What is it like to go somewhere for all of eternity?

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1 A problem

Say you did some bad things in life and go to hell. The devil there devices a punishment for you, asking you to walk from a mountain to a river where you can drink water. But the real punishment is that when you are about to reach the river, when only 10% of the total distance is left the devil makes your feet 10× slower. And this process keeps repeating, so that when you are just 10% of remaining distance away from the river, you are slowed 10× from the current speed. Will you ever reach the river?

Let's say distance from mountain to river is d. Speed at start is v, time taken to reach 90% of the distance is t. So, when you reach $0.9d$, speed becomes $\frac{v}{10}$. In next t time you travel $\frac{v\times t}{10} = \frac{0.9d}{10}$ which is 90% of the rest (90% of $d(1-0.9) = d(0.1) = d(0.09)$). In fact, after nt time, you travel $\frac{0.9d}{10^{n-1}} = \frac{9d}{10^n}$ more of the distance towards the river.

$$\text{total distance travelled, } t_n = \sum_{i=1}^{n} \frac{9d}{10^i}$$

2 Well what now?

What is $\lim_{n\to\infty} t_n$? Well let's define $d = 1$ and define the sequence $t_n, n \in \{1, 2, \dots\}$ alternatively. We can see that the t_n can be written as $0.999\cdots$ n times. Let's define $r_n = 1 - t_n = \frac{1}{10^n}$. Let's define $m = \frac{t_n+1}{2}$ and notice that $t_n \leq m \leq 1$.

$$\begin{aligned}
t_n &\leq m \leq 1 \\
\Rightarrow -t_n &\geq -m \geq -1 \\
\Rightarrow 1 - t_n &\geq 1 - m \geq 0 \\
\Rightarrow r_n &\geq 1 - m \geq 0 \\
\Rightarrow \frac{1}{10^n} &\geq 1 - m \geq 0
\end{aligned}$$

As $n \to \infty$, $1 - m$ gets sandwiched between 0 and 0. Thus at the limit, $1 - m = 0$ which means $m = 1$ which means $\frac{t_n+1}{2} = 1 \Rightarrow t_n + 1 = 2 \Rightarrow t_n = 1$. ■

Which means, once the eternity is over you'll be able to reach the river, drink the water and start a new life at the birth of the new universe.

3 Alternate proof?

Given a positive number $\varepsilon > 0$, we want to find an integer $N_0(\varepsilon)$ such that $\forall\ n \geq N_0(\varepsilon)$, $|t_n - 1| < \varepsilon$.

$$
\begin{aligned}
&|t_{N_0} - 1| < \varepsilon \\
&|1 - r_{N_0} - 1| < \varepsilon \\
&|r_{N_0}| < \varepsilon \\
&\left|\frac{1}{10^{N_0}}\right| < \varepsilon \\
&\frac{1}{10^{N_0}} < \varepsilon \quad \text{since it's always positive} \\
&10^{N_0} > \frac{1}{\varepsilon} \\
&\Rightarrow N_0 > \log_{10}\left(\frac{1}{\varepsilon}\right) \qquad \text{because log is an increasing function}
\end{aligned}
$$

Thus choosing $N_0 = \max\{0, \lceil\log_{10}\frac{1}{\varepsilon}\rceil + 1\}$ does the job. ■

This says that we can get arbitrarily close ($\varepsilon > 0$) to 1 and there will be an infinitely many points of the sequence within that ε-ball. This is the definition of convergence of a sequence to a point, or the definition of existence of a limit. This proves that $0.\bar{9} = 1$.

4 What about geometric progression?

$S_n = \frac{a(1-r^n)}{1-r}$. Identifying $a = \frac{9}{10}$ and $r = \frac{1}{10}$, $S_n = \dfrac{9}{10}\dfrac{1-\frac{1}{10^n}}{1-\frac{1}{10}}$ and $\lim\limits_{n\to\infty} S_n = \dfrac{\frac{9}{10}}{1-\frac{1}{10}} = 1$.

5 Why the hell did I make this?

Because in school we did a math problem where we proved that $0.\bar{9} = 1$ but it was wrong, because it can not be algebraically multiplied out. Instead, it's a "limit" of a sequence. Which is why in all my previous ways of getting to 1, I use limits. Just saying $0.\bar{9} = x$, $9.\bar{9} = 10x$, $9 = 9x$, $x = 1$ is wrong. The addition, multiplication, subtraction and division used here are not defined. High school math is much more difficult than we think.

This is the sin you committed because of which you in hell. You're a very bad man like Jerry.

And this is not the end of the story, what does the norm $|\cdot|$ mean. How do we know that at the limit, the sum of a GP is actually $\frac{a}{1-r}$ for $r < 1$. We need to define metrics, look at convergence, Cauchy sequences and much more to complete the proof.