Lecture - 4 10/1/2022

$A \in R^{m \times n}$

$A \in S^n$ $\quad R^{m \times m}$

$\Rightarrow$ PSD & PD

$\Rightarrow$ EVD

$S^n$

$A : [\quad]_{m \times n}$ $\quad m \times n$ $\quad m$

$$A = \sum_{i=1}^{r} \lambda_i u_i u_i^T$$

$r \leq n$

$\lambda \leftarrow r$

$$= \lambda_1 \boxed{u_1 u_1^T} + \lambda_2 u_2 u_2^T + \cdots$$

$A \in R^{m \times n}$ rank $= r$

Rank of a matrix Invertibility

- $Ax = b$ | $AC = I$ | $CA = I$

→ why a full rank matrix is invertible?

$A \in R^{m \times n}$

$A \in R^{m \times n}$

A rank 'r' $r \leq m$

$r \leq n$.

- $A \in R^{m \times n}$ $\hookrightarrow$ rank $= n$, — (i)
rank $= m$ — (ii)

$S_1 = \underline{AA^T}$

$S_1 : [\ ]_{m \times m}$

$S_2 = \underline{A^T A}$

$[\ ]_{n \times n}$

i) $m > n$, rank $= n$

$$A^{\dagger} = (A^T A)^{-1} A^T$$

$BA = I$

$B = (A^T A)^{-1} A^T$

ii) $n > m$, rank $= m$

$$A^{\dagger} = A^T (A A^T)^{-1}$$

$AC = I$

$C = A^T (A A^T)^{-1}$

$A \in R^{m \times n}$

rank$(A) \leq \min(m, n)$

## Glam Matrix $A \in R^{m \times n}$

$$S = A^T A$$

(i) 'S' is a symmetric and square.

(ii) S has same null space as 'A'

(iii) If 'A' has independent columns then $S = A^T A$ is invertible

iv) If $\det(A^T A) = \det(S) = 0$

→ columns are dependent.

v) Rank of Glam matrix is the dimension of column space.

vi) $A^T A$ is a positive semidefinite matrix, if not full column rank.

vii) If 'A' is full column rank, S is PD.

vii

viii) $\lambda_i(S = A^T A) \geq 0$

$> 0$ ($\rightarrow$ PD)

$\rightarrow$ A has full column rank)

$\lambda_i(S = A^T A) \geq 0$ ($\rightarrow$ PSD)

SVD

# Fundamental Subspaces

$$\boxed{Ax = b}$$

$A \in \mathbb{R}^{m \times n}$

$A$

$$A = \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}$$

$\in \mathbb{R}^m$

$Ax = 0$

$\underline{a_1}$

$$A = \begin{bmatrix} — & b_1 & — \\ — & b_2 & — \\ — & b_m & — \end{bmatrix}$$

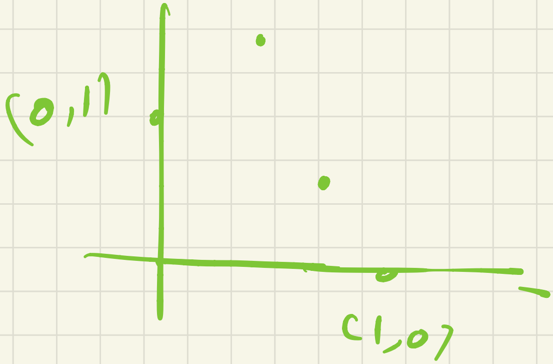
$A = [\;|\;\;]$

$A^T \underline{v}$

$A \underline{u}$

$m \times n$

$u \in \mathbb{R}^{n \times 1}$




$$a_1 : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\alpha_1 a_1 + \alpha_2 a_2$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Big[ \quad \Big] \qquad \Big[ A \Big] \Big[ \alpha \Big]$$

Column Space / Range space

$C(A)$ $\quad$ $R(A)$

$$R(A) = \{ \underline{v} \in \mathbb{R}^m \mid \underline{v} = A\underline{u} \ \forall\, u \in \mathbb{R}^n \}$$

$\Rightarrow$

$\underline{u}' \rightarrow u_1 a_1 + u_2 a_2 + \cdots + u_n a_n = \underline{v_1}$

$u' \longrightarrow$ $\quad \underline{v_2}$

- $R(A)$ is a Subspace..

$v_1 \in R(A)$ and $v_2 \in R(A)$

$\alpha_1, \alpha_2 \in R$

$\alpha_1 \underline{v_1} + \alpha_2 \underline{v_2} = v_3 \in R(A)$

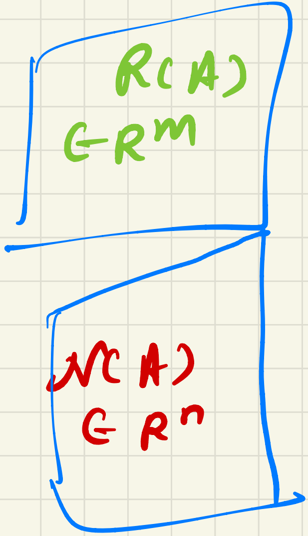
$\dim(R(A)) = \#$ no. of linearly independent columns

Null Space of $\underline{A}$

$$N(A) = \{ \underline{u} \in R^n \mid A\underline{u} = 0 \}$$

Range space $(A^T) = \{ y \in R^n \mid u = A^T \underline{v} \}$

$$N(A^T) = \{ \underline{v} \in R^m \mid A^T \underline{v} = 0 \}$$




$A \in \mathbb{R}^{m \times n}$

$R(A^T) \in \mathbb{R}^n$

$N(A^T) \in \mathbb{R}^m$

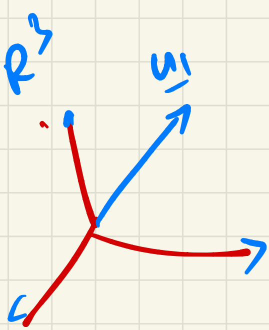
- See on selection

→ Pick any 'w' orthogonal to range space of A

$$\underline{w} \in R(A)^{\perp} \Rightarrow w^T(Au) = 0 \ \forall u$$

$$\Leftrightarrow u^T(A^T w) = 0 \quad \forall \ u \in \mathbb{R}^n$$

↳ this vector is orthogonal to all u!.




⇒ this is not possible until

$$A^T w = 0$$

$$\Rightarrow w \in N(A^T)$$

→ Any element orthogonal to range space of 'A'

lies in the null space of A'

$$\mathcal{R}(A)^{\perp} = \mathcal{N}(A^T)$$

$$\mathcal{N}(A)^{\perp} = \mathcal{R}(A^T)$$

## SVD

$A \in \mathbb{R}^{m\times n}$, $w$ $r = \text{rank}(A) \le \min\{m, n\}$

$$A \in \mathbb{R}^{m\times n} = \sum_{i=1}^{r} \sigma_i v_i u_i^T = V \Sigma U^T$$

$$V^T V = I$$
$$U^T U = I$$

$$= \sigma_1 v_1 u_1^T + \sigma_2 v_2 u_2^T + \dots + \sigma_r v_r u_r^T$$

$m\times 1$ $(1\times n)$

$$\sigma_r > 0$$

Sum

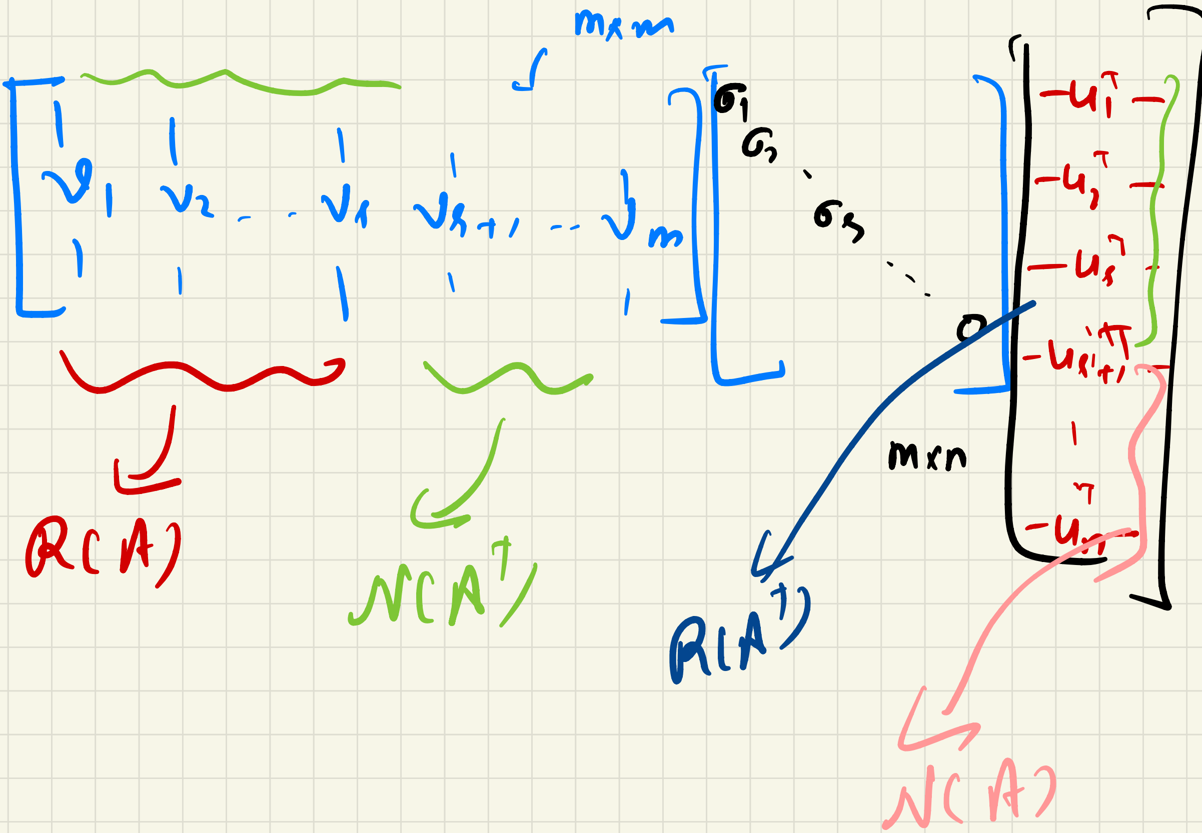
$v_i \in \mathbb{R}^m$, $v_1, v_2, \dots v_r \in R(A)$

$v_{r+1}, v_{r+2} \dots v_m \in N(A^T)$

$u_i \in \mathbb{R}^n$ $u_1, u_2, \dots u_r \in R(A^T)$

$u_{r+1}, u_{r+2} \dots u_n \in N(A)$

SVD =




$= \sigma_1 v_1 u_1^T + \sigma_2 v_2 u_2^T + --- + \sigma_r v_r u_r^T$

$\sigma_i > 0$

this decomposition does not depend on $\{v_{r+i}\}_{i=1}^{m-r}$ $\{u_{r+i}\}_{i=1}^{n-r}$

To conclude

$A \in R^{m \times r}$

SVD | EVD

$$A = V \Sigma U^T; \quad V^T V = I$$

$$U^T U = I$$

$[\sigma_1, \sigma_2 \ldots \sigma_r \ldots d]$

$$A^T A = U \Sigma^T \underbrace{V^T V}_{=I} \Sigma U^T$$

$$= U \Sigma^T \Sigma U^T$$

$$= U \Sigma^2 U^T$$

$A^T A = U \Sigma^2 U^T$

$S = A^T A$

$S = U \Sigma^2 U^T$

EVD

$\lambda_i(S) = \sigma_i^2$ $i = \ldots$

$(\sigma_i(A))^2 = \lambda_i(A^T A)$

$i = 1 \ldots n$

$V =$ Eigenvectors of $(A^T A) \leftarrow$ right singular vector

SVD: $A = U \Sigma V^T$

$A A^T = V \Sigma U^T U \Sigma V^T$

$= V \Sigma^2 V^T$

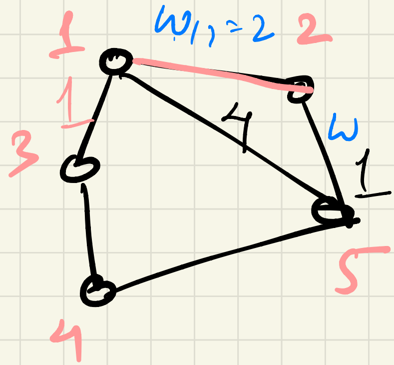
$\sigma_i(A) = (\lambda_i(A A^T))^{1/2}$ $i = 1 \ldots n$

$V$: left singular vectors

eigenvectors of $(A^T A)$

# Graph Matrices

$$G = (\mathcal{E}, W)$$




⇒ Undirected graph

$$C = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 & 0 & 1 \\ 5 & 1 & 1 & 0 & 1 & 0 \end{array}$$

n×n

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 & 4 \\ & & & & \end{bmatrix}$$

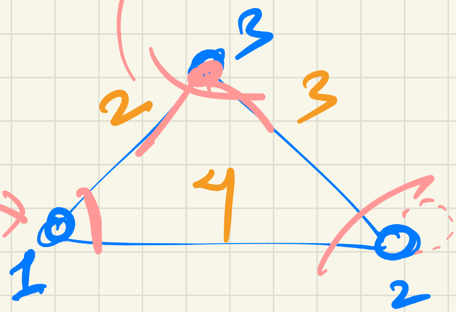
$W =$

$$C = [c_{ij}] = \begin{cases} c_{ij} = 1 & \text{if } i \sim j \\ c_{ij} = 0 & \text{if } i \nsim j \end{cases}$$

$A = \begin{bmatrix} \\ \end{bmatrix}$

$d_{33} = 5$

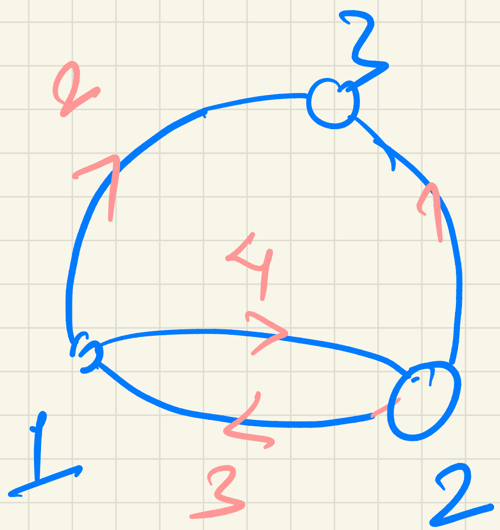
$d_{11}$ ~~6~~




$d_{22}$ 7

$$C = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$W = \cancel{A} = \begin{bmatrix} 0 & 4 & 2 \\ 4 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$




$$D = \begin{bmatrix} 6 & & \\ & 7 & \\ & & 5 \end{bmatrix}$$

Laplacian matrix $\boxed{L = TW}$

$$L = D - W$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 4 & 2 \\ 4 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$\boxed{L = \operatorname{diag}(W \cdot 1) - W}$$

$$D = \operatorname{diag}(W \cdot 1) \to \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix} \to \begin{bmatrix} 6 & & \\ & 7 & \\ & & 5 \end{bmatrix}$$

$$W = A = \begin{bmatrix} 0 & 4 & 2 \\ 4 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$L = D - W$$

$$L \cdot C \, | L \cdot 1 = 0$$

$$L = \begin{bmatrix} 6 & -4 & -2 \\ -4 & 7 & -3 \\ -2 & -3 & 5 \end{bmatrix}$$

(i) Diagonally dominated

$$\forall \; |L_{ii}| \geq \left|\sum_{i \neq j} L_{ij}\right|$$

(ii) M-matrix $L$

(iii) Laplacian matrix is PSD

iv) $L = U \Lambda V^T$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

v) $\lambda_1 = 0 \Rightarrow c = c \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

↳ first eigenvector

vi) K-comp

$$\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_k = 0$$

$\boxed{\text{Cheeger inequality,}}$

⇒ L: Laplacian Beltrami Operator

→ Dirichlet Energy of your system

→ Quantification of signals defined over graph

→ Graph Fourier Transform

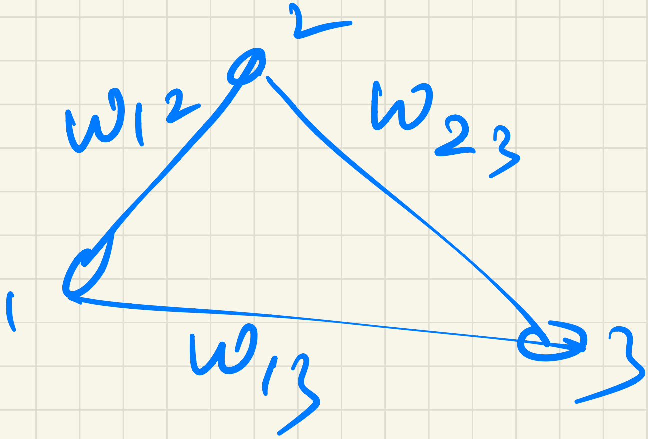
L is a PSD matrix

$x^T L x \geq 0 \quad \forall x$

H.W

$$W = \begin{bmatrix} 0 & w_{11} & \cdots & w_{1n} \\ & 0 & & \\ & & \ddots & \\ w_{1n} & \cdots & & 0 \; w_{nn} \end{bmatrix}$$

$$L = \begin{bmatrix} \sum_{i \neq j} w_{1j} & -w_{12} & \cdots & -w_{1n} \\ & & & \\ -w_{1n} & \cdots & & \sum_{i \neq j} w_{nj} \end{bmatrix}$$




$$W = \begin{bmatrix} 0 & w_{12} & w_{13} \\ w_{12} & 0 & w_{23} \\ w_{13} & w_{23} & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} w_{12}+w_{13} & -w_{12} & -w_{13} \\ -w_{12} & w_{12}+w_{23} & -w_{23} \\ -w_{13} & -w_{23} & w_{13}+w_{23} \end{bmatrix}$$

$L \in \mathbb{R}^{n \times n}$

$X \in \mathbb{R}^{n}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$X^T L X = \sum_{i<j} w_{ij} (x_i - x_j)^2 \geq 0$$

H.W