Bulletin of Economic Research 00:0, 2017, 0307-3378
DOI: 10.1111/boer.12125

A CLOSED-FORM SOLUTION FOR DETERMINING THE BURDEN OF PUBLIC DEBT IN NEOCLASSICAL GROWTH MODELS

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ABSTRACT

This paper develops a new approach, termed as the stock approach, to calculate the steady-state output loss caused by public debt in neoclassical growth models. The novelty of our stock approach is that it provides a closed-form solution to the steady-state output-debt relationship. The main conclusion of the paper is that the steady-state burden of public debt is country-specific in neoclassical growth models and it decreases with the private saving rate and increases with the population growth rate, with the exception of the special case where Ricardian equivalence holds.

Keywords: crowding out, debt fairy, neoclassical growth models, public debt

JEL Classification: O41, E13, H63

I. INTRODUCTION

This paper presents a closed-form solution to the effect of public debt on steady-state (long-run) output in neoclassical growth models. The calculation of the burden of public debt – that is, the output loss caused by the crowding out of physical capital – has a long tradition in neoclassical growth theory, dating back to the seminal article of Ball and Mankiw (1995). Ball and Mankiw introduced the famous debt fairy parable into the literature, which has been frequently used thereafter (e.g., Elmendorf and Mankiw, 1999; Engen and Hubbard, 2005; Laubach, 2009). Another known, although less widespread, method is the flow approach (e.g., Gale and Orszag, 2004). The problem with these conventional methods is that they either do not focus on the long run or do not present a closed-form solution to the output-debt relationship. By eliminating these deficiencies our stock approach provides a simple tool for policy analyses and empirical estimations.

The paper is organized as follows. Section II introduces the model. Section III discusses the conventional methods for calculating the burden of public debt in neoclassical growth models.

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Section IV presents the stock approach, our closed-form solution to the problem. Section V concludes the paper.

II. THE MODEL

The discussion is based on the human capital augmented neoclassical growth model, with a closed economy elaborated by Mankiw *et al.* (1992).[1] In this model output (Y) is

$$Y_t = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta} \quad (1)$$

where K is physical capital, H is human capital, L is labour, A is technology, $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$ and t is time. The steady-state output (per effective capita) is

$$\ln(\hat{y}^*) = \psi_1 \ln(s_K^*) + \psi_2 \ln(s_H^*) + \psi_3 \ln(n + g + \delta) \quad (2)$$

where $\hat{y} = Y/(AL)$, $\Psi_1 = \alpha/(1-\alpha-\beta)$, $\Psi_2 = \beta/(1-\alpha-\beta)$, $\Psi_3 = -(\Psi_1 + \Psi_2)$, s_K and s_H are the aggregate saving (investment) rates concerning physical and human capital, n is the population growth rate, g is the technology growth rate, δ is the depreciation rate, and n, g and δ are constant. The asterisk refers to the steady state.

Note, that the paper of Mankiw *et al.* (1992) operates with the Solow model by assuming constant and exogenous saving rates. Contrary to Mankiw *et al.*, we do not impose any constraint on households' consumption behaviour because of two interrelated reasons. First, we want to focus on neoclassical models in general, incorporating the whole range of consumption behaviour that might emerge on the continuum between the two extremes, the Solow model and the Ramsey-Cass-Koopmans (RCK) model. Second, despite the improvement in our knowledge over the last two decades about households' saving decision, there is still considerable confusion about the extent to which intergenerational links and dynamic optimization might characterize the behaviour of households.[2] Nevertheless, whatever assumptions are made about households' saving behaviour the long-run saving rate is constant and thus the long-run output is determined by equation (2) in either case (Barro and Sala-i-Martin, 2004, ch. 2.).

In order to facilitate the discussion, the model's parameters are set according to the experience of the developed countries (Mankiw *et al.*, 1992): $\alpha = \beta = 1/3$, $n = 0$, $g = 0.02$, $\delta = 0.04$.

III. CONVENTIONAL METHODS: THE DEBT FAIRY PARABLE AND THE FLOW APPROACH

Calculating the long-run burden of public debt, the literature conventionally resorts to either of two approaches, the debt fairy parable or the flow approach. This section briefly introduces these methods.

The *debt fairy parable* investigates by how much actual output would increase if debt were converted one-to-one into physical capital (Ball and Mankiw, 1995). As Elmendorf and Mankiw (1999, p. 1632–33) phrase it, 'Imagine that one night a debt fairy . . . were to travel around the economy and replaced every government bond with a piece of capital of equivalent value. How different would the economy be the next morning when everyone woke up?'.

[1] The equations in the paper are valid for the non-augmented neoclassical growth model, with $\beta = 0$, as well.

[2] For a discussion, see Romer (2012, ch. 8).

The calculation is based on the marginal product of physical capital (MPK). Assuming that production factors are rewarded by their marginal product, the MPK is $\alpha(Y/K)$. In developed countries Y/K is around 1/3, so MPK is around 1/9. The burden of debt is then calculated as

$$\Delta Y_t \big/ Y_{t-1} = -MPK_t \cdot b_t \tag{3}$$

where $b = B/Y$ is the debt-to-GDP ratio and B is public debt.[3]

For example, according to this approach, a 90 percent debt-to-GDP ratio decreases actual output by approximately 10 percent ($= -(1/9) \cdot 0.9$).[4]

The problem with the debt fairy approach is that it is a pure production function approach – as Engen and Hubbard (2005) have labelled it – without any direct link to long-run output.

Contrary to the debt fairy approach, the second conventional method focuses on the long run (e.g., Gale and Orszag, 2004). We call this method the *flow approach* because it calculates the burden of debt according to the budget deficit, the flow measure of government balance. The underlying idea is that the budget deficit reduces the aggregate saving rate and, thereby, the steady-state output as well. The calculation proceeds as follows: 1. determine the extent of crowding out caused by the deficit and calculate the aggregate saving rate accordingly, 2. calculate the induced percentage change in long-run output, 3. calculate the steady-state debt-to-GDP ratio. These steps necessitate an expression for both the aggregate saving rate and the steady-state debt-to-GDP ratio.

The aggregate savings are the sum of private and government savings. For simplicity we assume that government savings influence only the accumulation of physical capital:

$$s_{K,t} = s_{KP,t} - d \tag{4}$$

where s_{KP} is the private saving rate and d is the budget deficit compared to output ($d<0$ for surplus). We follow the conventional way in the literature and assume that the deficit is determined arbitrarily by the government and held constant thereafter (e.g., Ball and Mankiw 1995; Elmendorf and Mankiw 1999).

The burden of debt depends on how private savings respond to the budget deficit. Let p denote the extent to which budget deficit (surplus) crowds out (crowds in) private investments: $p = -\partial I/\partial D$, where $p \in [0, 1]$, I is private (physical capital) investments and D is the budget deficit in absolute terms. Conversely, $(1\text{-}p)$ reflects how private savings react to the budget deficit. Given this, the private saving rate can be expressed as the sum of an autonomous component ($\bar{s}_{KP}$), which is theoretically independent of the deficit, and the impact of the budget deficit:

$$s_{KP,t} = (1-p)d + \bar{s}_{KP,t}. \tag{5}$$

Equation (5) is a reaction function reflecting how private savings respond to budget deficit. Note, that the theoretical range for p encompasses all the possible consumption behaviours in relation to public debt. If $p = 0$ the impact of budget deficit on aggregate savings is counterbalanced by a rise of equal amount in private savings, therefore Ricardian equivalence holds. In contrast, if $p = 1$ the aggregate savings fall by the amount of budget deficit, therefore the crowding out is complete. With regard to neoclassical growth models, the case of $p = 0$ corresponds to the Ramsey-Cass-Koopmans model where intertemporal optimization and intergenerational links rule households' saving behaviour. The other limiting case, $p = 1$, holds

[3]GDP: Gross Domestic Product.

[4]The burden of public debt related to distortionary taxes is beyond the scope of this paper. However, its magnitude is negligible (Dedák and Dombi, 2016).

in the Solow model with a constant and exogenous private saving rate. Beyond the two extreme cases, the closer p is to zero the more intergenerational links and dynamic optimization dominate households' consumption behaviour and the smaller the burden of public debt is (see below).[5]

Combining equations (4) and (5), the aggregate saving rate can be reformulated as

$$s_{K,t} = \bar{s}_{KP,t} - pd. \tag{6}$$

Equations (4), (5) and (6) are macroeconomic identities, so they hold on both the transitional path and also in steady state.

The final step in the flow approach is the determination of the steady-state debt-to-GDP ratio. Differentiating the debt-to-GDP ratio with respect to time yields the equation of motion for b: $\partial b_t / \partial t = d - g_{Y,t} b_t$, where g_Y is the growth rate of output. Assuming an arbitrary constant deficit-to-GDP ratio, and using the equation of motion, the steady-state value for b is given by

$$b^* = \frac{d}{g_Y^*} = \frac{d}{g+n}, \tag{7}$$

where $g_Y^* = g + n$.[6]

Consider, for example, the initial steady-state characterized by a balanced budget ($d = 0$), zero debt ($b^* = 0$), complete crowding out ($p = 1$), and an autonomous private saving rate of 25 percent ($\bar{s}_{KP}^* = 0.25$). What happens if the government starts to run an annual deficit of 1.8 percent for an infinitely long time? First, the aggregate saving rate (s_K^*) falls to 23.2 percent (equation 6). Second, the long-run output is reduced by 7.5 percent: $\ln \hat{y}^*_{d=0.018} - \ln \hat{y}^*_{d=0} = \ln(0.232) - \ln(0.25)$ (equation 2). Third, the long-run debt-to-GDP ratio rises to 90 percent (equation 7). Consequently, this approach predicts that, under the above conditions, the long-run burden of a 90 percent debt-to-GDP ratio is 7.5 percent.

The disadvantage of the flow approach is that it does not present a closed-form solution to the long-run output-debt relationship.

IV. THE STOCK APPROACH: A CLOSED-FORM SOLUTION

This section introduces our approach to the problem, which is closely related to the conventional methods but avoids their disadvantages. First, it focuses on the long run. Second, it establishes a direct link between public debt and output. The key idea is to rewrite the long-run output equation so that the budget deficit would appear as a separate independent variable in an additive fashion on the right-hand side of equation 2. After that, the deficit can be easily substituted by the steady-state public debt according to equation 7. Thus, we couple debt dynamics and long-run output within one equation. We call our method the *stock approach* because the output loss is determined directly by public debt, the stock measure of government balance.

The derivation of the long-run output-debt equation begins with the observation that the budget deficit can be expressed as an arbitrary z fraction of the steady-state autonomous private saving rate: $d = z \cdot \bar{s}_{KP}^*$. Equation 6 can now be reformulated as $s_K^* = (1 - pz)\bar{s}_{KP}^*$. Substituting

[5]Note, that a p close to zero corresponds to the basic Blanchard model in which public debt has only marginal negative effect on long-run output (e.g., Evans 1991; Dedak and Dombi 2016).

[6]Inflation is neglected in this paper. In the case of non-zero inflation, d would stand for the inflation-adjusted budget deficit.

this new formula for s_K^* in equation 2, and using the fact that $\ln(1-pz)\approx -pz$ if pz is close to zero, which is probably the case, we get the following: [7]

$$\ln(\hat{y}^*) = \Psi_1 \ln(\bar{s}_{KP}^*) - \Psi_1 \frac{p}{\bar{s}_{KP}^*} d + \Psi_2 \ln(s_H^*) + \Psi_3 \ln(n+g+\delta) \tag{8}$$

Substituting equation 7 for d in equation 8, we arrive at the long-run output-debt equation, the key equation of our stock approach:

$$\ln(\hat{y}^*) = \Psi_1 \ln(\bar{s}_{KP}^*) - \Psi_4 b^* + \Psi_2 \ln(s_H^*) + \Psi_3 \ln(n+g+\delta) \tag{9}$$

where $\Psi_4 = (\Psi_1 p(g+n)/\bar{s}_{KP}^*$.

Four comments are worth noting concerning equation 9. First, if Ricardian equivalence fails ($p > 0$) the coefficient of b^* is negative, that is, public debt decreases long-run output.

Second, as a result of the $\ln(1-pz)\approx -pz$ linear approximation, b^* affects $\ln(\hat{y}^*)$ linearly. Of course, the long-run output-debt relationship is not log-linear in reality, but log-linearization is very useful both from the viewpoint of policy and regression analysis with regard to the fact that pz is close to zero.[8]

Third, the marginal effect of public debt on long-run output ($-\psi_4$) is country-specific and depends on the autonomous private saving rate, the extent of crowding out, the population growth rate, and other parameters (g, α, β). Indeed, ψ_4 decreases with $\bar{s}_{KP}^*$, and increases with n and p. Consequently, a high (autonomous) private saving rate paired with a low population growth rate results not only in high steady-state output but also in a low burden of public debt. The intuitive explanation is simple. With a higher private saving rate, the budget deficit related to b^* decreases the aggregate saving rate to a lesser extent in percentage terms (equation 6), and, hence, leads to a smaller decline in steady-state output (equation 2). Furthermore, with a higher population growth rate, the d deficit-to-GDP ratio will also be higher for a given b^* (equation 7), and thereby the aggregate saving rate and the long-run output will be lower.

Fourth, in addition to simplicity and tractability, a further advantage of having an output-debt equation is that it provides a theoretical basis for the empirical literature. So far, the empirical literature has mostly operated with arbitrarily constructed growth-debt regressions.[9] Although in their spirit these regressions are similar to equation 9, they lack any thorough underpinning by growth theory. Equation 9 can be used either as a structural regression on its own or as a part of an augmented regression.

In the following, we calculate the marginal effect of the debt-to-GDP ratio on long-run output and the burden of public debt at a 90 percent debt-to-GDP ratio according to equation 9. During the calculation we assume complete crowding out, so the results can be considered as the upper bounds for the burden of public debt. Figures 1 and 2 present the results as the function of steady-state autonomous private saving rate and population growth rate.

[7] It is worth briefly discussing why $pz \approx 0$ holds under realistic assumptions. First, the upper limit of p is one. However, in fact, p is well below zero because crowding out is far from complete. For example, Laubach (2009) calibrates it to 0.6. Second, assuming a debt-to-GDP ratio of 100 percent, which is well above the average pre-crisis level, and a long-run GDP growth of 2 percent, we arrive at a deficit-to-GDP ratio of 2 percent (equation 7). Since, compared to output, private savings (and thus autonomous private savings as well) are slightly above 20 percent in most developed countries, the value of $z = d/\bar{s}_{KP}^*$ is around 10 percent. According to the latter, if $p = 0.6$, pz is only 0.06. Of course, the calibration alters slightly for developing countries with probably more myopic consumption behaviour (higher p), lower private saving rate, and larger deficit. Consequently, in their cases, pz is larger in general, but not dramatically so. For example, if $p = 0.8$, $b^* = 0.7$, $g_Y{}^* = 0.04$ and $\bar{s}_{KP}^* = 0.15$ the value of pz is 0.149.

[8] To get the non-linear equivalent of equation 9, substitute equation 6 for s_K^* in equation 2 and then equation 7 for d: $\ln(\hat{y}^*) = \Psi_1 \ln(\bar{s}_{KP}^* - p(g+n)b^*) + \Psi_2 \ln(s_H^*) + \Psi_3 \ln(n+g+\delta)$.

[9] See Panizza and Presbitero (2013) for a survey.

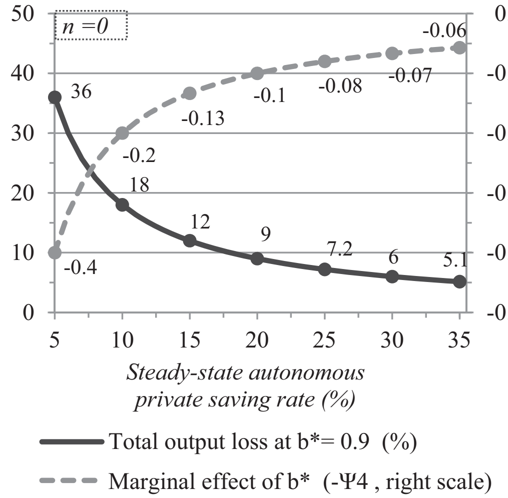

Fig. 1. The marginal effect of public debt and the output loss at a 90% debt-to-GDP ratio as the function of steady-state autonomous private saving rate.

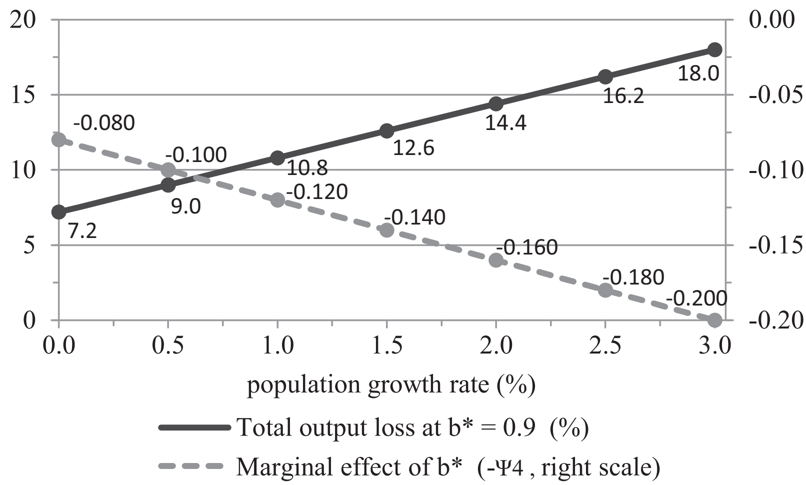

Fig. 2. The marginal effect of public debt and the output loss at a 90% debt-to-GDP ratio as the function of population growth rate.

Notes to figure 1 and figure 2: Total output loss is calculated as follows: $(100 \cdot 0.9 \cdot \psi_4)$. The parameters are set as follows: $g = 0.02, \alpha = \beta = 1/3, p = 1$. In figure 1, $n = 0$. In figure 2, $\bar{s}^*_{KP} = 0.25$.

The figures reflect our previous observations according to which the burden of debt decreases with $\bar{s}^*_{KP}$ and increases with n. Consider first Figure 1. If the steady-state autonomous private saving rate is 25 percent, $-\psi_4 = -0.08$. This means that a one percentage point increase in the debt-to-GDP ratio reduces the long-run output by 0.08 percent. In this case, the total loss

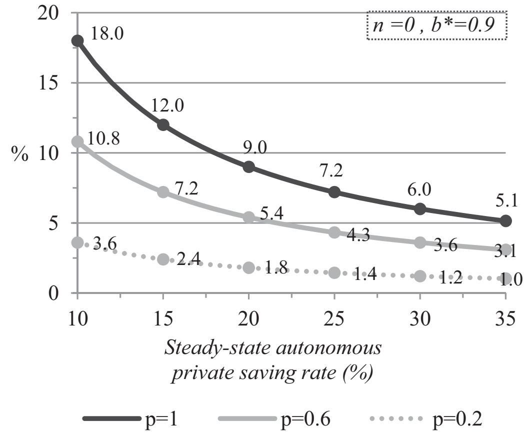

Fig. 3. The effect of the crowding out intensity (p) on total output loss at a 90% debt-to-GDP ratio as the function of steady-state autonomous private saving rate.

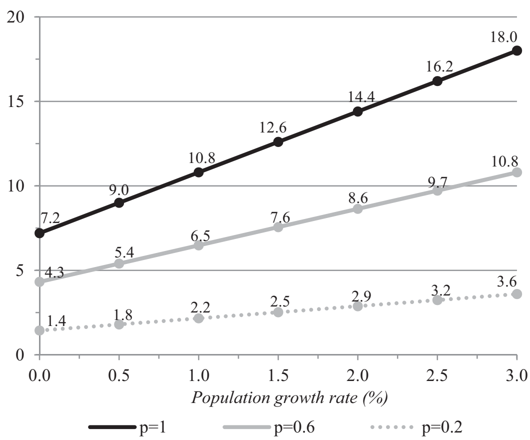

Fig. 4. The effect of the crowding out intensity (p) on total output loss at a 90% debt-to-GDP ratio as the function of population growth rate.

Notes to figure 3 and figure 4: Total output loss is calculated as follows: $(100 \cdot 0.9 \cdot \psi_4)$. The parameters are set as follows: $g = 0.02$, $\alpha = \beta = 1/3$. In Figure 3, $n = 0$. In Figure 4, $\bar{s}^*_{KP} = 0.25$. In order to save space, in Figure 3 we consider a reduced domain for $\bar{s}^*_{KP}$ compared to Figure 1.

of output emanating from a 90 percent debt-to-GDP ratio is 7.2 percent.[10] In contrast, at lower saving rates the output loss can be much higher. Regarding the results shown in Figure 2, the sensitivity of the burden of public debt to the population growth rate is discernible. For example, if the population growth rate increases from zero to 2 percent, the upper limit of the long-run

[10]Recall that under the same settings a 90 percent debt-to-GDP ratio entailed a decrease in long-run output of around 7.5 percent according to the flow approach. This minor difference between the results of the stock and flow approaches can be attributed to the $\ln(1 - pz) \approx -pz$ linear approximation. Performing the calculations according to the non-linearized output-debt equation (see footnote 8), one would arrive at the same result as the one provided by the flow approach.

burden of a 90 percent debt-to-GDP ratio, in the case of $\bar{s}^*_{KP} = 0.25$, doubles, jumping to 14.4 percent.

According to the results shown in Figures 1 and 2, as the (private) saving rate tends be higher and the population growth rate tends to be lower in developed countries than in most of the developing ones, the following general economic conclusion emerges: public debt is less detrimental in developed countries than in developing ones. Moreover, if we consider the typical population growth rate and private saving rate of developed economies (i.e., $n \in [0\%, 0.5\%]$ and $\bar{s}^*_{KP} \in [20\%, 25\%)$, we can conclude that the upper limit of the long-run output cost of public debt does not seem to be too disquieting in these countries. Returning to our example ($b^* = 0.9$), the (maximum) output loss is around, or below, 10 percent under the circumstances of developed economies, which is really not an alarming magnitude. Moreover, departing from the hypothesis of complete crowding out, the issue is further marginalized as we show below.

The country-specific debt coefficient and its theoretical trajectories, presented in Figures 1 and 2, have a special appeal. They are in line with the most recent findings of the empirical literature, according to which: 1. the effect of public debt on economic growth is country-specific, and 2. on average, public debt is more detrimental in developing countries than in developed ones (Kourtellos *et al.*, 2013; Eberhardt and Presbitero, 2015).

Figures 1 and 2 have depicted the case of complete crowding out, that is, the theoretical extreme of $p = 1$. In that sense the calculated debt burdens can be considered as the upper bound for a given population growth rate, and long-run autonomous private saving rate. Of course, p declines as households' saving behaviour increasingly obeys intertemporal optimization and intergenerational links become stronger. As p decreases and approaches to zero, the trajectory of the marginal effect of debt ($-\Psi_4$) shift upwards, whereas the trajectory of total output loss shift downwards. To demonstrate the mitigation of the burden of debt as p decreases, in Figures 3 and 4 we present the total output loss emanating from a 90 percent debt-to-GDP ratio in the case of different crowding out intensities. According to these results, for example, if ceteris paribus p is reset from 1 to 0.6, the burden of a 90 percent debt-to-GDP ratio decreases from 7.2 percent to 4.3 percent in the case of an autonomous private saving rate of 25 percent and a population growth rate of zero percent (figure 3). This is a considerable decrease that again underlines the importance of households' saving behaviour in cushioning the negative effect of government indebtedness.

V. CONCLUSION

Neoclassical growth models are frequently used to gauge the potential effect of public debt on steady-state output. However, the conventional methods used for calculating the burden of public debt do not provide a closed-form solution to the long-run output-debt relationship. Our stock approach fills this loophole in neoclassical growth theory by establishing a direct link between steady-state output and debt that holds independent of whatever assumptions are made about household behaviour.

Our results deliver two crucial implications. First, if Ricardian equivalence does not hold, which is surely the case, the enormous differences in private saving rates and population growth rates, observed across the world, lead to vast differences in the burden of public debt among countries. Second, the burden of debt in developed countries, characterized by a high saving rate and low population growth rate, is rather small even at a debt level of 90 percent and even if significant deviations from Ricardian equivalence exist. Therefore, paying down public debt in these countries does not seem to exert a serious impact on the evolution of long-run income, at least according to neoclassical growth models.

Furthermore, the implications of the closed form solution presented in this paper can help to explain the most recent findings of the empirical literature, according to which the burden of debt is country-specific, and on average, more detrimental for developing countries than developed ones. In addition, our output-debt equation can provide a further theoretical basis for future empirical research in the field.

Finally, since our country specific debt coefficient contains the degree of crowding out, the empirical application of the output-debt equation can contribute to the investigation of a further, greatly debated topic on the practical relevance of Ricardian equivalence.

ACKNOWLEDGEMENTS

We would like to thank two anonymous referees for their invaluable and helpful comments and suggestions which greatly contributed to the improvement of the paper. Of course, any remaining errors are our own responsibility.

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