UM EECS 270 F22
Introduction to Logic Design

5. Switching Functions

Switching Functions

- $f(x_1, x_2, \ldots, x_n)$ is a mapping from $B_2^n \rightarrow B_2$
- f can be specified by many equivalent expressions or by tables of combinations (truth tables)
- Elementary functions:
  - A minterm m_i is an AND term of n literals
  - A maxterm M_i is an OR term of n literals

  Ex: 4 variables A, B, C, D
    - $m_5(A, B, C, D) = A'BC'D$ (0101)
    - $M_5(A, B, C, D) = A + B' + C + D'$ (0101)
- $m_i = 1$ for exactly one combination of variables, and 0 for all others
- $M_i = 0$ for exactly one combination of variables, and 1 for all others
- $m_i = M_i'$

Canonical Forms

- Canonical Sum-of-Products (SOP)
  - Also known as Disjunctive Normal Form (DNF)
  - Sum of minterms (those for which $f = 1$)
  - Shorthand: $\sum(\ldots)$
- Canonical Product-of-Sums (POS)
  - Also known as Conjunctive Normal Form (CNF)
  - Product of maxterms (those for which $f = 0$)
  - Shorthand: $\prod(\ldots)$

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  - Shorthand: $\prod(\ldots)$

Decimal	$x\,y\,z$	f
0	000	1
1	001	0
2	010	1
3	011	1
4	100	0
5	101	0
6	110	1
7	111	1

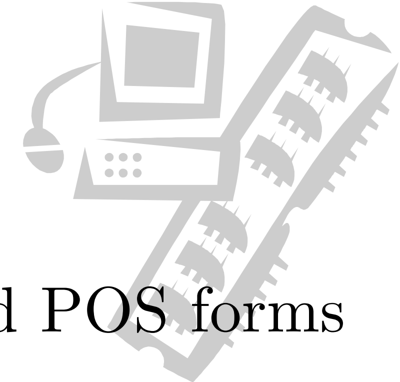
$$f(x,y,z) = \sum_{x,y,z}(0,2,3,6,7)$$

$$f(x,y,z) = x'y'z' + x'yz' + x'yz + xyz' + xyz$$

$$f(x,y,z) = \prod_{x,y,z}(1,4,5)$$

$$f(x,y,z) = (x + y + z')(x' + y + z)(x' + y + z')$$

In-Class Exercise



Express $f(a,b,c) = ab + ac + bc$ in canonical SOP and POS forms

$$
\begin{aligned}
f(a,b,c) &= ab + ac + bc \\
&= ab(c + c') + a(b + b')c + (a + a')bc \\
&= abc + abc' + abc + ab'c + abc + a'bc \\
&= a'bc + ab'c + abc' + abc \\
&= \sum\nolimits_{abc}(3,5,6,7) \\
&= \prod\nolimits_{abc}(0,1,2,4) \\
&= (a + b + c)(a + b + c')(a + b' + c)(a' + b + c)
\end{aligned}
$$

What are minterms???

Row Index	x y	m_0	m_1	m_2	m_3
0	0 0				
1	0 1				
2	1 0				
3	1 1				

Symbolically:

Don’t Cares

Decimal	***x y z***	***f***
0	000	1
1	001	0
2	010	d
3	011	1
4	100	0
5	101	d
6	110	1
7	111	1

$$f(x,y,z) = \sum_{x,y,z}(0,3,6,7) + d(2,5)$$

$$f(x,y,z) = \prod_{x,y,z}(1,4)d(2,5)$$

On-Set $= \{0,\ 3,\ 6,\ 7\}$
Off-Set $= \{1,\ 4\}$
Don’t-Care-Set $= \{2,\ 5\}$

Code Word Representation of Product Terms

Variables: $u\ v\ w\ x\ y\ z$

Code Word	Product Term	#minterms "covered"
001101	$u'v'wxy'z$	$1\ \{m_{13}\}$
0-1-01	$u'wy'z$	$2^2 = 4\ \{m_9, m_{13}, m_{25}, m_{29}\}$

$$\#\text{minterms covered by code word} = 2^{\#\text{missing literals}}$$

Boole's (Shannon's) Expansion Theorem

- Decomposition of a switching function of n variables into functions of $n-1$ variables
  - $f(x_1, x_2, \dots, x_n) = x_1' f(0, x_2, \dots, x_n) + x_1 f(1, x_2, \dots, x_n)$
  - $f(x_1, x_2, \dots, x_n) = [x_1 + f(0, x_2, \dots, x_n)][x_1' + f(1, x_2, \dots, x_n)]$
- Proof?

Boole's (Shannon's) Expansion Theorem

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  - $f(x_1, x_2, \dots, x_n) = [x_1' + f(1, x_2, \dots, x_n)][x_1 + f(0, x_2, \dots, x_n)]$
- The functions resulting from fixing x are referred to as co-factors
  - $f(0, x_2, \dots, x_n)$ is the negative cofactor of f wrt x_1
  - $f(1, x_2, \dots, x_n)$ is the positive cofactor of f wrt x_1
- Notation:
  - $f_{x_1'} = f(0, x_2, \dots, x_n)$
  - $f_{x_1} = f(1, x_2, \dots, x_n)$

Boole's (Shannon's) Expansion Theorem

- Decomposition of a switching function of n variables into functions of $n-1$ variables:

$$\boxed{\begin{aligned} f(x_1, x_2, \cdots, x_n) &= x_1' f_{x_1'} + x_1 f_{x_1} \\ f(x_1, x_2, \cdots, x_n) &= [x_1' + f_{x_1}] \cdot [x_1 + f_{x_1'}] \end{aligned}}$$

- Repeated application yields canonical forms (ex: $n = 2$)

$$f(x, y) = x' f(0, y) + x f(1, y)$$

$$f(x, y) = x'[y' f(0,0) + y f(0,1)] + x[y' f(1,0) + y f(1,1)]$$

$$f(x, y) = f(0,0) \cdot x'y' + f(0,1) \cdot x'y + f(1,0) \cdot xy' + f(1,1) \cdot xy$$

$$f(x, y) = f_{x'y'} \cdot x'y' + f_{x'y} \cdot x'y + f_{xy'} \cdot xy' + f_{xy} \cdot xy$$

$$f(x, y) = a_0 \cdot m_0 + a_1 \cdot m_1 + a_2 \cdot m_2 + a_3 \cdot m_3$$

Truth Table for *Arbitrary* 2-Variable Function $f(x, y)$

$$
\begin{aligned}
f(x,y) &= a_0 m_0 + a_1 m_1 + a_2 m_2 + a_3 m_3 \\
&= a_0 x'y' + a_1 x'y + a_2 xy' + a_3 xy
\end{aligned}
$$

Row Index	x	y	f
0	0	0	a_0
1	0	1	a_1
2	1	0	a_2
3	1	1	a_3

Think of each a_i as a *minterm selector*

$$
a_i = \begin{cases} 1, m_i \text{ is a minterm of } f \\ 0, m_i \text{ is not a minterm of } f \end{cases}
$$

Algebraic v. Set View of Functions

$$
\begin{aligned}
f(x,y) &= a_0 m_0 + a_1 m_1 + a_2 m_2 + a_3 m_3 \\
&= a_0 x'y' + a_1 x'y + a_2 xy' + a_3 xy
\end{aligned}
$$

$$
a_0 = a_2 = 1 \Rightarrow \quad
\begin{aligned}
f(x,y) &= m_0 + m_2 \\
&= x'y' + xy' \\
&= (x' + x)y' \\
&= y'
\end{aligned}
$$

$$
U = \{m_0, m_1, m_2, m_3\}
$$

$$
f(x,y) \subseteq U
$$

$$
f(x,y) = \{m_0, m_2\}
$$

Switching Functions of 2 Variables

$$\boxed{f(x,y) = a_0 m_0 + a_1 m_1 + a_2 m_2 + a_3 m_3}$$

$a_3 a_2 a_1 a_0$	$f(x,y)$	Name	Symbol	Unique?

Switching Functions of 2 Variables

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1111	1	Tautology		

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1001	$xy + x'y'$	XNOR / EQV	$x \odot y$	
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Functional Properties

- The canonical SOP (POS) form is unique, subject to commutativity
- Two functions are logically equivalent iff their canonical SOP (POS) form are identical
- The canonical SOP (POS) form contains 2^n coefficients each of which can be either 0 or 1. Thus, there are 2^{2^n} switching functions of n variables

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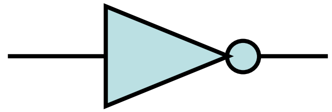
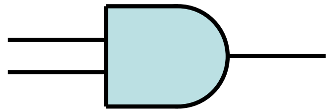
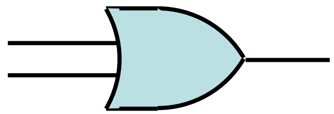
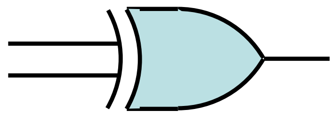
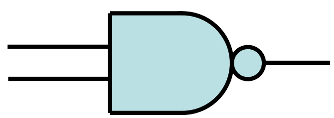
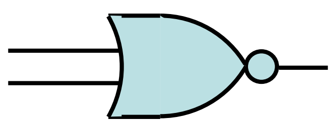
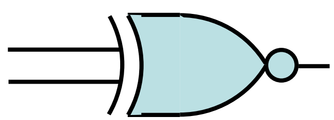
n	2^n	2^{2^n}	N_n
1	2	4	3
2	4	16	6
3	8	256	22
4	16	65,536	402
5	32	4,294,967,296	1,228,158
N_n = number of "types" of functions of n variables [Slepian 53][a]			

[a] Slepian, D., "On the Number of Symmetry Types of Boolean Functions of *n* Variables," Can. J. Math., 5(2):185–193, 1953.

Logic Gates

NOT	x		x'
AND	x, y		xy
OR	x, y		$x + y$
XOR	x, y		$x \oplus y = x'y + xy'$
NAND	x, y		$(xy)' = x' + y'$
NOR	x, y		$(x + y)' = x'y'$
XNOR	x, y		$x \odot y = x'y' + xy$

Properties of XOR (modulo-2 Addition)

- Commutativity: $x \oplus y = y \oplus x$
- Associativity: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
- Distributivity: $x(y \oplus z) = xy \oplus xz$
- Relationship to XNOR: $(x \oplus y)' = x \odot y$
  - XNOR is “equal”
  - XOR is “not equal”
- Conditional Complementation:

$$s \oplus x = \begin{cases} x & \text{if } s = 0 \\ x' & \text{if } s = 1 \end{cases}$$

- Parity: Value of $f = x_1 \oplus x_2 \oplus \ldots \oplus x_n$
  - Remains unchanged if an even number of variables are complemented
  - Is complemented if an odd number of variables are complemented
- Any identity $f(X) = g(X)$ can be re-expressed as $f(X) \oplus g(X) = 0$

Functional Completeness

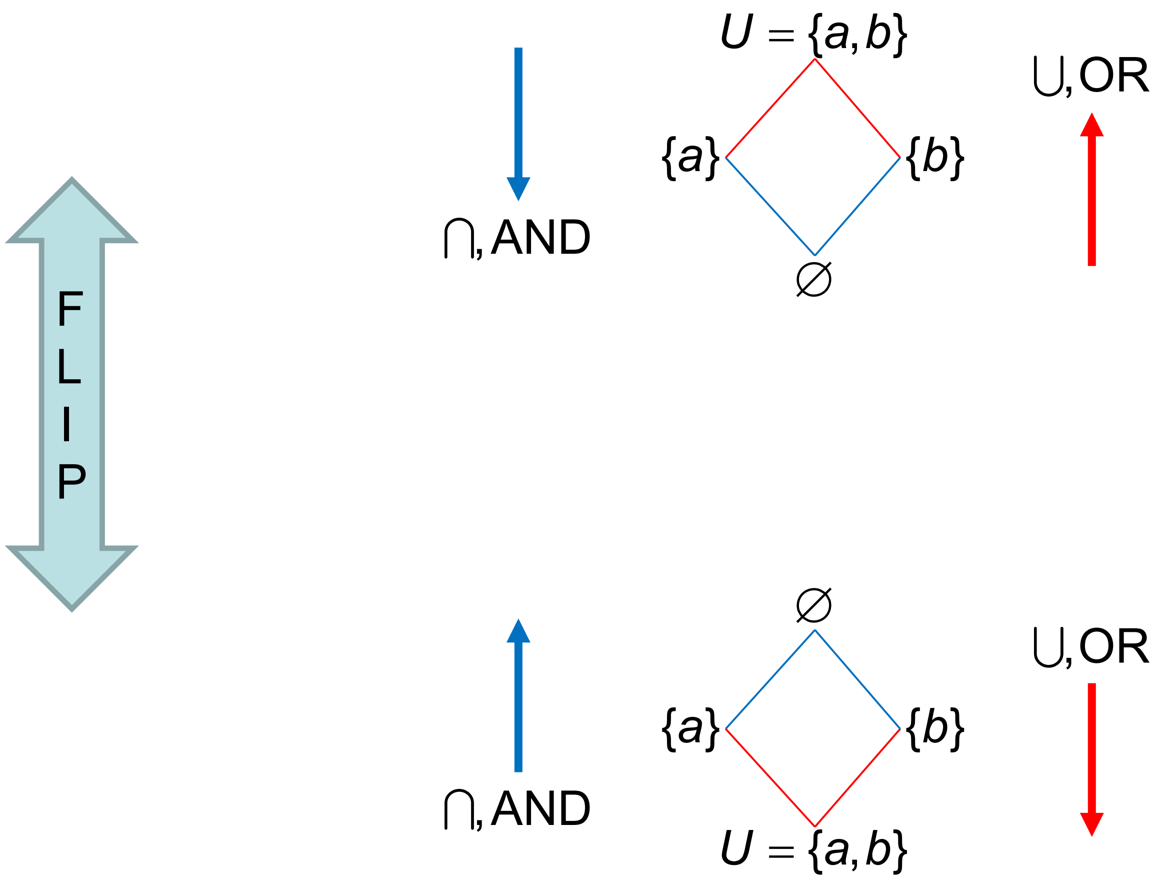
- A set of operations is functionally-complete (or universal) iff every switching function can be expressed entirely by means of operations from this set
- The following are functionally-complete operation sets
  - $\{+, \cdot, '\}$ (by definition)
  - $\{+, '\}$ (by De Morgan's theorem)
  - $\{\cdot, '\}$ (by De Morgan's theorem)
  - {NAND}: $x \uparrow y = x' + y'$
    Complement: $x \uparrow x = x' + x' = x'$
    AND: $(x \uparrow y) \uparrow (x \uparrow y) = (x \uparrow y)' = (x' + y')' = xy$
  - {NOR}

Isomorphic Systems

- Two algebraic systems are isomorphic if there is a one-to-one correspondence between elements and operations from one system to the other
- Examples of algebraic systems which are isomorphic to Switching algebra:
  - Series-parallel switching circuits
  - Propositional calculus
  - Algebra of sets

	Switching Algebra	**Algebra of Sets**
Elements	0 1	$\emptyset$ U
Unary Op	x'	$U - x$
Binary Ops	$x \cdot y$	$x \cap y$
	$x + y$	$x \cup y$

``Big'' Boolean Algebras: Duality



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