Mathematical Excalibur

Volume 12, Number 1 *March 2007 – April 2007*

Olympiad Corner

Below are the 2007 Asia Pacific Math Olympiad problems.

Problem 1. Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Problem 2. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H be the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

Problem 3. Consider n disks $C_1, C_2, \cdots, C_n$ in a plane such that for each $1 \le i < n$, the center C_i is on the circumference of C_{i+1}, and the center of C_n is on the circumference of C_1. Define the score of such an arrangement of n disks to be the number of pairs (i, j) for which C_i properly contains C_j. Determine the maximum possible score.

Problem 4. Let x, y and z be positive real numbers such that $\sqrt{x}+\sqrt{y}+\sqrt{z}$ = 1. Prove that

$$\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}}\ge 1.$$

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 31, 2007***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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From *How to Solve It* to Problem Solving in Geometry

K. K. Kwok
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Geometry is the science of
correct reasoning on
incorrect figures.

If you can't solve a problem,
then there is an easier
problem you can solve, find
it.
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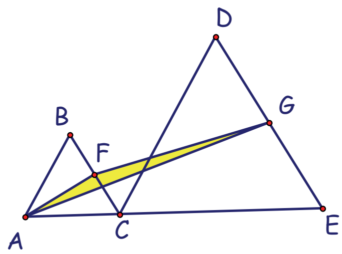
George Pölya

幾何是：在靜止中看出動態，從變幻中覓得永恆

數學愛好者，強

Example 1. In the figure below, C is a point on AE. ΔABC and ΔCDE are equilateral triangles. F and G are the midpoints of BC and DE respectively. If the area of ΔABC is 24 cm^2, the area of ΔCDE is 60 cm^2, find the area of ΔAFG.



Idea and solution outline:
This question is easy enough and can be solved by many different approaches. One of them is to recognize that the extensions of AF and CG are parallel. (Why? At what angles do they intersect line AE?) Thus $[AFC] = [AFG]$.

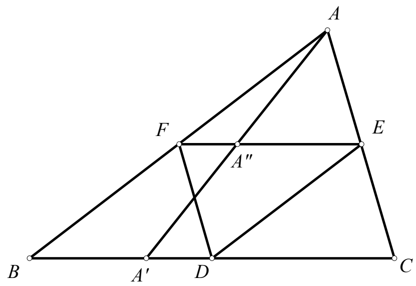
Example 2. In ΔABC, $AB = AC$. A point P on the plane satisfies $\angle ABP = \angle ACP$. Show that P is either on BC or on the perpendicular bisector of BC.

Solution:
Apply the sine law to ΔABP and ΔACP, we have

$$\sin\angle APB = \frac{AB\sin\angle ABP}{AP} = \frac{AC\sin\angle ACP}{AP} = \sin\angle APC.$$

Thus, either $\angle APB = \angle APC$ or $\angle APB + \angle APC = 180^\circ$. The first case implies $\Delta ABP \cong \Delta ACP$, so $BP = CP$ and P lies on the perpendicular bisector of BC. The second case implies P lies on BC.

Example 3.[Tournament of Towns1993] Vertices A, B and C of a triangle are connected to points A', B' and C' lying in their respective opposite sides of the triangle (not at vertices). Can the midpoints of the segments AA', BB' and CC' lie in a straight line?

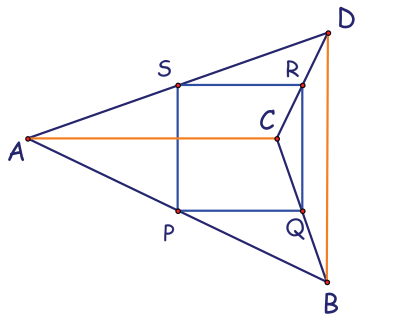


Solution outline:
Let D, E and F be midpoints of BC, AC, and AB respectively. Given any point A' on BC, let AA' intersect EF at A''. Then it is easy to see that A'' is indeed the midpoint of AA'.

Therefore, the midpoints of the segments AA', BB' and CC' lie respectively on EF, DF and DE, and cannot be collinear.

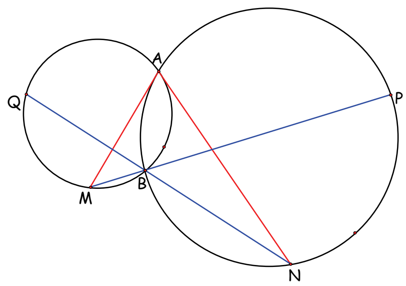
Example 4. [Tournament of Towns 1993] Three angles of a non-convex, non-self-intersecting quadrilateral are equal to 45 degrees (i.e. the last equals 225 degrees). Prove that the midpoints of its sides are vertices of a square.

Idea:
Do you know a similar, but easier problem? For example, the famous *Varignon Theorem*: By joining the midpoints of the sides of an arbitrary quadrilateral, a parallelogram is formed.



Solution outline:
Extend BC to cut AD at O. Then ΔOAB and ΔOCD are both isosceles right-angled triangle. It follows that a 90° rotation about O will map A into B and C into D, so that $AC = BD$ and they are perpendicular to each other.

Example 5. [Tournament of Towns 1994] Two circles intersect at the points A and B. Tangent lines drawn to both of the circles at the point A intersect the circles at the points M and N. The lines BM and BN intersect the circles once more at the points P and Q respectively. Prove that the segments MP and NQ are equal.



Idea:
MP and NQ are sides of the triangles ΔAQN and ΔAMP respectively, so it is natural for us to prove that the two triangles are congruent. It is easy to observe that the two triangles are similar, so what remains to prove is either $AQ = AM$ or $AP = AN$. Note that we can transmit the information between the two circles by using the theorem on alternate segment at A.

Solution outline:

(1) Observe that $\Delta AQN \sim \Delta AMP$.

(2) $AP = AN$ follows from computing

$\angle APN = \angle APB + \angle BPN$

$= \angle ANB + \angle BAN$ [$\angle$s in same segment]

$= \angle ANB + \angle AQN$ [$\angle$ in alt. segment]

$= 180^{\circ} - \angle QAN$

$= 180^{\circ} - \angle MAP$ [by step (1)]

$= \angle AMB + \angle APB$

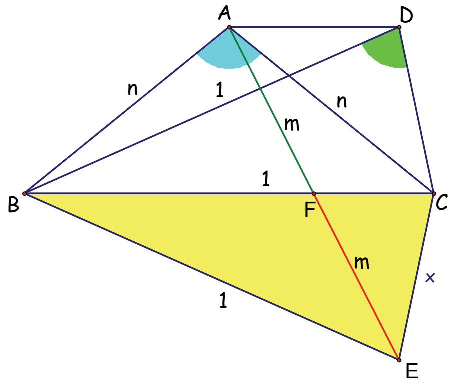
$= \angle AMB + \angle MAB$ [$\angle$ in alt. segment]

$= \angle ABP$ [ext. $\angle$ of Δ]

$= \angle ANP$ [$\angle$s in the same segment].

Example 6. $ABCD$ is a trapezium with $AD \,//\, BC$. It is known that $BC = BD = 1$, $AB = AC$, $CD < 1$ and $\angle BAC + \angle BDC = 180^{\circ}$, find CD.

Idea:
The condition $\angle BAC + \angle BDC = 180^{\circ}$ leads us to consider a cyclic quadrilateral. If we reflect ΔBDC across BC, a cyclic quadrilateral is formed.



Solution outline:
(1) Let E be the reflection of D across BC.

(2) $\angle BAC + \angle BDC = 180^{\circ}$
$\Rightarrow \angle BAC + \angle BEC = 180^{\circ}$
$\Rightarrow ABEC$ is cyclic,

$AD \,//\, BC \Rightarrow AF = FE$,

$AB = AC \Rightarrow \angle BEF = \angle FEC$

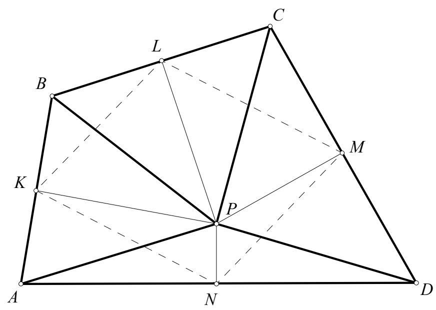
$$\Rightarrow \frac{FC}{BF} = \frac{EC}{BE} = EC .$$

(3) Let $AF = FE = m$, $AB = AC = n$ and $DC = EC = x$. It follows from Ptolemy's theorem that $AE\times BC = AC\times BE + AB\times EC$, i.e. $2m = n\,(1 + x)$. Now

$$\frac{2}{1+x} = \frac{n}{m} = \frac{AC}{AF} = \frac{BE}{BF} = \frac{BC}{BF} = \frac{BF+FC}{BF}$$
$$= 1 + \frac{FC}{BF} = 1 + \frac{EC}{BE} = 1 + x ,$$

i.e. $(1 + x)^2 = 2$. Therefore, $x = \sqrt{2} - 1$.

Example 7. [Tournament of Towns 1995] Let P be a point inside a convex quadrilateral $ABCD$. Let the angle bisector of $\angle APB$, $\angle BPC$, $\angle CPD$ and $\angle DPA$ meet AB, BC, CD and DA at K, L, M and N respectively. Find a point P such that $KLMN$ is a parallelogram.



Idea:
The *angle bisector theorem* enables us to replace the ratios that K, L, N and M divided the sides of the quadrilateral by the ratios of the distance from P to A, B, C and D. For instance, we have

$$\frac{AK}{KB} = \frac{AP}{BP} \text{ and } \frac{AN}{ND} = \frac{AP}{DP}$$

If $BP = DP$, we have $\frac{AK}{KB} = \frac{AN}{ND}$ and hence $KN//BD$. Similarly, we have $LM//BD$ and so $KN//LM$.

Therefore, we shall look for a point P such that $BP = DP$ and $AP = CP$.

Solution outline:
(1) Let P be the point of intersection of the perpendicular bisectors of the diagonals AC and BD. Then $AP = CP$ and $BP = DP$.

(2) By the angle bisector theorem, we have

$$\frac{AK}{KB} = \frac{AP}{BP} = \frac{AP}{DP} = \frac{AN}{ND}$$

and so $KN \,//\, BD$. Similarly, $LM//BD$, $KL//AC$ and $MN//AC$.

Hence $KN//LM$ and $KL//NM$, which means that $KLMN$ is a parallelogram.

Remark: Indeed, point P in the solution above is the only point that satisfies the condition given in the problem.

Example 8. [IMO 2001] Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$ac + bd = (b + d + a - c)(b + d - a + c)$.

Prove that $ab + cd$ is not prime.

Remark: This is a difficult problem in number theory. However, we would like to present a solution aided by geometrical insights!

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 31, 2007.***

Problem 271. There are 6 coins that look the same. Five of them have the same weight, each of these is called a good coin. The remaining one has a different weight from the 5 good coins and it is called a bad coin. Devise a scheme to weigh groups of the coins using a scale (not a balance) three times only to determine the bad coin and its weight.

Problem 272. ΔABC is equilateral. Find the locus of all point Q inside the triangle such that

$$\angle QAB + \angle QBC + \angle QCA = 90^\circ.$$

Problem 273. Let R and r be the circumradius and the inradius of triangle ABC. Prove that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\ge\frac{R}{r}.$$

(*Source: 2000 Beijing Math Contest*)

Problem 274. Let $n < 11$ be a positive integer. Let p_1, p_2, p_3, p be prime numbers such that $p_1+p_3^n$ is prime. If $p_1+p_2=3p$, $p_2+p_3=p_1^n(p_1+p_3)$ and $p_2>9$, then determine $p_1p_2p_3^n$.

(*Source: 1997 Hubei Math Contest*)

Problem 275. There is a group of children coming from 11 countries (at least one child from each of the 11 countries). Their ages are from 7 to 13. Prove that there are 5 children in the group, for each of them, the number of children in the group with the same age is greater than the number of children in the group from the same country.

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Solutions

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Problem 266. Let

$$N = 1+10+10^2+\cdots+10^{1997}.$$

Determine the 1000^{th} digit after the decimal point of the square root of N in base 10. (*Source: 1998 Putnam Exam*)

Solution. **Jeff CHEN** (Virginia, USA), **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina), **Anna Ying PUN** (HKU, Math, Year 1) and **Fai YUNG**.

The answer is the same as the unit digit of $10^{1000}\sqrt{N}$. We have

$$10^{1000}\sqrt{N}=10^{1000}\sqrt{\frac{10^{1998}-1}{9}}=\frac{\sqrt{10^{3998}-10^{2000}}}{3}.$$

Since

$$(10^{1999}-7)^2<10^{3998}-10^{2000}<(10^{1999}-4)^2,$$

so it follows that $10^{1000}\sqrt{N}$ is between $(10^{1999}-7)/3=33\cdots31$ and $(10^{1999}-4)/3=33\cdots32$. Therefore, the answer is 1.

Commended solvers: **Simon YAU** and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6).

Problem 267. For any integer a, set

$$n_a = 101a - 100\cdot 2^a.$$

Show that for $0 \le a, b, c, d \le 99$, if

$$n_a+n_b \equiv n_c+n_d \pmod{10100},$$

then $\{a,b\}=\{c,d\}$. (*Source: 1994 Putnam Exam*)

Solution. **Jeff CHEN** (Virginia, USA), **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina), **Anna Ying PUN** (HKU, Math, Year 1) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6).

If $n_a+n_b \equiv n_c+n_d \pmod{10100}$, then $a+b \equiv n_a+n_b \equiv n_c+n_d \equiv c+d \pmod{100}$ and $2^a+2^b \equiv n_a+n_b \equiv n_c+n_d \equiv 2^c+2^d \pmod{101}$.

By Fermat's little theorem, $2^{100}\equiv 1 \pmod{101}$ and so $2^{a+b}\equiv 2^{c+d} \pmod{101}$. Next

$$\begin{aligned}(2^a-2^c)(2^a-2^d) &= 2^a(2^a-2^c-2^d)+2^{c+d}\\ &\equiv 2^a(-2^b)+2^{a+b}\\ &= 0 \pmod{101}.\end{aligned}$$

So $2^a\equiv 2^c \pmod{101}$ or $2^a \equiv 2^d \pmod{101}$.

Now we claim that if $0\le s\le t\le 99$ and $2^s\equiv 2^t \pmod{101}$, then $s=t$. To see this, let k be the least positive integer such that $2^k\equiv 1 \pmod{101}$. Dividing 100 by k, we get $100 = kq+r$ with $0\le r<k$. Since $2^r = 2^{100-kq}\equiv 1 \pmod{101}$ too, so $r = 0$, then k is a divisor of 100.

Clearly, $1 < 2^1, 2^2, 2^4, 2^5 < 101$ and $2^{10} = 1024 \equiv 14 \pmod{101}$, $2^{20}\equiv 14^2 \equiv -6 \pmod{101}$, $2^{25}\equiv (-6)32 \equiv 10 \pmod{101}$, $2^{50}\equiv 10^2\equiv -1 \pmod{101}$. Hence $k=100$. Finally $2^{t-s}\equiv 1 \pmod{101}$ and $0\le t-s<100$ imply $t-s=0$, proving the claim.

By the claim, we get $a=c$ or $a=d$. From $a+b\equiv c+d \pmod{100}$ and $0 \le a, b, c, d \le 99$, we get $a = c$ implies $b = d$ and similarly $a = d$ implies $b = c$. The conclusion follows.

Problem 268. In triangle ABC, $\angle ABC = \angle ACB = 40^\circ$. Points P and Q are inside the triangle such that $\angle PAB = \angle QAC = 20^\circ$ and $\angle PCB = \angle QCA = 10^\circ$. Must B, P, Q be collinear? Give a proof. (*Source: 1994 Shanghai Math Competition*)

Solution. **Jeff CHEN** (Virginia, USA), **Courtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher), **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Kelvin LEE** (Winchester College, England) **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina) and **NG Ngai Fung** (STFA Leung Kau Kui College).

Since lines AP, BP, CP concur, by the trigonometric form of Ceva's theorem,

$$\frac{\sin\angle CBP\sin\angle BAP\sin\angle PCA}{\sin\angle PBA\sin\angle PAC\sin\angle PCB}=1,$$

which implies

$$\frac{\sin\angle CBP}{\sin\angle PBA}=\frac{\sin 80^\circ\sin 10^\circ}{\sin 20^\circ\sin 30^\circ}=\frac{\cos 10^\circ\sin 10^\circ}{\sin 20^\circ/2}=1.$$

So $\angle CBP=\angle PBA=20^\circ$. Replacing P by Q above, we similarly have

$$\frac{\sin\angle CBQ}{\sin\angle QBA}=\frac{\sin 20^\circ\sin 30^\circ}{\sin 80^\circ\sin 10^\circ}=1.$$

So $\angle QBA=\angle CBQ=20^\circ$. Then B, P, Q are on the bisector of $\angle ABC$.

Commended solvers: **CHIU Kwok Sing** (Belilios Public School), **FOK Pak Hei** (Pui Ching Middle School), **Anna Ying PUN** (HKU, Math, Year 1) and **Simon YAU**.

Problem 269. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$, $a_{n+1} = f(a_n)$ for all $n \ge 0$. Prove that if there exists a positive integer m for which $a_m = 0$, then either $a_1 = 0$ or

$a_2 = 0$. (*Source: 2000 Putnam Exam*)

Solution. **Irfan GLOGIC** (Sarajevo College, 4th grade, Sarajevo, Bosnia and Herzegovina), **Salem MALIKIĆ** (Sarajevo College, 3rd grade, Sarajevo, Bosnia and Herzegovina) and **Anna Ying PUN** (HKU, Math, Year 1).

Observe that for any integers m and n, $m-n$ divides $f(m)-f(n)$ since for all nonnegative integer k, m^k-n^k has $m-n$ as a factor. For nonnegative integer n, let $b_n=a_{n+1}-a_n$, then by the last sentence, b_n divides b_{n+1} for all n.

Since $a_0 = a_m = 0$, $a_1 = a_{m+1}$ and so $b_0=b_m$. If $b_0 = 0$, then $a_1 = a_{m+1} = b_m+a_m = 0$.

If $b_0 \neq 0$, then using b_n divides b_{n+1} for all n and $b_0=b_m$, we get $b_n = \pm\, b_0$ for $n=1,2,\cdots,m$. Since $b_0+b_1+\cdots+b_m = a_m-a_0 = 0$, half of the integers $b_0, \cdots, b_m$ are positive and half are negative. Then there is $k < m$ such that $b_{k-1} = -b_k$, which implies $a_{k-1}=a_{k+1}$. Then $a_m=a_{m+2}$ and so $0=a_m=a_{m+2}=f(f(a_m))=f(f(a_0))=a_2$.

Problem 270. The distance between any two of the points A, B, C, D on a plane is at most 1. Find the minimum of the radius of a circle that can cover these four points. (*Source 1998 Tianjin Math Competition*)

Solution. **Jeff CHEN** (Virginia, USA).

Case 1: (one of the point, say D, is inside or on a side of ΔABC) If ΔABC is acute, then one of the angle, say $\angle BAC \ge 60^\circ$. By the extended sine law, the circumcircle of ΔABC covers the four points with diameter

$$2R = \frac{BC}{\sin \angle BAC} \le \frac{2}{\sqrt{3}}.$$

(Note equality occurs in case ΔABC is equilateral.) If ΔABC is right or obtuse, then the circle using the longest side as diameter covers the four points with $R \le 1/2$.

Case 2: ($ABCD$ is a convex quadrilateral) If there is a pair of opposite angles, say angles A and C, are at least 90°, then the circle with BD as diameter will cover the four points with $R \le 1/2$. Otherwise, there is a pair of neighboring angles, say angles A and B, both of which are less than 90°.

If $\angle ADB \ge \angle ACB \ge 90^\circ$, then the circle with AB as diameter covers the four points and radius $R \le 1/2$.

If $\angle ADB \ge \angle ACB$ and $\angle ACB < 90^\circ$, then D is in or on the circumcircle of ΔABC with radius $R \le 1/\sqrt{3}$ as in case 1.

So summarizing all cases, we see the minimum radius that works for all possible arrangements of A, B, C *and* D is $R = 1/\sqrt{3}$.

Commended solvers: **NG Ngai Fung** (STFA Leung Kau Kui College) and **Anna Ying PUN** (HKU, Math, Year 1).

Olympiad Corner

(continued from page 1)

Problem 5. A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all possible positions of this light.

From *How to Solve It* to Problem Solving in Geometry

(continued from page 2)

Idea:
Observe that

$$ac + bd = (b + d + a - c)(b + d - a + c)$$
$$\Leftrightarrow a^2 + c^2 - ac = b^2 + d^2 + bd$$

The last equality suggests one to think about using the cosine law as follow:

$$\begin{aligned} &a^2 + c^2 - 2ac\cos 60^\circ \\ &= a^2 + c^2 - ac \\ &= b^2 + d^2 + bd \\ &= b^2 + d^2 - 2bd \cos 120^\circ. \end{aligned}$$

Solution:

(1) **Lemma**: Let x, y, and z be positive integers with $z < x$ and $z < y$. If xy/z is an integer, then xy/z is composite.
[Can you prove this lemma? Is there any trivial case you can see immediately? How about proving the lemma by mathematical induction in z?]

The case $z = 1$ is trivial. In case $z > 1$, inductively suppose the lemma is true for all positive integers z' less than z. Then z has a prime divisor p, say $z = pz'$. Since xy/z is an integer, either p divides x or p divides y, say p divides x. Then $x = px'$. So $xy/z=x'y/z'$ with $z' < x'$ and $z' < z < y$. By the induction hypothesis, $xy/z=x'y/z'$ is composite.

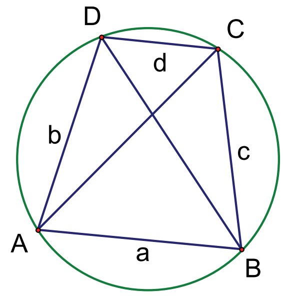
(2) The equality

$$ac + bd = (b + d + a - c)(b + d - a + c)$$

is equivalent to

$$a^2 + c^2 - ac = b^2 + d^2 + bd$$

In view of this, we can construct cyclic quadrilateral $ABCD$ with $AB = a$, $BC = c$, $CD = d$, $DA = b$, $\angle ABC = 60^\circ$ and $\angle ADC = 120^\circ$.



(3) Considering the ratios of areas and using Ptolemy's theorem, we have

$$\frac{AC}{BD} = \frac{ab+cd}{ac+bd} \text{ and } AC \times BD = ad + bc.$$

(4) Therefore,

$$\frac{ab+cd}{ac+bd} = \frac{AC}{BD} = \frac{AC^2}{AC \times BD} = \frac{a^2+c^2-ac}{ad+bc},$$

which implies

$$ab+cd = \frac{(ac+bd)(a^2+c^2-ac)}{ad+bc} \qquad (*).$$

(5) To get the conclusion from the lemma, it remains to show

$$ad + bc < ac + bd$$

and $$ad + bc < a^2 + c^2 - ac.$$

Now

$$\begin{aligned} &(ac + bd) - (ad + bc) \\ &= (a - b)(c - d) > 0 \\ \Rightarrow\ & ad + bc < ac + bd. \end{aligned}$$

Also,

$$\begin{aligned} &(ab + cd) - (ac + bd) \\ &= (a - d)(b - c) > 0 \\ \Rightarrow\ & ab + cd > ac + bd \\ \Rightarrow\ & ad + bc < a^2 + c^2 - ac \quad (\text{by } (*)). \end{aligned}$$

Now the result follows from the lemma.

Mathematical Excalibur

Volume 12, Number 2 *May 2007 – August 2007*

Olympiad Corner

Below are the problems of the 2006 Belarussian Math Olympiad, Final Round, Category C.

Problem 1. Is it possible to partition the set of all integers into three nonempty pairwise disjoint subsets so that for any two numbers a and b from different subsets,

a) there is a number c in the third subset such that $a + b = 2c$?
b) there are two numbers c_1 and c_2 in the third subset such that $a + b = c_1 + c_2$?

Problem 2. Points X, Y, Z are marked on the sides AB, BC, CD of the rhombus $ABCD$, respectively, so that $XY \| AZ$. Prove that XZ, AY and BD are concurrent.

Problem 3. Let a, b, c be real positive numbers such that $abc = 1$. Prove that

$$2(a^2+b^2+c^2)+a+b+c \geq 6+ab+bc+ca.$$

Problem 4. Given triangle ABC with $\angle A = 60^\circ$, $AB = 2005$, $AC = 2006$. Bob and Bill in turn (Bob is the first) cut the triangle along any straight line so that two new triangles with area more than or equal to 1 appear.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 20, 2007***.

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From *How to Solve It* to Problem Solving in Geometry (II)

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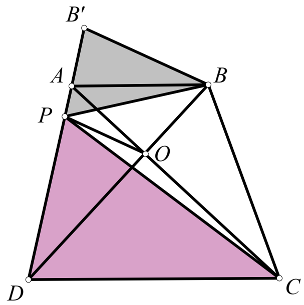
We will continue with more examples.

Example 9. In the trapezium $ABCD$, $AB \| CD$ and the diagonals intersect at O. P, Q are points on AD and BC respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Show that $OP = OQ$.

Idea:
We shall try to find OP in terms of "more basic" lengths, e.g. AB, CD, OA, OC, …. To achieve that, we can construct a triangle that is similar to ΔDPC.



Solution Outline:
(1) Extend DA to B' such that $BB' = BA$. Then $\angle PB'B = \angle B'AB = \angle PDC$. So $\Delta DPC \sim \Delta B'PB$.

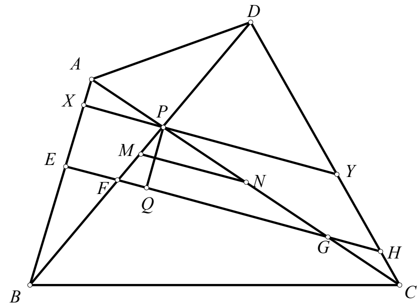
(2) It follows that
$$\frac{DP}{PB'} = \frac{CD}{BB'} = \frac{CD}{BA} = \frac{DO}{BO}$$
and so $PO \| BB'$.

(3) Since $\Delta DPO \sim \Delta DB'B$, we have
$$OP = BB' \times \frac{DO}{DB} = AB \times \frac{DO}{DB}.$$

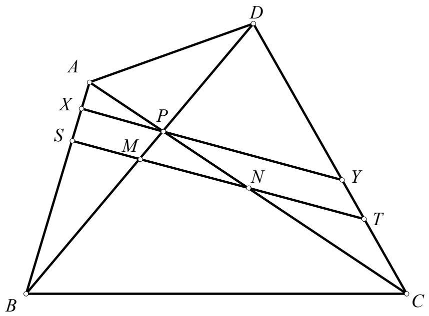
(4) Similarly, we have $OQ = AB \times \dfrac{CO}{CA}$

and the result follows.

Example 10. In quadrilateral $ABCD$, the diagonals intersect at P. M and N are midpoint of BD and AC respectively. Q is the reflected image of P about MN. The line through P and parallel to MN cuts AB and CD at X and Y respectively. The line through Q parallel to MN cuts AB, BD, AC and CD at E, F, G and H respectively. Prove that $EF = GH$.



Idea:
The diagram is not simple. We shall try to express the lengths involved in terms of "more basic" lengths, e.g. PA, PB, PC and PD.



Solution Outline:
(1) First observe that $PM = MF$ and $PN = NG$, hence $BF = PD$ and $CG = PA$.

(2) $\dfrac{EF}{XP} = \dfrac{BF}{BP} = \dfrac{PD}{BP}$, $EF = \dfrac{PD \times XP}{BP}$.

Similarly, we have $GH = \dfrac{PA \times YP}{CP}$.

(3) Let the line MN cuts AB and CD at S and T respectively. Then

$$\frac{SM}{XP} = \frac{BM}{BP} = \frac{BD}{2BP}, \frac{SN}{XP} = \frac{AN}{AP} = \frac{AC}{2AP}.$$

Subtracting the equalities get
$$\frac{MN}{XP} = \frac{1}{2}\left(\frac{AC}{AP} - \frac{BD}{BP}\right).$$

Similarly, we have
$$\frac{MN}{YP} = \frac{1}{2}\left(\frac{BD}{PD} - \frac{AC}{PC}\right).$$

(4) $EF = GH \Leftrightarrow \dfrac{PD\times XP}{BP} = \dfrac{PA\times YP}{CP}$

$\Leftrightarrow \dfrac{PD\times MN}{BP\times YP} = \dfrac{PA\times MN}{CP\times XP}$. By (3),

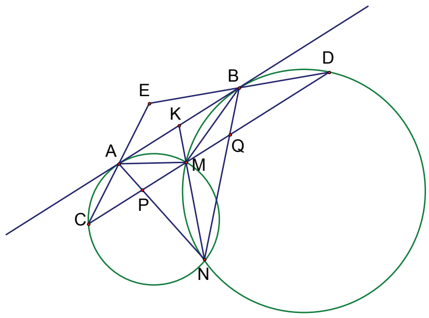
$\dfrac{PD}{BP}\left(\dfrac{BD}{PD}-\dfrac{AC}{PC}\right) = \dfrac{PA}{CP}\left(\dfrac{AC}{AP}-\dfrac{BD}{BP}\right)$

$\Leftrightarrow \dfrac{BD}{BP} - \dfrac{PD\times AC}{BP\times PC} = \dfrac{AC}{CP} - \dfrac{PA\times BD}{CP\times BP}$

$\Leftrightarrow \dfrac{BD}{BP} + \dfrac{PA\times BD}{CP\times BP} = \dfrac{AC}{CP} + \dfrac{PD\times AC}{BP\times PC}$.

By addition, both sides of the last equation equal $\dfrac{AC\times BD}{BP\times CP}$.

Example 11. [IMO 2000] Two circles Γ_1 and Γ_2 intersect at M and N. Let L be the common tangent to Γ_1 and Γ_2 so that M is closer to L than N is. Let L touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to L meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.



Idea:
First, note that if $EP = EQ$, then E lies on the perpendicular bisector of PQ.

Observe that $AB \parallel CD$ implies A and B are the midpoints of arc CAM and arc DBM respectively, from which we see ΔACM and ΔBDM are isosceles.

Second, we have $\angle EAB = \angle ECM = \angle AMC = \angle BAM$ and similarly, $\angle EBA = \angle ABM$. That means E is the reflected image of M about AB. In particular, $EM \perp AB$ and hence $EM \perp PQ$.

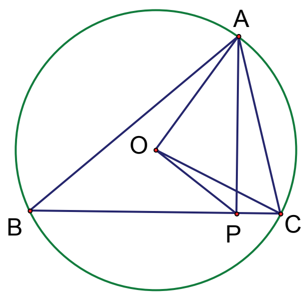
Therefore, the result follows if we can show that M is the midpoint of PQ.

Solution outline:
(1) Extend NM to meet AB at K.

(2) $AK^2 = KN\times KM = BK^2 \Rightarrow K$ is the midpoint of $AB \Rightarrow M$ is the midpoint of PQ.

(3) Following the steps discussed above, we get $EM \perp PQ$ and hence $EP = EQ$.

Example 12. [IMO 2001] Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A. Suppose that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.



Idea:
(1) Examine the conclusion $\angle CAB + \angle COP < 90^\circ$, which is equivalent to $2\angle CAB + 2\angle COP < 180^\circ$. That is,

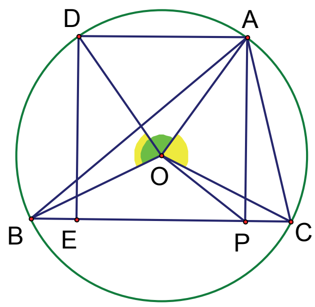
$$\angle COB + 2\angle COP < 180^\circ.$$

On the other hand, we have $\angle COB + 2\angle OCP = 180^\circ$. Therefore, we shall show $\angle COP < \angle OCP$ or $PC < OP$.

(2) Examine the condition $\angle BCA \geq \angle ABC + 30^\circ$, which is equivalent to $2\angle BCA - 2\angle ABC \geq 60^\circ$. That is,

$$\angle BOA - \angle AOC \geq 60^\circ.$$

What is the meaning of $\angle BOA - \angle AOC$?



Solution outline:
(1) Let D and E be the reflected image of A and P about the perpendicular bisector of BC respectively. Let R be the circumradius.

(2) $\angle BCA \geq \angle ABC + 30^\circ$
$\Rightarrow \angle BOA - \angle AOC \geq 60^\circ$
$\Rightarrow \angle DOA \geq 60^\circ$
$\Rightarrow EP = DA \geq R.$

(3) $OP + R = OP + OC = OE + OC$
$> EC = EP + PC \geq R + PC$
$\Rightarrow OP > PC \quad \Rightarrow \angle COP < \angle OCP.$

(4) $2\angle CAB + 2\angle COP$
$= \angle COB + 2\angle COP$
$< \angle COB + 2\angle OCP < 180^\circ$
and the result follows.

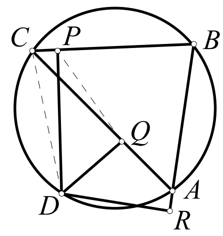
Example 13. [Simson's Theorem] The feet of the perpendiculars drawn from any point on the circumcircle of a triangle to the sides of the triangle are collinear.

Solution:
In the figure below, D is a point on the circumcircle of ΔABC, P, Q, and R are feet of perpendiculars from D to BC, AC, and BA respectively.

Note that $DQAR$, $DCPQ$, and $DPBR$ are cyclic quadrilaterals. So

$$\angle DQR = \angle DAR = \angle BCD = 180^\circ - \angle PQD,$$

i.e. $\angle DQR + \angle PQD = 180^\circ$. Thus, P, Q, and R are collinear.



Example 14. [IMO 2003] Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB respectively. Show that $PQ = QR$ if and only if the bisector of $\angle ABC$ and $\angle ADC$ meet on AC.

Solution :
From Simson's theorem, P, Q, and R are collinear. Now

$\angle DPC = \angle DQC = 90^\circ$
$\Rightarrow D, P, C$ and Q are concyclic
$\Rightarrow \angle DCA = \angle DPQ = \angle DPR.$

Similarly, since D, Q, R and A are concyclic, we get $\angle DAC = \angle DRP$. It follows that $\Delta DCA \sim \Delta DPR$.

Similarly, $\Delta DAB \sim \Delta DQP$ and $\Delta DBC \sim \Delta DRQ$. So,

$$\frac{DA}{DC} = \frac{DR}{DP} = \frac{DB\cdot\frac{QR}{BC}}{DB\cdot\frac{PQ}{BA}} = \frac{QR}{PQ}\cdot\frac{BA}{BC}.$$

Therefore, $PQ = QR \Leftrightarrow \dfrac{DA}{DC} = \dfrac{BA}{BC}$.

Example 15. [IMO 2001] In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$. What are the possible angles of triangle ABC?

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 20, 2007.***

Problem 276. Let n be a positive integer. Given a $(2n-1)\times(2n-1)$ square board with exactly one of the following arrows ↑, ↓, →, ← at each of its cells. A beetle sits in one of the cells. Per year the beetle creeps from one cell to another in accordance with the arrow's direction. When the beetle leaves the cell, the arrow at that cell makes a counterclockwise 90-degree turn. Prove that the beetle leaves the board in at most $2^{3n-1}(n-1)!-3$ years.
(*Source: 2001 Belarussian Math Olympiad*)

Problem 277. (*Due to Koopa Koo, Univ. of Washington, Seattle, WA, USA*) Prove that the equation

$$x^2+y^2+z^2+2xyz=1$$

has infinitely many integer solutions (then try to get all solutions – Editiors).

Problem 278. Line segment SA is perpendicular to the plane of the square $ABCD$. Let E be the foot of the perpendicular from A to line segment SB. Let P, Q, R be the midpoints of SD, BD, CD respectively. Let M, N be on line segments PQ, PR respectively. Prove that AE is perpendicular to MN.

Problem 279. Let R be the set of all real numbers. Determine (with proof) all functions $f: R\to R$ such that for all real x and y,

$$f\big(f(x)+y\big)=2x+f\big(f(f(y))-x\big).$$

Problem 280. Let n and k be fixed positive integers. A basket of peanuts is distributed into n piles. We gather the piles and rearrange them into $n+k$ new piles. Prove that at least $k+1$ peanuts are transferred to smaller piles than the respective original piles that contained them. Also, give an example to show the constant $k+1$ cannot be improved.

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Solutions

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Problem 271. There are 6 coins that look the same. Five of them have the same weight, each of these is called a <u>*good*</u> coin. The remaining one has a different weight from the 5 good coins and it is called a <u>*bad*</u> coin. Devise a scheme to weigh groups of the coins using a scale (not a balance) three times only to determine the bad coin and its weight.
(*Source: 1998 Zhejiang Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **St. Paul's College Math Team**, **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Number the coins 1 to 6. For the first weighting, let us weigh coins 1, 2, 3 and let the weight be $3a$. For the second weighting, let us weigh coins 1, 2, 4, 5 and let the weight be $4b$.

If $a = b$, then coin 6 is bad and we can use the third weighting to find the weight of this coin.

If $a \neq b$, then the bad coin is among coins 1 to 5. For the third weighting, let us weigh coins 2, 4 and let the weight be $2c$.

If coin 1 is bad, then c and $4b-3a$ are both the weight of a good coin. So $3a-4b+c=0$. Similarly, if coin 2 or 3 or 4 or 5 is bad, we get respective equations $3a-2b-c=0$, $b-c=0$, $a-2b+c=0$ and $a-c=0$.

We can check that if any two of these equations are satisfied simultaneously, then we will arrive at $a=b$, a contradiction. Therefore, exactly one of these five equations will hold.

If the first equation $3a-4b+c=0$ holds, then coin 1 is bad and its weight can be found by the first and third weightings to be $3a-2c$. Similarly, for $k = 2$ to 5, if the k-th equation holds, then coin k is bad and its weight can be found to be $3c-2b$, $3a-2c$, $4b-3a$ and $4b-3a$ respectively.

Problem 272. $\Delta\, ABC$ is equilateral. Find the locus of all point Q inside the triangle such that

$$\angle QAB+\angle QBC+\angle QCA=90^\circ.$$

(*Source: 2000 Chinese IMO Team Training Test*)

Solution. **Alex Kin-Chit O** (STFA Cheng Yu Tung Secondary School) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6).

We take the origin at the center O of $\Delta\, ABC$. Let $\omega \neq 1$ be a cube root of unity and A,B,C,Q correspond to the complex numbers 1, ω, $\omega^2=\overline{\omega}$, z respectively. Then

$$\angle QAB+\angle QBC+\angle QCA=90^\circ$$

if and only if

$$\frac{\omega-1}{z-1}\cdot\frac{\overline{\omega}-\omega}{z-\omega}\cdot\frac{1-\overline{\omega}}{z-\overline{\omega}}=\frac{(\omega-\overline{\omega})\,|\,\omega-1|^2}{z^3-1}$$

is purely imaginary, which is equivalent to z^3 is real. These are the complex numbers whose arguments are multiples of $\pi/3$. Therefore, the required locus is the set of points on the three altitudes.

Commended solvers: **Jeff CHEN** (Virginia, USA), **St. Paul's College Math Team, Simon YAU** and **YIM Wing Yin** (HKU, Year 1).

Problem 273. Let R and r be the circumradius and the inradius of triangle ABC. Prove that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\geq\frac{R}{r}.$$

(*Source: 2000 Beijing Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **Kelvin LEE** (Winchester College, England), **NG Eric Ngai Fung** (STFA Leung Kau Kui College), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6) and **YIM Wing Yin** (HKU, Year 1).

Without loss of generality, let a, b, c be the sides and $a \geq b \geq c$. By the extended sine law, $R = a/(2\sin A) = b/(2\sin B) = c/(2\sin C)$. Now the area of the triangle is $(bc \sin A)/2=abc/(4R)$ and is also rs, where $s = (a + b + c)/2$ is the semi- perimeter. So $abc=4Rrs$.

Next, observe that for any positive x and y, we have $(x^2-y^2)(1/x-1/y)\leq 0$, which after expansion yields

$$\frac{x^2}{y}+\frac{y^2}{x}\geq x+y. \qquad (*)$$

By the cosine law and the extended sine law, we get

$$\frac{\cos A}{\sin^2 A}=\frac{(b^2+c^2-a^2)/2bc}{(a/2R)^2}$$

$$=\frac{2R^2}{abc}\left(\frac{b^2+c^2-a^2}{a}\right)=\frac{R}{2rs}\left(\frac{b^2}{a}+\frac{c^2}{a}-a\right).$$

Adding this to the similar terms for B and C, we get

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}$$

$$=\frac{R}{2rs}\left(\frac{b^2}{a}+\frac{a^2}{b}+\frac{c^2}{b}+\frac{b^2}{c}+\frac{a^2}{c}+\frac{c^2}{a}-a-b-c\right)$$

$$\ge \frac{R}{2rs}(a+b+c)=\frac{R}{r} \quad \text{by (*)}.$$

Commended solvers: **CHEUNG Wang Chi** (Singapore).

Problem 274. Let $n < 11$ be a positive integer. Let p_1, p_2, p_3, p be prime numbers such that $p_1 + p_3^n$ is prime. If $p_1 + p_2 = 3p$, $p_2 + p_3 = p_1^n(p_1 + p_3)$ and $p_2 > 9$, then determine $p_1 p_2 p_3^n$.
(*Source: 1997 Hubei Math Contest*)

Solution. **CHEUNG Wang Chi** (Singapore), **NG Eric Ngai Fung** (STFA Leung Kau Kui College), **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Assume $p_1 \ge 3$. Then $p_1+p_2 > 12$ and $3p$ is even, which would imply p is even and at least 5, contradicting p is prime. So $p_1=2$ and $p_2=3p-2$.

Modulo 3, the given equation $p_2 + p_3 = p_1^n(p_1+p_3)$ leads to

$$\begin{aligned} 0 &\equiv 3p \\ &= p_2+2 = 2^n(2+p_3)+2 \\ &= 2^{n+1} + 2 + (2^n-1)p_3 \\ &\equiv (-1)^{n+1}+2+((-1)^n-1)p_3 \pmod 3. \end{aligned}$$

The case n is even results in the contradiction $0 \equiv 1 \pmod 3$. So n is odd and we get $0 \equiv p_3 \pmod 3$. So $p_3 = 3$.

Finally, the cases $n = 1, 3, 5, 7, 9$ lead to $p_1 + p_3^n = 5, 29, 245, 2189, 19685$ respectively. Since 245, 19685 are divisible by 5 and 2189 is divisible by 11, n can only be 1 or 3 for $p_1+p_3^n$ to be prime. Now $p_2 = p_1^n(p_1+p_3) - p_3 = 2^n 5 - 3 > 9$ implies $n = 3$. Then the answer is

$$p_1 p_2 p_3^n = 2 \cdot 37 \cdot 3^3 = 1998.$$

Problem 275. There is a group of children coming from 11 countries (at least one child from each of the 11 countries). Their ages are from 7 to 13. Prove that there are 5 children in the group, for each of them, the number of children in the group with the same age is greater than the number of children in the group from the same country.

Solution. **Jeff CHEN** (Virginia, USA).

For i =7 to 13 and j = 1 to 11, let a_{ij} be the number of children of age i from country j in the group. Then

$$b_i = \sum_{j=1}^{11} a_{ij} \ge 0 \quad \text{and} \quad c_j = \sum_{i=7}^{13} a_{ij} \ge 1$$

are the number of children of age i in the group and the number of children from country j respectively. Note that

$$c_j = \sum_{i=7}^{13} a_{ij} = \sum_{b_i \ne 0} a_{ij}, \text{ where } \sum_{b_i \ne 0} \text{ is}$$

used to denote summing i from 7 to 13 skipping those i for which b_i=0. Now

$$\sum_{b_i \ne 0}\sum_{j=1}^{11} a_{ij}\left(\frac{1}{c_j}-\frac{1}{b_i}\right)$$

$$=\sum_{j=1}^{11}\frac{\sum_{b_i\ne 0} a_{ij}}{c_j}-\sum_{b_i \ne 0}\frac{\sum_{j=1}^{11} a_{ij}}{b_i}$$

$$\ge \sum_{j=1}^{11} 1 - \sum_{i=7}^{13} 1 = 4.$$

Since $a_{ij}(1/c_j - 1/b_i) < a_{ij}/c_j \le 1$, there are at least five terms $a_{ij}(1/c_j - 1/b_i) > 0$. So there are at least five ordered pairs (i,j) such that $a_{ij} > 0$ (so we can take a child of age i from country j) and we have $b_i > c_j$.

Olympiad Corner

(continued from page 1)

Problem 4. (Cont.) After that an obtused-angled triangle (or any of two right-angled triangles) is deleted and the procedure is repeated with the remained triangle. The player loses if he cannot do the next cutting. Determine, which player wins if both play in the best way.

Problem 5. AA_1, BB_1 and CC_1 are the altitudes of an acute triangle ABC. Prove that the feet of the perpendiculars from C_1 onto the segments AC, BC, BB_1 and AA_1 lie on the same straight line.

Problem 6. Given real numbers a, b, k (k>0). The circle with the center (a,b) has at least three common points with the parabola $y = kx^2$; one of them is the origin (0,0) and two of the others lie on the line $y=kx+b$. Prove that $b \ge 2$.

Problem 7. Let x, y, z be real numbers greater than 1 such that

$$xy^2 - y^2 + 4xy + 4x - 4y = 4004,$$

and $$xz^2 - z^2 + 6xz + 9x - 6z = 1009.$$

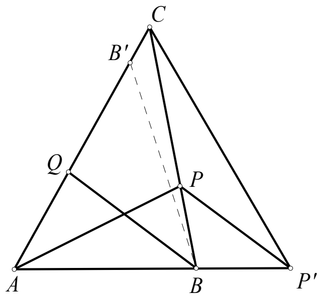
Determine all possible values of the expression $xyz+3xy+2xz-yz+6x-3y-2z$.

Problem 8. A $2n\times 2n$ square is divided into $4n^2$ unit squares. What is the greatest possible number of diagonals of these unit squares one can draw so that no two of them have a common point (including the endpoints of the diagonals)?

From *How to Solve It* to Problem Solving in Geometry (II)

(continued from page 2)

Idea:
By examining the conditions given, we may see that the point C is not too important.



We will focus on how to represent the condition $AB + BP = AQ + QB$ in the diagram. For that, we construct points P' and B' on AB and AQ extended respectively so that $PB = P'B$ and $QB' = QB$. Then

$AB + BP = AQ + QB$
$\Rightarrow AB + BP' = AQ + QB' \Rightarrow AP' = AB'$
$\Rightarrow AP'B'$ is equilateral (as $\angle B'AP' = 60^\circ$).

Solution outline:
(1) Let $\angle ABQ = \angle QBP = \theta$. Since $PB = P'B$, we have $\angle PP'B = \theta$.

(2) Since AP bisects $\angle QAB$ and $\Delta AB'P'$ is equilateral, it follows that B' is the reflected image of P' about AP. So, $PP' = PB'$ and $\angle QB'P = \angle AP'P = \theta$.

(3) Since $QB = QB'$ and $\angle QBP = \theta = \angle QB'P$, by Example 2, P lies on either BB' or the perpendicular bisector of BB'. If P does not lie on BB', we will have $PB = PB' = PP'$. This will imply $\Delta BPP'$ is equilateral, $\theta = 60^\circ$ and $\angle QAB + \angle ABP = 60^\circ + 2\theta = 180^\circ$, which is absurd. So, P must lie on BB'. Therefore, $B' = C$.

(4) Since $QB=QB'=QC$, $\angle QCB = \angle QBC = \theta$. So $\angle QAB + 2\theta + \theta = 180^\circ \Rightarrow 60^\circ + 3\theta = 180^\circ \Rightarrow \theta = 40^\circ$. Therefore, $\angle ABC = 80^\circ$, $\angle ACB = 40^\circ$.

Mathematical Excalibur

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Olympiad Corner

Below were the problems of the 2007 International Math Olympiad, which was held in Hanoi, Vietnam.

Day 1 (*July 25, 2007*)

Problem 1. Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i $(1 \le i \le n)$ define

$$d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : 1 \le j \le n\}$$

and let $d = \max\{d_i : 1 \le i \le n\}$.

(a) Prove that, for any real numbers $x_1 \le x_2 \le \ldots \le x_n$,

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \le x_2 \le \ldots \le x_n$ such that equality holds in (*).

Problem 2. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF=EG=EC$. Prove that ℓ is the bisector of angle DAB.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 25, 2007***.

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Convex Hull

Kin Yin Li

A set S in a plane or in space is convex if and only if whenever points X and Y are in S, the line segment XY must be contained in S. The intersection of any collection of convex sets is convex. For an arbitrary set W, the convex hull of W is the intersection of all convex sets containing W. This is the smallest convex set containing W. For a finite set W, the boundary of the convex hull of W is a polygon, whose vertices are all in W.

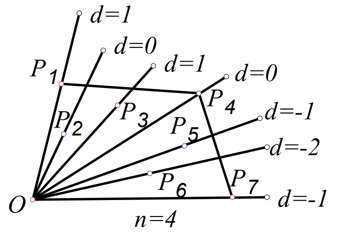
In a previous article (see *pp*. 1-2, *vol*. 5, *no*. 1 of Math. Excalibur), we solved problem 1 of the 2000 IMO using convex hull. Below we will discuss more geometric combinatorial problems that can be solved by studying convex hulls of sets.

Example 1. There are $n > 3$ coplanar points, no three of which are collinear and every four of them are the vertices of a convex quadrilateral. Prove that the n points are the vertices of a convex n-sided polygon.

Solution. Assume one of these points, say P, is inside the convex hull of the n points. Let Q be a vertex of the convex hull. The diagonals from Q divide the convex hull into triangles. Since no three points are collinear, P is inside some $\triangle QRS$, where RS is a side of the boundary. Then P,Q,R,S cannot be the vertices of a convex quadrilateral, a contradiction. So all n points can only be the vertices of the boundary polygon.

Example 2. (*1979 Putnam Exam*) Let A be a set of $2n$ points in the plane, no three of which are collinear, n of them are colored red and the other blue. Prove that there are n line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.

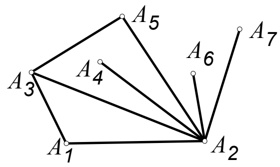
Solution. The case $n = 1$ is true. Suppose all cases less than n are true. For a vertex O on the boundary polygon of the convex hull of these $2n$ points, it is one of the $2n$ points, say its color is red. Let P_1, P_{2n-1} be adjacent vertices to O. If one of them, say P_1, is blue, then draw line segment OP_1 and apply induction to the other $2(n-1)$ points to finish. Otherwise,



let $d = 1$ and rotate the line OP_1 toward line OP_{2n-1} about O hitting the other $2n-3$ points one at a time. When the line hits a red point, increase d by 1 and when it hits a blue point, decrease d by 1. When the line hits P_{2n-1}, $d = (n-1) - n = -1$. So at some time, $d = 0$, say when the line hits P_j. Then $P_1, \ldots, P_j$ are on one side of line OP_j and $O, P_{j+1}, \ldots, P_{2n-1}$ are on the other side. The inductive step can be applied to these two sets of points, which leads to the case n being true.

Example 3. (*1985 IMO Longlisted Problem*) Let A and B be finite disjoint sets of points in the plane such that any three distinct points in $A \cup B$ are not collinear. Assume that at least one of the sets A, B contains at least five points. Show that there exists a triangle all of whose vertices are contained in A or in B that does not contained in its interior any point from the other set.

Solution. Suppose A has at least five points. Take a side A_1A_2 of the boundary of the convex hull of A. For any other A_i in A, let $\alpha_i = \angle A_1A_2A_i$, say $\alpha_3 < \alpha_4 < \cdots < 180°$. Then the convex hull H of A_1, A_2, A_3, A_4, A_5 contains no other points of A.



(continued on page 4)

Perpendicular Lines

Kin Yin Li

In geometry, sometimes we are asked to prove two lines are perpendicular. If the given facts are about right angles and lengths of segments, the following theorem is often useful.

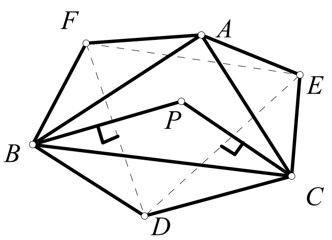
Theorem. On a plane, for distinct points R, S, X, Y, we have $RX^2-SX^2=RY^2-SY^2$ if and only if $RS \perp XY$.

Proof. Let P and Q be the feet of the perpendicular from X and Y to line RS respectively. If $RS \perp XY$, then $P=Q$ and $RX^2-SX^2=RP^2-SP^2=RY^2-SY^2$.

Conversely, if $RX^2-SX^2=RY^2-SY^2=m$, then $m=RP^2-(SR\pm RP)^2$. So $RP=\mp(SR^2+m)/2SR$. Replacing P by Q, we get $RQ=\mp(SR^2+m)/2SR$. Hence, $RP=RQ$. Interchanging R and S, we also get $SP=SQ$. So $P=Q$. Therefore, $RS \perp XY$.

Here are a few illustrative examples.

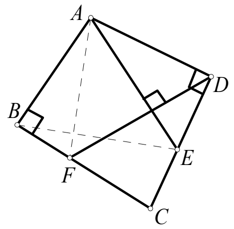
Example 1. (*1997 USA Math Olympiad*) Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent.



Solution. Let P be the intersection of the perpendicular line from B to FD with the perpendicular line from C to DE. Then $PB \perp FD$ and $PC \perp DE$. By the theorem above, we have $PF^2-PD^2=BF^2-BD^2$ and $PD^2-PE^2=CD^2-CE^2$.

Adding these and using $AF=BF$, $BD=CD$ and $CE=AE$, we get $PF^2-PE^2=AF^2-AE^2$. So $PA \perp EF$ and P is the desired concurrent point.

Example 2. (*1995 Russian Math Olympiad*) $ABCD$ is a quadrilateral such that $AB=AD$ and $\angle ABC$ and $\angle CDA$ are right angles. Points F and E are chosen on BC and CD respectively so that $DF \perp AE$. Prove that $AF \perp BE$.

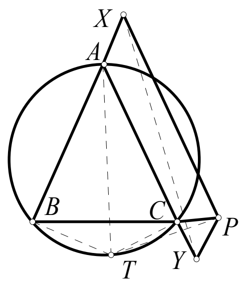


Solution. We have $AE \perp DF$, $AB \perp BF$ and $AD \perp DE$, which are equivalent to

$$AD^2-AF^2=ED^2-EF^2, \qquad \text{(a)}$$
$$AB^2-AF^2=-BF^2, \qquad \text{(b)}$$
$$AD^2-AE^2=-DE^2. \qquad \text{(c)}$$

Doing (a) − (b) + (c) and using $AD=AB$, we get $AB^2-AE^2=BF^2-EF^2$, which implies $AF \perp BE$.

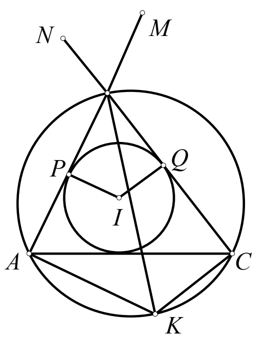
Example 3. In acute $\triangle ABC$, $AB=AC$ and P is a point on ray BC. Points X and Y are on rays BA and AC such that $PX \| AC$ and $PY \| AB$. Point T is the midpoint of minor arc BC on the circumcircle of $\triangle ABC$. Prove that $PT \perp XY$.



Since AT is a diameter, $\angle ABT=90°=\angle ACT$. Then $TX^2=XB^2+BT^2$ and $TY^2=TC^2+CY^2$. So $TX^2-TY^2=BX^2-CY^2$.

Since $PX \| AC$, we have $\angle ABC=\angle ACB=\angle XPB$, hence $BX=PX$. Similarly, $CY=PY$. Therefore, $TX^2-TY^2=PX^2-PY^2$, which is equivalent to $PT \perp XY$.

Example 4. (1994 Jiangsu Province Math Competition) For $\triangle ABC$, take a point M by extending side AB beyond B and a point N by extending side CB beyond B such that $AM=CN=s$, where s is the semiperimeter of $\triangle ABC$. Let the inscribed circle of $\triangle ABC$ have center I and the circumcircle of $\triangle ABC$ have diameter BK. Prove that $KI \perp MN$.



Solution. Let the incircle of $\triangle ABC$ touch side AB at P and side BC at Q. We will show $KM^2-KN^2=IM^2-IN^2$.

Now since $\angle MAK=\angle BAK=90°$ and $\angle NCK=\angle BCK=90°$, we get

$$\begin{aligned}KM^2-KN^2&=(KA^2+AM^2)-(KC^2+CN^2)\\&=KA^2-KC^2\\&=(KA^2-KB^2)+(KB^2-KC^2)\\&=BC^2-AB^2.\end{aligned}$$

Also, since $\angle MPI=\angle BPI=90°$ and $\angle NQI=\angle BQI=90°$, we get

$$\begin{aligned}IM^2-IN^2&=(IP^2+PM^2)-(IQ^2+QN^2)\\&=PM^2-QN^2\\&=(AM-AP)^2+(CN-QC)^2.\end{aligned}$$

Now

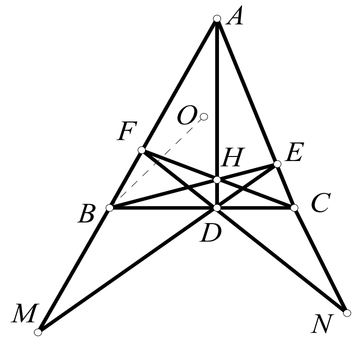
$$AM-AP=s-\frac{AB+CA-BC}{2}=BC$$

and

$$CN-QC=s-\frac{CA+BC-AB}{2}=AB.$$

So $IM^2-IN^2=BC^2-AB^2=KM^2-KN^2$.

Example 5. (*2001 Chinese National Senior High Math Competition*) As in the figure, in $\triangle ABC$, O is the circumcenter. The three altitudes AD, BE and CF intersect at H. Lines ED and AB intersect at M. Lines FD and AC intersect at N. Prove that (1) $OB \perp DF$ and $OC \perp DE$; (2) $OH \perp MN$.



Solution. (1) Since $\angle AFC=90°=\angle ADC$, so A,C,D,F are concyclic. Then $\angle BDF=\angle BAC$. Also,

$$\begin{aligned}\angle OBC&=½(180°-\angle BOC)\\&=90°-\angle BAC=90°-\angle BDF.\end{aligned}$$

So $OB \perp DF$. Similarly, $OC \perp DE$.

(2) Now $CH \perp MA$, $BH \perp NA$, $DA \perp BC$, $OB \perp DF=DN$ and $OC \perp DE=DM$. So

$$MC^2-MH^2=AC^2-AH^2 \qquad \text{(a)}$$
$$NB^2-NH^2=AB^2-AH^2 \qquad \text{(b)}$$
$$DB^2-DC^2=AB^2-AC^2 \qquad \text{(c)}$$
$$BN^2-BD^2=ON^2-OD^2 \qquad \text{(d)}$$
$$CM^2-CD^2=OM^2-OD^2. \qquad \text{(e)}$$

Doing (a) − (b) + (c) + (d) − (e), we get $NH^2-MH^2=ON^2-OM^2$. So $OH \perp MN$.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***November 25, 2007.***

Problem 281. Let N be the set of all positive integers. Prove that there exists a function $f : N \to N$ such that $f(f(n)) = n^2$ for all n in N. (*Source: 1978 Romanian Math Olympiad*)

Problem 282. Let a, b, c, A, B, C be real numbers, $a \neq 0$ and $A \neq 0$. For every real number x,

$$|ax^2+bx+c| \leq |Ax^2+Bx+C|.$$

Prove that $|b^2-4ac| \leq |B^2-4AC|$.

Problem 283. P is a point inside ΔABC. Lines AC and BP intersect at Q. Lines AB and CP intersect at R. It is known that $AR=RB=CP$ and $CQ=PQ$. Find $\angle BRC$ with proof. (*Source: 2003 Japanese Math Olympiad*)

Problem 284. Let p be a prime number. Integers x, y, z satisfy $0 < x < y < z < p$. If x^3, y^3, z^3 have the same remainder upon dividing by p, then prove that $x^2+y^2+z^2$ is divisible by $x+y+z$. (*Source: 2003 Polish Math Olympiad*)

Problem 285. Determine the largest positive integer N such that for every way of putting all numbers 1 to 400 into a 20×20 table (1 number per cell), one can always find a row or a column having two numbers with difference not less than N. (*Source: 2003 Russian Math Olympiad*)

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Solutions

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Problem 276. Let n be a positive integer. Given a $(2n-1)\times(2n-1)$ square board with exactly one of the following arrows ↑, ↓, →, ← at each of its cells. A beetle sits in one of the cells. Per year the beetle creeps from one cell to another in accordance with the arrow's direction. When the beetle leaves the cell, the arrow at that cell makes a counterclockwise 90-degree turn. Prove that the beetle leaves the board in at most $2^{3n-1}(n-1)!-3$ years.
(*Source: 2001 Belarussian Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA), **GRA20 Problem Solving Group** (Roma, Italy), **PUN Ying Anna** (HKU, Math Year 1) and **Fai YUNG**.

Let $a(n)$ be the maximum number of years that the beetle takes to leave the $(2n-1)\times(2n-1)$ board. Then $a(1) = 1$. For $n > 1$, apart from 1 year necessary for the final step, the beetle can stay

(1) in each of the 4 corners for at most 2 years (two directions that do not point outside)

(2) in each of the other $4(2n-3)$ cells of the outer frame for at most 3 years (three directions that do not point outside)

(3) in the inner $(2n-3)\times(2n-3)$ board for at most $a(n-1)$ years (when the starting point is inside the inner board) plus $4(2n-3)a(n-1)$ years (when the arrow in a cell of the outer frame points inward the beetle enters the inner board).

Therefore, $a(n) \leq 1 + 4\cdot 2 + 3\cdot 4(2n-3) + (4(2n-3)+1)a(n-1)$. Since $a(n-1) \geq 0$,

$$\begin{aligned} a(n)+3 &\leq 8(n-1)(a(n-1)+3) -3a(n-1) \\ &\leq 8(n-1)(a(n-1)+3). \end{aligned}$$

Since $a(1) + 3 = 4$, we get $a(n) + 3 \leq 2^{3n-1}(n-1)!$ and so $a(n) \leq 2^{3n-1}(n-1)!-3$.

Problem 277. (*Due to Koopa Koo, Univ. of Washington, Seattle, WA, USA*) Prove that the equation

$$x^2 + y^2 + z^2 + 2xyz = 1$$

has infinitely many integer solutions (then try to get all solutions – Editors).

Solution. **Jeff CHEN** (Virginia, USA), **FAN Wai Tong** and **GRA20 Problem Solving Group** (Roma, Italy).

It is readily checked that if n is an integer, then $(x, y, z) = (n, -n, 1)$ is a solution.

Comments: Trying to get all solutions, we can first rewrite the equation as

$$(x^2-1)(y^2-1) = (xy+z)^2.$$

For any solution (x, y, z), we must have $x^2-1 = du^2$, $y^2-1 = dv^2$, $xy+z = \pm duv$ for some integers d, u, v. The cases d is negative, 0 or 1 lead to trivial solutions. For $d > 1$, we may suppose it is square-free (that is, no square divisor greater than 1). Then we can find all solutions of Pell's equation $s^2-dt^2 = 1$ (see *vol.* 6, *no.* 3 of *Math Excalibur*, *page* 1). Any two solutions (s_0, t_0) and (s_1, t_1) of Pell's equation yield a solution $(x, y, z)=(s_0, s_1, \pm dt_0t_1-s_0s_1)$ of

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Commended solvers: **PUN Ying Anna** (HKU, Math Year 1) and **WONG Kam Wing** (TWGH Chong Ming Thien College).

Problem 278. Line segment SA is perpendicular to the plane of the square $ABCD$. Let E be the foot of the perpendicular from A to line segment SB. Let P, Q, R be the midpoints of SD, BD, CD respectively. Let M, N be on line segments PQ, PR respectively. Prove that AE is perpendicular to MN.

Solution 1. **Stephen KIM** (Toronto, Canada).

Below when we write $XY \perp IJK\ldots$, we mean line XY is perpendicular to the plane containing $I, J, K,\ldots$. Also, we write $XY \perp WZ$ for vectors XY and WZ to mean their dot product is 0.

Since $SA \perp ABCD$, so $SA \perp BC$. Since $AB \perp BC$, so $BC \perp SAB$. Since A,E are in the plane of SAB, $AE \perp BC$. This along with the given fact $AE \perp SB$ imply $AE \perp SBC$.

Since P, Q are midpoints of SD, BD respectively, we get $PQ \| SB$. Similarly, we have $QR \| BC$. Then the planes SBC and PQR are parallel. Since MN is on the plane PQR, so MN is parallel to the plane SBC. Since $AE \perp SBC$ from the last paragraph, so $AE \perp MN$.

Solution 2. **Kelvin LEE** (Winchester College, England) and **PUN Ying Anna** (HKU, Math Year 1).

Let A be the origin, AD be the x-axis, AB be the y-axis and AS be the z-axis. Let $B = (0, a, 0)$ and $S = (0, 0, s)$. Then $C = (a, a, 0)$ and $E = (0, rs, ra)$ for some r. The homothety with center D and ratio 2 sends P to S, Q to B and R to C. Let it send M to M' and N to N'. Then M' is on SB, N' is on SC and $M'N' \| MN$. So $M' = (0, 0, s) + (0, a, -s)u = (0, au, s(1-u))$ for some u and $N' = (0, 0, s) + (a, a, -s)v = (av, av, s(1-v))$ for some v. Now the dot product of AE and $M'N'$ is

$$(0, rs, ra)\cdot(av, a(v-u), s(u-v)) = 0.$$

So $AE \perp M'N'$. Therefore, $AE \perp MN$.

Commended solvers: **WONG Kam Wing** (TWGH Chong Ming Thien College).

Problem 279. Let R be the set of all real numbers. Determine (with proof) all functions f: $R \to R$ such that for all real x and y,

$$f(f(x)+y)=2x+f(f(f(y))-x).$$

Solution. **Jeff CHEN** (Virginia, USA), **Salem MALIKIĆ** (Sarajevo College, 3rd Grade, Sarajevo, Bosnia and Herzegovina) and **PUN Ying Anna** (HKU, Math Year 1).

Setting $y = 0$, we get

$$f(f(x)) = 2x + f(f(f(0)) - x). \qquad (1)$$

Then putting $x = 0$ into (1), we get

$$f(f(0)) = f(f(f(0))). \qquad (2)$$

In (1), setting, $x = f(f(0))$, we get

$$f(f(f(f(0)))) = 2f(f(0)) + f(0).$$

Using (2), we get $f(f(0)) = 2f(f(0)) + f(0)$. So $f(f(0)) = -f(0)$. Using (2), we see $f^{(k)}(0) = -f(0)$ for $k = 2,3,4,\ldots$.

In the original equation, setting $x = 0$ and $y = -f(0)$, we get

$$\begin{aligned} f(0) &= -2f(0) + f(f(f(-f(0)))) \\ &= -2f(0) + f^{(5)}(0) \\ &= -2f(0) - f(0) = -3f(0). \end{aligned}$$

So $f(0) = 0$. Then (1) becomes

$$f(f(x)) = 2x + f(-x). \qquad (3)$$

In the original equation, setting $x = 0$, we get $f(y) = f(f(f(y)))$. (4)

Setting $x = f(y)$ in (3), we get

$$f(y) = f(f(f(y))) = 2f(y) + f(-f(y)).$$

So $f(-f(y)) = -f(y)$. Setting $y = -f(x)$ in the original equation, we get

$$0 = 2x + f(f(f(-f(x))) - x).$$

For every real number w, setting $x = -w/2$, we see $w = f(f(f(-f(x))) - x)$. Hence, f is surjective. Then by (4), $w = f(f(w))$ for all w. By (3), setting $x = -w$, we get $f(w) = w$ for all w. Substituting this into the original equation clearly works. So the only solution is $f(w) = w$ for all w.

Commended solvers: **Kelvin LEE** (Winchester College, England),

Problem 280. Let n and k be fixed positive integers. A basket of peanuts is distributed into n piles. We gather the piles and rearrange them into $n + k$ new piles. Prove that at least $k + 1$ peanuts are transferred to smaller piles than the respective original piles that contained them. Also, give an example to show the constant $k + 1$ cannot be improved.

Solution. **Jeff CHEN** (Virginia, USA), **Stephen KIM** (Toronto, Canada) and **PUN Ying Anna** (HKU, Math Year 1).

Before the rearrangement, for each pile, if the pile has m peanuts, then attach a label of $1/m$ to each peanut in the pile. So the total sum of all labels is n.

Assume that only at most k peanuts were put into smaller piles after the rearrangement. Since the number of piles become $n + k$, so there are at least n of these $n + k$ piles, all of its peanuts are now in piles that are larger or as large as piles they were in before the rearrangement. Then the sum of the labels in just these n piles is already at least n. Since there are $k > 0$ more piles, this is a contradiction.

For an example to show $k + 1$ cannot be improved, take the case originally one of the n piles contained $k + 1$ peanuts. Let us rearrange this pile into $k + 1$ piles with 1 peanut each and leave the other $n - 1$ piles alone. Then only these $k + 1$ peanuts go into smaller piles.

Commended solvers: **WONG Kam Wing** (TWGH Chong Ming Thien College).

Olympiad Corner

(continued from page 1)

Problem 3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique in the other room.

Day 2 (*July 26, 2007*)

Problem 4. In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Problem 5. Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

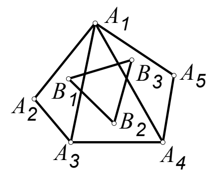
Problem 6. Let n be a positive integer. Consider

$$S = \{(x,y,z) : x,y,z \in \{0,1,\ldots,n\}, x+y+z>0\}$$

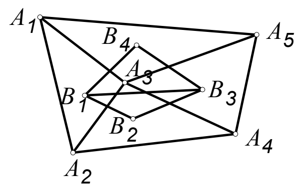
as a set of $(n + 1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

Convex Hull

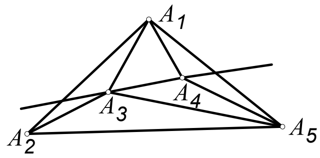
(continued from page 1)



Case 1: (The boundary of H is the pentagon $A_1A_2A_3A_4A_5$.) If $\triangle A_1A_2A_3$ or $\triangle A_1A_3A_4$ or $\triangle A_1A_4A_5$ does not contain any point of B in its interior, then we are done. Otherwise, there exist B_1, B_2, B_3 in their interiors respectively. Then we see $\triangle B_1B_2B_3$ is a desired triangle.



Case 2: (The boundary of H is a quadrilateral, say $A_1A_2A_4A_5$ with A_3 inside.) If $\triangle A_1A_3A_2$ or $\triangle A_2A_3A_4$ or $\triangle A_4A_3A_5$ or $\triangle A_5A_3A_1$ does not contain any point of B in its interior, then we are done. Otherwise, there exist B_1, B_2, B_3, B_4 in their interiors respectively. Then either $\triangle B_1B_2B_3$ or $\triangle B_3B_4B_1$ does not contain A_3 in its interior. That triangle is a desired triangle.



Case 3: (The boundary of H is a triangle, say $A_1A_2A_5$ with A_3, A_4 inside, say A_3 is closer to line A_1A_2 than A_4.) If $\triangle A_1A_2A_3$ or $\triangle A_1A_3A_4$ or $\triangle A_1A_4A_5$ or $\triangle A_2A_3A_5$ or $\triangle A_3A_4A_5$ does not contain any point of B in its interior, then we are done. Otherwise, there exists a point of B in each of their interiors respectively. Then three of these points of B lie on one side of line A_3A_4. The triangle formed by these three points of B is a desired triangle.

Mathematical Excalibur

Volume 12, Number 4 *November 2007 – December 2007*

Olympiad Corner

Below were the problems of the 10th China Hong Kong Math Olympiad, which was held on November 24, 2007. It was a three hour exam.

Problem 1. et D be a point on the side BC of triangle ABC such that $AB+BD = AC+CD$. The line segment AD cut the incircle of triangle ABC at X and Y with X closer to A. Let E be the point of contact of the incircle of triangle ABC on the side BC. Show that

(i) EY is perpendicular to AD,
(ii) XD is $2IA'$, where I is the incentre of the triangle ABC and A' is the midpoint of BC.

Problem 2. Is there a polynomial f of degree 2007 with integer coefficients, such that $f(n)$, $f(f(n))$, $f(f(f(n)))$, … are pairwise relatively prime for every integer n? Justify your claims.

Problem 3. In a school there are 2007 male and 2007 female students. Each student joins not more than 100 clubs in the school. It is known that any two students of opposite genders have joined at least one common club.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 15, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Inequalities with Product Condition

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There are many inequality problems that have n positive variables $a_1, a_2, \ldots, a_n$ (generally $n = 3$) such that their product is 1. There are several ways to solve this kind of problems. One common method is to change these variables by letting

$$a_1=\left(\frac{x_2}{x_1}\right)^{\alpha}, a_2=\left(\frac{x_3}{x_2}\right)^{\alpha}, \cdots, a_n=\left(\frac{x_1}{x_n}\right)^{\alpha},$$

where $x_1, x_2, \ldots, x_n$ are positive real numbers and generally α=1. Here are some examples on the usage of these substitutions.

Example 1. If a, b, c are positive real numbers such that $abc = 1$, then prove that

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\ge\frac{3}{2}.$$

Solution. Since $abc = 1$, we can find positive x, y, z such that $a = x/y$, $b = y/z$, $c = z/x$ (for example, x=1=abc, y=bc and z=c). Then

$$\begin{aligned}&\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\\ &=\frac{x/y}{(x/z)+1}+\frac{y/z}{(y/x)+1}+\frac{z/x}{(z/y)+1}\\ &=\frac{zx}{xy+yz}+\frac{xy}{zx+yz}+\frac{yz}{xy+zx}\ge\frac{3}{2},\end{aligned}$$

where the inequality follows from <u>*Nesbitt's inequality*</u> applied to zx, xy and yz. (*Editor*—Nesbitt's inequality asserts that if $r,s,t > 0$, then

$$\frac{r}{s+t}+\frac{s}{t+r}+\frac{t}{r+s}\ge\frac{3}{2}.$$

It follows by writing the left side as

$$\begin{aligned}&\frac{r+s+t}{s+t}+\frac{r+s+t}{t+r}+\frac{r+s+t}{r+s}-3\\ &=\left(\frac{s+t}{2}+\frac{t+r}{2}+\frac{r+s}{2}\right)\left(\frac{1}{s+t}+\frac{1}{t+r}+\frac{1}{r+s}\right)-3\\ &\ge\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)^2-3=\frac{3}{2},\end{aligned}$$

where the inequality sign is due to the Cauchy-Schwarz inequality.)

Equality occurs if and only if the three variables are equal.

Example 2. (*2004 Russian Math Olympiad*) Prove that if $n > 3$ and $x_1, x_2, \ldots, x_n > 0$ have product 1, then

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\cdots+\frac{1}{1+x_n+x_nx_1}>1.$$

Solution. Again we use the substitutions $x_1 = a_2/a_1$, $x_2 = a_3/a_2$, …, $x_n = a_1/a_n$ (say a_1=1 and for $i > 1$, a_i=$x_1x_2\cdots x_{i-1}$). Then the inequality is equivalent to

$$\begin{aligned}&\frac{a_1}{a_1+a_2+a_3}+\frac{a_2}{a_2+a_3+a_4}+\cdots+\frac{a_n}{a_n+a_1+a_2}\\ &>\sum_{i=1}^{n}\frac{a_i}{a_1+a_2+\cdots+a_n}=1,\end{aligned}$$

where the inequality sign is because $n > 3$ and $a_i > 0$ for all i so that $a_i+a_{i+1}+a_{i+2} < a_1+a_2+\cdots+a_n$.

Example 3. If $a, b, c > 0$ and $abc = 1$, then prove that

$$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Solution. Since $abc = 1$, we can find positive x, y, z such that $a = x/y$, $b = z/x$, $c = y/z$ (for example, $x = 1 = abc$, $y = bc$ and z=b). After doing the substitution, the inequality can be rewritten as

$$3+\frac{x^2}{yz}+\frac{z^2}{xy}+\frac{y^2}{zx}\ge\frac{x}{y}+\frac{z}{x}+\frac{y}{z}+\frac{y}{x}+\frac{x}{z}+\frac{z}{y}.$$

Multiplying by xyz on both sides, we get

$$\begin{aligned}&x^3+y^3+z^3+3xyz\\ &\quad\ge x^2y+xy^2+y^2z+yz^2+z^2x+zx^2,\end{aligned}$$

which is just <u>*Schur's inequality*</u> (see *vol.* 10, *no.* 5, *p.* 2 of <u>*Math Excalibur*</u>). Since x,y,z are positive, equality holds if and only if $x = y = z$, that is $a = b = c$.

Example 4. (*Mathlinks Contest*) Prove that if $a,b,c,d > 0$ and $abcd = 1$, then

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+d)}+\frac{1}{d(1+a)}\ge 2.$$

Solution. Let us perform the following substitutions

$$a=\frac{x}{y}, b=\frac{z}{x}, c=\frac{t}{z}, d=\frac{y}{t}$$

with $x,y,z,t>0$ (for example, $x=1=abcd$, $y=bcd$, $z=b$ and $t=bc$). Then after simple transformations, our inequality becomes

$$\frac{y}{x+z}+\frac{x}{z+t}+\frac{z}{y+t}+\frac{t}{x+y}\ge 2.$$

Let I be the left side of this inequality and

$$J=y(x+z)+x(z+t)+z(y+t)+t(x+y).$$

By the Cauchy-Schwarz inequality, we easily get $IJ\ge(x+y+z+t)^2$. Then

$$\frac{y}{x+z}+\frac{x}{z+t}+\frac{z}{y+t}+\frac{t}{x+y}=I$$
$$\ge\frac{(x+y+z+t)^2}{yx+yz+xz+xt+zy+zt+tx+ty}.$$

So it is enough to prove that

$$\frac{(x+y+z+t)^2}{yx+yz+xz+xt+zy+zt+tx+ty}\ge 2,$$

which is equivalent to

$$x^2+y^2+z^2+t^2\ge 2(yz+xt).$$

This one is equivalent to

$$(x-t)^2+(y-z)^2\ge 0,$$

which is obviously true.

For equality case to occur, we must have $x=t$ and $y=z$, which directly imply $a=c$ and $b=d$ so $ab=1$ and therefore $b=d=1/a=1/c$ is the equality case.

Example 5. (*Crux* 3147) Let $n\ge 3$ and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1x_2\cdots x_n=1$. For $n=3$ and $n=4$ prove that

$$\frac{1}{x_1^2+x_1x_2}+\frac{1}{x_2^2+x_2x_3}+\cdots+\frac{1}{x_n^2+x_nx_1}\ge\frac{n}{2}.$$

Solution. We consider the substitutions

$$x_1=\sqrt{\frac{a_2}{a_1}}, x_2=\sqrt{\frac{a_3}{a_2}}, \ldots, x_n=\sqrt{\frac{a_1}{a_n}}.$$

The inequality becomes

$$\frac{a_1}{a_2+\sqrt{a_1a_3}}+\frac{a_2}{a_3+\sqrt{a_2a_4}}+\cdots+\frac{a_n}{a_1+\sqrt{a_na_2}}\ge\frac{n}{2}.$$

Since

$$\sqrt{a_1a_3}\le\frac{a_1+a_3}{2}, \ldots, \sqrt{a_na_2}\le\frac{a_n+a_2}{2}$$

by the *AM-GM* inequality, it suffices to show that

$$\frac{a_1}{a_1+2a_2+a_3}+\frac{a_2}{a_2+2a_3+a_4}+\cdots+\frac{a_n}{a_n+2a_1+a_2}\ge\frac{n}{4}.$$

Let I be the left side of this inequality and

$$J=a_1(a_1+2a_2+a_3)+\cdots+a_n(a_n+2a_1+a_2).$$

By the Cauchy-Schwarz inequality, we have $IJ\ge(a_1+a_2+\cdots+a_n)^2$. Thus, to prove $I\ge n/4$, it suffices to show that $(a_1+a_2+\cdots+a_n)^2/J\ge n/4$, which is equivalent to

$$4(a_1+a_2+\cdots a_n)^2$$
$$\ge n\big(a_1(a_1+2a_2+a_3)+\cdots+a_n(a_n+2a_1+a_2)\big).$$

For $n=4$, by expansion, we can see the inequality is actually an identity. For $n=3$, the inequality is equivalent to

$$a_1^2+a_2^2+a_3^2\ge a_1a_2+a_2a_3+a_3a_1,$$

which is true because

$$2(a_1^2+a_2^2+a_3^2)-2(a_1a_2+a_2a_3+a_3a_1)$$
$$=(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2$$
$$\ge 0.$$

This completes the proof. Equality holds if and only if $x_i=1$ for all i.

NOTE: This problem appeared in the May 2006 issue of the Crux Mathematicorum. It was proposed by Vasile Cîrtoaje and Gabriel Dospinescu. No complete solution was received (except the above solution of the proposers).

Example 6. (*Crux* 2023) Let a,b,c,d,e be positive real numbers such that $abcde=1$. Prove that

$$\frac{a+abc}{1+ab+abcd}+\frac{b+bcd}{1+bc+bcde}+\frac{c+cde}{1+cd+cdea}$$
$$+\frac{d+dea}{1+de+deab}+\frac{e+eab}{1+ea+eabc}\ge\frac{10}{3}.$$

Solution. Again we consider the standard substitutions

$$a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{t}, d=\frac{t}{u}, e=\frac{u}{x},$$

where $x,y,z,t,u>0$.

Now we have

$$\frac{a+abc}{1+ab+abcd}=\frac{1/y+1/t}{1/x+1/z+1/u}.$$

Writing the other relations and letting

$$a_1=\frac{1}{x}, a_2=\frac{1}{y}, a_3=\frac{1}{z}, a_4=\frac{1}{t}, a_5=\frac{1}{u},$$

we have to show that if $a_1, a_2, a_3, a_4, a_5>0$, then

$$\sum_{cyclic}\frac{a_2+a_4}{a_1+a_3+a_5}\ge\frac{10}{3}. \qquad (*)$$

(*Editor*—The notation

$$\sum_{cyclic}f(a_1,a_2,\ldots,a_n)$$

for n variables $a_1, a_2, \ldots, a_n$ is a shorthand notation for

$$\sum_{i=1}^{n}f(a_i,a_{i+1},\ldots,a_{i+n}),$$

where $a_{i+j}=a_{i+j-n}$ when $i+j>n$.)

Let I be the left side of inequality (*),

$$J=\sum_{cyclic}(a_2+a_4)(a_1+a_3+a_5)$$

and $S=a_1+a_2+a_3+a_4+a_5$. Using the Cauchy-Schwarz inequality, we easily get

$$IJ\ge\left(\sum_{cyclic}(a_2+a_4)\right)^2=(2S)^2=4S^2.$$

So to prove $I\ge 10/3$, it is enough to show

$$\frac{4S^2}{J}\ge\frac{10}{3}. \qquad (**)$$

Now comparing S^2 and J, we can observe that $2S^2-J$ equals

$$T=(a_2+a_4)^2+(a_1+a_4)^2+(a_3+a_5)^2$$
$$+(a_2+a_5)^2+(a_1+a_3)^2.$$

Using this relation, (**) can be rewritten as

$$12S^2\ge 10J=10(2S^2-T)$$
$$=20S^2-10T.$$

This simplifies to $5T\ge 4S^2$. Finally, writing $5=1^2+1^2+1^2+1^2+1^2$, we can get $5T\ge 4S^2$ from the Cauchy-Schwarz inequality easily.

Again equality occurs if and only if all the a_i's are equal, which corresponds to the case $a=b=c=d=e=1$.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***January 15, 2008.***

Problem 286. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that there exists a real number y such that the sum of $\{x_1-y\}, \{x_2-y\}, \ldots, \{x_n-y\}$ is at most $(n-1)/2$. (Here $\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to x.)

Can y always be chosen to be one of the x_i's ?

Problem 287. Determine (with proof) all nonempty subsets A, B, C of the set of all positive integers $\mathbb{Z}^+$ satisfying

(1) $A\cap B=B\cap C=C\cap A=\varnothing$;
(2) $A\cup B\cup C=\mathbb{Z}^+$;
(3) for every $a\in A$, $b\in B$ and $c\in C$, we have $c+a\in A$, $b+c\in B$ and $a+b\in C$.

Problem 288. Let H be the orthocenter of triangle ABC. Let P be a point in the plane of the triangle such that P is different from A, B, C.

Let L, M, N be the feet of the perpendiculars from H to lines PA, PB, PC respectively. Let X, Y, Z be the intersection points of lines LH, MH, NH with lines BC, CA, AB respectively.

Prove that X, Y, Z are on a line perpendicular to line PH.

Problem 289. Let a and b be positive numbers such that $a+b<1$. Prove that

$$\frac{(a-1)^2+b(2a-b)}{(b-1)^2+a(2b-a)} \geq \min\left\{\frac{a}{b},\frac{b}{a}\right\}.$$

Problem 290. Prove that for every integer a greater than 2, there exist infinitely many positive integers n such that a^n-1 is divisible by n.

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Solutions

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Due to an editorial mistake in the last issue, solution to problems 279 by **Li ZHOU** (Polk Community College, Winter Haven, Florida USA) was overlooked and his name was not listed among the solvers. We express our apology to him.

Problem 281. Let N be the set of all positive integers. Prove that there exists a function $f: N\to N$ such that $f(f(n)) = n^2$ for all n in N. (*Source: 1978 Romanian Math Olympiad*)

***Solution* 1. George Scott ALDA, Jeff CHEN** (Virginia, USA), **NGOO Hung Wing** (HKUST, Math Year 1), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7) and **Fai YUNG**.

Let x_k be the k-th term of the sequence

2,3,5,6,7,8,10,11,12,13,14,15,17,18,19,…

of all positive integers that are not perfect squares in increasing order. By taking square roots (repeatedly) of an integer $n>1$, we will eventually get to one of the x_k's. So every integer $n>1$ is of the 2^m-th power of x_k for some nonnegative integer m and positive integer k.

We define $f(1)=1$. For $n>1$, if n is the 2^m-th power of x_k, then we define $f(n)$ as follow:

case 1: if k is odd, then $f(n)$ is the 2^m-th power of x_{k+1};

case 2: if k is even, then $f(n)$ is the 2^{m+1}-st power of x_{k-1}.

Observe that if n is under case 1, then $f(n)$ will be under case 2. Similarly, if n is under case 2, then $f(n)$ will be under case 1. In computing $f(f(n))$, we have to apply case 2 once so that m increases by 1 and the k value goes up once and down once. Therefore, we have $f(f(n)) = n^2$ for all n in N.

***Solution 2.* GRA20 Problem Solving Group** (Roma, Italy) and **Kelvin LEE** (Trinity College, Cambridge, England).

We first define a function $g: N\to N$ such that $g(g(n)) = 2n$. Let p be an odd prime and let $\mathrm{ord}_p(n)$ be the greatest nonnegative integer α such that $p^\alpha \mid n$. If $\mathrm{ord}_p(n)$ is even, then let $g(n)=2pn$, otherwise let $g(n)=n/p$.

Next we will check $g(g(n)) = 2n$. If $\mathrm{ord}_p(n)$ is even, then $\mathrm{ord}_p(g(n)) = \mathrm{ord}_p(2pn)$ is odd and so $g(g(n)) = g(2pn) = 2pn/p = 2n$.

If $\mathrm{ord}_p(n)$ is odd, then $\mathrm{ord}_p(g(n)) = \mathrm{ord}_p(n/p)$ is even and so $g(g(n)) = g(n/p) = 2p(n/p) = 2n$.

Define $f(1)=1$. For an integer $n>1$, let

$$n=\prod_{k=1}^{r} p_k^{\alpha_k}\text{ , where all }\alpha_k>0,$$

be the prime factorization of n, then we define

$$f(n)=\prod_{k=1}^{r} p_k^{g(\alpha_k)}.$$

Finally, we have

$$f(f(n))=\prod_{k=1}^{r} p_k^{g(g(\alpha_k))}=\prod_{k=1}^{n} p_k^{2\alpha_k}=n^2.$$

Problem 282. Let a, b, c, A, B, C be real numbers, $a\neq 0$ and $A\neq 0$. For every real number x,

$$|ax^2+bx+c| \leq |Ax^2+Bx+C|.$$

Prove that $|b^2-4ac| \leq |B^2-4AC|$. (*Source: 2003 Putnam Exam*)

***Solution.* Samuel Liló ABDALLA** (ITA, São Paulo, Brazil), **Jeff CHEN** (Virginia, USA), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7).

We have

$$|a|=\lim_{x\to\infty}\frac{|ax^2+bx+c|}{x^2}\leq\lim_{x\to\infty}\frac{|Ax^2+Bx+C|}{x^2}=|A|.$$

If $B^2-4AC>0$, then $Ax^2+Bx+C=0$ has two distinct real roots x_0 and x_1. By the given inequality, these will also be roots of $ax^2+bx+c=0$. So $b^2-4ac>0$. Then

$$|B^2-4AC|=A^2(x_0-x_1)^2$$
$$\geq a^2(x_0-x_1)^2=|b^2-4ac|.$$

If $B^2-4AC\leq 0$, then by replacing A by $-A$ or a by $-a$ if necessary, we may assume $A\geq a>0$. Since $A>0$ and $B^2-4AC\leq 0$, so for every real number x, $Ax^2+Bx+C\geq 0$. Then the given inequality implies for every real x,

$$Ax^2+Bx+C\geq \pm(ax^2+bx+c). \quad (*)$$

Then $(A-a)x^2+(B-b)x+(C-c)\geq 0$. This implies

$$(B-b)^2\leq 4(A-a)(C-c). \quad (**)$$

Similarly,

$$(B+b)^2\leq 4(A+a)(C+c). \quad (***)$$

Then

$$(B^2-b^2)^2\leq 16(A^2-a^2)(C^2-c^2)$$
$$\leq 16(AC-ac)^2,$$

which implies $B^2-b^2\leq 4|AC-ac|$.

Taking $x=0$ in (*), we get $C\geq |c|$. Since $A\geq a>0$, we get $B^2-b^2\leq$

$4(AC-ac)$. Hence,

$$4ac - b^2 \le 4AC - B^2. \qquad (\dagger)$$

Using (**) and (***), we have

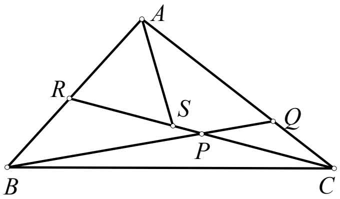
$$\begin{aligned} B^2+b^2 &= \frac{(B-b)^2+(B+b)^2}{2} \\ &\le 2\big((A-a)(C-c)+(A+a)(C+c)\big) \\ &= 4(AC+ac). \end{aligned}$$

Then $-(4ac - b^2) \le 4AC - B^2$. Along with (†), we have

$$\begin{aligned} |b^2-4ac| &= \pm(4ac - b^2) \\ &\le 4AC - B^2 = |B^2-4AC|. \end{aligned}$$

Problem 283. P is a point inside $\triangle ABC$. Lines AC and BP intersect at Q. Lines AB and CP intersect at R. It is known that $AR=RB=CP$ and $CQ=PQ$. Find $\angle BRC$ with proof. (*Source: 2003 Japanese Math Olympiad*)

Solution. **Stephen KIM** (Toronto, Canada).



Let S be the point on segment CR such that $RS=CP=AR$. Since $CQ=PQ$, we have

$$\angle ACS = \angle QPC = \angle BPR.$$

Also, since $RS=CP$, we have

$$SC = CR - RS = CR - CP = RP.$$

Considering line CR cutting $\triangle ABQ$, by Menelaus' theorem, we have

$$\frac{RB}{AR}\cdot\frac{PQ}{BP}\cdot\frac{AC}{CQ} = 1.$$

Since $AR=RB$ and $CQ=PQ$, we get $AC = BP$. Hence, $\triangle ACS \cong \triangle BPR$. Then $AS = BR = AR = CP = RS$ and so $\triangle ARS$ is equilateral. Therefore, $\angle BRC=120°$.

Commended solvers: **FOK Pak Hei** (Pui Ching Middle School, Form 6), **Kelvin LEE** (Trinity College, Cambridge, England), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), **NG Ngai Fung** (STFA Leung Kau Kui College, Form 5) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7).

Problem 284. Let p be a prime number. Integers x, y, z satisfy $0 < x < y < z < p$. If x^3, y^3, z^3 have the same remainder upon dividing by p, then prove that $x^2+y^2+z^2$ is divisible by $x+y+z$. (*Source: 2003 Polish Math Olympiad*)

Solution. **George Scott ALDA, José Luis DÍAZ-BARRERO** (Universitat Politècnica de Catalunya, Barcelona, Spain), **EZZAKI Mahmoud** (Omar Ibn Abdelaziz, Morocco), **Stephen KIM** (Toronto, Canada), **Kelvin LEE** (Trinity College, Cambridge, England) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Since $x^3 \equiv y^3 \equiv z^3 \pmod{p}$, so

$$p \mid x^3-y^3 = (x-y)(x^2+xy+y^2).$$

Since $0 < x < y < z < p$ and p is prime, we have $p \nmid x-y$ and hence

$$p \mid x^2+xy+y^2. \qquad (1)$$

Similarly,

$$p \mid y^2+yz+z^2 \qquad (2)$$

and

$$p \mid z^2+zx+x^2. \qquad (3)$$

By (1) and (2), p divides

$$(x^2+xy+y^2)-(y^2+yz+z^2)=(x-z)(x+y+z).$$

Since $0 < z-x < p$, we have $p \mid x+y+z$.

Also, $0 < x < y < z < p$ implies $x+y+z = p$ or $2p$ and $p > 3$. Now

$$x+y+z \equiv x^2+y^2+z^2 \pmod{2}.$$

Thus, it remains to show $p \mid x^2+y^2+z^2$.

Now $x^2+xy+y^2 = x(x+y+z)+ y^2-xz$. From (1), we get

$$p \mid y^2-xz. \qquad (4)$$

Similarly,

$$p \mid x^2-zy \qquad (5)$$

and

$$p \mid z^2-yx. \qquad (6)$$

Adding the right sides of (1) to (6), we get

$$p \mid 3(x^2+y^2+z^2).$$

Since $p > 3$ is prime, we get $p \mid x^2+y^2+z^2$ as desired.

Commended solvers: **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 7).

Problem 285. Determine the largest positive integer N such that for every way of putting all numbers 1 to 400 into a 20×20 table (1 number per cell), one can always find a row or a column having two numbers with difference not less than N. (*Source: 2003 Russian Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA) and **Stephen KIM** (Toronto, Canada).

The answer is 209. We first show $N \le 209$. Divide the table into a left and a right half, each of dimension 20×10. Put 1 to 200 row wise in increasing order into the left half. Similarly, put 201 to 400 row wise in increasing order into the right half. Then the difference of two numbers in the same row is at most $210-1=209$ and the difference of two numbers in the same column is at most $191-1=190$. So $N \le 209$.

Next we will show $N \ge 209$. Let $M_1 = \{1,2,\ldots,91\}$ and $M_2 = \{300, 301, \ldots, 400\}$.

Color a row or a column <u>red</u> if and only if it contains a number in M_1. Similarly, color a row or a column <u>blue</u> if and only if it contains a number in M_2. We claim that

(1) the number of red rows plus the number of red columns is at least 20 and
(2) the number of blue rows plus the number of blue columns is at least 21.

Hence, there is a row or a column that is colored red and blue. So two of the numbers in that row or column have a difference of at least $300-91=209$.

For claim (1), let there be i red rows and j red columns. Since the numbers in M_1 can only be located at the intersections of these red rows and columns, we have $ij \ge 91$. By the *AM-GM* inequality,

$$i + j \ge 2\sqrt{ij} \ge 2\sqrt{91} > 19.$$

Similarly, claim (2) follows from the facts that there are 101 numbers in M_2 and $2\sqrt{101} > 20$.

Olympiad Corner

(continued from page 1)

Problem 3. (*Cont.*) Show that there is a club with at least 11 male and 11 female members.

Problem 4. Determine if there exist positive integer pairs (m,n), such that

(i) the greatest common divisor of m and n is 1, and $m \le 2007$,

(ii) for any k=1,2, …, 2007,

$$\left[\frac{nk}{m}\right] = \left[\sqrt{2}\,k\right].$$

(Here $[x]$ stands for the greatest integer less than or equal to x.)

Mathematical Excalibur

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Olympiad Corner

Below were the problems of the 2007Estonian IMO Team Selection Contest.

First Day

Problem 1. On the control board of a nuclear station, there are n electric switches ($n > 0$), all in one row. Each switch has two possible positions: up and down. The switches are connected to each other in such a way that, whenever a switch moves down from its upper position, its right neighbor (if it exists) automatically changes position. At the beginning, all switches are down. The operator of the board first changes the position of the leftmost switch once, then the position of the second leftmost switch twice etc., until eventually he changes the position of the rightmost switch n times. How many switches are up after all these operations?

Problem 2. Let D be the foot of the altitude of triangle ABC drawn from vertex A. Let E and F be points symmetric to D with respect to lines AB and AC, respectively. Let R_1 and R_2 be

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 25, 2008***.

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Square It!

Pham Van Thuan
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Inequalities involving square roots of the form

$$\sqrt{A}+\sqrt{B}+\sqrt{C}+\sqrt{D}\le k$$

can be solved using the Cauchy-Schwarz inequality. However, solving inequalities of the following form

$$\sqrt{A}+\sqrt{B}+\sqrt{C}+\sqrt{D}\ge k$$

is far from straightforward. In this article, we will look at such problems. We will solve them by squaring and making more delicate use of the Cauchy-Schwarz inequality.

Example 1. Three nonnegative real numbers x, y and z satisfy $x^2+y^2+z^2=1$. Prove that

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2}+\sqrt{1-\left(\frac{y+z}{2}\right)^2}+\sqrt{1-\left(\frac{z+x}{2}\right)^2}\ge\sqrt{6}.$$

Solution. Squaring both sides of the inequality and simplifying, we get the equivalent inequality

$$\sum_{cyclic}\sqrt{1-\left(\frac{x+y}{2}\right)^2}\sqrt{1-\left(\frac{y+z}{2}\right)^2}\ge\frac{7}{4}+\frac{xy+yz+zx}{4},$$

where

$$\sum_{cyclic}f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y).$$

Notice that

$$1-\left(\frac{x+y}{2}\right)^2=\frac{x^2+y^2+(z^2+1)}{2}-\left(\frac{x+y}{2}\right)^2=\frac{(x-y)^2}{4}+\frac{z^2+1}{2}.$$

By the Cauchy-Schwarz inequality,

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2}\sqrt{1-\left(\frac{y+z}{2}\right)^2}$$
$$\ge\frac{(x-y)(z-y)}{4}+\frac{\sqrt{(z^2+1)(x^2+1)}}{2}$$
$$\ge\frac{y^2+xz-yz-xy}{4}+\frac{zx+1}{2}.$$

Similarly, we obtain two other such inequalities. Multiplying each of them by 2, adding them together, simplifying and finally using $x^2+y^2+z^2=1$, we get the equivalent inequality in the beginning of this solution.

Example 2. For a, b, $c > 0$, prove that

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\ge2\sqrt{1+\frac{abc}{(a+b)(b+c)(c+a)}}.$$

Solution. Multiplying both sides by $\sqrt{(a+b)(b+c)(c+a)}$, we have to show

$$\sum_{cyclic}\sqrt{a(c+a)(a+b)}\ge2\sqrt{(a+b+c)(ab+bc+ca)}.$$

Squaring both sides, we get the equivalent inequality

$$\sum_{cyclic}a^3+2\sum_{cyclic}(a+b)\sqrt{ab(a+c)(b+c)}\ge3\sum_{cyclic}ab(a+b)+9abc. \qquad (*)$$

By the Cauchy-Schwarz inequality and the AM-GM inequality, we have

$$\begin{aligned}&(a+b)\sqrt{ab(a+c)(b+c)}\\&\ge(a+b)\sqrt{ab(\sqrt{ab}+c)^2}\\&=(a+b)(\sqrt{ab}+c)\sqrt{ab}\\&=ab(a+b)+(a+b)c\sqrt{ab}\\&\ge ab(a+b)+2abc.\end{aligned}$$

Using this, we have

$$\sum_{cyclic}a^3+2\sum_{cyclic}(a+b)\sqrt{ab(c+a)(c+b)}\ge\sum_{cyclic}a^3+2\sum_{cyclic}ab(a+b)+12abc.$$

Comparing with (*), we need to show

$$\sum_{cyclic}a^3-\sum_{cyclic}ab(a+b)+3abc\ge0.$$

This is just Schur's inequality

$$\sum_{cyclic}a(a-b)(a-c)\ge0.$$

(See <u>*Math. Excalibur*</u>, vol.10, no.5, p.2)

From the last example, we saw that other than the Cauchy-Schwarz inequality, we might need to recall Schur's inequality

$$\sum_{cyclic} x^r(x-y)(x-z) \ge 0.$$

Here we will also point out a common variant of Schur's inequality, namely

$$\sum_{cyclic} x^r(y+z)(x-y)(x-z) \ge 0.$$

This variant can be proved in the same way as Schur's inequality (again see *Math. Excalibur*, vol.10, no.5, p.2). Both inequalities become equality if and only if either the variables are all equal or one of them is zero, while the other two are equal. In the next two examples, we will use these.

Example 3. Let a, b, c be nonnegative real numbers such that $a + b + c = 1$. Prove that

$$\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2} \ge \sqrt{3}.$$

When does equality occur?

Solution. Squaring both sides of the inequality and using

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 1-2(ab+bc+ca),$$

we get the equivalent inequality

$$\sum_{cyclic} \sqrt{a+(b-c)^2}\sqrt{b+(c-a)^2} \ge 3(ab+bc+ca).$$

By the Cauchy-Schwarz inequality,

$$\sqrt{a+(b-c)^2}\sqrt{b+(c-a)^2}$$
$$= \sqrt{(b-c)^2+(a+b+c)a}\sqrt{(c-a)^2+(a+b+c)b}$$
$$\ge |(b-c)(c-a)| + (a+b+c)\sqrt{ab}.$$

Similarly, we can obtain two other such inequalities. Adding them together, the right side is

$$\sum_{cyclic} |(b-c)(c-a)| + (a+b+c)\sum_{cyclic}\sqrt{ab}.$$

By the triangle inequality and the case $r = 0$ of Schur's inequality, we get

$$\begin{aligned} \sum_{cyclic} |(b-c)(c-a)| &\ge \left|\sum_{cyclic}(b-c)(c-a)\right| \\ &= \sum_{cyclic}(c-b)(c-a) \\ &= (a^2+b^2+c^2)-(ab+bc+ca). \end{aligned} \qquad (**)$$

Thus, to finish, it will be enough to show

$$a^2+b^2+c^2+(a+b+c)(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \ge 4(ab+bc+ca).$$

Now we make the substitutions

$$x=\sqrt{a}, \quad y=\sqrt{b} \quad \text{and} \quad z=\sqrt{c}.$$

In terms of x, y, z, the last inequality becomes

$$\sum_{cyclic}(x^4+x^3y+x^3z+x^2yz-4x^2y^2) \ge 0. \qquad (***)$$

Since the terms are of degree 4, we consider the case $r = 2$ of Schur's inequality, which is

$$\sum_{cyclic} x^2(x-y)(x-z)$$
$$= \sum_{cyclic}(x^4-x^3y-x^3z+x^2yz) \ge 0.$$

This is not quite equal to (***). So next (due to degree 4 consideration again), we will look at the case $r = 1$ of the variant

$$\sum_{cyclic} x(y+z)(x-y)(x-z)$$
$$= \sum_{cyclic}(x^3y+x^3z-2x^2y^2) \ge 0.$$

Readily we see (***) is just the sum of Schur's inequality with twice its variant.

Finally, tracing back, we see equality occurs if and only if $a = b = c = 1/3$ or one of them is 0, while the other two are equal to 1/2.

Example 4. Three nonnegative real numbers a, b, c satisfy $a + b + c = 2$. Prove that

$$\sqrt{\frac{a+b}{2}-ab}+\sqrt{\frac{b+c}{2}-bc}+\sqrt{\frac{c+a}{2}-ca} \ge \sqrt{2}.$$

Solution. Squaring both sides of the inequality and using $a + b + c = 2$, we get the equivalent inequality

$$\sum_{cyclic}\sqrt{\left(\frac{a+b}{2}-ab\right)\left(\frac{b+c}{2}-bc\right)} \ge \frac{ab+bc+ca}{2}.$$

Note that

$$\frac{a+b}{2}-ab = \frac{2(a+b)-(a+b)^2+(a-b)^2}{4}$$
$$= \frac{(a-b)^2+(2-a-b)(a+b)}{4}$$
$$= \frac{(a-b)^2}{4}+\frac{c(a+b)}{4}.$$

Applying twice the Cauchy-Schwarz inequality, we have

$$\sqrt{\left(\frac{a+b}{2}-ab\right)\left(\frac{b+c}{2}-bc\right)}$$
$$\ge \frac{|(a-b)(b-c)|}{4}+\frac{\sqrt{ca(a+b)(b+c)}}{4}$$
$$\ge \frac{1}{4}\left(|(a-b)(b-c)|+\sqrt{ca(b+\sqrt{ca})^2}\right)$$
$$= \frac{1}{4}\left(|(a-b)(b-c)|+\sqrt{abc}\sqrt{b}+ca\right).$$

Similarly, we can obtain two other such inequalities. Adding them together and using (**) in example 3, we get

$$4\sum_{cyclic}\sqrt{\left(\frac{a+b}{2}-ab\right)\left(\frac{b+c}{2}-bc\right)}$$
$$\ge \sum_{cyclic}\left(|(a-b)(b-c)|\right)+\sqrt{abc}\sum_{cyclic}\sqrt{b}+\sum_{cyclic}ca$$
$$\ge a^2+b^2+c^2+\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Substituting

$$x=\sqrt{a}, \quad y=\sqrt{b} \quad \text{and} \quad z=\sqrt{c}$$

and using Schur's inequality and its variant, we have

$$a^2+b^2+c^2+\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})$$
$$= x^4+y^4+z^4+x^2yz+xy^2z+xyz^2$$
$$\ge \sum_{cyclic}\left(x^3y+x^3z\right)$$
$$\ge 2\sum_{cyclic}x^2y^2 = 2(ab+bc+ca).$$

Combining this with the last displayed inequalities, we can obtain the equivalent inequality in the beginning of this solution.

To conclude this article, we will give two exercises for the readers to practice.

Exercise 1. Three nonnegative real numbers x, y and z satisfy $x^2+y^2+z^2=1$. Prove that

$$\sum_{cyclic}\sqrt{1-xy}\sqrt{1-yz} \ge 2.$$

Exercise 2. Three nonnegative real numbers x, y and z satisfy $x+y+z=1$. Prove that

$$x\sqrt{1-yz}+y\sqrt{1-zx}+z\sqrt{1-xy} \ge \frac{2\sqrt{2}}{3}.$$

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***February 25, 2008.***

Problem 291. Prove that if a convex polygon lies in the interior of another convex polygon, then the perimeter of the inner polygon is less than the perimeter of the outer polygon.

Problem 292. Let $k_1 < k_2 < k_3 < \cdots$ be positive integers with no two of them are consecutive. For every m = 1, 2, 3, …, let $S_m = k_1+k_2+\cdots+k_m$. Prove that for every positive integer n, the interval $[S_n, S_{n+1})$ contains at least one perfect square number.
(*Source: 1996 Shanghai Math Contest*)

Problem 293. Let CH be the altitude of triangle ABC with $\angle ACB = 90°$. The bisector of $\angle BAC$ intersects CH, CB at P, M respectively. The bisector of $\angle ABC$ intersects CH, CA at Q, N respectively. Prove that the line passing through the midpoints of PM and QN is parallel to line AB.

Problem 294. For three nonnegative real numbers x, y, z satisfying the condition $xy + yz + zx = 3$, prove that

$$x^2 + y^2 + z^2 + 3xyz \geq 6.$$

Problem 295. There are $2n$ distinct points in space, where $n \geq 2$. No four of them are on the same plane. If $n^2 + 1$ pairs of them are connected by line segments, then prove that there are at least n distinct triangles formed.
(*Source: 1989 Chinese IMO team training problem*)

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Solutions

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Problem 286. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that there exists a real number y such that the sum of $\{x_1-y\}, \{x_2-y\}, \ldots, \{x_n-y\}$ is at most $(n-1)/2$.

(Here $\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to x.)

Can y always be chosen to be one of the x_i's ?

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **HO Kin Fai** (HKUST, Math Year 3), **Anna Ying PUN** (HKU, Math Year 2), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **Fai YUNG**.

For i = 1, 2, …, n, let

$$S_i = \sum_{j=1}^{n} \{x_j - x_i\}.$$

For all real x, $\{x\} + \{-x\} \leq 1$ (since the left side equals 0 if x is an integer and equals 1 otherwise). Using this, we have

$$\sum_{i=1}^{n} S_i = \sum_{1 \leq i < j \leq n} (\{x_j - x_i\} + \{x_i - x_j\})$$

$$\leq \sum_{1 \leq i < j \leq n} 1 = \frac{n(n-1)}{2}.$$

So the average value of S_i is at most $(n-1)/2$. Therefore, there exists some $y = x_i$ such that S_i is at most $(n-1)/2$.

Problem 287. Determine (with proof) all nonempty subsets A, B, C of the set of all positive integers $\mathbb{Z}^+$ satisfying

(1) $A \cap B = B \cap C = C \cap A = \varnothing$;
(2) $A \cup B \cup C = \mathbb{Z}^+$;
(3) for every $a \in A$, $b \in B$ and $c \in C$, we have $c + a \in A$, $b + c \in B$ and $a + b \in C$.

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **HO Kin Fai** (HKUST, Math Year 3), **Anna Ying PUN** (HKU, Math Year 2), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **Fai YUNG**.

Let the minimal element of C be x. Then $\{1, 2, \ldots, x-1\} \subseteq A \cup B$. Since for every $a \in A$, $b \in B$, we have $x + a \in A$, $b + x \in B$. So all numbers not divisible by x are in $A \cup B$. Then every $c \in C$ is a multiple of x. By (3), the sum of every $a \in A$ and $b \in B$ is a multiple of x.

Assume $x = 1$. Then $a \in A$, $b \in B$ imply $a+1 \in A$, $b + 1 \in B$, which lead to $a + b \in A \cap B$ contradicting (1).

Assume $x = 2$. We may suppose $1 \in A$. Then by (3), all odd positive integers are in A. For $b \in B$, we get $1 + b \in C$. Then b is odd, which lead to $b \in A \cap B$ contradicting (1).

Assume $x \geq 4$. Then $\{1,2,3\} \subseteq A \cup B$, say $y, z \in \{1,2,3\} \cap A$. Taking a $b \in B$, we get $y+b$, $z+b \in C$ by (3). Then $(y+b) - (z+b) = y - z$ is a multiple of x. But $|y - z| < x$ leads to a contradiction.

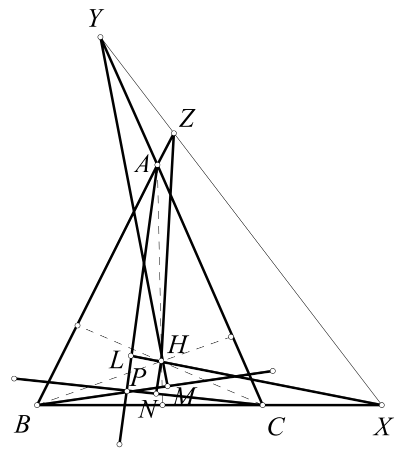
Therefore, $x = 3$. We claim 1 and 2 cannot both be in A (or both in B). If $1, 2 \in A$, then (3) implies $3k + 1$, $3k + 2 \in A$ for all $k \in \mathbb{Z}^+$. Taking a $b \in B$, we get $1 + b \in C$, which implies $b = 3k + 2 \in A$. Then $b \in A \cap B$ contradicts (1).

Therefore, either $1 \in A$ and $2 \in B$ (which lead to $A = \{1,4,7,\ldots\}$, $B = \{2,5,8,\ldots\}$, $C = \{3,6,9,\ldots\}$) or $2 \in A$ and $1 \in B$ (which similarly lead to $A = \{2,5,8,\ldots\}$, $B = \{1,4,7,\ldots\}$, $C = \{3,6,9,\ldots\}$).

Problem 288. Let H be the orthocenter of triangle ABC. Let P be a point in the plane of the triangle such that P is different from A, B, C.

Let L, M, N be the feet of the perpendiculars from H to lines PA, PB, PC respectively. Let X, Y, Z be the intersection points of lines LH, MH, NH with lines BC, CA, AB respectively.

Prove that X, Y, Z are on a line perpendicular to line PH.



Solution 1. **Jeff CHEN** (Virginia, USA) and **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England).

Since $XH = LH \perp PA$, $AH \perp CB = XB$, $BH \perp AC = AY$ and $YH = MH \perp BP$, we have respectively (see Math. Excalibur, vol.12, no.3, p.2)

$$XP^2 - XA^2 = HP^2 - HA^2 \quad (1)$$
$$AX^2 - AB^2 = HX^2 - HB^2 \quad (2)$$
$$BA^2 - BY^2 = HA^2 - HY^2 \quad (3)$$
$$YB^2 - YP^2 = HB^2 - HP^2 \quad (4)$$

Doing (1)+(2)+(3)+(4), we get

$$XP^2 - YP^2 = XH^2 - YH^2,$$

which implies $XY \perp PH$. Similarly, $ZY \perp PH$. So, X, Y, Z are on a line perpendicular to line PH.

Solution 2. **Anna Ying PUN** (HKU, Math Year 2) and **Stephen KIM** (Toronto, Canada).

Set the origin of the coordinate plane at H. For a point J, let (x_J, y_J) denote its coordinates. Since the slope of line PA is $(y_P - y_A)/(x_P - x_A)$, the equation of line HL is

$$(x_P - x_A)x + (y_P - y_A)y = 0. \qquad (1)$$

Since the slope of line HA is y_A/x_A, the equation of line BC is

$$x_A x + y_A y = x_A x_B + y_A y_B. \qquad (2)$$

Let $t = x_A x_B + y_A y_B$. Since point C is on line BC, we get $x_A x_C + y_A y_C = x_A x_B + y_A y_B = t$. Similarly, $x_B x_C + y_B y_C = t$.

Since X is the intersection of lines BC and HL, so the coordinates of X satisfy the sum of equations (1) and (2), that is

$$x_P x + y_P y = t.$$

(Since the slope of line PH is y_P/x_P, this is the equation of a line that is perpendicular to line PH.) Similarly, the coordinates of Y and Z satisfy $x_P x + y_P y = t$. Therefore, X, Y, Z lie on a line perpendicular to line PH.

Commended solvers: **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Problem 289. Let a and b be positive numbers such that $a + b < 1$. Prove that

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \geq \min\left\{\frac{a}{b}, \frac{b}{a}\right\}.$$

Solution. **Samuel Liló ABDALLA** (ITA, São Paulo, Brazil), **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Anna Ying PUN** (HKU, Math Year 2), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), **Simon YAU Chi Keung** (City University of Hong Kong) and **Fai YUNG**.

Since $0 < a,\ b < a + b < 1$, we have

$$(b-1)^2 + a(2b-a) = b^2 + 2(a-1)b - a^2 + 1$$
$$= (b+a-1)^2 + 2a(1-a) > 0.$$

In case $a \geq b > 0$, we have

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \geq \min\left\{\frac{a}{b}, \frac{b}{a}\right\} = \frac{b}{a}$$

$$\Leftrightarrow\ a(a-1)^2 + ab(2a-b) \geq b(b-1)^2 + ab(2b-a)$$

$$\Leftrightarrow\ (a-b)[(a+b-1)^2 + 2ab] \geq 0,$$

which is true. In case $b > a > 0$, we have

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \geq \min\left\{\frac{a}{b}, \frac{b}{a}\right\} = \frac{a}{b}$$

$$\Leftrightarrow\ b(a-1)^2 + b^2(2a-b) \geq a(b-1)^2 + a^2(2b-a)$$

$$\Leftrightarrow\ (b-a)(1 - a^2 - b^2) \geq 0,$$

which is also true as $a^2 + b^2 < a + b < 1$.

Problem 290. Prove that for every integer a greater than 2, there exist infinitely many positive integers n such that $a^n - 1$ is divisible by n.

Solution 1. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **GRA20 Problem Solving Group** (Roma, Italy) and **HO Kin Fai** (HKUST, Math Year 3).

We will show by math induction that $n = (a-1)^k$ for $k = 1, 2, 3, \ldots$ satisfy the requirement. For $k = 1$, since $a - 1 > 1$ and $a \equiv 1 \pmod{a-1}$, so

$$a^{a-1} - 1 \equiv 1^{a-1} - 1 = 0 \pmod{a-1}.$$

Next, suppose case k is true. Then $a^{(a-1)^k} - 1$ is divisible by $(a-1)^k$. For the case $k + 1$, all we need to show is

$$\frac{a^{(a-1)^{k+1}} - 1}{a^{(a-1)^k} - 1} \equiv 0 \pmod{a-1}.$$

Note $b = a^{(a-1)^k} \equiv 1 \pmod{a-1}$. The left side of the above displayed congruence is

$$\frac{b^{a-1} - 1}{b - 1} = \sum_{k=0}^{a-2} b^k \equiv \sum_{k=0}^{a-2} 1 = a - 1 \equiv 0 \pmod{a-1}.$$

This completes the induction.

Solution 2. **Anna Ying PUN** (HKU, Math Year 2) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Note $n = 1$ works. We will show if n works, then $a^n - 1 (> 2^n - 1 \geq n)$ also works. If n works, then $a^n - 1 = nk$ for some positive integer k. Then

$$a^{a^n - 1} - 1 = a^{nk} - 1 = (a^n - 1)\sum_{j=0}^{k-1} a^{nj},$$

which shows $a^n - 1$ works.

Comments: Cheung Wang Chi pointed out that interestingly $n = 1$ is the only positive integer such that <u>$2^n - 1$ is divisible by n</u> (denote this by $n \mid 2^n - 1$). [This fact appeared in the 1972 Putnam Exam.-*Ed*.] To see this, he considered a minimal $n > 1$ such that $n \mid 2^n - 1$. He showed if $a, b, q \in \mathbb{Z}^+$ and $a = bq + r$ with $0 \leq r < b$, then $2^a - 1 = ((2^b)^q - 1)2^r + (2^r - 1) = (2^b - 1)N + (2^r - 1)$ for some $N \in \mathbb{Z}^+$. Hence,

$$\gcd(2^a - 1, 2^b - 1) = \gcd(2^b - 1, 2^r - 1)$$
$$= \cdots = 2^{\gcd(a,b)} - 1$$

by the Euclidean algorithm. Since $n | 2^n - 1$ and $n | 2^{\varphi(n)} - 1$ by Euler's theorem, so $n | 2^d - 1$, where $d = \gcd(n, \varphi(n)) \leq \varphi(n) < n$. Then $n \mid 2^d - 1$ implies $d > 1$ and $d | 2^d - 1$, contradicting minimality of n.

Commended solvers: **Samuel Liló ABDALLA** (ITA, São Paulo, Brazil) and **Fai YUNG**.

Olympiad Corner

(continued from page 1)

Problem 2. (*Cont.*) the circumradii of triangles BDE and CDF, respectively, and r_1 and r_2 be the inradii of the same triangles. Prove that

$$|S_{ABD} - S_{ACD}| \geq |R_1 r_1 - R_2 r_2|,$$

where S_K is the area of figure K.

Problem 3. Let n be a natural number, $n \geq 2$. Prove that if $(b^n - 1)/(b - 1)$ is a prime power for some positive integer b, then n is prime.

<u>Second Day</u>

Problem 4. In square $ABCD$, points E and F are chosen in the interior of sides BC and CD, respectively. The line drawn from F perpendicular to AE passes through the intersection point G of AE and diagonal BD. A point K is chosen on FG such that $AK = EF$. Find $\angle EKF$.

Problem 5. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all reals x and y, $f(x + f(y)) = y + f(x+1)$.

Problem 6. Consider a 10×10 grid. On every move, we color 4 unit squares that lie in the intersection of some two rows and two columns. A move is allowed if at least one of the 4 squares is previously uncolored. What is the *largest* possible number of moves that can be taken to color the whole grid?