ELL780 ELL 457 Indian Institute of Technology Delhi
Lecture 12 Elements of Point Set Topology: Closed, Bounded and Compact Sets
Scribed by: *Pranjal Rai (2018EE10484)*
Instructors: Sandeep Kumar and Jayadeva

Contents

1 Introduction **2**

2 Closure and limit points **2**
2.1 Closure points . . . 2
2.2 Limit points . . . 3
2.3 Remarks on closure & limit points . . . 4

3 Closed set and it's properties **7**
3.1 Closed set . . . 7
3.2 Properties of closed sets . . . 10

4 Supremum and Infimum **11**
4.1 Supremum . . . 11
4.2 Infimum . . . 11

5 Bounded Set **12**

6 Compact Set **15**

7 Sequence in Metric Space **18**

1 Introduction

In the previous lecture the concepts like open ball, closed ball and interior of sets and openness of a set were defined and explored.
In the current lecture the concepts of closure/adherence points, limit/accumulation points will be discussed. Using the concepts of these points the closeness of a set will then be defined and characterized. Various properties of closed sets and relation between openness and closeness will also be discussed. Further, the ideas of supremum and infimum will be introduced for a metric space which will then be used to define the diameter of a set and hence boundedness. The discussion will also include Bolzano-Weistrauss theorem for the bounded sets. Another crucial concept which is a required to understand continuity is compactness. The definition and properties of compact sets will be discussed along with the Heine-Borel theorem. At last, sequences and convergence of sequences in a metric space will be introduced.

2 Closure and limit points

2.1 Closure points

Closure points are used to describe a closed set and are also known as Adherent points.

Definition 2.1 (Closure point) *Let A be a subset of X, and x a point in X, which may not necessarily be in A. Then x is said to be closure point of A if every open ball with positive radius and center at x, contains at least one point of A. Mathematically we can write the following :*

$$\forall\ r > 0, \quad \mathbb{U}(x, r) \bigcap A \neq \phi \tag{1}$$

Definition 2.2 (Closure of a set) *Let A be a subset of X. Then the set of all closure points of A is called the closure of A. The closure of any set, $A \subseteq X$ is denoted by $\overline{A}$ or $Cl(A)$. Mathematically we can write the following :*

$$\overline{A} = Cl(A) = \left\{ x \in X \;\middle|\; \mathbb{U}(x, r) \bigcap A \neq \phi, \quad \forall\ r > 0 \right\} \tag{2}$$

Let A be a subset of X and x be any point belonging to A. Then any open ball centered at x, will contain x and thus, will trivially have a non-null intersection with A. So, every point belonging to a set will always be it's closure point and further it will always belong to the closure of A ($\overline{A}$). We can also conclude that, a set will always be a subset of it's closure. Mathematically we can write and derive the above conclusion as follows :

$$\text{If } x \in A \implies \forall\ r > 0, \quad x \in \mathbb{U}(x, r) \implies \forall\ r > 0, \quad \mathbb{U}(x, r) \bigcap A \neq \phi \tag{3}$$

$$\text{So, If } x \in A \implies x \in \overline{A} \tag{4}$$

$$\implies A \subseteq \overline{A} \tag{5}$$

A figure illustrating points which satisfies, the above definition of Closure points is as follows:

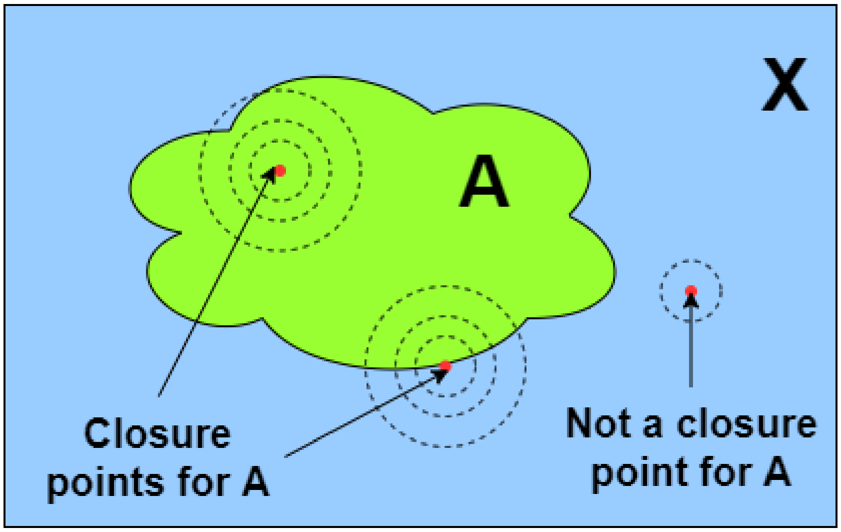


Figure 1: Closure and non-closure points for a set $A \subseteq X$

Some examples of closure point and closure are following:

Example 2.1 *For the set* $[0, 1]$, 0 *is a closure point and it's closure will be the set* $[0, 1]$.

Example 2.2 *For the set* $(0, 1)$, 0 *is a closure point and the closure will be the set* $[0, 1]$.

2.2 Limit points

Limit points are also useful in describing a closed set and are also known as Accumulation points.

Definition 2.3 (Limit point) *Let A be a subset of X, and y a point in X, which may not necessarily be in A. Then x is said to be limit point of A if every open ball with positive radius and center at y, contains at least one point of A distinct from y. Mathematically we can write :*

$$\forall\ r > 0, \quad \mathbb{U}(y, r) \bigcap [A - \{y\}] \neq \phi \tag{6}$$

Definition 2.4 (Derived set) *Let A be a subset of X. Then the set of all limit points of A is called the derived set of A. The derived set of any set, $A \subseteq X$ is denoted by A'. Mathematically we can write the following :*

$$A' = \left\{ y \in X \;\middle|\; U(y, r) \bigcap [A - \{y\}] \neq \phi, \quad \forall\ r > 0 \right\} \tag{7}$$

A figure illustrating points which satisfies, the above definition of Limit points is as follows:

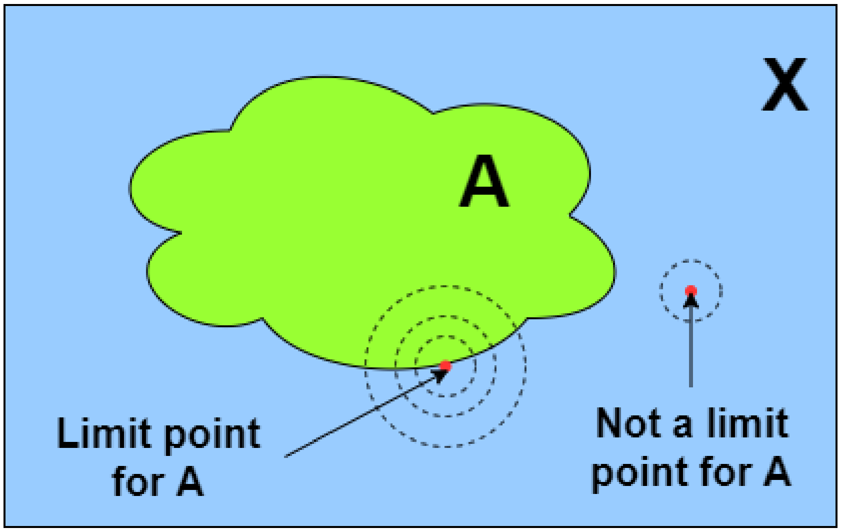


Figure 2: Limit and non-limit points for a set $A \subseteq X$

Some examples of limit point and the derived set are following:

Example 2.3 *For the set* $(0, 1)$*, 0 is a limit point and it's derived set will be* $[0, 1]$*.*

Example 2.4 *The set of numbers of the form 1/n, n = 1, 2, 3, ... , has 0 as a limit point and it's derived set will be a singleton set with* 0 *as it's element i.e,the set* $\{0\}$*.*

2.3 Remarks on closure & limit points

Consider a set $A \subseteq X$ and a point $x \in X$. Then we have the following remarks :

Remark 2.5 *If* $x \in A$ *and* $x \in \overline{A}$*, then it can be a closure point without being a limit point.*

It was shown in *Equation (4)*, that $x \in A \implies x \in \overline{A}$. And according to the above remark x may not be a limit point. For example, consider the set $B = \{1, 2, 3, 4, ..\}$ and the point $y = 1$.
Then clearly, $y \in B$ also $y \in \overline{B}$ but $y \notin B'$ because we can always have find an open ball centered at y, which does not share any element with B distinct from y. One of such open ball is $U(y, \frac{1}{2})$ which satisfies $U(y, \frac{1}{2}) \bigcap [B - y] = \phi$.

Remark 2.6 *Every limit point is a closure point but the converse in not true.*

The above remark is trivially based on the definitions of the closure and the limit points. A point $x \in X$ will be a limit point if $\forall\ r > 0,\ \ \mathbb{U}(x, r) \bigcap [A - \{x\}] \neq \phi$ and this in-turn implies that $\forall\ r > 0,\ \ \mathbb{U}(x, r) \bigcap A \neq \phi$, which is the condition for being a closure point. Thus, every limit point is a closure point.But, the converse is not always true and a closure point may or may not be a limit point, depending on the set. For example, for the set $[0, 1]$ the point 1 is both a closure and a limit point whereas for the set $\{1, 2, 3, 4, ...\}$ the point 1 is a closure but not a limit point.

Remark 2.7 *The relation between the closure and the derived set is* $\overline{A} = A \bigcup A'$

Let y be a point such that $y \in \left(A \bigcup A'\right)$. Then we have the following three possibilities:
a) $y \in A$ & $y \notin A'$ **Or** b) $y \notin A$ & $y \in A'$ **Or** c) $y \in A$ & $y \in A'$
In all the above three cases, y will also belong to $\overline{A}$ because using the above remarks, $y \in A \implies y \in \overline{A}$ and $y \in A' \implies y \in \overline{A}$. Thus, we conclude that $y \in \left(A \bigcup A'\right) \implies y \in \overline{A}$ or in other words $\left(A \bigcup A'\right) \subseteq \overline{A}$.
Now, Let y be a point such that $y \in \overline{A}$. Then according to the definition we have $\forall\ r > 0,\ \ \mathbb{U}(x, r) \bigcap A \neq \phi$ and based on this relation we can have following two possibilities:

a) $\forall\ r > 0,\ \ \mathbb{U}(y, r) \bigcap [A - \{y\}] \neq \phi \implies y \in A' \implies y \in \left(A \bigcup A'\right)$ **Or**
b) $\forall\ r > 0,\ \ \mathbb{U}(y, r) \bigcap [A - \{y\}] = \phi$ and $\mathbb{U}(y, r) \bigcap A \neq \phi \implies y \in A \implies y \in \left(A \bigcup A'\right)$

So, based on these possibilities we can say that $y \in \overline{A} \implies y \in \left(A \bigcup A'\right)$ or in other words, $\overline{A} \subseteq \left(A \bigcup A'\right)$. Now, we showed that $\left(A \bigcup A'\right) \subseteq \overline{A}$ as well as $\overline{A} \subseteq \left(A \bigcup A'\right)$. So, The above two sets are subsets of each other and thus, should be equal i.e, $\overline{A} = \left(A \bigcup A'\right)$.

Remark 2.8 *If* $x \in \overline{A}$ *but,* $x \notin A$ *then* x *is a limit point*

Using *Remark 2.7*,we can say that $\overline{A} = \left(A \bigcup A'\right)$. Then if x be a point such that $x \in \overline{A}$ but, $x \notin A$ then clearly it must be in the derived set i.e, $x \in A'$ and thus, x will surely be a limit point.

Remark 2.9 *If* $x \notin \overline{A}$ *then*
i) x *is an interior point of the set* $X - A$
ii) x *is also an exterior point of the set* A

Using the above definitions and remarks we know that $A \subseteq \overline{A} \implies X - \overline{A} \subseteq X - A$.
Now if x does not belong to the closure of the set A then we can say that $x \notin \overline{A} \implies x \in X - \overline{A}$.
Further using $A \subseteq \overline{A} \implies X - \overline{A} \subseteq X - A$ we can write $x \notin \overline{A} \implies x \in X - \overline{A} \implies x \in X - A$.
Now as $x \in X - A \implies \exists\ r > 0$ such that $\mathbb{U}(x, r) \bigcap A = \phi \implies \mathbb{U}(x, r) \subseteq X - A$.
So, all the points in the set $X - \overline{A}$ satisfy the definition of the interior of $X - A$ and thus, it is conclusive that $x - \overline{A} = Int(X - A)$.
Further using the definition of the exterior of the set we can also say that x belongs to the exterior of the set A.

Definition 2.10 (Isolated point) *The* x *is called an* ***isolated point*** *if it is an element of* A *but is not a limit point of* A*. Mathematically we can write that:*

$$x \in A \text{ but } x \notin A' \implies \exists\ r > 0 \text{ such that } \mathbb{U}(x, r) \bigcup [A - \{x\}] = \phi \tag{8}$$

So, if x is an isolated point of a set A then we can find an open ball centered around x which contains only a single element of A which in-turn will be x.
An example of an isolated point is as follows:

Example 2.5 *For the set* $A = \{1, 2, 3, 4, ..\}$ *the point* 1 *is an isolated point because we can find an open ball centered around* 1 *which contains only a single element of* A *which in-turn is* 1*. One of such open ball is* $\mathbb{U}(1, \frac{1}{4})$ *which satisfies* $\mathbb{U}(1, \frac{1}{4}) \bigcup [A - 1] = \phi$*.*

Theorem 2.6 *Let* A *be a subset of* X*, and* x *a point in* X*, which may not necessarily be in* A*. If* x *is a limit point of* A *then every open ball centered at* x *contains infinitely many points of* A*.*

Proof: Let us assume that there exists an open ball centered at x which contains finite number of elements of X say, $a_1, a_2, a_3, a_4, .., a_n$. If $r = min\{||x - a_1||, ||x - a_2||, ||x - a_3||, .., ||x - a_n||\}$ then, $\mathbb{U}(x, \frac{r}{2})$ will be an open ball which contains no elements of X but, x. Thus, this open ball satisfies $\mathbb{U}(x, \frac{r}{2}) \bigcap [A - \{x\}] = \phi \implies x \notin A'$. So, the conclusion is that x is not a limit point and this is direct contradiction of our initial assumption. So, this implies that our initial assumption is incorrect and thus, if x is a limit point then every open ball centered at x will contain infinitely many points of A. ∎

The inverse of the above theorem is not always true. For example, the set $A = \{1, 2, 3, 4, ...\}$ have infinitely many elements of X but no limit points exists for this set.

Corollary 2.7 *Let* A *be a subset of* X*. Then* A *cannot have a limit point unless it contains infinitely many points. However, the converse is not always true.*

Proof: Let the set A have a limit point x belonging to X and A have finite number of points. Then using the *Theorem 2.6* every open ball centered at x must contain an infinitely many points of A. But, this is not possible with the assumption of finite A thus, the assumption is incorrect and A must contain infinitely many points. ∎

The converse of the above corollary is not always true and can be illustrated using the same example that we used in the *Theorem 2.6* i.e, the set $\{1, 2, 3, 4, ...\}$, which contains infinitely many points but still, does not have any limit point.

Remark 2.11 *Let* x *be a point* $x \in X$ *and a* A *be a set* $A \subseteq X$*. Then* x *belongs to the closure of* A *if and only if the metric distance of* x *from the set* A *is zero i.e,* $x \in \overline{A} \iff d(x, A) = 0$

If $x \in \overline{A}$ then using the definition of closure we have that $\forall\ r > 0,\ \ \mathbb{U}(x, r) \bigcap A \neq \phi$. So, let $x_r \in \mathbb{U}(x, r) \bigcap A$ then, $d(x, A) \leq d(x, x_r) \leq r$. Thus, we have $d(x, r) \leq r \quad \forall\ r > 0$ and as r is an arbitrary and positive we can conclude that $x \in \overline{A} \implies d(x, A) = 0$.
For the converse, let $d(x, A) = 0$ then for any arbitrary $r > 0\ \exists\ x_r \in A$ such that $d(x, x_r) < r$ and thus, $\mathbb{U}(x, r) \bigcap A \neq \phi$ and using the definition of closure we can conclude that, $x \in \overline{A}$.

Example 2.8 *Consider the set* $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\} \bigcup \{0\}$. *For this set the point* 0 *lies in it and also satisfies the following:*

$$\exists\, r > 0 \text{ such that } U(0, r) \bigcap [A - \{0\}] \neq \phi$$

Thus, 0 *is a limit point of* A *and in fact it's the only limit point of* A. *Thus* $A' = \{0\}$ *and as* $\overline{A} = A \bigcup A' = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\} \bigcup \{0\} \bigcup \{0\} = A$.
Thus, this set is same as it's closure i.e, $A = \overline{A}$. *Now let* $\frac{1}{n}$ *represents all the points other than* $\{0\}$. *Then the point* $\frac{1}{n}$ *satisfies :*

$$\exists\, r > 0 \text{ such that } \mathbb{U}\Big(\frac{1}{n}, r\Big) \bigcap \Big[A - \Big\{\frac{1}{n}\Big\}\Big] = \phi$$

Thus, all the points of the from $\frac{1}{n}$ *are isolated points for the set* A. *This is also illustrated below:*

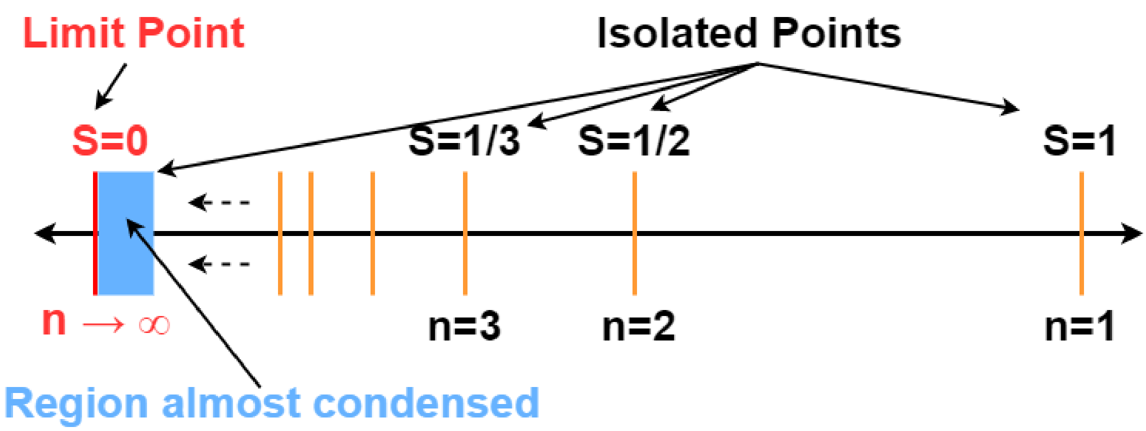


Figure 3: Limit and Isolated points for the set $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\} \bigcup \{0\} \subseteq \mathbb{R}$

3 Closed set and it's properties

3.1 Closed set

Definition 3.1 *Let* (X, d) *be a Metric Space and* d *be the distance metric. Then the set* $A \subseteq$ *is said to be closed if and only if it contains all it's closure points i.e,* $A = \overline{A}$.

Using the definition of closure of a set $A \subseteq \overline{A}$. Now if A is also a closed set then using the above definition, $A = \overline{A}$. Also as $A = \overline{A} \iff A \subseteq \overline{A}\ \&\ \overline{A} \subseteq A$ so, showing that $\overline{A} \subseteq A$ is sufficient to show that A is a closed set.

The figure illustrating a set which satisfies the above definition of closeness is as follows:

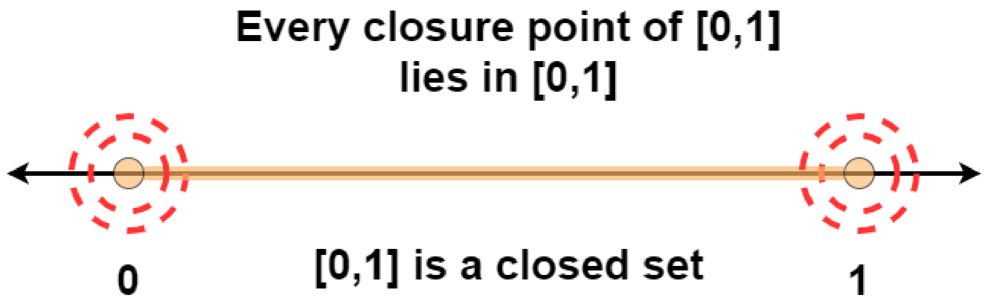


Figure 4: The set $[0, 1]$ is a closed set

The figure illustrating a set which does not satisfies the above definition of closeness is as follows:

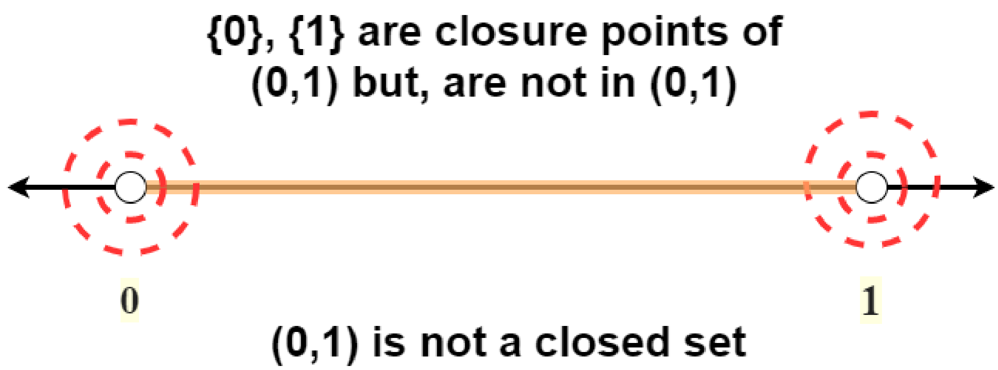


Figure 5: The set $(0, 1)$ is not a closed set

Theorem 3.1 *Let A be a subset of X then the following statements are equivalent :*
i) *The set A is closed in X*
ii) *The set A contains all it's closure points*
iii) *The set A contains all it's limit points*
iv) *The set A is same as it's closure i.e, $A = \overline{A}$*

Proof: To show that the above statements are equivalent we need to show that all the statements both way implies each other. Consider the statement *i)* which says A is a closed set then, Using the *Definition 3.1*, it can be said that $A = \overline{A}$.
Statement *i)* means $A = \overline{A}$ and this clearly implies $\overline{A} \subseteq A$ which is same as statement *ii)*. So, it had been shown that $i) \implies ii)$. Similarly, consider the statement $ii)$ i.e, the set A contains all it's closure points i.e, $\overline{A} \subseteq A$. Using the definition of the closure it is established that $A \subseteq \overline{A}$. So, coupling $\overline{A} \subseteq A$ and $A \subseteq \overline{A}$ it is conclusive that $A = \overline{A}$. So, it had been shown that $ii) \implies i)$. Now, coupling $ii) \implies i)$ with $i) \implies ii)$ gives further conclusion that $i)$ and $ii)$ are equivalent i.e, $i) \iff ii)$.
Statement *i)* means $A = \overline{A} \implies \overline{A} = A$. Using the property of closure points it is known that $\overline{A} = A \bigcup A'$. Thus, $\overline{A} = A$ means that $A = A \bigcup A' \implies A' \subseteq A$ and this is same as statement

iii). So, it had bee shown that $i) \implies iii)$. Similarly, consider statement *iii*) which means that $A' \subseteq A$. Now, using the property of closure i.e, $\overline{A} = A \bigcup A'$ and as $A' \subseteq A$ $\implies \overline{A} = A \bigcup A' = A \implies \overline{A} = A$ which is same as statement *i)* and means that A is a closed set. So, it had been shown that $iii) \implies i)$. Now, coupling $iii) \implies i)$ with $i) \implies iii)$ gives further conclusion that *i*) and *iii*) are equivalent i.e, $i) \iff iii)$
Statement *iv*) is a direct consequence of the *definition 3.1* and thus trivially *i*) and *iv*) are equivalent i.e, $i) \iff iv)$.
Thus, all the above 4 statements are equivalent to each other and implies closeness of set A.

■

Theorem 3.2 (Another definition of closed set) *Let A be a subset of X then A is called a closed set if and only if it's complement, $X - A$ is open.*

Proof: For this proof the properties of an open set are needed. As discussed in previous lectures, a set $A \subseteq X$ is an open set if and only if it is equal to it's interior i.e, $A = Int(A)$. The definition of the interior of a set establishes that for any set A it's interior completely lies in it i.e, $Int(A) \subseteq A$. Thus, to show that a set A is open (i.e, $A = Int(A)$) it is sufficient to show that A lies completely in it's interior i.e, $A \subseteq Int(A)$.
Suppose, A is a closed set i.e, $A = \overline{A}$. Now, using the result of *Remark 2.9*, If x is a point such that, $x \notin \overline{A}$ then $x \in Int(X - A) \implies X - \overline{A} = Int(X - A)$. As we have the condition $A = \overline{A}$, we can conclude that $X - \overline{A} = X - A = Int(X - A)$. Thus, we have shown that the set $X - A$ is an open set i.e, $X - A = Int(X - A)$.
Similarly, suppose that, A is a set such that it's complement is open i.e, $X - A = Int(X - A)$. Now, using the result of *Remark 2.9*, If x is a point such that, $x \notin \overline{A}$ then $x \in Int(X - A)$ which implies that $X - \overline{A} = Int(X - A)$. As we also have the condition that $X - A = Int(X - A)$ thus, it is conclusive that, $X - \overline{A} = X - A \implies A = \overline{A}$ which is same as saying that the set A is closed. Finally, combining $x - A = Int(X - A) \implies A = \overline{A}$ and $A = \overline{A} \implies X - A = Int(X - A)$ it is conclusive that a set is closed if and only if it's complement is open. Mathematically, it can be written as $A = \overline{A} \iff X - A = Int(X - A)$. ■

An example of a closed set is as follows :

Example 3.3 *A closed interval $[a, b]$ in $\mathbb{R}^1$ is always a closed set. The closeness of the interval $[0, 1]$ is illustrated in the Figure 4. Similarly, an open interval (a, b) in $\mathbb{R}^1$ is not a closed set. The non closeness of the interval $(0, 1)$ is illustrated in the Figure 5*

Example 3.4 *The Cartesian product of the form $[a_1, b_1] \times ... \times [a_n, b_n]$ of n one-dimensional closed intervals is also a closed set in $\mathbb{R}^n$ also called an n-dimensional closed interval $[a, b]$.*

Theorem 3.5 *Every closed ball in a metric space is a closed set. If a is a point in the metric space X with a metric d we can write that :*

$$a \in X, \text{ for any } \ r > 0 \ \text{ then } \ \mathbb{B}(a,r) = \{x \in X \mid d(a,x) \leq r\} \ \text{ is a closed set} \tag{9}$$

Proof: For the metric space X with metric d, let a be a point in X then consider a closed-ball, $\mathbb{B}(a,r)$ centered at a with an arbitrary radius r. The open ball $\mathbb{B}(a,r)$ is defined as $\{x \in X \mid d(a,x) \leq r\}$.
Consider any point $y \notin \mathbb{B}(a,r)$ then $d(a,y) > r$. Let $\delta < d(a,y) - r$ then there always exists an open ball $\mathbb{U}(u,r)$ which satisfy the condition that $\mathbb{U}(y,\delta) \bigcap \mathbb{B}(a,r) = \phi$ and thus, any point y which lies outside $\mathbb{B}(a,r)$ does not belong to the closure of $\mathbb{B}(a,\epsilon)$. Using the definition of the closure, any set belongs to it's closure thus, the set $\mathbb{B}(a,r)$ belongs to it's closure while no point outside it belongs to it's closure. Thus, it have been shown that any closed ball is always closed in a metric space i.e, $\mathbb{B}(a,r) = \overline{\mathbb{B}(a,r)}$ ∎
For an open ball the above conclusion cannot be made as the boundary points always lies outside the open ball but, are belongs to it's closure.

3.2 Properties of closed sets

Two of the important properties of the closed sets are as follows:

Property 3.1 *Suppose $\{F_\alpha \ : \ \alpha \in \Lambda\}$ be a family of closed sets where, Λ is the indexing set then the intersection of these sets i.e, $\bigcap\limits_{\alpha \in \lambda} F_\alpha$ is a closed set.*

Proof: Let $\{F_\alpha \ : \ \alpha \in \Lambda\}$ be an arbitrary collection of closed sets. Then using the Demorgan's law $\left(\bigcap_{\alpha\in\lambda} F_\alpha\right)^c = \bigcup_{\alpha\in\lambda}(F_\alpha)^c$. Thus, the complement of an intersection of closed sets is the union of the complements of those closed sets. Using *Theorem 3.2* since, the complement of a closed set always is open, so $\bigcup_{\alpha\in\lambda}(F_\alpha)^c$ is a union of open sets. Since, the union of open sets is open, we see that $\bigcup_{\alpha\in\lambda}(F_\alpha)^c = \left(\bigcap_{\alpha\in\lambda} F_\alpha\right)^c$ is open. Further the complement of $\left(\bigcap_{\alpha\in\lambda} F_\alpha\right)^c$ $= \bigcap_{\alpha\in\lambda} F_\alpha$ i.e, the intersection is also a closed set. ∎

Property 3.2 *Suppose $\{F_1, F_2, F_3, ..., F_n\}$ are n closed sets then the finite-union of the sets $F_1, F_2, F_3, ...,$ and F_n i.e, $\bigcup\limits_{i=1}^{n} F_i$ is a closed set.*

Proof: Let $F_1, F_2, F_3, ...F_n$ be n closed sets and let x be a limit point of $\bigcup_{i=1}^n F_i$. Now, if x is a limit point of F_1 or F_2 or ... F_n and it is clearly contained in $\bigcup_{i=1}^n F_i$. Now, suppose that x is not a limit point of any of $F_1, F_2, F_3..., F_n$ then using the definition of limit points, for each i in $1, 2, 3..., n$ there exists r_i such that the open ball $\mathbb{U}(x, r_i)$ does not intersects with F_i except possibly for x. But then if $r = min\{r_1, r_2, r_3, ..., r_n\}$ then the open ball $\mathbb{U}(x, r)$ does not intersect with $\bigcup_{i=1}^{n} F_i$ except possibly for x, which contradicts x being a limit point. This contradiction establishes the result that the finite union is closed. Trying to extend it to infinitely is not possible as then the min will be replaced by ∞ which is not necessarily positive. ■

4 Supremum and Infimum

The definitions of the supremum and infimum are very similar and are as follows:

4.1 Supremum

Definition 4.1 (Supremum) *Let X be a metric space and A be a set in X, such that $A \neq \phi$ and is bounded above then, $\alpha = \sup A$ is defined as the supremum of the set A if:*

1. *$\alpha \in X$ is an upper bound of A.*
2. *If $x < \alpha$, then x is not an upper bound of A.*

The supremum of a set always belongs to the closure of that set. For the metric space $\mathbb{R}^1$, the open ball centered around the supremum i.e, $\forall\ r > 0\ \mathbb{U}(\alpha, r)$ will lie on the real line and will have a non zero intersection with the corresponding set. Mathematically we can write :

$$\left(\ \alpha - \epsilon,\ \alpha + \epsilon\ \right) \bigcap A \neq \phi\ \forall\ \epsilon > 0 \implies \alpha \in \overline{A} \tag{10}$$

4.2 Infimum

Definition 4.2 (Infimum) *Let X be a metric space and A be a set in X, such that $A \neq \phi$ and is bounded below then, $\beta = \inf A$ is defined as the infimum of the set A if:*

1. *$\beta \in X$ is a lower bound of A.*
2. *If $x > \beta$, then x is not a lower bound of A.*

Similar to the supremum the infimum of a set also always belongs to the closure of that set. If we consider the metric space $\mathbb{R}^1$, the open ball centered around the infimum i.e, $\forall\ r > 0\ \mathbb{U}(\alpha, r)$ will lie on the real line and will have a non zero intersection with the corresponding set.
Mathematically we can again write :

$$\left(\ \beta - \epsilon,\ \beta + \epsilon\ \right) \bigcap A \neq \phi\ \forall\ \epsilon > 0 \implies \beta \in \overline{A} \tag{11}$$

The Supremum and Infimum of a set as it's closure points is illustrated in the following figure:

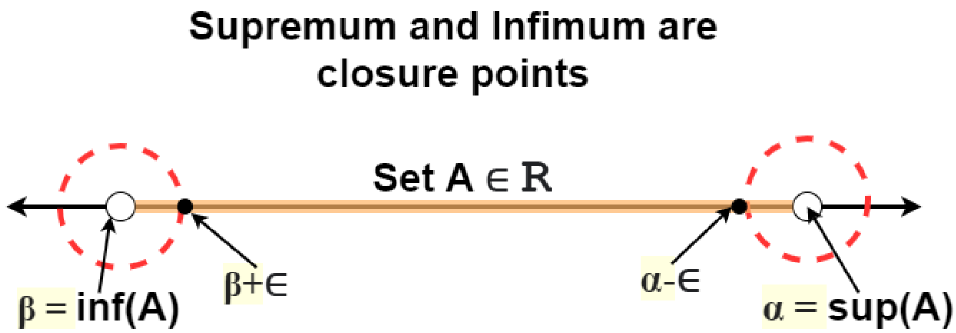


Figure 6: The Supremum and Infimum of a set are also it's closure points.

5 Bounded Set

The bounded set is defined using the diameter of a set. The definition is as follows:

Definition 5.1 (Bounded set) *Let X be a metric space and A be a set in that space then the set A is called bounded if it's diameter is finite i.e, $dia(A) < \infty$.*
If the diameter is finite than the set $A \subseteq X$ lies entirely within an open ball. Thus, the set A can also be defined to be bounded if, it lies entirely within an open ball of the form $\mathbb{U}(a, r)$ for some $r > 0$ and some $a \in X$.

Thus, to show that a set is bounded either, it can be shown that the diameter is finite or it can be shown that the set lies completely inside an open ball in the metric space.
A figure illustrating a set which satisfies, the above definition of boundedness if as follows:

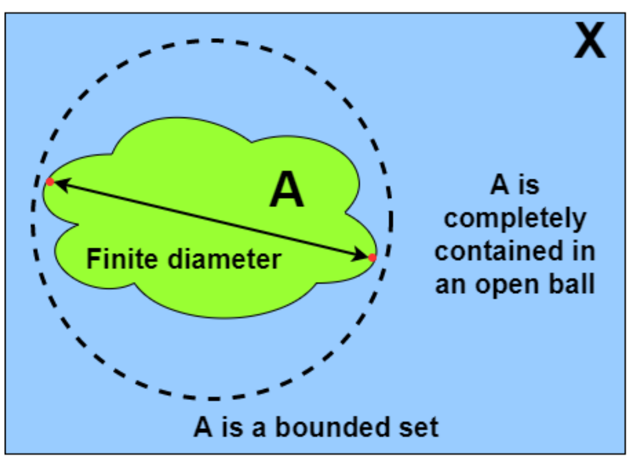


Figure 7: The set $A \subseteq X$ is bounded

Theorem 5.1 (Bolzano-Weistrauss theorem) *Let $\mathbb{R}^n$ be the finite dimensional Eucledian space and $A \subseteq \mathbb{R}n$ be a bounded set containing infinitely many points, then there is atleast one point in $\mathbb{R}n$ which is a limit point for the set A.*

Proof: First proceeding for $\mathbb{R}^1$. Since A is bounded, it lies in some interval $[-a, a]$. At least one of the subintervals $[-a, 0]$ or $[0, a]$ contains an infinite subset of A. Call one such subinterval $[a_1, b_1]$. Bisect $[a_1, b_1]$ to obtain a subinterval $[a_2, b_2]$ containing an infinite subset of A, and continue this process. In this way a countable collection of intervals is obtained, the n^{th} interval $[a_n, b_n]$ being of length $b_n - a_n = a/2^{n-1}$. Clearly, the sup of the left end points a_n and the inf of the right end points b_n must be equal, say to x. The point x will then be a limit point of A because, if r is any positive number, the interval $[a_n, b_n]$ will be contained in $\mathbb{U}(x, r)$ as soon as n is large enough so that $b_n - a_n < r/2$. The interval $\mathbb{U}(x, r)$ contains a point of A distinct from x and hence x is a limit point of A. This proves the theorem for $\mathbb{R}^1$.
Now, for $\mathbb{R}^n$, $n > 1$, by an extension of the ideas used in treating $\mathbb{R}^1$. (Refer to *Figure* 8). Since A is bounded, A lies in some open ball $\mathbb{U}(0, a)$, $a > 0$, and therefore within the n-dimensional interval J_1 defined by the inequalities

$$-a \ < \ x_k \ < \ a \quad (k = 1, 2, ..., n).$$

Here J_1 denotes the cartesian product

$$J_1 = I_1^{(1)} \times I_2^{(1)} \times ... \times I_n^{(1)}.$$

that is, the set of points $(x_1, ..., x_n)$, where $x_k \in I_k(1))$ and where each $I_k^{(1)}$ is a one-dimensional interval of the form $-a < x_k < a$. Each interval $I_k^{(1)}$ can be bisected to form two subintervals $I_{k,1}^{(1)}$ and $I_{k,1}^{(1)}$), defined by the inequalities:

$$I_{K,1}^{(1)} \ : \ -a < x_k < 0 \qquad I_{k,2}^{(1)} : 0 < x_k < a.$$

Next, we consider all possible cartesian products of the form

$$I_{1,K_1}^{(1)} \times I_{2,K_2}^{(1)} \times I_{3,K_3}^{(1)} \times ... \times I_{n,K_n}^{(1)} \tag{12}$$

where each $k_i = 1$ or 2. There are exactly 2^n such products and, of course, each such product is an n-dimensional interval. The union of these 2^n intervals is the original interval J_1, which contains A; and hence at least one of the 2^n intervals in *Equation 12* must contain infinitely many points of A. One of these we denote by J_2, which can then be expressed as

$$J_2 = I_1^{(2)} \times I_2^{(2)} \times ... \times I_n^{(2)},$$

where each $I_k^{(2)}$ is one of the subintervals of $I_k^{(1)}$ of length a. We now proceed with J_2 as we did with J_1, bisecting each interval $I_k^{(2)}$ and arriving at an n-dimensional interval J_3 containing an infinite subset of A. If we continue the process, we obtain a countable collection of n-dimensional intervals $J_1, J_2, J_3, ...$, where the m^{th} interval J_m has the property that it contains an infinite subset of A and can be expressed in the form

$$J_m = I_1^{(m)} \times I_2^{(m)} \times ... \times I_n^{(m)} \quad \text{where} \quad I_k^{(m)} \subset I_k^{(1)}.$$

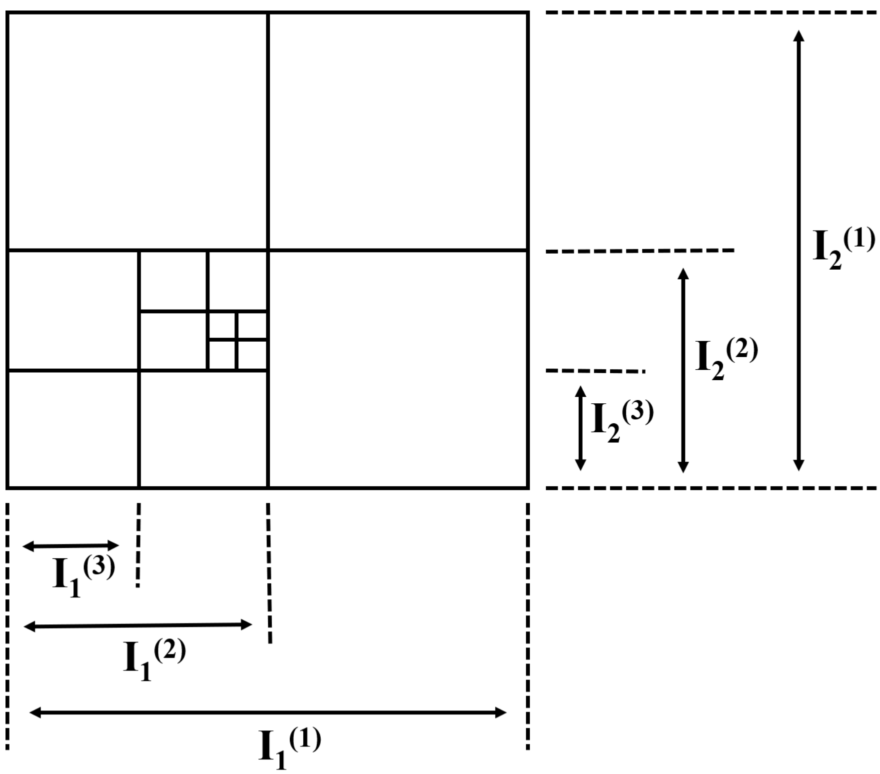


Figure 8: Figure for the proof of Belzano-Weistrauss theorem

$$\text{Writing} \quad I_k^{(m)} = [a_k^{(m)}, b_k^{(m)}]$$

$$\text{We have,} \quad b_k^{(m)} - a_k^{(m)} = \frac{a}{2^{m-1}} \qquad (k = 1, 2, 3, .., n).$$

For each fixed k, the sup of all left endpoints $a_k^{(m)}$, $(m = 1, 2, ...)$, must therefore be equal to the inf of all right endpoints $b_k^{(m)}$, $(m = 1, 2, ...)$, and their common value we denote by t_k. We now assert that the point $t = (t_1, t_2, ..., t_n)$ is a limit point of A. To see this, take any open ball $\mathbb{U}(t, r)$. The point t, of course, belongs to each of the intervals $J_1, J_2, ...$ constructed above, and when m is such that $a/2^{m-2} < r/2$, this neighborhood will include J_m contains infinitely many points of A, so will $\mathbb{U}(t,r)$, which proves that t is indeed a limit point of A.

■

6 Compact Set

The notion of a compact set is of great importance in understanding and analysing continuity. The definiton of compactness is as follows:

Definition 6.1 (Open cover) *Let X be a metric space and A be a set in X then, the collection of open subsets of X i.e, $\{G_\alpha\}$ such that $A \subset \bigcup_\alpha G_\alpha$ is defined as an open cover of set A.*

Definition 6.2 (Compact set) *Let X be a metric space then a set $A \subset X$, is compact if every open cover of A contains a finite subcover. Notably, the requirement for being a compact set is that if $\{G_\alpha\}$ is an open cover of A, then there are finitely many indices $\alpha_1, \alpha_2, ...\alpha_n$ such that,*

$$A \subset G_{\alpha_1} \bigcup ... \bigcup G_{\alpha_n} \tag{13}$$

The above definiton is a general and is applicable for any metric space. The finite dimensional Euclidean space is known for it's simplicity and the definition of compactness can be further simplified in the case of a finite dimensional Euclidean space.
The theorem characterizing compactness in case of a finite dimensional Euclidean space is is known as the Heine-Borel theorem and is as follows:

Theorem 6.1 (Heine-Borel theorem) *Let A be a subset of $\mathbb{R}^n$. Then the following three statements are equivalent:*
i) *A is compact*
ii) *A is closed and bounded*
iii) *Every infinite subset of A has a limit point in A*

Proof: As noted above, **(ii)** implies **(i)** . If we prove that **(i)** implies **(ii)** , then **(ii)** implies **(iii)** and that **(iii)** implies **(ii)** , this will establish the equivalence of all three statements.
Assume **(i)** holds. We shall prove first that A is bounded. Choose a point p in A. The collection of open balls $\mathbb{U}(p,k)$, $k = 1, 2, ...$, is an open cover of A. By compactness a finite subcollection also covers A and hence A is bounded. Next we prove that A is closed. Suppose A is not closed. Then there is a limit point y of A such that $y \notin S$. If $x \in S$, let $r_x = ||x-y||/2$. Each r_x is positive since $y \notin A$ and the collection $\mathbb{U}(x; r_x) : x \in A$ is an open cover of A. By compactness, a finite number of these neighborhoods cover A, say

$$A \subseteq \bigcup_{k=1}^{p} \mathbb{U}(x_k, r_k)$$

Let r denote the smallest of the radii $r_1, r_2, ..., r_p$ Then it is easy to prove that the open ball $\mathbb{U}(y, r)$ has no points in common with any of the open balls $\mathbb{U}(x_k, r_k)$. In fact, if $x \in \mathbb{U}(y, r)$, then $||x - y|| < r \leq r_k$, and by the triangle inequality we have,

$$||y - x_k|| \leq ||y - x|| + ||x - x_k|| \quad \text{So,}$$
$$||x - x_k|| \geq ||y - x_k|| - ||x - y|| = 2r_k - ||x - y|| > r_k$$

Hence $x \notin \mathbb{U}(x_k, r_k)$. Therefore $\mathbb{U}(y, r) \bigcap A$ is empty, contradicting the fact that y is a limit point of A. This contradiction shows that A is closed and hence (**i**) implies (**ii**).
Assume (**ii**) holds. In this case the proof of (**iii**) is immediate, because if T is an infinite subset of A then T is bounded (since A is bounded), and hence by the *Bolzano-Weierstrass theorem* T has a limit point x, say. Now x is also a limit point of A and hence $x \in S$, since A is closed. Therefore (**ii**) implies (**iii**).
Assume (**iii**) holds. We shall prove (**ii**). If A is unbounded, then for every $m > 0$ there exists a point $x_m \in A$ with $||x_m|| > m$. The collection $T = x_1, x_2, ...$ is an infinite subset of A and hence, by (**iii**), T has a limit point y in A. But form $m > 1 + ||y||$ we have

$$||x_m - y|| \geq ||x_m|| - ||y|| > m - ||y|| > 1,$$

contradicting the fact that y is a limit point of T. This proves that A is bounded. To complete the proof we must show that A is closed. Let x be a limit point of A. Since every neighborhood of x contains infinitely many points of A, we can consider the neighborhoods $\mathbb{U}(x, 1/k)$, $wherek = 1, 2, ...$, and obtain a countable set of distinct points, say $T = x_1, x_2...$, contained in A, such that $x_k \in \mathbb{U}(x, 1/k)$. The point x is also a limit point of T. Since T is an infinite subset of A, part (**iii**) of the theorem tells us that T must have a limit point in A. The theorem will then be proved if we show that x is the only limit point of T. To do this, suppose that $y \neq x$. Then by the triangle inequality we have

$$||y - x|| \leq ||y - x_k|| + ||x_k - x|| < ||y - x_k|| + 1/k, \quad \text{If } x_k \in T$$

If k_0 is taken so large that $1/k < \frac{1}{2}||y - x||$ whenever $k \geq k_0$, the last inequality leads to $\frac{1}{2}||y - x|| < ||y - x_k||$. This shows that $x_k \notin \mathbb{U}(y, r)$ when $k \geq k_0$, if $r = \frac{1}{2}||y - x||$. Hence y cannot be a limit point of T. This completes the proof that (**iii**) implies (**ii**). ■
The Heine-Borel, also inspires a way for another characterization of compactness in an arbitrary metric space (X, d). Every subset of set A does not have a limit point in A and this condition is related to compactness as shown in the following theorem :

Theorem 6.2 *Consider a metric space (X, d) and a set $A \subseteq X$. If A is a compact subset of X then the following conditions follows:*
i) *A is closed and bounded*
ii) *Every infinite subset of A has a limit point in A*

Proof: To prove **(i)** we refer to the proof of the *Heine-Borel theorem* and use that part of the argument which showed that **(i)** implies **(ii)**. The only change is that the Euclidean distance $||x-y||$ is to be replaced throughout by the metric $d(x, y)$.
To prove **(ii)** we argue by contradiction. Let T be an infinite subset of A and assume that no point of A is a limit point of T. Then for each point $x \in A$ there is an open ball $\mathbb{U}(x, r)$ which contains no point of T $(if x \notin T)$ or exactly one point of T (x itself, if $x \in T$). As x runs through A, the union of these balls $\mathbb{U}(x, r)$ is an open cover of A. Since A is compact, a finite subcollection covers A and hence also covers T. But this is a contradiction because T is an infinite set and each ball contains at most one point of T. ∎
For an arbitrary metric space (X, d) space the **(i)** condition is not equivalent to compactness but, the **(ii)** statement id. Whereas, for the finite dimensional Euclidean space statement, due to it's special nature **(i)** with the statement **(ii)** is equivalent to compactness.

A figure illustrating a set which satisfies the above requirements of compactness is as follows :

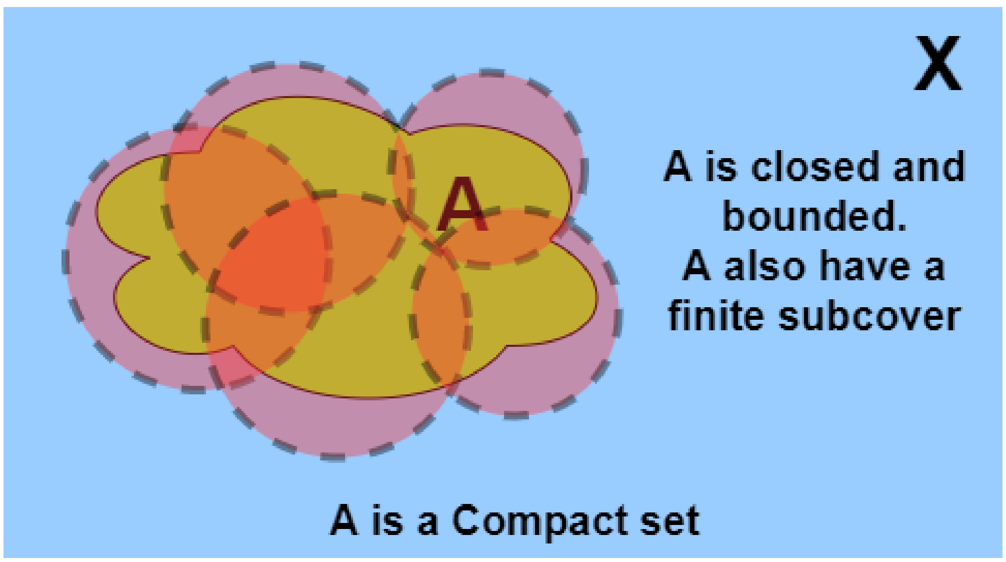
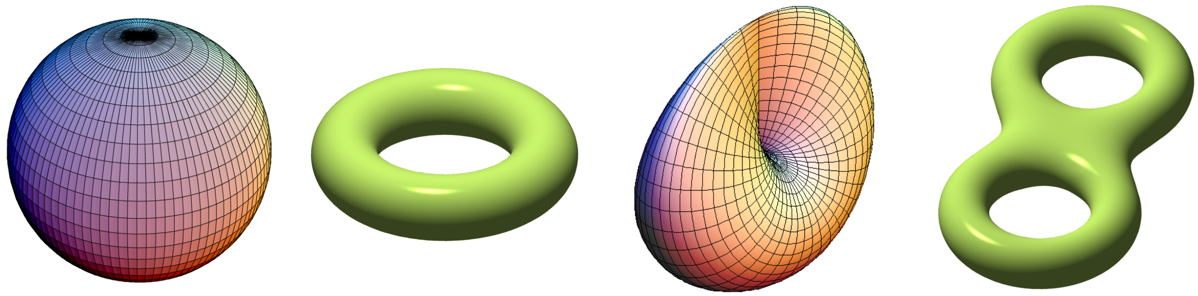


Figure 9: The set $A \subseteq X$, have a finite subcover, is closed and bounded and, it's every infinite subset contains all it's limit points. Thus, the set $A \subseteq X$ it is a compact set in a metric space X.

In the $3D$ Eucliden space the Sphere is trivially a compact set. Many other interesting topological structures are also identified as compact sets. The following figures show the Sphere, Torous, Double Torous and Cross Cap, which are compact sets in $\mathbb{R}^3$. Whereas the structures, Klein Bottle is compact sets in $\mathbb{R}^4$ i,e the $4D$ Euclidean space.



(a) Sphere (b) Torous (c) Cross Cap (d) Double Torous

Figure 10: Examples of compact sets in the $3D$ Eucledian space i,e. $\mathbb{R}^3$

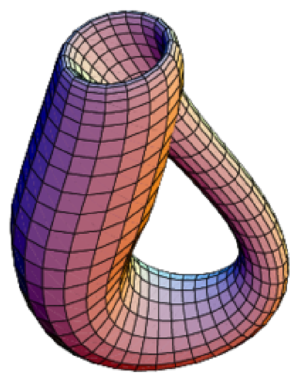


Figure 11: Klein Bottle, a compact set in $4D$ Eucledian space i,e. $\mathbb{R}^4$

7 Sequence in Metric Space

Let (X, d) be a metric space then the sequence in that metric space is defined as follows:

Definition 7.1 (Sequence) *A sequence in a metric space* (X, d) *is a function* $x : \mathbb{N} \to X$*. The sequence is denoted by the notation* x_n *which is the value of the function* x *at the integer* n*.*

Definition 7.2 (Limit of a sequence) *A sequence of points* x_n *in a metric space* X *is said to converge to a point* $a \in X$ *if for every* $\epsilon > 0$ *there exists an* $N_\epsilon \in N$ *such that for all* $n \geq N_\epsilon$ *following satisfies:* $x_n \in \mathbb{U}(a, \epsilon)$ *or the metric distance* $d(x_n, x) < \epsilon$*. The notation for the limit of a sequence is* $x_n \to a$*. Mathematically, the definition can be written as :*

$$\forall\ \epsilon > 0, \quad \exists\ N_\epsilon \in \mathbb{N} \ \text{ such that we have } \ x_n \in \mathbb{U}(a, \epsilon) \ \text{ or } \ d(x_n, x) < \epsilon, \quad \forall\ n \geq N_\epsilon \tag{14}$$

Using the definition, if any sequence x_n converges to a point a then, every open ball $\mathbb{U}(a, \epsilon)$ where $\epsilon > 0$ should contain all the points of the sequence for all n greater than some threshold.

Remark 7.3 *The limit of a sequence is always unique.*

To prove the above remark, let there be a sequence x_n with two distinct limits L and M. Let, $\epsilon = \frac{d(L,M)}{10} > 0$ then $\exists\ N_1$ such that $d(x_n, L) < \epsilon$ for all $n > N_1$ and $\exists\ N_2$ such that $d(x_n, M) < \epsilon$ for all $n > N_2$. Assume, $N = max\{N_1, N_2\}$ then for all $n > N$, $d(x_n, M) < \epsilon$ and $d(x_n, M) < \epsilon$. Now, using the triangle law $d(L, M) \leq d(L, x_n) + d(x_n, M) < 2 \times \epsilon = \frac{d(L,M)}{5}$. Thus, it is evident from the assumption that $d(L, M) < \frac{d(L,M)}{5}$ which is cannot be true as $d(L, M) > 0$ thus by contradiction, the initial assumption is incorrect and thus, $L = M$ and the limit of a sequence is always unique.

Remark 7.4 *If x_m is a convergent sequence in a finite dimensional metric space $(\mathbb{R}^n, d)$ converging to $a \in \mathbb{R}^n$ then $x_m = [x_m^1, x_m^2, ..., x_m^k, ...x_m^n]^T$ and $a = [a^1, a^2, ..., a^k, ...a^n]^T$ where, $x_m^k \in \mathbb{R}$ and $a^k \in \mathbb{R}$. Then each component of the vector x_m individually converges to the corresponding component of the vector a.*

$$\text{So,} \quad x_m \to a \text{ then} \quad \implies \begin{bmatrix} x_m^1 \\ x_m^2 \\ . \\ . \\ . \\ x_m^k \\ . \\ . \\ . \\ x_m^n \end{bmatrix} \to \begin{bmatrix} a^1 \\ a^2 \\ . \\ . \\ . \\ a^k \\ . \\ . \\ . \\ a^n \end{bmatrix} \implies \begin{bmatrix} x_m^1 & \to & a^1 \\ x_m^2 & \to & a^2 \\ & . & \\ & . & \\ & . & \\ x_m^k & \to & a^k \\ & . & \\ & . & \\ & . & \\ x_m^n & \to & a^n \end{bmatrix}$$

Thus, x_m converges to a if each x_m^i individually converges to a_i i.e, $x_m \to a$ if $x_m^i \to a^i\ \forall\ i$.

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