Junior problems

J151. Let $a \geq b \geq c > 0$. Prove that

$$(a-b+c)\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right) \geq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J152. Let $a, b, c > 0$. Prove that the following inequality holds

$$\frac{a+b}{a+b+2c}+\frac{b+c}{b+c+2a}+\frac{c+a}{c+a+2b}+\frac{2(ab+bc+ca)}{3\left(a^2+b^2+c^2\right)} \leq \frac{13}{6}.$$

Proposed by Andrei Răzvan Băleanu, "George Coşbuc" College, Motru, Romania

J153. Find all integers n such that $n^2+2010n$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J154. Let ABC be an acute triangle and let $MNPQ$ be a rectangle inscribed in the triangle such that $M, N \in BC, P \in AC, Q \in AB$. Prove that

$$\text{area} MNPQ \leq \frac{1}{2} \text{area} ABC.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

J155. Find all n for which there are n consecutive integers whose sum of squares is a prime.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J156. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) + f(x+y)$ is a rational number for all real numbers x and all $y > 0$. Prove that $f(x)$ is a rational number for all real numbers x.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Buzau, Romania

Senior problems

S151. Find all triples (x, y, z) of real numbers such that

$$x^2 + y^2 + z^2 + 1 = xy + yz + zx + |x - 2y + z|.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S152. Let $k \geq 2$ be an integer and let $m, n \geq 2$ be relatively prime integers. Prove that the equation

$$x_1^m + x_2^m + ... + x_k^m = x_{k+1}^n$$

has infinitely many solutions in distinct positive integers.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

S153. Let X be a point interior to a convex quadrilateral $ABCD$. Denote by P, Q, R, S the orthogonal projections of X onto AB, BC, CD, DA, respectively. Prove that

$$PA \cdot AB + RC \cdot CD = \frac{1}{2}(AD^2 + BC^2)$$

if and only if

$$QB \cdot BC + SD \cdot DA = \frac{1}{2}(AB^2 + CD^2).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S154. Let $k \geq 2$ be an integer and let $n_1, \ldots, n_k$ be positive integers. Prove that there are no rational numbers $x_1, \ldots, x_k, y_1, \ldots, y_k$ such that

$$(x_1 + y_1\sqrt{2})^{2n_1} + \cdots + (x_k + y_k\sqrt{2})^{2n_k} = 5 + 4\sqrt{2}.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

S155. Let a, b, c, d be the complex numbers corresponding to the vertices A, B, C, D of a convex quadrilateral $ABCD$. Given that $a\overline{c} = \overline{a}c$, $b\overline{d} = \overline{b}d$ and $a+b+c+d = 0$, prove that $ABCD$ is a parallelogram.

Proposed by Bogdan Enescu, "B.P.Hasdeu" National College, Buzau, Romania

S156. Let $f : \mathbf{N} \to [0, \infty)$ be a function satisfying the following conditions:

(a) $f(100) = 10$;

(b) $\frac{1}{f(0)+f(1)} + \frac{1}{f(1)+f(2)} + ... + \frac{1}{f(n)+f(n+1)} = f(n+1)$, for all nonnegative integers n.

Find $f(n)$ in closed form.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

Undergraduate problems

U151. Let n be a positive integer and let

$$f(x) = x^{n+8} - 10x^{n+6} + 2x^{n+4} - 10x^{n+2} + x^n + x^3 - 10x + 1.$$

Evaluate $f(\sqrt{2} + \sqrt{3})$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

U152. Prove that for $n \geq 3$,

$$\varphi(2) + \varphi(3) + \cdots + \varphi(n) \geq \frac{n(n-1)}{4} + 1,$$

where φ is the Euler's totient function.

Proposed by Yufei Zhao, Massachusetts Institute of Technology, USA

U153. Let a, b, c, d be non-zero complex numbers such that $ad - bc \neq 0$ and let n be a positive integer. Consider the equation

$$(ax + b)^n + (cx + d)^n = 0.$$

(a) Prove that for $|a| = |c|$ the roots of the equation are situated on a line.

(b) Prove that for $|a| \neq |c|$ the roots of the equation are situated on a circle.

(c) Find the radius of the circle when $|a| \neq |c|$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

U154. Find sufficient and necessary conditions on $a, b \in \mathbb{R}$ so that the set

$$S_{a,b} = \{(\{na\}, \{nb\}) | n \in \mathbb{N}\}$$

is dense in the unit square $[0, 1]^2$.

Proposed by Holden Lee, Massachusetts Institute of Technology, USA

U155. Evaluate

$$\int_{1/3}^{1/2} \frac{\tan 2x - \cot 3x}{x} dx.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U156. Let $f : [a, b] \to \mathbf{R}$ be a continuous function such that $\int_0^1 xf(x)dx = 0$. Prove that

$$\left| \int_0^1 x^2 f(x) dx \right| \leq \frac{1}{6} \max_{x \in [0,1]} |f(x)|.$$

Proposed by Duong Viet Thong, National Economics University, Hanoi, Vietnam

Olympiad problems

O151. Consider a triangle ABC and a point P in its interior. Lines PA, PB, PC intersect BC, CA, AB at A', B', C', respectively. Prove that

$$\frac{BA'}{BC}+\frac{CB'}{CA}+\frac{AC'}{AB}=\frac{3}{2}$$

if and only if at least two of the triangles PAB, PBC, PCA have the same area.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O152. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be sequences defined by $a_{n+3}=a_{n+2}+2a_{n+1}+a_n$, $n=0,1,...$, $a_0=1$, $a_1=2$, $a_2=3$ and $b_{n+3}=b_{n+2}+2b_{n+1}+b_n$, $n=0,1,...$, $b_0=3$, $b_1=2$, $b_2=1$. How many integers do the sequences have in common?

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

O153. Find all triples (x,y,z) of integers such that $x^2y+y^2z+z^2x=2010^2$ and $xy^2+yz^2+zx^2=-2010$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O154. A non-isosceles acute triangle ABC is given. Let O, I, H be the circumcenter, the incenter, and the orthocenter of the triangle ABC, respectively. Prove that $\angle OIH > 135^\circ$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O155. Prove that the equation

$$x^2+y^3=4z^6$$

is not solvable in integers.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O156. In a cyclic quadrilateral $ABCD$ with $AB=AD$ points M, N lie on the sides BC and CD, respectively so that $MN=BM+DN$. Lines AM and AN meet the circumcircle of $ABCD$ again at points P and Q, respectively. Prove that the orthocenter of the triangle APQ lies on the segment MN.

Proposed by Nairi Sedrakyan, Yerevan, Armenia