

c) The matrix IP

$$= \begin{bmatrix} 0 & 1 & 2 & 3 & \dots & N \\ 0 & 1/N & 0 & (N-1)/N & & 0 \\ 0 & 2/N & 0 & (N-2)/N & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1/N \\ N & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\tau = \tau IP$$

$$\tau(\lambda I - IP) = 0$$

$$|\lambda I - IP| = 0$$

$$\text{or } |IP - \lambda I| = 0$$

$$\therefore \begin{vmatrix} -\lambda & 1 & & & 0 \\ 1/N & -\lambda & N-1/N & & 0 \\ 2/N & -\lambda & N-2/N & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1/N \\ 0 & 0 & 0 & \dots & 1 - \lambda \end{vmatrix} = 0$$

Expanding across row 1,

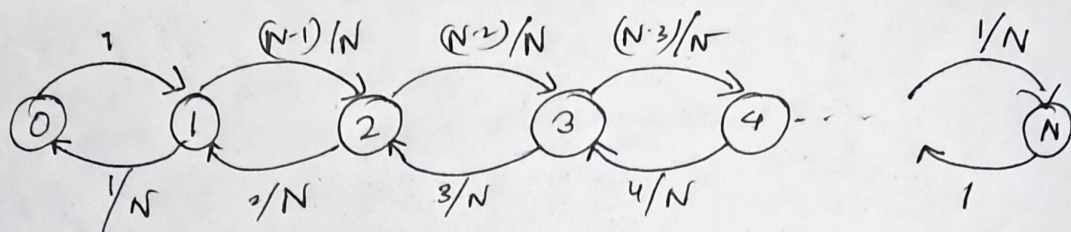
$$\Rightarrow -\lambda \begin{vmatrix} -\lambda & (N-1)/N & 0 \\ 2/N & -\lambda & \\ & & 1/N \\ & & 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1/N & N-1/N & 0 \\ 0 & -\lambda & N-2/N \\ & 3/N & \ddots & \ddots & 1/N \\ & & & & 1 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda^2 & (N-1)/N \\ +2\lambda/N & -\lambda^2 & (N-2)/N \\ & -\lambda^2 & 2/N \\ & +\lambda & -\lambda^2 \end{vmatrix} + 1 \begin{vmatrix} 1/N & (N-1)/N \\ 0 & -\lambda & N-2/N \\ & & \ddots & \ddots & 1/N \\ & & & & 1 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda^2 + 1/N & N-1/N(\lambda+1) \\ 2\lambda/N & -\lambda^2 - \lambda \\ & & (\lambda+1)/N \\ & & \lambda+1 & -\lambda^2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1) \begin{vmatrix} \lambda+1/N & N-1/N \\ 2\lambda/N & -\lambda \\ & & (\lambda+1)/N \\ & & -\lambda \end{vmatrix} = 0$$

Q1(3). Method 2



$$\left(\sum_{j \neq i} P_{ij} \right) \pi(i) = \sum_{j \neq i} P_{ji} \pi(j)$$

flux out flux in

~~$$(P_{i(i+1)} + P_{i(i-1)}) \pi(i) = (P_{(i-1)i} \pi(i-1) + P_{(i+1)i} \pi(i+1))$$~~

$$\Rightarrow \left(1 - \frac{1}{N} + \frac{1}{N} \right) \pi(i) = \left(\left(1 - \frac{i-1}{N} \right) \frac{i+1}{N} \pi(i+1) \right)$$

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$$\Rightarrow \pi(i) = \left(\frac{(N+1)-i}{N} \pi(i-1) + \frac{i+1}{N} \pi(i+1) \right)$$

for $i \neq 0, N$. (1)

for $i=0$

$$\pi(0) = \frac{1}{N} \pi(1)$$

for $i=N$

$$1 \cdot \pi(N) = \frac{1}{N} \cdot \pi(N-1)$$

① ...

$$N \pi(i) = (N+1) \pi(i-1) - i \pi(i-1) + i \pi(i+1) + \pi(i+1)$$

$$\Rightarrow N(\pi(i) - \pi(i-1)) = (\pi(i-1) + \pi(i+1)) + i(\pi(i+1) - \pi(i-1))$$

Using Equation of birth death process, —

$$\pi(x) P_{(x)(x+1)} = \pi(x+1) P_{(x+1)(x)}$$

$$\Rightarrow \pi(x) \left(1 - \frac{x}{N}\right) = \pi(x+1) \left(\frac{x+1}{N}\right)$$

$$\Rightarrow \frac{\pi(x)}{\pi(x+1)} = \frac{(x+1)/N}{(N-x)/N}$$

$$\Rightarrow \frac{\pi(x+1)}{\pi(x)} = \left(\frac{N-x}{x+1}\right)$$

$$\therefore \frac{\pi(x+1)}{\pi(x)} \cdot \frac{\pi(x)}{\pi(x-1)} \cdots \frac{\pi(1)}{\pi(0)} = \prod_{i=0}^x \left(\frac{N-i}{i+1}\right)$$

$$= \frac{(N-x)}{x+1} \cdot \frac{N-(x-1)}{(x-1)+1} \cdots \frac{N-0}{1}$$

$$\frac{\pi(x+1)}{\pi(0)} = \frac{N!}{(N-x)!} \cdot \frac{1}{(x+1)!} = \frac{N!}{(N-x-1)! (x+1)!}$$

$$\Rightarrow \boxed{\pi(x+1) = {}^N C_{x+1} \pi(0)} = {}^N C_{x+1}$$

$$\circ \circ \Pi = [\pi(0) \quad {}^N C_1 \pi(0) \quad \dots \quad {}^N C_N \pi(0)]$$

Since Π gives probability (at $t=\infty$) of remaining in a state, hence.

$$\sum_{i=0}^N \Pi(i) = 1 \Rightarrow \pi(0) ({}^N C_0 + {}^N C_1 + \dots + {}^N C_N) = 1$$

$$\Rightarrow \pi(0) (2^N) = 1 \Rightarrow \boxed{\pi(0) = \frac{1}{2^N}}$$

$$\begin{aligned} \Rightarrow \Pi &= [\pi(0) \quad {}^N C_1 \pi(0) \quad \dots \quad {}^N C_N \pi(0)] \\ &= \pi(0) [{}^N C_0 \quad {}^N C_1 \quad {}^N C_2 \quad \dots \quad {}^N C_N] \\ &= \frac{1}{2^N} [{}^N C_0 \quad {}^N C_1 \quad \dots \quad {}^N C_N] \\ &= \frac{1}{2^N} \left[\dots \quad {}^N C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{N-x} \quad \dots \right] \end{aligned}$$

\therefore The stationary distribution is binomial

$$\pi \in \Pi \in \text{Bin} \left(p = \frac{1}{2} \right)$$