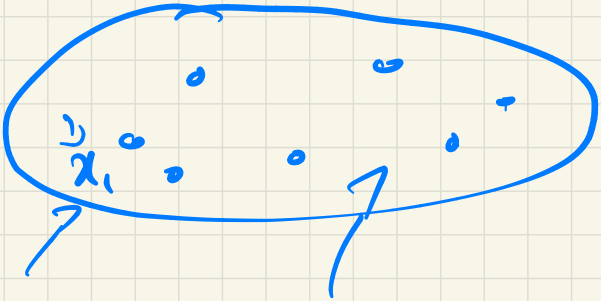
Lecture 3 — 7/11/2022

Matrix Algebra

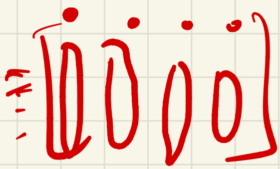


$x \in \mathbb{R}$

$x \in \mathbb{N}$

$x \in \mathbb{Z}$

$M = [\quad]$

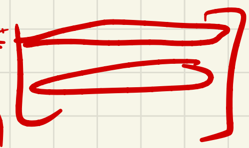
$M =$ 

100×5

x_2 o

$x_3 \in \mathbb{R}^{100}$

o

$M_{2 \times 5}$ 

150×100

x_4

o

x_1 o

x_4

o

5

$M = [\quad]$

- Four fundamental Spaces
- Rank
- Independence of vectors
- orthogonality
- Gram Schmidt
- Projection
- Decomposition

$$A = \begin{bmatrix} | & & & & | \\ c_1 & - & - & - & c_n \\ | & & & & | \end{bmatrix}^{n}$$

$$c_i \in \mathbb{R}^m$$

$$A = \begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_m & - \end{bmatrix}$$

$2x - y = 0$

$-x + 2y = 0$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$\boxed{Ax = b}$

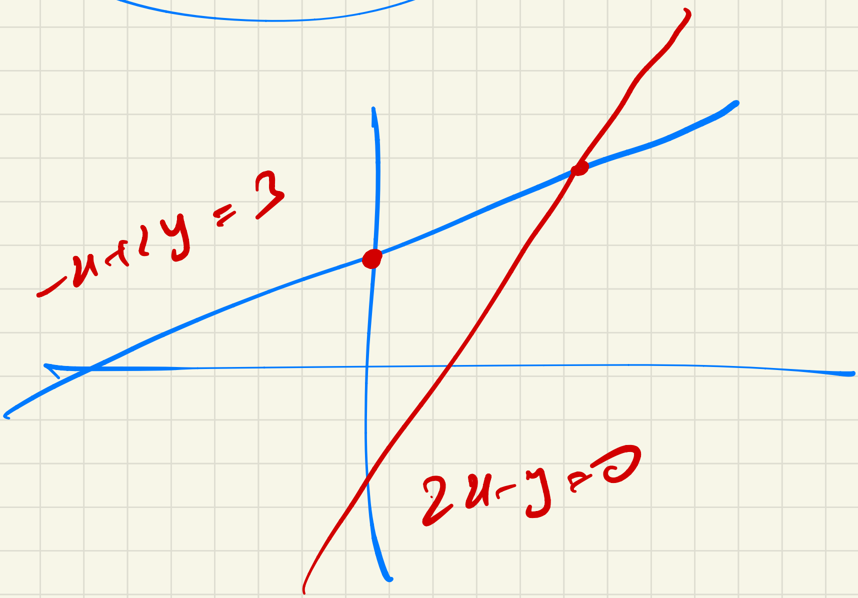
$Ax = b$

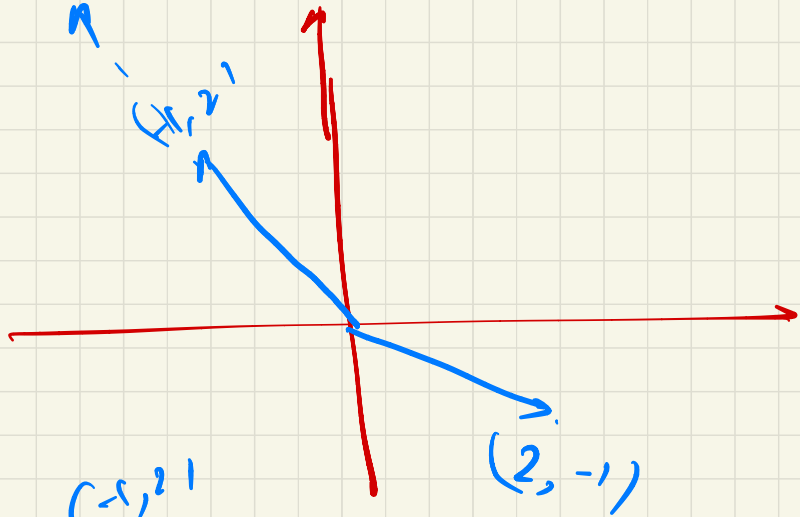
$\|Ax - b\|^2$

$Ax \approx \hat{b}$

$$2x - y = 0$$

$$-x + 2y = 0$$





$$\begin{array}{cc} (2,-1) & (-1,2) \\ \end{array}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

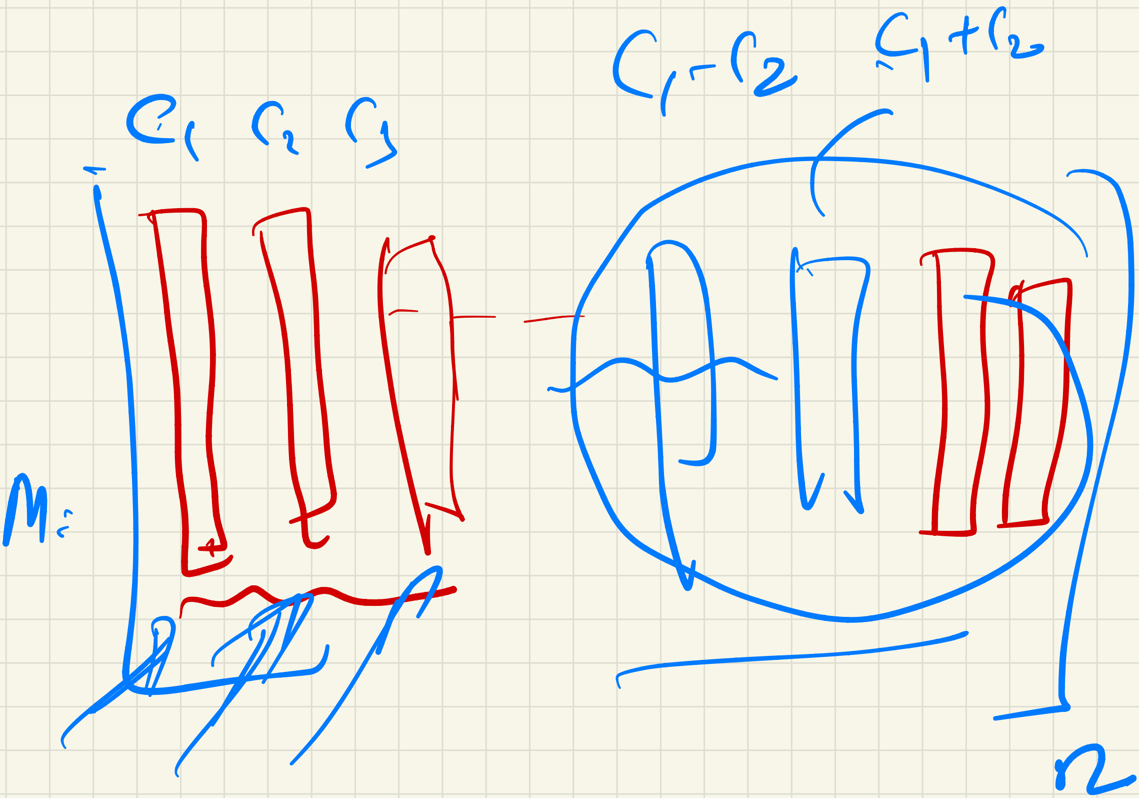
$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$x = 1, \ y = 2$

⇒ Column space

⇒ Row space

⇒ Rank of a matrix



Matrix Rank

→ Column Rank

→ Row Rank

Column R : no of independent columns

Row rank : no of independent rows

matrix rank ⟹ row rank = column rank

Matrix Decomposition

i) $A = LU$ ← Lower triangular and upper triangular

ii) $A = QR$ ← QR

iii) $A = Q\Lambda Q^T$ ← EVD ($A \in S^n$)

iv) $A = V\Sigma V^T$ ← SVD (

v) $A = PQ$ ← Matrix fact

($A \in R^{m \times n}$ (2)

($A = PQ$

$P \in R^{m \times k}$ $Q \in R^{k \times n}$

vi) $A = LL^T$

EVD

Weighted norms: ($\|\cdot\|_p$

$\|\cdot\|_p = \left(\sum |x_i|^p\right)^{1/p}$

→ Scalar: $a \in R^1$

→ Vector: $\underline{u} \in R^n$

→ Matrix: $P \in R^{m \times n}$

$\|\cdot\|_Q$

Weighted Norm

$\underline{v} \in \mathbb{R}^n$

$A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$

$\|\underline{v}\|_A^2 = \sum_{i=1}^{n} a_i v_i^2$; $a \in R^T$

$\|\underline{v}\|^2 = \sum_{i=1}^{n} v_i^2$

$\underline{v}^T A \underline{v}$

$\|\underline{v}\|_A^2$ a valid norm?

$\underline{a} = [a_1 \; \ldots \; a_n]$

a $\forall a_i \geq 0$ then also $\sum a_i v_i^2 > 0$

$$\underline{v} = \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \|v\|_A = \sum_{i=1}^{n} a_i v_i^2$$

$\underline{a} = [0, a_1, \ldots, a_n]$

$$= 0 \cdot v_1^2 + a_1 \cdot 0 + \ldots + 0$$

$$= 0$$

$\|v\|_A = 0$ even if $\underline{v} \neq 0$

$\forall a_i > 0$

$\| \underline{v} \|^2_A$

$A = \begin{bmatrix} a & \cdots & a_n \end{bmatrix}$

$P^2 = \begin{bmatrix} & \end{bmatrix}$

$\boxed{\underline{v}^T A \underline{v} > 0}$ $\forall \underline{v}$

Positive definite matrix 'A'

$\| \cdot \|^2_A$

Let's generalize $\underline{v} \in$

$$\|v\|_P^2 = v^T P v$$

EVD (Eigen Value Decomposition)

$A \in S^n$ (Symmetric)

$$\boxed{A = Q \Lambda Q^T}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$

$Q^T Q = I$, $\quad q_i^T q_j = 0 \quad \forall i \neq j$

$\qquad\qquad\quad q_i^T q_j = 1 \quad \forall i = j$

$Q Q^T = Q^T Q = I$, $\quad Q^{-1} = Q^T$

$q_i^T q_j^T = 0 \quad \forall i, j \qquad q_i^T q_j = 1$

① $A = \Sigma$

$A = Q \Lambda Q^T$

$$= \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}$$

$$A = \sum_{i=1}^{n} \lambda_i \boxed{q_i q_i^T}$$

↳ rank 1

$xx^T =$

$$= \lambda_1 \begin{bmatrix} | \\ q_1 \\ | \end{bmatrix} \begin{bmatrix} - & q_1^T & - \end{bmatrix} + \cdots + \lambda_n \begin{bmatrix} | \\ q_n \\ | \end{bmatrix} \begin{bmatrix} - & q_n & - \end{bmatrix}$$

$$A = \boxed{\lambda_1 \cdots \lambda_s}$$

① $\operatorname{tr}(A) = \operatorname{tr}(\Phi \Lambda \Phi^T) = \operatorname{tr}(\Phi^T \Phi \Lambda) = \operatorname{tr}(\Lambda)$

$$= \sum \lambda_i$$

② $\det(A) = \prod_{i=1}^{n} \lambda_i$

Laplacian

<u>Positive Definite Matrix</u>

- When $\|v\|_P^2 = v^T P v$ is a valid norm?

$\exists!$ Conditions on $P \in S^n$ such that

$v^T P v$ is a norm

$\|\underline{v}\|_P^2 > 0 \quad \forall\, \underline{v} \neq 0$

$$\boxed{P \text{ s.t. } \underline{v}^T P \underline{v} > 0 \quad \forall\, \underline{v} \neq 0}$$

Now we will relate this EVD

$\Rightarrow$ P: $\underline{v}^T P \underline{v} > 0$

$P = Q \underline{\Lambda} Q^T$

$\underline{v}^T Q \Lambda Q^T \underline{v}$

$\underline{\check{v}}$

$$\boxed{\underline{\check{v}} = Q^T \underline{v}}$$

$\underline{\check{v}}^T \underline{\Lambda}\, \underline{\check{v}} > 0$

$\Rightarrow \sum \lambda_i \check{v}_i^2 \geq 0$ $\qquad \lambda_i > 0$

$$\sum_{i=1}^{n} \lambda_i \tilde{v}_i^2 > 0 \Rightarrow \lambda_i > 0 \ \forall i$$

PD $\Rightarrow \forall$ eigenvalues should be positive.

$$\boxed{P > 0}$$

P.D

1) $\underline{v}^T P \underline{v} > 0 \quad \forall \underline{v}$

2) $\lambda_i(P) > 0 \quad \forall i$

3) $P = LL^T$ (Cholesky-decompos

$\rightarrow L$ is a lower triangular matrix

$P > 0 \qquad \underline{v}^T P \underline{v} = \underline{v}^T L L^T \underline{v} = \| L^T \underline{v} \|_2^2 > 0 \quad \forall v \neq 0$

> 0 ?

$$\underline{v}^T P \underline{v} = \|L^T \underline{v}\|^2 > 0 \quad \forall \ \underline{v} \neq 0$$

L is full rank

$L^T \underline{v} \neq 0$ for $\underline{v} \neq 0$