Tutorial Sheet 1
Nonlinear Programming

1. Consider the constraint set $(1-x_1-x_2)^3 \geq 0,\ x_1,\ x_2 \geq 0$. Show that at $x^* = (1/2, 1/2)^t$, $D(x^*) \neq \mathcal{D}(x^*)$.

2. Consider the constraint set $x_1 + x_2 \leq 3,\ x_1 \geq 0,\ x_1^2 + x_2^2 \leq 16$. For the extreme points x at the intersection of the constraints, determine the sets $D(x),\ \mathcal{D}(x)$.

3. The feasible region of some optimization problem is given by $\{x \in R^2 : x_1 \geq 2,\ x_2 \geq 0\}$. Which of the vectors $(-2,2)^t$, $(0,2)^t$, $(2,0)^t$ are feasible directions at $(4,1)^t$, $(2,3)^t$, $(1,4)^t$, respectively.

4. Show that $f(x_1,x_2) = (x_2 - x_1^2)^2 + x_1^5$ has only one stationary point which is neither the point of maxima nor the point of minima of f.

5. Use graphical approach to find the solution of the following problem:

$$\begin{aligned} \min\ \mathrm{f(x)} = &\ x_1^2 + x_2^2 + 2x_2 \\ \text{subject to}\ \mathrm{x}_1^2 + \mathrm{x}_2^2 = &\ 1 \\ x_1 + x_2 \geq &\ 0.5 \\ x_1,\ x_2 \geq &\ 0. \end{aligned}$$

6. Solve the following nonlinear problems graphically and verify the KKT conditions at an optimal point.

i.
$$\begin{aligned} \max\ \mathrm{v} = \mathrm{2x_1 + 3x_2} & \\ \text{subject to}\ \mathrm{x_1x_2} \leq &\ 8 \\ x_1^2 + x_2^2 \leq &\ 20 \\ x_1, x_2 \geq &\ 0 \end{aligned}$$

ii.
$$\begin{aligned} \max\ \mathrm{v} = \mathrm{x_2} & \\ \text{subject to}\ \mathrm{x}_1^2 + \mathrm{x}_2^2 \leq &\ 4 \\ x_1^2 - x_2 \geq &\ 0 \end{aligned}$$

iii.
$$\begin{aligned} \min\ \mathrm{f(x)} = (\mathrm{x_1} - 4)^2 + (\mathrm{x_2} - 4)^2 & \\ \text{subject to}\ \mathrm{x_1 + x_2} \leq &\ 4 \\ x_1 - x_2 \leq &\ 2 \\ x_1, x_2 \geq &\ 0 \end{aligned}$$

iv.
$$\begin{aligned} \max\ \mathrm{f(x)} = \mathrm{x_2} & \\ \text{subject to}\ \mathrm{x}_1^2 + \mathrm{x}_2^2 \leq &\ 4 \\ x_1^2 \geq &\ x_2. \end{aligned}$$

7. Find the KKT point of

$$\begin{aligned} \min\ \mathrm{f(x)} = &\ x^2 + y^2 + z^2 \\ \text{subject to}\ \mathrm{x}^2 \geq &\ 1 + z^2 \\ x, y, z \geq &\ 0. \end{aligned}$$

8. Determine if the KKT conditions are satisfied at the point $(1,1)^t$ for the problem

$$\begin{aligned} \min\ \mathrm{f(x)} = 100(\mathrm{x_2} - \mathrm{x}_1^2)^2 + &\ (1 - x_1)^2 \\ \text{subject to}\ \mathrm{x}_1^2 + \mathrm{x}_2^2 \leq &\ 2 \\ x_1 \geq &\ 0. \end{aligned}$$

9. Points $(0,0)^t$ and $(6,9)^t$ are probable points of minimizers of the optimization problem with $f(x_1,x_2) = x_1^3 - x_1^2 x_2 + 2x_2^2$ over the set $x_1, x_2 \geq 0$. Check whether the necessary KKT conditions are satisfied at the points.

10. Consider the following problem

$$\begin{aligned} \min\ \mathrm{f(x)} = \mathrm{x}_1^2 + \mathrm{x}_2^2 + &\ 2x_2 \\ \text{subject to}\ \mathrm{2x_1 + 3x_2} \geq &\ 1 \\ x_1^2 + x_2^2 \leq &\ 1 \\ x_1, x_2 \geq &\ 0 \end{aligned}$$

a. Are (1/2, 0) and $(1/2, \sqrt{3}/2)$ local minimum ?

b. Test if any of these is a global optimal solution ?

11. Check the following functions for convexity:
$f(x_1,x_2) = e^{x_1} + x_2^2 + 5$ $\quad f(x_1,x_2) = 3x_1^2 - 5x_1x_2 + x_2^2$ $\quad f(x_1,x_2) = (1/4)x_1^4 - x_1^2 + x_2^2$

12. If $f_1(x)$ and $f_2(x)$ are convex functions on $S \subset R^n$, then prove that $f(x) = \max\{f_1(x), f_2(x)\}$ is a convex function on S.