On the Origins of David Hilbert's "Grundlagen der Geometrie"

MICHAEL TOEPELL

Communicated by C. J. SCRIBA

Introduction

Toward the end of the 19th century a remarkable change came about in the field of the foundations of geometry. Whereas geometry had hitherto been based on empirical facts, it was now seen as a purely formal deductive system. HILBERT (1862–1943) perfected this method in his *Grundlagen der Geometrie*, which first appeared in June 1899 in a Festschrift commemorating the unveiling of the GAUSS-WEBER monument in Göttingen. The ninth edition appeared in 1962 for the HILBERT centenary, and the twelfth for the bicentenary of GAUSS' birth in 1977.

HILBERT'S method immediately gave a new direction to mathematical thought in the 20th century. Its impact on contemporaries has been studied and further developed in numerous publications, for instance in the works by A. SCHMIDT (S 1), FREUDENTHAL (F 1) and VAN DER WAERDEN (W 1) cited in the foreword of the newer editions of the *Grundlagen*. However, little is known about the origins of HILBERT'S *Grundlagen* or about the development that led him to it.

According to the biographies (T 1 ; D 2; B 1; B 2; B 3; W 3; R 1), HILBERT appeared to have worked almost exclusively with algebra and questions concerning the theory of numbers in the years prior to 1899. His publications on the theory of invariants suggest as much. With the *Grundlagen der Geometrie*, however, he presented to the public a thoroughly mature work about an entirely different subject. Referring to the break with the number-theoretical works of 1897/1898 (GA **1**), WEYL has remarked (W 3, 264):

> "There could not have been a more complete break than the one dividing Hilbert's last paper on the theory of number fields from his classical book, *Grundlagen der Geometrie*."

This work is a synopsis of the author's doctoral dissertation, which is to be published by Verlag Vandenhoeck & Ruprecht, Göttingen (T 2 in the bibliography at the end of the article).

In 1935 OTTO BLUMENTHAL wrote in his biographical sketch of HILBERT concerning the Festschrift (B 3, 402):

> "It has brought to its author a worldwide reputation, whereas up to that time he was appreciated only among experts. It is worth tracing the grounds for this success and the development of Hilbert's ideas."

He adds the suggestion:

> "This development seems to have begun very early."[†1]

This is the subject of the following research.

It may well be asked how HILBERT arrived at his creation? What kind of preparatory work did he find? What works did he study? To what extent had he already dealt with the foundations of geometry? Which problems had particularly stimulated him? The answers to these questions have been (and had to be) mostly suppositions, since the book itself offers no information on the subject.

BLUMENTHAL based his supposition on the following:

(1) verbal comments by HILBERT on a lecture of H. WIENER (1891; W 4);
(2) the letter from HILBERT to KLEIN *Über die gerade Linie als kürzeste Verbindung zweier Punkte* (=: BG; as Appendix I since the second edition of the *Grundlagen*) published in the Mathematische Annalen for 1895 – the sole publication on geometry by HILBERT prior to 1899;
(3) HILBERT's lectures on projective geometry delivered in the summer semester of 1891 and his lectures on non-Euclidean geometry delivered during the summer semester of 1894 (B 3, 402foll.).

1. Hilbert's Papers

The manuscripts of both HILBERT's lectures are still extant (Cod. 535 = PG and Cod. 541 = GG). In 1967 the mathematical institute of the University of Göttingen donated HILBERT's papers to the Niedersächsische Staats- und Universitätsbibliothek. They were set in order and have been accessible ever since. The catalogue of his papers covers 741 items, among them letters from roughly 500 correspondents and about 50 manuscripts of his own lectures. Additional unevaluated material is contained in the collection of Professor Dr. OTTO VOLK in Würzburg. All this material makes it possible for us not only to demonstrate the origins of HILBERT's contribution to the Festschrift but also to trace the steps that HILBERT had omitted from his publications. These concern, amongst other

[†] My translations are (I hope) correct according to sense, but they are sometimes not literal.

[1] (B 3, 402): „Sie hat seinem bis dahin nur in Fachkreisen gewürdigten Verfasser den Weltruf eingetragen. Es ist lohnend, dem Grund dieses Erfolges und der Entwicklung von HILBERTS Ideen nachzuspüren.

Diese Entwicklung scheint schon sehr früh eingesetzt zu haben."

things, questions of projective geometry. Thus the posthumous papers allow what hitherto was not possible: to see the master in his workshop. We have sifted and prepared the material with regard to the foundations of geometry (up to 1899). Combined with other material, this has led to the accumulation of a mass of detailed information which gives a comprehensive picture of HILBERT'S development. This synopsis is intended to provide an impression of that development.

The manuscripts of four HILBERT lectures form the basis of the study:
(1) *Projektive Geometrie* (summer semester 1891) =: PG
(2) *Die Grundlagen der Geometrie* (summer semester 1894) =: GG
(3) *Über den Begriff des Unendlichen* (Easter vacation-course 1898) =: FK
(4) *Grundlagen der Euklidischen Geometrie* (winter semester 1898/99) =: EG
The elaboration *Elemente der Euklidischen Geometrie* (=: SG), prepared by HILBERT'S *Assistent* VON SCHAPER, appeared in March 1899 from EG and from that HILBERT developed the Festschrift essay, *Grundlagen der Geometrie* (=: ^{1}GG), published in June 1899. The Easter vacation courses for teachers were arranged by KLEIN in 1892 and were held every second year thereafter in Göttingen (F 4, 279).

Something can also be learned about HILBERT'S geometrical work from letters. However, HILBERT'S own letters, if preserved at all, are widely scattered. The circumstances are not as favorable in HILBERT'S case as in EINSTEIN'S – though the HILBERT-EINSTEIN correspondence is preserved among EINSTEIN'S papers (E 1, 301). Only in rare instances are copies or drafts of HILBERT'S letters preserved from the period before 1900. From our viewpoint the principal correspondents are KLEIN, MINKOWSKI, LINDEMANN and HURWITZ. HILBERT made frequent comments about his geometrical studies in his letters to KLEIN (whose papers are also in Göttingen). This correspondence has been published recently: *Der Briefwechsel David Hilbert–Felix Klein* (1886–1918). Mit Anmerkungen herausgegeben von GÜNTHER FREI *Göttingen 1985*.

HILBERT also had an active correspondence with his friend MINKOWSKI. The roughly one hundred letters from MINKOWSKI to HILBERT (1888–1908) were published in 1973 (M 1). HILBERT'S half of the correspondence seems to be lost; BLUMENTHAL still had these letters when he wrote the biographical sketch (R 1, VIII). On the other hand, an extensive correspondence is preserved with his friend HURWITZ (UB Göttingen) and with his teacher, LINDEMANN, who was only ten years older than he (collection of OTTO VOLK). These letters have not yet been published. Besides this material, the present work also considers the authorities cited by HILBERT and the textbooks of his time.

2. Projective Geometry and Fundamental Questions before Hilbert

Before HILBERT concerned himself with axiomatically constructed geometry, he had worked on projective geometry. Projective geometry is about the invariant properties of projections. In EUCLID, APOLLONIUS and PAPPOS are there theorems to be found which are projectively invariant – though this was not realized at the time. Inspired by his investigations into the continuous transformations of conic sections, KEPLER extended the line through the point of infinity (K 1, 92). DESAR-

GUES showed that the cross ratio is invariant under projection and discovered a series of new projective theorems, among them the so-called "Theorem of DESARGUES" (ed. 1648).

The following theorem goes back to PASCAL (1640): The points of intersection of pairs of opposite sides of a hexagon inscribed in a conic lie on one line (cited by HILBERT in PG 62). PAPPOS had already proved the special case (PG 45) in which the six points lie on two lines (W 2, 477). If one extends to infinity the line on which the points of intersection lie, we get the affine form of the theorem, which HILBERT from 1898 termed PASCAL'S Theorem.

In 1827 MÖBIUS had introduced into analytic projective geometry homogeneous coordinates and the so-called MÖBIUS grid, *i.e.* the possibility of ascribing a real number to every point on a line by constructing harmonic points to three given points. HILBERT did not use this method for his lectures on synthetically based projective geometry PG (1891), but he did use it for GG (1894). In MÖBIUS' time projective geometry was still a part of Euclidean geometry: the Euclidean metric could still be used for it and one can say that the points constructed by the grid come arbitrarily close to every point.

In 1832 JAKOB STEINER summarised the projective theorems of Euclidean geometry from a systematic viewpoint, so that these theorems form a self-contained field with its own natural context. STEINER opens with the statement (S 4, 1, 237):

> "The basic concepts required in geometry are: space, the plane, the line, and the point."[2]

Moreover he emphasizes that everything depends on the relations of incidence of these elements. However, the cross ratio was still being seen as the ratio of two ratios of segments (S 4, 1, 244).

In 1847 the last remaining alien element was removed from projective geometry by VON STAUDT: the Euclidean metric. But in order to show that one can come arbitrarily close to every point by use of the methodically pure MÖBIUS grid, an axiom of continuity or completeness was required. One can thereby relate four points to a cross ratio, *i.e.* one can ascribe to them a real number, or provide each point with a projective coordinate in relation to three fixed points. If coordinates are used, a metric can be introduced. HILBERT was still working along these lines in 1894. Later, however, he did not see that as the object of the foundations. The foundations need go only so far that one can introduce coordinates. VON STAUDT'S primitive propositions about incidence and order in the first three paragraphs of his *Geometrie der Lage* are largely formulated as axioms (F 3, 193 foll.). But continuity, which VON STAUDT had silently premised in a proof, is lacking (1847; S 2, 50). KLEIN pointed out this gap in 1872 (K 2, 136 and 140).

Attention was drawn to non-Euclidean geometry both by the publication of the correspondence between GAUSS and SCHUMACHER (1860; K 2, 576) and also by the publications of HELMHOLTZ and BELTRAMI (1868; F 2, 4). KLEIN recognized

---

[2] (S 4, 1, 237): "Die in der Geometrie erforderlichen Grundvorstellungen sind: der Raum, die Ebene, die Gerade, und der Punct."

that a model of non-Euclidean geometry could be derived from CAYLEY's projective determination of metrics (1859; C 2; mentioned in SG 97). In 1875 HANKEL saw in the "neuere Geometrie" the royal road to all mathematics (H 1, 33).

In his papers of 1871/72 *Über die sogenannte Nicht-Euklidische Geometrie* (K 2; K 3) KLEIN investigated the "elliptic", "hyperbolic", and "parabolic" geometries (K 2, 577). By using CAYLEY's determination of metrics he derived these three geometries from projective geometry (K 2, 624). On the basis of the independence, now clarified, of the axiom of parallels, KLEIN in his second paper proceeded to undertake similar investigations

> "with regard to all other presuppositions that underly our geometrical notions" (K 3, 113).[3]

In the same year WORPITZKI, whom HILBERT likewise cited in GG, insisted that "it could be deduced (*abstrahiert*) just as well from another axiom" (W 5, 418). Thus by 1872 the goal was clear: A new system of axioms was needed for projective geometry, from which, by additional axioms, the various metrical geometries could be derived. Underlying this was the idea that a subset of the theorems in Euclidean geometry must be derivable from a subset of the axioms.

PASCH was the first to establish such an axiomatic projective geometry. His work *Vorlesungen über neuere Geometrie* (P 1), which appeared in 1882 derives from lectures delivered in the winter semester of 1873/74 at KLEIN's suggestion (in the foreword KLEIN is the only person mentioned; P 1, IV). PASCH had selected his fundamental propositions (*Grundsätze*) as the smallest possible subset of all the theorems in his geometry (P 1, 17). These theorems are empirically based "directly on observations" (P 1, 17 and 43). PASCH also tackled the question of continuity: he took as a fundamental proposition the Archimedean axiom, whose independence had been demonstrated by STOLZ in 1882 (S 5); it is PASCH's fourth axiom of congruence (P 1, 105). On the other hand, PASCH rejected the continuity axiom laid down by KLEIN ("... that a point which is produced by a convergent infinite process can be assumed to exist"; K 3, 136)[4], because "it is not in harmony with the views here maintained" (P 1, 126). PASCH argued "that an observation cannot be related to an infinite number of things" (P 1, 126). Two points, which cannot be empirically distinguished, should not be seen as different. Another problematical aspect of PASCH's work was that he introduced congruence axioms into projective geometry. (PASCH's *Neuere Geometrie* and its impact have been studied by CONTRO: C 5 and C 4, 11–22.)

Shortly before KLEIN wrote his second paper on non-Euclidean geometry, in 1872, the studies by CANTOR (1872; C 1) and DEDEKIND (1872; D 1, foreword), of fundamental importance for the axiomatisation of real numbers, appeared almost simultaneously. For HILBERT a fundamental aim was to study the importance of the continuity axioms.

---

[3] (K 3, 113): "... mit Bezug auf alle anderen Voraussetzungen, die unseren geometrischen Vorstellungen zu Grunde liegen."

[4] (K 3, 136): "... dass man einen Punkt, der durch einen convergenten unendlichen Process erzeugt werden soll, als wirklich existirend annehmen darf."

3. Hilbert's Lectures on "Projektive Geometrie" and his First Encounter with the Fundamental Questions (1880–1893)

As can be seen from one of his surviving exercise-books (Cod. 741, 47) HILBERT had already become aquainted with projective geometry in 1879, while still a schoolboy. In LINDEMANN's papers there are preserved, besides the aforementioned letters from HILBERT, the minute-books of the mathematical colloquia and seminars from his period in Königsberg. An impression can be gathered from these minute-books of how geometrical problems and the literature concerning them – *e.g.*, ERDMANN's *Axiome der Geometrie* (1877; E 2), REYE's *Geometrie der Lage* (1866; R 2) and the analytic geometry of space in the work of CLEBSCH/LINDEMANN (1891; C 3) – were constantly being discussed during the years 1884–1893, while HILBERT was there.

HILBERT "habilitated" in 1886 at the age of 24. Many manuscripts of the lectuıes given by him after that date are preserved. After he had lectured on line-geometry in 1889 and on the theory of the plane algebraic curves in 1890, in the summer of 1891 the subject of projective geometry was introduced into the geometric lectures. Up to that time he had delivered 14 lectures. The manuscript of HILBERT's lectures on projective geometry (PG) comprises 109 pages. HILBERT begins with a survey, dividing geometry into three parts, a division which he consistently followed in later years:

PG 3 "The divisions of geometry.
1. Intuitive geometry."

In this HILBERT included school geometry, projective geometry and the Analysis Situs (topology). He delivered a lecture of this kind in the winter semester of 1920/21 on *Anschauliche Geometrie*, and this he published, together with S. COHN-VOSSEN, in 1932 (H 3).

"2. Axioms of geometry.
(investigates which axioms are used in the established facts in intuitive geometry and confronts these systematically with geometries in which some of these axioms are dropped)".

These considerations led HILBERT to his investigations of independence (already begun in 1894) and to his own geometries in GG.

"3. Analytical geometry.
(in which from the outset a number is ascribed to the points in a line and thus reduces geometry to analysis)".[5]

---

[5] PG 3: "Eintheilung der Geometrie.
1. Geometrie der Anschauung.
2. Axiome der Geometrie.
(untersucht, welche Axiome bei den in der Geometrie der Anschauung gewonnenen

HILBERT lectured on analytical geometry in the winter semester of 1893/94 and again in the winter semester of 1894/95 (Cod. 543). The lecture itself begins:

PG 5 "Geometry is the theory about the properties of space. .../
7 It is based on the simplest experiment that can be carried out, namely drawing."[6]

HILBERT subsequently separated himself from these formulations.

There follow five pages presenting the historical development of geometrical methods (printed unabridged in T 2). As a comparison shows, HILBERT here follows HANKEL (H 1, 1–33 Introduction: *Historische Übersicht des Entwicklungsganges der neueren Geometrie*), though HILBERT does not mention that work. VON STAUDT's method served as a model for HILBERT, because in these

PG 12 "one neither reckons nor measures but simply constructs; one uses neither a compass nor a protractor, but simply the ruler."[7]

Nevertheless, VON STAUDT in his *Geometrie der Lage* has not a single drawing. HILBERT consciously wants to construct his lecture following these pure methods in order to keep projective geometry free from axiomatic or analytic influence. He knew, of course, the axiomatic construction of PASCH's projective geometry.

In the first part of HILBERT's work the contents follow REYE's *Geometrie der Lage*; in the second they follow JAKOB STEINER's lectures on synthetic geometry as elaborated by SCHRÖTER. HILBERT was not entirely satisfied with his own presentation. He decided for the future to recognize intuition as the basis of projective geometry and to use it to its fullest extent. Hereby also a special assumption of continuity (which is not yet mentioned in PG) would be dropped. The purer way turned out to be the harder one.

We learn something about the audience of HILBERT's lectures from a letter of his to KLEIN. HILBERT wrote on 30. 6. 1891 (Cod. Ms. F. KLEIN 9, 769):

> "Our audience consists in the main of two students, to whom may be added as a third man the director of the royal art school—a painter interested in geometry—who also attends my lectures on projective geometry."[8]

Thatsachen benutzt werden und stellt systematisch die Geometrien gegenüber, bei welchen einige dieser Axiome weggelassen werden)

3. Analytische Geometrie.
(ordnet den Punkten einer Geraden von vorneherein die Zahl zu, und führt so die Geometrie auf die Analysis zurück)."

[6] PG 5 and 7: "Die Geometrie ist die Lehre von den Eigenschaften des Raumes. .../ Sie beruht auf dem einfachsten Experiment, was man machen kann, nämlich auf dem Zeichnen."

[7] PG 12: "... weder rechnet noch misst, sondern nur construirt, weder Zirkel noch Winkelmaass benutzt, sondern nur das Lineal."

[8] Cod. Ms. F. KLEIN 9, 769: "Unser Zuhörerkreis besteht jetzt im Wesentlichen aus 2 Studenten, zu denen in meiner Vorlesung über projektive Geometrie noch als dritter Mann der Vorsteher der hiesigen königlichen Kunstschule — ein für Geometrie interessierter Maler — hinzukommt."

At the end of September 1891, HILBERT heard a lecture at the congress of natural scientists in Halle by HERMANN WIENER *Über Grundlagen und Aufbau der Geometrie* (W 4). There HILBERT heard about the general validity of the axiomatic method and in particular about the possibility of developing projective geometry from PASCAL's and DESARGUES' theorems taken as axioms. According to BLUMENTHAL, HILBERT is supposed to have uttered these famous words in a Berlin waiting-room on the return journey from Halle:

> "One should always be able to say, instead of 'points, lines, and planes', 'tables, chairs, and beer mugs'."[9]

Accordingly, in 1891, HILBERT saw the intuitive part of geometrical concepts as being mathematically irrelevant. But only gradually did he express that view in correspondingly radical written formulations.

4. Lectures on "Die Grundlagen der Geometrie" (summer semester 1894)

As can be seen from letters of his to KLEIN and MINKOWSKI, HILBERT intended lecturing about non-Euclidean geometry in the summer semester of 1893. But the lectures were first delivered in the summer semester of 1894. HILBERT's concept was to produce the purest possible exact system of an axiomatic, non-Euclidean geometry concluding with Euclidean geometry. HILBERT gave the lectures the title *Die Grundlagen der Geometrie.* A large part of the manuscript was written by a second, easily legible hand, that of his wife KÄTHE, as can be concluded from one of the letters.

HILBERT prefaced his manuscript with a bibliography of over 40 works in German. Amongst others he names PASCH, HELMHOLTZ, LOBAČEVSKIJ, RIEMANN, KLEIN, KILLING, LIE, CLEBSCH, LINDEMANN, MÖBIUS, VON STAUDT, REYE, ERDMANN and WIENER. The axiomatic studies of the Italians PEANO (1889), VERONESE (1891), and FANO (1892) are not mentioned. VERONESE's book had only just been translated.

In his introduction HILBERT made fundamental remarks about how mathematical objects are recognized. Contrary to what he had said in PG 5 he now avoided any explicit definition of geometry:

GG 7 "Among the phenomena, or facts of experience that we take into account observing nature. there is a particular group, namely the group of those facts which determine the external form of things. Geometry concerns itself withthese facts."[10]

---

[9] (B 3, 403): "Man muß jederzeit an Stelle von 'Punkte, Geraden, Ebenen' 'Tische, Stühle, Bierseidel' sagen können."

[10] GG7: "Unter den Erscheinungen oder Erfahrungsthatsachen, die sich uns bei der Betrachtung der Natur bieten, giebt es eine besonders ausgezeichnete Gruppe, nämlich die Gruppe derjenigen Thatsachen, welche die äußere Gestalt der Dinge bestimmen. Mit diesen Thatsachen beschäftigt sich die Geometrie."

He even regards the axioms as "facts":

> 8 "These unprovable facts have to be determined in advance and we term them axioms."

In this HILBERT stood at the same level with PASCH, who likewise derived his axioms from "experience". HILBERT, however, questions whether the axioms are complete and independent:

> 8 "Our colleague's problem is this: what are the necessary and sufficient conditions, independent of each other, which one must posit for a system of things, so that every property of these things corresponds to a geometrical fact and vice versa, so that by means of such a system of things a complete description and ordering of all geometrical facts is possible."[11]

HILBERT took up an idea of HEINRICH HERTZ (H 2, 1), which in geometry leads to the use of space intuition (*Raumanschauung*) only in the sense of a possible intuitive analogy. In the first part of his book HILBERT developed, like PASCH, the fundamental theorems of projective geometry. But whereas PASCH divided his axioms into eight axioms for lines and four for planes, HILBERT now arranged his axioms according to relations, an order he had touched on in PG (1891). He first separates the axioms of connection and order. Following VON STAUDT's and MÖBIUS' practice he now ascribes rational numbers of the kind $m/2^n$ to the points of a line with the help of the construction of the fourth harmonic element and he ascribes, conversely, a point to every such number. In order "to prevent a gap" in the transition to the real numbers (GG 38), HILBERT states an axiom of continuity. From PASCH he adopted the formulation given by WEIERSTRASS (P 1, 125foll.). Subsequently the fundamental theorems of projective geometry are derived, but it is not possible (so HILBERT wrote) "by means of the current theorems" (GG 60a) to derive a large number of phenomena, such as, *e.g.*, displacement.

Then he specializes the projective system. This leads to the axioms of congruence. By introducing a determination of metrics he was able to investigate hyperbolic and parabolic geometries. Elliptical geometry had been excluded by HILBERT through the axioms of order. The second part is based on the non-axiomatically constructed work of CLEBSCH/LINDEMANN (C 3).

In retrospect it is remarkable that HILBERT states the axiom of continuity straight after the first two groups of axioms. As can be seen from his manuscript, HILBERT had not anticipated this at the outset. In the second part of the lectures he often proceeded analytically—for his axiomatic system an impure method, which he wished to avoid in the future. In 1898 he added the axioms of continuity

[11] GG 8: "Diese nicht beweisbaren Thatsachen müssen wir von vorherein festsetzen und nennen sie Axiome. ...

Das Problem unseres Collegs lautet also: Welches sind die nothwendigen und hinreichenden und unter sich unabhängigen Bedingungen, die man an ein System von Dingen stellen muss, damit jeder Eigenschaft dieser Dinge eine geometrische Thatsache entspricht und umgekehrt, so dass also mittelst obigen Systems von Dingen ein vollständiges Beschreiben und Ordnen aller geometrischen Thatsachen möglich ist."

at the end and thus shows how dispensible they are. The intersection theorems enable HILBERT to establish a segment arithmetic without an axiom of continuity. In 1894 he had not even mentioned the intersection theorems and consequently had not examined their importance. Thus WIENER's contribution, which had been seen as decisive by BLUMENTHAL (B 3, 402), was applied in this way by HILBERT not before 1898.

Likewise contrary to what he had written in GG, HILBERT in 1898 proceeded on the basis of Euclidean geometry. As early as 1894, he had remarked in that regard:

GG 2 "If I lecture again, it will be on Euclidean geometry."[12]

Possible reasons for the transition from projective to Euclidean geometry are:
(1) the order relation of three elements had proved unsuitable;
(2) the principle of duality is not valid in the geometry discussed in GG.
In EG 35 and SG 147 HILBERT in addition viewed continuity as one of the premises of projective geometry. The projective studies thereby disappeared from his *Grundlagen der Geometrie.*

A direct product of GG was HILBERT's work *Über die gerade Linie als kürzeste Verbindung zweier Punkte* (= BG; 14. 8. 1894). Here HILBERT shows that a generalized Cayleyan determination of metrics fulfills the triangular inequality. In doing this he bases himself on the axioms of incidence, order and continuity. In the winter semester of 1894/95 HILBERT lectured on analytical geometry; in 1895 he accepted the chair at Göttingen and in the following years (until 1897) he concerned himself principally with number theory.

5. The Vacation-Course "Über den Begriff des Unendlichen" (Easter 1898)

HILBERT was inspired to take up anew the foundations of geometry by a letter from FRIEDRICH SCHUR to FELIX KLEIN dated 30. 1. 1898 (Cod. Ms. F. KLEIN 11, 920, printed in T 3). HILBERT writes in a letter of 16. 3. 1898 to HURWITZ (Cod. Ms. Math. Arch. Nr. 76/272):

> "Schur has recently shown in a letter to Klein, that with the aid of congruence axioms in space, Pascal's theorem can be proved in the plane for a pair of lines, *i.e.* without recourse to the Archimedean axiom. This letter, which Schönflies introduced to us in a lecture to the mathematical society, has given me the inspiration to take up again my old ideas about the foundations of Euclidean geometry."[13]

[12] GG 2: "Wenn ich wieder lese, so erst die Euklidische Geometrie."

[13] Cod. Ms. Math. Arch. Nr. 76/272: "Neulich hat SCHUR in einem Briefe an KLEIN gezeigt, dass mit Hülfe der Congruenzsätze im Raume der Pascalsche Satz in der Ebene für das Geradenpaar bewiesen werden kann, d.h. ohne das Archimedische Axiom. Dieser Brief, über den uns SCHÖNFLIES in der math. Gesellschaft vortrug, hat mir Anlass gegeben, meine älteren Überlegung(en) über die Grundlagen der Euklidischen Geometrie wieder aufzunehmen."

Noteworthy inspirations had been previously provided by WIENER. HILBERT now busied himself, amongst other things, with the connection between the two intersection theorems and with the theory of plane areas. In the 27-page manuscript for the Easter vacation-course of 1898 he introduced his audience to the most up-to-date research questions. For the first time he introduced the groups of axioms in what was subsequently to be their usual sequence, examined the resulting constructability of figures, the independence of the continuity-axiom, and asked which axioms were indespensible if, in place of PASCAL'S and DESARGUES' theorems, one premised some axioms that were used to prove these theorems.

The remainder of the manuscript is marked by its concern with proving PASCAL'S theorem. For this HILBERT had recourse to the earlier works and ideas of VON STAUDT (S 3; cited in GG) and STOLZ (S 6, **1**, 6th section). HILBERT investigated the theory of plane areas by introducing a product of segments which uses the Euclidean equality of content. Here too the independence of the Archimedean axiom is prominent.

Once HILBERT'S basic concept had been established, a number of individual problems came into focus on which he now worked intensively. That led him to the exhaustive system of his decisive lecture for the Festschrift.

6. Lectures on "Grundlagen der Euklidischen Geometrie" (winter semester 1898/99)

Contrary to the situation in Königsberg, in Göttingen it was not unusual to lecture on Euclidean geometry. Nevertheless BLUMENTHAL writes (B 3, 402):

> "Hilbert had announced a series of lectures on "Elements of Euclidean geometry" for the winter semester of 1898/99. This caused astonishment among the students, for even we older people, participants in the 'strolls through the number field' had never known Hilbert to concern himself with geometrical questions. He spoke to us only about number fields. Astonishment and wonder were caused when the lecture began and a completely novel content emerged."[14]

Clearly, BLUMENTHAL had not attended the Easter vacation-course for teachers; perhaps he had not suspected any geometrical content in the theme *Über den Begriff des Unendlichen.*

The lecture manuscript EG contains an exhaustive discussion of those areas that were mostly treated in brief in the vacation-course. The logical meaning of the axioms was studied by construction of arithmetic models. Amongst these were proofs of independence for axioms of the first two groups. A first step toward § 23

[14] (B 3, 402): "Für den Winter 1898 bis 1899 hatte HILBERT eine Vorlesung über 'Elemente der Euklidischen Geometrie' angekündigt. Das erregte bei den Studenten Verwunderung, denn auch wir älteren, Teilnehmer an den 'Zahlkörperspaziergängen' hatten nie gemerkt, daß HILBERT sich mit geometrischen Fragen beschäftigte: er sprach uns nur von Zahlkörpern. Staunen und Bewunderung aber erwachten, als die Vorlesung begann und einen völlig neuartigen Inhalt entwickelte."

in the Festschrift ([1]GG) was formed by demonstrating the impossibility of proving DESARGUES' theorem in the plane. In accordance with the theme of the lectures, HILBERT examined in detail the studies of congruence that were possible without using continuity. Thus the last part is devoted to the hitherto unprovable theorems. This is principally concerned with sections of circles and lines and with intersecting circles. Much of this was omitted in the Festschrift, including the historical survey of the parallel axiom that follows in the manuscript, two of the theorems equivalent to this axiom, the detailed presentation of a non-Euclidean geometry and the introduction of ideal elements. The theorem of PASCAL is proved in the plane with reference to the manuscript SG (elaboration by VON SCHAPER). On that basis HILBERT constructs a segment arithmetic and the theory of plane area without the use of continuity axioms. The significance of PASCAL'S intersection theorem thereby manifests itself—which for HILBERT at that time was the "most important result". The "Archimedean axiom" is presented in three formulations: in the original formulation, rediscovered by STOLZ, as well as in the projective formulation and in the WEIERSTRASS formulation. The idea of a segment arithmetic based on the theorem of DESARGUES—without involving the congruence axioms—was new.

> "Therefore it is suitable for the derivation of pure projective geometry in von Staudt's sense of the word, i.e. geometry which makes no use of metric in the ordinary sense." (H 4, 67)[15]

HILBERT pays particular attention to the introduction of numbers, to consistence and to the axiomatic method.

7. The Elaboration: "Elemente der Euklidischen Geometrie" (March 1899)

The elaboration SG, *Elemente der Euklidischen Geometrie*, prepared by VON SCHAPER, contains numerous remarks, some of them on the motivations of the theory, and examples, which were omitted in the Festschrift. But also compared with the lecture manuscript EG, the elaboration is the more extensive, due to transitional additions and to now completed proofs. The formulations are more exact. But HILBERT still sees in the axioms "very simple ... original facts", whose validity is experimentally provable in nature.

Here there appear for the first time studies of the theorems of LEGENDRE, which were only published thirty years later (beginning with [7]GG, using the Archimedean axiom). An eight-page introduction to projective geometry by means of ideal elements (*e.g.* the principle of duality with examples, specialized forms of the theorems of DESARGUES and PASCAL), was omitted in the Festschrift. The theory of plane area was also further elaborated in SG. Thus the difference be-

[15] (H 4, 67): "Sie eignet sich deshalb zur Herleitung der rein projektiven Geometrie im von Staudtschen Sinne des Worts, d.h. der Geometrie, die von dem im gewöhnlichen Sinn genommenen Maß keinen Gebrauch macht."

tween the equality of area and content (in more recent editions of the *Grundlagen*: "equidecomposability" and "equicomplementability") and the definition of the measure of area gain in significance. The theorem of KILLING-STOLZ ([7]GG § 21) was proved without the Archimedean axiom. The investigations leading to [1]GG Chap. VI are new, *i.e.* the provability of PASCAL's theorem without the axioms of congruence but with the aid of the Archimedean axiom. For that purpose the so-called "new" segment arithmetic, which is independent of the axioms of congruence, was employed as was also the clarification of the relation to the commutative law of multiplication. In SG HILBERT for the time introduces the real numbers with the aid of the so-called CANTOR's axiom. And here he now also expresses the idea of reducing the consistence of geometry to that of arithmetic. The manuscript SG ends with an outline of a chapter about elementary geometric constructions.

8. Festschrift "Grundlagen der Geometrie" (June 1899)

For the Festschrift, which appeared in June 1899, HILBERT had once more revised his lectures. He renounces an interpretation of the fundamental concepts ([1]GG § 1):

"Let us consider three distinct systems of things: ..."[16]

The groups of axioms III (axioms of congruence) and IV (axioms of parallels) of SG were transposed. Here the detailed angle arithmetic in SG was largely omitted. The investigations of congruence are immediately followed by the Archimedean axiom. With that chapter I ends. The collection into seperate chapters of the scattered investigations of independence and of the theorems of DESARGUES and PASCAL was new. The proof of consistence, hitherto only indicated, was demonstrated in detail ([1]GG § 9). In [1]GG § 34 HILBERT constructs a spacial "non-Pascalian" geometry, which is at the same time also a "non-Archimedean" geometry. The constructable segments of the last chapter correspond in analysis precisely to the numbers of the field Ω in § 9.

It seems that HILBERT's aim from the outset was the algebraisation of geometry. In 1894 (GG) he was still content with the introduction of coordinates by means of the MÖBIUS grid. At the end he established that it must also be possible to calculate with the numbers ascribed to geometrical objects. Hence the laws of fields are required. While PASCH still speaks of primitive propositions (P 1, 17) "directly based on observation", from which he derives all the remaining theorems, for HILBERT after his manucript GG (1894) the relations between the objects of intuition provide the starting-point. Having perceived both the starting-point and the aim, it remains only to find the way. HILBERT – like EUCLID – proceeds axiomatically. Here the question arises, which axioms are required.

The axioms of incidence are largely clear; the axioms of order are already somewhat problematical. Hence there are difficulties in projective geometry. Proceeding to Euclidean geometry, he introduces the concept of congruence

[16] ([1]GG 4): "Wir denken drei verschiedene Systeme von Dingen: ..."

without hesitation. But then appears the problem of the intersection theorems, of the axiom of parallels, of the Archimedean axiom, of continuity—all the statements connected with that are not independent from one another. As our investigation demonstrates, it was not easy for HILBERT to find a suitable way through this maze of axioms.

HILBERT omitted the continuity axioms of CANTOR and WEIERSTRASS in the Festschrift. He was content with a countable field. In the autumn of 1899 he published instead an entirely novel axiom, which also implies the real numbers: the axiom of completeness. Its relation to the axioms of CANTOR and WEIERSTRASS was vigorously discussed in the following decades. The results were considered in the thoroughly revised 7$^{\text{th}}$ edition (1930), which was the last edition to appear during HILBERT'S lifetime.

As we have seen, in 1891 HILBERT was already engaged with the problem of the foundation of geometry. He increasingly detached himself from the traditional standpoint, but he maintained close contact with other mathematicians, especially KLEIN, LINDEMANN, HURWITZ and MINKOWSKI. In 1894 he established an independent axiomatic construction of geometry. In 1898 he took up the subject anew and worked intensively at it, until in 1899 his own conception appeared in the Festschrift.

Bibliography

1. Manuscripts and Letters
(Niedersächsische Staats- und Universitätsbibliothek Göttingen):

Abbr.	Cod. Ms. D. HILBERT	
	741, 47	(exercise-book on projective geometry, 1879)
PG :=	535	Projektive Geometrie (summer semester 1891)
GG :=	541	Die Grundlagen der Geometrie (summer semester 1894)
FK :=	597	Über den Begriff des Unendlichen (Easter vacation-course 1898)
EG :=	551	Grundlagen der Euklidischen Geometrie (winter semester 1898/99)
SG :=	552	Elemente der Euklidischen Geometrie (elaboration by HANS VON SCHAPER, March 1899)

Cod. Ms. F. KLEIN 9, 769: Letter from HILBERT to KLEIN (30. 6. 1891)
Cod. Ms. Math. Arch. Nr. 76/272: Letter from HILBERT to HURWITZ (16. 3. 1898)

2. Published References:

B 1 BERNAYS, PAUL: Die Bedeutung Hilberts für die Philosophie der Mathematik. Die Naturwissenschaften **10** (1922) 93–99.

B 2 BLUMENTHAL, OTTO: David Hilbert. Die Naturwissenschaften **10** (1922) 67–72.

B 3 BLUMENTHAL, OTTO: Lebensgeschichte. In: David Hilberts Gesammelte Abhandlungen **3** (1935) 388–429.

C 1 CANTOR, GEORG: Über die Ausdehnung eines Satzes aus der Theorie der trigonometr. Reihen. Math. Ann. **5** (1872) 123–132.

C 2 CAYLEY, ARTHUR: A sixth memoir upon quantics. Lond. Trans. 1859, V. 149. In: Coll. Math. Papers, Cambridge 1889–1897, Vol. **2**, pp. 561–592.

C 3 CLEBSCH, ALFRED: Vorlesungen über Geometrie. Hrsg. v. F. LINDEMANN. (1. Bd.: 1876) 2. Bd. 1. Theil: Leipzig 1891, S. 433–637: Die Grundbegriffe der projektivischen und metrischen Geometrie.

C 4 CONTRO, WALTER S.: Die Entwicklung der Geometrie zum hypothetisch-deduktiven System. Ein Beitrag zur Geschichte der Grundlagen der Geometrie. Dissertation Frankfurt 1970.

C 5 CONTRO, WALTER S.: Von Pasch zu Hilbert. Archive for Hist. of Exact Sciences **15** (1976) 283–295.

D 1 DEDEKIND, RICHARD: Stetigkeit und irrationale Zahlen. 1872. Braunschweig ^{6}1960.

D 2 DEHN, MAX: Hilberts geometrisches Werk. Die Naturwissenschaften **10** (1922) 77–82.

E 1 EARMAN, JOHN, & GLYMOUR, CLARK: Einstein and Hilbert: Two Months in the History of General Relativity. Archive for Hist. of Exact Sciences **19** (1978) 291–308.

E 2 ERDMANN, BENNO: Die Axiome der Geometrie. Leipzig 1877.

F 1 FREUDENTHAL, HANS: Zur Geschichte der Grundlagen der Geometrie. Nieuw Archief voor Wiskunde (4) **5** (1957) 105–142.

F 2 FREUDENTHAL, HANS: Die Grundlagen der Geometrie um die Wende des 19. Jahrhunderts. Math.-Phys. Sem.-Ber. **7** (1960) 2–25.

F 3 FREUDENTHAL, HANS: The Impact of von Staudt's Foundations of Geometry. In: For Dirk Struik, R. S. COHEN etc. (eds.), Dordrecht 1974. pp. 189–200.

F 4 FRICKE, ROBERT: Felix Klein zum 25. April 1919, seinem siebzigsten Geburtstage. Die Naturwissenschaften **7** (1919) 275–280.

H 1 HANKEL, HERMANN: Die Elemente der projektivischen Geometrie in synthetischer Behandlung. Leipzig 1875.

H 2 HERTZ, HEINRICH: Die Prinzipien der Mechanik. Leipzig 1894. Nachdruck Darmstadt 1963.

BG HILBERT, DAVID: Über die gerade Linie als kürzeste Verbindung zweier Punkte. Math. Ann. **46** (1895) 91–96 = Anh. I ab ^{2}GG.

^{1}GG HILBERT, DAVID: Grundlagen der Geometrie. Leipzig ^{1}1899. ^{2}1903. ^{7}1930. Stuttgart ^{12}1977.
The Foundations of Geometry. Transl. by E. J. TOWNSEND. Chicago 1902. Reprint 1947.

GA HILBERT, DAVID: Gesammelte Abhandlungen. 3 Bde. Berlin 1932, 1933, 1935. Reprint New York 1965.

H 3 HILBERT, DAVID, & COHN-VOSSEN, STEFAN: Anschauliche Geometrie. Berlin 1932. Nachdruck Darmstadt 1973.

H 4 HÖLDER, OTTO: Streckenrechnung und projektive Geometrie. Ber. ü. d. Verh. d. Königl. Sächs. Ges. d. Wiss. zu Leipzig, Math.-Phys. Kl. **63** (1911) 65–183.

K 1 KEPLER, JOHANNES: Ad Vitellionem paralipomena. 1604. In: Ges. Werke **2**, Astronomiae pars optica. Hrsg. v. F. HAMMER. München 1939.

K 2 KLEIN, FELIX: Über die sogenannte Nicht-Euklidische Geometrie. Math. Ann. **4** (1871) 573–625.

K 3 KLEIN, FELIX: Über die sogenannte Nicht-Euklidische Geometrie. Math. Ann. **6** (1873) 112–145.

M 1 MINKOWSKI, HERMANN: Briefe an David Hilbert. Hrsg. v. L. RÜDENBERG u. H. ZASSENHAUS. Berlin 1973.

P 1 PASCH, MORITZ: Vorlesungen über neuere Geometrie. Leipzig ^{1}1882. (Berlin ^{2}1926. Nachdruck ^{2}1976).

R 1 REID, CONSTANCE: Hilbert. New York/Berlin 1970.

R 2 REYE, THEODOR: Geometrie der Lage. Hannover ^{1}1866/67.

S 1 SCHMIDT, ARNOLD: Zu Hilberts Grundlegung der Geometrie. In: GA **2**, 404–414.

S 2 VON STAUDT, GEORG KARL CHR.: Geometrie der Lage. Nürnberg 1847.

S 3 VON STAUDT, GEORG KARL CHR.: Beiträge zur Geometrie der Lage. 3 Hefte. Nürnberg 1856, 1857, 1860.

S 4 STEINER, JAKOB: Systematische Entwicklung der Abhängigkeit geometrischer Gestalten voneinander. Berlin 1832. In: Ges. Werke **1**. Berlin 1881. Reprint 1971. S. 229–460.

S 5 STOLZ, OTTO: Zur Geometrie der Alten, insbesondere über ein Axiom des Archimedes. Math. Ann. **22** (1883) 504–519.

S 6 STOLZ, OTTO: Vorlesungen über Allgemeine Arithmetik. 2 Bde. Leipzig 1885, 1886.

T 1 TOEPLITZ, OTTO: Der Algebraiker Hilbert. Die Naturwissenschaften **10** (1922) 73–77.

T 2 TOEPELL, MICHAEL: Über die Entstehung von David Hilberts "Grundlagen der Geometrie". Dissertation Göttingen 1986.

T 3 TOEPELL, MICHAEL: Zur Schlüsselrolle Friedrich Schurs bei der Entstehung von David Hilberts "Grundlagen der Geometrie." In: Mathemata. Festschrift für Helmuth Gericke. Hrsg. von M. FOLKERTS & U. LINDGREN. Stuttgart 1985. S. 637–649.

W 1 VAN DER WAERDEN, BARTEL L.: De logische Grondslagen der Euklidische Meetkunde. Christiaan Huygens **13** (1934/1935) 65–84, 257–274 und **14** (1935/1936) 183–212.

W 2 VAN DER WAERDEN, BARTEL L.: Erwachende Wissenschaft. Basel ([1]1956) [2]1966.

W 3 WEYL, HERMANN: David Hilbert and His Mathematical Work. Anhang zu R 1, S. 245–283.

W 4 WIENER, HERMANN: Über Grundlagen und Aufbau der Geometrie. Verh. d. Ges. dtsch. Naturf. u. Ärzte **2** (1892) 8–9.

W 5 WORPITZKI, JULIUS: Über die Grundbegriffe der Geometrie. Archiv für Mathematik und Physik **55** (1873) 405–421.

Burgkmairstr. 42
D-8000 München 21
Federal Republic of Germany

(Received October 14, 1985)