Solutions: Assignment 1

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MAINAK BISWAS
PhD student, IISc

1) (i) $I_j = \begin{cases} 1 & : \text{if the learner succeeds in the } j^{th} \text{ attempt} \\ 0 & \text{otherwise} \end{cases}$

$P(I_j = 1) = p \quad ; \quad I_j \sim \text{Bern}(p)$

(ii) X : number of attempts to pass the exam.

$K \geq N$: else, not possible $P(X = K;\ (K \leq N)) = 0$

$P(X = K) = ?$

$|\leftarrow \quad K \text{ attempts} \quad \rightarrow|$

$\underbrace{F\ S\ F\ F\ S\ \dots\dots}_{N-1 \text{ successes},\ K-N \text{ failures}}\ \underline{S}$

(assume iid.)

(iii) $P(X = K) = P(\underbrace{N-1 \text{ success}, K-N \text{ failures in } K-1 \text{ trials}}_{\sim \text{Bin}(K-1,\ N-1)}) \cdot \underbrace{p}_{P(I_K = 1)}$

$$= \binom{K-1}{N-1} p^{N-1} (1-p)^{K-1-(N-1)} \cdot p$$

$$= \binom{K-1}{N-1} p^{N} (1-p)^{K-N}$$

(iv) $\underbrace{\boxed{\_\ \_\ \_\ S}}_{Y_1}\ \underbrace{\boxed{\_\ \_\ S}}_{Y_2}\ \underbrace{\boxed{S}}_{Y_3}\ \dots$

$$\mathbb{E}[X] = \mathbb{E}[Y_1 + Y_2 + \dots Y_N]$$

Y_1 : number of trials for 1^{st} success $\sim \text{Geo}(p)$

$$= N\, \mathbb{E}\, Y_1 \leftarrow \text{geo}(p)$$

$$= N/p$$

(v) $\text{Var}(X) = N\, \text{Var}(Y_1)$: (independent) $\{Y_1\ Y_2\ \dots\ Y_N\}$ each are independent $\text{geo}(p)$

$$= N \cdot \frac{(1-p)}{p^2}$$

2)

geo(p)

2)

$F_X(x) = \mathbb{P}(X \le x)$

$Y = F_X(x) \xrightarrow{\text{for continuous distr. with inv. } F} Y \sim Unif(0,1)$

Here $X \sim Bern(p)$

$\mathbb{P}(Y \le y) = \mathbb{P}(F_X(X) \le y) = \mathbb{P}(X \le \tilde{F}^{-1}(y))$

F^{-1} is not defined.

We define $X = \tilde{F}^{-1}(u)$: s.t $X \sim F_X(x)$

$\Downarrow$

(generalized inverse)

$\tilde{F}^{-1}(y) = \min\{x : F_X(x) \geq y\} \quad 0 < y < 1$

$\tilde{F}^{-1}(0) = -\infty$

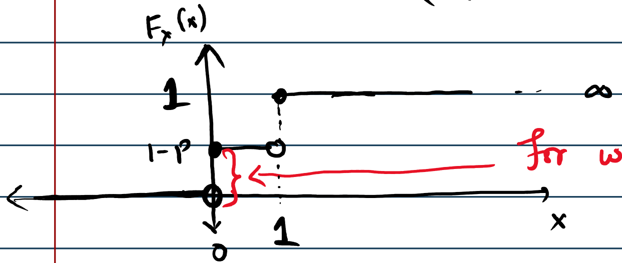
$\tilde{F}^{-1}(1) = \infty$

$\tilde{F}^{-1} : \underline{\underline{[0,1]}} \to \mathbb{R}$

(i) $F_X(x) = \mathbb{P}(X \le x)$: for $X \sim Bern(p)$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & x \geq 1 \end{cases}$$

Fig 1



for what values of x is the $F_X(x)$ greater than this range?

$\Rightarrow \forall x \geq 0$

$\Downarrow$

min = 0

(Reason)

(ii) $\tilde{F}_X^{-1}(y) = \begin{cases} -\infty & y = 0 \\ 0 & 0 < y < 1-p \\ 1 & 1-p \le y < 1 \\ \infty & y = 1 \end{cases}$

$\parallel$

$\min\{x : F_X(x) \geq y\}$

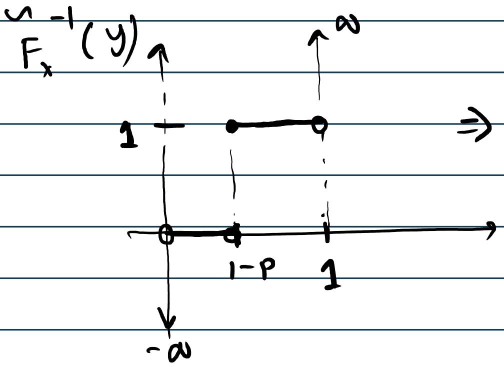
Similarly $\tilde{F}^{-1}(1-p+\epsilon) = \min\{x : F_X(x) \geq 1-p+\epsilon\}$

Similarly $\tilde{F}^{-1}(1-p+\epsilon) = \min_{x} \{ x : F_X(x) \geq 1-p+\epsilon \} \quad \forall \epsilon > 0$

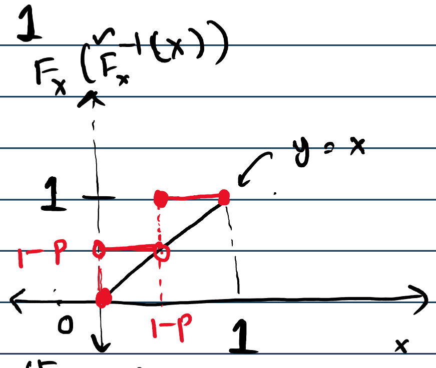
$= \min_{x} \{ x : x \geq 1 \}$

$= 1$

$\therefore$



$\Rightarrow$



(Fig 2)

idea as distribution become cont.

(becomes close to $y=x$)

$$F_X(\tilde{F}_X^{-1}(x)) = \begin{cases} 0 & \tilde{F}_X^{-1}(x) < 0 \\ 1-p & 0 \leq \tilde{F}_X^{-1}(x) < 1 \\ 1 & \tilde{F}_X^{-1}(x) \geq 1 \end{cases}$$

$$= \begin{cases} 0 & x = 0 \Rightarrow \tilde{F}_X^{-1}(0) = -\infty < 0 \\ 1-p & 0 < x < 1-p : 0 \leq \tilde{F}_X^{-1}(x) = 0 < 1 \\ 1 & 1-p \leq x \leq 1 : \tilde{F}_X(x) = 1 \geq 1 \end{cases}$$

(iii) Now to sample from $X \sim Bern(p)$

Step 1 : $U \sim Unif(0,1)$

Step 2 : $X = \tilde{F}^{-1}(U)$ — in fig 2

Argument: $(1-p)$ prob of sampling a 0.

p prob of sampling a 1

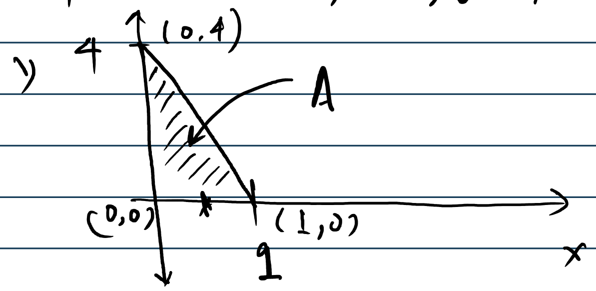
So uniform can also be used to sample discrete.

p prob of samples a 1

so uniform can also be used to sample discrete distribution.

(iv) Check the python code file. - Q2 (Algo in 3 implemented)

3) $x + y/4 \leq 1$, $x, y \geq 0 \in A$



$$|A| = \frac{1}{2} \times 1 \cdot 4 = 2$$

(i) $$f_{x,y}(x,y) = \begin{cases} 1/2 & x \in A \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $$f_x(x) = \int_y f_{x,y}(x,y)\, dy$$

$\Rightarrow x + y/4 \leq 1$

$\Rightarrow y \leq (1-x) \cdot 4$

$$= \int_0^{4(1-x)} \frac{1}{2} \cdot dy = 2(1-x)$$

$$f_{y|x}(x,y) = \frac{f_{x,y}(x,y)}{f_x(x)} = \frac{1}{2} \cdot \frac{1}{2(1-x)}$$

$$= \frac{1}{4(1-x)}$$

$$f_{Y|X}(x,y) = \begin{cases} \frac{1}{4(1-x)} & 0 \leq y \leq (1-x)4 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore f_{Y|X}(x,y) = \text{Unif}(0, 4(1-x))$$

(iii) $$\mathbb{E}[Y|X] = \int_0^{4(1-x)} y \, f_{Y|X}(x,y) \, dy$$

$$= \frac{1}{4(1-x)} \cdot \frac{16(1-x)^2}{2}$$

$$= 2(1-x)$$

(iv) $$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$$

$$= \mathbb{E}[2(1-X)]$$

$$= 2[1 - \mathbb{E}X]$$

$$= 2\left[1 - \int_0^1 x \, f_X(x) dx\right]$$

$$= 2\left[1 - 2\int_0^1 x(1-x) \, dx\right]$$

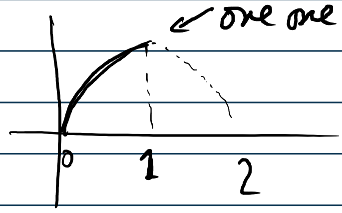
$$= 2\left[1 - 2\cdot\left[\frac{1}{2} - \frac{1}{3}\right]\right]$$

$$= 2\cdot\left[1 - \frac{1}{3}\right] = 4/3$$

(v) $$F_X(x) = \int_0^x 2(1-x)\,dx$$

$$F_X(x) = 2\left[x - \frac{x^2}{2}\right] \qquad \underline{\underline{0 \le x \le 1}}$$

$$= x(2-x)$$



Sampling algo: –

need to sample F_X : need F_X^{-1}

$$F_X(x) = -(1 - 2x + x^2) + 1$$

$$\Rightarrow (x-1)^2 = 1 - F_X(x)$$

$$x = 1 \pm \sqrt{1 - \underbrace{F_X(x)}_{\text{replace w/ } u \sim Unif(0,1)}}$$

$$x = 1 - \sqrt{u}$$

1) step 1 : sample $U \sim Unif(0,1)$

2) step 2 : Hence: $x = 1 - \sqrt{u}$

3) step 3 : Hence $Y|X \sim Unif(0, 4(1-x))$

[see implementation in the notebook- jupyter]

4) (i)

$$M_Z(t) = \mathbb{E}[e^{tZ}]$$

$$= \underset{N}{E}\left[\mathbb{E}\left[e^{tZ} \mid N=n\right]\right]$$

$$= \underset{N}{\mathbb{E}}\left[\underbrace{\mathbb{E}\left[e^{tX_1}\right]}^{n}\right] \quad \text{[independent and identical]}$$

$$= \underset{N}{E}\left\{\left[M_X(t)\right]^n\right\}$$

$$= \underset{N}{E}\left[e^{N\{\log[M_X(t)]\}}\right]$$

$$= M_N\left(\log M_X(t)\right) \quad \underline{\text{Hence proved}}$$

(ii)

1. $Y_1 = Z^3$. {one-one}

$$\frac{dy_1}{dz} = 3z^2 \qquad z = y_1^{1/3}$$

$$\frac{dz}{dy_1} = \frac{1}{3z^2}$$

$$f_{Y_1}(y) = \underbrace{f_Z(z)}\ \frac{dz}{dy} \qquad = \frac{1}{3y_1^{2/3}}$$

$$= f_Z(y_1^{1/3}) \cdot \frac{1}{3z^2}$$

$$= \frac{1}{3y^{2/3}\sqrt{2\pi}} e^{-\frac{1}{2}y^{2/3}}$$

2. $Y_2 = Z^4$

2. $Y_2 = Z^4$

$$P(Y_2 \le y) = P(Z^4 \le y)$$
$$= P(-\sqrt[4]{y} \le Z \le \sqrt[4]{y})$$
$$= 2\,\mathbb{P}(0 \le Z \le \sqrt[4]{y})$$

$$F_{Y_2}(y) = 2 \cdot \int_0^{y^{1/4}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

Diff. both sides

$$f_{Y_2}(y) = \sqrt{\frac{2}{\pi}} \frac{1}{4} y^{-3/4} \cdot e^{-y^{1/2}/2}$$

$$= \frac{1}{\sqrt{8\pi}\, y^{3/4}} e^{-\frac{1}{2} y^{1/2}}$$

(iii) $\mathbb{P}(X = k)$ $\qquad X \mid N \sim Bin(N, p)$; $N \sim Poiss(\lambda)$

$$P(X = k) = \sum_{n \geq k} P(X = k, N = n)$$

$$= \sum_{n \geq k} P(X = k \mid N = n)\, P(N = n)$$

$$= \sum_{n \geq k} \binom{n}{k} \cdot p^k (1-p)^{n-k} \frac{e^{-\lambda} \lambda^n}{n!} \qquad \lambda^k \lambda^{n-k}$$

$-\lambda \qquad k \qquad n-k$

$$= e^{-\lambda} \sum_{n \geq k} \frac{(\lambda p)^k}{k!} \frac{[\lambda(1-p)]^{n-k}}{(n-k)!}$$

$$P(X=k) = e^{-\lambda} \frac{(\lambda p)^k}{k!} \sum_{n \geq k} \frac{[\lambda(1-p)]^{n-k}}{(n-k)!}$$

$$n-k = m$$

$$\sum_{m \geq 0} \frac{\{\lambda(1-p)\}^m}{m!} = e^{\lambda(1-p)}$$

$$\therefore \underbrace{\boxed{P(X=k) = e^{-\lambda p} \frac{(\lambda p)^k}{k!}}}_{\text{Pois}(\lambda p)}$$

5) (i)

Markov

$$P(X \geq \epsilon) \leq \frac{E f(X)}{f(\epsilon)}$$

Assume $X : \Omega \to R_+$

$$f(x) = e^{\theta x} \longrightarrow \theta > 0 \quad \text{(increasing funct)}$$

$$\therefore P(X \geq \epsilon) \leq \frac{E\, e^{\theta X}}{e^{\theta \epsilon}} = \left[E e^{\theta X}\right] \cdot e^{-\theta \epsilon}.$$

Hence pro

(ii) 2 ... 2 ...

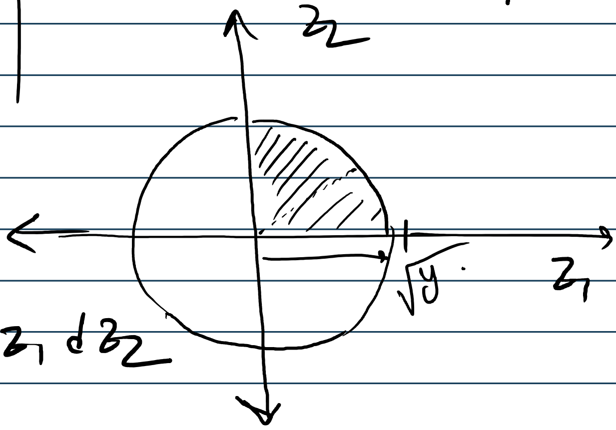
(ii) $\chi^2(2) = Z_1^2 + Z_2^2$: $f_{\chi^2(2)}(y) = ?$

$P(\chi^2(2) \leq y)$

$\mathbb{P}(Z_1^2 + Z_2^2 \leq y)$

$f_{Z_1 Z_2}(z_1, z_2) = \frac{1}{2\pi} e^{-(z_1^2 + z_2^2)/2}$

(independent)



$$= \iint_{(z_1^2+z_2^2 \leq y)} \frac{1}{2\pi} e^{-(z_1^2+z_2^2)/2} \, dz_1 \, dz_2$$

(polar coords)

$$= \int_0^{2\pi} \int_0^{\sqrt{y}} \frac{r \, e^{-r^2/2}}{2\pi} \, dr \, d\theta$$

$dz_1 \, dz_2 = r \, dr \, d\theta$

$r^2 = z_1^2 + z_2^2$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{y}} r e^{-r^2/2} \, dr$$

$$= \frac{2\pi}{2\pi} \cdot \int_0^{\sqrt{y}} e^{-r^2/2} \, d(r^2/2)$$

$\left[-e^{-r^2/2} \right|_0^{\sqrt{y}}$

$$= \left[1 - e^{-y/2} \right]$$

$$F_{\chi^2(2)}(y) = 1 - e^{-y/2}$$

$$\boxed{f_{\chi^2(2)}(y) = \frac{1}{2} e^{-y/2}}$$

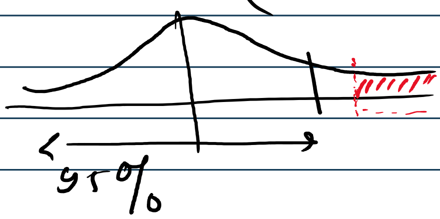
(iii)

(a) Z-test : (as described σ known)

$$\hat{\sigma} = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{40}} \text{ Kg.}$$

$$\text{Z-score} : \frac{66-58}{5/\sqrt{40}} = \left(\frac{8\sqrt{40}}{5}\right) \simeq 10.11$$

(one side test)



$$p = P(|z-0| > \text{Z-score}) = P\left(Z > 10\right) < 0.001$$

(10 sd away)

H_0 : discarded as $p < 0.05$

(b) $\sigma \to$ true not known.

$$\hat{\sigma} = \frac{7}{\sqrt{40}}$$

degrees of freedom = 40 - 1

$$T_{39} = \frac{Z}{\sqrt{\frac{1}{39}\sum_{i=1}^{39} Z_i}}$$

$$t = \frac{66-58}{7/\sqrt{40}} = \frac{8\sqrt{40}}{7}$$

$$P(T > 7) < 0.001$$

$P(T > 7) < 0.001$

$P < 0.05$

H_0 : discarded. (significant difference)