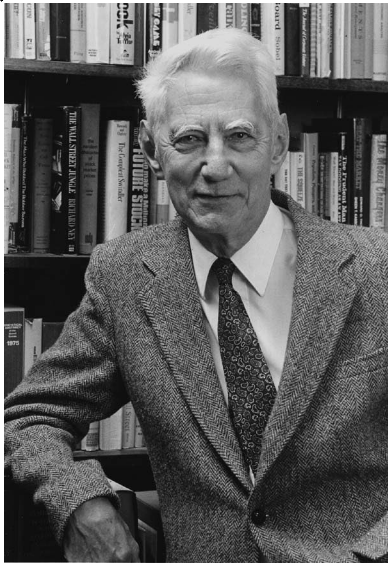
UM EECS 270 F22
Introduction to Logic Design

4. Boolean (Switching) Algebra

Boolean (Switching) Algebra

- Basic "language" for combinational and sequential switching circuits
- History:
  - Boole (1854) developed "the science of logic" to "give expression . . . to the fundamental laws of reasoning in the symbolic language of a Calculus."
  - Huntington (1904) formally introduced an axiomatic definition (one of several) of Boolean algebras. His axioms became known as the Huntington postulates.
  - Birkhoff (1940s) discussed Boolean algebras in the context of lattice theory
  - Shannon (1938) developed the 2-valued algebra of switching (relay) circuits and showed its relation to Boolean algebra



Claude Shannon

Huntington Postulates

A Boolean Algebra is a set B with two binary operators $+$ and $\cdot$
And the equivalence relation $=$
that satisfies the following properties:

- Closure
  - with respect to $+$
  - with respect to $\cdot$
- Identity elements
  - 0 with respect to $+$
  - 1 with respect to $\cdot$
- Commutative
  - $x \cdot y = y \cdot x$
  - $x + y = y + x$
- Distributive
  - $\cdot$ is distributive over $+$
  - $+$ is distributive over $\cdot$
- Complements: $\forall x \in B, \exists x' \in B$ (called the complement of x) such that
  - $x + x' = 1$ and
  - $x \cdot x' = 0$
- There are at least 2 distinct elements in B

Formal Definition of Switching Algebra

- Base set: B_2={0, 1}
- One unary operation: NOT or COMPLEMENT: $(x', \bar{x}, \neg x)$
- Two binary operations: AND $(\cdot,\ \wedge)$, OR $(+,\ \vee)$
- Postulates (axioms):

Postulate	**Defines**	**A**	**B**
P1	Switching Variables	$x = 0$ iff $x \neq 1$	$x = 1$ iff $x \neq 0$
P2	NOT	$0'$=1	$1'$= 0
P3		$0 \cdot 0 = 0$	$1 + 1 = 1$
P4	AND / OR	$1 \cdot 1 = 1$	$0 + 0 = 0$
P5		$0 \cdot 1 = 1 \cdot 0 = 0$	$0 + 1 = 1 + 0 = 1$

- Duality: $0 \leftrightarrow 1, \cdot \leftrightarrow\ +$

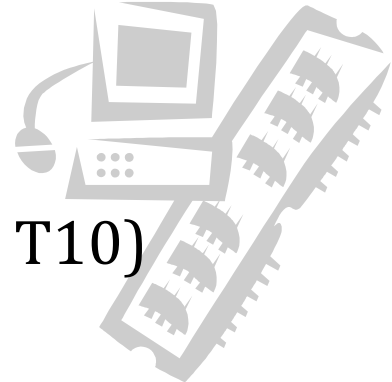
Properties (Theorems) of Switching Algebra

- From postulates, we can derive many theorems which can be used to manipulate switching expressions
- Recursive definition of switching expressions:
  - Any switching constant or variable is a switching expression
  - If E and F are switching expressions, then so are E', F', $E \cdot F$, and $E + F$
- A literal is a variable x or its complement x'
- Theorems can be proved by:
  - Perfect induction (enumerating all possible combinations of variables)
  - Finite induction
  - Algebraic manipulation (using postulates and already proved theorems)
  - Use of duality

(Some) Theorems

	A	Name	B
T1	$x \cdot 1 = x$	Identities	$x + 0 = x$
T2	$x \cdot 0 = 0$	Null Elements	$x + 1 = 1$
T3	$x \cdot x = x$	Idempotency	$x + x = x$
T4		Involution $(x')' = x$	
T5	$x \cdot x' = 0$	Complements	$x + x' = 1$
T6	$x \cdot y = y \cdot x$	Commutativity	$x + y = y + x$
T7	$x \cdot (x + y) = x$	Absorption	$x + (x \cdot y) = x$
T8	$x \cdot (x' + y) = x \cdot y$	No Name	$x + (x' \cdot y) = x + y$
T9	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	Associativity	$(x + y) + z = x + (y + z)$
T10	$x \cdot (y + z) = x \cdot y + x \cdot z$	Distributivity	$x + (y \cdot z) = (x + y) \cdot (x + z)$
T11	$x \cdot y + x' \cdot z + y \cdot z$ $= x \cdot y + x' \cdot z$	Consensus	$(x + y) \cdot (x' + z) \cdot (y + z)$ $= (x + y) \cdot (x' + z)$
T12		De Morgan's $f(x_1, \dots, x_n, 0, 1, \cdot, +)' = f(x_1', \dots, x_n', 1, 0, +, \cdot)$	

Proof by Perfect Induction



Prove the truth of the distributive law (theorem T10)

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

	A	**B**
P3	$0 \cdot 0 = 0$	$1 + 1 = 1$
P4	$1 \cdot 1 = 1$	$0 + 0 = 0$
P5	$0 \cdot 1 = 1 \cdot 0 = 0$	$1 + 0 = 0 + 1 = 1$

$x\,y\,z$	$x \cdot y$	$x \cdot z$	$y + z$	$x \cdot (y + z)$	$x \cdot y + x \cdot z$	**Using**
0 0 0	0	0	0	0	0	P3A, P4B
0 0 1	0	0	1	0	0	P3A, P5A, P5B, P4B
0 1 0	0	0	1	0	0	P5A, P3A, P5B, P4B
0 1 1	0	0	1	0	0	P5A, P3B, P4B
1 0 0	0	0	0	0	0	P5A, P4B
1 0 1	0	1	1	1	1	P5A, P4A, P5B
1 1 0	1	0	1	1	1	P4A, P5A, P5B
1 1 1	1	1	1	1	1	P4A, P3B

Proof by Finite Induction

Prove that $(x_1 + x_2 + \ldots + x_n)' = x_1' \cdot x_2' \cdot \ldots \cdot x_n'$

- Basis: Establish truth for $n = 2$ by perfect induction:

$$\boxed{(x_1 + x_2)' = x_1' \cdot x_2'}$$

- Induction: Assume statement is true for $n = k, k \geq 2$ and prove its truth for $n = k + 1$

Induction Hypothesis: $\boxed{(x_1 + x_2 + \ldots + x_k)' = x_1' \cdot x_2' \cdot \ldots \cdot x_k'}$

$$\begin{aligned}
(\underbrace{x_1 + x_2 + \ldots + x_k}_{y} + x_{k+1})' &= [y + x_{k+1}]' = [(x_1 + x_2 + \ldots + x_k) + x_{k+1}]' \\
&= y' \cdot x_{k+1}' = (x_1 + x_2 + \ldots + x_k)' \cdot x_{k+1}' \\
&= x_1' \cdot x_2' \cdot \ldots \cdot x_k' \cdot x_{k+1}'
\end{aligned}$$

Proof by Algebraic Manipulation

Prove the consensus theorem (T11)

$$xy + x'z + yz = xy + x'z$$

$$\begin{aligned}
xy + x'z + yz &= xy + x'z + yz1 && \text{(Identity)} \\
&= xy + x'z + yz(x + x') && \text{(Complement)} \\
&= xy1 + x'z1 + xyz + x'yz && \text{(Id., Dist., Assoc.)} \\
&= xy(1 + z) + x'z(1 + y) && \text{(Assoc., Dist.)} \\
&= xy1 + x'z1 && \text{(Null Element)} \\
&= xy + x'z && \text{(Identity)}
\end{aligned}$$

Additional Comments about Theorems

- The following properties are peculiar to switching algebra and are not true for the algebra of real numbers:
  - Idempotency
  - All properties involving complements
  - Distributivity of sum over product
- Associativity allows the extension of the two binary operators AND and OR to three or more variables
- Simplification of switching expressions is facilitated by the absorption and consensus theorems
- The involution property and De Morgan’s laws provide the rules for complementing switching expressions