Computational Physics 08/22/2020

HW 7 Sunday

1a) $p = p_0 - \alpha x$; $u_x = \alpha Re (1-y^2)/2$ $\forall \alpha$

$u_y = 0$

$$\vec{\nabla} \cdot \vec{u} = \frac{du_x}{dx} + \frac{du_y}{dy} = \frac{d}{dx}\left[\alpha Re (1-y^2)/2\right] + 0 = 0$$

$$\dot{\vec{u}} = \frac{\partial \vec{u}}{\partial t} = \frac{\partial u_x}{\partial t}\hat{i} + \frac{\partial u_y}{\partial t}\hat{j} = \frac{\alpha Re}{\cancel{2}} \left(-2y \frac{dy}{dt}\right)\hat{i} + 0$$

$$= -\alpha Re \, y \, u_y \hat{i} = 0$$

$$(\vec{u} \cdot \vec{\nabla})\vec{u} = \frac{\partial u}{\partial x} u_x + \frac{\partial u}{\partial y} \cancel{u_y}^{0}$$

$$= \cancel{\frac{\partial u}{\partial x}}^{0} u_x = 0$$

$\therefore$ LHS $= \dot{\vec{u}} + (\vec{u} \cdot \vec{\nabla})\vec{u} = 0$

$$-\nabla p = -\frac{\partial p}{\partial x}\hat{i} - \frac{\partial p}{\partial y}\hat{j} = \alpha \hat{i} \quad \Big| \, p = p_0 - \alpha x$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{\frac{\partial^2 u_x}{\partial x^2}}^{0}\hat{i} + \cancel{\frac{\partial^2 u_y}{\partial x^2}}^{0}\hat{j}$$

$$+ \frac{\partial^2 u_x}{\partial y^2}\hat{i} + \cancel{\frac{\partial^2 u_y}{\partial y^2}}^{0}\hat{j}$$

$$= \frac{\partial^2}{\partial y^2}\left[\alpha Re (1-y^2)/2\right]\hat{i} = -\alpha Re \, \hat{i}$$

$\therefore$ $Re^{-1} \nabla^2 \vec{u} = -\alpha \hat{i}$

$\therefore$ RHS $= -\nabla p + Re^{-1} \nabla^2 u = 0$

$\therefore$ $p = p_0 - \alpha x$, $u_x = \alpha Re (1-y^2)/2$, $u_y = 0$

is a sol^n to NS eqn for any α (proved)

b) $\vec{\omega} = \vec{\nabla} \times \vec{u}$

NS eqn : $\dot{\vec{u}} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\nabla p + Re^{-1} \nabla^2 \vec{u}$

$$\vec{\nabla} \times \left[\dot{\vec{u}} + (\vec{u} \cdot \vec{\nabla})\vec{u}\right] = \nabla \times \left[-\nabla p + Re^{-1} \nabla^2 \vec{u}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{u}) + (\vec{u} \cdot \vec{\nabla})(\vec{\nabla} \times \vec{u}) = -\nabla \cancel{\times} \overset{0}{\vec{\nabla}} p + Re^{-1} \nabla^2 (\vec{\nabla} \times \vec{u})$$

$$\Rightarrow \dot{\vec{\omega}} + (\vec{u} \cdot \vec{\nabla})\vec{\omega} = Re^{-1} \nabla^2 \vec{\omega}$$

Now, $\vec{\omega} = (0, 0, \omega)$ (2D)

$$\therefore \dot{\omega} + (\vec{u} \cdot \vec{\nabla})\omega = Re^{-1} \nabla^2 \omega$$

(proved)

- Boundary conditions on ω are same as on $\vec{u}$.

c) Since, $\vec{\omega} = \vec{\nabla} \times \vec{u}$ and $\vec{u}$ depends on pressure, first we have to find $\vec{u}$ from NS eqn and use it to find $\vec{\omega}$.

Simple method to find $\vec{u}$:

$$u^{n+1/2} = u^n + \Delta t \left(-\vec{\nabla} p^n + \nu \nabla^2 \vec{u}^n\right)$$

$$\frac{u^{n+1} - u^{n+1/2}}{\Delta t} = -\vec{\nabla} p^{n+1}$$

$\vec{\nabla} \cdot u^{n+1} = 0$ (from continuity eqn)

$$\therefore \nabla^2 p^{n+1} = \frac{1}{\Delta t} \vec{\nabla} \cdot \vec{u}^{n+1/2}$$

$\therefore \vec{u}^{n+1} = u^{n+1/2} - \Delta t \, \nabla p^{n+1}$.

Now, from the vorticity eqn, we see that ω^{n+1} has a dependence on u^n. So, to find ω^{n+2} we ~~would~~ need u^{n+1} value as well.

$$\frac{\omega^{n+2} - \omega^{n+1}}{\Delta t} = -(u^{n+1} \cdot \nabla)\, \omega^{n+1} + Re^{-1} \nabla^2 \omega^{n+1}$$

$$\therefore \omega^{n+2} = \omega^{n+1} + \Delta t \left\{ -(u^{n+1} \cdot \nabla)\omega^{n+1} + Re^{-1} \nabla^2 \omega^{n+1} \right\}$$

2a) $A[n(r)] = \int dr'\, dr'' \, \frac{n(r')\, n(r'')}{|r' - r''|}$

$$\frac{\delta A[n(r)]}{\delta n(r)} = \lim_{\varepsilon \to 0} \int \frac{dr'\, dr''}{\varepsilon |r' - r''|} \left[n(r') + \varepsilon \delta(r' - r) \right] \cdot \left[n(r'') + \varepsilon \delta(r'' - r) \right]$$

$$= \lim_{\epsilon \to 0} \left[\int \frac{dr'\, dr''}{\varepsilon |r' - r''|} \left[n(r') \varepsilon \delta(r'' - r) \right] + \int \frac{dr'\, dr''}{\varepsilon |r' - r''|} \left[\varepsilon n(r'') \delta(r' - \right. \right.$$

$$= \cancel{2 \int dr' (n} \quad 2 \int \frac{dr'\, n(r')}{|(r) - r')|}$$

($\because r'$ and r'' are interchang in the 2 terms)

b) $A[n(r)] = \int dr' \, |\nabla n(r')|^2$

$= \int dr' \, \nabla n(r) \cdot \nabla n(r)$

We can write:

$$\frac{\delta A[n(r)]}{\partial n(r)} = \frac{\partial \, \nabla n(r) \cdot \nabla n(r)}{\partial n(r)} - \nabla \cdot \frac{\partial \, \nabla n(r) \cdot \nabla n(r)}{\partial \nabla n(r)}$$

$$= -\frac{\nabla n(r) \cdot \nabla n(r)}{n(r)} - \left(\frac{2 \nabla^2 n(r)}{n(r)} - 2 \frac{\nabla n(r) \, n(r)}{n(r)} \right.$$

$$= \frac{\nabla n(r) \cdot \nabla n(r)}{n(r)} - \frac{2 \nabla^2 n(r)}{n(r)}$$

Scanned with CamScanner