ELL780 ELL 457 Indian Institute of Technology Delhi
Lecture 14 Continuous Function
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2 Introduction

In this chapter, we want to look at functions on metric spaces. In particular, we want to see how mapping metric spaces to metric spaces relates to properties of subsets of the metric spaces. Furthermore, we develop better understanding of operations and properties of such mappings and look closely at homeomorphism, uniform continuity and kinds of discontinuities in them.

3 Continuous Functions on Metric Spaces

Recall that a function $f : R \to R$ is said to be continuous at a point $a \in R$ if points close to a are mapped close to $f(a)$. Formally, the condition is :

$$(\forall \epsilon > 0)(\exists \delta > 0)|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Since this is a metric condition, in other words it is defined in terms of the (usual) metric $d(x, y) = |x - y|$ on R, we can generalize it directly to arbitrary metric spaces.

Definition 3.1 *Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y$ a function, and $a \in X$. We say that f is continuous at a (with respect to the metrics d and ρ) if*

$$(\forall \epsilon > 0)(\exists \delta > 0)d(x, a) < \delta \implies \rho(f(x), f(a)) < \epsilon$$

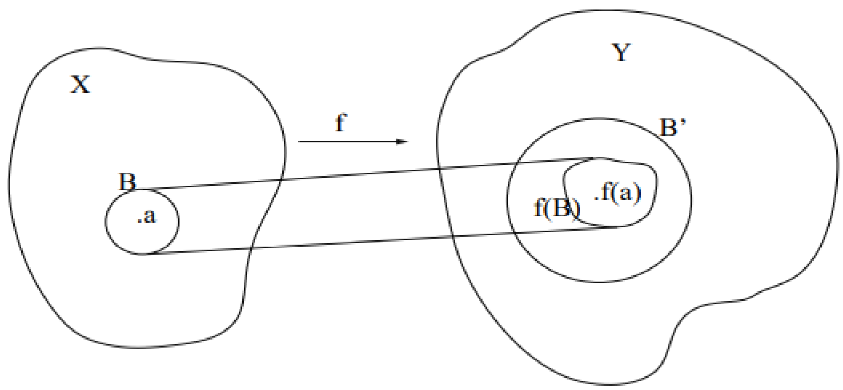
Equivalently:

$$(\forall \epsilon > 0)(\exists \delta > 0)f(B_\delta(a)) \subseteq B_\epsilon(f(a))$$

Theorem 3.1 *Suppose that (X, d) and (Y, ρ) are metric spaces, $A \subset X$, $f : A \to Y$ and $p \in A$ and p is a limit point of A. Then f is continuous at p if and only if*

$$\lim_{x \to p} f(x) = f(p)$$

Definition 3.2 *Suppose that (X, d) and (Y, ρ) are metric spaces, $A \subset X$ and $f : A \to Y$. Then f is continuous on A if and only if f is continuous at $p \in A$.*



Remark 3.3 *It follows, from limit theorems concerning the algebraic manipulations of functions for which the limits exist, the all real-valued polynomials in k real variables are continuous in R^k*

Definition 3.4 *Suppose that (X, d) and (Y, ρ) are metric spaces, $A \subset X$ and $f : A \to Y$. Let f be a bijective mapping, then f^{-1} exists:*
a)$f^{-1}(G)$ is open in X, $\forall$ open sets $G \subseteq Y$
b)A is an open set, $A \subseteq X \nRightarrow f(A)$ is open

Example 3.2 *Consider the function $f : R \to R$,*

$$f(x) = c, \forall x \in R$$

Then range $(f) = \{c\}$, which implies that $f : R(open) \to \{c\}(closed)$.

Example 3.3 *Consider the function $f : R \to R$,*

$$f(x) = tan^{-1}(x), \forall x \in R$$

Then, range $(f) = (-\pi/2, \pi/2)$, which implies that $f^{-1} : (-\pi/2, \pi/2)(open) \to R(open)$.

Theorem 3.4 *Let (X, d) and (Y, ρ) be two metric spaces and $f : X \to Y$ be a function. Then, f is continuous if and only if*

$$\{x_n\} \to x_0 \implies \{f(x_n)\} \to f(x_0)$$

In other words, f is continuous if and only if, for all convergent sequences $\{x_n\}$ in X which converge to x_0, the resulting sequence $\{f(x_n)\}$ converges to $f(x_0)$.

Proof: First suppose that f is continuous at x, and let $\{x_n\}$ be a sequence converging to x in (X, d). Continuity of f means

$$(\forall \epsilon > 0)(\exists \delta > 0) d(x_0, x) < \delta \implies \rho(f(x_0), f(x)) < \epsilon$$

Convergence of the sequence means

$$(\forall \epsilon > 0)(\exists\ N \in n)(\forall n \geq N) d(x_n, x) < \delta$$

. Combining above 2 equations (with the same δ, and putting $x_0 = x_n$) gives:

$$(\forall \epsilon > 0)(\exists\ N \in n)(\forall n \geq N) \rho(f(x_n), f(x)) < \epsilon$$

. In other words, the sequence $\{f(x_n)\}$ converges in (Y, ρ) to $f(x)$, as required. Conversely, suppose that f is discontinuous at x. Then there is a positive real number ϵ such that, for all $\delta > 0$, there is an $x_0 \in X$ with $d(x_0, x) < \delta$ but $\rho(f(x_0), f(x)) > \epsilon$. For each $n \in N$ put $\delta = \frac{1}{n}$ in the above, and let x_n be a suitable choice of x_0 . In other words, we have a sequence $\{x_n\}$ in X with $d(x_n, x) < \frac{1}{n}$ for all n, but $\rho(f(x_n), f(x)) > \epsilon$ for all n. It follows that $\{x_n\}$ converges to x in (X, d), but that $\{f(x_n)\}$ does not converge to $f(x)$. [Note: it is possible that the sequence $\{f(x_n)\}$ does converge in (Y, ρ), but if so its limit is not $f(x)$.] ■

Example 3.5 *Prove that the function $f : RxR \to R$, given by*

$$f(x) = \begin{cases} \frac{xy}{x^3+y^3} & (x, y) \neq (0, 0) \\ 0 & x = y = 0 \end{cases}$$

is not continuous at $(0, 0)$. Let $p_n = (\frac{1}{n}, \frac{1}{n})$. Then $\{p_n\}_{(n=1)}^{\infty}$ converges to $(0, 0)$. But,

$$\lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} \frac{\frac{1}{n}.\frac{1}{n}}{(\frac{1}{n})^3 + (\frac{1}{n}^3)} = \lim_{n\to\infty} \frac{n}{2} = \infty \neq 0$$

Hence, by Sequence Characterization, we conclude that f is not continuous at $(0, 0)$.

4 Continuity of Operations on continuous functions

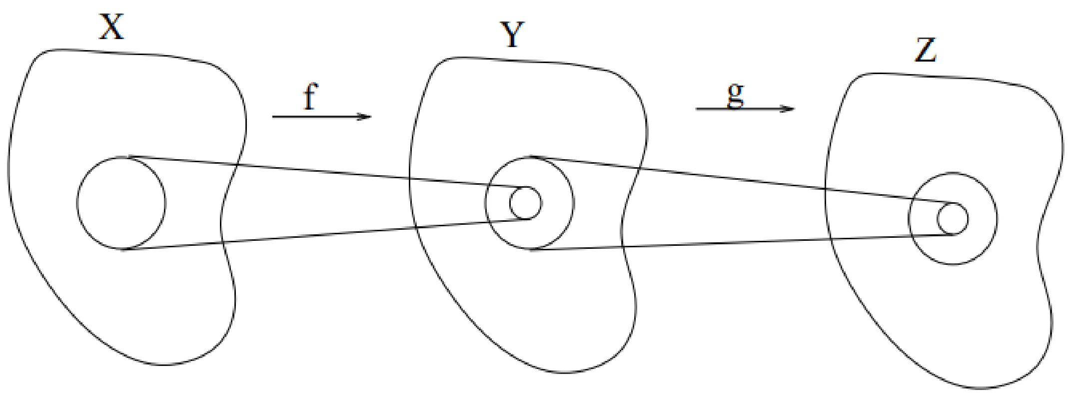
Theorem 4.1 *(Composite of continuous maps is continuous) Suppose that (X, d), (Y, ρ) and (Z, σ) are metric spaces, $f : X \to Y$ is continuous (with respect to the metrics d, ρ) at a point $x \in X$, $g : Y \to Z$ is continuous (with respect to the metrics ρ, σ) at $f(x) \in Y$, and $h = g(f) : X \to Z$ is their composite. Then h is continuous at x with respect to the metrics d, σ.*

Proof: Suppose that $\epsilon > 0$. Since g is continuous at $f(x)$, we can find $\delta > 0$ such that $g(B_\delta(f(x))) \subset B_\epsilon(g(f(x)))$. Now, since f is continuous at x, we can find $\gamma > 0$ such that $f(B_\gamma(x)) \subset B_\delta(f(x))$. Hence $h(B_\gamma(x)) = g(f(B_\gamma(x))) \subset g(B_\delta(f(x)) \subset B_\epsilon(g(f(x))) = B_\epsilon(h(x))$. By definition, h is continuous at x. ■

Example 4.2 *If $h : R^2 \to R^2$ is defined by*

$$h(x, y) = (\cos(x + y), \sin(x + y))$$

then h is the composite of two continuous functions $f : R^2 \to R(f(x, y) = x + y)$ and $g : R \to R^2(g(\theta) = (cos\theta, sin\theta))$, with respect to the usual metrics on R and R^2 . Hence h is continuous with respect to the usual metric on R^2 .



Remark 4.1 *Let f, g be two continuous functions defined $f : X \to R$ and $g : X \to R$, then*
a)$f \pm g$ is continuous on X
b)$f.g$ is continuous on X
c)$\frac{f}{g}$ is continuous on X (assume $g(x) \neq 0\ \forall x \in X$)

5 Cauchy Sequences and their Mapping

Definition 5.1 *A sequence $\{x_n\}$ is said to be Cauchy sequence, if given $\epsilon > 0$ there exists N such that if $m, n > N$ then $|x_m - x_n| < \epsilon$.*

Definition 5.2 *A relation $\sim$ on the set of Cauchy sequences in (X, d) is defined by*

$$\{x_n\} \sim \{y_n\} \implies d(x_n, y_n) \to 0, in R$$

.

Proposition 5.1 *Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y$ a function and suppose $\{x_n\}$ is a sequence, then*
a) $\{x_n\}$ is Cauchy $\not\Longrightarrow$ $\{f(x_n)\}$is Cauchy
b) $\{x_n\}$ is convergent $\implies$ $\{f(x_n)\}$is Convergent.

6 Compact Sets in Metric Spaces

Definition 6.1 *A set A in a metric space (X, d) is compact if every sequence in A has a subsequence that converges (in (X, d)) to a limit that belongs to A.*

Theorem 6.1 *Let (X, d) and (Y, ρ) be metric spaces, let A be a compact set with $A \subseteq X$, and $f : X \to Y$ a map that is continuous with respect to d and ρ. Then $f(A)$ is a compact set in (Y, ρ).*

Proof: Let $\{y_n\}$ be a sequence in $f(A)$. For each $n \in N$, choose $x_n \in A$ such that $f(x_n) = y_n$. Then $\{x_n\}$ is a sequence in A. Since A is compact, we can choose a subsequence $\{x_{n_m}\}$ that converges in (X, d) to a limit $x \in A$. By continuity, the subsequence $\{y_{n_m}\} = \{f(\{x_{n_m}\})\}$ of $\{y_n\}$ converges in (Y, ρ) to $f(x) \in f(A)$. Hence each sequence in $f(A)$ has a subsequence that converges in (Y, ρ) to a limit in $f(A)$. In other words, $f(A)$ is compact. ■

Definition 6.2 *For a set X, a function $f : X \to R^k$ is said to be bounded if and only if*

$$(\exists M)(M \in R^{k(}\forall x))(x \in X \implies ||f(x)|| \leq M))$$

Theorem 6.2 *(Boundedness Theorem) Let A be a compact subset of a metric space (X, d) and suppose that $f : A \to R^k$ is continuous. Then $f(A)$ is closed and bounded.*

Remark 6.3 *In particular, $f(A)$ is bounded as claimed in the Boundedness Theorem.*

Proof: By Heine-Borel Theorem, we know that compactness of any set A in R_k is equivalent to A being closed and bounded. Hence, from Theorem 6.1, if A is a compact metric space, and function $f : A \to R^k$, then $f(A)$ is compact. But we know that $f(A) \subset R^k$, hence $f(A)$ being compact yields that it is closed and bounded. ■

Theorem 6.3 *(Extreme value theorem) Suppose that f is a continuous function from a compact subset A of a metric space X into R_1, with*

$$M = sup_{p \in A} f(p)$$

$$m = inf_{p \in A} f(p)$$

Then there exist points u and v in A such that $f(u) = M$ and $f(v) = m$.

Proof: From Heine Borel Theorem Theorem 6.1, $f(A) \subset R$ and continuity of f implies that $f(A)$ is closed and bounded. The Least Upper and Greatest Lower Bound Properties for the reals yields the existence of finite real numbers M and m such that $M = sup_{p \in A} f(p)$ and $m = inf_{p \in A} f(p)$. Since $f(A)$ is closed, we have $M \in f(A)$ and $m \in f(A)$. Hence, there exists u and v in A such that $f(u) = M$ and $f(v) = m$; i.e., $f(u) = sup_{p \in A} f(p)$ and $f(v) = inf_{p \in A} f(p)$. ■

7 Homeomorphisms and equivalent metrics

We are already familiar with one-to-one mapping, or invertible functions and it is known that unique inverse function exists for such mapping. These functions also known as bijective mapping satisfy that $f^{-1}(y)$ contains precisely one element, for each $y \in Y$.

Theorem 7.1 *Suppose that f is a continuous one-to-one mapping of a compact metric space X onto a metric space Y . Then the inverse mapping f^{-1} which is defined by $f^{-1}(f(x)) = x$ for all $x \in X$ is a continuous mapping that is a one-to-one correspondence from Y to X.*

Proof: Suppose that f is a continuous one-to-one mapping of a compact metric space X onto a metric space Y . Because f is one-to-one, the inverse f^{-1} is a function from range $(f) = Y in X$. From the Open Set Characterization of Continuous Functions, we know that f^{-1} is continuous in Y if $f(U)$ is open in Y for every U that is open in X. Suppose that $U \subset X$ is open. Then, we know that, U_c is compact as a closed subset of the compact metric space X. From Theorem 6.1 , $f(U_c)$ is compact. Since every compact subset of a metric space is closed, we conclude that $f(U_c)$ is closed. Because f is one-to-one, $f(U_c) = f(X) - f(U)$, then f onto yields that $f(U_c) = Y - f(U) = f(U_c)$. Therefore, $f(U_c)$ is closed which is equivalent to $f(U)$ being open. Since U was arbitrary, for every U open in X, we have that $f(U)$ is open in Y . Hence, f^{-1} is continuous in Y. ■

Definition 7.1 *Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ is called a homeomorphism if it is bijective and both $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous (with respect to $d and \rho$). In this case the spaces (X, d) and (Y, ρ) R are said to be homeomorphic.*

Remark 7.2 *Such a function $f : X \to Y$ defined on homeomorphic spaces (X, d) and (Y, ρ) is known as Topological Mapping. For such functions,*

f is continuous $\implies$ f^{-1} is continuous.
In general, this is not true.

Definition 7.3 *Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ is called an isometry if it is bijective and satisfies://*

$$\rho(f(x), f(x')) = d(x, x') \forall x, x' \in X \tag{1}$$

Two metric spaces are said to be isometric if there exists an isometry between them.

Remark 7.4 *It is not difficult to show that the notion of isometry is an equivalence relation on metric spaces. (The identity function $X \to Y$ is an isometry; the inverse of an isometry is an isometry; the composite of two isometries is an isometry).*

Exercise 7.2 *Show that every isometry $X \to Y$ is continuous.*

Note that homeomorphism is a weaker equivalence relation than isometry, defined on metric spaces using continuous maps. Homeomorphic spaces are not in general isometric. Nevertheless, many properties like convergence of sequences, open sets, and closed sets are similar for both of them.

8 Equivalent Metrics

If X is a fixed set with two metrics d_1 and d_2, one can consider the identity map $X \to X$ as a map between the two metric spaces (X, d_1) and (X, d_2). If this is a homeomorphism from (X, d_1) to (X, d_2), then we say that the metrics d_1 and d_2 on X are equivalent.

Example 8.1 *1. The max-metric, the Manhattan metric, and the euclidean metric on $_n$ are all equivalent.*
2. If d is a metric on X and λ is a positive real number, then $\lambda.d$ is a metric equivalent to d.
3. If X is a finite set, then every metric on X is equivalent to the discrete metric.

To see this, note that any one-element set x is closed, so any subset of X is a union of finitely many closed sets, so is closed. This is true for all metrics on X, so the identity map on X is continuous with respect to any two metrics d, d' on X, and so a homeomorphism between (X, d) and (X, d').

8.1 Connectedness

A metric space X is called disconnected if $X = A \cup B$. Where A and B are disjoint non-empty sets in X.
If X is not disconnected $\implies$ X is connected.

Example 8.2 *1. $S = R - \{0\}$ is disconnected, 2. Every open interval is connected.*

9 Uniform Continuity

Definition 9.1 *Given metric spaces $(X, d_{X)} and (Y, d_Y)$, a function $f : X \to Y$ is uniformly continuous on X if and only if*

$$(\forall \epsilon > 0)(\exists \delta > 0)[(\forall p)(\forall q)(p, q \in X_x^d(p, q) < \delta \implies d_y(f(p), f(q)) < \epsilon)]$$

Example 9.1 *The function $f(x) = x^2 : R \to R$ is uniformly continuous on $[1, 3]$.*

For $\epsilon > 0$ let $\delta = \epsilon/6$. For $x_1, x_2 \in [1, 3]$, the triangular inequality yields that

$$|x_1 + x_2| \leq |x_1| + |x_2| \leq 6$$

Hence, $x_1, x_2 \in [1, 3]$ and $|x_1 + x_2| < \delta$ implies that

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 - x_2||x_1 + x_2| < \delta.6 = \epsilon.$$

Since, $\epsilon > 0$ and $x_1, x_2 \in [1, 3]$ were arbitrary, we conclude that f is uniformly continuous on $[1, 3]$.

Theorem 9.2 *If f is a continuous mapping from a compact metric space X to a metric space Y , then f is uniformly continuous on X*

10 Types of Discontinuities

Given two metric spaces (X, dX) and (Y, dY) and a function f from a subset A of X into Y. If $p \in X$ and f is not continuous at p, then we can conclude that f is not defined at $p(p \notin A = dom(f)), \lim_{x \to p} f(x), x \in A$ does not exist, and/or $p \in A$ and $\lim_{x \to p} f(x)$ exists but $f(p) \neq \lim_{x \to p} f(x)$; a point for which any of the three conditions occurs is called a point of discontinuity.

Definition 10.1 *A function f is discontinuous at a point $c \in dom(f)$ or has a discontinuity at c if and only if either $\lim_{x\to c} f(x)$ doesn't exist or $\lim_{x\to c} f(x)$ exists and is different from $f(c)$.*

Example 10.1 *The domain of $f(x) = \frac{|x|}{x}$ is $R-\{0\}$. Consequently, f has no points of discontinuity on its domain.*

Definition 10.2 Let f be a function that is defined on the segment (a, b). Then, for any point $x \in [a,b)$, the right-hand limit is denoted by f (x+) and

$$f(x+) = q \Leftrightarrow (\forall \{t_n\}_{n=1}^{\infty}) \left[\left(\{t_n\} \subset (x,b) \lim_{n\to\infty} t_n = x \right) \Rightarrow \lim_{n\to\infty} f(t_n) = q \right]$$

and, for any $x \in (a,b]$, the left-hand limit is denoted by f (x-) and

$$f(x-) = q \Leftrightarrow (\forall \{t_n\}_{n=1}^{\infty}) \left[\left(\{t_n\} \subset (a,x) \lim_{n\to\infty} t_n = x \right) \Rightarrow \lim_{n\to\infty} f(t_n) = q \right].$$

10.1 Existence of Limit

If $\lim_{t\to x} f(x)$ exists then,

$$\lim_{t\to x^+} f(x) = l = \lim_{t\to x^-} f(x)$$

Remark 10.2 *If both $\lim_{t\to x^+} f(x)$ and $\lim_{t\to x^-} f(x)$ exists, but are unequal, then it is considered as Jump – type discontinuity.*

Example 10.2 *1. Consider the function*

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

We evaluate left-hand and right-hand limits for $f(x)$ at $x = 0$, and

$$f(0^+) = 1$$

$$f(0^-) = (-1)$$

So, this is a jump-type discontinuity.

2. Consider the function

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We observe that left-hand limit and right-hand limit are not equal, but can be externally made equal, if we define f (x) = 1 for x = 0. So, this type of discontinuity is known as removable discontinuity.

3. *Consider the function* $f(x) = \frac{1}{x}, (x \neq 0)$*. It has non-existent left-hand and right-hand limits. So in this case, as the limit approaches infinity, this type of discontinuity is termed as infinite type discontuity. Similar case exists for*

$$f(x) = \sin \frac{1}{x}, (x \neq 0).$$

References

[1] Cormen TH, Leiserson CE, Rivest RL, Stein C. *Introduction to algorithms.*. MIT press. 2009 Jul 31