ELL780 ELL 457 Indian Institute of Technology Delhi

Lecture 11 Open Sets and Closed Sets

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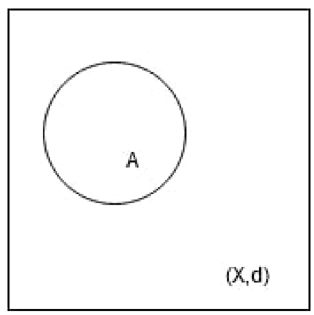
Contents

1 Introduction

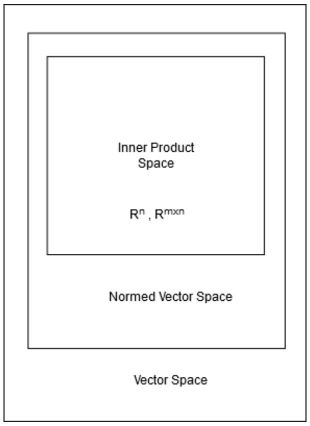
In the previous lecture we learnt about the conditions metric space, normed space and inner product space satisfy. We studied the behavior of different types of norms like $L_1, L_2....L_\infty$ and also varios types of inner products. In these lecture scribes we will explore different elements of point set topology. This will help us understand and get a feeling of how sets are represented in a higher dimensional space as compared to a real line. We are going to discuss the following topics: Open Ball; Closed Ball; Interior Point; Relative Interior; Exterior Point; Boundary Points; Closure Points; Limit Points; Open Sets; Closed Sets; We need to understand these fundamental point sets before beginning study of functions.

2 Recap

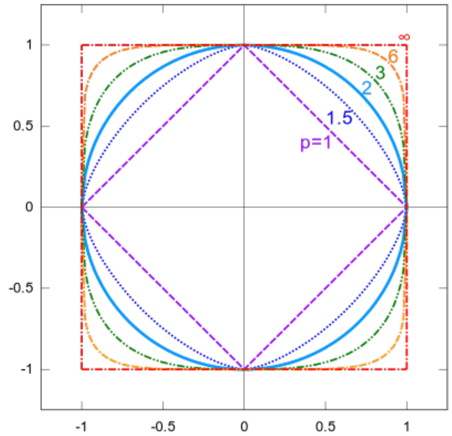
In these lecture scribes our discussion will revolve around a metric space (X, d) where X is a vector space and d is a distance metric on X. Further we will focus on a subset of this metric space A such that $A \subseteq X$ and A is non empty. Pictorially we can represent it as shown below:-



The more general type of space we talk about is vector space. If we a able to define a metric on this vector space it is called a metric space. Further if we are able to define a norm on a vector space then is called a normed vector space. Normed vector space is also a metric space as norm induces a metric. If we go even further and we are able to define an inner product on a vector space then it is called a inner product space. Inner product space is also a normed space as inner product induces a norm. If we take a set of all vector spaces, say V, normed vector spaces, say N, and inner product spaces, say I. Then from the above we can clearly deduce that $I \subseteq N \subseteq \; V$ We can pictorially observe the following as shown below:-



Note: There does not exist any inner product that can lead to L_1 norm.
Let us consider a Ln norm in the vector space $\mathbb{R}^2$. Then for $x \in \mathbb{R}^2$ the plot for $||x||_n = 1$ is as follows:-



The region between L_2 and L_∞ is covered by all n such that $n \in (2, \infty)$ and the region between L_1 and L_2 norm is covered by n such that $n \in (1, 2)$. All norms such that $n \in [1, \infty)$ are convex. For $n < 1$ the norm is not convex.

3 Open Ball

Definition 3.1 (Open Ball) *For a metric space (X, d), Let a be a given point in metric space (X,d) and let r be a positive number. The set of all points x in metric space(X,d) such that*

$$U(a, r) = U_r(a) = \{\ x \in X\ ;\ d(a, x) < r\ \}$$

Open Ball is characterized by two things: Center(a) and Radius(r). The ball U(a,r) consists of all points whose distance from a is less than r. On a real line it is an open interval with centre at a. On a real plane it is a circular disk. In real space it is a spherical solid with centre at a and radius r.

There are balls of three different shapes:

Example 3.1 *Single-point balls:* $U[(0,0), r)] = (0,0), \forall\ r \in (0,1]$

Example 3.2 *Balls that are same as in Euclidean Metric:* $U[(0,0), 2] = \{\ x \in \mathbb{R}^2\ ;\ x_1^2 + x_2^2 < 4\ \}$

Example 3.3 *Balls that are punctured at origin :* $U[(1/2, 1/2), 1] = \{\ x \in \mathbb{R}^2\ ;\ (x_1 - 1/2)^2 + (x_2 - 1/2)^2 < 1\ \}$

4 Closed Ball

Definition 4.1 *For a metric space (X, d), Let a be a given point in metric space (X,d) and let r be a positive number. The set of all points x in metric Space(X,d) such that*

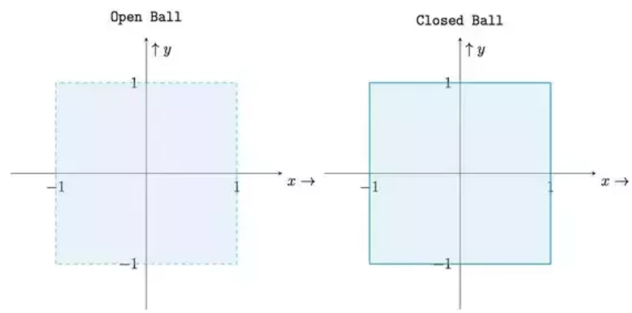
$$B(a, r) = B_r(a) = \{\ x \in X\ ;\ d(a, x) \leq r\ \}$$

Closed Ball is characterized by two things: Center(a) and Radius(r). The ball B(a,r) consists of all points whose distance from a is less than or equal to r. On a real line it is an closed interval with centre at a. On a real plane it is a circular disk. In real space it is a spherical solid with centre at a and radius r.

Example 4.1 *For the set $(x, y) \in \mathbb{R}^2\ : x^2 + y^2 \leq 1$ is a closed set. So the closed ball of radius r centered at a' is the set defined as:*

$$B[a', r] = \{\ x \in X :\ d(a', x) \leq r\ \}$$

Example 4.2 *If the metric is $d((x1, y1), (x2, y2)) = max(|x1 - x2|, |y1 - y2|)$. Then visual representation of open and closed ball in form of a plot is given below:*



5 Sphere

Definition 5.1 *For a metric space (X, d), Let a be a given point in metric space (X,d) and let r be a positive number. The set of all points x in metric space(X,d) such that*

$$S(a, r) = \{ \ x \in X \ ; \ d(a, x) = r \ \}$$

Sphere is characterized by two things: Center(a) and Radius(r). The sphere S(a,r) consists of all points whose distance from a is equal to r. On a real line it is two points with centre at a. On a real plane it is a circular ring. In real space it is a hollow sphere with centre at a and radius r.

Note: Open Ball and Closed Ball can never be empty but sphere can be empty.

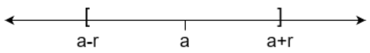
Example 5.1 *Let (X, d) be a discrete metric space Discrete metric is defined as*

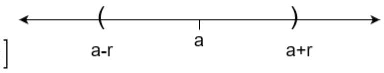
$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & otherwise \end{cases} \tag{1}$$

Now we observe the following different values of radius for for open ball, closed ball and sphere:- Let a be a point such that $a \in X$ then

$$U(a, 1) = \{a\} \ (As\ there\ is\ only\ one\ point\ such\ that\ d(a, x) < 1\ such\ that\ x \in \ X\ that\ is\ a\ itself)$$
$$U(a, 1/2) = \{a\} \ (Similarly\ we\ can\ observe\ for\ d(x, y) < 1/2)$$
$$U(a, 3) = X \ (We\ can\ clearly\ see\ that\ d(x, y) < 3\ for\ all\ x \in X)$$
$$B(a, 1) = X \ (We\ can\ clearly\ see\ that\ d(x, y) \leq 1\ for\ all\ x \in X\ by\ definition\ of\ the\ metric)$$
$$B(a, 1/2) = \{a\}$$
$$S(a, 1) = X - \{a\}(As\ all\ points\ except\ a\ itself\ is\ at\ distance\ of\ 1\ from\ a)$$
$$S(a, 1/2) = \phi(As\ no\ point\ exists\ such\ that\ d(a, x) = 1/2\ where\ x)$$

Thus we can see in the above case that a sphere can be an empty set but open ball and closed ball are never empty.

[Fig 1(a)] 

[Fig 1(b)] 

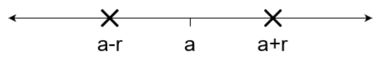


Fig 1(c)]

Figure 1: Figure 1(a) represents the set U(a,r) for example 5.2.Figure 1(b) represents the set B(a,r) for example 5.2. Figure 1(c) represents the set S(a,r) for example 5.2

Example 5.2 *For a metric space* $(\mathbb{R}, d)$ *where* $d(x, y) = |y - x|$; $a \in \mathbb{R}$ *and* $r > 0$

$$U(a, r) = \{\ x \in \mathbb{R}\ ;\ |x - a| < r\ \} => x \in (a - r, a + r)$$
$$B(a, r) = \{\ x \in \mathbb{R}\ ;\ |x - a| \leq r\ \} => x \in [a - r, a + r]$$
$$S(a, r) = \{\ x \in \mathbb{R}\ ; a - r, a + r\}$$

Now we have to generalize this concept of intervals to a higher dimension.
For visualising this, we take the following example:
For a Normed space also knows as Euclidean Space $(\mathbb{R}^2, ||.||_2)$
Let a=(0,0) and r=1

$$U(a, 1) = \{\ x \in \mathbb{R}^2\ ;\ (x_1^2 + x_2^2)^{1/2} < 1\ \}$$

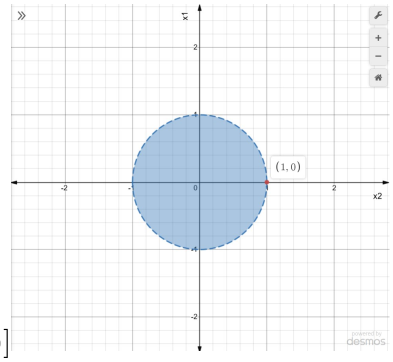
All points in the circle belong to the Set but all points on the circle do not belong to the Set.

$$B(a, 1) = \{\ x \in \mathbb{R}^2\ ;\ (x_1^2 + x_2^2)^{1/2} \leq 1\ \}$$

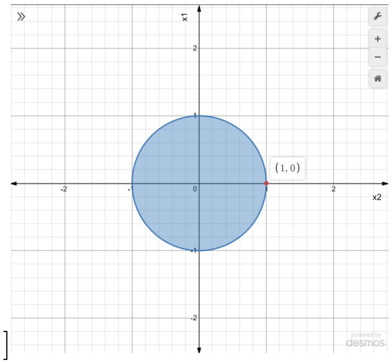
All points in and on the circle belong to the Set

$$S(a, 1) = \{\ x \in \mathbb{R}^2\ ; (x_1^2 + x_2^2)^{1/2} = 1\}$$

All points on the circle belong to the Set



[Fig 2(a)]



[Fig 2(b)]

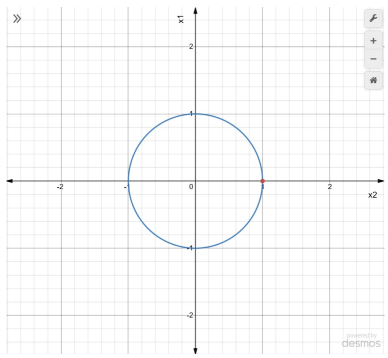


Fig 2(c)]

Figure 2: Figure 2(a) represents the set U(a,1) for example 5.2.Figure 2(b) represents the set B(a,1) for example 5.2. Figure 2(c) represents the set S(a,1) for example 5.2

6 Interior Point

Definition 6.1 *Let A be a subset of X, Let $a \in A$.Then a is called an interior point of A if there is an open ball with center at a, all of whose points belong to A.*

Mathematically it is defined as:

$$int(A) = \{a \mid \exists\, r > 0 \text{ such that } U(a, r) \subseteq A\}$$

We can say that every interior point a of A can be surrounded by an ball $B(a) \subseteq A$. The set of all interior points of A is called the interior of A and is denoted by $int(A)$. Any set containing a ball with center a is sometimes called a neighborhood of a.

6.1 Properties of Interior set

- int(A) is an open set in X
- int(int(A))=int(A)
- A is an open subset of X if and only if A = int(A)

- Let S be an open subset of X then $S \subseteq A$ if and only if $S \subseteq int(A)$

Open set is defined in section 10 to understand the above properties of open sets read the said section.

Example 6.1 *Let* $A = [0, 1] => int(A) = (0, 1)$ *as for all* $x \in A$ *except* $\{0, 1\}$ *we can make an open ball such that* $U(x, r) \subseteq A$

Example 6.2 $A = [0, 1) => int(A) = (0, 1)$ *here also we can observe for* $x = 0$ *no* $r > 0$ *exists such that an open ball centered at 0 satisfies* $U(0, r) \subseteq A$

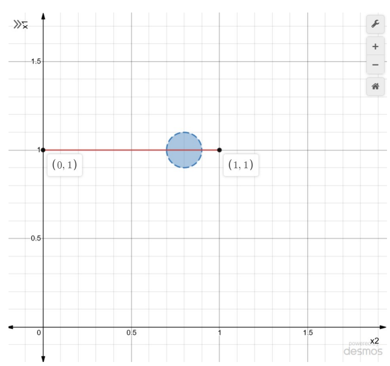
Example 6.3 *Let us consider* $\mathbb{N} => int(\mathbb{N}) = \phi$ *Since* $\mathbb{N}$ *is a discrete set hence for any* r *we can observe that* $U(x, r) \not\subseteq \mathbb{N}$

7 Relative Interior

Definition 7.1 *Let* A *be a subset of* X. *We can say that* x *is a relative interior point of* A *if* $x \in B(x, \epsilon) \bigcap aff(A) \subset A, for\ some\ \epsilon > 0$. *The set of all relative interior point of* A *is called the relative interior of* A *and is denoted by* $r_i(A)$

In optimization problems, we generally encounter sets like L which is defined below. in sets like these we can clearly see $int(L) = \phi$.

$$L = \{X \in \mathbb{R} \mid x_1 = 1\ ; 0 \leq x_2 \leq 1\}$$



We quite often encounter sets like these in today's machine learning problems
For this reason we define the concept of relative interior point

$$r_i(A) = \{a \in A \mid \exists\ r > 0\ such\ that\ U(a, r) \bigcap affine(A) \subseteq A\}$$

$$where\ affine(A)\ is\ the\ affine\ hull(A)\ which\ is\ defined\ as\ follows:$$

$$aff(X) = \{\sum_{i=1}^{k} \theta_i x_i \mid K > 0,\ x_i \in X;\ \theta_i \in \mathbb{R}; \sum_{i=1}^{k} \theta_i = 1\}$$

For the same set L the set of relative interior points we obtain are $r_i(L) = (0, 1)$

Property 1. Line Segment Property: Let A be a non-empty convex set.

$$If\ x \in r_i(A), \overline{x} \in cl(A),\ then\ \alpha x + (1-\alpha)\overline{x} \in r_i(A),\ for\ \alpha \in (0, 1]$$

Property 2. Prolongation Lemma : Let A be a non-empty convex set.Then we have

$$x \in r_i(A) \Leftrightarrow \forall\ \overline{x} \in A, \exists\gamma > 0\ such\ that\ x + \gamma(x - \overline{x}) \in A$$

We can also say that, x is a relative interior point iff every line segment in A having x as one of the endpoints can be prolonged beyond x without leaving A.

8 Exterior Point

Definition 8.1 *Let A be a subset of X, Let $a \in A$.Then a is called an exterior point of A if there is an open ball with center at a, all of whose points belong to $X - A$.*

Mathematically it is defined as:

$$Ext(A) = \{a \mid \exists\ r > 0\ such\ that\ U(a, r) \subseteq X - A\}$$
$$Ext(A) = int(X - A)$$

We can say that every exterior point a of A can be surrounded by a ball $B(a) \subseteq X - A$. The set of all exterior points of A is called the exterior of A and is denoted by $ext(A)$. Any set containing a ball with center a is sometimes called a neighborhood of a.

8.1 Properties of Exterior set

- ext(A) is open subset of X that is disjoint from A
- The union of all open subsets of X that are disjoint from A is equal to ext(A)
- The largest open subset of X that is disjoint from A is equal to ext(A)
- $int(A) \subseteq ext(ext(A))$

Example 8.1 *Let us consider the set $A = [2, 8) \in \mathbb{R}$ the exterior set of A is ext(A) = $(-\infty, 2) \bigcup [8, \infty)$*

9 Boundary Points

Definition 9.1 *Let A be a subset of X. A point x in X is called a boundary point of A if every ball $B_M(x; r)$ contains at least one point of A and at least one point of $X - A$. The set of all boundary points of A is called the boundary of A and is denoted by ∂A.*

Mathematically defined as:

$$Boundary(A) = \{a \mid \exists\, r > 0 \text{ such that } U(a,r) \bigcap (A) \neq \phi \text{ and } U(a,r) \bigcap (X - A) \neq \phi\}$$

9.1 Properties of Boundary sets

- Boundary of a set is always a closed set
- The interior of the boundary of a closed set is an empty set
- $Boundary(A) = Boundary(A^C)$ => Boundary of a set is equal to the boundary of the compliment of the set
- The boundary of the interior of a set as well as boundary of closure of a set are both contained in the boundary of the set

Example 9.1 *Let us consider the set $A = [0, 1] \in \mathbb{R}$ then the interior, exterior and boundary points are as follows:-*
Int(A) = (0,1) for all these points there exists an open ball s.t. $r > 0$ that belongs to A
$Ext(A) = (-\infty, 0) \bigcup (1, \infty)$ for all these points there exists an open ball s.t. $r > 0$ that totally lies outside A
Boundary(A) = {0,1} as no matter how arbitrarily small we take r for 0 or 1 a part of any open ball s.t. $r > 0$ lies inside A.

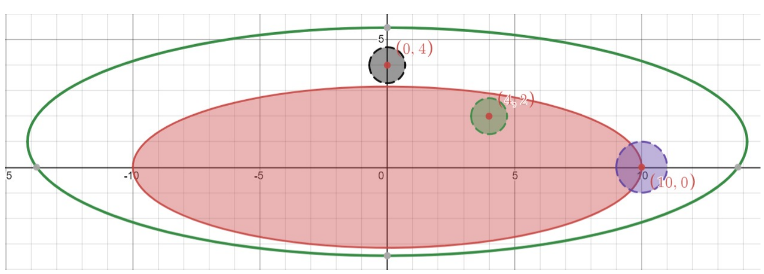


Figure 3: Let us consider the above where A is the set of points in the red ellipse and X is the set of points in the green ellipse we can clearly observe the following about the points in the figure:- $(4, 2) \in int(A), (0, 4) \in ext(A)$ and $(10, 0) \in Boundary(A)$

10 Open Sets

Let A be a subset of X. Then the set A is open if every point in A has a neighborhood lying in the set. An open set of radius r and center a is the set of all points x such that $|x-a|<r$, and is denoted $D_r(a)$. In one-space, the open set is an open interval. In two-space, the open set is a disk. In three-space, the open set is a ball. Mathematically defined as:

$$\forall\ a \in A; a \in Int(A)$$

In words, a set A is open if and only if $A = int(A)$

Example 10.1 *(0,1] is not an open set as* $\{1\} \in A$ *but* $\{1\}$ *is not an interior point of A*

Example 10.2 *(a,b) is an open set as all points of A belong to int(A)*

Theorem 10.3 *Let* (X,d) *be a metric space*

1. *If* $\{G_\alpha; \alpha \in \wedge\}$ *is a family of open sets in X then* $\bigcup_{\alpha \in\ \wedge} G_\alpha$ *is open*
2. *If* $G_1, G_2, G_3, ...G_n$ *are open sets, then* $\bigcap_{i=1,2,...n} G_i$ *is open (Finite Intersection)*

Proof:

1. Let $\{G_\alpha; \alpha \in \wedge\}$ be a collection of open sets and let S denote their union, $S = \bigcup_{\alpha \in\ \wedge} G_\alpha$. Assume $x \in S$.Then x must belong to at least one of the sets in G_α, say $x \in G_\alpha$. Since G_α is open, there exists an open ball $U(x,r) \subseteq S$. But $G_\alpha \subseteq S$, so $U(x,r) \subseteq S$ and hence x is an interior point of S. Since every point of S is an interior point, S is open.
   Mathematically shown as:
   $$G_\alpha\ \exists\ r > 0\ such\ that\ U(a,r) \subseteq G_\alpha => U(a,r) \subseteq G = \bigcup_\alpha G_\alpha$$
2. Let $S = \bigcap_{i=1,2,...n} G_i$ where each G_α is open. Assume $x \in S$(If S is empty then we don't need to prove it). Then $x \in G_\alpha$ for every $k = 1,2,...m$, and hence there is an open ball $U(x, r_k) \subseteq G_\alpha$. Let r be the smallest of the positive numbers $r_1, r_2, ...r_m$. Then $x \in U(x,r) \subseteq S$.That is x is an interior point, so S is open.
   We can also show it mathematically for easier understanding. Below is the mathematical explanation:
   $G_1, G_2, ...G_n$ are given open sets
   To prove: $\bigcap_{j=1,2,...n} G_j$ is open.
   Let $a \in \bigcap_{j=1}^{n}$G=>a $\in G_j$ $\forall$ j=1,2,...n(all G_j are open)
   $\forall$ j $\exists$ $(r_j > 0)$ such that $U(a, r_j) \subseteq G_j$
   Let $r = min(r_1, r_2, r_3, ...r_j, r_n)$; $[r_j > 0\ \forall\ 1 \leq j \leq n]$
   $r = min(r_1, r_2, ...r_n) > 0$; $[r > 0]$
   Take $U(a,r);\ Since\ r < r_j\ \forall\ j = 1, 2....n$
   $U(a,r) \subseteq U(a, r_j) \subseteq G_j\ \forall\ j = 1, 2...n$

$U(a,r) \subseteq \bigcap_{j=1}^{n} G_j$
Thus we have proved that intersection of a finite number of open sets is open.
We do not take the case of infinite intersection as it may lead to the condition below:
$min(r_1, r_2, ...r_n) = 0$
We can in the following case that a infinite intersection of open sets leads to a singleton which is a closed set. Hence we have a counter example
$G_n = (-1/n, 1/n)\ \forall\ n \in \mathbb{N}$ are open sets
$\bigcap_{n\in\mathbb{N}} G_n = \{0\}$

■

11 Closure Point

Definition 11.1 *Let A be a subset of X, Let $x \in A$.let r be a positive number. Then x is said to be closure point of A if every open ball with radius r and center at x,and has a non empty intersection with A where $A \subseteq X$ and $x \in X$. Mathematically defined as:*

$$\overline{A} = Cl(A)\ = \{x \mid \forall\ r > 0\ ;\ U(x,r) \bigcap\ A \neq \phi\}$$

The set of all closure points of A is called a closure of A ($\overline{A} = Cl(A)$)
$The\ closure\ of\ A\ can\ alternatively\ be\ defined\ as\ the\ intersection\ of\ all\ closed\ sets\ containing\ A$
$Every\ point\ of\ A\ is\ the\ closure\ point\ of\ A => A \subseteq \overline{A}$

Example 11.1 *Let us consider the set A=[2,3] then the closure set for A is* $\overline{A} = [2,3]$ *as an open ball* $for\ all\ elements\ of\ A\ such\ that\ r > 0\ of\ A\ have\ a\ not\ empty\ intersection\ with\ A$

Example 11.2 *Let us consider the set A=(1,5] then the closure set for A is* $\overline{A} = [1,5]$ *as an open ball* $for\ 2\ and\ all\ elements\ of\ A\ such\ that\ r > 0\ of\ A\ have\ a\ not\ empty\ intersection\ with\ A$

Theorem 11.3 *Let A be a subset of a metric space X. Then $X - \overline{A} = int(X - A)$ and $X - int(A) = \overline{X - A}$*

Proof: We begin by proving $X - \overline{A} = int(X - A)$ If $x \in X$ is not in $\overline{A}$, there must exist some $B_{1/2^n}(x)$ not meeting A, for otherwise we would have some $x_n \in B_{1/2^n}(X) \bigcap A$ for all n, so clearly $x_n \to x$, contrary to the fact that $x \notin \overline{A}$ is not a limit of a sequence of elements of A. This shows

$$X - \overline{A} \subseteq int(X - A)$$

Conversely, if x is in the interior of $X - A$ then some $B_r(x)$ lies in $X - A$ and hence is disjoint from A. It follows that no sequence in A can possibly converge to x because for $\varepsilon = r \geq 0$ we would run into problems(i.e.,there's nothing in A within a distance of less than ε from x, since $B_\varepsilon(x) \subseteq X - A$)
Applying the general equality

$$X - \overline{A} = int(X - A)$$

for arbitrary subsets A to X to the subset $X - A$ in the role of A, we get

$$X - \overline{X - A} = int(A)$$

Taking complements of both sides within X yields

$$\overline{X - A} = X - int(A)$$

$\blacksquare$

Theorem 11.4 *Let A be a subset of a metric space X. Then A is closed if and only if it contains ∂A, and in general*

$$\partial A = \overline{A} \bigcap \overline{X - A} = \partial(X - A)$$

Proof: The boundary ∂A is defined as $\overline{A} - int(A)$. Thus,

$$\overline{A} = int(A) \bigcup \partial A \subseteq A \bigcup \partial A$$

so when $\partial A \subseteq A$ we get $\overline{A} \subseteq A$ and therefore that $A = \overline{A}$, so A is closed. Conversely, if A is closed then since $\partial A \subseteq \overline{A}$ by definition and $\overline{A} = A$ for closed A we get $\partial A \subseteq A$

$$\partial A = \overline{A} \bigcap \overline{X - A}$$

Since the right side is unaffected by replacing A with $X - A$ everywhere(because $X - (X - A) = A$, it follows that $\partial A = \partial(X - A)$, As for verifying that ∂A is the intersection of the closures of A and $X - A$, we use the definition of ∂A to rewrite this as:

$$\overline{A} - int(A) = \overline{A} \bigcap \overline{X - A}$$

Since $\overline{A} - int(A) = \overline{A} \bigcap (X - int(A))$, it suffices to check that

$$X - int(A) = \overline{X - A}$$

$\blacksquare$

12 Limit Points

Definition 12.1 *Let A be a subset of X, Let $y \in X$. Let r be a positive number. Then x is said to be limit point of A if every open ball with radius r and center at y, has a non empty intersection with $[A - \{y\}]$ where $A \subseteq X$ and $y \in X$. Mathematically defined as :*

$$L(A) \ = \{y \mid \forall \ r > 0 \ ; \ U(y, r) \bigcap [A - \{y\}] \neq \phi\}$$

Example 12.1 *Let us consider A = (2,3) then L(A) = [2,3]. 2 and 3 are included as open balls around both for* $r > 0$ *have a non empty intersection with A*

Example 12.2 *Let us consider* $A = \{1/n : n \in \mathbb{N}\}$ *then we can see that L(A) =* $\{0\}$ *as 0 is the only element that has infinitely many point of* A *around it*

Theorem 12.3 *Let* $x \in \Re$ *and* $A \subseteq \Re$
1. If x *has a neighborhood which only contains finitely many members of* A *then* x *cannot be a limit point of* A*.*
2. If x *is a limit point of* A *then any neighborhood of* x *contains infinitely many members of* A*.*
3. No finite set can have a limit point.

Proof: Let U be a neighborhood of x which contains only a finite number of points of A that is $U \bigcap A$ is finite. Then, $U \bigcap A$ is also finite. Suppose $U \bigcap A / \{x\} = \{y_1, y_2, .., y_n\}$. We show there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \bigcap U$ does not contain any member of $A/\{x\}$. Since both $(x - \epsilon, x + \epsilon)$ and U are neighborhood of x, so is their intersection. This will prove there is a neighborhood of x containing no element of $A/\{x\}$ hence proving x is not a limit point of A. Let $\epsilon = min\{|x - y_1|, |x - y_2|, ..., |x - y_n|\}$. Since x is not equal to y_i, $\epsilon \geq 0$. Then $(x - \epsilon, x + \epsilon) \bigcap U$ contains no points of A other than x thus proving our claim. ■

Theorem 12.4 *Let* $A \subseteq \Re$

$$\overline{A} = A \bigcup L(A)$$

Proof:
1. $A \cup L(A) \subseteq \overline{A}$
We already know that $A \subseteq \overline{A}$. We now show that $L(A) \subseteq \overline{A}$. This will imply the result. Suppose that $x \in L(A)$. Then $\forall\ \epsilon > 0, (x - \epsilon, x + \epsilon) \cap A \neq \phi$, since $A/\{x\} \subseteq A$. Therefore, x is in the closure of A.
2. $\overline{A} \subseteq A \cup L(A)$
Let $x \in \overline{A}$. We need to prove that $x \in A \cup L(A)$. Either $x \in A$ or $x \notin A$. If $x \in A$ then $x \in A \cup L(A)$. If $x \notin A$ then x is close to A. Therefore, $\forall \epsilon > 0$, $(x - \epsilon, x + \epsilon) \cap A \neq \phi$. Since $x \notin A, A = A/\{x\}$, therefore $(x - \epsilon, x + \epsilon) \cap A \neq \phi$. It follows that $x \in L(A)$ and therefore $x \in A \cup L(A)$. So we see that in all cases, if we assume that $x \in \overline{A}$ then we must have $x \in A \cup L(A)$ ■

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