MTL 704: Tutorial Sheet-3
Gradient Based Optimization Methods

1. Let f be quadratic , $f(x) = \frac{1}{2}ax^2 - bx + c,\ a, b, c$ are constants, $a > 0$.
   - (i) Write the value of $x^\star$ that minimizes f.
   - (ii) Write recursive equation for the derivative descent search (DDS) algorithm for f.
   - (iii) Assuming DDS algorithm converges, show that it converges to the optimal value $x^\star$ (found in (i)).
   - (iv) Find the order of convergence.
   - (v) Find the range of values of α for which the algorithm converges for all starting points $x^{(0)}$.

2. Consider the function
$$f(x) = 3(x_1^2 + x_2^2) + 4x_1x_2 + 5x_1 + 6x_2 + 7,\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$$
Suppose we use a fixed step size gradient algorithm to find the minimizer of f. Find the largest range of α for which the algorithm is globally convergent.

3. Consider the function $f : R^2 \to R$ given by $f(x) = \frac{3}{2}(x_1^2 + x_2^2) + (1 + a)x_1x_2 - (x_1 + x_2) + b$, where a, b are unknown real parameters.
   - (i) Write f in usual quadratic form.
   - (ii) Find the largest set of values of a and b such that the unique global minimizer(in terms of a and b).
   - (iii) Consider the algorithm $x^{(k+1)} = x^{(k)} - \frac{2}{5}\nabla f(x^{(k)})$
     Find the largest set of values of a and b for which the algorithm converges to the global minimizer of f for any initial point $x^{(0)}$.

4. Consider the optimization problem
$$\text{Minimize}||Ax - b||^2,\ A \in R^{m\times n},\ m \geq n,\ b \in R^m.$$
   - (i) Show that the objective function is a quadratic function.
   - (ii) Write the fixed step size gradient algorithm for solving this problem
   - (iii) Suppose $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$
     Find the largest range of α such that the algorithm in part (ii) converges to the solution of the problem.

5. Use the method of steepest descent to find the minimizer of
$$f(x_1, x_2, x_3) = (x_1 - 4)^4 + (x_2 - 3)^2 + 4(x_3 + 5)^4$$
The initial point is $x^{(0)} = (4, 2, -1)^t$. Perform two iterations.

6. Perform two iterations of steepest descent algorithm for the function
$$f = (x_1 + 2x_2 - 6)^2 + (2x_1 + x_2 - 5)^2,\ x^{(0)} = (0, 0)^T.$$

7. Find the mutually conjugate directions with respect to the matrices
(i) $\mathrm{G}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}$, (ii) $\mathrm{G}_2 = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 6 \end{pmatrix}$

8. For the conjugate gradient method applied on minimizing $f(x) = \dfrac{1}{2}x^TQx - b^tx$, prove that
   - (a) $\nabla f(x^k)^T d^i = 0, \quad \forall\ i = 0, 1, \ldots, k - 1;$
   - (b) $(d^k)^T Qd^i = 0, \quad \forall\ i = 0, 1, \ldots, k - 1.$

9. Find the minimizer of
$$f(x_1, x_2) = \frac{1}{2}x^T \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} x - x^T \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad x \in R^2,$$
using the conjugate direction method with initial point $x^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and Q-conjugate directions $d^0 = (1, 0)^T,\ d^1 = (-3/8, -3/4)^T$.

10. For the quadratic function,
$$f(x_1, x_2, x_3) = \frac{3}{2}x_1^2 + 2x_2^2 + \frac{3}{2}x_3^2 + x_1x_3 + 2x_2x_3 - 3x_1 - x_3,$$
find the minimizer using the conjugate gradient method taking the initial point as origin.

11. Solve the system of linear equations: $-x_1 + 4x_2 = 2,\ \ 4x_1 - x_2 = 7$, by the Newton's method.

12. Can the problem: $\min\ (x_1 + x_2)^2 + (x_2 + x_3)^2$, be solved by the Newton's method starting with $(-4,\ 1,\ 1)^T$. Justify. If yes, get the optimal solution of the problem.

13. Consider minimization of the function

$$f(x_1, x_2) = (x_1 + x_2)^2 + \left(2(x_1^2 + x_2^2 - 1) - \frac{1}{3}\right)^2.$$

Find the largest open ball about the optimal solution $x^* = 0$ in which the Hessian matrix, $\nabla^2 f(x)$, is a positive definite matrix. For what initial point $x^0 = (x_1^0, x_1^0)^T$ in this ball does the Newton's method converge?

14. Prove that if the matrix S_k is positive definite in the DFP method then so is the matrix S_{k+1}.

15. Starting with $x_1 = (-2, 4)^T$, minimize
$$f(x_1, x_2) = \frac{3}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1x_2 - 2x_1$$
by the DFP method.

16. Starting from $(1, 1)^T$, perform two iterations of the DFP algorithm for minimizing the function $f(x_1, x_2) = x_1^2 + 10x_2^2$.