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Abstract The foundations of probability are viewed through the lens of the subjectivist interpretation. This article surveys conditional probability, arguments for probabilism, probability dynamics, and the evidential and subjective interpretations of probability.

Keywords Foundations of probability · Bayesianism · Chance · Logical probability · Evidential probability · Conditionalisation

1 Introduction

Survey articles on the foundations of probability often begin by canvassing various "interpretations", or analyses, of probability. Probabilities are variously understood as relative frequencies, measures of evidential support, graded dispositions; and so forth. I will begin by assuming the subjectivist interpretation, on which probabilities are individual degrees of belief, or credences. I do not claim that subjectivistism is the uniquely correct interpretation of probability—in fact, I use it to introduce other interpretations in Sections 4 and 5. Rather, subjectivism is a compelling starting point that offers valuable tools for understanding other interpretations of probability.

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2 The Logic of Probability

Alongside the interpretation of probability, we must consider the mathematical structure of probability. Most authors use the probability calculus developed by Kolmogorov [55]. Kolmogorov posits *probability spaces*, i.e., triples $\langle \Omega, F, P \rangle$, with Ω a set of possibilities, F a Boolean algebra on Ω, and P a function ranging over the real numbers and satisfying the following axioms.

Tautology $P(\Omega) = 1$
Zero $P(\emptyset) = 0$
Finite Additivity Where $A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$

For cases where Ω contains more than finitely many possibilities, Kolmogorov proposes the stronger principle of

Countable Additivity Where $A_1 \cap A_2 \cap \ldots = \emptyset$, $P \vee_{i=1}^{\infty} (A_i) = \sum_{i=1}^{\infty} P(A_i)$

However, it is controversial whether we should accept countable as well as finite additivity [5, 40, 86].

Kolmogorov proposes an additional axiom governing *conditional* probability.

Ratio Formula $P(B|A) = \frac{P(A \wedge B)}{P(A)}$

Hájek [28] argues against taking the ratio formula as a definition of conditional probability. In cases where $P(A)$ is 0, the right-hand side of the Ratio Formula is undefined, yet we can attribute probabilities conditional on A. Consider a point-sized dart thrown at a continuous dart board, where the dart's probability of landing in a region is proportional to the region's area. The dart's probability of landing on the vertical centre line is 0, but its probability of landing on the top half of the line, given that it lands on the line, is 1/2.

There are three ways of extending the definition of conditional probability to account for Hájek's dart and similar examples. The first approach [78, 79], takes conditional probability as conceptually primitive and defines unconditional probability as probability conditional on the tautology. The second approach [7–9] is to model conditional probability using *lexical probabilities*, or sequences of unconditional probability functions. For a given sequence, the probability of B given A is defined as $\frac{P'(A \wedge B)}{P'(A)}$ for the first probability function P' in the sequence such that $P'(A) > 0$. The third approach [6, 34, 35, 60, 110] is to deny that the point-sized dart has probability 0 of hitting the centre line, and instead claim that the probability is infinitesimal.

The three types of solutions are closely formally related to each other; Halpern [33] discusses conditions under which they are equivalent. While I will discuss defences of the Kolmogorov axioms below—since theorists of probability are generally committed to at least that much—any complete account of probability will need to defend additional formal principles. Exactly which principles, however, may turn on substantive questions about the interpretation of probability in question [17].

2.1 Defending the Formalism

Given an interpretation of probability, such as the subjectivist interpretation, and a mathematical formalism, such as Kolmogorov's probability calculus, we must show that they line up—that the proposed interpretation satisfies the proposed formalism. Much of the literature focuses on of P, but Ω and F deserve attention. What are the objects in P's domain?

To begin with, I will assume that Ω is a set of possible worlds, and F is a set of propositions. However, we will see in Section 2 that subjectivists have reasons to model Ω as a set of *centred* worlds, and in Section 4 that there are sometimes reasons to treat the possibilities as repeatable event types.

Whatever the nature of Ω, Kolmogorov's axioms stand in need of positive defense. Actual people seem to violate them—for instance, by assigning positive credence to logical impossibilities that they fail to recognise as impossible. Subjectivists often fall back to the position that they describe a rational ideal, similar to the rational idea of logical consistency for full belief.

But why think the Kolmogorov axioms capture logical consistency? The literature contains three prominent types of response to this challenge: Dutch book arguments, accuracy arguments, and representation theorems. Each has been marshalled to defend additional constraints beyond the Kolmogorov axioms. Since different argumentative strategies sometimes lead to different conclusions, it is important to settle whether (and under what circumstances) each type of argument is sound.

2.2 Dutch Book Arguments

Dutch book arguments (or DBAs) begin with a premise linking betting odds to credences. If an agent believes A to degree x, the argument goes, her credences license her in accepting a bet that pays $\$Sx$ if $\neg A$ and $\$S(x - 1)$ if A, for any positive or negative real number S. Furthermore, betting is assumed to be additive: if an agent's degrees of belief license her in buying each bet in the set individually, then they license her in buying all of the bets together.

The centrepiece of the DBA is a Dutch book theorem i.e., a mathematical proof that, given premises 1 and 2, any agent who violates a purported constraint (such as Kolmogorov's axioms) will be licensed in accepting a Dutch book—a set of betting transactions that jointly add up to a sure loss. The argument also requires a converse Dutch book theorem—a mathematical proof that obeying the constraint provides immunity to Dutch books. Given the additional claim that vulnerability to a Dutch book is grounds for rational criticism, it seems to follow that agents are rationally obligated to obey the probability calculus.

If the idea of a bet is taken literally, DBAs are a disastrous flop [29, 102]. Bets made with real currency, rather than some more abstract form of utility, satisfy neither additivity nor the initial assumption about when bets are licensed. It is unclear why dispositions that would result in a sure loss in the presence of a Dutch bookie should convict an agent of irrationality in ordinary circumstances, where Dutch bookies are rare. Finally, the aim of DBAs is to establish the Kolmogorov axioms as canons of rational *belief*, and it is unclear what the *pragmatic* norm of avoiding a sure loss has to do with *epistemic* norms of rationality.

Some defenders of DBAs reply that these objections take the notion of betting too literally. The problem with vulnerability to Dutch books is not the monetary loss, but rather an inconsistency in the evaluation of bets [14, 36, 41, 88]. However, Maher [70] argues that the relevant evaluative concepts cannot be understood without presupposing that partial beliefs are probabilities.

Problems with DBAs are likely to have serious repercussions, since DBAs have been used to defend principles beyond the Kolmogorov axioms: Countable Additivity [108], Conditionalisation [65, 95], Jeffrey Conditionalisation [2], Reflection [99], and the thirder solution to the Sleeping Beauty problem [15, 37].

As Hájek and Hitchcock [31] point out, these additional DBAs raise a puzzle. Converse Dutch book theorems [51, 59] purport to show that any agent who obeys the probability calculus is immune to a Dutch book. Yet other Dutch book theorems purport to show that any agent who violates Conditionalisation, Reflection, etc. is vulnerable to a Dutch book. Since one can obey the probability calculus while violating Conditionalisation, Reflection, etc., how can both sets of theorems be true?

The answer is that philosophers implicitly disagree about what counts as a Dutch book. The converse Dutch book theorem for the Kolmogorov axioms assumes that collections of bets are finite, while the DBA for Countable Additivity relies on so-called Dutch books that contain infinite numbers of bets. Arntzenius et al. [4] develop examples which call the additivity assumption into question for such sets. The converse Dutch book for Kolmogorov's axioms assumes that all bets in a collection happen at a single time, while the DBAs for Conditionalisation and Reflection require *diachronic* Dutch books, containing bets made at different times. Many authors question whether vulnerability to diachronic Dutch books is grounds for rational criticism—granted that I am rationally required to be consistent at a time, am I also rationally required to be consistent across times? So a key task, for those who aim to put the foundations of DBAs in order, is to settle which DBAs are sound and why.

2.3 Accuracy Arguments

DBAs relied heavily on a dominance argument—it is irrational to choose a course of action that will leave you worse off no matter what. However, DBAs relied on a pragmatic goal—avoiding monetary loss. What if we interpret the goal of partial belief as epistemic from the beginning as Joyce [47] suggests?

Beliefs, philosophers generally agree, ought to match the way the world is. So, we might conjecture, should partial beliefs. Let us use the word *accuracy* for the requisite type of matching. In the case of full belief, it is easy to say what accuracy amounts to: a belief in p is accurate iff p is true, while disbelief in p is accurate iff p is false. Furthermore, a person's entire belief state is better insofar as more of the attitudes in it are accurate.

But what counts as accuracy for partial beliefs, which don't straightforwardly admit of truth or falsehood? Joyce suggests that accuracy comes in degrees: the closer a degree of belief is to the truth, the more accurate it is. Furthermore, in addition to measuring the local accuracy of individual beliefs, we can measure the degree to which an entire credence function is accurate. Joyce suggests that a rational agent's credences should avoid being *dominated* with respect to accuracy—i.e., there should be no credence function that is more accurate than it in every possible world.

While it is difficult to pinpoint a unique measure of accuracy, Joyce formulates a set of constraints that any acceptable measure must satisfy, and shows that on any measure satisfying these constraints, obeying the Kolmogorov axioms is necessary and sufficient for avoiding accuracy-domination.[1] Leitgeb and Pettigrew [62] appeal to an additional norm of minimising expected inaccuracy (where expectation is defined in a way that does not suppose that credences are probabilities) to narrow Joyce's class of measures still further, to the set of quadratic scoring rules.

Like DBAs, accuracy arguments have been used to defend principles that extend beyond the probability calculus. Pettigrew [77] uses an accuracy argument to defend the Principal Principle. His argument replaces Joyce's definition of accuracy, in terms of proximity to the truth, with a definition of accuracy as proximity to the chances. Greaves and Wallace [27], Leitgeb and Pettigrew [61], and Easwaran [18] use expected accuracy considerations to argue for Conditionalisaton; Easwaran also argues for principles of Reflection and Conglomerability, while Leitgeb and Pettigrew develop an expected accuracy argument against Jeffrey Conditionalisation and in favour of an alternative principle. Kierland and Monton [53] argue that one version of an accuracy-domination argument justifies the halfer solution to the Sleeping Beauty problem, while another version justifies the thirder solution, and argue that the halfer version is more plausible in one class of cases.

The amount of agreement between DBAs and accuracy-dominance arguments is striking, and is not a coincidence. Williams [106] shows that both are corollaries of the separating hyperplane theorem, and that for each Dutch book, there is a corresponding accuracy-domination result. Thus, accuracy-domination arguments can be used to provide a purely epistemic grounding for principles typically supported with DBAs. Note, however, that Williams' argument concerns only results that appeal to dominance, and not those that appeal to expectation.

2.4 Representation Theorems

Another type of argument relies on representation theorems, which link credences to qualitative states of an agent assumed to be more fundamental than credences. Drawing on a distinction from Konek [56], we can divide representation theorems into the *decision theoretic*, which take the basic states to be binary *preferences* [1, 45, 81], and the *epistemic*, which take the basic states to be *comparative probabilities* of the form 'A is more likely than B' [83, 101].

Both types of representation theorems begin by setting up constraints on the qualitative states—where these constraints may be interpreted either as requirements of rationality, or as true generalisations about human beings. They then designate an isomorphism between preferences or comparative belief orderings, on one hand, and probability-utility pairs or probabilities on the other. In particular, let us say that a

[1] [71] objects that the constraints are implausible; [49] responds by developing a revised set of constraints.

pair $\langle P, U \rangle$, consisting of a probability function P and a utility function U, *represents* a weak preference ordering $\succsim$ just in case for all A and B in the domain of P, $A \succsim B$ iff the expected value of A (according to P and U) is greater than the expected value of B (according to P and U). Likewise, let us say that a probability function P *represents* a comparative belief ordering $\succsim$ just in case for all A and B in P's domain, $A \succsim B$ iff $P(A) \geq P(B)$.

The representation theorem itself consists of an existence component, which shows that every preference ordering (comparative belief ordering) is represented by some pair $\langle P, U \rangle$ (probability function P), and a uniqueness component, which shows that if $\langle P, U \rangle$ and and $\langle P', U' \rangle$ represent the same preference ordering (P and P' represent the same comparative belief ordering), then $P = P'$.[2]

Since the constraints on the qualitative comparisons were meant to be canons of rationality (or true generalisations about human beings), it is meant to follow that rational agents (or actual human beings) have partial beliefs satisfying the probability calculus.

Meacham and Weisberg [73] raise a dilemma for decision theoretic representation theorems. On the one hand, if the constraints on basic comparisons are taken as claims about actual people then—at least in the case of the usual constraints on preferences—there is a wealth of empirical evidence against them [50].[3] On the other hand, if the constraints on preferences are taken as ideals of rationality, the representation theorem fails to establish that ideal agents have credences that obey the probability calculus. Rather, the representation theorems establish only that the preference of ideal agents are *as if* those agents had credences that obeyed the probability calculus, and acted so as to maximise expected utility. But there is more to beliefs and desires (or credences and utilities) than the preferences they give rise to—a person could be irrational by adopting preferences that failed to maximise expected utility according to her credence function, even though they maximised expected utility according to some credence function or other (see also [30, 112]).

Another problem for decision theoretic representation theorems is that, like DBAs, they seem to involve pragmatic rather than epistemic rationality. What does the pragmatic consistency of preferences have to do with the logical consistency of beliefs?

Epistemic representation theorems are on more secure footing. It may be implausibly pragmatic to claim that partial beliefs are nothing over and above the preferences they (and utilities) give rise to. However, it is more plausible that there is nothing

[2] In decision theoretic representation theorems, the uniqueness component also requires that the utility function be unique up to positive linear transformation, i.e., if both $\langle P, U \rangle$ and $\langle P', U' \rangle$ represent the same preference ordering, then there exit some positive real number a and real number b such that $U = aU' + b$.

The decision theoretic representation theorem in [45] famously fails to secure the uniqueness half of the theorem; some preference orderings are represented by a $\langle P, U \rangle$ and a $\langle P'U' \rangle$ where $P \neq P'$. Joyce [48] shows we can regain uniqueness by supplementing Jeffrey's constraints on preferences with additional constraints on comparative belief, and on the relationship between comparative belief and preference.

[3] Note, however, that one can to weaken the constraints so that agents are no longer representable as expected utility maximisers, while still allowing that they have a representation that includes a unique *credence* function [10].

more to degrees of belief than the comparative probabilities they encode—that utilities simply measure certain features of comparative belief. And unlike constraints on preference, constraints on comparative belief are straightforwardly epistemic.

3 Dynamics

Beliefs change as agents learn new information. How rationally ought they to change? More formally, given a prior credal state C and a new constraint E imposed by experience, what is the belief distribution C_E that the agent ought to adopt? Answers typically capture the idea of a minimally disruptive change to C, so that C_E does not change C any more than it must to accommodate E.

The orthodox account of rational belief change, Conditionalisation, presupposes that evidence comes in the form of a constraint that a proposition is learned with certainty. (With harmless ambiguity, I will call this proposition E as well.) Conditionalisation is then a recipe for revising one's credence function—the P part of the triple $\langle \Omega, F, P \rangle$. Where P is the agent's prior credence function and P_E is her posterior credence function revised to accommodate E, and H is an arbitrary proposition, the principle is as follows

Conditionalisation $\quad P_E(H) = P(H|E)$

However, two phenomena raise trouble for conditionalisation.

First, there are cases involving uncertain evidence—e.g., where an observer tries to judge the colour of a swatch under dim light. Jeffrey [45] proposes that in these cases, evidence comes not in the form of a proposition learned with certainty, but in the form of a probability shift over a partition $\{E_1, E_2, \ldots, E_n\}$. (In the example of the coloured swatch, the propositions in the partition might be claims about the colour of the swatch.) Where P is the agent's prior credence function, P_E is her credence function revised to accommodate a shift in probability over the partition $\{E_1, E_2, \ldots, E_n\}$, q_i is the new probability of E_i, and H is an arbitrary proposition, Jeffrey defends

Jeffrey Conditionalisation $\quad P(H) = \sum_{i=1}^{n} q_i P(H|E_i)$

Conditionalisation is just the special case of Jeffrey Conditionalisation where one member of the partition has its probability raised to 1.

Jeffrey Conditionalisation raises puzzles about how to understand evidence. First, how can we identify the relevant partition? (Allowing any partition seems to trivialise the principle.) Second, when two updates involve the same partition, how can we identify whether they involve the same evidence? If we characterise evidence using the q_is, as Jeffrey does, we get the result that Jeffrey conditionalisation fails to commute—the order in which two pieces of evidence are received matters to the end result. Field [22] proposes an alternative definition of evidence using the parameter

$$\alpha_i =_{df} \log \frac{q/p}{(1-q)/(1-p)}$$

where p_i is the agent's prior credence evidence proposition E_i before updating, and q_i is her credence in E_i after updating. Under Field's definition of evidence, Jeffrey conditionalisation commutes—in fact, under certain weak assumptions, Field's is the *only* measure of evidence that yields this result [103]. However, it's not entirely clear that commutativity is the *right* result; Weisberg [105] argues it is incompatible with an appealing type of confirmational holism.

A second set of problem cases for Conditionalisation involves self-locating belief. In Adam Elga's Sleeping Beauty problem [20], an experimental subject, Beauty, is put to sleep on Sunday night, and a fair coin is tossed. If the coin lands tails, Beauty is woken on Monday, subjected to memory erasure, and woken again on Tuesday with no memory of the Monday waking. If it lands heads, Beauty is woken once on Monday, and and sleeps through Tuesday. Beauty knows all this on Sunday, before she is put to sleep. When she wakes on Monday, what credence should she have in the proposition that the coin lands heads (call this proposition 'Heads')?

Elga argues for the answer 1/3 (called the *thirder solution*), while an opposing camp, following [66], argues for the answer 1/2 (the *halfer solution*). Elga argues that the thirder solution contradicts Conditionalisation, since Beauty changes her credences between Sunday and Monday even though she "neither receives new information nor suffers a cognitive mishap" [146]. Arntzenius [3] surveys a variety of similar counterexamples to Conditionalisation, all involving agents who lose track of their location.

Some have argued *contra* Elga that the thirder solution is compatible with Conditionalisation, since Beauty receives new information between Sunday and Monday [104], or since losing track of one's location is a cognitive mishap [39]. However, even before we evaluate any solutions, Sleeping Beauty poses a *prima facie* challenge to Conditionalisation. On Sunday, Beauty knows the self-locating proposition *that it is Sunday*, and presumably assigns that proposition credence 1. On Monday, she learns *that it is not Sunday*. How is she to conditionalise on this new information, which she previously assigned credence 0? Furthermore, the problem threatens to generalise. As people move through time and update their beliefs about their locations, it seems that they are always learning propositions that are incompatible with what they previously assigned credence 1.

The problem is not just that the relevant conditional credences are undefined—we have seen that there are ways of extending the concept of conitional probability cases where the condition has probability 0. The problem is to work out how Beauty's Sunday credences constrain her Monday credences. And a fully adequate solution requires not merely a verdict about the example, but a general rule for updating self-locating beliefs from which the specific verdict can be derived. Titelbaum [97] gives a helpful overview of various proposals in the literature. Once again, the concept of evidence is a central issue—in particular, how to understand and use self-locating evidence.

4 Evidential Probability

Partial beliefs are subject not only to norms of consistency (like the Kolmogorov axioms) and inference (like Conditionalisation), but also to norms of justification.

What credences should a partial believer adopt in light of her total evidence? The relevant credences are not subjective probabilities (since they may not be held by any particular subject), but evidential probabilities. The project of pinning down a unique evidential probability for every state of evidence remains radically incomplete.

A key principle underpinning the study of evidential probability is that probabilities should avoid being more opinionated than they are entitled to be. The simplest version of this idea is the Principle of Indifference, formulated as followed by Keynes [52, 42].

PI If there is no known reason for predicating of our subject one rather than another of several alternatives, then relatively to such knowledge the assertions of each of these alternatives have an equal probability.

The continuous generalisation of the PI is the injunction to adopt a uniform probability measure over all the possibilities. Unfortunately, both the discrete and the continuous versions of the PI yield different answers when the same problem is specified using different parameters. The continuous version of the problem can be illustrated by the following example from van Fraassen [100, 303],

> A precision tool factory produces iron cubes with edge length ≤ 2 cm. What is the probability that a cube has length ≤ 1 cm, given that it was produced by that factory?

In the discrete case, dividing the possible side lengths into $\{(1, 1](1, 2]\}$ yields the answer 1/2, while dividing them into $\{(1, 1/2](1/2, 1], (1, 2]\}$ yields the answer 1/3. In the continuous case, a uniform distribution over edge length yields the answer 1/2, a uniform distribution over side area yields a probability of 1/4, and a uniform distribution over cube volume yields the answer 1/8.

Jeffreys [46] suggests that problem specifications should include a set of allowed transformations of the parameters, and that evidential probabilities must be invariant under the appropriate transformations. In van Fraassen's example, the uniform distribution is not invariant under the transformations of squaring and cubing, but the log uniform measure is. North [75] argues that physics provides empirical knowledge of the allowable transformations, which correspond to symmetries in the dynamic laws.

Jaynes [44] proposes another way of formalising the idea that evidential probabilities should avoid being more opinionated than the evidence warrants. His principle of maximising entropy (henceforth *Maxent*) requires that evidential probabilities be chosen so as to maximise entropy within whatever additional constraints our evidence places on our credence function—where entropy is defined as $-\sum_i p_i log p_i$, with the p_is denoting the probabilities of individual possibilities in Ω. Both the discrete Maxent and its continuous generalisation yield different results when fed different specifications of the same problem [85].

Kullback and Leibler [57] propose a measure of entropy relative to an initial probability distribution, which is not sensitive to changes in problem specificiation. However, the Kullback-Leibler definition only characterises entropy relative to a prior probability distribution. We might interpret this either as a way of making explicit a free parameter, or as a way of rewriting Maxent as a dynamic rule constraining how credence should change in the light of evidence. This second, dynamic interpretation raises questions about the relationship between Maxent and Conditionalisation.

Although Maxent and Conditionalisation agree in a range of cases [85, 89] , there are also cases in which they conflict [84, 85]. What is the appropriate philosophical response to this conflict? Seidenfeld and Skyrms [89] argue that Conditionalisation is the more basic constraint, while Maxent is merely a useful heuristic. Williams [107], on the other hand, argues that Conditionalisation is compatible with—and is in fact a special case of—Maxent. Uffink [98] points out that to judge the relationship between Maxent and Condtionalization, we need a bridge principle connecting empirical evidence to constraints on probability distributions, and that different bridge principles yield different results.

The concept of *exchangeability* captures yet another sense in which a probability distribution may be uninformative. A sequence of probabilistic trials is exchangeable just in case any two finite subsequences containing the same number of each outcome type receive the same probability—in intuitive terms, just in case the order in which trials are labeled carries no information about their outcomes. Carnap [11] uses the concept of exchangeability to characterise probability measure m^*, and an associated confirmation function c^* that assigns probabilities to the outcomes of unobserved conditional on observed frequencies. Later, Carnap [12] generalises c^* to a class of c_λ functions, where the parameter λ measures how quickly the functions learn from new evidence. (See [111] for an overview of subsequent generalizations.) Like the PI and Maxent, the class c_λ is dependent on the language we choose to describe outcomes. (See [96] for an argument that this language-dependence runs deep.) Even fixing the language does not pin down a unique evidential probability function, since there is an additional degree of freedom for λ, the rate of learning.

5 Chance

Belief should respond not just to symmetries, but to the chancy dispositions of physical objects. The Principal Principle Lewis [63, 87] is a nuanced account of how chance should constrain partial belief.

> Let C be any reasonable initial credence function. Let t be any time. Let x be any real number in the unit interval. Let X be the proposition that the chance, at time t, of A's holding equals x. Let E be any proposition compatible with X that is admissible at time t. Then
>
> $$\textbf{(PP)}\ C(A|XE) = x$$

Note that C is not the credence function of any real individual, but a *rational initial* credence function—i.e., an evidential probability function. Lewis does not assume that there is a uniquely rational initial credence function; 'rational' here means 'rationally permissible'. A person's credences are rational iff they coincide with the credences assigned conditional on her total evidence by some rational initial credence function C. Since the PP holds for all propositions E (including propositions that might constitute an agent's total evidence), it entails that a rational agent's credence, conditional on the chance of A being x will be x—provided her total evidence is admissible.

The PP does not require that the credences of rational agents match the chances. Rather, it requires that the credences of rational agents match those agents' subjective estimates of the chances. Furthermore, this requirement is in place only if the rational agents have no better source of information that trumps the chances; information is admissible iff it is 'the sort of information whose impact on credence about outcomes comes entirely by way of credence about the chances of those outcomes' [63, 92]. So the PP is not a claim about accuracy, but a claim about justification.

Having introduced chance via its connection to partial belief, we can ask additional questions about its nature—in particular, about its relationship to concepts such as determinism, frequency, and type chance.

Say that there is *deterministic chance* just in case there are events that are both determined—that is, guaranteed by the past, together with the fundamental laws of nature—and chancy—that is, having chances between 0 and 1. Could there be deterministic chances? We encounter objective probabilities in many deterministic settings: statistical mechanics [87], evolutionary theory [90], the physics of gambling devices [93], and deterministic interpretations of quantum mechanics [26]. Schaffer [82] argues that these probabilities are not chances, since they fail to fit the conceptual role of chance. Other authors have been more sympathetic to deterministic chances, emphasising the links between chance, and such concepts as ability [16], special science laws [23], and the best-system analysis of scientific theories [67, 68]. Lyon [69] argues that deterministic objective probabilities constitute a third type of probability alongside credence and chance. Whether we count deterministic probabilities with the chances, or place them in a category of their own, we must give some account of what they are, and how they can guide scientific investigation.

A second question concerns the relationship between chance and frequency. Could there be a world where the chance of an outcome associated with a type of event (say, heads on fair coin tosses) diverged dramatically from the long-run frequency of like outcomes among all the trials? According to the Humean view advocated by Lewis [64], they could not, since the chances are are by the theory that does the best job of systematically summarising patterns in the actual world (where 'best' is a matter of balancing the competing theoretical virtues of simplicity, strength, and fit). On the other hand, if the world contains a finite number of IID trials, there will be a positive *chance* of a so-called *undermining future*, in which long-run frequencies diverge from the (actual) chances. When combined with the Principal Principle, the possibility of undermining futures generates a contradiction [64]. Unless we can break the connection between Humeanism and undermining, we must either give up on Humeanism about chance, or replace the Principal Principle with one of the alternatives proposed in the literature [32, 42, 64, 80].

A third question concerns the relation between single-case chances that attach to propositions, like the chances that figure in the PP, and 'general' or 'type' chances. Can one of these conceptions be reduced to the other, and if not, what is the relation between them? Fetzer [21] proposes that attributions of chance to types of trials are merely generalisations about tokens belonging to those types. Ismael [43] proposes an analysis of token chance in terms of type chance, while Meacham [72] and Hoefer

[38] suggest that we do entirely without token chances altogether, and rewrite the PP in terms of type chances.

A set of techniques that deserves mention is *Bayesian networks*, which encode relations of probabilistic dependence among event types using graphs [54, 76, 91]. The graphs contain nodes representing random variables, and arrows representing relations of conditional dependence and independence between variables. The arrows can also be interpreted as representing relations of direct *causal* dependence between variables. Many of the applications of Bayesian networks require that, under suitable conditions, the causal and probabilistic interpretations of the arrows line up. These assumptions are useful, since they enable researchers to infer causal claims from data about correlation, and to use causal hypotheses to predict correlations. Their status, and their significance for the foundations of probability, are less clear. While some have argued for an analysis of causation in terms of probability [19, 24, 25, 58, 74, 92, 94], others argue that the links between causation and probability are at best defeasible methodological assumptions [13, 109].

6 Conclusion

Probabilities are (among other things) subjective degrees of belief, and we can learn much about the foundations of probability by considering the norms that govern degrees of belief. Norms of logical consistency include Tautology, Zero, Finite Additivity, and perhaps additional principles governing infinite additivity and conditional probability. These norms may be justified either on pragmatic or on purely epistemic grounds. Each of the major pragmatic justifications (DBAs and decision theoretic representation theorems) corresponds closely to a more persuasive, purely epistemic justification (accuracy arguments and epistemic representation theorems).

The orthodox norm governing probabilistic reasoning is Conditionalisation. However, uncertain evidence and self-locating evidence pose challenges to Conditionalisation, and highlight the need for a clearer philosophical account of evidence.

Norms of justification for partial belief motivate the evidential interpretation of probability. The idea of uninformative probability distributions has appeared in various guises: the Principle of Indifference, invariant probability distributions, the Maxent principle, and exchangeability. All of these concepts require a choice of parameter, whether it be a set of transformations under which the distribution is to remain invariant, a prior probability distribution relative to which entropy is measured, or a preferred language for describing the outcomes of trials. When Maxent is understood as a dynamic rule, the need for a clear account of evidence re-emerges.

Finally, the Principal Principle is a norm of justification that connects credence to chance. Chance, in turn, bears interesting conceptual relations to determinism, frequency, repeatable trials, and causation. Sorting out these relationships in detail will help us to sort out the nature of scientific confirmation. Two particularly promising applications are objective probabilities in the special sciences, and the apparatus of Bayesian networks.

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