Let $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ be an isometry i.e. $\phi \in \mathrm{Isom}(\mathbb{H}^2)$. Let also that $A \in PSL(2, \mathbb{R})$ such that $A \circ \phi(I) = I$ where I is the imaginary axis minus $\{0\}$.Also let $B(z) = \frac{1}{\lambda} z \; \forall z \in \mathbb{C}$ where λ such that $A \circ \phi(i) = \lambda i$ for $i = \sqrt{-1}$. Then $B \circ A \circ \phi(i) = i$ and $B \circ A \circ \phi(I) = I$ as $B(I) = I$. Then we have $B \circ A \circ \phi$ preserves the $\{0, \infty\}$ and i why?

Reason:

$B \in \mathrm{Isom}(\mathbb{H}^2)$, hence

$$B \circ A \circ \phi \in \mathrm{Isom}(\mathbb{H}^2)$$

Hence $B \circ A \circ \phi$ is a continuous map from $\mathbb{H}^2$ to $\mathbb{H}^2$ hence we can extend it to the boundary of $\mathbb{H}^2$ i.e. $\mathbb{C} \cup \{\infty\}$. Hence $B \circ A \circ \phi$ preserves the $\{0, \infty\}$ and i.

Now, consider a map $C(z) = -\frac{1}{z}$, this map is an isometry of $\mathbb{H}^2$.

Reason:

Corresponidng matrix of C is

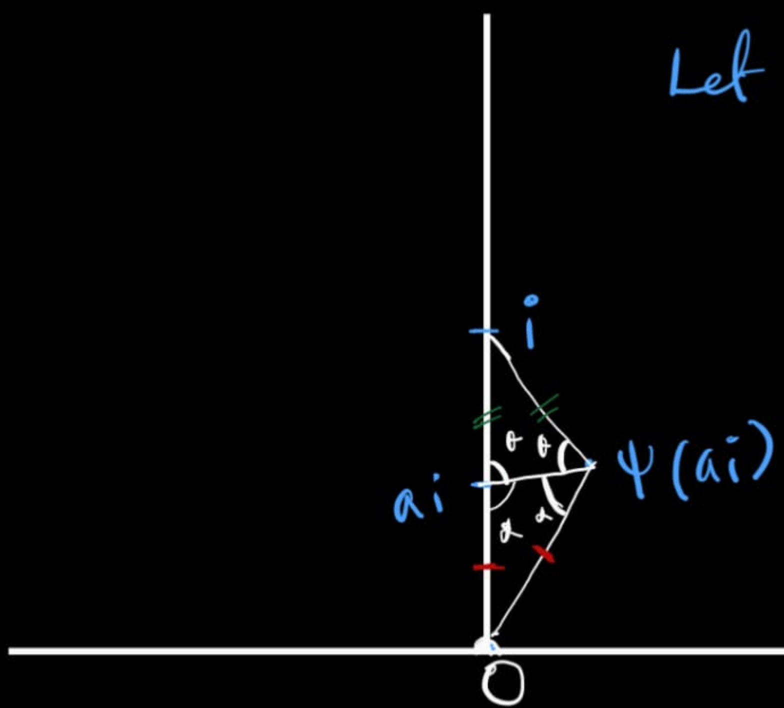
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is in $PSL(2, \mathbb{R})$ and hence $C \in \mathrm{Isom}(\mathbb{H}^2)$.

Now, $C \circ B \circ A \circ \phi(0) = 0$ and $C \circ B \circ A \circ \phi(\infty) = \infty$ and $C \circ B \circ A \circ \phi(i) = i$ and hence $C \circ B \circ A \circ \phi(I) = I$.

Reason:

Let $C \circ B \circ A \circ \phi = \psi$



$\alpha + \theta = \pi$

$\Rightarrow \psi(ai) = ai \Rightarrow \psi(I) = I$

pointwise.