

## Junior problems

J337. Prove that for each integer  $n \geq 0$ ,  $16^n + 8^n + 4^{n+1} + 2^{n+1} + 4$  is the product of two numbers greater than  $4^n$ .

*Proposed by Titu Andreescu, University of Texas at Dallas*

*Solution by Moubinool Omarjee, Lycée Henri IV, Paris, France*

We have

$$\begin{aligned} 16^n + 8^n + 4^{n+1} + 2^{n+1} + 4 &= (4^n + 2)^2 + 2^n(4^n + 2) = \\ &= (4^n + 2)(4^n + 2^n + 2). \end{aligned}$$

and the conclusion follows.

*Also solved by Daniel Lasasoa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Milan, Italy; Apratim Dey, Indian Statistical Institute, Kolkata, India; Arber Avdullahu, Mehmet Akif College, Kosovo; Corneliu Mănescu-Avram, Technological Transportation High School, Ploiești, Romania; Daniel López-Aguayo, Centro de Ciencias Matemáticas UNAM, Morelia, Mexico; David E. Manes, Oneonta, NY, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Haimoshri Das, South Point High School, India; Henry Ricardo, New York Math Circle; Joel Schlosberg Bayside, NY, USA; José Hernández Santiago, México; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Samridhi Singha Roy, Kolkata, India; Neculai Stanciu and Titu Zvonaru, Romania; Polyhedra, Polk State College, FL, USA; Phillip Ahn, Concord Academy, Concord, MA, USA; Thomas Choi, Phillips Academy Andover, Andover, MA, USA; Michael Tang, Edina High School, MN, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Sonny An; S. Viswanathan, Center for Scientific Learning, Nagpur, Maharashtra, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nick Iliopoulos, Music Junior HS, Trikala, Greece; Rade Krenkov, SOU Goce Delcev, Valandovo, Macedonia; Paul Revenant, Lycée Champollion, Grenoble, France.*

J338. Consider lattice points  $P_1, P_2, \dots, P_{n^2}$  with coordinates  $(u, v)$ , where  $1 \leq u, v \leq n$ . For points  $P_i = (u_i, v_i)$  and  $P_j = (u_j, v_j)$ , define  $d(P_i P_j) = |u_i - u_j| + |v_i - v_j|$ . Evaluate

$$\sum_{1 \leq i < j \leq n^2} d(P_i P_j).$$

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Note that each combination  $u_i \neq u_j$  appears exactly  $n^2$  times in the sum (as many as possible combinations of values  $v_i, v_j$  not necessarily distinct), whereas each combination  $u_i = u_j$  appears exactly  $\binom{n}{2}$  times in the sum (as many as possible combinations of values  $v_i \neq v_j$ ), for a total of  $\binom{n}{2}n^2 + n\binom{n}{2} = \binom{n^2}{2}$  elements in the sum, as expected. Considering that the terms with  $u_i = u_j$  do not contribute to the sum, we can readily see that a value  $u_i$  contributes with positive sign to the sum  $n^2(u_i - 1)$  times (as many times as it appears together with any of the  $u_j < u_i$ ), and with negative sign  $n^2(n - u_i)$  times (as many times as it appears together with any of the  $u_j > u_i$ ). Therefore, the net contribution to the sum of a given  $k \in \{1, 2, \dots, n\}$  is  $kn^2(2k - n - 1)$ , ie the total contribution to the sum of the  $|u_i - u_j|$  is

$$\begin{aligned} \sum_{k=1}^n kn^2(2k - n - 1) &= 2n^2 \sum_{k=1}^n k^2 - n^2(n+1) \sum_{k=1}^n k = \\ &= 2n^2 \frac{n(n+1)(2n+1)}{6} - n^2(n+1) \frac{n(n+1)}{2} = \frac{n^3(n+1)(n-1)}{6}. \end{aligned}$$

Since the contribution to the sum is the same for the  $|v_i - v_j|$  by symmetry, we conclude that

$$\sum_{1 \leq i < j \leq n^2} d(P_i P_j) = \frac{n^3(n+1)(n-1)}{3}.$$

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J339. Solve in positive integers the equation

$$\frac{x-1}{y+1} + \frac{y-1}{z+1} + \frac{z-1}{x+1} = 1.$$

*Proposed by Titu Andreescu, University of Texas at Dallas*

*Solution by Michael Tang, Edina High School, MN, USA*

Make the substitutions  $a = x + 1$ ,  $b = y + 1$ ,  $c = z + 1$ , so that  $a, b, c \geq 2$ . Then we have

$$\frac{a-2}{b} + \frac{b-2}{c} + \frac{c-2}{a} = 1 \quad (*)$$

or

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1 + 2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

By the AM-GM inequality,  $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$ , and equality holds iff  $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$ , or  $a = b = c$ . If  $a = b = c$ , then we get  $3 \cdot (a-2)/a = 1$ , so  $a = b = c = 3$ , or  $x = y = z = 2$ , which is a solution to (\*). Otherwise, the inequality is strict, so

$$1 + 2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} > 3 \implies \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1.$$

We determine all possible triples  $(a, b, c)$  with  $a, b, c \geq 2$  that satisfy this inequality, and then check them against (\*). Without loss of generality, let  $a \leq b \leq c$ , as our inequality is symmetric in  $a, b, c$ . If  $a \geq 3$ , then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

a contradiction, so  $a = 2$ . Then we must have  $\frac{1}{b} + \frac{1}{c} > \frac{1}{2}$ . If  $4 \leq b \leq c$ , then

$$\frac{1}{b} + \frac{1}{c} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

a contradiction, so  $b \leq 3$ . If  $b = 2$ , then  $\frac{1}{c} > 0$ , so  $c$  can be any positive integer at least 2. Therefore, we have  $(a, b, c) = (2, 2, t)$  for some  $t \geq 2$ , as well as its permutations. If  $(a, b, c) = (2, 2, t)$ , then (\*) becomes  $(t-2)/2 = 1$ , so  $t = 4$ , and  $(a, b, c) = (2, 2, 4) \iff (x, y, z) = (1, 1, 3)$  satisfies the equation. Similarly,  $(x, y, z) = (1, 3, 1)$  and  $(x, y, z) = (3, 1, 1)$  satisfy the equation.

If  $b = 3$ , then

$$\frac{1}{c} > \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

so  $c \leq 5$ , and the triples  $(a, b, c) = (2, 3, 3), (2, 3, 4), (2, 3, 5)$  and their permutations satisfy our inequality. However, it turns out that none of these triples satisfy (\*), so we find no new solutions.

Therefore, the solutions are  $(x, y, z) = (2, 2, 2), (1, 1, 3), (1, 3, 1), (3, 1, 1)$ .

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J340. Let  $ABC$  be a triangle with incircle  $\omega$  and circumcircle  $\Gamma$ . Let  $I$  be the incenter of  $ABC$ ,  $D$  the tangency point of  $\omega$  with  $BC$ ,  $M$  the midpoint of  $ID$ , and  $N$  the antipode of  $A$  with respect to  $\Gamma$ . Let  $A'$  be the tangency point of  $\Gamma$  with the circle tangent to rays  $AB, AC$ , and externally to  $\Gamma$ . Prove that lines  $A'D$  and  $MN$  intersect on  $\Gamma$ .

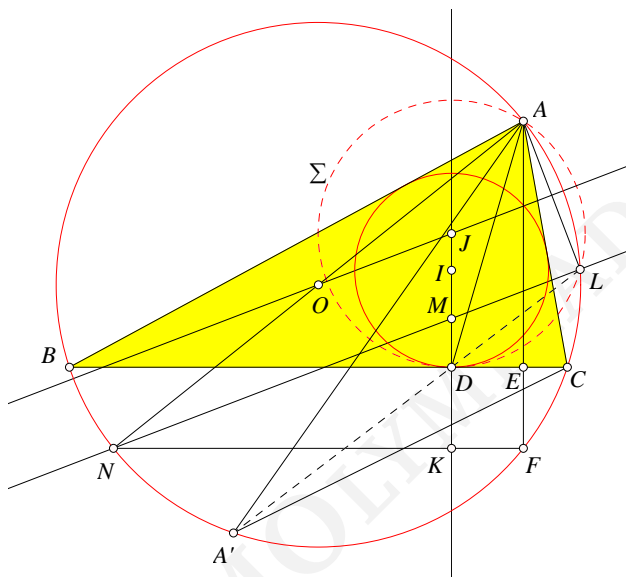
Proposed by Marius Stănean, Zalău, Romania

*Solution by Polyhedra, Polk State College, FL, USA*

Suppose that  $NM$  intersects  $\Gamma$  at  $L$ . We first prove a key lemma,

*Lemma.* The circumcircle  $\Sigma$  of  $\triangle ADL$  is tangent to  $BC$  at  $D$ .

*Proof.* Let  $O$  be the center of  $\Gamma$ . Suppose that the line through  $O$  and parallel to  $NM$  intersects  $ID$  at  $J$ . The lemma follows from the claim that  $J$  is the center of  $\Sigma$ . Let  $E$  be the foot of the altitude on  $BC$  and  $h = AE$ . It suffices to show that  $AD^2 = 2hJD$ .



Referring to the figure, we see that  $JD = MD + JM$  and  $JM = \frac{AL \cdot NM}{2NK}$ . Also,

$$AL = AN \sin(\angle ANF - \angle MNK) = AF \cdot \frac{NK}{NM} - NF \cdot \frac{MK}{NM}.$$

Hence

$$JM = \frac{1}{2} \left[ AF - \frac{(NK + DE)MK}{NK} \right] = \frac{1}{2} \left( h - \frac{r}{2} - \frac{DE \cdot MK}{NK} \right), \quad JD = \frac{1}{2} \left( h + \frac{r}{2} - \frac{DE \cdot MK}{NK} \right),$$

and it suffices to show that

$$DE^2 = h \left( \frac{r}{2} - \frac{DE}{NK} \cdot MK \right).$$

Now

$$DE = s - c - b \cos C = \frac{a + b - c}{2} - \frac{a^2 + b^2 - c^2}{2a} = \frac{(c - b)(s - a)}{a},$$

$$\frac{1}{2}hr = \frac{\Delta^2}{as} = \frac{(s - a)(s - b)(s - c)}{a},$$

$$NK = NF - DE = 2R \sin(C - B) - DE = \frac{c^2 + a^2 - b^2}{2a} - \frac{a^2 + b^2 - c^2}{2a} - \frac{(c - b)(s - a)}{a} = \frac{(c - b)s}{a},$$

$$\begin{aligned} hMK &= h(MD + EF) = \frac{1}{2}hr + 2hR \cos B \cos C \\ &= \frac{(s - a)(s - b)(s - c)}{a} + \frac{(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)}{4a^2}. \end{aligned}$$

Substituting in these expressions we finally get

$$\begin{aligned} \frac{1}{2}hr - \frac{DE}{NK} \cdot hMK &= (s - a) \left[ \frac{(s - b)(s - c)}{s} - \frac{(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)}{4a^2s} \right] \\ &= \frac{(s - a)[a^2(a^2 - b^2 - c^2 + 2bc) - (a^4 - b^4 - c^4 + 2b^2c^2)]}{4a^2s} \\ &= \frac{(c - b)^2(s - a)^2}{4a^2} = DE^2, \end{aligned}$$

establishing the lemma.

It is well known and easy to prove, by an inversion with center  $A$  and power  $AB \cdot AC$ , that  $AA'$  is the reflection of  $AD$  across  $AI$ . For details, see the solution to O306 in Issue 4 (2014). The lemma then implies that

$$\angle CDL = \angle LAD = \angle LAC + \angle CAD = \angle LA'C + \angle BAA' = \angle LA'C + \angle BCA',$$

which forces the collinearity of  $A', D, L$ .

*Also solved by Daniel Lasaosa, Pamplona, Spain; George Gavrilopoulos, Nea Makri High School, Athens, Greece.*

J341. Let  $ABC$  be a triangle and let  $P$  be a point in its interior. Let  $PA = x$ ,  $PB = y$ ,  $PC = z$ . Prove that

$$\frac{(x + y + z)^9}{xyz} \geq 729a^2b^2c^2.$$

*Proposed by Marcel Chiriță, Bucharest, Romania*

*Solution by Polyhedra, Polk State College, USA*

Let  $\angle BPC = 2\alpha$ ,  $\angle CPA = 2\beta$ , and  $\angle APB = 2\gamma$ . Without loss of generality, assume that  $\alpha \leq 90^\circ$ . Then

$$\begin{aligned} \cos 2\alpha + \cos 2\beta + \cos 2\gamma &= 2\cos^2 \alpha - 1 + 2\cos(\beta + \gamma)\cos(\beta - \gamma) \\ &= 2\cos^2 \alpha - 1 - 2\cos \alpha \cos(\beta - \gamma) \\ &= 2\left(\cos \alpha - \frac{1}{2}\right)^2 + 2[1 - \cos(\beta - \gamma)]\cos \alpha - \frac{3}{2} \geq -\frac{3}{2}. \end{aligned}$$

Now, by the law of cosines,

$$a^2 = y^2 + z^2 - 2yz \cos 2\alpha = (y + z)^2 - 2yz(1 + \cos 2\alpha).$$

Hence,

$$\begin{aligned} xa^2 + yb^2 + zc^2 &= x(y + z)^2 + y(z + x)^2 + z(x + y)^2 - 2xyz(3 + \cos 2\alpha + \cos 2\beta + \cos 2\gamma) \\ &\leq x(y + z)^2 + y(z + x)^2 + z(x + y)^2 - 3xyz \\ &= (x + y + z)(xy + yz + zx) \leq \frac{1}{3}(x + y + z)^3. \end{aligned}$$

Finally, by the AM-GM inequality,

$$3^6 xa^2 yb^2 zc^2 \leq 3^3 (xa^2 + yb^2 + zc^2)^3 \leq (x + y + z)^9,$$

completing the proof.

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J342. Solve in positive real numbers the system of equations

$$\frac{x^3}{4} + y^2 + \frac{1}{z} = \frac{y^3}{4} + z^2 + \frac{1}{x} = \frac{z^3}{4} + x^2 + \frac{1}{y} = 2.$$

*Proposed by Titu Andreescu, University of Texas at Dallas*

*Solution by Alessandro Ventullo, Milan, Italy*

Adding the three equations, we get

$$\left(\frac{x^3}{4} + x^2 + \frac{1}{x} - 2\right) + \left(\frac{y^3}{4} + y^2 + \frac{1}{y} - 2\right) + \left(\frac{z^3}{4} + z^2 + \frac{1}{z} - 2\right) = 0.$$

Let  $f(t) = \frac{t^3}{4} + t^2 + \frac{1}{t} - 2$ . We have

$$f(t) = \frac{t^4 + 4t^3 - 8t + 4}{4t} = \frac{(t^2 + 2t - 2)^2}{4t} \geq 0$$

for all  $t \in \mathbb{R}^+$  and  $f(t) = 0$  if and only if  $t^2 + 2t - 2 = 0$ , i.e.  $t = \sqrt{3} - 1$ . It follows that  $f(x) + f(y) + f(z) = 0$  if and only if  $x = y = z = \sqrt{3} - 1$ .

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## Senior problems

S337. Let  $a, b, c$  be positive real numbers such that  $ab + bc + ca = 3$ . Prove that

$$(a^3 + 1)(b^3 + 1)(c^3 + 1) \geq 8.$$

*Proposed by Titu Andreescu, University of Texas at Dallas*

*First solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India*

Notice that from Hölder's Inequality,

$$(a^3 + 1 + a^3 + 1)(b^3 + b^3 + 1 + 1)(1 + c^3 + c^3 + 1) \geq (ab + bc + ca + 1)^3 = 64,$$

from which implies the result.

*Second solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India*

Notice that from Hölder's and Cauchy-Schwarz Inequality,

$$\begin{aligned} (a^3 + 1 + a^3 + a^3 + 1 + 1)(b^3 + b^3 + 1 + 1 + b^3 + 1)(1 + c^3 + 1 + c^3 + 1 + c^3) &\geq (ab + bc + ca + a + b + c)^3 \\ &\geq \left(ab + bc + ca + \sqrt{3(ab + bc + ca)}\right)^3 = 6^3, \end{aligned}$$

*Also solved by Russelle Guadalupe, Institute of Mathematics, College of Science, University of the Philippines-Diliman, Quezon City, Philippines; Daniel Lasaosa, Pamplona, Spain; Kwon Il Kobe Ko, Cushing Academy, Ashburnham, MA, USA; Phillip Ahn, Concord Academy, Concord, MA, USA; Thomas Choi, Phillips Academy Andover, Andover, MA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Sonny An; S. Viswanathan, Center for Scientific Learning, Nagpur, Maharashtra, India; Albert Stadler, Herliberg, Switzerland; Arber Avdullahu, Mehmet Akif College, Kosovo; Arkady Alt, San Jose, CA, USA; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Paul Revenant, Lycée Champollion, Grenoble, France; Neculai Stanciu and Titu Zvonaru, Romania; Li Zhou, Polk State College, Winter Haven, FL, USA.*



S338. Let  $ABC$  be a triangle with  $\angle A$  being the largest angle. Let  $D, E, F$  be points on sides  $BC, CA, AB$ , such that  $AD$  is the altitude from  $A$ ,  $DE$  is the internal angle bisector of  $\angle ADC$ , and  $DF$  is the internal angle bisector of  $\angle ADB$ . If  $AE = AF$ , prove that  $ABC$  is isosceles or right.

*Proposed by Titu Zvonaru and Neculai Stanciu, Romania*

*Solution by Li Zhou, Polk State College, Winter Haven, FL, USA*

Since  $DE$  and  $DF$  are angle bisectors of  $\angle ADC$  and  $\angle ADB$ , we have  $AE = \frac{AC \cdot AD}{AD + DC}$  and  $AF = \frac{AB \cdot AD}{AD + BD}$ . Hence the condition  $AE = AF$  yields  $AB(AD + DC) = AC(AD + BD)$ . Squaring both sides and applying the Pythagorean theorem, we get

$$AB^2(AC^2 + 2AD \cdot DC) = AC^2(AB^2 + 2AD \cdot BD).$$

Thus  $(AD^2 + BD^2)DC = (AD^2 + DC^2)BD$ , which is equivalent to  $(BD - DC)(BD \cdot DC - AD^2) = 0$ . If  $BD - DC = 0$ , then  $AB = AC$ ; if  $BD \cdot DC - AD^2 = 0$ , then  $\angle A = 90^\circ$ .

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S339. Let  $p$  be a prime congruent to 2 mod 7. Solve in nonnegative integers the system of equations

$$\begin{aligned}7(x + y + z)(xy + yz + zx) &= p(2p^2 - 1), \\70xyz + 21(x - y)(y - z)(z - x) &= 2p(p^2 - 4).\end{aligned}$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Note first that

$$\begin{aligned}(2x - y)(2y - z)(2z - x) &= 2xy^2 + 2zx^2 + 2yz^2 - 4y^2z - 4x^2y - 4z^2x + 7xyz = \\&= 10xyz + 3(xy^2 + zx^2 + yz^2 - y^2z - x^2y - z^2x) - \\&\quad - (xy^2 + yz^2 + zx^2 + x^2y + y^2z + z^2x + 3xyz) = \\&= 10xyz + 3(x - y)(y - z)(z - x) - (x + y + z)(xy + yz + zx) = \\&= \frac{2p(p^2 - 4) - p(2p^2 - 1)}{7} = -p,\end{aligned}$$

and since  $p$  is prime, it follows that one of  $2x - y, 2y - z, 2z - x$  is either  $p$  or  $-p$ , and the other two are either 1 or  $-1$ . Now, note that the first equation is invariant under exchange of any two variables, and the second equation is invariant under cyclic permutation of the variables, or we may choose wlog  $|2x - y| = p$  and  $|2y - z| = |2z - x| = 1$ . Now, note that  $(2x - y) + (2y - z) + (2z - x) = x + y + z$ , or if  $2x - y = -p$ , then  $2y - z = 2z - x = \pm 1$ , and  $x + y + z = \pm 2 - p \leq 0$ , and since  $x, y, z$  are non-negative, we would have  $x = y = z = 0$ , absurd. It follows that  $2x - y = p$ , and  $2y - z = -(2z - x) = \pm 1$ , yielding two possible cases.

*Case 1:*  $2y - z = 1$  and consequently  $2z - x = -1$ , or  $z = \frac{x-1}{2}$  and  $y = \frac{x+1}{4}$ . It follows that  $x \equiv 3 \pmod{4}$  for  $y, z$  to be integers, or some integer  $k$  exists such that  $x = 4k + 3$ ,  $y = k + 1$ ,  $z = 2k + 1$ . But then  $p = x + y + z = 7k + 5 \equiv 5 \pmod{7}$ , in contradiction with the problem statement. No solution exists in this case.

*Case 2:*  $2y - z = -1$  and consequently  $2z - x = 1$ , or  $z = \frac{x+1}{2}$  and  $y = \frac{x-1}{4}$ . It follows that  $x \equiv 1 \pmod{4}$  for  $y, z$  to be integers, or some integer  $k$  exists such that  $x = 4k + 1$ ,  $y = k$ ,  $z = 2k + 1$ , yielding indeed  $p = x + y + z = 7k + 2 \equiv 2 \pmod{7}$ , as desired. Direct substitution shows that this form for  $x, y, z$  is indeed a solution for any non-negative integer  $k$ , including  $k = 0$  which corresponds to  $p = 2$ .

Restoring generality, we conclude that

$$(x, y, z) = (4k + 1, k, 2k + 1), (k, 2k + 1, 4k + 1), (2k + 1, 4k + 1, k),$$

where  $k$  is the non-negative integer integer such that  $p = 7k + 2$ , are all solutions, and there can be no others when  $p$  is a prime of this form.

*Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India; David E. Manes, Oneonta, NY, USA; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Paul Revenant, Lycée Champollion, Grenoble, France.*

S340. Prove that in a triangle with sides  $a, b, c$  and semiperimeter  $s$ ,

$$\sum_{\text{cyc}} \frac{a}{\sqrt{(s-b)(s-c)}} + \frac{5}{4} \sum_{\text{cyc}} \frac{(a-b)^2}{(s-a)(s-b)} \geq \frac{3R}{r}.$$

*Proposed by Mircea Lascu and Titu Zvonaru, Romania*

*Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India*

Using the formulae,

$$R = \frac{abc}{4\sqrt{(s-a)(s-b)(s-c)}}, \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

we have

$$\frac{3R}{r} = \frac{3abc}{4(s-a)(s-b)(s-c)}.$$

The idea is to get rid of denominators by making them equal on both the sides:

$$\sum_{\text{cyc}} \frac{a}{\sqrt{(s-b)(s-c)}} + \frac{5}{4} \sum_{\text{cyc}} \frac{(a-b)^2}{(s-a)(s-b)} = \sum_{\text{cyc}} \frac{4a(s-a)\sqrt{(s-b)(s-c)}}{4(s-a)(s-b)(s-c)} + \sum_{\text{cyc}} \frac{5(a-b)^2(s-c)}{4(s-a)(s-b)(s-c)}.$$

Thus we need to show that

$$\sum_{\text{cyc}} 4a(s-a)\sqrt{(s-b)(s-c)} + \sum_{\text{cyc}} 5(a-b)^2(s-c) \geq 3abc.$$

Using Ravi's Substitution,  $a = x+y, b = y+z, c = z+x$ , for positive real numbers  $x, y, z$  we get an equivalent inequality to prove that

$$\sum_{\text{cyc}} 4z(x+y)\sqrt{xy} + \sum_{\text{cyc}} 5y(z-x)^2 \geq 3(x+y)(y+z)(z+x).$$

From the AM-GM Inequality, we get

$$\sum_{\text{cyc}} 4z(x+y)\sqrt{xy} \geq \sum_{\text{cyc}} 4z \cdot 2\sqrt{xy} \cdot \sqrt{xy} = \sum_{\text{cyc}} 8xyz = 24xyz.$$

Also,

$$\sum_{\text{cyc}} 5y(z-x)^2 = 5 \left( \sum_{\text{cyc}} xy(x+y) \right) - 30xyz.$$

Thus we need to prove that

$$5 \left( \sum_{\text{cyc}} xy(x+y) \right) - 6xyz \geq 3 \left( \sum_{\text{cyc}} xy(x+y) \right) + 6xyz \Leftrightarrow \sum_{\text{cyc}} xy(x+y) \geq 6xyz,$$

which follows easily from the AM-GM Inequality. Equality holds iff  $a = b = c$ .

*Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Lasasosa, Pamplona, Spain; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou, Polk State College, Winter Haven, FL, USA; Arkady Alt, San Jose, CA, USA; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Rao Yiyi, Wuhan, China.*

S341. Let  $a, b, c, d$  be nonnegative real numbers such that  $a + b + c + d = 6$ . Find the maximum value of  $4a + 2ab + 2abc + abcd$ .

*Proposed by Marius Stănean, Zalău, Romania*

*Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy*

Lagrange multipliers. For  $(a, b, c, d) \neq (0, 0, 0, 0)$  let's define the function

$$F(a, b, c, d, \lambda) = f(a, b, c, d) - \lambda(a + b + c + d - 6) = 4a + 2ab + 2abc + abcd - \lambda(a + b + c + d - 6)$$

$$\begin{aligned} F_a = 0 &\iff 4 + 2b + 2bc + bcd = \lambda \iff 4a + 2ab + 2abc + abcd = \lambda a \\ F_b = 0 &\iff 2a + 2ac + acd = \lambda \iff 2ab + 2abc + abcd = \lambda b \\ F_c = 0 &\iff 2ab + abd = \lambda \iff 2abc + abcd = \lambda c \\ F_d = 0 &\iff abc = \lambda \end{aligned}$$

which is

$$\begin{aligned} F_a = 0 &\iff 4a + 2ab + 2\lambda + \lambda d = \lambda a \\ F_b = 0 &\iff 2ab + 2\lambda + \lambda d = \lambda b \\ F_c = 0 &\iff 2\lambda + \lambda d = \lambda c \\ F_d = 0 &\iff abc = \lambda \end{aligned}$$

By summing the first three we get

$$4a + 4ab + 6\lambda + 3\lambda d = 6\lambda - \lambda d \iff 4a + 4ab = -5\lambda d$$

which implies  $\lambda < 0$  but this conflicts with  $abc = \lambda$ .

This means that in the open set  $(0, 6)^4$  there is no constrained critical point of  $f(a, b, c, d)$  and then the maximum of  $f(a, b, c, d)$  as well as its minimum of the set  $[0, 6]^4$ , belongs to the boundary.

If  $a = 0$  we have  $f(0, b, c, d) \equiv 0$ .

If  $b = 0$  we have  $f(a, 0, c, d) = 4a \leq 24$ .

If  $c = 0$  we have  $f(a, b, 0, d) = 4a + 2ab$  and  $a + b \leq 6$ . It follows  $4a + 2ab \leq 4a + 2a(6 - a)$ ,  $0 \leq a \leq 6$ . Easy computations show that the maximum occurs for  $a = 6$  and the value is 24.

If  $d = 0$  we have  $f(a, b, c, 0) = 4a + 2ab + 2abc$ ,  $a + b + c = 6$ . Clearly

$$f(a, b, c, 0) \doteq g(a, b) \leq 4a + 2ab + 2ab(6 - a - b) = 4a + 14ab - 2a^2b - 2ab^2, \quad 0 \leq a + b \leq 6$$

To find the maximum of  $g(a, b)$  we compute the gradient

$$g_a = 0 = 4 + 14b - 4ab - 2b^2 = 0, \quad g_b = a(14 - 2a - 4b) = 0$$

The second yields  $a = 7 - 2b$  which inserted into the first equation gives  $6b^2 - 14b + 4 = 0$  and then

$$(a, b) = \left(\frac{19}{3}, \frac{1}{3}\right) \doteq P_1, (a, b) = (3, 2) \doteq P_2.$$

The determinant

$$g_{a,a}g_{b,b} - (g_{a,b})^2$$

is positive at  $P_1$  and negative at  $P_2$ . Since  $g_{a,a}(3, 2) = -8$ , it follows that  $P_1$  is a point of local maximum while  $P_2$  is a saddle.

To prove that  $P_1$  is a point of absolute maximum in the set  $0 \leq a + b \leq 6$ , we need to compute the maximum of  $g(a, b)$  on the boundary. The boundary is made of three intervals. The first one is  $a = 0$ ,  $0 \leq b \leq 6$ . In such a case  $g(0, b) \equiv 0$ .

The second one is  $0 \leq a \leq 6$  and  $b = 0$ . In such a case  $g(a, 0) = 4a \leq 24$ . Finally if  $a + b = 6$ , we have  $g(a, 6 - a) = 16a - a^2 \leq 32 = g(4, 2)$ .

It follows that the maximum is

$$f(3, 2, 1, 0) = 36.$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Paul Revenant, Lycée Champollion, Grenoble, France; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; S. Viswanathan, Center for Scientific Learning, Nagpur, Maharashtra, India; Li Zhou, Polk State College, Winter Haven, FL, USA; Rao Yiyi, Wuhan, China; Adnan Ali, Student in A.E.C.S-4, Mumbai, India.*

S342. Let  $a \geq b \geq c$  be positive real numbers. Prove that for all  $t \in [0, \frac{\pi}{4}]$ ,

$$\frac{a-b}{a \sin t + b \cos t} + \frac{b-c}{b \sin t + c \cos t} + \frac{c-a}{c \sin t + a \cos t} \geq 0.$$

*Proposed by Titu Andreescu, University of Texas at Dallas*

*Solution by Daniel Lasasosa, Pamplona, Spain*

Multiplying throughout by the product of the (clearly positive) denominators and rearranging terms, the proposed inequality is equivalent to

$$\begin{aligned} & (a^2b + b^2c + c^2a - 3abc) \sin t (\cos t - \sin t) + \\ & + (ab^2 + bc^2 + ca^2 - 3abc) \cos t (\cos t - \sin t) + \\ & + (a-b)(b-c)(a-c) \sin t \cos t \geq 0. \end{aligned}$$

Since  $t \in [0, \frac{\pi}{4}]$ , we have  $\cos t - \sin t \geq 0$ , with equality iff  $t = \frac{\pi}{4}$ , whereas  $\sin t \geq 0$  with equality iff  $t = 0$ , and  $\cos t > 0$ . Moreover, by the AM-GM inequality we have  $a^2b + b^2c + c^2a - 3abc \geq 0$  and  $ab^2 + bc^2 + ca^2 - 3abc \geq 0$ , with equality in either case iff  $a = b = c$ . Clearly  $a-b, b-c, a-c$  are non-negative, their product being zero iff at least two out of  $a, b, c$  are equal. The conclusion follows. Equality holds for all  $t \in [0, \frac{\pi}{4}]$  iff  $a = b = c$ , or whenever  $a, b, c$  are not all equal, iff  $t = \frac{\pi}{4}$  (otherwise the second term cannot be zero) and simultaneously at least two out of  $a, b, c$  are equal (otherwise the third term cannot be simultaneously zero).

*Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Li Zhou, Polk State College, Winter Haven, FL, USA; Rao Yiyi, Wuhan, China; Arkady Alt, San Jose, CA, USA; Nermin Hodzic, University of Tuzla at Tuzla, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Neculai Stanciu and Titu Zvonaru, Romania.*

## Undergraduate problems

U337. Let  $n$  be a positive integer. Find all real polynomials  $f$  and  $g$  such that

$$(x^2 + x + 1)^n f(x^2 - x + 1) = (x^2 - x + 1)^n g(x^2 + x + 1),$$

for all real numbers  $x$ .

*Proposed by Marcel Chiriță, Bucharest, Romania*

*Solution by Alessandro Ventullo, Milan, Italy*

Observe that  $x^2 - x + 1 = (1 - x)^2 - (1 - x) + 1$ . Hence, by substitution  $x \mapsto 1 - x$ , we get

$$(x^2 - 3x + 3)^n f(x^2 - x + 1) = (x^2 - x + 1)^n g(x^2 - 3x + 3)$$

and dividing by the original equation, we have

$$\frac{g(x^2 - 3x + 3)}{g(x^2 + x + 1)} = \left( \frac{x^2 - 3x + 3}{x^2 + x + 1} \right)^n. \quad (1)$$

Moreover, since  $x^2 + x + 1 = (-1 - x)^2 - (-1 - x) + 1$ , by substitution  $x \mapsto -1 - x$ , we get

$$(x^2 + x + 1)^n f(x^2 + 3x + 3) = (x^2 + 3x + 3)^n g(x^2 + x + 1)$$

and dividing by the original equation, we have

$$\frac{f(x^2 + 3x + 3)}{f(x^2 - x + 1)} = \left( \frac{x^2 + 3x + 3}{x^2 - x + 1} \right)^n.$$

By substitution  $x \mapsto -x$ , we obtain

$$\frac{f(x^2 - 3x + 3)}{f(x^2 + x + 1)} = \left( \frac{x^2 - 3x + 3}{x^2 + x + 1} \right)^n. \quad (2)$$

Comparing (1) and (2), we obtain

$$\frac{f(x^2 - 3x + 3)}{g(x^2 - 3x + 3)} = \frac{f(x^2 + x + 1)}{g(x^2 + x + 1)}$$

for all  $x \in \mathbb{R}$ . It follows that  $\frac{f(x^2 - 3x + 3)}{g(x^2 - 3x + 3)}$  and  $\frac{f(x^2 + x + 1)}{g(x^2 + x + 1)}$  have the same derivative for all  $x \in \mathbb{R}$ . So,

$$\begin{aligned} & (2x - 3) \frac{f'(x^2 - 3x + 3)g(x^2 - 3x + 3) - f(x^2 - 3x + 3)g'(x^2 - 3x + 3)}{(g(x^2 - 3x + 3))^2} = \\ & = (2x + 1) \frac{f'(x^2 + x + 1)g(x^2 + x + 1) - f(x^2 + x + 1)g'(x^2 + x + 1)}{(g(x^2 + x + 1))^2}. \end{aligned}$$

for all  $x \in \mathbb{R}$ . Since  $(1 - x)^2 + (1 - x) + 1 = x^2 - 3x + 3$ , by substitution  $x \mapsto 1 - x$  into the right-hand side, we have

$$(2x - 3) \frac{f'(x^2 - 3x + 3)g(x^2 - 3x + 3) - f(x^2 - 3x + 3)g'(x^2 - 3x + 3)}{g^2(x^2 - 3x + 3)} = 0$$

for all  $x \in \mathbb{R}$ . It follows that  $\frac{f(x^2 - 3x + 3)}{g(x^2 - 3x + 3)}$  is constant on  $\mathbb{R}$ , i.e.  $f(x) = \lambda g(x)$  for some  $\lambda \in \mathbb{R}$ . Substituting into the original equation, we get  $\lambda = 1$ . Now, from equation (2), we get

$$\frac{f(x^2 - 3x + 3)}{(x^2 - 3x + 3)^n} = \frac{f(x^2 + x + 1)}{(x^2 + x + 1)^n}$$

for all  $x \in \mathbb{R}$ . It follows that  $\frac{f(x^2 - 3x + 3)}{(x^2 - 3x + 3)^n}$  and  $\frac{f(x^2 + x + 1)}{(x^2 + x + 1)^n}$  have the same derivative for all  $x \in \mathbb{R}$ . So,

$$\begin{aligned} & (2x - 3) \frac{f'(x^2 - 3x + 3)(x^2 - 3x + 3) - nf(x^2 - 3x + 3)}{(x^2 - 3x + 3)^{n+1}} = \\ &= (2x + 1) \frac{f'(x^2 + x + 1)(x^2 + x + 1) - nf(x^2 + x + 1)}{(x^2 + x + 1)^{n+1}}. \end{aligned}$$

for all  $x \in \mathbb{R}$ . By substitution  $x \mapsto 1 - x$  into the right-hand side, we get once again that

$$\frac{d}{dx} \frac{f(x^2 - 3x + 3)}{(x^2 - 3x + 3)^n} = 0$$

for all  $x \in \mathbb{R}$ , so  $f(x) = cx^n$  for some  $c \in \mathbb{R}$ . It follows that  $f(x) = g(x) = cx^n$ , where  $c \in \mathbb{R}$ .

*Also solved by Daniel Lasasosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Moubinool Omarjee, Lycée Henri IV, Paris, France.*

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U338. Determine the number of pairs  $(a, b)$  such that the equation  $ax = b$  is solvable in the ring  $(\mathbb{Z}_{2015}, +, \cdot)$ .

*Proposed by Dorin Andrica, Cluj-Napoca, Romania*

*Solution by Joel Schlosberg Bayside, NY, USA*

For  $a \in \mathbb{Z}_{2015}$ , let  $d_a = \gcd(a, 2015)$  and  $d'_a = \frac{2015}{d_a}$ . Since 2015 is squarefree,  $d_a$  and  $d'_a$  are relatively prime. Suppose some prime  $p$  divides both  $a/d_a$  and  $d'_a$ . Then  $pd_a|a$  and  $pd_a|d'_a d_a = 2015$ , making  $pd_a$  a common divisor of  $a$  and 2015; yet  $pd_a > d_a = \gcd(a, 2015)$ , a contradiction. Thus,  $a/d_a$  and  $d'_a$  are relatively prime.

If  $ax = b$  is solvable in  $\mathbb{Z}_{2015}$ , then  $b \equiv ax \equiv 0 \pmod{d_a}$ . Conversely, if  $d_a|b$ , let  $(a/d_a)^{-1}$  be the multiplicative inverse of  $a/d_a$  modulo  $d'_a$  (which exists by relative primality) and  $x = (a/d_a)^{-1}(b/d_a)$ . Then

$$(a/d_a)x \equiv (b/d_a) \pmod{d'_a}$$

$$ax \equiv b \pmod{d'_a d_a = 2015},$$

solving  $ax = b$  in  $\mathbb{Z}_{2015}$ .

Therefore, for a given  $a$ , the number of  $b$  such that  $ax = b$  is solvable in  $\mathbb{Z}_{2015}$  is the number of  $b$  in  $\mathbb{Z}_{2015}$  divisible by  $d_a$ , which is  $2015/d_a$ .

For  $d|2015$ ,  $a \in \mathbb{Z}_{2015}$  has  $d_a = d$  iff  $d|a$  and  $a/d$  is relatively prime to  $2015/d$ . Such  $a$  are of the form  $a = da'$  where  $a'$  is invertible in  $\mathbb{Z}_{2015/d}$ . By the definition of Euler's totient function  $\phi$ , there are  $\phi(2015/d)$  such elements  $a'$  in  $\mathbb{Z}_{2015/d}$ , and since each distinct  $a'$  corresponds to a distinct  $a$ , exactly as many  $a \in \mathbb{Z}_{2015}$  with  $d_a = d$ .

Therefore, the total number of pairs  $(a, b)$  is

$$\sum_{d|2015} \phi(2015/d) \frac{2015}{d} = \sum_{d'|2015} \phi(d') d'.$$

Since the prime factorization of 2015 is  $5 \cdot 13 \cdot 31$ , and since  $\phi$  is multiplicative, this becomes

$$\begin{aligned} & 1\phi(1) + 5\phi(5) + 13\phi(13) + 31\phi(31) + 5 \cdot 13\phi(5 \cdot 13) + 5 \cdot 31\phi(5 \cdot 31) \\ & + 13 \cdot 31\phi(13 \cdot 31) + 5 \cdot 13 \cdot 31\phi(5 \cdot 13 \cdot 31) \\ & = (1 + 5\phi(5)) (1 + 13\phi(13)) (1 + 31\phi(31)) \\ & = (1 + 5(5 - 1)) (1 + 13(13 - 1)) (1 + 31(31 - 1)) \\ & = 3069507. \end{aligned}$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Alessandro Ventullo, Milan, Italy.*

U339. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that (i)  $f$  is continuous on  $\mathbb{Q}$  and (ii)  $f(x) < f(x + \frac{1}{n})$  for all  $x \in \mathbb{R}$  and  $n \in \mathbb{Z}^+$ . Prove that  $f$  is increasing over  $\mathbb{R}$ .

*Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania*

*Solution by Joel Schlosberg Bayside, NY, USA*

Consider any  $x, y \in \mathbb{R}$  with  $x < y$ . For  $m \in \mathbb{Z}^+$ ,

$$f(x) < f(x + \frac{1}{n}) < f(x + \frac{2}{n}) < \cdots < f(x + \frac{m}{n}),$$

so  $f(x + q) \in (f(x), \infty)$  for all  $q \in \mathbb{Q}^+$ .

Since  $f$  is continuous, the preimage of the open set  $(f(x), \infty)$  under  $f$  is open. Since the preimage contains  $\{x + q | q \in \mathbb{Q}^+\}$ , which is a dense subset of  $(x, \infty)$ , the preimage contains  $(x, \infty) \ni y$ , so  $f(y) > f(x)$ . Therefore,  $f$  is increasing over  $\mathbb{R}$ .

*Also solved by Daniel Lasasosa, Pamplona, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.*

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U340. Define

$$A_n = \frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \cdots + \frac{n}{n^2 + n^2}.$$

Find

$$\lim_{n \rightarrow \infty} n \left( n \left( \frac{\pi}{4} - A_n \right) - \frac{1}{4} \right).$$

*Proposed by Yong Xi Wang, China*

*Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA*

Let  $f$  be three-times differentiable on the interval  $[0, 1]$ , and let  $[a, b] \subset [0, 1]$ . By Taylor's Theorem,

$$f(x) = f(b) + f'(b)(x - b) + \frac{1}{2}f''(b)(x - b)^2 + \frac{1}{6}f'''(\xi)(x - b)^3,$$

for some  $\xi$  between  $x$  and  $b$ . It follows that

$$\int_a^b f(x) dx = f(b)(b - a) - \frac{1}{2}f'(b)(b - a)^2 + \frac{1}{6}f''(b)(b - a)^3 + O((b - a)^4).$$

Now, let  $n$  be a positive integer. Then, for  $k = 0, 1, 2, \dots, n - 1$ ,

$$\int_{(k-1)/n}^{k/n} f(x) dx = \frac{1}{n}f\left(\frac{k}{n}\right) - \frac{1}{2n^2}f'\left(\frac{k}{n}\right) + \frac{1}{6n^3}f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^4}\right).$$

Summing over  $k$  then yields

$$\int_0^1 f(x) dx = \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) - \frac{1}{2n^2} \sum_{k=1}^n f'\left(\frac{k}{n}\right) + \frac{1}{6n^3} \sum_{k=1}^n f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^3}\right). \quad (3)$$

Similarly

$$f(1) - f(0) = \int_0^1 f'(x) dx = \frac{1}{n} \sum_{k=1}^n f'\left(\frac{k}{n}\right) - \frac{1}{2n^2} \sum_{k=1}^n f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^2}\right) \quad (4)$$

and

$$f'(1) - f'(0) = \int_0^1 f''(x) dx = \frac{1}{n} \sum_{k=1}^n f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n}\right). \quad (5)$$

Combining (1), (2), and (3) then yields

$$\int_0^1 f(x) dx = \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) - \frac{1}{2n}[f(1) - f(0)] - \frac{1}{12n^2}[f'(1) - f'(0)] + O\left(\frac{1}{n^3}\right).$$

Now, let

$$f(x) = \frac{1}{1 + x^2}.$$

Then

$$\begin{aligned} \int_0^1 f(x) dx &= \frac{\pi}{4}; \\ \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) &= \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + (k/n)^2} = \sum_{k=1}^n \frac{n}{n^2 + k^2} = A_n; \\ f(1) - f(0) &= \frac{1}{2} - 1 = -\frac{1}{2}; \\ f'(1) - f'(0) &= -\frac{1}{2} - 0 = -\frac{1}{2}; \end{aligned}$$

so that

$$\frac{\pi}{4} = A_n + \frac{1}{4n} + \frac{1}{24n^2} + O\left(\frac{1}{n^3}\right).$$

Finally,

$$\lim_{n \rightarrow \infty} n \left( n \left( \frac{\pi}{4} - A_n \right) - \frac{1}{4} \right) = \frac{1}{24}.$$

*Also solved by Daniel Lasasa, Pamplona, Spain; Alessandro Ventullo, Milan, Italy; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Li Zhou, Polk State College, Winter Haven, FL, USA.*

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U341. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a non-decreasing function such that the sequence  $\{I_n\}_{n \geq 1}$  defined by

$$I_n = \int_0^1 f(x)e^{nx} dx$$

is bounded. Prove that  $f(x) = 0$  for all  $x \in (0, 1)$ .

*Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania*

*Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia*

$$\begin{aligned} I_n &= \int_0^1 f(x)e^{nx} dx \geq f(0) \cdot \int_0^1 e^{nx} dx = f(0) \cdot \frac{e^n - 1}{n} \text{ and } I_n \text{ is bounded} \\ &\Rightarrow f(0) = 0 \Rightarrow f(x) \geq 0, \forall x \in [0, 1]. \end{aligned}$$

Assume by contradiction there exists  $\exists x_0 \in (0, 1) : f(x_0) > 0$ . Then we have

$$\begin{aligned} I_n &= \int_0^{x_0} f(x)e^{nx} dx + \int_{x_0}^1 f(x)e^{nx} dx \\ &\geq \int_{x_0}^1 f(x)e^{nx} dx \geq f(x_0) \int_{x_0}^1 e^{nx} dx = f(x_0) \cdot \frac{e^n - e^{nx_0}}{n} \rightarrow \infty. \end{aligned}$$

hence we have a contradiction.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.*

U342. Let  $A$  be an  $m \times n$  matrix with real entries, and let  $w_1, \dots, w_n$  be a nonnegative real numbers associated with the columns of  $A$ . For every  $j = 1, \dots, n$  we say that  $w_j$  represents the *weight* of column  $A_j$ . Describe a polynomial time algorithm for finding a subset of linearly independent columns of  $A$  whose sum of weights is maximum.

*Proposed by Cosmin Pohoata, Columbia University, USA*

*Solution by the author*

The greedy approach works (and gives polynomial time).

Relabel the weights  $w_1, \dots, w_n$  such that  $w_1 \geq \dots \geq w_n$  - this step requires a sorting of the nonnegative reals  $w_1, \dots, w_n$  and so it can be easily done in polynomial time.

Let the set of columns of  $A$  be  $\mathcal{C}$ . Initialize the subset of columns  $S = \emptyset$ . At iteration  $i$  of the algorithm, choose a column  $C$  from  $\mathcal{C} - S$  of maximal weight that is linearly independent of the columns from  $S$  and update  $S$  to  $S \cup \{C\}$ . When an iteration finds no such column, stop and output  $S$ .

It is easy to check that the algorithm works and that it is of polynomial time (provided the fact that checking linear independence can be done in polynomial time).

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## Olympiad problems

O337. Does there exist an irreducible polynomial  $P(x)$  with integer coefficients such that  $P(n)$  is a perfect power greater than 1 for all  $n \in \mathbb{N}$ ?

*Proposed by Oleksiy Klurman, Université de Montréal, Canada*

*Solution by the author*

The answer is "No". First observe that since  $P(x)$  is irreducible then  $\gcd(P(n), P'(n)) \leq M$  for some  $M > 0$  and all  $n$ .

This implies that there exist infinitely many primes  $p$ , such that  $p|P(n)$  and  $p$  does not divide  $P'(n)$  for some  $n \in \mathbb{Z}$ . The result then follows from the following lemma.

*Lemma.* Suppose  $p^k \parallel P(n_0)$  and  $p \nmid P'(n_0)$ . Then, there exist  $n_p$  such that  $p \parallel P(n_p)$ .

*Proof.* We proceed in the spirit of Hensel's lemma. For  $t \in \mathbb{N}$  consider the Taylor expansion

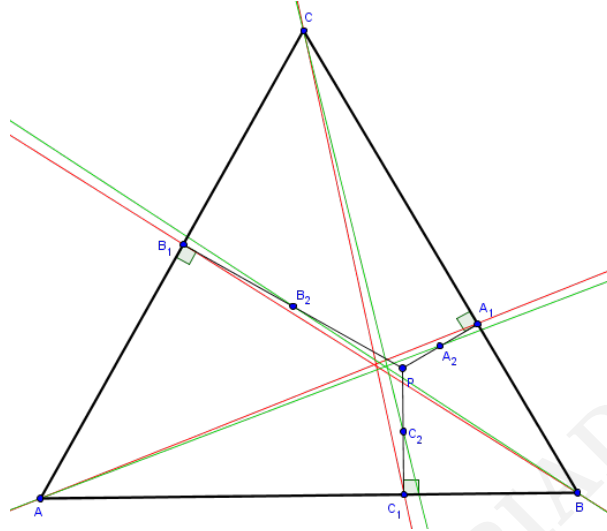
$$P(n_0 + tp) = P(n_0) + tpP'(n_0) + p^2G(n_0, t),$$

for some  $G(n_0, t) \in \mathbb{Z}$ . Now if  $k = 1$ , we may take  $t = 0$  and  $n_0 = n_p$ . Alternatively, if  $k \geq 2$  take  $t = 1$  and set  $n_p = n_0 + p$ .

O338. Let  $P$  be a point inside equilateral triangle  $ABC$ . Denote by  $A_1, B_1, C_1$  the projections of  $P$  onto triangle's sides and denote by  $A_2, B_2, C_2$  the midpoints of  $PA_1, PA_2, PA_3$ , respectively. Prove that  $AA_1, BB_1, CC_1$  are concurrent if and only if  $AA_2, BB_2, CC_2$  are concurrent.

*Proposed by Nairi Sedrakyan, Armenia*

*Solution by Andrea Fanchini, Cantù, Italy*



Using barycentric coordinates we denote  $P(u : v : w)$  and using Conway's notation being the triangle equilateral we have  $a = b = c$  and  $S_A = S_B = S_C = \frac{a^2}{2}$ .

Furthermore the infinite perpendicular points of the sides are

$$BC_{\infty\perp}(-a^2 : S_C : S_B) = (-2 : 1 : 1), \quad CA_{\infty\perp}(S_C : -b^2 : S_A) = (1 : -2 : 1), \quad AB_{\infty\perp}(S_B : S_A : -c^2) = (1 : 1 : -2)$$

then line  $PBC_{\infty\perp}$  have equation

$$\begin{vmatrix} -2 & 1 & 1 \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \Rightarrow PBC_{\infty\perp} \equiv (w - v)x + (2w + u)y - (2v + u)z = 0$$

$$\text{so } A_1 = PBC_{\infty\perp} \cap BC = (0 : 2v + u : 2w + u), \Rightarrow \left(0, \frac{2v+u}{2(u+v+w)}, \frac{2w+u}{2(u+v+w)}\right).$$

Now line  $AA_1$  have equation

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2v + u & 2w + u \\ x & y & z \end{vmatrix} = 0 \Rightarrow AA_1 \equiv -(2w + u)y + (2v + u)z = 0$$

Furthermore the midpoint  $A_2$  will have coordinates  $A_2 = (2u : 4v + u : 4w + u)$ .

Then the line  $AA_2$  have equation

$$\begin{vmatrix} 1 & 0 & 0 \\ 2u & 4v + u & 4w + u \\ x & y & z \end{vmatrix} = 0 \Rightarrow AA_2 \equiv -(4w + u)y + (4v + u)z = 0$$

Ciclically we find that the lines  $BB_1, CC_1$  and  $BB_2, CC_2$  have equation

$$BB_1 \equiv (2w + v)x - (2u + v)z = 0, \quad CC_1 \equiv -(2v + w)x + (2u + w)y = 0$$

$$BB_2 \equiv (4w + v)x - (4u + v)z = 0, \quad CC_2 \equiv -(4v + w)x + (4u + w)y = 0$$



Now three lines  $p_ix + q_iy + r_iz = 0$ , with  $i = 1, 2, 3$ , are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

so  $AA_1, BB_1, CC_1$  are concurrent if and only if

$$\begin{vmatrix} 0 & -(2w + u) & 2v + u \\ 2w + v & 0 & -(2u + v) \\ -(2v + w) & 2u + w & 0 \end{vmatrix} = 0$$

that is if  $u^2(w - v) + v^2(u - w) + w^2(v - u) = 0$ .

Now  $AA_2, BB_2, CC_2$  are concurrent if and only if

$$\begin{vmatrix} 0 & -(4w + u) & 4v + u \\ 4w + v & 0 & -(4u + v) \\ -(4v + w) & 4u + w & 0 \end{vmatrix} = 0$$

that is if  $u^2(w - v) + v^2(u - w) + w^2(v - u) = 0$ , so we prove that  $AA_1, BB_1, CC_1$  are concurrent if and only if  $AA_2, BB_2, CC_2$  are concurrent, q.e.d.

*Also solved by Sardor Bozorboyev, S.H.Sirojiddinov lyceum, Tashkent, Uzbekistan; Daniel Lasaosa, Pamplona, Spain.*

O339. Let  $a, b, c$  be positive real numbers satisfying

$$\frac{1}{a^3 + b^3 + 1} + \frac{1}{b^3 + c^3 + 1} + \frac{1}{c^3 + a^3 + 1} \geq 1.$$

Prove that

$$(a + b)(b + c)(c + a) \leq 6 + \frac{2}{3}(a^3 + b^3 + c^3).$$

*Proposed by Nguyen Viet Hung, Hanoi, Vietnam*

*Solution by Li Zhou, Polk State College, Winter Haven, FL, USA*

By clearing the denominators, we see that the given condition becomes

$$2(1 + a^3 + b^3 + c^3) \geq (a^3 + b^3)(b^3 + c^3)(c^3 + a^3).$$

Since  $a^3 + b^3 \geq 2(\frac{a+b}{2})^3$ , we further get that  $2(1 + a^3 + b^3 + c^3) \geq 8x^3$ , where  $x = \frac{1}{8}(a + b)(b + c)(c + a)$ . Therefore

$$2(a^3 + b^3 + c^3) + 18 - 24x \geq 8x^3 - 24x + 16 = 8(x - 1)^2(x + 2) \geq 0,$$

which is equivalent to the claimed inequality.

*Also solved by Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Daniel Lasasosa, Pamplona, Spain; S.Viswanathan, Center for Scientific Learning, Nagpur, Maharashtra, India; George Gavrilopoulos, Nea Makri High School, Athens, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.*

O340. Let  $ABC$  be a triangle and let  $BB'$  and  $CC'$  be the altitudes from  $B$  and  $C$ , respectively. Ray  $C'B'$  intersects the circumcircle of triangle  $ABC$  at  $B''$ . Let  $\alpha_A$  be the measure of angle  $ABB''$ . Similarly, define angles  $\alpha_B$  and  $\alpha_C$ . Prove that

$$\sin \alpha_A \sin \alpha_B \sin \alpha_C \leq \frac{3\sqrt{3}}{32}.$$

*Proposed by Dorin Andrica, Cluj-Napoca, Romania*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Note first that  $AB'C'$  is similar to  $ABC$ , with proportionality factor equal to  $\cos A$  (it suffices to note that  $BC'B'C$  is cyclic with circumcircle with diameter  $BC$  since  $\angle BB'C = \angle BC'C = 90^\circ$ ). Moreover, since  $AB'C'$  can be obtained from  $ABC$  by reflecting on the internal bisector of angle  $A$ , and then performing a homothety with center  $A$ , it follows that the altitude from  $A$  in  $AB'C'$  is the symmetric of  $AA'$  with respect to the internal bisector of angle  $A$ , hence the altitude from  $A$  in  $AB'C'$  passes through the circumcenter  $O$  of  $ABC$ . Or the triangle whose vertices are  $A$ , and the intersections of  $B'C'$  with the circumcircle of  $ABC$ , is isosceles at  $A$ , and  $\angle AB''B' = \angle ABB'' = \alpha_A$ . It follows that the length of the altitude from  $A$  onto  $B'C'$  is  $AB'' \sin \alpha_A = 2R \sin^2 \alpha_A = 2R \sin B \sin C \cos A$ , or  $\sin \alpha_A = \sqrt{\sin B \sin C \cos A}$ , and similarly for  $\alpha_B$  and  $\alpha_C$ , or

$$\sin \alpha_A \sin \alpha_B \sin \alpha_C = \sin A \sin B \sin C \sqrt{\cos A \cos B \cos C}.$$

Now,  $ABC$  is acute (if for example angle  $A$  is obtuse,  $C'B'$  does not intersect the circumcircle of  $ABC$  and  $B''$  is not defined), hence

$$\begin{aligned} \cos A \cos B \cos C &= \frac{\cos C (\cos(A - B) - \cos C)}{2} \leq \frac{\cos C - \cos^2 C}{2} = \\ &= \frac{1 - (1 - 2 \cos C)^2}{8} \leq \frac{1}{8}, \end{aligned}$$

with equality iff  $C = 60^\circ$  and simultaneously  $A = B$ , i.e. iff  $ABC$  is equilateral. On the other hand, and denoting by  $S$  the area of  $ABC$ , we have

$$\sin A \sin B \sin C = \frac{abc}{8R^3} = \frac{S}{2R^2} \leq (\sin 60^\circ)^3 = \frac{3\sqrt{3}}{8},$$

since it is well known that for fixed circumradius, the area of  $ABC$  is largest when  $ABC$  is equilateral. It follows that

$$\sin \alpha_A \sin \alpha_B \sin \alpha_C = \sin A \sin B \sin C \sqrt{\cos A \cos B \cos C} \leq \frac{3\sqrt{3}}{8} \sqrt{\frac{1}{8}} = \frac{3\sqrt{6}}{32},$$

with equality iff  $ABC$  is equilateral.

*Also solved by Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan; Li Zhou, Polk State College, Winter Haven, FL, USA.*

O341. Let  $a$  be a positive integer. Find all nonzero integer polynomials  $P(X), Q(X)$  such that

$$P(X)^2 + aP(X)Q(X) + Q(X)^2 = P(X).$$

*Proposed by Mircea Becheanu, Bucharest, Romania*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Assume that either  $\deg(P)$  or  $\deg(Q)$  is at least 1. Note that if  $\deg(P) > \deg(Q)$ , then the LHS has degree  $2\deg(P)$ , while the RHS has degree  $\deg(P)$ , absurd. If  $\deg(Q) > \deg(P)$ , then the LHS has degree  $2\deg(Q)$ , but  $2\deg(Q) > 2\deg(P) \geq \deg(P)$ , absurd again. We therefore need either  $\deg(P) = \deg(Q) \geq 1$ , or  $P(x), Q(x)$  both constant.

When  $P(x), Q(x)$  are not both constant, let  $\deg(P) = \deg(Q) = u \geq 1$ . Let  $A, B$  be the (nonzero by definition) leading coefficients of  $P(x), Q(x)$ , or the coefficients of  $x^{2u}$  in the LHS and the RHS are respectively  $A^2 + aAB + B^2$  and 0. Note therefore that

$$\frac{A}{B} = \frac{-a \pm \sqrt{a^2 - 4}}{2},$$

where  $\sqrt{a^2 - 4}$  must be at the same time rational (because  $A, B$  are nonzero integers) and the square root of an integer (because  $a$  is an integer). Hence  $a^2 - 4$  must be a perfect square, or since the only two squares which differ by 4 are 0, 4, it follows that  $a^2 = 4$ , and  $a = 2$  because  $a$  is positive by definition. Therefore,  $P(x) = (P(x) + Q(x))^2$  is the square of a polynomial  $p(x) = P(x) + Q(x)$  with integer coefficients, or when  $P(x), Q(x)$  are not constant, solutions exist only iff  $a = 2$ , in which case for any nonconstant polynomial  $p(x)$  with integer coefficients,

$$P(x) = (p(x))^2, \quad Q(x) = p(x) - (p(x))^2,$$

is a solution, and there can be no solutions of any other form, or for any other positive integer  $a$ .

Consider now the case where  $P(x), Q(x)$  are both constant, or the problem reduces to finding all integer solutions to the equation  $p^2 + apq + q^2 = p$  for positive integer  $a$ . When  $a = 1$ , the quadratic equation  $q^2 + pq + p^2 - p = 0$  in unknown  $q$  has discriminant  $4p - 3p^4$ , always negative when either  $p < 0$  or  $p \geq 2$ , hence the only nonzero solution can be  $p = 1$ , which in turn results in  $q = -1$  for  $q$  to be nonzero. There can be no other solution than  $(p, q) = (1, -1)$  when  $a = 1$ .

Consider now  $a \geq 2$ . Note now that quadratic equation  $p^2 + (aq - 1)p + q^2 = 0$  in unknown  $p$  has two roots with product  $q^2$ . If both roots are equal, then  $|p| = |q|$ , for  $p^2(a + 2) = |p|$ , or  $|p|(a + 2) = 1$ , absurd since  $a + 2 \geq 3$ . It follows that if  $(p, q)$  is a solution in nonzero integers, then so is  $(p', q) = \left(\frac{q^2}{p}, q\right)$ , where  $\frac{q^2}{p} = 1 - aq - p$  is clearly a nonzero integer. Similarly, quadratic equation  $q^2 + apq + p^2 - p = 0$  can only have two equal roots if  $p^2 - p = p(p - 1)$  is a perfect square, or since  $p - 1, p$  are coprime, both must be perfect squares. But the only two consecutive integers which are perfect squares are 0, 1, or both roots can only be equal for nonzero  $p$  when  $p = 1$ , yielding  $q(q + a) = 0$ , which clearly has two distinct roots, and only  $q = -a$  is nonzero. In any other case, note that if  $(p, q)$  is a solution, then so is  $(p, q') = \left(p, \frac{p^2 - p}{q}\right)$ , where  $\frac{p^2 - p}{q} = -ap - q$  is clearly a nonzero integer unless  $p = 1$ . Since  $|p| = |q|$  has been ruled out, note that if  $|p| > |q|$  then  $|p'| < |q|$ , whereas if  $|q| > |p|$ , then  $|q'| < |p - 1| < |p|$ , or starting from any solution  $(p, q)$ , we may generate a new solution with lower value of  $|pq|$ . Since  $|pq|$  is an integer, the process cannot continue indefinitely, and by the previous results we will eventually end up in a solution with  $p = 1$ , and consequently  $q = -a$ . Note that from this solution, we may generate back the initial solution, since the steps are reversible. In other words, there are infinitely many solutions for any  $a$ , which can be generated from  $(p_0, q_0) = (1, -a)$  by

$$(p_{2n+1}, q_{2n+1}) = \left(\frac{q_{2n}^2}{p_{2n}}, q_{2n}\right), \quad (p_{2n+2}, q_{2n+2}) = \left(p_{2n+1}, \frac{p_{2n+1}^2 - p_{2n+1}}{q_{2n+1}}\right).$$

This is true for all  $a \geq 2$ , since it can easily be proved by induction that  $p_{2n+2} = p_{2n+1} > p_{2n} > 0$  for all  $n \geq 0$ , and similarly  $q_{2n+2} > q_{2n+1} = q_{2n}$  for all  $n \geq 0$ . In the case  $a = 1$ , which was treated separately earlier on, we find  $(p_1, q_1) = (p_0, q_0) = (1, -1)$ , and  $q_2 = 0$ , or no other solutions are generated by this recursive method. All other values of  $a \geq 2$  have infinitely many solutions, and the previously describe recursion indeed generates solutions with increasing product  $|p_n q_n|$ .

Note that we can give a closed form for this sequence of solutions, by defining  $r_0 = 1$ ,  $r_1 = a$ , and for all  $n \geq 0$ ,  $r_{n+2} = ar_{n+1} - r_n$ , and then taking

$$p_{2n-1} = p_{2n} = r_n^2, \quad q_{2n} = q_{2n+1} = -r_n r_{n+1}.$$

Indeed,

$$p_{2n}^2 + ap_{2n}q_{2n} + q_{2n}^2 - p_{2n} = (r_{n-1}^2 - r_n r_{n-2} - 1) r_n^2$$

and it can be trivially found by backward induction that

$$r_{n-1}^2 - r_n r_{n-2} - 1 = r_{n-2}^2 - r_{n-1} r_{n-3} - 1 = r_1^2 - r_2 r_0 - 1 = a^2 - (a^2 - 1) - 1 = 0.$$

Solving the recursive relation by the usual methods yields, for  $a \geq 3$ ,

$$\begin{aligned} r_n &= \frac{1}{\sqrt{a^2 - 4}} \left( \left( \frac{a + \sqrt{a^2 - 4}}{2} \right)^{n+1} - \left( \frac{a - \sqrt{a^2 - 4}}{2} \right)^{n+1} \right), \\ p_{2n-1} = p_{2n} &= \frac{1}{a^2 - 4} \left( \left( \frac{a + \sqrt{a^2 - 4}}{2} \right)^{2n+2} + \left( \frac{a - \sqrt{a^2 - 4}}{2} \right)^{2n+2} - 2 \right), \\ q_{2n} = q_{2n+1} &= -\frac{1}{a^2 - 4} \left( \left( \frac{a + \sqrt{a^2 - 4}}{2} \right)^{2n+3} + \left( \frac{a - \sqrt{a^2 - 4}}{2} \right)^{2n+3} - a \right). \end{aligned}$$

For  $a = 2$ , the characteristic equation of the recursive relation becomes  $\rho^2 - 2\rho + 1 = (\rho - 1)^2 = 0$ , or the solution is of the form  $r_n = An + B$ , hence for the initial conditions  $r_0 = 1$  and  $r_1 = a = 2$  to be satisfied we find  $r_n = n + 1$ , yielding

$$p_{2n-1} = p_{2n} = (n + 1)^2, \quad q_{2n} = q_{2n+1} = -(n + 1)(n + 2).$$

*Also solved by Sardor Bozorboyev, S.H.Sirojjidinov lyceum, Tashkent, Uzbekistan.*

O342. Let  $n \geq 2015$  be a positive integer and let  $A \subset \{1, \dots, n\}$  such that  $|A| \geq \lfloor \frac{n+1}{2} \rfloor$ . Prove that  $A$  contains a three term arithmetic progression.

*Proposed by Oleksiy Klurman, Université de Montréal, Canada*

*First solution by the author*

We begin with the following simple lemma.

*Lemma.* If  $B \subset \{1, 2, \dots, 8\}$  and  $|B| \geq 5$ , then  $B$  contains a three term arithmetic progression.

*Proof.* Suppose that this is not the case and consider the sets  $B_1 = \{1, 2, 3, 4\}$  and  $B_2 = \{5, 6, 7, 8\}$ . One of them must contain at least three elements. Without loss of generality, we may assume that  $|B_1 \cap A| \geq 3$ . We now have two possibilities:  $1, 2, 4 \in B$  or  $1, 3, 4 \in B$  (in the other cases we readily have the arithmetic progression in  $B_1$ ). In the first case, numbers 6 and 7 cannot belong to  $B_2$ , and thus  $B_2 = \{5, 8\}$ . This is also impossible since we then have a progression of the form  $2, 5, 8$ . The second case is totally analogous. This completes the proof of the lemma.

We now turn to the solution of the main problem. Let  $n = 8k + r \geq 1000$  and  $n$  is odd. We may assume that  $|A| = 4k + \lfloor \frac{r+1}{2} \rfloor$ . For any  $0 \leq k_0 \leq k$  consider partition of the set

$$\{1, 2, \dots, n\} \setminus [8k_0 + 1, 8k_0 + r] = [1, 8] \cup \dots \cup [8(k_0 - 1) + 1, 8k_0] \cup [8k_0 + r + 1, 8(k_0 + 1) + r] \cup \dots$$

Each interval from this partition contains at most 4 elements  $A$  and therefore the interval  $[8k_0 + 1, 8k_0 + r]$  must contain at least  $\lfloor \frac{r+1}{2} \rfloor$  elements from  $A$ .

Consider the interval  $[8t + 1, 8t + r]$  where  $t$  is an even number and assign to it the array  $(x_1, x_2, \dots, x_r)$  of 0 and 1 in the following way:  $x_i = 1$  iff  $8t + i \in A$  and 0 otherwise. Observe, that for  $r$  elements we have at most  $\binom{r}{r+1/2} \leq \binom{7}{4} = 35$  different arrays. On the other hand, the number of intervals  $[8t + 1, 8t + r]$  with even  $t$  is  $\lfloor \frac{k+2}{2} \rfloor = \lfloor \frac{n-r}{16} + 1 \rfloor \geq 50$ . By the pigeonhole principle there exist intervals  $[8m + 1, 8m + r]$  and  $[8n + 1, 8n + r]$  that have identical corresponding arrays.

This implies that the interval  $[4(m+n) + 1, 4(m+n) + r]$  cannot contain more than  $r - \lfloor \frac{r+1}{2} \rfloor$  elements from  $A$ . This contradiction finishes the proof for odd  $n \geq 1013$ .

If  $n = 2k$ , then  $[1, 2, \dots, n] = [1, 2, \dots, 2\lfloor k/2 \rfloor - 1] \cup [2\lfloor k/2 \rfloor, \dots, 2k] = A_1 \cup A_2$  and each subset contains odd number of elements. Clearly,  $|A_2| > |A_1| \geq 1011$  and we can use the same arguments as in the previous case.

*Second solution by Daniel Lasaosa, Pamplona, Spain*

We first note that given a set  $K$  of integers  $k$ , there exists a three-term arithmetic progression in  $K$  iff there exists a three-term arithmetic progression in  $K - d$ , iff there exists a three-term arithmetic progression in  $d - K$  for any integer  $d$ . Denote now by  $m(n)$  the maximum number  $m$  of elements of  $\{1, 2, \dots, n\}$  such that no three of them are in arithmetic progression. The observation  $m(n_1 + n_2) \leq m(n_1) + m(n_2)$  is clearly true, since if we can find  $m$  elements in  $\{1, 2, \dots, n_1 + n_2\}$  such that no three of them are in arithmetic progression, then there are  $m_1 \leq m(n_1)$  of those elements in  $\{1, 2, \dots, n_1\}$ , and  $m_2 \leq m(n_2)$  of those elements in  $\{n_1 + 1, n_1 + 2, \dots, n_1 + n_2\}$ , with  $m(n_1 + n_2) = m_1 + m_2$ . As a trivial consequence,  $m(n)$  is non-decreasing.

*Claim 1:*  $m(8) \leq 4$ .

*Proof 1:* Assume that a 5-element subset  $K$  of  $\{1, 2, \dots, 8\}$  exists such that no three of them are in arithmetic progression. Considering  $9 - K$ , wlog three of them are in  $\{1, 2, 3, 4\}$ , or since  $1, 2, 3$  and  $2, 3, 4$  are arithmetic progressions,  $K$  must include either  $1, 2, 4$  or  $1, 3, 4$ . Now  $1, 2, 4$  rules out  $6, 7$  but  $2, 5, 8$  is an arithmetic progression, while  $1, 3, 4$  rules out  $5, 7$  but  $4, 6, 8$  is an arithmetic progression, with contradiction in both cases. The Claim 1 follows.

*Claim 2:*  $m(13) = 7$  with equality for the subsets  $\{1, 2, 4, 5, 10, 11, 13\}$ ,  $\{1, 2, 4, 5, 10, 12, 13\}$ ,  $\{1, 2, 4, 8, 10, 11, 13\}$ ,  $\{1, 2, 4, 9, 10, 12, 13\}$ ,  $\{1, 3, 4, 6, 10, 12, 13\}$ ,  $\{1, 3, 4, 9, 10, 12, 13\}$ .

*Proof 2:* Assume that a 7-element subset  $K$  of  $\{1, 2, \dots, 13\}$  exists such that 7 is not in  $K$ , or wlog 4 elements of  $K$  must be in  $\{1, 2, \dots, 6\}$ . It is not hard to see that these 4 elements must be  $\{1, 2, 4, 5\}$ ,  $\{1, 2, 5, 6\}$ ,  $\{1, 3, 4, 6\}$  or  $\{2, 3, 5, 6\}$ . These subsets rule out respectively  $3, 6, 7, 8, 9$ , and  $3, 4, 7, 8, 9, 10, 11$ , and  $2, 5, 7, 8, 9, 11$ , and  $1, 4, 7, 8, 9, 10$ . In the second case there are only 2 elements left to choose, while in the fourth case there are exactly 3 elements left to choose,  $11, 12, 13$ , which are already in arithmetic progression. In the third case we obtain  $\{1, 3, 4, 6, 10, 12, 13\}$ , and in the first case we must choose 3 elements not in arithmetic progression from  $\{10, 11, 12, 13\}$ , which can be done in exactly two ways, for sets  $\{1, 2, 4, 5, 10, 11, 13\}$ ,  $\{1, 2, 4, 5, 10, 12, 13\}$ . Once we add sets  $14 - K$  for each one of the  $K$ 's found, the remaining sets must contain 7, and by our previous argument, must contain exactly 3 elements in  $\{1, 2, \dots, 6\}$ , and exactly 3 elements in  $\{8, 9, \dots, 13\}$ , where moreover  $k, 14 - k$  cannot simultaneously appear. Therefore, if  $5, 7$  are in the set,  $3, 6, 9$  are not in the set, or  $11, 8$  are in the set, contradiction since  $5, 8, 11$  are an arithmetic progression. We reach a similar contradiction if  $7, 9$  are in the set. There are therefore no more sets for  $m(13) = 7$ . The Claim 2 follows.

*Claim 3:*  $m(17) \leq 8$ .

*Proof 3:* Assume that a 9-element subset  $K$  of  $\{1, 2, \dots, 17\}$  can be found which contains no three-element arithmetic progression. All of the subsets in Claim 2 rule out all but at most one of the elements in  $\{14, 15, 16, 17\}$ , or there are at most 6 elements in  $\{1, 2, \dots, 13\}$ , hence at least 3 elements in  $\{14, 15, 16, 17\}$ , and similarly at least 3 elements in  $\{1, 2, 3, 4\}$ . Clearly all 4 element subsets of  $\{1, 2, 3, 4\}$  contain a three-element arithmetic progression, or either  $\{1, 2, 4\}$  or  $\{1, 3, 4\}$  must be in  $K$ . Similarly, either  $\{14, 16, 17\}$  or  $\{14, 15, 17\}$  must be in  $K$ . Now,  $1, 2, 4$  rule out  $6, 7$  and  $1, 3, 4$  rule out  $5, 7$ , while  $14, 16, 17$  rule out  $11, 12$  and  $14, 15, 17$  rule out  $11, 13$ . Note further that neither 7 nor 11 can be in  $K$ , while  $1, 17$  are both in  $K$ , or 9 cannot be in  $K$ . Note finally that since 4 is in  $K$ ,  $8, 12$  cannot be simultaneously in  $K$ , and since 14 is in  $K$ ,  $6, 10$  cannot be simultaneously in  $K$ . It follows that, wlog by the symmetry around 9, 5 must be in  $K$ , or  $\{1, 2, 4\}$  must be in  $K$ , hence  $6, 8$  cannot be in  $K$ . This means that two out of  $10, 12, 13$  must be in  $K$ , which rules out 14, contradiction. The Claim 3 follows.

*Claim 4:*  $m(19) \leq 9$ .

*Proof 4:* Assume that a 10-element subset  $K$  of  $\{1, 2, \dots, 19\}$  can be found which contains no three-element arithmetic progression. All of the subsets in Claim 2 rule out all but at most one element in  $\{14, 15, \dots, 19\}$ , or there are at most 6 elements in  $\{1, 2, \dots, 13\}$ , hence at least 4 elements in  $\{14, 15, \dots, 19\}$ , and similarly there are at least 4 elements in  $\{1, 2, \dots, 6\}$ . Note that every 5-element subset of  $\{1, 2, \dots, 6\}$  contains one of the two arithmetic progressions  $1, 2, 3$  or  $4, 5, 6$ , or there are exactly four elements in  $\{1, 2, \dots, 6\}$ , and four elements in  $\{14, 15, \dots, 19\}$ . The 4-element subsets of  $\{1, 2, \dots, 6\}$  which do not contain a three-term arithmetic progression are  $\{1, 2, 4, 5\}$ ,  $\{1, 2, 5, 6\}$ ,  $\{1, 3, 4, 6\}$  and  $\{2, 3, 5, 6\}$ . They respectively rule out  $7, 8, 9$ , and  $7, 8, 9, 10, 11$ , and  $7, 8, 9, 11$ , and  $7, 8, 9, 10$ . Similarly,  $\{15, 16, 18, 19\}$ ,  $\{14, 15, 18, 19\}$ ,

$\{14, 16, 17, 19\}$  and  $\{14, 15, 17, 18\}$  respectively rule out 11, 12, 13, and 9, 10, 11, 12, 13, and 9, 11, 12, 13, and 10, 11, 12, 13. It follows that 7, 8, 9, 11, 12, 13 can never be in  $K$ , or 10 must be in  $K$ , which means that  $20 - k$  cannot be in  $K$  as long as  $k$  is in  $K$ . But if 1 is in  $K$ , 19 is not in  $K$ , or 14, 15, 17, 18 are in  $K$ , hence 2, 3, 5, 6 are not in  $K$ , absurd.

*Claim 5:*  $m(21) \leq 10$ .

*Proof 5:* Assume that an 11-element subset  $K$  of  $\{1, 2, \dots, 21\}$  can be found which contains no three-element arithmetic progression. All of the subsets in Claim 2 rule out all but at most one element in  $\{14, 15, \dots, 21\}$ , or there are at most 6 elements in  $\{1, 2, \dots, 13\}$ , hence at least 5 elements in  $\{14, 15, \dots, 21\}$ , and similarly there are at least 5 elements in  $\{1, 2, \dots, 8\}$ , absurd because of Claim 1. The Claim 5 follows.

It follows from Claims 3, 4 and 5 and by the fact that  $m(n)$  is non-decreasing, that

$$\begin{aligned} m(33) &\leq m(16) + m(17) \leq 16, & m(34) &\leq 2m(17) \leq 16, \\ m(35) &\leq m(17) + m(18) \leq 17, & m(36) &\leq m(17) + m(19) \leq 17, \\ m(37) &\leq m(18) + m(19) \leq 18, & m(38) &\leq 2m(19) \leq 18, \\ m(39) &\leq m(19) + m(20) \leq 19, & m(40) &\leq m(19) + m(21) \leq 19. \end{aligned}$$

Or  $m(n) \leq \lfloor \frac{n-1}{2} \rfloor$  for 8 consecutive values of  $n$  starting at 33, or since  $m(8) = 4$ , this result is by trivial induction clearly true for all  $n \geq 33$ . Therefore, for any  $n \geq 33$ , taking  $\lfloor \frac{n+1}{2} \rfloor$  elements out of  $\{1, 2, \dots, n\}$ , we guarantee that there will be at least one three-term arithmetic progression. The conclusion follows, the lower bound having been improved from 2015 to 33.

*Note:* Using numerical computation, the bound for  $m(19)$  can be lowered to 8, the bound for  $m(21)$  to 9, and so on, or the bound  $\lfloor \frac{n+1}{2} \rfloor$  can also be significantly reduced. In fact, for  $n \geq 70$  we have  $m(n) < \frac{n}{3}$ , and increasing the base case for the induction we can obtain still higher values of  $u$  such that  $m(n) < \frac{n}{u}$  for  $n \geq 2015$ .



*Third solution by Li Zhou, Polk State College, USA*

This is a well-known topic started by Erdős and Turan in 1936. See, for example, Arun Sharma, Sequences of Integers Avoiding 3-term Arithmetic Progressions, *Electronic J. of Combinatorics* **19** (2012), which gives  $r(27) = 11$ , where  $r(n)$  is the cardinality of the largest subsets of  $\{1, \dots, n\}$  avoiding 3-term arithmetic progression (It is easier to prove the weaker  $r(27) \leq 13$  by writing  $1, 2, \dots, 27$  in a  $3 \times 9$  array.). Hence, for  $n \geq 2015$ ,

$$r(n) \leq \frac{13n}{27} + 26 < \frac{n-1}{2} < \left\lfloor \frac{n+1}{2} \right\rfloor,$$

completing the proof.

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