ELL780 ELL 457 Indian Institute of Technology Delhi
Lecture 16 Matrix Analysis and Linear Algebra
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2 Introduction

Variety of the problems can be easily solved if we organize the relevant information in a systematic way. Linear algebra is the analysis of the sets of linear equations and their transformations using the columns called vectors and array of vectors called matrices. It mainly consists of theory and applications of linear system of equations. The four fundamental sub-spaces associated with a matrix help us in the analysis of linear system. Linear algebra allow us to review the lines, planes, rotations in N-dimensional space, projections, least square fittings, etc., all with the help of matrices. Dimensionality matching is one of the most important tool when dealing with matrix analysis.
In the sections below we will learn ways to organize the linear equations in a mathematical structure so as to obtain the solution in an easy and fast manner. Matrix analysis is used to solve the equations of the type $AX = B$ by organising the equations in the form of matrices and vectors.

3 Introduction to Matrices

A matrix of order $m \times n$ is defined as a system of 'mn' numbers which are arranged in the form of rectangular array with 'm' number of rows and 'n' number of columns usually bounded in square brackets [ ].
A $m \times n$ matrix is generally written as

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \tag{1}$$

In compact form it is also written as $A = [a_{ij}]_{m \times n}$ or $A = [a_{ij}]$ where, $1 \leq i \leq m$ and $1 \leq j \leq n$. Matrix should not be considered as the collection of numbers rather it is a single entity with a number of elements which cannot be reduced to a single number.

Examples of matrices:

1. $\begin{bmatrix} 2 & 7 \\ 8 & 11 \end{bmatrix}$ consists of 2 rows and 2 columns and hence is a 2×2 matrix having elements $a_{11} = 2$, $a_{12} = 7$, $a_{21} = 8$, $a_{22} = 11$.
2. $\begin{bmatrix} x & y & z \end{bmatrix}$ consists of three elements which represents the coordinates of a point in solid geometry.

4 Rank of the Matrix

A matrix is said to be of rank r, if it has at least one non-zero minor of order r and all minors of order $> r$ are zero.
Rank gives the number of independent columns (or rows) of the given matrix. For a matrix A, rank of the matrix is denoted by $Rank(A)$ or simply r.

4.1 Properties of rank of matrix:

- Rank of a matrix is the number of linearly independent column vectors as well as the number of linearly independent row vectors.
- For any matrix A_{mn}, $rank(A) \leq min(m, n)$.
- $Rank(AB) \leq min(Rank(A), Rank(B))$.
- $Rank(A^T) = Rank(A)$ where A^T denotes the transpose of matrix A.
- Rank of a matrix is the number of non-zero rows in its echelon form.
- Elementary transformation do not change the rank of the matrix.
- Rank of the matrix is always greater than or equal to one except for the null matrix which has rank zero.

4.2 Calculation of rank using Echelon form of matrix

Echelon Form of matrix:
A matrix is said to be in echelon form if it satisfies the following properties:

- Leading non zero element of every row must be behind leading non zero element of all the previous rows.
- All the zero rows should be below the non zero rows.

Examples Consider the following matrix

$$\begin{bmatrix} 4 & 5 & 2 \\ 8 & 12 & 4 \\ 1 & 4 & 2 \end{bmatrix}$$

$R_2 R_2 - 2R_1,\ R_3 \leftarrow R_3 - 4R_1$

$$\begin{bmatrix} 4 & 5 & 2 \\ 0 & 2 & 0 \\ 0 & -16 & -6 \end{bmatrix}$$

$R_3 \leftarrow R_3 + 8R_2$

$$\begin{bmatrix} 4 & 5 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

Now, the number of non-zero rows in the echelon form matrix is 3. Hence ,the rank of the matrix is 3.

Example: Consider the following matrix

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Reducing the above matrix in the echelon form, we get the following matrix,

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of non-zero rows in the echelon matrix is 2, hence the rank of the matrix is 2, *i.e* **Rank(A)=2**

Note:

- If $r = m$ and $r < n$, matrix A is short and wide,
- If $r < m$ and $r = n$, matrix A is tall and thin.

Rank of a matrix can be associated with the degree of freedom of system which the matrix represents. If the matrix has low rank, then it means that it has less number of independent vectors and hence low degree of freedom, i.e. the data points of the system of equations lies in the small subspace.

Rank of a matrix can be used to compute the number of solutions for a system of equations. Similarly, in control theory rank of a matrix is used to find the controllability and observability of the system.

5 System of Linear equations

The basic idea of studying the linear algebra is to solve the system of equations given in the form of $AX = B$. To solve the linear equations we make use of the structured matrices

$$AX = B \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times 1}$ and $B \in \mathbb{R}^{m\times 1}$.
AX=B can also be written in a more elaborated form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix} \tag{3}$$

which can be further simplified and written in the form of set of equations as below:

$$\begin{gathered} a_{11}x_1 + a_{12}x_1 + ... + a_{1n}x_1 = b_1 \\ a_{21}x_1 + a_{22}x_1 + ... + a_{2n}x_1 = b_2 \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_1 + ... + a_{mn}x_1 = b_m \end{gathered}$$

Homogeneous system of equations
When $b_1 = b_2 = ... = b_m = 0$ then the above system of linear equations is known as the homogeneous system of equations.

5.1 Solution of system of linear equations

The system of equations is said to be consistent if there exist a vector X which satisfies the equation $AX = B$. If there is no vector X which satisfies the equation $AX = B$, the system of equations is said to be inconsistent. For the consistent system of equations, we can have a unique solution or infinite solution depending upon the properties of the matrix A.
Thus, the system of equations can be broadly classified in the following ways:

1. Inconsistent systems
2. Consistent systems
   (a) Unique solution
   (b) Infinite solution.

The examples for each set of equations is given below along with the corresponding graphs (obtained using [4]) which can be further analysed to draw some important properties of the same.

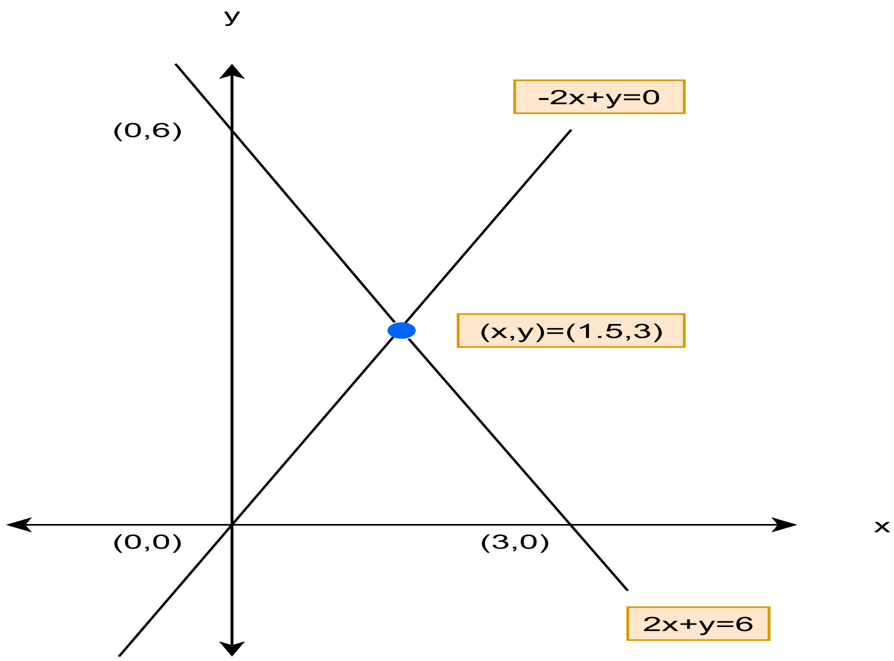

Figure 1: System of equations showing unique solution

- **Unique solution**:
  Consider the set of equations:

$$-2x + y = 0$$
$$2x + y = 6$$

  The two set of equations represent line which intersect each others at a point which gives unique solution ($x = 1.5, y = 3$). This is also shown using the graph given in the Figure 1.

- **No solution**:
  Consider the set of equations:

$$2x + y = 6$$
$$4x + 2y = 4$$

  The two set of equations represent line which are parallel to each other and do not intersect at any point. There is no solution for such type of system of equations which is also shown by the plot in Figure 2.

- **Infinite Solution**:
  Again consider the set of equations:

$$4x + 2y = 4$$

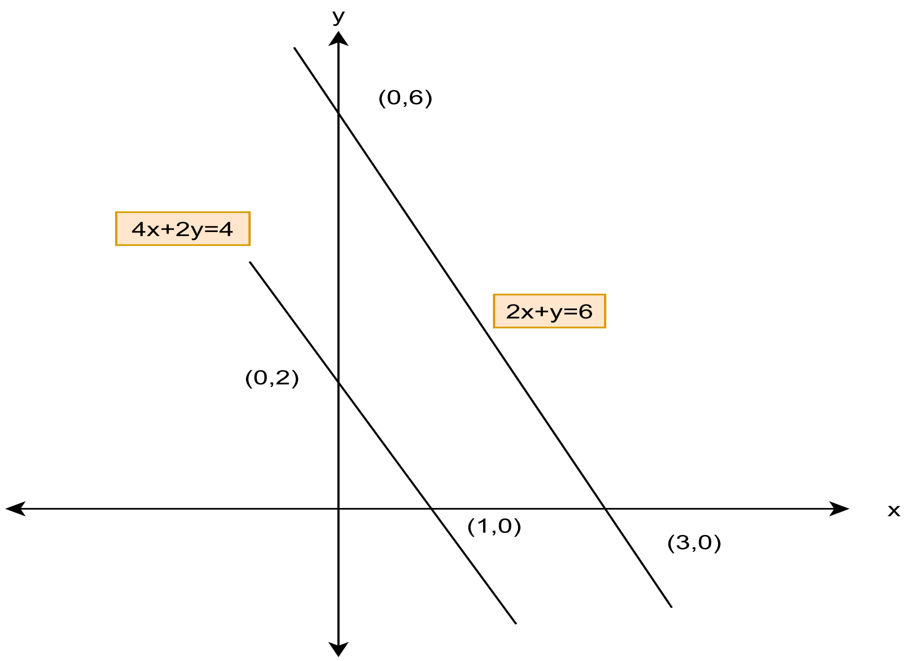

Figure 2: System of equations showing no solution

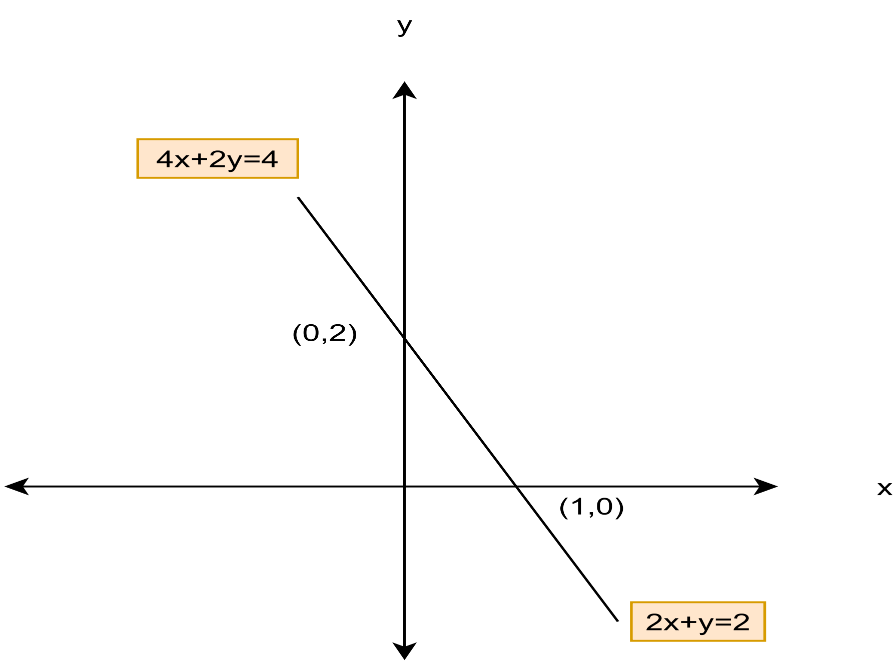

Figure 3: System of equations showing infinite solution

$$2x + y = 2$$

The two set of equations represent line which lies on each other and hence, have infinite solution. The solutions are all set of points lying on the line. The corresponding plot is shown in Figure 3.

General procedure to test the consistency of system of linear equations based on the rank.

Let r be the rank of matrix A and let r_a be the rank of Augmented matrix $[A|B]$.

- If $r \neq r_a$ then the equations are inconsistent and there is no solution.
- If $r = r_a = m$ then the equations are consistent and there is a unique solution.
- If $r = r_a < m$ then the equations are consistent and there are infinitely many solutions.

For system of linear homogeneous equations

- If $r = m$, the system of equations have only trivial solution i.e. $x_1, x_2, ...x_m = 0$.
- $r < m$, the system of equations have $m - r$ linearly independent solutions and infinite number of solutions.
- If $m < n$, the solution is always other than $x_1 = x_2 = ... = x_m = 0$ and the no. of solutions are infinite.
- If $m = n$, system of equations are consistent and have non-trivial solutions.

Example: Solve the homogeneous system of linear equations $AX = 0$ where $\begin{bmatrix} -1 & -1 & 1 \\ -3 & 2 & 1 \\ -1 & 3 & -1 \end{bmatrix}$.

Check if the system of linear equations have no solution, infinite no. of solutions or a unique solution.

Solution: Transforming given system of equations in row echelon form

$$\begin{bmatrix} -1 & -1 & 1 \\ -3 & 2 & 1 \\ -1 & 3 & -1 \end{bmatrix}.$$

Using row operation $R_3 \leftarrow R_3 + R_1$

$$\begin{bmatrix} -1 & -1 & 1 \\ -3 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Using row operation $R_2 \leftarrow R_2 - 3R_1$

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & 5 & -2 \\ 0 & 2 & 0 \end{bmatrix}.$$

From here, we can deduce that the rank of matrix is three. Hence, there are no free variable and the system of equations can be written as

$$\begin{aligned} -x_1 - x_2 + x_3 &= 0 \\ 5x_2 - 2x_3 &= 0 \\ 2x_2 &= 0 \end{aligned}$$

Hence, the above system of equations have trivial solution i.e. $x_1 = x_2 = x_3 = 0$.

5.2 Methods to solve system of liner equation

1. **Elimination method**
   The idea of elimination method is to eliminate $(n-1)$ variable from $X \in \mathbb{R}^{n\times 1}$ to get the value of one variable of the vector X.
   e.g. consider the following system of equations

   $$5x + 7y = 14 \tag{4}$$
   $$3x + 2y = 6 \tag{5}$$

   Multiplying equation (4) by 3 and subtracting from the result of equation (5) multiplied by 5, we get $y = 12/11$ and on back substitution of the value of y in equation (4), we get $x = 14/11$

2. **Row view of the matrix analysis**
   This is the general geometrical method of solving the system of linear equations. We are exposed to this in terms of matrix multiplications or solving equations. Consider the same set of equations given in (4) and (5) i.e.

   $$\begin{aligned} 5x + 7y &= 14 \\ 3x + 2y &= 6 \end{aligned}$$

   Solving the above system of equations by solving two equations simultaneously using graphical method we obtain $x = 14/11$ and $y = 12/11$ as shown in Figure 4 (obtained using [2]).

   This method looks easy to solve the linear system of equations but they become quite difficult when we move to the higher dimensions in which order of the matrix increases. Visualising in terms of rows, each row define a plane. Intersection of 3 planes result in a point which represents a solution to the three dimensional system of equations.
   Now let us see an example of solving system of linear equations with three variables.

   $$-2x_1 + x_3 = -8 \tag{6}$$
   $$x_1 - 2x_2 + x_3 = -2 \tag{7}$$
   $$x_1 - x_2 + 4x_3 = -3 \tag{8}$$

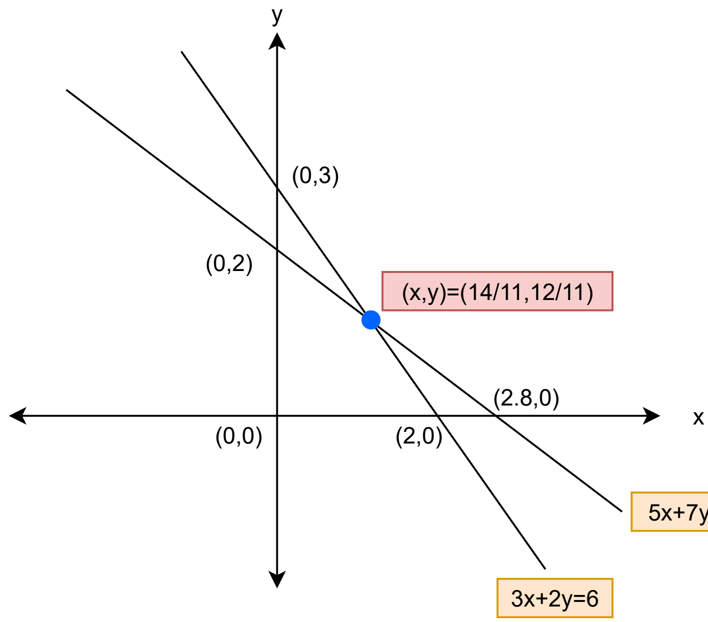

Figure 4: Row view (Geometrical representation of two dimensional system)

Writing them in matrix form we get

$$\begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ -2 \\ -3 \end{bmatrix}$$

Here, each equation represents the equation of a plane with equation 6 representing a plane extending from $-\infty$ to ∞ in the second dimension i.e. x_2. The intersection of these three plane is shown in Figure 5. On solving the above set of equations ,we get the solution as $x_1 = \frac{52}{15}$, $x_2 = \frac{11}{5}$ and $x_3 = \frac{-16}{15}$.

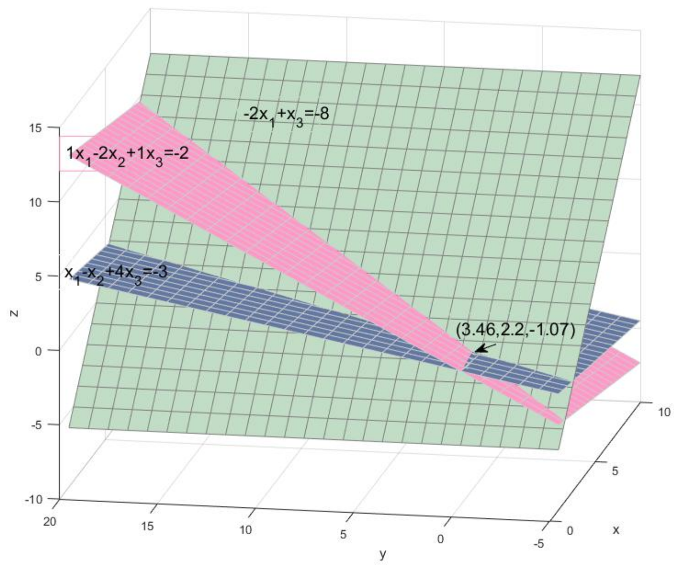

Figure 5: Row view (Geometrical representation of three dimensional system)

3. **Column views of the matrix analysis**
This method gives us more flexibility and abstractness in the analysis which helps us to understand matrix analysis in higher dimensions easily. Consider the same set of equations given in (4) and (5) i.e.

$$5x + 7y = 14$$
$$3x + 2y = 6$$

The following system of equations can be written in the column form as follows

$$AX = x \begin{bmatrix} 5 \\ 3 \end{bmatrix} + y \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \end{bmatrix} \tag{9}$$

where x and y are unknowns which scales the column vector.

For the above problem ,the linear combination of the vector $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ should produce the vector $\begin{bmatrix} 14 \\ 6 \end{bmatrix}$.

The solution for the above problem is to scale the vector $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ by $(\frac{14}{11})$ and $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ by $(\frac{12}{11})$ which

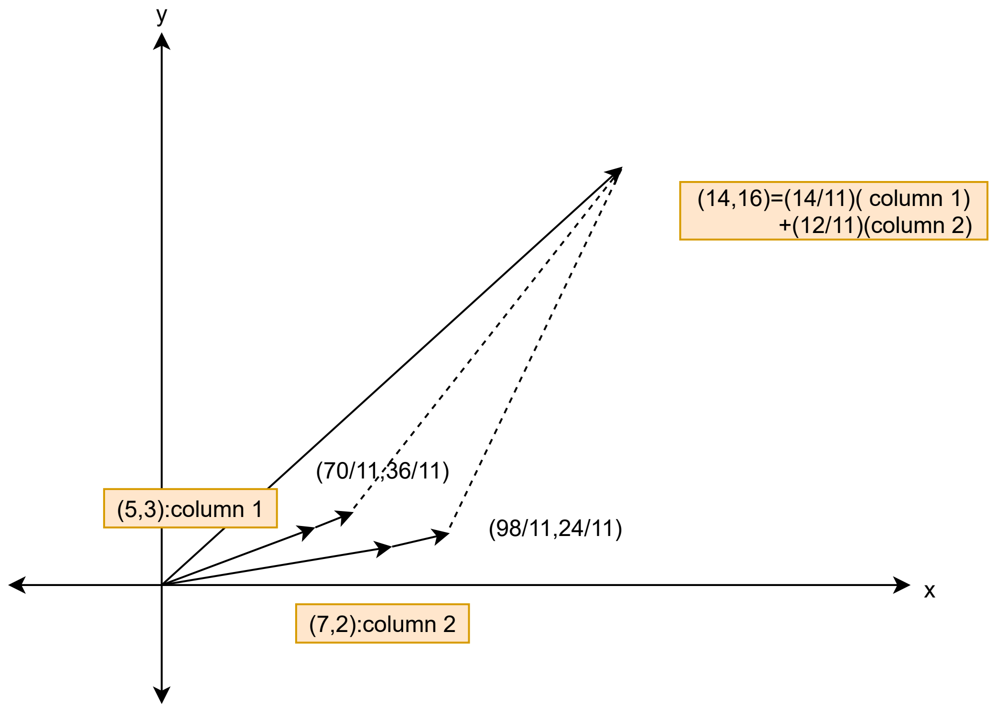

Figure 6: Column view of two dimensional system (Linear combination of columns gives B)

gives the vector $\begin{bmatrix} 14 \\ 6 \end{bmatrix}$. This is pictorially shown in Figure 6.

Let's consider the same set of three equations again as taken in the equation (6-8) and try to solve it using column picture

$$-2x_1 + x_3 = -8 \tag{10}$$

$$x_1 - 2x_2 + x_3 = -2 \tag{11}$$

$$x_1 - x_2 + 4x_3 = -3 \tag{12}$$

Writing them in matrix form we get

$$\begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ -2 \\ -3 \end{bmatrix}$$

In general terms, we can say that in column picture of matrix analysis we scale and add vectors in such a manner that their addition given the third vector. On solving the above set of equations, we get the solution as $x_1 = \frac{52}{15}$, $x_2 = \frac{11}{5}$ and $x_3 = \frac{-16}{15}$.
Figure 7 shows the scalled addition on column 1 and column 2 to produce an intermediate vector $\begin{bmatrix} .-6.9 \\ -0.93 \\ 1.27 \end{bmatrix}$, which is further added to the scaled version on column 3 to produce the vector $\begin{bmatrix} -8 \\ -2 \\ 3 \end{bmatrix}$,

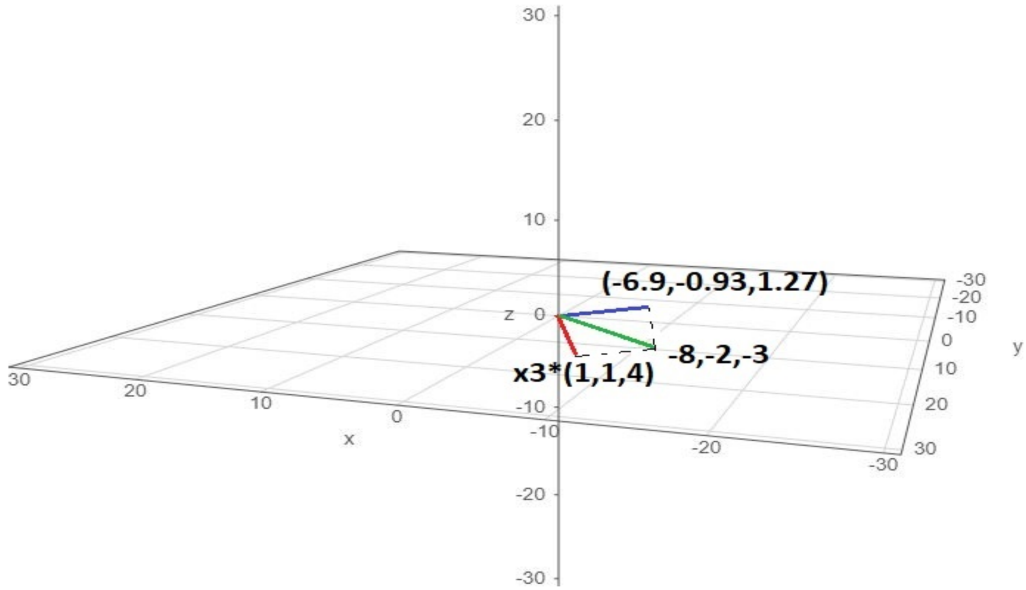

Figure 7: Addition of first two vectors

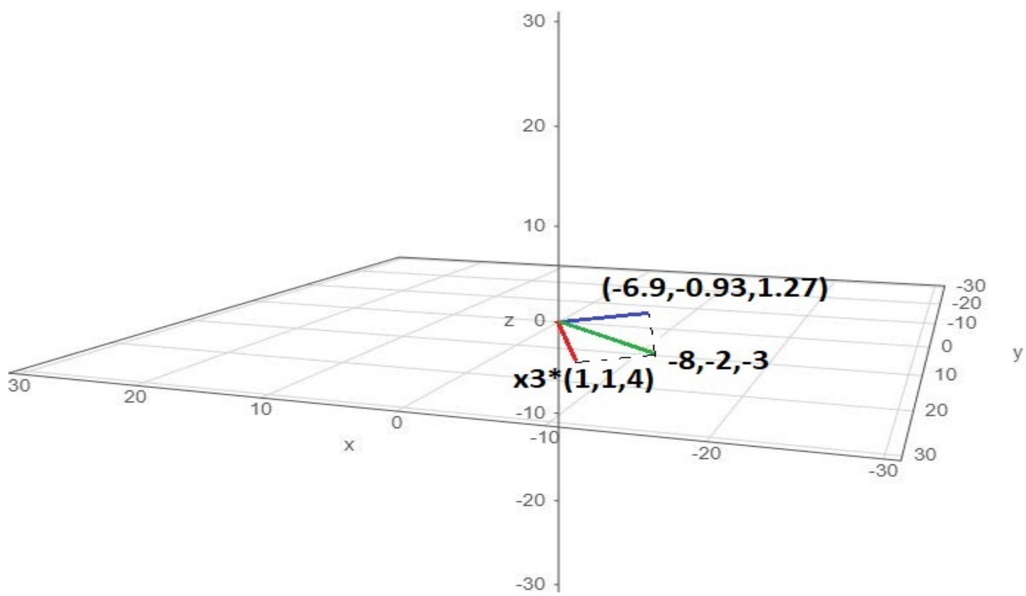

Figure 8: Column view of three dimensional system (Linear combination of columns gives B)

as shown in figure 8.

6 Matrix-Vector Multiplication

There are two ways by which we can perform matrix multiplication.

1. **Inner Product**
   This is also commonly known as row by column method. Consider the matrix multiplication

$$AX = B$$

   Now, B_{ij} is obtained by multiplying the i^{th} row of A to the j^{th} column of X.
   e.g. consider the following example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{13}$$

   which can be simplified by dot product in the following ways

$$x_1 + 2x_2 + 3_3 = b_1$$

$$4x_1 + 5x_2 + 6_3 = b_2$$

$$7x_1 + 8x_2 + 9_3 = b_3$$

2. **Column view of multiplication**
   In this method, we multiply the columns of A with the rows of X to get B.

$$AX = B$$

   e.g. Consider the same matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   The above matrix can be expressed as follows

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{14}$$

   The linear combination of the left hand side vectors should yield $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

7 Column space

The column space of matrix A, denoted by $Col(A)$ is defined as the set of all possible linear combination of its column vectors. For a $m \times n$ matrix, it the the subspace of R^m.
e.g. consider the following matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The $Col(A)$ of above matrix is $\mathbb{R}^3$.
Generally, the column space of a matrix can be used to solve the system of linear equation. Consider the linear equation AX=B where $A \in \mathbb{R}^{m \times n}$, $X \in \mathbb{R}^{n \times 1}$ and $B \in \mathbb{R}^{m \times 1}$ such that

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix} \tag{15}$$

Now, this system of equations can be solved only if the vector 'B' can be written as the combination of columns of A and we say that the vector 'B' is in the column space of A. If the vector B doesn't lie in the column space of A, we say that the system of equation is not exactly solvable.
More precisely, we can write the above problem as following:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ a_{m2} \end{bmatrix} \begin{matrix} . & . \\ . & . \\ + . & . \\ . & . \\ . & . \end{matrix} + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$$

Now, we have to find the value of x_1,x_2...,x_n which will multiply the 1^{st},2^{nd},...,n^{th} column respectively such that the combination of left side yields vector B, and that vector 'x' is the solution of the given system of equations. If such vector doesn't exist then we can interpret that vector B is not in the column space of A, and the given system of equations are not solvable.

Example
Consider the linear system given by AX=b where $A \in \mathbb{R}^{3 \times 3}$, $X \in \mathbb{R}^{3 \times 1}$ and $B \in \mathbb{R}^{3 \times 1}$ such that

$$\begin{bmatrix} 0 & 4 & 3 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}$$

which can be further written as the combination of columns as follows:

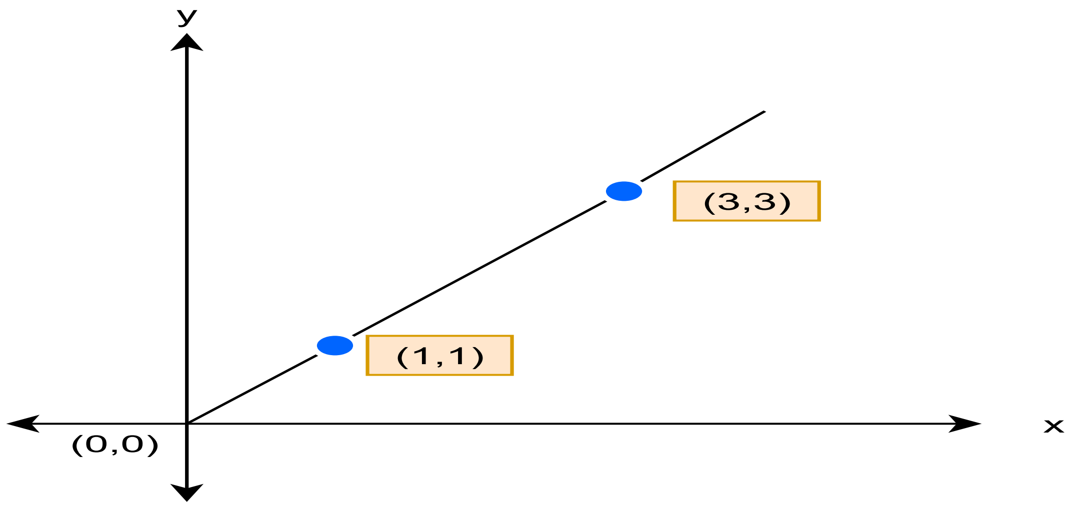

Figure 9: This figure shows that the column space of the matrix P is a line.

$$x_1 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}$$

Now,no combination of x_1,x_2 and x_3 on multiplying column $1st$, $2nd$ and $3rd$ respectively will give vector B and hence, the vector B doesn't lie in the column space of A. Hence the system of equations is not solvable.

Consider a matrix $P = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$. It has two column vectors but they are dependent on each other. i.e. column 2 is a multiple of column 1. Hence, linear combination of these two vectors does not span the whole plane R^2 but defines a line as shown in Figure (9).

Now consider a matrix $Q = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 2 & 6 \end{bmatrix}$. This matrix has three columns. We can observe the column 3 is the linear combination of column 1 and column 2. Therefore it will result in a plane and not full R^3 as shown in Figure (10).

The dimension of the column space is generally given by the rank of the matrix.

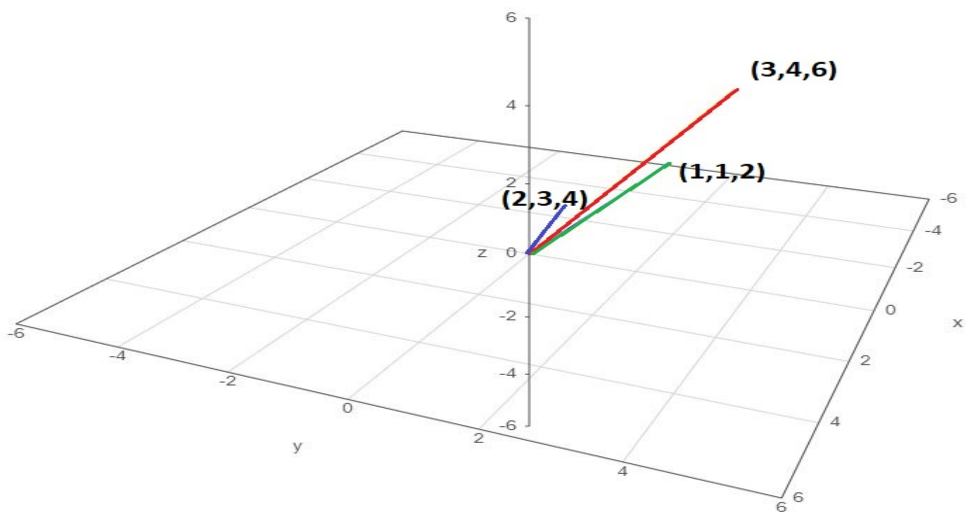

Figure 10: This figure shows that the column space of the matrix Q is a plane

8 Matrix-Matrix Multiplication

Consider the matrix multiplication $AB = C$. where, $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{m \times p}$. (Here, we have assumed matrix element s to be real. But they may be complex also.)

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & . & . & b_{1p} \\ b_{21} & b_{22} & . & . & b_{2p} \\ . & . & . & . & . \\ . & . & . & . & . \\ b_{n1} & b_{n2} & . & . & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & . & . & c1p \\ c_{21} & c_{22} & . & . & c_{2p} \\ . & . & . & . & . \\ . & . & . & . & . \\ c_{m1} & c_{m2} & . & . & c_{mp} \end{bmatrix} \tag{16}$$

If matrix B consist of a single column then it is equivalent to the matrix-vector multiplication. Following are some general facts about the matrix multiplication

1. a_{ij} element of the product matrix AB is the product of i^{th} row of A and j^{th} column of B.
2. Every column of AB is a combination of the column of A.
3. Matrix multiplication is not commutative $i.e$ $AB \neq BA$.
4. Matrix multiplication is associative $i.e$ $(AB)C = A(BC)$.
5. Matrix multiplication is distributive $i.e$ $A(B + C) = AB + AC$.

There are two ways for matrix multiplication which are described in the following.

8.1 Inner Product

This method of matrix multiplication involves the row times columns operation. For instance, to obtain the element c_{34} we multiply each element of third row of A to the corresponding elements of B as shown:

$$c_{43} = a_{31}b_{14} + a_{32}b_{24} + ... + a_{3n}b_{n4}$$

In general for product of a matrix of dimension $m \times n$ with the matrix of dimension $n \times p$ gives the resultant matrix off dimension $m \times p$.
Each element of the resultant matrix can be written as

$$[c]_{kl} = \begin{bmatrix} a_{k1} & a_{k2} & . & . & a_{kn} \end{bmatrix} \begin{bmatrix} a_{1l} \\ a_{2l} \\ . \\ . \\ a_{ml} \end{bmatrix}$$

$$[c]_{kl} = \sum_{i=1}^{n} a_{ki}b_{il} \tag{17}$$

Consider A and B to be 3×3 matrices as shown

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} c_{11} & c_{12} & c13 \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \tag{20}$$

Example:
Consider the matrix A and B with each having dimension of (3×3) given as

$$\text{A=}\begin{bmatrix} 1 & 4 & 0 \\ 2 & 6 & 9 \\ 1 & 3 & 4 \end{bmatrix} \text{ and B=}\begin{bmatrix} 7 & 1 & 2 \\ 0 & 4 & 6 \\ 2 & 2 & 1 \end{bmatrix}$$

The matrix multiplication of C=AB is given by

$$C = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 6 & 9 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & 1 & 2 \\ 0 & 4 & 6 \\ 2 & 2 & 1 \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} 7+0+0 & 1+16+0 & 2+24+0 \\ 14+0+18 & 2+24+18 & 4+36+9 \\ 7+0+8 & 1+12+8 & 2+18+4 \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} 7 & 17 & 26 \\ 32 & 44 & 49 \\ 15 & 21 & 24 \end{bmatrix} \tag{23}$$

8.2 Outer Product

This method of matrix multiplication involves the column times row operation.

$$\begin{bmatrix} | & | & . & . & | \\ | & | & . & . & | \\ col1 & col2 & . & . & coln \\ | & | & . & . & | \\ | & | & . & . & | \end{bmatrix} \begin{bmatrix} - & - & row1 & - & - \\ - & - & row2 & - & - \\ . & . & . & . & . \\ . & . & . & . & . \\ - & - & rown & - & - \end{bmatrix} \tag{24}$$

Multiplication of a column of matrix A by the row of matrix B produces a matrix of rank 1.

$$\begin{bmatrix} a_{m1} \\ a_{m2} \\ . \\ . \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{1n} & b_{12} & . & . & b_{kp} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & . & . & c1p \\ c_{21} & c_{22} & . & . & c_{2p} \\ . & . & . & . & . \\ . & . & . & . & . \\ c_{m1} & c_{m2} & . & . & c_{mp} \end{bmatrix}$$

Multiplication of n rows by the n columns gives n rank 1 matrices.
Consider A and B to be 3×3 matrices as shown

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \begin{bmatrix} b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \begin{bmatrix} b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} \end{bmatrix} + \begin{bmatrix} a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\ a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{32}b_{21} & a_{32}b_{22} & a_{32}b_{23} \end{bmatrix} + \begin{bmatrix} a_{13}b_{31} & a_{13}b_{32} & a_{13}b_{33} \\ a_{23}b_{31} & a_{23}b_{32} & a_{23}b_{33} \\ a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} \end{bmatrix}$$

(The three matrices in the above step are all rank 1 matrices.)

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} \tag{26}$$

On comparison of matrix in equation (19) and (23) we can see that the resultant matrix is same, whether we calculate matrix product by inner product or by outer product.
Example Consider the same set of example as taken in the previous case *i.e*

$$\text{A=}\begin{bmatrix} 1 & 4 & 0 \\ 2 & 6 & 9 \\ 1 & 3 & 4 \end{bmatrix} \text{ and B=}\begin{bmatrix} 7 & 1 & 2 \\ 0 & 4 & 6 \\ 2 & 2 & 1 \end{bmatrix}.$$

Now, the product of the matrix $C = AB$ using outer product can be done using the following ways:

$$\begin{aligned} C &= \begin{bmatrix} 1 & 4 & 0 \\ 2 & 6 & 9 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & 1 & 2 \\ 0 & 4 & 6 \\ 2 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 7 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 9 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 1 & 2 \\ 14 & 2 & 4 \\ 7 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 16 & 24 \\ 0 & 24 & 36 \\ 0 & 12 & 18 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 18 & 18 & 9 \\ 8 & 8 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 17 & 26 \\ 32 & 44 & 49 \\ 15 & 21 & 24 \end{bmatrix} \end{aligned}$$

Examples on application of Linear algebra analysis:
Example 1:
Consider the circuit shown in the figure (11) below. Using loop analysis, find the current through each loop.

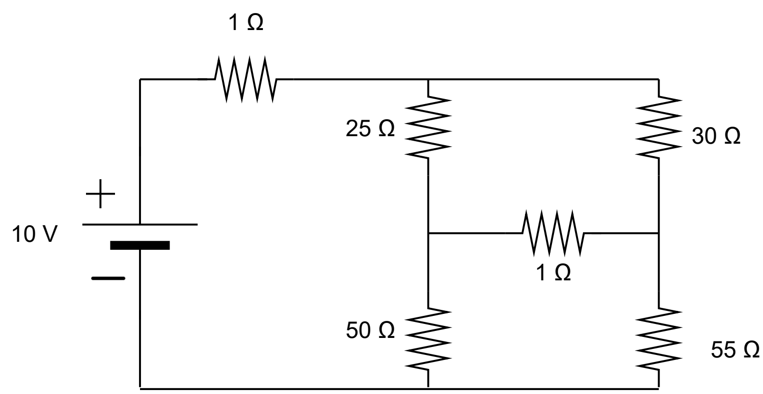

Figure 11: Electrical Circuit

Solution: Consider the amount of current flowing be $i1$, $i2$ and $i3$ through the three loop as shown in the figure (12). Now,applying the Kirchhoff's Voltage Law (KVL) in each loop, we get

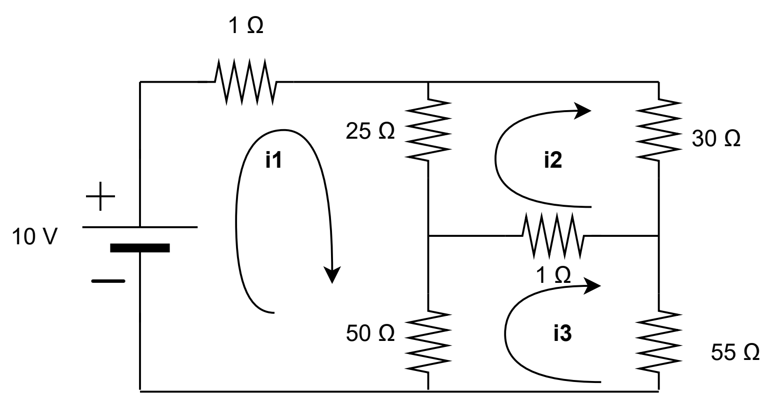

Figure 12: Circuit showing current through each loop

the following three set of equations:

$$i1 + 25(i1 - i2) + 50(i1 - i3) = 10$$

$$25(i2 - i1) + 30i2 + 1(i2 - i1) = 0$$

$$50(i3 - i1) + 1(i3 - i2) + 55i3 = 0$$

On re-arranging the above set of equations,we get the following set of equations

$$76i1 - 25i2 - 50i3 = 10$$

$$-25i1 + 56i2 - i3 = 0$$

$$-50i1 - i2 + 106i3 = 0$$

By simplifying and manipulating the above set of equations, eventually all the unknowns of the following equations can be solved provided the numbers of equations is equal to number of unknowns. However, as the circuit size grows, we need to solve for large number of unknowns which will become very hectic as well as complex analysis. Hence, solving such type of problems using linear algebra analysis will become easy task.
Now arranging the equations of the above circuit in the matrix form as follows

$$\begin{bmatrix} 76 & -25 & -50 \\ -25 & -56 & -1 \\ -50 & -1 & 106 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

Writing the above in the form of augmented matrix, we have the following

$$\begin{bmatrix} 76 & -25 & -50 & 10 \\ -25 & 56 & -1 & 0 \\ -50 & -1 & 106 & 0 \end{bmatrix}$$

We can now perform some row operation for solving the value of unknowns by simple back substitution method,

$$\begin{bmatrix} 1 & 0 & 0 & 0.245 \\ 0 & 1 & 0 & 0.111 \\ 0 & 0 & 0 & 0.117 \end{bmatrix}$$

Now, by back substitution, we can have the value of current as follows

$$i1 = 0.245$$

$$i2 = 0.111$$

$$i3 = 0.117$$

Example 2:
Consider three children C1, C2 and C3 who desire to buy some biscuits, breads, cake and chocolates. Each of them want all these commodities in different amount and can buy them from two shops S1 and S2. Find the shop which is best for every child such that each of them pays the least amount. Price of each commodity in two shops and the amount of every commodity which each child wishes to buy is tabulated below.

Table 1: Prices in shops

	S1	**S2**
Biscuits	5	6
Bread	12	15
Cake	12	10
Chocolate	20	25

Table 2: Demanded quantity of commodity

	Biscuit	**Bread**	**Cake**	**Chocolate**
C1	5	4	2	3
C2	2	4	7	6
C3	1	6	5	6

Solution: This problem can be solved easily if we write the given information in matrix form as follows:

We can write demand matrix as $D = \begin{bmatrix} 5 & 4 & 2 & 3 \\ 2 & 4 & 7 & 6 \\ 1 & 6 & 5 & 6 \end{bmatrix}$.

Similarly we can write the price matrix as $P = \begin{bmatrix} 5 & 6 \\ 12 & 15 \\ 12 & 10 \\ 20 & 25 \end{bmatrix}$.

The product matrix is given by $R = DP = \begin{bmatrix} 157 & 185 \\ 262 & 202 \\ 257 & 296 \end{bmatrix}$.

The first row of matrix R represents the amount of money spent by C1 in shop S1 and S2. Similarly second and third rows of matrix R represents the amount of money spent by C2 and C3 in shops S1 ans S2 respectively. Hence, it is optimal for C1 to buy from shop S1 , C2 to buy from shop S2 and C3 to buy from shop S1.

Problems for practice:

Consider the following matrices:

$$A = \begin{bmatrix} -1 & 23 & 10 \\ 0 & -2 & -11 \end{bmatrix}, B = \begin{bmatrix} -6 & 2 & 10 \\ 3 & -3 & 4 \\ -5 & -11 & 9 \\ 1 & -1 & 9 \end{bmatrix}, C = \begin{bmatrix} -3 & 2 & 9 & -5 & 7 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 6 \\ -5 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 3 \end{bmatrix}, \quad F = \begin{bmatrix} 3 \\ 5 \\ -11 \\ 7 \end{bmatrix}, \quad G = \begin{bmatrix} -6 & -4 & 23 \\ -4 & -3 & 4 \\ 23 & 4 & 9 \end{bmatrix}$$

1. What is the dimension of each matrix?
2. Which matrices are square?
3. Which matrices are symmetric?
4. Which are column matrices?
5. Determine the column space and rank space of each matrix.
6. Find rank of A and B.
7. Find A× C using row view of matrix multiplication.
8. Find A × G using column view of matrix multiplication.

References

[1] Linear Algebra and its Applications by Gilbert Strang 4th edition.

[2] MATLAB, MATLAB production server R2015a

[3] https://academo.org/demos/3d-vector-plotter/

[4] https://app.diagrams.net/

Note: The four fundamental sub-spaces which are the essential and central part of the linear system analysis are beyond the scope of lecture 16.