Numerical Methods / Scientific Computing

Project: GPS positioning

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Project: GPS positioning

GPS fundamentals

- GPS is based on time and the known position of 24 GPS specialized satellites.
- Satellites carry atomic clocks that are synchronized with one another. The satellite locations are also tracked with great precision.
- Satellites continuously transmit data about their current time and location.
- A receiver monitors multiple satellites and solves equations to determine the precise (x, y, z) position of the receiver. (At any time, 5–8 satellites are visible).

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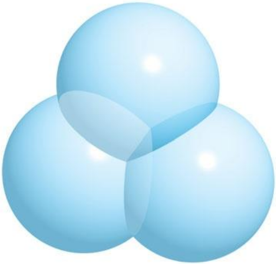
GPS: detailed description

- At a given instant, the receiver collects signals from 3 satellites and determines their transmission times t_i = the difference between their time of transmission and time of arrival
- The distance of the satellite from the receiver is $r_i = ct_i$, where $c = 299,792.458$ km/sec is the speed of light.
- The receiver is on the surface of a sphere centered at the (known) position of the satellite (A_i, B_i, C_i) with radius r_i.
- Three spheres are known, whose intersection consists of two points. One intersection is the location (x, y, z) of the receiver.



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GPS equations

- Let (A_i, B_i, C_i) be the location of satellite i.
- Let t_i be the transmission time of satellite i.
- Then the intersection point (x, y, z) of the three spheres satisfies

$$\begin{aligned}\sqrt{(x-A_1)^2+(y-B_1)^2+(z-C_1)^2} &= ct_1 \\ \sqrt{(x-A_2)^2+(y-B_2)^2+(z-C_2)^2} &= ct_2 \\ \sqrt{(x-A_3)^2+(y-B_3)^2+(z-C_3)^2} &= ct_3.\end{aligned} \tag{1}$$

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GPS: Time correction

- GPS receiver clocks are much less precise. Thus, there is a drift d between the receiver and satellite clocks.
- To fix this problem, introduce one more equation using a fourth satellite:

$$
\begin{aligned}
\sqrt{(x-A_1)^2+(y-B_1)^2+(z-C_1)^2} &= c(t_1-d) \\
\sqrt{(x-A_2)^2+(y-B_2)^2+(z-C_2)^2} &= c(t_2-d) \\
\sqrt{(x-A_3)^2+(y-B_3)^2+(z-C_3)^2} &= c(t_3-d) \\
\sqrt{(x-A_4)^2+(y-B_4)^2+(z-C_4)^2} &= c(t_4-d).
\end{aligned}
\tag{2}
$$

- Solve for the unknowns (x, y, z) (position of receiver) and d (time drift of receiver).

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GPS: Solving the system

- Rewrite the system in a more convenient form and solve for (x, y, z, d)

$$\begin{aligned}
(x - A_1)^2 + (y - B_1)^2 + (z - C_1)^2 &= \left[c(t_1 - d)\right]^2 \\
(x - A_2)^2 + (y - B_2)^2 + (z - C_2)^2 &= \left[c(t_2 - d)\right]^2 \\
(x - A_3)^2 + (y - B_3)^2 + (z - C_3)^2 &= \left[c(t_3 - d)\right]^2 \\
(x - A_4)^2 + (y - B_4)^2 + (z - C_4)^2 &= \left[c(t_4 - d)\right]^2
\end{aligned} \tag{3}$$

- Algebraic solution:
  - Subtract the last three equations from the first, obtaining three linear equations in x, y, z
  - Solve the linear systems for x, y, z and substitute into any of the original equations to obtain a quadratic equation in d
- However, in practice the system is ill-conditioned and can only be solved numerically

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GPS: Project tasks

(1) Solve the system by using Multivariate Newtons Method. Find the receiver position (x, y, z) and time correction d for simultaneous satellite positions (A_i, B_i, C_i) equal to $(15600, 7540, 20140)$, $(18760, 2750, 18610)$, $(17610, 14630, 13480)$, $(19170, 610, 18390)$ km, and measured time intervals $t_i = 0.07074, 0.07220, 0.07690, 0.07242$ sec, respectively. Initial vector $(x_0, y_0, z_0, d_0) = (0, 0, 6370, 0)$.

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GPS: Project tasks

(2) Set up a test of the conditioning of the GPS problem. Define satellite positions (A_i, B_i, C_i) from spherical coordinates (ρ, ϕ_i, θ_i) as

$$\begin{aligned} A_i &= \rho\cos(\phi_i)\cos(\theta_i) \\ B_i &= \rho\cos(\phi_i)\sin(\theta_i) \\ C_i &= \rho\sin(\phi_i) \end{aligned} \tag{4}$$

where $\rho = 26570$ km, $0 \leqslant \phi_i \leqslant \pi/2$ and $0 \leqslant \theta_i \leqslant 2\pi$ for $i = 1, \ldots, 4$ are chosen arbitrarily. Set $x = 0$, $y = 0$, $z = 6370$, $d = 0.0001$, and calculate the corresponding satellite ranges $R_i =$ and travel times $t_i = d + R_i/c$, where $c = 299,792.458$ km/sec. Define an error magnification factor as below. The atomic clocks aboard the satellites are correct up to 10^{-8} second. Study the effect of changes in the transmission time of this magnitude. Let the backward, or input error be the input change in meters. At the speed of light, $\Delta t_i = 10^{-8}$ second corresponds to $10^{-8}c \approx 3$ meters. Let the forward, or output error be the change in position $\|(\Delta x, \Delta y, \Delta z)\|_\infty$, caused by such a change in t_i, in meters.

$$\text{Error magnification factor} = \frac{\|(\Delta x, \Delta y, \Delta z)\|_\infty}{c\|(\Delta t_1, \ldots \Delta t_4)\|_\infty}$$

$$\text{Condition number} = \text{maximum error magnification factor for all small } \Delta t_i, 10^{-8} \text{ or less}$$

Change each Δt_i by $\Delta t_i = +10^{-8}$ or -10^{-8}, not all the same. Denote the new solution of the equations (2) by $(\bar{x}, \bar{y}, \bar{z}, \bar{d})$.

- Compute $\|(\Delta x, \Delta y, \Delta z)\|_\infty$, and the error magnification factor, by taking different Δt_i's.
- What is the maximum position error found, in meters?
- Estimate the condition number of the problem.

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GPS: Project tasks

(3) Repeat previous step with a more tightly grouped set of satellites. Choose all ϕ_i's within 5% of one another and all θ_i's within 5% of one another

- Solve with and without the same input error as in previous step
- Find the maximum position error and error magnification factor
- Compare the conditioning of the GPS problem when the satellites are tightly or loosely bunched