The Cauchy–Schwarz inequality (also called Cauchy–Bunyakovsky–Schwarz inequality)

Statement:

Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \in \mathbb{C}$ then we have the following:

$$\left|\sum_{i=1}^{n} a_i \bar{b}_i\right|^2 \leq \left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right)$$

and equality holds iff $a_i = tb_i$ for some $t \in \mathbb{C}\ \forall i \in \{1, 2, 3, \cdots n\}$

Proof:

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \sum_{i=1}^{n} |a_i - tb_i|^2 \quad (Remember) \tag{1}$$

Note:

$$|z|^2 = z\overline{z} \tag{2}$$

Using (2) in (1) we get,

$$\begin{aligned} f(t) &= \sum_{i=1}^{n} (a_i - tb_i)(\overline{a_i - tb_i}) \\ &= \sum_{i=1}^{n} (a_i - tb_i)(\overline{a_i} - \overline{t}\overline{b_i}) \\ &= \sum_{i=1}^{n} \left(|a_i|^2 - (a_i\overline{b_i} + \overline{a_i}b_i)t + |b_i|t^2\right) \\ &= \sum_{i=1}^{n} \left(|a_i|^2 - Re(a_i\overline{b_i})t + |b_i|t^2\right) \\ &= \sum_{i=1}^{n} |a_i|^2 - 2Re\left(\sum_{i=1}^{n} a_i\overline{b_i}\right)t + \left(\sum_{i=1}^{n} |b_i|\right)t^2 \end{aligned}$$

Hence we get,

$$f(t) = \sum_{i=1}^{n} |a_i|^2 - 2Re\left(\sum_{i=1}^{n} a_i\overline{b_i}\right)t + \left(\sum_{i=1}^{n} |b_i|\right)t^2 \tag{3}$$

Note: Notice that $f(t) \geq 0 \; \forall t \in \mathbb{R}$, hence by the property of quadratic equation we get, $b^2 - 4ac \leq 0$. Then apply this to equation (3) we get,

$$\left(2Re\left(\sum_{i=1}^{n} a_i\overline{b_i}\right)\right)^2 \leq 4\left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right) \tag{4}$$

Now, making $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \sum_{j=1}^{n} |a_j - itb_j|^2 \quad (i = \sqrt{-1})$$

And proceeding similarly we get,

$$\left(2Im\left(\sum_{i=1}^{n} a_i\overline{b_i}\right)\right)^2 \leq 4\left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right) \tag{5}$$

Adding (4) and (5) we get,

$$\left|\sum_{i=1}^{n} a_i\bar{b}_i\right|^2 \leq \left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right)$$

Now, for the equality we have, $f(t) = 0$ happens only if two of its roots are same, otherwise it could n't be always non-negative.
Hence we have, $b^2 - 4ac = 0$ and this happens iff the equality happens in C-S inequality.

But, from definition of f we have $a_i = tb_i \; \forall i = 1, 2, \cdots, n$ or