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NOMAD: N-th Order M-Dimensional Nonlinear Diff Eq Solver

This document is a copy of most of my raw notes from my senior year of college and shortly after where NOMAD was first derived in 2017.

Page 2 shows a simple way of describing a multivariate polynomial equation of arbitrarily high order. This was the foundation for NOMAD. After realizing I could use such an equation to describe ODE's, I was off to the races to derive its analytical Taylor series.

Pages 3-9 show a high-level review of NOMAD that I wrote up shortly after graduating college when I was attempting to expand the technique to cover some basic transcendental functions.

Pages 10-57 show my raw notes where I actually derived everything shown in the review. It also includes many of the details of the actual implementation details, and an efficient algorithm to compute the n-th derivative. The algorithm includes what I believe to be a novel math operator, somewhat similar to convolution. The final result is an efficient, iterative solution to compute the n-th derivative for a set of multivariate polynomial ODEs. There is a lot of intense math here, but that's what it took.

Pages 58-64 show the beginning of an attempt to extend this to some basic transcendental functions. The derivatives never converged to something with any kind of pattern here, so I am not sure if there is a similarly elegant solution. At least, I could not see one. I've included this mostly because it contains some cool math to describe any sum of trigonometric functions in a succinct way, similar to the arbitrary polynomial equation on page 2. More math behind my attempt at this extension could be provided upon request.

$$p(x) = \sum_{i=1}^{(N+1)^M} \left[a_i \prod_{j=1}^{M} \left(x_j^{k_{ji}} \right) \right]; \quad k \in \mathbb{Z}^{M \times (N+1)^M}, \quad x := \{x_1, \dots, x_M\}$$

k_{ji} is exponent for x_j in i-th term of summation.

$\vec{x} \in \mathbb{R}^M$ $\quad a_{(i)}$ $\quad a \in \mathbb{R}^{(N+1)^M}$, a_i is coefficient of i-th term of summation.

N is order of polynomial.

M is number of dimensions of $\vec{x}$. $\quad J \in \{1, \dots, M\}$

$$\frac{\partial p(x)}{\partial x_w} = \sum_{i=1}^{N(N+1)^{M-1}} \left[b_{wi} \, x^{(k_{wi}-1)} \prod_{j \in J/\{w\}} \left(x_j^{k_{ji}} \right) \right]$$

where $b_w = \{ b_{wi} = a_m k_{wm} \mid \forall m \in \{1, \dots, (N+1)^M\} \text{ s.t. } k_{wm} \neq 0 \}$

$$\nabla p(x) = \begin{bmatrix} \frac{\partial p(x)}{\partial x_1} \\ \vdots \\ \frac{\partial p(x)}{\partial x_M} \end{bmatrix}$$

$$\frac{\partial p(x)}{\partial x_y \partial x_w} = \begin{cases} \sum_{i=1}^{(N+1)^{M-1}(N-1)} \left[C_{ywi} \, x_w^{(k_{wi}-2)} \prod_{j \in J/\{y,w\}} \left(x_j^{k_{ji}} \right) \right], & y = w \\ \sum_{i=1}^{(N+1)^{M-2}(N)^2} \left[C_{ywi} \, x_w^{(k_{wi}-1)} \, x_y^{(k_{yi}-1)} \prod_{j \in J/\{y,w\}} \left(x_j^{k_{ji}} \right) \right], & y \neq w \end{cases}$$

where $C_{ywi} = \begin{cases} \{ C_{ywi} = b_{wm}(k_{wm} - 1) \mid \forall m \in \{1, \dots, N(N+1)^{M-1}\} \text{ s.t. } k_{wm} - 1 \neq 0 \} & \text{if } y = w \\ \{ C_{ywi} = b_{wm}(k_{ym}) \mid \forall m \in \{1, \dots, N(N+1)^{M-1}\} \text{ s.t. } k_{ym} \neq 0 \} & \text{if } y \neq w \end{cases}$

$$\nabla^2 p(x) = H(p(x)) = \begin{bmatrix} \frac{\partial^2 p(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 p(x)}{\partial x_1 \partial x_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 p(x)}{\partial x_M \partial x_1} & \cdots & \frac{\partial^2 p(x)}{\partial x_M^2} \end{bmatrix}$$

Review (8/27/17): Notation description

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} \in \mathbb{R}^{M \times 1}$, where M is the dimension of $\vec{x}$

Let $K \in \mathbb{Z}_{\geq 0}^{M \times (N+1)^M}$, where N is the largest power of any $x \in \vec{x}$ in the expression $\frac{d\vec{x}}{dt}$

↳ Form K as:

$$K = \begin{bmatrix} \hat{k}(M, (N+1)^0) \\ \hat{k}(M-1, (N+1)^1) \\ \vdots \\ \hat{k}(1, (N+1)^{(M-1)}) \end{bmatrix}, \text{ where } \hat{k}(i,j) = [N \cdot \mathbb{1}, (N-1) \cdot \mathbb{1}, \cdots, 1 \cdot \mathbb{1}, 0 \cdot \mathbb{1}]$$

concatenated j times, and $\mathbb{1} \in [1,1]^{(1 \times (N+1)^{(i-1)})}$

⟹ Then $K_{j,i}$ is the exponent of x_j in the i-th term of the sum (for all $\frac{dx_j}{dt}$). $(j \in \{1, \ldots, M\}, i \in \{1, \ldots, (N+1)^M\})$

Let $a \in \mathbb{R}^{M \times (N+1)^M}$, where a_{ji} is the coefficient of the i-th term of the sum describing $\frac{dx_j}{dt}$.

↳ a is selected to include or exclude terms. If $a_{ji} = 0$, then the term described by the i-th column of K is irrelevant (i.e. does not contribute) to the sum describing $\frac{dx_j}{dt}$. $(j \in \{1, \ldots, M\}, i \in \{1, \ldots, (N+1)^M\})$.

Review (8/27/17): More Notation Description

With the notation of the previous page, $\frac{d\vec{X}}{dt}$ may be succintly described as follows (provided that $\frac{d\vec{X}}{dt}$ is an M-dimensional set of autonomous, multivariate polynomial differential equations).

$$\frac{dx_\ell}{dt} = \sum_{\alpha=1}^{(N+1)^M} \left[a_{\ell,\alpha} \left(\prod_{\beta=1}^{M} x_\beta^{(K_{\beta,\alpha})} \right) \right]$$

Taking more derivatives w.r.t. time, we have:

$$\frac{d^2x_\ell}{dt^2} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \left[\left(a_{\ell,\alpha} \cdot K_{j_1,\alpha} \cdot a_{j_1,i_1} \right) \left(\prod_{\beta=1}^{M} x_\beta^{(K_{\beta,i_1} + K_{\beta,\alpha} - \delta_{\beta,j_1})} \right) \right]$$

$$\frac{d^3x_\ell}{dt^3} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \sum_{j_2=1}^{M} \sum_{i_2=1}^{(N+1)^M} \left[\left(a_{\ell,\alpha} K_{j_1,\alpha} a_{j_1,i_1} (K_{j_2,i_1} + K_{j_2,\alpha} - \delta_{j_2,j_1}) a_{j_2,i_2} \right) \cdot \cdots \right.$$
$$\left. \cdots \cdot \left(\prod_{\beta=1}^{M} x_\beta^{(K_{\beta,i_2} + K_{\beta,i_1} + K_{\beta,\alpha} - \delta_{\beta,j_1} - \delta_{\beta,j_2})} \right) \right]$$

where $\delta_{i,j}$ is the Kronecker delta:

$$\delta_{i,j} \equiv \delta[i-j] = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

As shown above, the repeated derivatives are very complicated, and the repeating summations are cumbersome to write.

Review (8/27/17): <u>More notation description</u>

To ease the notation, define:

$$J(n) \equiv \{ j_1, j_2, \cdots, j_{n-1} \}$$

$$I(n) \equiv \{ i_1, i_2, \cdots, i_{n-1} \}$$

with:

$$\sum_{j \in J(n)} \equiv \sum_{j_1=1}^{M} \sum_{j_2=1}^{M} \cdots \sum_{j_{n-1}=1}^{M}, \text{ where } \sum_{j \in J(1)} \equiv (1) \text{ (i.e. no summation)}$$

$$\sum_{i \in I(n)} \equiv \sum_{i_1=1}^{(N+1)^M} \sum_{i_2=1}^{(N+1)^M} \cdots \sum_{i_{n-1}=1}^{(N+1)^M}, \text{ where } \sum_{i \in I(1)} \equiv (1) \text{ (i.e. no summation)}$$

$$\sum_{\alpha=1} \equiv \sum_{\alpha=1}^{(N+1)^M}$$

Thus, we write:

$$\sum_{\substack{\alpha=1 \\ j \in J(n) \\ i \in I(n)}} \equiv \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \cdots \sum_{j_{n-1}=1}^{M} \sum_{i_{n-1}=1}^{(N+1)^M}$$

with this, we may rewrite the previous derivatives:

Review (8/27/17): Notation complete, solution:

$$\frac{d^{(1)}x_\ell}{dt^{(1)}} = \sum_{\substack{\alpha=1\\ j\in J(1)\\ i\in I(1)}} \left[a_{\ell,\alpha}\left(\prod_{\beta=1}^{M} x_\beta^{(K_{\beta,\alpha})}\right)\right]$$

$$\frac{d^{(2)}x_\ell}{dt^{(2)}} = \sum_{\substack{\alpha=1\\ j\in J(2)\\ i\in I(2)}} \left[(a_{\ell,\alpha}a_{j_1,i_1})(K_{j_1,\alpha})\left(\prod_{\beta=1}^{M} x_\beta^{(K_{\beta,\alpha}+K_{\beta,i_1}-\delta_{\beta,j_1})}\right)\right]$$

$$\frac{d^{(3)}x_\ell}{dt^{(3)}} = \sum_{\substack{\alpha=1\\ j\in J(3)\\ i\in I(3)}} \left[(a_{\ell,\alpha}a_{j_1,i_1}a_{j_2,i_2})(K_{j_1,\alpha}(K_{j_2,\alpha}+K_{j_2,i_1}-\delta_{j_2,j_1}))\left(\prod_{\beta=1}^{M} x_\beta^{(K_{\beta,\alpha}+K_{\beta,i_2}-\delta_{\beta,j_2}+K_{\beta,i_1}-\delta_{\beta,j_1})}\right)\right]$$

$\hookrightarrow$ * exponent on x_β is $(K_{\beta,\alpha}+K_{\beta,i_2}-\delta_{\beta,j_2}+K_{\beta,i_1}-\delta_{\beta,j_1})$

Repeating this process, one finds that the n-th derivative of the function $X_\ell(t)$ with the first derivative $\frac{d^{(1)}x_\ell(t)}{dt}$ is given by:

$$\frac{d^{(n)}}{dt^{(n)}}(X_\ell(t)) = \sum_{\substack{\alpha=1\\ j\in J(n)\\ i\in I(n)}} \left[\left(\prod_{\beta=0}^{n-1} a_{j_\beta,i_\beta}\right)\left(\prod_{\beta=1}^{n-1}\left(\sum_{\gamma=0}^{\beta-1}\left(K_{j_\beta,i_\gamma}-\delta_{j_\beta,j_\gamma}u[\gamma-1]\right)\right)\right)\left(\prod_{\beta=1}^{M} x_\beta^{\left(\sum_{\gamma=0}^{n-1}\left(K_{\beta,i_\gamma}-\delta_{\beta,j_\gamma}u[\gamma-1]\right)\right)}\right)\right]$$

Review (8/27/17): Series Solution

→ With a generic formula for the n-th derivative, it is easy to construct a Taylor Series solution for $x(t)$:

$$x(t) = x(t_0) + \sum_{n=1}^{\infty}\left[\left(\frac{d^{(n)}}{dt^{(n)}}(x(t))\Big|_{t=t_0}\right)\left(\frac{(t-t_0)^n}{n!}\right)\right]$$

In theory, this works perfectly. However, if $\frac{dx_\ell}{dt}$ is a polynomial with order N and dimension M, then, <u>at most</u>, $\frac{d^2x_\ell}{dt^2}$ is of order $2N$ and dimension M. Similary, $\frac{d^3x_\ell}{dt}$ & $\frac{d^4x_\ell}{dt^4}$ are of order $4N$ & $8N$, and thus $\frac{d^nx_\ell}{dt^n}$ is of order $(2^{(n-1)})N$.

* This means that, <u>at most</u>, $\frac{d^nx_\ell}{dt^n}$ has $\left[(2^{(n-1)})N+1\right]^M$ terms.

Of course, this is <u>computationally impossible</u> as n grows large, even for small M.

* An algorithm that minimizes the size of the exponent matrix "K" based on the coefficients in "a" is absolutely necessary. Fortunately, the "a" matrix is often sparse (i.e., many zeros!) for most differential equations of order N. →

Review (8/27/2017): Efficient algorithm for Series Computation

→ The equation for any derivative of $x_\ell(t)$, by this definition, can be boiled down to determining the "a" & "K" matrix of each derivative. This can be done iteratively as follows:

1) Parse an "a" matrix from the input differential equations $\frac{d\vec{x}}{dt}$. Determine the appropriate "K" matrix. For all columns of "a" equal to $\vec{\emptyset}$, remove this column & the corresponding column of "K". Denote these $\hat{a}(1)$ & $\hat{K}(1)$.

2) Let $\hat{a}(n)$ be the coefficient matrix of the n-th deriv, $\hat{K}(n)$ be the exponent matrix of the n-th deriv, and $\hat{\alpha}(n)$ be the summation index of the n-th deriv. Then,

$$\hat{a}(n)_{\ell,\hat{\alpha}(n)} = \hat{a}(n-1)_{\ell,\hat{\alpha}(n-1)} \cdot \hat{K}(n-1)_{\ell,\hat{\alpha}(n-1)} \cdot \hat{a}(1)_{j,i}$$

$$\hat{K}(n)_{\beta,\hat{\alpha}(n)} = \hat{K}(n-1)_{\beta,\hat{\alpha}(n-1)} + \hat{K}(1)_{\beta,i} - \delta_{\beta,j}$$

where →

Review: Alg. Continued

$$\rightarrow \frac{d^{(n)}}{dt^{(n)}}(x_\ell) = \sum_{\hat{\alpha}(n)=1}^{|\hat{a}(n)_{:,\ell}|} \hat{a}(n)_{\ell,\hat{\alpha}(n)} \left(\prod_{\beta=1}^{M} x_\beta^{\hat{k}(n)_{\beta,\alpha(n)}} \right)$$

Of course, computing $\hat{k}(n)$ & $\hat{a}(n)$ term by term is horrendously inefficient, but can be greatly simplied by an operation similar to convolution:

* See other notebook for example. *

Finally, to remain in range double representation, when the taylor series is being constructed $\hat{a}(n)$ should be computed as $\frac{\hat{a}(n)}{n}$

* This concludes review of original NOMAD.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} \in \mathbb{R}^{M\times 1}$$

$$p(\vec{x}) = \sum_{i=1}^{(N+1)^M} \left[a_i \prod_{j=1}^{M} \left(x_j^{(k_{ji})} \right) \right] ; \quad k \in \mathbb{Z}_{\geq 0}^{M\times(N+1)^M}$$

k_{ji} is exponent for x_j in i-th term of sum.

$a \in \mathbb{R}^{(N+1)^M}$, a_i is coefficient of i-th term in sum

~~let $\mathbb{1} \in [1,1]^{(1\times(N+1))}$~~

Let $\hat{k}(i,j) = [N\cdot\mathbb{1} \quad (N-1)\mathbb{1} \quad \cdots \quad 1\mathbb{1} \quad 0\mathbb{1}]$ concatenated j times.

where $\mathbb{1} \in [1,1]^{(1\times(N+1)^{(i-1)})}$.

Then:

$$k = \begin{bmatrix} \hat{k}(M, (N+1)^0) \\ \hat{k}(M-1, (N+1)^1) \\ \vdots \\ \hat{k}(1, (N+1)^{M-1}) \end{bmatrix}.$$

"a" may then be selected to include or exclude terms. If $a_i = 0$, then the ith column of k is irrelevant, as all terms related to it go to 0/vanish.

Things are about to get messy.

Let $\frac{dx_1}{dt} = \sum_{i=1}^{(N+1)^M} \left[a_i^1 \prod_{j=1}^{M} x_j^{k_{ji}} \right]$

$$\frac{dx_2}{dt} = \sum_{i=1}^{(N+1)^M} \left[a_i^2 \prod_{j=1}^{M} x_j^{k_{ji}} \right]$$

$$\vdots$$

$$\frac{dx_M}{dt} = \sum_{i=1}^{(N+1)^M} \left[a_i^M \prod_{j=1}^{M} x_j^{k_{ji}} \right]$$

superscript l is row index to $a \in \mathbb{R}^{(N+1)^M \times M}$

$$\boxed{\text{find } \frac{d^{\gamma} x_l}{dt^{\gamma}}, \text{ when } \frac{dx_l}{dt} = \sum_{i=1}^{(N+1)^M} \left[a_i^l \prod_{j=1}^{M} x_j^{k_{ji}} \right]}$$

$f(\gamma, l)$

problem statement

$$\frac{dx_l}{dt} = \sum_{i=1}^{(N+1)^M} \left[a_i^l \prod_{j=1}^{M} x_j^{k_{ji}} \right]$$

↳ Sufficient to examine how $\left(\prod_{j=1}^{M} x_j^{k_{ji}}\right)$ evolves w/ repeated differentiation.

$$\prod_{j=1}^{M} x_j^{k_{ji}} = x_1^{k_{1i}} x_2^{k_{2i}} \cdots x_M^{k_{Mi}}$$

↳

$$\frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_{ji}}\right) = x_1^{k_{1i}}\left[\frac{d}{dt}\left(x_2^{k_{2i}} \cdots x_M^{k_{Mi}}\right)\right] + k_{1i} x_1^{(k_{1i}-1)}\left(\frac{dx_1}{dt}\right)\left(x_2^{k_{2i}} \cdots x_M^{k_{Mi}}\right)$$

$$= x_1^{k_{1i}}\left[x_2^{k_{2i}}\left[\frac{d}{dt}\left(x_3^{k_{3i}} \cdots x_M^{k_{Mi}}\right)\right] + k_{2i} x_2^{(k_{2i}-1)}\left(\frac{dx_2}{dt}\right)\left(x_3^{k_{3i}} \cdots x_M^{k_{Mi}}\right)\right] + \cdots$$

$$\cdots + k_{1i} x_1^{(k_{1i}-1)}\left(\frac{dx_1}{dt}\right)\left(x_2^{k_{2i}} \cdots x_M^{k_{Mi}}\right)$$

↳

$$= x_1^{k_{1i}} x_2^{k_{2i}}\left[\frac{d}{dt}\left(x_3^{k_{3i}} \cdots x_M^{k_{Mi}}\right)\right] + k_{1i} x_1^{(k_{1i}-1)}\left(\frac{dx_1}{dt}\right)\left(x_2^{k_{2i}} \cdots x_M^{k_{Mi}}\right) + \cdots$$

$$\cdots + k_{2i} x_1^{k_{1i}} x_2^{(k_{2i}-1)}\left(\frac{dx_2}{dt}\right)\left(x_3^{k_{3i}} \cdots x_M^{k_{Mi}}\right)$$

↳ Appears to converge to:

$$\frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_{ji}}\right) = x_1^{k_{1i}} x_2^{k_{2i}} x_3^{k_{3i}} \cdots x_M^{k_{Mi}} + k_{1i} x_1^{(k_{1i}-1)} x_2^{k_{2i}} \cdots x_M^{k_{Mi}}\left(\frac{dx_1}{dt}\right) + \cdots$$

$$\cdots + k_{2i} x_1^{k_{1i}} x_2^{(k_{2i}-1)} x_3^{k_{3i}} \cdots x_M^{k_{Mi}}\left(\frac{dx_2}{dt}\right) + \cdots$$

$$\cdots + k_{3i} x_1^{k_{1i}} x_2^{k_{2i}} x_3^{(k_{3i}-1)} \cdots x_M^{k_{Mi}}\left(\frac{dx_3}{dt}\right) + \cdots$$

$$\cdots + \cdots + \cdots + \cdots + k_{Mi} x_1^{k_{1i}} x_2^{k_{2i}} x_3^{k_{3i}} \cdots x_M^{(k_{Mi}-1)}\left(\frac{dx_M}{dt}\right) \longrightarrow A$$

$$x_1^{k_i^1} x_2^{k_i^2} \cdots x_{M-2}^{k_i^{M-2}} \left(\frac{d}{dt}\left(x_{M-1}^{k_i^{M-1}} x_M^{k_i^M} \right) \right) + \Big(k_i^1 x_1^{k_i^1 - 1} x_2^{k_i^2} \cdots x_M^{k_i^M} \frac{dx_1}{dt} + \cdots$$

$$\underbrace{\cdots + k_i^2 x_1^{k_i^1} x_2^{k_i^2 - 1} x_3^{k_i^3} \cdots x_M^{k_i^M} \frac{dx_2}{dt} + \cdots + k_i^{M-2} x_{M-2}^{k_i^{M-2}-1} x_1^{k_i^1} x_2^{k_i^2} \cdots x_{M-1}^{k_i^{M-1}} x_M^{k_i^M} \Big)}_{\Rightarrow = F}$$

$$= x_1^{k_i^1} x_2^{k_i^2} \cdots x_{M-2}^{k_i^{M-2}} \left(\frac{d}{dt}\left(x_{M-1}^{k_i^{M-1}} x_M^{k_i^M} \right) \right) + F$$

$$= x_1^{k_i^1} \cdots x_{M-2}^{k_i^{M-2}} \left(x_{M-1}^{k_i^{M-1}} \frac{d}{dt}\left(x_M^{k_i^M} \right) + x_M^{k_i^M} \frac{d}{dt}\left(x_{M-1}^{k_i^{M-1}} \right) \right) + F$$

$\Rightarrow$ whoops... $\frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_i^j} \right)$ converges to:

$$= k_i^1 x_1^{(k_i^1 - 1)} x_2^{k_i^2} \cdots x_M^{k_i^M} \frac{dx_1}{dt} + k_i^2 x_1^{k_i^1} x_2^{(k_i^2 - 1)} \cdots x_M^{k_i^M} \frac{dx_2}{dt} + \cdots$$

$$\cdots + \cdots + k_i^{M-1} x_1^{k_i^1} x_2^{k_i^2} \cdots x_{M-1}^{(k_i^{M-1} - 1)} x_M^{k_i^M} \frac{dx_{M-1}}{dt} + k_i^M x_1^{k_i^1} x_2^{k_i^2} \cdots x_M^{(k_i^M - 1)} \frac{dx_M}{dt}$$

$$= k_i^1 x_1^{(k_i^1 - 1)} \cdots x_M^{k_i^M} \left(\sum_{i_1=1}^{(N+1)^M} \left[a_{i_1}^1 \prod_{j=1}^{M} x_j^{k_{i_1}^j} \right] \right) + \cdots$$

$$\cdots + \cdots + k_i^M x_1^{k_i^1} \cdots x_M^{(k_i^M - 1)} \left(\sum_{i_M=1}^{(N+1)^M} \left[a_{i_M}^1 \prod_{j=1}^{M} x_j^{k_{i_M}^j} \right] \right)$$

$$= \boxed{\sum_{\alpha=1}^{M} k_i^\alpha \left(\sum_{i_\alpha=1}^{(N+1)^M} \left[a_{i_\alpha}^1 \prod_{j=1}^{M} x_j^{k_{i_\alpha}^j} \right] \right) \prod_{j=1}^{M} x_j^{(k_i^j - \delta_\alpha^j)} = \frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_i^j} \right)}$$

$$\rightarrow \frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_{ji}}\right) = \cancel{x_1^{k_{1i}} x_2^{k_{2i}} \cdots x_M^{k_{Mi}}} + k_{1i} x_1^{(k_{1i}-1)} x_2^{k_{2i}} \cdots x_M^{k_{Mi}} \left(\sum_{i_1=1}^{(N+1)^M} \left[a^1_{i_1} \prod_{j=1}^{M} x_j^{k_{ji_1}}\right]\right) + \cdots$$

$$\cdots + k_{2i} x_1^{k_{1i}} x_2^{(k_{2i}-1)} \cdots x_M^{k_{Mi}} \left(\sum_{i_2=1}^{(N+1)^M} \left[a^2_{i_2} \prod_{j=1}^{M} x_j^{k_{ji_2}}\right]\right) + \cdots$$

$$\cdots + \cdots + \cdots + \cdots + k_{Mi} x_1^{k_{1i}} x_2^{k_{2i}} \cdots x_M^{(k_{Mi}-1)} \left(\sum_{i_M=1}^{(N+1)^M} \left[a^M_{i_M} \prod_{j=1}^{M} x_j^{k_{ji_M}}\right]\right)$$

$$\Rightarrow \cancel{\prod_{j=1}^{M} x_j^{k_{ji}}} + \sum_{\alpha=1}^{M} \left[\underbrace{k_{\alpha i} \left(\sum_{i_\alpha=1}^{(N+1)^M} \left[a^\alpha_{i_\alpha} \prod_{j=1}^{M} x_j^{k_{ji_\alpha}}\right]\right)}_{i_\alpha \text{ is confusing notation} \rightarrow \text{ rewriting!}} \prod_{j=1}^{M} x_j^{(k_{ji} - \delta[j-\alpha])} \right]$$

$$\Rightarrow \cancel{\prod_{j=1}^{M} x_j^{k_{ji}}} + \sum_{\alpha=1}^{M} \left[k_{\alpha i} \left(\sum_{\beta=1}^{(N+1)^M} \left[a^\alpha_{\beta} \prod_{j=1}^{M} x_j^{k_{j\beta}}\right]\right) \left(\prod_{j=1}^{M} x_j^{(k_{ji} - \delta[j-\alpha])}\right) \right]$$

$$= \frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_{ji}}\right)$$

$$= \sum_{\alpha=1}^{M} k_i^\alpha \left(\sum_{i_\alpha=1}^{(N+1)^M} \left[a^1_{i_\alpha} \prod_{j=1}^{M} x_j^{k^j_{i_\alpha}}\right]\right) \left(\prod_{j=1}^{M} x_j^{(k_i^j - \delta^j_\alpha)}\right)$$

$$\frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_{ji}}\right) = \cancel{\frac{d}{dt}\prod_{j=1}^{M} x_j^{k_{ji}}} + \sum_{\alpha=1}^{M}\left[k_{\alpha i}\left(\sum_{\beta=1}^{(N+1)^M}\left[a_\beta^\alpha \prod_{j=1}^{M} x_j^{k_{j\beta}}\right]\right)\left(\prod_{j=1}^{M} x_j^{(k_{ji}-\delta[j-\alpha])}\right)\right]$$

$$\rightarrow \frac{d^2 x_\ell}{dt^2} = \sum_{i=1}^{(N+1)^M}\left[\cancel{a_i^\ell \frac{d}{dt}\prod_{j=1}^{M} x_j^{k_{ji}}} + a_i^\ell\left(\sum_{\alpha=1}^{M}\left[k_{\alpha i}\left(\sum_{\beta=1}^{(N+1)^M}\left[a_\beta^\alpha \prod_{j=1}^{M} x_j^{k_{j\beta}}\right]\right)\left(\prod_{j=1}^{M} x_j^{(k_{ji}-\delta[j-\alpha])}\right)\right]\right)\right]$$

when $\frac{dx_\ell}{dt} = \sum_{i=1}^{(N+1)^M}\left[a_i^\ell \prod_{j=1}^{M} x_j^{k_{ji}}\right]$

$$\frac{d^3 x_\ell}{dt^3}\ ? \rightarrow = \cancel{\frac{d}{dt}\left(\sum_{i=1}^{(N+1)^M} a_i^\ell \prod_{j=1}^{M} x_j^{k_{ji}}\right)} + \frac{d}{dt}\left(\sum_{i=1}^{(N+1)^M} a_i^\ell \sum_{\alpha=1}^{M}\left[k_{\alpha i}\left(\sum_{\beta=1}^{(N+1)^M}\left[a_\beta^\alpha \prod_{j=1}^{M} x_j^{k_{j\beta}}\right]\right)\left(\prod_{j=1}^{M} x_j^{(k_{ji}-\delta[j-\alpha])}\right)\right]\right)$$

$\cancel{\frac{d^2 x_\ell}{dt^2}}$

Simplify

~~Sufficient to examine:~~

$$\rightarrow \left(\sum_{\beta=1}^{(N+1)^M}\left[a_\beta^\alpha \prod_{j=1}^{M} x_j^{k_{j\beta}}\right]\right)\left(\prod_{j=1}^{M} x_j^{(k_{ji}-\delta[j-\alpha])}\right)$$

$$= \left(a_1^\alpha x_1^{k_{11}} x_2^{k_{21}} \cdots x_M^{k_{M1}} + a_2^\alpha x_1^{k_{12}} x_2^{k_{22}} \cdots x_M^{k_{M2}} + \cdots + a_{(N+1)^M}^\alpha x_1^{k_{1(N+1)^M}} \cdots x_M^{k_{M\beta}}\right)\left(\prod_{j=1}^{M} x_j^{(k_{ji}-\delta[j-\alpha])}\right)$$

$$= \left(a_1^\alpha x_1^{k_{1}+k_{1i}-\delta[1-\alpha]} x_2^{k_{21}+k_{2i}-\delta[2-\alpha]} \cdots x_M^{k_{M1}+k_{Mi}-\delta[M-\alpha]} + \cdots + \cdots\right.$$

$$\left. \cdots + a_{\underbrace{(M+1)^M}_{\beta}}^\alpha x_1^{k_{1\beta}+k_{1i}-\delta[1-\alpha]} x_2^{k_{2\beta}+k_{2i}-\delta[2-\alpha]} \cdots x_M^{k_{M\beta}+k_{Mi}-\delta[M-\alpha]}\right)$$

$$= \left(\sum_{\beta=1}^{(N+1)^M}\left[a_\beta^\alpha \prod_{j=1}^{M}\left(x_j^{(k_{j\beta}+k_{ji}-\delta[j-\alpha])}\right)\right]\right)$$

$$\frac{dx_l}{dt} = \sum_{i=1}^{(N+1)^M} \left[a_i^l \prod_{j=1}^{M} x_j^{k_{ji}} \right]$$

$$\frac{d^2 x_l}{dt^2} = \sum_{i=1}^{(N+1)^M} \left[\cancel{\ldots} + a_i^l \left(\sum_{\alpha=1}^{M} \left[K_{\alpha i} \left(\sum_{\beta=1}^{(N+1)^M} \left[a_\beta^\alpha \prod_{j=1}^{M} \left(x_j^{(k_{j\beta}+k_{ji}-\delta[j-\alpha])} \right) \right] \right) \right] \right) \right]$$

$$\frac{d^3 x_l}{dt^3} = \underbrace{\cancel{\ldots}}_{\cancel{\ldots}} + \frac{d}{dt}\left(\underbrace{\sum_{i=1}^{(N+1)^M} a_i^l \left(\sum_{\alpha=1}^{M} K_{\alpha i} \right)}_{\text{time indep}} \left(\underbrace{\sum_{\beta=1}^{(N+1)^M} \left[a_\beta^\alpha \prod_{j=1}^{M} x_j^{k_{j\beta}+k_{ji}-\delta[j-\alpha]} \right]}_{\text{time dep}} \right) \right)$$

Sufficient to examine: $\frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{(k_{j\beta}+k_{ji}-\delta[j-\alpha])} \right)$ $= \delta_\alpha^j$

$$= \frac{d}{dt}\left(x_1^{k_{1\beta}+k_{1i}-\delta[1-\alpha]} x_2^{k_{2\beta}+k_{2i}-\delta[2-\alpha]} \cdots x_M^{k_{M\beta}+k_{Mi}-\delta[M-\alpha]} \right)$$

$$= x_1^{(k_{1\beta}+k_{1i}-\delta_\alpha^1)} \left(\frac{d}{dt}\left(x_2^{k_{2\beta}+k_{2i}-\delta_\alpha^2} \cdots x_M^{k_{M\beta}+k_{Mi}-\delta_\alpha^M} \right) \right) + \cdots$$

$$\cdots (k_{1\beta}+k_{1i}-\delta_\alpha^1)\, x_1^{(k_{1\beta}+k_{1i}-\delta_\alpha^1-1)} \left(x_2^{(k_{2\beta}+k_{2i}-\delta_\alpha^2)} \cdots x_M^{(k_{M\beta}+k_{Mi}-\delta_\alpha^M)} \right) \left(\frac{dx_1}{dt} \right)$$

$$= x_1^{(k_\beta^1+k_i^1-\delta_\alpha^1)} \left(x_2^{k_\beta^2+k_i^2-\delta_\alpha^2} \left(\frac{d}{dt}\left(x_3^{k_\beta^3+k_i^3-\delta_\alpha^3} \cdots x_M^{k_\beta^M+k_i^M-\delta_\alpha^M} \right) \right) + \cdots \right.$$

$$\cdots + (k_\beta^2+k_i^2-\delta_\alpha^2)\, x_2^{(k_\beta^2+k_i^2-\delta_\alpha^2-1)} \left(x_3^{k_\beta^3+k_i^3-\delta_\alpha^3} \cdots x_M^{k_\beta^M+k_i^M-\delta_\alpha^M} \right) \left(\frac{dx_2}{dt} \right) \Bigg) + \cdots$$

$$\cdots + (k_\beta^1+k_i^1-\delta_\alpha^1)\, x_1^{(k_\beta^1+k_i^1-\delta_\alpha^1-1)} \left(x_2^{k_\beta^2+k_i^2-\delta_\alpha^2} \cdots x_M^{k_\beta^M+k_i^M-\delta_\alpha^M} \right) \left(\frac{dx_1}{dt} \right)$$

$$\frac{dx_l}{dt} = \sum_{i=1}^{(N+1)^M} \left[a_i^l \prod_{j=1}^{M} x_j^{k_i^j} \right] = \sum_{i=1}^{(N+1)^M} a_i^l \sum_{\alpha=1}^{M} \left[k_i^\alpha \sum_{i_\alpha=1}^{(N+1)^M} \left[a_{i_\alpha}^\alpha \prod_{j=1}^{M} x_j^{k_{i_\alpha}^j + k_i^j - \delta_\alpha^j} \right] \right]$$

$$\frac{d^2x_l}{dt^2} = \sum_{i=1}^{(N+1)^M} a_i^l \sum_{\alpha=1}^{M} \left[k_i^\alpha \sum_{\beta=1}^{(N+1)^M} \left[a_\beta^\alpha \prod_{j=1}^{M} x_j^{k_\beta^j + k_i^j - \delta_\alpha^j} \right] \right]$$

$$\frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_\beta^j + k_i^j - \delta_\alpha^j} \right) = x_1^{k_\beta^1 + k_i^1 - \delta_\alpha^1} x_2^{k_\beta^2 + k_i^2 - \delta_\alpha^2} \left(\frac{d}{dt}\left(x_3^{k_\beta^3 + k_i^3 - \delta_\alpha^3} \cdots x_M^{k_\beta^M + k_i^M - \delta_\alpha^M} \right) \right) + \cdots$$

$$\cdots + \left(k_\beta^1 + k_i^1 - \delta_\alpha^1\right) x_1^{(k_\beta^1 + k_i^1 - \delta_\alpha^1 - 1)} x_2^{k_\beta^2 + k_i^2 - \delta_\alpha^2} \cdots x_M^{k_\beta^M + k_i^M - \delta_\alpha^M} \left(\frac{dx_1}{dt}\right) + \cdots$$

$$\cdots + \left(k_\beta^2 + k_i^2 - \delta_\alpha^2\right) x_1^{k_\beta^1 + k_i^1 - \delta_\alpha^1} x_2^{(k_\beta^2 + k_i^2 - \delta_\alpha^2 - 1)} \cdots x_M^{k_\beta^M + k_i^M - \delta_\alpha^M} \left(\frac{dx_2}{dt}\right)$$

w/ $\beta = i_\alpha$

$$\frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_{i_\alpha}^j + k_i^j - \delta_\alpha^j} \right) = x_1^{k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1} x_2^{k_{i_\alpha}^2 + k_i^2 - \delta_\alpha^2} \left(\frac{d}{dt}\left(x_3^{k_{i_\alpha}^3 + k_i^3 - \delta_\alpha^3} \cdots x_M^{k_{i_\alpha}^M + k_i^M - \delta_\alpha^M} \right) \right) + \cdots$$

$$\cdots + \left(k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1\right) x_1^{(k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1 - 1)} x_2^{k_{i_\alpha}^2 + k_i^2 - \delta_\alpha^2} \cdots x_M^{k_{i_\alpha}^M + k_i^M - \delta_\alpha^M} \left(\frac{dx_1}{dt}\right) + \cdots$$

$$\cdots + \left(k_{i_\alpha}^2 + k_i^2 - \delta_\alpha^2\right) x_1^{(k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1)} x_2^{(k_{i_\alpha}^2 + k_i^2 - \delta_\alpha^2 - 1)} \cdots x_M^{k_{i_\alpha}^M + k_i^M - \delta_\alpha^M} \left(\frac{dx_2}{dt}\right)$$

$$= \left(k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1\right) x_1^{(k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1 - 1)} x_2^{k_{i_\alpha}^2 + k_i^2 - \delta_\alpha^2} \cdots x_M^{k_{i_\alpha}^M + k_i^M - \delta_\alpha^M} \left(\frac{dx_1}{dt}\right) + \cdots$$

$$\cdots + \left(k_{i_\alpha}^2 + k_i^2 - \delta_\alpha^2\right) x_1^{k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1} x_2^{(k_{i_\alpha}^2 + k_i^2 - \delta_\alpha^2 - 1)} \cdots x_M^{k_{i_\alpha}^M + k_i^M - \delta_\alpha^M} \left(\frac{dx_2}{dt}\right) + \cdots$$

$$\cdots + \cdots + \cdots + \left(k_{i_\alpha}^{M-1} + k_i^{M-1} - \delta_\alpha^{M-1}\right) x_1^{k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1} \cdots x_{M-1}^{(k_{i_\alpha}^{M-1} + k_i^{M-1} - \delta_\alpha^{M-1} - 1)} x_M^{k_{i_\alpha}^M + k_i^M - \delta_\alpha^M} \left(\frac{dx_{M-1}}{dt}\right) + \cdots$$

$$\cdots + \left(k_{i_\alpha}^M + k_i^M - \delta_\alpha^M\right) x_1^{k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1} \cdots x_{M-1}^{k_{i_\alpha}^{M-1} + k_i^{M-1} - \delta_\alpha^{M-1}} x_M^{(k_{i_\alpha}^M + k_i^M - \delta_\alpha^M - 1)} \left(\frac{dx_M}{dt}\right)$$

$= \sum_{\beta=1}^{M}$

not the same before. Thats now i_α

$$\Rightarrow \frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_{i_\alpha}^j + k_i^j - \delta_\alpha^j}\right) = \cdots$$

$$\cdots = (k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1)\, x_1^{(k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1 - 1)}\, x_2^{k_{i_\alpha}^2 + k_i^2 - \delta_\alpha^2} \cdots x_M^{k_{i_\alpha}^M + k_i^M - \delta_\alpha^M} \left(\sum_{i_1=1}^{(N+1)^M}\left[a_{i_1}^1 \prod_{j=1}^{M} x_j^{k_{i_1}^j}\right]\right)$$

$$\cdots + (k_{i_\alpha}^M + k_i^M - \delta_\alpha^M)\, x_1^{k_{i_\alpha}^1 + k_i^1 - \delta_\alpha^1}\, x_2^{k_{i_\alpha}^2 + k_i^2 - \delta_\alpha^2} \cdots x_M^{(k_{i_\alpha}^M + k_i^M - \delta_\alpha^M - 1)} \left(\sum_{i_M=1}^{(N+1)^M}\left[a_{i_M}^M \prod_{j=1}^{M} x_j^{k_{i_M}^j}\right]\right)$$

$$= \sum_{\beta=1}^{M}\left(k_{i_\alpha}^\beta + k_i^\beta - \delta_\alpha^\beta\right)\left(\sum_{i_\beta=1}^{(N+1)^M} a_{i_\beta}^\beta \prod_{j=1}^{M} x_j^{k_{i_\beta}^j}\right)\left(\prod_{j=1}^{M} x_j^{(k_{i_\alpha}^j + k_i^j - \delta_\alpha^j - \delta_\beta^j)}\right)$$

$$\Rightarrow \boxed{\begin{aligned} &= \sum_{\beta=1}^{M}\left(k_{i_\alpha}^\beta + k_i^\beta - \delta_\alpha^\beta\right)\left[\sum_{i_\beta=1}^{(N+1)^M} a_{i_\beta}^\beta \prod_{j=1}^{M} x_j^{(k_{i_\beta}^j + k_{i_\alpha}^j + k_i^j - \delta_\alpha^j - \delta_\beta^j)}\right] \\ &= \frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{k_{i_\alpha}^j + k_i^j - \delta_\alpha^j}\right) \end{aligned}}$$

$$\Rightarrow \text{thus: } \frac{d^3 x_\ell}{dt^3} = \frac{d}{dt}\left(\sum_{i=1}^{(N+1)^M} a_i^\ell \left[\sum_{\alpha=1}^{M} k_i^\alpha \left[\sum_{i_\alpha=1}^{(N+1)^M} a_{i_\alpha}^\alpha \left[\prod_{j=1}^{M} x_j^{(k_{i_\alpha}^j + k_i^j - \delta_\alpha^j)}\right]\right]\right]\right)$$

$$\Rightarrow \frac{d^3 x_\ell}{dt^3} = \sum_{i=1}^{(N+1)^M} a_i^\ell \left[\sum_{\alpha=1}^{M} k_i^\alpha \left[\sum_{i_\alpha=1}^{(N+1)^M} a_{i_\alpha}^\alpha \left[\frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{(k_{i_\alpha}^j + k_i^j - \delta_\alpha^j)}\right)\right]\right]\right]$$

$$\Rightarrow \boxed{\frac{d^3 x_\ell}{dt^3} = \sum_{i=1}^{(N+1)^M} a_i^\ell \sum_{\alpha=1}^{M} k_i^\alpha \sum_{i_\alpha=1}^{(N+1)^M} a_{i_\alpha}^\alpha \sum_{\beta=1}^{M}\left(k_{i_\alpha}^\beta + k_i^\beta - \delta_\alpha^\beta\right) \sum_{i_\beta=1}^{(N+1)^M} a_{i_\beta}^\beta \prod_{j=1}^{M} x_j^{(k_{i_\beta}^j + k_{i_\alpha}^j + k_i^j - \delta_\alpha^j - \delta_\beta^j)}}$$

$$\frac{dx_\ell}{dt} = \sum_{i=1}^{(N+1)^M} \left[a_i^\ell \prod_{j=1}^{M} x_j^{(k_i^j)} \right]$$

var $j \to$ row index
var $i \to$ column index.
$\to$ ~~technically~~ except
$x_j^{k_i^j}$
$\to$ should technically be $x_j^{k_i^j}$, but that makes exponent notation inconvenient

$$\frac{d^2x_\ell}{dt^2} = \sum_{i=1}^{(N+1)^M} a_i^\ell \sum_{\alpha=1}^{M} k_i^\alpha \sum_{i_\alpha=1}^{(N+1)^M} \left[a_{i_\alpha}^\alpha \prod_{j=1}^{M} x_j^{(k_{i_\alpha}^j + k_i^j - \delta_\alpha^j)} \right]$$

$\to$ ~~$\frac{d^2x_\ell}{dt^2} = \sum_{i=1}^{(N+1)^M} \sum_{\alpha=1}^{M} \sum_{i_\alpha=1}^{(N+1)^M}$~~

$$\to \frac{d^2x_\ell}{dt^2} = \sum_{i=1}^{(N+1)^M} \sum_{\alpha=1}^{M} \sum_{i_\alpha=1}^{(N+1)^M} \left[\left(a_i^\ell k_i^\alpha a_{i_\alpha}^\alpha \right) \prod_{j=1}^{M} x_j^{(k_{i_\alpha}^j + k_i^j - \delta_\alpha^j)} \right]$$

$$\to \frac{d^3x_\ell}{dt^3} = \sum_{i=1}^{(N+1)^M} \sum_{\alpha=1}^{M} \sum_{i_\alpha=1}^{(N+1)^M} \sum_{\beta=1}^{M} \sum_{i_\beta=1}^{(N+1)^M} \left[\left(a_i^\ell k_i^\alpha a_{i_\alpha}^\alpha (k_{i_\alpha}^\beta + k_i^\beta - \delta_\alpha^\beta) a_{i_\beta}^\beta \right) \prod_{j=1}^{M} x_j^{(k_{i_\beta}^j + k_{i_\alpha}^j + k_i^j - \delta_\alpha^j - \delta_\beta^j)} \right]$$

$(k_{i_\beta}^j + k_{i_\alpha}^j + k_i^j - \delta_\alpha^j - \delta_\beta^j)$

~~prediction!~~ $\to$ correct!

$$\to \frac{d^4x_\ell}{dt^4} = \sum_{i=1}^{(N+1)^M} \sum_{\alpha=1}^{M} \sum_{i_\alpha=1}^{(N+1)^M} \sum_{\beta=1}^{M} \sum_{i_\beta=1}^{(N+1)^M} \sum_{\gamma=1}^{M} \sum_{i_\gamma=1}^{(N+1)^M} \left[\left(a_i^\ell k_i^\alpha a_{i_\alpha}^\alpha (k_{i_\alpha}^\beta + k_i^\beta - \delta_\alpha^\beta) a_{i_\beta}^\beta (k_{i_\beta}^\gamma + k_{i_\alpha}^\gamma + k_i^\gamma - \delta_\alpha^\gamma - \delta_\beta^\gamma) a_{i_\gamma}^\gamma \right) \cdot \cdots \right.$$
$$\left. \cdots \cdot \prod_{j=1}^{M} x_j^{(k_{i_\gamma}^j + k_{i_\beta}^j + k_{i_\alpha}^j + k_i^j - \delta_\alpha^j - \delta_\beta^j - \delta_\gamma^j)} \right]$$

~~Lets find out~~

$\to$ found out! good!

$$\frac{d^3 x_l}{dt^3} = \sum_{i,\alpha,i_\alpha,\beta,i_\beta=1}^{\{(N+1)^M,M,(N+1)^M,M,(N+1)^M\}} \left[a_i^l k_i^\alpha a_{i_\alpha}^\alpha \left(k_{i_\alpha}^\beta + k_i^\beta - \delta_\alpha^\beta\right) a_{i_\beta}^\beta \prod_{j=1}^{M} x_j^{\left(k_{i_\beta}^j + k_{i_\alpha}^j + k_i^j - \delta_\alpha^j - \delta_\beta^j\right)} \right]$$

$$\frac{d}{dt}\left(\frac{d^3 x_l}{dt^3}\right) \xrightarrow{\text{examine}} \frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{\left(k_{i_\beta}^j + k_{i_\alpha}^j + k_i^j - \delta_\alpha^j - \delta_\beta^j\right)}\right) = \cdots$$

Let $\hat{k}^j = k_{i_\beta}^j + k_{i_\alpha}^j + k_i^j - \delta_\alpha^j - \delta_\beta^j$

$$= \frac{d}{dt}\left(\prod_{j=1}^{M} x_j^{(\hat{k}^j)}\right).$$

$$= x_1^{\hat{k}^1}\left(\frac{d}{dt}\left(x_2^{\hat{k}^2}\cdots x_M^{\hat{k}^M}\right)\right) + \hat{k}^1 x_1^{(\hat{k}^1-1)}\left(x_2^{\hat{k}^2}\cdots x_M^{\hat{k}^M}\right)\frac{dx_1}{dt}$$

$$= x_1^{\hat{k}^1} x_2^{\hat{k}^2}\cdots\left(\frac{d}{dt}\left(x_{M-1}^{\hat{k}^{M-1}} x_M^{\hat{k}^M}\right)\right) + \hat{k}^1 x_1^{(\hat{k}^1-1)} x_2^{\hat{k}^2}\cdots x_M^{\hat{k}^M}\frac{dx_1}{dt} + \cdots$$

$$\cdots + \hat{k}^2 x_1^{\hat{k}^1} x_2^{(\hat{k}^2-1)} x_3^{\hat{k}^3}\cdots x_M^{\hat{k}^M}\frac{dx_2}{dt} + \cdots + \cdots$$

$$\cdots + \hat{k}^{M-2} x_1^{\hat{k}^1} x_2^{\hat{k}^2}\cdots x_{M-2}^{(\hat{k}^{M-2}-1)} x_{M-1}^{\hat{k}^{M-1}} x_M^{\hat{k}^M}\frac{dx_{M-2}}{dt}$$

$$= \hat{k}^1 x_1^{(\hat{k}^1-1)} x_2^{\hat{k}^2}\cdots x_M^{\hat{k}^M}\left(\frac{dx_1}{dt}\right) + \hat{k}^2 x_1^{\hat{k}^1} x_2^{(\hat{k}^2-1)}\cdots x_M^{\hat{k}^M}\left(\frac{dx_2}{dt}\right) + \cdots$$

$$\cdots + \cdots + \hat{k}^M x_1^{\hat{k}^1} x_2^{\hat{k}^2}\cdots x_M^{(\hat{k}^M-1)}\left(\frac{dx_M}{dt}\right)$$

$$= \hat{k}^1 x_1^{(\hat{k}^1-1)} x_2^{\hat{k}^2}\cdots x_M^{\hat{k}^M}\left(\sum_{i_1=1}^{(N+1)^M}\left[a_{i_1}^1 \prod_{j=1}^{M} x_j^{(k_{i_1}^j)}\right]\right) + \cdots + \cdots$$

$$\cdots + \hat{k}^M x_1^{\hat{k}^1} x_2^{\hat{k}^2}\cdots x_M^{(\hat{k}^M-1)}\left(\sum_{i_M=1}^{(N+1)^M}\left[a_{i_M}^M \prod_{j=1}^{M} x_j^{(k_{i_M}^j)}\right]\right) \longrightarrow$$

$$\Rightarrow = \sum_{\varepsilon=1}^{M} \left(\hat{k}^{\varepsilon}\right) \left(\sum_{i_\varepsilon=1}^{(N+1)^M} a^{\varepsilon}_{i_\varepsilon} \prod_{j=1}^{M} x_j^{\left(k^j_{i_\varepsilon}\right)}\right) \left(\prod_{j=1}^{M} x_j^{\left(\hat{k}^j - \delta^j_\varepsilon\right)}\right)$$

$$= \sum_{\varepsilon=1}^{M} \hat{k}^{\varepsilon} \left(\sum_{i_\varepsilon=1}^{(N+1)^M} a^{\varepsilon}_{i_\varepsilon} \left(\prod_{j=1}^{M} x_j^{\left(k^j_{i_\varepsilon} + \hat{k}^j - \delta^j_\varepsilon\right)}\right)\right)$$

but, $\hat{k}^j = k^j_{i_\beta} + k^j_{i_\alpha} + k^j_i - \delta^j_\alpha - \delta^j_\beta$

$$= \sum_{\varepsilon=1}^{M} \left(k^{\varepsilon}_{i_\beta} + k^{\varepsilon}_{i_\alpha} + k^{\varepsilon}_i - \delta^{\varepsilon}_\alpha - \delta^{\varepsilon}_\beta\right) \left(\sum_{i_\varepsilon=1}^{(N+1)^M} a^{\varepsilon}_{i_\varepsilon} \left(\prod_{j=1}^{M} x_j^{\left(k^j_{i_\varepsilon} + k^j_{i_\beta} + k^j_{i_\alpha} + k^j_i - \delta^j_\alpha - \delta^j_\beta - \delta^j_\varepsilon\right)}\right)\right)$$

$$\Rightarrow \frac{d^4 x_l}{dt^4} = \sum_{i,\alpha,i_\alpha,\beta,i_\beta=1}^{(N+1)^M, M, (N+1)^M, M, (N+1)^M} \left(a^l_i k^\alpha_i a^\alpha_{i_\alpha} \left(k^\beta_{i_\alpha} + k^\beta_i - \delta^\beta_\alpha\right) a^\beta_{i_\beta}\right) \left(\sum_{\varepsilon=1}^{M} \sum_{i_\varepsilon=1}^{(N+1)^M} \left(k^{\varepsilon}_{i_\beta} + k^{\varepsilon}_{i_\alpha} + k^{\varepsilon}_i - \delta^{\varepsilon}_\alpha - \delta^{\varepsilon}_\beta\right) a^{\varepsilon}_{i_\varepsilon} \left(\cdots \right.\right.$$

$$\left.\left. \cdots \prod_{j=1}^{M} x_j^{\left(k^j_{i_\varepsilon} + k^j_{i_\beta} + k^j_{i_\alpha} + k^j_i - \delta^j_\alpha - \delta^j_\beta - \delta^j_\varepsilon\right)} \right)\right)$$

pattern [redacted] found!

the whole thing!

testing before going farther

for $\dot{x} = y^2 - x$, $\dot{y} = x^2 - y$

$$k = \begin{array}{c} x \\ y \end{array} \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$(\dot{y}\text{'s } x^2)$ $(\dot{x}\text{'s } x)$ $(\dot{x}\text{'s } y^2)$ $(\dot{y}\text{'s } y)$

$$\boxed{\begin{array}{l} \text{for } \dot{x}: a_7 = 1,\ a_6 = -1,\ a_{i \setminus \{6,7\}} = 0. \\ \text{for } \dot{y}: a_3 = 1,\ a_8 = -1,\ a_{i \setminus \{3,8\}} = 0. \end{array}}$$

~~for $\frac{d^n x_l}{dt^n}$: let~~

~~Let J:=~~

For $\frac{d^n x_\ell}{dt^n}$, there is $2n-1$ summations.

last $2(n-1)/2$ are defined by indices

in the sets $J(n)$, $I(n)$; $\{j_1, j_2, j_3, \dots, j_{n-1}\} =: J(n)$

$\{i_1, i_2, i_3, \dots, i_{n-1}\} =: I(n)$

The ~~last~~ first sum is indexed by $alpha = 1 : (N+1)^M$

* All sums w/ an index $j \in J(n)$ go up to M
* All sums w/ index $i \in I(n)$ go up to $(N+1)^M$

To ease notation, the equations will be rewritten as follows:

$i \to \alpha$; $j \to \beta$; $\underbrace{l \to l}_{\text{same}}$;

$\alpha \to j_1$; $\beta \to j_2$; $\varepsilon \to j_3$; ....

$i_\alpha \to i_1$; $i_\beta \to i_2$; $i_\varepsilon \to i_3$; ...

$$\frac{dx_l}{dt} = \sum_{\alpha=1}^{(N+1)^M} \left[a_\alpha^l \prod_{\beta=1}^{M} x_\beta^{(K_\alpha^\beta)} \right]$$

$$\frac{d^2x_l}{dt^2} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \left[\left(a_\alpha^l K_\alpha^{j_1} a_{i_1}^{j_1} \right) \prod_{\beta=1}^{M} x_\beta^{(K_{i_1}^\beta + K_\alpha^\beta - \delta_{j_1}^\beta)} \right]$$

$$\frac{d^3x_l}{dt^3} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \sum_{j_2=1}^{M} \sum_{i_2=1}^{(N+1)^M} \left[\left(a_\alpha^l K_\alpha^{j_1} a_{i_1}^{j_1} \left(K_{i_1}^{j_2} + K_\alpha^{j_2} - \delta_{j_1}^{j_2} \right) a_{i_2}^{j_2} \right) \prod_{\beta=1}^{M} x_\beta^{(K_{i_2}^\beta + K_{i_1}^\beta + K_\alpha^\beta - \delta_{j_1}^\beta - \delta_{j_2}^\beta)} \right]$$

$$\frac{d^4x_l}{dt^4} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \sum_{j_2=1}^{M} \sum_{i_2=1}^{(N+1)^M} \sum_{j_3=1}^{M} \sum_{i_3=1}^{(N+1)^M} \left[\left(a_\alpha^l K_\alpha^{j_1} a_{i_1}^{j_1} \left(K_{i_1}^{j_2} + K_\alpha^{j_2} - \delta_{j_1}^{j_2} \right) a_{i_2}^{j_2} \left(K_{i_2}^{j_3} + K_{i_1}^{j_3} + K_\alpha^{j_3} - \delta_{j_1}^{j_3} - \delta_{j_2}^{j_3} \right) a_{i_3}^{j_3} \right) \cdots \right.$$

$$\left. \cdots \left(\prod_{\beta=1}^{M} x_\beta^{(K_{i_3}^\beta + K_{i_2}^\beta + K_{i_1}^\beta + K_\alpha^\beta - \delta_{j_1}^\beta - \delta_{j_2}^\beta - \delta_{j_3}^\beta)} \right) \right]$$

$$X = \begin{bmatrix} x_1^1 & x_2^1 & \cdots & x_M^1 \\ x_1^2 & x_2^2 & \cdots & x_M^2 \\ \vdots & \vdots & & \vdots \\ x_1^M & x_2^M & \cdots & x_M^M \end{bmatrix} \in \mathbb{R}^{M\times(N+1)^M}, \quad \underline{K \in \mathbb{R}^{M\times(N+1)^M}}.$$

then X_α^β is element @ $(\overset{\text{row}}{\beta}, \overset{\text{col}}{\alpha})$; $\boxed{\text{note: } x_1^\beta(t) = x_2^\beta(t) = \cdots = x_M^\beta(t). \text{ structure only for convenience of code!}}$

then $X^{\cdot K} \in \mathbb{R}^{M\times(N+1)^M}$ when expo is element w.se.

let $\underline{X} = \left[\left(\prod_{\beta=1}^{M} (x_1^\beta)^{k_1^\beta}\right), \left(\prod_{\beta=1}^{M} (x_2^\beta)^{k_2^\beta}\right), \ldots, \left(\prod_{\beta=1}^{M} (x_M^\beta)^{k_M^\beta}\right)\right]$

then $\underline{X}_\alpha = \left(\prod_{\beta=1}^{M} (x_\alpha^\beta)^{k_\alpha^\beta}\right)$!

$$\boxed{\text{Thus, } \frac{d(x_\alpha^l)}{dt} = a_\alpha^l \underline{X}_\alpha \text{ and } \frac{d(x^l)}{dt} = \sum_{\alpha=1}^{(N+1)^M} \left[a_\alpha^l \underline{X}_\alpha\right]!}$$

$$\frac{dx_l}{dt} = \sum_{\alpha=1}^{(N+1)^M} \left[a_\alpha^l \prod_{\beta=1}^{M} x_\beta^{(k_\alpha^\beta)} \right] = \sum_{\alpha=1}^{(N+1)^M} \left[a_\alpha^l \, \bar{x}_\alpha \right]$$

$$\frac{d^2x_l}{dt^2} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j\in J(2),\, i\in I(2)} \left[\left(a_\alpha^l a_{i_1}^{j_1} \right) \left(k_\alpha^{j_1} \right) \left(\prod_{\beta=1}^{M} x_\beta^{\left(k_\alpha^\beta + k_{i_1}^\beta - \delta_{j_1}^\beta\right)} \right) \right]$$

$$\frac{d^3x_l}{dt^3} = \sum_{\alpha=1,\, j\in J(3),\, i\in I(3)} \left[\left(a_\alpha^l a_{i_1}^{j_1} a_{i_2}^{j_2} \right) \left(k_\alpha^{j_1} \left(k_\alpha^{j_2} + k_{i_1}^{j_2} - \delta_{j_1}^{j_2} \right) \right) \left(\prod_{\beta=1}^{M} x_\beta^{\left(k_\alpha^\beta + k_{i_2}^\beta - \delta_{j_2}^\beta + k_{i_1}^\beta - \delta_{j_1}^\beta\right)} \right) \right]$$

$$\frac{d^4x_l}{dt^4} = \sum_{\alpha=1,\, j\in J(4),\, i\in I(4)} \left[\left(a_\alpha^l a_{i_1}^{j_1} a_{i_2}^{j_2} a_{i_3}^{j_3} \right) \left(k_\alpha^{j_1} \left(k_\alpha^{j_2} + k_{i_1}^{j_2} - \delta_{j_1}^{j_2} \right) \left(k_\alpha^{j_3} + k_{i_2}^{j_3} - \delta_{j_2}^{j_3} + k_{i_1}^{j_3} - \delta_{j_1}^{j_3} \right) \left(\cdots \right.\right.\right.$$

$$\left.\left.\left. \cdots \prod_{\beta=1} (x^\beta)^{\left(k_\alpha^\beta + k_{i_3}^\beta - \delta_{j_3}^\beta + k_{i_2}^\beta - \delta_{j_2}^\beta + k_{i_1}^\beta - \delta_{j_1}^\beta\right)} \right) \right] .$$

$$\frac{d^2x^l}{dt^2} = \sum_{\substack{\alpha=1 \\ j\in J(2) \\ i\in I(2)}} \left[\left(a_\alpha^l a_{i_1}^{j_1} \right) \left(k_\alpha^{j_1} \right) \overbrace{\left(\prod_{\beta=1}^{M} (x_\alpha^\beta)^{(k_\alpha^\beta)} \right)}^{\bar{x}_\alpha} \overbrace{\left(\prod_{\beta=1}^{M} (x_{i_1}^\beta)^{(k_{i_1}^\beta)} \right)}^{\bar{x}_{i_1}} \left(\prod_{\beta=1}^{M} (x_{i_1}^\beta)^{(-\delta_{j_1}^\beta)} \right) \right]$$

$$= \sum \left[\left(a_\alpha^l \bar{x}_\alpha \right) \left(a_{i_1}^{j_1} \bar{x}_{i_1} \right) \left(k_\alpha^{j_1} \prod_{\beta=1}^{M} (x_{i_1}^\beta)^{(-\delta_{j_1}^\beta)} \right) \right]$$

$$\boxed{\frac{d^2x^l}{dt^2} = \sum \left[\left(\frac{d(x_\alpha^l)}{dt} \right) \left(\frac{d(x_{i_1}^{j_1})}{dt} \right) \left(k_\alpha^{j_1} \prod_{\beta=1}^{M} (x_{i_1}^\beta)^{(-\delta_{j_1}^\beta)} \right) \right]}$$

$$\frac{d^2x^l}{dt^2} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \left[\frac{d(x_{i_1}^{j_1})}{dt} \right] \left[\frac{d(x_\alpha^l)}{dt} \right] \left(k_\alpha^{j_1} \prod_{\beta=1}^{M} (x_{i_1}^\beta)^{(-\delta_{j_1}^\beta)} \right)$$

subscript redundant w/ x's

changing allows removal of i_1 sum!

$$\boxed{= \frac{d^2x^l}{dt^2} = \left(\sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \left[\left(k_\alpha^{j_1} \prod_{\beta=1}^{M} (x_{j_1}^\beta)^{(-\delta_{j_1}^\beta)} \right) \frac{d(x_\alpha^l)}{dt} \right] \left(\frac{d(x^{j_1})}{dt} \right) \right)}$$

$$\frac{d^3x^l}{dt^3} = \sum \left(a^l_\alpha a^{j_1}_{i_1} a^{j_2}_{i_2}\right)\left(K^{j_1}_\alpha\left(K^{j_2}_\alpha + K^{j_2}_{i_1} - \delta^{j_2}_{j_1}\right)\right)\left(\prod_{\beta=1}^{M} \overbrace{(x^\beta_\alpha)^{(K^\beta_\alpha)}}^{\bar{x}_\alpha} \overbrace{(x^\beta_{i_1})^{(K^\beta_{i_1})}}^{\bar{x}_{i_1}} (x^\beta_{j_1})^{(-\delta^\beta_{j_1})} \overbrace{(x^\beta_{i_2})^{(K^\beta_{i_2})}}^{\bar{x}_{i_2}} (x^\beta_{j_2})^{(-\delta^\beta_{j_2})}\right)$$

$$= \sum \Big[\left(a^l_\alpha \bar{x}_\alpha\right)\left(a^{j_1}_{i_1}\bar{x}_{i_1}\right)\left(K^{j_1}_\alpha \prod_{\beta=1}^{M} (x^\beta_{j_1})^{(-\delta^\beta_{j_1})}\right)\left(a^{j_2}_{i_2}\bar{x}_{i_2}\right)\Big(\cdots$$

$$\cdots \left(K^{j_2}_\alpha + K^{j_2}_{i_1} - \delta^{j_2}_{j_1}\right)\left(\prod_{\beta=1}^{M} (x^\beta_{j_2})^{(-\delta^\beta_{j_2})}\right)\Big]$$

$$= \sum_{\alpha=1}\sum_{j_1=1}\sum_{i_1=1}\sum_{j_2=1}\sum_{i_2=1}\left[\frac{d(x^l_\alpha)}{dt}\right]\left[\frac{d(x^{j_1}_{i_1})}{dt}\right]\left(K^{j_1}_\alpha \prod_{\beta=1}^{M}(x^\beta_{j_1})^{(-\delta^\beta_{j_1})}\right)\left(\frac{d(x^{j_2}_{i_2})}{dt}\right)\left(K^{j_2}_\alpha + K^{j_2}_{i_1} - \delta^{j_2}_{j_1}\right)\left(\prod_{\beta=1}^{M}(x^\beta_{j_2})^{(-\delta^\beta_{j_2})}\right)$$

Notation Guide.

Let $X = \begin{bmatrix} x_1^1 & x_2^1 & \cdots & x_M^1 \\ x_1^2 & x_2^2 & \cdots & x_M^2 \\ \vdots & \vdots & & \vdots \\ x_1^M & x_2^M & \cdots & x_M^M \end{bmatrix} \in \mathbb{R}^{M \times (N+1)^M}$

then (x_α^β) is the element in X @ the β-th row & α-th column.

critical importance

* Note: $x_1^\beta = x_2^\beta = \cdots = x_M^\beta$, $\forall \beta \in \{1, \ldots, M\}$. The structure of X here is only for convenience of code. *

Recall that $K \in \mathbb{R}^{M \times (N+1)^M}$, Thus, $(X^{\cdot K}) \in \mathbb{R}^{M \times (N+1)^M}$ (expo is element wise!)

Let $\Sigma = \left[\left(\prod_{\beta=1}^{M} (x_1^\beta)^{k_1^\beta} \right), \left(\prod_{\beta=1}^{M} (x_2^\beta)^{k_2^\beta} \right), \ldots, \left(\prod_{\beta=1}^{M} (x_M^\beta)^{k_M^\beta} \right) \right]$.

Then $\Sigma_\alpha = \left(\prod_{\beta=1}^{M} (x_\alpha^\beta)^{k_\alpha^\beta} \right)!$ So, the α-th element in the sum expressed by $\frac{d(x^l)}{dt}$ is given by Σ_α! i.e.

$$\boxed{\frac{d(x_\alpha^l)}{dt} = a_\alpha^l \Sigma_\alpha \quad \text{and} \quad \frac{d(x^l)}{dt} = \sum_{\alpha=1}^{(N+1)^M} \left[a_\alpha^l \Sigma_\alpha \right]}$$ critical importance

Let the indexes $J(n) := \{j_1, j_2, \ldots, j_{n-1}\}$ be indexes summing from $[1, M]$.

$I(n) := \{i_1, i_2, \ldots, i_{n-1}\}$

Let indexes $I(n)$ be indexes summing from $[1, (N+1)^M]$.

Let α be ~~the~~ a index summing from $[1, (N+1)^M]$.

W/ Notation on last page.

$$\frac{dx^l}{dt} = \sum_{\substack{\alpha=1\\ j\in J(1)\\ i\in I(1)}} \left[a^l_\alpha \prod_{\beta=1}^{M} (x^\beta_{\cdot})^{(k^\beta_\alpha)}\right] = \sum\left[a^l_\alpha \bar{x}_\alpha\right] = \sum\left[\frac{d(x^l_\alpha)}{dt}\right]$$

$(x^\beta_{\cdot})$ ← subscript optional!

$$\frac{d^2(x^l)}{dt^2} = \sum_{\substack{\alpha=1,\\ j\in J_2, i\in I_2}} \left[\left(a^l_\alpha a^{j_1}_{i_1} k^{j_1}_\alpha\right) \prod_{\beta=1}^{M} (x^\beta_{\cdot})^{(k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1})}\right]$$

$$\frac{d^3(x^l)}{dt^3} = \sum_{\substack{\alpha=1, j\in J_3,\\ i\in J_3}} \left[\left(\underbrace{a^l_\alpha a^{j_1}_{i_1} a^{j_2}_{i_2}}_{(\hat{a}^l_{\hat\alpha} \hat{k}^{j_2}_{\hat\alpha} a^{j_2}_{i_2})} \underbrace{k^{j_1}_\alpha}_{\text{from } d^2 = a^l_{\hat\alpha}} \underbrace{\left(k^{j_2}_\alpha + k^{j_2}_{i_1} - \delta^{j_2}_{j_1}\right)}_{= \hat{k}^{j_2}_{\hat\alpha}}\right) \prod_{\beta=1}^{M} (x^\beta_{\cdot})^{\underbrace{(k^\beta_\alpha + k^\beta_{i_2} - \delta^\beta_{j_2} + k^\beta_{i_1} - \delta^\beta_{j_1})}_{\hat{k}^\beta_{\hat\alpha}}}\right]$$

$(\hat{k}^\beta_{\hat\alpha} + k^\beta_{i_2} - \delta^\beta_{j_2})$

$(\hat{a}^l_{\hat\alpha} a^{j_3}_{i_3} k^{j_3}_\alpha)$

$$\frac{d^4(x^l)}{dt^4} = \sum_{\substack{\alpha=1, j\in J(4)\\ i\in J(4)}} \left[\left(a^l_\alpha a^{j_1}_{i_1} a^{j_2}_{i_2} a^{j_3}_{i_3} k^{j_1}_\alpha \left(k^{j_2}_\alpha + k^{j_2}_{i_1} - \delta^{j_2}_{j_1}\right)\left(k^{j_3}_\alpha + k^{j_3}_{i_2} - \delta^{j_3}_{j_2} + k^{j_3}_{i_1} - \delta^{j_3}_{j_1}\right)\right)\left(\cdots\right.\right.$$

$$\left.\left.\cdots \prod_{\beta=1} (x^\beta_{\cdot})^{(k^\beta_\alpha + k^\beta_{i_3} - \delta^\beta_{j_3} + k^\beta_{i_2} - \delta^\beta_{j_2} + k^\beta_{i_1} - \delta^\beta_{j_1})}\right]$$

$(\hat{k}^\beta_{\hat\alpha} + k^\beta_{i_3} - \delta^3_{j_3})$

$\vdots$

$$\cancel{\frac{d^n(x^l)}{dt^n} = \sum_{\substack{\alpha=1, j\in J(n)\\ i\in I(n)}}}$$

*let $j_0 = l$, $i_0 = \alpha$

by starting $\gamma = 1$, when $n=1$ the limit is 0, the term product is ignored, probably bad notation.

$$\frac{d^n(x^l)}{dt^n} = \sum_{\substack{\alpha=1, j\in J(n),\\ i\in I(n)}} \left[\left(\prod_{\gamma=0}^{n-1} a^{j_\gamma}_{i_\gamma}\right)\left(\prod_{\gamma=1}^{n-1}\left(\sum_{\gamma=0}^{(\gamma-1)}\left(k^{j_\gamma}_{i_\gamma} - \delta^{j_\gamma}_{j_\gamma} u[\gamma-1]\right)\right)\right)\left(\cdots\right.\right.$$

$$\left.\left.\cdots\left(\prod_{\beta=1} (x^\beta_{\cdot})^{\left(\sum_{\gamma=0}^{n-1}\left(k^\beta_{i_\gamma} - \delta^\beta_{j_\gamma} u[\gamma-1]\right)\right)}\right)\right]$$

rewriting nicely.

$$\frac{d^n(x^l)}{dt^n} = \sum_{\substack{\alpha=1,\\ j\in J(n),\\ i\in I(n)}} \left[\left(\prod_{\beta=0}^{n-1} a_{i_\beta}^{j_\beta} \right) \left(\prod_{\beta=1}^{n-1} \left(\sum_{\gamma=0}^{\beta-1} \left(k_{i_\gamma}^{j_\beta} - \delta_{j_\gamma}^{j_\beta} u[\gamma-1] \right) \right) \right) \cdots \right.$$

$$\left. \cdots \left(\prod_{\beta=1}^{M} (x_\cdot^\beta)^{\left(\sum_{\gamma=0}^{n-1} \left(k_{i_\gamma}^{\beta} - \delta_{j_\gamma}^{\beta} u[\gamma-1] \right) \right)} \right) \right]$$

w/ $j_0 = l$ and $i_0 = \alpha$, and the assumption that

$$\prod_{\beta=0}^{n-1} (\cdot) = 1 \text{ when } (n-1) < \beta.$$

n-th deriv of function $x^l(t)$ w/ first derivatve of

$$\frac{d(x^l)}{dt} = \sum_{\alpha=1}^{(N+1)^M} \left[a_\alpha^l \prod_{\beta=1}^{M} (x_\cdot^\beta)^{k_\alpha^\beta} \right]$$

$$\frac{d^2(x^l)}{dt^2} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \left(a^l_\alpha a^{j_1}_{i_1} k^{j_1}_\alpha\right)\left(\prod_{\beta=1}^{M} (x_\cdot^\beta)^{(k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1})}\right)$$

$$= \sum_\alpha a^l_\alpha \left(\sum_{j_1} \sum_{i_1} (a^{j_1}_{i_1} k^{j_1}_\alpha) \prod_{\beta=1}^{M} (x_\cdot^\beta)^{(k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1})}\right)$$

$$= \sum_\alpha a^l_\alpha \left(\sum_{j_1=1} k^{j_1}_\alpha \prod_{\beta=1}^{M} (x_\cdot^\beta)^{(k^\beta_\alpha - \delta^\beta_{j_1})} \underbrace{\sum_{i_1} \left(a^{j_1}_{i_1} \prod_{\beta=1}^{M} (x_{i_1}^\beta)^{k^\beta_{i_1}}\right)}_{\left(\frac{d(x^{j_1})}{dt}\right)}\right)$$

$$= \sum_{\alpha=1} a^l_\alpha \left(\sum_{j_1=1} k^{j_1}_\alpha \underbrace{\left(\prod_{\beta=1}^{M} (x_\cdot^\beta)^{(k^\beta_\alpha - \delta^\beta_{j_1})}\right)}_{\frac{\partial}{\partial(x_\cdot^{j_1})}\left(\frac{d(x^l_\alpha)}{dt}\right)} \left(\frac{dx^{j_1}}{dt}\right)\right)$$

need a^l_α in there.

$$= \sum_{\alpha=1} \sum_{j_1=1} \left(\frac{\partial}{\partial x^{j_1}_\cdot}\left(\frac{d(x^l_\alpha)}{dt}\right)\right)\left(\frac{dx^{j_1}_\cdot}{dt}\right)$$

operation seems straight forward.

have this

have this

* Need recursive construction of higher order

$$\frac{d^3(x^\ell)}{dt^3} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \left[\left(a^\ell_\alpha a^{j_1}_{i_1} k^{j_1}_\alpha a^{j_2}_{i_2} \left(k^{j_2}_\alpha + k^{j_2}_{i_1} - \delta^{j_2}_{j_1} \right) \right) \prod_{\beta=1}^{M} (x^\beta)^{\left(k^\beta_\alpha + k^\beta_{i_2} + k^\beta_{i_1} - \delta^\beta_{j_1} - \delta^\beta_{j_2}\right)} \right]$$

$$= \sum_{\alpha=1} \sum_{j_1=1} \sum_{i_1=1} \left[\left(a^\ell_\alpha a^{j_1}_{i_1} k^{j_1}_\alpha a^{j_2}_{i_2} \left(k^{j_2}_\alpha + k^{j_2}_{i_1} - \delta^{j_2}_{j_1} \right) \right) \prod_{\beta=1}^{M} (x^\beta_\alpha)^{\left[\left(k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1}\right) - \delta^\beta_{j_2}\right]} (x^\beta_{i_2})^{(k^\beta_{i_2})} \right]$$

$$= \sum_{\alpha=1} \sum_{j_1} \sum_{i_1} \sum_{j_2} \left[\left(k^{j_2}_\alpha + k^{j_2}_{i_1} - \delta^{j_2}_{j_1} \right) \left(a^\ell_\alpha a^{j_1}_{i_1} k^{j_1}_\alpha \right) \left(\prod_{\beta=1}^{M} \left[x^\beta_\alpha \right]^{\left[\left(k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1}\right) - \delta^\beta_{j_2}\right]} \right) \left(\cdots \right.\right.$$

$$\left.\left. \sum_{i_2} \underbrace{a^{j_2}_{i_2} \left(\prod_{\beta=1}^{M} (x^\beta_{i_2})^{(k^\beta_{i_2})} \right)}_{\frac{d(x^{j_2})}{dt}} \right] \right.$$

$$= \sum_{\alpha=1} \sum_{j_1} \sum_{i_1} \sum_{j_2} \left[\underbrace{\left(k^{j_2}_\alpha + k^{j_2}_{i_1} - \delta^{j_2}_{j_1} \right)}_{} \underbrace{\left(a^\ell_\alpha a^{j_1}_{i_1} k^{j_1}_\alpha \right) \left(\prod_{\beta=1}^{M} (x^\beta_\alpha)^{\left[\left(k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1}\right) - \delta^\beta_{j_2}\right]} \right)}_{\frac{d^2(x^\ell_\alpha)}{dt^2}} \left(\frac{d(x^{j_2})}{dt} \right) \right]$$

yes, these depend on i_1 and j_1 ..but for any j_2, the exponent is already known due to the recursive nature of calculation.

$$\frac{\partial}{\partial(x^{j_2})} \left(\frac{d^2(x^\ell_\alpha)}{dt^2} \right)$$

Thus, $\sum_{j_1}$ & $\sum_{i_1}$ can effectively be absorbed into $\sum_\alpha$, by expressing $(k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1})$ as a new k^β_α! Recomputing the K matrix on each iteration can be done. Similarly, $(a^\ell_\alpha a^{j_1}_{i_1} k^{j_1}_\alpha)$ becomes the new a^ℓ_α! → Sensible, as polynomial order will increase w/ each iteration.

w/ observation on last page:

$$\hat{a}^{l}_{\alpha} = (a^{l}_{\alpha} a^{j_1}_{i_1} K^{j_1}_{\alpha}), \quad \hat{K}^{\beta}_{\alpha} = (K^{\beta}_{\alpha} + K^{\beta}_{i_1} \delta^{\beta}_{j_1})$$

something! → not $(N+1)^M$ anymore!, greater than!

$$= \sum_{\alpha=1} \sum_{j_2=1}^{M} \left[\underbrace{\left(\hat{K}^{j_2}_{\alpha} \hat{a}^{l}_{\alpha} \prod_{\beta=1}^{M} (x^{\beta}_{\alpha}) \left[\hat{K}^{\beta}_{\alpha} - \delta^{\beta}_{j_2} \right] \right)}_{\frac{\partial}{\partial(x^{j_2})}\left(\frac{d^2(x^{l}_{\alpha})}{dt^2}\right)} \left(\frac{d(x^{j_2})}{dt} \right) \right]$$

$$\frac{d^3 x^{l}}{dt^3} = \sum_{\alpha=1}^{(\text{New \# terms})} \sum_{j_2=1}^{M} \left[\left(\frac{\partial}{\partial(x^{j_2})} \left(\frac{d^2(x^{l}_{\alpha})}{dt^2} \right) \right) \left(\frac{d(x^{j_2})}{dt} \right) \right]$$

→ only 2 sums! w/ recursive computation!

```
(start recursiveMethod(n))
d^n/dt^n :  for α = 1:1:(num terms for n-th deriv)
            for j = 1:1:(num dims)
                recursiveMethod(n-1)
            end
        end.
```

Really, the whole algorithm can be boiled down to ~~recursive~~ iterative computation of a coefficient matrix, because the highest order of the polynomial can only double @ max, per deriv.

So, if $\frac{dx^\ell}{dt}$ is of order N and dimension M, then, <u>at most</u>, $\frac{d^2(x^\ell)}{dt^2}$ is of order $\underset{2^1}{2N}$ and dimension M.

$\frac{d^3(x^\ell)}{dt^3}$ of order $\underset{2^2}{4N}$ & dim M

$\frac{d^4(x^\ell)}{dt^4}$ of order $\underset{2^3}{8N}$ & dim M.

$\vdots$

$\frac{d^\gamma(x^\ell)}{dt^\gamma}$ of order $(2^{\gamma-1})N$ and dim M.

This of course implies $\left[(2^{\gamma-1})N+1\right]^M$ terms, <u>at most</u>, in the γ-th derivative.

<u>Computationally [redacted] impossible.</u>

Need an adaptive algorithm that minimizes the size of K based on coefficients a.

i.e. if $a_i^j = 0$, don't include that column of K, and remove that column a_i if $a_i^j = 0 \ \forall j$

description:

given a set of N-th order M-dimensional diffeq's, parse a from the set in compliance w/ the structure of (M,N)-k matrix.

↳ parser needs some more work, good for now.

if $a_i^j = 0 \; \forall j$, remove the i-th column of a & k.

↳ significantly ~~reduces~~ slows growth of a & k as $n \to \infty$.

find relation between $\hat{a}$, $\hat{k}$ and n.

$$\frac{dx^l}{dt} = \sum_{\alpha=1}^{(N+1)^M} a_\alpha^l \prod_{\beta=1}^{M} (x^\beta)^{k_\alpha^\beta}$$

$$\frac{d^2x^l}{dt^2} = \sum_{\alpha=1}^{(N+1)^M} a_\alpha^l \left[\frac{d}{dt}\left((x^1)^{k_\alpha^1} x^{2\,k_\alpha^2} \cdots x^{M\,k_\alpha^M}\right)\right]$$

$$= \sum_{\alpha=1}^{(N+1)^M} a_\alpha^l \left[(x^1)^{k_\alpha^1}\left(\frac{d}{dt}\left(x^{2\,k_\alpha^2} \cdots x^{M\,k_\alpha^M}\right)\right) + k_\alpha^1 (x^1)^{k_\alpha^1 - 1}\left(\frac{dx^1}{dt}\right) x^{2\,k_\alpha^2} \cdots x^{M\,k_\alpha^M}\right]$$

$$= \sum_{\alpha=1}^{(N+1)^M} a_\alpha^l \left[x_1^{k_\alpha^1} x_2^{k_\alpha^2}\left(\frac{d}{dt}\left(x^{3\,k_\alpha^3} \cdots x^{M\,k_\alpha^M}\right)\right) + x_1^{k_\alpha^1} k_\alpha^2 x^{2\,k_\alpha^2 - 1} \frac{dx^2}{dt}\left(x_3^{3\,k^3} \cdots x^{M\,k^M}\right) + \cdots \right.$$

$$\left. \cdots + k_\alpha^1 x^{1\,k_\alpha^1 - 1} x^{2\,k_\alpha^2} \cdots x^{M\,k_\alpha^M}\left(\frac{dx^1}{dt}\right)\right]$$

$$= \sum_{\alpha=1}^{(N+1)^M} a_\alpha^l \left[\sum_{j_1=1}^{M} \left(k_\alpha^{j_1} \prod_{\beta=1}^{M} (x^\beta)^{k_\alpha^\beta - \delta_{j_1}^\beta} \right)\left(\frac{dx^{j_1}}{dt}\right)\right]$$

for $\frac{d^2(x^l)}{dt^2} = \sum_{\alpha=1}^{(N+1)^M} \sum_{j_1=1}^{M} \sum_{i_1=1}^{(N+1)^M} \left(a^l_\alpha a^{j_1}_{i_1} K^{j_1}_\alpha \right) \prod_{\beta=1}^{M} (x^\beta)^{(K^\beta_\alpha + K^\beta_{i_1} - \delta^\beta_{j_1})}$

$\Rightarrow$ ~~$\forall \alpha, j_1$ compos:~~ we have

$\alpha=1 \rightarrow$
$j_1=1 \rightarrow$ $a^l_1 a^1_1 K^1_1 (x^\beta)^{K^\beta_1 + K^\beta_1 - \delta^\beta_1} + a^l_1 a^1_2 K^1_1 x^{\beta\, K^\beta_1 + K^\beta_2 - \delta^\beta_1} + \cdots$

$\cdots a^l_1 a^1_3 K^1_1 (x^\beta)^{K^\beta_1 + K^\beta_3 - \delta^\beta_1} + \cdots + a^l_1 a^1_{(N+1)^M} K^1_1 (x^\beta)^{K^\beta_1 + K^\beta_{(N+1)^M} - \delta^\beta_1} + \cdots$

$\alpha=1$
$j_1=2$: $\cdots + a^l_1 a^2_1 K^2_1 (x^\beta)^{K^\beta_1 + K^\beta_1 - \delta^\beta_2} + a^l_1 a^2_2 K^2_1 (x^\beta)^{K^\beta_1 + K^\beta_2 - \delta^\beta_2} + \cdots$

$\cdots + \cdots + a^l_1 a^2_{(N+1)^M} K^2_1 (x^\beta)^{K^\beta_1 + K^\beta_{(N+1)^M} - \delta^\beta_2} + \cdots$

The question is... when is

$$K^\beta_{\alpha_1} + K^\beta_{i_{1,1}} - \delta^\beta_{j_{1,1}} = K^\beta_{\alpha_2} + K^\beta_{i_{1,2}} - \delta^\beta_{j_{1,2}}, \quad \forall \beta$$

i.e., what are $\alpha_1, \alpha_2, i_{1,1}, j_{1,1}, i_{1,2}, j_{1,2}$?

and what are the values of $K^\beta_{\alpha_1} + K^\beta_{i_{11}} - \delta^\beta_{j_{11}}$?

$\Rightarrow$ When this happens, ~~you have~~ x^β's can be grouped together, resulting in sums of $a^l_\alpha a^{j_1}_{i_1} K^{j_1}_\alpha$'s, which is $\hat{a}$ for the $\hat{K}$ value of $K^\beta_{\alpha_1} + K^\beta_{i_1} - \delta^\beta_{j_1}$.

$\cancel{(M\times N)(N\times M)(M\times N) = M\times N}$

$a \in M\times(N+1)^M$, $K \in M\times(N+1)^M$

let $N = (N+1)^M$

aK^Ta

$$\cancel{\begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_{(N+1)^M}^1 \\ a_1^2 & a_2^2 & & a_{(N+1)^M}^2 \\ \vdots & \vdots & & \vdots \\ a_1^M & a_2^M & & a_{(N+1)^M}^M \end{bmatrix} \begin{bmatrix} K_1^1 & K_1^2 & \cdots & K_1^M \\ K_2^1 & K_2^2 & \cdots & K_M^2 \\ \vdots & \vdots & & \vdots \\ K_1^{(N+1)^M} & K_2^{(N+1)^M} & \cdots & K_M^{(N+1)^M} \end{bmatrix} \begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_{(N+1)^M}^1 \\ a_1^2 & a_2^2 & \cdots & a_{(N+1)^M}^2 \\ \vdots & \vdots & & \vdots \\ a_1^M & a_2^M & \cdots & a_{(N+1)^M}^M \end{bmatrix}}$$

$$\Rightarrow \cancel{\begin{bmatrix} (a_1^1 k_1^1 + a_2^1 k_1^2 + \cdots + a_N^1 k_1^N) \end{bmatrix}}$$

Note*: transposing K does not change how K_x^β is indexed. only moves location, index the same...

$$\cancel{\begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_N^1 \\ a_1^2 & a_2^2 & \cdots & a_N^2 \\ \vdots & \vdots & & \vdots \\ a_1^M & a_2^M & \cdots & a_N^M \end{bmatrix} \begin{bmatrix} K_1^1 & K_1^2 & \cdots & K_1^M \\ K_2^1 & K_2^2 & \cdots & K_2^M \\ \vdots & \vdots & & \vdots \\ K_N^1 & K_N^2 & \cdots & K_N^M \end{bmatrix} \begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_N^1 \\ a_1^2 & a_2^2 & & a_N^2 \\ \vdots & \vdots & & \vdots \\ a_1^M & a_2^M & \cdots & a_N^M \end{bmatrix}}$$

$$\Rightarrow = \cancel{\begin{bmatrix} (a_1^1 k_1^1 + a_2^1 k_2^1 + \cdots + a_N^1 k_N^1) \end{bmatrix}}$$

$i = (N+1)^{(M-1)}$ $\quad 2(N+1)^{(M-1)}$ $\quad N\cdot(N+1)^{M-1}$ $\quad (N+1)(N+1)^{M-1}$

$$K = \begin{bmatrix} N & N \cdots N & (N-1) & (N-1) \cdots (N-1) & 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 \\ N & \cdots\cdots 0 & N & \cdots\cdots 0 & \cdots \\ N \cdots 0 & & & & \end{bmatrix}$$

$$K_i^j = \left(K^j_{i+(N+1)^{(M-j)}} + 1\right).$$ ✓ true.

$$\cancel{K_i^j \neq K_{i+1}}$$

$$\cancel{K_1^1 = K^1_{1+(N+1)^{(M-1)}+1}}$$

$$\boxed{K_i^j = K^j_{i+\gamma(N+1)^{(M-j)}} + \gamma} \longrightarrow \text{true}$$

$i=1$
$j=1$ $\rightarrow$ $\cancel{K_1^1 = K^1_{1+2(N+1)^{(M-1)}}+2}$
$j=2$

$$\cancel{K_i^j = K^j_{i+(1+}}$$

How to combine $a \in M \times (N+1)^M$ and $k \in M \times (N+1)^M$ to get $\hat{a} \in M \in (2N+1)^M$?

Known size of $\hat{a}$ & $\hat{k}$

$(x^\beta)^{k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1}} \rightarrow$ for any β, $\exists j_1$ s.t. $\delta^\beta_{j_1} = 1$

~~and max$(k^\beta_\alpha) = $ max$(k^\beta_{i_1}) = N$.~~

Let k^β_α and $k^\beta_{i_1}$ be maximized @ N. $\rightarrow$ $\boxed{\text{thus, } \alpha = 1 \text{ \& } i_1 = 1}$

~~Then, for $\beta = 1$, $1 \leq \alpha, i_1 \leq (N+1)^{(M-1)}$.~~

~~and~~ we have $(x^\beta)^{N+N-\delta^\beta_{j_1}} = (x^\beta)^{2N-\delta^\beta_{j_1}}$.

for any β, $\exists$ a j_1 s.t. $\delta^\beta_{j_1} = 1$ since $\beta, j_1 \in \{1, \ldots, M\}$.

Thus, $\nexists$ a term in $\frac{d^2 x^\ell}{dt^2}$ w/ exponents

$(x^1)^{2N}(x^2)^{2N} \cdots (x^M)^{2N} \rightarrow$ Such a term would correspond to $\hat{a}^\beta_1$, but since this term cannot exist, $\hat{a}^\beta_1 = 0 \ \forall \beta$ by construction of the problem

$\rightarrow$ $\boxed{\hat{a}^\beta_1 = 0 \ \forall \beta}$

$\boxed{\alpha = 1, i_1 = 1}$

Let K^{β}_{α} and $K^{\beta}_{i_1}$ be ~~maximized~~ minimized @ 0. $\forall \beta$.

Then, $\alpha = i_1 = (N+1)^M$, and we have

$$\prod_{\beta=1}^{M} \Big[(X^{\beta}) \underbrace{K^{\beta}_{(N+1)^M} + K^{\beta}_{(N+1)^M} - \delta^{\beta}_{j_1}}_{\hat{K}.} \Big] \underbrace{\left(a^{l}_{(N+1)^M} a^{j_1}_{(N+1)^M} K^{j_1}_{(N+1)^M} \right)}_{\hat{a}_{(2N+1)^M}}$$

but $K^{\beta}_{(N+1)^M} = K^{\beta}_{(N+1)^M} = 0 \; \forall \beta$

$\hookrightarrow \prod \left[(X^{\beta}) - \delta^{\beta}_{j_1} \right] \hat{a}_{(2N+1)^M} \longrightarrow a_{(2N+1)^M} = 0$ by necessity $\forall$ rows, since deriv of constants vanish.

$$\frac{d^2(x^l)}{dt^2} = \sum_{\alpha=1}^{\overset{N}{\overset{\shortparallel}{(N+1)^M}}} \sum_{j_1=1}^{M} \sum_{i_1=1}^{\overset{N}{\overset{\shortparallel}{(N+1)^M}}} \left(a^l_\alpha a^{j_1}_{i_1} K^{j_1}_\alpha\right) \prod_{\beta=1}^{M} (x^\beta)^{(K^\beta_\alpha + K^\beta_{i_1} - \delta^\beta_{j_1})}$$

$\alpha=1$, $j_1=1$: $\Rightarrow a^l_1 K^1_1 a^1_1 (\Pi(x^\beta)^{K^\beta_1 + K^\beta_1 - \delta^\beta_1}) + a^l_1 K^1_1 a^1_2 (x^{\beta K^\beta_1 - \delta^\beta_1 + K^\beta_2}) + \cdots + a^l_1 K^1_1 a^1_N (x^{\beta K^\beta_1 - \delta^\beta_1 + K^\beta_N}) + \cdots$

$\alpha=1$, $j_1=2$: $\cdots + a^l_1 K^2_1 a^2_1 ((x^\beta)^{K^\beta_1 - \delta^\beta_2 + K^\beta_1}) + a^l_1 K^2_1 a^2_2 (x^{\beta K^\beta_1 - \delta^\beta_2 + K^\beta_2}) + \cdots + a^l_1 K^2_1 a^2_N (x^{\beta K^\beta_1 - \delta^\beta_2 + K^\beta_N}) + \cdots$

$\alpha=1$, $j_1=M$: $\cdots + \underbrace{\cdots}_{j_1=3 \text{ to } M-1} + a^l_1 K^M_1 a^M_1 (x^{\beta K^\beta_1 - \delta^\beta_M + K^\beta_1}) + a^l_1 K^M_1 a^M_2 (x^{\beta K^\beta_1 - \delta^\beta_M + K^\beta_2}) + \cdots + a^l_1 K^M_1 a^M_N (x^{\beta K^\beta_1 - \delta^\beta_M + K^\beta_N}) + \cdots$

$\alpha=2$, $j_1=1$: $\cdots + a^l_2 K^1_2 a^1_1 (x^{\beta K^\beta_2 - \delta^\beta_1 + K^\beta_1}) + a^l_2 K^1_2 a^1_2 (x^{\beta K^\beta_2 - \delta^\beta_1 + K^\beta_2}) + \cdots + a^l_2 K^1_2 a^1_N (x^{\beta K^\beta_2 - \delta^\beta_1 + K^\beta_N}) + \cdots$

$\alpha=2$, $j_1=2$: $\cdots + a^l_2 K^2_2 a^2_1 (x^{\beta K^\beta_2 - \delta^\beta_2 + K^\beta_1}) + a^l_2 K^2_2 a^2_2 (x^{\beta K^\beta_2 - \delta^\beta_2 + K^\beta_2}) + \cdots + a^l_2 K^2_2 a^2_N (x^{\beta K^\beta_2 - \delta^\beta_2 + K^\beta_N}) + \cdots$

$\alpha=2$, $j_1=M$: $\cdots + \underset{j_1=3 \text{ to } M-1}{\cdots} + a^l_2 K^M_2 a^M_1 (x^{\beta K^\beta_2 - \delta^\beta_M + K^\beta_1}) + a^l_2 K^M_2 a^M_2 (x^{\beta K^\beta_2 - \delta^\beta_M + K^\beta_2}) + \cdots + a^l_2 K^M_2 a^M_N (x^{\beta K^\beta_2 - \delta^\beta_M + K^\beta_N}) + \cdots$

$\alpha=N-1$, $j_1=1$: $\cdots + \underset{\alpha=3 \text{ to } N-1}{\cdots} + a^l_{N-1} K^1_{N-1} a^1_1 (x^{K^\beta_{N-1} + K^\beta_1 - \delta^\beta_1}) + a^l_{N-1} K^1_{N-1} a^1_2 (x^{K^\beta_{N-1} - \delta^\beta_1 + K^\beta_2}) + \cdots + a^l_{N-1} K^1_{N-1} a^1_N (x^{K^\beta_{N-1} - \delta^\beta_1 + K^\beta_N}) + \cdots$

$\alpha=N-1$, $j_1=2$: $\cdots + a^l_{N-1} K^2_{N-1} a^2_1 (x^{K^\beta_{N-1} + K^\beta_1 - \delta^\beta_2}) + a^l_{N-1} K^2_{N-1} a^2_2 (x^{K^\beta_{N-1} - \delta^\beta_2 + K^\beta_2}) + \cdots + a^l_{N-1} K^2_{N-1} a^2_N (x^{K^\beta_{N-1} - \delta^\beta_2 + K^\beta_N}) + \cdots$

$\alpha=N-1$, $j_1=M$: $\cdots + \underset{j_1=3 \text{ to } M-1}{\cdots} + a^l_{N-1} K^M_{N-1} a^M_1 (x^{K^\beta_{N-1} - \delta^\beta_M + K^\beta_1}) + a^l_{N-1} K^M_{N-1} a^M_2 (x^{K^\beta_{N-1} - \delta^\beta_M + K^\beta_2}) + \cdots + a^l_{N-1} K^M_{N-1} a^M_N (x^{K^\beta_{N-1} - \delta^\beta_M + K^\beta_N}) + \cdots$

$\alpha=N$, $j_1=1$: $+ (0)$.

$K^\beta_N = 0 \ \forall \beta \Rightarrow \hat{a}^\beta_{2N} = 0 \ \forall \beta.$

$\Rightarrow$

$$= a_1^{\ell} k_1^1 a_1^1 \left(x_1^{k_1^1+k_1^1-1} x_2^{k_1^2+k_1^2} \cdots x_M^{k_1^M+k_1^M}\right) + a_1^{\ell} k_1^1 a_2^1 \left(x_1^{k_1^1+k_2^1-1} x_2^{k_1^2+k_2^2} \cdots x_M^{k_1^M+k_2^M}\right) + \cdots$$

$\alpha=1$, $j_1=1$

$$\cdots + a_1^{\ell} k_1^1 a_N^1 \left(x_1^{k_1^1+k_N^1-1} x_2^{k_1^2+k_N^2} \cdots x_M^{k_1^M+k_N^M}\right) + \cdots$$

$\alpha=1$, $j_1=2$

$$\cdots + a_1^{\ell} k_1^2 a_1^2 \left(x_1^{k_1^1+k_1^1} x_2^{k_1^2+k_1^2-1} \cdots x_M^{k_1^M+k_1^M}\right) + a_1^{\ell} k_1^2 a_2^2 \left(x_1^{k_1^1+k_2^1} x_2^{k_1^2+k_2^2-1} \cdots x_M^{k_1^M+k_2^M}\right) + \cdots$$

$$\cdots + a_1^{\ell} k_1^2 a_N^2 \left(x_1^{k_1^1+k_N^1} x_2^{k_1^2+k_N^2-1} \cdots x_M^{k_1^M+k_N^M}\right) + \cdots + \cdots$$

$j_1 = 3$ to $M-1$

$\alpha=1$, $j_1=M$

$$\cdots + a_1^{\ell} k_1^M a_1^M \left(x^{k_1^1+k_1^1} x_2^{k_1^2+k_1^2} \cdots x_M^{k_1^M+k_1^M-1}\right) + a_1^{\ell} k_1^M a_2^M \left(x_1^{k_1^1+k_2^1} x_2^{k_1^2+k_2^2} \cdots x_M^{k_1^M+k_2^M-1}\right) + \cdots$$

$$\cdots + a_1^{\ell} k_1^M a_N^M \left(x^{k_1^1+k_N^1} x_2^{k_1^2+k_N^2} \cdots x_M^{k_1^M+k_N^M-1}\right) + \cdots$$

$\alpha=2$, $j_1=1$

$$\cdots + a_2^{\ell} k_2^1 a_1^1 \left(x^{k_2^1+k_1^1-1} x_2^{k_2^2+k_1^2} \cdots x_M^{k_2^M+k_1^M}\right) + a_2^{\ell} k_2^1 a_2^1 \left(x_1^{k_2^1+k_2^1-1} x_2^{k_2^2+k_2^2} \cdots x_M^{k_2^M+k_2^M}\right) + \cdots$$

$$\cdots + a_2^{\ell} k_2^1 a_N^1 \left(x_1^{k_2^1+k_N^1-1} x_2^{k_2^2+k_N^2} \cdots x_M^{k_2^M+k_N^M}\right) + \cdots$$

i.e (α, i_1, j_1) is alright

$$a^l_\alpha a^{j_1}_{i_1} k^{j_1}_\alpha + \sum a^l_e a^g_f k^g_e$$

exactly

$\hat{a}^l_\alpha =$ $\forall$ $e, f, g \neq (\alpha, i_1, j_1)$ s.t.

$$\prod_{\beta=1}^{M} (x_\beta)^{(k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1})} = \prod_{\beta=1}^{M} (x_\beta)^{(k^\beta_e + k^\beta_f - \delta^\beta_g)}$$

$$\Rightarrow \log\left[\prod_{\beta=1}^{M} (x_\beta)^{(k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1})}\right] = \log\left[\prod_{\beta=1}^{M} (x_\beta)^{(k^\beta_e + k^\beta_f - \delta^\beta_g)}\right]$$

$\Rightarrow =$ i.e. $k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1} = k^\beta_e + k^\beta_f - \delta^\beta_g \quad \forall \beta$

α, i_1, j_1 are known and $\Rightarrow \boxed{k^\beta_\alpha + k^\beta_i - \delta^\beta_{j_1} = V^\beta \in \{0, \dots, 2N\}}$

Case 1: let $e = \alpha$. $\Rightarrow \boxed{k^\beta_i - \delta^\beta_{j_1} = V^\beta - k^\beta_\alpha}$

then $k^\beta_\alpha - k^\beta_\alpha + k^\beta_{i_1} - \delta^\beta_{j_1} = (k^\beta_{i_1} - \delta^\beta_{j_1} = k^\beta_f - \delta^\beta_g) \quad \forall \beta.$

thus, $\boxed{k^\beta_f = k^\beta_{i_1} - \delta^\beta_{j_1} + \delta^\beta_g \quad \forall \beta}$.

$\boxed{j_1 \neq g} \quad \boxed{k^{j_1}_{i_1} \geq 1} \quad \boxed{k^g_{i_1} \leq N-1}$

$\Rightarrow 0 \leq k^\beta_f \leq N \Rightarrow 0 \leq k^\beta_{i_1} - \delta^\beta_{j_1} + \delta^\beta_g \leq N \; ; \; j_1 \neq g.$

$\Rightarrow$ thus since $j_1 \neq g$, $k^{j_1}_{i_1} \geq 1$ and $k^g_{i_1} \leq N-1$

otherwise $k^{j_1}_f < 0$ and $k^g_f > N$

$$K_\alpha = \begin{bmatrix} N & N & N & N & \cdots & N & N-1 & N-1 & N-1 & N-1 & \cdots & N-1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ N & N & N & \cdots & N & N-1 & N-1 & N-1 & N-1 & \cdots & 0 \end{bmatrix}$$

in 2d, 2 order

$$K_\alpha = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$K_i = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\hat{K} = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

$\hat{\alpha}=0$ highest, unique highest, unique. $\hat{\alpha}=0.$

when

1) pick $\hat{K}_\alpha$; $\rightarrow \hat{\alpha} = 2:1:(2N$

~~if $K_1^{M} = 2N$, then~~
~~and $K_2^{M} = 2N-1$, then~~

$K \in M \times (N+1)^M$; $K_i^j = K_{i+(N+1)^{(M-j)}}$

$K_i^1 = K^1_{i+(N+1)^{(M-1)}} - (N+i)$

$K_i^j = K^j_{i+(N+1)^{(M-j)}}$

$K_1^1 = K_2^1 = \cdots = K_{N+1}^1 \rightarrow K^1_{i(N+1)+1} = K^1_{i(N+1)+2} = \cdots K^1_{i(N+1)+(N+1)}$

$$\boxed{K^1_{u(N+1)+i} = K^1_{u(N+1)+j}, \quad \forall\, u \in \{1,\dots,M\},\ i,j \in \{1,\dots,N+1\}}$$

$= N-u$

K^2

$$K_\alpha = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$K_{i_1} \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$K_\alpha = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$K_{i_1} - \delta_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & a_6^1 & a_7^1 & a_8^1 & a_9^1 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & a_6^2 & a_7^2 & a_8^2 & a_9^2 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow 2 \text{ unique spots in } \hat{k}$$

$2\ \hat{\alpha}$'s

↳ lookup → store

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \text{corresp to unique } \hat{\alpha} \text{ in } \hat{k}_{\hat{\alpha}}$$

$$k_\alpha \oplus (k_{i_1} - \delta_{j_1})$$

$\alpha = (N+1)^M$

$i_1 = 1$

$j_1 = 1$

$$\hat{a}_{\hat{\alpha}} = \begin{bmatrix} (a^1_{(N+1)^M})(a^1_1)(k^1_{(N+1)^M}) \\ (a^2_{(N+1)^M})(a^1_1)(k^1_{(N+1)^M}) \end{bmatrix}$$

↳ do manually w/ $\dot{x} = y^2 - x$, $\dot{y} = x^2 - y$ reduced a & k!

4 3 4 2 3 4 1 2 3 4 1 2 3 1 2 1

1 1 2 1 2 3 1 2 3 4 2 3 4 3 4 4

$\hat{\alpha}$ 1 5 → 45 → 345 → 2345 → 12345

2 1 → 12 → 123 → 1234 → 12345

$(a^l_\alpha a^{j_1}_{i_1} k^{j_1}_\alpha)$

$\dot{x}=y^2-x,\ \dot{y}=x^2-y$ $\quad \alpha = 1,2,3,4$

$\alpha = 1,2,3,4$

$$a=\begin{bmatrix}0 & -1 & 1 & 0\\ 1 & 0 & 0 & -1\end{bmatrix},\quad k=\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 0 & 2 & 1\end{bmatrix}$$

$(k_\alpha)=\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 0 & 2 & 1\end{bmatrix} \oplus$: $\alpha=4$ $\rightarrow \begin{bmatrix}\alpha\\ i\end{bmatrix}=\begin{bmatrix}4\\ 1\end{bmatrix}$

$(k_{i_1}-\delta_{i_1})$ $\quad \begin{bmatrix}1 & 0 & -1 & -1\\ 0 & 0 & 2 & 1\end{bmatrix}$: $i_1=1, j_1=1$

$=\begin{bmatrix}1\\ 1\end{bmatrix}=\hat{k}_{\hat{\alpha}},\ \hat{a}_{\hat{\alpha}}=\begin{bmatrix}+(0\cdot 0\cdot 0)\\ +(-1\cdot 0\cdot 0)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 0 & 2 & 1\end{bmatrix} \oplus$: $\alpha=3,4$ $\rightarrow \begin{bmatrix}\alpha\\ i\end{bmatrix}=\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}$

$\begin{bmatrix}1 & 0 & -1 & -1\\ 0 & 0 & 2 & 1\end{bmatrix}$: $i=1,2$, $j_1=1$

$=\begin{bmatrix}1\\ 2\end{bmatrix},\begin{bmatrix}0\\ 1\end{bmatrix}=\hat{k}_{\hat{\alpha}},\ \hat{a}_{\hat{\alpha}}=\begin{bmatrix}+(1\cdot 0\cdot 0)\\ +(0\cdot 0\cdot 0)\end{bmatrix},\begin{bmatrix}+(0)\\ +(-1)(-1)\cdot 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},\begin{bmatrix}0\\ 0\end{bmatrix}$

$\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 0 & 2 & 1\end{bmatrix} \oplus \begin{bmatrix}\alpha\\ i\end{bmatrix}=\begin{bmatrix}1,2\\ 3,4\end{bmatrix}$

$\begin{bmatrix}1 & 0 & -1 & -1\\ 0 & 0 & 2 & 1\end{bmatrix}$

$=\begin{bmatrix}1\\ 2\end{bmatrix},\begin{bmatrix}0\\ 1\end{bmatrix}=\hat{k}_{\hat{\alpha}}$

$\hat{a}_{\hat{\alpha}}=\begin{bmatrix}0\\ 1\cdot 1\cdot 2\end{bmatrix},\begin{bmatrix}(0)\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 2\end{bmatrix},\begin{bmatrix}0\\ 0\end{bmatrix}$

$\hat{k}_{\hat{\alpha}}=\begin{bmatrix}1\\ 1\end{bmatrix} \rightarrow \begin{bmatrix}\alpha\\ i_1\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}$

$\hat{a}_{\hat{\alpha}}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 0 & 2 & 1\end{bmatrix} \oplus \begin{bmatrix}\alpha\\ i\end{bmatrix}=\begin{bmatrix}2,3,4\\ 1,2,3\end{bmatrix}$ $\quad j_1=1$

$\begin{bmatrix}1 & 0 & -1 & -1\\ 0 & 0 & 2 & 1\end{bmatrix}$

$=\begin{bmatrix}2\\ 0\end{bmatrix},\begin{bmatrix}0\\ 2\end{bmatrix},\cancel{\begin{bmatrix}-1\\ 3\end{bmatrix}}=\hat{k}_{\hat{\alpha}},\ \hat{a}_{\hat{\alpha}}=\begin{bmatrix}(-1)\cdot 0\\ 0\end{bmatrix},\begin{bmatrix}0\\ 0\end{bmatrix},\cancel{\begin{bmatrix}0\\ 0\end{bmatrix}}$

invalid!

$\hat{k}_{\hat{\alpha}}=\begin{bmatrix}1\\ 0\end{bmatrix},\ \hat{a}_{\hat{\alpha}}=\begin{bmatrix}1\\ 0\end{bmatrix}$

$\hat{k}_{\hat{\alpha}}=\begin{bmatrix}0\\ 2\end{bmatrix},\ \hat{a}_{\hat{\alpha}}=\begin{bmatrix}-1\\ 0\end{bmatrix}$

$\hat{k}_{\hat{\alpha}}=\begin{bmatrix}1\\ 2\end{bmatrix},\ \hat{a}_{\hat{\alpha}}=\begin{bmatrix}0\\ 2\end{bmatrix}$

$\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 0 & 2 & 1\end{bmatrix} \oplus \begin{bmatrix}\alpha\\ i\end{bmatrix}=\begin{bmatrix}1,2,3,4\\ 1,2,3,4\end{bmatrix}$

$\begin{bmatrix}1 & 0 & -1 & -1\\ 0 & 0 & 2 & 1\end{bmatrix}$ $\quad j_1=1$

$=\begin{bmatrix}3\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\end{bmatrix},\cancel{\begin{bmatrix}-1\\ 4\end{bmatrix},\begin{bmatrix}-1\\ 2\end{bmatrix}}=\hat{k}_{\hat{\alpha}},\ \hat{a}_{\hat{\alpha}}=\begin{bmatrix}0\\ 0\end{bmatrix},\begin{bmatrix}(-1)(-1)(1)\\ 0\end{bmatrix},\cancel{\begin{bmatrix}0\\ 0\end{bmatrix}}$

invalid invalid

$\hat{k}_{\hat{\alpha}}=\begin{bmatrix}2\\ 0\end{bmatrix},\ \hat{a}_{\hat{\alpha}}=\begin{bmatrix}0\\ -2\end{bmatrix}$

$\hat{a}_{\hat{\alpha}}=\begin{bmatrix}0\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\end{bmatrix}$

$=\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 0 & 2 & 1\end{bmatrix} \oplus \begin{bmatrix}\alpha\\ i\end{bmatrix}=\begin{bmatrix}1,2,3\\ 2,3,4\end{bmatrix}$

$\begin{bmatrix}1 & 0 & -1 & -1\\ 0 & 0 & 2 & 1\end{bmatrix}$ $\quad j_1=1$

$\hat{a}_{\alpha}=\begin{bmatrix}0\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 0\end{bmatrix}$

$=$

$=\begin{bmatrix}2\\ 0\end{bmatrix},\begin{bmatrix}0\\ 2\end{bmatrix},\cancel{\begin{bmatrix}-1\\ 3\end{bmatrix}}=\hat{k}_{\alpha}$ $\quad \begin{bmatrix}0\\ -2\end{bmatrix}$

invalid

$[1\ 1\ 1\ 1\ 1\ 1]$

$2\times 1 \quad 1\times N$

$$\hat{K}_x = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 2 & 1 \end{bmatrix}; \quad \hat{a} = \begin{bmatrix} 2 & 0 & 0 & 1 & -3 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$K_i = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}; \quad a = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

sliding K across $\hat{K}$: $\begin{bmatrix} x \\ i \end{bmatrix} = \left[\begin{array}{c|c|c|

$a^l_\alpha a^j_i k^j_\alpha$

$K_\alpha = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad a = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

$(K_i - \delta_2) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \end{bmatrix} \; ; j = 2$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \overset{=\hat{k}}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}, \begin{bmatrix} \alpha \\ i \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \to \hat{a} = \begin{bmatrix} 0 \\ -1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \to$

$\begin{bmatrix} 0 & 0 \\ 2 & 1 \\ 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{k} \to \hat{a} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} \alpha \\ i \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \hat{k} \to \hat{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} \alpha \\ i \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \overset{=\hat{k}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \to \hat{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} \alpha \\ i \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \hat{k} \to \hat{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} \alpha \\ i \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{k} \to \hat{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} \alpha \\ i \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \hat{k} \to \hat{a} = \cancel{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\begin{bmatrix} \alpha \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \to \hat{a} = \begin{bmatrix} 0 \\ 1 \cdot 0 \end{bmatrix}.$

this page

$\hat{k} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \hat{a} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\hat{k} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \hat{a} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\hat{k} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \hat{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\hat{k} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \hat{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

last page

$\hat{k} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\hat{k} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \hat{a} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\hat{k} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \hat{a} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\hat{k} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \hat{a} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

combining, into valid k-struct.

$\hat{k} = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 2 & 1 \end{bmatrix}$

$\hat{a} = \begin{bmatrix} 2 & 0 & 0 & 1 & -3 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{bmatrix}$

reading off.

$$\frac{d^2x}{dt^2} = 2x^2y + x - 3y^2$$

$$\frac{d^2y}{dt^2} = -3x^2 + 2y^2x + y$$

Correct!

$2\left(\sum_{i=1}^{M-1} i\right) + M(N-M+1)$ cols in ali

but $\sum_{i=1}^{M-1} i = \frac{(M-1)(M)}{2} \rightarrow 2\left(\frac{(M-1)(M)}{2}\right) + M(N-M+1)$

$\rightarrow = \cancel{M^2} - \cancel{M} + MN - \cancel{M^2} + \cancel{M} = MN \rightarrow$ #cols "ali" $= MN$

$\left(\sum_{i=1}^{M-1} i\right) + M(N-M+1) = \frac{(M-1)(M)}{2} + MN - M^2 + M$

$= \frac{M^2}{2} - \frac{M}{2} + MN - M^2 + M = MN + \frac{M}{2} - \frac{M^2}{2} = MN - \frac{(M)(M-1)}{2}$

$= \frac{2MN - M(M-1)}{2} = \frac{M(2N-M+1)}{2}$ index offset

$M = N = 4 \rightarrow = \frac{4(8-4+1)}{2} = 10$

$N=6, M=4 \rightarrow = \frac{4(12-4+1)}{2} = \frac{4(9)}{2} = \frac{36}{2} = 18$

$24 + 2 - \frac{16}{2} = 26 - 8 = 18$

$\cancel{M-1}$ entries + $N-M+1$ entries

$= N$ entries

algorithm for (2nd, or all?) deriv.

given $\hat{K}^{\beta}_{\hat{\alpha}} = K^{\beta}_{\alpha} + K^{\beta}_{i} - \delta^{\beta}_{j}$, w/ $\hat{a}_{\hat{\alpha}} = a^{l}_{\alpha} a^{j}_{i} K^{j}_{\alpha}$,

and the reduced! matrices K & a, compute $\hat{K}$ & $\hat{a}$

by:

$$(K_\alpha) = \begin{bmatrix} K^1_1 & K^1_2 & K^1_3 & \cdots & K^1_{(N+1)^m} \\ K^2_1 & K^2_2 & K^2_3 & \cdots & K^2_{(m+1)^m} \\ \vdots & \vdots & \vdots & & \vdots \\ K^M_1 & K^M_2 & K^M_3 & \cdots & K^M_{(N+1)^m} \end{bmatrix} \oplus$$

$$\begin{bmatrix} K^1_1 - \delta^1_j & K^1_2 - \delta^1_j & \cdots \\ K^2_1 - \delta^2_j & K^2_2 - \delta^2_j & \cdots \\ \vdots & \vdots & \end{bmatrix} = (K^i_i - \delta_j), \ \forall j$$

Sliding $(K^{\beta}_{i} - d^{\beta}_{j})$ across K^{β}_{α} $\forall j$, keeping track of $\begin{bmatrix} \alpha \\ i \end{bmatrix}$ during the slide. If the sum is valid $\forall \beta$ (no negative values)

then compute $\hat{a}_{\hat{\alpha}}$ using the indices.

$(K_\alpha + K_i - \delta_j)$ will repeat values, so sum $\hat{a}_{\hat{\alpha}}$'s up if it repeats. At the end, ~~sort $\hat{K}$ to be a valid exponent~~ any $\hat{K}_{\hat{\alpha}}$'s for which $\hat{a}_{\hat{\alpha}} = \vec{0}$ are removed. Algo finishes by sorting $\hat{K}$ into valid K-struct, while maintaining the pairs $(\hat{K}_{\hat{\alpha}}, \hat{a}_{\hat{\alpha}})$.

* Numerical issues when number of derivatives get large! This is due to sparse representation of large number w/ type double.

Constructing a taylor series:

$$X(t) = X(t_0) + \sum_{n=1}^{\infty} \left(\frac{d^n x}{dt^n}\bigg|_{t=t_0} \right) \frac{(t-t_0)^n}{n!} \; ;$$

~~[illegible]~~ w/ $\frac{d^n x}{dt^n}$ formula:

$$X(t) = x(t_0) + \sum_{n=1}^{\infty} \left[\sum_{\substack{\alpha \\ j \in J(n) \\ i \in I(n)}} \left[\left(\prod_{\beta=0}^{n-1} a_{i_\beta}^{j_\beta} \right) \left(\prod_{\beta=1}^{n-1} \left(\sum_{\gamma=0}^{\beta-1} \left(K_{i_\gamma}^{j_\gamma} - \delta_{j_\gamma}^{j_\beta} u[\gamma-1] \right) \right) \right)^{u[n-2]} \left(\prod_{\beta=1}^{M} (x_\beta)^{\sum_{\gamma=0}^{n-1} (K_{i_\gamma}^{\beta} - \delta_{j_\gamma}^{\beta} u[\gamma-1])} \right) \cdots \right] \Bigg|_{t=t_0} \right] \frac{(t-t_0)^n}{n!}$$

$\hookrightarrow x = x(t_0)$

Now, ~~[illegible]~~ Let ${}^n\hat{a}_{\hat{\alpha}}$ be the set of coefficients for the n-th derivative, ${}^n\hat{K}_{\hat{\alpha}}$ be the set of ~~coefficients~~ exponents for the n th deriv.

Let ${}^n\tilde{a}_{\tilde{\alpha}}$ ~~and~~ / ${}^n\tilde{K}_{\tilde{\alpha}}$ be the set of coefficients/exponents for the (n-1)th deriv.

Let a_α / K_α be the a / k corresponding to the first derivative. Then, the ~~taylor series~~ n-th deriv can be expressed as:

$$\cancel{x(t) = x_0 + \ldots}$$

$$\frac{d^n x^\ell}{dt^n} = \sum_{\hat{\alpha}=1}^{|\hat{a}^\ell|} \hat{a}^\ell_{\hat{\alpha}} \left(\prod_{\beta=1}^{M} (x_\beta)^{k^\beta_{\hat{\alpha}}} \right)$$

$$\frac{d^n x^\ell}{dt^n} = \sum_{\tilde{\alpha}=1}^{|\tilde{a}^\ell|} \sum_{j=1}^{M} \sum_{i=1}^{(N+1)^M} \left(\tilde{a}^\ell_{\tilde{\alpha}} \tilde{k}^j_{\tilde{\alpha}} a^j_i \right) \left(\prod_{\beta=1}^{M} (x_\beta)^{(\tilde{k}^\beta_{\tilde{\alpha}} + k^\beta_i - \delta^\beta_j)} \right)$$

Thus, the taylor series can be expressed as:

$$x(t) = x(t_0) + \sum_{n=1}^{\infty} \left[\sum_{\tilde{\alpha}=1}^{|\tilde{a}^\ell|} \sum_{j=1}^{M} \sum_{i=1}^{(N+1)^M} \left({}^{(n)}\tilde{a}^\ell_{\tilde{\alpha}} \, {}^{(n)}\tilde{k}^\ell_{\tilde{\alpha}} \, a^j_i \right) \left(\prod_{\beta=1}^{M} (x_\beta)^{({}^{(n)}\tilde{k}^\beta_{\tilde{\alpha}} + k^\beta_i - \delta^\beta_j)} \right) \right] \Bigg|_{t=t_0} \frac{(t-t_0)^n}{n!}$$

Notice, $\forall n \geq 1$, ${}^{(n)}\hat{a}^\ell_{\hat{\alpha}} = {}^{(n)}\tilde{a}^\ell_{\tilde{\alpha}} \, {}^{(n)}\tilde{k}^\ell_{\tilde{\alpha}} \, a^j_i$ & ${}^{(n)}\hat{a}^\ell_{\hat{\alpha}} = {}^{(n-1)}\hat{a}^\ell_{\hat{\alpha}_{n-1}} \, {}^{(n-1)}\hat{k}^\ell_{\hat{\alpha}_{n-1}} \, a^j_i$

or

${}^{(n)}\hat{k}^\beta_{\hat{\alpha}} = {}^{(n)}\tilde{k}^\beta_{\tilde{\alpha}} + k^\beta_i - \delta^\beta_j$. ${}^{(n)}\hat{k}^\beta_{\hat{\alpha}_n} = {}^{(n-1)}\hat{k}^\beta_{\hat{\alpha}_{n-1}} + k^\beta_i - \delta^\beta_j$

It is clear that, regardless of a_α and k_α, elements of ${}^{(n)}\hat{a}_{\hat{\alpha}}$ get <u>very</u> large as n gets large (by large, I mean $> 50-100$), which leads to numerical instability. Furthermore, $\frac{1}{n!}$, or specifically $n!$, becomes unimaginably large as n grows, possibly faster than ${}^{(n)}\hat{a}$. However, $\frac{{}^{(n)}\hat{a}}{n!}$ is a reasonable number $\forall n$, well within the range of non-sparse representation. Thus, $\longrightarrow$

Thus, ...

$$= \frac{{}^{(n-1)}\hat{a}\,{}^{(n-1)}\hat{k}a}{n!}$$

Notice: $$\frac{{}^{(n)}\hat{a}}{n!} = \frac{{}^{(n)}\hat{a}\,{}^{(n)}\hat{k}a}{n!} = \left(\left[\frac{{}^{(n)}\hat{a}}{(n-1)!}\right]\right)\left(\frac{{}^{(n)}\hat{k}a}{n}\right) = \cdots$$

$$\cdots = \left(\left[\frac{{}^{(n-1)}\hat{a}\,{}^{(n-1)}\hat{k}a}{(n-1)!}\right]\frac{{}^{(n)}\hat{k}a}{n}\right) = \cdots$$

$$\cdots = \left(\left[\left(\frac{{}^{(n-1)}\hat{a}}{(n-2)!}\right)\frac{{}^{(n-1)}\hat{k}a}{n-1}\right]\left(\frac{{}^{(n)}\hat{k}a}{n}\right)\right) = \cdots$$

$$= \left(\left[\left(\frac{{}^{(n-2)}\hat{a}\,{}^{(n-2)}\hat{k}a}{(n-2)!}\right)\frac{{}^{(n-1)}\hat{k}a}{n-1}\right]\left(\frac{{}^{(n)}\hat{k}a}{n}\right)\right) = \cdots$$

$$= \left(\left[\left(\left[\frac{{}^{(n-2)}\hat{a}}{(n-3)!}\right]\frac{{}^{(n-2)}\hat{k}a}{(n-2)}\right)\frac{{}^{(n-1)}\hat{k}a}{(n-1)}\right]\frac{{}^{(n)}\hat{k}a}{n}\right) = \cdots$$

in terms of $\hat{a}$ makes more sense

$$= \left(\left[\left(\left[\frac{{}^{(n-3)}\hat{a}}{(n-3)!}\right]\frac{{}^{(n-3)}\hat{k}a}{(n-2)}\right)\frac{{}^{(n-2)}\hat{k}a}{(n-1)}\right]\frac{{}^{(n-1)}\hat{k}a}{n}\right)$$

$$= \left[\frac{{}^{(1)}\hat{a}}{(1)!}\right]\left(\frac{{}^{(1)}\hat{k}a}{2}\right)\left(\frac{{}^{(2)}\hat{k}a}{3}\right)\left(\frac{{}^{(3)}\hat{k}a}{4}\right)\cdots\left(\frac{{}^{(n-2)}\hat{k}a}{n-1}\right)\left(\frac{{}^{(n-1)}\hat{k}a}{n}\right).$$

Thus, the Taylor series will still be accurate if one defines

$${}^{(n)}\hat{a} = \frac{{}^{(n-1)}\hat{a}\,{}^{(n-1)}\hat{k}a}{(n-1)!}$$ and computes it as:

$$X_\ell(t) = X_\ell(t_0) + \sum_{n=1}^{\infty}\left[\left(\sum_{\hat{\alpha}=1}^{|\hat{\alpha}^\ell|} {}^{(n)}\hat{a}^\ell_{\hat{\alpha}}\left(\prod_{\beta=1}^{M}(x_\beta)^{{}^{(n)}\hat{k}^\beta_{\hat{\alpha}}}\right)\right)\Bigg|_{x=x(t_0)}\frac{(t-t_0)^n}{n}\right],$$

where ${}^{(n)}\hat{a} = \dfrac{{}^{(n-1)}\hat{a}\ {}^{(n-1)}\hat{k}\ a}{(n-1)!}$.

~~By computing each~~

By including a division by n in the computation of ${}^{(n)}\hat{a}$ each iteration, the factorial will happen gradually. Furthermore, the division by n each iteration will help control the growth of the elements in ${}^{(n)}\hat{a}$. The overall effect of this algorithmic change will (hopefully) keep elements in ${}^{(n)}\hat{a}$ from growing so large that they fall outside the range of nonsparse double representation.

$$\frac{dx_\ell}{dt} = \sum_{\alpha=1}^{(N+1)^M} \left[a_\alpha^\ell \prod_{\beta=1}^{M} (x_\beta)^{k_\alpha^\beta} \right] ; \quad \text{polynomial case.}$$

regular & hyperbolic

Want to expand to ~~transcendentals~~ → trigs.

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} = \left(\frac{1}{2j}\right) e^{(j)x} - \left(\frac{1}{2j}\right) e^{(-j)x}$$

$$\hookrightarrow \sinh(x) = -j \sin(jx) = \frac{e^{x} - e^{-x}}{2}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\hookrightarrow \cosh(x) = \cos(jx) = \frac{e^{x} + e^{-x}}{2}$$

$$\tan x = \frac{e^{jx} - e^{-jx}}{j(e^{jx} + e^{-jx})}$$

$$\hookrightarrow \tanh(x) = -j \tan(jx) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

All hyperbolic transcendentals can be expressed as weighted linear combinations of "e".

Thus, it suffices to add a new term to the derivative:

$$b_\alpha^\ell \left(\frac{e^{Y_1} + (-1)^{r_\alpha^\ell} e^{-Y_1}}{e^{Y_2} + (-1)^{s_\alpha^\ell} e^{-Y_2}} \right)^{h_\alpha^\ell}$$

with $Y_1 = \sum_{\gamma=1} \left[c_\gamma^\ell \prod_{\beta=1}^{M} (x_\beta)^{p_\gamma^\beta} \right]$,

$$Y_2 = \sum_{\eta=1} \left[d_\eta^\ell \prod_{\beta=1}^{M} (x_\beta)^{q_\eta^\beta} \right],$$

$r_\alpha^\ell, s_\alpha^\ell \in \{0, 1\}$, for selection of trig function!

New term:

$$b_\alpha^l \left(\frac{e^{\left(\sum_{\gamma=1}^{(\hat{N}+1)^{\hat{M}}} c_\gamma^l \prod_{\beta=1}^{\hat{M}} (x_\beta)^{p_\gamma^\beta}\right)} + (-1)^{r_\alpha^l} e^{\left(\sum_{\gamma=1}^{(\hat{N}+1)^{\hat{M}}} + d_\gamma^l \prod_{\beta=1}^{\hat{M}} (x_\beta)^{p_\gamma^\beta}\right)}}{e^{\left(\sum_{n=1}^{(\hat{N}+1)^{\hat{M}}} f_n^l \prod_{\beta=1}^{\hat{M}} (x_\beta)^{q_n^\beta}\right)} + (-1)^{s_\alpha^l} e^{\left(\sum_{n=1}^{(\hat{N}+1)^{\hat{M}}} g_n^l \prod_{\beta=1}^{\hat{M}} (x_\beta)^{q_n^\beta}\right)}} \right)^{h_\alpha^l}$$

n

eta.

~~To select sin, let $b_\alpha^l = \frac{\text{real}(b_\alpha^l)}{j}$,~~

~~$g_n^l = f_n^l = 0 \ \forall n$,~~

~~$c_\gamma^l = d_\gamma^l \ \forall \gamma$, and $r_\alpha^l = 1$.~~

~~to make hyperbolic~~

~~$c_\gamma^l =$~~

cardinality of u^l

Let $Y(x, u, v) = \sum_{\gamma=1}^{|u^l|} u_\gamma^l \prod_{\beta=1}^{|v_\gamma|} (x_\beta)^{v_\gamma^\beta}$ then $\longrightarrow$

$$Y(x,u,v) = \sum_{\gamma=1}^{|u^\ell|} u_\gamma^\ell \prod_{\beta=1}^{|v_\gamma|} (x_\beta)^{v_\gamma^\beta} \equiv Y(u,v)$$

new term is: $$b_\alpha^\ell \left(\frac{e^{Y(c,p)} + (-1)^{r_\alpha^\ell} e^{Y(d,p)}}{e^{Y(f,q)} + (-1)^{s_\alpha^\ell} e^{Y(g,q)}} \right)$$

To add no transcendental to α-th term, select $b_\alpha^\ell = 0$.

To add sin; Select $\text{real}\{b_\alpha^\ell\} = 0$, $\underbrace{\text{imag}\{b_\alpha^\ell\} \neq 0}_{\text{since } -\gamma_j = \frac{\gamma}{j}.}$

Select $f_\gamma^\ell = g_\gamma^\ell = 0 \ \forall \gamma$. → γ is coefficient of sin term. because the two comes from there

↳ gives denom of $(1) + (-1)^{s_\alpha^\ell}(1)$

Select $s_\alpha^\ell = 0$ → gives denom of $(1) + (-1)^0(1) = 2$.

Select $r_\alpha^\ell = 1$ → gives numerator of $\gamma\left(e^{Y(c,p)} - e^{Y(d,p)}\right)$

Select [redacted] gives [redacted] → $\gamma \sin(Y(c,p))$

↳ $\text{real}\{c_\gamma^\ell\} = \text{real}\{d_\gamma^\ell\} = 0 \ \forall \gamma$,

$\text{imag}\{c_\gamma^\ell\} = -\text{imag}\{d_\gamma^\ell\} \ \forall \gamma$;

Also, $\exists \gamma$ s.t. $\text{imag}\{c_\gamma^\ell\} \neq 0$.

To add sinh: select $\text{imag}\{b_\alpha^l\}=0$, $\text{real}\{b_\alpha^l\}\neq 0$.

$$\left\{\begin{array}{l} \text{Select } f_\gamma^l = g_\gamma^l = 0 \ \forall \gamma, \ s_\alpha^l = 0. \\ r_\alpha^l = 1, \ \text{imag}\{c_\gamma^l\} = \text{imag}\{d_\gamma^l\} = 0 \ \forall \gamma. \\ \text{real}\{c_\gamma^l\} = -\text{real}\{d_\gamma^l\}, \ \forall \gamma; \exists \gamma \text{ s.t. } c_\gamma^l \neq 0. \end{array}\right.$$

$\Rightarrow$ gives $\sinh(y(c,p))$.

To add cos or cosh:

Select $f_\gamma^l = g_\gamma^l = 0 \ \forall \gamma$, $s_\alpha^l = 0$, $\text{imag}\{b_\alpha^l\}=0$, $r_\alpha^l = 0$. $\text{real}\{b_\alpha^l\}\neq 0$.

for cos:	for cosh:
$\text{real}\{c_\gamma^l\} = \text{real}\{d_\gamma^l\} = 0$ $\forall \gamma$	$\text{imag}\{c_\gamma^l\} = \text{imag}\{d_\gamma^l\} = 0$ $\forall \gamma$
$\text{imag}\{c_\gamma^l\} = -\text{imag}\{d_\gamma^l\}$	$\text{real}\{c_\gamma^l\} = -\text{real}\{d_\gamma^l\}$

$\exists \gamma$ s.t. $c_\gamma^l \neq 0$.

gives $b_\alpha^l \cos(y(c,p))$; gives $b_\alpha^l \cosh(y(c,p))$

* to add sech or sec:

Select: $c_\gamma^\ell = d_\gamma^\ell = 0 \ \forall \gamma$, $r_\alpha^\ell = 0$, $s_\alpha^\ell = 0$

$\text{imag}\{b_\alpha^\ell\} = 0$, $\text{real}\{b_\alpha^\ell\} \neq 0$.

for sec:	for sech:
$\text{real}\{f_\gamma^\ell\} = \text{real}\{g_\gamma^\ell\} = 0 \ \forall \gamma$	$\text{imag}\{f_\gamma^\ell\} = \text{imag}\{g_\gamma^\ell\} = 0 \ \forall \gamma$
$\text{imag}\{f_\gamma^\ell\} = -\text{imag}\{g_\gamma^\ell\} \ \forall \gamma$	$\text{real}\{f_\gamma^\ell\} = -\text{real}\{g_\gamma^\ell\} \ \forall \gamma$

$\exists \gamma$ s.t. $f_\gamma^\ell \neq 0$.

gives $b_\alpha^\ell \sec(h)(y(f,g))$.

* to add csc or csch:

Select: $c_\gamma^\ell = d_\gamma^\ell = 0 \ \forall \gamma$, $r_\alpha^\ell = 0$; $s_\alpha^\ell = 1$.

for csc:	for csch:
$\text{imag}\{b_\alpha^\ell\} \neq 0$.	$\text{imag}\{b_\alpha^\ell\} = 0$.
$\text{real}\{b_\alpha^\ell\} = 0$.	$\text{real}\{b_\alpha^\ell\} \neq 0$.
$\text{real}\{f_\gamma^\ell\} = \text{real}\{g_\gamma^\ell\} = 0 \ \forall \gamma$	$\text{real}\{f_\gamma^\ell\} = \text{real}\{g_\gamma^\ell\} \ \forall \gamma$
$\text{imag}\{f_\gamma^\ell\} = -\text{imag}\{g_\gamma^\ell\} \ \forall \gamma$	$\text{imag}\{f_\gamma^\ell\} = \text{imag}\{g_\gamma^\ell\} = 0. \ \forall \gamma$

for tan, tanh; cot, coth:

$c^l_\gamma = f^l_\gamma \ \forall \gamma$

$c^l_\gamma = -d^l_\gamma$ and $f^l_\gamma = -g^l_\gamma \ \forall \gamma$.

$p^\beta_\gamma = q^\beta_\gamma \ \forall \gamma, \beta$ i.e. $p = q$.

for tan, tanh:	for cot, coth:
$r^l_\alpha = 1, \ s^l_\alpha = 0$	$r^l_\alpha = 0, \ s^l_\alpha = 1$

for tan, cot:	for tanh, coth:
$\text{real}\{b^l_\alpha\} = 0$	$\text{real}\{b^l_\alpha\} \neq 0$
$\text{imag}\{b^l_\alpha\} \neq 0$	$\text{imag}\{b^l_\alpha\} = 0$
$\text{real}\{c^l_\gamma\} = 0 \ \forall \gamma$.	$\text{real}\{c^l_\gamma\} \neq 0 \ \forall \gamma$
$\text{imag}\{c^l_\gamma\} \neq 0 \ \forall \gamma$	$\text{imag}\{c^l_\gamma\} = 0 \ \forall \gamma$.

All typical forward regular & hyperbolic trig functions can be expressed w/ this new term!

Nth order polynomial & transcendental, M-dim general equation:

$$\frac{dx_\ell}{dt} = \sum_{\alpha=1}^{(N+1)^M} \left[a_\alpha^\ell + b_\alpha^\ell \left(\frac{e^{\left(\sum_{\gamma=1}^{(F+1)^M} c_\gamma^\ell \prod_{\beta=1}^{M} (x_\beta)^{p_\gamma^\beta}\right)} + (-1)^{r_\alpha^\ell} e^{\left(\sum_{\gamma=1}^{(F+1)^M} d_\gamma^\ell \prod_{\beta=1}^{M} (x_\beta)^{p_\gamma^\beta}\right)}}{e^{\left(\sum_{\gamma=1}^{(W+1)^M} f_\gamma^\ell \prod_{\beta=1}^{M} (x_\beta)^{q_\gamma^\beta}\right)} + (-1)^{s_\alpha^\ell} e^{\left(\sum_{\gamma=1}^{(W+1)^M} g_\gamma^\ell \prod_{\beta=1}^{M} (x_\beta)^{q_\gamma^\beta}\right)}} \right)^{h_\alpha^\ell} \left(\prod_{\beta=1}^{M} (x_\beta)^{k_\alpha^\beta} \right) \right]$$

$$\frac{dx_\ell^*}{dt} = \sum_{\alpha=1}^{(N+1)^M} a_\alpha^\ell \prod_{\beta=1}^{M} x_\beta^{k_\alpha^\beta} \rightarrow \frac{d^n x_\ell^*}{dt^n} \text{ is known \& computable.}$$

want $\frac{d^n x_\ell}{dt^n}$ for:

$$\frac{dx_\ell}{dt} = \sum_{\alpha=1}^{(N+1)^M} b_\alpha^\ell \left(\frac{e^{\left(\sum_{\gamma=1}^{(F+1)^M} c_\gamma^\ell \prod_{\beta=1}^{M} x_\beta^{p_\gamma^\beta}\right)} + (-1)^{r_\alpha^\ell} e^{\left(\sum_{\gamma=1}^{(F+1)^M} d_\gamma^\ell \prod_{\beta=1}^{M} x_\beta^{p_\gamma^\beta}\right)}}{e^{\left(\sum_{\gamma=1}^{(W+1)^M} f_\gamma^\ell \prod_{\beta=1}^{M} x_\beta^{q_\gamma^\beta}\right)} + (-1)^{s_\alpha^\ell} e^{\left(\sum_{\gamma=1}^{(W+1)^M} g_\gamma^\ell \prod_{\beta=1}^{M} (x_\beta)^{q_\gamma^\beta}\right)}} \right)^{h_\alpha^\ell} \left(\prod_{\beta=1}^{M} x_\beta^{k_\alpha^\beta} \right)$$

If $Y(x,u,V) \equiv Y(u,v) = \sum_{\gamma=1}^{|u^\ell|^E} u_\gamma^\ell \prod_{\beta=1}^{M} (x_\beta)^{V_\gamma^\beta}$, then

$$\frac{dx_\ell}{dt} = \sum_{\alpha=1}^{(N+1)^M} b_\alpha^\ell \left(\frac{e^{Y(c,p)} + (-1)^{r_\alpha^\ell} e^{Y(d,p)}}{e^{Y(f,q)} + (-1)^{s_\alpha^\ell} e^{Y(g,q)}} \right)^{h_\alpha^\ell} \prod_{\beta=1}^{M} x_\beta^{k_\alpha^\beta}$$

*note: $Y(u,v) = \left.\frac{dx_\ell^*}{dt}\right|_{\substack{a=u \\ k=V}} \rightarrow \frac{d^{(n-1)} Y(u,v)}{dt^{(n-1)}} = \left.\frac{d^n x_\ell^*}{dt^n}\right|_{\substack{a=u \\ k=V}}$