ELL780 ELL 457 Indian Institute of Technology Delhi
Lecture 8 Sequences
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1 Introduction

We begin by discussing the motivation for analysis of real sequences. It becomes interesting if we view sequences as function of natural numbers. Let us take an example, assume you need to determine the total amount of CO_2 in the atmosphere as a function of the day, then you can model it as a function of natural numbers where the numbers denote the day number. Once this is done, you can find answers to questions like will the amount of CO_2 be bounded or not? Will it converge to a particular value or not?

1.1 Example

Another interesting application is called *"The Zeno's Paradox"*. Zeno noted that in order to reach a place, one must first travel half of the distance to that place, then half of the left of the distance and so on, according to him, if we travel each half path in a finite time, we can never reach the goal in any finite time. This paradox is a direct reformulation of the following problem:

$$\text{Find } x \text{ such that } \sum_{n=1}^{x} \frac{1}{2^n} = 1$$

A geometrical interpretation of this problem could be if we divide a square into half, and then the remaining half into half and so on, we would cover the entire square as shown in the figure-1 below.

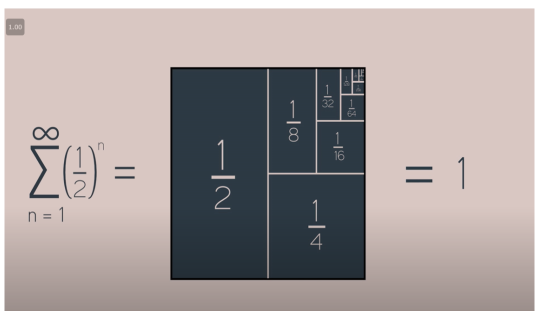

Figure 1: Figure depcting geometrical interpretation of zeno's paradox

For the curiosity to solve all these questions, we shall study the behaviour of real sequences in the following sections.

2 Subsequence

2.1 Definition

Let $\{a_n\}$ be a sequence and $\{n_1, n_2, ...\}$ be a sequence of positive integers such that $i > j$ implies $n_i > n_j$. Then the sequence $\{a_{n_i}\}_{i=1}^{\infty}$ is called a subsequence of $\{a_n\}$. Thus, $\{a_{n_i}\}$ is a selection of some (possibly all) of the s_n's taken in order.

For a more precise definition, we can view a sequence as a function a with domain $\mathbb{N}$. For a subset, there can be a natural function σ such that $\sigma(k) = n_k$ for $k \in \mathbb{N}$. The function σ selects an infinite subset of $\mathbb{N}$, in order. The subsequence of a is the composition of σ and a, i.e. $t = a \circ \sigma$. That is, $t_k = t(k) = a \circ \sigma = a(\sigma(k)) = a(n_k) = a_{n_k}$.

2.1.1 Examples

- We take the sequence $\{x_n\} = \frac{1}{n} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$
  (For $n_k = 2k$) subsequence is $\{x_{n_k}\} = \frac{1}{n_k} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, ...\}$
  (For $n_k = 2k+1$) subsequence is $\{x_{n_k}\} = \frac{1}{n_k} = \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, ...\}$
  Note: $= \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, \frac{1}{5}, ...\}$ cannot be a subsequence of $\{x_n\}$

- Let $\{s_n\}$ be a sequence defined by $s_n = n^2(-1)^n$ then the positive terms of a sequence define a subsequence.The sequence is

  $$(1, 4, 9, 16, 25, 36, 49, 64, ...)$$

  and the subsequence is

  $$(4, 16, 36, 64, 100, 144, ...)$$

  Thinking from the second definition, $s(k) = k^2(-1)^k$ and $\sigma(k) = 2 * k$ hence, $s_{n_k} = t(k) = s(\sigma(k)) = (2k)^2(-1)^{2k} = 4k^2$.

- Consider a sequence $\{a_n\}$ where, $a_n = \sin(\frac{n\pi}{3})$. Then the sequence a_n is:

  $$\{\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}, 0, -\tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2}\sqrt{3}, 0, ...\}$$

  And the desired sequence is:

  $$\{\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}, 0, 0, \tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}, 0, ...\}$$

  It is evident that $\sigma(k) = k + 2\lfloor \frac{k}{4} \rfloor$.

- Consider the following sequence $\{a_n\}$, sub-sequence 1 and sub-sequence 2 represent 2 different sub-sequences of $\{a_n\}$

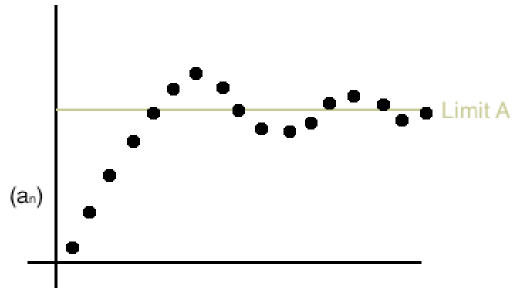

Figure 2: sequence $\{a_n\}$

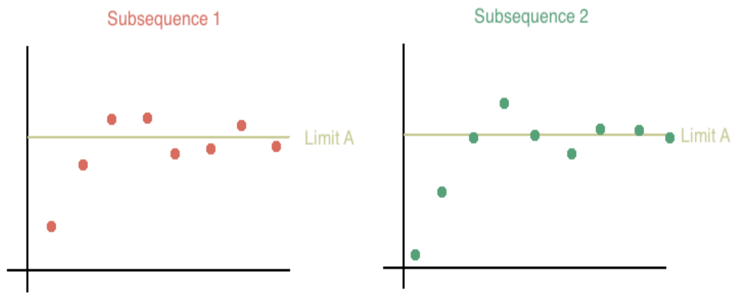

Figure 3: Sub sequences of $\{a_n\}$

2.2 Convergence of a subsequence

Theorem:
If a sequence of real numbers $\{a_n\}_{i=1}^{\infty}$, is convergent to L, then any subsequence of $\{a_n\}$ is also convergent to L.

Proof. Let $\{n_i\}_{i=1}^{\infty}$ be a sequence of positive integers such that $\{a_{n_i}\}_{i=1}^{\infty}$ forms a subsequence of $\{a_n\}$. Let $\epsilon > 0$ be given. As $\{a_n\}$ converges to L, there exists $N \in \mathbb{N}$ such that

$$\mid a_n - L \mid < \epsilon, \quad \forall\, n \geq N$$

Choose $M \in \mathbb{N}$ such that $n_i \geq N$ for $i \geq M$. Then

$$\mid a_{n_i} - L \mid < \epsilon, \quad \forall\, i \geq M$$

Note: It may happen that the signal sequence is not convergent, but the subsequence are convergent.For example, $\{a_n\} = \{1, -1, 1, -1, 1, -1, ...\}$ in not convergent and $\{a_{2n}\} = \{1, 1, 1, 1, ...\}$ which in convergent

3 Convergence of a sequence

3.1 Theorem

If a sequence converges then its limit point is unique.

$$\text{if } \{x_n\} \to x \text{ and } \{x_n\} \to y$$

$$\implies x = y$$

Proof. Prove by contradiction, if $x \neq y$. Taking $0 < \epsilon < \frac{|x-y|}{2}$

$$\exists\ n_1 \in \mathbb{N} \text{ s.t. } \forall\ n > n_1, n \in \mathbb{N}, \mid x_n - x \mid < \epsilon$$
$$\exists\ n_2 \in \mathbb{N} \text{ s.t. } \forall\ n > n_2, n \in \mathbb{N}, \mid x_n - y \mid < \epsilon$$

but these conditions can never be satisfied for $n > max(n_1, n_2)$.
Hence we reach a contradiction if $x \neq y$.

3.2 Properties

Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences and $\{x_n\} \to x$ and $\{x_n\} \to y$ Then the following sequences have the convergences:

- $\{x_n + y_n\} \to x + y$
- for $k \in \mathbb{R}$, $\{kx_n\} \to kx$
- $\{x_n y_n\} \to xy$
- $\{\frac{x_n}{y_n}\} \to \frac{x}{y}$ (if $y_n \neq 0$, $\forall$ n and $y \neq 0$)

4 Infinite Series

4.1 Summation Notation

The notation $\sum_{k=m}^{n}$ is shorthand notation for $a_m + a_{m+1}...a_n$, The symbol $\sum$ instructs to sum and the limits m, n tell the starting and the ending points of the summands. For example, $\sum_{k=2}^{4} \frac{1}{k^2+k}$ is the shorthand for:

$$\frac{1}{2^2+2} + \frac{1}{3^2+3} + \frac{1}{4^2+4}$$

and $\sum_{k=0}^{n} 2^{-k}$ is shorthand for

$$1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n}$$

4.2 Infinite Series

To assign meaning to $\sum_{n=m}^{\infty}$, we consider s_n to be called partial sums, where $s_n = \sum_{k=m}^{n}$, shorthand for:

$$s_n = a_m + a_{m+1} + ... + a_n$$

We say that $\sum_{n=m}^{\infty}$ converges if the partial sums converge to a real number S.

4.2.1 Examples

1. If $0 < r < 1$. then $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$.
Solution: Since it is sum of a GP, $\sum_{n=0}^{m} r^n = \frac{1-x^{n+1}}{1-r}$.
As, $\lim_{m\to\infty}(\sum_{n=0}^{m} r^n) = \sum_{n=0}^{m} r^n(\frac{1-r^{n+1}}{1-r}) = \frac{1}{1-r}$. Thus the given series converges to $\frac{1}{1-r}$.
This can be solved graphically as well as shown in the figure.

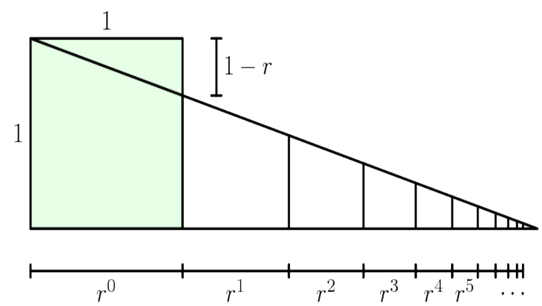

Figure 4: Solving a geometric series graphically

by using the properties of similar triangles, we can show that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

2. The series $\sum_{n=0}^{\infty} \frac{1}{x}$ diverges.

Solution: Consider the sequence of partial sums $\{s_n\}$, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Now, let us examine the subsequence s_{2^n} of $\{s_n\}$. Here,

$$s_2 = 1 + \frac{1}{2} = \frac{3}{2},$$
$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2.$$

Suppose $s_{2^n} > \frac{(n+2)}{2}$, then

$$\begin{aligned} s_{2^n} &= s_{2^n} + \textstyle\sum_{k=1}^{2^n} \frac{1}{2^n+k} \\ &> \frac{n+2}{2} + \textstyle\sum_{k=1}^{2^n} \frac{1}{2^{n+1}} \\ &= \frac{n+3}{2}. \end{aligned}$$

Since the sequence is unbounded, it is divergent.

4.3 Convergence of infinite series

4.3.1 Cauchy condensation test

Let $\{a_n\}_1^{\infty}$ be an decreasing sequence of positive numbers. Then $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Proof: Let s_n and t_n be the sequence of partial sums of $\sum a_n$ and $\sum 2^n a_{2^n}$ respectively. Then s_n and t_n are monotonically increasing sequences. We know that such sequences converge if they are bounded from above.
Proof follows from the observation that:

$$s_{2^n} = \sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + ... + (a_{2^{n-1}+1} + ... + a_{2^n})$$
$$\leq a_1 + a_2 + 2a_4 + 4a_8 + 8a_{16} + ...2^{n-1}a_{2^n} = a1 + 1/2t_n.$$

Therefore, if $\{s_n\}$ converges then $\{s_{2^n}\}$ converges and hence bounded from above.

On the other hand,

$$s_{2^n 1} = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + ...a_{15}) + (a_{2^{n-1}} + ... + a_{2^n 1})$$
$$\leq a_1 + 2a_2 + 4a_4 + 8a_8 + ... + 2^{n-1}a_{2^{n-1}} = a_1 + tn - 1$$

So if $\{t_n\}$ converges, then $\{s_{2^n 1}\}$ converges. Now the conclusion follows from $s_n \leq s_{2^{n+1}1}$ and the fact that $\{s_n\}$ is monotonically increasing sequence.

4.3.2 Absolute Convergence test

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. If $\sum_{n=1}^{\infty} |a_n|$ converges, we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ does not, then we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

4.3.3 Comparison test

Let $\sum a_n$ be a series of real numbers. Then, $\sum a_n$ converges absolutely if there is an absolutely convergent series $\sum c_n$ with $|a_n| \leq |c_n|$ for all $n \geq N, N \in \mathbb{N}$.

Proof As, we know that $|a_n| \leq |c_n|$, hence, $\sum |a_n| \leq \sum |c_n|$ and since $|a_n| \geq 0 \forall n \in \mathbb{N}$, hence $\sum |a_n|$ is an increasing sequence, and we know an increasing sequence, bounded above converges, hence if $\sum |c_n|$ converges, $\sum |c_n| \leq L$ then, $\sum |a_n| \leq L$, hence $\sum a_n$ converges.

Theorem:
If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

4.3.4 Limit Comparision test

Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers, then
1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, then both $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then if $\sum b_n$ converges, $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then if $\sum b_n$ diverges, $\sum a_n$ diverges.

4.3.5 Examples

1. The series, $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.
Solution: From Cauchy Condensation Test,

$$\sum_{n=2}^{\infty} \frac{2^n}{2^n \log 2^n} = \frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n}$$

Which diverges.

2. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.
Solution: The sequence, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which as we know converges.

3. Let $\{a_n\}$ be a series of real numbers and let $p_n = max\{0, a_n\}$, then if $\sum a_n$ converges absolutely then $\sum p_n$ converges.
Solution: We can write,

$$p_n = \frac{a_n + |a_n|}{2}, \text{ Thus, } p_n = \frac{a_n}{2} + \frac{|a_n|}{2} \leq |a_n|$$

Hence if $\sum a_n$ converges absolutely, then $\sum p_n$ converges.

4. The series $\sum \frac{1}{n!}$ converges.
Solution: For, $n \geq 4, n! \geq 2^n$, hence, $\sum \frac{1}{n!} \leq \sum \frac{1}{2^n}$, which as we know converges, hence $\sum \frac{1}{n!}$ also converges.

5. The series, $\sum e^{-n}/n^2$ converges.
Solution: let us take $a_n = e^{-n}/n^2$ and $b_n = 1/n^2$, now, $\lim_{n\to\infty} \frac{a_n}{b_n} = e^{-n} = 0$, now as we know $\sum b_n$ converges, hence by comparison test, $\sum a_n$ also converges.

4.4 Another interpretation of $\sum_{n=1}^{\infty} 1/n^2$

As we all know that the series $\sum_{n=1}^{\infty} 1/n^2$ converges, but where does it converge to? There are many arithmetic ways to answer this question, but we shall explore a geometric way. From physics, we know that if we are standing at a distance r from a light source, then the intensity of light is proportional to $1/r^2$. Now, Let us consider an infinite row of light sources like the one shown in the figure 5 below.

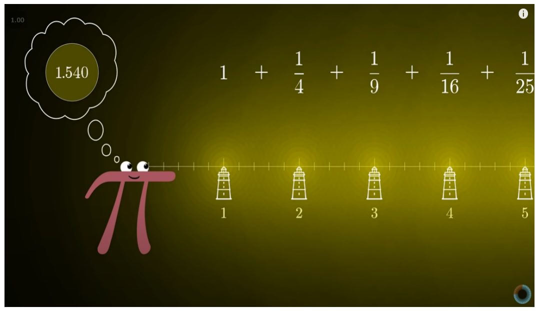

Figure 5: Problem Interpretation

Considering the intensity of each light to be 1 at r=1, if we sum the intensities of all these light sources at the origin, we will get $\sum_{n=1}^{\infty} 1/n^2$. We also need to note that for a triangle like shown in the figure 6,we have a property, that $1/h^2 = 1/a^2 + 1/b^2$. This is called the inverse Pythagorean theorem. Using this, we can manipulate the positions of the light sources to arrive at our answer.

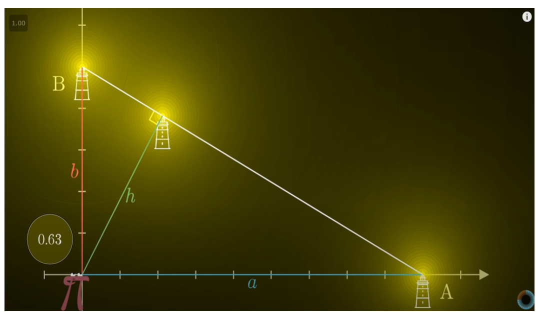

Figure 6: Inverse Pythagorean Theorem

Now, let us consider a light source at the top of circle of circumference 2 as shown in the Figure 7. The intensity observed by observer at the bottom would be $\pi^2/4$.

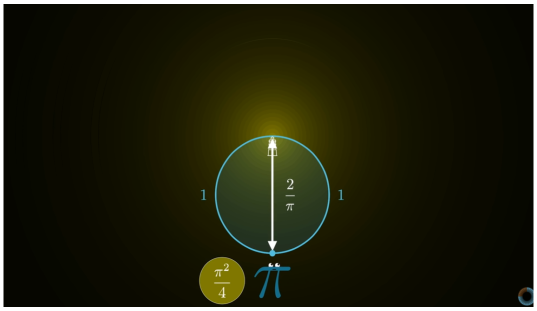

Figure 7: Light Source Depiction

Let us draw a circle of circumference 4 as shown in the Figure 8.

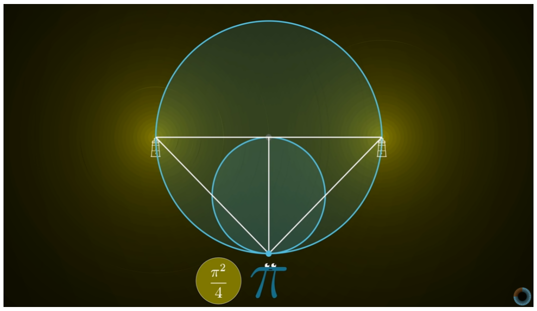

Figure 8: Extending the circle

We can decompose this light source into two sources using the inverse Pythagorean theorem by making a tangent to the inner circle and placing the light sources where the tangent touches the circle.
Now, we repeat the process, making a bigger circle of circumference 8 and for each of the 2 light sources, we draw a line from the top of the circle of circumference 4 to the first source and extend it to touch the outer circle as shown in the Figure 9.

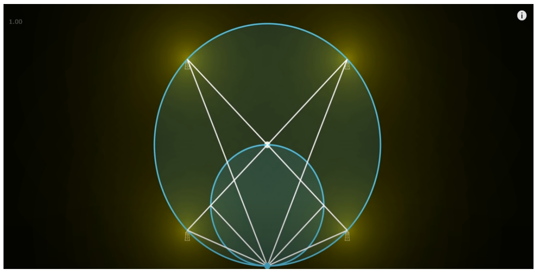

Figure 9: Placing sources

Since it makes an angle of 90° at the observer's eyes, we can again apply inverse Pythagorean

theorem and place the light sources at the ends of the line segment touching the outer circle. Repeating this for the second source, we get 4 evenly spaced sources at a distance of 2.
Now, if we repeat this process infinitely, we will get a circle with infinite radius which is equivalent to a straight line as show in the Figure 10.

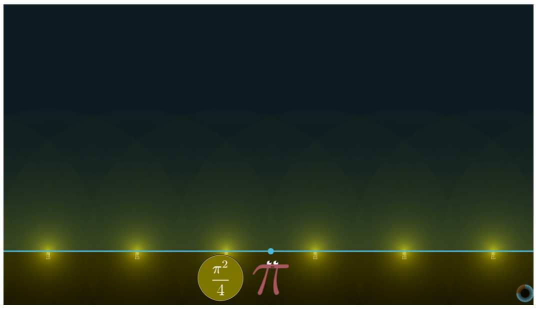

Figure 10: Limiting case

Observe that the sum of intensities still remains same, i.e. $\pi^2/4$.
We know that,

$$\ldots + \frac{1}{-3}^2 + \frac{1}{-2}^2 + \frac{1}{-1}^2 + \frac{1}{1}^2 + \frac{1}{2}^2 + \frac{1}{3}^2 + \ldots. = \pi^2/4$$
$$\implies \frac{1}{1}^2 + \frac{1}{2}^2 + \frac{1}{3}^2 + \ldots. = \pi^2/8.$$

To get the total sum of inverse squares, we need to consider the sum of even squares too. For that, we can consider the even numbers to be at a distance of twice the natural numbers, i.e. their intensity becomes a fourth of that of the required value, Let x be the answer required, then,

$$x = \frac{x}{4} + \frac{\pi^2}{8}$$
$$\implies x = \frac{\pi^2}{8} * \frac{4}{3}$$
$$\implies x = \frac{\pi^2}{6}$$

This proves the fact that $\sum_{n=1}^{\infty} = \frac{\pi^2}{6}$

5 Bounded Sequence

5.1 Definition

A sequence $\{x_n\}$ is bounded if $\exists$ M, s.t. $x_n \leq M \ \forall \ n \in \mathbb{N}$

5.2 Theorem

Every convergent sequence is bounded.
Proof As the sequence $\{x_n\}$ is convergent.
For $\epsilon > 0$
$\exists \ n_0 \in \mathbb{N}$ s.t. $\forall \ n > n_0, n \in \mathbb{N}$,

$| x_n - x |< \epsilon$
hence, $x_n < \epsilon + x \ \forall \ n > n_0, n \in \mathbb{N}$
Taking $\alpha = max(x_0, x_1, ..., x_{n_0-1}, x + \epsilon)$
$x_n \leq \alpha \ \forall n, \ n \in \mathbb{N}$
Hence the sequence in bounded.

6 Monotonic Sequence

6.1 Definition

A sequence x_n is monotonically increasing if $x_i \leq x_{i+1} \ \forall \ i \geq 1$ and monotonically decreasing if $x_i \geq x_{i+1} \ \forall \ i \geq 1$.

6.1.1 Examples

- The sequences $\{1 - 1/n\}$, $\{n^3\}$ are non-decreasing sequences.
- The sequences $\{1/n\}, \{1/n^2\}$ are non-increasing sequences.
- The sequences $\{(-1)^n\}$, $\{cos(n\pi)\}$, $\{(-1)^n n\}$, $\{(-1)^n/n\}$ and $\{n^{1/n}\}$ are not monotonic sequences.

Completeness axiom
If a non empty set X is bounded above, then it has a supremum. Similarly, if a non empty set X is bounded below, then it has an infimum.

6.2 Theorem

If a sequence $\{x_n\}$ is monotonically increasing and bounded above, then it is convergent. Similarly, if a sequence $\{x_n\}$ is monotonically decreasing and bounded below, then also it converges.

Proof: We shall prove it for monotonically increasing sequence only. The proof for decreasing sequences is analogous to it.
Since the sequence is bounded, $\sup \{x_n\}$ exists. Let $s = \sup \{x_n\}$. Since s is the supremum, $s - \epsilon$ is not an upper-bound for sequence $\{x_n\}$ and hence there exists N such that $s - \epsilon < x_N$.
Since the sequence is monotonically increasing, $x_N <= x_n \ \forall n > N$.
$\implies \ s - \epsilon \leq x_N \leq x_n \leq s \leq s + \epsilon \ \forall n > N$
$\implies \ |x_n - s| < \epsilon \ \forall \ n \geq N$
Hence the sequence is convergent.

6.2.1 Example

The sequence $\{(1 + 1/n)^n\} \, 1^\infty$ is convergent.

Solution.

Let $a_n = (1+1/n)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\frac{1}{n}\right)^k$. For $k = 1, 2, \ldots, n$, the $(k+1)^{th}$ term in the expansion is

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdots k}\frac{1}{n^k} = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

Similarly, if we expand a_{n+1}, then we obtain $(n+2)$ terms in the expansion and for $k = 1, 2, 3, \ldots$, the $(k+1)^{\text{th}}$ term is

$$\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right) < \frac{1}{k!}$$

It is clear that (1.2) is greater than or equal to (1.1) and hence $a_n \leq a_{n+1}$ which implies that $\{a_n\}$ is non decreasing. Further,

$$a_n = (1+1/n)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\frac{1}{n}\right)^k < 1 + \sum_{k=1}^{n} \frac{1}{k!} < 1 + 2 = 3.$$

$\left(k! > 2^{k-1} \implies \sum_{k=1}^{n} \frac{1}{k!} < \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 2\right)$ for each n. Thus $\{a_n\}$ is a bounded monotone sequence and hence convergent.

6.3 Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent sub-sequence.
Proof:
We first prove the following lemma.
Lemma: Every sequence has a monotone sub-sequence.
Proof: Considerx_n to be an arbitrary real sequence. Pick x_{N_1} such that $x_n \leq x_{N_1}$ for all $n > N_1$. We call such x_N as "peak". If we are able to pick infinitely many x_{N_i}'s, then x_{N_i} is decreasing and we are done. If there are only finitely many x_N's and let x_{n_1} be the last peak. Then for $n_2 \geq n_1$, x_{n_2} is not a peak. That means we can choose n_3 such that $x_{n_3} \geq x_{n_2}$. Again x_{n_3} is not a peak. So we can choose x_{n_4} such that $x_{n_4} \geq x_{n_3}$. Proceeding this way, we get a non-decreasing sub-sequence x_{n_2}, x_{n_3}, x_{n_4}, ...
Now, using Theorem 6.2, we can deduce that every bounded sequence has a convergent sub-sequence.

6.3.1 Example

It can be shown that the set $\mathbb{Q}$ of rational numbers can be listed as a sequence $\{r_n\}$, as $\mathbb{Q}$ is countable. This sequence has an amazing property: given any real number a there exists a sub-sequence r_{n_k} of r_n that converges to a. Since there are infinitely many rational numbers in every interval $(a-\epsilon, a+\epsilon)$ by the property of denseness of real numbers,we can find a sub-sequence of r_n converges to a.

7 Cauchy Sequence

A sequence $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence if $\forall \epsilon > 0$, there exists $N > 0$ such that

$$n, m > N \implies | x_n - x_m | < \epsilon$$

7.0.1 Example

$x_n = \frac{1}{2^n}$ is a cauchy sequence.
Try proving yourself!

7.0.2 Example

Let $\{a_n\}$ be defined as $a_1 = 1, a_{n+1} = 1 + \frac{1}{a_n}$. The show that $\{a_n\}$ is Cauchy.

Solution.
Note that $a_n > 1$ and $a_n a_{n-1} = a_{n-1} + 1 > 2$. Then

$$|a_{n+1} - a_n| = \left| \frac{a_{n-1} - a_n}{a_n a_{n-1}} \right| \leq \frac{1}{2} |a_n - a_{n-1}| \leq \frac{1}{2^{n-1}} |a_2 - a_1|, \forall n \geq 2.$$

Hence

$|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \ldots. + |a_{n+1} - a_n| \leq |a_2 - a_1| \frac{\alpha^{n-1}}{1-\alpha}, \alpha = \frac{1}{2}$
So given, $\epsilon > 0$, we can choose N such that $\frac{1}{2^{N-1}} < \frac{\epsilon}{2}$.
Indeed the following holds.

7.1 Theorem:

Any convergent sequence is a Cauchy Sequence but not every Cauchy sequence is convergent.

Proof: Since the sequence is Cauchy, $\exists N\ \forall \epsilon$ such that $|x_n - x_m| < \epsilon\ \forall n, m > N$. Since the sequence is convergent, $\lim x_n = x$.
$\implies |x_n - x| < \epsilon/2$, $\forall n > N$
$\implies |x_m - x| < \epsilon/2$, $\forall m > N$
Using triangle inequality,
$\implies |x_n - x_m| \leq |x_n - x| + |x - x_m|$
$\implies |x_n - x_m| \leq \epsilon/2 + \epsilon/2$
$\implies |x_n - x_m| \leq \epsilon$
Hence the sequence is Cauchy.

7.1.1 Important Note

In a general metric space, the converse of the above theorem is not true. For example, consider the sequence $a_n = \{ (1 + \frac{1}{n})^n\}$ in $\mathbb{Q}$ is not Cauchy. However, any Cauchy sequence in $\mathbb{R}$ is always convergent. For example, the same sequence a_n is convergent in $\mathbb{R}$ to the limit point e.

8 Limit Supremum and Limit Infimum

Let $a_n$be a bounded sequence. Then the limit superior of the sequence denoted by $\limsup a_n$, is defined as

$$\limsup_{n\to\infty} a_n := \inf_{k\in\mathbb{N}} \sup_{n\to\infty} a_n$$

Similarly, limit inferior of the sequence a_n, denoted by $\liminf a_n$, is defined as

$$\liminf_{n\to\infty} a_n := \sup_{k\in\mathbb{N}} \inf_{n\to\infty} a_n$$

Thus limit supremum is the infimum of the set of all supremums of the sequences starting from $x_i \ \forall \ i \geq 1$. Similarly, limit infimum is the supremum of the set of all infimums of the sequences starting from $x_i \ \forall \ i \geq 1$.

Corollary

Let α =$\limsup \{x_n\}$, and β =$\liminf \{x_n\}$, α_k =$\limsup \{x_n\}$ $n > k$ and β_k =$\liminf \{x_n\}$ $n > k$. Then, $\alpha \leq \alpha_k \leq \beta_k \leq \beta$.

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