Short Note on Heat Equation

Heat Equation

The heat equation is a fundamental partial differential equation (PDE) that describes the distribution of heat (or variation in temperature) in a given region over time. The general form of the heat equation in one spatial dimension is given by:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where:

- $u = u(x, t)$ represents the temperature at position x and time t
- α is the thermal diffusivity constant

Higher-dimensional Heat Equation

In n dimensions, the heat equation can be written as:

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u \tag{2}$$

where ∇^2 denotes the Laplacian operator, given by:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \tag{3}$$

Separation of Variables

For solving the heat equation, the method of separation of variables assumes:

$$u(x, t) = X(x)T(t) \tag{4}$$

Substituting into the 1D heat equation gives:

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{5}$$

This leads to two ordinary differential equations:

$$T'(t) = -\lambda \alpha T(t) \tag{6}$$
$$X''(x) = -\lambda X(x) \tag{7}$$

Example Solution

A common solution in a bounded domain $0 < x < L$ with Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$ is:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2 \pi^2 \alpha t}{L^2}} \sin\left(\frac{n\pi x}{L}\right) \tag{8}$$