

$$c) \text{ The matrix } P = \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & N \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \frac{1}{N} & 0 & \cdots & 0 \\ 2 & \frac{1}{N} & 0 & \frac{(N-1)}{N} & \cdots & 0 \\ 3 & 0 & \frac{2}{N} & 0 & \frac{(N-2)}{N} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$T = T_2 P$$

$$T(\lambda I - P) = 0$$

$$|\lambda I - P| = 0$$

$$\text{or } |P - \lambda I| = 0$$

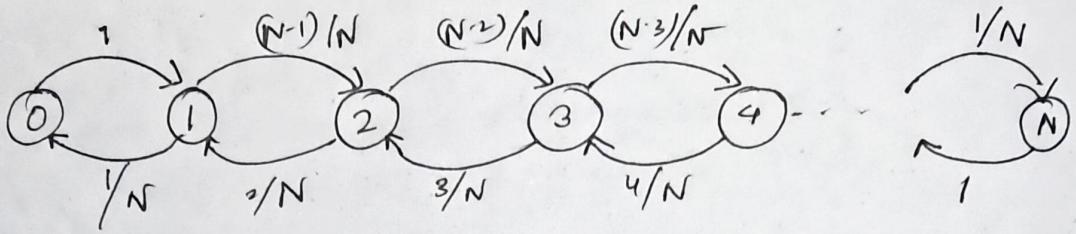
$$\left| \begin{array}{ccccc} -\lambda & 1 & 0 & \cdots & 0 \\ \frac{1}{N} & -\lambda & \frac{N-1}{N} & \cdots & 0 \\ 0 & \frac{2}{N} & -\lambda & \frac{N-2}{N} & \cdots \\ 0 & 0 & \frac{3}{N} & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & -\lambda & \frac{1}{N} \\ 0 & 0 & 0 & \cdots & 1 & -\lambda \end{array} \right| = 0$$

Expanding across row 1,

$$\begin{aligned} & \rightarrow \left| \begin{array}{ccccc} -\lambda & \frac{N-1}{N} & 0 & \cdots & 0 \\ \frac{2}{N} & -\lambda & & & \\ 0 & & 1 & -\lambda & \\ & & & & \\ & & & & \end{array} \right| - 1 \left| \begin{array}{ccccc} \frac{1}{N} & \frac{N-1}{N} & 0 & \cdots & 0 \\ 0 & -\lambda & \frac{N-2}{N} & \cdots & 0 \\ \frac{3}{N} & & 1 & -\lambda & \\ 0 & & & & \\ & & & & \end{array} \right| = 0 \\ & \rightarrow \left| \begin{array}{ccccc} -\lambda^2 & \frac{(N-1)\lambda}{N} & & & \\ + \frac{2\lambda}{N} & -\lambda^2 & \frac{(N-2)\lambda}{N} & & \\ & -\lambda^2 & \frac{2}{N} & & \\ & + \lambda & -\lambda^2 & & \\ & & & & \end{array} \right| + 1 \left| \begin{array}{ccccc} \frac{1}{N} & \frac{N-1}{N} & & & \\ 0 & -\lambda & & & \\ & & 1 & -\lambda & \\ & & & & \\ & & & & \end{array} \right| = 0 \\ & \Rightarrow \left| \begin{array}{ccccc} -\lambda^2 + \frac{1}{N} & \frac{N-1}{N}(\lambda+1) & & & \\ \frac{2\lambda}{N} & -\lambda^2 - \lambda & & & \\ & & (\lambda+1) & & \\ & & & & \\ & & & & \end{array} \right| = 0 \end{aligned}$$

$$\Rightarrow (\lambda+1) \left| \begin{array}{ccccc} \frac{\lambda^2 + 1}{N} & \frac{N-1}{N} & & & \\ \frac{2\lambda}{N(\lambda+1)} & -\lambda & & & \\ & & (\lambda+1)/N & & \\ & & & & \end{array} \right| = 0.$$

Q1(3). Method 2



$$\left(\sum_{j \neq i} P_{ij} \right) \pi(i) = \sum_{j \neq i} P_{ji} \pi(j)$$

~~flux out~~ ~~flux in~~

~~$$(P_{i(i+1)} + P_{i(i-1)}) \pi(i) = (P_{(i-1)i} \pi(i-1) + P_{(i+1)i} \pi(i+1))$$~~

$$\Rightarrow \left(1 - \frac{x}{N} + \frac{x}{N} \right) \pi(i) = \left(\left(1 - \frac{1}{N} \right) \pi(i-1) + \frac{i+1}{N} \pi(i+1) \right)$$

$$\Rightarrow \pi(i) = \left(\frac{(N+1)-i}{N} \pi(i-1) + \frac{i+1}{N} \pi(i+1) \right)$$

for $i \neq 0, N$.. ①

for $i=0$

$$\pi(0) = \frac{1}{N} \pi(1)$$

for $i=N$

$$1 \cdot \pi(N) = \frac{1}{N} \cdot \pi(N-1)$$

$$\textcircled{1} \quad N\pi(i) = (N+1)\pi(i-1) - i\pi(i-1) + i\pi(i+1)$$

$$\Rightarrow N(\pi(i) - \pi(i-1)) = (\pi(i-1) + \pi(i+1)) + i(\pi(i+1) - \pi(i-1))$$

Using Equation of birth death process, —

$$\Pi(x) P_{(x)(x+1)} = \Pi(x+1) P_{(x+1)(x)}$$

$$\Rightarrow \Pi(x) \left(1 - \frac{x}{N}\right) = \Pi(x+1) \left(\frac{x+1}{N}\right)$$

$$\Rightarrow \frac{\Pi(x)}{\Pi(x+1)} = \frac{(x+1)/N}{(N-x)/N}$$

$$\Rightarrow \frac{\Pi(x+1)}{\Pi(x)} = \left(\frac{N-x}{x+1}\right)$$

$$\therefore \frac{\Pi(x+1)}{\Pi(x)} \cdot \frac{\Pi(x)}{\Pi(x-1)} \cdots \frac{\Pi(1)}{\Pi(0)} = \prod_{i=0}^x \left(\frac{N-x}{i+1}\right)$$
$$= \frac{(N-x)}{x+1} \cdot \frac{N-(x-1)}{(x-1)+1} \cdots \frac{N-0}{1}$$

$$\frac{\Pi(x+1)}{\Pi(0)} = \frac{N!}{(N-x)!} \cdot \frac{1}{(x+1)!} = \frac{N!}{(N-x-1)!(x+1)!}$$

$$\rightarrow \boxed{\Pi(x+1) = N C_{x+1} \Pi(0)}$$

$$\stackrel{\circ}{\circ} \Pi = \begin{bmatrix} \pi(0) & N_{C_1} \pi(0) & \dots & \dots & \dots & \dots & N_{C_N} \pi(0) \end{bmatrix}$$

Since Π gives probability (at $t=\infty$) of remaining in a state, hence.

$$\sum_{i=0}^N \Pi(i) = 1 \Rightarrow \pi(0)(N_{C_0} + N_{C_1} + \dots + N_{C_N}) = 1$$

$$\Rightarrow \pi(0) (2^N) = 1 \Rightarrow \boxed{\pi(0) = \frac{1}{2^N}}$$

$$\Rightarrow \Pi = \begin{bmatrix} \pi(0) & N_{C_1} \pi(0) & \dots & \dots & \dots & \dots & N_{C_N} \pi(0) \end{bmatrix}$$

$$\Rightarrow \pi(0) \left[N_{C_0} \quad N_{C_1} \quad N_{C_2} \quad \dots \quad N_{C_N} \right]$$

$$\Rightarrow \frac{1}{2^N} \left[N_{C_0} \quad N_{C_1} \quad \dots \quad \dots \quad N_{C_N} \right]$$

$$\Rightarrow \frac{1}{2^N} \left[N_{C_2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{N-2} \right]$$

\therefore The stationary distribution is binomial

$$\boxed{\pi(C_i) \sim \text{Bin}(p=\frac{1}{2})}$$