest p_i is at least 47, which leads to some $a_i\geq p_i^2\geq 47^2>2016$, a contradiction.

Otherwise, $1\in A$. For the 15 numbers in A that is not 1, let a_i be their maximum, then $a_i\geq p_i^2\geq 47^2>2016$, again contradiction. So the least n is 16.

Other commended solvers: **Joe SPENCER.**

Problem 482. On $\triangle ABD$, C is a point on side BD with $C\neq B,D$. Let K_1 be the circumcircle of $\triangle ABC$. Line AD is tangent to K_1 at A. A circle K_2 passes through A and D and line BD is tangent to K_2 at D. Suppose K_1 and K_2 intersect at A and E with E inside $\triangle ACD$. Prove that $EB/EC=(AB/AC)^3$.

Solution. **Jafet Alejandro BACA OBANDO** (IDEAS High School, Nicaragua), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece), **Vijaya Prasad NALLURI** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Line AD tangent to K_1 at A implies $\angle DAC = \angle DBA$. With $\angle ADC=\angle BDA$, we see $\triangle DAC$ is similar to $\triangle DBA$. Now $BD/CD = [DBA]/[DAC] = AB^2/CA^2$. Then we have

$$\left(\frac{AB}{AC}\right)^3 = \frac{BD}{CD}\cdot\frac{AB}{AC} = \frac{BD/AC}{CD/AB}. \quad (*)$$

Next, $\angle DBE = \angle CBE = \angle CAE$ and $\angle BDE = \angle DAE = \angle ACE$ implies $\triangle DBE$ is similar to $\triangle CAE$. Similarly, $\angle ECD = \angle EAB$ and $\angle EDC = \angle EAD = \angle EBA$ implies $\triangle ECD$ is similar to $\triangle EAB$. Then

$$\frac{BD/CA}{CD/AB} = \frac{EB/AE}{EC/EA} = \frac{EB}{EC}. \quad (**)$$

Therefore, combining (*) and (**), we have $EB/EC=(AB/AC)^3$.

Other commended solvers: **BOBOJONOVA Latofat** (academic lycuem S. H. Sirojiddinov, Tashkent, Uzbekistan) and **WONG Yat**.

Problem 483. In the open interval (0,1), n distinct rational numbers a_i/b_i (i=1,2,…,n) are chosen, where n>1 and a_i, b_i are positive integers. Prove that the sum of the b_i's are at least $(n/2)^{3/2}$.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

Without loss of generality, we may suppose the numbers a_i/b_i are sorted so that the denominators are in ascending order. We have the following lemma.

Lemma. Let k be an integer in $[1,n]$ and b be the denominator of the k-th number. Then we have

$$b \geq \left(\frac{k}{2}\right)^{3/2} - \left(\frac{k-1}{2}\right)^{3/2}.$$

Proof. We first consider the number of denominators that are at most b. For every i = 1, 2, …, b, the number of denominators equal to i is at most $i-1$. Thus,

$$k \leq \sum_{i=1}^{b}(i-1) = \frac{b(b-1)}{2} \leq \frac{b^2}{2}.$$

This implies $b\geq\sqrt{2k}$. We will show

$$\sqrt{2k} \geq \left(\frac{k}{2}\right)^{3/2} - \left(\frac{k-1}{2}\right)^{3/2}.$$

It is equivalent to

$$4\sqrt{k} \geq k\sqrt{k} - (k-1)\sqrt{k-1}$$

or $(k-1)\sqrt{k-1} \geq (k-4)\sqrt{k}$.

For k=1,2,3,4, the left hand side is greater than the right hand side is non-positive. For $k \ge 5$, squaring the inequality, it is equivalent to $(k-1)^3 \ge (k-4)^2 k$ or $5k^2-13k+1 \ge 0$. The larger roots of the left hand side is $(13+\sqrt{149})/10$, which is less than 2.6. Then the left hand side is always positive for $k \ge 5$. QED

Using the lemma and summing the cases k=1, 2, …,n, we get the result.

Other commended solvers: **Jim GLIMMS, Joe SPENCER** and **WONG Yat**.

Problem 484. In a multiple choice test, there are four problems. For each problem, there are choices *A, B* and *C.* For any three students who took the test, there exist a problem the three students selected distinct choices. Determine the maximum number of students who took the test.

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

More generally, suppose there are n problems with $n \ge 4$. Let S_n be the maximum number of students who took the test with n problems. If $S_1>3$, then there would exist 2 students with the same choice and 1 problem cannot distinguish these 2 students. Now $S_1=3$ is certainly possible by given condition. In general if there is a problem which 3 students have different choices, then we say the problem *distinguish* them.

By pigeonhole principle, for problem 1, there is a choice among *A, B, C*, which at most $[S_n/3]$ selected. For the remaining at least $S_n-[S_n/3]$ students, problem 1 does not distinguish any 3 of them. So problem 2 to n will be used to distinguish these remaining students. Then $S_{n-1} \ge S_n-[S_n/3] \ge 2S_n/3$. Hence, $S_n \le 3S_{n-1}/2$. So $S_2 \le 4$, $S_3 \le 6$ and $S_4 \le 9$.

The following table will show S_4=9 :

Student\problem	I	II	III	IV
1	A	A	A	A
2	A	B	B	B
3	A	C	C	C
4	B	A	C	B
5	B	B	A	C
6	B	C	B	A
7	C	A	B	C
8	C	B	C	A
9	C	C	A	B

Other commended solvers: **Joe SPENCER**.

Problem 485. Let m and n be integers such that $m>n>1$, $S=\{1,2,\ldots,m\}$ and $T=\{a_1, a_2, \ldots, a_n\}$ is a subset of S. It is known that every two numbers in T do not both divide any number in S. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{n}.$$

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

For i=1,2,…,n, let

$$T_i=\{\, k\in S : k \text{ is divisible by } a_i \,\}.$$

Then T_i has $[m/a_i]$ elements. Since every pair of numbers in T do not both divide any number in S, so if $i \ne k$, then T_i and T_k are disjoint. Now the number of elements in the union of the sets $T_1, T_2, \ldots, T_n$ is

$$\left[\frac{m}{a_1}\right]+\left[\frac{m}{a_2}\right]+\cdots+\left[\frac{m}{a_n}\right]\le m.$$

Using $m/a_i < [m/a_i]+1$, we have

$$\sum_{i=1}^{n}\frac{m}{a_i}\le\sum_{i=1}^{n}\left[\frac{m}{a_i}\right]+\sum_{i=1}^{n}1\le m+n.$$

Then $m\sum_{i=1}^{n}\frac{1}{a_i}<m+n.$ Therefore,

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{m}<\frac{m+n}{n}.$$

Other commended solvers: **Joe SPENCER**.

Olympiad Corner

(*Continued from page 1*)

Problem 3. (Continued) Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

Problem 4. The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest integer k such that no matter how Starways establishes its flights, the city can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

Problem 5. Find all functions f : $\mathbb{R}^+\to\mathbb{R}^+$ such that

$$(z+1)f(x+y)=f(xf(z)+y)+f(yf(z)+x),$$

for all positive real numbers x, y, z.

Inequalities of Sequences

(*Continued from page 2*)

Solution. For every integer n, we have

$$\begin{aligned}\sqrt{2}n-3&<[\sqrt{2}(n-1)]\\&<a_n=[\sqrt{2(n^2-n+1/2)}]\quad(*)\\&\le[\sqrt{2}n]<\sqrt{2}n.\end{aligned}$$

From this, we get

$$n^2+(n+1)^2-(n-1)^2-n^2=4n>2a_n+1.$$

Hence,

$$\begin{aligned}a_{n+1}&=[\sqrt{n^2+(n+1)^2}]\ge[\sqrt{a_n^2+4n}]\\&\ge[\sqrt{a_n^2+2a_n+1}]=a_n+1\end{aligned}$$

for n=1,2,3,…. If (a) is false, then there exists N such that

$$a_{k+1}-a_k=1 \quad \text{for all } k\ge N. \qquad (**)$$

So $a_{N+k}=a_N+k$ for k=0,1,2,3,…. By (*), for k=0,1,2,3,…, we have

$$\sqrt{2}(N+k)-3<a_{N+k}=a_N+k,$$

i.e. $(\sqrt{2}-1)k<a_N+3-\sqrt{2}N$. Since N is constant, when k is large, this leads to a contradiction. So (a) must be true.

Next assume (b) is false. By (**), we can see there exists N such that

$$a_{k+1}-a_k\ge 2 \quad \text{for all } k\ge N.$$

Then $a_{N+k}\ge a_N+2k$ for k = 0,1,2,3,…. By (*), we have

$$a_N+2k<\sqrt{2}(N+k),$$

which is the same as

$$(2-\sqrt{2})k<\sqrt{2}N-a_N.$$

This leads to a contradiction when k is large. So (b) must be true.

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the *2016 IMO Team Selection Contest I* for Estonia.

Problem 1. There are k heaps on the table, each containing a different positive number of stones. Jüri and Mari make moves alternatively; Jüri starts. On each move, the player making the move has to pick a heap and remove one or more stones in it from the table; in addition, the player is allowed to distribute any number of the remaining stones from that heap in any way between other non-empty heaps. The player to remove the last stone from the table wins. For which positive integers k does Jüri have a winning strategy for any initial state that satisfies the conditions?

Problem 2. Let p be a prime number. Find all triples (a,b,c) of integers (not necessarily positive) such that

$$a^b b^c c^a = p.$$

Problem 3. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the equality $f(2^x+2y) = 2^y f(\,f(x)\,)\,f(y)$ for every $x,y\in\mathbb{R}$.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 21, 2016***.

For individual subscription for the next five issues for the 16-17 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2016

Kin Y. Li

This year Hong Kong served as the host of the International Mathematical Olympiad (IMO), which was held from July 6 to 16. Numerous records were set. Leaders, deputy leaders and contestants from 109 countries or regions participated in this annual event. A total of 602 contestants took part in this world class competition. Among the contestants, 71 were female and 531 were male.

After the two days of competition on July 11 and 12, near 700 contestants and guides from more than 100 countries or regions went to visit Mickey Mouse at the Hong Kong Disneyland for an excursion. That was perhaps the happiest moment in the IMO.

For Hong Kong, due to the hard work of the 6 team members and the strong coaching by Dr. Leung Tat Wing, Dr. Law Ka Ho and our deputy leader Cesar Jose Alaban along with the support of the many trainers and former team members, the team received 3 gold, 2 silver and 1 bronze medals, which was the best performance ever. Also, for the first time since Hong Kong participated in the IMO, we received a top 10 team ranking.

The Hong Kong IMO team members (in alphabetical order) are as follows:

(HKG1) Cheung Wai Lam, Queen Elizabeth School, Silver Medalist,

(HKG2) Kwok Man Yi, Baptist Lui Ming Choi Secondary School, Bronze Medalist,

(HKG3) Lee Shun Ming Samuel, CNEC Christian College, Gold Medalist,

(HKG4) Leung Yui Hin Arvin, Diocesan Boys' School, Silver Medalist,

(HKG5) Wu John Michael, Hong Kong International School, Gold Medalist and

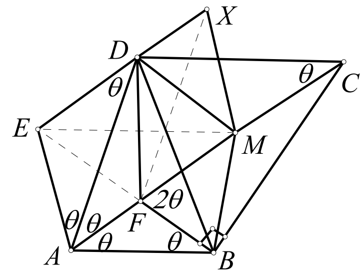
(HKG 6) Yu Hoi Wai, La Salle College, Gold Medalist.

The top 10 teams in IMO 2016 are (1) USA, (2) South Korea, (3) China, (4) Singapore, (5) Taiwan, (6) North Korea, (7) Russia and UK, (9) Hong Kong and (10) Japan.

The cutoffs for gold, silver and bronze medals were 29, 22 and 16 marks respectively. There were 44 gold, 101 silver, 135 bronze and 162 honourable mentions awardees.

Next, we will look at the problems in IMO 2016.

Problem 1. Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA=FB$ and F lies between A and C. Point D is chosen such that $DA=DC$ and AC is the bisector of $\angle DAB$. Point E is chosen such that $EA=ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM||EX$ and $AE||MX$). Prove that lines BD, FX, and ME are concurrent.



From the statement of the problem, we get a whole bunch of equal angles as labeled in the figure. We have $\triangle ABF\sim\triangle ACD$. Then $AB/AC=AF/AD$. With, $\angle BAC=\theta=\angle FAD$, we get $\triangle ABC\sim\triangle AFD$.

(continued on page 2)

Then $\angle AFD = \angle ABC = 90^\circ + \theta = 180^\circ - \frac{1}{2}\angle AED$. Hence, F is on the circle with center E and radius EA. Then $EF=EA=ED$ and $\angle EFA = \angle EAF = 2\theta = \angle BFC$. So B, F, E are collinear. Also, $\angle EDA = \angle MAD$ implies $ED||AM$. Hence E,D,X are collinear. From M is midpoint of CF and $\angle CBF=90^\circ$, we get $MF=MB$. Next the isosceles triangles EFA and MFB are congruent due to $\angle EFA=\angle MFB$ and $AF=BF$. Then $BM=AE=XM$ and $BE = BF+FE=AF+FM=AM=EX$. So $\triangle EMB \cong \triangle EMX$. As F and D lie on EB and EX respectively and $EF=ED$, we see lines BD and XF are symmetric respect to EM. Therefore, BD, XF, EM are concurrent.

Problem 2. Find all positive integer n for which each cell of an $n\times n$ table can be filled with one of the letters I, M and O in such a way that:

- in each row and each column, one third of the entries are I, one third are M and one third are O; and

- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Note: The rows and columns of an $n\times n$ table are each labeled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integers (i, j) with $1\le i, j \le n$. For $n > 1$, the table has $4n-2$ *diagonals* of two types. A diagonal of the first type consists of all cells (i, j) for which $i+j$ is a constant, and a diagonal of the second type consists of all cells (i,j) for which $i-j$ is a constant.

For n=9, it is not difficult to get an example such as

$$\begin{pmatrix} I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \end{pmatrix}$$

For $n=9m$, we can divide the $n\times n$ table into $m\times m$ blocks, where in each block we use the 9×9 table above.

Next suppose a $n\times n$ table satisfies the conditions. Then n is a multiple of 3, say $n=3k$. Divide the $n\times n$ into $k\times k$ blocks of 3×3 tables. Call the center entry of the 3×3 tables a *vital entry* and call any row, column or diagonal passing through a vital entry a *vital line.* The trick here is to do double counting on the number N of all ordered pairs (L,c), where L is a vital line and c is an entry on L that contains the letter M. On one hand, there are k occurrences of M in each vital row and each vital column. For vital diagonals, there are

$$1+2+\cdots+(k-1)+k+(k-1)+\cdots+2+1=k^2$$

occurrences of M. So $N=4k^2$. On the other hand, there are $3k^2$ occurrences of M in the whole table. Note each entry belongs to exactly 1 or 4 vital lines. Hence $N \equiv 3k^2 \pmod 3$, making k a multiple of 3 and n a multiple of 9.

Problem 3. Let $P=A_1A_2\ldots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

This is the hardest problem. 548 out of 602 contestants got 0 on this problem.

That $2S$ is an integer follows from the well-known *Pick's formula*, which asserts $S=I+B/2-1$, where I and B are the numbers of interior and boundary points with integral coordinates respectively.

Below we will outline the cleverest solution due to Dan Carmon, the leader of Israel. It suffices to consider the case $n=p^t$ with p prime, $t\ge 1$. By multiplying the denominator and translating, we may assume the center O is a point with integral coordinates, which we can move to the origin. We can further assume the x, y coordinates of the vertices are coprime and there exists i with x_i, y_i not both multiples of p. Then we make two claims:

(1) For $\triangle ABC$ with integral coordinates, suppose $n \mid AB^2, BC^2$ and let S be its area. Then $n \mid 2S$ if and only if $n \mid AC^2$.

(2) For those i such that x_i, y_i not both multiples of p, let Δ be twice the area of triangle $A_{i-1}A_iA_{i+1}$. Then p^t divides Δ.

For (1), note that $2S = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$,

$$AC^2 = AB^2 + BC^2 - 2\overrightarrow{BA}\cdot\overrightarrow{BC}$$
$$\equiv -2\overrightarrow{BA}\cdot\overrightarrow{BC} \pmod n$$

and $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|^2 + \left|\overrightarrow{BA}\cdot\overrightarrow{BC}\right|^2 = AB^2BC^2 \equiv 0 \pmod{n^2}$.

For (2), assume p^t does not divide Δ. Note O is defined by the intersection of the perpendicular bisectors, which can be written as the following system of vectors:

$$\overrightarrow{A_iA_{i+1}}\cdot\overrightarrow{A_iO} = \tfrac{1}{2}A_iA_{i+1}^2,\quad \overrightarrow{A_iA_{i-1}}\cdot\overrightarrow{A_iO} = \tfrac{1}{2}A_iA_{i-1}^2.$$

Say $\overrightarrow{A_iA_{i+1}} = (u_1, v_1)$, $\overrightarrow{A_iA_{i-1}} = (u_2, v_2)$. Using the fact that p^t does not divide $\Delta = |u_1v_2-u_2v_1|$, one can conclude that x_i, y_i are divisible by p by Cramer's rule. The rest of the solution follows by induction on the number of sides of the polygon and the two claims.

Problem 4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n)=n^2+n+1$. What is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

One can begin by looking at facts like

1. $\gcd(P(n),P(n+1))=1$ for all n
2. $\gcd(P(n),P(n+2))=1$ for $n\not\equiv 2 \pmod 7$
3. $\gcd(P(n),P(n+2))=7$ for $n\equiv 2 \pmod 7$
4. $\gcd(P(n),P(n+3))=1$ for $n\not\equiv 1 \pmod 3$
5. $3|\gcd(P(n),P(n+3))$ for $n\equiv 1 \pmod 3$.

Assume $P(a)$, $P(a+1)$, $P(a+2)$, $P(a+3)$, $P(a+4)$ is fragrant. By 1, $P(a+2)$ is coprime to $P(a+1)$ and $P(a+3)$. Next assume $\gcd(P(a),P(a+2)) > 1$. By 3, $a\equiv 2 \pmod 7$. By 2, $\gcd(P(a+1),P(a+3))=1$. In order for the set to be fragrant, we must have both $\gcd(P(a),P(a+3))$ and $\gcd(P(a+1),P(a+4))$ be greater than 1. By 5, this holds only when a and $a+1 \equiv 1 \pmod 3$, which is a contradiction.

For a fragrant set with 6 numbers, we can use the Chinese remainder theorem to solve the system $a \equiv 7 \pmod{19}$, $a+1\equiv 2 \pmod 7$ and $a+2\equiv 1 \pmod 3$. For example, a=197. By 3, $P(a+1)$ and $P(a+3)$ are divisible by 7. By 5, $P(a+2)$ and $P(a+5)$ are divisible by 3. Using $19|P(7)=57$ and $19|P(11)=133$, we can check $19|P(a)$ and $19|P(a+4)$. Then $P(a)$, $P(a+1)$, $P(a+2)$, $P(a+3)$, $P(a+4)$, $P(a+5)$ is fragrant.

Problem 5. The equation

$$(x-1)(x-2)\cdots(x-2016)$$
$$= (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***October 21, 2016.***

Problem 491. Is there a prime number p such that both p^3+2008 and p^3+2010 are prime numbers? Provide a proof.

Problem 492. In convex quadrilateral $ADBE$, there is a point C within $\triangle ABE$ such that

$\angle EAD+\angle CAB=180°=\angle EBD+\angle CBA.$

Prove that $\angle ADE=\angle BDC$.

Problem 493. For $n\geq 4$, prove that $x^n-x^{n-1}-x^{n-2}-\cdots-x-1$ cannot be factored into a product of two polynomials with rational coefficients, both with degree greater than 1.

Problem 494. In a regular n-sided polygon, either 0 or 1 is written at each vertex. By using non-intersecting diagonals, Bob divides this polygon into triangles. Then he writes the sum of the numbers at the vertices of each of these triangles inside the triangle. Prove that Bob can choose the diagonals in such a way that the maximal and minimal numbers written in the triangles differ by at most 1.

Problem 495. The lengths of each side and diagonal of a convex polygon are rational. After all the diagonals are drawn, the interior of the polygon is partitioned into many smaller convex polygonal regions. Prove that the sides of each of these smaller convex polygons are rational numbers.

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Solutions

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Problem 486. Let $a_0=1$ and

$$a_n=\frac{\sqrt{1+a_{n-1}^2}-1}{a_{n-1}}.$$

for $n=1,2,3,\ldots$. Prove that $2^{n+2}a_n>\pi$ for all positive integers n.

Solution. **Charles BURNETTE** (Graduate Student, Drexel University, Philadelphia, PA, USA), **Prithwijit DE** (HBCSE, Mumbai, India), **FONG Ho Leung** (Hoi Ping Chamber Secondary School), **Mustafa KHALIL** (Instituto Superior Tecnico, Syria), **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania), **Toshihiro SHIMIZU** (Kawasaki, Japan), **WONG Yat** and **YE Jeff York**, **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

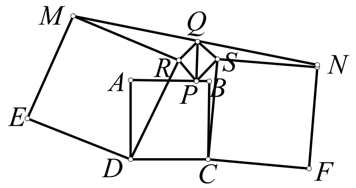
Let $a_n=\tan\theta_n$, where $0\leq\theta_n<\pi/2$. Then $a_0=1$ implies $\theta_0=\pi/4$. By the recurrence relation of a_n, we get

$$\tan\theta_n=\frac{\sec\theta_{n-1}-1}{\tan\theta_{n-1}}=\frac{1-\cos\theta_{n-1}}{\sin\theta_{n-1}}$$
$$=\frac{2\sin^2(\theta_{n-1}/2)}{2\cos(\theta_{n-1}/2)\sin(\theta_{n-1}/2)}=\tan\frac{\theta_{n-1}}{2}.$$

Then $a_n=\tan\theta_n=\tan\frac{\theta_0}{2^n}=\tan\frac{\pi}{2^{n+2}}>\frac{\pi}{2^{n+2}}$,

which is the desired inequality.

Problem 487. Let $ABCD$ and $PSQR$ be squares with point P on side AB and $AP>PB$. Let point Q be outside square $ABCD$ such that $AB\perp PQ$ and $AB=2PQ$. Let $DRME$ and $CSNF$ be squares as shown below. Prove Q is the midpoint of line segment MN.



Solution. **FONG Ho Leung** (Hoi Ping Chamber Secondary School), **Tran My LE** (Sai Gon University, Ho Chi Minh City, Vietnam) and **Duy Quan TRAN** (University of Medicine and Pharmacy, Ho Chi Minh City, Vietnam), **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Mihai STOENESCU** (Bischwiller, France), **WONG Yat** and **YE Jeff York**.

Let Q be the origin, P be $(0,-2)$ and $B=(x,-2)$. Since $AB\perp PQ$ and $PSQR$ is a square, so $S=(1,-1)$. Using $AB=2PQ=4$, we get $C=(x,-6)$. Since $CS=NS$ and $\angle CSN=90°$, we get $N=(6,2-x)$.

Similarly, $R=(-1,-1)$, $D=(x-4,-6)$ and $\angle DRM=90°$, so $M=(-6,x-2)$. Then the midpoint of MN is $(0,0)=Q$.

Other commended solvers: **Andrea FANCHINI** (Cantù, Italy), **Apostolos MANOLOUDIS** (4 High School of Korydallos, Piraeus, Greece) and **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, India).

Problem 488. Let $\mathbb{Q}$ denote the set of all rational numbers. Let $f:\mathbb{Q}\to\{0,1\}$ satisfy $f(0)=0, f(1)=1$ and the condition $f(x)=f(y)$ implies $f(x)=f((x+y)/2)$. Prove that if $x\geq 1$, then $f(x)=1$.

Solution. **Jon GLIMMS.**

We first show $f(n)=1$ for $n=1,2,3,\ldots$ by induction. The case $n=1$ is given. For $n>1$, suppose case $n=k-1$ is true. If $f(k)=0=f(0)$, then $f(k)=f((0+k)/2)=f((1+(k-1))/2)=f(k-1)=1$, which is a contradiction.

Assume there exists rational $r>1$ such that $f(r)=0$. Suppose $r=s/t$, where s, t are coprime positive integers. Define $g:\mathbb{Q}\to\{0,1\}$ by $g(x)=1-f(w(x))$, where $w(x)=(r-[r])x+[r]$. Observe that the graph of w is a line. So $w((x+y)/2)=(w(x)+w(y))/2$.

If $g(x)=g(y)$, then $f(w(x))=f(w(y))$, which implies

$$f(w(x))=f\left(\frac{w(x)+w(y)}{2}\right)=f\left(w\left(\frac{x+y}{2}\right)\right).$$

So $g(x)=g((x+y)/2)$. Then $g(n)=1$ by induction as f above. Finally, $s>t$ implies $w(t)=(r-[r])t+[r]=s-[r]t+[r]$ is a positive integer. Then $g(t)=1-f(w(t))=0$, contradiction.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan), **WONG Yat** and **YE Jeff York**,

Problem 489. Determine all prime numbers p such that there exist positive integers m and n satisfying $p=m^2+n^2$ and m^3+n^3-4 is divisible by p.

Solution. **Prithwijit DE** (HBCSE, Mumbai, India), **Jon GLIMMS, WONG Yat** and **YE Jeff York**.

Clearly, the case $p=2$ works. For such prime $p>2$, we get $m>1$ or $n>1$. Now we have

$$\begin{aligned}&(3m+3n)p-2(m^3+n^3-4)\\&=(m+n)^3+8\\&=(m+n+2)((m+n)^2-2(m+n)+4)\\&=(m+n+2)(p+2((m-1)(n-1)+1)).\end{aligned}$$

Observe that $p<p+2((m-1)(n-1)+1)<p+2mn\leq p+m^2+n^2=2p$. Then p divides $m+n+2$. So $m^2+n^2\leq m+n+2$, i.e. $(m-1/2)^2+(n-1/2)^2\leq(3/2)^2$. Then

$(m,n)=(1,2)$ or $(2,1)$ and $m^3+n^3-4 = 5 = p$. So $p = 2$ and 5 are the solutions.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 490. For a parallelogram $ABCD$, it is known that $\triangle ABD$ is acute and $AD=1$. Prove that the unit circles with centers A, B, C, D cover $ABCD$ if and only if

$$AB \le \cos\angle BAD + \sqrt{3}\sin\angle BAD.$$

Solution. **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

We first show that the unit circles with centers A, B, C, D cover $ABCD$ if and only if the circumradius R of $\triangle ABD$ is not greater than 1. Since $\triangle ABD$ is acute, its circumcenter O is inside the triangle. Then at least one of B or D is closer than (or equal to) C to O, since the region in $\triangle CDB$ that is closer to C than both B and D is the quadrilateral $CMO'N$, where M is the midpoint of CD, O' is the circumcenter of $\triangle CDB$ and N is the midpoint of BC. So for any point P in $\triangle ABD$, $\min\{PA,PB,PD\} \le PC$ and the maximal value of $\min\{PA,PB,PD\}$ is attained when $P=O$. So the unit circles with centers A, B, C, D cover $ABCD$ is equivalent to they cover O, which is equivalent to $R\le 1$.

Let $\alpha = \angle BAD$, $\beta = \angle ADB$ and $\gamma = \angle DBA$. By sine law, $AB/\sin\beta=1/\sin\gamma = 2R$. Then, we have

$$\begin{aligned} AB &= \frac{\sin\beta}{\sin\gamma} = \frac{\sin(\alpha+\gamma)}{\sin\gamma} \\ &= \frac{\sin\alpha\cos\gamma+\cos\alpha\sin\gamma}{\sin\gamma} \\ &= \cos\alpha + \cot\gamma\sin\alpha. \end{aligned}$$

Moreover, $R\le 1$ is equivalent to $1 \ge 1/(2\sin\gamma)$ or $\sin\gamma\ge 1/2=\sin 30^\circ$ or $\gamma\ge 30^\circ$ or $\cot\gamma \le \sqrt{3}$. Therefore, it is equivalent to $AB \le \cos\alpha + \sqrt{3}\sin\alpha$.

Other commended solvers: **WONG Yat** and **YE Jeff York**.

Olympiad Corner

(*Continued from page 1*)

Problem 4. Prove that for any positive integer n, $2\cdot\sqrt{3}\cdot\sqrt[3]{4}\cdots\sqrt[n-1]{n} > n$.

Problem 5. Let O be the circumcenter of the acute triangle ABC. Let c_1 and c_2 be the circumcircles of triangles ABO and ACO. Let P and Q be points on c_1 and c_2 respectively, such that OP is a diameter of c_1 and OQ is a diameter of c_2. Let T be the intersection of the tangent to c_1 at P and the tangent to c_2 at Q. Let D be the second intersection of the line AC and the circle c_1. Prove that points D, O and T are collinear.

Problem 6. A circle is divided into arcs of equal size by n points ($n\ge 1$). For any positive integer x, let $P_n(x)$ denote the number of possibilities for coloring all those points, using colors from x given colors, so that any rotation of the coloring by $i\cdot 360^\circ/n$, where i is a positive integer less than n, gives a coloring that differs from the original in at least one point. Prove that the function $P_n(x)$ is a polynomial with respect to x.

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(*Continued from page 2*)

For this problem, observe we need to erase at least 2016 factors. Consider erasing all factors $x-k$ with $k\equiv 2,3 \pmod 4$ on the left and $x-k$ with $k\equiv 0,1 \pmod 4$ on the right to get the equation

$$\prod_{j=0}^{503}(x-4j-1)(x-4j-4)=\prod_{j=0}^{503}(x-4j-2)(x-4j-3)$$

There are 4 cases we have to check.

(1) For $x=1,2,\cdots,2016$, one side is 0 and the other nonzero.

(2) For $x\in(4k+1,4k+2)\cup(4k+3,4k+4)$ where $k=0,1,\ldots,503$, if $j=0,1,\ldots,503$ and $j\ne k$, then $(x-4j-1)(x-4j-4)>0$, but if $j=k$, then $(x-4k-1)(x-4k-4) < 0$ so that the left side is negative. However, on the right side, each product $(x-4j-2)(x-4j-3)$ is positive, which is a contradiction.

(3) For $x<1$ or $x>2016$ or $x\in(4k,4k+1)$, where $k=0,1,\ldots,503$, dividing the left side by the right, we get

$$1=\prod_{j=0}^{503}\left(1-\frac{2}{(x-4j-2)(x-4j-3)}\right).$$

Note $(x-4j-2)(x-4j-3)>2$ for $j=0,1,\ldots,503$. Then the right side is less than 1, contradiction.

(4) For $x\in(4k+2,4k+3)$, where $k = 0, 1, \ldots, 503$, dividing the left side by the right, we get

$$1=\frac{x-1}{x-2}\frac{x-2016}{x-2015}\prod_{j=1}^{503}\left(1+\frac{2}{(x-4j+1)(x-4j-2)}\right).$$

The first two factors on the right are greater than 1 and the factor in the parenthesis is greater than 1, which is a contradiction.

Problem 6. There are $n>2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hand $n-1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if n is odd.

(b) Prove that Geoff can never fulfill his wish if n is even.

Unlike previous years, this problem 6 was not as hard as problem 3. There were 474 out of 602 contestants, who got 0 on this problem.

Take a disk containing all segments. Extend each segment to cut the boundary of the disk at points A_i, B_i.

(a) For odd n, go along the boundary and mark all these points 'in' and 'out' alternately. For each A_iB_i rename the 'in' point as A_i and 'out' point as B_i. Geoff can put a frog on each of the 'in' points. Let $A_iB_i\cap A_kB_k=P$. There are $n-1$ points on the open segment A_iB_i for every i. On the *open* arc A_iA_k, there is an <u>odd</u> number of points due to the alternate naming of the boundary points. Each of the points on *open* arc A_iA_k is a vertex of some A_xB_x, which intersects a unique point on either open segment A_iP or A_kP. So the number of points on open segments A_iP and A_kP are of opposite parity. Then the frogs started at A_i and A_k cannot meet at P.

(b) For even n, let Geoff put a frog on a vertex of a A_iB_i segment, say the frog is at A_i, which is the 'in' point and B_i is the 'out' point. As n is even, there will be two neighboring points labeled A_i and A_k. Let $A_iB_i\cap A_kB_k=P$. Then any other segment A_mB_m intersecting one of the open segments A_iP or A_kP must intersect the other as well. So the number of intersection points by the other segments on open segments A_iP and A_kP are the same. Then the frogs started at A_i and A_k will meet at P.