MAINAK BISWAS

Assignment 2: Linear Algebra

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[15 + 10 = 25 points]
(3 hours)

(3x Theory) (1 x Coding)

A Theory :- 15

1) [Matrix operations, elimination] -(5 points)

(i) $\text{trace}(A_{n\times p} B_{p\times n})$ $\text{trace}(B_{p\times n} A_{n\times p})$

(Hence $\text{trace}(ABC) = tr(CAB) = \text{trace}(BCA)$
using induction)
[All cyclic orders]

(ii) Decompose $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 8 \\ 4 & 6 & 18 \end{bmatrix}$

Into $A = LU$

(iii) Now, say you want to convert $A \rightarrow \check{A} = \begin{bmatrix} 2 & 7 & 8 \\ 4 & 6 & 18 \\ 1 & 2 & 3 \end{bmatrix}$

These are called permutation matrices (that you used to convert $A \rightarrow \check{A}$: $\check{A} = PA$)

Find the particular P. here

(iv) How many row permutation matrices are possible for a $m \times n$ matrix?

(v) Show that $P^{-1} = P^T$ for all these matrices

2) [Vector spaces, subspaces, more.] -(5 points)

(a) Understanding coordinates

Let $B_n = \{v_1, v_2, \cdots v_n\}$ be basis set for $\mathbb{R}^n$.

Then $[x]_B \in \mathbb{R}^n$ is defined as the coordinates of a vector. If:

$$x = \sum_{i=1}^{n} \alpha_i v_i$$

$$[x]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Example:

$$B_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(cartesian coordinates)

If $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Downarrow$

$\{[v]_B :$ if cartesian Basis is used$\}$

Find the coordinates of $v = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ under the Basis set: $\tilde{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

i.e. find $[v]_{\tilde{B}}$.

(b) $A \in \mathbb{R}^{m \times n}$, and $\text{rank}(A) = n$, and a subspace $S \subseteq \mathbb{R}^n$. Let $A(S) = \{Ax \mid x \in S\}$

Let $B = \{s_1, s_2, \cdots s_k\}$ be the basis of S. Find Basis of $A(S)$ in terms of B. Explain why? (i.e. prove it is a basis)

(c) Let rank(A) = 2 ; rank(B) = 1. Prove / disprove:

$1 \leq \text{rank}(A+B) \leq 3$

(d) Show that $X_1 \cup X_2$; $X_1, X_2 \subseteq \mathcal{F}^n$;

is not a subspace necessarily (unlike $X_1 \cap X_2$).

(Counter-examples are considered correct proofs in maths) / formally disprove.

3) <u>Eigen-values and Eigen-vectors</u> – (5 points)

(a) <u>Left-eigen vector</u>

$\Rightarrow y \neq 0$; $\{ y^T A = \mu y^T \}$ $A \in \mathbb{R}^{n \times n}$ [Def^n]

Now, Suppose $Ax = \lambda x$ (right e-vector) ; Show that if $\lambda \neq \mu$ then

$y^H x = 0$.

(b) Let A and B be 2 square matrices ($\mathbb{R}^{n \times n}$) an $B \sim A$ (similar to A : i.e. $\exists S$: s.t. $B = S^{-1} A S$)

Now: $x \in \mathbb{R}^n$ is a right eigenvector of B associated with λ [$Bx = \lambda x$].

(i) Then show that Sx is right eigen-vector of A associated with λ.

(ii) Let $y \in \mathbb{R}^n$ be a left eigen-values of B associated with μ. Then $(S^{-1})^T y$ is left-eigen-vector of A associated

μ. Then $(S^{-1})'y$ is left-eigen-vector of A associated with μ.

(C) Let $A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$; where $A_{11}, A_{22} \in \mathbb{R}^{n \times n}$

(Block diagonal matrix)

[consider the notn: $\sigma(A) = \{\lambda \mid \text{s.t. } Ax = \lambda x\}$]

(set of all eigen-values)

Show that $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$

(d) Let $A \in \mathbb{R}^{n \times n}$.

(i) Explain why $P_A(t) = \det(tI - A)$ [characteristic polynomial] have real coefficients.

(ii) Hence show the eigenvalues of A occur in conjugate pairs.

B Programming — (10 points)

Write a python script using "numpy" and and "sympy" to find the basis for

(a) Row space and Null space of any given real matrix

(b) Use the same function for colspace and rowspace of A^T.

(c) Print the rank of the matrix.

Hint: you need not write code for row elimination —

Sympy does it for you.

Test the correction on

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 3 & 6 & 1 & 9 & 6 \\ 2 & 4 & 1 & 7 & 5 \end{bmatrix}$$

Submit 1 .ipynb file for this (use google colab / Jupyter notebook).