Solutions: Assignment 2

24 March 2024 15:53

MAINAK BISWAS
IISc.

A

1) -

(i) $\text{trace}(A_{n \times p} B_{p \times n})$

$$= \sum_{i=1}^{n} (AB)_{ii}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{p} a_{ip} \, b_{pi} \right)$$

$$= \sum_{j=1}^{p} \sum_{i=1}^{n} a_{ip} \, b_{pi}$$

$$= \sum_{i=1}^{n} \sum_{i=1}^{n} b_{pi} \, a_{ip} = tr(BA)$$

Thus; $tr((AB)C) = tr(C(AB)) = tr((CA)B) = tr(B(CA))$

(ii)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 8 \\ 4 & 6 & 18 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 - 4R_1$

$$A \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & -2 & 6 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$\downarrow \; R_3 \leftarrow R_3 + \frac{2}{3} \cdot R_2$

$$E_2 E_1 A = U \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 22/3 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}$$

$$E_2 E_1 \quad \begin{bmatrix} 0 & 0 & 22/3 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}$$

$$A = E^{-1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 22/3 \end{bmatrix}$$

$$E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

(multiplying -1 cols)

$$(E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -2/3 & 1 \end{bmatrix}$$

$\frac{-4}{3} + 4$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 22/3 \end{bmatrix}$$

(iii)

$R_1^{(new)} \leftarrow R_2 (old)$

$R_2^{(new)} \leftarrow R_3 (old) \Rightarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$R_3^{(new)} \leftarrow R_1 (old)$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A .$$

e_1
e_2
e_3

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}$$

(iv) m rows

$P_{m \times m}$: are shuffling of I matrix

($m!$) ways to permute the rows. (i.e. reorder $e_1\ e_2\ e_3$ in any order and place them row wise.

(v) P^T (for above)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^T_{m \times m} P = I$$

If $P_{ij} = 1$: $(P^T)_{ji} = 1$

$P_{ik} = 0$ $\quad (P^T)_{ki} = 0 \ \forall i$

$$(PP^T)_{ii} = \sum_{l=1}^{m} P_{il} P_{li} = \underset{(\forall l \neq j)}{0} + 1 = 1$$

$$(PP^T)_{ij} = \sum P_{il} P_{lj}$$

$$= P_{ij} P_{jj}$$

$\hookrightarrow 0$ as $(P^T_{ji} = 1)$

$$= 0$$

$= PP^T = I$

$= 0$

$= PP^T = \underline{I}$

2)

(a) $$v = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$[v]_{\tilde{B}} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & -1 & -2 \end{array}\right]$$ – Solve using gauss-$\underline{\text{Jordan}}$.

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & -3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$$

$$\Rightarrow [v]_{\tilde{B}} = \begin{bmatrix} 1/2 \\ -1 \\ 3/2 \end{bmatrix}$$

(b) $B = \{s_1, s_2 \dots s_K\} \rightarrow$ Basis.

What's the basis of $A(S) = \{Ax \mid x \in S\}$

$\underset{\mathbb{R}^{m\times n}}{\text{rank}(A)} = n$ (full column rank : i.e. all the columns are independent}

Choose for Basis $B_{A(S)} = \{As_1, As_2 \dots, As_K\}$

$= A(B)$

Prove:

(i) It spans $A(S)$:

Let $z \in A(S)$; Then $\exists x \in S$

$\Rightarrow z = A\underline{\underline{x}}$ (definition)

$x = \sum_{i=1}^{K} \beta_i s_i$ [using basis of S]

$= A\left[\sum_{i=1}^{K} \beta_i s_i\right]$

$= \sum_{i=1}^{K} \beta_i \underbrace{(As_i)}$ Then (As_i) span the set.

Proved

(ii) They are independent :- (By contra)

$\sum_{i=1}^{K} \alpha_i As_i$

$= A\left[\sum_{i=1}^{K} \alpha_i s_i\right]$

$\in S$

$A(S) \neq 0$ [as A is full column rank]

$\dim(N(A)) = 0$

(c)

$$\text{rank}(A+B) \leq \overset{\uparrow 2}{\text{rank}(A)} + \overset{\uparrow 1}{\text{rank}(B)}$$

-(Shown in class)

$$\leq 3$$

$$\text{rank}(A) = \text{rank}(A-B+B)$$

$$\leq \text{rank}(A+B) + \text{rank}(-B)$$

$$\text{rank}(A) - \text{rank}(-B) \leq \text{rank}(A+B)$$

$$\Rightarrow 2 - 1 \leq \text{rank}(A+B)$$

Hence proven

$\text{rank}(A) = \text{rank}(-A)$

[linear comb. of column space doesn't change on taking negative of vector]

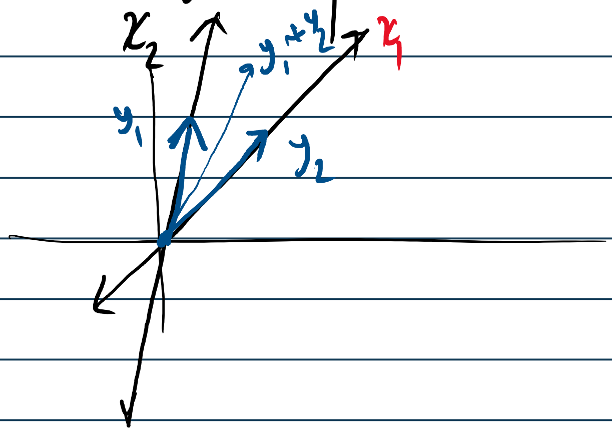
(d)

Let $X_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y = x \right\}$

$X_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y = 2x \right\}$

$X_1 \cup X_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y = x \text{ or } y = 2x \right\}$

$X_1 \cup X_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; y = x \text{ or } y = 2x \right\}$



Let $y_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $y_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$y_1 + y_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \notin X_1 \cup X_2$$

(Hence proved using counter example)

3)

(a)

$$y^T A = \mu y^T \quad ; \text{ and } Ax = \lambda x \quad (\mathrm{I})$$

$$\Downarrow$$

$$y^T \underbrace{Ax} = \mu y^T x$$

$\Rightarrow$ (from I)

$$y^T (\lambda x) = \mu y^T x$$

$$\Rightarrow (\lambda - \mu) y^T x = 0$$

either $y^T x = 0$ ($y \perp x$) or $\lambda = \mu$

Hence proved

(b)

$$B = S^{-1} A S$$

(i)

$$Bx = S^{-1} A S x = \lambda x$$

(i)

$$Bx = S^{-1}ASx = \lambda x$$

$$= SS^{-1}ASx = \lambda Sx$$

$$= A(Sx) = \lambda(Sx)$$ [Hence proved]

(ii)

$$y^T B = \mu^T y$$

$$y^T(S^{-1}AS) = \mu^T y$$

$$\Rightarrow (y^T S^{-1})A = \mu^T(yS^{-1})$$

$$(((S^{-1})^T y)^T A = \mu[(S^{-1})^T y)$$

(Hence proved)

(C) $A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$

Direct: Let $\lambda \in \sigma(A_{11})$ assume $X \in \mathbb{R}^n$

and $A_{11}X = \lambda X$

$$[A_{11}X, A_{22} \cdot 0]^T = [\lambda x, 0]^T$$

$$= \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} X \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} X \\ 0 \end{bmatrix}$$

$\therefore \lambda \in \sigma(A)$

Similarly true for $\mu \in \sigma(A_{22})$

$$= \begin{bmatrix} A_{11} & 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} = \lambda \begin{bmatrix} 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ x \end{bmatrix}$$

$\therefore$ If $\lambda \in \sigma(A_{11}) \cup \sigma(A_{22}) \Rightarrow \lambda \in \sigma(A)$

Proved

Direct 2

$$\lambda \in \sigma(A)$$

$$\Rightarrow \quad Ax = \lambda x$$

$$\Rightarrow \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_{11} x_1 \\ A_{22} x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$\therefore \lambda \in \sigma(A_{11}) \cup \sigma(A_{22})$$

From D1, D2: $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$

Hence proved

(d) (i) $\det(tI - A)$

$$= \begin{vmatrix} (t-a_{11}) & a_{12} & a_{13} & a_{nn} \\ & (t-a_{22}) & & \\ & & \ddots & \\ & & & (t-a_{nn}) \end{vmatrix}$$

all the terms are real. Then the polynomial all the a_{ij} are real. then in p_A polynomial.

$$p_A(t) = t^n - \left(\sum_{i=1}^{n} a_{ii}\right) t + \dots + (-1)^n |\det A|$$

(all coeff are real)

(ii) $\displaystyle\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} \in \mathbb{R}$

$\downarrow$

(hence occurs in conjugate pairs)

B Programming

Used the logic.

$$E \left[A^T_{n\times m} \,\middle|\, I_{n\times n} \right] = \left[EA^T \,\middle|\, E \right]$$

$$\begin{array}{c} r \\ n-r \end{array} \left[\begin{array}{c|c} U & E_1 \\ \hline O & E_2 \end{array} \right]$$

rows of E_2 : Basis of null space.

rows of U : Basis of Row Space.

Similarly perform elimination of $[A \mid I]$ to get $Col(A)$, $N(A^T)$ basis

(Find implementation in the Ass^n 2 . ipynb file)

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