2.a) $Q_{n+1}(a) = Q_n(a) + \alpha [R_{n+1} - Q_n(a)]$

$Q_{n+1}(a) = Q_n(a)(1-\alpha) + \alpha R_{n+1}$

let $Q_n(a) \equiv x_n$ $\quad R_n \equiv y_n$

$\therefore x_{n+1} = (1-\alpha) x_n + \alpha y_{n+1}$

$= (1-\alpha)((1-\alpha) x_{n-1} + \alpha y_n) + \alpha y_{n+1}$

$= (1-\alpha)^2 x_{n-1} + \alpha(y_{n+1} + (1-\alpha) y_n)$

$\vdots$

$= (1-\alpha)^{n+1} x_0 + \alpha (y_{n+1} + \cdots (1-\alpha)^n y_1)$

[(note $(1-\alpha)^{2} x_{n-1}$ $\quad n-1+2 = n+1$

$\downarrow$

$(1-\alpha)^{n+1} x_0$ $\quad n+1+0 = n+$)]

$\because x_0 = 0$

$\therefore x_{n+1} = \alpha \sum_{i=0}^{n} (1-\alpha)^{n-i} y_{i+1}$ $\Rightarrow$ $\boxed{Q_{n+1}(a) = \alpha \sum_{i=0}^{n} (1-\alpha)^{n-i} R_{i+1}}$

$\therefore Q_{n+1}(a) = \alpha[R_{n+1} + (1-\alpha) R_n + (1-\alpha)^2 R_{n-1} \cdots]$

2b)

$= \alpha R_{n+1} + (\alpha)(1-\alpha) R_n + \alpha (1-\alpha)^2 R_{n-1} \cdots]$

$\Uparrow$ $\quad \alpha > \alpha(1-\alpha) > \alpha(1-\alpha)^2 > \cdots$

$\therefore$ more weights on recent rewards

Since the distribution changes with time, we rely on / put more weightage on recent happenings, thus tracking ~~going along with~~ change of environment with time.

2c) $\alpha_n = \dfrac{\log(n+2)}{(n+2)}$. R-M algo :-

1> $\boxed{\alpha_n > 0}$ [$\log(n+2) > 0$ as; $n \geq 1 \therefore n+2 \geq 3$

$\therefore \log(n+2) \geq \log 3 > 0$ [monotonic inc.)

$\Rightarrow \log(n+2)/(n+2) > 0$]

2> $\boxed{\sum \alpha_n = \infty}$ (To show).

$S = \sum_{n=1}^{\infty} \dfrac{\log(n+2)}{(n+2)}$ $\quad S_k = \sum_{n=1}^{k} \dfrac{\log(n+2)}{(n+2)}$

$\boxed{k=1 \Rightarrow S_1 = \dfrac{\log 3}{3}}$ $\quad$ $\boxed{k=2 \Rightarrow S_2 = \dfrac{\log 3}{3} + \dfrac{\log 4}{4}}$ $\quad$ $\boxed{\therefore S = \dfrac{\log 3}{3} + \dfrac{\log 4}{4} + \dfrac{\log 5}{5} + \cdots}$

~~integral test~~ let $f(x) = \frac{\log(x+2)}{(x+2)}$

$$\int_1^{t} f(x)\,dx = \int_3^{t} \frac{\log y}{y}\,dy = \left[\frac{1}{2}(\log y)^2\right]_3^{t} = \frac{(\log t)^2 - (\log 3)^2}{2}.$$

$t \to \infty$, $\int_1^{t} f(x)\,dx \to \infty$.

Since integral diverges, hence $\sum_{n=1}^{\infty} f(n)$ also diverges.

[By integral test] [$f(x)$ is monotone decreasing as —

Let $f_1(x) = \frac{\log x}{x} \Rightarrow f_1'(x) = \frac{x\frac{1}{x} - (\log x)\cdot 1}{x^2} = \frac{1 - \log x}{x^2}$

$$= \frac{1 - \log x}{x^2}$$

$f_1'(x) = 0 \Rightarrow \log x = 1$ or, $\boxed{x = e}$

Also. $f_1''(x) = \frac{d}{dx}\left(\frac{1}{x^2} - \frac{\log x}{x^2}\right) = \frac{-2}{x^3} - \frac{x^2\frac{1}{x} - 2x\log x}{x^4}$

$$= -\frac{2}{x^3} - \frac{1}{x^3} + \frac{2\log x}{x^3}.$$

hence $x = e$ maxima.

$$= -\left.\frac{3 + 2\log x}{x^3}\right|_{x=e} < 0$$

Thus after $x > e$ we have the fn as ~~[illegible]~~ decreasing.

2> monotone decreasing

$$f_1'(x) = \frac{1-\log x}{x^2} < 0 \Rightarrow \log x > 1 \Rightarrow x > e$$

hence, after $x>e$ we have monotonic decreasing.

why $x>e$ satisfies $f(x)$?

$$f(x) = \frac{\log(x+2)}{(x+2)} \quad x \geq 1.$$

$3>e$. hence, $f(x)$ satisfies properties of $f_1(x)$ for, $x \geq 3 > e$.

So, $f(x)$ diverges.

(iii) $\sum_1 \alpha_n^2 < \infty$

$$f_1(x) = \frac{(\log x)^2}{x^2} \quad \int_1^\infty f_1(x)\,dx = \int_1^\infty \underset{\mathrm{I}}{\frac{(\log x)^2}{\underset{\mathrm{II}}{x^2}}}dx = \cancel{\int_1^\infty \frac{1}{x}(\log x)^2 d(\log x)}$$

$$= \boxed{\cancel{\frac{1}{x}\int (\log x)^2 d(\log x)}}$$

$$= (\log x)^2 \cdot \int \frac{dx}{x^2} - 2\int\left(\log x \int \frac{dx}{x^2}\right)dx$$

$$= (\log x)^2 \frac{x^{-1}}{-1} + 2\int \frac{\log x}{x}\cdot\frac{1}{x}\,dx = -\frac{(\log x)^2}{x} + 2\int \frac{(\overset{\mathrm{I}}{\log x})}{\underset{\mathrm{II}}{x^2}}dx$$

$$= -\frac{(\log x)^2}{x} + 2\,\frac{\log x(-1)}{x} + 2\int \frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$= -\frac{(\log x)^2}{x} - \frac{2\log x}{x} - \frac{2}{x} = -\left[\frac{(\log x)^2 + 2\log x + 2}{x}\right]$$

$$\therefore \int_1^\infty f_1(x)\,dx = -\left[\frac{(\log x)^2 + 2\log n + 2}{x}\right]_1^\infty = -(0-2)$$

$$= \underline{2}$$

$$\left[\lim_{x\to\infty} \frac{(\log x)^2 + 2\log x + 2}{x} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{2\log x \frac{1}{x} + 2\frac{1}{x}}{1} \right.$$

$$= \lim_{x\to\infty} \frac{2\log x}{x} + \cancel{\frac{2}{x}}^{\,0}$$

$$\left. = \lim_{x\to\infty} \frac{2\log x}{x}\left(\frac{\infty}{\infty}\right) = 2\lim_{x\to\infty} \frac{1}{x} = \underline{0}\right]$$

By integral test, $\sum_{n=1}^\infty \frac{\log(n+2)}{(n+2)^2}$ converges $(<\infty)$. (monotonic decreasing → apparent).