Multivariable Calculus
Semester 1: Linear Algebra

Anderson Trimm
Gwinnett School of Mathematics, Science and Technology

These are the notes for the Fall Semester 2020 of Multivariable Calculus at GSMST, which covers linear algebra. They will updated frequently throughout the semester. The latest PDF can always be accessed at https://github.com/atrimm/mvc/blob/master/Course%20Notes/linear_algebra_2019.pdf. Please email me with comments and corrections, or send them to me directly as pull requests to the source repository hosted at https://github.com/atrimm/mvc.

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1 Vectors and geometry

1.1 Physical motivation

The earliest notion of a *vector* comes from physics. In nature, we encounter certain physical quantities which cannot be uniquely specified by a number alone, but also depend on a direction in space.

Example 1.1. If the distance from town A to town B is 400 miles and we leave A and travel at 50 miles per hour, then we will arrive at B in 8 hours, but only if we travel in the direction from A to B! Thus, displacement (400 mi, from A to B) and velocity (50 mi/hr, from A to B) are two examples of such physical quantities.

To distinguish physical quantities which depend on a numerical value alone from those which also depend on a direction, we make the following definitions.

Definition 1.2 (Vectors and scalars).

(a) A *scalar* is a physical quantity which is uniquely specified by a numerical value alone.

(b) A *vector* is a physical quantity which is uniquely specified by a numerical value, called its *magnitude* or *norm*, and a direction.

Exercise 1.1. Classify each of the following quantities are vector or scalar:

(a) Force

(b) Temperature

(c) Mass

(d) Volume

(e) Acceleration

(f) Electric Charge

(g) Density

Solution. (a) and (e) are vectors. The rest are scalars. □

In the following sections, we will develop a mathematical model of vector and scalar quantities capable of modeling physical phenomena. As we will see, this model will have applications beyond physics as well.

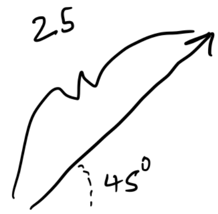
1.2 Mathematical model

1.2.1 Scalars

While electric charges are observed in nature to take only integer values, other scalar quantities, such as mass and temperature, are found to take any *real* value. Thus, in our model a scalar is simply a real number. We denote the set of all real numbers by $\mathbb{R}$. We will denote a scalar by a lower case latin letter, such as x.

1.2.2 Vectors

A vector quantity is uniquely specified by two pieces of information: a magnitude (which is a nonnegative real number) and a direction in three-dimensional space. We can therefore represent a vector geometrically as an *arrow* (directed line segment) in space. The arrow points in the direction specifying the vector while the length of the arrow represents the magnitude of the vector. For example, a force of 25 N directed at an angle of 45° with respect to the positive x-axis is represented by the arrow



We will denote a vector by a boldface latin letter (either upper or lowercase), such as $\mathbf{x}$. When writing by hand, a more common notation is $\vec{x}$.

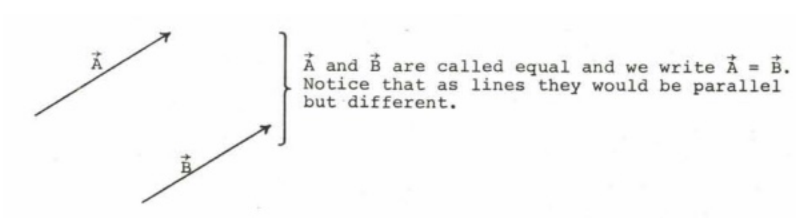
1.2.3 Equality of vectors

We will now discuss the notion of equivalence of two vectors. Recall first the equivalence of plane figures, say triangles. We consider two distinct triangles. Since the data defining a triangle are the side lengths and interior angle measures, any two triangles related by a transformation which leaves these unchanged represents an equivalent triangle; the only difference between them is the location in the plane. As you learned in basic algebra, any transformation of the plane which preserves lengths and angles can be written as a finite sequence of reflections, translations, and rotations and any two triangles related by such a transformation are said to be *congruent*.

Since the defining data of a vector is the magnitude and direction, we will agree that

Definition 1.3 (Equality of vectors). Two vectors are equal if they have the same magnitude and direction.

That is, two vectors are equal if they are represented by parallel line segments which have the same length and orientation.



If two vectors $\mathbf{x}$ and $\mathbf{y}$ are equal, we will denote this by $\mathbf{x} = \mathbf{y}$.

It is important to note that we have defined equality of vectors so that *location* in space does *not* matter. As line segments, the two parallel arrows are congruent, [1] but still considered distinct

[1] They are congruent simply because they have the same length; the fact that they are parallel or have the same orientation does not matter as far as congruence is concerned.

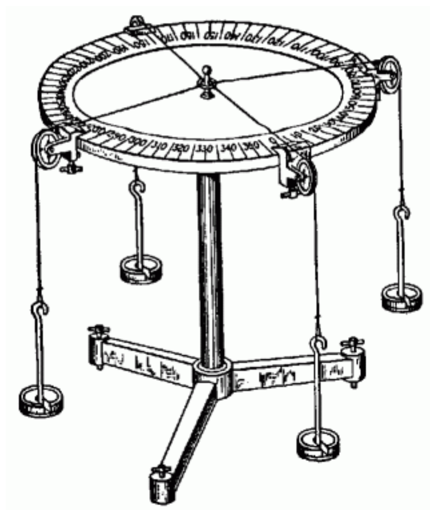
due to their difference in location; on the other hand, as *vectors* they are regarded as exactly the same vector.

1.3 Vector arithmetic

We will now define operations involving vectors, whose motivation will come from physics.

1.3.1 Vector addition

The most basic operation one can define on a set is a *binary operation*, which is a rule for combining any two elements in the set to produce a third element in the set. [2] There is a natural binary operation on the set of vectors, which is suggested by the *force table* experiment in mechanics.

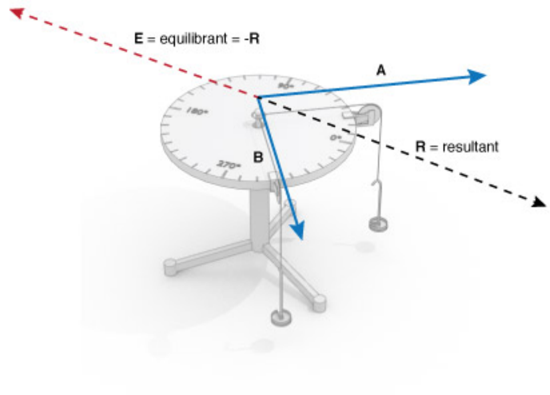


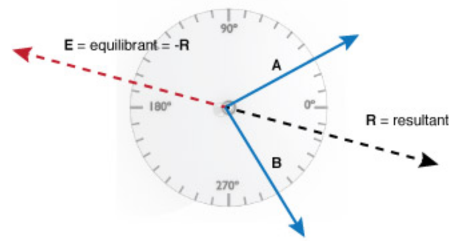
In a force table experiment, strings are tied to a metal ring which is positioned at the center of the table. The strings are then suspended over pulleys which are fixed at known angles, and known masses hung from the ends of the strings. The pull of gravity on a given mass creates tension in the string which pulls on the ring.

In an experiment in which *two* strings are tied to the ring, the tension in each string gives rise to two forces pulling on the washer in different directions. However, the washer ultimately accelerates in a single direction, which is the direction of the *net* (or *total*) force acting on the ring. The rule for combining the two tension force vectors to produce the net force vector is exactly the binary operation we seek to define.

To determine the net force, a third string is connected to the ring with mass and pulley position chosen so that the ring is in *static equilibrium* (i.e., it does not move at all under the influence of these three forces). This vector is called the *equilibrant* vector. By Newton's third law, the net force vector (also called the *resultant* vector) is then the *opposite* of the equilibrant vector, that is, it has the same magnitude and is directed along the same line, but with the opposite orientation.

[2] More formally, we write this as a map $V \times V \to V$, where $V \times V = \{(v, w) \mid v, w \in V\}$ denotes the set of all ordered pairs of elements of V, called the *Cartesian product* of V with itself.



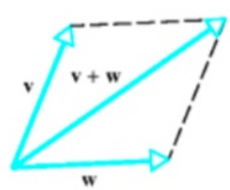


We therefore define the *sum* of two vectors as follows:

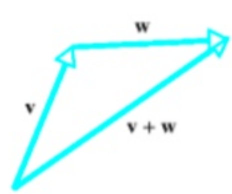
Definition 1.4 (Vector addition). The *sum* $\mathbf{v} + \mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ is the resultant vector of $\mathbf{v}$ and $\mathbf{w}$.

Note that when the two tension forces are along the *same* direction (e.g., just add another mass on the same string), the resultant vector points in this same direction and has magnitude given by the sum of the magnitudes of the two tension vectors, and hence the addition of $\mathbf{v}$ and $\mathbf{w}$ reduces to addition of ordinary numbers in this special case. This is why we have decided to call this binary operation *addition* and to continue to denote it by $+$; it can therefore be thought of as a *generalization* of the ordinary addition of scalars.

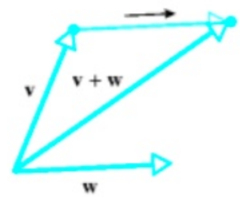
In terms of our geometric representation of vectors, the magnitude and direction of $\mathbf{v} + \mathbf{w}$ is determined as follows: Since the location of a vector is of no consequence (by our definition of equality of vectors), we may position the two vectors so that their initial points coincide. Then $\mathbf{v}$ and $\mathbf{w}$ form adjacent sides of a parallelogram, and the vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram, directed from the common initial point of $\mathbf{v}$ and $\mathbf{w}$ to the opposite vertex of the parallelogram, as shown below.



This is called the *parallelogram rule* for vector addition. Since the opposite sides of a parallelogram are congruent and parallel, we can equivalently view $\mathbf{v} + \mathbf{w}$ as the result of positioning the initial point of $\mathbf{w}$ at the terminal point of $\mathbf{v}$ and drawing the arrow connecting the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$.



This is called the *triangle rule* or *"tip to tail" rule* for vector addition. These two points of view are related by *parallel translation*. To go from the first point of view to the second, we translate the initial point of $\mathbf{w}$ along $\mathbf{v}$, keeping $\mathbf{w}$ parallel to its original direction at all times. Accordingly, $\mathbf{v}+\mathbf{w}$ is also called the *translation of* $\mathbf{w}$ *by* $\mathbf{v}$.



Example 1.5. There is another physical interpretation of this addition rule which agrees with our intuition. Suppose a person walks 10 steps in a north-easterly direction, and then turns and walks another 5 steps to the east. The vector $\mathbf{v}$ then represents his *displacement* from his initial position, with the length of $\mathbf{v}$ being his distance from where he started, and the direction of $\mathbf{v}$ pointing in the direction in which he moved. Similarly, the vector $\mathbf{w}$ represents his displacement from his position after he traveled along the vector $\mathbf{v}$. Their sum, added according to the tip to tail rule, is his *total* displacement from his initial position.

Let us now use this geometric picture to determine the properties of vector addition. Recall that the addition operation defined on the set of real numbers satisfies the following properties: [3]

(i) Associativity: $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{R}$.

(ii) Existence of an additive identity: $\mathbb{R}$ contains an element 0 such that $0+x = x$ for every $x \in \mathbb{R}$.

(iii) Existence of additive inverses: For every $x \in \mathbb{R}$, there exists and element $y \in \mathbb{R}$ such that $x+y=0$.

(iv) Commutativity: $x+y = y+x$ for all $x, y \in \mathbb{R}$.

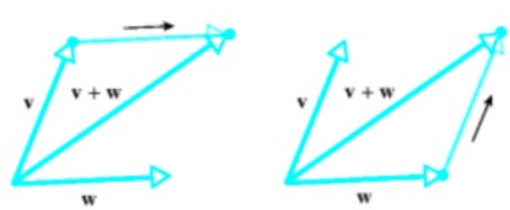
We now consider each of these in turn. We consider commutativity first, since it is the simplest to analyze.

Proposition 1.6 (Vector addition is commutative). Vector addition is commutative. That is,

$$\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$$

for any two vectors $\mathbf{v}$ and $\mathbf{w}$, since each of these is the diagonal of the parallelogram whose edges are formed by $\mathbf{v}$ and $\mathbf{w}$.

Proof. We see from the two diagrams below that the translation of $\mathbf{w}$ by $\mathbf{v}$ is the same vector as the translation of $\mathbf{v}$ by $\mathbf{w}$.



[3]Any set G on which there is a binary operation $*$ which satisfies the first three of these properties is said to form a *group* under $*$. One also says that $(G, *)$ is a group, or just that G is a group if $*$ is understood. If the fourth property (commutativity) also holds, G is said to form a *commutative* (or *abelian*) under $*$.

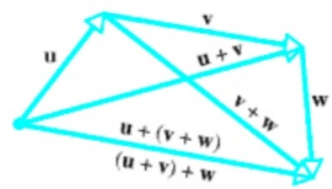
□

Proposition 1.7 (Vector addition is associative). Vector addition is associative. That is, for any three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$, we have

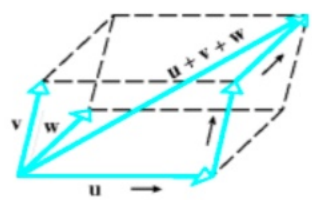
$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

We therefore denote both expressions by $\mathbf{u} + \mathbf{v} + \mathbf{w}$.

Proof. One can construct $\mathbf{u} + \mathbf{v} + \mathbf{w}$ by placing the vectors "tip to tail" in succession and then drawing the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{w}$. If the three vectors lie in the same plane, one can verify associativity from the diagram below. □



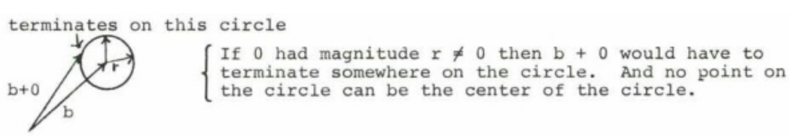
If the three vectors do not all lie in the same plane, then when placed at the same initial point the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ from adjacent edges of a *parallelepiped*. [4] Translating these vectors and adding tip to tail, we see that $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is the diagonal of this parallelepiped.



Corollary 1.8. The sum $\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k$ is independent of how the expression is bracketed.

Proof. We postpone the proof until we discuss vectors in coordinates in section 1.4, as the geometry is very complicated and difficult to analyze. In coordinates, this is seen to hold as a simple consequence of the fact that it holds for real numbers. □

Let us now check whether there is a vector which plays the role of an additive identity. That is, given any vector $\mathbf{b}$, is there a vector $\mathbf{0}$ such that $\mathbf{b} + \mathbf{0} = \mathbf{b}$? Let us suppose there is such a vector $\mathbf{0}$ and denote its magnitude by r. Let us now add $\mathbf{0}$ to $\mathbf{b}$ by the tip-to-tail method. Since $\mathbf{0}$ has magnitude r, $\mathbf{b} + \mathbf{0}$ must lie on a circle of radius r centered on the tip of $\mathbf{b}$. However, the condition $\mathbf{b} + \mathbf{0} = \mathbf{b}$ means that $\mathbf{b} + \mathbf{0}$ must have the same magnitude and direction as $\mathbf{b}$, and there is no point on the circle for which this is true. This shows that there is no such vector $\mathbf{0}$ with nonzero length.

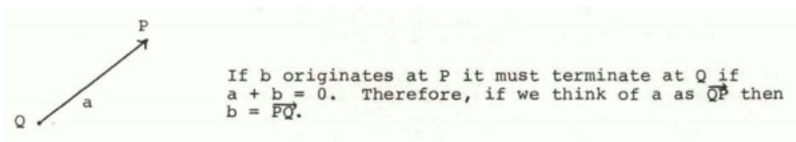


[4] A parallelepiped is a polygon whose faces are parallelograms, with each pair of opposite sides parallel.

A vector of length zero, does not have a defined direction. Thus, there is a unique vector which acts as an additive identity with respect to vector addition.

Definition 1.9 (The zero vector). The zero vector, which we denote by $\mathbf{0}$, is the unique vector of magnitude zero.

Finally, we want to investigate whether every vector has an additive inverse. That is, we want to determine if, when given any vector $\mathbf{a}$, we can find another vector $\mathbf{b}$ such that $\mathbf{a} + \mathbf{b} = \mathbf{0}$. Since the zero vector has zero length and since we add vectors from "tip to tail", it follows that if $\mathbf{a} + \mathbf{b} = \mathbf{0}$, then the tail of $\mathbf{a}$ and the tip of $\mathbf{b}$ must coincide. Thus, any vector $\mathbf{a}$ has a *unique* inverse, which we denote by $-\mathbf{a}$.



Definition 1.10 (Inverse of a vector). The additive inverse of a vector $\mathbf{a}$ is the unique vector $-\mathfrak{a}$, which has the same magnitude as $\mathbf{a}$ and opposite orientation.

In analogy with the equation $x + (-x) = 0$ for real numbers, we denote the additive inverse of $\mathbf{a}$ by $-\mathbf{a}$, so that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ for any vector $\mathbf{a}$. We may again agree, as in the case of numerical addition, to abbreviate $\mathbf{a} + (-\mathbf{b})$ as $\mathbf{a} - \mathbf{b}$, which allows us to define vector subtraction:

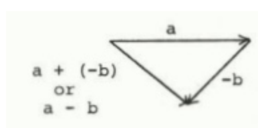
Definition 1.11 (Vector subtraction). The difference $\mathbf{a} - \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the sum

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

From this definition, we may view vector subtraction geometrically as follows: To form $\mathbf{a} - \mathbf{b}$,

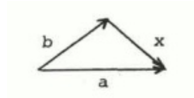
(1) Obtain $(-\mathbf{b})$ from $\mathbf{b}$ by reversing the direction of $\mathbf{b}$.

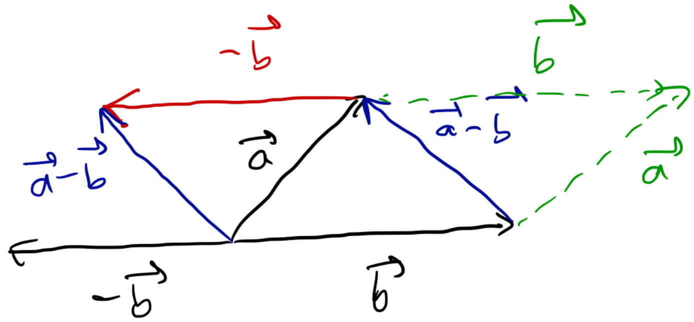
(2) Add $\mathbf{a}$ and $(-\mathbf{b})$ in the usual way, by placing the tail of $(-\mathbf{b})$ at the tip of $\mathbf{a}$. [5]



[5] Like numerical subtraction, vector subtraction is *not* commutative, so the order matters here.

Exercise 1.2. Show that if $\mathbf{a}$ and $\mathbf{b}$ are placed such that their initial points coincide, then $\mathbf{x} = \mathbf{a} - \mathbf{b}$ is the vector which extends from the tip of $\mathbf{b}$ to the tip of $\mathbf{a}$.





Solution.

□

In this section we have shown that our definition of vector addition satisfies the same properties as ordinary addition of real numbers. That is, the additive structures are exactly the same for vectors as for numbers. [6] Therefore, any results that hold for numbers also hold for vectors, for exactly the same reasons. For example,

Theorem 1.12 (Cancellation law). If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors such that $\mathbf{a} + \mathbf{b} = \mathbf{a} + \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

Proof. The proof for real numbers is as follows: let x, y, z be real numbers such that

$$x + y = x + z$$

Adding $-x$ to both sides of this equation, we have

$$\begin{aligned} -x + (x + y) &= -x + (x + z) \\ (-x + x) + y &= (-x + x) + z \text{ (since } + \text{ is associative)} \\ 0 + y &= 0 + z \text{ (since } -x \text{ is the inverse of } x\text{)} \\ y &= z \text{ (since 0 is the additive identity)} \end{aligned}$$

Since addition of vectors obeys exactly the same properties as addition of real numbers, this same proof holds for vectors simply by drawing arrows over x, y and z! □

1.3.2 Scalar multiplication

In physics, we observe that an unbalanced force on a body causes an acceleration in the direction of the force. It is also observed that the magnitude of acceleration of different bodies, when subjected to the same force, varies according to their mass. These observations are formalized in Newton's second law of motion

$$\mathbf{F} = m\mathbf{a}. \tag{1.1}$$

On the right side of this equation, we see a new operation: the product of a scalar and a vector.

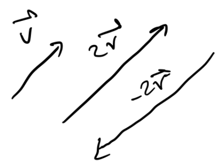
[6] To state this more formally, they both form an abelian group.

Definition 1.13 (Scalar multiplication). Let $\mathbf{v}$ be a vector and k a scalar. The *scalar multiple* of $\mathbf{v}$ by k is a vector $\mathbf{w}$, defined as follows:

- The length of $\mathbf{w}$ is $|k|$ times the length of $\mathbf{v}$. If $|k| = 0$, then $\mathbf{w}$ is the zero vector.
- $\mathbf{w}$ is parallel to $\mathbf{v}$.
- If $k > 0$, then $\mathbf{w}$ has the same orientation as $\mathbf{v}$. If $k < 0$, then $\mathbf{w}$ and $\mathbf{v}$ have opposite orientation.

If $\mathbf{w}$ is the scalar multiple of $\mathbf{v}$ by k, we write $\mathbf{w} = k\mathbf{v}$. [7]

Example 1.14. The vector $2\mathbf{v}$ has the same direction as $\mathbf{v}$ but twice its length, while $-2\mathbf{v}$ is oppositely directed to $\mathbf{v}$ and twice its length.



Theorem 1.15 (Properties of scalar multiplication). For any scalars c, d and vectors $\mathbf{v}, \mathbf{w}$,

(i) $0\mathbf{v} = \mathbf{0}$,

(ii) $1\mathbf{v} = \mathbf{v}$,

(iii) $(-1)\mathbf{v} = -\mathbf{v}$,

(iv) $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$,

(v) $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$,

(vi) $(dc)\mathbf{v} = c(d\mathbf{v})$.

Proof. We will prove pars (i) and (ii) and leave the rest as an exercise. In each case, we need to show that the vector on the left hand side has the same magnitude and direction as the vector on the right hand side.

(i) The vector $1\mathbf{v}$ has length $|1| = 1$ times the length of $\mathbf{v}$, and therefore has the same length as $\mathbf{v}$. The vector $1\mathbf{v}$ is parallel to $\mathbf{v}$, and since $1 > 0$ it has the same orientation as $\mathbf{v}$. This shows $1\mathbf{v}$ has the same magnitude and direction as $\mathbf{v}$, and therefore $1\mathbf{v} = \mathbf{v}$.

(ii) The length of $0\mathbf{v}$ is zero times the length of $\mathbf{v}$, which is zero. Hence, $0\mathbf{v}$ is the zero vector.

□

[7]Note that scalar multiplication is not a binary operation on V, since it does not take two vectors to a vector, but rather a scalar and a vector to a vector. More formally, scalar multiplication is a map $\mathbb{R} \times V \to V$.

Exercise 1.3. Prove parts (iii)-(vi) of Theorem 1.15.

Solution. In each case, we need to show that the vector on the left hand side has the same magnitude and direction as the vector on the right hand side. Each property involves checking several cases of signs.

(iii) $||(-1)\mathbf{v}|| = |-1| \cdot ||\mathbf{v}|| = ||\mathbf{v}||$. Since $-1 < 0$, the vector points in the opposite direction of $\mathbf{v}$. These are exactly the properties of $-\mathbf{v}$, so these vectors are equal.

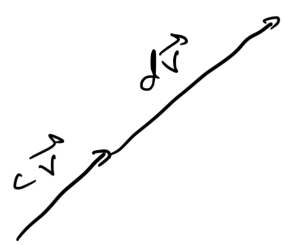
(iv) If $c = 0$, then the left hand side is

$$(c+0)\mathbf{v} = c\mathbf{v}$$

while the right hand side is

$$\begin{aligned} c\mathbf{v} + 0\mathbf{v} &= c\mathbf{v} + \mathbf{0} \\ &= c\mathbf{v} \end{aligned}$$

so equality holds. Similarly if $d = 0$. If $c = d = 0$, then both sides are equal to $\mathbf{0}$. Suppose now that neither c nor d are zero. If $c > 0$ and $d > 0$, then $c\mathbf{v}, d\mathbf{v}$, and $(c+d)\mathbf{v}$ all point in the direction of $\mathbf{v}$. The vector on the left hand side, in units of $||\mathbf{v}||$, has length $c+d$. Adding the vectors on the right hand side tip to tail, by the segment addition postulate the vector



on the right hand side, in units of $||\mathbf{v}||$, has length $c+d$. Since these vectors have the same magnitude and direction, they are equal. The case of $c < 0$ and $d < 0$ is virtually the same. If c and d have opposite signs, without loss of generality we can take $c > 0$ and $d < 0$. Then we need to check the cases $c > d$ and $c < d$. We leave this to the reader.

(v) If $c = 0$, then both sides are equal to $\mathbf{0}$. Assume now that $c \neq 0$. If $\mathbf{w} = \mathbf{0}$, then the left hand side is

$$c(\mathbf{v} + \mathbf{0}) = c\mathbf{v}$$

while the right hand side is

$$c\mathbf{v} + c\mathbf{0} = c\mathbf{v}$$

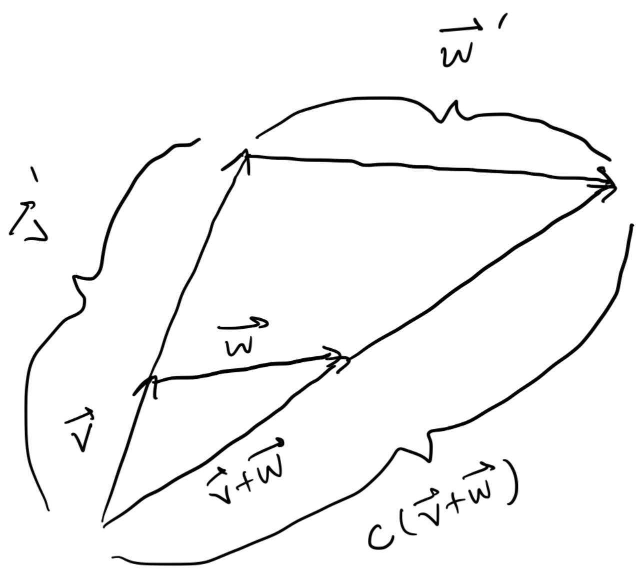
so equality holds. Similarly if $\mathbf{v} = \mathbf{0}$. If both $\mathbf{v} = \mathbf{0}$ and $\mathbf{w} = \mathbf{0}$, then both sides equal $\mathbf{0}$.

Suppose now that $\mathbf{v}$ and $\mathbf{w}$ are both nonzero. Suppose first that these are parallel. Then $\mathbf{w} = \lambda\mathbf{v}$ for some nonzero scalar λ. Suppose that $\lambda > 0$. Then $\mathbf{v} + \lambda\mathbf{v} = (1+\lambda)\mathbf{v}$ by (i), so the vector on the left hand side has length

$$c(1+\lambda) = c + c\lambda$$

in units of $||\mathbf{v}||$. Since $c\mathbf{v} + c\lambda\mathbf{v} = (c + c\lambda)\mathbf{v}$ by (1), the right hand side also has length $c + c\lambda$. Since $c + c\lambda > 0$, both vectors point in the same direction and are therefore equal. We leave it to the reader to check the case $\lambda < 0$.

Let us now suppose that $\mathbf{v}$ and $\mathbf{w}$ are not parallel and that $c > 0$. The figure below shows the sum $\mathbf{v} + \mathbf{w}$ and its scalar multiple $c(\mathbf{v} + \mathbf{w})$. Scalar multiple $\mathbf{v}$ and $\mathbf{w}$ by some unknown positive scalars to obtain $\mathbf{v}'$ and $\mathbf{w}'$ to form a larger triangle.



Since $\mathbf{v}'$ is parallel to $\mathbf{v}$ and $\mathbf{w}'$ is parallel to $\mathbf{w}$, the two triangles are similar, and therefore

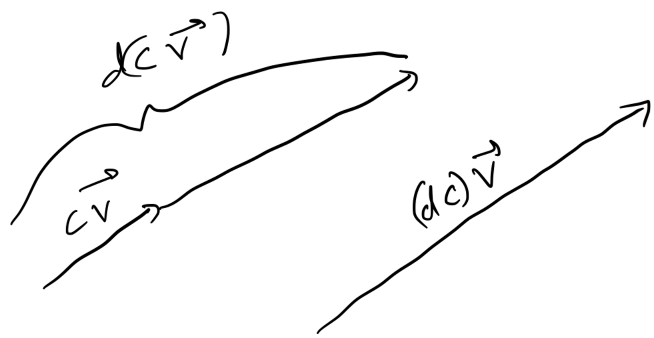
$$\begin{aligned}\frac{||\mathbf{w}'||}{||\mathbf{w}||} = \frac{||\mathbf{v}'||}{||\mathbf{v}||} &= \frac{||c(\mathbf{v}+\mathbf{w})||}{||\mathbf{v}+\mathbf{w}||} \\ &= \frac{c||\mathbf{v}+\mathbf{w}||}{||\mathbf{v}+\mathbf{w}||} \\ &= c\end{aligned}$$

and therefore

$$||\mathbf{w}'|| = c||\mathbf{w}|| \text{ and } ||\mathbf{v}'|| = c||\mathbf{v}||.$$

Since $c > 0$, the primed and unprimed vectors point in the same direction, so indeed $\mathbf{w}' = c\mathbf{w}$ and $\mathbf{v} = c\mathbf{v}$. The case $c < 0$ is similar and left to the reader.

(vi) If either $c = 0$ or $d = 0$ (or both), of if $\mathbf{v} = \mathbf{0}$, then both sides are equal to $\mathbf{0}$. Assume now that $c > 0$ and $d > 0$. The the vector on the left hand side and the vector on the right hand side point in the same direction. The vector on the left, in units of $||\mathbf{v}||$, has length cd. Since the vector $c\mathbf{v}$ has length $c||\mathbf{v}||$ and $d(c\mathbf{v})$ has length $d||c\mathbf{v}|| = dc||\mathbf{v}||$, this vector also has length cd (in units of $||\mathbf{v}||$). Since the vector on the left hand side has the same magnitude and direction as the vector on the right hand side, these are equal. It remains to check the other combinations of signs of c and d, which we leave to the reader.



□

The diligent reader who has worked through the exercises in this section has most likely found many of them to be quite tedious. While the geometric picture of vectors we have developed in this section will be essential for visualization purposes, it is indeed far from optimal for practical computations. This situation will be rectified in the next section by introducing Cartesian coordinate systems in which to describe vectors. [8] As we will see, all properties we deduced geometrically in this section will be seen to hold as simple consequences of the properties of real numbers. Despite this simplification of computations, it is important to remember that vectors are geometrical objects whose properties (length and magnitude) do not depend on any particular coordinate system which we may use to describe them.

1.4 Vectors in coordinate systems

Let us now describe vectors with respect to a Cartesian coordinate system.

1.4.1 Vectors in one dimension

A one-dimensional Cartesian coordinate system is provided by the real number line. Consider a nonzero vector inside the real line; that is, an arrow originating at some point x_1 and terminating at a distinct point x_2. [9] This vector can point in only one of two directions: either left or right, depending on whether $x_2 < x_1$, or $x_1 < x_2$, respectively. Since the location of the vector does not matter, we are free to consider the vector to have initial point at the origin. In this case, the vector

[8] As should be familiar from freshman geometry, a coordinate system is called Cartesian (or rectangular) if its coordinate axes are all mutually perpendicular.

[9] The zero vector has length zero and is therefore just a point.

originates at the origin, and terminates at some point x. The length of the vector is then $|x|$, and its orientation is determined by the sign of x, which we denote by $\operatorname{sgn}(x)$, which can be written as $\operatorname{sgn}(x) = \frac{x}{|x|}$.

Exercise 1.4. Prove that

$$\frac{x}{|x|} = \begin{cases} +1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Solution. If $x > 0$, then $|x| = x$, so $\frac{x}{|x|} = \frac{x}{x} = 1$. If $x < 0$, then $|x| = -x$, then $\frac{x}{|x|} = \frac{x}{-x} = -1$. □

If we have two such vectors terminating at x_1 and x_2, respectively, then these vectors are equal if and only if $|x_1| = |x_2|$ and $\operatorname{sgn}(x_1) = \operatorname{sgn}(x_2)$. Since for any $x \neq 0$, we can write

$$\begin{aligned} x &= \frac{x}{|x|}|x| \\ &= \operatorname{sgn}(x)|x|, \end{aligned}$$

these vectors are equal if and only if $x_1 = x_2$. This shows that there is a 1-1 correspondence between vectors in one-dimension and real numbers, with the correspondence given by

$$x \leftrightarrow \mathbf{x} = (\operatorname{sgn}(x), |x|).$$

We see that there is no essential difference between vectors in one dimension and scalars.

1.4.2 Vectors in two dimensions

A two-dimensional Cartesian coordinate system is given by $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. Each point $(x, y) \in \mathbb{R}^2$ specifies the coordinates of a point in a plane. Again, we are free to parallel translate all vectors so that their initial point is at the origin. If the terminal point of such a vector $\mathbf{x}$ is (x, y), then the magnitude of $\mathbf{x}$, which we denote by $||\mathbf{x}||$, [10] is given by the Pythagorean theorem

$$||\mathbf{x}|| = \sqrt{x^2 + y^2}.$$

We specify the direction of this vector by giving the angle θ with respect to the positive x-axis, which is given by

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

where $\theta \in [0, 2\pi)$. Note that this is just the usual change of coordinates from Cartesian to polar coordinates. Since two vectors are equal if and only if $||\mathbf{x_1}|| = ||\mathbf{x_2}||$ and $\theta_1 = \theta_2$, inverting the formulas above

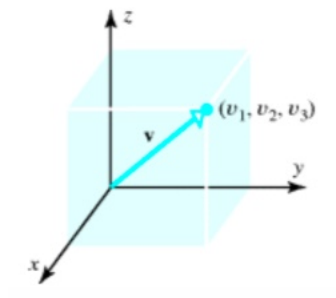
$$\begin{aligned} x &= ||\mathbf{x}|| \cos\theta, \\ y &= ||\mathbf{x}|| \sin\theta, \end{aligned}$$

shows that $(||\mathbf{x}_1||, \theta_1) = (||\mathbf{x}_2||, \theta_2)$ implies $(x_1, y_1) = (x_2, y_2)$. Thus, two-dimensional vectors are in 1-1 correspondence with points in $\mathbb{R}^2$. The coordinates of the end point of the vector are called the *components* of the vector. We will write a two-dimensional vector $\mathbf{v}$ in terms of its components by writing $\mathbf{v} = (v_1, v_2)$. As we have just seen, this expression for $\mathbf{v}$ in terms of its coordinates is unique.

[10] We use double bars to denote the magnitude of a vector to distinguish this from the absolute value of a real number. Note that, when $\mathbf{x}$ is one dimensional, then $||\mathbf{x}|| = \sqrt{x^2} = |x|$.

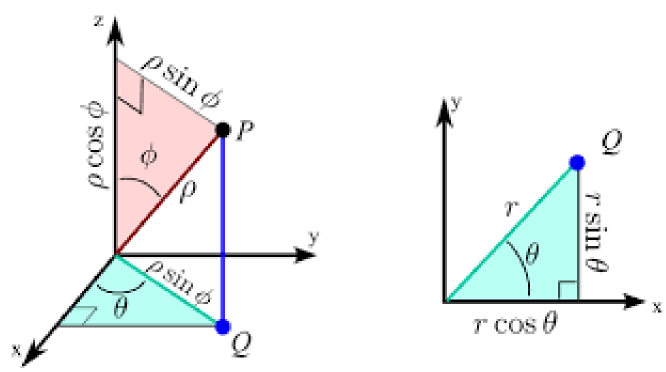
1.4.3 Vectors is three dimensions

A three-dimensional Cartesian coordinate system is shown below.



The coordinate axes are labeled x, y, and z, and are arranged such that if you take your right hand and curl your fingers from the positive x-axis toward the positive y-axis, your thumb points along the positive z-axis. Such a coordinate system is called *right-handed*. Again, we find the set of all vectors in three dimensions is in 1-1 correspondence with the set $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$. If φ is the angle the vector makes with the positive z-axis and θ is the angle the projection of the vector onto the xy-plane makes with the positive x-axis (i.e., it is the usual polar angle in the plane), then the direction of a vector $\mathbf{x}$ is specified by the two angles (φ, θ), where we take $\varphi \in [0, \pi], \theta \in [0, 2\pi)$. A little trigonometry shows that the correspondence between $(||\mathbf{x}||, (\varphi, \theta))$ and (x, y, z) is given by

$$x = ||\mathbf{x}|| \sin\varphi \cos\theta, \tag{1.2}$$
$$y = ||\mathbf{x}|| \sin\varphi \sin\theta, \tag{1.3}$$
$$z = ||\mathbf{x}|| \cos\varphi. \tag{1.4}$$



Exercise 1.5. Work out these formulas from the diagram.

Solution. Since $r = ||\mathbf{v}|| \sin\varphi$,

$$x = r\cos\theta = ||\mathbf{v}|| \sin\varphi \cos\theta$$
$$y = r\sin\theta = ||\mathbf{v}|| \sin\varphi \sin\theta$$

We see also from the diagram that $z = ||\mathbf{v}|| \cos\varphi$. $\square$

As in two-dimensions, we write a three-dimensional vector $\mathbf{v}$ in terms of its components as $\mathbf{v} = (v_1, v_2, v_3)$, and this expression is unique. From the formulas (1.2)-(1.4), we see that the length of a vector $\mathbf{v}$ is given by an obvious generalization of the Pythagorean formula

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{1.5}$$

Exercise 1.6. Use the formulas (1.2)-(1.4) above to prove this.

Solution. We have

$$\begin{aligned} v_1^2 + v_2^2 + v_3^2 &= ||\mathbf{v}||^2 \sin^2\varphi\cos^2\theta + ||\mathbf{v}||^2 \sin^2\varphi\sin^2\theta + ||\mathbf{v}||^2\cos^2\varphi \\ &= ||\mathbf{v}||^2(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi) \\ &= ||\mathbf{v}||^2(\sin^2\varphi(\underbrace{\cos^2\theta + \sin^2\theta}_{=1})) + \cos^2\varphi) \\ &= ||\mathbf{v}||^2(\underbrace{\sin^2\varphi + \cos^2\varphi}_{=1}) \\ &= ||\mathbf{v}||^2. \end{aligned}$$

Since both sides are ≥ 0, this implies that

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

□

1.4.4 Vector arithmetic in coordinates

We now derive the rule for vector addition in terms of components. Given vectors $\mathbf{v}$ and $\mathbf{w}$, we wish to find the components of $\mathbf{v} + \mathbf{w}$. According to the tip-to-tail rule, the $\mathbf{v} + \mathbf{w}$ is given by translating the vector $\mathbf{w}$ so that its initial point coincides with the terminal point of $\mathbf{v}$. This translation moves the x-coordinate of $\mathbf{w}$ by v_1 units in the x-direction, the y-coordinate by v_2 units in the y-direction, and the z-coordinate by v_3 units in the z-direction. That is,

$$(w_1, w_2, w_2) \mapsto (v_1 + w_1, v_2 + w_2, v_3 + w_3)$$

Thus, the coordinates of the endpoint of $\mathbf{v} + \mathbf{w}$ are given by

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3).$$

Definition 1.16 (Vector addition in components). In terms of components, the rule for vector addition is given by:

(i) Vectors in one-dimension:

$$(v_1) + (w_1) = (v_1 + w_1)$$

(ii) Vectors in two-dimensions:

$$(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2).$$

(iii) Vectors in three-dimensions:

$$(v_1, v_2, v_3) + (w_1, w_2, w_3) = (v_1 + w_1, v_2 + w_2, v_3 + w_3).$$

This formula is illustrated in the diagram on the left in Figure 1.4.4.

Example 1.17. Let $\mathbf{v} = (1, 2, 3)$ and $\mathbf{w} = (-3, 1, 7)$. Then

$$\mathbf{v} + \mathbf{w} = (-2, 3, 10).$$

Exercise 1.7. (a) Use the formula in Definition 1.16 to show that vector addition is commutative.

(b) Use the formula in Definition 1.16 to show that vector addition is associative.

(c) What are the components of the zero vector?

(d) Given a vector $\mathbf{v} = (v_1, v_2, v_3)$, what are the components of $-\mathbf{v}$ (the additive inverse of $\mathbf{v}$)?

Solution.

(a) For any $\mathbf{v}$ and $\mathbf{w}$, we have

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= (v_1, v_2, v_3) + (w_1, w_2, w_3) \\
&= (v_1 + w_1, v_2 + w_2, v_3 + w_3) \\
&= (w_1 + v_1, w_2 + v_2, w_3 + v_3) \\
&= \mathbf{w} + \mathbf{v}
\end{aligned}$$

hence, vector addition is commutative.

(b) For any $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, we have

$$\begin{aligned}
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ((u_1, u_2, u_3) + (v_1, v_2, v_3)) + (w_1, w_2, w_3) \\
&= (u_1 + v_1, u_2 + v_2, u_3 + v_3) + (w_1, w_2, w_3) \\
&= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3) \\
&= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3)) \\
&= (u_1, u_2, u_3) + (v_1 + w_1, v_2 + w_2, v_3 + w_3) \\
&= (u_1, u_2, u_3) + ((v_1, v_2, v_3) + (w_1, w_2, w_3)) \\
&= \mathbf{u} + (\mathbf{v} + \mathbf{w})
\end{aligned}$$

hence, vector addition is associative.

(c) The zero vector, $\mathbf{b} = (b_1, b_2, b_3)$, satisfies

$$\mathbf{v} + \mathbf{b} = \mathbf{v}$$

for any vector $\mathbf{v}$. In components, this becomes

$$(v_1 + b_1, v_2 + b_2, v_3 + b_3) = (b_1, b_2, b_3)$$

Since two vectors are equal if and only if their corresponding components are equal, we have

$$\begin{aligned} v_1 + b_1 &= v_1, \\ v_2 + b_2 &= v_2, \\ v_3 + b_3 &= v_3, \end{aligned}$$

and therefore $(b_1, b_2, b_3) = (0, 0, 0)$.

(d) Let $\mathbf{b} = (b_1, b_2, b_3)$ denote the inverse of $\mathbf{v}$. Then we have

$$\mathbf{v} + \mathbf{b} = \mathbf{0}$$

In components, this becomes

$$(v_1 + b_1, v_2 + b_2, v_3 + b_3) = (0, 0, 0)$$

which implies

$$\begin{aligned} v_1 + b_1 &= 0, \\ v_2 + b_2 &= 0, \\ v_3 + b_3 &= 0, \end{aligned}$$

and therefore $(b_1, b_2, b_3) = (-v_1, -v_2, -v_3)$.

$\square$

Proposition 1.18. If $\mathbf{v} = (v_1, v_2, v_3)$ is a vector and k a scalar, then the components of the scalar multiple $k\mathbf{v}$ of $\mathbf{v}$ by k are given by

(i) Vectors in one dimension:

$$k(v_1) = (kv_1)$$

(ii) Vectors in two dimensions:

$$k(v_1, v_2) = (kv_1, kv_2)$$

(iii) Vectors in three dimensions:

$$k(v_1, v_2, v_3) = (kv_1, kv_2, kv_3)$$

Example 1.19. Let $\mathbf{v} = (1, 2, 3)$ and $k = 3$. Then

$$k\mathbf{v} = (3, 6, 9).$$

The formula in (1.18) is illustrated in the diagram on the right in Figure 1.4.4.

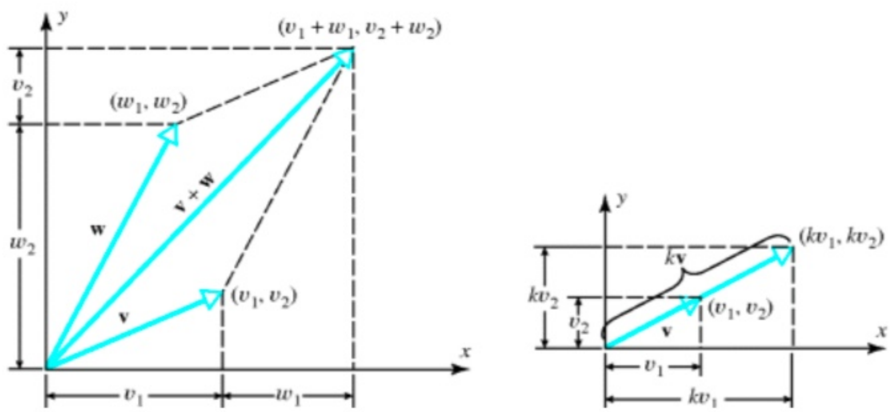


Figure 1: Vector operations in components.

Exercise 1.8. (a) Prove the formulas in Proposition 1.18 for vectors in one and two dimensions.

(b) Use this formula to prove that $||k\mathbf{v}|| = |k| \cdot ||\mathbf{v}||$ for vectors in three dimensions.

(c) Show that if $k > 0$, then $k\mathbf{v}$ has the same direction as $\mathbf{v}$. [11]

Solution.

(a) In one dimension, it is a property of the absolute value function that $|kv| = |k||v|$. If $k > 0$, then $kv > 0$ if $v > 0$ and $kv < 0$ if $v < 0$, so kv points in the same direction as v. This shows kv is the scalar multiple of v by k.

In two dimensions,

$$\begin{aligned} ||k\mathbf{v}|| &= ||(kv_1, kv_2)|| \\ &= \sqrt{(kv_1)^2 + (kv_2)^2} \\ &= \sqrt{k^2(v_1^2 + v_2^2)} \\ &= \sqrt{k^2}\sqrt{v_1^2 + v_2^2} \\ &= |k| \cdot ||\mathbf{v}||. \end{aligned}$$

We can write the components of $\mathbf{v}$ in terms of its magnitude and direction as

$$(v_1, v_2) = (||\mathbf{v}|| \cos\theta, ||\mathbf{v}|| \sin\theta).$$

Now let $v_1 \mapsto kv_1$ and $v_2 \mapsto kv_2$. Then

$$\begin{aligned} \cos\theta &\mapsto \frac{kv_1}{|k| \cdot ||\mathbf{v}||} = \operatorname{sgn}(k)\frac{v_1}{||\mathbf{v}||} = \operatorname{sgn}(k)\cos\theta \\ \sin\theta &\mapsto \frac{kv_2}{|k| \cdot ||\mathbf{v}||} = \operatorname{sgn}(k)\frac{v_2}{||\mathbf{v}||} = \operatorname{sgn}(k)\sin\theta, \end{aligned}$$

[11] This formula still holds if $k < 0$, but checking this case involves some care since φ must stay in the range $[0, \pi)$ and θ in $[0, 2\pi)$.

If $\text{sgn}(k) > 0$, $(\cos\theta, \sin\theta) \mapsto (\cos\theta, \sin\theta)$ and since $\theta \in [0, 2\pi)$, this implies $\theta \mapsto \theta$. If $\text{sgn}(k) < 0$, then $(\cos\theta, \sin\theta) \mapsto (-\cos\theta, -\sin\theta) = (\cos(\theta \pm \pi), \sin(\theta \pm \pi))$ (where either $+$ or $-$ is taken to keep $\theta \in [0, 2\pi)$), and therefore $k\mathbf{v}$ points in the opposite of the direction of $\mathbf{v}$. This proves that (kv_1, kv_2) is indeed equal to $k\mathbf{v}$.

(b) In three dimensions,

$$\begin{aligned} ||k\mathbf{v}|| &= ||(kv_1, kv_2, kv_3)|| \\ &= \sqrt{(kv_1)^2 + (kv_2)^2 + (kv_3)^2} \\ &= \sqrt{k^2(v_1^2 + v_2^2 + v_3^2)} \\ &= \sqrt{k^2}\sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= |k| \cdot ||\mathbf{v}||. \end{aligned}$$

(c) We can write a three-dimensional vector $\mathbf{v} = (v_1, v_2, v_3)$ in terms of its magnitude $||\mathbf{v}||$ and direction (φ, θ) (where $\varphi \in [0, \pi)$ and $\theta \in [0, 2\pi)$) as

$$\mathbf{v} = ||\mathbf{v}||(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

Inverting these formulas to solve for φ, θ in terms of the components gives

$$\begin{aligned} \cos\varphi &= \frac{v_3}{||\mathbf{v}||}, \\ \cos\theta &= \frac{v_1}{||\mathbf{v}||\sin\varphi} \end{aligned}$$

If $k > 0$, we have

$$\cos\varphi \mapsto \frac{k}{|k|}\frac{v_3}{||\mathbf{v}||} = \text{sgn}(k)\cos\varphi = \cos\varphi$$

and since $\varphi \in [0, \pi)$, this implies $\varphi \mapsto \varphi$. Therefore

$$\cos\theta \mapsto \frac{k}{|k|}\frac{v_1}{||\mathbf{v}||\sin\varphi} = \text{sgn}(k)\cos\theta = \cos\theta.$$

Since $\theta \in [0, 2\pi)$, this implies $\theta \mapsto \theta$. Hence, $(\varphi, \theta) \mapsto (\varphi, \theta)$, so the direction is unchanged. This shows that if $k > 0$, then $k\mathbf{v} = (v_1, v_2, v_3)$.

$\square$

Exercise 1.9. Use the formula in Proposition (1.18) to prove Theorem (1.15).

Solution. Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. Then

(i) $0\mathbf{v} = 0(v_1, v_2, v_3) = (0 \cdot v_1, 0 \cdot v_2, 0 \cdot v_3) = (0, 0, 0) = \mathbf{0}$.

(ii) $1\mathbf{v} = 1(v_1, v_2, v_3) = (1 \cdot v_1, 1 \cdot v_2, 1 \cdot v_3) = (v_1, v_2, v_3) = \mathbf{v}$.

(iii) $(-1)\mathbf{v} = -1(v_1, v_2, v_3) = (-1 \cdot v_1, -1 \cdot v_2, -1 \cdot v_3) = (-v_1, -v_2, -v_3) = -\mathbf{v}$.

(iv) Let c, d be any scalars. Then

$$\begin{aligned}
(c+d)\mathbf{v} &= (c+d)(v_1, v_2, v_3) \\
&= ((c+d)v_1, (c+d)v_2, (c+d)v_3) \\
&= (cv_1 + dv_1, cv_2 + dv_2, cv_3 + dv_3) \\
&= (cv_1, cv_2, cv_3) + (dv_1, dv_2, dv_3) \\
&= c(v_1, v_2, v_3) + d(v_1, v_2, v_3) \\
&= c\mathbf{v} + d\mathbf{v}.
\end{aligned}$$

(v) Let c be any scalar. Then

$$\begin{aligned}
c(\mathbf{v}+\mathbf{w}) &= c((v_1, v_2, v_3) + (w_1, w_2, w_3)) \\
&= c(v_1 + w_1, v_2 + w_2, v_3 + w_3) \\
&= (c(v_1 + w_1), c(v_2 + w_2), c(v_3 + w_3)) \\
&= (cv_1 + cw_1, cv_2 + cw_2, cv_3 + cw_3) \\
&= (cv_1, cv_2, cv_3) + (cw_1, cw_2, cw_3) \\
&= c\mathbf{v} + c\mathbf{w}.
\end{aligned}$$

(vi) Let c, d be any scalars. Then

$$\begin{aligned}
(dc)\mathbf{v} &= (dc)(v_1, v_2, v_3) \\
&= ((dc)v_1, (dc)v_2, (dc)v_3) \\
&= (d(cv_1), d(cv_2), d(cv_3)) \\
&= d(cv_1, cv_2, cv_3) \\
&= d(c\mathbf{v}).
\end{aligned}$$

□

Exercise 1.10. Prove that if $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$, then

$$\mathbf{v} - \mathbf{w} = (v_1 - w_1, v_2 - w_2, v_3 - w_3).$$

Solution. By definition of vector subtraction,

$$\begin{aligned}
\mathbf{v} - \mathbf{w} &= \mathbf{v} + (-\mathbf{w}) \\
&= (v_1, v_2, v_3) + (-w_1, -w_2, -w_3) \\
&= (v_1 - w_1, v_2 - w_2, v_3 - w_3).
\end{aligned}$$

□

The distance between two points (v_1, v_2, v_3) and (w_1, w_2, w_3) in $\mathbb{R}^3$ is therefore given by

$$||\mathbf{v} - \mathbf{w}|| = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2}.$$

Exercise 1.11. (a) Find the components of the vector $\overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$.

(b) Find the distance between $P_1(2, -1, 4)$ and $P_2(7, 5, -8)$.

Solution:

(a) The components of $\overrightarrow{P_1P_2}$ are given by

$$\overrightarrow{P_1P_2} = (7 - 2, 5 - (-1), -8 - 4) = (5, 6, -12)$$

(b) The distance between P_1 and P_2 is

$$||\overrightarrow{P_1P_2}|| = \sqrt{5^2 + 6^2 + (-12)^2} = \sqrt{205}.$$

1.4.5 Unit vectors

Definition 1.20 (Unit vector). A *unit vector* is a vector of unit length. That is, a vector $\mathbf{v}$ such that $||\mathbf{v}|| = 1$.

Proposition 1.21 (Normalizing a vector). If $\mathbf{v} \neq 0$, then $\mathbf{v}/||\mathbf{v}||$ is a unit vector in the direction of $\mathbf{v}$. The unit vector $\mathbf{v}/||\mathbf{v}||$ is often denoted $\hat{\mathbf{v}}$ and the process of forming $\hat{\mathbf{v}}$ from $\mathbf{v}$ is called *normalizing* $\mathbf{v}$.

Proof. Since $\mathbf{v} \neq 0$, $||\mathbf{v}|| \neq 0$, so we can multiply $\mathbf{v}$ by $1/||\mathbf{v}||$. Computing the length of $||\frac{\mathbf{v}}{||\mathbf{v}||}||$, we see that

$$||\frac{\mathbf{v}}{||\mathbf{v}||}|| = \frac{||\mathbf{v}||}{||\mathbf{v}||} = 1,$$

hence $\mathbf{v}/||\mathbf{v}||$ is a unit vector. Since $\mathbf{v}/||\mathbf{v}||$ is a scalar multiple of $\mathbf{v}$ (by $k = 1/||\mathbf{v}||$) it is parallel to $\mathbf{v}$, and since $k > 0$, $\mathbf{v}/||\mathbf{v}||$ has the same orientation as $\mathbf{v}$. □

Example 1.22. We can express any nonzero vector $\mathbf{v}$ as a product of its magnitude and direction by means of the formula [12]

$$\mathbf{v} = ||\mathbf{v}||\frac{\mathbf{v}}{||\mathbf{v}||}.$$

For example, taking $\mathbf{v} = (1, -2, 3)$,

$$||\mathbf{v}|| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$$

and thus

$$\mathbf{v} = \sqrt{14}(\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}).$$

[12] Note that this is a generalization of the formula $x = |x|\frac{x}{|x|}$ for a nonzero real number x.

Definition 1.23 (Standard unit vectors). The *standard unit vectors* for $\mathbb{R}^3$ are the vectors

$$\begin{aligned}\mathbf{i} &= (1,0,0),\\ \mathbf{j} &= (0,1,0),\\ \mathbf{k} &= (0,0,1).\end{aligned}$$

These vectors are unit vectors pointing along the x-, y-, and z-axes, respectively. [13]

Using the standard unit vectors, we may write the vector $\mathbf{v} = (v_1, v_2, v_3)$ as

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

A sum of this form is said to be a *linear combination* of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. You should be comfortable working with both expressions for a vector $\mathbf{v}$ in terms of its components.

Example 1.24. Find a unit vector in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

Solution: We find $\overrightarrow{P_1P_2}$ and normalize:

$$\begin{aligned}\overrightarrow{P_1P_2} &= (3-1, 2-0, 0-1) = (2,2,-1),\\ ||\overrightarrow{P_1P_2}|| &= \sqrt{2^2+2^2+(-1)^2} = 3,\end{aligned}$$

and therefore the desired unit vector is given by

$$\frac{\overrightarrow{P_1P_2}}{||\overrightarrow{P_1P_2}||} = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}).$$

Example 1.25. Find a vector 6 units long in the direction of $\mathbf{v} = (2,2,-1)$.

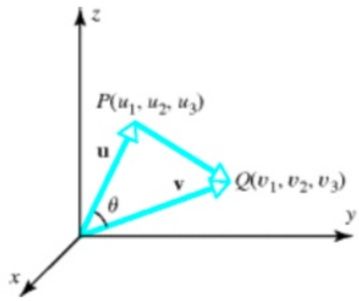
Solution: The vector we want is

$$6\frac{\mathbf{v}}{||\mathbf{v}||} = 6\frac{(2,2,-1)}{\sqrt{2^2+2^2+(-1^2)}} = 6\frac{(2,2,-1)}{3} = (4,4,-2).$$

1.4.6 The dot product

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ positioned so that their initial points coincide determine an angle $\theta \in [0, \pi]$, which is the angle between the two vectors.

[13] The standard unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are also commonly denoted as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ or $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$.



Note that the information about θ is encoded in $\mathbf{u} - \mathbf{v}$, since if we fix the magnitudes of $\mathbf{u}$ and $\mathbf{v}$ and open the angle, then $\mathbf{u} - \mathbf{v}$ will also change. The fundamental relation satisfied by $\mathbf{u}, \mathbf{v}$ and θ is the law of cosines, which says that if $\mathbf{w} = \mathbf{u} - \mathbf{v}$, then

$$||\mathbf{w}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta. \tag{1.6}$$

Let us make the following definition.

Definition 1.26 (Dot product). The *dot product* of two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta. \tag{1.7}$$

The angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is then given in terms of their dot product by

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}\right), \tag{1.8}$$

and we see that

- θ is acute if $\mathbf{u} \cdot \mathbf{v} > 0$.
- θ is obtuse if $\mathbf{u} \cdot \mathbf{v} < 0$.
- θ is right if $\mathbf{u} \cdot \mathbf{v} = 0$.

Since the dot product takes as input two vectors and returns a scalar, it is also called the *scalar product*. Note that from Eq. (1.6), we can write $\mathbf{u} \cdot \mathbf{v}$ in terms of magnitudes only, as

$$\mathbf{u} \cdot \mathbf{v} = \frac{||\mathbf{u}||^2 + ||\mathbf{v}||^2 - ||\mathbf{u} - \mathbf{v}||^2}{2}. \tag{1.9}$$

Note that, unlike (1.7), the right hand side of (1.9) is defined if either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, in which case $\mathbf{u} \cdot \mathbf{v} = 0$. Thus, we define the dot product of any vector with the zero vector to be zero:

$$\mathbf{v} \cdot \mathbf{0} = 0 \tag{1.10}$$

for every vector $\mathbf{v}$.

Since (1.9) involves only lengths of segments and angles between segments, it holds in *all* coordinate systems. However, it takes a particularly simple form in Cartesian coordinates. Writing out the norm of each vector in terms of its components gives

$$\mathbf{u} \cdot \mathbf{v} = \frac{u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2 - (u_3 - v_3)^2}{2} \tag{1.11}$$

Expanding the binomials and cancelling like terms, we find

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{1.12}$$

Exercise 1.12. Simplify equation (1.11) to obtain (1.12).

Exercise 1.13. Compute the angle between the vectors $\mathbf{u} = (0, 0, 3)$ and $\mathbf{v} = (\sqrt{2}, 0, \sqrt{2})$.

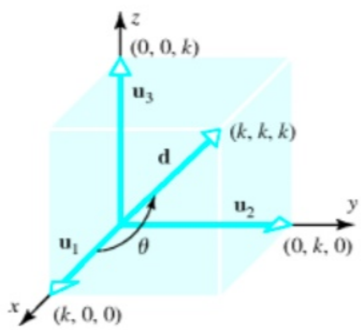
Solution. The angle between these vectors is

$$\theta = \cos^{-1}\left(\frac{0(\sqrt{2} + 0(0) + 3(\sqrt{2}))}{3(2)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

□

Example 1.27. Find the angle between a diagonal of a cube and one of its edges.

Solution: Let s be the length of an edge and place the cube in the first octant so that one vertex is at the origin and two edges are along the x- and y-axes.



If we let $\mathbf{u}_1 = (s, 0, 0)$, $\mathbf{u}_1 = (0, s, 0)$, and $\mathbf{u}_3 = (0, 0, s)$, then the vector

$$\mathbf{d} = (s, s, s) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

is a diagonal of the cube. The angle between $\mathbf{d}$ and $\mathbf{u}_1$ is

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\mathbf{u}_1 \cdot \mathbf{d}}{||\mathbf{u}_1||\ ||\mathbf{d}||}\right) \\ &= \cos^{-1}\left(\frac{s^2}{(s)(\sqrt{3s^2})}\right) \\ &= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &\approx 54.74^\circ. \end{aligned}$$

Proposition 1.28 (Properties of the dot product). Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be vectors and k any scalar. Then

(1) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(2) $(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$

(3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(4) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$

(5) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$

(6) $\mathbf{a} \cdot \mathbf{a} = ||\mathbf{a}||^2$

Proof. Each of these can be proved by writing out the vectors in components and using (1.12). For example, to prove (1)

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= b_1a_1 + b_2a_2 + b_3a_3 \\ &= \mathbf{b} \cdot \mathbf{a}\end{aligned}$$

Properties (2)-(6) are proved similarly. Note that (5) follows from (3) and (4), but is included for emphasis since it is used frequently. □

Exercise 1.14. Prove properties (2)-(6) in Proposition 1.28.

Note, however, the differences between the dot product and ordinary multiplication. For instance, one might ask, "Is the dot product associative?". This question doesn't even make sense for the dot product, as expressions such as $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ are not defined, since $\mathbf{a}$ is a vector and $(\mathbf{b} \cdot \mathbf{c})$ is a scalar, and one can only form the dot product between two vectors. Thus, even though we found that the additive structure on the set of vectors was exactly the same as that of the real numbers, the multiplicative structure induced by the dot product is very different from that of the real numbers.

1.4.7 Orthogonal vectors

While writing vectors in component form has the advantage of facilitating many computations, this form seems to obscure geometric relations between vectors. For instance, the two vectors

$$\begin{aligned}\mathbf{u} &= (3, -2, 1) \\ \mathbf{v} &= (0, 2, 4).\end{aligned}$$

are perpendicular (see Fig. 1.4.7 below), but this does not seem obvious from looking at the components of the two vectors.

We know from trigonometry that right angles are special, so it would be nice to have a way to check if two vectors written in component form are perpendicular. Fortunately, the dot product gives us an easy way to determine this.

Definition 1.29 (Orthogonal vectors). Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *orthogonal* if $\mathbf{a} \cdot \mathbf{b} = 0$.

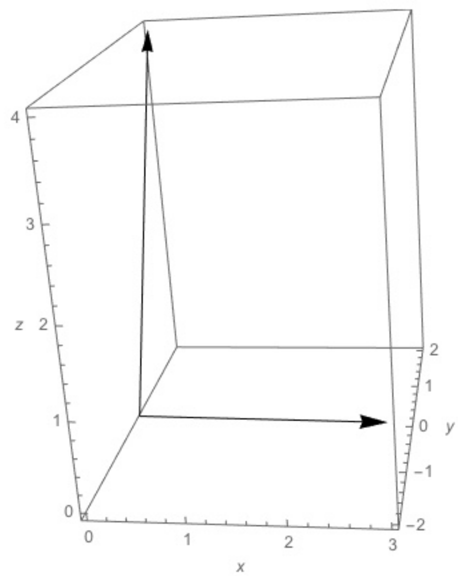


Figure 2: Orthogonal vectors $\mathbf{u} = (3, -2, 1)$ and $\mathbf{v} = (0, 2, 4)$.

To see the significance of this definition, if $\mathbf{a}$ and $\mathbf{b}$ are non-zero then their dot product can be written as

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \, ||\mathbf{b}|| \cos \theta.$$

Since $||\mathbf{a}||$ and $||\mathbf{b}||$ are both greater than zero, $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\theta = \frac{\pi}{2}$, that is, if and only if $\mathbf{a}$ and $\mathbf{b}$ are *perpendicular*. Therefore for nonzero vectors being orthogonal (zero dot product) is the same as being perpendicular (intersecting at a right angle). Since $\mathbf{v} \cdot \mathbf{0} = 0$ for every vector $\mathbf{v}$, the zero vector is orthogonal to every vector (including itself).

Example 1.30. The two vectors in the example above

$$\begin{aligned} \mathbf{u} &= (3, -2, 1) \\ \mathbf{v} &= (0, 2, 4). \end{aligned}$$

are orthogonal (and therefore indeed perpendicular) since $\mathbf{u} \cdot \mathbf{v} = 3(0) - 2(2) + 1(4) = -4 + 4 = 0$.

Definition 1.31 (Orthogonal set). A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be *orthogonal* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.

Example 1.32. The standard unit vectors for $\mathbb{R}^3$ form an orthogonal set. One can easily verify that

$$\mathbf{e}_i \cdot \mathbf{e_j} = \delta_{ij}$$

where

$$\delta_{ij} \equiv \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j \end{cases}$$

is called the *Kronecker delta symbol*. Since $\mathbf{e}_i \cdot \mathbf{e_j} = 0$ whenever $i \neq j$, the set $\{e_1, e_2, e_3\}$ is indeed orthogonal.

In the orthogonal set in the previous example, each vector in the set is a unit vector. Such a set has a special name:

Definition 1.33 (Orthonormal set). An orthogonal set of vectors is said to be an *orthonormal set* if each vector in the set is a unit vector.

The examples above illustrate another difference between the dot product of two vectors and the ordinary product of two numbers. For two real numbers, if $ab = 0$, then either $a = 0$ or $b = 0$. These examples clearly show that if $\mathbf{a} \cdot \mathbf{b} = 0$, then it need not be true that either $\mathbf{a} = 0$ or $\mathbf{b} = 0$.

1.4.8 Projection of a vector

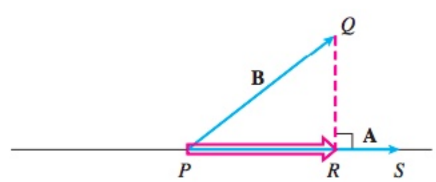
Let $\mathbf{a}$ be a nonzero vector, $\hat{\mathbf{a}} = \frac{\mathbf{a}}{||\mathbf{a}||}$ the unit vector obtained by normalizing $\mathbf{a}$, and $\mathbf{b}$ another vector. Then Eq. (1.7) shows that

$$\hat{\mathbf{a}} \cdot \mathbf{b} = \frac{\mathbf{a}}{||\mathbf{a}||} \cdot \mathbf{b} = ||\mathbf{b}|| \cos\theta$$

is the component of $\mathbf{b}$ in the direction of $\mathbf{a}$. Multiplying by $\hat{\mathbf{a}}$, we get a vector parallel to $\mathbf{a}$ whose magnitude is the component of $\mathbf{b}$ along $\mathbf{a}$.

Definition 1.34 (Vector projection). The vector $\text{proj}_{\mathbf{a}}\mathbf{b} \equiv (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}} = ||\mathbf{b}|| \cos\theta \hat{\mathbf{a}}$ is called the *projection of* $\mathbf{b}$ *onto* $\mathbf{a}$.

Geometrically, the projection of $\mathbf{b} = \overrightarrow{PQ}$ onto $\mathbf{a} = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by connecting a perpendicular segment from Q to the line PS. [14]

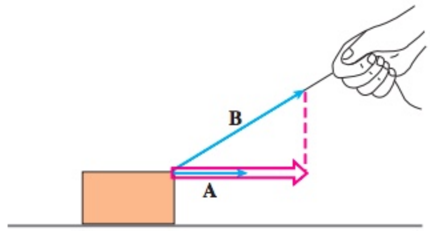


Physically, if $\mathbf{b}$ represents a force, then $\text{proj}_{\mathbf{a}}\mathbf{b}$ is the "effective" force in the $\mathbf{a}$ direction; that is, the component of the force along $\mathbf{a}$.

[14] Note that by using the definitions of the dot product and the norm of $\mathbf{a}$, one may produce many equivalent expressions for $\text{proj}_{\mathbf{a}}\mathbf{b}$:

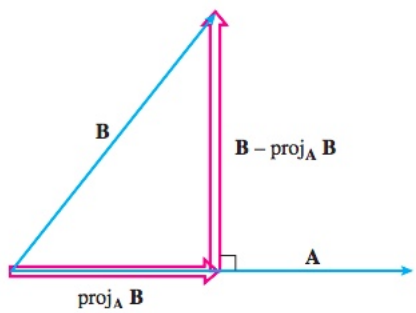
$$\begin{aligned}\text{proj}_{\mathbf{a}}\mathbf{b} &= (||\mathbf{b}|| \cos\theta)\hat{\mathbf{a}} \\ &= (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}} \\ &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||}\right) \frac{\mathbf{a}}{||\mathbf{a}||} \\ &= \left(\frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}\end{aligned}$$

The first of these is perhaps the easiest to remember, as it makes most transparent the relation to elementary right triangle trigonometry.



It is often desirable to express a vector $\mathbf{b}$ as a sum of two orthogonal vectors. For instance, in mechanics we frequently decompose forces in this way so that we may treat a two-dimensional problem as two one-dimensional problems. We can easily express a vector $\mathbf{b}$ as such a sum of two vectors, one parallel to some nonzero vector $\mathbf{a}$ and one orthogonal to $\mathbf{a}$, in terms of the projection of $\mathbf{b}$ along $\mathbf{a}$:

$$\begin{aligned} \mathbf{b} &= \mathbf{b}_{\parallel} + \mathbf{b}_{\perp} \\ &= \mathrm{proj}_{\mathbf{a}}\mathbf{b} + (\mathbf{b} - \mathrm{proj}_{\mathbf{a}}\mathbf{b}). \end{aligned} \tag{1.13}$$



Example 1.35. Express $\mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3$ as the sum of a vector parallel to $\mathbf{a} = 3\mathbf{e}_1 - \mathbf{e}_2$ and a vector orthogonal to $\mathbf{a}$.

Solution: Since $\hat{\mathbf{a}} \equiv \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{3\mathbf{e}_1 - \mathbf{e}_2}{\sqrt{10}}$, we can write $\mathbf{b} = \mathbf{b}_{\parallel} + \mathbf{b}_{\perp}$ with

$$\begin{aligned} \mathbf{b}_{\parallel} &= (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}} = \frac{1}{2}(3\mathbf{e}_1 - \mathbf{e}_2) = \frac{1}{2}\mathbf{a} \\ \mathbf{b}_{\perp} &= \mathbf{b} - \mathbf{b}_{\parallel} = \frac{1}{2}\mathbf{e}_1 + \frac{3}{2}\mathbf{e}_2 - 3\mathbf{e}_3. \end{aligned}$$

1.5 Equations of lines and planes

1.5.1 Lines in space

The coordinate systems of analytic geometry allow us to consider geometric objects such as lines and planes in terms of vectors. These geometric ideas will give us valuable intuition later on in the course when we take a more abstract point of view toward vectors.

First let us recall that any two points define a line. Equivalently, we can also determine a line if we know one point on the line and the slope of the line.

Let us now work in Cartesian coordinates. Suppose L is a line passing through a point $P_0(x_0, y_0, z_0)$ and parallel to a vector $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$. Now let $P(x, y, z)$ be any point in space. In which case will $P(x, y, z)$ to be on the line? This will be the case if and only if the vector $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$, that is, if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$. Therefore,

Definition 1.36 (Vector equation for a line). The line through $P_0(x_0, y_0, z_0)$ and parallel of $\mathbf{v}$ is the set of all points $P(x, y, z)$ such that $\overrightarrow{P_0P} = t\mathbf{v}$, with $-\infty < t < \infty$. This equation is called the *vector equation* of the line.

In terms of Cartesian coordinates, the vector equation for the line becomes

$$\begin{aligned}(x - x_0)\mathbf{e}_1 + (y - y_0)\mathbf{e}_2 + (z - z_0)\mathbf{e}_3 &= t(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\ &= tv_1\mathbf{e}_1 + tv_2\mathbf{e}_2 + tv_3\mathbf{e}_3\end{aligned}$$

which implies

$$(x - x_0 - tv_1)\mathbf{e}_1 + (y - y_0 - tv_2)\mathbf{e}_2 + (z - z_0 - tv_3)\mathbf{e}_3 = \mathbf{0}$$

and hence

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3. \tag{1.14}$$

Thus, the vector equation of the line is equivalent to the three scalar equations in Eq. (1.14), each of which is the usual equation for a line with slope v_i in one variable t.

Definition 1.37 (Parametric equations for a line). The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ and parallel to $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ is given by

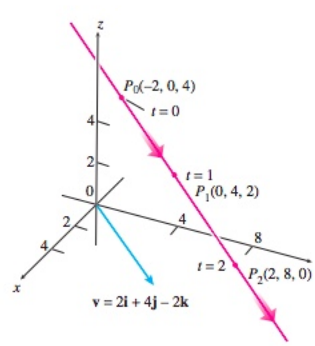
$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

These equations are called the (standard) *parametric equations* for the line.

Example 1.38. Find the parametric equations for the line through $(-2, 0, 4)$ and parallel to $\mathbf{v} = 2\mathbf{e}_1 + 4\mathbf{e}_2 - 2\mathbf{e}_3$.

Solution: Plugging into Eq. (1.14) gives

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t.$$



Example 1.39. Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution: The vector from P to Q is

$$\begin{aligned}\overrightarrow{PQ} &= (1-(-3), -1-2, 4-(-3)) \\ &= (4, -3, 7).\end{aligned}$$

We take this vector to be our "$\mathbf{v}$". The point P_0 could be either P or Q. Arbitrarily choosing it to be Q, Eq. (1.14) gives

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

Example 1.40. Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution: We have seen in the previous exercise that the parametric equations

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

describe an infinite line containing P and Q when we take $-\infty < t < \infty$. To describe the line segment joining P and Q, we simply restrict the domain of t. We see that the line passes through P at $t = -1$ and $Q = 0$. So the line segment joining P and Q is given by

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

with $-1 < t < 0$.

In $\mathbb{R}^2$, there is a unique normal direction to a given line. Let $\mathbf{n} = (n_1, n_2)$ be a normal vector to a line in $\mathbb{R}^2$ and $P_0(x_0, y_0)$ any point on the line (see Fig. 1.5.1 below). If $Q(x, y)$ is any other point on the line, then we must have

$$\overrightarrow{P_0Q} \cdot \mathbf{n} = 0.$$

Since $\overrightarrow{P_0Q} = (x - x_0, y - y_0)$,

$$\overrightarrow{P_0Q} \cdot \mathbf{n} = n_1(x - x_0) + n_2(y - y_0) = 0$$

or

$$n_1 x + n_2 y = c \tag{1.15}$$

where $c = n_1 x_0 + n_2 y_0$. Equation (1.15) is called the *point normal* equation of the line.

Example 1.41. In $\mathbb{R}^2$ the equation

$$6(x-3) + (y+7) = 0$$

represents a line through the point $(3, -7)$ with normal vector $\mathbf{n} = (6, 1)$.

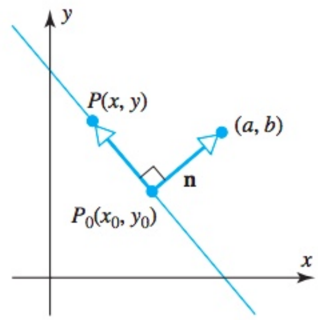


Figure 3: Vectors involved in point normal equation of line.

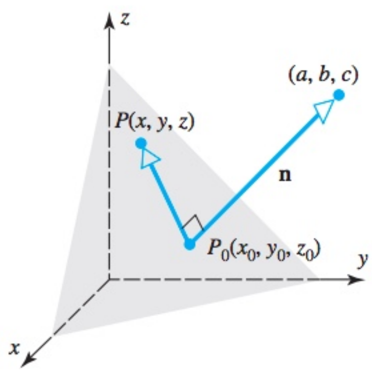


Figure 4: Vectors involved in point normal equation of plane.

1.5.2 Planes in space

Similar to a line in $\mathbf{R}^2$, a plane in $\mathbb{R}^3$ has a unique normal direction. We will now derive a point normal equation for a given plane. Let $\mathbf{n} = (n_1, n_2, n_3)$ be a normal vector to the plane. Given a point $P_0(x_0, y_0, z_0)$ on the plane and a vector $\mathbf{n}$ normal to the plane, what is the condition for an arbitrary point in space $P(x, y, z)$ to lie on the plane? If P lies in the plane, the $\overrightarrow{P_0P}$ is a vector lying in the plane (see Fig. **??** below). Then, since $\mathbf{n}$ is normal to the plane, we must have that $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

Expanding the dot product in terms of components of these vectors, we obtain

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0. \tag{1.16}$$

or

$$n_1x + n_2y + n_3z = c \tag{1.17}$$

where $c = n_1x_0 + n_2y_0 + n_3z_0$. This is the *point normal* equation of the plane.

Example 1.42. Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = (5, 2, -1)$.

Solution: The component equation is

$$5(x-(-3))+2(y-0)+(-1)(z-7)=0,$$

Simplifying, we obtain

$$\begin{aligned} 5x+15+2y-z+7&=0 \\ 5x+2y-z&=-22. \end{aligned}$$

Example 1.43. Find the point where the line

$$x=\frac{8}{3}+2t, \quad y=2t, \quad z=1+t$$

intersects the plane $3x+2y+6z=6$.

Solution: The point $(\frac{8}{3}+2t, 2t, 1+t)$ lies in the plane if its coordinates satisfy the equation of the plane; that is, if

$$3(\frac{8}{3}+2t)+2(-2t)+6(1+t)=6$$

This has a solution at $t=-1$, so the point of intersection is

$$(x,y,z)|_{t=-1}=(\frac{2}{3},2,0).$$

Exercise 1.15. (a) Find a vector parallel to the line of intersection of the planes $3x-6y-2z=15$ and $2x+y-2z=5$.

(b) Find parametric equations for the line in which these planes intersect.

1.6 Some useful distance formulas

Theorem 1.44 (Distance from point to line or plane).

(a) In $\mathbb{R}^2$, the distance D between the point $P_0(x_0,y_0)$ and the line $ax+by+c=0$ is

$$D=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}} \tag{1.18}$$

(b) In $\mathbb{R}^3$ the distance D between the point $P_0(x_0,y_0,z_0)$ and the plane $ax+by+cz+d=0$ is

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}. \tag{1.19}$$

Proof.

(a) Left as an exercise. The steps are virtually identical to the proof in part (b).

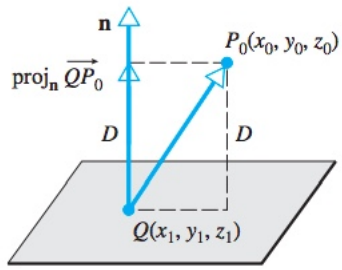


Figure 5: Distance from P_0 to the plane.

(b) Let $Q(x_1, y_1, z_1)$ be any point in the plane. Translate the normal vector $\mathbf{n} = (a, b, c)$ so that its initial point is at Q.

The distance D is then the length of the projection $\text{proj}_{\mathbf{n}}\overrightarrow{QP_0}$:

$$\begin{aligned} D &= ||\text{proj}_{\mathbf{n}}\overrightarrow{QP_0}|| \\ &= \left|\left|\frac{\overrightarrow{QP_0}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right|\right| \\ &= \left|\frac{\overrightarrow{QP_0}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right| ||\mathbf{n}|| \\ &= \frac{|\overrightarrow{QP_0}\cdot\mathbf{n}|}{||\mathbf{n}||^2}||\mathbf{n}|| \\ &= \frac{|\overrightarrow{QP_0}\cdot\mathbf{n}|}{||\mathbf{n}||} \end{aligned}$$

Now

$$\begin{aligned} \overrightarrow{QP_0} &= (x_0 - x_1, y_0 - y_1, z_0 - z_1), \\ \overrightarrow{QP_0}\cdot n &= a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1), \\ ||\mathbf{n}|| &= \sqrt{a^2 + b^2 + c^2}. \end{aligned}$$

and therefore

$$\begin{aligned} D &= \frac{|\overrightarrow{QP_0}\cdot\mathbf{n}|}{||\mathbf{n}||} && (1.20) \\ &= \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2}}. && (1.21) \end{aligned}$$

Since the point $Q(x_1, y_1, z_1)$ lies in the plane, its coordinates satisfy the equation of the plane; thus

$$ax_1 + by_1 + cz_1 + d = 0$$

or

$$d = -ax_1 - by_1 - cz_1.$$

Substituting this expression into (1.20) yields (1.19).

□

Example 1.45. By (1.19), the distance D between the point $P_0(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$ is

$$D = \frac{|2(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}.$$

The formula (1.20) also allows us to compute the distance between parallel planes.

Example 1.46. The planes

$$x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7$$

are parallel since their normal vectors $(1, 2, -2)$ and $(2, 4, -4)$ are parallel vectors. To find the distance D between the planes, we just select an arbitrary point P_0 on one of the planes and then compute its distance to the other plane using (1.20). By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane. The distance between P_0 and the plane $2x + 4y - 4z = 7$ is then

$$D = \frac{|2(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6}.$$

2 Systems of linear equations

We will now change gears and turn to a seemingly distinct topic: that of finding solutions to systems of linear equations. However, we will quickly see that this task is intimately related to the vector operations we have just studied in the previous section.

2.1 Basic definitions

Definition 2.1 (Linear equation). A *linear equation* in the variables $x_1, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b \tag{2.1}$$

where b and the *coefficients* $a_1, \cdots, a_n$ are real numbers.

Exercise 2.1. Which of the following equations are linear?

(a) $4x_1 - 5x_2 + 2 = x_1$

(b) $x_2 = 2\sqrt{x_1} - 6$

(c) $2x_1 + x_2 - x_3 = 2\sqrt{6}$

(d) $4x_1 - 5x_2 = x_1x_2$

Definition 2.2 (System of linear equations). A *system of linear equations* (or a *linear system*) is a collection of one or more linear equations involving the *same* variables $x_1, \cdots, x_n$.

Example 2.3. The following is a system of two linear equations in three variables x_1, x_2, x_3:

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - \ 4x_3 &= -7 \end{aligned} \tag{2.2}$$

Definition 2.4 (Solution). Any n-tuple $(s_1, \cdots, s_n)$ of numbers which satisfies *each* equation in a linear system when $s_1, \cdots, s_n$ are substituted for $x_1, \cdots, x_n$ is called a *solution* of the system.

Example 2.5 (Testing a solution). The 3-tuple $(5, 6.5, 3)$ is a solution of the system (2.2) since

$$\begin{aligned} 2(5) - 6.5 + 1.5(3) &= 8 \\ 5 \qquad - \ 4(3) &= -7 \end{aligned}$$

Definition 2.6 (Solution set). The set of all solutions is called the *solution set* of the linear system.

Definition 2.7 (Consistent and inconsistent systems). If a system of equations has at least one solution it is said to be *consistent*. If the system has no solutions, it is said to be *inconsistent*.

Exercise 2.2. Is $(3, 4, -2)$ a solution of the following linear system?

$$\begin{aligned} 5x_1 - x_2 + 2x_3 &= 7 \\ -2x_1 + 6x_2 + 9x_3 &= 0 \\ -7x_1 + 5x_2 - 3x_3 &= -7 \end{aligned}$$

We have just seen how to check if a given point is a solution to a linear system. Obviously checking points at random is not going to be an effective strategy to find the solution set of a given linear system. We will therefore need a systematic method of finding solution sets of systems of linear equations. But first, let us consider what kind of solution sets we might hope to find.

Example 2.8 (Two linear equations in two unknowns). Consider the most general linear system of two equations in two unknowns

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 &= b_1, \\ A_{21}x_1 + A_{22}x_2 &= b_2. \end{aligned} \tag{2.3}$$

Rewriting each equation in slope-intercept form, (2.3) becomes

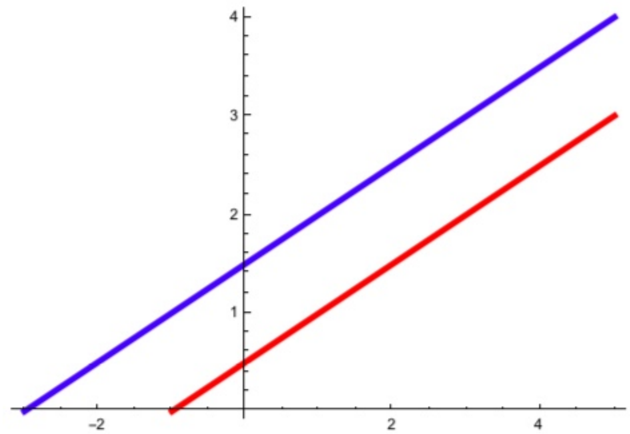
$$\begin{aligned} x_2 &= -\frac{A_{11}}{A_{12}}x_1 + \frac{b_1}{A_{12}}, \\ x_2 &= -\frac{A_{21}}{A_{22}}x_1 + \frac{b_2}{A_{22}}. \end{aligned} \tag{2.4}$$

Geometrically, each equation describes a line in the plane. Since these equations are in the same variables, the two lines lie in the *same* plane. A solution to the linear system corresponds to a point which lies on *both* lines at the same time, i.e., it is a point of intersection of the two lines. It is geometrically evident that there are three possibilities for the possible solution sets of the linear system (2.3), depending on the coefficients:

(1) The linear system (2.3) has *no solution* when

$$\begin{aligned} \frac{A_{11}}{A_{12}} &= \frac{A_{21}}{A_{22}}, \\ b_1 &\neq b_2, \end{aligned}$$

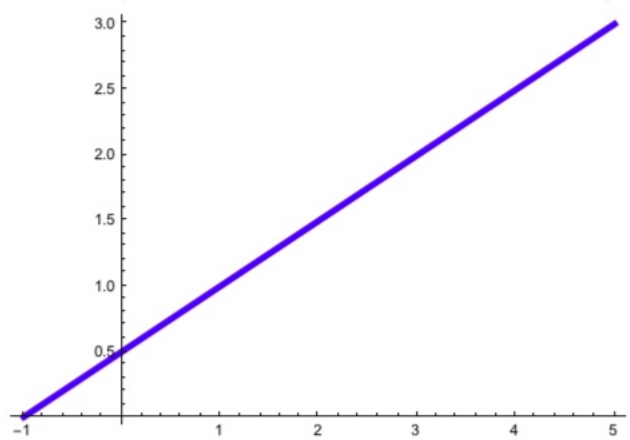
i.e., when the lines are parallel (same slope) but non-overlapping (different y-intercepts).



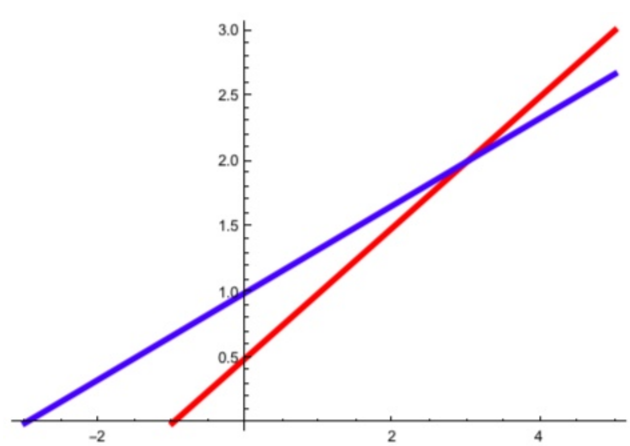
(2) The linear system (2.3) has *infinitely many solutions* when

$$\frac{A_{11}}{A_{12}} = \frac{A_{21}}{A_{22}},$$
$$b_1 = b_2,$$

i.e., when the lines are overlapping (both described by exactly the same equation).



(3) The linear system (2.3) has a *unique solution* in all other cases (i.e., when the lines are non-parallel).



In general, we will consider a linear system of m equations in n unknowns:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= y_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\;\; \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= y_m \end{aligned} \tag{2.5}$$

where $y_1, \cdots, y_m$ and $A_{ij}, 1 \le i \le m, 1 \le j \le n$ are given. In (2.5), we denote the coefficient of x_j in the ith equation as A_{ij}.

It turns out (see Theorems 2.27 and 2.32) that the three possibilities in Example 2.8 are the only possibilities in general; that is, the general linear system in (2.5) has either

1. No solution,
2. A unique solution, or
3. Infinitely many solutions.

We can again understand this claim geometrically in the case of three linear equations in three unknowns:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= y_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= y_1 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= y_1. \end{aligned}$$

These are the point-normal equations of three planes in $\mathbb{R}^3$. The possible intersections are shown below, giving rise to the types of solution sets mentioned above:

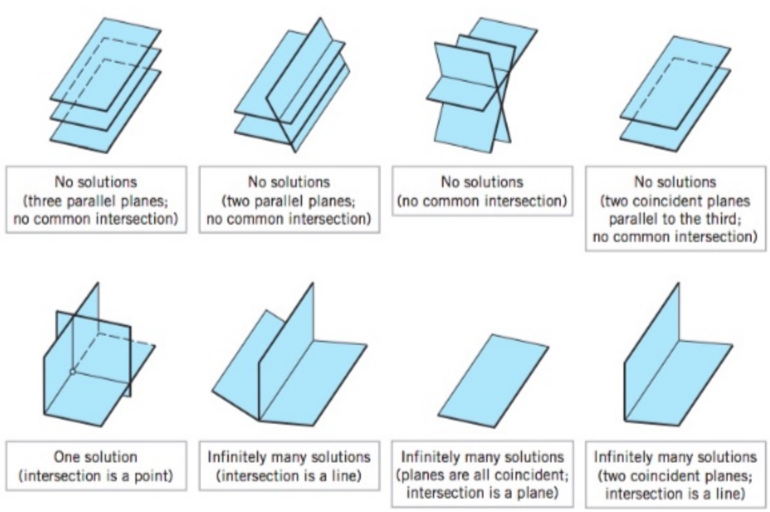


Figure 6: The intersections of three planes give the solution sets of a linear system of three equations in three unknowns.

Exercise 2.3. To get comfortable with the notation in (2.5), write out the system (2.5) when

(a) $n = 2, m = 3$,

(b) $m = 3, n = 2$, and

(c) $m = n = 3$.

Definition 2.9 (Homogeneous and inhomogeneous sytems). If $y_1 = y_2 = \cdots = y_m = 0$ in (2.3), the system is said to be *homogeneous*. Otherwise, it is said to be *inhomogeneous*.

Exercise 2.4. Write down a homogeneous system of 3 linear equations in 2 unknowns.

2.2 Elimination

A fundamental technique for finding the solutions of a system of linear equations is that of *elimination* of variables. Roughly, this technique involves multiplying the equations in the system by numbers and then adding the resulting equations together so that some of the variables drop out, leading to a simpler system of equations.

Example 2.10 (Solving by elimination). To illustrate this technique, consider the homogeneous system

$$2x_1 - x_2 + x_3 = 0 \tag{2.6}$$
$$x_1 + 3x_2 + 4x_3 = 0. \tag{2.7}$$

Adding $(-2)\cdot$ (2.7) + (2.6) gives $-7x_2 - 7x_3 = 0$ or

$$x_2 = -x_3. \tag{2.8}$$

Adding 3 (2.6) + (2.7) gives $7x_1 + 7x_3 = 0$ or

$$x_1 = -x_3. \tag{2.9}$$

Thus, a solution of this system is obtained by setting $x_3 = t$, where t is any real number, and then solving for x_1 and x_2 in terms of t using (2.8) and (2.9). The solution set can therefore be written as $\{(-t, -t, t) : t \in \mathbb{R}\}$, which is said to be written in *parametric form*. We see that there is a solution for each $t \in \mathbb{R}$, so this system has an infinite number of solutions.

Exercise 2.5. Verify that $(-t, -t, t)$ is indeed a solution of the system of equations in the previous example for any $t \in \mathbb{R}$.

We now begin to formalize the elimination process in order to carry it out in a systematic way and to understand why it works. Consider again the general linear system of m equations in n unknowns in (2.5). If we select m scalars $c_1, \cdots, c_m$, multiply the jth equation by c_j for $j = 1, \cdots, m$, and add all of the equations together, we obtain a new linear equation, given by [15]

$$\begin{aligned}(c_1A_{11} + c_2A_{21} + \cdots + c_mA_{m1})x_1 + \cdots + (c_1A_{1n} + c_2A_{21} + \cdots + c_mA_{mn})x_n \\ = c_1y_1 + \cdots + c_my_m.\end{aligned} \tag{2.10}$$

[15] In this language, equations (2.8) and (2.9) in the previous example were obtained in exactly this way, as linear combinations of equations (2.6) and (2.7).

Definition 2.11 (Linear combination). The equation (2.10) is said to be a *linear combination* of the equations in (2.5).

Theorem 2.12 (Linear combinations of linear equations preserve solutions). If $(s_1, \cdots, s_n)$ is a solution of the linear system (2.5), then it is also a solution of (2.10).

Proof. Substituting $(x_1, \cdots, x_n) = (s_1, \cdots, s_n)$ in (2.10) gives

$$\begin{aligned}
&(c_1 A_{11} + c_2 A_{21} + \cdots + c_m A_{m1}) s_1 + \cdots + (c_1 A_{1n} + c_2 A_{21} + \cdots + c_m A_{mn}) s_n \\
&= c_1(A_{11} s_1 + \cdots + A_{1n} s_n) + \cdots + c_m (A_{m1} s_1 + \cdots + A_{mn} s_n) \\
&= c_1 y_1 + \cdots + c_m y_m,
\end{aligned}$$

where we have used the fact that $A_{i1} s_1 + \cdots + A_{in} s_n = y_i$ for $i = 1, \ldots, m$, that is, that $(s_1, \ldots, s_n)$ is a solution to the original system of equations. [16] $\square$

Theorem 2.12 is the fundamental idea of the elimination process: given the linear system (2.5), if we have another linear system

$$\begin{aligned}
B_{11} x_1 + B_{12} x_2 + \cdots + B_{1n} x_n &= z_1 \\
B_{21} x_1 + B_{22} x_2 + \cdots + B_{2n} x_n &= z_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\;\;\vdots \\
B_{k1} x_1 + B_{k2} x_2 + \cdots + B_{kn} x_n &= z_k
\end{aligned} \tag{2.11}$$

in which each of the k equations is a linear combination of the equations in (2.5), then *every solution of* (2.5) *is a solution of this new system*. Thus, *the elimination process allows us to solve a linear system by forming linear combinations of the given equations to produce an system of equations which has the same solutions, but is easier to solve.*

Of course, it may happen that some solutions of (2.11) are not solutions of (2.5). As a simple example, consider the following two linear systems:

$$x_1 - 2x_2 = -1 \tag{2.12}$$
$$-x_1 + 3x_2 = 3 \tag{2.13}$$

and

$$x_1 - 2x_2 = -1 \tag{2.14}$$
$$-x_1 + 2x_2 = 1 \tag{2.15}$$

Using the elimination technique, the former system has a unique solution $(3, 2)$ whereas the latter has an infinite number of solutions, which we can parametrize as $\{(2t-1, t) : t \in \mathbb{R}\}$. By taking $t = 2$, we see that $(3, 2)$ is a solution of the second system, but any solution of the second system with $t \neq 2$ is not a solution of the first.

This can be understood as follows: Both equations in the second system can be written as linear combinations of the equations in the first system, since

$$\begin{aligned}
(2.14) &= 1 \cdot (2.12) + 0 \cdot (2.13) \\
(2.15) &= -1 \cdot (2.12) + 0 \cdot (2.13)
\end{aligned}$$

However, the second equation in the first system cannot be written as a linear combination of the equations in the second system.

[16] This is exactly what you verified in Exercise 2.5.

Exercise 2.6. Show that

$$c_1(2.14) + c_2(2.15) = (2.13)$$

requires

$$\begin{aligned} c_1 - c_2 &= -1 \\ c_1 - c_2 &= -3, \end{aligned}$$

which has no solution.

This motivates the following definition:

Definition 2.13 (Equivalent linear systems). Two systems of linear equations are said to be *equivalent* if they have the same set of solutions.

We have just seen that two linear systems might fail to be equivalent if the equations in one system cannot be written as linear combinations of the equations in the other system. We thus have the following theorem:

Theorem 2.14 (Equivalent linear systems). Two systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the other system.

Proof. The proof follows immediately from Theorem 2.12. □

Exercise 2.7. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} x_1 - x_2 &= 0 \qquad & 3x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 0 \qquad & x_1 + x_2 &= 0 \end{aligned}$$

If the elimination process is to be effective in finding the solutions of the system (2.5), then we must see how to form linear combinations of the given equations to produce an equivalent system of equations which is easier to solve. In the next section, we discuss a method to systematically do this. We begin by developing a more convenient notation.

2.3 Matrices

Given the general system of m linear equations in n unknowns in (2.5), we wish to form linear combinations of these equations in such a way that we are guaranteed to produce an equivalent system which is easier to solve. In this section, we will formalize this process and make precise the kind of system at which we wish to arrive.

In forming linear combinations of the equations in (2.5), notice that we are actually only computing with the coefficients A_{ij} and scalars y_i, with the variables x_j more or less acting as placeholders. We shall therefore abbreviate the system by

$$AX = Y \tag{2.16}$$

where

$$A = \begin{bmatrix} A_{11} \cdots A_{1n} \\ \vdots \qquad \vdots \\ A_{m1} \cdots A_{mn} \end{bmatrix} \tag{2.17}$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \tag{2.18}$$

Definition 2.15 (Coefficient matrix). A rectangular array of numbers as in (2.17) is called a *matrix*. In (2.17), the matrix A is called the *coefficient matrix* of the system. The mn numbers A_{ij} (which are the coefficients of the equations in (2.5)) are called the *entries* (or *matrix elements*) of the matrix A. Since A has m rows and n columns, it is said to be an $m \times n$ matrix. Note that the number of *rows* is always listed first. [17]

Example 2.16 (Coefficient matrix of a linear system). The matrix of coefficients for the linear system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ x_1 + 3x_2 + 4x_3 &= 0 \end{aligned}$$

is

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & 4 \end{bmatrix}.$$

Exercise 2.8. Write the coefficient matrix for the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 0 \\ 5x_1 - 5x_3 &= 0. \end{aligned}$$

2.4 Elementary row operations

We now consider operations on the rows of the matrix A which correspond to forming linear combinations of the equations in the system $AX = Y$. We only wish to consider operations which lead to an *equivalent* system of equations. As we will see, any such operation can built out of three *elementary row operations*:

Definition 2.17 (Elementary row operations). The three *elementary row operations* are the following:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a *nonzero* constant.

More formally, we can view each elementary row operation as a *function* e which takes an $m \times n$ matrix A to an $m \times n$ matrix $e(A)$. The function is specified on the matrix elements A_{ij} of A explicitly in each of the three cases above as follows:

[17] For now $AX = Y$ is simply a shorthand notation for the system (2.5). We will soon define a multiplication operation for matrices such that the product of A and X is Y.

1. $e(A)_{ij} = \begin{cases} A_{ij} \text{ if } i \neq r, \\ A_{rj} + cA_{sj} \text{ if } i = r. \end{cases}$

2. $e(A)_{ij} = \begin{cases} A_{ij} \text{ if } i \neq r, s, \\ A_{sj} \text{ if } i = r, \\ A_{rj} \text{ if } i = s. \end{cases}$

3. $e(A)_{ij} = \begin{cases} A_{ij} \text{ if } i \neq r, \\ cA_{rj} \text{ if } i = r. \end{cases}$

One reason we restrict to these three elementary row operations is that they each have an inverse (which is itself an elementary row operation of the same type), allowing us to recover the original matrix A from $e(A)$. This is crucial in making sure the resulting linear system is equivalent to the original one.

Exercise 2.9. For each pair of matrices below, find the elementary row operation that transforms the first matrix into the second, and then find the inverse row operation that transforms the second matrix into the first.

1. $\begin{bmatrix} 0 & -2 & 5 \\ 1 & 4 & -7 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 4 & -7 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -5 & 9 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 9 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Theorem 2.18 (Elementary row operations are invertible). To each elementary row operation e there corresponds an elementary row operation e^{-1} of the *same* type as e, such that $e^{-1}(e(A)) = e(e^{-1}(A)) = A$ for all A.

Proof. We consider each type of elementary row operation in turn.

(1) If e is the operation which replaces row r by row r plus c times row s ($r \neq s$), then e^{-1} is the operation which replaces row r by row r plus $(-c)$ times row s. To see this, note that composing these operations leaves A_{ij} unchanged if $i \neq r$ and sends $A_{rj} \mapsto A_{rj} + cA_{sj} \mapsto (A_{rj} + cA_{sj}) - cA_{sj} = A_{rj}$.

(2) If e interchanges rows r and s, then $e^{-1} = e$.

(3) If e be the operation which multiplies the rth row of a matrix by the non-zero scalar c, then e^{-1} is the operation which multiplies the rth row by $\frac{1}{c}$, since composing these operations sends $A_{rj} \mapsto cA_{rj} \mapsto \frac{1}{c}(cA_{rj}) = A_{rj}$ and leaves A_{ij} unchanged if $i \neq r$.

□

Definition 2.19 (Row-equivalent matrices). Two $m \times n$ matrices are said to be *row-equivalent* if one can be obtained from the other by a finite sequence of elementary row operations.

Lemma 2.20 (Row-equivalence is an equivalence relation). Row-equivalence is an equivalence relation. That is, if A, B and C are any $m \times n$ matrices, then they satisfy the following properties

(i) (Relfexivity) A is row-equivalent to itself;

(ii) (Symmetry) If A is row-equivalent to B, then B is row-equivalent to A;

(iii) (Transitivity) If A is row-equivalent to B, and B is row-equivalent to C, then A is row-equivalent to C.

Proof. (i) (Reflexivity) A is equal to itself by an empty sequence of elementary row operations, hence A is row-equivalent to itself.

(ii) (Symmetry) If A is row-equivalent to B, then $B = (e_n \circ e_{n-1} \circ \cdots e_1)(A)$. Then $A = (e_1^{-1} \circ \cdots \circ e_{n-1}^{-1} \circ e_n^{-1})(B)$ (since $e_j^{-1}(e_j(A)) = A$ and $e_j(e_j^{-1}(B)) = B$ for all $j = 1, \ldots, n$), hence B is row-equivalent to A.

(iii) (Transitivity) If A is row-equivalent to B and B is row-equivalent to C, then $B = (e_n \circ \cdots \circ e_1)(A)$ and $C = (\tilde{e}_m \circ \cdots \circ \tilde{e}_1)(B)$. We therefore have $C = (\tilde{e}_m \circ \cdots \circ \tilde{e}_1 \circ e_n \cdots \circ e_1)(A)$, hence A is row-equivalent to C.

□

Theorem 2.21 (Row-equivalence implies equivalence). If A and B are row-equivalent $m \times n$ matrices, then the linear systems $AX = 0$ and $BX = 0$ are equivalent (have exactly the same solutions).

Proof. Since we pass from A to B by a finite sequence of elementary row operations

$$A = A_0 \to A_1 \to \cdots \to A_k = B,$$

by transitivity (see Lemma 2.20) it suffices to prove that the systems $A_j X = 0$ and $A_{j+1} X = 0$ have the same solutions, i.e., that one elementary row operation does not disturb the set of solutions.

Suppose now that B is obtained from A by a single elementary row operation. For each of the three types of elementary row operations, each equation in the system $BX = 0$ will be a linear combination of the equations in the system $AX = 0$. Since the inverse of an elementary row operation is an elementary row operation, each equation in $AX = 0$ will also be a linear combination of the equations in $BX = 0$. Hence, these two systems are equivalent. □

Example 2.22 (Solving a homogeneous system of equations by elementary row operations). We now demonstrate how to use Theorem 2.21 to solve a homogeneous system of liner equations by elementary row operations.

Consider the homogeneous system of linear equations

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 2x_4 &= 0 \\ x_1 + 4x_2 - x_4 &= 0 \\ 2x_1 + 6x_2 - x_3 + 5x_4 &= 0. \end{aligned} \tag{2.19}$$

We write the coefficient matrix of the system and apply the following sequence of elementary row operations:

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{R3 \to -R1+R3} \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 0 & 7 & -4 & 3 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{2}R1}$$

$$\begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 1 & 4 & 0 & -1 \\ 0 & 7 & -4 & 3 \end{bmatrix} \xrightarrow{R2 \to -R1+R2} \begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 9/2 & -3/2 & -2 \\ 0 & 7 & -4 & 3 \end{bmatrix} \xrightarrow{R2 \to \frac{2}{9}R2}$$

$$\begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1/3 & -4/9 \\ 0 & 7 & -4 & 3 \end{bmatrix} \xrightarrow{R3 \to -7R2+R3} \begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1/3 & -4/9 \\ 0 & 0 & -5/3 & 55/9 \end{bmatrix} \xrightarrow{R3 \to -\frac{3}{5}R3}$$

$$\begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1/3 & -4/9 \\ 0 & 0 & 1 & -11/3 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{3}R3+R2} \begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & 0 & -5/3 \\ 0 & 0 & 1 & -11/3 \end{bmatrix} \xrightarrow{R1 \to -\frac{3}{2}R3+R1}$$

$$\begin{bmatrix} 1 & -1/2 & 0 & 13/2 \\ 0 & 1 & 0 & -5/3 \\ 0 & 0 & 1 & -11/3 \end{bmatrix} \xrightarrow{R1 \to \frac{1}{2}R2+R1} \begin{bmatrix} 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \\ 0 & 0 & 1 & -11/3 \end{bmatrix}$$

The final matrix is the coefficient matrix of the system

$$\begin{aligned} x_1 + \frac{17}{3}x_4 &= 0 \\ x_2 - \frac{5}{3}x_4 &= 0 \\ x_3 - \frac{11}{3}x_4 &= 0 \end{aligned} \tag{2.20}$$

whose solution set is given by taking $x_4 = t$ to be any real number, and then using (2.20) to solve for x_1, x_2, and x_3 in terms of t:

$$\{(-\frac{17}{3}t, \frac{5}{3}t, \frac{11}{3}t, t) : t \in \mathbb{R}\}. \tag{2.21}$$

Since the coefficient matrices of (2.19) and (2.20) are row-equivalent, by Theorem 2.21 the systems (2.19) and (2.20) are equivalent, and hence (2.21) is also the solution set of the original system (2.19).

In the previous example we were obviously not performing row operations at random. Instead, our choice of row operations was motivated by a desire to simplify the coefficient matrix in a manner analogous to 'eliminating unknowns' in the system of linear equations. Roughly speaking, we use the x_1 term in the first equation of the system to eliminate the x_1 terms in the other equations. Then we use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until we obtain the simplest possible equivalent system of equations. In the following sections, we will discuss an algorithm for carrying out this process.

2.5 Echelon matrices

We now make a formal definition of the type of matrix at which we are attempting to arrive. In the following definitions, a *nonzero* row of a matrix means a row that contains at least one nonzero entry; a *leading entry* of a nonzero row is the leftmost nonzero entry in that row.

Definition 2.23 (Row echelon form and reduced row echelon form). An $m \times n$ matrix is said to be in *row echelon form* (REF) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in row echelon form satisfies the following additional conditions, then it is said to be in *reduced row echelon form* (RREF):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Example 2.24. The matrix

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row echelon form and the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is in reduced row echelon form.

Exercise 2.10. State whether each matrix is in REF, RREF, or neither. Justify your answers.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -3 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 4 & 0 & 0 & 0 & -3 & 7/2 & 0 & 26 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3/2 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 & -1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 22 \end{bmatrix}$

It will be proved in Theorem 2.33 that every $m \times n$ matrix A is row-equivalent to a *unique* row-reduced echelon matrix U, called the *reduced row echelon form of A*.

In the next section we will see a row-reduction algorithm which will put *any* $m \times n$ matrix A into its unique reduced row echelon form. [18]

[Explain here why RREF is what we want to arrive at by performing elementary row operations. Columns with no leading 1s are called free variables. These will be parameters, and the leading 1s allow us to solve for the remaining variables in terms of the free ones, giving us the parametric description of the solution set.]

2.6 Pivots

Note that when row operations on a matrix reduce it to REF, further row operations to obtain the RREF *do not change the positions of the leading entries.* Since the RREF is unique, *the leading entries are always in the same positions in any echelon form obtained from a given matrix*. These leading entries correspond to leading 1's in the RREF. This motivates the following definitions:

Definition 2.25 (Pivots, Pivot positions, pivot columns).

(i) A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the RREF of A.

(ii) A **pivot column** is a column of A that contains a pivot position.

(iii) A **pivot** is a non-zero number in a pivot position.

Example 2.26 (Locating pivot columns and pivot positions). Row reduce the matrix A below to echelon form, and locate the pivot positions and pivot columns.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

[18] Note that this algorithm will prove that a RREF of A exists, while Theorem 2.33 shows that it is unique.

SOLUTION Use the same basic strategy as in Section 1.1. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position. A good choice is to interchange rows 1 and 4 (because the mental computations in the next step will not involve fractions).

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Pivot column} \end{array}$$

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix (1) below. The pivot position in the second row must be as far left as possible—namely, in the second column. Choose the 2 in this position as the next pivot.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Next pivot column} \end{array} \tag{1}$$

Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \tag{2}$$

The matrix in (2) is different from any encountered in Section 1.1. There is no way to create a leading entry in column 3! (We can't use row 1 or 2 because doing so would destroy the echelon arrangement of the leading entries already produced.) However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{General form: } \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

Pivot positions

$$A = \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \tag{3}$$

Pivot columns

2.7 Gauss-Jordan elimination

In this section we introduce the *Gauss-Jordan elimination* algorithm which will allow us to systematically reduce any matrix A to its unique RREF, U.

The algorithm consists of four steps, and it produces a matrix in REF. A fifth step produces a matrix in RREF. We illustrate the algorithm by an example.

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

SOLUTION

STEP 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{array}{l}\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \\ \quad\text{Pivot column}\end{array}$$

STEP 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\begin{array}{l}\quad\text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}\end{array}$$

STEP 3

Use row replacement operations to create zeros in all positions below the pivot.

As a preliminary step, we could divide the top row by the pivot, 3. But with two 3's in column 1, it is just as easy to add -1 times row 1 to row 2.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

> **STEP 4**
>
> Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the "top" row to the row below. This produces

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. If we want the reduced echelon form, we perform one more step.

STEP 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} + (-6)\cdot\text{row 3} \\ \leftarrow \text{Row 2} + (-2)\cdot\text{row 3} \\ \end{array}$$

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{Row scaled by } \frac{1}{2} \\ \end{array}$$

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} + (9)\cdot\text{row 2} \\ \\ \end{array}$$

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row scaled by } \frac{1}{3} \\ \\ \end{array}$$

This is the reduced echelon form of the original matrix. ■

The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

Exercise 2.11. Row reduce the following matrices to RREF. Circle the pivot positions and pivot columns in the final matrix.

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$

2.8 Existence and uniqueness of solutions

In this section we will answer the following two fundamental questions for any given linear system:

1. Is the system consistent; that is, does at least one solution *exist*.

2. If a solution exists, is it the *only* one; that is, is the solution *unique*.

2.8.1 Homogeneous systems of linear equations

A homogeneous system of linear equations $AX = 0$ always has at least one solution, the *trivial* solution, given by $x_1 = x_2 = \cdots = x_n = 0$. Thus, *a homogeneous system of linear equations is always consistent*. The fundamental question for a homogeneous system of linear equations is whether there exists a non-trivial solution.

Consider the system $RX = 0$, where R is an $m \times n$ matrix in reduced row echelon form. Let $1 \le r \le m$ and let $1, \cdots, r$ be the nonzero rows of R. The system $RX = 0$ therefore consists of r non-trivial equations. Letting $x_1, \cdots, x_r$ denote the first r variables, and $u_i = x_{r+i}, i = 1, \cdots, n-r$ denote the remaining $n-r$ free variables, the non-trivial equations take the form

$$x_1 + \sum_{j=1}^{n-r} C_{1j} u_j = 0 \tag{2.22}$$

$$\vdots \qquad\qquad \vdots \tag{2.23}$$

$$x_r + \sum_{j=1}^{n-r} C_{rj} u_j = 0 \tag{2.24}$$

where each $x_i, i = 1, \cdots, r$ occurs (with non-zero coefficient) only in the ith equation. All solutions to the system $RX = 0$ are obtained by assigning any real numbers to $u_1, \cdots, u_{n-r}$ and then computing the values of $x_1, \cdots, x_r$ using (2.22) - (2.24). This shows that *if $r < n$, the system $RX = 0$ has an infinite number of solutions*. [19] If $r = n$, then R is the $n \times n$ identity matrix and the system $RX = 0$ has only the trivial solution.

We thus have the following theorem:

Theorem 2.27 (Solution sets of homogeneous linear systems).

(a) If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution (in fact, an infinite number of them).

(b) If A is an $n \times n$ (square) matrix, then A is row-equivalent to the $n \times n$ identity matrix if and only if the system of equations $AX = 0$ has only the trivial solution. [20]

Proof. (a) Let R be the unique RREF of the matrix A. Since A and R are row-equivalent, the systems $AX = 0$ and $RX = 0$ have exactly the same solutions. As before, let r be the number of non-zero rows in R. Then $r \le m$, and since $m < n$, we have $r < n$. We will therefore have $n - r > 0$ free variables, so $AX = 0$ has a non-trivial solution.

[19] The $n \times n$ identity matrix is the $n \times n$ matrix for which each diagonal entry is 1, and each off-diagonal entry is 0. See Exercise 2.12 below for an example of this case.

[20] If $m > n$, the system has a unique solution if and only if the reduced row echelon form the coefficient matrix is of the form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) ($\Longrightarrow$) If A is row-equivalent to I, then $AX = 0$ and $IX = 0$ have the same solutions, hence $AX = 0$ has only the trivial solution $X = 0$.
($\Longleftarrow$) Suppose $AX = 0$ has only the trivial solution. Let R be the unique RREF of R, and let r be the number of non-zero rows of R. Since R is row-equivalent to A, $RX = 0$ has only the trivial solution. Thus $r \geq n$. But since R has n rows, $r \leq n$, and therefore $r = n$. Since R is in RREF, it is the $n \times n$ identity matrix.

□

Exercise 2.12. Consider the system $RX = 0$ with coefficient matrix

$$R = \begin{bmatrix} 1 & 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Verify the following:

(a) The system $RX = 0$ consists of r non-trivial equations and $m - r$ trivial equations. Write out these equations.

(b) Show that the non-trivial equations take the form (2.22) - (2.24). Indentify the basic variables $x_1, \cdots, x_r$, the free variables $u_1, \cdots, u_{n-r}$, and the coefficients C_{ij} in these equations.

(c) Find the solution set and express it in parametric form.

2.8.2 Inhomogeneous systems of linear equations

So far we have used elementary row operations to solve homogeneous systems of linear equations. What, then, do elementary row operations do toward solving an *inhomogeneous* system of linear equations? Happily, it turns out that we solve an inhomogeneous system of linear equations in exactly the same way as a homogeneous one, with one minor modification.

Definition 2.28 (Augmented matrix of an inhomogeneous linear system). The *augmented matrix* of an inhomogeneous system of linear equations $AX = Y$ is the $m \times (n+1)$ matrix whose first n columns are the columns of the coefficient matrix A and whose last column is Y. We denote this matrix as $A' = [A|Y]$.

Example 2.29. The augmented matrix of the inhomogeneous linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \tag{2.25}$$

is given by [21]

$$A' = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

[21] The vertical bar offsetting the final column in A' has no meaning and is only there as a visual device to remind us that the matrix is an augmented matrix. Many authors do not typeset this vertical bar, and instead simply denote an augmented matrix exactly in the same way as a coefficient matrix. In this case, one determines whether the matrix is a coefficient matrix or augmented matrix from context.

Suppose we perform a sequence of elementary row operations on the coefficient matrix A of an inhomogeneous linear system $AX = Y$, arriving at it's unique reduced row echelon form, R. If we perform the same operations on A', we will arrive at a matrix R' in reduced row echelon form, whose first n columns are those of R and whose last column is the $m \times 1$ matrix Z which results from applying the same sequence of elementary row operations to the matrix Y. By the same arguments as before, the system $RX = Z$ is equivalent to the original system $AX = Y$, and therefore we can read off the solution set from $RX = Z$.

Exercise 2.13. Verify that performing Gauss-Jordan elimination on the augmented matrix A' of Example 2.29 gives

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This shows that the inhomogeneous system (2.25) has the unique solution $(x_1, x_2, x_3) = (1, 0, -1)$.

Therefore, *to solve an inhomogeneous linear system $AX = Y$, we simply reduce the augmented matrix $A' = [A|Y]$ to its unique reduced row-echelon form, $R' = [R|Z]$, using Gauss-Jordan elimination (exactly as we do for the coefficient matrix A for a homogeneous system). We then read off the solution set from the equivalent system $RX = Z$.*

While a homogeneous system of linear equations is always consistent, this need not be the case for an inhomogeneous system, even if the number of equations is fewer than the number of unknowns.

Example 2.30. Consider the inhomogeneous linear system

$$x_1 + 3x_2 + 7x_3 = 2 \tag{2.26}$$
$$-2x_1 - 6x_2 - 14x_3 = -3 \tag{2.27}$$

Replacing (2.27) by 2(2.26) + (2.27) gives $0 = 1$, which is false for any (x_1, x_2, x_3), so the system is inconsistent.

Example 2.31. Consider now the inhomogeneous linear system $AX = Y$ with augmented matrix

$$A' = \left[\begin{array}{ccc|c} 1 & -2 & 1 & y_1 \\ 2 & 1 & 1 & y_2 \\ 0 & 5 & -1 & y_3 \end{array}\right] \tag{2.28}$$

We would like to know:

1. Under what condition does the solution exist?
2. If a solution exists, is it unique?

Performing elementary row operations on A', we arrive at the row echelon matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & -2y_1 + y_2 \\ 0 & 0 & 0 & 2y_1 - y_2 + y_3 \end{array}\right]. \tag{2.29}$$

We see that the system is consistent only if $2y_1 - y_2 + y_3 = 0$. In this case, x_3 is a free variable, so the system has an infinite number of solutions.

This example is illustrative of the general case.

Theorem 2.32 (Existence and uniqueness for inhomogeneous systems).

(a) An inhomogeneous linear system is consistent if and only if the rightmost column of its augmented matrix is *not* a pivot column - that is, if and only if any echelon form of the matrix has *no* row of the form

$$[0 \ \cdots \ 0 \ b] \quad \text{with } b \text{ nonzero.}$$

(b) If an inhomogeneous linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Proof. Left to the reader. The details are very similar to the homogeneous case, with the obvious modifications. □

Before we close this section, let us take care of a piece of unfinished business and prove that every matrix A has a unique RREF.

Theorem 2.33 (Uniqueness of reduced row echelon form). Every $m \times n$ matrix A is row-equivalent to a *unique* row-reduced echelon matrix U, called the *reduced row echelon form of A*

Proof. [22] (By contradiction.) Suppose A can be reduced by finite sequences of elementary row operations to two distinct R and S, both in RREF. Since $R \neq S$, there must be a pair of indices (i,j) such that $R_{ij} \neq S_{ij}$. Corresponding to R and S, respectively, form new $m \times k$ matrices $(k \leq n)$ R' and S' by selecting the first (leftmost) column for which R and S differ along with all pivot columns to the left of this column. [23] Noting that the leftmost column in which R and S differ must be a non-pivot column, the matrices R' and S' must therefore take the form

$$R' = \left(\begin{array}{c|c} I_n & \mathbf{r}' \\ \hline O & \mathbf{0} \end{array}\right) \quad \text{and} \quad S' = \left(\begin{array}{c|c} I_n & \mathbf{s}' \\ \hline O & \mathbf{0} \end{array}\right). \tag{2.30}$$

Note that R' and S' are both row-equivalent to A (since R and S are row-equivalent to A and deleting columns does not affect row-equivalence) and therefore to each other.

Now interpret the matrices in (2.30) as augmented matrices. The system for R' has a unique solution $\mathbf{r}'$, while the system for S' has a unique solution $\mathbf{s}'$. Since the linear systems corresponding to row-equivalent matrices are equivalent, we must have $\mathbf{r}' = \mathbf{s}'$, which means $R' = S'$, and therefore $R = S$, which contradicts our assumption that these matrices are distinct. Hence, the RREF of A must be unique. □

[22]The following proof is due to W.H. Holzmann.

[23]For instance, if $R = \begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 2 & 0 & 7 & 9 \\ 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, then $R' = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$ and $S' = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{pmatrix}$.

3 Matrices

3.1 Matrix operations

In the preceding section, we viewed each elementary row operation as a function which takes as input a matrix and produces a new matrix. In the following sections, we will see that elementary row operations can equivalently be viewed as a function which takes as input *two* matrices and returns a third matrix. This will lead us to define operations of addition and multiplication on *entire matrices*, rather than on the just the rows of a matrix. We will also find it useful to define other operations on the set of matrices, which are not analogous to any familiar operations on the set of real numbers.

3.1.1 Formal definition of a matrix

In order to define operations on the set of matrices, it will help to take a slightly more formal view of a matrix. First, recall that a *sequence* is just a function $f : \mathbb{N} \to \mathbb{R}$. [24] Denoting $f(n)$ by f_n, we visualize a sequence as a list of real numbers $(f_1, f_2, f_3, \cdots)$, indexed by $n \in \mathbb{N}$.

Example 3.1. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \frac{1}{n}$. Then we write the sequence $(f_1, f_2, f_3, \dots) = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

Definition 3.2 (Matrix). Let $\overline{n} = \{1, 2, \cdots, n\}$. A *matrix* is a function $A : \overline{m} \times \overline{n} \to \mathbb{R}$.

As we did with sequences, we denote $A(i, j)$ by A_{ij}. We then visualize the matrix A as the rectangular array of the numbers A_{ij} with m rows and n columns. Two matrices are *equal* if they have the same size and if each of the corresponding entries are equal.

3.1.2 Matrix addition and scalar multiplication

The definition of a matrix in 3.2 makes it clear how to define addition and scalar multiplication of matrices; namely, we define these operations pointwise, the same way we always do for any function:

$$(f + g)(a) = f(a) + g(a)$$
$$(cf)(a) = cf(a)$$

Definition 3.3 (Matrix addition). If A and B are two $m \times n$ matrices, then the matrix $A + B$ has elements $(A + B)_{ij} = A_{ij} + B_{ij}$.

Note that matrix addition is only defined if A and B are two matrices of the same size.

Exercise 3.1. Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$. Determine whether $A + B$ and $A + C$ are defined. If so, compute them.

Definition 3.4 (Scalar multiplication of a matrix). If A is an $m \times n$ matrix and c is any scalar, then the matrix cA has elements $(cA)_{ij} = cA_{ij}$.

[24] The notation $f : A \to B$ denotes a function f from domain A (the set of "inputs") to codomain B (the set of "outputs") defined by $b = f(a)$ for all $a \in A$.

Exercise 3.2. Let A and B be the same as in the previous exercise. Compute $2B$ and $A - 2B$.

The usual rules of algebra apply to sums and scalar multiples of matrices, as the next theorem shows.

Theorem 3.5 (Properties of matrix addition and scalar multiplication). Let A, B, and C be matrices of the same size, and let r and s be scalars.

(i) $(A + B) + C = A + (B + C)$ (matrix addition is associative)

(ii) $A + 0 = A$ (the zero matrix is an additive identity)

(iii) Each matrix A has an additive inverse $-A$ such that $A - A = 0$.

(iv) $A + B = B + A$ (matrix addition is commutative)

(v) $r(A + B) = rA + rB$ (scalar multiplication distributes over matrix addition)

(vi) $(r + s)A = rA + sA$ (scalar multiplication distributes over scalar addition)

(vii) $r(sA) = (rs)A$ (associativity of scalar multiplication)

These properties are exactly the same properties of addition and scalar multiplication of vectors. Indeed, we can view a vector in $\mathbb{R}^n$ as an $n \times 1$ (or $1 \times n$) matrix. [25]

Proof. For each of these we need to show that (1) the matrix on the left and right hand side of each equation has the same size, and (2) each of the corresponding entries are equal. Condition (1) holds for each since A, B, and C are all the same size. Condition (2) holds in each case because of the corresponding properties of real numbers. For example, property (iv) holds for matrices A and B since $A_{ij} + B_{ij} = B_{ij} + A_{ij}$ holds for real numbers A_{ij} and B_{ij}. The remaining properties are checked similarly and are left as an exercise. □

Exercise 3.3. Prove properties (ii)-(vii) in Theorem 3.5.

3.1.3 Matrix multiplication

We have seen in the previous sections that the process of forming linear combinations of the rows of a matrix is a fundamental one. We now introduce a systematic way of doing this.

Let B be an $m \times n$ matrix with rows $\beta_1, \beta_2, \cdots, \beta_m$. Denote the jth entry of the ith row by B_{ij}. For example, if

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

then $m = 2, n = 3, \beta_1 = (1, 2, 3), B_{12} = 2$, etc. Suppose we form a new $m \times n$ matrix C whose rows $\gamma_1, \gamma_2, \cdots, \gamma_m$ are linear combinations of the rows of B. For example, if we take B as above and let [26]

$$\begin{aligned} \gamma_1 &= 2\beta_1 - \beta_2, \\ \gamma_2 &= \beta_1 + 2\beta_2, \end{aligned}$$

[25] In Section 5, we will see other sets which admit operations of addition and scalar multiplication which behave exactly like those of vectors. This will lead to the notion of an abstract *vector space*.

[26] These are not elementary row operations. We are just taking arbitrary linear combinations here.

then

$$C = \begin{bmatrix} 2(1,2,3) - 1(4,5,6) \\ (1,2,3) + 2(4,5,6) \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 9 & 12 & 15 \end{bmatrix}.$$

Denoting the ith multiple of jth row of B by A_{ij}, from the matrix above on the left we see that the rows of C take the form

$$\gamma_1 = A_{11}\beta_1 + A_{12}\beta_2 = \sum_{j=1}^{2} A_{1j}\beta_j,$$
$$\gamma_2 = A_{21}\beta_1 + A_{22}\beta_2 = \sum_{j=1}^{2} A_{2j}\beta_j.$$

Thus, the rows of C are determined by 4 scalars which are themselves entries in a 2×2 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Letting C_{ik} denote the kth entry of the ith row of C, we see that $C_{ik} = \sum_{j=1}^{2} A_{ij}B_{jk}$. This is precisely the formula for the dot product of the ith row of A with the jth column of B, viewing each as a vector in $\mathbb{R}^2$. For example, $C_{11} = A_{11}B_{11} + A_{12}B_{21} = 2(1) + (-1)(4) = -2$.

Generalizing this example, we wish to define the *product* of two matrices A and B to be the matrix C whose ij-entry C_{ij} is the dot product of the ith row of A with the jth column of B. We note immediately that, for this to make sense, the matrices A and B must be compatible sizes, in the sense that the set of rows of A and the set of columns of B must each consist of vectors of the same length, otherwise the dot products will not be defined. This leads us to the following definition.

Definition 3.6 (Matrix multiplication). Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. The *product* of A and B is the $m \times p$ matrix C whose (i,j)-entry is $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$, which is the dot product of the ith row of A with the jth column of B.

Example 3.7 (Row-column rule for matrix multiplication). One can remember this definition by the following "row-column" rule for computing AB: the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. That is, the (i,j)-entry of AB is given by

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj} \tag{3.1}$$

Example 3.8. Taking A and B to be the matrices

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -2 & 6 \end{bmatrix},$$
$$B = \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix},$$

their product is

$$\begin{aligned} C &= AB \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix} \\ &= \begin{bmatrix} -5 & -6 & -4 \\ 50 & 11 & 26 \\ 80 & 22 & 44 \end{bmatrix} \end{aligned}$$

As discussed above, the product of these matrices is defined since the length of the rows (or, equivalently, the number of *columns*) in the first matrix coincides with the length of the columns (or, equivalently, the number of *rows*) in the second matrix. In this case, multiplying an 3×2 matrix and a 2×3 matrix gave a 3×3 matrix.

Exercise 3.4. If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

Exercise 3.5. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$. Compute AB.

Note that this definition of matrix multiplication agrees with our notation $AX = Y$ for a linear system of equations: A is an $m \times n$ matrix, X is an $n \times 1$ matrix, and Y, which is the product of A and X is an $m \times 1$ matrix.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Theorem 3.9 (Properties of matrix multiplication). Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then the following hold for all such matrices A, B, and C.

(i) $(AB)C = A(BC)$ (associative law of multiplication)

(ii) $A(B + C) = AB + AC$ (left distributive law)

(iii) $(B + C)A = BA + CA$ (right distributive law)

(iv) $r(AB) = (rA)B = A(rB)$ for any scalar r

(v) $I_m A = A = AI_n$ (identity for matrix multiplication)

Proof. As before, for each of these we need to check that (1) both sides are matrices of the same size, and (2) the corresponding matrix elements on each side are all equal.

(i) For AB to be defined, B must be an $n \times p$ matrix, for some p. Then, for BC to be defined, C must be $p \times q$ matrix. We therefore have

$$\begin{aligned} AB &: (m \times n)(n \times p) = m \times p \\ BC &: (n \times p)(p \times q) = n \times q \end{aligned}$$

so

$$\begin{aligned}(AB)C &: (m \times p)(p \times q) = m \times q\\ A(BC) &: (m \times n)(n \times q) = m \times q\end{aligned}$$

hence both sides have the same size.

Applying the definition of matrix multiplication, we have

$$\begin{aligned}[(AB)C]_{ij} &= \sum_{\ell=1}^{p}(AB)_{i\ell}C_{\ell j}\\ &= \sum_{\ell=1}^{p}\sum_{k=1}^{n}(A_{ik}B_{k\ell})C_{\ell j}\\ &= \sum_{\ell=1}^{p}\sum_{k=1}^{n}A_{ik}(B_{k\ell}C_{\ell j}) \text{ (by associativity in } \mathbb{R})\\ &= \sum_{k=1}^{n}A_{ik}(BC)_{kj}\\ &= [A(BC)]_{ij},\end{aligned}$$

so the corresponding entries are equal, showing that (i) holds. Properties (ii) - (v) are checked similarly, and are left as exercises. □

Definition 3.10 (Power of a matrx). If A is an $n \times n$ matrix and k is a positive integer, then the *kth power of* A, A^k, is the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k \text{ times}}.$$

Since matrix multiplication is associative, there is no need to insert parenthesis into the expression above.

We have seen in theorem Theorem 3.9 that some of the properties of multiplication of real numbers also hold for matrices (e.g., associativity). However, it is not true that *all* such properties hold for matrix multiplication, as the next three examples show.

Example 3.11 (Matrix multiplication is not commutative). If A is a 3×2 matrix and B is a 2×4 matrix, then their product is a 3×4 matrix. However, the product BA is not even defined, since the sizes are not compatible! This shows that matrix multiplication is not commutative. Even if we choose A and B such that both AB and BA are defined, these might not be equal. For instance, let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Then $AB = \begin{bmatrix} 16 & 9 \\ -18 & -15 \end{bmatrix}$ and $BA = \begin{bmatrix} 4 & 6 \\ 11 & -3 \end{bmatrix}$. Hence AB and BA are both defined, but $AB \neq BA$.

Example 3.12 (Matrix multiplication does not satisfy the cancellation laws). For real numbers, the cancellation law says that if $AB = AC$, then $B = C$. This does not hold, in general, for matrices. As an example, let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. We see that

$AB = BC = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$, but $B \neq C$. One can show similarly that the other cancellation law ($BA = CA \implies B = C$) [27] also does not hold in general for matrices.

Example 3.13 (Zero products do not imply one matrix is zero). For real numbers, if $AB = 0$ then either $A = 0$ or $B = 0$. To see that this does not hold, in general, for matrices, let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. Then $AB = 0$, but neither A nor B is the zero matrix.

We now define two operations which are not analogous to any familiar operations on the set of real numbers.

3.1.4 Transpose of a matrix

Definition 3.14 (Transpose of a matrix). Given an $m \times n$ matrix A, the *transpose* of A is the $n \times m$ matrix, denoted A^T, whose columns are formed from the corresponding rows of A, that is, given A, A^T is the matrix with elements $(A^T)_{ij} = A_{ji}$.

Example 3.15. If $A = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}$, then $A^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$.

Theorem 3.16 (Properties of A^T). Let A and B be matrices whose sizes are appropriate for the following sums and products to be defined. Then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) For any scalar r, $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$

Proof. Clearly the matrices on each side are the same size since A and B are the same size. All that is left is to check that the corresponding matrix elements are equal in each case. We will do this for (iv). The proofs for (i)-(iii) are checked similarly, and are left as exercises.

$$\begin{aligned} [(AB)^T]_{ij} &= (AB)_{ji} \\ &= \sum_{k=1}^{n} A_{jk} B_{ki} \\ &= \sum_{k=1}^{n} B_{ki} A_{jk} \text{ (by commutativity in } \mathbb{R}) \\ &= \sum_{k=1}^{n} B^T_{ik} A^T_{kj} \\ &= [B^T A^T]_{ij} \end{aligned}$$

This proves property (iv). □

[27] Note that the two cancellation laws are not equivalent here since matrix multiplication is not commutative.

Exercise 3.6. Prove properties (i)-(iii) of Theorem 3.16.

3.1.5 Trace of a matrix

There is another useful property defined for *square* ($n \times n$) matrices.

Definition 3.17 (Trace of a matrix). The *trace* of an $n \times n$ matrix A is the sum of its diagonal entries:

$$\operatorname{Tr} A = \sum_{i=1}^{n} A_{ii}.$$

Example 3.18. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then

$$\operatorname{Tr} A = 1 + 5 + 9 = 15.$$

Theorem 3.19 (Properties of the trace). Let A, B be $n \times n$ matrices and c a scalar. Then

(i) $\operatorname{Tr}(A + B) = \operatorname{Tr} A + \operatorname{Tr} B$,

(ii) $\operatorname{Tr}(cA) = c \operatorname{Tr} A$,

(iii) $\operatorname{Tr} A^T = \operatorname{Tr} A$,

(iv) $\operatorname{Tr} AB = \operatorname{Tr} BA$,

Exercise 3.7. Prove Theorem 3.19.

3.2 Invertible matrices

3.2.1 Elementary matrices

We will now see that elementary row operations can be represented by multiplication by matrices.

Definition 3.20 (Elementary matrix). An $n \times n$ (square) matrix is said to be an *elementary matrix* if it can be obtained from the $n \times n$ identity matrix by means of a single elementary row operation.

Exercise 3.8. Let c be a constant. Verify that the complete list of 2×2 elementary matrices is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} (c \neq 0), \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} (c \neq 0).$$

Theorem 3.21 (Elementary row operations are represented by elementary matrices). If e is an elementary row operation and E the $n \times n$ elementary matrix $E = e(I)$, then for every $m \times n$ matrix A, $e(A) = EA$.

Proof. We consider each type of elementary row operation separately.

(i) Suppose $r \neq s$ and let e be the elementary row operation which replaces row r by row r plus c times row s. We need to show that $EA = e(A)$. We can write the elements of the corresponding elementary matrix E as [28]

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ \delta_{rk} + c\delta_{sk}, & i = r. \end{cases}$$

Then, if $i \neq r$,

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik}A_{kj} = \sum_{k=1}^{m} \delta_{ik}A_{kj} = A_{ij}$$

and for $i = r$,

$$(EA)_{rj} = \sum_{k=1}^{m} E_{rk}A_{kj} = \sum_{k=1}^{m} (\delta_{rk} + c\delta_{sk})A_{kj} = A_{rj} + cA_{sj}$$

which shows $EA = e(A)$, as desired.

(ii) Suppose $r \neq s$ and let e be the elementary row operation which exchanges rows r and s. The elements of the corresponding elementary matrix are given by

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r, s \\ \delta_{rk}, & i = s \\ \delta_{sk}, & i = r. \end{cases}$$

Then, if $i \neq r, s$,

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik}A_{kj} = \sum_{k=1}^{m} \delta_{ik}A_{kj} = A_{ij}$$

if $i = r$,

$$(EA)_{rj} = \sum_{k=1}^{m} E_{rk}A_{kj} = \sum_{k=1}^{m} \delta_{sk}A_{kj} = A_{sj}$$

and if $i = s$,

$$(EA)_{sj} = \sum_{k=1}^{m} E_{sk}A_{kj} = \sum_{k=1}^{m} \delta_{rk}A_{kj} = A_{rj}$$

which shows $EA = e(A)$.

(iii) Let e be the elementary row operation which multiplies rows r by c, $c \neq 0$. We need to show that $EA = e(A)$. The corresponding elementary matrix can be written as

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ c\delta_{rk}, & i = r. \end{cases}$$

[28] In the following δ_{ij} denotes the ij-component of the identity matrix; that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Then, if $i \neq r$,

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik}A_{kj} = \sum_{k=1}^{m} \delta_{ik}A_{kj} = A_{ij}$$

if $i = r$,

$$(EA)_{rj} = \sum_{k=1}^{m} E_{rk}A_{kj} = \sum_{k=1}^{m} (c\delta_{rk})A_{kj} = cA_{rj}$$

which shows $EA = e(A)$. □

Exercise 3.9. (a) Let $E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ be the elementary matrix obtained from the 2×2 identity matrix by replacing $R2 \mapsto R2 - 3R1$. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Show that the matrix product EA is equal to the result of performing the row replacement $R2 \mapsto R2 - 3R1$ on A.

(b) Let $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be the elementary matrix obtained from the 2×2 identity matrix by the interchange $R1 \leftrightarrow R2$. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Show that the matrix product EA is equal to the result of interchanging the two rows of A.

Corollary 3.22. Let A and B be $m \times n$ matrices. Then B is row-equivalent to A if and only if $B = PA$, where P is a product of $m \times m$ elementary matrices.

Proof. ($\Longrightarrow$) Suppose B is row-equivalent to A. Let $E_1, E_2, \cdots, E_s$ be the elementary matrices corresponding to some sequence of elementary row operations which carries A into B. Then $B = (E_s \cdots E_2E_1)A$.

($\Longleftarrow$) Suppose $B = PA$ where $P = E_s \cdots E_2E_1$ and the E_i are $m \times m$ elementary matrices. Then E_1A is row-equivalent to A, and $E_2(E_1A)$ is row-equivalent to E_1A, so by transitivity E_2E_1A is row-equivalent to A. It follows by induction that $(E_s \cdots E_1)A$ is row-equivalent to A. □

Exercise 3.10. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, and let A be an arbitrary 3×3 matrix. Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

3.2.2 Invertible matrices

Suppose P is an $m \times m$ matrix which is a product of elementary matrices. Then, for each $m \times n$ matrix A, the matrix $B = PA$ is row-equivalent to A. By symmetry (see Lemma 2.20), A is therefore row-equivalent to B and there is a product Q of elementary matrices such that $A = QB$. In particular, this is true when A is the $m \times m$ identity matrix.

In other words, there is an $m \times m$ matrix Q, which is itself a product of elementary matrices, such that $QP = I$. As we shall soon see, the existence of a Q with $QP = I$ is equivalent to the fact that P is a product of elementary matrices.

Definition 3.23 (Invertible matrix). Let A be an $n \times n$ matrix. An $n \times n$ matrix B such that $BA = I$ is called a *left inverse* of A; an $n \times n$ matrix B such that $AB = I$ is called a *right inverse* of A. If $AB = BA = I$, then B is called a *two-sided inverse* (or just *inverse*) of A and A is said to be *invertible* or *non-singular*. A matrix which is not invertible is called a *singular* matrix.

Exercise 3.11. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$. Show that B is a two-sided inverse for A.

Theorem 3.24 (Uniqueness of inverses). If A has a left inverse B and a right inverse C, then $B = C$.

Proof. Suppose $BA = I$ and $AC = I$. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

□

Theorem 3.24 says that if a matrix A has a left inverse and a right inverse, then these must be equal. In particular, if A has a two-sided inverse, this theorem shows that it is unique. We denote the unique two-sided inverse by A^{-1} and call it *the* inverse of A.

Theorem 3.25 (Properties of A^{-1}).

(i) If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

(ii) If both A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

(iii) If A is invertible, so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

Proof. (i) Follows immediately from the symmetry of the definition.

(ii) $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$. By uniqueness of inverses, $(AB)^{-1} = B^{-1}A^{-1}$.

(iii) $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$. By uniqueness of inverses, $(A^{-1})^T = (A^T)^{-1}$.

Generalizing (ii), any finite product of invertible matrices is invertible, with inverse

$$(A_1 A_2 \cdots A_s) = A_s^{-1} \cdots A_2^{-1} A_1^{-1}$$

□

Theorem 3.26 (Elementary matrices are invertible). An elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Proof. Let E be the elementary matrix corresponding to the elementary row operation e. If e' is the inverse operation of e and $E' = e'(I)$, then $EE' = e(E') = e(e'(I)) = I$, so E is invertible and $E' = E^{-1}$. □

Example 3.27. (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$

(d) When $c \neq 0$, $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} c^{-1} & 0 \\ 0 & 1 \end{bmatrix}$, $\quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & c^{-1} \end{bmatrix}$.

Theorem 3.28 (Invertible matrix theorem). If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

Proof. We will show that $(i) \implies (ii) \implies (iii) \implies (i)$. Assume (i) is true. Let R be the RREF of A. Then $R = E_k \cdots E_2 E_1 A$, where $E_1, \cdots, E_k$ are elementary matrices. Since each E_j is invertible, we have $A = E_1^{-1} \cdots E_k^{-1} R$. Since products of invertible matrices are invertible, we see that A is invertible if and only if R is invertible. Since R is a square matrix in RREF, R is invertible iff it is the $n \times n$ identity matrix (otherwise, it would have a row of zeros; we will prove in section 4 that any such matrix is singular). Hence, $(i) \implies (ii)$. (iii) follows immediately from (ii). (i) follows immediately from (iii) by Theorem 3.25. □

Corollary 3.29 (How to compute A^{-1}). If A is an invertible $n \times n$ matrix and if a sequence of elementary row operations reduces A to the identity, then that same sequence of operations when applied to I yields A^{-1}.

Proof. If $E_k \cdots E_1 A = I$, where each E_j is an elementary matrix, then multiplying both sides on the right by A^{-1} gives $E_k \cdots E_1 I = A^{-1}$. □

Corollary 3.29 gives an algorithm to compute A^{-1}: we simply row reduce the augmented matrix $[A|I]$. If A is row-equivalent to I, then $[A|I]$ is row-equivalent tp $[I|A^{-1}]$. Otherwise, A does not have an inverse.

Example 3.30. We find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ by performing Gauss-Jordan elimination on the augmented matrix

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right],$$

which gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$.

Exercise 3.12. Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix}$, if it exists.

Exercise 3.13. Let c be a constant. Find the inverse of each elementary matrix and check your answer by multiplying it by the original matrix.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$,

(d) $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$, $(c \neq 0)$

Theorem 3.31 (Properties of A^{-1}). Let A be an $n \times n$ matrix.

(a) If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$.

(b) If both A and B are invertible, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

(c) If A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

Property (b) generalizes to any finite product of matrices:

$$(A_1 A_2 \ldots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}.$$

Exercise 3.14. Prove Theorem 3.31.

3.2.3 Relation to linear systems

Theorem 3.32 (Solution sets of n linear equations in n unknowns). For an $n \times n$ matrix A, the following are equivalent.

(a) A is invertible.

(b) The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.

(c) The system of equations $AX = Y$ has a unique solution X for each $n \times 1$ matrix Y.

Proof. It is clear that $(a) \implies$ both (b) and (c) by multiplying on the left by A^{-1}. If (b) is true, then A is row-equivalent to the identity matrix, hence A is invertible by the previous theorem. This shows (a) and (b) are equivalent. Finally, suppose (c) holds. Let R be the RREF of A. We need to show that $R = I$. Let $Y = (0, \cdots, 1)$. If $RX = E$ can be solved for this Y, then the last row of R cannot be zero. Since R is a square matrix in RREF, we must have $R = I$. This shows (a) and (c) are equivalent. (b) and (c) are equivalent, since they are both equivalent to (a). $\square$

Note that Theorem 3.32 adds the following condition to the invertible matrix theorem:

Theorem 3.33 (Invertible matrix theorem). If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

(iv) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Example 3.34. Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17. \end{aligned}$$

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$, which has inverse $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$, so the unique solution is given by

$$X = A^{-1}Y = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

3.3 Some special matrices

3.3.1 Diagonal matrices

In this unit we will discuss *square* matrices that have various special forms. These matrices arise in a wide variety of applications and will also play an important role in subsequent sections.

An $n \times n$ (square) matrix whose entries A_{ij} all vanish when $i \neq j$ is called a *diagonal matrix*. Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \tag{3.2}$$

where each $d_j \in \mathbb{R}$.

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; that is, D is invertible if and only if $d_{ii} \neq 0$ for all $i = 1, \ldots, n$. Then

$$D^{-1} = \begin{bmatrix} d_1^{-1} & 0 & \cdots & 0 \\ 0 & d_2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^{-1} \end{bmatrix}. \tag{3.3}$$

The kth power of a diagonal matrix is

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}. \tag{3.4}$$

Exercise 3.15. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Compute A^{-1}, A^3, and A^{-3}.

Matrix products that involve diagonal factors are especially easy to compute. For example,

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

In words, to multiply a matrix A on the left by a diagonal matrix D, one can multiply successive rows of A by the successive diagonal entries of D, and to multiply A on the right by D, one can multiply successive columns of A by the successive diagonal entries of D.

Exercise 3.16. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 0 \\ 5 & 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 & -2 & 3 \\ 1 & 1 & -1 & 0 \\ 6 & 2 & 3 & -4 \end{bmatrix}$. Compute BA and AC.

3.3.2 Triangular matrices

A square matrix in which all the entries above the main diagonal are zero is called ***lower triangular***, and a square matrix in which all the entries below the main diagonal are zero is called ***upper triangular***. A matrix that is either upper triangular or lower triangular is called ***triangular***.

EXAMPLE 2 Upper and Lower Triangular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4 × 4 upper triangular matrix

A general 4 × 4 lower triangular matrix

In terms of the matrix elements a_{ij}:

- A square matrix is lower triangular if $a_{ij} = 0$ for $i < j$.
- A square matrix is upper triangular if $a_{ij} = 0$ for $i > j$.

Theorem 3.35 (Properties of triangular matrices).

(a) If A is lower (upper) triangular, then A^T is upper (lower) triangular.

(b) If $A_1, \cdots, A_k$ are lower (upper) triangular, then so is the product $A_1 \cdots A_k$.

(c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

(d) If A is an invertible lower (upper) triangular matrix, then so is A^{-1}.

Proof. (a) A lower triangular means $a_{ij} = 0$ for $i < j$. Since $(A^T)_{ij} = a_{ji}$, $(A^T)_{ij} = 0$ for $i > j$, hence A^T is upper triangular.

(b) Suppose A and B are lower triangular. Then

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{j-1} A_{ik} \underbrace{B_{kj}}_{=0\ \forall k} + \sum_{k=j}^{n} \underbrace{A_{ik}}_{=0\ \forall k} B_{kj} = 0.$$

The fact that $A_1A_2 \cdots A_k$ is lower (upper) triangular if $A_1, A_2, \cdots, A_k$ are follows by induction on k.

(c) If $d_j = 0$ for some j, then the jth column would not be a pivot column, so A would not be row-equivalent to I and hence would not be invertible.

(d) We will prove this later once we have a formula for A^{-1} for a general $n \times n$ invertible matrix. □

Exercise 3.17. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Compute A^{-1}, B^{-1}, AB, and BA.

3.3.3 Symmetric matrices

A square matrix A is **symmetric** if $A = A^T$. Some examples are

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

In terms of the matrix elements a_{ij}, a matrix is symmetric if $a_{ij} = a_{ji}$ for all i, j.

Theorem 3.36 (Properties of symmetric matrices). If A and B are symmetric matrices with the same size, and if k is any scalar, then

(a) A^T is symmetric.

(b) $A + B$ and $A - B$ are symmetric.

(c) kA is symmetric.

(d) If A and B are symmetric, then AB is symmetric if and only if A and B commute.

(e) If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Proof. (a) $(A^T)^T = A = A^T$ (since A is symmetric), hence A^T is symmetric.
(b) $(A+B)^T = A^T + B^T = A + B$.
(c) $(kA)^T = kA^T = kA$.
(d) $(AB)^T = B^T A^T = BA$. Then $BA = AB \iff A$ and B commute.
(e) We will prove this later once we have a formula for A^{-1} for a general invertible matrix. □

3.3.4 AA^T and A^TA

Matrix products of the form AA^T and A^TA arise in a variety of applications.

Theorem 3.37 (Properties of AA^T and A^TA). (a) The products AA^T and A^TA are both square matrices. They are the same size if and only if A is a square matrix.

(b) The products AA^T and A^TA are both symmetric.

(c) If A is invertible, then AA^T and A^TA are also invertible.

Proof. (a) If A is $m \times n$, A^T is $n \times m$. Therefore AA^T is $m \times m$ and A^TA is $n \times n$. They are the same size iff $n = m$, i.e., if A is a square matrix.

(b) $(AA^T)^T = (A^T)^T(A)^T = AA^T$. $(A^TA)^T = (A)^T(A^T)^T = A^TA$.

(c) $(AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^TA^{-1}$. $(A^TA)^{-1} = A^{-1}(A^T)^{-1} = A^{-1}(A^{-1})^T$.

□

Exercise 3.18. Let $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$. Show explicitly that A^TA and AA^T are symmetric matrices.

4 Determinants

4.1 Condition for invertibility

We have seen that not all $n \times n$ matrices have an inverse. We would now like to ask, given an arbitrary $n \times n$ matrix A, what is the set of conditions on the entries of A which are both necessary and sufficient for A to be invertible?

In the case of a 1×1 matrix

$$A = \begin{bmatrix} a \end{bmatrix},$$

we see immediately that A is invertible if and only if $a \neq 0$.

Consider now a general 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

If the first column of A is zero, then A is not invertible, so the first column of A must have at least one nonzero entry. Interchanging the two rows if necessary, without loss of generality we may assume that $a_{11} \neq 0$. If $a_{21} = 0$, then A is of the form

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

and is therefore invertible if and only if $a_{22} \neq 0$, in which case the second column will also be a pivot column.

If $a_{21} \neq 0$, then perform the following sequence of elementary row operations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow[R2 \mapsto a_{11} R2]{R1 \mapsto a_{21} R1} \begin{bmatrix} a_{11}a_{21} & a_{12}a_{21} \\ a_{11}a_{21} & a_{11}a_{22} \end{bmatrix} \xrightarrow{R2 \mapsto R2 - R1} \begin{bmatrix} a_{11}a_{21} & a_{12}a_{21} \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}.$$

Since $a_{11}a_{21} \neq 0$ by assumption, the first column is a pivot column. The second column is a pivot column if and only if the quantity $a_{11}a_{22} - a_{21}a_{12} \neq 0$. Note that the quantity $a_{11}a_{22} - a_{21}a_{12}$ is zero if any row or column of A is zero, so the *single* condition $a_{11}a_{22} - a_{21}a_{12} \neq 0$ is actually both necessary *and sufficient* for A to be invertible.

Let $|A| \equiv a_{11}a_{22} - a_{21}a_{12}$. Assuming $|A| \neq 0$, we can continue to sequence of elementary row operations above on the augmented matrix $[A|I]$ to obtain a general formula for A^{-1}:

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right] \xrightarrow[R2 \mapsto a_{11} R2]{R1 \mapsto a_{21} R1} \left[\begin{array}{cc|cc} a_{11}a_{21} & a_{12}a_{21} & a_{21} & 0 \\ a_{11}a_{21} & a_{11}a_{22} & 0 & a_{11} \end{array}\right] \xrightarrow{R2 \mapsto R2 - R1} \left[\begin{array}{cc|cc} a_{11}a_{21} & a_{12}a_{21} & a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} & -a_{21} & a_{11} \end{array}\right]$$

$$\xrightarrow{R2 \mapsto \frac{1}{|A|} R2} \left[\begin{array}{cc|cc} a_{11}a_{21} & a_{12}a_{21} & a_{21} & 0 \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] \xrightarrow{R1 \mapsto -a_{12}a_{21} R2 + R1} \left[\begin{array}{cc|cc} a_{11}a_{21} & 0 & a_{21} + \frac{a_{12}a_{21}^2}{|A|} & -\frac{a_{11}a_{12}a_{21}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right]$$

$$\xrightarrow{R1 \mapsto \frac{1}{a_{11}a_{21}} R1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}}{a_{11}a_{21}} + \frac{a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{11}a_{12}a_{21}}{a_{11}a_{21}|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}|A|}{a_{11}a_{21}|A|} + \frac{a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}(a_{11}a_{22} - a_{12}a_{21})}{a_{11}a_{21}|A|} + \frac{a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}(a_{11}a_{22} - a_{12}a_{21}) + a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}a_{11}a_{22} - a_{12}a_{21}^2 + a_{12}a_{21}^2}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{21}a_{11}a_{22}}{a_{11}a_{21}|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{a_{22}}{|A|} & -\frac{a_{12}}{|A|} \\ 0 & 1 & -\frac{a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{array}\right]$$

We have therefore proved the following:

Proposition 4.1 (Formula for inverse of a 2×2 matrix). A 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is invertible if and only if $|A| = a_{11}a_{22}a_{12}a_{21} \neq 0$, in which case A^{-1} is given by the formula

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Exercise 4.1. Consider a general 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

For A to be invertible, the first column must have a nonzero entry. By exchanging rows, if necessary, we may assume without loss of generality that $a_{11} \neq 0$.

(a) Perform a sequence of elementary row operations to put A into the row-equivalent form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}.$$

(b) For A to be invertible, the second column must be a pivot column. Without loss of generality, we may assume the $(2,2)$-entry is nonzero. Continue performing elementary row operations to show that A can be put in the row-equivalent form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}|A| \end{bmatrix}$$

where now $|A|$ is given by

$$|A| \equiv a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Conclude that the single condition $|A| \neq 0$ is both necessary and condition for A to be invertible.

We have just seen that for a 1×1, 2×2, or 3×3 matrix A, invertibility of A is determined by a single number, $|A|$. We call this number the *determinant* of the matrix A, which is also denoted $\det A$. [29]

Looking at the determinant formulas for each case above, we notice that these can be written in the following recursive fashion:

$$\begin{vmatrix} a_{11} \end{vmatrix} = a_{11} \tag{4.1}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{4.2}$$

$$= a_{11}|a_{22}| - a_{12}|a_{21}|, \tag{4.3}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{4.4}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \tag{4.5}$$

We can use this to recursively define the determinant of an $n \times n$ matrix.

[29] Note that the notation $|A|$ has nothing to do with absolute value.

Definition 4.2 (Determinant of an $n \times n$ matrix). Let A be an $n \times n$ matrix, and let A_{ij} denote the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of A. Define the (i, j)-*cofactor* of A to be the number $C_{ij} = (-1)^{i+j} \det A_{ij}$. [30] We then define the determinant of A to be the number

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}. \tag{4.6}$$

This formula is said to be a *cofactor expansion* along the first row of A.

It remains to be shown that an $n \times n$ matrix A is invertible if and only if $\det A \neq 0$ when $n > 3$. This will be done in Section 4.4. Therefore, we have added yet another equivalent condition to the Invertible Matrix Theorem:

Theorem 4.3 (Invertible matrix theorem). If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

(iv) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(v) $\det A \neq 0$.

Exercise 4.2. Show that the formula for the determinant of a 2×2 and 3×3 matrix are given by the formula in Equation (4.6).

Exercise 4.3. Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$.

(a) Compute the $(1, 1)$ and $(1, 2)$ minor determinants and cofactors of A.

(b) Compute the determinant of A.

Let us now consider again our $n \times n$ matrix A to be the coefficient matrix of a system of linear equations. The Invertible Matrix Theorem (4.3) tells us that the system has a unique solution if and only if A is invertible, that is, if and only if $\det A = 0$. The solution set of our system of course does not depend on the order in which we list the equations. However, the formula

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}$$

singles out the first row of our matrix A as special. This choice is immaterial, and only reflects one possible convention for the steps in the Gauss-Jordan elimination algorithm. Similarly, our ordering of the variables was also merely a choice, and could have been chosen differently (e.g., could have chosen the first column of the matrix to correspond to x_5 rather than x_1, etc.) Hence, we have the following theorem

[30] The determinant $\det A_{ij}$ is called the (i, j) *minor determinant* of A.

Theorem 4.4 (Cofactor expansion along any row or column). The determinant of an $n \times n$ matrix A can be computed by cofactor expansion along any row or column. The cofactor expansion across the ith row is given by

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

and the cofactor expansion down the jth column is given by

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

Exercise 4.4. Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

by cofactor expansion along

(a) The second row.

(b) The third row.

(c) The third column.

Exercise 4.5. Consider the sign $(-1)^{i+j}$ appearing in the (i, j)-cofactor C_{ij} as an $n \times n$ matrix. Work out the entries for $n = 2, 3, 4$. What do you notice? How can this help you work out the signs of the cofactors quickly when you are computing a determinant?

Exercise 4.6. Compute the determinant of the matrix

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$

Hint: Compute $\det A$ *by cofactor expansion along the first column. Why is this the best choice?*

Exercise 4.7. We have argued previously that an $n \times n$ matrix with a row or column of zeros is not invertible. What is the determinant of a matrix a row or column of zeros?

Exercise 4.8. Compute the determinant of each of the following matrices:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Theorem 4.5 (Determinant of a diagonal matrix). Let D be a diagonal $n \times n$ matrix and let d_{ii} denote the (i,i)-entry of A. Then $\det D = \prod_{i=1}^{n} d_{ii}$.

Proof. (By induction on n.) For a 2×2 diagonal matrix $D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}$, the formula in Equation (4.2) gives $\det D = d_{11}d_{22}$. Suppose now that the proposition is true for a $(k-1) \times (k-1)$ diagonal matrix and consider a $k \times k$ diagonal matrix

$$D = \begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{kk} \end{bmatrix}.$$

Computing $\det D$ by cofactor expansion along the first row, gives

$$\det D = d_{11} \begin{vmatrix} d_{22} & 0 & 0 & \cdots & 0 \\ 0 & d_{33} & 0 & \cdots & 0 \\ 0 & 0 & d_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{kk} \end{vmatrix}$$

. Since $\begin{bmatrix} d_{22} & 0 & 0 & \cdots & 0 \\ 0 & d_{33} & 0 & \cdots & 0 \\ 0 & 0 & d_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{kk} \end{bmatrix}$ is a $(k-1) \times (k-1)$ matrix, by the inductive hypothesis we have $\begin{vmatrix} d_{22} & 0 & 0 & \cdots & 0 \\ 0 & d_{33} & 0 & \cdots & 0 \\ 0 & 0 & d_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{kk} \end{vmatrix} = \prod_{i=2}^{k} d_{ii}$ and therefore $\det D = \prod_{i=1}^{k} d_{ii}$, as desired. ☐

Corollary 4.6 (Invertibility of a diagonal matrix). A diagonal matrix is invertible if and only if each element on the main diagonal is nonzero.

Exercise 4.9. Prove Corollary 4.6.

Exercise 4.10. Compute the determinant of each of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 2 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 2 & 4 & -5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Theorem 4.7 (Determinant of a triangular matrix). Let T be an $n \times n$ triangular matrix. Let d_{ii} denote the diagonal entries of T. Then $\det T = \prod_{i=1}^{n} d_{ii}$.

Exercise 4.11. Prove Theorem 4.7 by induction. *Hint:* Copy the steps in the proof of Theorem 4.5.

Corollary 4.8 (Invertibility of a triangular matrix). A triangular matrix is invertible if and only if each element on the main diagonal is nonzero.

Exercise 4.12. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 9 & -4 \end{bmatrix}$. Compute $\det A$ and $\det A^T$. What do you notice?

Theorem 4.9 (Determinant of a Transpose). For any $n \times n$ matrix, $\det A^T = \det A$.

Proof. We prove the theorem by induction on n. The theorem holds trivially for $n = 1$ and is easily verified for $n = 2$. Suppose now that the theorem holds for $n = k$. Let A be a $(k+1) \times (k+1)$ matrix. Computing the determinant of A^T along the first row gives

$$\det A^T = \sum_{i=1}^{n} (a^T)_{1i} C_{1i}$$

where $C_{1i} = (-1)^{1+i} \det A^T_{1i}$. Noting that $A^T_{1i} = (A_{i1})^T$, [31] we therefore have

$$\det A^T = \sum_{i=1}^{n} (a^T)_{1i} (-1)^{1+i} \det(A_{i1})^T$$

Since $(A_{i1})^T$ is a $k \times k$ matrix, by the inductive hypothesis we have $\det(A_{i1})^T = \det A_{i1}$. Since we also have $(a^T)_{1i} = a_{i1}$, we find

$$\begin{aligned} \det A^T &= \sum_{i=1}^{n} (a^T)_{1i} C_{1i} \\ &= \sum_{i=1}^{n} (a^T)_{1i} (-1)^{1+i} \det A^T_{1i} \\ &= \sum_{i=1}^{n} (a^T)_{1i} (-1)^{1+i} \det(A_{i1})^T \\ &= \sum_{i=1}^{n} a_{i1} (-1)^{i+1} \det A_{i1} \end{aligned}$$

which is the cofactor expansion of $\det A$ along the first *column*. Since the determinant of A can be computed by cofactor expansion along any row or column, we have shown that $\det A^T = \det A$, as desired. $\square$

[31] This can be seen by thinking through the definitions of each side. To help with this, consider a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$. Taking $i = 3$, we have $(A^T)_{31} = \begin{bmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{bmatrix}$. On the other hand, $A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$, which is the transpose of $(A^T)_{31}$, so indeed $(A^T)_{31} = (A_{31})^T$.

4.2 Row Operations and Determinants

In the previous section, we saw that the determinant of an $n \times n$ matrix A can be computed by cofactor expansion by the formula

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

across any row or column. By choosing a row or column with many zero entries, the computation of the determinant is greatly simplified. However, for a matrix which has no zero entries, the computation quickly becomes very tedious, as cofactor expansion requires $\mathcal{O}(n!)$ operations. To see how quickly the complexity of the computation grows, for $n = 25$, $n! \cong 1.5 \times 10^{25}$. A computer performing 10^{12} operations per second would take 500,000 years to compute the determinant of a 25×25 matrix by this method! By today's standards, a 25×25 matrix is *very small*, so we clearly need a more practical way to compute determinants.

It was proved in Theorem 4.7 that the determinant of a triangular $n \times n$ matrix T requires only n multiplications, since the determinant is simply the product of the diagonal entries of T. Since any row-echelon form, U, of an $n \times n$ matrix A is an upper triangular matrix, a strategy to compute the determinant of A efficiently would be to row-reduce A to U, and then compute the determinant of U. To do so, we will need to know how $\det U$ is related to $\det A$.

Exercise 4.13. Consider a general 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Let B be a matrix row-equivalent to A by the following elementary row operations. In each case, compute $\det B$ and compare it to $\det A$.

(a) $R1 \mapsto cR_1, c \neq 0$.

(b) $R1 \leftrightarrow R2$.

(c) $R1 \mapsto R1 + cR2$.

Theorem 4.10 (Elementary row operations and determinants). Let A be an $n \times n$ matrix. Then

(a) Row replacements do not change $\det A$.

(b) Each interchange changes the sign of $\det A$.

(c) Multiplying a row by c multiplies $\det A$ by c.

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	The first row of A is multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	The first and second rows of A are interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	A multiple of the second row of A is added to the first row.

To prove this theorem, it will be useful to rephrase it in terms of elementary matrices, as follows.

Theorem 4.11 (Elementary matrices and determinants). If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det EA = (\det E)(\det A)$$

where

$$\det E = \begin{cases} 1 \text{ if } E \text{ is a row-replacement,} \\ -1 \text{ if } E \text{ is an interchange,} \\ c \text{ if } E \text{ is scaling by } c. \end{cases}$$

Proof. (By induction on n.) Using the list of 2×2 elementary matrices from Exercise 3.8, we check that the determinant of each is indeed equal to $1, -1$, or c, depending on the elementary row-operation. Taking a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we now check that the theorem holds in each case. For instance, if $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, then we have

$$\begin{aligned} \det EA &= \begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} \\ &= (a+kc)d - c(b+kd) \\ &= ad + kcd - cb - ckd \\ &= ad - bc \\ &= \det A. \end{aligned}$$

This establishes the base case. Now suppose the theorem holds for $n = k$ and let A be a $k \times k$ matrix. Expand $\det EA$ along a row which is unaffected by E, say row i:

$$detEA = \sum_{j=1}^{k+1} a_{ij}(-1)^{i+j} \det(EA)_{ij}.$$

Since the matrix $(EA)_{ij}$ is $k \times k$, by the inductive hypothesis $\det(EA)_{ij} = \alpha \det A_{ij}$, where $\alpha = 1, -1$, or c, depending on E. Thus

$$\begin{aligned}
\det EA &= \sum_{j=1}^{k+1} a_{ij}(-1)^{i+j} \det(EA)_{ij} \\
&= \sum_{j=1}^{k+1} a_{ij}(-1)^{i+j}(\alpha \det A_{ij}) \\
&= \alpha \sum_{j=1}^{k+1} a_{ij}(-1)^{i+j} \det A_{ij} \\
&= \alpha \det A,
\end{aligned}$$

proving the theorem. In particular, taking $A = I$ to be the $n \times n$ identity matrix, we see that $\det E = 1, -1$, or c, depending on E. □

Suppose now that an $n \times n$ matrix A has been reduced to a row-echelon form U by r interchanges and any number of row-replacements. [32] Since U is triangular, $\det U = \prod_{i=1}^{n} u_{ii}$ is the product of its diagonal entries u_{ii}. If A is invertible, then each of these is nonzero. Otherwise, at lease one of the u_{ii} is zero. Since we can write $U = PA$, where P is a product of elementary matrices corresponding to the elementary row operations taking A to U, by Theorem 4.11 we have

Corollary 4.12 (Determinant of a matrix by row operations). Let A be an $n \times n$ matrix which has been reduced to a row-echelon form U by r interchanges and any number of row-replacements. Then

$$\det A = \begin{cases} (-1)^r \prod_{i=1}^{n} u_{ii} & \text{(if } A \text{ is invertible),} \\ 0 & \text{(if } A \text{ is not invertible),} \end{cases}$$

where $\prod_{i=1}^{n} u_{ii} = \det U$.

[Need to prove that, while U is not unique, $\det U$ is. Or is this obvious?]

Corollary 4.13 (Invertibility condition). An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

Most computers use the method of 4.12 to compute $\det A$: when A is $n \times n$, it can be shown that the computation of $\det A$ by row operations requires only $\mathcal{O}(n^3)$ operations. For $n = 25$, this is only about 15,000 operations, which any modern computer can carry out in a fraction of a second.

Exercise 4.14. Use row-reduction to compute the determinant of

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}.$$

Proposition 4.14 (Matrices with proportional rows or columns). If an $n \times n$ matrix A has two rows or columns that are scalar multiples of each other, then $\det A = 0$.

[32]Note that we do not need to scale any of the rows, since we only need to reduce U to any REF, and not RREF.

Proof. If A has two proportional rows, then by a row replacement it has a row of zeros and therefore $\det A = 0$. If A has two columns which are proportional, then A^T has two rows which are proportional. By the same comments above applied to A^T, $\det A^T = 0$. Since $\det A^T = \det A$ (by Theorem 4.9), $\det A = 0$, completing the proof. □

4.3 Properties of Determinants

Exercise 4.15. Let $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Compute $\det A, \det B, \det AB, \det(A+B)$. What do you notice?

Theorem 4.15 (Determinant of a product). If A and B are matrices, then $\det AB = (\det A)(\det B)$.

Proof. If either A or B is not invertible, then neither is AB. The theorem is true in this case since then $\det AB = (\det A)(\det B) = 0$. Suppose now that both A and B are invertible. Since A is invertible, then by the Invertible Matrix Theorem A is row-equivalent to I, so we can write $A = E_k E_{k-1} \cdots E_2 E_1 I = E_{k-1} \cdots E_2 E_1$, where each E_i is an elementary matrix. By repeated use of Theorem 4.11, we see that

$$\begin{aligned}
\det AB &= \det(E_k E_{k-1} \cdots E_2 E_1 B) \\
&= \det E_k \det(E_{k-1} \cdots E_2 E_1 B) \\
&= \det E_k \det E_{k-1} \cdots \det E_2 \det E_1 \det B \\
&= \det E_k \det E_{k-1} \cdots \det E_3 \det(E_2 E_1) \det B \\
&= \det(E_k E_{k-1} \cdots E_2 E_1) \det B \\
&= \det A \det B.
\end{aligned}$$

□

Exercise 4.16. Prove by induction that Theorem 4.15 holds for arbitrary products of $n \times n$ matrices. That is, show that if $A_1, \ldots, A_k$ are $n \times n$ matrices, then

$$\det(\prod_{i=1}^{k} A_i) = \prod_{i=1}^{k} \det A_i.$$

Exercise 4.17. Let A and B be $n \times n$ matrices. We have seen that, in general, $AB \neq BA$. Show that, despite this, it is *always* true that $\det(AB) = \det(BA)$.

Exercise 4.18. Let U be a square matrix such that $U^T U = I$. Show that the possible values of $\det U$ are ± 1.

Exercise 4.19. Let A and P be $n \times n$ matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$.

Theorem 4.16 (Determinant of a scalar multiple). If A is an $n \times n$ matrix and c a scalar, then

$$\det(cA) = c^n \det A.$$

Proof. (By induction on n.) Let A be a 2×2 matrix. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\begin{aligned} |cA| &= \begin{vmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{vmatrix} \\ &= (ca_{11})(ca_{22}) - (ca_{12})(ca_{21}) \\ &= c^2(a_{11}a_{22} - a_{12}a_{21}) \\ &= c^2 \det A. \end{aligned}$$

Suppose now that the theorem holds for $n = k - 1$. Let A be a $k \times k$ matrix. Then

$$\det cA = \sum_{j=1}^{n} (ca_{ij})(-1)^{i+j} \det(cA)_{ij}.$$

Since $(cA)_{ij}$ is $(k-1) \times (k-1)$, by the inductive hypothesis $\det(cA)_{ij} = c^{k-1} \det A_{ij}$, and therefore

$$\begin{aligned} \det cA &= \sum_{j=1}^{n} (ca_{ij})(-1)^{i+j} \det(cA)_{ij} \\ &= c \sum_{j=1}^{n} a_{ij}(-1)^{i+j} c^{k-1} \det A_{ij} \\ &= cc^{k-1} \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij} \\ &= c^k \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij} \\ &= c^k \det A, \end{aligned}$$

as desired. □

Theorem 4.17 (Determinant of an Inverse). If A is an invertible $n \times n$ matrix, then

$$\det A^{-1} = \frac{1}{\det A}.$$

Proof. If A is invertible, then $A^{-1}A = I$. Taking the determinant of both sides gives

$$\begin{aligned} 1 = \det I &= \det(A^{-1}A) \\ &= \det A^{-1} \det A. \end{aligned}$$

Since A is invertible, $\det A \neq 0$, so we can divide both sides by $\det A$ to obtain

$$\det A^{-1} = \frac{1}{\det A}.$$

□

4.4 Cramer's Rule

Let's take a closer look at the formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, which is given by the formula

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{4.7}$$

Using the definition of the (i, j)-cofactor $C_{ij} = (-1)^{i+j} \det A_{ij}$ and arranging these as a 2×2 matrix, we find

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}.$$

Therefore, we see that we can identify the matrix in Equation (4.7) as the transpose of the matrix of cofactors of A.

Definition 4.18 (Adjugate matrix). Let A be an $n \times n$ matrix. The transpose of the matrix of cofactors of a A is called the *adjugate* of A, denoted adj A. [33]

The formula for A^{-1} can therefore be written as

$$A^{-1} = \frac{1}{\det A} \text{adj } A.$$

Lemma 4.19. Let A be an $n \times n$ matrix. Then

$$\text{adj } A^T = (\text{adj } A)^T.$$

Proof. Since $A^T_{ij} = (A_{ji})^T$, we have

$$\begin{aligned} (-1)^{i+j} \det A^T_{ij} &= (-1)^{j+i} \det(A_{ji})^T \\ &= (-1)^{j+i} \det A_{ji}, \end{aligned}$$

which says that the (i, j)-cofactor of A^T is the (j, i)-cofactor of A. The former is the (j, i)-entry of adj (A^T), while the latter is the (i, j)-entry of adj (A), or, the (j, i)-entry of $[\text{adj } (A)]^T$. Since this is true for all $i, j = 1, \ldots, n$, adj $(A^T) = [\text{adj } (A)]^T$. □

Theorem 4.20 (Formula for A^{-1}). If A is an invertible $n \times n$ matrix, then $A^{-1} = (\det A)^{-1} \text{adj } A$.

Proof. We need to show that $(\det A)^{-1}\text{adj } A$ is a two-sided inverse for A; that is, we need to show that $\frac{\text{adj } A}{\det A} A = A \frac{\text{adj } A}{\det A} = I$. Multiplying through by $\det A$, this becomes $A(\text{adj } A) = (\text{adj } A)A = (\det A)I$, which is what we will now prove. First, recall the cofactor expansion for $\det A$ along the jth column:

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det A_{ij}.$$

[33] The matrix adj A is also sometimes called the *classical adjoint* of A. We will see another adjoint in Section 8.1, which has nothing to do with adj A. We reserve the term "adjoint" for the latter.

We now claim that if $j \neq k$, then the expression

$$\sum_{i=1}^{n} a_{ik} C_{ij} = 0.$$

To see this, let B be the matrix obtained from A by replacing the jth column of A by the kth column of A. Since B has two equal columns, $\det B = 0$. Since $B_{ij} = A_{ij}$, computing $\det B$ by cofactor expansion along the jth column gives

$$\begin{aligned} 0 = \det B &= \sum_{i=1}^{n} b_{ij}(-1)^{i+j} \det B_{ij} \\ &= \sum_{i=1}^{n} a_{ik}(-1)^{i+j} \det A_{ij} \\ &= \sum_{i=1}^{n} a_{ik} C_{ij}, \end{aligned}$$

as claimed. These two properties of cofactors can be summarized as

$$\sum_{i=1}^{n} a_{ik} C_{ij} = \delta_{kj}(\det A). \tag{4.8}$$

Since left hand side of this equation is equal to $\sum_{i=1}^{n} a_{ik} C_{ji}^{T} = \sum_{i=1}^{n} C_{ji}^{T} a_{ik}$, we see that Equation (4.8) is the kj-component of the matrix equation

$$(\text{adj } A)A = (\det A)I. \tag{4.9}$$

Applying Equation (4.9) to A^T gives

$$\begin{aligned} (\text{adj } A^T)A^T &= [(\det A^T)I] \\ &= [(\det A)I]^T. \end{aligned}$$

Taking the transpose of both sides then gives

$$A(\text{adj } A^T)^T = (\det A)I.$$

Applying Lemma 4.19, this becomes Taking the transpose of both sides then gives

$$A(\text{adj } A) = (\det A)I,$$

completing the proof. □

Exercise 4.20. Use the formula $A^{-1} = (\det A)^{-1}\text{adj } A$ to compute the inverse of each of the following matrices:

(a) $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$,

(b) $B = \begin{bmatrix} 5 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & -1 \end{bmatrix}$.

Theorem 4.21 (Cramer's Rule). Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of the linear system $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \ldots, n, \tag{4.10}$$

where $A_i(\mathbf{b})$ is the matrix obtained from A by replacing the ith column of A by $\mathbf{b}$. The formula in Equation (4.10) for computing the entries of $\mathbf{x}$ is called *Cramer's rule*.

Proof. By the Invertible Matrix Theorem, if A is invertible then the linear system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b} = (\det A)^{-1}(\operatorname{adj} A)\mathbf{b}$, where we have applied the formula for A^{-1} from Theorem 4.20. The jth component of the vector $\mathbf{x}$ is then given by

$$\begin{aligned} x_j &= (\det A)^{-1} \sum_{i=1}^{n} C^T_{ji} b_i \\ &= (\det A)^{-1} \sum_{i=1}^{n} C_{ij} b_i \\ &= (\det A)^{-1} \sum_{i=1}^{n} b_i (-1)^{i+j} \det A_{ij} \\ &= (\det A)^{-1} \det A_j(\mathbf{b}). \end{aligned}$$

□

Exercise 4.21. Use Cramer's rule to solve each of the following systems:

(a)

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8. \end{aligned}$$

(b)

$$\begin{aligned} -5x_1 + 2x_2 &= 9, \\ 3x_1 - x_2 &= -4. \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 &= 3, \\ -3x_1 + 2x_3 &= 0, \\ x_2 - 3x_3 &= 2. \end{aligned}$$

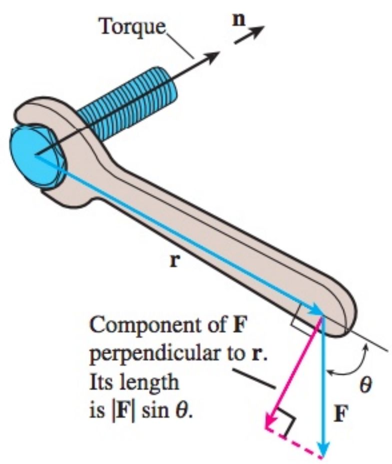
The formula $A^{-1} = (\det A)^{-1} \operatorname{adj} A$ is mainly useful for theoretical purposes, as it allows one to deduce properties of A^{-1} for a general invertible matrix A. To compute A^{-1} for a concrete matrix A, the method of row-reducing $[A|I]$ is much more efficient. Similarly, Cramer's rule is very inefficient for solving linear systems, as computing just *one* determinant takes about as much work as solving $A\mathbf{x} = \mathbf{b}$ by row-reduction. Cramer's rule is also mostly useful as a theoretical tool. For instance, it can be used to study how sensitive the solution of $A\mathbf{x} = \mathbf{b}$ is to changes in an entry in $\mathbf{b}$ or in A.

4.5 The Cross Product

4.5.1 Motivation and Definition

We will now change gears and define a new type of product between two vectors in three-dimensional space, motivated by physics. We will find that this product is related to determinants.

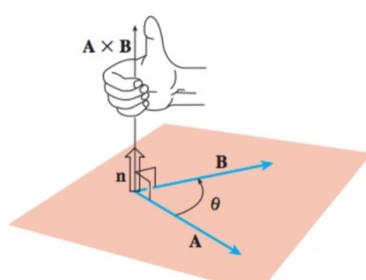
Suppose we wish to tighten a bolt using a wrench. Applying a force **F** to the handle of the wrench produces a torque, $\vec{\tau}$, which acts along the axis of the bolt to drive the bolt forward. One observes that $\vec{\tau} = ||\mathbf{r}||\ ||\mathbf{F}|| \sin\theta \hat{\mathbf{n}}$, where **r** a vector stretching from the bolt to the point on the handle where the force is applied, **F** is the applied force vector, θ is the angle between the vectors **r** and **F**, and $\hat{\mathbf{n}}$ is a unit vector pointing along the axis of the bolt. These are shown in the figure below:



Definition 4.22 (Cross product). The *cross product* of two nonzero vectors **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}|| \sin\theta \hat{\mathbf{n}}$$

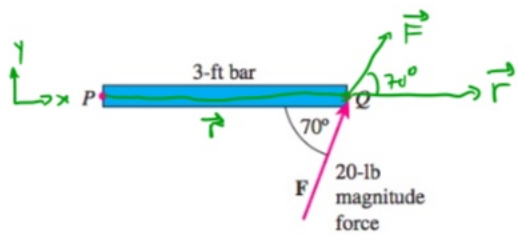
where θ is the angle between **a** and **b** and $\hat{\mathbf{n}}$ is a unit vector in the direction determined by the *right hand rule*, as illustrated in the figure below:



If either **a** or **b** is the zero vector, then we define $\mathbf{a} \times \mathbf{b}$ to be $\mathbf{0}$.

Example 4.23. A 3 ft metal bar is hinged at one of its ends. If a 20 lb force is applied at the other endpoint at an angle of 70° to the bar, then, choosing xy-coorindates centered at the hinge, the resulting torque is

$$||\vec{\tau}|| = (3)(2)\sin(70°)\hat{\mathbf{k}}.$$



4.5.2 Basic Properties of the Cross Product

Proposition 4.24 (Geometry of the cross product). Let $\mathbf{a}$ and $\mathbf{b}$ be two nonzero vectors. Then

(a) $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

(b) $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Proof. (a) By the right hand rule, $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. In particular, it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

(b) Since $\mathbf{a}$ and $\mathbf{b}$ are nonzero, $||\mathbf{a}|| \neq 0$ and $||\mathbf{b}|| \neq 0$, and therefore $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta = 0$ if and only if $\sin\theta = 0$. Since $0 \leq \theta \leq \pi$, $\sin\theta = 0$ if and only if $\theta = 0$ or π, that is, when the vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel.

□

Remark 4.25 (The cross product is neither commutative nor associative).

(a) If $\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta\hat{\mathbf{n}}$, then $\mathbf{b} \times \mathbf{a} = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta(-\hat{\mathbf{n}}) = -\mathbf{a} \times \mathbf{b}$, so the cross product is not commutative.

(b) Consider $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. By part (a) of Proposition 4.24, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. Since $\mathbf{b} \times \mathbf{c}$ is, in turn, orthogonal to both $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is therefore perpendicular to a perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$, and his hence parallel to this plane. [Insert figure.] Repeating the same argument for $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, we find it is parallel to the plane containing $\mathbf{a}$ and $\mathbf{b}$. In general, $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ will not lie in the same plane, so the cross products $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ will not lie in the same plane, and hence cannot be equal. This shows that the cross product is not associative.

We will now see that vector and scalar distributive laws *do* hold for the cross product.

Proposition 4.26 (Scalar distributive law). Scalar multiplication distributes over the cross product. That is if $\mathbf{a}, \mathbf{b}$ are vectors and r, s scalars, then

$$(r\mathbf{a}) \times (s\mathbf{b}) = (rs)(\mathbf{a} \times \mathbf{b}).$$

Proof. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$, and let φ be the angle between $r\mathbf{a}$ and $s\mathbf{b}$. Also, let $\hat{\mathbf{n}}$ be the unit vector determined by the right hand rule from $\mathbf{a}$ and $\mathbf{b}$, and $\hat{\mathbf{m}}$ be the unit vector determined by the right hand rule from $r\mathbf{a}$ and $s\mathbf{b}$. Then

$$\begin{aligned}(r\mathbf{a}) \times (s\mathbf{b}) &= ||s\mathbf{b}||\ ||s\mathbf{b}|| \sin\varphi\ \hat{\mathbf{m}} \\ &= |r|\ |s|\ ||\mathbf{a}||\ ||\mathbf{b}|| \sin\varphi\ \hat{\mathbf{m}} \\ &= |rs|\ ||\mathbf{a}||\ ||\mathbf{b}|| \sin\varphi\ \hat{\mathbf{m}}\end{aligned}$$

We now have several cases to check, depending on r and s.

(1) If either $r = 0$ or $s = 0$, then both sides are $\mathbf{0}$, so equality holds.

(2) If $r > 0$ and $s > 0$, then $\varphi = \theta$, $\hat{\mathbf{m}} = \hat{\mathbf{n}}$, and $|rs| = rs$, so equality holds.

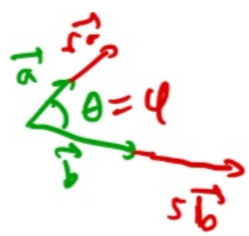


Figure 7: Case (2).

(3) If either $r < 0$ and $s > 0$ or $r > 0$ and $s < 0$, then $\varphi = \pi - \theta$, $\hat{\mathbf{m}} = -\hat{\mathbf{n}}$, and $|rs| = -rs$, and therefore

$$\begin{aligned}(r\mathbf{a}) \times (s\mathbf{b}) &= -rs||\mathbf{a}||\ ||\mathbf{b}|| \sin(\pi - \theta)\ (-\hat{\mathbf{n}}) \\ &= rs||\mathbf{a}||\ ||\mathbf{b}|| \sin\theta \hat{\mathbf{n}} \\ &= rs(\mathbf{a} \times \mathbf{b})\end{aligned}$$

where we have used the fact that $\sin(\pi - \theta) = \sin\theta$.

Figure 8: Case (3).

(4) If $r < 0$ and $s < 0$, then $\varphi = \theta$, $\hat{\mathbf{m}} = \hat{\mathbf{n}}$, and $|rs| = rs$, and therefore

$$\begin{aligned}(r\mathbf{a}) \times (s\mathbf{b}) &= rs||\mathbf{a}||\ ||\mathbf{b}|| \sin\theta\ \hat{\mathbf{n}} \\ &= rs(\mathbf{a} \times \mathbf{b})\end{aligned}$$

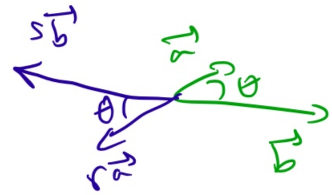


Figure 9: Case (4).

This exhausts the possibilities for r and s, completing the proof. □

Proposition 4.27 (Vector distributive law). The cross product distributes over vector addition. That is,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

Proof. Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ be vectors in three-dimensional space. If any of these are $\mathbf{0}$, then equality holds, so assume now that none of these are $\mathbf{0}$. We will prove the vector distributive law by constructing $\mathbf{a} \times \mathbf{b}$ in a clever way, as illustrated in the diagram below:

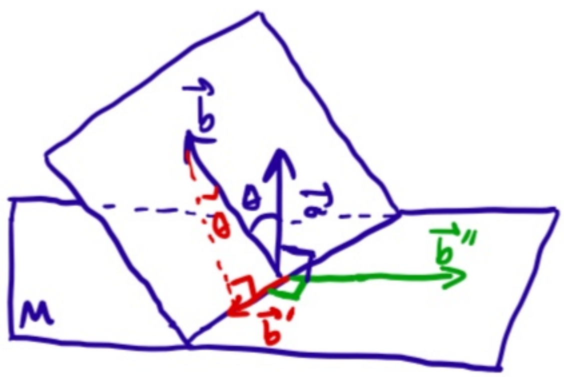


Figure 10: Steps to show that $\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||\mathbf{b}''$.

(1) Let M be the plane perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$, and let $\mathbf{b}'$ be the projection of $\mathbf{b}$ onto M. Letting θ denote the angle between $\mathbf{a}$ and $\mathbf{b}$, as usual, we have $||\mathbf{b}'|| = ||\mathbf{b}|| \sin\theta$.

(2) Now rotate $\mathbf{b}'$ by $90°$ counterclockwise about $\mathbf{a}$ to obtain $\mathbf{b}''$. Scalar multiplying $\mathbf{b}''$ by $||\mathbf{a}||$, we obtain the vector $||\mathbf{a}||\mathbf{b}''$. Noting that

$$||\,(||\mathbf{a}||\mathbf{b}'')\,|| = ||\mathbf{a}||\,||\mathbf{b}''|| = ||\mathbf{a}||\,||\mathbf{b}'|| = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta$$

and that $||\mathbf{a}||\mathbf{b}''$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ and points in the direction given by the right hand rule, we see that

$$||\mathbf{a}||\mathbf{b}'' = \mathbf{a} \times \mathbf{b} \tag{4.11}$$

as these two vectors have the same magnitude and direction.

(3) Repeating the exact same steps with $\mathbf{c}$, we obtain

$$||\mathbf{a}||\mathbf{c}'' = \mathbf{a} \times \mathbf{c}. \tag{4.12}$$

(4) Consider now a triangle with legs $\mathbf{b}, \mathbf{c}$, and $\mathbf{b}+\mathbf{c}$.

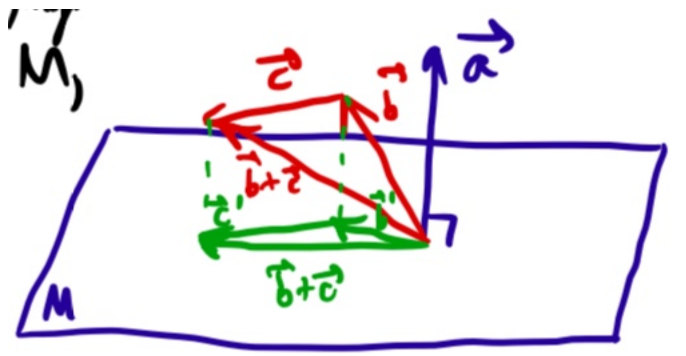


Figure 11: Triangle from step (4), when $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ do not all lie in the same plane. [Crop out writing in the corner of graphic.]

If the plane of the triangle does not contain $\mathbf{a}$, then after projecting this triangle onto M, rotating by 90° counterclockwise about $\mathbf{a}$, and multiplying each leg by $||\mathbf{a}||$, we obtain a triangle in the plane M with legs $\mathbf{b}'', \mathbf{c}''$, and $(\mathbf{b}+\mathbf{c})''$, which satisfy

$$||\mathbf{a}||\mathbf{b}'' + ||\mathbf{a}||\mathbf{c}'' = ||\mathbf{a}||(\mathbf{b}+\mathbf{c})''. \tag{4.13}$$

Substituting (4.11) and (4.12) into (4.13) then gives

$$\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{a} \times (\mathbf{b}+\mathbf{c}),$$

as desired.

(5) Finally, if the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ all lie in the same plane, then the triangle with legs $\mathbf{b}, \mathbf{c}$, and $\mathbf{b}+\mathbf{c}$ when projected onto M projects to a line segment.

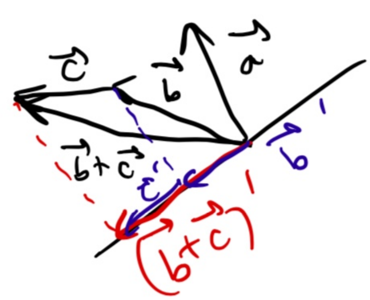


Figure 12: Triangle from step (5), when $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ all lie in the same plane.

It then follows from the segment addition postulate that $(\mathbf{b}+\mathbf{c})' = \mathbf{b}' + \mathbf{c}'$. The rest of the proof then follows the same steps above, so we have now shown that equality holds for all vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$.

□

4.5.3 Cross Product in Coordinates

As we have seen in Section 1, the study of the properties of vectors in three-dimensional space is greatly facilitated by choosing a Cartesian coordinate system. To write the cross product of two vectors

$$\begin{aligned}\mathbf{a} &= a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}},\\ \mathbf{b} &= b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}},\end{aligned}$$

in terms of their components, we begin by working out the cross products of the standard unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. We note immediately that $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$ by part (b) of Proposition 4.24. Since the distinct vectors are all mutually orthogonal, by the right hand rule we find

$$\begin{aligned}\hat{\mathbf{i}} \times \hat{\mathbf{j}} &= \hat{\mathbf{k}},\\ \hat{\mathbf{j}} \times \hat{\mathbf{k}} &= \hat{\mathbf{i}},\\ \hat{\mathbf{k}} \times \hat{\mathbf{i}} &= \hat{\mathbf{j}}\end{aligned}$$

while

$$\begin{aligned}\hat{\mathbf{j}} \times \hat{\mathbf{i}} &= -\hat{\mathbf{k}},\\ \hat{\mathbf{k}} \times \hat{\mathbf{j}} &= -\hat{\mathbf{i}},\\ \hat{\mathbf{i}} \times \hat{\mathbf{k}} &= -\hat{\mathbf{j}}.\end{aligned}$$

[Insert wheel graphic.]

We can now use the scalar and vector distributive laws from Propositions 4.26 and 4.27, as well as the cross products of the standard unit vectors, to work out a formula for the components of $\mathbf{a} \times \mathbf{b}$. Let

$$\begin{aligned}\mathbf{a} &= a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}},\\ \mathbf{b} &= b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}},\end{aligned}$$

be any two vectors. Then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \times (b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}})\\ &= a_1b_1\underbrace{(\hat{\mathbf{i}} \times \hat{\mathbf{i}})}_{=\mathbf{0}} + a_1b_2\underbrace{(\hat{\mathbf{i}} \times \hat{\mathbf{j}})}_{=\hat{\mathbf{k}}} + a_1b_3\underbrace{(\hat{\mathbf{i}} \times \hat{\mathbf{k}})}_{=-\hat{\mathbf{j}}}\\ &+ a_2b_1\underbrace{(\hat{\mathbf{j}} \times \hat{\mathbf{i}})}_{=-\hat{\mathbf{k}}} + a_2b_2\underbrace{(\hat{\mathbf{j}} \times \hat{\mathbf{j}})}_{=\mathbf{0}} + a_2b_3\underbrace{(\hat{\mathbf{j}} \times \hat{\mathbf{k}})}_{=\hat{\mathbf{i}}}\\ &+ a_3b_1\underbrace{(\hat{\mathbf{k}} \times \hat{\mathbf{i}})}_{=\hat{\mathbf{j}}} + a_3b_2\underbrace{(\hat{\mathbf{k}} \times \hat{\mathbf{j}})}_{=-\hat{\mathbf{i}}} + a_3b_3\underbrace{(\hat{\mathbf{k}} \times \hat{\mathbf{k}})}_{=\mathbf{0}}\end{aligned}$$

and collecting like terms gives

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}. \tag{4.14}$$

Note that the formula in Equation (4.14) is exactly what one would obtain by *formally* taking the determinant of the 'matrix'

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where the 'matrix' above has the standard unit vectors as entries in the first row, rather than numbers. [34]

Proposition 4.28 (Determinant formula for the cross product). The cross product of two vectors

$$\begin{aligned} \mathbf{a} &= a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}, \\ \mathbf{b} &= b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}, \end{aligned}$$

has components given by the following formal determinant:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}. \end{aligned}$$

Exercise 4.22. (a) Use the determinant formula in Proposition 4.28 to find the cross product $\mathbf{a} \times \mathbf{b}$ when $\mathbf{a} = (1, 2, -2)$ and $\mathbf{b} = (3, 0, 1)$.

(b) Compute $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ if $\mathbf{a} = (2, 1, 1)$ and $\mathbf{b} = (-4, 3, 1)$.

Theorem 4.29 (Properties of the cross product). If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are any vectors in three-dimensional space and k is any scalar, then:

(a) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$,

(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c})$,

(c) $(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$

(d) $\mathbf{a} \times \mathbf{a}$

Exercise 4.23. We have proved Theorem 4.29 already. Reprove it using the determinant formula in Proposition 4.28.

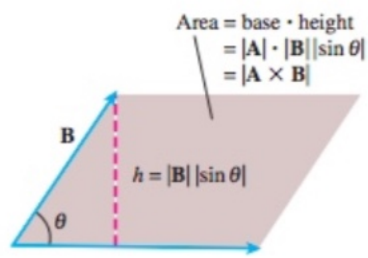
[34]The use of the word 'formally' here reflects the fact that the determinant is a function which is really only defined for matrices whose entries are numbers, and not a matrix whose entries are a mix of numbers and vectors, as we have not actually defined such a beast.

4.5.4 Applications to Geometry

Note that, geometrically, the magnitude of the cross product of two nonzero vectors **a** and **b**

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}||\sin\theta$$

is the area of the parallelogram whose adjacent sides are formed by **a** and **b**:



Example 4.30. To find the area of the triangle with vertices $A(2,0), B(3,4)$, and $C(-1,2)$, we first form the vectors $\overrightarrow{AB} = (3-2, 4-0) = (1,4)$ and $\mathbf{AC} = (-1-2, 2-0) = (-3,2)$. Since the area of the triangle whose legs are $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AB} - \overrightarrow{AC}$ is half the area of the parallelogram whose adjacent sides are formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$, we compute the area by taking half the magnitude of the cross product of these two vectors:

$$\begin{aligned}
\text{Area} &= \frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| \\
&= \frac{1}{2}\left|\left|\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 4 & 0 \\ -3 & 2 & 0 \end{vmatrix}\right|\right| \\
&= \frac{1}{2}||(2+12)\hat{\mathbf{k}}|| \\
&= \frac{1}{2}|14|\;||\hat{\mathbf{k}}|| \\
&= 7\ (\text{units})^2.
\end{aligned}$$

Exercise 4.24. Find the area of the triangle with vertices $A(1,1), B(2,2), C(3,-3)$. *Hint: To compute the cross product, write each of these vectors as a vector in $\mathbb{R}^3$, whose z-component is 0.*

Exercise 4.25. Find a vector perpendicular to the plane defined by the points $P(1,-1,0), Q(2,1,-1)$, and $R(-1,1,2)$. Find a unit normal to this plane.

Exercise 4.26. Find a vector parallel to the line of intersection of the planes

$$\begin{aligned} 3x - 6y - 2z &= 15, \\ 2x + y - 2z &= 5. \end{aligned}$$

4.6 Triple Scalar Product

Since $\mathbf{a} \times \mathbf{b}$ is a vector, the product $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ is defined. This product actually comes up frequently, so we will discuss its properties in this section.

Definition 4.31 (Triple scalar product). For any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ the *triple scalar product* is defined by

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = ||\mathbf{a} \times \mathbf{b}||\, ||\mathbf{c}|| \cos\theta,$$

where θ is the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$.

Proposition 4.32 (Determinant formula for the triple scalar product). The triple scalar product is given by

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2 b_3 - a_3 b_2) c_1 - (a_1 b_3 - a_3 b_1) c_2 + (a_1 b_2 - a_2 b_1) c_3.\end{aligned}$$

Exercise 4.27. Use the determinant formula for the cross product from Proposition 4.28 to prove this.

Proposition 4.33 (Cyclic symmetry of the triple scalar product). The triple scalar product is invariant under cyclic permutation of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}.$$

Exercise 4.28. Prove this using the fact that a row interchange changes the sign of the determinant.

Proposition 4.34 (Interchange of dot and cross product in triple scalar product). The triple scalar product is invariant under exchanging the dot and cross product, in the sense that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

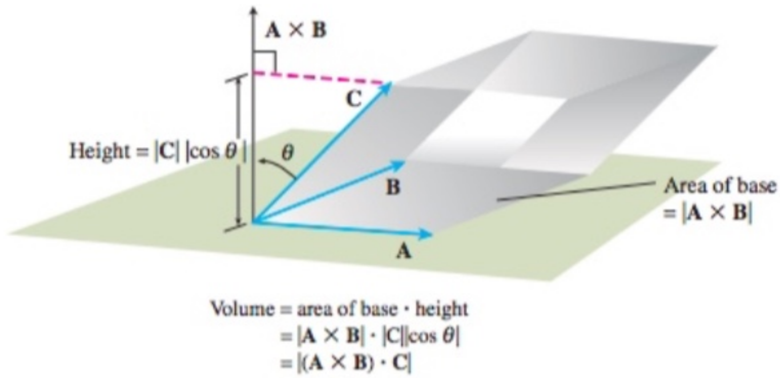
Exercise 4.29. Prove this using the cyclic symmetry of the cross product together with the symmetry of the dot product.

Proposition 4.35 (Relationships between dot and cross product). We have the following relationships between the dot and cross products:

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(*a*)	$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$	($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$)
(*b*)	$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$	($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$)
(*c*)	$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$	(*Lagrange's identity*)
(*d*)	$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$	(*relationship between cross and dot products*)
(*e*)	$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$	(*relationship between cross and dot products*)

Exercise 4.30. Prove these.



Geometrically, the magnitude of $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is the volume of the parallelepiped whose adjacent sides are $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$. The number $||\mathbf{a} \times \mathbf{b}||$ is the area of the base parallelogram, while $||\mathbf{c}|| \cos\theta$ is the height of the paralleleliped. For this reason, the triple scalar product is also sometimes referred to as the *box product* of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$.

Exercise 4.31. Use the triple scalar product to show that the volume of the parallelepiped with adjacent sides $\mathbf{a} = (1, 2, -1), \mathbf{b} = (-2, 0, 3)$, and $\mathbf{c} = (0, 7, -4)$ is 23 (units)^3.

Proposition 4.36 (Geometric interpretation of determinants). We have the following geometric interpretation of determinants:

THEOREM 3.5.4

(*a*) *The absolute value of the determinant*

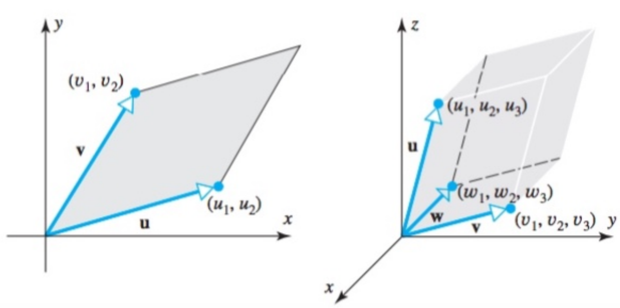
$$\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} \right\| = \|\vec{u} \times \vec{v}\|$$

is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$. *(See Figure 3.5.7a.)*

(*b*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = |(\vec{v} \times \vec{w}) \cdot \vec{u}|$$

is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$. *(See Figure 3.5.7b.)*



5 Vector Spaces

We have seen previously that various diverse applications, such as analyzing electrical circuits and balancing chemical equations, could all be solved in exactly the same way: since the solution of the problem involved forming linear combinations of a fixed number of unknown quantities, this gave rise to a system of linear equations which we were able to solve by our row-reduction algorithm. We therefore find it useful to generalize our concept of a vector, abstracting the essential properties which make it possible to solve any system which shares these properties. This gives rise to the notion of a *vector space*.

5.1 Basic Definitions and Examples

Definition 5.1 (Vector space). Let V be a set and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. [Explain why we need to let the scalars be complex. Add appendix on field properties for $\mathbb{C}$.] Define a function

$$\begin{aligned} + : V \times V &\to V \\ (\mathbf{v}, \mathbf{w}) &\mapsto \mathbf{v} + \mathbf{w} \end{aligned}$$

called *vector addition* (or just *addition*) and a function

$$\begin{aligned} \cdot : \mathbb{F} \times V &\to V \\ (x, \mathbf{v}) &\mapsto x\mathbf{v} \end{aligned}$$

called *scalar multiplication*. [35] The ordered 4-tuple $(V, \mathbf{F}, +, \cdot)$ is called a *vector space over* $\mathbb{F}$ (or just a *vector space*) if the following axioms hold:

A1. (Associativity of addition) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
A2. (Commutativity of addition) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$.
A3. (Existence of an additive identity) There exists an element $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
A4. (Existence of additive inverses) For every $\mathbf{v} \in V$ there exists a $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

S1. (Associativity of multiplication) $(xy)\mathbf{v} = x(y\mathbf{v})$ for all $x, y \in \mathbb{F}, \mathbf{v} \in V$.
S2. (Distributivity over scalar addition) $(x + y)\mathbf{v} = x\mathbf{v} + y\mathbf{v}$ for all $x, y \in \mathbb{F}, \mathbf{v} \in V$
S3. (Distributivity over vector addition) $x(\mathbf{v} + \mathbf{w}) = x\mathbf{v} + x\mathbf{w}$ for all $x \in \mathbb{F}, \mathbf{v}, \mathbf{w} \in V$.
S4. (Multiplication by 1 fixes each vector) $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

If the operations $+, \cdot$ are clear from context, then it is common to simply refer to V as a vector space. When $\mathbb{F} = \mathbb{R}$, then V is also said to be *real vector space*. When $\mathbb{F} = \mathbb{C}$, then V is also said to be *complex vector space*.

Elements of a vector space V are called *vectors* and elements of $\mathbb{F}$ are called *scalars*. [Digression on fields.]

Let us now consider various examples of vector spaces. We begin with the simplest possible example of a vector space:

Example 5.2 (The zero vector space). Let $V = \{0\}$ and define $0 + 0 := 0$ and $x0 := 0$ for all $x \in \mathbb{F}$. It is easy to check that A1-S4 are satisfied, so this is a vector space. This is called the *zero vector space* or the *trivial vector space*.

[35] Rather than write $x \cdot \mathbf{v}$, it is more common to just write $x\mathbf{v}$.

Exercise 5.1. Check that the zero vector space is indeed a vector space.

Exercise 5.2. Is $\emptyset$ a vector space?

Proof. The empty set fails axiom A3 since it does not contain a zero vector, so $\emptyset$ is not a vector space. $\square$

Of course, $\mathbb{F}^n$ is another example of a vector space, since we *defined* a vector space to have the algebraic properties of $\mathbb{F}^n$.

Example 5.3 ($\mathbb{F}^n$). Let $V = \mathbb{F}^n := \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{F}\}$ and define for $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n, x \in \mathbb{F}$.

$$\begin{aligned} \mathbf{v} + \mathbf{w} &:= (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n) \\ x\mathbf{v} &:= (xv_1, xv_2, \cdots, xv_n). \end{aligned}$$

The vector space axioms for $\mathbb{F}^n$ hold since *we based these axioms* on known properties of $\mathbb{F}^n$.

We can also consider *infinite* sequences of elements of $\mathbb{F}$:

Example 5.4 ($\mathbb{F}^\omega$). Let $\mathbb{F}^\omega$ denote the set of infinite sequences of elements of $\mathbb{F}$. An element of $\mathbb{F}^\omega$ is therefore of the form

$$\mathbf{v} = (v_1, v_2, \cdots, v_n, \cdots)$$

with $v_i \in \mathbb{F}$, and where we take $\mathbf{v} = \mathbf{w}$ if and only if $v_i = w_i$ for all i. Defining addition and scalar multiplication componentwise by

$$\begin{aligned} \mathbf{v} + \mathbf{w} &:= (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n, \cdots) \\ x\mathbf{v} &:= (xv_1, xv_2, \cdots, xv_n, \cdots) \end{aligned}$$

one can verify that $\mathbb{F}^\omega$ is a vector space. [36]

We have seen that matrices are composed of vectors (rows and columns). However, as we will now see, we can also view the *entire* matrix itself as a vector.

Example 5.5 ($M^{m\times n}(\mathbb{F})$). Let $M^{m\times n}(\mathbb{F})$ denote the set of $m \times n$ matrices with entries in $\mathbb{F}$. This is a vector space under the usual componentwise addition and scalar multiplication:

$$\begin{aligned} (a+b)_{ij} &= a_{ij} + b_{ij} \\ (xa)_{ij} &= xa_{ij} \end{aligned}$$

All of these examples so far were obtained by replacing $\mathbb{F}^n$ by another set V, but keeping essentially the same operations. In the next example, we illustrate the flexibility in choice of vector addition and scalar multiplication.

[36] For the reader wondering about the notation, the symbol ω denotes the first infinite *ordinal*, which is most likely an unfamiliar concept. For those who are interested, I encourage you to go ask Michael Rawlins about ordinal numbers.

Example 5.6 (An "unusual" vector space). Let V be the set positive real numbers, $\mathbb{R}^+$, and define vector addition to be ordinary multiplication and scalar multiplication to be exponentiation. That is, for any $u, v \in \mathbb{R}^+$ and $k \in \mathbb{R}$,

$$u + v = uv$$
$$ku = u^k$$

With these definitions, we have, for instance, $1 + 1 = (1)(1) = 1$ and $(2)(1) = 1^2 = 1$. If this is to be a vector space, we see that the zero vector must be $\mathbf{0} = 1$, since

$$u + 1 = (u)(1) = u$$

and that the additive inverse of u is its reciprocal $\frac{1}{u}$, since

$$u + \frac{1}{u} = (u)(\frac{1}{u}) = 1 = \mathbf{0}$$

One can check that all the vector space axioms are satisfied. For instance, axiom S4 holds due to the properties of exponents:

$$k(u + v) = (uv)^k = u^k v^k = (ku) + (kv).$$

Exercise 5.3. Verify that Example 5.6 is a vector space.

While the previous example shows that it is possible to give the same set different vector space structures, not every choice of operations will give rise to a vector space.

Exercise 5.4. Let $V = \mathbb{R}^2$ and define vector addition as usual, but scalar multiplication as

$$k(v_1, v_2) = (kv_1, 0).$$

Show that this is not a vector space. Which axiom(s) fail to hold?

Example 5.7. Let $V = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ and define vector addition and scalar multiplication as the usual ones for $\mathbb{R}^2$.

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$
$$k(x, y) := (kx, ky).$$

Vector addition is a well-defined function $V \times V \to V$, since $x_1 \geq 0$ and $x_2 \geq 0$ together imply that $x_1 + x_2 \geq 0$. However, if $k = -1$, then $(-x, -y)$ is not an element of V if $x > 0$. Thus, this is not a vector space since scalar multiplication is not a well-defined function $\mathbb{R} \times V \to V$.

5.2 Vector Space Properties

There are additional properties of any vector space which follow directly from Definition 5.1.

Theorem 5.8 (Properties of vector spaces). Let V be a vector space, $\mathbf{v} \in V$, and $x \in \mathbb{F}$. Then:

(a) The element $\mathbf{0} \in V$ of A3 is unique.

(b) $x\mathbf{0} = \mathbf{0}$ for all $x \in \mathbb{F}$

(c) $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$.

(d) For each $\mathbf{v}$ the $\mathbf{w}$ of A4 is unique. We denote this vector by $-\mathbf{v}$.

(e) $-\mathbf{v} = (-1)\mathbf{v}$ for all $\mathbf{v} \in V$.

(f) If $x\mathbf{v} = \mathbf{0}$, then either $x = 0$ or $\mathbf{v} = \mathbf{0}$.

Proof. (a) Suppose $\mathbf{0}, \mathbf{0}' \in V$ such that

$$\mathbf{0} + \mathbf{v} = \mathbf{v} \tag{5.1}$$
$$\mathbf{0}' + \mathbf{v} = \mathbf{v} \tag{5.2}$$

for all $\mathbf{v} \in V$. Then

$$\begin{aligned} \mathbf{0}' &= \mathbf{0} + \mathbf{0}' \text{ (by (1))} \\ &= \mathbf{0} \text{ (by (2))} \end{aligned}$$

Hence, $\mathbf{0}$ is unique.

(b) We have, by A3, x$\mathbf{0}$=x$\mathbf{0}$+$\mathbf{0}$. We also have

$$\begin{aligned} x\mathbf{0} &= x(\mathbf{0} + \mathbf{0}) \text{ (since } \mathbf{0} = \mathbf{0} + \mathbf{0} \text{ by A3)} \\ &= x\mathbf{0} + x\mathbf{0} \text{ (by S3)} \end{aligned}$$

and therefore $x\mathbf{0} + x\mathbf{0} = x\mathbf{0} + \mathbf{0}$. Adding the inverse of $x\mathbf{0}$ to both sides (A4), and using (A1), we find $x\mathbf{0} = \mathbf{0}$. (c) Applying A3, we have $0\mathbf{v} = 0\mathbf{v} + \mathbf{0}$. We also have $0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$ by S2. Letting $\mathbf{w}$ be the inverse of $0\mathbf{v}$ (which exists by A4), we have

$$\begin{aligned} 0\mathbf{v} + \mathbf{0} &= 0\mathbf{v} + 0\mathbf{v} \\ \mathbf{w} + 0\mathbf{v} + \mathbf{0} &= \mathbf{w} + 0\mathbf{v} + 0\mathbf{v} \\ (\mathbf{w} + 0\mathbf{v}) + \mathbf{0} &= (\mathbf{w} + 0\mathbf{v}) + 0\mathbf{v} \text{ (by A1)} \\ 0 + \mathbf{0} &= 0 + 0\mathbf{v} \text{ (by A4)} \end{aligned}$$

and therefore $0\mathbf{v} = \mathbf{0}$.

(d) Suppose for some $\mathbf{v} \in V$ there exist $\mathbf{w}, \mathbf{w}' \in V$ such that

$$\mathbf{v} + \mathbf{w} = \mathbf{0} \tag{5.3}$$
$$\mathbf{v} + \mathbf{w}' = \mathbf{0} \tag{5.4}$$

Then

$$\begin{aligned} \mathbf{w}' &= \mathbf{w}' + \mathbf{0} \text{ (by A3)} \\ &= \mathbf{w}' + (\mathbf{v} + \mathbf{w}) \text{ (by (3))} \\ &= (\mathbf{w}' + \mathbf{v}) + \mathbf{w} \text{ (by A1)} \\ &= \mathbf{0} + \mathbf{w} \text{ (by (4))} \\ &= \mathbf{w} \text{ (by A3).} \end{aligned}$$

Hence, for each $\mathbf{v}$, $\mathbf{w} \equiv -\mathbf{v}$ is unique.

(e) We have

$$\begin{aligned}\mathbf{v} + (-1)\mathbf{v} &= 1\mathbf{v} + (-1)\mathbf{v} \text{ (by S4)} \\ &= (1 + (-1))\mathbf{v} \text{ (by S2)} \\ &= 0\mathbf{v} \\ &= \mathbf{0} \text{ by (c)}\end{aligned}$$

By (d), for each $\mathbf{v} \in V$ there exists a unique $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = 0$. Hence, $-\mathbf{v} = (-1)\mathbf{v}$.

(f) Suppose $x\mathbf{v} = \mathbf{0}$. Then $x\mathbf{v} = 0\mathbf{v}$ by (c) and therefore $x\mathbf{v} + (-0\mathbf{v}) = 0\mathbf{v} + (-0\mathbf{v}) = 0$ by A4. Applying (e), we have $(x - 0)\mathbf{v} = \mathbf{0}$. If $x = 0$, we are done. If $x \neq 0$, we can multiply both sides by $\frac{1}{x-0} = \frac{1}{x}$ and we have $\mathbf{v} = \mathbf{0}$. □

We have just seen that the vector space structure establishes a connection between such diverse mathematical objects such as geometric vectors, vectors in $\mathbb{F}^n$, infinite sequences, matrices, and real-valued functions, to name a few. Consequently, whenever we prove a new theorem which holds for a general vector space (such as Theorem 5.8), it will apply to *all* of these vector spaces.

5.3 $\mathbb{F}^A$

We can obtain a large number of examples of vector spaces in the following way. Note that, in Examples 5.3, 5.4, and 5.5 above, the reason why the vector space axioms were satisfied was essentially because they are satisfied for real numbers. We can exploit this to systematically find other examples of vector spaces.

Let $\overline{n}$ denote the first n positive integers; that is, $\overline{n} := \{1, 2, \ldots, n\}$. We can view an ordered n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$ as a *function*

$$\mathbf{x} : \overline{n} \to \mathbb{F}$$

defined by $\mathbf{x}(i) = x_i$. From this point of view, vector addition and scalar multiplication are just the usual definitions of pointwise addition and scalar multiplication of functions:

$$\begin{aligned}(\mathbf{x} + \mathbf{y})(i) &= \mathbf{x}(i) + \mathbf{y}(i) \\ &= x_i + y_i \\ (c\mathbf{x})(i) &= c\mathbf{x}(i) \\ &= cx_i.\end{aligned}$$

Now, replace $\overline{n} \equiv \{1, 2, \cdots, n\}$ by *any* set A. Denote by $\mathbb{R}^A$ the set of all functions from A into $\mathbb{F}$, that is, $\mathbb{F}^A \equiv \{f | f : A \to \mathbb{F}\}$. This set is again a vector space under the operations

$$(f + g)(a) = f(a) + g(a) \text{ for all } a \in A \tag{5.5}$$

$$(xf)(a) = xf(a) \text{ for all } a \in A, x \in \mathbb{F} \tag{5.6}$$

It then follows immediately that $\mathbb{F}^A$ is a vector space because $\mathbb{F}$ is (we will prove this in Proposition 5.10 below).

Example 5.9. Let $f(x), g(x) \in \mathbb{F}^{\mathbb{F}}$ (the vector space of all functions $\mathbb{F} \to \mathbb{F}$), where $f(x) = \sin(x)$ and $g(x) = e^x$. Then their vector sum is given by

$$(f + g)(x) = f(x) + g(x) = \sin(x) + e^x$$

and the scalar multiple of $f(x)$ by $\sqrt{\pi}$ is given by

$$(\sqrt{\pi} f)(x) = \sqrt{\pi} f(x) = \sqrt{\pi} \sin(x).$$

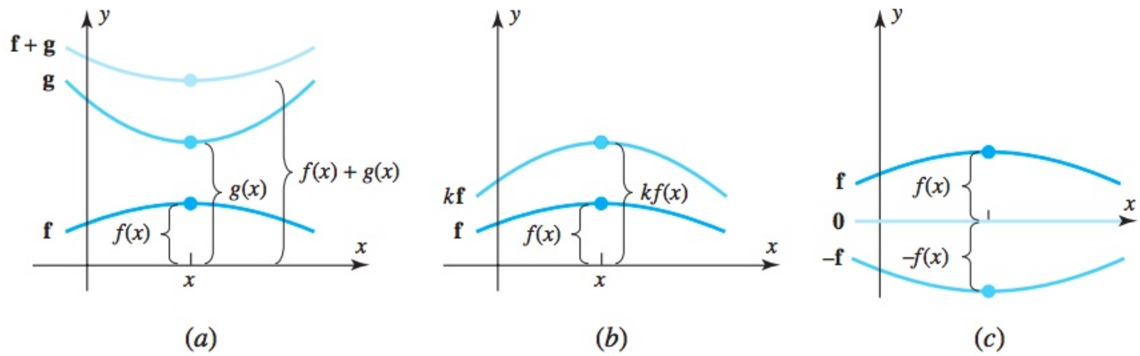


Figure 13: Visualization of the operations (5.5) and (5.6) for continuous functions from $\mathbb{F} \to \mathbb{F}$. The figure in (a) illustrates a vector sum, (b) illustrates a scalar multiple, and (c) an additive inverse.

Proposition 5.10 ($\mathbb{F}^A$ is a vector space). The set $\mathbb{F}^A$ is a vector space under the operations (5.5) and (5.6).

Proof. We will show that A1-A4, S1-S4 of Definition 5.1 hold. The key fact is that the values $f(a)$ of a function $f \in \mathbb{F}^A$ are real numbers. The reader should be able to justify each step in what follows.

A1. Let $f, g, h \in \mathbb{F}^A$. Then, for all $a \in A$,

$$\begin{aligned}
[(f+g)+h](a) &= (f+g)(a) + h(a) \\
&= (f(a)+g(a)) + h(a) \\
&= f(a) + (g(a)+h(a)) \text{ (Since addition of real numbers is associative.)} \\
&= f(a) + (g+h)(a) \\
&= [f+(g+h)](a).
\end{aligned}$$

This shows that the functions $(f+g)+h$ and $f+(g+h)$ have the same value for every $a \in A$. Since a function is defined by its values, they are therefore the same function. This proves that vector addition is associative.

A2. Let $f, g \in \mathbb{F}^A$. Then, for all $a \in A$,

$$\begin{aligned}
(f+g)(a) &= f(a) + g(a) \\
&= g(a) + f(a) \text{ (Since addition of real numbers is commutative.)} \\
&= (g+f)(a).
\end{aligned}$$

Hence, vector addition is commutative.

A3. Let 0 denote the zero function in $\mathbb{F}^A$, defined as the function which maps every element of A to the real number 0; that is, $0(a) = 0$ for all $a \in A$. The zero function is then the additive identity in $\mathbb{F}^A$, since for all $f \in \mathbb{F}^A$,

$$\begin{aligned}(f+0)(a) &= f(a) + 0(a) \\ &= f(a) + 0 \\ &= f(a) \text{ (since the number 0 is the additive identity in } \mathbb{F}\text{).}\end{aligned}$$

Hence, $\mathbb{F}^A$ has an additive identity.

A4. Given $f \in \mathbb{F}^A$, let $-f$ denote the function whose value at each $a \in A$ is given by the negative of the value of f at a; that is, $(-f)(a) = -f(a)$. Then $-f$ is the additive inverse of f, since

$$\begin{aligned}(f+(-f))(a) &= f(a) + (-f)(a) \\ &= f(a) - f(a) \\ &= 0 \text{ (since subtracting a real number from itself gives 0).}\end{aligned}$$

This shows that $(f + (-f))(a) = 0$ for all $a \in A$, hence it is the zero function (which is the additive identity in $\mathbb{F}^A$). This shows that $-f$ is the additive inverse of f.

S1. Let $x, y \in \mathbb{F}$ and $f \in \mathbb{F}^A$. Then for every $a \in A$,

$$\begin{aligned}[(xy)f](a) &= (xy)f(a) \\ &= x(yf(a)) \text{ Since multiplication in } \mathbb{F} \text{ is associative.} \\ &= x((yf)(a)) \\ &= [x(yf)](a).\end{aligned}$$

Hence, scalar multiplication is associative.

S2. Let $x, y \in \mathbb{F}$ and $f \in \mathbb{F}^A$. Then for all $a \in A$,

$$\begin{aligned}[(x+y)f](a) &= (x+y)f(a) \\ &= xf(a) + yf(a) \text{ (By the distributive property in } \mathbb{F}.) \\ &= (xf)(a) + (yf)(a) \\ &= (xf + yf)(a).\end{aligned}$$

Hence, scalar multiplication distributes over scalar addition.

S3. Let $x \in \mathbb{F}$ and $f, g \in \mathbb{F}^A$. Then for every $a \in A$,

$$\begin{aligned}[x(f+g)](a) &= x(f+g)(a) \\ &= x(f(a) + g(a)) \\ &= xf(a) + xg(a) \text{ (By the distributive property in } \mathbb{F}.) \\ &= (xf + xg)(a).\end{aligned}$$

Hence, scalar multiplication distributes over vector addition.

S4. Let $f \in \mathbb{F}^A$. Then for every $a \in A$,

$$\begin{aligned}(1f)(a) &= 1f(a) \\ &= f(a) \text{ (Since 1 is the multiplicative identity in } \mathbb{F}.)\end{aligned}$$

Hence scalar multiplication by 1 fixes every function $f \in \mathbb{F}^A$. This completes the proof that $\mathbb{F}^A$ is a vector space. □

Note that all of the examples of vector spaces in section 5.1 are examples of $\mathbb{F}^A$ for various choices of A:

(i) $\mathbb{F}^n$ (the vector space of all ordered n-tuples of real numbers) is the same as $\mathbb{F}^{\overline{n}}$ (the vector space of all functions $\overline{n} \to \mathbb{F}$), since, as noted at the beginning of this section, the set of n values of such a function can be taken to be the n entries of a vector in $\mathbb{F}^n$.

(ii) $\mathbb{F}^\omega$ (the vector space of infinite sequences of real numbers) is the same as $\mathbb{F}^{\mathbb{N}}$ (the set of all functions $\mathbb{N} \to \mathbb{F}$), since the set of values of such a function is a sequence in $\mathbb{F}$.

(iii) $M^{m\times n}(\mathbb{F})$ (the vector space of $m \times n$ matrices) is the same as $\mathbb{F}^{\overline{m}\times\overline{n}}$ (the vector space of all functions $\overline{m} \times \overline{n} \to \mathbb{F}$), since the mn values of such a function can then be taken to be the mn entries of an $m \times n$ matrix.

(iv) $\{0\}$ (the zero vector space) is the same as $\mathbb{F}^{\emptyset}$ (the vector space of all functions from $\emptyset \to \mathbb{F}$). To see this, note that there is a unique function from the empty set into any non-empty set, B. [37] Letting 0 denote the only element of $\mathbb{F}^{\emptyset}$, we give $\mathbb{F}^{\emptyset}$ the structure of the zero vector space under the operations $0 + 0 = 0$ and $x0 = 0$ of example 5.2.

We therefore obtain infinitely many examples of vector spaces by varying the set A. Here are a few more familiar examples:

- $V = \mathbb{F}^{\{a\}}$, where $\{a\}$ is a singleton set, is $\mathbb{F}$ itself.
- $V = \mathbb{F}^{\mathbb{F}}$ is the set of all $\mathbb{F}$-valued functions of one variable in $\mathbb{F}$.
- $V = \mathbb{F}^{\mathbb{F}\times\mathbb{F}}$ is the set of all $\mathbb{F}$-valued functions of two variables in $\mathbb{F}$.
- $V = \mathbb{F}^{\overbrace{\mathbb{F}\times\mathbb{F}\times\cdots\times\mathbb{F}}^{n \text{ times}}}$ is the set of all $\mathbb{F}$-valued functions of n variables in $\mathbb{F}$.

[37] For any non-empty set B, the Cartesian product $A \times B = \emptyset$ if $A = \emptyset$, since in this case there are no ordered pairs (a, b) with $a \in A$ and $b \in B$ since there are no $a \in A$. Since the only subset of $\emptyset$ is $\emptyset$, the only relation between $\emptyset$ and a non-empty set B is the empty relation. All that is left is to show that this is a function. A function from A to B is a relation $R \subset A \times B$ satisfying two conditions:

(1) (Existence of images) For each $a \in A$, there exists a $b \in B$ such that $(a, b) \in R$

(2) (Uniqueness of images) If $(a, b) \in R$ and $(a, b') \in R$, then $b = b'$.

Both of these conditions are vacuously true for the empty relation on $\emptyset \times B$ (since there is no $a \in \emptyset$ for which they could fail), so the empty relation $\emptyset \subset \emptyset \times B$ is indeed a function. This shows that there is indeed a unique function from $\emptyset$ to any non-zero set B.

5.4 Subspaces

Given a vector space $(V, +, \cdot)$, one way to get a new vector space is consider a nonempty subset $W \subseteq V$ and the restriction of the operations $+, \cdot$ to W.

Definition 5.11 (Subspace). Let $(V, +, \cdot)$ be any vector space, and let W be a nonempty subset of V. Define vector addition and scalar multiplication on W as the *restrictions* of those on V. Then, if $(W, +|_W, \cdot|_W)$ satisfies axioms A1-A4 and S1-S4, it is a vector space in its own right. In this case, W is called a *subspace* of V. [38]

Proposition 5.12 (Trivial subspaces). Every vector space V has at least two subspaces: V and $\{0\}$.

Proof. By the definition of a subset ($A \subseteq B$ if, for every $a \in A$, $a \in A \implies a \in B$), every set is a subset of itself. Hence $V \subseteq V$, and the restriction of the vector operations of V to itself are the same operations. Since V is a vector space by assumption, this shows that V is a subspace of itself.

Now, since every vector space V has a zero vector (by definition of a vector space), it is always possible to take $W = \{0\}$. Let us now consider the restriction of the vector addition and scalar multiplication operations of V to W. Since $0 + v = 0$ for every vector $v \in V$, this must also be true, in particular, for $v = 0$. Therefore $0 + 0 = 0$ gives the restriction of $+$ to W. Now by part (b) of Theorem 5.8, $x0 = 0$ for every $x \in \mathbb{R}$. Hence, the restriction of the vector operations of V to W are exactly those of Example 5.2, and therefore W is a zero vector space, and hence a subspace of V. $\square$

In principle, we need to show that all the vector space axioms of Definition 5.1 hold for W to show that it is a subspace. However, since one already knows that V is a vector space, showing that $W \subseteq V$ is actually much easier: since the operations of vector addition and scalar multiplication on W are the restrictions of those of V, certain axioms are satisfied *automatically* since they are already satisfied for all vectors in V and, in particular, those in W. For instance, if $+_V$ is commutative and associative for all vectors in V, then $+_W$ is also commutative and associative on W since every vector in W is also in V. We therefore only need to check the axioms that are not inherited from V in this way. So which axioms are not inherited?

First, note that Definition 5.11 of a subspace requires W to be *closed* under the operations of V. That is, the sum of two vectors in W must be another vector in W, and, similarly, multiplying any vector w in W by a scalar k must give another vector in W. As the following example shows, this is not automatically satisfied if we restrict to an arbitrary subset of V.

Example 5.13. Let $V = \mathbb{R}^2$ with the usual operations, and take $W \subseteq \mathbb{R}^2$ to be the set of points inside the unit disc; that is, $W = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Consider $v = (1, 0)$. Since $v + v = (2, 0) \notin W$ (since $2^2 + 0^2 = 4 > 1$), W is not closed under vector addition. Similarly, $2(1, 0) = (2, 0)$, so W is also not closed under scalar multiplication. Hence, W is not a subspace of V.

Assume now that we have chosen a subset W of a vector space V which is closed under the operations of V. Going through the axioms in Definition 5.1, we see that all but two hold automatically, simply because they hold for all vectors in V, and in particular those in W. The two do not hold automatically are

- A3 - Existence of an additive identity $\mathbf{0}$ in W.

[38] We must require that W be nonempty since if $W = \emptyset$, then it does not have a zero vector and therefore cannot be a vector space.

- A4 - Existence of an additive inverse for each $\mathbf{w} \in W$.

Indeed, consider the following example:

Example 5.14. Let $V = \mathbb{R}$ and W be the positive reals; that is, $W = (0, \infty)$. The zero vector of V is the real number 0, which is not in W, so axiom A3 fails. Axiom A4 also fails for this choice of W: The additive inverse of the real number 1 in V is -1. Now 1 belongs to W, but -1 does not, so we see that 1 has no additive inverse in W.

Thus, given a vector space V and a subset $W \subseteq V$, to check that W is a subspace it would seem that we need to

1. check that W is closed under the vector addition and scalar multiplication of V, and if so,
2. check that the zero vector is in W, and finally
3. check that every vector in W has an additive inverse in W.

However, the following lemma shows that if W is closed under addition and scalar multiplication, then A3 and A4 actually follow *automatically*.

Lemma 5.15. If W is a nonempty subset of a vector space V, then W is a subspace if and only if it is closed under the addition and scalar multiplication of V when restricted to W.

Proof. The proof of necessity is trivial (if W is a subspace, then it is closed under addition and multiplication by definition). To prove sufficiency, suppose W is a non-empty subset of V which is closed under the addition and scalar multiplication of V when restricted to W. As noted above, all the axioms of Definition 5.1 are guaranteed to hold for the restricted operations, except for A3 and A4, so we need to check each of these. Since W is not empty, there exists $\mathbf{w} \in W$. Since W is closed under scalar multiplication, $0\mathbf{w} = \mathbf{0} \in W$, so W contains the zero vector. Hence A3 holds for W. Similarly, for every $\mathbf{w} \in W$, $(-1)\mathbf{w} = -\mathbf{w} \in W$, so A4 holds for W. Thus, all the axioms of Definition 5.1 hold for W, proving that it is a subspace of V. □

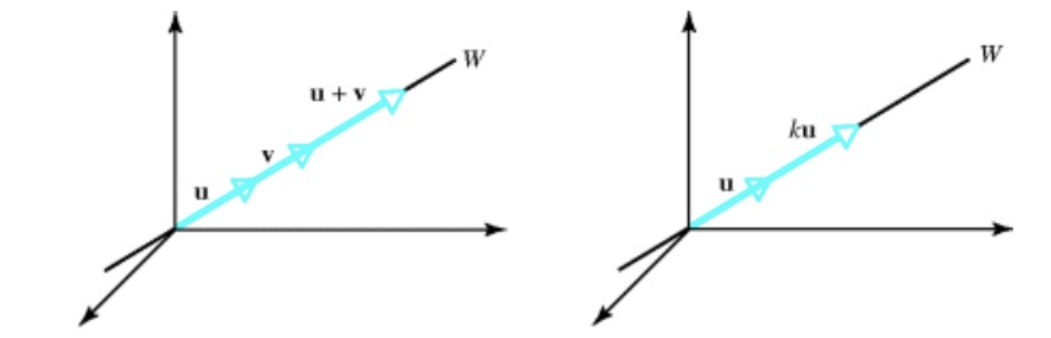
Checking for closure of W under the vector addition and scalar multiplication of V can actually be combined into a single condition:

Theorem 5.16 (Subspace criterion). A nonempty subset W of V is a subspace if and only if for each pair of vectors $\mathbf{v}, \mathbf{w} \in W$ and each scalar $x \in \mathbb{R}$, the vector $x\mathbf{v} + \mathbf{w} \in W$.

Proof. The proof of necessity is again trivial, since if W is a subspace then by definition it is closed under vector addition and scalar multiplication, so $x\mathbf{v} \in W$ and therefore $x\mathbf{v} + \mathbf{w} \in W$. To prove sufficiency, we need to show that if W is a nonempty subset of V satisfying the stated hypothesis, then W is closed under addition and scalar multiplication. Since W is not empty, we can choose $\mathbf{v}, \mathbf{w} \in W$. By assumption, $1\mathbf{v} + \mathbf{w} = \mathbf{v} + \mathbf{w} \in W$, which shows W is closed under addition. We also have, by assumption, $(-1)\mathbf{v} + \mathbf{v} = -\mathbf{v} + \mathbf{v} = \mathbf{0} \in W$. It follows that for any $x \in \mathbb{R}$ and $\mathbf{v} \in W$, $x\mathbf{v} + \mathbf{0} = x\mathbf{v} \in W$, which shows W is closed under scalar multiplication. It then follows from Lemma 5.15 that W is a subspace. □

Exercise 5.5. Prove Proposition 5.12 using Theorem 5.16.

Example 5.17 (Lines through the Origin in $\mathbb{F}^n$). Let $\mathbf{v} \in \mathbb{F}^n$. Then the line $L_{\mathbf{v}}$ through the origin and $\mathbf{v}$ is given by $L_{\mathbf{v}} = \{\mathbf{w} \in \mathbb{F}^n : \mathbf{w} = t\mathbf{v} \text{ for some } t \in \mathbb{F}\}$. Then, if $\mathbf{v}_1, \mathbf{v}_2 \in L_{\mathbf{v}}$ and $x \in \mathbb{F}$, $x\mathbf{v}_1 + \mathbf{v}_2 = xt_1\mathbf{v} + t_2\mathbf{v} = (xt_1 + t_2)\mathbf{v} \in L_{\mathbf{v}}$, hence $L_{\mathbf{v}}$ is a subspace of $\mathbb{F}^n$ by the subspace criterion.



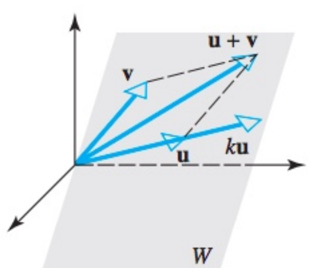
(a) W is closed under addition.

(b) W is closed under scalar multiplication.

Exercise 5.6. Let $L_{\mathbf{v},\mathbf{b}} = \{\mathbf{w} \in \mathbf{F}^n : \mathbf{w} = t\mathbf{v} + \mathbf{b} \text{ for some } t \in \mathbb{R} \text{ and } \mathbf{b} \neq \mathbf{0}\}$ be a line which does not pass through the origin in $\mathbb{F}^n$. Show that such a line is *not* a subspace of $\mathbb{F}^n$.

Example 5.18 (Planes through the origin in $\mathbb{F}^n$). Let $\mathbf{v}, \mathbf{w}$ be any two non-proportional vectors in $\mathbb{F}^n$. Then $W = \{t_1\mathbf{v} + t_2\mathbf{w} : t_1, t_2 \in \mathbb{F}\}$ is a plane through the origin in $\mathbb{F}^n$. If $\mathbf{v}_1, \mathbf{v}_2 \in W$ and $x \in \mathbb{F}$, then $x\mathbf{v_1} + \mathbf{v}_2 = x(t_1\mathbf{v} + t_2\mathbf{w}) + (t_1'\mathbf{v} + t_2'\mathbf{w}) = (xt_1 + t_1')\mathbf{v} + (xt_2 + t_2')\mathbf{w} \in W$, hence W is a subspace of $\mathbb{F}^n$.

[Higher dimensional subspaces. Comment on the converse, i.e., that these are the only subspaces of $\mathbb{F}^n$.]



Exercise 5.7. Show that a plane that does not pass through the origin is *not* a subspace of $\mathbb{F}^n$.

Exercise 5.8. Let $W = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$. Is W a subspace of $\mathbb{R}^2$?

Example 5.19 (Subspaces of $\mathbb{F}^{\overline{n} \times \overline{m}}$). The following sets of matrices are subspaces of $\mathbb{F}^{\overline{m} \times \overline{n}}$:

- Symmetric matrices
- Upper triangular matrices
- Lower triangular matrices
- Diagonal matrices

Exercise 5.9. Show that each of these is a subspace of $\mathbb{F}^{\overline{m}\times\overline{n}}$.

Note that the space of invertible matrices is *not* a subspace of $\mathbb{F}^{\overline{m}\times\overline{n}}$. This set clearly fails to be closed under scalar multiplication: for any invertible matrix, A, $0A$ is the zero matrix, so it is not invertible. Similarly $A + (-A) = 0$, which is not invertible.

Example 5.20 (Function spaces). Subspaces of $\mathbb{F}^A$ are called *function spaces*.

- Take $V = \mathbb{R}^{(a,b)}$ and let $W = \mathscr{C}(a,b)$ denote the set of all *continuous* real-valued functions on the open interval (a,b). If f, g are continuous on (a,b) and $x \in \mathbb{R}$, then from elementary calculus we know that $xf + g$ is also continuous on (a,b), so $W = \mathscr{C}(a,b)$ is a subspace of $\mathbb{R}^{(a,b)}$.
- The set $\mathscr{C}^1(a,b)$ of continuously differentiable functions (functions with a continuous first derivative) on (a,b) is a also subspace of $\mathbb{R}^{(a,b)}$, as are $\mathscr{C}^m(a,b)$ (the set of functions whose mth derivative exists and is continuous on (a,b)) and $\mathscr{C}^\infty(a,b)$ (the set of functions whose derivatives all exist and are continuous on (a,b)).

Example 5.21 (Polynomials). The set of all polynomials of degree n is *not* a subspace of $\mathbb{F}^\mathbb{F}$ as it is not closed under addition, e.g. $((1 + x) + (1 - x) = 2)$.

However, the set of all polynomials of degree $\leq n$ *is* a subspace of $\mathbb{F}^\mathbb{F}$, denoted P_n.

The set of all polynomials, denoted P_∞, is also a subspace of $\mathbb{F}^\mathbb{F}$.

Exercise 5.10. Verify that P_n and P_∞ are subspaces of $\mathscr{C}^\infty(\mathbb{R}) \equiv \mathscr{C}^\infty(-\infty,\infty)$.

Example 5.22 (Solution sets of homogeneous linear systems). The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in n unknowns is a subspace of $\mathbf{R}^n$. To see this, let $\mathbf{x}_1, \mathbf{x}_2$ be solutions of this linear system and let c be any scalar. By the rules of matrix multiplication, we then have

$$\begin{aligned} A(c\mathbf{x}_1 + \mathbf{x}_2) &= A(c\mathbf{x}_1) + A\mathbf{x}_2 \\ &= c(A\mathbf{x}_1) + A\mathbf{x}_2 \\ &= c0 + 0 \\ &= 0, \end{aligned}$$

which shows that $c\mathbf{x}_1 + \mathbf{x}_2$ is another solution of the linear system. Hence, the solution set is a subspace by the subspace criterion.

Exercise 5.11. The solution set of an inhomogeneous linear system in n unknowns is *not* a subspace of $\mathbb{R}^n$.

Solution. Let $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b} \neq \mathbf{0}$, be an inhomogeneous linear system. Since $A\mathbf{0} = \mathbf{0} \neq \mathbf{b}$, we see that the zero vector of $\mathbb{R}^n$ is not in the solution set, hence the solution set cannot be a subspace of $\mathbb{R}^n$. □

5.5 The Lattice of Subspaces

Since differentiability implies continuity, the subspaces in Examples 5.20 and 5.21 have a nested structure, as shown in the figure below: [39]

[39] Here $F(-\infty,\infty)$ is what your book calls $\mathbb{R}^{(-\infty,\infty)}$. The latter is the more common notation.

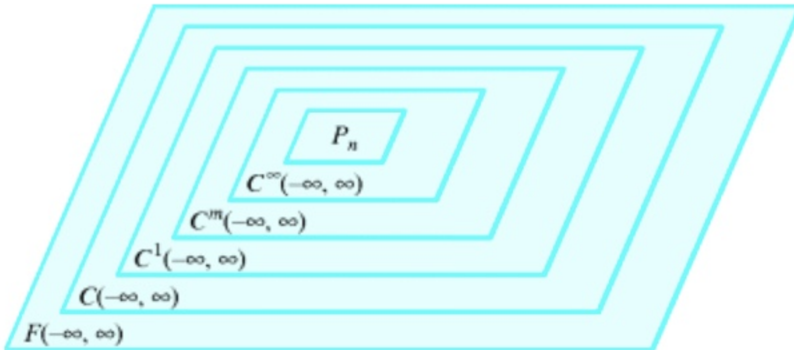


Figure 14: Nested structure of the function spaces of Examples 5.20 and 5.21.

We may similar study the relation among arbitrary subspaces of a given vector space V. To this end, let $\mathcal{S}(V)$ denote the collection of all subspaces of a vector space V. The reader is advised to review Appendix A before reading this section.

Exercise 5.12. Show that $\mathcal{S}(V)$ is a partially ordered set by inclusion.

Proof. We need to prove that $\subseteq$ is reflexive, antisymmetric, and transitive. Recall that $A \subseteq B$ means that $x \in A \implies x \in B$.

(1) For all $W \in \mathcal{S}(V)$, $W \subseteq W$ (since $x \in W \implies x \in W$), so reflexivity holds.

(2) For all $W_1, W_2 \in \mathcal{S}(V)$, $W_1 \subseteq W_2$ and $W_2 \subseteq W_1 \implies W_1 = W_2$ (since $x \in W_1 \implies x \in W_2$ and $x \in W_2 \implies x \in W_1$, W_1 and W_2 have exactly the same elements), so antisymmetry holds.

(3) For all $W_1, W_2, W_3 \in \mathcal{S}(V)$, $W_1 \subseteq W_2$ and $W_2 \subseteq W_3 \implies W_1 \subseteq W_3$ (since $x \in W_1 \implies x \in W_2$ and $x \in W_2 \implies x \in W_3$ implies $x \in W_1 \implies x \in W_3$), so transitivity holds.

This proves that $(\mathcal{S}(V), \subseteq)$ is a partially ordered set. □

Exercise 5.13. Prove that $\{0\}$ is the minimum element in $\mathcal{S}(V)$ and V is the maximum element in $\mathcal{S}(V)$.

Proof. We proved in Proposition 5.12 that $\{0\}, V \in \mathcal{S}(V)$. Let $U \in \mathcal{S}(V)$. Since U is a subspace of V, $0 \in U$. Thus, $\{0\} \subseteq U$ for every $U \in \mathcal{S}(V)$, which proves that $\{0\}$ is the minimum element of $\mathcal{S}(V)$. By definition, every subspace of V is a subset of V. Thus, $U \subseteq V$ for all $U \in \mathcal{S}(V)$. This proves that V is the maximum element of $\mathcal{S}(V)$. □

Exercise 5.14. If $S, T \in \mathcal{S}(V)$, prove that $S \cap T = \inf\{S, T\}$. That is, $S \cap T$ is the largest subspace of V contained in both S and T.

Proof. We first show that $S \cap T$ is a subspace of V. Since S, T are subspaces of V, they both contain 0, so $0 \in S \cap T$. This shows that $S \cap T$ is not empty. Let $\mathbf{v}, \mathbf{w} \in S \cap T$ and $x \in \mathbb{F}$. In particular, $\mathbf{v}, \mathbf{w} \in S$ so $x\mathbf{v} + \mathbf{w} \in S$ since S is a subspace. Similarly, $\mathbf{v}, \mathbf{w} \in T$, so $x\mathbf{v} + \mathbf{w} \in T$ since T is a subspace. Since $x\mathbf{v} + \mathbf{w}$ is in both S and T, $x\mathbf{v} + \mathbf{w} \in S \cap T$, proving that $S \cap T \in \mathcal{S}(V)$ by the subspace criterion. Since, by definition, $S \cap T \subseteq S$ and $S \cap T \subseteq T$, this shows that $S \cap T$ is a lower bound for $\{S, T\}$. We now show it is the *greatest* lower bound; that is, if U is another lower bound for $\{S, T\}$, then $U \subseteq S \cap T$. This is immediate, since if U is a lower bound for $\{S, T\}$, then $U \subseteq S$ and $U \subseteq T$, and therefore $U \subseteq S \cap T$. This proves that $S \cap T = \inf\{S, T\}$. $\square$

Exercise 5.15. If $\{S_i \mid i \in K\}$ is any collection of subspaces of V, then

$$\bigcap_{i \in K} S_i = \inf\{S_i \mid i \in K\}.$$

Proof. The proof is essentially the same as that of the previous exercise. Since each S_i is a subspace of V, $0 \in S_i$ for all $i \in K$. Now let $\mathbf{v}, \mathbf{w} \in \bigcap_{i \in K} S_i$ and $x \in \mathbb{F}$. Then $x\mathbf{v} + \mathbf{w} \in S_i$ for all $i \in K$, since each S_i is a subspace, and so $x\mathbf{v} + \mathbf{w} \in \bigcap_{i \in K} S_i$, so $\bigcap_{i \in K} S_i \in \mathcal{S}(V)$ by the subspace criterion. Since, by definition, $\bigcap_{i \in K} S_i \subseteq S_i$ for all $i \in K$, this proves that $\bigcap_{i \in K} S_i$ is a lower bound for $\{S_i \mid i \in K\}$. If U is another lower bound for $\{S_i \mid i \in K\}$, then $U \subseteq S_i$ for all $i \in K$, and therefore $U \subseteq \bigcap_{i \in K} S_i$. Thus, $\bigcap_{i \in K} S_i = \inf\{S_i \mid i \in K\}$. $\square$

We have just proved that, given $S, T \in \mathcal{S}(V)$, that $S \cap T \in \mathcal{S}(V)$ and $S \cap T = \inf\{S, T\}$. The reader may therefore guess that $\sup\{S, T\} = S \cup T$. However, as the next proposition shows, $S \cup T$ is not even guaranteed to be a subspace of V.

Proposition 5.23. Let $S, T \in \mathcal{S}(V)$. Then $S \cup T \in \mathcal{S}(V)$ is a subspace if and only if $S \subseteq T$ or $T \subseteq S$.

Proof. If $S \subseteq T$, then $S \cup T = T$, which is a subspace of V by assumption. Similarly, if $T \subseteq S$, then $S \cup T = S$, which is a subspace of V by assumption. Now suppose $S \nsubseteq T$ and $T \nsubseteq S$. We will show that $S \cup T$ is not a subspace of V. Since $S \nsubseteq T$, there exists $\mathbf{u} \in S - T$. Since $T \nsubseteq S$, there exists $\mathbf{w} \in T - S$. Consider the vector $\mathbf{u} + \mathbf{w} \in S \cup T$. Suppose $\mathbf{u} + \mathbf{w} \in S$. Since $-\mathbf{u} \in S$ and since S is a subspace, then $\mathbf{u} + \mathbf{w} - \mathbf{u} = \mathbf{w} \in S$, which is a contradiction. Thus, $\mathbf{u} + \mathbf{w} \notin S$. Suppose now that $\mathbf{u} + \mathbf{w} \in T$. Since $-\mathbf{w} \in T$ and since T is a subspace, then $\mathbf{u} + \mathbf{w} - \mathbf{w} = \mathbf{u} \in T$, which is a contradiction. Thus, $\mathbf{u} + \mathbf{w}$ is in neither S nor T, and therefore is not in $S \cup T$. Since $\mathbf{u} \in S \cup T$ and $\mathbf{w} \in S \cup T$, but $\mathbf{u} + \mathbf{w} \notin S \cup T$, this shows that $S \cup T$ is not closed under addition and therefore is not a subspace of V. $\square$

To determine the smallest subspace of V containing the subspaces S and T, we make the following definition.

Definition 5.24 (Sum of subspaces). Let S and T be subspaces of V. The *sum* $S + T$ is defined by

$$S + T = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in S, \mathbf{v} \in T\}.$$

More generally, the *sum* of any collection $\{S_i \mid i \in K\}$ of subspaces is the set of all *finite* sums of vectors from the union $\bigcup S_i$:

$$\sum_{i \in K} S_i = \left\{ s_1 + \cdots + s_n \mid s_j \in \bigcup_{i \in K} S_i \right\}.$$

Exercise 5.16. If $S, T \in \mathcal{S}(V)$, then $S + T \in \mathcal{S}(V)$ and

$$S + T = \sup\{T, S\}.$$

Proof. Since S and T are subspaces of V, $0 \in S \cap T$, so we can write

$$0 = \underbrace{0}_{\in S} + \underbrace{0}_{\in T} \in S + T.$$

Thus, $S + T$ is not empty. Let $\mathbf{v}, \mathbf{w} \in S + T$ and $x \in \mathbb{F}$. Then $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_1 \in S$ and $\mathbf{v}_2 \in T$. Similarly, $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ with $\mathbf{w}_1 \in S$ and $\mathbf{w}_2 \in T$. Then

$$\begin{aligned} x\mathbf{v} + \mathbf{w} &= (x\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{w}_1 + \mathbf{w}_2) \\ &= \underbrace{(x\mathbf{v}_1 + \mathbf{w}_1)}_{\in S} + \underbrace{(x\mathbf{v}_2 + \mathbf{w}_2)}_{\in T} \in S + T. \end{aligned}$$

Thus, $S + T$ is a subspace of V by the subspace criterion. Since

$$\begin{aligned} S = S + 0 &:= \{\mathbf{v} + 0 \mid \mathbf{v} \in S\} \subseteq S + T, \\ T = 0 + T &:= \{0 + \mathbf{v} \mid \mathbf{v} \in T\} \subseteq S + T, \end{aligned}$$

$S + T$ is an upper bound for $\{S, T\}$. We now show that $S + T$ is the *least* upper bound for $\{S, T\}$; that is, we need to show that if U is another upper bound for $\{S, T\}$, then $S + T \subseteq U$. If U is an upper bound for $\{S, T\}$, then $S \subseteq U$ and $T \subseteq U$. Since U is a subspace of V, it is closed under addition, so all elements of the form $\mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in S$ and $\mathbf{w} \in T$ are in U, therefore $S + T \subseteq U$. Hence,

$$S + T = \sup\{S, T\}.$$

□

Exercise 5.17. If $\{S_i \mid i \in K\}$ is any collection of subspaces of V, then

$$\sum_{i \in K} S_i = \sup\{S_i \mid i \in K\}.$$

Proof. Since $0 \in S_i$ for all $i \in K$, we can write 0 as a finite sum

$$0 = \underbrace{0}_{\in S_1} + \underbrace{0}_{\in S_2} + \cdots + \underbrace{0}_{\in S_n}.$$

Thus, $\sum_{i \in K} S_i$ is not empty. Let $\mathbf{v}_i, \mathbf{w_i} \in S_i$ for $i = 1, \ldots, n$ and let $x \in \mathbb{F}$. Then

$$x\left(\sum_{i=1}^{n} \mathbf{v}_i\right) + \sum_{i=1}^{n} \mathbf{w}_i = \sum_{i=1}^{n} (x\mathbf{v}_i + \mathbf{w}_i) \in S_1 + \cdots + S_n.$$

Thus, $\sum_{i \in K} S_i$ is a subspace by the subspace criterion. For any $i \in K$,

$$\begin{aligned} S_i = 0 + \cdots + S_i + \cdots + 0 &= \{0 + \cdots + \mathbf{v} + \cdots + 0 \mid \mathbf{v} \in S_i\} \\ &\subseteq \sum_{i \in K} S_i. \end{aligned}$$

Thus, $\sum_{i \in K} S_i$ is an upper bound for $\{S_i \mid i \in K\}$. If U is another upper bound for $\{S_i \mid i \in K\}$, then $S_i \subseteq U$ for all $i \in K$. Since U is a subspace of V, it is closed under addition, so every finite sum

$$s_1 + \cdots + s_m$$

with $s_i \in S_i$ is in U. This shows that $\sum_{i \in K} \subseteq U$ and therefore

$$\sum_{i \in K} S_i = \sup\{S_i \mid i \in K\}.$$

$\square$

If a partially ordered set P has the property that every pair of elements has a least upper bound and a greatest lower bound, then P is called a *lattice*. If P has a smallest element and a largest element and has the property that every collection of elements has a least upper bound and a greatest lower bound, then P is called a *complete lattice*. The least upper bound of a collection is also called the *join* of the collection and the greatest lower bound is called the *meet*.

We have proved the following theorem.

Theorem 5.25 ($\mathcal{S}(V)$ is a complete lattice). The set $\mathcal{S}(V)$ of all subspaces of a vector space V is a complete lattice under set inclusion, with smallest element $\{0\}$, largest element V, meet

$$\inf\{S_i \mid i \in K\} = \bigcap_{i \in K} S_i$$

and join

$$\sup\{S_i \mid i \in K\} = \sum_{i \in K} S_i$$

Exercise 5.18. Let

$$\begin{aligned} U &= \{(x,0,0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}, \\ V &= \{(0,y,0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}, \\ W &= \{(x,y,0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}. \end{aligned}$$

Show that U, V, and W are subspaces of $\mathbb{F}^3$ and that

$$W = U + V.$$

Proof. The sets U, V, and W are easily verified to be subspaces of $\mathbb{F}^3$ by the subspace criterion. Since

$$(x,0,0) + (0,y,0) = (x,y,0),$$

$U + V \subseteq W$. To show the converse, given $(x,y,0) \in W$, we need to find $(a,0,0) \in U$ and $(0,b,0) \in V$ such that

$$\begin{aligned} (x,y,0) &= (a,0,0) + (0,b,0) \\ &= (a,b,0). \end{aligned}$$

Which has the unique solution $x = a$ and $y = b$. Thus, $W \subseteq U + V$ and therefore $W = U + V$. $\square$

Exercise 5.19. Let

$$\begin{aligned} U &= \{(x,x,y,y) \in \mathbb{F}^4 \mid x,y \in \mathbb{F}\}, \\ V &= \{(x,x,x,y) \in \mathbb{F}^4 \mid x,y \in \mathbb{F}\}, \\ W &= \{(x,x,y,z) \in \mathbb{F}^4 \mid x,y,z \in \mathbb{F}\}. \end{aligned}$$

Show that U, V, and W are subspaces of $\mathbb{F}^3$ and that

$$W = U + V.$$

Proof. The sets U, V, and W are easily verified to be subspaces of $\mathbb{F}^4$ by the subspace criterion. The equation

$$(a,a,a,b) + (c,c,d,e) = (a+c, a+c, a+d, a+e)$$

is of the form (x,x,y,z) with $x,y,z \in \mathbb{F}$, so $U + V \subseteq W$. To show that converse, let $(x,x,y,z) \in W$. We need to find $(a,a,b,b) \in U$ and $(c,c,c,d) \in V$ such that

$$\begin{aligned} (x,x,y,z) &= (a,a,b,b) + (c,c,c,d) \\ &= (a+c, a+c, b+c, b+d). \end{aligned}$$

This leads to the system of equations

$$\begin{aligned} a + 0 + c + 0 &= x, \\ 0 + b + c + 0 &= y, \\ 0 + b + 0 + d &= z. \end{aligned}$$

Row-reducing the corresponding augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 1 & 0 & 1 & z \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & x-y+z \\ 0 & 1 & 0 & 1 & z \\ 0 & 0 & 1 & -1 & y-z \end{array}\right],$$

we find an infinite number of solutions

$$\begin{aligned} a &= -t + x - y + z \\ b &= -t + z \\ c &= t + (y - z) \\ d &= t. \end{aligned}$$

For instance, when $t = 0$ we have

$$(x,x,y,z) = (x-y+z, x-y+z, z, z) + (y-z, y-z, y-z, 0).$$

Thus, $W = U + V$. $\square$

5.6 Direct Sums

In Exercise 5.18, we found that each each element $\mathbf{w} \in W$ could be written uniquely as $\mathbf{u} + \mathbf{v}$ with $\mathbf{u} \in U$ and $\mathbf{v} \in V$. In this case we say that W is the *direct sum* of U and V.

Definition 5.26 (Direct Sum). If $U_1, \ldots, U_m$ are subspaces of V, then the sum $U_1 + \cdots + U_m$ is called a *direct sum* if each element of $U_1 + \cdots + U_m$ can be written uniquely as a sum $u_1 + \cdots + u_m$ with $u_j \in U_j$ for all $j = 1, \ldots, m$. In this case, we write $U_1 + \cdots + U_m$ as

$$\bigoplus_{i=1}^{m} U_i := U_1 \oplus \cdots \oplus U_m.$$

Exercise 5.20. Let

$$\begin{aligned} U &= \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}, \\ V &= \{(0, 0, z) \in \mathbb{F}^3 \mid z \in \mathbb{F}\}. \end{aligned}$$

Prove that $\mathbb{F}^3 = U \oplus V$.

Exercise 5.21. Let U_j be the subspace of $\mathbb{F}^n$ consisting of vectors whose coordinates are all 0, except possibly the jth entry (thus, for example, $U_2 := \{(0, x, 0, \ldots, 0) \in \mathbb{F}^n \mid x \in \mathbb{F}\}$). Prove that

$$\mathbb{F}^n = \bigoplus_{i=1}^{n} U_i.$$

Example 5.27. Let

$$\begin{aligned} U_1 &= \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}, \\ U_2 &= \{(0, 0, z) \in \mathbb{F}^3 \mid z \in \mathbb{F}\}, \\ U_3 &= \{(0, y, y) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}. \end{aligned}$$

Then $\mathbb{F}^3 = U_1 + U_2 + U_3$, because every vector $(x, y, z) \in \mathbb{F}^3$ can be written as

$$(x, y, z) = \underbrace{(x, y, 0)}_{\in U_1} + \underbrace{(0, 0, z)}_{\in U_2} + \underbrace{(0, 0, 0)}_{\in U_3}.$$

However, $\mathbb{F}^3 \neq U_1 \oplus U_2 \oplus U_3$, since, for example we can write

$$(0, 0, 0) = \underbrace{(0, 1, 0)}_{\in U_1} + \underbrace{(0, 0, 1)}_{\in U_2} + \underbrace{(0, -1, -1)}_{\in U_3}$$

as well as

$$(0, 0, 0) = \underbrace{(0, 0, 0)}_{\in U_1} + \underbrace{(0, 0, 0)}_{\in U_2} + \underbrace{(0, 0, 0)}_{\in U_3}.$$

The next theorem shows that when deciding whether a sum of subspaces is a direct sum, we need only consider whether 0 can be uniquely written as an appropriate sum.

Theorem 5.28 (Condition for a direct sum). Let $U_1, \ldots, U_m$ be subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where $u_j \in U_j$ for all $j = 1, \ldots, m$, is by taking each $u_j = 0$.

Proof. If $U_1 + \cdots + U_m$ is a direct sum, then each vector in V can be written uniquely as $u_1 + \cdots + u_m$, where $u_j \in U_j$ for all $j = 1, \ldots, m$. In particular, noting that $0 \in u_j$ for all $j = 1, \ldots, m$, 0 can therefore be written uniquely as $0 = 0 + \cdots + 0$.

Now suppose that the only way to write 0 as a sum $u_1 + \cdots + u_m$, where $u_j \in U_j$ for all $j = 1, \ldots, m$, is by taking each $u_j = 0$. To show that $U_1 + \cdots + U_m$ is a direct sum, let $v \in U_1 + \cdots + U_m$. Suppose we can write

$$v = v_1 + \cdots + v_m$$

with $v_j \in U_j$ for all $j = 1, \ldots, m$ and

$$v = v_1' + \cdots + v_m'$$

with $v_j' \in U_j$ for all $j = 1, \ldots, m$. Subtracting these two equations gives

$$0 = (v_1 - v_1') + \cdots + (v_m - v_m').$$

Since $v_j - v_j' \in U_j$ for all $j = 1, \ldots, m$, we must therefore have $v_j = v_j'$ for all $j = 1, \ldots, m$. This shows that each vector in $U_1 + \cdots + U_m$ can be written uniquely as $u_1 + \cdots + u_m$ where $u_j \in U_j$ for all $j = 1, \ldots, m$, so the sum is direct. □

The next result gives a simple criterion for testing which pairs of subspaces give a direct sum.

Proposition 5.29 (Direct sum of two subspaces). Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof. Suppose that $U + W$ is a direct sum. If $\mathbf{v} \in U \cap W$, then $0 = \mathbf{v} + (-\mathbf{v})$, where $\mathbf{v} \in U$ and $-\mathbf{v} \in W$ ($-\mathbf{v} \in W$ since $\mathbf{v} \in W$ and W is closed under scalar multiplication). By Theorem 5.28, we must have $\mathbf{v} = \mathbf{0}$. Thus, $U \cap W = \{\mathbf{0}\}$.

Now suppose $U \cap W = \{\mathbf{0}\}$. Suppose that

$$\mathbf{0} = \mathbf{u} + \mathbf{w}$$

for some $\mathbf{u} \in U$ and $\mathbf{w} \in W$. This equation implies that $\mathbf{w} = -\mathbf{u}$ and therefore $\mathbf{u} \in U \cap W = \{\mathbf{0}\}$, hence $\mathbf{u} = \mathbf{0}$. By Theorem 5.28, $U + W$ is a direct sum. □

Example 5.30. Let $U = \{(x, 0) \in \mathbb{F}^2 \mid x \in \mathbb{F}\}$ and $V = \{(0, y) \in \mathbb{F}^2 \mid x \in \mathbb{F}\}$. Since

$$(x, 0) = (0, y)$$

implies $x = y = 0$, $U \cap V = \{\mathbf{0}\}$, so $\mathbb{F}^2 = U \oplus V$.

Exercise 5.22. Let $M^{n \times n}(\mathbb{F})$ be the vector space of $n \times n$ matrices with entries in $\mathbb{F}$. Let $S^{n \times n}(\mathbb{F})$ and $A^{n \times n}(\mathbb{F})$ denote the subspaces of symmetric and antisymmetric $n \times n$ matrices, respectively. Prove that $M^{n \times n}(\mathbb{F}) = S^{n \times n}(\mathbb{F}) \oplus A^{n \times n}(\mathbb{F})$.

Proof. If $A \in M^{n\times n}(\mathbb{F})$, then we can write

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\in S^{n\times n}(\mathbb{F})} + \underbrace{\frac{1}{2}(A - A^T)}_{\in A^{n\times n}(\mathbb{F})},$$

so $M^{n\times n}(\mathbb{F}) = S^{n\times n}(\mathbb{F}) + A^{n\times n}(\mathbb{F})$. If $A \in S^{n\times n}(\mathbb{F}) \cap A^{n\times n}(\mathbb{F})$, then $A = A^T$ and $A = -A^T$, which implies that $A = -A$ and therefore $A = 0$. By Proposition 5.29, the sum is direct. □

Exercise 5.23. Recall that a function $f : \mathbb{F} \to \mathbb{F}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbb{F}$ and is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{F}$. Let U_e denote the set of even functions and U_o the set of odd functions. Show that

$$\mathbb{F}^{\mathbb{F}} = U_e \oplus U_o.$$

Proof. The subsets U_e and U_o are easily verified to be subspaces of $\mathbb{F}^{\mathbb{F}}$ by the subspace criterion. Since any function $f \in \mathbb{F}^{\mathbb{F}}$ can be written as

$$f(x) = \underbrace{\frac{1}{2}(f(x) + f(-x))}_{\in U_e} + \underbrace{\frac{1}{2}(f(x) - f(-x))}_{\in U_o},$$

$\mathbb{F}^{\mathbb{F}} = U_e + U_o$. If $f \in U_e \cap U_o$, then for all $x \in \mathbb{F}$, $f(-x) = f(x)$ and $f(-x) = -f(x)$, which implies f is the zero map. Thus, $U_e \cap U_o = \{\mathbf{0}\}$, so by Proposition 5.29, the sum is direct. □

Proposition 5.29 deals only with the case of two subspaces. If one has more than two subspaces, it is not enough to test that each pair intersect only at $\mathbf{0}$. For instance, in Example 5.27 we have

$$U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{\mathbf{0}\}$$

but the sum is not direct. For an arbitrary collection of subspaces, one has the following generalization of Proposition 5.29:

Exercise 5.24. Let

$$U = \{(x, x, y, y) \in \mathbb{F}^4 \mid x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^4$ other than $\{\mathbf{0}\}$ such that $\mathbb{F}^4 = U \oplus W$.

Proof. Consider $W = \{(0, z, w, 0) \in \mathbb{F}^4 \mid z \in \mathbb{F}\}$. If $(\alpha, \beta, \gamma, \delta) \in \mathbb{F}^4$, then

$$\begin{aligned}(\alpha, \beta, \gamma, \delta) &= (x, x, y, y) + (0, z, w, 0)\\ &= (x, z + x, y + w, y)\end{aligned}$$

implies the system of equations

$$\begin{aligned} x &= \alpha, \\ x + z &= \beta, \\ y + w &= \gamma, \\ y &= \delta. \end{aligned}$$

which has the unique solution

$$\begin{aligned} x &= \alpha, \\ y &= \delta, \\ z &= -\alpha + \beta, \\ w &= -\delta + \gamma. \end{aligned}$$

Since every $(\alpha, \beta, \gamma, \delta)$ can be written uniquely as the sum of an element of U and an element of W, $\mathbb{F}^4 = U \oplus W$. $\square$

Proposition 5.31. If $U_1, \ldots, U_m$ is a finite collection of subspaces of a vector space V, then $U_1 + \cdots + U_m$ is a direct sum of and only if

$$U_i \cap \left(\sum_{j \neq i} U_j \right) = \{\mathbf{0}\}$$

for each $i = 1, \ldots, m$.

Proof. Suppose $U_1 + \cdots + U_m$ is a direct sum, and suppose

$$\mathbf{0} \neq \mathbf{v} \in U_i \cap \left(\sum_{j \neq i} U_j \right).$$

Then $\mathbf{v} = \mathbf{u}_i \in U_i$ and

$$\mathbf{u}_i = \mathbf{u}_{j_1} + \cdots + \mathbf{u}_{j_{m-1}} \in \sum_{j \neq i} U_j$$

where at least one $\mathbf{u}_{j_k} \neq \mathbf{0}$. It follows that

$$\mathbf{0} = -\mathbf{u}_i + \mathbf{u}_{j_1} + \cdots + \mathbf{u}_{j_{m-1}},$$

which contradicts Theorem 5.28. Thus, if $U_1 + \cdots + U_m$ is a direct sum, then

$$U_i \cap \left(\sum_{j \neq i} U_j \right) = \{\mathbf{0}\}$$

for each $i = 1, \ldots, m$.

We now show that if

$$U_i \cap \left(\sum_{j \neq i} U_j \right) = \{\mathbf{0}\}$$

for each $i = 1, \ldots, m$, then the only way to write the zero vector as

$$\mathbf{0} = \mathbf{u}_1 + \cdots + \mathbf{u}_m$$

with $\mathbf{u}_j \in U_j$ for all $j = 1, \ldots, m$ is by taking $\mathbf{u}_1 = \cdots = \mathbf{u}_m = \mathbf{0}$. Suppose that

$$\mathbf{0} = \mathbf{u}_1 + \cdots + \mathbf{u}_m$$

with $\mathbf{u}_j \in U_j$ for all $j = 1, \ldots, m$ and with some $\mathbf{u}_j \neq \mathbf{0}$. Then at least two $\mathbf{u}_j \neq \mathbf{0}$, which by reordering the terms if necessary we can take to be $\mathbf{u}_1$ and $\mathbf{u}_2$. We then thave

$$-\mathbf{u}_1 = \mathbf{u}_2 + \cdots + \mathbf{u}_m$$

which shows that $\mathbf{u}_1 \in U_1 \cap \left(\sum_{i \neq j} U_j\right)$ and therefore $\left(\sum_{i \neq j} U_j\right) \neq \{\mathbf{0}\}$. □

Note that the condition in Proposition 5.31 is *stronger* than the condition that $U_i \cap U_j = \{\mathbf{0}\}$ for all $i \neq j$.

Example 5.32. Returning to Example 5.27, since

$$U_1 + U_3 = \{(x, y, z) \in \mathbb{F}^3 \mid x, y, z \in \mathbb{F}\},$$

we have

$$U_2 \cap (U_1 + U_3) = \{(0, 0, z) \in \mathbb{F}^3 \mid z \in \mathbb{F}\}$$

for $z \neq 0$, so $U_2 \cap (U_1 + U_3) \neq \{\mathbf{0}\}$ and Proposition 5.31 implies that $U_1 + U_2 + U_3$ is not a direct sum.

5.7 Subspace generated by a set

Suppose we are given a random nonempty subset W of a vector space V. In general, W will not be a subspace, since it will not be closed under addition and scalar multiplication. Of course, the set W will always be *contained* is some subspace of V' of V (since we could always just take $V' = V$ itself if there is no proper subspace containing W). However, we will frequently be interested in the *smallest* subspace containing W.

Let $S = \{U_i \mid i \in K\}$ denote the set of subspaces of V which contain W. This is a partially ordered set by inclusion. The smallest subspace of V containing W is the minimum element of S. Since W is contained in U_i for all $i \in K$, W is contained in $\cap_{i \in K} U_i$. Moreover, $\cap_{i \in I} U_i \subseteq U_j$ for every subspace U_j containing W, so $\cap_{i \in I} U_i$ is the minimum element of S.

In the following we will give an explicit construction of this subspace. By the subspace criteria, to make a given subset W into a subspace of V (if it is not already a subspace of V), we need to enlarge W by adding appropriate vectors in V so that the enlarged set is closed under addition and scalar multiplication. To form the smallest such subspace, we need to add the *minimum* number of additional vectors needed to accomplish this. We will now show how to do this.

Definition 5.33 (Linear combination). A vector $\mathbf{v}$ is called a *linear combination* of a subset $W = \{\mathbf{v}_i\}_{i \in I}$ of V if $\mathbf{v}$ is a *finite* sum $\sum_{i=1}^{n} x_i \mathbf{v}_i$, where the vectors $\mathbf{v}_i$ are all in W and the scalars $x_i \in \mathbb{F}$.

Example 5.34. If $W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$, then a linear combination of W is any vector of the form

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} c_1 - 3c_2 \\ 2c_1 + 7c_2 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{F}.$$

Example 5.35. If $W = \{\sin t, \cos t, e^t\} \subset \mathbb{R}^{\mathbb{R}}$, then a linear combination of W is any function of the form $f(t) = c_1 \sin t + c_2 \cos t + c_3 e^t$ with $c_1, c_2, c_3 \in \mathbb{R}$.

Example 5.36. If $W = \{t^n\}_{n=0}^{\infty} \subset \mathbb{F}^{\mathbb{F}}$, then a function $f(t)$ is a linear combination of W if and only if it is a polynomial function $f(t) = \sum_{j=1}^{n} x_j t^j$.

Definition 5.37 (Linear span). Let $W = \{\mathbf{w}_i\}_{i \in I}$ be a nonempty subset of a vector space V. The set of all linear combinations of W is called the *linear span* (or just *span*) of W, and denoted $\operatorname{Span} W$.

Theorem 5.38 ($\operatorname{Span} W$ **is a subspace).**

(a) $\operatorname{Span} W$ is a subspace of V containing W.

(b) Let $\cap_{i \in I} W_i$ be the intersection of all subspaces W_i of V containing W. Then $\operatorname{Span} W = \cap_{i \in I} W_i$.

(c) If W is a subspace of V, then $\operatorname{Span} W = W$.

Proof. (a) Suppose first that $W = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ is finite. Then, for $\mathbf{u}, \mathbf{v} \in SpanW$ and $c \in \mathbb{F}$, we have $c\mathbf{u} + \mathbf{v} = c(\sum_{i=1}^{n} x_i \mathbf{w}_i) + \sum_{i=1}^{n} y_i \mathbf{w}_i = \sum_{i=1}^{n} (cx_i + y_i)\mathbf{w}_i \in SpanW$, hence $SpanW$ is a subspace of V by the subspace criterion. Now suppose that $W = \{\mathbf{w}_i\}_{i \in I}$ is infinite, and let $\mathbf{u}, \mathbf{v} \in SpanW$. Then

$$\begin{aligned} \mathbf{u} &= x_1 \mathbf{w}_{\alpha_1} + x_2 \mathbf{w}_{\alpha_2} + \cdots + x_n \mathbf{w}_{\alpha_n}, \\ \mathbf{v} &= y_1 \mathbf{w}_{\beta_1} + y_2 \mathbf{w}_{\beta_2} + \cdots + y_m \mathbf{w}_{\beta_m}, \end{aligned}$$

for some *finite* collections $\{\mathbf{w}_{\alpha_1}, \mathbf{w}_{\alpha_2}, \ldots, \mathbf{w}_{\alpha_n}\}$ and $\{\mathbf{w}_{\beta_1}, \mathbf{w}_{\beta_2}, \ldots, \mathbf{w}_{\beta_n}\}$ of vectors in W. Then, for c any scalar, we have

$$\begin{aligned} c\mathbf{u} + \mathbf{v} &= c(x_1 \mathbf{w}_{\alpha_1} + x_2 \mathbf{w}_{\alpha_2} + \cdots + x_n \mathbf{w}_{\alpha_n}) + y_1 \mathbf{w}_{\beta_1} + y_2 \mathbf{w}_{\beta_2} + \cdots + y_m \mathbf{w}_{\beta_m} \\ &= (cx_1)\mathbf{w}_{\alpha_1} + (cx_2)\mathbf{w}_{\alpha_2} + \cdots + (cx_n)\mathbf{w}_{\alpha_n} + y_1 \mathbf{w}_{\beta_1} + y_2 \mathbf{w}_{\beta_2} + \cdots + y_m \mathbf{w}_{\beta_m}, \end{aligned}$$

which is a linear combination of W (since this sum is *finite*), hence $\operatorname{Span} W$ is a subspace of V by the subspace criterion.

In either case, each vector $\mathbf{w}_i$ in W can be written as a linear combination $1\mathbf{w}_i + \sum_{k=1}^{n} c_k \mathbf{w}_k$, where $\{\mathbf{w}_k\}_{k=1}^{n}$ is any finite subset of W and the coefficients c_k are all taken to be zero. Therefore, $W \subseteq \operatorname{Span} W$.

(b) By part (a), $\operatorname{Span} W$ is a subspace of V containing W, so $\operatorname{Span} W = W_i$ for some $i \in I$ and therefore $\cap_{i \in I} W_i \subseteq \operatorname{Span} W$ by definition of the intersection.

Conversely, since $\cap_{i \in I} W_i$ is a subspace of V containing W, it must contain $\operatorname{Span} W$ since it is closed under addition and scalar multiplication. Hence $\operatorname{Span} W \subseteq \cap_{i \in I} W_i$, and therefore $\operatorname{Span} W = \cap_{i \in I} W_i$.

(c) Suppose W be a subspace of V. We have seen in part (a) that $W \subseteq \operatorname{Span} W$. Conversely, since W is a subspace, it is closed under addition and scalar multiplication, so $\operatorname{Span} W \subseteq W$ and therefore $\operatorname{Span} W = W$.

$\square$

Therefore, Span W can be directly characterized as the uniquely determined smallest subspace of V which contains the set W. This subspace is also frequently called the *subspace generated by* W.

Exercise 5.25. Let $W = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{F} \right\}$. Show that W is a subspace of $\mathbb{F}^4$.

Solution. Since $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, we see that $W = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, which is a subspace of $\mathbb{R}^4$. □

There are two important questions regarding spanning sets which are encountered frequently:

1. Given a subset W of vectors in a vector space V and a fixed vector $\mathbf{v} \in V$, determine whether $\mathbf{v}$ is a linear combination of the vectors in W (that is, whether $\mathbf{v} \in \operatorname{Span} W$).
2. Given a vector space V, find a subset W of V such that $V = \operatorname{Span} W$ (that is, find a spanning set for V).

Example 5.39. Let $\mathbf{v} = (1, 2, -1)$ and $\mathbf{w} = (6, 4, 2)$ in $\mathbb{R}^3$. Determine whether each of the following vectors is a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

(a) $\mathbf{b} = (9, 2, 7)$.

Solution. This is true if there exist some $c_1, c_2 \in \mathbb{R}$ such that $c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{b}$. This leads to the system of equations

$$\begin{aligned} c_1 + 6c_2 &= 9 \\ 2c_1 + 4c_2 &= 2 \\ -c_1 + 2c_2 &= 7. \end{aligned}$$

Since $\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right]$ row reduces to $\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$, this system has the unique solution $c_1 = -3, c_2 = 2$. Therefore, $\mathbf{b} = -3\mathbf{v} + 2\mathbf{w} \in \operatorname{Span}\{\mathbf{v}, \mathbf{w}\}$. □

(b) $\mathbf{b} = (4, -1, 8)$.

Solution. In this case we arrive at the system

$$\begin{aligned} c_1 + 6c_2 &= 4 \\ 2c_1 + 4c_2 &= -1 \\ -c_1 + 2c_2 &= 8. \end{aligned}$$

Since, $\left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array}\right]$ is row equivalent to $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$, the system is inconsistent. Hence, $\mathbf{b}$ is not a linear combination of $\mathbf{v}$ and $\mathbf{w}$. □

Exercise 5.26. Let $\mathbf{v} = (1, 2, -1)$ and $\mathbf{w} = (6, 4, 2)$ in $\mathbb{R}^3$. What are all the vectors $\mathbf{b} = (b_1, b_2, b_3)$ in $\text{Span}\{\mathbf{v}, \mathbf{w}\}$?

Solution. If $\mathbf{b} \in \text{Span}\{\mathbf{v}, \mathbf{w}\}$, then $\mathbf{b} = c_1\mathbf{v} + c_2\mathbf{w}$ for some scalars c_1, c_2. That is, the inhomogeneous linear system

$$\begin{aligned} c_1 + 6c_2 &= b_1 \\ 2c_1 + 4c_2 &= b_2 \\ -c_1 + 2c_2 &= b_3 \end{aligned}$$

is consistent. Since $\left[\begin{array}{cc|c} 1 & 6 & b_1 \\ 2 & 4 & b_2 \\ -1 & 2 & b_3 \end{array}\right]$ is row equivalent to $\left[\begin{array}{cc|c} 1 & 6 & b_1 \\ 0 & -8 & b_2 - 2b_1 \\ 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$, we see that the system is consistent if and only if $-b_1 + b_2 + b_3 = 0$. Thus,

$$\text{Span}\{\mathbf{v}, \mathbf{w}\} = \{(b_1, b_2, b_3) \in \mathbb{R}^3 : b_1 = b_2 + b_3\}.$$

The reader can check that this condition indeed holds for the vector in part (a) of Example 5.39, but not the vector in part (b). □

Example 5.40 (Standard Unit Vectors in $\mathbb{F}^n$). Let us now find a spanning set for $\mathbb{F}^n$. Let $e_i = (0, \cdots, 1, \cdots, 0)$ be the ith standard unit vector in $\mathbb{F}^n$. By Theorem 5.38, $\text{Span}\{e_i\}_{i=1}^n$ is a subspace of $\mathbb{F}^n$. Since

$$(x_1, \cdots, x_i, \cdots, x_n) = x_1\mathbf{e}_1 + \cdots + x_i\mathbf{e}_i + \cdots + x_n\mathbf{e}_n$$

for any $(x_1, \cdots, x_i, \cdots, x_n) \in \mathbb{F}^n$, we see that $\mathbb{R}^n \subseteq \text{Span}\{e_i\}_{i=1}^n$ and hence $\text{Span}\{e_i\}_{i=1}^n = \mathbb{F}^n$.

Exercise 5.27. Show that the monomials $1, x, x^2, \cdots, x^n$ span P_n (the vector space of all polynomials of degree $\leq n$).

Solution. By Theorem 5.38, $\text{Span}\{x^i\}_{i=0}^n$ is a subspace of P_n. Since any polynomial $p \in P_n$ can be written as

$$p = a_0 + a_1x + \cdots + a_nx^n,$$

we see that $P_n \subseteq \text{Span}\{x^i\}_{i=0}^n$ and hence $P_n = Span\{x^i\}_{i=0}^n$. □

[Hold of on this next example until we have more theory?]

Example 5.41. Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2), \mathbf{v}_2 = (1, 0, 1), \mathbf{v}_3 = (2, 1, 4)$ span $\mathbb{R}^3$.

Solution. We must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ can be expressed uniquely as a linear combination $\sum_{i=1}^3 c_i\mathbf{v}_i$. If so, then given any $\mathbf{b}$ the coefficients c_1, c_2, c_3 in the equation $\sum_{i=1}^3 c_i\mathbf{v}_i = \mathbf{b}$ must be a solution to the linear system

$$\begin{aligned} c_1 + c_2 + 2c_3 &= b_1 \\ c_1 + c_3 &= b_2 \\ 2c_1 + c_2 + 4c_3 &= b_3. \end{aligned}$$

Since the determinant of the coefficient matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ is nonzero, the linear system has a unique solution for all $\mathbf{b} \in \mathbb{R}^3$. Hence, the set $\{\mathbf{v}_i\}_{i=1}^3$ spans $\mathbb{R}^3$. □

Since we have previously seen that the standard unit vectors $\{e_i\}_{i=1}^3$ span $\mathbb{R}^3$ (see Example 5.40), Example 5.41 shows that spanning sets are not unique. We may then ask "When do two subsets W, W' of a vector space V span the same subspace of V?" The answer is given in the following theorem:

Theorem 5.42 (Condition for W, W' to span the same subspace). If W and W' are two nonempty subsets of a vector space V, then $SpanW = SpanW'$ if and only if each vector in W is a linear combination of those in W' and vice versa.

Proof. ($\Longleftarrow$) $W \subseteq SpanW' \implies SpanW \subseteq SpanW'$ and $W' \subseteq SpanW \implies SpanW' \subseteq SpanW$, hence $SpanW = SpanW'$. ($\implies$) Each $w \in W$ belongs to $SpanW$ and hence to $SpanW'$ (since these are equal), hence each $w \in W$ can be written as a linear combination of vectors in W'. By the same reasoning, each vector in W' can be written as a linear combination of vectors in W. □

Example 5.43. Consider again the vectors $\mathbf{v}_1 = (1,1,2), \mathbf{v}_2 = (1,0,1), \mathbf{v}_3 = (2,1,4)$. It is obvious that each of these is a linear combination of the standard unit vectors $\{e_i\}_{i=1}^n$. To express $\mathbf{e}_1$, say, as a linear combination of $\{\mathbf{v}_i\}_{i=1}^3$, we must find scalars c_1, c_2, c_3 such that $\mathbf{e}_1 = \sum_{i=1}^3 c_i \mathbf{v}_i$; that is, we must find a solution to the linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right].$$

Since this matrix is row-equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right],$$

we see that $e_1 = \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$. Of course, rather than repeating the same row operations for each of the e_i, in practice one simply repeats Exercise 5.26 and row reduces

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 4 & b_3 \end{array}\right]$$

for a general $\mathbf{b} = (b_1, b_2, b_3) \in \mathbf{R}^3$, and then plugs in each e_i to obtain each as a linear combination of $\{\mathbf{v}_i\}_{i=1}^3$.

5.8 Linear Mappings

Recall that we are taking our prototypical example of a real vector space to be the general function space $\mathbb{F}^A$, which is the vector space of all $\mathbb{F}$-valued functions on a set A.[40] Note that, in

[40]We saw in Section 5.3 that all the examples of vector spaces we have studied so far are realized as $\mathbb{F}^A$ for particular choices of the set A.

addition to the vector operations, $\mathbb{F}^A$ is also closed under the operation of (pointwise) multiplication of two functions:

$$(fg)(a) = f(a)g(a)\ \forall a \in A.$$

This is also true for the subspaces $\mathscr{C}^m([a,b])$ for $m = 0, \cdots, \infty$ of continuous real-valued functions on the interval $[a,b]$.

With respect to these three operations, $\mathbb{F}^A$ and $\mathscr{C}^m([a,b])$ are examples of *algebras*, which are vector spaces which are also closed under multiplication (satisfying certain axioms, of course). Why then do we bother with the notion of vector spaces? Why not study all three operations?

The answer is that the vector operations are exactly the operations that are "preserved" by many of the most important mappings of sets of functions.

Example 5.44. Consider the laws of integration. The definite integral of a continuous function on $[a,b]$ is a mapping $T : \mathscr{C}([a,b]) \to \mathbb{R}$ defined by $T(f) = \int_a^b f(t)dt$. The laws of integration tell us that

$$T(f+g) = T(f) + T(g)$$
$$T(cf) = cT(f)$$

Thus T "preserves" the vector operations, in the sense that performing the vector operations followed by T is the same as performing T and then performing the vector operations. However, T does *not* preserve multiplication: it is *not* true in general that $T(fg) = T(f)T(g)$ (e.g., $\int_a^b x dx \int_a^b x^2 dx \neq \int_a^b x^3 dx$).

Example 5.45. We may similarly view differentiation of a continuously differentiable function as a mapping $T : \mathscr{C}^1(a,b) \to \mathscr{C}(a,b)$ defined by $T(f) = \frac{df}{dx}$. The laws of differentiation tell us that

$$T(f+g) = T(f) + T(g)$$
$$T(cf) = cT(f).$$

However, $\frac{d}{dx}(x)\frac{d}{dx}(x^2) \neq \frac{d}{dx}(x^3)$, so again we see that differentiation preserves the vector operations of $\mathscr{C}^1(a,b)$, but not the multiplication operation.

Exercise 5.28. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x_1, x_2, x_3) = (y_1, y_2) = (2x_1 - x_2 + x_3, x_1 + 3x_2 - 5x_3)$$

Verify that

(a) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$

(b) $T(c\mathbf{x}) = cT(\mathbf{x})$

Solution. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Then

(a)

$$\begin{aligned}
T(\mathbf{x}+\mathbf{y}) &= T(x_1+y_1, x_2+y_2, x_3+y_3) \\
&= (2(x_1+y_1) - (x_2+y_2) + x_3 + y_3, x_1 + y_1 + 3(x_2+y_2) - 5(x_3+y_3)) \\
&= (2x_1 + 2y_1 - x_2 - y_2 + x_3 + y_3, x_1 + y_1 + 3x_2 + 3y_2 - 5x_3 - 5y_3) \\
&= (2x_1 - x_2 + x_3 + 2y_1 - y_2 + y_3, x_1 + 3x_2 - 5x_3 + y_1 + 3y_2 - 5y_3) \\
&= (2x_1 - x_2 + x_3, x_1 + 3x_2 - 5x_3) + (2y_1 - y_2 + y_3, y_1 + 3y_2 - 5y_3) \\
&= T(\mathbf{x}) + T(\mathbf{y}).
\end{aligned}$$

(b)

$$\begin{aligned}
T(c\mathbf{x}) &= T(cx_1, cx_2, cx_3) \\
&= (2cx_1 - cx_2 + cx_3, cx_1 + 3cx_2 - 5cx_3) \\
&= c(2x_1 - x_2 + x_3, x_1 + 3x_2 - 5x_3) \\
&= cT(\mathbf{x}).
\end{aligned}$$

□

Definition 5.46 (Linear mapping). Let V and W be vector spaces. A mapping $T : V \to W$ is said to be $\mathbb{F}$-*linear* (or just *linear*, if $\mathbb{F}$ is understood) if $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v} \in V$ and all $c \in \mathbb{F}$.[41]

If $W = V$, then a linear mapping $T : V \to V$ is said to be a *linear operator* on V.

The two conditions in Definition 5.46 can be combined into a single one:

Proposition 5.47 (Test for linearity). A mapping $T : V \to W$ of vector spaces is linear if and only if $T(c\mathbf{v}_1 + \mathbf{v}_2) = cT(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and all $c \in \mathbb{F}$.

Proof. The proof is very similar to that of Theorem 5.16. The proof of necessity is trivial. To prove sufficiency, note that

$$\begin{aligned}
T(\mathbf{v}_1 + \mathbf{v}_2) &= T(1\mathbf{v}_1 + \mathbf{v}_2) \\
&= 1T(\mathbf{v}_1) + T(\mathbf{v}_2) \\
&= T(\mathbf{v}_1) + T(\mathbf{v}_2),
\end{aligned}$$

so T preserves addition. To prove that T preserves scalar multiplication, first note that, since T preserves addition, it follows that

$$\begin{aligned}
T(\mathbf{0}) &= T(-\mathbf{v} + \mathbf{v}) \\
&= T((-1)\mathbf{v} + \mathbf{v}) \\
&= (-1)T(\mathbf{v}) + T(\mathbf{v}) \\
&= -T(\mathbf{v}) + T(\mathbf{v}) \\
&= \mathbf{0}.
\end{aligned}$$

[41]Since our vectors are often themselves functions, we use the word "mapping" here rather than function, even though these two words are being used synonymously.

Therefore,

$$\begin{aligned} T(c\mathbf{v}) &= T(c\mathbf{v} + \mathbf{0}) \\ &= cT(\mathbf{v}) + T(\mathbf{0}) \\ &= cT(\mathbf{v}) + \mathbf{0} \\ &= cT(\mathbf{v}), \end{aligned}$$

which shows that T preserves scalar multiplication. Hence, T is linear. □

Exercise 5.29. Determine whether each of the following mappings are linear:

(a) $T_1 : M^{n\times n}(\mathbb{F}) \to M^{n\times n}(\mathbb{F})$ defined by $T_1(A) = A^T$,

(b) $T_2 : M^{n\times n}(\mathbb{F}) \to \mathbb{F}$ defined by $T_2(A) = \det(A)$.

Solution. (a) Let A, B be $n \times n$ matrices and $c \in \mathbb{R}$. Then

$$\begin{aligned} T_1(cA + B) &= (cA + B)^T \\ &= (cA)^T + B^T \\ &= cA^T + B^T \\ &= cT_1(A) + T_2(B). \end{aligned}$$

Hence, T_1 is linear.

(b) Let A be an $n \times n$ matrix and let $c \in \mathbb{R}$. Then

$$\begin{aligned} T_2(cA) &= \det(cA) \\ &= c^n \det(A) \neq c \det(A) \end{aligned}$$

if $n > 1$. Hence, T_2 is not linear if $n > 1$.

□

Example 5.48 (The zero map). Define $T : V \to W$ by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. Then this is linear, since for all $\mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{R}$ we have

$$\begin{aligned} T(c\mathbf{v} + \mathbf{w}) &= \mathbf{0} \\ &= \mathbf{0} + \mathbf{0} \\ &= c\mathbf{0} + \mathbf{0} \\ &= cT(\mathbf{v}) + T(\mathbf{w}). \end{aligned}$$

Example 5.49 (The identity map). Let V be any vector space. The map $I_V : V \to V$ defined by $I_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$ is called the *identity map* on V. This is a linear operator on V, since for any $\mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{R}$ we have

$$\begin{aligned} I_V(c\mathbf{v} + \mathbf{w}) &= c\mathbf{v} + \mathbf{w} \\ &= cI_V(\mathbf{v}) + I_V(\mathbf{w}). \end{aligned}$$

Exercise 5.30. Let $\varphi : A \to B$ be any mapping from a *set* A to a set B. Show that composition by φ is a linear mapping from $\mathbb{F}^B$ to $\mathbb{F}^A$. That is, show that $T : \mathbb{F}^B \to \mathbb{F}^A$ defined by $T(f) = f \circ \varphi$ is linear.

Solution. Let $f, g \in \mathbb{R}^B$ and $c \in \mathbb{F}$. Note that $f \circ \varphi, g \circ \varphi : A \to \mathbb{F}$. For all $a \in A$ we have

$$\begin{aligned}
T((cf+g)(a)) &= [(cf+g)\circ\varphi](a) \text{ (by def of } T) \\
&= (cf+g)(\varphi(a)) \text{ (by def of composition)} \\
&= cf(\varphi(a)) + g(\varphi(a)) \text{ (by def of addition and scalar multiplication in } \mathbb{F}^B) \\
&= c(f\circ\varphi)(a) + (g\circ\varphi)(a) \text{ (by def of composition)} \\
&= cT(f(a)) + T(g(a)) \text{ (by def of } T) \\
&= [cT(f) + T(g)](a) \text{ (by def of addition and scalar multiplication in } \mathbb{F}^A)
\end{aligned}$$

hence

$$T(cf+g) = cT(f) + T(g)$$

and therefore T is linear. □

Proposition 5.50 (Linear maps preserve linear combinations). If $T : V \to W$ is linear, then

$$T\left(\sum_{i=1}^{n} c_i \mathbf{v}_i\right) = \sum_{i=1}^{n} c_i T(\mathbf{v}_i)$$

for any linear combination $\sum_{i=1}^{n} c_i \mathbf{v}_i$ of V.

Proof. (By induction on n.) The base case was established in Proposition 5.47. Suppose now this holds for any linear combination of n vectors $\sum_{i=1}^{n} c_i \mathbf{v}_i$ in V. Then

$$\begin{aligned}
T(\sum_{i=1}^{n} c_i \mathbf{v}_i + c_{n+1}\mathbf{v}_{n+1}) &= T(\sum_{i=1}^{n} c_i \mathbf{v}_i) + T(c_{n+1}\mathbf{v}_{n+1}) \\
&= \sum_{i=1}^{n} c_i T(\mathbf{v}_i) + c_{n+1}T(\mathbf{v}_{n+1}),
\end{aligned}$$

completing the proof. □

Example 5.51. For any $f_i \in \mathscr{C}([a,b])$ and $c_i \in \mathbb{R}$,

$$\int_a^b (\sum_{i=1}^{n} c_i f_i) = \sum_{i=1}^{n} c_i \int_a^b f_i.$$

Example 5.52. Every linear mapping $\mathbb{F} \to \mathbb{F}$ is of the form $x \mapsto kx$ for some fixed $k \in \mathbb{F}$. To see this, for fixed $k \in \mathbb{F}$ denote this mapping by T_k. Then

$$\begin{aligned}
T_k(cx+y) &= k(cx+y) \\
&= kcx + ky \\
&= ckx + ky \\
&= cT_k(x) + T_k(y)
\end{aligned}$$

so T_k is indeed linear. Conversely, let $T : \mathbb{F} \to \mathbb{F}$ be any linear mapping. Then, since $1x = x$ for every $x \in \mathbb{F}$, we have

$$\begin{aligned} T(x) &= T(1x) \\ &= xT(1) \\ &= T(1)x \end{aligned}$$

and therefore $T(x) = kx$, for $k = T(1)$, which is a fixed element of $\mathbb{F}$.

Example 5.52 is a special case of the following theorem:

Theorem 5.53 (Every linear mapping $\mathbb{F}^n \to \mathbb{F}^m$ can be represented by a matrix). Let A be an $m \times n$ matrix $T_A : \mathbb{F}^n \to \mathbb{F}^m$ be defined by multiplication by A:

$$T_A(\mathbf{x}) = A\mathbf{x}.$$

Then T_A is linear. Moreover, every linear mapping $T : \mathbb{F}^n \to \mathbb{F}^m$ is of this form for some fixed matrix A.

Proof. To show that T_A is linear, let $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ and $c \in \mathbb{F}$. Then

$$\begin{aligned} T_A(c\mathbf{x} + \mathbf{y}) &= A(c\mathbf{x} + \mathbf{y}) \\ &= A(c\mathbf{x}) + A\mathbf{y} \\ &= cA\mathbf{x} + A\mathbf{y} \\ &= cT_A(\mathbf{x}) + T_A(\mathbf{y}), \end{aligned}$$

hence T_A is linear.

Conversely, let $T : \mathbb{F}^n \to \mathbb{F}^m$ be any linear mapping. By Example 5.40, we can write any vector $\mathbf{x} \in \mathbb{F}^n$ as $\mathbf{x} = \sum_{i=1}^{n} x_i e_i$, where $\{e_i\}_{i=1}^{n}$ are the standard unit vectors for $\mathbb{F}^n$. By Proposition 5.50 we then have

$$\begin{aligned} T(\mathbf{x}) &= T\left(\sum_{i=1}^{n} x_i e_i\right) \\ &= \sum_{i=1}^{n} x_i T(e_i), \end{aligned}$$

which is exactly the result of multiplying the vector $\mathbf{x} \in \mathbb{F}^n$ by the $m \times n$ matrix A whose ith column is given by $T(e_i)$, which is a fixed vector in $\mathbb{F}^m$. $\square$

While Theorem 5.53 proves that every linear mapping $T : \mathbb{F}^n \to \mathbb{F}^m$ can be represented by a matrix, this matrix is not unique.

Example 5.54. Consider the linear mapping in Exercise 5.28. Since

$$\begin{aligned} T(1,0,0) &= (2,1) \\ T(0,1,0) &= (-1,3) \\ T(0,0,1) &= (1,-5), \end{aligned}$$

by Theorem 5.53 the linear mapping T is multiplication by the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -5 \end{bmatrix}.$$

However, we saw in Example 5.43 that the vectors $\mathbf{v}_1 = (1,1,2), \mathbf{v}_2 = (1,0,1), \mathbf{v}_3 = (2,1,4)$ also span $\mathbb{R}^3$. Thus, for any $\mathbf{x} \in \mathbb{R}^3$ we can find coefficients c_i such that $\mathbf{x}$ can also be written as $\mathbf{x} = \sum_{i=1}^3 c_i \mathbf{v}_i$. Applying Proposition 5.50 now gives

$$\begin{aligned} T(\mathbf{x}) &= T(\sum_{i=1}^3 c_i \mathbf{v}_i) \\ &= \sum_{i=1}^3 c_i T(\mathbf{v}_i) \end{aligned}$$

which is the result of multiplying $\mathbf{x}$ by the matrix A' whose ith column is given by $T(\mathbf{v}_i)$. Since

$$\begin{aligned} T(1,1,2) &= (2-1+2, 1+3-10) = (3,-6) \\ T(1,0,1) &= (2-0+1, 1+0-5) = (3,-4) \\ T(2,1,4) &= (4-1+4, 2+3-20) = (7,-15), \end{aligned}$$

the matrix A' given by

$$A' = \begin{bmatrix} 3 & 3 & 7 \\ -6 & -3 & -15 \end{bmatrix},$$

represents the *same* linear mapping T. For example, let $\mathbf{x} = (1,2,3)$. Then, we may write $\mathbf{x} = 1e_1 + 2e_2 + 3e_3$ and then we have

$$\begin{aligned} T(\mathbf{x}) &= 1T(e_1) + 2T(e_2) + 3T(e_3) \\ &= 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -8 \end{bmatrix}. \end{aligned}$$

But since

$$\begin{aligned} e_1 &= \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 \\ e_2 &= 2\mathbf{v}_1 - \mathbf{v}_3 \\ e_3 &= -\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 \end{aligned}$$

we can also write

$$\begin{aligned} \mathbf{x} &= 1e_1 + 2e_3 + 3e_3 \\ &= (\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3) + 2(2\mathbf{v}_1 - \mathbf{v}_3) + 3(-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3) \\ &= 2\mathbf{v}_1 - \mathbf{v}_2 \end{aligned}$$

and therefore

$$\begin{aligned} T(\mathbf{x}) &= 2T(\mathbf{v}_1) - T(\mathbf{v}_2) + 0T(\mathbf{v}_3) \\ &= 2\begin{bmatrix} 3 \\ -6 \end{bmatrix} - \begin{bmatrix} 3 \\ -4 \end{bmatrix} + 0\begin{bmatrix} 7 \\ -15 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 & 7 \\ -6 & -4 & -15 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -8 \end{bmatrix}, \end{aligned}$$

which is the same as above.

We have just seen that studying linear mappings $\mathbb{F}^n \to \mathbb{F}^m$ is essentially the same as studying $m \times n$ matrices. However, we have also seen that different matrices might represent the same linear mapping, so in practice we will have to make a particular choice for which matrix to use. We will see how to deal with keeping track of this choice later in section 5.14. For now, the standard choice will be the matrix whose columns are formed by the images of the standard unit vectors under T.

Definition 5.55 (Standard matrix of a linear mapping $\mathbb{F}^n \to \mathbb{F}^m$). Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear mapping. The matrix

$$A = [T(e_1), T(e_2), \ldots, T(e_n)]$$

is called the *standard matrix* of the linear mapping T.

Exercise 5.31. Find the standard matrix of the linear mapping $T : \mathbb{F}^2 \to \mathbb{F}^3$ defined by $T(\mathbf{x}) = \mathbf{w}$, where

$$\begin{aligned} w_1 &= -x_1 + x_2 \\ w_2 &= 3x_1 - 2x_2 \\ w_3 &= 5x_1 - 7x_2. \end{aligned}$$

Solution. Since

$$\begin{aligned} T(1,0) &= (-1,3,5) \\ T(0,1) &= (1,-2,7), \end{aligned}$$

the standard matrix A of T is given by

$$A = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & 7 \end{bmatrix}.$$

□

When a linear mapping is 1-1, it is usually referred to as a *linear transformation*. Linear transformations have important applications in geometry. The case of linear transformations of the plane is treated in detail in section 4.9 of your textbook.

5.9 Properties of Linear Mappings

In section 5.3 we proved that the set of all $\mathbb{F}$-valued functions on a common domain A, which we denoted $\mathbb{F}^A$, is a vector space under the operations

$$(f+g)(a) = f(a) + g(a)$$
$$(xf)(a) = xf(a).$$

We can generalize $\mathbb{F}^A$ by replacing $\mathbb{F}$ by any other vector space W.

Proposition 5.56. Let W be a vector space, and let $W^A \equiv \{f | f : A \to W\}$ denote the set of all W-valued functions on a common domain A. Then W^A is a vector space under

$$(f+g)(a) = f(a) + g(a)$$
$$(xf)(a) = xf(a)$$

where the operations on the right hand side are those of W.

Proof. As before, the vector space axioms hold for W^A because they hold for W. The proof is exactly the same as that of Proposition 5.10, *mutatis mutandis*. $\square$

Example 5.57. The analog of $\mathbb{F}^n$ is W^n, the set of all n-tuples $(\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n)$ of vectors in W. We can view this space as a function space $W^{\overline{n}} = \{f | f : \overline{n} \to W\}$.

For instance, if W is the vector space of 2×2 matrices, then

$$\mathbf{w} = (\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} e & \pi \\ \sqrt{2} & -4 \end{bmatrix}) \in W^3.$$

In general, we will be most interested in the case where $A = V$ is another vector space. Then W^V is the set of all functions from V to W. The set of all *linear* maps is naturally singled out, as they preserve the vector operations.

Proposition 5.58. The set of all $\mathbb{F}$-linear maps $T : V \to W$ is a subspace of W^V. We will denote this subspace by $\mathrm{Hom}_{\mathbb{F}}(V, W)$. If $\mathbb{F}$ is understood, then we may instead write simply $\mathrm{Hom}_{\mathbb{F}}(V, W)$. [42] If $V = W$, we will simply write $\mathrm{Hom}(V)$ instead of $\mathrm{Hom}(V, V)$.

Proof. Let $T_1, T_2 \in \mathrm{Hom}(V, W)$ and let $x \in \mathbb{F}$. Then for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $c \in \mathbb{F}$,

$$\begin{aligned}
(xT_1 + T_2)(c\mathbf{v}_1 + \mathbf{v}_2) &= xT_1(c\mathbf{v}_1 + \mathbf{v}_2) + T_2(c\mathbf{v}_1 + \mathbf{v}_2) \text{ (by def of addition and scalar mult in } W^V) \\
&= x(cT_1(\mathbf{v}_1) + T_1(\mathbf{v}_2)) + cT_2(\mathbf{v}_1) + T_2(\mathbf{v}_2) \text{ (since } T_1 \text{ and } T_2 \text{ are linear)} \\
&= xcT_1(\mathbf{v}_1) + xT_1(\mathbf{v}_2) + cT_2(\mathbf{v}_1) + T_2(\mathbf{v}_2) \text{ (by S3, Def. 5.1 for } W) \\
&= xcT_1(\mathbf{v}_1) + cT_2(\mathbf{v}_1) + xT_1(\mathbf{v}_2) + T_2(\mathbf{v}_2) \text{ (by commutativity of addition in } W) \\
&= cxT_1(\mathbf{v}_1) + cT_2(\mathbf{v}_1) + xT_1(\mathbf{v}_2) + T_2(\mathbf{v}_2) \text{ (by commutativity of multiplication in } \mathbb{F}) \\
&= c(xT_1 + T_2)(\mathbf{v}_1) + (xT_1 + T_2)(\mathbf{v}_2) \text{ (by def of addition in } W^V).
\end{aligned}$$

[42] In mathematics, once one defines a particular class of objects of interest it is customary to immediately define maps between objects which preserve the objects' defining properties. Such structure-preserving maps are called *homomorphisms*, which comes from Greek words "homos" meaning "same" and "morphe" meaning "form". The set of all homomorphisms from an object A to another object B is then customarily denoted $\mathrm{Hom}(A, B)$. A class of objects together with all the Hom-sets between objects forms a *category*. The collection of all vector spaces together with all linear maps therefore gives us the category Vect of vector spaces, where Obj(Vect) is the collection of all vector spaces, and for any $V, W \in$ Obj(Vect), $\mathrm{Hom}(V, W)$ is the set of linear maps from $V \to W$.

Hence, $xT_1 + T_2$ is linear and therefore $\mathrm{Hom}(V, W)$ is a subspace of W^V by the subspace criterion. □

We will now study some important properties of linear maps.

Theorem 5.59 (Linear maps preserve subspaces). Let $T : V \to W$ be a linear map, and A a subset of V. Denote by $T(A)$ the image of A under T; that is, $T(A)$ is the set $T(A) = \{T(\mathbf{v}_i) : \mathbf{v}_i \in A\}$. Then

(a) $T(\mathrm{Span}(A)) = \mathrm{Span}(T(A))$. In particular, if A is a subspace of V, then $T(A)$ is a subspace of W.

(b) Furthermore, if Y is a subspace of W, then $T^{-1}(Y)$ is a subspace of V.

Proof. (a) Let $A = \{\mathbf{v}_i\}_{i\in I}$. Then $T(A) = \{T(\mathbf{v}_i)\}_{i\in I}$. Since T is linear, by Proposition 5.50

$$T\left(\sum_{i=1}^{n} c_i\mathbf{v}_i\right) = \sum_{i=1}^{n} c_i T(\mathbf{v}_i)$$

for any linear combination $\sum_{i=1}^{n} c_i\mathbf{v}_i$ in V. This formula shows that $\mathbf{w} \in T(\mathrm{Span}(A))$ if and only if $\mathbf{w} \in \mathrm{Span}(T(A))$, hence $T(\mathrm{Span}(A)) = \mathrm{Span}(T(A))$. If A is a subspace of V, then by part (c) of Theorem 5.38 $A = \mathrm{Span}(A)$. Then by the preceding arguments $T(A) = T(\mathrm{Span}(A)) = \mathrm{Span}(T(A))$, which is a subspace of W.

(b) Let Y be a subspace of W. Since T is linear, $\mathbf{0} \in T^{-1}(\mathbf{0})$, so $T^{-1}(Y)$ is not empty. Let $\mathbf{v}_1, \mathbf{v}_2 \in T^{-1}(Y)$ and $c \in \mathbb{R}$. Then there exist $\mathbf{w}_1, \mathbf{w}_2 \in Y$ such that $\mathbf{v}_1 = T(\mathbf{w}_1)$ and $\mathbf{v}_2 = T(\mathbf{w}_2)$, and therefore

$$\begin{aligned} c\mathbf{v}_1 + \mathbf{v}_2 &= cT(\mathbf{w}_1) + T(\mathbf{w}_2) \\ &= T(c\mathbf{w}_1 + \mathbf{w}_2) \text{ (since } T \text{ is linear)}, \end{aligned}$$

which shows that $c\mathbf{w}_1 + \mathbf{w}_2 \in T^{-1}(Y)$. Thus, $T^{-1}(Y)$ is a subspace of V by Theorem 5.16. □

Example 5.60. Let $T \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^4)$ have standard matrix

$$A = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

and let

$$V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3.$$

Then

$$\begin{aligned} \mathrm{Span}\, V &= \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} c_1 + 2c_2 \\ 2c_1 - c_2 \\ 0 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \end{aligned}$$

and therefore

$$\begin{aligned}
T(\text{Span}(A)) &= \left\{ \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix} \begin{bmatrix} c_1 + 2c_2 \\ 2c_1 - c_2 \\ 0 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} 11c_1 - 3c_2 \\ 14c_1 - 2c_2 \\ 17c_1 - c_2 \\ 20c_1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\
&= \left\{ c_1 \begin{bmatrix} 11 \\ 14 \\ 17 \\ 20 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\
&= \text{Span} \left\{ \begin{bmatrix} 11 \\ 14 \\ 17 \\ 20 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \end{bmatrix} \right\} \\
&= \text{Span} \left\{ T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right), T\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right) \right\} \\
&= \text{Span}(T(A)).
\end{aligned}$$

Theorem 5.61 (Kernel and Image of a Linear Map). Let $T : V \to W$ be a linear mapping between two vector spaces. Then

(a) the set $T^{-1}(\mathbf{0}) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$ is a subspace of V, called the *kernel* of T, denoted $\ker T$.

(b) The set $T(V) = \{\mathbf{W} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$ is a subspace of W called the *image of V under T*, denoted $\operatorname{Im} T$.

Proof. (a) $\{\mathbf{0}\}$ is a subspace of W, so by part (b) of Theorem 5.59 $T^{-1}(\mathbf{0})$ is a subspace of V.

(b) By part (a) of Theorem 5.59 , linear maps preserve subspaces so $\operatorname{Im} V$ is a subspace of W.

□

Example 5.62. (a) The zero map $0 : V \to W$ defined by $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$ has

$$\ker 0 = V, \ \operatorname{Im} V = \{\mathbf{0}\}.$$

(b) The identity map $I_V : V \to V$ defined by $I_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$ has

$$\ker I_V = \{\mathbf{0}\}, \ \operatorname{Im} V = V.$$

Example 5.63. Consider the matrix mapping $T : \mathbb{R}^2 \to \mathbb{R}^3$ whose standard matrix is

$$A = \begin{bmatrix} 1 & 4 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}.$$

By definition, $\ker T = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{0}\}$ = the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ (which we indeed saw was always a subspace of V in Example 5.22). Since

$$\begin{bmatrix} 1 & 4 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$$

is row-equivalent to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we see that this system has only the trivial solution and therefore $\ker T = \{\mathbf{0}\}$.

Now $\operatorname{Im} T = \{\mathbf{b} \in \mathbb{R}^3 : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^2\}$ = the set of all $\mathbf{b} \in \mathbb{R}^3$ such that the inhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ is consistent. Since

$$\begin{bmatrix} 1 & 4 & b_1 \\ -1 & 1 & b_2 \\ 0 & 2 & b_3 \end{bmatrix}$$

is row-equivalent to

$$\begin{bmatrix} 1 & 4 & b_1 \\ 0 & 5 & b_1 + b_2 \\ 0 & 0 & -\frac{2}{5}b_1 - \frac{2}{5}b_2 + b_3 \end{bmatrix},$$

the system is consistent if and only if $-\frac{2}{5}b_1 - \frac{2}{5}b_2 + b_3 = 0 \implies b_3 = \frac{2}{5}(b_1 + b_2)$. Therefore

$$\begin{aligned} \operatorname{Im} V &= \left\{ \begin{bmatrix} b_1 \\ b_2 \\ \frac{2}{5}(b_1 + b_2) \end{bmatrix} : b_1, b_2 \in \mathbb{R} \right\} \\ &= \left\{ b_1 \begin{bmatrix} 1 \\ 0 \\ \frac{2}{5} \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix} : b_1, b_2 \in \mathbb{R} \right\} \\ &= \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{2}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix} \right\}. \end{aligned}$$

Theorem 5.64 (Kernel of an injective linear map). A linear mapping T is injective (1-1) if and only if $\ker T = \{\mathbf{0}\}$.

Proof. Suppose T is injective. Since T is linear $T(\mathbf{0}) = \mathbf{0}$. Now suppose $T(\mathbf{x}) = \mathbf{0}$ for some $\mathbf{x} \in V$. Since T is injective, $T(\mathbf{x}) = T(\mathbf{0}) \implies \mathbf{x} = \mathbf{0}$, hence $\ker T = \{\mathbf{0}\}$.

Now suppose $\ker T = \{\mathbf{0}\}$, and suppose $T(\mathbf{x}) = T(\mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in V$. Since T is linear,

$$\begin{aligned} & T(\mathbf{x}) = T(\mathbf{y}) \\ \implies & T(\mathbf{x}) - T(\mathbf{y}) = \mathbf{0} \\ \implies & T(\mathbf{x} - \mathbf{y}) = \mathbf{0}, \end{aligned}$$

which says that $\mathbf{x} - \mathbf{y} \in \ker T = \{\mathbf{0}\}$ and therefore $\mathbf{x} - \mathbf{y} = \mathbf{0}$ and thus $\mathbf{x} = \mathbf{y}$, which proves that T is injective. □

[Include example.]

5.10 Isomorphisms

We have now seen a variety of examples of vector spaces. Some of these, however, are not essentially different. For instance, any singleton set $\{a\}$ can be given the structure of the zero vector space by defining $a + a = a$ and $xa = a$ for all $x \in \mathbb{F}$. There is no real difference, then, between any two zero vector spaces $\{a\}$ and $\{b\}$; all we have done is label the only vector in the set by a different symbol. Algebraically, these are exactly the same. In general, two vector spaces are the same in this sense if

1. there is a 1-1 correspondence (i.e., a bijective map) between the underlying sets, and
2. this 1-1 correspondence preserves vector addition and scalar multiplication in the sense of Definition 5.46; that is, it is linear.

Definition 5.65 (Isomorphism). Let V and W be vector spaces. A bijective linear map $T : V \to W$ is called an *isomorphism*. If such a map exists, then V and W are said to be *isomorphic* as vector spaces, and we denote this by $V \cong W$.

Example 5.66. We have noted before that $\mathbb{F}^2$ is not a subspace of $\mathbb{F}^3$. Indeed, $\mathbb{F}^2$ is not even a *subset* of $\mathbb{F}^3$, since elements of the former are ordered pairs of real numbers while elements of the latter are ordered triples of real numbers. However, the set $W = \{(x, y, 0)\} \subseteq \mathbb{F}^3$ is a subspace of $\mathbb{F}^3$ which is *isomorphic* to $\mathbb{F}^2$. An isomorphism is given by the map

$$\psi : W \to \mathbb{F}^2$$
$$\psi(x, y, 0) = (x, y).$$

To verify, note that this map is surjective since $(x, y) = \psi(x, y, 0)$ for all $(x, y) \in \mathbb{F}^2$ and is injective since if $\psi(x_1, y_1, 0) = \psi(x_2, y_2, 0)$ then $(x_1, y_1) = (x_2, y_2) \implies x_1 = x_2$ and $y_1 = y_2$ and therefore $(x_1, y_1, 0) = (x_2, y_2, 0)$. Finally, it is linear since for any $(x_1, y_1, 0), (x_2, y_2, 0) \in W$ and $c \in \mathbb{F}$ we have

$$\begin{aligned}
\psi(c(x_1, y_1, 0) + (x_2, y_2, 0)) &= \psi(cx_1 + x_2, cy_1 + y_2, 0) \\
&= (cx_1 + x_2, cy_1 + y_2) \\
&= c(x_1, y_1) + (x_2, y_2) \\
&= c\psi(x_1, y_1, 0) + \psi(x_2, y_2, 0).
\end{aligned}$$

Example 5.67 ($P_{n-1} \cong \mathbb{F}^n$). Let P_{n-1} denote the vector space of all polynomials of degree $\leq n - 1$ as in Example 5.21. For each $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}^n$, define $T : \mathbb{F}^n \to P_{n-1}$ by $T(\mathbf{c}) = \sum_{i=0}^{n-1} c_i x^i$. We will show this mapping is an isomorphism. Let $\mathbf{c}, \mathbf{d} \in \mathbb{F}^n$ and let $k \in \mathbb{F}$. Then

$$\begin{aligned}
T(k\mathbf{c} + \mathbf{d}) &= \sum_{i=0}^{n-1} (k\mathbf{c} + \mathbf{d})_i x^i \\
&= \sum_{i=0}^{n-1} (kc_i + d_i) x^i \\
&= \sum_{i=0}^{n-1} kc_i x^i + \sum_{i=0}^{n-1} d_i x^i \\
&= k \sum_{i=0}^{n-1} c_i x^i + \sum_{i=0}^{n-1} d_i x^i \\
&= kT(\mathbf{c}) + T(\mathbf{d}),
\end{aligned}$$

hence T is linear.

The map T is clearly surjective, since for any polynomial $\sum_{i=0}^{n-1} c_i x^i \in P_{n-1}$, $\sum_{i=0}^{n-1} c_i x^i = T(\mathbf{c})$. To see that T is injective, suppose $T(\mathbf{c}) = T(\mathbf{d})$ for some $\mathbf{c}, \mathbf{d} \in \mathbb{F}^n$. Then, for all $x \in \mathbb{F}$,

$$\begin{aligned} \sum_{i=0}^{n-1} c_i x^i &= \sum_{i=0}^{n-1} cd_i x^i \\ \implies \sum_{i=0}^{n-1} (c_i - d_i) x^i &= 0. \end{aligned}$$

By the fundamental theorem of algebra, a degree $n-1$ polynomial has at most $n-1$ roots. Therefore, the only way this can vanish for all $x \in \mathbb{F}$ is if $\mathbf{c} = \mathbf{d}$. Hence, T is also injective, and therefore an isomorphism. We have proved that $P_{n-1} \cong \mathbb{F}^n$.

Example 5.68. Consider the "unusual" vector space V in Example 5.6, defined by $V = \mathbb{R}^+$ (the positive reals) with addition and multiplication defined by

$$\begin{aligned} x + y &:= xy \\ kx &:= x^k \end{aligned}$$

for all $x, y \in V$ and all $k \in \mathbb{R}$. Let b be any fixed positive real number, and define $T_b : \mathbb{R} \to \mathbb{R}^+$ by $T_b(x) = b^x$ for all $x \in \mathbb{R}$. This mapping is linear, since for all $x, y, k \in \mathbb{R}$,

$$\begin{aligned} T_b(kx + y) &= b^{kx+y} \\ &= b^{kx} b^y \\ &= (b^x)^k b^y \\ &= kT_b(x) + T_b(y). \end{aligned}$$

Of course, $T_b(x) = b^x$ has a two-sided inverse given by $T_b^{-1}(x) = \log_b(x)$, so T_b is a bijection, and therefore T_b is an isomorphism. Thus, $V \cong \mathbb{R}$. This shows that this "unusual" vector space is actually not unusual at all; it is the same as the "ordinary" vector space $\mathbb{R}$.

Exercise 5.32. Show that the mapping $T : M^{2\times 2}(\mathbb{F}) \to \mathbb{F}^4$ defined by

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = (a_{11}, a_{12}, a_{21}, a_{22})$$

is an isomorphism.

Solution. Since

$$\begin{aligned} T\left(c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) &= T\left(\begin{bmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} \\ ca_{21} + b_{21} & ca_{22} + b_{22} \end{bmatrix}\right) \\ &= (ca_{11} + b_{11}, ca_{12} + b_{12}, ca_{21} + b_{21}, ca_{22} + b_{22}) \\ &= c(a_{11}, a_{12}, a_{21}, a_{22}) + (b_{11}, b_{12}, b_{21}, b_{22}) \\ &= T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) + T\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right), \end{aligned}$$

T is linear. Let $(x, y, z, w) \in \mathbb{F}^4$. Then $(x, y, z, w) = T\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)$, so T is surjective. Finally, suppose

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = T\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right).$$

Then

$$(a_{11}, a_{12}, a_{21}, a_{22}) = (b_{11}, b_{12}, b_{21}, b_{22}).$$

Two vectors are equal if and only if their corresponding entries are equal, so this implies

$$a_{11} = b_{11}, a_{12} = b_{12}, a_{21} = b_{21}, a_{22} = b_{22}$$

and therefore

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Thus, T is also injective, and therefore an isomorphism. Hence $M^{2\times 2}(\mathbb{F}) \cong \mathbb{F}^4$. The same argument shows that $M^{m\times n}(\mathbb{F}) \cong \mathbb{F}^{mn}$. The reader is invited to fill in the details of the proof. $\square$

We will now prove that any two isomorphic vector spaces are equivalent. We will need the following lemma.

Lemma 5.69 (Inverse, composition of isomorphisms).

(a) If $\varphi : V \to W$ is an isomorphism, then so is $\varphi^{-1} : W \to V$.

(b) If $\varphi : U \to V$ and $\psi : V \to W$ are isomorphisms, then so is $\psi \circ \varphi : U \to W$.

(c) If $\varphi : U \to V$ and $\psi : V \to W$ are isomorphisms, then $(\psi \circ \varphi)^{-1} : W \to U$ is an isomorphism and

$$(\psi \circ \varphi)^{-1} = \varphi^{-1} \circ \psi^{-1}.$$

Proof. (a) Let $\varphi : V \to W$ be an isomorphism. Since φ is bijective, its inverse $\varphi^{-1} : W \to V$ exists (which is also bijective, with inverse φ). Let $\mathbf{w}_1, \mathbf{w}_2 \in W$ and let $c \in \mathbb{R}$. Then since φ is surjective, there exists $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $\mathbf{w}_1 = \varphi(\mathbf{v}_1)$ and $\mathbf{w}_2 = \varphi(\mathbf{v}_2)$. We have then

$$\begin{aligned}
\varphi^{-1}(c\mathbf{w}_1 + \mathbf{w}_2) &= \varphi^{-1}(c\varphi(\mathbf{v}_1) + \varphi(\mathbf{v}_2)) \\
&= \varphi^{-1}(\varphi(c\mathbf{v}_1 + \mathbf{v}_2) \text{ (since } \varphi \text{ is linear)} \\
&= (\varphi^{-1} \circ \varphi)(c\mathbf{v}_1 + \mathbf{v}_2) \\
&= c\mathbf{v}_1 + \mathbf{v}_2 \text{ (since } \varphi^{-1} \circ \varphi = \mathrm{id}_V) \\
&= c\varphi^{-1}(\mathbf{w}_1) + \varphi^{-1}(\mathbf{w}_2) \text{ (since } \varphi \text{ is injective)}
\end{aligned}$$

which shows that φ^{-1} is also linear, and hence an isomorphism.

(b) Suppose $\varphi : U \to V$ and $\psi : V \to W$ are isomorphisms, and let $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}(\psi \circ \varphi)(c\mathbf{u}_1 + \mathbf{u}_2) &= \psi(\varphi(c\mathbf{u}_1 + \mathbf{u}_2)) \\ &= \psi(c\varphi(\mathbf{u}_1) + \varphi(\mathbf{u}_2)) \text{ (since } \varphi \text{ is linear)} \\ &= c\psi(\varphi(\mathbf{u}_1)) + \psi(\varphi(\mathbf{u}_2)) \text{ (since } \psi \text{ is linear)} \\ &= c(\psi \circ \varphi)(\mathbf{u_1}) + (\psi \circ \varphi)(\mathbf{u_2})\end{aligned}$$

which shows that $\psi \circ \varphi$ is linear. To show that $\psi \circ \varphi$ is injective, suppose that

$$\psi(\varphi(\mathbf{u}_1)) = \psi(\varphi(\mathbf{u}_2))$$

for some $\mathbf{u}_1, \mathbf{u}_2 \in U$. Since ψ is injective, this implies that

$$\varphi(\mathbf{u}_1) = \varphi(\mathbf{u}_2)$$

and since φ is injective, this implies that

$$\mathbf{u}_1 = \mathbf{u}_2,$$

proving that $\psi \circ \varphi$ is injective. Finally, to show that $\psi \circ \varphi$ is surjective, let $\mathbf{w} \in W$. Then, since φ is surjective, there exists $\mathbf{v} \in V$ such that $\mathbf{w} = \varphi(\mathbf{v})$. Since ψ is surjective, there exists $\mathbf{u} \in U$ such that $\psi(\mathbf{u}) = \mathbf{v}$ and therefore such that

$$\psi(\varphi(\mathbf{u})) = \mathbf{w},$$

which proves that $\psi \circ \varphi$ is also surjective, and hence an isomorphism.

(c) By part (a) $\psi^{-1} : W \to V$ and $\varphi^{-1} : V \to U$ are isomorphisms and therefore by part (b) so is $\varphi^{-1} \circ \psi^{-1} : W \to U$. Now for any $\mathbf{w} \in W$ we have

$$\begin{aligned}(\psi \circ \varphi) \circ (\varphi^{-1} \circ \psi^{-1})(\mathbf{w}) &= \psi \circ (\varphi \circ \varphi^{-1}) \circ \psi^{-1}(\mathbf{w}) \text{ (by associativity of composition)} \\ &= \psi \circ (id_V) \circ \psi^{-1}(\mathbf{w}) \text{ (since } \varphi \text{ and } \varphi^{-1} \text{ are inverses)} \\ &= \psi \circ \psi^{-1}(\mathbf{w}) \text{ (by def of the identity map)} \\ &= id_W(\mathbf{w}) \text{ (by def of the identity map)} \\ &= \mathbf{w}\end{aligned}$$

and for any $\mathbf{u} \in U$ we have

$$\begin{aligned}(\varphi^{-1} \circ \psi^{-1}) \circ (\psi \circ \varphi)(\mathbf{u}) &= \varphi^{-1} \circ (\psi^{-1} \circ \psi) \circ \varphi(\mathbf{u}) \text{ (by associativity of composition)} \\ &= \varphi^{-1} \circ id_V \circ \varphi(\mathbf{u}) \text{ (since } \psi \text{ and } \psi^{-1} \text{ are inverses)} \\ &= \varphi^{-1} \circ \varphi(\mathbf{u}) \text{ (by def of the identity map)} \\ &= id_U(\mathbf{u}) \\ &= \mathbf{u}.\end{aligned}$$

This shows that

$$(\psi \circ \varphi) \circ (\varphi^{-1} \circ \psi^{-1}) = id_W$$

and

$$(\varphi^{-1} \circ \psi^{-1}) \circ (\psi \circ \varphi)(\mathbf{u}) = id_U,$$

and therefore $\varphi^{-1} \circ \psi^{-1}$ is the unique inverse of $\psi \circ \varphi$.

□

Theorem 5.70 (Isomorphism is an equivalence relation). Isomorphism is an equivalence relation on the collection of vector spaces.

Proof. We need to show reflexivity, symmetry, and transitivity. Let V be any vector space. Then the identity map (Example 5.49) is a linear bijection from $V \to V$. Hence, $V \cong V$. Now suppose that $V \cong W$. Then there exists a linear bijection $\varphi : V \to W$. By part (a) of Lemma 5.69, $\varphi^{-1} : W \to V$ is also an isomorphism, hence $W \cong V$. Finally, suppose that $U \cong V$ and $V \cong W$. Then there exist isomorphisms $\varphi : U \to V$ and $\psi : V \to W$. By part (b) of Lemma 5.69, $\psi \circ \varphi : U \to W$ is an isomorphism, and therefore $U \cong W$. Thus, isomorphism is an equivalence relation. $\square$

5.11 Dimension

An *invariant* of a vector space V is any quantity associated to V which is unchanged under an isomorphism. Thus, it is a quantity which is the same for all isomorphic vector spaces. In this section we will define an important invariant of any vector space, called the *dimension* of the vector space.

5.11.1 Finite-dimensional vector spaces

Definition 5.71 (Finite-dimensional vector space). A vector space is called *finite-dimensional* if it admits a finite spanning set. Otherwise, it is called *infinite-dimensional*.

Example 5.72.

(a) $\mathbb{R}^n$ is finite-dimensional, since $\mathbb{R}^n = \text{Span}\{e_1, e_2, \ldots, e_n\}$.

(b) P_{n-1} is also finite-dimensional, since $P_{n-1} = \text{Span}\{1, x, x^2, \ldots, x^{n-1}\}$.

(c) The vector space P_∞ of all polynomials is infinite-dimensional. To see this, suppose that $S = \{p_1, p_2, \ldots, p_k\}$ is a finite spanning set for P_∞. It is immediately clear that any polynomial of degree $> \max\{\deg p_i\}_{i=1}^k$ cannot be in the span of S, so P_∞ cannot be spanned by any finite set of polynomials. Hence, it is infinite-dimensional. [43]

Geometrically, $\mathbb{R}^1$ is a line, $\mathbb{R}^2$ is a plane, and $\mathbb{R}^3$ is physical space, so it makes sense to say that $\mathbb{R}^n$ is n-dimensional, as this agrees with the sense in which the word "dimension" is used in geometry (as the number of coordinates needed to specify a point in the space). While we have no geometric picture of P_{n-1}, we saw in Example 5.67 that $P_{n-1} \cong \mathbb{R}^n$, so we should really think of these as the same vector space. Hence, P_{n-1} it makes sense to say that P_{n-1} is also n-dimensional. This motivates the need for a more precise definition of dimension which does not rely on geometry, and which will hold for any vector space. The treatment of infinite-dimensional vector spaces is beyond the scope of this course, so we will restrict ourselves to finite-dimensional vector spaces.

In the examples of P_{n-1} and $\mathbb{R}^n$, their dimension was given by the number of vectors in the given spanning set. The number of vectors in a spanning set is therefore a good candidate for the dimension of a finite-dimensional vector space. However, we have seen that spanning sets are not

[43] We will prove later in Theorem 5.97 that every subspace of a finite-dimensional vector space is finite-dimensional, so this implies that $\mathbb{R}^{\mathbb{R}}$ and $\mathscr{C}^m(\mathbb{R})$ for $m = 0, 1, \ldots, \infty$ are all infinite-dimensional (since P_∞ is an infinite-dimensional subspace of each of these).

unique (see, e.g., Example 5.41). Indeed, they might not even have the same number of vectors! For instance, consider the set $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Since the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \end{array}\right]$$

is already in reduced row echelon form, we see that it is consistent for all $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$, proving that $\operatorname{Span} S = \mathbb{R}^2$. In fact, any set of the form $\{e_1, e_2, \mathbf{v}_1, \mathbf{v}_2, \dots\}$ will span $\mathbb{R}^2$ for exactly the same reason. However, it is not possible for any set with fewer than 2 vectors to span $\mathbb{R}^2$, since any augmented matrix

$$\left[\begin{array}{c|c} v_1 & b_1 \\ v_2 & b_2 \end{array}\right]$$

can have at most one pivot in the first column, so the corresponding system will always be inconsistent for some $\in \mathbb{R}^2$. Indeed, geometrically the span of a single vector in $\mathbb{R}^2$ is a line, and not a plane. Thus, we expect that the number of vectors in a *minimal* spanning set will provide a good definition of dimension.

5.11.2 Linearly Independent Sets and Bases

Before we discuss how to find minimal spanning sets, we introduce some terminology.

Definition 5.73 (Linear combination map). Let V be a vector space and $S = \{\mathbf{v}_i\}_{i=1}^n$ a finite subset of V. The map

$$L_S : \mathbb{R}^n \to V$$

defined by $L_S(\mathbf{x}) = \sum_{i=1}^n x_i \mathbf{v}_i$ is called the *linear combination map* corresponding to S.

Example 5.74. (a) Let $S = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$. Then $L_S : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\begin{aligned} L_S(\mathbf{x}) &= x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} x_1 - 3x_2 \\ -2x_1 + x_2 \end{bmatrix} \end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^2$.

(b) Let $S = \{e^t, \sin t, \cos t\} \subseteq \mathscr{C}^\infty(\mathbb{R})$. Then $L_S : \mathbb{R}^3 \to \mathscr{C}^\infty(\mathbb{R})$ is given by

$$L_S(\mathbf{x}) = x_1 e^t + x_2 \sin t + x_3 \cos t$$

for all $\mathbf{x} \in \mathbb{R}^3$.

Proposition 5.75 (Linear combination map is linear). The linear combination map is linear.

Proof. Let V be a vector space, $S = \{\mathbf{v}_i\}_{i=1}^n$ a finite subset of V, and $L_S : \mathbb{R}^n \to V$ the corresponding linear combination map. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} L_S(c\mathbf{x} + \mathbf{y}) &= \sum_{i=1}^n (c\mathbf{x} + \mathbf{y})_i \mathbf{v}_i \\ &= \sum_{i=1}^n (cx_i + y_i)\mathbf{v}_i \\ &= c\sum_{i=1}^n x_i \mathbf{v}_i + \sum_{i=1}^n y_i \mathbf{v}_i \\ &= cL_S(\mathbf{x}) + L_S(\mathbf{y}). \end{aligned}$$

□

Notice that $\operatorname{Span} S = \operatorname{Im} L_S$. If L_S is surjective, then $\operatorname{Im} L_S = V$, and therefore $\operatorname{Span} S = V$. We saw in Section 5.7 several examples of how to determine whether L_S is surjective. Let us now consider injectivity of L_S.

Definition 5.76 (Linearly independent sets, bases). Let V be a vector space, $S = \{\mathbf{v}_i\}_{i=1}^n$ a finite subset of V, and $L_S : \mathbb{R}^n \to V$ the corresponding linear combination map.

(a) The set S is *linearly independent* if L_S is injective. Otherwise, S is said to be *linearly dependent*.

(b) The set S is a *basis* for V if L_S is an isomorphism.

Proposition 5.77 (A useful characterization of linear independence). Let V be a vector space and $S = \{\mathbf{v}_i\}_{i=1}^n$ a finite subset of V. The set S is linearly independent if and only if $\sum_{i=1}^n x_i \mathbf{v}_i = \mathbf{0}$ implies $x_1 = x_2 = \cdots = x_n = 0$.

Proof. If $L_S(\mathbf{x}) = \sum_{i=1}^n x_i \mathbf{v}_i = \mathbf{0}$, then $\mathbf{x} \in \ker L_S$. By Theorem 5.64, L_S is injective if and only if its kernel contains only $\mathbf{0}$. □

We first make some general observations.

Example 5.78 (Some special sets). Let V be any vector space.

(a) If $S = \emptyset$, then the condition that $\sum_{i=1}^n x_i \mathbf{v}_i = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$ is vacuously true. So $\emptyset$ is linearly independent.

(b) If $S = \{\mathbf{v}\}$, then $x\mathbf{v} = \mathbf{0}$ has only the trivial solution if and only if $\mathbf{v} \neq \mathbf{0}$. Thus, a set with a single vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v}$ is not the zero vector.

(c) In fact, if S is any set of vectors which contains $\mathbf{0}$, then it is linearly dependent. To see this, write $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{0}\}$. Then

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k + x_{k+1}\mathbf{0} = \mathbf{0}$$

has a nontrivial solution, given by taking $x_1 = x_2 = \cdots = x_k = 0$ and $x_{k+1} \neq 0$.

(d) Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then S is linearly independent if and only if each vector is not a scalar multiple of the other. To see this, suppose that $\mathbf{v}_2 = c\mathbf{v}_1$ for some $c \neq 0$. Then

$$\begin{aligned} x_1\mathbf{v}_1 + x_2\mathbf{v}_2 &= \mathbf{0} \\ x_1\mathbf{v}_1 + x_2 c\mathbf{v}_1 &= \mathbf{0} \\ (x_1 + x_2 c)\mathbf{v}_1 &= \mathbf{0} \end{aligned}$$

has a nontrivial solution, given by choosing $x_1 = -cx_2$. Conversely, suppose that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$$

has a nontrivial solution. Without loss of generality we may assume $x_1 \neq 0$. Then we may divide this equation by x_1 to obtain

$$\mathbf{v}_1 = -\frac{x_2}{x_1}\mathbf{v}_2$$

so $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$. [44] Thus, the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k + x_{k+1}\mathbf{0} = \mathbf{0}$$

has a nontrivial solution if and only if one of the vectors in S can be written as a linear combination of the other one.

Generalizing part (d) of the previous example, we can give an equivalent condition for linear independence.

Theorem 5.79 (Equivalent definition of linear independence). Let V be any vector space. A finite set $S = \{\mathbf{v}_i\}_{i=1}^n \subseteq V$ is linearly independent if and only if no vector in S can be written as a linear combination of the others.

Proof. We will show that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

has a nontrivial solution if and only if some vector in S can be written as a linear combination of the others. First, suppose that $x_1 = c_2\mathbf{v_2} + \cdots + c_n\mathbf{v}_n$ for some $c_2, \ldots, c_n \in \mathbb{R}$. Then

$$\begin{aligned} x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n &= \mathbf{0} \\ x_1(c_2\mathbf{v_2} + \cdots + c_n\mathbf{v}_n) + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n &= \mathbf{0} \\ (x_1c_2 + x_2)\mathbf{v}_2 + (x_1c_3 + x_3)\mathbf{v}_3 + \cdots + (x_1c_n + x_n)\mathbf{v}_n &= \mathbf{0} \end{aligned}$$

has a nontrivial solution given by taking $x_i = -x_1c_i$ for $i = 2, \ldots, n$. Now suppose the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

has a nontrivial solution. Without loss of generality, we may assume that $x_1 \neq 0$. Then we may divide by x_1 to obtain

$$\mathbf{v}_1 = -\frac{x_2}{x_1}\mathbf{v}_2 - \frac{x_3}{x_1}\mathbf{v}_3 - \cdots - \frac{x_n}{x_1}\mathbf{v}_n$$

so we see that $\mathbf{v}_1$ is a linear combination of the remaining vectors in S. □

Exercise 5.33. Determine whether each of the following sets of vectors in $\mathbb{R}^3$ is linearly independent:

[44] This is true also when $x_2 = 0$, for then $\mathbf{v}_1 = \mathbf{0}$ and by part (c) S is linearly dependent.

(a) $S = \emptyset$

(b) $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$

(c) $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix} \right\}$

(d) $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -7 \end{bmatrix} \right\}$

(e) $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(f) $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

(g) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} \right\}$

(h) $S = \left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \\ 3 \end{bmatrix} \right\}$

Solution. (a) Yes, since the empty set is linearly independent.

(b) Yes, since a set containing a single nonzero vector is linearly independent.

(c) No, since the second vector is a scalar multiple of the first.

(d) Yes, since neither vector is a scalar multiple of the other.

(e) No, since any set containing the zero vector is linearly dependent.

(f) The equation $\sum_{i=1}^{3} \mathbf{v}_i = \mathbf{0}$ is equivalent to the homogeneous linear system

$$\begin{aligned} x_1 + 5x_2 + 3x_3 &= 0 \\ -2x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 - x_2 + x_3 &= 0. \end{aligned}$$

By the Invertible Matrix Theorem, the system has only the trivial solution if and only if the determinant of the coefficient matrix

$$\begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

is nonzero. By direct computation, we find

$$\begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

so S is linearly dependent.

(g) Since

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1 \neq 0,$$

S is linearly independent. The Invertible Matrix Theorem further implies that $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^3$, so S is a basis for $\mathbb{R}^3$.

(h) The coefficient matrix

$$\begin{bmatrix} -2 & 5 & 1 & -7 \\ 3 & -4 & 6 & 1 \\ 1 & 0 & 2 & 3 \end{bmatrix}$$

is row-equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & \frac{37}{5} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -\frac{11}{5} \end{bmatrix}$$

we see that the equation

$$x_1 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has the nontrivial solution

$$\begin{aligned} x_1 &= -\frac{37}{5}t \\ x_2 &= -2t \\ x_3 &= \frac{11}{5}t \\ x_4 &= t. \end{aligned}$$

Thus, S is linearly dependent.

□

Example 5.80. Let $S = \{e_i\}_{i=1}^n$ be the standard unit vectors for $\mathbb{R}^n$. Recall that $(e_i)_j$ (the jth component of the ith standard unit vector) is given by $(e_i)_j = \delta_{ij}$. [45] The jth component of $L_S(\mathbf{x})$ is then

[45] Here δ_{ij} is the *Kronecker delta symbol*, defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

These are exactly the entries of the $n \times n$ identity matrix.

given by

$$\begin{aligned}(L_S(\mathbf{x}))_j &= (\sum_{i=1}^{n} x_i e_i)_j \\ &= \sum_{i=1}^{n} x_i (e_i)_j \\ &= \sum_{i=1}^{n} x_i \delta_{ij} \\ &= x_j,\end{aligned}$$

so $L_S(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. We see that L_S is the identity map on $\mathbb{R}^n$, which is an isomorphism. Thus, $\{e_i\}_{i=1}^n$ is a basis for $\mathbb{R}^n$ called the *standard basis*.

Exercise 5.34. Show that the matrices $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ form a basis for $M^{2\times 2}(\mathbb{R})$. In general, the matrices $\{e_{ij}\}_{i=1,\ldots,m; j=1,\ldots,n}$, where e_{ij} has (i,j)-entry 1 and 0s for all other entries, form a basis for $M^{m\times n}(\mathbb{R})$ called the *standard basis*.

Solution. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M^{2\times 2}(\mathbb{R})$. Since

$$x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix},$$

the equation

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has the unique solution $x_1 = a, x_2 = b, x_3 = c, x_4 = d$. Hence, this set is a basis for $M^{2\times 2}(\mathbb{R})$. □

Exercise 5.35. Show that the polynomials $\{x^i\}_{i=0}^n$ form a basis for P_n.

Solution. Let $a_0 + a_1 x + \cdots + a_n x^n \in P_n$. Then

$$\begin{aligned} c_0 + c_1 x + \cdots + c_n x^n &= a_0 + a_1 x + \cdots + a_n x^n \\ \implies (c_0 - a_0) + (c_1 - a_1)x + \cdots + (c_n - a_n)x^n &= 0 \end{aligned}$$

for all $x \in \mathbb{R}$. By the Fundamental Theorem of Algebra, a degree n polynomial has only n roots, so for this to hold for all $x \in \mathbb{R}$ we must have $c_i = a_i$ for $i = 0, \ldots, n$. Thus, every polynomial in P_n can be expressed uniquely as a linear combination of $\{x^i\}_{i=0}^n$, so this set forms a basis for P_n. □

Exercise 5.36. Determine whether each of following sets of polynomials form a linearly independent set in P_2. If so, determine whether they form a basis for P_2.

(a) $p_1(x) = 1 - x,\ p_2(x) = 1 + x^2$

(b) $p_1(x) = 2 - x,\ p_2(x) = 5 + 3x - 2x^2,\ p_3(x) = 1 + 3x - x^2$

Solution. (a) Since neither polynomial is a scalar multiple of the other, this set is linearly independent. Now suppose $a_0 + a_1x + a_2x^2$ is any polynomial in P_2. Then

$$c_1(1-x) + c_2(1+x^2) = c_1 + c_2 - c_1x + c_2x^2 = a_0 + a_1x + a_2x^2$$

implies that

$$\begin{aligned} c_1 + c_2 &= a_0 \\ -c_1 &= a_1 \\ c_2 &= a_2. \end{aligned}$$

Upon substituting the second two equations into the first, we see that this system is consistent if and only if $-a_1 + a_2 = a_0$. Thus, these polynomials do not span P_2. For instance, the polynomial $1 + x + x^2 \notin \text{Span}\{1-x, 1+x^2\}$ since $-a_1 + a_2 = -1 + 1 = 0 \neq 1 = a_0$.

(b) Let $a_0 + a_1x + a_2x^2$ be a polynomial in P_2. Then

$$\begin{aligned} c_1(2-x) + c_2(5+3x-2x^2) + c_3(1+3x-x^2) &= (2c_1 + 5c_2 + c_3) + (-c_1 + 3c_2 + 3c_3)x + (-2c_2 - c_3)x^2 \\ &= a_0 + a_1x + a_2x^2 \end{aligned}$$

is equivalent to the inhomogeneous linear system

$$\begin{aligned} 2c_1 + 5c_2 + c_3 &= a_0 \\ -c_1 + 3c_2 + 3c_3 &= a_1 \\ -2c_2 - c_3 &= a_2. \end{aligned}$$

Since the coefficient matrix

$$\begin{bmatrix} 2 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix}$$

has determinant

$$\begin{vmatrix} 2 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{vmatrix} = 3 \neq 0,$$

the system has a unique solution. Hence, $\{2-x, 5+3x-2x^2, 1+3x-x^2\}$ is a basis for P_2. $\square$

5.11.3 Bases and Coordinates

If $B = \{\mathbf{v}_i\}_{i=1}^n$ is a basis for a vector space V, then every vector $\mathbf{v} \in V$ can be written uniquely as a linear combination of B; that is,

$$\mathbf{v} = \sum_{i=1}^n c_i \mathbf{v}_i$$

for a unique vector of coefficients $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$. If $V = \mathbb{R}^n$ and B is the standard basis, then if we order the vectors in B as $e_1, e_2, \ldots, e_n$, the entries of $\mathbf{c}$ are just the components of the vector $\mathbf{v}$, or equivalently, the coordinates of the tip of the vector. The same is true for any vector space in which one has a basis: the existence of a basis means the vector space is isomorphic to $\mathbb{R}^n$. The directions along which the basis vectors point (after arranging these in a fixed order) can be viewed as coordinate axes, [46] and the coefficients of any vector $\mathbf{v}$, when written as a linear combination of these basis vectors, give the coordinates of the tip of $\mathbf{v}$ with respect to this coordinate system.

Definition 5.81 (Ordered basis). A basis for a vector space V in which the bases vectors have been given a fixed ordering is called an *ordered basis* for V. When we say that $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an ordered basis for V, it is understood that the vectors are listed in their given order.

Definition 5.82 (Basis and coordinate isomorphisms). Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space V.

(a) The unique vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{v} = \sum_{i=1}^n x_i \mathbf{v}_i$ is called the *coordinate vector of* $\mathbf{v}$ *with respect to* B. We denote $\mathbf{x}$ by $[\mathbf{v}]_B$. The ith entry of $[\mathbf{v}]_B$ is called the *ith coordinate of* $\mathbf{v}$.

(b) The linear combination map $L_B : \mathbb{R}^n \to V$ is called the *basis isomorphism with respect to B*, and its inverse $L_B^{-1} : V \to \mathbb{R}^n$ which takes $\mathbf{v} \mapsto [\mathbf{v}]_B$ is called the *coordinate isomorphism with respect to B*.

Example 5.83. Let B be the standard ordered basis basis for $\mathbb{R}^n$. Then the basis and coordinate isomorphisms are both the identity map.

Example 5.84. Consider the ordered basis $B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. The basis isomorphism is then multiplication by the matrix $\begin{bmatrix} 2 & 1 \\ 1 & -8 \end{bmatrix}$, and the coordinate isomorphism is multiplication by

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 1 & -8 \end{bmatrix}^{-1} &= \frac{1}{-16-1} \begin{bmatrix} -8 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{8}{17} & \frac{1}{17} \\ \frac{1}{17} & \frac{-2}{17} \end{bmatrix}. \end{aligned}$$

The coordinates of the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ with respect to this basis are therefore given by

$$\begin{aligned} [\mathbf{v}]_B &= L_B^{-1}(\mathbf{v}) \\ &= \begin{bmatrix} \frac{8}{17} & \frac{1}{17} \\ \frac{1}{17} & \frac{-2}{17} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} \frac{29}{17} \\ \frac{-7}{17} \end{bmatrix}. \end{aligned}$$

Exercise 5.37. Let

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} \right\}$$

be an ordered basis for $\mathbb{R}^3$.

[46] Note that these coordinate axes need not be orthogonal.

(a) Find the vector $\mathbf{v} \in \mathbb{R}^3$ whose coordinate vector relative to B is $[\mathbf{v}]_B = (-1, 3, 2)$.

(b) Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to B.

Solution. (a)

$$\begin{aligned}\mathbf{v} &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 11 \\ 31 \\ 7 \end{bmatrix}.\end{aligned}$$

(b)

$$\begin{aligned}[\mathbf{v}]_B &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} -36 & 8 & 21 \\ 5 & -1 & -3 \\ 9 & -2 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.\end{aligned}$$

□

Exercise 5.38. Consider the ordered basis

$$p_1(x) = 2 - x,\ p_2(x) = 5 + 3x - 2x^2,\ p_3(x) = 1 + 3x - x^2$$

for P_2 from part (b) of Example 5.36.

(a) Find the polynomial $p(x) \in P_2$ whose coordinate vector relative to B is $(-1, 3, 2)$.

(b) Find the coordinate vector of $p(x) = 5 - x + 9x^2$ with respect to B.

Solution. (a)

$$\begin{aligned}p(x) &= L_B(-1, 3, 2) \\ &= -p_1(x) + 3p_2(x) + 2p_3(x) \\ &= -(2 - x) + 3(5 + 3x - 2x^2) + 2(1 + 3x - x^2) \\ &= -2 + x + 15 + 9x - 6x^2 + 2 + 6x - 2x^2 \\ &= (-2 + 15 + 2) + (1 + 9 + 6)x + (-6 - 2)x^2 \\ &= 15 + 16x - 8x^2.\end{aligned}$$

(b) Let $[p(x)]_B = (c_1, c_2, c_3)$. Then

$$\begin{aligned}5 - x + 9x^2 &= c_1p_1(x) + c_2p_2(x) + c_3p_3(x)\\ &= c_1(2-x) + c_2(5+3x-2x^2) + c_3(1+3x-x^2)\\ &= (2c_1+5c_2+c_3) + (-c_1+3c_2+3c_3)x + (-2c_2-c_3)x^2\end{aligned}$$

and therefore (c_1, c_2, c_3) is the solution to the linear system

$$\begin{aligned}2c_1 + 5c_2 + c_3 &= 5\\ -c_1 + 3c_2 + 3c_3 &= -1\\ -2c_2 - c_3 &= 9.\end{aligned}$$

Since

$$\begin{bmatrix} 2 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 4 \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{7}{3} \\ \frac{2}{3} & \frac{4}{3} & \frac{11}{3} \end{bmatrix},$$

the unique solution is given by

$$\begin{aligned}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 4 \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{7}{3} \\ \frac{2}{3} & \frac{4}{3} & \frac{11}{3} \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}\\ &= \begin{bmatrix} 40 \\ -22 \\ 35 \end{bmatrix}.\end{aligned}$$

□

5.11.4 Dimension of a Vector Space

We have defined a vector space V to be finite-dimensional if it admits a finite spanning set. Here, we will see that we can assign to each such V a unique integer, called the *dimension* of V, which satisfies our intuitive requirements about dimensionality and which turns out to be an important tool in deeper studies of the properties of such spaces. We will also see that a finite-dimensional vector space is one which can be characterized as a vector space isomorphic to $\mathbb{R}^n$. This isomorphism allows a linear mapping between two such spaces to be uniquely represented by a matrix. The theory of linear mappings between finite-dimensional vector spaces is therefore completely equivalent to the theory of matrices.

Our first task will be to show that every finite-dimensional vector space has a basis. We will need the following facts.

Lemma 5.85 (Plus/minus lemma). Let S be a nonempty subset of a vector space V.

(a) If S is linearly independent and $\mathbf{v} \in V - \text{Span}(S)$, then $S \cup \{\mathbf{v}\}$ is linearly independent.

(b) If $\mathbf{v} \in S$ can be expressed as a linear combination of vectors in $S - \{\mathbf{v}\}$, then $\text{Span}(S - \{\mathbf{v}\}) = \text{Span}(S)$.

Proof. (a) Suppose $S = \{\mathbf{v}_i\}_{i=1}^n \subset V$ is linearly independent and let $\mathbf{v} \in V - \text{Span}(S)$. Suppose

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n + x\mathbf{v} = \mathbf{0}$$

If $x = 0$, then we have

$$\begin{aligned} x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n + \mathbf{0} &= \mathbf{0} \\ x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n &= \mathbf{0} \end{aligned}$$

and therefore $x_i = 0$ for all $i = 1, \cdots, n$ since S is linearly independent. If $x \neq 0$, we can divide by x and solve for $\mathbf{v}$ to obtain $\mathbf{v} = -\frac{x_1}{x}\mathbf{v}_1 - \cdots - \frac{x_n}{x}\mathbf{v}_n$, which says $\mathbf{v} \in \text{Span}(S)$, which is a contradiction. We therefore have that $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n + x\mathbf{v} = \mathbf{0} \implies x_1 = \cdots = x_n = x = 0$, therefore $S \cup \{\mathbf{v}\}$ is linearly independent.

(b) Let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_{r-1}, \mathbf{v}\}$ and suppose there exist $c_1, \cdots, c_{r-1}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_{r-1}\mathbf{v}_{n-1}$. Let $\mathbf{w} \in \text{Span}(S - \{\mathbf{v}\})$. Then $\mathbf{w} = x_1\mathbf{v}_1 + \cdots + x_{r-1}\mathbf{v}_{r-1} = x_1\mathbf{v}_1 + \cdots + x_{r-1}\mathbf{v}_{r-1} + 0\mathbf{v} \in \text{Span}(S)$. Now let $\mathbf{w} \in \text{Span}(S)$. Then

$$\begin{aligned} \mathbf{w} &= x_1\mathbf{v}_1 + \cdots + x_{r-1}\mathbf{v}_{r-1} + x\mathbf{v} \\ &= x_1\mathbf{v}_1 + \cdots + x_{r-1}\mathbf{v}_{r-1} + x(c_1\mathbf{v}_1 + \cdots + c_{r-1}\mathbf{v}_{r-1}) \\ &= (x_1 + xc_1)\mathbf{v}_1 + \cdots + (x_{r-1} + xc_{r-1})\mathbf{v}_{r-1} \end{aligned}$$

so $\mathbf{w} \in \text{Span}(S - \{\mathbf{v}\})$, and therefore $\text{Span}(S) = \text{Span}(S - \{\mathbf{v}\})$.

□

Theorem 5.86. Let V be a finite-dimensional vector space and $S = \{\mathbf{v}_i\}_{i=1}^n$ be a subset of V. Then

(a) If $\text{Span}(S) = V$, then S contains a subset B which is a basis for V.

(b) If S is linearly independent, then S can be extended to a basis for V.

Proof. (a) If S spans V but is not a basis, then S is linearly dependent. Then S contains some vector $\mathbf{v}$ which can be written as a linear combination of the vectors in $S - \{\mathbf{v}\}$. By Lemma 5.85, $\text{Span}(S - \{\mathbf{v}\}) = \text{Span}(S) = V$. If $S - \{\mathbf{v}\}$ is linearly independent, we are done. If not, continue this process until we arrive at a set which is linearly independent and therefore a basis for V. This process must eventually terminate on a linearly independent set, since the empty set is linearly independent.

(b) If S is linearly independent but not a basis, then S does not span V. Since V is finite-dimensional, there exists a finite spanning set $S' = \{\mathbf{w}_k\}_{k=1}^n$ for V. First, note that S' contains at least one vector which is not in $\text{Span}\, S$. To see this, suppose otherwise. Then $S' \subset \text{Span}\, S$, and therefore $\text{Span}\, S' = V \subset \text{Span}\, S$, which is a contradiction, since S does not span V. Hence, S' must have some vector, $\mathbf{w}_{i_1}$, which is not in $\text{Span}\, S$. By Lemma 5.85, the set $S \cup \{\mathbf{w}_{i_1}\}$ is linearly independent. If this set now spans V, we are done. Otherwise, there must exist exist some vector $\mathbf{w}_{i_2} \in S'$ which is not in the span of $S \cup \{\mathbf{w}_{i_1}\}$, so that $S \cup \{\mathbf{w}_{i_1}, \mathbf{w}_{i_2}\}$ is linearly independent. Continuing in this way, we eventually obtain a spanning set for V, since if we all vectors of S' to S then $S \cup S'$ will span V.

□

Corollary 5.87. Every finite-dimensional vector space has a basis.

Proof. Since V is finite-dimensional, there exists some finite spanning set S for V. By part (a) of Theorem 5.86, S contains a basis for V. $\square$

We will now show that every basis has the same number of elements.

Theorem 5.88. Let V be a finite-dimensional vector space, $B = \{\mathbf{v}_i\}_{i=1}^n$ a basis for V, and $S = \{\mathbf{w}_i\}_{i \in I}$ any subset of V. Then

(a) If S has more than n vectors, it is linearly dependent.

(b) If S has fewer than n vectors, then it does not span V.

Proof. (a) Suppose $|S| = m > n$. Since B is a basis, we can write each vector $\mathbf{w}_k \in S$ uniquely as

$$\mathbf{w}_k = \sum_{i=1}^{n} x_{k,i}\mathbf{v}_i \tag{5.7}$$

for some constants $x_{k,i}$. The set S is linearly independent if

$$\sum_{k=1}^{m} y_k \mathbf{w}_k = \mathbf{0} \tag{5.8}$$

has only the trivial solution $y_1 = y_2 = \cdots = y_m = 0$. Substituting in (5.7) into (5.8), (5.8) becomes

$$\sum_{k=1}^{m}\sum_{i=1}^{n} y_k x_{k,i}\mathbf{v}_i = \mathbf{0}.$$

Since the $\mathbf{v}_i$ form a basis for V, they are linearly independent and we therefore have

$$\sum_{k=1}^{m} y_k x_{k,i} = 0 \tag{5.9}$$

for each $i = 1, \cdots, n$. This is a homogeneous linear system of n equations in $m > n$ unknowns, so it has nontrivial solutions for the y_ks. Thus, S is linearly dependent.

(b) Suppose $|S| = m < n$. Suppose $\mathrm{Span}(S) = V$. Then each $\mathbf{v} \in V$ can be written as a linear combination of S. In particular, this is true for each vector in B:

$$\mathbf{v}_i = \sum_{k=1}^{m} x_{i,k}\mathbf{w}_k. \tag{5.10}$$

Consider now the equation

$$\sum_{i=1}^{n} y_i \mathbf{v}_i = \mathbf{0}. \tag{5.11}$$

Substituting (5.10) into (5.11), (5.11) becomes

$$\sum_{i=1}^{n} y_i x_{i,k}\mathbf{w}_k = \mathbf{0}. \tag{5.12}$$

Since B is a basis, it is linearly independent so Eq. (5.11) implies that $y_i = 0$ for all $i = 1, \cdots, n$ and therefore

$$\sum_{i=1}^{n} y_i x_{i,k} = 0 \tag{5.13}$$

for all $k = 1, \cdots, m$. But this is a homogeneous linear system of m equations in $n > m$ unknowns so it has nontrivial solutions for the y_i, which is a contradiction. Thus, the set S does not span V.

□

Corollary 5.89. Every basis for a finite-dimensional vector space V has the same number of elements.

Proof. Let B be a basis for V with $|B| = n$ and let S be any subset of V. By the previous theorem, if $|S| > n$, it is not linearly independent and therefore not a basis. If $|S| < n$ it does not span V and is therefore not a basis. Therefore, if S is a basis it must have exactly n elements. □

Definition 5.90. Let V be a finite-dimensional vector space. The number of elements in any basis for V is called the *dimension* of V, denoted dim V.

Example 5.91. The set $\{e_i\}_{i=1}^{n}$ forms a basis for $\mathbb{R}^n$, so $\mathbb{R}^n$ is n-dimensional.

Proposition 5.92. The trivial vector space $\{\mathbf{0}\}$ has dimension 0.

Proof. We will show that the empty set is a basis for $\{\mathbf{0}\}$. We have seen that $\emptyset$ is linearly independent, since it is vacuously true that $\sum_i c_i \mathbf{v}_i = \mathbf{0}$ has only the trivial solution for any empty collection of vectors $\mathbf{v}_i$. We then define $\operatorname{Span} \emptyset = \{\mathbf{0}\}$ so that $\emptyset$ is a basis for $\{\mathbf{0}\}$. This definition is consistent with the characterization of $\operatorname{Span} W$ for a nonempty subset $W \subseteq V$ in Theorem 5.38: If V is any vector space, then we define $\operatorname{Span}(\emptyset)$ to be equal to the intersection of all subspaces of V which contain $\emptyset$, which is the intersection of all subspaces of V (it is vacuously true that empty set is a subset of every set), which is precisely $\{\mathbf{0}\}$. Since $\emptyset$ is linearly independent and spans $\{\mathbf{0}\}$, it is a basis for $\{\mathbf{0}\}$. Thus $\dim(\{\mathbf{0}\}) = |\emptyset| = 0$. □

We will now classify all finite-dimensional vector spaces up to isomorphism.

Lemma 5.93. The composition of two isomorphisms is an isomorphism.

Proof. Let U, V, W be vector spaces and let $T \in Hom(U, V)$ and $S \in Hom(V, W)$ be isomorphisms. Consider now. $S \circ T \in Hom(U, W)$. To show that $S \circ T$ is linear, let $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $x \in \mathbb{R}$. Then

$$\begin{aligned}
(S \circ T)(x\mathbf{u}_1 + \mathbf{u}_2) &= S(T(x\mathbf{u}_1 + \mathbf{u}_2)) \\
&= S(xT(\mathbf{u}_1) + T(\mathbf{u}_2)) && \text{(since } T \text{ is linear)} \\
&= xS(T(\mathbf{u}_1)) + S(T(\mathbf{u}_2)) && \text{(since } S \text{ is linear)} \\
&= x(S \circ T)(\mathbf{u}_1) + (S \circ T)(\mathbf{u}_2),
\end{aligned}$$

so $S \circ T$ is linear. To show that $S \circ T$ is surjective, let $\mathbf{w} \in W$. Since S is surjective, there exists $\mathbf{v} \in V$ such that $\mathbf{w} = S(\mathbf{v})$. Since T is surjective, there exists $\mathbf{u} \in U$ such that $\mathbf{v} = T(\mathbf{u})$ and therefore $\mathbf{w} = S(T(\mathbf{u})) = (S \circ T)(\mathbf{u})$, which shows $S \circ T$ is surjective.

To see $S \circ T$ is injective, suppose $\mathbf{u} \in \ker S \circ T$. Then $(S \circ T)(\mathbf{u}) = S(T(\mathbf{u})) = \mathbf{0}$. This says that $T(\mathbf{u}) \in \ker S$. Since S is injective, $\ker S = \{\mathbf{0}\}$, and therefore $T(\mathbf{u}) = \mathbf{0}$. But this says that $\mathbf{u} \in \ker T$. Since T is injective, $\ker T = \{\mathbf{0}\}$, $\mathbf{u} = \mathbf{0}$. This shows that $\ker S \circ T = \{\mathbf{0}\}$, and hence $S \circ T$ is injective, completing the proof that $S \circ T$ is an isomorphism. □

Theorem 5.94. If $T \in Hom(V, W)$ is an isomorphism and $B = \{\mathbf{v}_i\}_{i=1}^n$ is a basis for V, then $T(B) = \{T(\mathbf{v}_i)\}_{i=1}^n$ is a basis for W.

Proof. Since B is a basis for V, the corresponding linear combination mapping $L_B \in Hom(\mathbb{R}^n, V)$ is an isomorphism. By the previous lemma $T \circ L_B \in Hom(\mathbb{R}^n, W)$ is also an isomorphism. The set $T(B)$ is therefore a basis for W. $\square$

Theorem 5.95. Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

Proof. Let V, W be two finite-dimensional vector spaces. Choose bases B_V for V and B_W for W. ($\implies$) Suppose $T \in Hom(V, W)$ is an isomorphism. By the previous theorem, $T(B_V)$ is a basis for W. Therefore $\dim V = |B| = |T(B)| = \dim W$. ($\impliedby$) Now suppose $\dim V = \dim \mathrm{W} = n$. Then V and W are each isomorphic to $\mathbb{R}^n$, and hence to each other. $\square$

Corollary 5.96 (Classification of finite-dimensional vector spaces). Up to isomorphism, there is a unique finite-dimensional vector space V for each nonnegative integer $n = 0, 1, 2, \cdots$. If $n = 0$, $V \cong \{\mathbf{0}\}$. If $n \in \mathbb{N}$, then $V \cong \mathbb{R}^n$.

We will now see that every subspace of a finite-dimensional vector space is also finite-dimensional.

Theorem 5.97 (Dimension of a subspace). Let W be a subspace of a finite-dimensional vector space V. Then

(a) W is finite-dimensional.

(b) $\dim W \leq \dim V$.

(c) $W = V$ if and only if $\dim W = \dim V$.

Proof. (a) Since V is finite-dimensional, it has a finite spanning set S. Since $W \subset V = \operatorname{Span} S$, W is finite-dimensional.

(b) Any basis of W is a linearly independent set of vectors in V By by part (b) of Theorem 5.86, this set can therefore be extended to a basis for V. Hence, the number of vectors in this set is less than or equal to the number of vectors in a basis for V. Thus, $\dim W \leq \dim V$.

(c) If $W = V$, then $\dim W = \dim V$. Conversely, suppose that $\dim W = \dim V$ and let $B = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ be a basis for W. If B is not also a basis for V, then since B is linearly independent, it can be extended to a basis for V. But this implies $\dim V > \dim W$, which is a contradiction. Thus, B must also be a basis for V. Thus, $W = \operatorname{Span} B = V$.

$\square$

If we are given a vector space V with known dimension n, then the following theorem shows that if we find a set S of n vectors which is *either* linearly independent *or* spans V, then S must already be a basis for V.

Theorem 5.98. Let V be a finite-dimensional vector space of dimension $\dim V = n$, and let S be a set of n vectors in V. If either S is linearly independent or $\operatorname{Span} S = V$, then S is a basis for V.

Proof. Suppose S is linearly independent. By the $+/-$ lemma (Lemma 5.85), S can be extended to a basis for V. Thus, $\dim V \geq |S|$. But $|S| = \dim V$ so S must already be a basis for V.

Now suppose $\operatorname{Span} S = V$. By the $+/-$ lemma (Lemma 5.85), S contains a basis for V. Thus, $\dim V \leq |S|$. But $|S| = \dim V$, so S must already be a basis for V. $\square$

Example 5.99. Consider the set $S = \left\{ \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 21 \\ 16 \end{bmatrix} \right\} \subset \mathbb{R}^2$. Since these two vectors are not scalar multiples of each other, S is linearly independent and hence a basis for $\mathbb{R}^2$ by Theorem 5.98.

Proposition 5.100 (Dimension of a Direct Sum). If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Proof. Since U_1 and U_2 are subspaces of V, $U_1 \cap U_2$ is a subspace of V by Exercise 5.14. By Theorem 5.97, $U_1 \cap U_2$ is finite-dimensional, so $\dim(U_1 \cap U_2) = m \leq n$. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a basis for $U_1 \cap U_2$. Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is, in particular, a linearly independent set of vectors in U_1, by part (b) of Theorem 5.86, it can be extended to a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_j\}$ $(0 \leq j \leq n - m)$ for U_1. Thus, $\dim U_1 = m + j$. Similarly, $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ can be extended to a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_k\}$ $(0 \leq k \leq n - m)$ for U_2. Thus, $\dim U_2 = m + k$.

We will show that

$$\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}_1, \ldots, \mathbf{w}_k\}$$

is a basis for $U_1 + U_2$. This will complete the proof, because then we will have

$$\begin{aligned} \dim(U_1 + U_2) &= m + j + k \\ &= m + j + k + m - m \\ &= (m + j) + (k + m) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2). \end{aligned}$$

Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}_1, \ldots, \mathbf{w}_k\}$ contains bases for U_1 and U_2,

$$U_1, U_2 \subseteq \operatorname{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}_1, \ldots, \mathbf{w}_k\},$$

and hence

$$U_1 + U_2 \subseteq \operatorname{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}_1, \ldots, \mathbf{w}_k\},$$

since $\operatorname{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is closed under addition.

All that remains is to show linear independence. Suppose that

$$a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m + b_1\mathbf{v}_1 + \cdots + b_j\mathbf{v}_j + c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k = 0. \tag{5.14}$$

Then

$$c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k = -a_1\mathbf{u}_1 - \cdots - a_m\mathbf{u}_m - b_1\mathbf{v}_1 - \cdots - b_j\mathbf{v}_j,$$

which shows that $c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k \in U_1$. But $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \in U_2$, so $c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k \in U_1 \cap U_2$. Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a basis for $U_1 \cap U_2$, this implies

$$c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k = d_1\mathbf{u}_1 + \cdots + d_m\mathbf{u}_m$$

for some scalars $d_1, \ldots, d_m$ and therefore

$$c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k - d_1\mathbf{u}_1 - \cdots - d_m\mathbf{u}_m = 0.$$

Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis for U_2, it is a linearly independent set so $c_1 = \cdots = c_k = d_1 = \cdots = d_m = 0$. Equation (5.14) then becomes

$$a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m + b_1\mathbf{v}_1 + \cdots + b_j\mathbf{v}_j = 0.$$

Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_j\}$ is a basis for U_1, it is a linearly independent set, so this implies $a_1 = \cdots = a_m = b_1 = \cdots = b_j = 0$, proving that

$$\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}_1, \ldots, \mathbf{w}_k\}$$

is linearly independent. $\square$

Corollary 5.101 (Dimension of a Direct Sum). If U_1 and U_2 be subspaces of a finite-dimensional vector space, then

$$\dim(U_1 \oplus U_2) = \dim U_1 + \dim U_2.$$

Proof. If $U_1 + U_2$ is a direct sum, then $U_1 \cap U_2 = \{0\}$, and therefore $\dim(U_1 \cap U_2) = 0$. $\square$

Corollary 5.102. If $U_1, \ldots, U_m$ are subspaces of a finite-dimensional vector space V such that $U_1 + \cdots + U_m$ is a direct sum, then

$$\dim(U_1 \oplus \cdots \oplus U_m) = \dim U_1 + \cdots + \dim U_m.$$

Proof. (By induction on m.) We have proved this holds for $m = 2$ in Corollary 5.101. Suppose this holds for $m = k$ and consider

$$U_1 \oplus \cdots \oplus U_{k+1}.$$

Let $V = U_1 \oplus \cdots \oplus U_k$. Then

$$U_1 \oplus \cdots \oplus U_{k+1} = V \oplus U_{k+1}.$$

By Corollary 5.101,

$$\dim(U_1 \oplus \cdots \oplus U_{k+1}) = \dim V + \dim U_{k+1}.$$

By the inductive hypothesis,

$$\dim V = \dim U_1 + \cdots + \dim U_k,$$

and therefore

$$\dim(U_1 \oplus \cdots \oplus U_{k+1}) = \dim U_1 + \cdots + \dim U_k + \dim U_{k+1}.$$

$\square$

5.12 Fundamental Theorem of Linear Maps

Theorem 5.103 (Fundamental Theorem of Linear Maps). Let V and W be finite-dimensional vector spaces and let $T \in \operatorname{Hom}(V, W)$. Then

$$\dim V = \dim \ker T + \dim \operatorname{im} T.$$

Proof. We have seen that $\ker T$ and $\operatorname{im} T$ are subspaces of V and W, respectively. By Theorem **??**, they are finite-dimensional and $\dim \ker T \leq \dim V$ and $\dim \operatorname{im} T \leq \dim W$. Let $\{u_1, \ldots, u_m\}$ be a basis for $\ker T$. By **??**, this can be extended to a basis $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ for V. Then $\dim V = m + n$. We will now prove that $\{Tv_1, \ldots, Tv_n\}$ is a basis for $\operatorname{im} T$.

Let $v \in V$. Since $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ is a basis for V, we can write

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n,$$

for some scalars $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$. Applying T to both sides of this equation gives

$$\begin{aligned} T(v) &= T(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) \\ &= T(a_1 u_1 + \cdots + a_m u_m) + T(b_1 v_1 + \cdots + b_n v_n) \text{ (since } a_1 u_1 + \cdots + a_m u_m \in \ker T) \\ &= 0 + T(b_1 v_1 + \cdots + b_n v_n) \\ &= T(b_1 v_1 + \cdots + b_n v_n) \\ &= b_1 T(v_1) + \cdots + b_n T(v_n), \end{aligned}$$

which shows that $\{T(v_1), \ldots, T(v_n)\}$ spans $\operatorname{im} T$. All that remains is to show linear independence. Suppose

$$c_1 T(v_1) + \cdots + c_n T(v_n) = 0,$$

then by linearity

$$T(c_1 v_1 + \cdots + c_n v_n) = 0.$$

This shows that $c_1 v_1 + \cdots + c_n v_n \in \ker T$, so we can write

$$c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m$$

for some scalar $d_1, \ldots, d_m$. Then

$$c_1 v_1 + \cdots + c_n v_n - d_1 u_1 - \cdots - d_m u_m = 0,$$

which implies $c_1 = \cdots = c_n = d_1 = \cdots = d_m = 0$ since $\{v_1, \ldots, v_n, d_1, \ldots, d_m\}$ is linearly independent. This shows that $\{T(v_1), \ldots, T(v_n)\}$ is also linearly independent, and therefore is a basis for $\operatorname{im} T$. $\square$

Corollary 5.104. If V and W are finite-dimensional vector spaces and $T \in \operatorname{Hom}(V)$.

(a) If $\dim V > \dim W$, then T is not injective.

(b) If $\dim V < \dim W$, then T is not surjective.

Proof. (a) Assume $\dim V > \dim W$. By Theorem 5.103,

$$\dim \ker T = \dim V - \dim \operatorname{im} T.$$

Since $\operatorname{im} T$ is a subspace of W, $\dim \operatorname{im} T \leq \dim W$, so

$$\begin{aligned} \dim \ker T &= \dim V - \dim \operatorname{im} T \\ &\geq \dim V - \dim W \\ &> 0. \end{aligned}$$

By **??**, T is injective if and only $\dim \ker T = 0$. Thus, T is not injective.

(b) Since $\operatorname{im} T$ is a subspace of W, $\dim \operatorname{im} T \leq \dim W$. Then

$$\begin{aligned} \dim \operatorname{im} T &= \dim V - \dim \ker T \text{ (since } \dim \ker T \geq 0) \\ &\leq \dim V \\ &< \dim W. \end{aligned}$$

$\square$

5.13 Change of Basis

Sometimes a given basis is not the most convenient for a particular problem, and it becomes desirable to change basis. Consider, for example, the motion of a box sliding down an inclined plane:

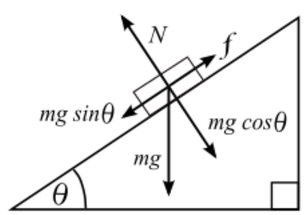


Figure 15: The forces on a box sliding down an inclined plane.

With respect to the standard basis for $\mathbb{R}^2$, the equations of motion are

$$N \sin\theta - f \cos\theta = ma_{e_1} \tag{5.15}$$
$$N \cos\theta - mg = ma_{e_2}. \tag{5.16}$$

Since we have components of acceleration in both the x and y directions to solve for, this obscures the fact that the motion is one-dimensional. It is therefore more convenient to choose a basis $B' = \{e_1', e_2'\}$ where these basis vectors are the image of e_1 and e_2, respectively, under a counterclockwise rotation by θ radians. With respect to B', the equations of motion are

$$mg \sin\theta - f = ma_{e_1'} \tag{5.17}$$
$$N - mg \cos\theta = 0. \tag{5.18}$$

Now the frictional force f is proportional to the normal force

$$f = \mu N,$$

where μ is the coefficient of kinetic friction, so solving (5.18) for N and substituting into (5.17) gives

$$mg\sin\theta - \mu mg\cos\theta = ma_{e_1'}$$

or

$$g(\sin\theta - \mu\cos\theta) = a_{e_1'},$$

which is the acceleration along the plane.

How are the coordinates (x, y) and (x', y') with respect to these two bases related? This is most easily determined by writing (x, y) in polar coordinates (r, α):

$$x = r\cos\alpha \tag{5.19}$$
$$y = r\sin\alpha. \tag{5.20}$$

The polar coordinates of (x', y') are then

$$\begin{aligned} x' &= r\cos(\alpha - \theta) \\ &= r\cos\alpha\cos\theta + r\sin\alpha\sin\theta \\ &= x\cos\theta + y\sin\theta \\ y' &= r\sin(\alpha - \theta) \\ &= r\sin\alpha\cos\theta - r\cos\alpha\sin\theta \\ &= -x\sin\theta + y\cos\theta, \end{aligned}$$

which we can write as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the two coordinate vectors are related by multiplication by the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

In general, given a finite-dimensional vector space V and a vector $\mathbf{v} \in V$, if we change basis from B to B', we would like to know how $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ are related. To this end, let B, B' be two bases for V, and let $L_B : \mathbb{R}^n \to V$ and $L_{B'} : \mathbb{R}^n \to V$ be the corresponding basis isomorphisms. Then $L_{B'}^{-1} \circ L_B \in \mathrm{Hom}(\mathbb{R}^n)$ is the isomorphism [47] taking $[\mathbf{v}]_B \mapsto [\mathbf{v}]_{B'}$, which makes the following diagram commute: [48]

$$\begin{tikzcd} & V \arrow[dr, "L_{B'}^{-1}"] & \\ \mathbb{R}^n \arrow[ur, "L_B"] \arrow[rr, "L_{B'}^{-1}\circ L_B"'] & & \mathbb{R}^n \end{tikzcd}$$

[47] We proved in Lemma 5.69 that the composition of two isomorphisms is an isomorphism, so $L_{B'}^{-1} \circ L_B$ is an isomorphism.

[48] A diagram showing various mappings together with their respective domains and codomains is said to *commute* if one always obtains the same value independent of the path taken.

Definition 5.105 (Change of coordinates isomorphism, transition matrix). This isomorphism is called the *change of coordinates isomorphism*. We have seen in Theorem 5.53 that every element of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ can be represented by a matrix. In fact, $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \cong M^{m \times n}(\mathbb{R})$, as we will prove in Section 5.14. The matrix representation of $L_{B'}^{-1} \circ L_B$ is called the *transition matrix* from $B \to B'$, and is denoted $P_{B \to B'}$.

We will now work out explicitly the form of the transition matrix. Let $B = \{\mathbf{u}_i\}_{i=1}^n$ and $B' = \{\mathbf{u}_i'\}_{i=1}^n$ be two ordered bases for $\mathbb{R}^n$. For any vector $\mathbf{v} \in V$, we can write the coordinate vector of $\mathbf{v}$ with respect to the ordered basis B uniquely as

$$[\mathbf{v}]_B = (x_1, x_2, \ldots, x_n)$$

for some $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Then

$$\begin{aligned}
[\mathbf{v}]_{B'} &= (L_{B'}^{-1} \circ L_B)([\mathbf{v}]_B) \\
&= L_{B'}^{-1}(L_B([\mathbf{v}]_B)) \text{ (by def of composition)} \\
&= L_{B'}^{-1}(x_1\mathbf{u}_1 + x_2\mathbf{u}_2) \text{ (by def } L_B) \\
&= x_1 L_{B'}^{-1}(\mathbf{u}_1) + x_2 L_{B'}^{-1}(\mathbf{u}_2) \text{ (since } L_{B'}^{-1} \text{ is linear)} \\
&= x_1[\mathbf{u}_1]_{B'} + x_2[\mathbf{u}_2]_{B'} \text{ (by def of } L_{B'}^{-1}). \\
&= P_{B \to B'}[\mathbf{v}]_B \text{ (by def of matrix multiplication)}
\end{aligned}$$

where $P_{B \to B'} := [[\mathbf{u}_1]_{B'}, [\mathbf{u}_2]_{B'}, \ldots, [\mathbf{u}_n]_{B'}]$ is the $n \times n$ matrix whose ith row is $[\mathbf{u}_i]_{B'}$, the coordinate vector of the ith basis vector in the ordered basis B.

Example 5.106. Let $B = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$, and let $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ be the ordered basis given by

$$\mathbf{u}_1' = (1, 1), \ \mathbf{u}_2' = (2, 1).$$

We need the coordinate vectors of e_1 and e_2 with respect to B',. Writing

$$\begin{aligned}
e_1 &= c_1\mathbf{u}_1' + c_2\mathbf{u}_2' \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\end{aligned}$$

and therefore

$$\begin{aligned}
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\end{aligned}$$

Similarly, writing

$$e_2 = c_1' \mathbf{u}_1' + c_2' \mathbf{u}_2'$$

we obtain

$$\begin{aligned} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} &= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \end{aligned}$$

and therefore

$$\begin{aligned} [e_1]_{B'} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\ [e_2]_{B'} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \end{aligned}$$

The transition matrix from $B \to B'$ is therefore given by

$$P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Exercise 5.39. Let B, B' be the two ordered bases in Example 5.106. If a vector $\mathbf{v} \in \mathbb{R}^2$ has coordinates $[\mathbf{v}]_B = (-3, 5)$ with respect to B, what are the coordinates of $\mathbf{v}$ with respect to B'?

Solution.

$$\begin{aligned} [\mathbf{v}]_{B'} &= P_{B \to B'} [\mathbf{v}]_B \\ &= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 13 \\ -8 \end{bmatrix}. \end{aligned}$$

□

Since the ordered bases B and B' are arbitrary, the transition matrix $P_{B \to B'}$ should be invertible and equal to $P_{B' \to B}$. We will now prove that this is indeed the case.

Proposition 5.107 ($P_{B' \to B} \to P^{-1}_{B \to B'}$). $P_{B' \to B} \to P^{-1}_{B \to B'}$

Proof. By part (c) of Lemma 5.69, the inverse of $L_{B'}^{-1} \circ L_B \in \mathrm{Hom}(\mathbb{R}^n)$ is an isomorphism and equal to $L_B^{-1} \circ L_{B'} = P_{B' \to B}$. □

Example 5.108. Let B, B' be the bases in Example 5.106. Since $P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$, by Proposition 5.107, the transition matrix from $B' \to B$ is

$$\begin{aligned} P_{B' \to B} &= P^{-1}_{B \to B'} \\ &= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Note that the columns of the transition matrix from B' to the standard basis in Example 5.108 are simply the basis vectors of B' themselves. This is always the case for a transition matrix to the standard basis of $\mathbb{R}^n$, as we will prove in Proposition 5.110.

5.13.1 Computing Transition Matrices Efficiently

We now discuss an algorithm for computing transition matrices efficiently. To save writing, we go through the derivation for a three-dimensional vector space V, but the derivation is exactly the same for any finite-dimensional vector space.

Given two bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3'\}$ for V, we have seen that the transition matrix, $P_{B\to B'}$ is given by

$$P_{B\to B'} = [[\mathbf{u}_1]_{B'}, [\mathbf{u}_2]_{B'}, [\mathbf{u}_3]_{B'}]$$

where the ith column, $[\mathbf{u}_i]_{B'}$, is the coordinate of the "old" basis vector $\mathbf{u}_i$ with respect to the "new" basis B':

$$\begin{aligned}
\mathbf{u}_1 &= ([\mathbf{u}_1]_{B'})_1\mathbf{u}_1' + ([\mathbf{u}_1]_{B'})_2\mathbf{u}_2' + ([\mathbf{u}_1]_{B'})_3\mathbf{u}_3' \\
\mathbf{u}_2 &= ([\mathbf{u}_2]_{B'})_1\mathbf{u}_1' + ([\mathbf{u}_2]_{B'})_2\mathbf{u}_2' + ([\mathbf{u}_2]_{B'})_3\mathbf{u}_3' \\
\mathbf{u}_3 &= ([\mathbf{u}_3]_{B'})_1\mathbf{u}_1' + ([\mathbf{u}_3]_{B'})_2\mathbf{u}_2' + ([\mathbf{u}_3]_{B'})_3\mathbf{u}_3'.
\end{aligned}$$

To solve for $[\mathbf{u}_1]_{B'}$, we need to solve the linear system

$$\begin{aligned}
\begin{bmatrix}(\mathbf{u}_1)_1\\(\mathbf{u}_1)_2\\(\mathbf{u}_1)_3\end{bmatrix} &= ([\mathbf{u}_1]_{B'})_1\begin{bmatrix}(\mathbf{u}_1')_1\\(\mathbf{u}_1')_2\\(\mathbf{u}_1')_3\end{bmatrix} + ([\mathbf{u}_1]_{B'})_2\begin{bmatrix}(\mathbf{u}_2')_1\\(\mathbf{u}_2')_2\\(\mathbf{u}_2')_3\end{bmatrix} + ([\mathbf{u}_1]_{B'})_3\begin{bmatrix}(\mathbf{u}_3')_1\\(\mathbf{u}_3')_2\\(\mathbf{u}_3')_3\end{bmatrix} \\
&= \begin{bmatrix}(\mathbf{u}_1')_1 & (\mathbf{u}_2')_1 & (\mathbf{u}_3')_1\\(\mathbf{u}_1')_2 & (\mathbf{u}_2')_2 & (\mathbf{u}_3')_2\\(\mathbf{u}_1')_3 & (\mathbf{u}_2')_3 & (\mathbf{u}_3')_3\end{bmatrix}\begin{bmatrix}([\mathbf{u}_1]_{B'})_1\\([\mathbf{u}_1]_{B'})_2\\([\mathbf{u}_1]_{B'})_3.\end{bmatrix}
\end{aligned}$$

As we have seen many times, we can solve for the components of $[\mathbf{u}_1]_{B'}$ by row-reducing the augmented matrix

$$\left[\begin{array}{ccc|c}(\mathbf{u}_1')_1 & (\mathbf{u}_2')_1 & (\mathbf{u}_3')_1 & (\mathbf{u}_1)_1\\(\mathbf{u}_1')_2 & (\mathbf{u}_2')_2 & (\mathbf{u}_3')_2 & (\mathbf{u}_1)_2\\(\mathbf{u}_1')_3 & (\mathbf{u}_2')_3 & (\mathbf{u}_3')_3 & (\mathbf{u}_1)_3\end{array}\right]$$

Note that the corresponding systems of equations for $[\mathbf{u}_2]_{B'}$ and $[\mathbf{u}_3]_{B'}$ have *exactly the same coefficient matrix*. Thus, rather than repeating the Gauss-Jordan procedure three times, we might as well append the other two basis vectors in B as well:

$$\left[\begin{array}{ccc|ccc}(\mathbf{u}_1')_1 & (\mathbf{u}_2')_1 & (\mathbf{u}_3')_1 & (\mathbf{u}_1)_1 & (\mathbf{u}_2)_1 & (\mathbf{u}_3)_1\\(\mathbf{u}_1')_2 & (\mathbf{u}_2')_2 & (\mathbf{u}_3')_2 & (\mathbf{u}_1)_2 & (\mathbf{u}_2)_2 & (\mathbf{u}_3)_3\\(\mathbf{u}_1')_3 & (\mathbf{u}_2')_3 & (\mathbf{u}_3')_3 & (\mathbf{u}_1)_3 & (\mathbf{u}_2)_2 & (\mathbf{u}_3)_3\end{array}\right]$$

Since B and B' are bases for V, we know there exists a unique solution for each $[\mathbf{u}_1]_{B'}$, $[\mathbf{u}_2]_{B'}$, and $[\mathbf{u}_3]_{B'}$. Thus, the 3×3 submatrix whose columns are given by the $\mathbf{u}_i'$ is row-equivalent to the 3×3

identity matrix and the other three columns of the reduced matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & ([\mathbf{u}_1]_{B'})_1 & ([\mathbf{u}_2]_{B'})_1 & ([\mathbf{u}_3]_{B'})_1 \\ 0 & 1 & 0 & ([\mathbf{u}_1]_{B'})_2 & ([\mathbf{u}_2]_{B'})_2 & ([\mathbf{u}_3]_{B'})_3 \\ 0 & 0 & 1 & ([\mathbf{u}_1]_{B'})_3 & ([\mathbf{u}_2]_{B'})_2 & ([\mathbf{u}_3]_{B'})_3 \end{array}\right]$$

give us the components of the $[\mathbf{u}_i]_{B'}$ (as functions of of the entries of the basis vectors $\mathbf{u}_j$ and $\mathbf{u}'_k$). But these are exactly the columns of $P_{B\to B'}$. This therefore gives us an algorithm for computing $P_{B\to B'}$:

1. Form the matrix $[B'|B]$, where $B = [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n]$ and similarly for B'.
2. Row-reduce this matrix to RREF. The resulting matrix will be $[I|P_{B\ \to B'}]$, from which we read off the transition matrix $P_{B\ \to B'}$.

Example 5.109. Let B, B' be bases for $\mathbb{R}^3$ where $\mathbf{u}_1 = (2,1,1)$, $\mathbf{u}_2 = (2,-1,1)$, $\mathbf{u}_3 = (1,2,1)$ and $\mathbf{u}'_1 = (3,1,-5)$, $\mathbf{u}'_2 = (1,1,-3)$, $\mathbf{u}'_3 = (-1,0,2)$. To compute $P_{B\to B'}$, we simply row-reduce

$$\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & \frac{5}{2} \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

from which we read off $P_{B\to B'} = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}$.

We can check, e.g., that

$$\begin{aligned} \mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} &= 3\begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \\ &= 3\mathbf{u}'_1 - 2\mathbf{u}'_2 + 5\mathbf{u}'_3, \end{aligned}$$

so that $[\mathbf{u}_1]_{B'} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ is indeed the first column of $P_{B\to B'}$, and similarly for $[\mathbf{u}_2]_{B'}$ and $[\mathbf{u}_3]_{B'}$.

Proposition 5.110 (Transition matrix to the standard basis of $\mathbb{R}^n$). Any nonsingular $n \times n$ matrix A is a transition matrix from ordered basis B' to the standard basis of $\mathbb{R}^n$, where the basis vectors in B' are the columns of the matrix A.

Proof. If B is any ordered basis for $\mathbb{R}^n$ and B' is the standard basis, then $[B'|B] = [I|B]$, where I is the $n \times n$ identity matrix. By the above, $B = P_{B\to B'}$. ☐

Example 5.111. Consider the ordered basis $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 15 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. If B' denotes the standard basis, then

$$P_{B\to B'} = \begin{bmatrix} 1 & -7 \\ 2 & 15 \end{bmatrix}.$$

5.14 Matrix of a Linear Mapping

We will show here that every linear mapping between finite dimensional vector spaces can be represented by a matrix. Therefore, the theory of linear mappings on these spaces is completely mirrored by the theory of matrices.

Let V be a finite-dimensional vector space, $B = \{\mathbf{v}_i\}_{i=1}^n$ be a basis for V, and $L_B : \mathbf{x} \mapsto \sum_{i=1}^n x_i \mathbf{v}_i$ in $Hom(\mathbb{R}^n, V)$ the corresponding basis isomorphism.

Definition 5.112. Let V be a vector space. The n-fold Cartesian product

$$V^n = \{(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) : \mathbf{v}_i \in V \text{ for all } i = 1, \cdots, n\}$$

is a vector space under the operations

$$\begin{aligned}(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) + (\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n) &= (\mathbf{v}_1 + \mathbf{w}_1, \mathbf{v}_2 + \mathbf{w}_2, \cdots, \mathbf{v}_n + \mathbf{w}_n) \\ k(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) &= (k\mathbf{v}_1, k\mathbf{v}_2, \cdots, k\mathbf{v}_n)\end{aligned}$$

Proof. We have seen that $W^A = \{f | f : A \to W\}$ is a vector space for any common domain A. $V^n \cong V^{\overline{n}}$ (the RHS being the set of all maps from $\overline{n} \to V$), so this is a vector space. □

Proposition 5.113 ($Hom(\mathbb{R}^n, V) \cong V^n$). $Hom(\mathbb{R}^n, V) \cong V^n$.

Proof. Let $\mathbf{v} = (\mathbf{v}_1, \cdots, \mathbf{v}_n) \in V^n$ and consider the mapping $\psi : V^n \to Hom(\mathbb{R}^n, V)$ defined by $\psi(\mathbf{v}) = L_{\mathbf{v}}$, where $L_{\mathbf{v}}(\mathbf{x}) = \sum_{i=1}^n x_i \mathbf{v}_i$. This is linear since for every $\mathbf{v}$ and $\mathbf{w}$ in V^n and every $k \in \mathbb{R}$ we have

$$\begin{aligned}\psi(k\mathbf{v} + \mathbf{w})(\mathbf{x}) &= L_{k\mathbf{v}+\mathbf{w}}(\mathbf{x}) \\ &= \sum_{i=1}^n x_i (k\mathbf{v} + \mathbf{w})_i \\ &= \sum_{i=1}^n x_i (k\mathbf{v}_i + \mathbf{w}_i) \\ &= k \sum_{i=1}^n x_i \mathbf{v}_i + \sum_{i=1}^n x_i \mathbf{w}_i \\ &= kL_{\mathbf{v}}(\mathbf{x}) + L_{\mathbf{w}}(\mathbf{x}) \\ &= k\psi(\mathbf{v})(\mathbf{x}) + \psi(\mathbf{w})(\mathbf{x})\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$ and therefore

$$\psi(k\mathbf{v} + \mathbf{w}) = k\psi(\mathbf{v}) + \psi(\mathbf{w}),$$

hence ψ is linear.

To show that ψ is surjective, let $T \in Hom(\mathbb{R}^n, W)$. Then, for any $\mathbf{x} \in \mathbb{R}^n$, we have

$$\begin{aligned}T(\mathbf{x}) &= T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \\ &= L_{\mathbf{v}}(\mathbf{x})\end{aligned}$$

where $\mathbf{v} = (T(e_1), T(e_2), \cdots, T(e_n))$. Therefore $T = L_{\mathbf{v}} = \psi(\mathbf{v})$, hence ψ is surjective.

Finally, to show that ψ is injective, suppose $\psi(\mathbf{v}) = \psi(\mathbf{w})$. Then, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned} L_{\mathbf{v}}(\mathbf{x}) &= L_{\mathbf{w}}(\mathbf{x}) \\ \sum_{i=1}^{n} x_i \mathbf{v}_i &= \sum_{i=1}^{n} x_i \mathbf{w}_i \\ \implies \sum_{i=1}^{n} x_i(\mathbf{v}_i - \mathbf{w}_i) &= \mathbf{0}. \end{aligned}$$

This can only be zero for all $\mathbf{x} \in \mathbb{R}^n$ if $\mathbf{v}_i = \mathbf{w}_i$ for all $i = 1, \cdots, n$, i.e., if $\mathbf{v} = \mathbf{w}$. Thus, ψ is also injective and hence an isomorphism. $\square$

Proposition 5.114 ($Hom(\mathbb{R}^n, \mathbb{R}^m) \cong M^{m\times n}(\mathbb{R})$). $Hom(\mathbb{R}^n, \mathbb{R}^m) \cong M^{m\times n}(\mathbb{R})$.

Proof. By the previous proposition, $Hom(\mathbb{R}^n, \mathbb{R}^m) \cong (\mathbb{R}^m)^n$. Each element of the latter is an n-tuple of m-tuples, which we can view as an $m \times n$ matrix. Thus $Hom(\mathbb{R}^n, \mathbb{R}^m) \cong M^{m\times n}(\mathbb{R})$. $\square$

Lemma 5.115 (Composition by a fixed linear map). Composition on the right by a fixed $T \in Hom(V, W)$ is a linear mapping from the vector space $Hom(W, X)$ to the vector space $Hom(V, X)$. It is an isomorphism if T is an isomorphism. Similar statements hold for composition on the left by T.

Proof. Let $\psi : Hom(W, X) \to Hom(V, X)$ be defined by $\psi(S) = S \circ T$. To show that ψ is linear, we need to show that $\psi(kS_1 + S_2) = k\psi(S_1) + \psi(S_2)$ for all $S_1, S_2 \in Hom(W, X)$ and all $k \in \mathbb{R}$. Letting $\mathbf{v} \in V$, we have

$$\begin{aligned} \psi(kS_1 + S_2)(\mathbf{v}) &= [(kS_1 + S_2) \circ T](\mathbf{v}) && \text{(def of } \psi) \\ &= (kS_1 + S_2)(T(\mathbf{v})) && \text{(def of } \circ) \\ &= kS_1(T(\mathbf{v})) + S_2(T(\mathbf{w})) && \text{(def of + for } Hom(W, X)) \\ &= k(S_1 \circ T)(\mathbf{v}) + (S_2 \circ T)(\mathbf{v}) && \text{(def of } \circ) \\ &= k\psi(S_1)(\mathbf{v}) + \psi(S_2)(\mathbf{v}) && \text{(def of } \psi) \end{aligned}$$

This holds for all $\mathbf{v} \in V$, so $\psi(kS_1 + S_2) = k\psi(S_1) + \psi(S_2)$, hence ψ is linear.

Now suppose $T \in Hom(V, W)$ is an isomorphism and let $T^{-1} \in Hom(W, V)$ be its inverse. Let $\varphi : Hom(V, X) \to Hom(W, X)$ be defined by $\varphi(R) = S \circ T^{-1}$. Then $\varphi \circ \psi : Hom(W, X) \to Hom(W, X)$ gives

$$\begin{aligned} (\varphi \circ \psi)(S) &= \varphi(\psi(S)) \\ &= \varphi(S \circ T) \\ &= (S \circ T) \circ T^{-1} \\ &= S \circ (T \circ T^{-1}) \\ &= S \circ I_V \\ &= S \end{aligned}$$

Similarly, one can check that $\psi \circ \varphi : Hom(V, X) \to Hom(V, X)$ is the identity map on V. Hence φ is a two-sided inverse for ψ, which shows ψ is a bijection and hence an isomorphism. $\square$

Theorem 5.116. If V and W are finite-dimensional vector spaces, then $Hom(V, W) \cong M^{m\times n}(\mathbb{R})$.

Proof. We will prove that $Hom(V,W) \cong Hom(\mathbb{R}^n, \mathbb{R}^m)$, and theorem will then follow from Proposition 5.114. Let $B = \{\mathbf{v}_i\}_{i=1}^n$ be a basis for V, $B' = \{\mathbf{w}_i\}_{i=1}^n$ a basis for W, and $L_B \in Hom(\mathbb{R}^n, V)$, $L_{B'} \in Hom(\mathbb{R}^m, W)$ the corresponding basis isomorphisms. Then $L_{B'}^{-1} \circ T \circ L_B \in Hom(\mathbb{R}^n, \mathbb{R}^m)$. By the previous lemma, the mapping $T \mapsto T \circ L_B$ is an isomorphism from $Hom(V,W)$ to $Hom(\mathbb{R}^n, W)$ and the mapping $T \circ L_B \mapsto L_{B'}^{-1} \circ T \circ L_B$ is an isomorphism from $Hom(\mathbb{R}^n, W)$ to $Hom(\mathbb{R}^n, \mathbb{R}^m)$. Since the composition of isomorphisms is an isomorphism, the mapping $T \mapsto L_{B'}^{-1} \circ T \circ L_B$ is an isomorphism from $Hom(V,W)$ to $Hom(\mathbb{R}^n, \mathbb{R}^m)$. Thus, $Hom(V,W) \cong Hom(\mathbb{R}^n, \mathbb{R}^m) \cong M^{m\times n}(\mathbb{R})$. □

5.14.1 Explicit Form of the Matrix of a Linear Mapping

We now explicitly compute the entries of the matrix of a given linear transformation, given bases B and B' for V and W. Let B and B' be bases for V and W, respectively, and let $L_B : \mathbb{R}^n \to V$ and $L_{B'} : \mathbb{R}^m \to V$ be the corresponding basis isomorphisms. Then, given $T \in \mathrm{Hom}(V,W)$, $L_{B'}^{-1} \circ T \circ L_B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is the unique linear map making the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle L_B}\big\uparrow & & \big\downarrow{\scriptstyle L_{B'}} \\ \mathbb{R}^n & \xrightarrow[L_{B'}^{-1}\circ T\circ L_B]{} & \mathbb{R}^m \end{array}$$

Note that, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned} L_{B'}^{-1} \circ T \circ L_B(\mathbf{x}) &= L_{B'}^{-1}(T(L_B(\mathbf{x}))) \\ &= L_{B'}^{-1}(T(\sum_{i=1}^n x_i \mathbf{v}_i)) \\ &= \sum_{i=1}^n x_i L_{B'}^{-1}(T(\mathbf{v}_i)) \\ &= \sum_{i=1}^n x_i [T(\mathbf{v}_i)]_{B'} \\ &= [[T(\mathbf{v}_1)]_{B'}, [T(\mathbf{v}_2)]_{B'}, \ldots, [T(\mathbf{v}_n)]_{B'}]\mathbf{x}, \end{aligned}$$

where $[[T(\mathbf{v}_1)]_{B'}, [T(\mathbf{v}_2)]_{B'}, \ldots, [T(\mathbf{v}_n)]_{B'}]$ is the matrix whose ith column is given by $[T(\mathbf{v}_i)]_{B'}$, the coordinate vector with respect to B' of the image under T of the ith basis vector of B.

Definition 5.117 (Matrix of a linear mapping). The matrix $T_{B,B'} = [[T(\mathbf{v}_1)]_{B'}, [T(\mathbf{v}_2)]_{B'}, \ldots, [T(\mathbf{v}_n)]_{B'}]$ is called the *matrix of the linear mapping T*.

Example 5.118. Let $T \in \mathrm{Hom}(P_1, P_2)$ be defined by $T(p(x)) = xp(x)$. Let $B = \{\mathbf{v}_1, \mathbf{v}_2\} = \{1, x\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{1, x, x^2\}$ be the standard bases for P_1 and P_2, respectively. Since

$$\begin{aligned} T(\mathbf{v}_1) &= x \cdot 1 = x = \mathbf{u}_2 \\ T(\mathbf{v}_2) &= x \cdot x = x^2 = \mathbf{u}_3, \end{aligned}$$

we have

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and therefore

$$T_{B,B'} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For example, if $p(x) = 2 - x$, then $T(p(x)) = x(2-x) = 2x - x^2$. Note that $[p(x)]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and

$$\begin{aligned} T_{B,B'}[p(x)]_B &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \\ &= [T(p(x))]_{B'}. \end{aligned}$$

5.14.2 Matrix of a Composition of Linear Mappings

In Lemma 5.69 it was shown that if $S \in \mathrm{Hom}(U, V)$ and $T \in \mathrm{Hom}(V, W)$, then $T \circ S \in \mathrm{Hom}(U, W)$. Choosing bases for these vector spaces, let us now see how the matrix representing $T \circ S$ is related to those representing T and S.

Let $B = \{\mathbf{u}_i\}_{i=1}^{\ell}$, $B' = \{\mathbf{v}_i\}_{i=1}^{m}$, and $B = \{\mathbf{w}_i\}_{i=1}^{n}$ be bases for U, V, and W, respectively. Consider the diagram below:

$$\begin{tikzcd} U \arrow[r, "S"] \arrow[rr, bend left, "T\circ S"] & V \arrow[r, "T"] & W \\ \mathbb{R}^{\ell} \arrow[u, "L_B"] \arrow[r, "L_{B'}^{-1}\circ S\circ L_B"'] \arrow[rr, bend right, "L_{B''}^{-1}\circ T\circ L_{B'}\circ L_{B'}^{-1}\circ S\circ L_B"'] & \mathbb{R}^{m} \arrow[u, "L_{B'}"] \arrow[r, "L_{B''}^{-1}\circ T\circ L_{B'}"'] & \mathbb{R}^{n} \arrow[u, "L_{B''}"] \end{tikzcd}$$

From the previous section, given $T \circ S : U \to W$, the matrix representing $T \circ S$ is the $n \times \ell$ matrix whose ith column is $[T(S(\mathbf{u}_i))]_{B''}$. On the other hand, since $L_{B''}^{-1} \circ T \circ S \circ L_B = L_{B''}^{-1} \circ T \circ L_{B'} \circ L_{B'}^{-1} \circ$

$S \circ L_B$, for any vector $\mathbf{x} \in \mathbb{R}^\ell$ we have

$$\begin{aligned}
L_{B''}^{-1} \circ T \circ S \circ L_B(\mathbf{x}) &= L_{B''}^{-1} \circ T \circ L_{B'} \circ L_{B'}^{-1} \circ S \circ L_B(\mathbf{x}) \\
&= L_{B''}^{-1}(T(L_{B'}(L_{B'}^{-1}(S(L_B(\mathbf{x})))))) \\
&= L_{B''}^{-1}(T(L_{B'}(L_{B'}^{-1}(S(\sum_{i=1}^{\ell} x_i \mathbf{u}_i))))) \\
&= \sum_{i=1}^{\ell} x_i L_{B''}^{-1}(T(L_{B'}(L_{B'}^{-1}(S(\mathbf{u}_i))))) \\
&= \sum_{i=1}^{\ell} x_i L_{B''}^{-1}(T(L_{B'}([S(\mathbf{u}_i)]_{B'}))) \\
&= \sum_{i=1}^{\ell} x_i L_{B''}^{-1}(T(\sum_{j=1}^{m}([S(\mathbf{u}_i)]_{B'})_j \mathbf{v}_j)) \\
&= \sum_{i=1}^{\ell} \sum_{j=1}^{m}([S(\mathbf{u}_i)]_{B'})_j x_i L_{B''}^{-1}(T(\mathbf{v}_j)) \\
&= \sum_{i=1}^{\ell} \sum_{j=1}^{m}([S(\mathbf{u}_i)]_{B'})_j x_i [T(\mathbf{v}_j)]_{B''} \\
&= \sum_{i=1}^{\ell} \left(\sum_{j=1}^{m} [T(\mathbf{v}_j)]_{B''}([S(\mathbf{u}_i)]_{B'})_j \right) x_i \\
&= \left[\sum_{j=1}^{m} [T(\mathbf{v}_j)]_{B''}([S(\mathbf{u}_i)]_{B'})_j \right] \mathbf{x}
\end{aligned}$$

and therefore

$$[T(S(\mathbf{u}_i))]_{B''} = \sum_{j=1}^{m} [T(\mathbf{v}_j)]_{B''}([S(\mathbf{u}_i)]_{B'})_j; \tag{5.21}$$

that is, the ith column of the matrix representing $T \circ S$ is given by the ith column of the *product* of the matrices representing T and S. [49]

Example 5.119. Let $S \in \mathrm{Hom}(P_2, P_3)$ be defined by $S(p(x)) = xp(x)$ for all $p(x) \in P_2$ and let $T \in \mathrm{Hom}(P_3, M^{2\times 2}(\mathbb{R}))$ be defined by $T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}$. Let us choose the standard basis for each space, but let us choose a nonstandard ordering for the basis vectors in B' just to make things interesting:

$$\begin{aligned}
B &= \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{1, x, x^2\} \\
B' &= \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \{1, x, x^2, x^3\} \\
B'' &= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.
\end{aligned}$$

[49] Note that the matrix representing T is $n \times m$ and the matrix representing S is $m \times \ell$, so their product is indeed well-defined and gives and $n \times \ell$ matrix.

Then

$$\begin{aligned} S(\mathbf{u}_1) &= x \cdot 1 = x = \mathbf{v}_2 \\ S(\mathbf{u}_2) &= x \cdot x = x^2 = \mathbf{v}_3 \\ S(\mathbf{u}_3) &= x \cdot x^2 = x^3 = \mathbf{v}_4 \end{aligned}$$

and therefore

$$[S(\mathbf{u}_1)] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; [S(\mathbf{u}_2)] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; [S(\mathbf{u}_3)] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so the matrix representing S with respect to these bases is

$$S_{B,B'} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since

$$\begin{aligned} T(\mathbf{v}_1) &= \mathbf{w}_3 \\ T(\mathbf{v}_2) &= \mathbf{w}_2 \\ T(\mathbf{v}_3) &= \mathbf{w}_4 \\ T(\mathbf{v}_4) &= \mathbf{w}_1 \end{aligned}$$

we have

$$[T(\mathbf{v}_1)]_{B''} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; [T(\mathbf{v}_2)]_{B''} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; [T(\mathbf{v}_3)]_{B''} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; [T(\mathbf{v}_4)]_{B''} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and therefore the matrix representing T with respect to these basis is

$$T_{B',B''} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Finally,

$$\begin{aligned} T(S(\mathbf{u}_1)) &= T(\mathbf{v}_2) = \mathbf{w}_2, \\ T(S(\mathbf{u}_2)) &= T(\mathbf{v}_3) = \mathbf{w}_4, \\ T(S(\mathbf{u}_3)) &= T(\mathbf{v}_4) = \mathbf{w}_1, \end{aligned}$$

so

$$[T(S(\mathbf{u}_1))]_{B''} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ [T(S(\mathbf{u}_2))]_{B''} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\ [T(S(\mathbf{u}_3))]_{B''} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and therefore the matrix representing $T \circ S$ is given by

$$(T \circ S)_{B,B''} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We see that

$$(T \circ S)_{B,B''} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T_{B',B''} S_{B,B'},$$

as expected.

Exercise 5.40. Show that in the special case where $T_A \in Hom(\mathbb{R}^n, \mathbb{R}^m)$ is multiplication by A, and where B and B' are the *standard bases* for $\mathbb{R}^n$ and $\mathbb{R}^m$, that $T_{B,B'} = A$.

[Add solution.]

It is worth noting that the considerations of this section give *independent* motivation for Definition 3.6 of matrix multiplication, as we have arrived at Equation 5.21 by *completely different* considerations than those in Section 3.1.3 (where we were studying linear combinations of rows of matrices). There is further motivation for this definition of the matrix product: if we take $W = V$, then $Hom(V)$ is closed under composition as well as vector addition, and moreover forms an *algebra* with respect to the two operations. Taking Definition 3.6 of multiplication, $M^{n\times n}(\mathbb{R})$ also forms an algebra and we find that $\text{Hom}(V)$ and $M^{n\times n}(\mathbb{R})$ are isomorphic at the level of *algebras*, and not just as vector spaces. We will study some of the implications of this additional structure in Section 6.

5.15 The Dual Space

Let V be a finite-dimensional (real) vector space.

Definition 5.120. The vector space $V^* \equiv \text{Hom}(V, \mathbb{R})$ is called the *dual space of* V. An element of V^* is called a *covector on* V.

We now show that $\dim V^* = \dim V$. We will need the following lemma.

Lemma 5.121. Let V and W be vector spaces, and let $(E_1, \ldots, E_n)$ be a basis for V. For any n elements $w_1, \ldots, w_n \in W$, there is a unique linear map $T : V \to W$ satisfying $T(E_i) = w_i$ for $i = 1, \ldots, n$.

Proof. Any $v \in V$ can be written uniquely as $v^i E_i$. Define $T(v) = v^i w_i$, where $w_i = T(E_i)$. This map is linear by definition. Suppose T' is any other linear map satisfying $T'(E_i) = w_i$ for $i = 1, \ldots, n$. Then for all $v \in V$,

$$\begin{aligned} T'(v) &= T'(v^i E_i) \\ &= v^i T'(E_i) \\ &= v^i w_i \\ &= T(v). \end{aligned}$$

Thus, T is unique. □

Proposition 5.122. Given any basis $(E_1, \ldots, E_n)$ for V, let $\varepsilon^1, \ldots, \varepsilon^n \in V^*$ be the covectors defined by

$$\varepsilon^i(E_j) = \delta^i_j,$$

where δ^i_j is the Kronecker delta symbol. Then $(\varepsilon^1, \ldots, \varepsilon^n)$ is a basis for V^*, called the *dual basis to* (E_j). Therefore, $\dim V^* = \dim V$.

Proof. By Lemma 5.121, $\varepsilon^i(E_j) = \delta^i_j$ defines a unique linear map for each $i = 1, \ldots, n$. To show that (ε^i) is linearly independent, suppose $c_i \varepsilon^i = 0$ (that is, $c_i \varepsilon^i$ is the zero map). This means that $c_i \varepsilon^i(v) = 0$ for all $v \in V$. In particular, when $v = E_j$ we have

$$\begin{aligned} 0 &= c_i \varepsilon^i(E_j) \\ &= c_i \delta^i_j \\ &= c_j \end{aligned}$$

This shows that $c_i = 0$ for all $i = 1, \ldots, n$, so the set (ε^i) is linearly independent.

To show that span $\{\varepsilon^i\} = V^*$, let $\omega \in V^*$. We need to show that $\omega = \omega_i \varepsilon^i$ for some $\omega_i \in \mathbb{R}$, $i = 1, \ldots, n$. Define $\omega_i \equiv \omega(E_i)$. Then for any $v \in V$, writing $v = v^j E_j$ we have

$$\begin{aligned} \omega_i \varepsilon^i(v) &= \omega_i \varepsilon^i(v^j E_j) \\ &= \omega_i v^j \varepsilon^i(E_j) \\ &= \omega_i v^j \delta^i_j \\ &= \omega_i v^i \\ &= v^i \omega_i \\ &= v^i \omega(E_i) \\ &= \omega(v^i E_i) \\ &= \omega(v). \end{aligned}$$

Thus, $\omega = \omega(E_i)\varepsilon^i$, which shows that span $\{\varepsilon^i\} = V^*$, and is therefore a basis for V^*. Since $|\{\varepsilon^i\}| = n$, this shows that $\dim V^* = n = \dim V$. □

Example 5.123. If $(e_1, \ldots, e_n)$ is the standard basis for $\mathbb{R}^n$, then the dual basis is denoted $(e^1, \ldots, e^n)$ and is called the *standard dual basis*. These basis covectors are the linear functionals on $\mathbb{R}^n$ given by

$$\begin{aligned} e^i(v) &= e^i(v^j e_j) \\ &= v^j e^i(e_j) \\ &= v^j \delta^i_j \\ &= v^i. \end{aligned}$$

In other words, e^i is the linear functional that picks out the ith component of a vector. Recall that an element of $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ can be represented by a $1 \times n$ matrix (a row matrix). The basis covectors can therefore be represented by the row matrices

$$e^1 = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & 1 & \ldots & 0 \end{pmatrix}, \quad \ldots, \quad e^n = \begin{pmatrix} 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Exercise 5.41. Show that if (E_j) is any basis for a vector space V and (ε^i) is its dual basis, then for any vector $v = v^j E_j \in V$, $\varepsilon^i(v) = v^i$. Thus, just as in the case of $\mathbb{R}^n$, the ith basis covector ε^i picks out the ith component of a vector with respect to the basis (E_j).

Proof.

$$\varepsilon^i(v) = \varepsilon^i(v^j E_j) = v^j \varepsilon^i(E_j) = v^j \delta^i_j = v^i.$$

□

Exercise 5.42. More generally, Proposition 5.122 shows that we can express an arbitrary covector $\omega \in V^*$ in terms of the dual basis as

$$\omega = \omega_i \varepsilon^i,$$

where the components are determined by $\omega_i = \omega(E_i)$. Show that the action of ω on a vector $v = v^j E_j$ is

$$\omega(v) = \omega_i v^i.$$

Proof.

$$\begin{aligned} \omega(v) &= \omega_i \varepsilon^i(v^j E_j) \\ &= \omega_i v^j \varepsilon^i(E_j) \\ &= \omega_i v^j \delta^i_j \\ &= \omega_i v^i. \end{aligned}$$

□

Definition 5.124. Given a linear map $A : V \to W$ between finite dimensional vector spaces V, W, we define a linear map $A^* : W^* \to V^*$, called the *dual map* or *transpose of A*, by

$$(A^*\omega)(v) = \omega(Av) \qquad \text{for } \omega \in W^*, v \in V.$$

Exercise 5.43. Show that this definition makes sense. (Does it make sense to plug an element of V into $A^*\omega$? Does it make sense to plug Av into ω?)

Proof. Since $A^*\omega \in V^*$, $A^*\omega$ takes v to the real number $A^*\omega(v)$. Since $Av \in W$ and $\omega \in W^*$, ω takes Av to the real number $\omega(Av)$. □

Exercise 5.44. Show that $A^*\omega$ is a linear functional on V, and that A^* is a linear map.

Proof. For any $c \in \mathbb{R}$, $v_1, v_2 \in V$, we have

$$\begin{aligned} A^*\omega(cv_1 + v_2) &= \omega(A(cv_1 + v_2)) \qquad \text{(by def of } A^*\text{)} \\ &= \omega(cAv_1 + Av_2) \qquad \text{(since } A \text{ is linear)} \\ &= c\omega(Av_1) + \omega(Av_2) \qquad \text{(since } \omega \text{ is linear)} \\ &= cA^*\omega(v_1) + A^*\omega(v_2) \qquad \text{(by def of } A^*\text{).} \end{aligned}$$

This shows that $A^*\omega$ is a linear functional on V.

Now let $c \in \mathbb{R}$ and $\omega_1, \omega_2 \in W^*$. Then, for any $v \in V$, we have

$$\begin{aligned} A^*(c\omega_1 + \omega_2)(v) &= (c\omega_1 + \omega_2)(Av) \qquad \text{(by def of } A^*\text{)} \\ &= (c\omega_1)(Av) + \omega_2(Av) \qquad \text{(by def of } + \text{ on } V^*\text{)} \\ &= c\omega_1(Av) + \omega_2(Av) \qquad \text{(by def of } \cdot \text{ on } V^*\text{)} \\ &= cA^*\omega_1(v) + A^*\omega_2(v) \qquad \text{(by def of } A^*\text{)} \end{aligned}$$

and therefore

$$A^*(c\omega_1 + \omega_2) = cA^*\omega_1 + A^*\omega_2.$$

This shows that A^* is a linear map. □

Proposition 5.125. The dual map satisfies the following properties: [50]

(a) $(A \circ B)^* = B^* \circ A^*$.

(b) $(\mathrm{id}_V)^* : V^* \to V^*$.

Proof. Let U, V, W be vector spaces and let $B : U \to V$ and $A : V \to W$ be linear maps:

$$\begin{tikzcd} U \arrow[r, "B"] \arrow[rr, bend right, "A\circ B"'] & V \arrow[r, "A"] & W \end{tikzcd}$$

The corresponding dual maps are then

$$\begin{tikzcd} U^* & V^* \arrow[l, "B"'] & W^* \arrow[l, "A"'] \arrow[ll, bend left, "(A\circ B)^*"] \end{tikzcd}$$

[50]These properties show that the assignment sending a vector space to its dual and a linear map to its dual map is a contravariant functor from the category of real vector spaces to itself.

(a) Let $\omega \in W^*$. Then for any $v \in U$ we have

$$\begin{aligned}(A \circ B)^*\omega(v) &= \omega((A \circ B)v) \\ &= \omega(ABv).\end{aligned}$$

Meanwhile,

$$\begin{aligned}B^* \circ A^*\omega(v) &= A^*\omega(Bv) \text{ (note that } Bv \in V \text{ and } A^*\omega \in V^*\text{, so this makes sense)} \\ &= \omega(ABv) \\ &= (A \circ B)^*\omega(v).\end{aligned}$$

Hence,

$$(A \circ B)^* = B^* \circ A^*.$$

(b) Let $\omega \in V^*$. Then for any $v \in V$ we have

$$\begin{aligned}\mathrm{Id}_V^*\omega(v) &= \omega(\mathrm{Id}_V v) \\ &= \omega(v).\end{aligned}$$

Hence $\mathrm{Id}_V^* = \mathrm{Id}_{V^*}$.

□

Let $(V^{**}) = (V^*)^*$ denote the *second dual space*. We will now show that, for any finte-dimensional vector space, V^{**} is canonically isomorphic to V. [51] For each $v \in V$, define a linear functional $\xi(v) : V^* \to \mathbb{R}$ by

$$\xi(v)(\omega) = \omega(v) \text{ for } \omega \in V^*. \tag{5.22}$$

Exercise 5.45.

(a) For any $v \in V$, show that $\xi(v)(\omega)$ depends linearly on ω, so $\xi(v) \in V^{**}$.

(b) Show that the map $\xi : V \to V^{**}$ is linear.

Proof. (a) Let $c \in \mathbb{R}, \omega_1, \omega_2 \in V^*$. Then

$$\begin{aligned}\xi(v)(c\omega_1 + \omega_2) &= (c\omega_1 + \omega_2)(v) \\ &= c\omega_1(v) + \omega_2(v) \\ &= c\xi(v)(\omega_1) + \xi(v)(\omega_2).\end{aligned}$$

This shows that $\xi(v)$ is a linear functional from $V^* \to \mathbb{R}$, so, by definition, $\xi(v) \in V^{**}$.

[51] What we mean by this, is that there is an isomorphism $\xi : V^{**} \to V$ which does not depend on any choices, such as a choice of basis.

(b) Let $c \in \mathbb{R}$, $v_1, v_2 \in V$, and $\omega \in V^*$. Then

$$\begin{aligned} \xi(cv_1 + v_2)(\omega) &= \omega(cv_1 + v_2) \\ &= c\omega(v_1) + \omega(v_2) \\ &= c\xi(v_1)(\omega) + \xi(v_2)(\omega). \end{aligned}$$

Hence, $\xi : V \to V^{**}$ is linear.

□

Proposition 5.126. For any finite-dimensional vector space V, the map $\xi : V \to V^{**}$ is an isomorphism. [52]

Proof. Since $\dim V = \dim V^{**}$, it suffices to show that ξ is injective. (♣ Cite appropriate theorem here.) Suppose $v \in V$ is not zero. Extend v to a basis $(v = E_1, \ldots, E_n)$ for V, and let $(\varepsilon^1, \ldots, \varepsilon^n)$ denote the dual basis for V^*. Then $\xi(v) \neq 0$ because

$$\xi(v)(\varepsilon^1) = \varepsilon^1(v) = \varepsilon^1(E_1) = 1.$$

This shows that $\ker \xi = \{0\}$, so ξ is linear. □

The preceding proposition shows that when V is finite-dimensional, we can unambiguously identify V^{**} with V itself, because the map ξ is canonically defined, without reference to any basis. It is important to observe that although V^* is also isomorphic to V (for the simple reason that any two finite-dimensional vector spaces of the same dimension are isomorphic), there is no *canonical* isomorphism $V \cong V^*$.

Exercise 5.46. (a) (♣ Explain why this is what we mean by *canonical*.) Suppose V and W are finite-dimensional vector spaces and $A : V \to W$ is any linear map. Show that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\xi_V"'] & W \arrow[d, "\xi_W"] \\ V^{**} \arrow[r, "(A^*)^*"'] & W^{**}, \end{tikzcd}$$

where ξ_V and ξ_W denote the isomorphisms by (5.22) for V and W, respectively.

(b) Show that there does *not* exist a way to assign to each finite-dimensional vector space V an isomorphism $\beta_V : V \to V^*$ such that for every map $A : V \to W$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\beta_V"'] & W \arrow[d, "\beta_W"] \\ V^* & W^*, \arrow[l, "A^*"] \end{tikzcd}$$

Proof. (a) We need to show that $(\xi_W \circ A)(v)(\omega) = ((A^*)^* \circ \xi_V)(v)(\omega)$ for all $v \in V$ and $\omega \in W^*$. First, note that

$$(\xi_W \circ A)(v)(\omega) = \xi_W(Av)(\omega) = \omega(Av).$$

[52] If V is infinite-dimensional, then V is not isomorphic to V^{**}.

Following the other direction, we have

$$\begin{aligned}((A^*)^* \circ \xi_V)(v)(\omega) &= \xi_V(v)(A^*\omega)\\ &= A^*\omega(v)\\ &= \omega(Av)\\ &= (\xi_W \circ A)(v)(\omega),\end{aligned}$$

so the diagram commutes. [53]

(b) Assume there exist isomorphisms β_V, β_W so that the diagram commutes for every linear map $A : V \to W$. Take A to be the zero map, which is linear, and consider $v \neq 0$ in V. Since β_V is an isomorphism, $\beta_V(v) : V^* \to \mathbb{R}$ is not the zero map, so there exists $v' \in V$ such that $\beta_V(v)(v') \neq 0$. However, $Av = 0$ (since A is the zero map), and $\beta_W(0) : W^* \to \mathbb{R}$ is the zero map (since β_W is linear), and therefore for every $v' \in V$

$$\begin{aligned}A^*(\beta_W(Av))(v') &= A^*(0)(v')\\ &= 0(Av')\\ &= 0,\end{aligned}$$

so the diagram does not commute.

□

Because of Proposition 5.126, the real number $\omega(v)$ obtained by applying a covector ω to a vector v is sometimes denoted by either of the more symmetric-looking notations $\langle \omega, v\rangle = \langle v, \omega\rangle$; both expressions can be thought of either as the action of the covector $\omega \in V^*$ on the vector $v \in V$, or as the action of the linear functional $\xi(v) \in V^{**}$ on the element $\omega \in V^*$. There should be no cause for confusion with the use of the same angle bracket notation for inner products: whenever one of the arguments is a vector and the other a covector, the notation $\langle \omega, v\rangle$ is always to be interpreted as the natural pairing between vectors and covectors, not as an inner product. We typically omit any mention of the map ξ, and think of $v \in V$ either as a vector or as a linear functional on V^*, depending on the context.

There is also a symmetry between bases and dual bases for a finite-dimensional vector space V: any basis for V determines a dual basis for V^*, and conversely, any basis for V^* determines a dual basis for $V^{**} = V$.

Exercise 5.47. If (ε^i) is the basis for V^* dual to a basis (E_j) for V, show that $(\xi(E_j))$ is the basis for $V^{**} \cong V$ dual to (ε^i).

Proof.

$$\xi(E_j)(\varepsilon^i) = \varepsilon^i(E_j) = \delta^i_j.$$

□

[53]If this is confusing, let $B = A^*$, $T = W^*$, and $U = V^*$. Then $B : T \to U$ and $B^* : U^* \to T^*$ and $(B^*\alpha)(t) = \alpha(Bt)$ for all $\alpha \in U^*$ and $t \in T$. Now substitute $\alpha = \xi_V(v), B = A^*$, and $t = \omega$ to obtain $\xi_V(v)(A^*\omega)$. By def of ξ_V, $\xi_V(v)(A^*\omega) = A^*\omega(v)$, and by definition of A^*, $A^*\omega(v) = \omega(Av)$.

5.16 Fundamental Matrix Spaces

We have just seen that for any finite-dimensional vector spaces V and W, $\mathrm{Hom}(V, W) \cong M^{m\times n}(\mathbb{R})$. We have also seen that given any $T \in \mathrm{Hom}(V, W)$, there are important subspaces associated to T, namely the kernel $\ker T \subseteq V$ and image $T(V) \subseteq W$ (see Theorem 5.61). In this section we study the corresponding subspaces in $\mathbb{R}^n$ and $\mathbb{R}^m$. These are together called the *Fundamental Matrix Spaces*. The first is the analog of $\ker T$. For the rest of this section, let A be an $m \times n$ matrix.

Definition 5.127 (Null space). The *null space* of A is the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. If we denote this by Nul A, then

$$\text{Nul } A := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Proposition 5.128 (Nul A is a subspace of $\mathbb{R}^n$). Nul A is a subspace of $\mathbb{R}^n$.

Proof. Nul $A = \ker T_A$, where $T_A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is the corresponding matrix transformation, which by Theorem 5.61 is a subspace of $\mathbb{R}^n$.

We can also prove this directly: for any $\mathbf{x}, \mathbf{y} \in$ Nul A and $c \in \mathbb{R}$, we have

$$A(c\mathbf{x} + \mathbf{y}) = cA\mathbf{x} + \mathbf{y} = c \cdot \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so $c\mathbf{x} + \mathbf{y} \in$ Nul A. Hence, Nul A is a subspace of $\mathbb{R}^n$ by the subspace criterion. □

Definition 5.129 (Column space). The *column space* of A is the set of all linear combinations of the columns of A. We denote this set by Col A. If we denote the ith column of A by $\mathbf{a}_i$, then $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ and

$$\text{Col } A := \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$

Proposition 5.130 (Col A is a subspace of $\mathbb{R}^m$). Col A is a subspace of $\mathbb{R}^m$

Proof. By Theorem 5.38 if $S \subseteq \mathbb{R}^m$, then Span S is a subspace of $\mathbb{R}^m$. □

5.16.1 Finding Bases for Nul A and Col A

We will now illustrate a method of finding bases for the fundamental matrix spaces through examples.

Example 5.131. Consider the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

We can find a parametric description of Nul A by row reducing this matrix. The reduced row echelon form of A is

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that x_2, x_4, x_5 are free variables, so we set

$$\begin{aligned} x_2 &= t_1 \\ x_4 &= t_2 \\ x_5 &= t_3. \end{aligned}$$

A vector $\mathbf{x} \in \mathbb{R}^5$ is therefore a solution to this system of equations if it is of the form

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} &= \begin{bmatrix} 2t_1 + t_2 - 3t_3 \\ t_1 \\ -2t_2 + 2t_3 \\ t_2 \\ t_3 \end{bmatrix} \\ &= t_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Therefore

$$\operatorname{Nul} A = \operatorname{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

To see that these vectors are linearly independent, apply the $+/-$ lemma (Lemma 5.85): We think of building up this set one vector at a time. The first vector is not the zero vector, so the set containing just that vector is linearly independent. The second vector is not a scalar multiple of the first, so the set containing just the first two vectors is linearly independent. Finally, since the fifth entry of both of the first two vectors is zero, the third vector cannot be a linear combination of the first two. Hence, the set containing all three vectors is linearly independent. Since the span of three linearly independent vectors is three-dimensional, by Theorem 5.98 these vectors therefore form a basis for Nul A.

Theorem 5.132. Let A be an $m \times n$ matrix and R its reduced row echelon form. Then the pivot columns of R form a basis for Nul A.

Proof. By the definition of reduced row echelon form, the pivot column vectors are linearly independent. Since Nul A is the span of these vectors, they form a basis for Nul A. $\square$

Example 5.133. Let us first find a basis for the column space of a matrix which is already in reduced row echelon form. Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By definition,

$$\operatorname{Col} A = \operatorname{Span}\left\{ \underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{\mathbf{b}_1}, \underbrace{\begin{bmatrix}4\\0\\0\\0\end{bmatrix}}_{\mathbf{b}_2}, \underbrace{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}_{\mathbf{b}_3}, \underbrace{\begin{bmatrix}2\\-1\\0\\0\end{bmatrix}}_{\mathbf{b}_4}, \underbrace{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}_{\mathbf{b}_5} \right\}$$

Note that $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$. By the $+/-$ Lemma (Lemma 5.85), we can remove these two vectors are the remaining vectors still span Col A:

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\}.$$

This set of vectors is linearly independent, and therefore form a basis for Col A.

Theorem 5.134 (Basis for Col A when A is in RREF). If A is in reduced row echelon form, then its pivot columns form a basis for Col A.

Proof. Since A is in RREF, each non-pivot column is a linear combination of the pivot columns to the left of it, and can therefore be removed from the set of column vectors without affecting their span. □

Now consider a matrix A which is not in reduced row echelon form. We first note that *elementary row operations change the column space*. For instance, the matrices

$$A = \begin{bmatrix}1 & 3\\2 & 6\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & 3\\0 & 0\end{bmatrix}$$

are row equivalent, being related by a row-replacement. However,

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix}1\\2\end{bmatrix} \right\}$$

while

$$\operatorname{Col} B = \operatorname{Span}\left\{ \begin{bmatrix}1\\0\end{bmatrix} \right\},$$

which are different spaces. However, *elementary row operations do not change the linear dependence relationships* among the columns of A.

Lemma 5.135 (Elementary row operations do not change the linear dependence relationships among the columns of A). Elementary row operations do not change the linear dependence relationships among the columns of A

Proof. A linear dependence relationship among the columns of A is a (non-trivial) solution to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. If A and B are row-equivalent, then $B\mathbf{x} = \mathbf{0}$ has exactly the same solution set as $A\mathbf{x} = \mathbf{0}$. □

Theorem 5.136 (Basis for Col A). The pivot columns of A form a basis for Col A.

Proof. Let B be any row echelon form of A. The set of pivot columns of B is linearly independent, since the leftmost pivot column is nonzero and no vector in the set is a linear combination of the vectors that precede it (by definition of reduced row echelon form). Since A is row-equivalent to B, by Lemma 5.135 the pivot columns of A are linearly independent as well. Since the set of non-pivot columns of B are linear combinations of the pivot columns of B, the same is true for A. By the $+/-$ lemma (Lemma 5.85), removing the non-pivot columns of A therefore does not change Col A. This shows that the pivot columns of A span Col A and are linearly independent, and hence form a basis for Col A. □

Thus, to find a basis for Col A for *any* matrix A,

1. Row-reduce A to REF, U.
2. Locate the pivot columns.
3. The corresponding columns in A form a basis for Col A.

We illustrate this procedure in the next example.

Example 5.137. Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

Its reduced row echelon form is

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that columns 1,3 and 5 of B are linearly independent. Thus, the corresponding columns of A form a basis for Col A:

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}.$$

Exercise 5.48. Find a basis for the null space and column space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution. The matrix A is row-equivalent to

$$B = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned}
\text{Nul } A &= \left\{ \begin{bmatrix} 2t_1 + t_2 - 3t_3 \\ t_1 \\ -2t_2 + 2t_3 \\ t_2 \\ t_3 \end{bmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\} \\
&= \left\{ t_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\} \\
&= \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.
\end{aligned}$$

Thus, $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A.

Since only the first and third columns of A are pivot columns, a basis for Col A is given by

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}.$$

□

5.16.2 Dimensions of Nul A and Col A

Definition 5.138 (Rank and nullity).

(a) The dimension of Col A is called the *rank* of A, denoted rank A.

(b) The dimension of Nul A is called the *nullity* of A, denoted nullity A.

Theorem 5.139 (Dimensions of Nul A and Col A). Given any matrix A.

(a) rank $A =$ the number of pivot columns of A,

(b) nullity $A =$ the number of non-pivot columns of A.

Proof. (a) By Theorem 5.136, the pivot columns of A form a basis for Col A, so the dimension of Col A is the number of pivot columns of A.

(b) By Theorem 5.132, the dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, which is equal to the number of non-pivot columns of A.

□

Theorem 5.140 (The rank theorem). Let A be an $m \times n$ matrix. Then $\operatorname{rank} A + \operatorname{nullity} A = n$.

Proof. The number of pivot columns plus the number of non-pivot columns equals the total number of columns. □

Exercise 5.49. Determine the rank and nullity of the matrix A in Exercise 5.48 and verify Theorem 5.140.

Solution. In Exercise 5.48, we found the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\operatorname{Nul} A$, so nullity $A =$ 3. We found that the set $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ is a basis for $\operatorname{Col} A$, so rank $A = 2$. Indeed, $3 + 2 = 5$, which is the number of columns of A. □

Exercise 5.50. (a) If A is a 7×9 matrix with nullity 2, what is the rank of A?

(b) Could a 6×9 matrix have nullity 2?

Solution. (a) By Theorem 5.140,

$$\begin{aligned} \operatorname{rank} A + \operatorname{nullity} A &= \text{number of columns} \\ \operatorname{rank} A + 2 &= 9 \end{aligned}$$

therefore rank $A = 7$.

(b) Since the matrix A has 9 columns, if nullity $A = 2$, then by Theorem 5.140 rank A must satisfy

$$\operatorname{rank} A + 2 = 9$$

and therefore rank $A = 7$. But A has only 6 rows, and therefore at most 6 pivot columns, which implies rank $A \leq 6$. Thus, it is not possible for A to have nullity 2. □

5.16.3 The Row Space of a Matrix

It is also frequently useful to consider the span of the *rows* of an $m \times n$ matrix A.

Definition 5.141 (Row Space). The set of all linear combinations of the row vectors of A is called the *row space* of A, denoted Row A. If $\mathbf{r}_1, \ldots, \mathbf{r}_m$ denote the rows of A, then

$$\operatorname{Row} A = \operatorname{Span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}.$$

Proposition 5.142 (Row A is a subspace of $\mathbb{R}^n$). Row A is a subspace of $\mathbb{R}^n$.

Proof. The span of a subset of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. □

Lemma 5.143 (Elementary row operations do not change Row A). If A is row-equivalent to B, then Row $A =$ Row B.

Proof. By transitivity of row-equivalence, it suffices to show that this is true for a single elementary row operation of each type. Since an elementary row operation of any of the three types produces a matrix B whose rows are linear combinations of the rows of A, then any linear combination of the rows of B is a linear combination of the rows of A. Hence Row $B \subseteq$ Row A. Since elementary row operations are invertible, the same argument shows that Row $A \subseteq$ Row B, and therefore that Row $A =$ Row B. ☐

Corollary 5.144 (Basis for Row A). If B is any row echelon form of a matrix A, then the non-zero rows of B form a basis for Row A.

Proof. Since B is in row echelon form, the first row is nonzero. By definition of reduced row echelon form, the remaining nonzero rows are not linear combinations of the preceding ones, so these are linearly independent and hence they form a basis for Row B. By Lemma 5.143, Row $B =$ Row A. ☐

Exercise 5.51. Find a basis for the row space of the matrix in Exercise 5.48.

Solution. In Exercise 5.48 we found that the RREF of A is

$$B = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus a basis for Row A is given by

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

☐

Theorem 5.145 (dim Row A = dim Col A). For any matrix A, $\dim \text{Row } A = \dim \text{Col } A$.

Proof. Let B be any REF of A. The nonzero rows of B form a basis for Row B. Since B is in REF, each nonzero row contains a pivot, hence

$$\dim \text{Row } A = \dim \text{Row } B = \text{ number of nonzero rows } = \text{ number of pivots } = \dim \text{Col } A.$$

☐

In the following example we will find a bases for all three fundamental matrix spaces.

Example 5.146. Let

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}.$$

The reduced row echelon form of A is

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nonzero rows of B give a basis for Row A, so

$$\text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}.$$

We see that the first, second, and fourth columns of B are pivot columns, so the corresponding columns of A form a basis for Col A:

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

Finally, Nul A is the solution set of the corresponding homogeneous linear system $A\mathbf{x} = \mathbf{0}$, so

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

5.16.4 Invertible Matrix Theorem, Revisited

Note that the considerations of this section adds several additional equivalent conditions for an $n \times n$ matrix A to be invertible:

Theorem 5.147 (Invertible Matrix Theorem, Revisited).

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) *A is invertible.*
(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*
(c) *The reduced row echelon form of A is I_n.*
(d) *A is expressible as a product of elementary matrices.*
(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix $\mathbf{b}$.*
(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.*
(g) $\det(A) \neq 0$.
(h) *The column vectors of A are linearly independent.*
(i) *The row vectors of A are linearly independent.*
(j) *The column vectors of A span R^n.*
(k) *The row vectors of A span R^n.*
(l) *The column vectors of A form a basis for R^n.*
(m) *The row vectors of A form a basis for R^n.*
(n) *A has rank n.*
(o) *A has nullity 0.*

6 Eigenvalues and Eigenvectors

6.1 Similarity

Let V and W be finite dimensional vector spaces, and let $T : V \to W$ be a linear mapping. By choosing ordered bases $B = \{\mathbf{v}_i\}_{i=1}^n$ for V and $B' = \{\mathbf{v}'_i\}_{i=1}^n$ for W, we can represent T by the matrix $[T]_{B',B}$ whose ith column is $[T(\mathbf{v}_i)]_{B'}$, where $\mathbf{v}_i$ is the ith basis vector in the ordered basis B (see Section 5.14). In the case where $W = V$ (in which case we say that T is a *linear operator* on V), then it is most convenient to take $B' = B$. In this case, we will denote the matrix of the linear operator T with respect to the basis B by $[T]_B$, rather than $[T]_{B,B}$.

Example 6.1. Let $D : P_3 \to P_3$ be the differentiation operator, defined by

$$D(c_0 + c_1x + c_2x^2 + c_3x^3) = c_1 + 2c_2x + 3c_3x^2.$$

Choosing the standard basis $B = \{1, x, x^2, x^3\}$ for P_3, the ith column of $[D]_B$ (the matrix representing D) is given by (see section 5.14) $[D(\mathbf{v}_i)]_B$. Since

$$\begin{aligned} D(\mathbf{v}_1) &= 0 = \mathbf{0} \\ D(\mathbf{v}_2) &= 1 = \mathbf{v}_1 \\ D(\mathbf{v}_3) &= 2x = 2\mathbf{v}_2 \\ D(\mathbf{v}_4) &= 3x^2 = 3\mathbf{v}_3 \end{aligned}$$

we have

$$[D(\mathbf{v}_1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ [D(\mathbf{v}_2)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ [D(\mathbf{v}_3)]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix},\ [D(\mathbf{v}_4)]_B = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

and therefore

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ 0 \end{bmatrix}$$

and $L_B(c_1, 2c_2, 3c_3, 0) = c_1 + 2c_2x + 3c_3x^2 = D(c_0 + c_1x + c_2x^2 + c_3x^3)$.

Given a linear operator T on an n-dimensional vector space V, the entries of the matrix $[T]_B$ representing T depend on the chosen basis B. If we choose another basis B' for V, how are the matrices $[T]_B$ and $[T]_{B'}$ related?

Given any $\mathbf{v} \in V$, we have seen previously that

$$[\mathbf{v}]_B = P_{B' \to B}[\mathbf{v}]_{B'}. \tag{6.1}$$

Now
$$[T(\mathbf{v})]_B = [T]_B[\mathbf{v}]_B \tag{6.2}$$
and by Equation (6.1) we also have
$$[T(\mathbf{v})]_B = P_{B'\to B}[T(\mathbf{v})]_{B'} = P_{B'\to B}[T]_{B'}[\mathbf{v}]_{B'}. \tag{6.3}$$
Setting the right hand sides of Equations (6.2) and (6.3) equal to each other and using Equation (6.1), we find
$$\begin{aligned} P_{B'\to B}[T]_{B'}[\mathbf{v}]_{B'} &= [T]_B P_{B'\to B}[\mathbf{v}]_{B'} \\ [T]_{B'}[\mathbf{v}]_{B'} &= P^{-1}_{B'\to B}[T]_B P_{B'\to B}[\mathbf{v}]_{B'}. \end{aligned}$$
Since this is true for all $\mathbf{v} \in V$, we must have
$$[T]_{B'} = P^{-1}_{B'\to B}[T]_B P_{B'\to B}.$$

Definition 6.2 (Similar matrices). Let $A, B \in M^{n\times n}(\mathbb{R})$. The matrix B is said to be *similar* to A if there exists an invertible matrix $P \in M^{n\times n}(\mathbb{R})$ such that $B = P^{-1}AP$. If B is similar to A, we indicate this by writing $B \sim A$.

Proposition 6.3. Similarity is an equivalence relation on $M^{n\times n}(\mathbb{R})$.

Proof. (i) (Reflexivity): Let $A \in M^{n\times n}(\mathbb{R})$. Taking $P = I$, $A = I^{-1}AI = IAI = A$. Therefore $A \sim A$ for all $A \in M^{n\times n}(\mathbb{R})$.

(ii) (Symmetry): Suppose $B \sim A$. Then there exists an invertible matrix P such that $B = P^{-1}AP$. Multiplying this equation on the left by P and on the right by P^{-1}, we find that $PBP^{-1} = PP^{-1}APP^{-1} = A$. Hence, $B \sim A \implies A \sim B$.

(iii) (Transitivity): Suppose $C \sim B$ and $B \sim A$. Then there exist invertible matrices $P, Q \in M^{n\times n}(\mathbb{R})$ such that $C = Q^{-1}BQ$ and $B = P^{-1}AP$. Then
$$C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$$
so $C \sim A$.

□

Thus, if B is similar to A, we will just say that A and B are similar. If there exists an invertible $n \times n$ matrix P such that $[T]_{B'} = P^{-1}[T]_B P$, we see that $[T]_{B'}$ and $[T]_B$ represent exactly the same linear operator T.

Example 6.4. Consider the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, 0)$. Consider the bases
$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad B' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$
for $\mathbb{R}^2$. The matrices representing T with respect to each of these bases are
$$\begin{aligned} [T]_B &= [[T(e_1)]_B, [T(e_2)]_B] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ [T]_{B'} &= [[T(\mathbf{u}_1')]_{B'}, [T(\mathbf{u}_2')]_{B'}] = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}. \end{aligned}$$

Since $P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, we can check that

$$P_{B' \to B}^{-1} [T]_B P_{B' \to B} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} = [T]_{B'}.$$

Definition 6.5 (Similarity invariant). Let $A \in M^{n \times n}(\mathbb{R})$. A property of A which holds for all matrices similar to A is called a *similarity invariant*.

Theorem 6.6 (Similarity invariants). Let A, B be similar matrices in $M^{n \times n}(\mathbb{R})$. Then

(a) $\det A = \det B$;

(b) A is invertible if and only if B is invertible;

(c) $\operatorname{rank} A = \operatorname{rank} B$;

(d) nullity $A =$ nullity B;

(e) $\operatorname{tr} A = \operatorname{tr} B$.

Proof. Let $B = P^{-1}AP$.

(a) $\det B = \det(P^{-1}AP) = \det P^{-1} \det A \det P = \frac{1}{\det P} \det A \det P = \det A$.

(b) Suppose the nullity of B is k. Choose a basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ for Nul B = Nul $(P^{-1}AP)$. We will show that the nullity of A is equal to the nullity of B by constructing a basis for the null space of A and showing that it has the same number of elements as $\mathcal{B}$. For all $\mathbf{v}_i \in \mathcal{B}$, we have $B\mathbf{v}_i = P^{-1}AP\mathbf{v}_i = \mathbf{0}$. Multiplying this equation on the left by P, we have $A(P\mathbf{v}_i) = \mathbf{0}$, which shows the vectors $P\mathcal{B} = \{P\mathbf{v}_1, \cdots, P\mathbf{v}_k\} \subset$ Nul A. Suppose that $c_1P\mathbf{v}_1 + \cdots + c_kP\mathbf{v}_k = \mathbf{0}$ for some scalars $c_1, \cdots, c_k$. Multiplying both sides of this equation on the left by P^{-1}, it becomes $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ and so we must have $c_1 = \cdots = c_k = 0$ by linear independence of $\mathcal{B}$. This shows $P\mathcal{B}$ is also linearly independent. We will now show that these vectors also span Nul A, and are therefore a basis for Nul A. Let $\mathbf{v} \in$ Nul A. Then $A\mathbf{v} = PBP^{-1}\mathbf{v} = \mathbf{0}$. Multiplying both sides of this equation on the left by P^{-1} gives $B(P^{-1}\mathbf{v}) = \mathbf{0}$, which shows that $P^{-1}\mathbf{v} \in$ Nul B. We can therefore write $P^{-1}\mathbf{v}$ in the basis $\mathcal{B}$ as

$$P^{-1}\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

for some scalars $c_1, \cdots, c_k$. Multiplying both sides on the left by P then gives

$$\mathbf{v} = c_1(P\mathbf{v}_1) + \cdots + c_k(P\mathbf{v}_k)$$

which shows $P\mathcal{B}$ spans Nul A, and is therefore a basis for Nul A. Thus, nullity A = nullity B.

(c) Since A and B are both $n \times n$ matrices, we have

$$\begin{aligned} \text{rank } B &= n - \text{nullity } B \\ &= n - \text{nullity } A \\ &= \text{rank } A. \end{aligned}$$

(d) First note that, for any two matrices A and B,

$$\begin{aligned} \text{tr } AB &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} \\ &= \text{tr } BA. \end{aligned}$$

Then, if $B = P^{-1}AP$, we have

$$\begin{aligned} \text{tr } B &= \text{tr } P^{-1}AP \\ &= \text{tr } P^{-1}(AP) \\ &= \text{tr } (AP)P^{-1} \\ &= \text{tr } A(PP^{-1}) \\ &= \text{tr } AI \\ &= \text{tr } A. \end{aligned}$$

□

Theorem 6.6 allows us to define the following:

Definition 6.7 (Similarity invariants of a linear operator). Let V be a vector space, $T \in Hom(V)$, and B any basis for V. Then

(a) $\det T = \det\,[T]_B$

(b) nullity $T =$ nullity $[T]_B$

(c) rank $T =$ rank $[T]_B$

(d) tr $T =$ tr $[T]_B$

That is the determinant (resp. rank, nullity, trace) of a linear operator T is the determinant (resp. rank, nullity, trace) of any matrix representing T.

We will see many examples in the following section.

6.2 Eigenvalues and Eigenvectors

We saw in the previous section that

- If V is a finite-dimensional vector space, $T \in Hom(V)$, and B, B' bases for V, then $[T]_{B'}$ and $[T]_B$ are similar matrices:

$$[T]_{B'} = P_{B' \to B}^{-1} [T]_B P_{B' \to B}$$

- Similar matrices all have the same determinant, trace, rank, and nullity, which allowed us to define these quantities for $T \in Hom(V)$ as the corresponding quantity for $[T]_B$ in any basis.

Given $T \in Hom(V)$, we therefore seek a basis in which $[T]_B$ takes a form which is as simple as possible. If we can find a basis in which $[T]_B$ is diagonal, then the determinant, trace, rank, and nullity can all be found immediately by inspection.

Example 6.8. Suppose a linear operator T can be represented in some basis by the matrix

$$[T]_B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see immediately that

(a) $\det T = 0$;

(b) tr $T = 1$;

(c) rank $T = 2$;

(d) nullity $T = 1$;

(e) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a basis for Col T and Row T.

(f) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul T.

The previous example showed that if we can find a basis in which T is represented by a diagonal matrix, we can readily answer any question about the properties of T. This leads to the following questions:

(1) Can each linear operator $T \in Hom(V)$ be represented by a diagonal matrix in some basis?

(2) If not, for which operators T does such a basis exist?

(3) How can be find such a basis if there is one?

(4) If no such basis exists, what is the simplest type of matrix by which we can represent T?

We begin by noting the following.

Theorem 6.9 (Diagonality condition). Let V be a finite-dimensional vector space, $B = \{\mathbf{v}_i\}_{i=1}^n$ a basis for V, and $T \in Hom(V)$. Then

$$[T]_B = \begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & c_n \end{bmatrix}$$

is diagonal if and only if $T(\mathbf{v}_i) = c_i \mathbf{v}_i$ for all $i = 1, \cdots, n$.

Proof. Recall that the ith column of $[T]_B$ is given by $[T(\mathbf{v}_i)]_B$. ($\Longrightarrow$) If $[T]_B$ is diagonal, then $[T(\mathbf{v}_i)]_B = (0, \ldots, 0, c_i, 0, \ldots, 0)$. Then $L_B([T(\mathbf{v}_i)]_B) = T(\mathbf{v}_i) = L_B(0, \ldots, 0, c_i, 0, \ldots, 0) = c_i \mathbf{v}_i$. ($\Longleftarrow$) If $T(\mathbf{v}_i) = c_i \mathbf{v}_i$, then the ith column of $T(\mathbf{v})$ is $[T(\mathbf{v}_i)]_B = L_B^{-1}(c_i \mathbf{v}_i) = (0, \ldots, 0, c_i, 0, \ldots, 0)$, hence $[T]_B$ is diagonal. ☐

Definition 6.10 (Eigenvalue of a linear operator). Let V be a vector space and $T \in Hom(V)$. An *eigenvalue* of T is a scalar λ such that there is a non-zero vector $\mathbf{v} \in V$ such that

$$T(\mathbf{v}) = \lambda \mathbf{v}.$$

The vector $\mathbf{v}$ is called an *eigenvector* of T associated with the eigenvalue λ.

Note that, if $\mathbf{v}$ is an eigenvector of T associated with eigenvalue λ and if B is a basis for V, then

$$[T(\mathbf{v})]_B = [T]_B[\mathbf{v}]_B = [\lambda \mathbf{v}]_B = \lambda [\mathbf{v}]_B.$$

Definition 6.11 (Eigenvalue of an $n \times n$ matrix). An eigenvalue of an $n \times n$ matrix A is a scalar λ such that there exists a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \lambda \mathbf{x}$$

The vector $\mathbf{x}$ is called an *eigenvector* of A associated with the eigenvalue λ.

Exercise 6.1. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A? If so, find the corresponding eigenvalues.

Solution. $A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$, so $\mathbf{u}$ is an eigenvector of A with eigenvalue -4. $A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$ is not a scalar multiple of $\mathbf{v}$, so $\mathbf{v}$ is not an eigenvalue of A. □

Theorem 6.12 (Similar matrices have the same eigenvalues). Similar matrices have the same eigenvalues.

Proof. Suppose $B = P^{-1}AP$ for some invertible matrix P. Suppose $\mathbf{v}$ is an eigenvector of B with eigenvalue λ; that is, $\mathbf{v} \neq \mathbf{0}$ such that $B\mathbf{v} = \lambda\mathbf{v}$. Then

$$\begin{aligned} B\mathbf{v} &= \lambda\mathbf{v} \\ P^{-1}AP\mathbf{v} &= \lambda\mathbf{v} \\ PP^{-1}AP\mathbf{v} &= P\lambda\mathbf{v} \\ AP\mathbf{v} &= \lambda P\mathbf{v}, \end{aligned}$$

which shows that $P\mathbf{v}$ is an eigenvector of A if eigenvalue λ. Thus, every eigenvalue of B is an eigenvalue of A. Since $A = PBP^{-1}$, the exact same argument shows that every eigenvalue of A is also an eigenvalue of B. Thus, A and B have exactly the same eigenvalues. □

Corollary 6.13. Let $T \in \mathrm{Hom}(V)$. Then any matrix representing T has the same eigenvalues as T.

We now discuss how to systematically find eigenvalues of a given matrix. A scalar λ is an eigenvalue of A if and only if $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution. This is equivalent to the homogeneous linear system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}, \tag{6.4}$$

where I is the $n \times n$ identity matrix.

Definition 6.14 (Eigenspace). The set of all solutions of Equation (6.4) is called the *eigenspace* of A associated to the eigenvalue λ, which we will denote by E_λ.

Proposition 6.15. Suppose $A \in M^{n\times n}(\mathbb{R})$ has an eigenvalue λ. Then the eigenspace of A corresponding to λ is a subspace of $\mathbb{R}^n$.

Proof. By definition, $E_\lambda = \mathrm{Nul}\,(A - \lambda I)$, which is a subspace of $\mathbb{R}^n$. Note that E_λ consists of the zero vector together with all the eigenvectors corresponding to λ. □

Note that the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if $\det(A - \lambda I) = 0$, or, equivalently, if $\det(\lambda I - A) = 0$.

Theorem 6.16. Let A be an $n \times n$ matrix. Then $\det(\lambda I - A)$ is a monic (leading coefficient is 1), degree n polynomial.

Proof. (By induction on n). Let A be a 2×2 matrix. Then

$$\begin{aligned} \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} &= (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

is a monic, degree 2 polynomial. Suppose the assertion is true for $n = k$, and let A be a $(k+1) \times (k+1)$ matrix. Then

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & a_{12} & \dots & a_{1,k+1} \\ a_{21} & \lambda - a_{22} & \dots & a_{a,k+1} \\ \vdots & & & \\ a_{k+1,1} & a_{k+2,2} & \dots & \lambda - a_{k+1,k+1} \end{vmatrix}.$$

Evaluating the determinant by cofactor expansion along the first row, we obtain

$$\det(\lambda I - A) = (\lambda - 1_{11})C_{11} + a_{12}C_{12} + \dots a_{k+1}C_{1,k+1}.$$

Since $C_{1j} = (-1)^{1+j} \det A_{1j}$ with A_{1j} $k \times k$, by the inductive hypothesis $C_{1j} = (-1)^{1+j}(\lambda^n + \dots)$, where the $\dots$ denote lower-degree terms in λ. Thus,

$$\det(\lambda I - A) = (\lambda - a_{11})(\lambda^k + \dots) + a_{12}(-\lambda^k + \dots) + \dots a_{k+1}(-1)^{k+1}(\lambda^k + \dots) = \lambda^{k+1} + \dots,$$

which proves the assertion holds for all $n \in \mathbb{N}$. $\square$

Definition 6.17 (Characteristic equation). Given any $n \times n$ matrix A, the monic, degree n polynomial $\det(\lambda I - A)$ is called the *characteristic polynomial* of A. The equation

$$\det(\lambda I - A) = 0$$

is called the *characteristic equation* of A.

Example 6.18. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then the characteristic equation for A is

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1,$$

which has solution $\pm i$.

Theorem 6.19. If V is a complex vector space, then every linear operator has an eigenvalue (and thus an eigenvector).

Proof. By the Fundamental Theorem of Algebra, the polynomial equation $\det(\lambda I - A) = 0$ has a root in $\mathbb{C}$ for any matrix A representing T. $\square$

We see from Example 6.18 that this theorem is false if V is a real vector space.

Corollary 6.20. An $n \times n$ matrix A has at most n distinct eigenvalues.

Proof. By the Fundamental Theorem of Algebra, every non-constant polynomial has n roots in $\mathbb{C}$ (counted with multiplicity). $\square$

The next example shows how to find the eigenvalues and corresponding eigenspaces of a given $n \times n$ matrix.

Example 6.21. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. The characteristic equation of A is given by

$$\begin{aligned} 0 = \begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} &= (1-\lambda)(2-\lambda) - 30 \\ &= 2 - 3\lambda + \lambda^2 - 30 \\ &= \lambda^2 - 3\lambda - 28 \\ &= (\lambda - 7)(\lambda + 4) \end{aligned}$$

so A has two eigenvalues $\lambda = 7, -4$. To find a parametric description of W_7, we row reduce the matrix

$$A - 7I = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

giving

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

which shows that $E_7 = \text{Span}\{(1,1)\}$. Geometrically, this is the line $y = x$ in the plane.

Exercise 6.2. Work out E_{-4} from the previous example.

Proposition 6.22. Similar matrices have the same characteristic equation.

Proof. Suppose $B = P^{-1}AP$ for some invertible matrix P. Then

$$\begin{aligned} \det(\lambda I - B) &= \det(\lambda P^{-1} I P - P^{-1}AP) \\ &= \det(P^{-1}(\lambda I - A)P) \\ &= \det(P^{-1}) \det(\lambda I - A) \\ &= \det(P^{-1}) \det(P) \det(\lambda I - A) \\ &= \frac{1}{\det P} \det P \det(\lambda I - A) \\ &= P \det(\lambda I - A). \end{aligned}$$

□

WARNING! We have seen that similar matrices have the same eigenvalues. However, the converse is not true: two matrices which have the same eigenvalues are not necessarily similar, as the next example shows.

Example 6.23. Consider the matrices $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. We note that $B = 2I$. If C is any matrix similar to B, then there exists an invertible matrix P such that $C = P^{-1}BP$. But then $C = P^{-1}BP = P^{-1}2IP = 2P^{-1}IP = 2I = B$. Thus, the only matrix similar to B is B itself. In particular, A is not similar to B. However, A and B have the same characteristic polynomials and therefore the same eigenvalues.

Computing eigenvalues of larger matrices require solving polynomial equations of degree n. If a matrix A is triangluar, then this is easy.

Proposition 6.24. The eigenvalues of a triangular matrix are the entries on the main diagonal.

Proof. Let

$$A = \begin{bmatrix} a_{11} & * & * & \dots & * \\ 0 & a_{22} & * & \dots & * \\ 0 & 0 & \ddots & \dots & * \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Then the characteristic equation

$$0 = \det(\lambda I - A) = \begin{vmatrix} a_{11} & * & * & \dots & * \\ 0 & a_{22} & * & \dots & * \\ 0 & 0 & \ddots & \dots & * \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix} = \prod_{i=1}^{n}(\lambda - a_{ii})$$

has solutions $a_{11}, a_{22}, \dots, a_{nn}$. □

Example 6.25. The distinct eigenvalues of

$$A = \begin{bmatrix} 5 & -2 & 6 & 1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are 1,3,5.

Exercise 6.3. Show that the eigenspaces associated to the eigenvalues in the previous example are

$$W_1 = \text{Span}\left\{\begin{bmatrix} -1/4 \\ -4 \\ -1 \\ 1 \end{bmatrix}\right\}, \; W_3 = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}, \; W_5 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Unfortunately, solving for the roots of an arbitrary polynomial is a hard problem, so in general eigenvalues are difficult to compute. However, computers can approximate the eigenvalues of a given matrix to very good precision.

Finding eigenvalues of 3×3 matrices or larger (which may not be triangular) with *integer* entries can sometimes be accomplished with the help of the following theorem:

Theorem 6.26 (Rational root theorem). Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a degree n polynomial with integer coefficients. If $x = \frac{p}{q}$ is a *rational* root of $p(x)$, then $p \mid a_0$ and $q \mid a_n$. [54]

[54] We write $a \mid b$ if a divides b; that is, if there exists a positive integer k such that $b = ka$.

Proof. Assume $x = \frac{p}{q}$ is a root of $p(x) = 0$. Assume also that $\frac{p}{q}$ is in lowest terms, so that p and q have no common factors. Then

$$a_n \frac{p^n}{q^n} + a_{n-1} \frac{p^{n-1}}{q^{n-1}} + \cdots + a_1 \frac{p}{q} + a_0 = 0.$$

Multiplying both sides by q^n, we obtain

$$a_n p^n + a_{n-1} p^{n-1} q + a_{n-2} q^{n-2} q^2 + \cdots + a_1 p q^{n-1} + a_0 q^n = 0. \tag{6.5}$$

and therefore

$$a_n p^n + a_{n-1} p^{n-1} q + a_{n-2} q^{n-2} q^2 + \cdots + a_1 p q^{n-1} = -a_0 q^n.$$

The left hand side is divisible by p (since each term has at least one factor of p), so the right hand side must be as well. Since p does not divide q, it also does not divide q^n, and therefore p must divide a_0.

We can also rewrite Equation (6.5) as

$$a_{n-1} p^{n-1} q + a_{n-2} q^{n-2} q^2 + \cdots + a_1 p q^{n-1} + a_0 q^n = -a_n p^n + .$$

Since the left-hand side is divisible by q, the right-hand side must be as well. Since q does not divide p, it also does not divide p^n and therefore q must divide a_n, completing the proof. □

Note that Theorem 6.26 does *not* say that every polynomial with integer coefficients has a rational root. It only says that *if* such a polynomial has a rational root, then it must be of the form in the theorem.

Example 6.27. Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$. The characteristic equation of A is given by

$$0 = \det(\lambda I - A) = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 2 & 2-\lambda & -1 \\ 2 & 2 & -\lambda \end{vmatrix} = \lambda^3 - 5\lambda^2 + 8\lambda - 4.$$

By Theorem 6.26, the possible rational roots are $\pm 1, \pm 2$, and ± 4. We can test these by synthetic division to check if any is a root of the characteristic polynomial. Doing this shows that 1 is a root, so we have

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda^2 - 4\lambda + 4) = (\lambda - 1)(\lambda - 2)^2 = 0$$

and therefore the distinct eigenvalues of A are $\lambda = 1, 2$.

Exercise 6.4. Show that the eigenspaces corresponding to the eigenvalues found in the previous example are

$$W_1 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}, \ W_2 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

6.3 Diagonalizable Operators

We proved in Theorem 6.9 that if V admits a basis B in which every vector in B is an eigenvector of T, then $[T]_B$ is diagonal.

Definition 6.28. A linear operator $T \in Hom(V)$ is said to be *diagonalizable* if there exists a basis consisting of eigenvectors of T. Such a basis is called an *eigenbasis*.

We saw in the previous section how to find the eigenvalues and eigenvectors of any linear operator T. We will now see when such a basis for V exists and how to find it.

Example 6.29. (a) The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda = \pm i$. The corresponding eigenvectors are $\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}$. These are linearly independent, so they are a basis for $\mathbb{C}^2$. Thus A is diagonalizable, in the ordered basis $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$ is given by $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. If we view A as an operator on $\mathbb{R}^2$, then it is not diagonalizable, since it has no real eigenvalues.

(b) The matrix $B = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ has eigenvalues $1, 2$. The eigenspace corresponding to each of these eigenvalues is 1 dimensional:

$$E_1 = \operatorname{Span}\{(1,0,2)\}, \quad W_2 = \operatorname{Span}\{(1,1,2)\}$$

While we see that any vector in E_1 is not a scalar multiple of any vector in E_2, it is not possible to find 3 linearly independent eigenvectors of B, so V does not have a basis consisting of eigenvectors of B. Hence, B is not diagonalizable.

Part (b) of the previous example showed that, even for a complex vector space, not every linear operator is diagonalizable. We now give sufficient conditions for a matrix to be diagonalizable. We will need the following lemma.

Lemma 6.30. Let V be an n-dimensional vector space. Then any linearly independent set of n vectors is a basis for V. Thus, an $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvalues.

Proof. Let S be a linearly independent set of n vectors. By the $+/-$ lemma (Lemma 5.85), S can be extended to a basis for V. Since V is n-dimensional, by Theorem 5.88 S is a basis for V. If S consists of eigenvectors of A, then it is an eigenbasis for V. □

Theorem 6.31. Eigenvectors corresponding to different eigenvalues are linearly independent.

Proof. (By contradiction.) Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of A, and $\{\mathbf{v}_i\}_{i=1}^m$ an ordered set of corresponding eigenvectors (where $\mathbf{v}_i$ corresponds to λ_i for each $i = 1, \ldots, m$). Assume $\{\mathbf{v}_i\}_{i=1}^m$ is linearly dependent. Since $\mathbf{v}_1 \neq \mathbf{0}$ by definition, $\{\mathbf{v}_1\}$ is linearly independent. Let r be the largest integer $1 \leq r < m$ such that $\{\mathbf{v}_i\}_{i=1}^r$ is linearly independent (after reindexing the vectors, if necessary). Then $\{\mathbf{v}_i\}_{i=1}^{r+1}$ is linearly dependent (by definition of r), so there exist scalars $c_1, \ldots, c_{r+1}$ not all zero such that

$$c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r + c_{r+1}\mathbf{v}_{r+1} = \mathbf{0}. \tag{6.6}$$

Multiplying both sides of Equation (6.6) by the matrix A gives

$$c_1\lambda_1\mathbf{v}_1 + \cdots + \cdots + c_r\lambda_r\mathbf{v}_r + c_{r+1}\lambda_{r+1}\mathbf{v}_{r+1} = \mathbf{0}. \tag{6.7}$$

Multiplying Equation (6.6) by λ_{r+1} and subtracting Equation (6.6) from Equation (6.7) then gives

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{v}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{v}_r = \mathbf{0}.$$

Since the set $\{\mathbf{v}_i\}_{i=1}^r$ is linearly independent, we must have $c_i(\lambda_i - \lambda_{r+1}) = 0$ for all $i = 1, \ldots, r$. Since the λ_i are all distinct, this implies $c_i = 0$ for all $i = 1, \ldots, r$. Equation (6.6) then becomes $c_{r+1}\mathbf{v}_{r+1} = \mathbf{0}$ which implies $c_{r+1} = 0$ (since $\mathbf{v}_{r+1} \neq \mathbf{0}$), which is a contradiction since the set $c_1, \ldots, c_{r+1}$ are not all zero. Thus, $\{\mathbf{v}_i\}_{i=1}^m$ is linearly independent. □

Corollary 6.32 (An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.). An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

What is an $n \times n$ matrix A does not have n distinct eigenvalues? Let $\lambda_1, \ldots, \lambda_m$, denote the distinct eigenvalues of A, where $m < n$.

Definition 6.33. (a) The multiplicity of λ_k as a root of the characteristic polynomial of A is called the *algebraic multiplicity* of λ_k.

(b) The dimension of the eigenspace corresponding to λ_k is called the *geometric multiplicity* of λ_k.

Lemma 6.34. For each $1 \leq k \leq m$, the geometric multiplicity of λ_k is less than or equal to the algebraic multiplicity of λ_k.

Proof. Let $d_k = \dim E_{\lambda_k}$. Choose d_k linearly independent eigenvectors with eigenvalue λ_k. If we choose a basis for V where the first d_k vectors are these eigenvectors, it follows from Theorem 6.9 that A will take the form

$$A = \begin{bmatrix} \lambda_k I_k & B \\ 0 & C \end{bmatrix}$$

where I_k is the $d_k \times d_k$ identity matrix. The characteristic polynomial of A, being a similarity invariant, can therefore be factored as

$$(\lambda - \lambda_k)^{d_k} \det(\lambda I - C) \tag{6.8}$$

so the algebraic multiplicity of λ_k is at least d_k. □

Exercise 6.5. Show that the characteristic polynomial of the matrix

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & a & b \\ 0 & \lambda_1 & 0 & 0 & 0 & c & d \\ 0 & 0 & \lambda_2 & 0 & 0 & e & f \\ 0 & 0 & 0 & \lambda_2 & 0 & g & h \\ 0 & 0 & 0 & 0 & \lambda_2 & i & j \\ 0 & 0 & 0 & 0 & 0 & k & l \\ 0 & 0 & 0 & 0 & 0 & m & n \end{pmatrix}$$

takes the form in Equation (6.8).

Theorem 6.35 (Diagonalization theorem). Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_m$. Let $d_k = \dim E_{\lambda_k}$. Then the following are equivalent.

(a) A is diagonalizable.

(b) The characteristic polynomial for A is

$$f(\lambda) = (\lambda - \lambda_1)^{d_1} \cdots (\lambda - \lambda_m)^{d_m}$$

and $\dim E_{\lambda_i} = d_i$ for all $i = 1, \ldots, m$.

(c) $\sum_{i=1}^{m} \dim E_{\lambda_i} = \dim V$.

Proof. If A is diagonalizable, then there is a basis for V in which A takes the form

$$A = \begin{bmatrix} \lambda_1 I_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 I_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_m I_m \end{bmatrix},$$

where I_k is the $d_k \times d_k$ identity matrix. The characteristic polynomial of A is then $\prod_{i=1}^{m} (\lambda - \lambda_i)^{d_i}$. Thus, (a) $\implies$ (b). Now suppose (b) holds. Then $\sum_{i=1}^{m} d_i = \deg(f) = \dim V$. Since $d_i = \dim W_{\lambda_i}$ for all $i = 1, \ldots, m$, we arrive at (c). Finally, suppose (c) holds. Let B_i be a basis for E_{λ_i}. Since eigenvectors corresponding to different eigenvalues are linearly independent, $\cup_{i=1}^{m} B_i$ is a linearly independent set, and hence a basis for V. Therefore (c) $\implies$ (a). ☐

Theorem 6.36. Suppose A is diagonalizable with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Let B_i be a basis for the ith eigenspace E_{λ_i}. Then

$$B = \cup_{i=1}^{m} B_i$$

is a basis for V.

Proof. By Theorem 6.35, if A is diagonalizable, then $\sum_{i=1}^{m} \dim E_{\lambda_i} = \dim V$. Since eigenvectors corresponding to distinct eigenvalues are linearly independent, $\cup_{i=1}^{m} B_i$ consists of $\dim V$ linearly independent vectors and is therefore a basis for V. ☐

Example 6.37. Let

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

The characteristic equation of A is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2 = 0$$

and therefore the distinct eigenvalues of A are $\lambda = 1, 2$. We find

$$E_1 = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}, \; W_2 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Since $\dim E_1 + \dim E_2 = 1 + 2 = 3 = \dim V$, by Theorem 6.35 A is diagonalizable. Choosing the basis

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Exercise 6.6. Let

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$$

Diagonalize A, if possible.

Solution. The eigenvalues of A are $2, 1$. The corresponding eigenspaces are

$$W_2 = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}, \ E_1 = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} \right\}.$$

Since $\dim e_2 + \dim E_1 = \dim V$, by Theorem 6.35, A is diagonalizable. By choosing the eigenbasis

$$B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} \right\}$$

we have

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

Theorem 6.38 (Sum of eigenspaces is a direct sum). Let V be a finite-dimensional vector space and $T \in \operatorname{Hom}(V)$. If $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of T, then

$$E_{\lambda_1} + \cdots + E_{\lambda_m}$$

is a direct sum. Furthermore,

$$\dim E_{\lambda_1} + \cdots + \dim E_{\lambda_m} \leq \dim V,$$

where equality holds if and only if T is diagonalizable, in which case

$$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}.$$

Proof. To show that $E_{\lambda_1} + \cdots + E_{\lambda_m}$ is a direct sum, suppose

$$u_1 + \cdots + u_m = 0$$

where each $u_j \in E_j$. Because eigenvalues corresponding to distinct eigenvalues are linearly independent (by Theorem 6.31), this implies each $u_j = 0$ (otherwise this contradicts the fact that $\{u_j\}$ is linearly independent, since the coefficient of each vector is equal to 1). By Theorem 5.28, $E_{\lambda_1} + \cdots + E_{\lambda_m}$ is a direct sum. By Corollary 5.102,

$$\dim(E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}) = \dim E_{\lambda_1} + \cdots + \dim E_{\lambda_m}.$$

Since $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ is a subspace of V, by

$$\dim(E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}) = \dim E_{\lambda_1} + \cdots + \dim E_{\lambda_m} \leq V$$

by part (b) of Theorem 5.97. If T is diagonalizable, then equality holds by part (c) of Theorem 6.35, in which case $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ by part (c) of Theorem 5.97. □

6.4 Orthonormal Bases

Recall that two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are said to be *orthogonal* if their dot product is zero:

$$\mathbf{x} \cdot \mathbf{y} = 0.$$

Recall that, for nonzero vectors, the dot product is given by

$$\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}||\,||\mathbf{y}|| \cos\theta$$

two nonzero vectors are orthogonal if and only if they are perpendicular.

Definition 6.39.

(a) Let $S = \{\mathbf{v}_1, \cdots \mathbf{v}_k\}$ be a collection of vectors in $\mathbb{R}^n$. Then S is said to be an *orthogonal set* if all the vectors in S are mutually orthogonal, that is, if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ when $i \neq j$.

(b) If, in addition, we have $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for all $i = 1, \cdots, k$, we say that S is an *orthonormal set*.

Example 6.40. Let

$$\mathbf{u}_1 = (0,1,0),\ \mathbf{u}_2 = (1,0,1),\ \mathbf{u}_3 = (1,0,-1).$$

The set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$.

Note that, from any orthogonal set S of nonzero vectors in $\mathbb{R}^3$, we can obtain an orthonormal set by normalizing each vector in S.

Example 6.41. The lengths of the vectors in the previous example are given by

$$||\mathbf{u}_1|| = 1,\ ||\mathbf{u}_2|| = \sqrt{2},\ ||\mathbf{u}_3|| = \sqrt{2}.$$

Normalizing these vectors yields

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{||\mathbf{u}_1||} = (0,1,0),\ \mathbf{v}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}),\ \mathbf{v}_3 = \frac{\mathbf{u}_3}{||\mathbf{u}_3||} = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}).$$

This set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthonormal since $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$ and $||\mathbf{v}_1|| = ||\mathbf{v}_2|| = ||\mathbf{v}_3|| = 1$.

Note that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ from the example above is a basis for $\mathbb{R}^3$.

Definition 6.42. Let B be a basis for $\mathbb{R}^n$. If B is also orthogonal, it is said to be an *orthogonal basis*. If B is orthonormal, it is said to be an *orthonormal basis*.

Example 6.43. The standard basis for $\mathbb{R}^n$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

is an orthonormal basis.

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal basis for $\mathbb{R}^3$ and let $\mathbf{x} \in \mathbb{R}^3$. Since B is a basis for $\mathbb{R}^3$, there exist scalars c_1, c_2, c_3 such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

As we have seen before, to find the coefficient vector $\mathbf{c} \equiv [\mathbf{x}]_B$, we need to find the inverse of the matrix whose ith column is $\mathbf{v}_i$. Then $c_i = (L_B^{-1}(\mathbf{x}))_i$.

Example 6.44. Let B be the orthonormal basis from Example 6.41 and let $\mathbf{x} = (2, 1, 3)$. Find $[\mathbf{x}]_B$.

$$\begin{aligned} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} &= c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + c_3 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} [\mathbf{x}]_B \equiv \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 5/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \end{aligned}$$

We will now see that the fact that the basis is orthonormal allows us to find $[\mathbf{x}]_B$ much more easily. Again, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$ and write $\mathbf{x} \in \mathbb{R}^3$ as

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

Taking the dot product of both sides with $\mathbf{v}_1$, we find

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{x} &= c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + c_2\mathbf{v}_1 \cdot \mathbf{v}_2 + c_3\mathbf{v}_1 \cdot \mathbf{v}_3 \\ &= c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 \\ &= c_1. \end{aligned}$$

Similarly, we find $\mathbf{v}_2 \cdot \mathbf{x} = c_2$ and $\mathbf{v}_3 \cdot \mathbf{x} = c_3$. Thus, by choosing an orthonormal basis, we find an explicit formula for the coefficients c_i in terms of $\mathbf{x}$ and the basis vectors $\mathbf{v}_i$:

$$\mathbf{x} = (\mathbf{v}_1 \cdot \mathbf{x})\mathbf{v}_1 + (\mathbf{v}_2 \cdot \mathbf{x})\mathbf{v}_2 + (\mathbf{v}_3 \cdot \mathbf{x})\mathbf{v}_3 \tag{6.9}$$

Exercise 6.7. Use Eq. (7.10) to find $[\mathbf{x}]_B$ for the vector in Example 7.32.

Solution. If $\mathbf{x} = (2, 1, 3)$, then

$$[\mathbf{x}]_B = \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{x} \\ \mathbf{v}_2 \cdot \mathbf{x} \\ \mathbf{v}_3 \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 \\ 5/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

□

The previous example shows that it is much more convenient to work with orthonormal bases. We will now show that one can always take a given basis for a subspace of $\mathbb{R}^n$ and construct an orthonormal basis for the same subspace. We will use the following two results.

Lemma 6.45 (An orthogonal set of nonzero vectors is linearly independent.). An orthogonal set of nonzero vectors is linearly independent.

Proof. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal set and suppose that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = 0. \tag{6.10}$$

If we take the dot product of both sides of Equation (6.10) with $\mathbf{v}_1$, since S is an orthogonal set we obtain

$$\begin{aligned} c_1 \underbrace{\mathbf{v}_1 \cdot \mathbf{v}_1}_{=||\mathbf{v}_1||^2} + c_2 \underbrace{\mathbf{v}_1 \cdot \mathbf{v}_2}_{=0} + \cdots + c_n \underbrace{\mathbf{v}_1 \cdot \mathbf{v}_n}_{=0} &= 0 \\ c_1||\mathbf{v}_1||^2 &= 0. \end{aligned}$$

Since $\mathbf{v}_1 \neq 0$, we must have $c_1 = 0$. Dotting Equation (6.10) with $\mathbf{v}_2, \ldots, \mathbf{v}_n$, we similarly find that $c_2 = \cdots = c_n = 0$. Thus, S is linearly independent. □

Note that the converse to Lemma 6.45 is *false*. For instance, consider the set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Since

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

S is a linearly independent subset of $\mathbb{R}^2$, but since

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \neq 0,$$

S is not an orthogonal set.

Lemma 6.46. Any subset of a linearly independent set is linearly independent.

Proof. Let $S = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a linearly independent set, and let S' be any subset. If S' is empty, then it is linearly independent (this is vacuously true, since there are no vectors in S' for which the zero linear combination could have a non-trivial solution). If S' is all of S, then it is linearly independent by assumption. Suppose now that S' is a proper subset of S, and list the elements (reordering the vectors, if necessary) as $S' = \{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$, where $1 \leq r < n$. Suppose that S' is linearly dependent. Then there exist scalars $c_1, \cdots, c_r$ not all zero such that

$$c_1\mathbf{u}_1 + \cdots + c_r\mathbf{u}_r = \mathbf{0}.$$

But then we have

$$c_1\mathbf{u}_1 + \cdots + c_r\mathbf{u}_r + 0 \cdot \mathbf{u}_{r+1} + \cdots + 0 \cdot \mathbf{u}_n = \mathbf{0},$$

which is a contradiction since S is linearly independent. Thus, S' must be linearly independent as well. $\square$

Theorem 6.47. Given a basis B for a subspace of $\mathbb{R}^m$, one can convert B into an orthonormal basis for the same subspace.

Proof. Let V be an n-dimensional subspace of $\mathbb{R}^m$, and let $B = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis for V. We will obtain an orthonormal basis $B' = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ by means of a construction known as the *Gram-Schmidt orthogonalization algorithm*, which we now describe. The first part of the algorithm successively builds an orthogonal basis. One then simply normalizes each vector to obtain an orthonormal basis.

First, let $\mathbf{v}_1 = \mathbf{u}_1$. Then $\mathbf{v}_1 \neq \mathbf{0}$, so the set $\{\mathbf{v}_1\}$ is linearly independent. Now define the other vectors inductively, as follows: Suppose $\mathbf{v}_1, \cdots, \mathbf{v}_m$ $(1 \leq m < n)$ have been chosen such that for every k in $1 \leq k \leq m$, the set

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$$

is an orthogonal basis for the subspace of V spanned by $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ (we have just seen that this is true for $m = 1$, since $\{\mathbf{u}_1\}$ is trivially an orthogonal basis for the subspace of V spanned by $\mathbf{u}_1$, so this establishes the base case). We will show how to construct the next vector, $\mathbf{v}_{m+1}$, so that the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ is an orthogonal basis for the subspace of $\mathbb{R}^n$ spanned by $\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$. This will complete the inductive proof that we can continue this process until we have an orthogonal basis for V.

To construct the next vector $\mathbf{v}_{m+1}$, let

$$\begin{aligned} \mathbf{v}_{m+1} &= \mathbf{u}_{m+1} - \sum_{k=1}^{m} \operatorname{proj}_{\mathbf{v}_k} \mathbf{u}_{m+1} \\ &= \mathbf{u}_{m+1} - \sum_{k=1}^{m} \frac{\mathbf{u}_{m+1} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \mathbf{v}_k. \end{aligned} \tag{6.11}$$

We need to show that $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ is an orthogonal set of nonzero vectors and that $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\} \subset \operatorname{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$. Since an orthogonal set is linearly independent by Lemma 6.45, $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ will then be an orthogonal basis for $\operatorname{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$, since a set of $m+1$ linearly independent vectors in an $(m+1)$-dimensional vector space is automatically a basis for that vector space.

First, we note that $\mathbf{v}_{m+1} \neq 0$, since otherwise we would have

$$\mathbf{u}_{m+1} = \sum_{k=1}^{m} \frac{\mathbf{u}_{m+1} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \mathbf{v}_k$$

which says $\mathbf{u}_{m+1}$ is a linear combination of $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ and hence a linear combination of $\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$, [55] which contradicts the fact that $\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$ is linearly independent. [56] Thus, $\mathbf{v}_{m+1} \neq 0$.

All that is left is to show that $\mathbf{v}_{m+1}$ is orthogonal to every vector in the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$. Let $1 \leq j \leq m$. Then

$$\begin{aligned}
\mathbf{v}_{m+1} \cdot \mathbf{v}_j &= \mathbf{u}_{m+1} - \sum_{k=1}^{m} \frac{\mathbf{u}_{m+1} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \mathbf{v}_k \cdot \mathbf{v}_j \\
&= \mathbf{u}_{m+1} \cdot \mathbf{v}_j - \sum_{k=1}^{m} \frac{\mathbf{u}_{m+1} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \mathbf{v}_k \cdot \mathbf{v}_j
\end{aligned}$$

Since $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ is an orthogonal set (by our inductive hypothesis), every term in the sum is zero except for $k = j$, so we find

$$\begin{aligned}
\mathbf{v}_{m+1} \cdot \mathbf{v}_j &= \mathbf{u}_{m+1} \cdot \mathbf{v}_j - \sum_{k=1}^{m} \frac{\mathbf{u}_{m+1} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \mathbf{v}_k \cdot \mathbf{v}_j \\
&= \mathbf{u}_{m+1} \cdot \mathbf{v}_j - \frac{\mathbf{u}_{m+1} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j} \mathbf{v}_j \cdot \mathbf{v}_j \\
&= \mathbf{u}_{m+1} \cdot \mathbf{v}_j - \mathbf{u}_{m+1} \cdot \mathbf{v}_j \\
&= 0.
\end{aligned}$$

Thus we have shown that $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ is an orthogonal set consisting of $m+1$ nonzero vectors.

All that remains is to show that this set is a basis for $\mathrm{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$. As mentioned above, it will suffice to show that $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\} \subset \mathrm{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$, since a linearly independent subset of an $(m+1)$-dimensional vector space consisting of $m+1$ vectors is automatically a basis. The set $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subset \mathrm{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_m\} \subset \mathrm{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$ by the inductive hypothesis. Eq. (7.11) then shows that $\mathbf{v}_{m+1} \in \mathrm{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$. Thus, we have shown that the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ is an orthogonal basis for $\mathrm{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$, completing the proof by induction that $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an orthogonal basis for V.

Finally, since none of these vectors are zero, we obtain an orthonormal basis for V by normalizing each vector as $\mathbf{v}_i \mapsto \frac{\mathbf{v}_i}{||\mathbf{v}_i||}$. □

Example 6.48. Let $\mathbf{u}_1 = (1,1,1)$, $\mathbf{u}_2 = (0,1,1)$, $\mathbf{u}_3 = (0,0,1)$ be a basis for $\mathbb{R}^3$. Taking the standard inner product on $\mathbb{R}^3$, we now follow the Gram-Schmidt algorithm to obtain an orthogonal basis

[55] Since, by assumption, $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ is a basis for $\mathrm{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$ and, in particular, $\mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_m\} = \mathrm{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$

[56] Note that we have used Lemma 7.33 here.

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 = (1,1,1) \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (0,1,1) - \frac{2}{3}(1,1,1) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \\
&= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{2}(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \\
&= (0, -\frac{1}{2}, \frac{1}{2})
\end{aligned}$$

Noting that

$$||\mathbf{v}_1|| = \sqrt{3},\ ||\mathbf{v}_2|| = \sqrt{\frac{2}{3}},\ ||\mathbf{v}_3|| = \frac{1}{\sqrt{2}}$$

normalizing each vector as $\mathbf{v}_i \mapsto \mathbf{q}_i \equiv \frac{\mathbf{v}_i}{||\mathbf{v}_i||}$ results in the orthonormal basis $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, where

$$\begin{aligned}
\mathbf{q}_1 &= (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \\
\mathbf{q}_2 &= (-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}), \\
\mathbf{q}_3 &= (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).
\end{aligned}$$

Exercise 6.8. Let $V = \mathbb{R}^3$ equipped with the standard inner product. Consider the basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = (3,0,4),\ \mathbf{u}_2 = (-1,0,7),\ \mathbf{u}_3 = (2,9,11).$$

Apply the Gram-Schmidt algorithm to construct an orthonormal basis $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$. If $\mathbf{x} = (1,2,3)$, find $[\mathbf{x}]_{B'}$.

Solution.

$$\begin{aligned}
\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} &= \left\{ \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -4/5 \\ 0 \\ 3/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \\
[\mathbf{x}]_{B'} &= \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{x} \\ \mathbf{v}_2 \cdot \mathbf{x} \\ \mathbf{v}_3 \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.
\end{aligned}$$

□

7 Inner Product Spaces

7.1 Basic Definitions

We defined a vector space V be abstracting properties of $\mathbb{R}^n$. We have so far made no mention of the dot product, which played a prominent role in geometry in $\mathbb{R}^n$. We now abstract the properties of the dot product to obtain an "inner product" on a vector space V.

The dot product is a mapping $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which takes two vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ to the real number

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

It is straightforward to verify that the dot product on $\mathbb{R}^n$ has the following properties for all $\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n$ and all $c \in \mathbb{R}$:

(i) $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$;

(ii) $\mathbf{x} \cdot (c\mathbf{y}_1 + \mathbf{y}_2) = c\mathbf{x} \cdot \mathbf{y}_1 + \mathbf{x} \cdot \mathbf{y}_2$;

(iii) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.

Recall also that the length of $\mathbf{x} \in \mathbb{R}^n$ is given by

$$||\mathbf{x}||^2 = \sum_{i=1}^{n} x_i^2 = \mathbf{x} \cdot \mathbf{x}.$$

We begin by defining an analogous product on $\mathbb{C}^n$. For $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define the length of $\mathbf{z}$ by

$$\begin{aligned} ||\mathbf{z}||^2 &= \sum_{i=1}^{n} |z_i|^2 \\ &= \sum_{i=1}^{n} z_i \overline{z}_i \end{aligned}$$

This suggests we define a "dot product" on $\mathbb{C}^n$ by

$$\mathbf{z} \cdot \mathbf{w} = \sum_{i=1}^{n} z_i \overline{w}_i. \tag{7.1}$$

Note that this agrees with the definition of the dot product on $\mathbb{R}^n$ when $\mathbf{z}, \mathbf{w} \in \mathbb{R}^n$.

Exercise 7.1. Show that the "dot product" on $\mathbb{C}^n$ defined above satisfies the following properties for all $\mathbf{z}, \mathbf{w}, \mathbf{u}, \in \mathbb{C}^n$ and all $c \in \mathbb{C}$:

(i) $\mathbf{z} \cdot \mathbf{z} \geq 0$ and $\mathbf{z} \cdot \mathbf{z} = 0$ if and only if $\mathbf{z} = \mathbf{0}$; [57]

(ii) $(c\mathbf{w} + \mathbf{u}) \cdot \mathbf{z} = c\mathbf{w} \cdot \mathbf{z} + \mathbf{u} \cdot \mathbf{z}$;

(iii) $\mathbf{z} \cdot \mathbf{w} = \overline{\mathbf{w} \cdot \mathbf{z}}$.

[57] Note that $\mathbf{z} \cdot \mathbf{z}$ is always real, so the inequality $\mathbf{z} \cdot \mathbf{z} \geq 0$ makes sense.

Solution.

(i) $\mathbf{z} \cdot \mathbf{z} = \sum_{i=1}^{n} |z_i|^2$ is a sum of nonnegative terms and is therefore nonnegative.

(ii)

$$\begin{aligned}
(c\mathbf{w} + \mathbf{u}) \cdot \mathbf{z} &= \sum_{i=1}^{n} (cw + u)_i z_i \\
&= \sum_{i=1}^{n} (cw_i + u_i) z_i \\
&= \sum_{i=1}^{n} cw_i z_i + \sum_{i=1}^{n} u_i z_i \\
&= c \sum_{i=1}^{n} w_i z_i + \sum_{i=1}^{n} u_i z_i \\
&= c\mathbf{w} \cdot \mathbf{z} + \mathbf{u} \cdot \mathbf{z}
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathbf{z} \cdot \mathbf{w} &= \sum_{i=1}^{n} z_i \overline{w}_i \\
&= \sum_{i=1}^{n} \overline{\overline{z}_i w_i} \\
&= \sum_{i=1}^{n} \overline{w_i \overline{z}_i} \\
&= \overline{\sum_{i=1}^{n} w_i \overline{z}_i} \\
&= \overline{\mathbf{w} \cdot \mathbf{z}}.
\end{aligned}$$

□

Definition 7.1 (Inner product). Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An *inner product* on V is a mapping $\langle , \rangle : V \times V \to \mathbb{F}$ with the following properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all $c \in \mathbb{F}$:

(i) (Positive definiteness) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0, = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

(ii) (Linearity in the first coordinate) $\langle c\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$;

(iii) (Conjugate symmetry) $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

Definition 7.2 (Inner Product Space). An *inner product space* is an ordered pair $(V, \langle , \rangle)$, where V is a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\langle , \rangle$ is an inner product on V. When $\langle , \rangle$ is understood, we just say "V is an inner product space." When $\mathbb{F} = \mathbb{R}$ we call V a *real inner product space* and when $\mathbb{F} = \mathbb{C}$ we call V a *complex inner product space*.

Proposition 7.3 (An inner product is conjugate-linear in the second argument). If $\langle , \rangle$ is an inner product on V, then for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{F}$,

$$\langle \mathbf{w}, c\mathbf{u} + \mathbf{v} \rangle = \overline{c}\langle \mathbf{w}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle.$$

Proof.

$$\begin{aligned} \langle \mathbf{w}, c\mathbf{u} + \mathbf{v} \rangle &= \overline{\langle c\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle} \text{ (by Property (iii))} \\ &= \overline{c\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle} \text{ (by Property (ii))} \\ &= \overline{c\langle \mathbf{u}, \mathbf{w} \rangle} + \overline{\langle \mathbf{v}, \mathbf{w} \rangle} \\ &= \bar{c}\overline{\langle \mathbf{u}, \mathbf{w} \rangle} + \overline{\langle \mathbf{v}, \mathbf{w} \rangle} \\ &= \bar{c}\langle \mathbf{w}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle \text{ (by Property (iii)).} \end{aligned}$$

□

Remark 7.4. (a) When $\mathbb{F} = \mathbb{R}$, Property (iii) becomes $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (symmetry). When $\mathbb{F} = \mathbb{C}$, the complex conjugation in Property (iii) is needed for consistency: if we omitted the complex conjugation from Property (iii), then for $\mathbf{v} \neq 0$ we would have

$$\langle i\mathbf{v}, i\mathbf{v} \rangle > 0$$

by Property (i). However, we would also have

$$\begin{aligned} \langle i\mathbf{v}, i\mathbf{v} \rangle &= i\langle \mathbf{v}, i\mathbf{v} \rangle \text{ (by Property (ii))} \\ &= i\langle i\mathbf{v}, \mathbf{v} \rangle \text{ (by Property (iii), omitting the complex conjugation)} \\ &= i^2\langle \mathbf{v}, \mathbf{v} \rangle \text{ (by Property (ii))} \\ &= -\langle \mathbf{v}, \mathbf{v} \rangle < 0, \end{aligned}$$

which is a contradiction.

(b) In the physics literature Property (ii) is usually taken to be linearity in the *second* coordinate, rather than the first. This choice is immaterial, however one should always check which convention is being used.

(c) When $\mathbb{F} = \mathbb{R}$, Proposition 7.3 is simply linearity in the second argument. A mapping $V \times V \to \mathbb{R}$ which is linear in each argument when the other is held fixed is called a *bilinear form* on V. When $\mathbb{F} = \mathbb{C}$, a mapping $V \times V \to \mathbb{C}$ which is linear in one argument and conjugate-liner in the second is called a *sesquilinear form* on V. [58] In this language a real inner product space is a real vector space together with a symmetric, positive-definite bilinear form, and a complex inner product space is a complex vector space together with a positive-definite, complex-symmetric sesquilinear form.

(d) There are various technical reasons why it is necessary to restrict the base field $\mathbb{F}$ to $\mathbb{R}$ and $\mathbb{C}$ in the definition of an inner product. This was one of the reasons for restricting $\mathbb{F}$ to $\mathbb{R}$ or $\mathbb{C}$ throughout the notes; as we will primarily be interested in inner product spaces, we would have had to make this restriction at this point anyway. [59]

Example 7.5. (1) We define the *standard inner product* on $\mathbb{R}^n$ to be the dot product:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$$

We denote this inner product space by $\mathbb{R}^n$ (with the standard inner product understood). This inner product space is often called *n-dimensional Euclidean space*.

[58] *Sesqui* means "one and a half times".

[59] For a discussion, see here.

(2) We define the *standard inner product* on $\mathbb{C}^n$ to be the "dot product" defined in Equation (7.1):

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{i=1}^{n} z_i \overline{w}_i.$$

We denote this inner product space by $\mathbb{C}^n$ (with the standard inner product understood). This inner product space is often called *n-dimensional unitary space*.

(3) The vector space $\mathscr{C}([a,b])$ of all continuous complex-valued functions on the closed interval $[a,b]$ is a complex inner product space under the inner product

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

Definition 7.6 (Orthogonal). Two vectors $\mathbf{v}, \mathbf{w}$ in an inner product space V are said to be *orthogonal* if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

Exercise 7.2. Let V be an inner product space. Then $\langle \mathbf{0}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{0} \rangle = 0$ for all $\mathbf{u} \in V$. That is, the zero vector is orthogonal to every vector. Property (i) in the definition of an inner product shows that the zero vector is the only vector orthogonal to itself.

Proof. For any $\mathbf{u} \in V$,

$$\begin{aligned}\langle \mathbf{0}, \mathbf{u} \rangle &= \langle 0\mathbf{u}, \mathbf{u} \rangle \\ &= 0\langle \mathbf{u}, \mathbf{u} \rangle \\ &= 0.\end{aligned}$$

It then follows by conjugate-symmetry that $\langle \mathbf{u}, \mathbf{0} \rangle = \overline{\langle \mathbf{0}, \mathbf{u} \rangle} = \overline{0} = 0$. □

We now show that the Pythagorean theorem holds in any inner product space.

Theorem 7.7 (Pythagorean theorem). Let V be an inner product space. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$

Proof.

$$\begin{aligned}||\mathbf{u} + \mathbf{v}||^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + 0 + 0 + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2.\end{aligned}$$

□

Exercise 7.3. Prove that the converse of the Pythagorean theorem holds in any real inner product space, but is in general false for complex inner product spaces.

Proof. If $||\mathbf{u}^2 + \mathbf{v}^2|| = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ then from the above we must have

$$\begin{aligned} 0 &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \overline{\langle \mathbf{u}, \mathbf{v} \rangle} \\ &= 2\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle. \end{aligned}$$

If V is real, then $\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$, so this implies $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. If V is complex, then the Pythagorean theorem holds as long as $\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle = 0$, so if the inner product of these two vectors has a nonzero imaginary part, then they are not orthogonal. Thus, the converse of the Pythagorean theorem is false for complex inner product spaces. □

As in $\mathbb{R}^n$, if we have two vectors $\mathbf{u}, \mathbf{v}$ in an inner product space V with $\mathbf{v} \neq \mathbf{0}$, then we can write $\mathbf{u}$ uniquely as the sum of a scalar multiple of $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

Proposition 7.8 (Orthogonal decomposition). Let V be an inner product space, and let $\mathbf{u}, \mathbf{v} \in V$ with $\mathbf{v} \neq \mathbf{0}$. Then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v} + (\mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v})$$

and $\langle \mathbf{v}, (\mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v}) \rangle = 0$.

Proof. The uniqueness of this decomposition is obvious, since the scalar $\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}$ is uniquely determined by $\mathbf{u}$ and $\mathbf{v}$. We only need to prove the last statement:

$$\begin{aligned} \langle \mathbf{v}, (\mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v}) \rangle &= \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, -\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v} \rangle \\ &= \langle \mathbf{v}, \mathbf{u} \rangle - \overline{\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}}\langle \mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{v}, \mathbf{u} \rangle - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{||\mathbf{v}||^2}\langle \mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{v}, \mathbf{u} \rangle - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{||\mathbf{v}||^2}||\mathbf{v}||^2 \\ &= \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle \\ &= 0. \end{aligned}$$

□

The following simple result is quite useful.

Lemma 7.9. Let V be an inner product space. If $\langle \mathbf{u}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{x} \rangle$ for all $\mathbf{x} \in V$, then $\mathbf{u} = \mathbf{v}$.

The next result points out one of the main differences between real and complex inner product spaces and will play a key role in later work.

Theorem 7.10. Let V be an inner product space and let $T \in \mathrm{Hom}(V)$.

(a) If $\langle T\mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{v}, \mathbf{w} \in V$, then $T = 0$ (the zero map).

(b) If V is a complex inner product space, then $\langle T\mathbf{v}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$ implies that $T = 0$, but this does not hold in general for real inner product spaces.

Proof. (a) Suppose $\langle T\mathbf{v}, \mathbf{w}\rangle = 0$ for all $\mathbf{v}, \mathbf{w} \in V$. Then

$$\begin{aligned} 0 &= \langle T\mathbf{v}, \mathbf{w}\rangle \\ \langle \mathbf{0}, \mathbf{w}\rangle &= \langle T\mathbf{v}, \mathbf{w}\rangle \text{ (by Exercise 7.2)} \end{aligned}$$

for all $\mathbf{w} \in V$ and therefore $T\mathbf{v} = \mathbf{0}$ by Lemma 7.9. Since $\mathbf{v}$ was an arbitrary vector in V, this shows that T is the zero map.

(b) Let $\mathbf{v} = c\mathbf{x} + \mathbf{y}$, for $\mathbf{x}, \mathbf{y} \in V$ and $c \in \mathbb{F}$. Then

$$\begin{aligned} 0 &= \langle T\mathbf{v}, \mathbf{v}\rangle \\ &= \langle T(c\mathbf{x} + \mathbf{y}), c\mathbf{x} + \mathbf{y}\rangle \\ &= \langle cT\mathbf{x} + T\mathbf{y}, c\mathbf{x} + \mathbf{y}\rangle \\ &= \langle cT\mathbf{x}, c\mathbf{x}\rangle + \langle cT\mathbf{x}, \mathbf{y}\rangle + \langle T\mathbf{y}, c\mathbf{x}\rangle + \langle T\mathbf{y}, \mathbf{y}\rangle \\ &= |c|^2\langle T\mathbf{x}, \mathbf{x}\rangle + c\langle T\mathbf{x}, \mathbf{y}\rangle + \bar{c}\langle T\mathbf{y}, \mathbf{x}\rangle + \langle T\mathbf{y}, \mathbf{y}\rangle \\ &= |c|^2 \cdot 0 + c\langle T\mathbf{x}, \mathbf{y}\rangle + \bar{c}\langle T\mathbf{y}, \mathbf{x}\rangle + 0 \\ &= c\langle T\mathbf{x}, \mathbf{y}\rangle + \bar{c}\langle T\mathbf{y}, \mathbf{x}\rangle. \end{aligned}$$

Assume now that $\mathbb{F} = \mathbb{C}$. Setting $c = 1$ gives

$$\langle T\mathbf{x}, \mathbf{y}\rangle + \langle T\mathbf{y}, \mathbf{x}\rangle = 0$$

and therefore

$$\langle T\mathbf{x}, \mathbf{y}\rangle = -\langle T\mathbf{y}, \mathbf{x}\rangle, \tag{7.2}$$

while setting $c = i$ implies

$$\langle T\mathbf{x}, \mathbf{y}\rangle - \langle T\mathbf{y}, \mathbf{x}\rangle = 0$$

and therefore

$$\langle T\mathbf{x}, \mathbf{y}\rangle = \langle T\mathbf{y}, \mathbf{x}\rangle. \tag{7.3}$$

Equations (7.2) and (7.3) together show that $\langle T\mathbf{x}, \mathbf{y}\rangle = 0$.

To see that this is false in a real inner product space, let $\mathbb{R}^2$ be the Euclidean plane and let $T \in \mathrm{Hom}(\mathbb{R}^2)$ be rotation by $90°$. Then T is not the zero map, but $\langle T\mathbf{v}, \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in \mathbb{R}^2$. □

7.2 Norm and Distance

As with the dot product on $\mathbb{R}^n$, we can use an inner product to define the *length* or *norm* of a vector.

Definition 7.11 (Norm). Let V be an inner product space. We define the *norm* of $\mathbf{v} \in V$ to be the real number

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}.$$

A vector $\mathbf{v}$ is a *unit vector* if $||\mathbf{v}|| = 1$.

Exercise 7.4. Prove that the norm has the following properties for all $\mathbf{v} \in V$ and $c \in \mathbb{F}$:

(a) $||\mathbf{v}|| \geq 0$ and $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

(b) $||c\mathbf{v}|| = |c| \, ||\mathbf{v}||$.

Proof.

(a) $||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and is equal to zero if and only if $\mathbf{v} = \mathbf{0}$. The proposition follows by taking the positive square root.

(b)

$$\begin{aligned} ||c\mathbf{v}|| &= \sqrt{\langle c\mathbf{v}, c\mathbf{v} \rangle} \\ &= \sqrt{c\bar{c}\langle \mathbf{v}, \mathbf{v} \rangle} \\ &= \sqrt{|c|^2 \langle \mathbf{v}, \mathbf{v} \rangle} \\ &= \sqrt{|c|^2}\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\ &= |c| \, ||\mathbf{v}||. \end{aligned}$$

□

The norm gives us the notion of the length of a vector. Using the inner product, we can also define the notion of the angle between two vectors, which we would like to define as

$$\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{||\mathbf{v}|| \, ||\mathbf{w}||}.$$

However, since $-1 \leq \cos\theta \leq 1$, this only makes sense if $-1 \leq \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{||\mathbf{v}|| \, ||\mathbf{w}||} \leq 1$ or, equivalently, if

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq ||\mathbf{v}|| \, ||\mathbf{w}||.$$

We now show that this is indeed the case.

Lemma 7.12 (Cauchy-Schwarz inequality). Let V be an inner product space. For all $\mathbf{v}, \mathbf{w} \in V$,

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq ||\mathbf{v}|| \, ||\mathbf{w}||$$

with equality if and only if one of $\mathbf{u}$ and $\mathbf{v}$ is a scalar multiple of the other.

Proof. If $\mathbf{v} = \mathbf{0}$, then $||\mathbf{v}|| = 0$ by Exercise 7.4 and $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ by Exercise 7.2, so equality holds. Suppose now that $\mathbf{v} \neq 0$. By Proposition 7.8 we can write

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v} + \mathbf{w},$$

where $\mathbf{w} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v}$ is orthogonal to $\mathbf{v}$. Since

$$\langle \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v}, \mathbf{w} \rangle = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\langle \mathbf{v}, \mathbf{w} \rangle = 0,$$

by the Pythagorean theorem (Theorem 7.7),

$$\begin{aligned}||\mathbf{u}||^2 &= \left|\left|\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{||\mathbf{v}||^2}\mathbf{v}\right|\right|^2 + ||\mathbf{w}||^2 \\ &= \frac{|\langle \mathbf{u}, \mathbf{v}\rangle|^2}{||\mathbf{v}||^4}||\mathbf{v}||^2 + ||\mathbf{w}||^2 \\ &= \frac{|\langle \mathbf{u}, \mathbf{v}\rangle|^2}{||\mathbf{v}||^2} + ||\mathbf{w}||^2 \\ &\geq \frac{|\langle \mathbf{u}, \mathbf{v}\rangle|^2}{||\mathbf{v}||^2},\end{aligned}$$

since $||\mathbf{w}||^2 \geq 0$ by part (a) of Exercise 7.4. Multiplying both sides by $||\mathbf{v}||^2$ gives

$$||\mathbf{u}||^2||\mathbf{v}||^2 \geq |\langle \mathbf{u}, \mathbf{v}\rangle|^2,$$

and since both $||\mathbf{u}||\ ||\mathbf{v}||$ and $|\langle \mathbf{u}, \mathbf{v}\rangle|$ are nonnegative, it follows that

$$||\mathbf{u}||\ ||\mathbf{v}|| \geq |\langle \mathbf{u}, \mathbf{v}\rangle|.$$

From the above, we see that equality holds if and only if $\mathbf{w} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{||\mathbf{v}||^2}\mathbf{v} = 0$ in which case $\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{||\mathbf{v}||^2}\mathbf{v}$ is a scalar multiple of $\mathbf{v}$. ☐

Example 7.13 (Examples of the Cauchy-Schwarz inequality).

(a) For $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$,

$$|\sum_{i=1}^{n} x_i y_i|^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

(b) If f, g are continuous real-valued functions on an interval $[a, b]$, then

$$\left|\int_a^b f(x)g(x)dx\right|^2 \leq \left(\int_a^b (f(x))^2 dx\right)\left(\int_a^b (g(x))^2 dx\right).$$

The next result, called the Triangle Inequality, has the geometric interpretation that the length of each side of a triangle is less than the sum of the lengths of the other two sides.

Theorem 7.14 (Triangle Inequalitya). Let V be an inner product space. Then for $\mathbf{u}, \mathbf{v} \in V$,

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||,$$

where equality holds if and only if either $\mathbf{u}$ or $\mathbf{v}$ is a nonnegative multiple of the other (i.e., a scalar multiple by a nonnegative *real* scalar).

Proof. Equality holds if either $\mathbf{u}$ or $\mathbf{v}$ is the zero vector. Assume now that both are nonzero. We have

$$\begin{aligned}
||\mathbf{u}+\mathbf{v}||^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\
&= \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle + \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle \\
&= \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle + \langle \mathbf{u},\mathbf{v}\rangle + \overline{\langle \mathbf{u},\mathbf{v}\rangle} \\
&= \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle + 2\mathrm{Re}\langle \mathbf{u},\mathbf{v}\rangle \\
&\le \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle + 2|\langle \mathbf{u},\mathbf{v}\rangle| \text{ (since Re} z \le |z| \text{ for all } z \in \mathbb{F}) \\
&\le \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle + 2||\mathbf{u}||\,||\mathbf{v}|| \text{ (by the Cauchy-Schwarz inequality)} \\
&= (||\mathbf{u}|| + ||\mathbf{v}||)^2.
\end{aligned}$$

Since both $||\mathbf{u}+\mathbf{v}||$ and $||\mathbf{u}|| + ||\mathbf{v}||$ are nonnegative, it follows that

$$||\mathbf{u}+\mathbf{v}|| \le ||\mathbf{u}|| + ||\mathbf{v}||.$$

If equality holds, then we must have

$$\mathrm{Re}\langle \mathbf{u},\mathbf{v}\rangle = |\langle \mathbf{u},\mathbf{v}\rangle| = ||\mathbf{u}||\,||\mathbf{v}||.$$

The first equality implies that $\langle \mathbf{u},\mathbf{v}\rangle$ is real and positive. We learned in the proof of the Cauchy-Schwarz inequality (Lemma 7.12) that the second equality implies $\mathbf{u} = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{||\mathbf{v}||^2}\mathbf{v}$ (up to relabeling $\mathbf{u}$ and $\mathbf{v}$), so $\mathbf{u}$ is a multiply of $\mathbf{v}$ by a positive scalar.

Conversely, if $\mathbf{u} = c\mathbf{v}$, where $c > 0$, then $\langle \mathbf{u},\mathbf{v}\rangle = \langle c\mathbf{v},\mathbf{v}\rangle = c\langle \mathbf{v},\mathbf{v}\rangle = c||\mathbf{v}||^2$, so $\mathrm{Re}\langle \mathbf{u},\mathbf{v}\rangle = |\langle \mathbf{u},\mathbf{v}\rangle| = ||\mathbf{u}||\,||\mathbf{v}||$, and therefore $||\mathbf{u}+\mathbf{v}|| = ||\mathbf{u}|| + ||\mathbf{v}||$. ☐

Theorem 7.15 (Reverse Triangle Inequality). Let V be an inner product space. Then for all $\mathbf{u},\mathbf{v} \in V$,

$$|\,||\mathbf{u}|| - ||\mathbf{v}||\,| \le ||\mathbf{u}-\mathbf{v}||.$$

Proof. We have

$$\begin{aligned}
||\mathbf{u}|| &= ||\mathbf{u}-\mathbf{v}+\mathbf{v}|| \\
&\le ||\mathbf{u}-\mathbf{v}|| + ||\mathbf{v}|| \text{ (by the Triangle Inequality)}
\end{aligned}$$

and therefore

$$||\mathbf{u}|| - ||\mathbf{v}|| \le ||\mathbf{u}-\mathbf{v}||. \tag{7.4}$$

Similarly,

$$\begin{aligned}
||\mathbf{v}|| &= ||\mathbf{v}-\mathbf{u}+\mathbf{u}|| \\
&\le ||\mathbf{v}-\mathbf{u}|| + ||\mathbf{u}|| \text{ (by the Triangle Inequality)} \\
&= ||\mathbf{u}-\mathbf{v}|| + ||\mathbf{u}||
\end{aligned}$$

and therefore

$$||\mathbf{v}|| - ||\mathbf{u}|| \le ||\mathbf{u}-\mathbf{v}||$$

or equivalently

$$-(||\mathbf{u}|| - ||\mathbf{v}||) \leq ||\mathbf{u} - \mathbf{v}||. \tag{7.5}$$

Equations (7.4) and (7.5) are equivalent to

$$|\,||\mathbf{u}|| - ||\mathbf{v}||\,| \leq ||\mathbf{u} - \mathbf{v}||.$$

□

The next result is called the parallelogram law because of its geometric interpretation: in every parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the four sides.

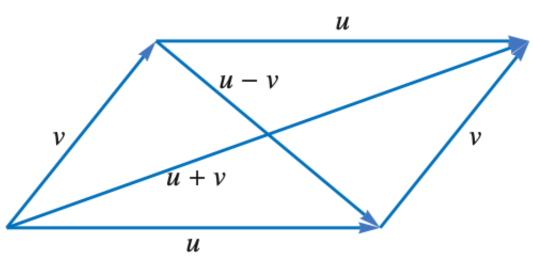


Figure 16: The parallelogram law.

Theorem 7.16 (Parallelogram Law). Let V be an inner product space and let $\mathbf{u}, \mathbf{v} \in V$. Then

$$||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2).$$

Exercise 7.5. Prove Theorem 7.16.

Proof. We have

$$\begin{aligned} ||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle + ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle \\ &= 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2). \end{aligned}$$

□

Remark 7.17. Any vector space V together with a function $||\ || : V \to \mathbb{R}$ which satisfies

(i) (Positive-definiteness) $||\mathbf{v}|| \geq 0$ and $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$;

(ii) (Absolute homogeneity) For all $\mathbf{v} \in V$ and $c \in \mathbb{F}$,

$$||c\mathbf{v}|| = |c|\ ||\mathbf{v}||;$$

(iii) (Triangle inequality) For all $\mathbf{u}, \mathbf{v} \in V$,

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||.$$

is called a *normed vector space* and the function $|| \, ||$ is called a *norm*. Thus, any inner product space is a normed vector space with norm $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

It is interesting to observe that the inner product on V can be recovered from the norm. Thus, knowing the length of all vectors in V is equivalent to knowing all inner products on V. In the complex case, we will need the following lemma.

Lemma 7.18. Let V be a complex inner product space. Then for all $\mathbf{u}, \mathbf{v} \in V$, we can write

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle + i\mathrm{Re}\langle \mathbf{u}, i\mathbf{v} \rangle. \tag{7.6}$$

This shows that the inner product is completely determined by its real part.

Exercise 7.6. Prove Lemma 7.18.

Proof. Since $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{C}$, we can write

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle + i\mathrm{Im}\langle \mathbf{u}, \mathbf{v} \rangle$$

For any complex number $z = \mathrm{Re} z + i\mathrm{Im} z$, we have

$$\begin{aligned} -iz &= -i(\mathrm{Re} z + i\mathrm{Im} z) \\ &= -i\mathrm{Re} z - i^2\mathrm{Im} z \\ &= -i\mathrm{Re} z + \mathrm{Im} z \end{aligned}$$

and so $\mathrm{Im} z = \mathrm{Re}(-iz)$. Therefore

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= \mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle + i\mathrm{Im}\langle \mathbf{u}, \mathbf{v} \rangle \\ &= \mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle + i\mathrm{Re}(-i\langle \mathbf{u}, \mathbf{v} \rangle) \\ &= \mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle + i\mathrm{Re}\langle \mathbf{u}, i\mathbf{v} \rangle. \end{aligned}$$

$\square$

Theorem 7.19 (Polarization identities).

(1) If V is a real inner product space, then

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}(||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2). \tag{7.7}$$

(2) If V is a complex inner product space, we have

$$\langle v, w \rangle = \frac{1}{4}(||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 + i||i\mathbf{v}+\mathbf{w}||^2 - i||i\mathbf{v}-\mathbf{w}||^2). \tag{7.8}$$

Proof. (1) If V is a real vector space, then $\mathrm{Re}\langle v, w \rangle = \langle v, w \rangle$, so

$$\begin{aligned} ||\mathbf{v}+\mathbf{w}||^2 &= ||\mathbf{v}||^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + ||\mathbf{w}||^2 \\ ||\mathbf{v}-\mathbf{w}||^2 &= ||\mathbf{v}||^2 - 2\langle \mathbf{v}, \mathbf{w} \rangle + ||\mathbf{w}||^2. \end{aligned}$$

Subtracting these two expressions gives

$$||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 = 4\langle \mathbf{v}, \mathbf{w} \rangle$$

and therefore

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}(||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2).$$

(2) For a complex inner product space we have

$$\begin{aligned} ||\mathbf{v}+\mathbf{w}||^2 &= ||\mathbf{v}||^2 + 2\mathrm{Re}\langle \mathbf{v}, \mathbf{w} \rangle + ||\mathbf{w}||^2 \\ -||\mathbf{v}-\mathbf{w}||^2 &= -||\mathbf{v}||^2 + 2\mathrm{Re}\langle \mathbf{v}, \mathbf{w} \rangle - ||\mathbf{w}||^2 \\ i||i\mathbf{v}+\mathbf{w}||^2 &= i||\mathbf{v}||^2 + 2i\mathrm{Re}\langle i\mathbf{v}, \mathbf{w} \rangle + i||\mathbf{w}||^2 \\ -i||i\mathbf{v}-\mathbf{w}||^2 &= -i||\mathbf{v}||^2 + 2i\mathrm{Re}\langle \mathbf{v}, \mathbf{w} \rangle - i||\mathbf{w}||^2 \end{aligned}$$

adding all of these together gives

$$\begin{aligned} ||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 + i||i\mathbf{v}+\mathbf{w}||^2 - i||i\mathbf{v}-\mathbf{w}||^2 &= 4(\mathrm{Re}\langle \mathbf{v}, \mathbf{w} \rangle + i\mathrm{Re}\langle i\mathbf{v}, \mathbf{w} \rangle) \\ &= 4\langle \mathbf{v}, \mathbf{w} \rangle \qquad \text{(by Equation (7.6))} \end{aligned}$$

and therefore

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}(||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 + i||i\mathbf{v}+\mathbf{w}||^2 - i||i\mathbf{v}-\mathbf{w}||^2).$$

□

Exercise 7.7. Show that Equation (7.8) may also be written as

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}\sum_{n=1}^{4} i^n ||\mathbf{v} + i^n \mathbf{w}||^2.$$

Proof.

$$\begin{aligned}\frac{1}{4}\sum_{n=1}^{4} i^n||\mathbf{v}+i^n\mathbf{w}||^2 &= \frac{1}{4}(i||\mathbf{v}+i\mathbf{w}||^2 + i^2||\mathbf{v}+i^2\mathbf{w}||^2 + i^3||\mathbf{v}+i^3\mathbf{w}||^2 + i^4||\mathbf{v}+i^4\mathbf{w}||^2)\\ &= \frac{1}{4}(i||\mathbf{v}+i\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 - i||\mathbf{v}-i\mathbf{w}||^2 + ||\mathbf{v}+\mathbf{w}||^2)\\ &= \frac{1}{4}(||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 + i||\mathbf{v}+i\mathbf{w}||^2 - i||\mathbf{v}-i\mathbf{w}||^2),\end{aligned}$$

which is equal to Equation (7.8). □

The norm can be used to define the distance between any two vectors in an inner product space.

Definition 7.20 (Distance). Let V be an inner product space. The *distance* $d(\mathbf{u},\mathbf{v})$ between any two vectors $\mathbf{u}$ and $\mathbf{v}$ in V is

$$d(\mathbf{u},\mathbf{v}) = ||\mathbf{u}-\mathbf{v}||.$$

The basic properties of the distance are the following, all of which follow from the corresponding properties of the norm $||\ ||$.

Theorem 7.21 (Properties of distance function).

(i) (Positive-definiteness) For all $\mathbf{u},\mathbf{v}\in V$, $d(\mathbf{u},\mathbf{v})\geq 0$ and $d(\mathbf{u},\mathbf{v})=0$ if and only if $\mathbf{u}=\mathbf{v}$.

(ii) (Symmetry) For all $\mathbf{u},\mathbf{v}\in V$

$$d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u}).$$

(iii) (Triangle inequality) For all $\mathbf{u},\mathbf{v},\mathbf{w}$ in V,

$$d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v}).$$

Exercise 7.8. Prove Theorem 7.21.

Proof.

(i) Recall that $||\mathbf{w}||\geq 0$ for all $\mathbf{w}\in V$ and $=0$ if and only if $\mathbf{w}=\mathbf{0}$. The proof therefore follows by taking $\mathbf{w}=\mathbf{u}-\mathbf{v}$, since $d(\mathbf{u},\mathbf{v})=||\mathbf{u}-\mathbf{v}||$.

(ii) We have

$$\begin{aligned}||\mathbf{u}-\mathbf{v}||^2 &= \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle\\ &= \langle -(\mathbf{v}-\mathbf{u}), -(\mathbf{v}-\mathbf{u})\rangle\\ &= (-1)^2\langle \mathbf{v}-\mathbf{u}, \mathbf{v}-\mathbf{u}\rangle\\ &= \langle \mathbf{v}-\mathbf{u}, \mathbf{v}-\mathbf{u}\rangle\\ &= ||\mathbf{v}-\mathbf{u}||^2.\end{aligned}$$

Since $||\mathbf{u}-\mathbf{v}||$ and $||\mathbf{v}-\mathbf{u}||$ are both nonnegative, this implies

$$d(\mathbf{u},\mathbf{v}) = ||\mathbf{u}-\mathbf{v}|| = ||\mathbf{v}-\mathbf{u}|| = d(\mathbf{v},\mathbf{u}).$$

(iii) We have

$$\begin{aligned} d(\mathbf{u},\mathbf{v}) &= ||\mathbf{u}-\mathbf{v}|| \\ &= ||\mathbf{u}-\mathbf{w}+\mathbf{w}-\mathbf{v}|| \\ &\leq ||\mathbf{u}-\mathbf{w}|| + ||\mathbf{w}-\mathbf{v}|| \text{ (by the Triangle Inequality for } ||\ ||) \\ &= d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v}). \end{aligned}$$

□

Definition 7.22 (Metric space). Any nonempty set V together with a function $d : V \times V \to \mathbb{R}$ satisfying the properties of Theorem 7.21 is called a *metric space* and the function d is called a *metric* on V. Thus, any inner product space is a metric space under the metric $d(\mathbf{u},\mathbf{v}) = ||\mathbf{u}-\mathbf{v}||$. The presence of an inner product, and hence a metric, allows us to define a *topology* on V, which gives rise to notions of continuity, convergence, and related concepts.

We have now encountered several spaces. The following graphic shows how these are related to each other:

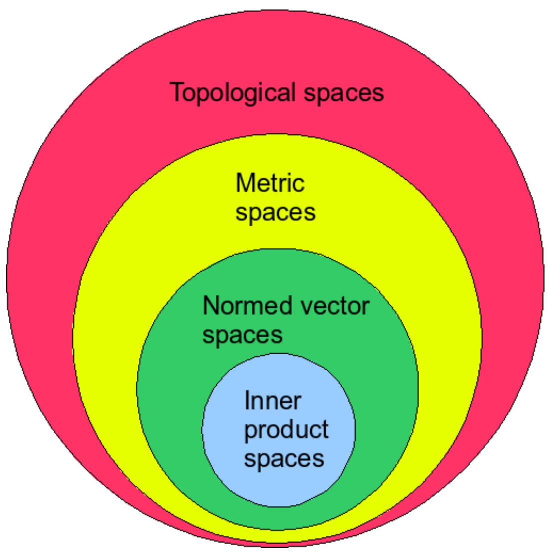


Figure 17: The relation between various spaces we have encountered so far. In this course, we will work almost exclusively with inner product spaces.

Problems

1. Consider the function $\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by
$$\langle \mathbf{x}, \mathbf{y} \rangle = |x_1 y_1| + |x_2 y_2|.$$
Is this an inner product on $\mathbb{R}^2$? Why or why not?

2. Let V be a real inner product space.
   (a) Show that $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = ||\mathbf{u}||^2 - ||\mathbf{v}||^2$ for every $\mathbf{u}, \mathbf{v} \in V$.
   (b) Show that if $\mathbf{u}, \mathbf{v} \in V$ have the same norm, then $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{u} - \mathbf{v}$.
   (c) Use part (b) to prove that the diagonals of a rhombus are perpendicular to each other.

3. Let $\mathbf{u}, \mathbf{v} \in V$. Prove that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ if and only if
$$||\mathbf{u}|| \le ||\mathbf{u} + c\mathbf{v}||$$
for all $c \in \mathbb{F}$.

4. Suppose $\mathbf{u}, \mathbf{v} \in V$ are such that
$$||\mathbf{u}|| = 3, \quad ||\mathbf{u} + \mathbf{v}|| = 4, \quad ||\mathbf{u} - \mathbf{v}|| = 6.$$
Compute $||\mathbf{v}||$.

5. Prove or disprove: there is an inner product on $\mathbb{R}^2$ such that the associated norm is given by
$$||(x_1, x_2)|| = \max\{x_1, x_2\}$$
for all $(x_1, x_2) \in \mathbb{R}^2$.

[Add some computational problems.]

7.3 Isometries

An isomorphism of vector spaces preserves the vector space operations. The corresponding concept for inner product spaces is *isometry*.

Definition 7.23 (Isometry). Let V and W be inner product spaces. A linear mapping $T \in \mathrm{Hom}(V, W)$ is an *isometry* if it preserves the inner product; that is, if [60]

$$\langle T\mathbf{u}, T\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \tag{7.9}$$

for all $\mathbf{u}, \mathbf{v} \in V$. A bijective isometry is called an *isometric isomorphism*. If an isometric isomorphism from V to W exists, we say that V and W are *isometrically isomorphic*.

Exercise 7.9. (a) Show that an isometry is injective. It follows that any surjective isometry is an isometric isomorphism.

[60] Note that the inner product on the left hand side of (7.9) is that of W while the one on the right hand side is that of V. Since this is clear from context, we will use the same notation for both inner products.

(b) Show that if V and W are finite-dimensional vector spaces with $\dim V = \dim W$, then $T \in \mathrm{Hom}(V, W)$ is an isometry if and only if it is an isometric isomorphism. (This is false for infinite-dimensional vector spaces.)

Proof.

(a) Let $T \in \mathrm{Hom}(V, W)$ is an isometry. Suppose $\mathbf{u} \in \ker T$. Then

$$\begin{aligned} ||\mathbf{u}||^2 &= \langle \mathbf{u}, \mathbf{u} \rangle \\ &= \langle T\mathbf{u}, T\mathbf{u} \rangle \\ &= \langle \mathbf{0}, \mathbf{0} \rangle \\ &= 0, \end{aligned}$$

which implies $\mathbf{u} = \mathbf{0}$. Thus $\ker T = \{\mathbf{0}\}$, so T is injective by **??**.

(b) Any isometric isomorphism is, in particular, an isometry. If V and W are finite-dimensional vector space with $\dim V = \dim W$, then injectivity implies surjectivity by **??**. By part (a), and isometry is automatically injective, so in this case it is automatically bijective, and hence an isometric isomorphism.

□

Since the inner product determines the norm, the following should not come as a surprise.

Theorem 7.24 (Isometries preserve the norm). A linear mapping $T \in \mathrm{Hom}(V, W)$ is an isometry if and only if it preserves the norm; that is, if and only if

$$||T\mathbf{v}|| = ||\mathbf{v}||$$

for all $\mathbf{v} \in V$.

Proof. If T is an isometry, then

$$\begin{aligned} ||T\mathbf{v}||^2 &= \langle T\mathbf{v}, T\mathbf{v} \rangle \\ &= \langle \mathbf{v}, \mathbf{v} \rangle \\ &= ||\mathbf{v}||. \end{aligned}$$

Conversely, suppose that $||T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v} \in V$. If V, W are real inner product spaces, then by Theorem 7.19

$$\begin{aligned} \langle T\mathbf{u}, T\mathbf{v} \rangle &= \frac{1}{4}(||T\mathbf{u} + T\mathbf{v}||^2 - ||T\mathbf{u} - T\mathbf{v}||^2) \\ &= \frac{1}{4}(||T(\mathbf{u} + \mathbf{v})||^2 - ||T(\mathbf{u} - \mathbf{v})||^2) \\ &= \frac{1}{4}(||\mathbf{u} + \mathbf{v}||^2 - ||\mathbf{u} - \mathbf{v}||^2) \\ &= \langle \mathbf{u}, \mathbf{v} \rangle. \end{aligned}$$

The proof for the case of a complex inner product space is left as an exercise. □

Exercise 7.10. Prove Theorem 7.24 for a complex inner product space.

Proof. Suppose that $||T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v} \in V$. If V, W are real inner product spaces, then by Theorem 7.19

$$\begin{aligned}
\langle T\mathbf{u}, T\mathbf{v}\rangle &= \frac{1}{4}(||T\mathbf{u}+T\mathbf{v}||^2 - ||T\mathbf{u}-T\mathbf{v}||^2 + i||T\mathbf{u}+iT\mathbf{v}||^2 - i||T\mathbf{u}-iT\mathbf{v}||^2) \\
&= \frac{1}{4}(||T(\mathbf{u}+\mathbf{v})||^2 - ||T(\mathbf{u}-\mathbf{v})||^2 + i||T(\mathbf{u}+i\mathbf{v})||^2 - i||T(\mathbf{u}-i\mathbf{v})||^2) \\
&= \frac{1}{4}(||\mathbf{u}+\mathbf{v}||^2 - ||\mathbf{u}-\mathbf{v}||^2 + i||\mathbf{u}+i\mathbf{v}||^2 - i||\mathbf{u}-i\mathbf{v}||^2) \\
&= \langle \mathbf{u}, \mathbf{v}\rangle.
\end{aligned}$$

□

7.4 Orthonormal Bases

Definition 7.25 (Orthonormal set). Let V be an inner product space. A subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V is said to be *orthogonal* if $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = 0$ for all $\mathbf{v}_i, \mathbf{v}_j \in S$ with $i \neq j$. If additionally $||\mathbf{v}_i|| = 1$ for all $\mathbf{v}_i \in S$, we say S is *orthonormal*. In other words, S is an orthonormal set if

$$\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \delta_{ij},$$

where

$$\delta_{ij} := \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

Example 7.26. (a) The standard basis in $\mathbb{F}^n$ is an orthonormal set.

(b) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$ is an orthonormal set in $\mathbb{F}^3$.

(c) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is an orthonormal set in $M^{2\times 2}(\mathbb{F})$ with respect to the inner product $\langle A, B\rangle = \operatorname{tr}(\overline{A}^T B)$.

(d) $\{\frac{1}{\sqrt{\pi}} \sin t, \frac{1}{\sqrt{\pi}} \cos t\}$ is an orthonormal set in $\mathscr{C}([0, 2\pi])$ with respect to the inner product $\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)dt$.

The next result shows that orthogonality is stronger than linear independence.

Theorem 7.27 (An orthonormal set of nonzero vectors is linearly independent.). An orthogonal set of nonzero vectors is linearly independent. In particular, an orthonormal set is linearly independent.

Proof. Let V be an inner product space and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal set of nonzero vectors in V. Suppose

$$\sum_{i=1}^{n} c_i \mathbf{v}_i = \mathbf{0}$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. Taking the inner product of both sides of this equation with the vector $\mathbf{v}_j$ gives

$$\begin{aligned} 0 &= \langle \mathbf{0}, \mathbf{v}_j \rangle \\ &= \langle \sum_{i=1}^{n} c_i \mathbf{v}_i, \mathbf{v}_j \rangle \\ &= \sum_{i=1}^{n} c_i \langle \mathbf{v}_i, \mathbf{v}_j \rangle \\ &= \sum_{i=1}^{n} c_i \langle \mathbf{v}_i, \mathbf{v}_j \rangle \\ &= \sum_{i=1}^{n} c_i \delta_{ij} ||\mathbf{v}_j||^2 \\ &= c_i ||\mathbf{v}_i||^2. \end{aligned}$$

Since $\mathbf{v}_i \neq \mathbf{0}$, $||\mathbf{v}_i||^2 > 0$, so we must have $c_i = 0$. Thus, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent. □

Remark 7.28. Note that the converse to Theorem 7.27 is *false*. For instance, consider the set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Since

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

S is a linearly independent subset of $\mathbb{R}^2$, but since

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \neq 0,$$

S is not an orthogonal set.

Definition 7.29 (Orthonormal basis). Let V be an inner product space. An *orthonormal basis* for V is an orthonormal set of vectors in V which are also a basis for V.

Example 7.30. The standard basis is an orthonormal basis for $\mathbb{F}^n$.

Corollary 7.31. Let V be a finite-dimensional inner product space. Any orthonormal set of $\dim V$ vectors is a basis for V.

Proof. By Theorem 7.27, any orthonormal set is linearly independent. By Theorem **??**, any linearly independent set of $\dim V$ vectors is a basis for V. □

Exercise 7.11. Show that

$$\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$$

is an orthonormal basis for $\mathbb{F}^4$.

Proof. Since there are $\dim \mathbb{F}^4 = 4$ vectors in the set, by the previous corollary it suffices to check that the set is orthonormal. $\square$

From a computational point of view, orthonormal bases have a practical advantage over arbitrary bases, as we will now see.

Example 7.32. Consider the orthonormal basis

$$B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

for $\mathbb{R}^3$ and let $\mathbf{v} = (2, 1, 3)$. Find $[\mathbf{v}]_B$.

Solution. As we have seen before, to find the coefficient vector $\mathbf{c} \equiv [\mathbf{v}]_B$, we need to find the inverse of the matrix whose ith column is $\mathbf{v}_i$. Then $c_i = (L_B^{-1}(\mathbf{v}))_i$. We then have

$$\begin{aligned} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} &= c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + c_3 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{aligned}$$

and therefore

$$\begin{aligned} [\mathbf{v}]_B \equiv \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 5/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \end{aligned}$$

$\square$

We will now see that the fact that the basis is orthonormal allows us to find $[\mathbf{v}]_B$ much more easily. More generally, let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for an inner product space V and write $\mathbf{v} \in V$ as

$$\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{v}_i.$$

Taking the inner product of both sides with $\mathbf{v}_j$, we find

$$\begin{aligned}\langle \mathbf{v}, \mathbf{v}_j \rangle &= \langle \sum_{i=1}^{n} c_i \mathbf{v}_i, \mathbf{v}_j \rangle \\ &= \sum_{i=1}^{n} c_i \langle \mathbf{v}_i, \mathbf{v}_j \rangle \\ &= \sum_{i=1}^{n} c_i \delta_{ij} \\ &= c_j\end{aligned}$$

Thus, by choosing an orthonormal basis, we find an explicit formula for the coefficients c_i in terms of $\mathbf{v}$ and the basis vectors $\mathbf{v}_i$:

$$\mathbf{v} = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{v}_i \rangle \mathbf{v}_i. \tag{7.10}$$

This is why it is most convenient to work with orthonormal bases.

Exercise 7.12. Use Eq. (7.10) to find $[\mathbf{v}]_B$ for the vector in Example 7.32.

Solution. If $\mathbf{v} = (2, 1, 3)$, then

$$[\mathbf{v}]_B = \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{x} \\ \mathbf{v}_2 \cdot \mathbf{x} \\ \mathbf{v}_3 \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 \\ 5/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

□

Now that we understand the usefulness of orthonormal bases, how do we go about finding them? For example, does the vector space $P_n(\mathbb{R})$ of all polynomials of degree at most n in the real variable x with real coefficients with the inner product

$$\langle p_1(x), p_2(x) \rangle = \int_{-1}^{1} p_1(x) p_2(x) dx$$

have an orthonormal basis? We will now show that one can always take a given basis for a subspace of an inner product space V and construct an orthonormal basis for the same subspace. We will use the following fact.

Lemma 7.33. Any subset of a linearly independent set is linearly independent.

Proof. Let $S = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a linearly independent set of vectors in an inner product space V, and let S' be any subset of S. If S' is empty, then it is linearly independent (this is vacuously true, since there are no vectors in S' for which the zero linear combination could have a non-trivial solution). If S' is all of S, then it is linearly independent by assumption. Suppose now that S' is a proper subset of S, and list the elements (reordering the vectors, if necessary) as $S' = \{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$, where $1 \leq r < n$. Suppose that S' is linearly dependent. Then there exist scalars $c_1, \cdots, c_r$ not all zero such that

$$c_1 \mathbf{u}_1 + \cdots + c_r \mathbf{u}_r = \mathbf{0}.$$

But then we have

$$c_1\mathbf{u}_1 + \cdots + c_r\mathbf{u}_r + 0\cdot\mathbf{u}_{r+1} + \cdots + 0\cdot\mathbf{u}_n = \mathbf{0},$$

which is a contradiction since S is linearly independent. Thus, S' must be linearly independent as well. $\square$

Theorem 7.34. Let V be a finite-dimensional inner product space. Given a basis B for a subspace of V, one can convert B into an orthonormal basis for the same subspace.

Proof. Let W be an n-dimensional subspace of V, and let $B = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis for W. We will obtain an orthonormal basis $B' = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ by means of a construction known as the *Gram-Schmidt orthogonalization algorithm*, which we now describe. The first part of the algorithm successively builds an orthogonal basis. One then simply normalizes each vector to obtain an orthonormal basis.

First, let $\mathbf{v}_1 = \mathbf{u}_1$. Then $\mathbf{v}_1 \neq \mathbf{0}$, so the set $\{\mathbf{v}_1\}$ is linearly independent. Now define the other vectors inductively, as follows: Suppose $\mathbf{v}_1, \cdots, \mathbf{v}_m$ $(1 \leq m < n)$ have been chosen such that for every k in $1 \leq k \leq m$, the set

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$$

is an orthogonal basis for the subspace of W spanned by $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ (we have just seen that this is true for $m = 1$, since $\{\mathbf{u}_1\}$ is trivially an orthogonal basis for the subspace of W spanned by $\mathbf{u}_1$, so this establishes the base case). We will show how to construct the next vector, $\mathbf{v}_{m+1}$, so that the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ is an orthogonal basis for the subspace of V spanned by $\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$. This will complete the inductive proof that we can continue this process until we have an orthogonal basis for W.

To construct the next vector $\mathbf{v}_{m+1}$, let

$$\begin{aligned} \mathbf{v}_{m+1} &= \mathbf{u}_{m+1} - \sum_{k=1}^{m} \operatorname{proj}_{\mathbf{v}_k}\mathbf{u}_{m+1} \\ &= \mathbf{u}_{m+1} - \sum_{k=1}^{m} \frac{\langle \mathbf{u}_{m+1}, \mathbf{v}_k\rangle}{\langle \mathbf{v}_k, \mathbf{v}_k\rangle}\mathbf{v}_k. \end{aligned} \tag{7.11}$$

We need to show that $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ is an orthogonal set of nonzero vectors and that $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\} \subseteq \operatorname{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$. Since an orthogonal set of nonzero vectors is linearly independent by Theorem 7.27, $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ will then be an orthogonal basis for $\operatorname{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$, since a set of $m+1$ linearly independent vectors in an $(m+1)$-dimensional vector space is automatically a basis for that vector space.

First, we note that $\mathbf{v}_{m+1} \neq 0$, since otherwise we would have

$$\mathbf{u}_{m+1} = \sum_{k=1}^{m} \frac{\langle \mathbf{u}_{m+1}, \mathbf{v}_k\rangle}{\langle \mathbf{v}_k, \mathbf{v}_k\rangle}\mathbf{v}_k$$

which says $\mathbf{u}_{m+1}$ is a linear combination of $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ and hence a linear combination of $\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$, [61] which contradicts the fact that $\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$ is linearly independent. [62] Thus, $\mathbf{v}_{m+1} \neq 0$.

[61] Since, by assumption, $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ is a basis for $\operatorname{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$ and, in particular, $\operatorname{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_m\} = \operatorname{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$

[62] Note that we have used Lemma 7.33 here.

All that is left is to show that $\mathbf{v}_{m+1}$ is orthogonal to every vector in the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$. Let $1 \leq j \leq m$. Then

$$\begin{aligned}\langle \mathbf{v}_{m+1}, \mathbf{v}_j \rangle &= \mathbf{u}_{m+1} - \sum_{k=1}^{m} \frac{\langle \mathbf{u}_{m+1}, \mathbf{v}_k \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle} \langle \mathbf{v}_k, \mathbf{v}_j \rangle \\ &= \langle \mathbf{u}_{m+1}, \mathbf{v}_j \rangle - \sum_{k=1}^{m} \frac{\langle \mathbf{u}_{m+1}, \mathbf{v}_k \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle} \langle \mathbf{v}_k, \mathbf{v}_j \rangle\end{aligned}$$

Since $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ is an orthogonal set (by our inductive hypothesis), every term in the sum is zero except for $k = j$, so we find

$$\begin{aligned}\langle \mathbf{v}_{m+1}, \mathbf{v}_j \rangle &= \langle \mathbf{u}_{m+1}, \mathbf{v}_j \rangle - \sum_{k=1}^{m} \frac{\langle \mathbf{u}_{m+1}, \mathbf{v}_k \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle} \langle \mathbf{v}_k, \mathbf{v}_j \rangle \\ &= \langle \mathbf{u}_{m+1}, \mathbf{v}_j \rangle - \frac{\langle \mathbf{u}_{m+1}, \mathbf{v}_j \rangle}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle} \langle \mathbf{v}_j, \mathbf{v}_j \rangle \\ &= \langle \mathbf{u}_{m+1}, \mathbf{v}_j \rangle - \langle \mathbf{u}_{m+1}, \mathbf{v}_j \rangle \\ &= 0.\end{aligned}$$

Thus we have shown that $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ is an orthogonal set consisting of $m+1$ nonzero vectors.

All that remains is to show that this set is a basis for $\text{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$. As mentioned above, it will suffice to show that $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\} \subseteq \text{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$, since a linearly independent subset of an $(m+1)$-dimensional vector space consisting of $m+1$ vectors is automatically a basis. The set $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subseteq \text{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_m\} \subseteq \text{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$ by the inductive hypothesis. Eq. (7.11) then shows that $\mathbf{v}_{m+1} \in \text{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$. Thus, we have shown that the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}\}$ is an orthogonal basis for $\text{Span}\{\mathbf{u}_1, \cdots, \mathbf{u}_{m+1}\}$, completing the proof by induction that $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an orthogonal basis for V.

Finally, since none of these vectors are zero, we obtain an orthonormal basis for V by normalizing each vector as $\mathbf{v}_i \mapsto \frac{\mathbf{v}_i}{||\mathbf{v}_i||}$. □

Corollary 7.35. Every finite-dimensional inner product space has an orthonormal basis.

Proof. Let V be a finite-dimensional inner product space and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ a basis for V. Apply the Gram-Schmidt process to construct an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. Then to obtain an orthonormal basis simply replace each vector $\mathbf{u}_k$ by $\frac{\mathbf{u}_k}{||\mathbf{u}_k||}$. □

Example 7.36. Consider the basis for $\mathbb{R}^3$ given by $\{\mathbf{u}_1 = (1,1,1),\ \mathbf{u}_2 = (0,1,1),\ \mathbf{u}_3 = (0,0,1)\}$. Taking the standard inner product on $\mathbb{R}^3$, we now follow the Gram-Schmidt algorithm to obtain an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{u}_1 = (1,1,1) \\ \mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = (0,1,1) - \frac{2}{3}(1,1,1) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \\ \mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\ &= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \\ &= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{2}(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \\ &= (0, -\frac{1}{2}, \frac{1}{2})\end{aligned}$$

Noting that

$$||\mathbf{v}_1|| = \sqrt{3},\ ||\mathbf{v}_2|| = \sqrt{\frac{2}{3}},\ ||\mathbf{v}_3|| = \frac{1}{\sqrt{2}}$$

normalizing each vector as $\mathbf{v}_i \mapsto \mathbf{q}_i \equiv \frac{\mathbf{v}_i}{||\mathbf{v}_i||}$ results in the orthonormal basis $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, where

$$\begin{aligned}
\mathbf{q}_1 &= (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}),\\
\mathbf{q}_2 &= (-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}),\\
\mathbf{q}_3 &= (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).
\end{aligned}$$

Exercise 7.13. Let $V = \mathbb{R}^3$ equipped with the standard inner product. Consider the basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = (3, 0, 4),\ \mathbf{u}_2 = (-1, 0, 7),\ \mathbf{u}_3 = (2, 9, 11).$$

Apply the Gram-Schmidt algorithm to construct an orthonormal basis $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$. If $\mathbf{x} = (1, 2, 3)$, find $[\mathbf{x}]_{B'}$.

Solution.

$$\begin{aligned}
\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} &= \left\{ \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -4/5 \\ 0 \\ 3/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},\\
[\mathbf{x}]_{B'} &= \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{x} \\ \mathbf{v}_2 \cdot \mathbf{x} \\ \mathbf{v}_3 \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.
\end{aligned}$$

□

Example 7.37. Find an orthonormal basis for $P_2(\mathbb{R})$ (the vector space of polynomials of degree at most 2 in the real variable x with real coefficients), where the inner product is given by $\langle p_1(x), p_2(x) \rangle = \int_{-1}^{1} p_1(x) p_2(x) dx$.

Solution. We will apply the Gram-Schmidt procedure to the basis $\{\mathbf{u}_1 = 1, \mathbf{u}_2 = x, \mathbf{u}_3 = x^2\}$.

To begin, take $\mathbf{v}_1 = \mathbf{u}_1 = 1$. Then

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2} \mathbf{v}_1.$$

Since

$$\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = \langle x, 1 \rangle = \int_{-1}^{1} x dx = 0,$$

$\mathbf{v}_2 = \mathbf{u}_2 = x$. Next,

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2.$$

We have

$$\begin{aligned}
\langle \mathbf{u}_3, \mathbf{v}_1 \rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3}, \\
\langle \mathbf{u}_3, \mathbf{v}_2 \rangle &= \int_{-1}^{1} x^3 dx = 0, \\
||\mathbf{v}_1||^2 &= \int_{-1}^{1} dx = 2, \\
||\mathbf{v}_2||^2 &= \int_{-1}^{1} x^2 dx = \frac{2}{3},
\end{aligned}$$

so

$$\begin{aligned}
\mathbf{v}_3 &= x^2 - \frac{2/3}{2} \cdot 1 - 0 \\
&= x^2 - \frac{1}{3}.
\end{aligned}$$

Now

$$||\mathbf{v}_3||^2 = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx,$$

so we have

$$||\mathbf{v}_1|| = \sqrt{2}, \quad ||\mathbf{v}_2|| = \sqrt{\frac{2}{3}}, \quad ||\mathbf{v}_3|| = \sqrt{\frac{8}{45}}.$$

Normalizing each of these vectors, our orthonormal basis for $P_2(\mathbb{R})$ is given by

$$\left\{ \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3}) \right\}.$$

Since $\{1, x, x^2, x^3, \ldots, x^n\}$ is a basis for $P_n(\mathbb{R})$, we could continue this sequence to obtain an orthonormal basis for $P_n(\mathbb{R})$. The resulting polynomials are called *Legendre polynomials*. □

7.5 Linear Functionals on Inner Product Spaces

Recall that a *linear functional* on a vector space V is a linear map from V to $\mathbb{F}$; i.e., an element of the dual space $V^* = \text{Hom}(V, \mathbb{F})$. If V is finite-dimensional, then vector spaces V and V^* are isomorphic simply by virtue of having the same dimension. In the case where V is endowed with an inner product, each element of V^* takes a simple form, as we now show.

Exercise 7.14. For any $\mathbf{u} \in V$, show that the map $\varphi_{\mathbf{u}} : V \to \mathbb{F}$ defined by $\varphi_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$ is linear. Thus, for each $\mathbf{u} \in V$, $\varphi_{\mathbf{u}} \in V^*$.

Example 7.38. Consider the linear functional $f : \mathbb{F}^3 \to \mathbb{F}$ on $\mathbb{F}^3$ defined by

$$f(z_1, z_2, z_3) = 2z_1 - 5z_2 + z_3.$$

We note that $f = \varphi_{\mathbf{u}}$ with with $\mathbf{u} = (2, -5, 1)$.

We will prove that for every linear functional $f \in V^*$ there exists a unique vector $\mathbf{u} \in V$ such that $f(\mathbf{v}) = \varphi_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle$ for every $\mathbf{v} \in V$. The next example illustrates that finding such a $\mathbf{u} \in V$ is nontrivial.

Example 7.39. Consider the linear map $f : P_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$f(p) = \int_{-1}^{1} p(t)(\cos(\pi t))dt.$$

It is not at all obvious that there exists $u \in P_2(\mathbb{R})$ such that

$$f(p) = \varphi_u(p) = \langle p, u \rangle$$

for every $p \in P_2(\mathbb{R})$. (Note that we cannot take $u(t) = \cos(\pi t)$ since $\cos(\pi t) \notin P_2(\mathbb{R})$.)

Theorem 7.40 (Riesz Representation Theorem).

(1) The map $V \to V^*$ taking $\mathbf{u} \mapsto \varphi_{\mathbf{u}}$ is a conjugate isomorphism. [63] In particular, for each $f \in V^*$, there exists a unique vector $\mathbf{u} \in V$ such that $f = \varphi_{\mathbf{u}}$; that is,

$$f(\mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$$

for all $\mathbf{v} \in V$. We call $\mathbf{u}$ the *Riesz vector* for f. [64]

(2) The map $R : V^* \to V$ taking $f \mapsto \mathbf{u}$ is also a conjugate isomorphism, being the inverse of the map in part (1). We call this map the *Riesz map*.

Proof.

(1) Let $\mathbf{u}_1, \mathbf{u}_2 \in V$ and $c \in \mathbb{F}$. Then for every $\mathbf{v} \in V$ we have

$$\begin{aligned} \varphi_{c\mathbf{u}_1 + \mathbf{u}_2}(\mathbf{v}) &= \langle \mathbf{v}, c\mathbf{u}_1 + \mathbf{u}_2 \rangle \\ &= \bar{c}\langle \mathbf{v}, \mathbf{u}_1 \rangle + \langle \mathbf{v}, \mathbf{u}_2 \rangle \\ &= \bar{c}\varphi_{\mathbf{u}_1} + \varphi_{\mathbf{u}_2}, \end{aligned}$$

which shows that the mapping $\mathbf{u} \mapsto \varphi_{\mathbf{u}}$ is conjugate linear. If $\mathbf{u}$ is an element of the kernel, then $\varphi_{\mathbf{u}}$ is the zero map, so for every $\mathbf{v} \in V$ we have

$$\begin{aligned} 0 &= \varphi_{\mathbf{u}}(\mathbf{v}) \\ &= \langle \mathbf{v}, \mathbf{u} \rangle. \end{aligned}$$

[63] Recall that a mapping $f : V \to W$ is *conjugate linear* if $f(c\mathbf{v}_1 + \mathbf{v}_2) = \bar{c}f(\mathbf{v}_1) + f(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$. If f is also bijective, then f is a *conjugate isomorphism*. [Need to introduce this earlier and explain significance. See Roman's book.]

[64] Riesz is pronounced "Reese".

In particular, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$, so we must have $\mathbf{u} = \mathbf{0}$, proving that $\mathbf{u} \mapsto \varphi_{\mathbf{u}}$ is injective. To prove surjectivity, let $f \in V^*$ and let $\{e_i\}$ be an orthonormal basis for V. Then $f = \varphi_{\mathbf{u}}$ with $\mathbf{u} = \sum_{i=1}^n \overline{f(e_i)} e_i$, since for any $\mathbf{v} = \sum_{i=1}^n c_i e_i \in V$ we have

$$\begin{aligned} f(\mathbf{v}) &= f(\sum_{i=1}^n c_i e_i) \\ &= \sum_{i=1}^n c_i f(e_i) \\ &= \sum_{i=1}^n \sum_{j=1}^n c_i f(e_j) \delta_{ij} \\ &= \sum_{i=1}^n \sum_{j=1}^n c_i f(e_j) \langle e_i, e_j \rangle \\ &= \langle \sum_{i=1}^n c_i e_i, \sum_{j=1}^n \overline{f(e_j)} e_j \rangle \\ &= \langle \mathbf{v}, \mathbf{u} \rangle. \end{aligned}$$

This proves that $\mathbf{u} \mapsto \varphi_{\mathbf{u}}$ is a conjugate isomorphism.

(2) Let $f_1, f_2 \in V^*$ and $c \in \mathbb{F}$. Then

$$\begin{aligned} R(cf_1 + f_2) &= \sum_{i=1}^n \overline{(cf_1 + f_2)(e_i)} e_i \\ &= \sum_{i=1}^n (\bar{c}\overline{f_1(e_i)} + \overline{f_2(e_i)}) e_i \\ &= \bar{c} \sum_{i=1}^n \overline{f_1(e_i)} e_i + \sum_{i=1}^n \overline{f_2(e_i)} e_i \\ &= \bar{c}\mathbf{u}_1 + \mathbf{u}_2 \\ &= \bar{c} R(f_1) + R(f_2) \end{aligned}$$

which proves that R is conjugate linear and hence a conjugate isomorphism.

□

Using the formula for the Riesz vector in the above proof, we can now solve Exercise 7.39. Take $\{e_1, e_2, e_3\} = \left\{ \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3}) \right\}$ to be the orthonormal basis of Legendre polynomials obtained in Example 7.37. Then the Riesz vector $\mathbf{u}$ for f is given by

$$\begin{aligned} u(x) &= \sum_{i=1}^3 \overline{f(e_i)} e_i \\ &= \left(\int_{-1}^1 \sqrt{\frac{1}{2}} (\cos(\pi t)) dt \right) \sqrt{\frac{1}{2}} + \left(\int_{-1}^1 \sqrt{\frac{3}{2}} t (\cos(\pi t)) dt \right) \sqrt{\frac{3}{2}} x \\ &+ \left(\int_{-1}^1 \sqrt{\frac{45}{8}} \left(t^2 - \frac{1}{3} \right) (\cos(\pi t)) dt \right) \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right). \end{aligned}$$

Evaluating the integrals gives

$$u(x) = -\frac{45}{2\pi^2}\left(x^2 - \frac{1}{3}\right).$$

Thus the linear map $f : P_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$f(p) = \int_{-1}^{1} p(t)(\cos(\pi t))dt.$$

is equal to

$$\varphi_u(p) = \langle p, u\rangle = -\frac{45}{2\pi^2}\int_{-1}^{1} p(t)\left(t^2 - \frac{1}{3}\right)dt.$$

Problems

1. Let n be a positive integer. Prove that

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \ldots, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \ldots, \frac{\sin(nx)}{\sqrt{\pi}}\right\}_{k=0}^{n}$$

is an orthonormal set of vectors in $\mathscr{C}([-\pi, \pi])$ (the vector space of continuous real-valued functions on $[-\pi, \pi]$) with respect to the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

2. Let $V = P_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3 together with the inner product

$$\langle p, q\rangle = \int_{-\infty}^{\infty} p(x)q(x)e^{-x^2}dx.$$

Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$, thereby computing the first four *Hermite polynomials*.

7.6 Orthogonal Complements

Definition 7.41. Let V be a finite-dimensional inner product space and let U be a subset of V. The *orthogonal complement* of U, denoted $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{u} \rangle = 0 \text{ for every } \mathbf{u} \in U\}.$$

Example 7.42. If U is a line in $\mathbb{R}^3$, then $U^\perp$ is the plane containing the origin that is perpendicular to U. If U is a plane in $\mathbb{R}^3$, then $U^\perp$ is the line containing the origin that is perpendicular to U.

Proposition 7.43 (Properties of orthogonal complements).

(a) If U is a subset of V, then $U^\perp$ is a subspace of V.

(b) $\{\mathbf{0}\}^\perp = V$.

(c) $V^\perp = \{\mathbf{0}\}$.

(d) If U is a subset of V, then $U \cap U^\perp \subseteq \{\mathbf{0}\}$.

(e) If U and W are subsets of V and $U \subseteq W$, then $W^\perp \subseteq U^\perp$.

Proof.

(a) Suppose U is a subset of V. By Exercise 7.2, $\langle \mathbf{0}, \mathbf{u} \rangle = 0$ for every $\mathbf{u} \in U$; thus $\mathbf{0} \in U^\perp$. Thus, $U^\perp$ is not empty. If $\mathbf{v}_1, \mathbf{v}_2 \in U^\perp$ and $c \in \mathbb{F}$, then

$$\begin{aligned}
\langle c\mathbf{v}_1 + \mathbf{v}_2, \mathbf{u} \rangle &= c\langle \mathbf{v}_1, \mathbf{u} \rangle + \langle \mathbf{v}_2, \mathbf{u} \rangle \\
&= c \cdot 0 + 0 \\
&= 0,
\end{aligned}$$

showing that $c\mathbf{v}_1 + \mathbf{v}_2 \in U^\perp$. Thus, $U^\perp$ is a subspace of V by the subspace criterion.

(b) By definition, $\{\mathbf{0}\}^\perp \subseteq V$. If $\mathbf{v} \in V$, then $\langle \mathbf{v}, \mathbf{0} \rangle = 0$ by Exercise 7.2. Since $\mathbf{v}$ is orthogonal to every vector in $\{\mathbf{0}\}$, $\mathbf{v} \in \{\mathbf{0}\}^\perp$. Since $\mathbf{v}$ was an arbitrary vector in V, this shows that $V \subseteq \{\mathbf{0}\}^\perp$ and therefore $\{\mathbf{0}\}^\perp = V$.

(c) Since $\{\mathbf{0}\}$ is the only vector orthogonal to every vector in V (again by Exercise 7.2), $V^\perp = \{\mathbf{0}\}$.

(d) If $U \cap U^\perp$ is empty, then $U \cap U^\perp = \varnothing \subseteq \{\mathbf{0}\}$. Otherwise, if $v \in U \cap U^\perp$, then $\langle \mathbf{v}, \mathbf{v} \rangle = 0$, which implies $\mathbf{v} = \mathbf{0}$, so again $U \cap U^\perp \subseteq \{\mathbf{0}\}$.

(e) Let $\mathbf{v} \in W^\perp$. Then $\mathbf{v}$ is orthogonal to every vector in W. Since $U \subseteq W$, $\mathbf{v}$ is, in particular, orthogonal to every vector in U. Thus, $\mathbf{w} \in U^\perp$, which shows that $W^\perp \subseteq U^\perp$.

$\square$

Theorem 7.44 (Direct sum of a subspace and its orthogonal complement). If U is a subspace of a finite-dimensional vector space V, then

$$V = U \oplus U^\perp.$$

Proof. We have seen in Proposition 7.8 that given $\mathbf{v} \in V$ and $\mathbf{u} \in U$, then

$$\mathbf{u} = \underbrace{\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v}}_{\in U} + \underbrace{\left(\mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v}\right)}_{\in U^\perp}.$$

By part (d) of Proposition 7.43, $U \cap U^\perp = \{\mathbf{0}\}$ (since U is a subspace of V), so by **??** $V = U \oplus U^\perp$. ☐

Corollary 7.45 (Dimension of $U^\perp$). If V is finite-dimensional and U a subspace of V, then

$$\dim U^\perp = \dim V - \dim U.$$

Proof. This follows immediately from Theorem 7.44 and Corollary 5.101. ☐

The next result is an important consequence of Theorem 7.44.

Theorem 7.46 (Orthogonal complement of orthogonal complement). If U is a subspace of a finite-dimensional vector space V, then

$$U = (U^\perp)^\perp.$$

Proof. Let $\mathbf{u} \in U$ and $\mathbf{v} \in U^\perp$. Then by definition of $U^\perp$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

This shows that $\mathbf{u}$ is orthogonal to every vector $\mathbf{v} \in U^\perp$, so $U \subseteq (U^\perp)^\perp$.

To prove the opposite inclusion, let $\mathbf{v} \in (U^\perp)^\perp$. Since $\mathbf{v} \in V$, by Theorem 7.44 we can write $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in U$ and $\mathbf{w} \in U^\perp$. Then $\mathbf{w} = \mathbf{v} - \mathbf{u} \in U^\perp$. Since $\mathbf{v} \in (U^\perp)^\perp$ and $\mathbf{u} \in (U^\perp)^\perp$ (from the above), we have $\mathbf{v} - \mathbf{u} \in (U^\perp)^\perp$ (since $(U^\perp)^\perp$ is a subspace of V and hence closed under subtraction). Thus $\mathbf{v} - \mathbf{u} \in U^\perp \cap (U^\perp)^\perp$, so $\mathbf{v} - \mathbf{u} = \mathbf{0}$ (by part (d) of Proposition 7.43) and therefore $\mathbf{v} = \mathbf{u}$, which implies $\mathbf{v} \in U$. Thus $(U^\perp)^\perp \subseteq U$ and therefore $(U^\perp)^\perp = U$. ☐

We now define a linear operator P_U for each subspace of a finite-dimensional vector space V.

Definition 7.47 (Orthogonal projection). Let V be a finite-dimensional inner product space and U a subspace of V. The *orthogonal projection* of V onto U is the operator $P_U \in \mathrm{Hom}(V)$ defined as follows: For $\mathbf{v} \in V$, write $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in U$ and $\mathbf{w} \in U^\perp$. Then $P_U\mathbf{v} = \mathbf{w}$.

The direct sum decomposition $V = U \oplus U^\perp$ given by Theorem 7.44 shows that P_U is well-defined.

Example 7.48. For any $\mathbf{u} \in V$ where $\mathbf{u} \neq \mathbf{0}$ and $U = \mathrm{Span}\{\mathbf{u}\}$, we have seen that we can write

$$\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{||\mathbf{u}||^2}\mathbf{u} + \left(\mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{||\mathbf{u}||^2}\mathbf{u}\right),$$

where the term on the left is in U and the term on the right in $U^\perp$. Thus

$$P_U\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{||\mathbf{u}||^2}\mathbf{u}$$

for every $\mathbf{v} \in V$.

Proposition 7.49 (Properties of P_U). Let V be a finite-dimensional inner product space, U a subspace of V and $\mathbf{v} \in V$. Then

(a) $P_U \in \mathrm{Hom}(V)$;

(b) $P_U\mathbf{u} = \mathbf{u}$ for every $\mathbf{u} \in U$;

(c) $P_U\mathbf{w} = \mathbf{0}$ for every $\mathbf{w} \in U^\perp$;

(d) $\mathrm{im}\, P_U = U$;

(e) $\ker P_U = U^\perp$;

(f) $\mathbf{v} - P_U\mathbf{v} \in U^\perp$;

(g) $P_U^2 = P_U$;

(h) $||P_U\mathbf{v}|| \leq ||\mathbf{v}||$;

(i) For every orthonormal basis $\{e_1, \ldots, e_m\}$ of U,

$$P_U\mathbf{v} = \sum_{i=1}^{m} \langle \mathbf{v}, e_i \rangle e_i.$$

Proof.

(a) Let $\mathbf{v}_1, \mathbf{v}_2 \in V$ and write

$$\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{w}_1 \qquad \text{and } \mathbf{v}_2 = \mathbf{u}_2 + \mathbf{w}_2$$

with $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in U^\perp$. Then for $c \in \mathbb{F}$

$$\begin{aligned} P_U(c\mathbf{v}_1 + \mathbf{v}_2) &= c(\mathbf{u}_1 + \mathbf{w}_1) + \mathbf{u}_2 + \mathbf{w}_2 \\ &= P_U(c\mathbf{u}_1 + \mathbf{u}_2 + c\mathbf{w}_1 + \mathbf{w}_2) \\ &= c\mathbf{u}_1 + \mathbf{u}_2 \\ &= cP\mathbf{v}_1 + P\mathbf{v}_2, \end{aligned}$$

where we have used the fact that $c\mathbf{u}_1 + \mathbf{u}_2 \in U$ and $c\mathbf{w}_1 + \mathbf{w}_2 \in U^\perp$ since $U, U^\perp$ are subspaces and hence closed under addition and scalar multiplication. Thus, $P_U \in \mathrm{Hom}(V)$.

(b) Let $\mathbf{u} \in U$. Then we can write $\mathbf{u} = \mathbf{u} + \mathbf{0}$ with $\mathbf{u} \in U$ and $\mathbf{0} \in U^\perp$. Thus $P_U\mathbf{u} = \mathbf{u}$.

(c) If $\mathbf{w} \in U^\perp$, we can write $\mathbf{w} = \mathbf{0} + \mathbf{w}$ with $\mathbf{0} \in U$ and $\mathbf{w} \in U^\perp$. Thus $P_U\mathbf{w} = 0$.

(d) The definition of P_U implies that $\mathrm{im}\, P_U \subseteq U$. Part (b) implies that $U \subseteq \mathrm{im}\, P_U$. Thus $\mathrm{im}\, P_U = U$.

(e) Part (c) implies that $U^\perp \subseteq \ker P_U$. To prove the inclusion in the other direction, note that if $\mathbf{v} \in \ker P_U$ then the decomposition given by Theorem 7.44 must be $\mathbf{v} = \mathbf{0} + \mathbf{v}$, where $\mathbf{0} \in U$ and $\mathbf{v} \in U^\perp$. Thus $\ker P_U \subseteq U^\perp$ and $\ker P_U = U^\perp$.

(f) If $\mathbf{v} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$ and $\mathbf{w} \in U^\perp$, then

$$\mathbf{v} - P_U\mathbf{v} = \mathbf{v} - \mathbf{u} = \mathbf{w} \in U^\perp.$$

(g) If $\mathbf{v} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$ and $\mathbf{w} \in U^\perp$, then

$$P_U^2\mathbf{v} = P_U(P_U\mathbf{v}) = P_U\mathbf{u} = \mathbf{u} = P_U\mathbf{v},$$

where we have used part (b).

(h) If $\mathbf{v} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$ and $\mathbf{w} \in U^\perp$, then

$$||P_U\mathbf{v}||^2 = ||\mathbf{u}||^2 \leq ||\mathbf{u}||^2 + ||\mathbf{w}||^2 = ||\mathbf{v}||^2,$$

where we have used the Pythagorean Theorem (Theorem 7.7).

(i) This follows from the fact that $P_U\mathbf{v} \in U$ and by Equation (7.10).

□

7.7 Minimization Problems

[Add section.]

Problems

[Add problems.]

8 Operators on Inner Product Spaces

In this section we study the structure of certain special types of linear operators on finite-dimensional inner product spaces.

8.1 The Adjoint of a Linear Operator

Theorem 8.1. Let V and W be finite-dimensional inner product spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $T \in \mathrm{Hom}(V, W)$. Then there is a unique function $T^* \in \mathrm{Hom}(W, V)$, called the *adjoint* of T, defined by the condition

$$\langle T\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, T^*\mathbf{w}\rangle$$

for all $\mathbf{v} \in V$ and all $\mathbf{w} \in W$.

Proof. Let $T \in \mathrm{Hom}(V, W)$. For each $\mathbf{w} \in W$, consider the linear functional $f_\mathbf{w} \in V^*$ defined by

$$f_\mathbf{w}(\mathbf{v}) = \langle T\mathbf{v}, \mathbf{w}\rangle$$

for all $\mathbf{v} \in V$. By the Riesz Representation Theorem (Theorem 7.40), there exists a unique vector $\mathbf{u} \in V$ such that $f_\mathbf{w} = \varphi_\mathbf{u}$; i.e.,

$$\begin{aligned} f_\mathbf{w}(\mathbf{v}) &= \varphi_\mathbf{u}(\mathbf{v}) \\ \langle T\mathbf{v}, \mathbf{w}\rangle &= \langle \mathbf{v}, \mathbf{u}\rangle \end{aligned}$$

for all $\mathbf{v} \in V$. Letting $f : V \to V^*$ be the conjugate-linear map taking $\mathbf{w} \mapsto f_{\mathbf{w}}$ and $R : V^* \to V$ the Riesz map taking $f_{\mathbf{w}} \mapsto \mathbf{u}$, we define $T^* : W \to V$ by $T^* = R \circ f$, so that

$$\begin{aligned} T^*\mathbf{w} &= R \circ f(\mathbf{w}) \\ &= R(f_{\mathbf{w}}) \\ &= \mathbf{u}. \end{aligned}$$

Since T^* is the composition of two conjugate-linear maps, it is linear, so $T^* \in \mathrm{Hom}(W, V)$. [Add Exercise showing this in section where conjugate linear maps are introduced. This might be a section on complexification.] □

Remark 8.2. In the physics literature the adjoint T^* is often called the *Hermitian conjugate* of T and is denoted by $T^\dagger$.

Example 8.3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1).$$

If $(y_1, y_2) \in \mathbb{R}^2$, then for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ we have

$$\begin{aligned} \langle (x_1, x_2, x_3), T^*(y_1, y_2) \rangle &= \langle T(x_1, x_2, x_3), (y_1, y_2) \rangle \\ &= \langle (x_2 + 3x_3, 2x_1), (y_1, y_2) \rangle \\ &= x_2 y_1 + 3x_3 y_1 + 2x_1 y_2 \\ &= \langle (x_1, x_2, x_3), (2y_2, y_1, 3y_1) \rangle. \end{aligned}$$

Thus $T^* : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$T^*(y_1, y_2) = (2y_2, y_1, 3y_1).$$

Proposition 8.4 (Properties of the adjoint). Let U, V, W be finite-dimensional inner product spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then for all $S, T \in \mathrm{Hom}(V, W)$ and all $c \in \mathbb{F}$,

(a) $(S + T)^* = S^* + T^*$;

(b) $(cT)^* = \bar{c}T^*$;

(c) $(T^*)^* = T$;

(d) $\mathrm{id}_V^* = \mathrm{id}_V$;

(e) $(ST)^* = T^*S^*$ for all $T \in \mathrm{Hom}(V, W)$ and $S \in \mathrm{Hom}(W, U)$ (with U another finite-dimensional inner product space over $\mathbb{F}$).

Proof. (a) For all $\mathbf{v} \in V$ and $\mathbf{w} \in W$ we have

$$\begin{aligned} \langle \mathbf{v}, (S + T)^*\mathbf{w} \rangle &= \langle (S + T)\mathbf{v}, \mathbf{w} \rangle \\ &= \langle S\mathbf{v}, \mathbf{w} \rangle + \langle T\mathbf{v}, \mathbf{w} \rangle \\ &= \langle \mathbf{v}, S^*\mathbf{w} \rangle + \langle \mathbf{v}, T^*\mathbf{w} \rangle \\ &= \langle \mathbf{v}, (S^* + T^*)\mathbf{w} \rangle. \end{aligned}$$

By uniqueness of the adjoint, we therefore have $(S + T)^* = S^* + T^*$.

(b) For all $\mathbf{v} \in V$ and $\mathbf{w} \in W$ we have

$$\begin{aligned} \langle \mathbf{v}, (cT)^* \mathbf{w} \rangle &= \langle cT\mathbf{v}, \mathbf{w} \rangle \\ &= c\langle T\mathbf{v}, \mathbf{w} \rangle \\ &= c\langle \mathbf{v}, T^* \mathbf{w} \rangle \\ &= \langle \mathbf{v}, \bar{c} T^* \mathbf{w} \rangle. \end{aligned}$$

By uniqueness of the adjoint, we therefore have $(cT)^* = \bar{c}T^*$.

(c) For all $\mathbf{v} \in V$ and $\mathbf{w} \in W$ we have

$$\begin{aligned} \langle \mathbf{v}, (T^*)^* \mathbf{w} \rangle &= \langle T^*\mathbf{v}, \mathbf{w} \rangle \\ &= \overline{\langle \mathbf{w}, T^*\mathbf{v} \rangle} \\ &= \overline{\langle T\mathbf{w}, \mathbf{v} \rangle} \\ &= \langle \mathbf{v}, T\mathbf{w} \rangle. \end{aligned}$$

By uniqueness of the adjoint, we therefore have $(T^*)^* = T$.

(d) For all $\mathbf{v} \in V$ and $\mathbf{w} \in W$ we have

$$\begin{aligned} \langle \mathbf{v}, \mathrm{id}_V^* \mathbf{w} \rangle &= \langle \mathrm{id}_V \mathbf{v}, \mathbf{w} \rangle \\ &= \langle \mathbf{v}, \mathbf{w} \rangle \\ &= \langle \mathbf{v}, \mathrm{id}_V \mathbf{w} \rangle \end{aligned}$$

By uniqueness of the adjoint, we therefore have $\mathrm{id}_V^* = \mathrm{id}_V$.

(e) For all $\mathbf{v} \in V$ and $\mathbf{w} \in W$ we have

$$\begin{aligned} \langle \mathbf{v}, (ST)^* \mathbf{w} \rangle &= \langle ST\mathbf{v}, \mathbf{w} \rangle \\ &= \langle T\mathbf{v}, S^*\mathbf{w} \rangle \\ &= \langle \mathbf{v}, T^*S^*\mathbf{w} \rangle \end{aligned}$$

By uniqueness of the adjoint, we therefore have $(ST)^* = T^*S^*$.

□

Exercise 8.1. Let V be a finite-dimensional inner product space and let $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of $T \in \mathrm{Hom}(V)$ if and only if $\overline{\lambda}$ is an eigenvalue of T^*. (*Hint*: Note that λ is an eigenvalue of T if and only if $T - \lambda I$ is not invertible, where I is the identity map on V.)

Proof. We will prove the contrapositive. Suppose λ is not an eigenvalue of T. Then $T - \lambda I$ is invertible, so there exists $S \in \mathrm{Hom}(V)$ such that

$$(T - \lambda I)S = S(T - \lambda I) = I.$$

Taking the adjoint of both sides and using Proposition 8.4 gives

$$\begin{aligned} ((T-\lambda I)S)^* = (S(T-\lambda I))^* &= I^* \\ S^*(T-\lambda I)^* = (T-\lambda I)^* S^* &= I \\ S^*(T^*-\overline{\lambda} I) = (T^*-\overline{\lambda} I)^* S^* &= I \end{aligned}$$

This shows that $T-\lambda I$ is invertible if and only if $T^*-\overline{\lambda}I$ is invertible. It follows that T has eigenvalue λ if and only if T^* has eigenvalue $\overline{\lambda}$. □

The next result shows the relationship between the kernel and image of a linear map and its adjoint.

Theorem 8.5 (Kernel and image of T^*). Let $T \in \mathrm{Hom}(V, W)$. Then

(a) $\ker T^* = (\mathrm{im}\, T)^\perp$;

(b) $\mathrm{im}\, T^* = (\ker T)^\perp$;

(c) $\ker T = (\mathrm{im}\, T^*)^\perp$;

(d) $\mathrm{im}\, T = (\ker T^*)^\perp$.

Proof. We first prove (a). Let $\mathbf{w} \in W$. Then

$$\begin{aligned} \mathbf{w} \in \ker T^* &\iff T^*\mathbf{w} = \mathbf{0} \\ &\iff \langle \mathbf{v}, T^*\mathbf{w}\rangle = 0 \text{ for all } \mathbf{v} \in V \\ &\iff \langle T\mathbf{v}, \mathbf{w}\rangle = 0 \text{ for all } \mathbf{v} \in V \\ &\iff \mathbf{w} \in (\mathrm{im}\, T)^\perp. \end{aligned}$$

Thus $\ker T^* = (\mathrm{im}\, T)^\perp$, proving (a). Taking the orthogonal complement of both sides of (a) (and using Theorem 7.46) gives (d). Replacing T with T^* in (a) (and using part (c) of Proposition 8.4) gives (c). Finally, replacing T with T^* in (d) gives (b). □

We now consider the matrix of T^*. Given $T \in \mathrm{Hom}(V, W)$, let $B = \{e_1, \ldots, e_n\}$ and $B' = \{f_1, \ldots, f_m\}$ be orthonormal (ordered) bases for V and W, respectively. Since $Te_i \in W$, we can expand Te_i in the B' basis as

$$Te_i = \sum_{j=1}^{m} \langle Te_i, f_j\rangle f_j,$$

where we have used Equation 7.10. Letting $A = [T]_{B,B'}$, we have

$$A_{ij} = \langle Te_i, f_j\rangle.$$

Similarly, since $T^* f_j \in V$, we can expand $T^* f_j$ in the B basis (again using Equation 7.10) as

$$T^* f_j = \sum_{i=1}^{n} \langle T^* f_j, e_i\rangle e_i.$$

Letting $A^* := [T^*]_{B',B}$, we have

$$\begin{aligned}(A^*)_{ji} &= \langle T^* f_j, e_i\rangle \\ &= \langle f_j, Te_i\rangle \\ &= \overline{\langle Te_i, f_j\rangle} \\ &= \overline{A}_{ij} \\ &= (\overline{A}^T)_{ji}.\end{aligned}$$

We have therefore proved the following theorem.

Theorem 8.6 (Matrix of T^*). Let $T \in \mathrm{Hom}(V, W)$. Then the matrix of T^* with respect to orthonormal bases for V and W is the conjugate transpose of the matrix of T.

Example 8.7. The conjugate transpose of the matrix $A = \begin{bmatrix} 2 & 3+4i & 7 \\ 6 & 5 & 8i \end{bmatrix}$ is the matrix

$$\overline{A}^T = \begin{bmatrix} 2 & 6 \\ 3-4i & 5 \\ 7 & -8i \end{bmatrix}.$$

Remark 8.8. (a) If V, W are real inner product spaces, then the conjugate transpose of a matrix is the same as the transpose.

(b) Note that the result $A^* = \overline{A}^T$ applies only when we are using orthonormal bases for V and W. Using arbitrary bases, the relation between A^* and A is not so simple.

8.2 Self-Adjoint Operators

We now focus on the case $V = W$ and consider linear operators on V.

Definition 8.9 (Self-adjoint operator). A linear operator $T \in \mathrm{Hom}(V)$ is called *self-adjoint* if $T = T^*$. In other words, $T \in \mathrm{Hom}(V)$ is self-adjoint if and only if

$$\langle T\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, T\mathbf{w}\rangle$$

for all $\mathbf{v}, \mathbf{w} \in V$.

It is clear that a linear operator T is self-adjoint if and only if the matrix of T with respect to any orthonormal basis for V satisfies $A = \overline{A}^T$.

Remark 8.10. In the physics literature, a self-adjoint operator is called *Hermitian*.

Exercise 8.2. Show that the linear operators on $\mathbb{C}^2$ whose standard matrices are given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are self-adjoint. These matrices are known as the *Pauli spin matrices*.

Proposition 8.11.

(a) If $S, T \in \operatorname{Hom}(V)$ are self-adjoint, then so is $S + T$.

(b) If $T \in \operatorname{Hom}(V)$ is self-adjoint and $c \in \mathbb{R}$, then (cT) is also self-adjoint.

Exercise 8.3. Prove Proposition 8.11.

Proof.

(a) If $S = S^*$ and $T = T^*$, then

$$\begin{aligned}(S+T)^* &= S^* + T^* \text{ (by part (a) of Proposition 8.4)}\\ &= S + T.\end{aligned}$$

(b) If $T = T^*$ and $c \in \mathbb{R}$, then

$$\begin{aligned}(cT)^* &= \bar{c}T^* \text{ (by part (b) of Proposition 8.4)}\\ &= cT^* \text{ (since } c \text{ is real)}\\ &= cT.\end{aligned}$$

□

The adjoint on $\operatorname{Hom}(V)$ plays a role analogous to complex conjugation on $\mathbb{C}$. Recall that $z \in \mathbb{C}$ is real if and only if $z = \bar{z}$. The next theorem shows that if $T = T^*$, then the eigenvalues are real.

Theorem 8.12 (Self-adjoint operators have real eigenvalues).

(a) The eigenvalues of a self-adjoint operator are real.

(b) Eigenvectors of a self-adjoint operator corresponding to distinct eigenvalues are orthogonal.

Proof. Suppose $T \in \operatorname{Hom}(V)$ is self-adjoint. Suppose $T\mathbf{v}_2 = \lambda_1\mathbf{v}_1$ and $T\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ and for some nonzero vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then

$$\lambda_1\langle \mathbf{v}_1, \mathbf{v}_2\rangle = \langle T\mathbf{v}_1, \mathbf{v}_2\rangle = \langle \mathbf{v}_1, T\mathbf{v}_2\rangle = \overline{\lambda}_2\langle \mathbf{v}_1, \mathbf{v}_2\rangle.$$

It follows that

$$(\lambda_1 - \overline{\lambda}_2)\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0. \tag{8.1}$$

If $\mathbf{v}_1 = \mathbf{v}_2$, then

$$(\lambda_1 - \overline{\lambda}_1)||\mathbf{v}_1||^2 = 0$$

and therefore $\lambda_1 = \overline{\lambda}_1$, proving (a). It follows that $\overline{\lambda}_2 = \lambda_2$ so Equation (8.1) becomes

$$(\lambda_1 - \lambda_2)\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0.$$

If $\lambda_1 \neq \lambda_2$, we must have $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$, proving (b). □

The next several theorems illustrate some differences between real and complex inner product spaces. [Merge with Theorem 7.10 here?]

Theorem 8.13. Let V be a complex inner product space and let $T \in \operatorname{Hom}(V)$. If $\langle T\mathbf{v}, \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in V$, then $T = 0$ (the zero map).

Proof. We have

$$\begin{aligned}\langle T\mathbf{u}, \mathbf{w}\rangle = {} & \frac{\langle T(\mathbf{u}+\mathbf{w}), \mathbf{u}+\mathbf{w}\rangle - \langle T(\mathbf{u}-\mathbf{w}), \mathbf{u}-\mathbf{w}\rangle}{4} \\ & + i\frac{\langle T(\mathbf{u}+i\mathbf{w}), \mathbf{u}+i\mathbf{w}\rangle - \langle T(\mathbf{u}-i\mathbf{w}), \mathbf{u}-i\mathbf{w}\rangle}{4}\end{aligned}$$

for all $\mathbf{u}, \mathbf{w}$, as can be verified by computing the right side. Note that each term on the right hand side is of the form $\langle T\mathbf{v}, \mathbf{v}\rangle$ for appropriate $\mathbf{v} \in V$. Thus our hypothesis implies that $\langle T\mathbf{u}, \mathbf{w}\rangle = 0$ for all $\mathbf{u}, \mathbf{w} \in V$. In particular, this implies that $\langle T\mathbf{u}, T\mathbf{u}\rangle = 0$ (by taking $\mathbf{w} = T\mathbf{u}$), and therefore $T\mathbf{u} = \mathbf{0}$ for all $\mathbf{u} \in V$, which shows that T is the zero map. □

Note that the previous theorem is false for real inner product spaces. For example, take $V = \mathbb{R}^2$ and let $T \in \operatorname{Hom}(\mathbb{R}^2)$ be a counterclockwise rotation by 90° about the origin; i.e., $T(x, y) = (-y, x)$. Then $T\mathbf{v}$ is orthogonal to $\mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^2$, even though $T \neq 0$.

Theorem 8.14. Let V be a complex inner product space and $T \in \operatorname{Hom}(V)$. Then T is self-adjoint if and only if

$$\langle T\mathbf{v}, \mathbf{v}\rangle \in \mathbb{R}$$

for every $\mathbf{v} \in V$.

Proof. For $\mathbf{v} \in V$, we have

$$\langle T\mathbf{v}, \mathbf{v}\rangle - \overline{\langle T\mathbf{v}, \mathbf{v}\rangle} = \langle T\mathbf{v}, \mathbf{v}\rangle - \langle \mathbf{v}, T\mathbf{v}\rangle = \langle T\mathbf{v}, \mathbf{v}\rangle - \langle T^*\mathbf{v}, \mathbf{v}\rangle = \langle (T - T^*)\mathbf{v}, \mathbf{v}\rangle.$$

If $\langle T\mathbf{v}, \mathbf{v}\rangle \in \mathbb{R}$ for every $\mathbf{v} \in V$, then the left hand side of this equation is 0, so $\langle (T - T^*)\mathbf{v}, \mathbf{v}\rangle = 0$ for every $\mathbf{v} \in V$. This implies $T - T^* = 0$ (by Theorem 8.13) and therefore $T = T^*$, so T is self-adjoint.

Conversely, if T is self-adjoint, then the right hand side equals 0, so $\langle T\mathbf{v}, \mathbf{v}\rangle = \overline{\langle T\mathbf{v}, \mathbf{v}\rangle}$ for every $\mathbf{v} \in V$, and therefore $\langle T\mathbf{v}, \mathbf{v}\rangle \in \mathbb{R}$ for every $\mathbf{v} \in V$. □

On a real inner product space V, a nonzero operator T might satisfy $\langle T\mathbf{v}, \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in V$ (again, this is true for rotation by 90° about the origin in $\mathbb{R}^2$). However, the next result shows that this cannot happen for a self-adjoint operator.

Theorem 8.15. If T is a self-adjoint operator on V such that

$$\langle T\mathbf{v}, \mathbf{v}\rangle = 0$$

for all $\mathbf{v} \in V$, then $T = 0$.

Proof. We already proved this in Theorem 8.13 for all linear operators on a complex inner product space, so in particular it holds for self-adjoint operators. Assume now that V is a real inner product space. If $\mathbf{u}, \mathbf{w} \in V$, then

$$\langle T\mathbf{u}, \mathbf{w}\rangle = \frac{\langle T(\mathbf{u}+\mathbf{w}), \mathbf{u}+\mathbf{w}\rangle - \langle T(\mathbf{u}-\mathbf{w}), \mathbf{u}-\mathbf{w}\rangle}{4},$$

which is proved by computing the right hand side using the equation

$$\langle T\mathbf{w}, \mathbf{u}\rangle = \langle \mathbf{w}, T\mathbf{u}\rangle = \langle T\mathbf{u}, \mathbf{w}\rangle, \tag{8.2}$$

where the first equality holds because T is self-adjoint and the second equality holds because we are working in a real inner product space.

Each term on the right hand side of Equation (8.2) is of the form $\langle T\mathbf{v}, \mathbf{v}\rangle$ for appropriate $\mathbf{v}$. Thus, by our hypothesis $\langle T\mathbf{u}, \mathbf{w}\rangle = 0$ for all $\mathbf{u}, \mathbf{w} \in V$. As before, by taking $\mathbf{w} = T\mathbf{u}$, this in particular implies $\langle T\mathbf{u}, T\mathbf{u}\rangle = 0$, and therefore $T\mathbf{u} = \mathbf{0}$ for all $\mathbf{u} \in V$, so T is the zero map. □

8.3 Normal Operators

Definition 8.16 (Normal operators). Let V be a finite-dimensional inner product space. A linear operator $T \in \mathrm{Hom}(V)$ is said to be normal if it commutes with its adjoint; that is, if

$$TT^* = T^*T.$$

Example 8.17. Every self-adjoint operator is normal, since if T is self-adjoint then $T = T^*$.

Exercise 8.4. Show that the linear operator T on $\mathbb{R}^2$ whose matrix is $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ is normal but not self-adjoint.

Theorem 8.18. A linear operator $T \in \mathrm{Hom}(V)$ is normal if and only if

$$||T\mathbf{v}|| = ||T^*\mathbf{v}||$$

for all $\mathbf{v} \in V$.

Proof. Let $T \in Hom(V)$. Then

$$\begin{aligned} T \text{ normal} &\iff T^*T - TT^* = 0 \text{ (the zero map)} \\ &\iff \langle (T^*T - TT^*)\mathbf{v}, \mathbf{v}\rangle = 0 \text{ for all } \mathbf{v} \in V \\ &\iff \langle T^*T\mathbf{v}, \mathbf{v}\rangle = \langle TT^*\mathbf{v}, \mathbf{v}\rangle \text{ for all } \mathbf{v} \in V \\ &\iff \langle T^*T\mathbf{v}, \mathbf{v}\rangle = \langle TT^*\mathbf{v}, \mathbf{v}\rangle \text{ for all } \mathbf{v} \in V \\ &\iff \langle T\mathbf{v}, T\mathbf{v}\rangle = \langle T^*\mathbf{v}, T^*\mathbf{v}\rangle \text{ for all } \mathbf{v} \in V \\ &\iff ||T\mathbf{v}||^2 = ||T^*\mathbf{v}||^2 \text{ for all } \mathbf{v} \in V. \end{aligned}$$

where we used Theorem 8.15 to establish the second equivalence (note that the operator $T^*T - TT^*$ is self-adjoint). □

Recall that Exercise 8.1 showed that the set of eigenvalues of T^* is equal to the set of complex conjugates of the eigenvalues of T. The exercise said nothing about eigenvectors, and indeed an operator and its adjoint may have different eigenvectors. However, the next corollary implies that a normal operator and its adjoint have the same eigenvectors.

Corollary 8.19. If $T \in \mathrm{Hom}(V)$ is normal and $\mathbf{v} \in V$ is an eigenvector of T with eigenvalue λ, then $\mathbf{v}$ is also an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

Proof. Suppose T is normal. Then

$$\begin{aligned}(T-\lambda I)(T-\lambda I)^* &= (T-\lambda I)(T^*-\overline{\lambda} I)\\ &= TT^* - \lambda I T^* - T\overline{\lambda} I + \lambda I \overline{\lambda} I\\ &= T^*T - T^*\lambda I - \overline{\lambda} I T + \overline{\lambda} I \lambda I\\ &= (T-\lambda I)^*(T-\lambda I)\end{aligned}$$

which shows that $T-\lambda I$ is normal as well. If $\mathbf{v}$ is an eigenvalue of T, then by Theorem 8.18

$$0 = ||(T-\lambda I)\mathbf{v}|| = ||(T-\lambda I)^*\mathbf{v}|| = ||(T^*-\overline{\lambda} I)\mathbf{v}||$$

which shows that $\mathbf{v}$ is an eigenvalue of T^* with eigenvalue $\overline{\lambda}$. □

We saw in part (b) of Theorem 8.12 that eigenvectors of self-adjoint operators corresponding to distinct eigenvalues are orthogonal. The same is true for normal operators, as we now show.

Corollary 8.20. If $T \in \text{Hom}(V)$ is normal, then the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

Proof. Assume $T \in \text{Hom}(V)$ is normal and assume $T\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $T\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ for distinct eigenvalues λ_1, λ_2. By Corollary 8.19, we have $T * \mathbf{v}_1 = \overline{\lambda_1}\mathbf{v}_1$ and $T^*\mathbf{v}_2 = \overline{\lambda_2}\mathbf{v}_2$. It follows that

$$\begin{aligned}(\lambda_1-\lambda_2)\langle \mathbf{v}_1, \mathbf{v}_2\rangle &= \lambda_1\langle \mathbf{v}_1, \mathbf{v}_2\rangle - \lambda_2\langle \mathbf{v}_1, \mathbf{v}_2\rangle\\ &= \langle \lambda_1\mathbf{v}_1, \mathbf{v}_2\rangle - \langle \mathbf{v}_1, \overline{\lambda}_2\mathbf{v}_2\rangle\\ &= \langle T\mathbf{v}_1, \mathbf{v}_2\rangle - \langle \mathbf{v}_1, T^*\mathbf{v}_2\rangle\\ &= \langle T\mathbf{v}_1, \mathbf{v}_2\rangle - \langle T\mathbf{v}_1, \mathbf{v}_2\rangle\\ &= 0.\end{aligned}$$

By assumption, $\lambda_1 \neq \lambda_2$, so we must have $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$. Thus, $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. □

Problems

[Add some problems.]

8.4 The Spectral Theorem

We saw in section 6.3 that a linear operator on V has a diagonal matrix with respect to a basis if and only if the basis consists of eigenvectors of V.

The nicest operators on V are those for which there is an *orthonormal* basis for V with respect to which the operator has a diagonal matrix. These are precisely the operators $T \in \text{Hom}(V)$ such that there is an orthonormal basis consisting of eigenvectors of T. Our goal in this section is to prove the Spectral Theorem, which characterizes these operators as the normal operators when $\mathbb{F} = \mathbb{C}$ and as the self-adjoint operators when $\mathbb{F} = \mathbb{R}$. The Spectral Theorem is probably the most useful tool in the study of operators on inner product spaces.

Because the conclusion of the Spectral Theorem depends on $\mathbb{F}$, we will break the Spectral Theorem into two pieces, called the Complex Spectral Theorem and the Real Spectral Theorem. As is often the case in linear algebra, complex vector spaces are easier to deal with than real vector spaces. Thus we present the Complex Spectral Theorem first.

8.4.1 The Complex Spectral Theorem

The key part of the Complex Spectral Theorem (**??**) states that if $\mathbb{F} = \mathbb{C}$ and $T \in \mathrm{Hom}(V)$ is normal, then T has a diagonal matrix with respect to some orthonormal basis for V. The next example illustrates this conclusion.

Example 8.21. Consider the normal operator $T \in \mathrm{Hom}(\mathbb{C}^2)$ whose standard matrix is

$$\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}.$$

The reader can verify that

$$\left\{ \begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

is an (ordered) orthonormal basis for $\mathbb{C}^2$ consisting of eigenvectors of T, and with respect to this basis the matrix of T is the diagonal matrix

$$\begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}.$$

Theorem 8.22 (Complex Spectral Theorem). Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathrm{Hom}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof. The equivalence of (b) and (c) follows from Theorem 6.9. Thus we only need to prove equivalence of (a) and (c).

First suppose (c) holds. The matrix of T^* with respect to the same basis is the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix. Since any two diagonal matrices commute, T commutes with T^*, so T is normal. Thus, (c) implies (a).

[To show (a) implies (c), we need Schur's Theorem (see Axler).] □

8.4.2 The Real Spectral Theorem

Theorem 8.23 (Real Spectral Theorem). Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathrm{Hom}(V)$. Then the following are equivalent:

(a) T is self-adjoint;

(b) V has an orthonormal basis consisting of eigenvectors of T;

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof. [Technical, lots of preliminaries.] □

Example 8.24. Consider the self-adjoint operator T on $\mathbb{R}^3$ whose standard matrix is

$$\begin{bmatrix} 14 & -13 & 8 \\ -13 & 14 & 8 \\ 8 & 8 & -7 \end{bmatrix}.$$

The reader may verify that

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \right\}$$

is an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of T, and with respect to this basis, the matrix of T is the diagonal matrix

$$\begin{bmatrix} 27 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -15 \end{bmatrix}.$$

If $\mathbb{F} = \mathbb{C}$, then the Complex Spectral Theorem (Theorem 8.22) gives a complete description of the normal operators on V. A complete description of the self-adjoint operators on V then easily follows (since they are the normal operators on V whose eigenvalues are all real).

If $\mathbb{F} = \mathbb{R}$, then the Real Spectral Theorem (Theorem 8.23) gives a complete description of the self-adjoint operators on V. [For a complete description of the normal operators on V, see Ch 9 of Axler.]

8.5 Positive Operators and Isometries

[Add section. Merge previous content on isometries.]

8.6 Polar Decomposition and Singular Value Decomposition

[Do I want to add a section on this?]

8.7 Orthogonal Matrices

[Need to merge with previous section.] Consider the following orthonormal bases for $\mathbb{R}^3$:

$$B = \left\{ \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right\},$$
$$B' = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

The transition matrix $P_{B \to B'}$ is obtained by row-reducing the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1/\sqrt{3} & -2/\sqrt{6} & 0 & 2/3 & 2/3 & 1/3 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} & -2/3 & 1/3 & 2/3 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} & 1/3 & -2/3 & 2/3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3\sqrt{3}} & \frac{1}{3\sqrt{3}} & \frac{5}{3\sqrt{3}} \\ 0 & 1 & 0 & -\frac{5}{3\sqrt{6}} & -\frac{5}{3\sqrt{6}} & \frac{\sqrt{2}}{3\sqrt{3}} \\ 0 & 0 & 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{array}\right]$$

and therefore

$$P_{B\to B'} = \begin{bmatrix} \frac{1}{3\sqrt{3}} & \frac{1}{3\sqrt{3}} & \frac{5}{3\sqrt{3}} \\ -\frac{5}{3\sqrt{6}} & -\frac{5}{3\sqrt{6}} & \frac{\sqrt{2}}{3\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \tag{8.3}$$

Similarly, one can compute $P_{B'\to B}$, which is given by

$$P_{B'\to B} = \begin{bmatrix} \frac{1}{3\sqrt{3}} & -\frac{5}{3\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{5}{3\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{5}{3\sqrt{3}} & \frac{2}{3\sqrt{3}} & 0 \end{bmatrix} \tag{8.4}$$

We have seen before that $P_{B'\to B} = P^{-1}_{B\to B'}$. However, note from the matrices above that $P_{B'\to B} = P^{T}_{B\to B'}$.

Proposition 8.25. If B and B' are orthonormal bases for $\mathbb{R}^n$, then $P_{B'\to B} = P^{T}_{B\to B'}$.

Proof. Let $B = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ and $B' = \{\mathbf{u}'_1, \cdots, \mathbf{u}'_n\}$ be two orthonormal bases for $\mathbb{R}^n$. Recall that the ith column of the transition matrix $P_{B'\to B}$ is given by $[\mathbf{u}'_i]_B$. Writing $\mathbf{u}'_i$ in the B-basis and using the orthonormality of B,

$$\begin{aligned} \mathbf{u}'_i &= c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n \\ \mathbf{u}'_i &= (\mathbf{u}'_i \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{u}'_i \cdot \mathbf{u}_n)\mathbf{u}_n \end{aligned}$$

so we find

$$[\mathbf{u}'_i]_B = \begin{bmatrix} \mathbf{u}'_i \cdot \mathbf{u}_1 \\ \mathbf{u}'_i \cdot \mathbf{u}_2 \\ \vdots \\ \mathbf{u}'_i \cdot \mathbf{u}_n \end{bmatrix}.$$

Thus, $(P_{B'\to B})_{ij} = \mathbf{u}'_i \cdot \mathbf{u}_j$. Since $P^{-1}_{B'\to B} = P_{B\to B'}$, we find

$$\begin{aligned} (P^{-1}_{B'\to B})_{ij} &= \mathbf{u}_i \cdot \mathbf{u}'_j \\ &= \mathbf{u}'_j \cdot \mathbf{u}_i \\ &= (P_{B'\to B})_{ji} \\ &= (P^{T}_{B'\to B})_{ij}. \end{aligned}$$

□

Definition 8.26. An $n \times n$ matrix A is said to be an **orthogonal matrix** if A is invertible and $A^{-1} = A^T$. This is equivalent to the condition that A satisfies $AA^T = A^TA = I$.

We have just proven above the following proposition.

Proposition 8.27. The transition matrix P between orthonormal bases is an orthogonal matrix.

Note from Equations (8.3) and (8.4) above that the rows and columns each form orthonormal bases for $\mathbb{R}^n$.

Theorem 8.28. The following are equivalent for an $n \times n$ matrix A.

(a) A is orthogonal.

(b) The row vectors of A form an orthonormal basis for $\mathbb{R}^n$ with respect to the Euclidean inner product.

(c) The column vectors of A form an orthonormal basis for $\mathbb{R}^n$ with respect to the Euclidean inner product.

Proof. Suppose A is orthogonal. Then $AA^T = I$. Letting $\mathbf{r}_i$ denote the ith row of A and $\mathbf{c}_i$ denote the ith row of A^T, we have $(AA^T)_{ij} = \mathbf{r}_i \cdot \mathbf{c}_j = \mathbf{r}_i \cdot \mathbf{r}_j = \delta_{ij}$, where we have used the fact that the jth column of A^T is equal to the jth row of A. Thus, the row vectors of an orthogonal matrix form an orthonormal set of n vectors in $\mathbb{R}^n$, and therefore an orthonormal basis for $\mathbb{R}^n$. This shows (a) implies (b). Now let $\mathbf{r}_i$ denote the ith row of A^T and $\mathbf{c}_i$ the ith row of A. Since $A^TA = I$, $(A^TA)_{ij} = \mathbf{r}_i \cdot \mathbf{c}_j = \mathbf{c}_i \cdot \mathbf{c}_j = \delta_{ij}$. Thus, (a) implies (c). These equations in reverse direction shows that (b) and (c) imply (a). This also shows that (b) $\iff$ (c), since we have seen that (b) implies (a) and (a) implies (c) and vice versa. $\square$

Exercise 8.5. Show that the matrix

$$A = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$$

is orthogonal.

We have seen that, in general, multiplication of a vector $\mathbf{x} \in \mathbb{R}^n$ by an $n \times n$ matrix A changes both the length and direction of A. One reason why orthogonal matrices are important is that the preserve the lengths of vectors as well as the angle between any two vectors in $\mathbb{R}^n$.

Theorem 8.29. If A is an $n \times n$ matrix, then the following are equivalent.

(a) A is orthogonal.

(b) $||A\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$

Proof. Since

$$\begin{aligned} A\mathbf{x} \cdot A\mathbf{y} &= (A\mathbf{x})^T A\mathbf{y} \\ &= \mathbf{x}^T A^T A\mathbf{y} \end{aligned}$$

$A\mathbf{x} \cdot A\mathbf{y}$ if and only if $A^TA = I$, that is, if and only if A is orthogonal. This shows (a) $\iff$ (c). By taking $\mathbf{y} = \mathbf{x}$, the same computation shows that (a) $\iff$ (b). We then have also that (b) $\iff$ (c) since (b) implies (a) and (a) implies (c) and vice versa. $\square$

The next theorem shows that the set of orthogonal matrices forms a *group* under matrix multiplication.

Theorem 8.30.

(a) If A is an orthogonal matrix, then $\det(A) = \pm 1$.

(b) If A is an orthogonal matrix, then A^{-1} is also orthogonal.

(c) If A and B are orthogonal matrices, then so is AB.

Proof. (a) Taking the determinant of both sides of $AA^T = I$ gives

$$\begin{aligned} \det(AA^T) &= \det(I) \\ \det(A)\det(A^T) &= 1 \\ \det(A)^2 &= 1 \end{aligned}$$

and therefore $\det(A) = \pm 1$.

(b) $A^{-1}(A^{-1})^T = A^T(A^T)^T = A^TA = I$, so A^{-1} is invertible.

(c) $(AB)(AB)^T = A(BB^T)A^T = AA^T = I$, which shows AB is orthogonal.

□

Note that the converse of Theorem 8.30 is *false*. For instance, the matrix $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ has determinant 1, but is not orthogonal.

Example 8.31. A rotation in three-dimensions can be represented by an orthogonal matrix. For instance, a counter-clockwise rotation by an angle θ about the z-axis is represented by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8.7.1 Orthogonal Diagonalization

Definition 8.32. (a) Two square matrices A and B are *orthogonally similar* if there exists and orthogonal matrix P such that $B = P^TAP$.

(b) If A is orthogonally similar to a diagonal matrix, then we say that A is **orthogonally diagonalizable**.

Theorem 8.33. If A is an $n \times n$ matrix, then the following are equivalent.

(a) A is orthogonally diagonalizable.

(b) A has an orthonormal set of n eigenvectors.

(c) A is symmetric.

Proof. Suppose A is orthogonally diagonalizable. Then there exists an orthogonal matrix P such that P^TAP is diagonal. The n column vectors of P are eigenvectors of A. Since P is orthogonal, these column vectors are orthonormal, so A has n orthonormal eigenvectors. Thus, (a) implies (b).

Now suppose A is an $n \times n$ matrix with an orthonormal set of n eigenvectors. The matrix P whose column vectors are these eigenvectors diagonalizes A. Since these eigenvectors are orthonormal, P is orthogonal and thus orthogonally diagonalizes A. This shows (b) implies (a).

If A is orthogonally diagonalizable, then there exists an orthogonal matrix P and a diagonal matrix D such that $D = P^TAP$. Multiplying both sides of this equation on the left by P and on the right by P^T gives $A = PDP^T$. Taking the transpose gives $A^T = (PDP^T)^T = (P^T)^TD^TP^T = PDP^T = A$ (where we have used the fact that $D^T = D$ since D is diagonal). Thus, A is symmetric, proving (a) implies (c).

We will now prove by induction on n that a symmetric $n \times n$ matrix A is orthogonally diagonalizable. [Finish.] □

Theorem 8.34. If A is a symmetric matrix, then

(a) The eigenvalues of A are real.

(b) Eigenvectors from different eigenspaces are orthogonal.

Proof. [Finish.] □

To orthogonally diagonalize a symmetric $n \times n$ matrix A:

1. Find a basis for each eigenspace of A.
2. Apply the Gram-Schmidt algorithm to each of these bases to obtain an orthonormal basis for each eigenspace.
3. Form the matrix P whose columns are the basis vectors from each of these orthonormal bases. Then $D = P^T AP$ will be diagonal, and the entries on the main diagonal will be the eigenvalues of A. [65]

Exercise 8.6. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

[65] Recall that eigenvectors from different eigenspaces are orthogonal, and applying the Gram-Schmidt algorithm ensures that the eigenvectors within the *same* eigenspace are orthonormal. It follows that the *entire* set of eigenvectors obtained by this procedure will be orthonormal.

9 Quantum Mechanics

[Edit, rewrite, change notation, add exercises.] In the late 19th century, the study of light emitted by gas discharge tubes led to observations which contradict the predictions of classical mechanics. This led to the formulation of the laws of quantum mechanics. In this section we review some of the experiments that lead to the development of quantum mechanics

9.1 Blackbody Radiation

A *blackbody* is an object that is a perfect absorber of radiation. In the ideal case, it absorbs all light that falls on it. While such an object does not reflect any light, if we heat it up it will radiate light. The *blackbody spectrum* is a plot, at fixed temperature, of the energy density of light emitted at each frequency.

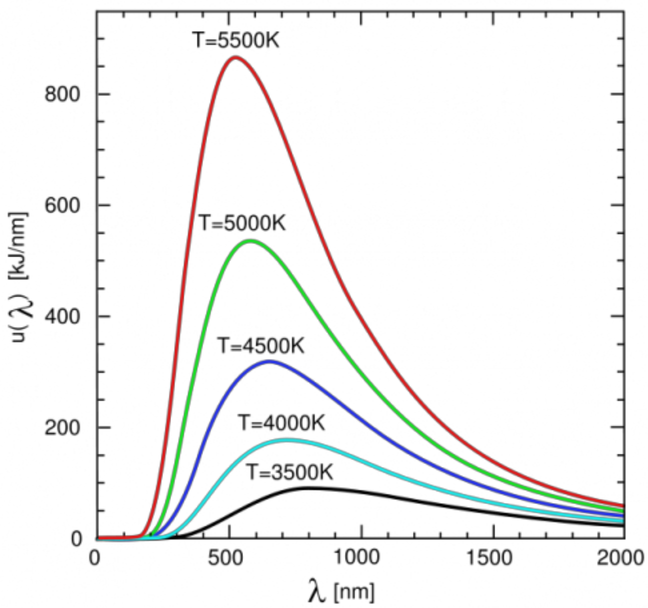


Figure 18: Blackbody spectrum.

We see that, as the temperature is increased, more light is emitted at shorter wavelengths. These plots were obtained experimentally, and it was a challenge in the late 19th century to reproduce these plots theoretically.

Using classical thermodynamics, one obtains the *Rayleigh-Jeans* formula for the energy density

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3}kT \tag{9.1}$$

where ν is the frequency of the radiation, c is the speed of light in vacuum, and k is Boltzmann's constant.

At low frequencies (long wavelengths) this formula fits the data well, but predicts an infinite energy density at high frequencies, which is clearly in conflict with observation. This became known as the *ultraviolet catastrophe*.

An idealized blackbody can be modeled as a metallic cavity with a small hole through which radiation can escape. From this model, the blackbody spectrum was reproduced theoretically in 1900 by Max Planck, who made the following radical assumption: the exchange of energy between the wall of the cavity and the electromagnetic radiation can only occur in discrete amounts, given by

$$E = nh\nu, \ n = 0, 1, 2, \ldots \tag{9.2}$$

where the constant $h = 6.62 \times 10^{-34} J - s$ is called *Planck's constant*. With this assumption, one obtain's *Planck's formula* for the energy density of the blackbody radiation

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} \tag{9.3}$$

For $h\nu << kT$, Planck's formula reduces to the Rayleigh-Jeans law:

$$\begin{aligned} u(\nu, T) &= \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} \\ &\cong \frac{8\pi\nu^2}{c^3} \frac{h\nu}{1 + h\nu/kT - 1} \\ &= \frac{8\pi\nu^2}{c^3} kT \end{aligned} \tag{9.4}$$

while for $h\nu >> kT$

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} h\nu \, e^{-h\nu/kT} \tag{9.5}$$

avoiding the ultraviolet catastrophe.

This *quantization* of the energy of the radiation is the origin of the name *quantum mechanics*.

9.2 Photoelectric Effect

When light shines on a metal surface, the surface emits electrons. For example, you can start a current in a circuit just by shining light on a metal plate. This happens because the energy carried by the incident light is transferred to the electrons, knocking them out from the surface. This is called the *photoelectric effect*, and the emitted electrons are called *photoelectrons*.

Using the classical wave theory of light, increasing the intensity of incident light should lead to greater kinetic energy of the ejected electrons. Furthermore, sufficiently dim light should result in a lag time between the initial shining of light and the subsequent emission of electrons.

However, experiments in the late 19th century did not agree with either of these predictions of the theory. Instead, it was observed that

1. When light strikes a metal surface, a current flows *instantaneously*, even for very weak light.

2. There is a threshold frequency, ν_0, below which there is no current, no matter how intense the incident light.

3. At fixed frequency, ν, the current is directly proportional to the intensity of the light, but the maximum kinetic energy of the emitted electrons is *constant*, given by

$$K_{\max} = h(\nu - \nu_0) \tag{9.6}$$

   where h is Planck's constant.

These observations were explained in 1905 by Albert Einstein, who proposed that light consisted of particles called *photons*. Each photon carries energy

$$E = h\nu \tag{9.7}$$

and momentum

$$p = h\nu. \tag{9.8}$$

The photon picture explains 1-3 as follows:

1. Each photon is absorbed by a single electron, ejecting it from the metal instantaneously.
2. Photons below the threshold frequency are not energetic enough to emit photoelectrons.
3. As one increases the intensity of the light, there are more ejected electrons and therefore a higher current, but their maximum kinetic energy does not increase.

9.3 The Bohr Theory of the Hydrogen Atom

By the early 20th century, it was well-known that light emitted by isolated atoms takes the form of a discrete series of lines called *spectral lines*. These lines occur at specific frequencies for each type of atom. It was also observed that atoms absorb light at specific frequencies as well. The

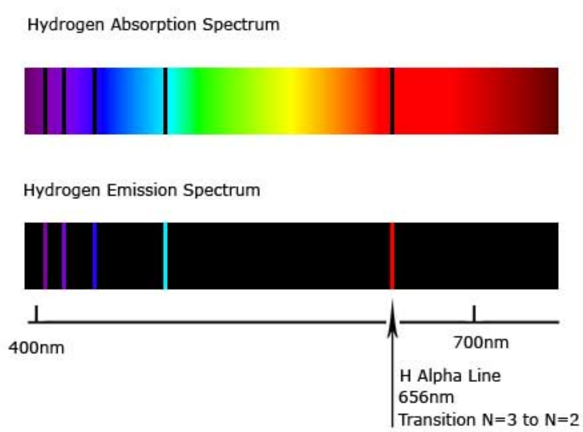


Figure 19: Absorption and emission spectra of hydrogen.

line spectrum of hydrogen is given by Rydberg's formula

$$\lambda^{-1} = R_H \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right), \tag{9.9}$$

where $R_H \simeq 1.098 \times 10^7 m^{-1}$ is the Rydberg constant and n_1, n_2 are integers such that $1 \leq n_1 < n_2$.

This formula was reproduced theoretically be Niels Bohr in 1913. In his theory, he made two assumptions about the behavior of the electron in the hydrogen atom:

1. The angular momentum of the electron orbiting the nucleus may only take the values
$$L \equiv mvr = h\hbar, n = 1, 2, \ldots \tag{9.10}$$
where m is the mass of the electron and $\hbar = h/2\pi$.

2. Electrons only radiate during transitions between orbits of fixed energy. A transition from an orbit of energy E_i to one of E_f is accompanied by the emission of a photon of energy
$$h\nu = E_i - E_f \tag{9.11}$$

Bohr calculated the spectrum as follows. The Coulomb force between the positively charged nucleus and the negatively charged electron holds the electron in orbit. Assuming the orbit is a circle of radius r, Newton's second law gives
$$\begin{aligned} F &= ma_c \\ \frac{ke^2}{r^2} &= \frac{mv^2}{r}. \end{aligned} \tag{9.12}$$

Multiplying both sides by $r^3 m$ and using the assumption that $rmv = n\hbar$, one finds the radius of the orbit takes the values
$$r_n = \frac{n^2\hbar^2}{mke^2}, \; n = 1, 2, \ldots \tag{9.13}$$
The smallest radius ($n = 1$) is called the *Bohr radius*, and is given by
$$a_0 \equiv r_1 \cong 0.529 \times 10^{-10} m. \tag{9.14}$$

The allowed velocities are then given by
$$\begin{aligned} v_n &= \frac{n\hbar}{mr} \\ &= \frac{n\hbar}{m}\left(\frac{mke^2}{n^2\hbar^2}\right) \\ &= \frac{ke^2}{n\hbar}, \; n = 1, 2, \ldots \end{aligned} \tag{9.15}$$

Using this, we can compute the energy levels of the electron in the hydrogen atom
$$\begin{aligned} E_n &= K + U \\ &= \frac{1}{2}mv_n^2 - \frac{ke^2}{r_n} \\ &= \frac{mk^2e^4}{2n^2\hbar^2} - \frac{mk^2e^4}{n^2\hbar^2} \\ &= -\frac{mk^2e^4}{2n^2\hbar^2} \end{aligned} \tag{9.16}$$

The state of lowest energy is called the *ground state* and is given by taking $n = 1$. In units of electron volts ($1eV \equiv 1.6 \times 10^{-19} J$), this is
$$\begin{aligned} E_1 &= -\frac{mk^2e^4}{2\hbar^2} \\ &\cong -13.6eV \end{aligned} \tag{9.17}$$

Using Bohr's assumption that $\hbar\nu = E_i - E_f$, we obtain the Rydberg formula

$$\frac{\hbar c}{\lambda} = -\frac{mk^2e^4}{2\hbar^2}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) \tag{9.18}$$

where the Rydberg constant is

$$R_H = \frac{mk^2e^4}{2c\hbar^3}. \tag{9.19}$$

While Bohr's model was a major step toward understanding the quantum theory of atoms, it still has major shortcomings, which we will see later.

9.4 de Broglie's Hypothesis

In his Ph.D. thesis in 1923, Louis de Broglie proposed that Einstein's relation for photons

$$\lambda = \frac{h}{p} \tag{9.20}$$

should also hold for material particles. The wavelength λ associated with a particle is called the *de Broglie wavelength*. This relation was later confirmed in electron diffraction experiments.

To get an idea of the size of this wavelength, consider a 60 kg man running at 5 m/s vs an electron moving at 5 m/s:

$$\begin{aligned}\lambda_{\text{man}} &= \frac{6.62 \times 10^{-34}}{60 \cdot 5} \cong 2.2 \times 10^{-36} m \\ \lambda_{\text{electron}} &= \frac{6.62 \times 10^{-34}}{9.11 \times 10^{-31} \cdot 5} \cong 1.5 \times 10^{-5} m\end{aligned}$$

Roughly, a system behaves quantum mechanically when its size is comparable to its de Broglie wavelength.

9.5 The Stern-Gerlach Experiment

9.5.1 Experimental Setup

An experiment that demonstrates very clearly the radical departure that quantum mechanics takes from the concepts of classical mechanics was carried out in Frankfurt by O. Stern and W. Gerlach in 1922. Their experimental setup is shown in Figure 9.5.1 below.

Silver atoms are heated in an oven with a small hole through which some can escape. They are then passed through a collimating slit before passing through a region of inhomogeneous magnetic field on their way to a collection screen. Each silver atom consists of a nucleus and 47 electrons. The first 46 form a spherical electron cloud with no net angular momentum. The total angular momentum of the atom is due to the spin of the 47th (5s) electron. Each silver atom therefore has a magnetic moment proportional to the spin

$$\vec{\mu} \propto \vec{S} \tag{9.21}$$

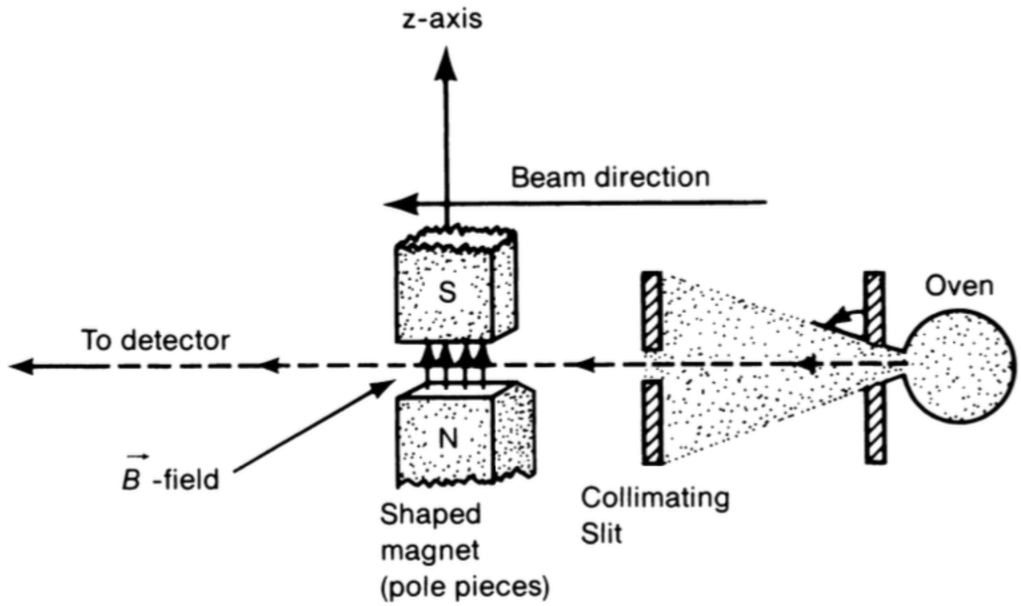


Figure 20: The Stern-Gerlach experiment.

As the atoms pass through the magnetic field, they experience a force given by

$$F_z = \frac{\partial}{\partial z}(\vec{\mu} \cdot \vec{B}) \simeq \mu_z \frac{\partial B}{\partial z} \tag{9.22}$$

so those atoms with $\mu_z > 0$ $(S_z < 0)$ are deflected downward, while those with $\mu_z < 0$ $(S_z > 0)$ are deflected upward. The beam is therefore split according to the value of μ_z and in this sense the SG apparatus measures S_z (up to the proportionality constant).

Since atoms in the oven are randomly oriented with no preferred direction for the orientation of $\vec{\mu}$, one expects all values of μ_z to be realized between $|\vec{\mu}|$ and $-|\vec{\mu}|$, as in Figure 9.5.1(a). Instead, what is actually observed is that the SG apparatus splits the beam into *two distinct components*, as shown in Figure 9.5.1(b).

To the extent that $\vec{\mu}$ can be identified within a proportionality factor with $\vec{S}$, *only 2 possible values* of S_z are observed to be possible:

$$S_z = \pm\frac{\hbar}{2}. \tag{9.23}$$

Our first takeaway from the Stern-Gerlach experiment is then

Lesson 1: Electron spin is quantized.

Of course, there was nothing special about the z-direction. We would have obtained the same results by orienting the SG apparatus along some other direction, say the x-direction.

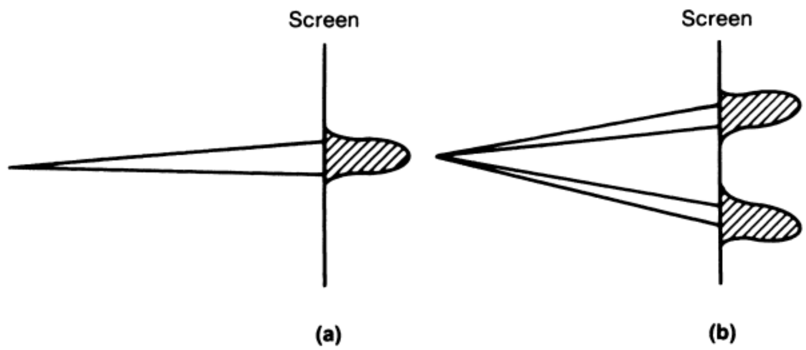


Figure 21: Beams from the SG apparatus; (a) is expected from classical physics, while (b) is actually observed.

9.5.2 Sequential Stern-Gerlach Experiments

Consider a beam that goes through two more SG apparatuses in sequence. In the experiment in Figure 9.5.2 below, the beam is passed through an SG apparatus oriented in the z-direction (which we denote SG$\hat{z}$), and the S_z- beam is blocked while the S_z+ beam is then passed through a second SG$\hat{z}$ apparatus. It is observed that an S_z+ beam comes out of the second apparatus, while

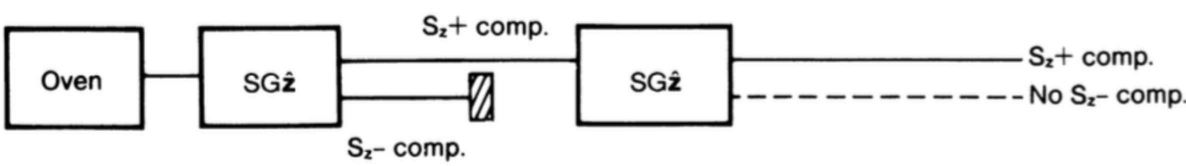


Figure 22: Sequential SG$\hat{z}$ experiments.

there is no S_z-. This is unsurprising - if all the electron spins are oriented up, they will remain so as long as no external magnetic field rotates them between the first and second SG apparatus.

A more interesting arrangement is shown in Figure 9.5.2 below.

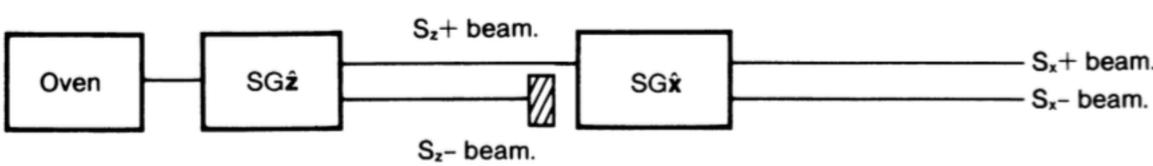


Figure 23: An SG$\hat{z}$ apparatus followed by SG$\hat{x}$.

In this setup, the original beam is passed through an SG$\hat{z}$ apparatus and again the S_z- beam is blocked. The S_z+ beam is then passed through an SG$\hat{x}$ apparatus. It is observed that two beams come out of the second apparatus: one with S_x+ and one with S_x-. The natural interpretation of this result is that 50% of the atoms in the S_z+ beam are characterized as (S_z+, S_x+) and the other 50% have (S_z+, S_x-). It turns out that this classical picture runs into difficulty, as we now show.

Consider now a third setup, as shown in Figure 9.5.2. It is this arrangement that will most dramatically illustrate the peculiarities of quantum-mechanical systems.

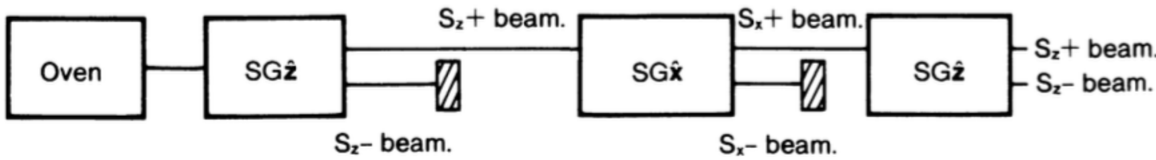


Figure 24: An SG$\hat{z}$ apparatus followed by SG$\hat{x}$ followed by another SG$\hat{z}$.

In this setup, we repeat the previous experiment and block the S_x- beam while letting the S_x+ beam pass through another SG$\hat{x}$ apparatus. It is observed that *two* components emerge from the third apparatus, not one! The emerging beams have *both* an S_z+ and an S_z- component. This is a complete surprise because after the atoms emerged from the first apparatus, we made sure that the S_z- component was completely blocked. How is it possible that the S_z- component, which we thought we eliminated earlier, reappears?

In quantum mechanics, we simply cannot determine S_z and S_x simultaneously. The measurement of S_x completely destroys any previous information we had about S_z. This is radically different from classical mechanics - for instance, there is no difficulty in measuring S_x and S_z for some macroscopic spinning object. This limitation is not due to incompetence of the experimenter - it is inherent in microscopic phenomena.

9.5.3 Analogy with Polarized Light

To model this system mathematically, it helps to make an analogy with a familiar classical system: that of polarized light. Consider a monochromatic light wave propagating in the z-direction.

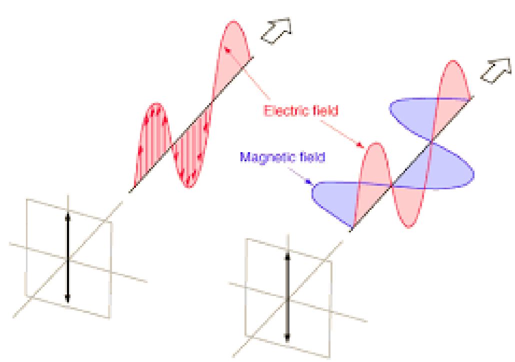


Figure 25: Linearly polarized light.

A linearly polarized (or plane polarized) wave with a polarization vector in the x-direction has a spacetime dependent electric field oscillating in the x-direction.

$$\vec{E} = E_0 \hat{x} \cos(kz - \omega t). \tag{9.24}$$

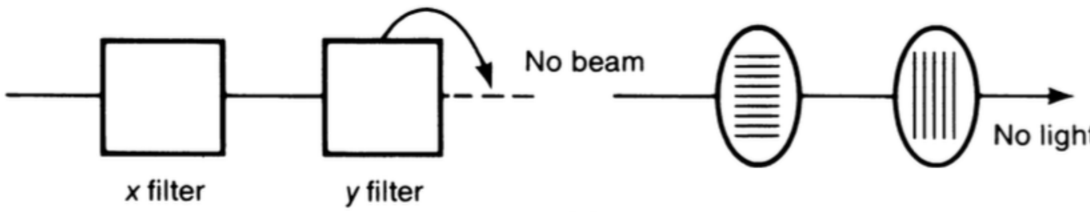


Figure 26: Successive polaroid filters.

Similarly, a y-polarized beam propagating in the z-direction is given by

$$\vec{E} = E_0\hat{y}\cos(kz - \omega t). \tag{9.25}$$

Polarized beams of this type can be obtained by letting an unpolarized beam pass through a polaroid filter. If one passes a beam through an x-filter followed by a y-filter (which is an x-filter rotated by 90°), no beam emerges.

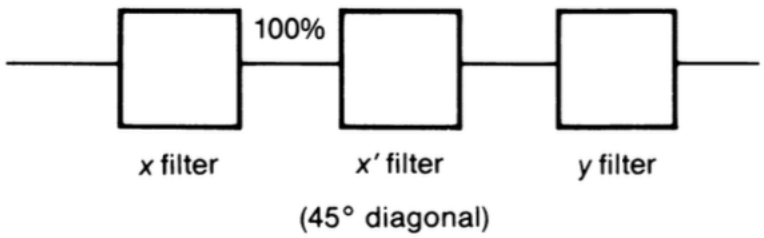


Figure 27: Insertion of an x'-filter.

The situation is more interesting if we insert between the x and y-filters another polaroid that selects only a beam polarized at an angle of 45° with respect to the x-axis (call this the x'-direction).

This time, there is a light beam coming out of the y-filter despite the fact that the beam emerging from the x-filter did not have any polarization component in the y-direction.

Note that this is analogous to the SG experiment in Figure 9.5.2 if we make the following correspondence:

$$\begin{aligned} S_z \pm \text{ atoms } &\leftrightarrow x\text{-}, y\text{-polarized light} \\ S_x \pm \text{ atoms } &\leftrightarrow x'\text{-}, y'\text{-polarized light} \end{aligned}$$

In classical electrodynamics, the x' and y' polarized beams are linear combinations of the x and y polarized light:

$$\begin{aligned} E_0\hat{x}' \cos(kz - \omega t) &= \frac{1}{\sqrt{2}}(\hat{x} + \hat{y}) \cos(kz - \omega t) \\ E_0\hat{y}' \cos(kz - \omega t) &= \frac{1}{\sqrt{2}}(-\hat{x} + \hat{y}) \cos(kz - \omega t) \end{aligned}$$

Similarly, the x and y polarized beam are linear combinations of x' and y' polarized beams. This explains the setup above: the x' filter picks the x' component of the x-polarized light, and the y filter then picks out the y-component of the x' polarized light.

Applying the correspondence to the Stern-Gerlach experiment, we can represent the spin state of a silver atom as a vector in an abstract two-dimensional vector space: $|S_z;+\rangle, |S_z;-\rangle$. The S_x states are then linear combinations of the S_z states:

$$\begin{aligned} |S_x;+\rangle &= \frac{1}{\sqrt{2}}(|S_z;+\rangle + |S_z;-\rangle) \\ |S_x;-\rangle &= \frac{1}{\sqrt{2}}(-|S_z;+\rangle + |S_z;-\rangle) \end{aligned}$$

This explains why two components emerge from the final SG apparatus in Figure 9.5.2. Since there is nothing special about the x-direction, we should be able to replace the SG$\hat{x}$ apparatuses with SG$\hat{y}$ ones, so $|S_y;\pm\rangle$ should also be linear combinations of $|S_z;\pm\rangle$, but we seem to have already used up the available possibilities. How can our vector space formalism distinguish $S_y\pm$ from $S_x\pm$ states.

Again, our analogy with polarized light comes to the rescue. This time, we consider circularly polarized light, which is obtained by letting a linearly polarized beam pass through a quarter-wave plate. When circularly polarized light passes through an x or y-filter we obtain either an x-

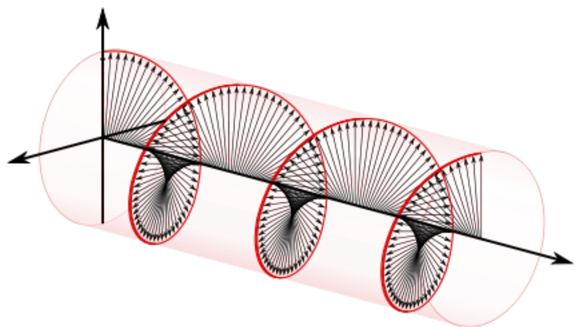


Figure 28: Electric field of circularly polarized light.

polarized or y-polarized beam of equal intensity, yet circularly polarized light is different from linearly polarized light. In the case of circular polarization, the electric field is again a superposition of x and y-polarized light, but where the y-component is 90° out of phase with the x-component. For right-circular polarization, this is

$$\vec{E} = \frac{E_0}{\sqrt{2}} \left[\hat{x} \cos(kz - \omega t) + \hat{y} \cos(kz - \omega t + \frac{\pi}{2}) \right]. \tag{9.26}$$

In complex notation, where $Re(\vec{\epsilon}) = \frac{\vec{E}}{E_0}$, we have

$$\begin{aligned} \vec{\epsilon} &= \frac{1}{\sqrt{2}} \left[\hat{x} e^{i(kz-\omega t)} + \hat{y} e^{i(kz-\omega+\frac{\pi}{2})} \right] \\ &= \frac{1}{\sqrt{2}} \left[\hat{x} e^{i(kz-\omega t)} + \hat{y} e^{i\frac{\pi}{2}} e^{i(kz-\omega)} \right] \\ &= \frac{1}{\sqrt{2}} (\hat{x} + i\hat{y}) e^{i(kz-\omega t)} \end{aligned}$$

A left circularly polarized wave is correspondingly

$$\vec{\epsilon} = \frac{1}{\sqrt{2}} (\hat{x} - i\hat{y}) e^{i(kz-\omega t)}. \tag{9.27}$$

The analogy with silver atoms is them

$$\begin{aligned} S_y+ \text{ atoms } &\leftrightarrow \text{right circular polarized light} \\ S_y- \text{ atoms } &\leftrightarrow \text{left circular polarized light} \end{aligned}$$

We see that if we allow our coefficients to be *complex*, then we can acommodate $S_y\pm$ states in our formalism:

$$|S_y; \pm\rangle = \frac{1}{\sqrt{2}} (|S_z; +\rangle \pm i|S_z; -\rangle). \tag{9.28}$$

We see that the vector space needed to describe the spin states of the silver atoms must be a *complex* vector space. A general state can therefore be written as

$$|\Psi\rangle = c_1|S_z;+\rangle + c_2|S_z;-\rangle,\ c_1, c_2 \in \mathbb{C}. \tag{9.29}$$

Our second takeaway from the Stern-Gerlach experiment is therefore

Lesson 2: Quantum states are represented by vectors in an abstract complex vector space.

We have so far made an analogy with classical electromagnetic waves. In quantum mechanics, the energy density of a wave of frequency ω, given classically by

$$\varepsilon = \frac{|E_x|^2 + |E_y|^2}{8\pi} \tag{9.30}$$

is not arbitrary, but must be an integral multiple of $\hbar\omega$:

$$\varepsilon = N\hbar\omega \tag{9.31}$$

where N is the number of photons in the wave. For x'-polarized light, say, we have $|E_x| = |E_y|$, so when x'-polarized light passes through an x-filter (which sets $E_y = 0$), the energy of the beam decreases by half:

$$\varepsilon_{x'} = \frac{2|E_x|^2}{8\pi} \to \varepsilon_x = \frac{|E_x|^2 + 0}{8\pi}. \tag{9.32}$$

From (9.31), we see that when the energy is halved by the filter, what must happen is that half the photons pass through and half don't. This is very strange - all the photons are identical and see identical conditions, so, classically, if any photon passes through, they all should. The only way to interpret what happens at the polaroid is to say that each photon has a *probability* $1/2$ of passing through. This leads us to our third takeaway:

Lesson 3: We are forced into a probabilistic point of view by the fact that the energy of electromagnetic radiation is quantized.

By our analogy, each silver atom therefore has probability $1/2$ of ending up in the S_z+ beam or in the S_z- beam.

9.6 Postulates of Quantum Mechanics

1. Physical states are represented by rays in a Hilbert space.

   - A Hilbert space is a kind of complex vector space; that is, if Φ and Ψ are vectors in the space, then so is $\xi\Phi + \eta\Psi$ for $\xi, \eta \in \mathbb{C}$.
   - A Hilbert space has a norm: for any pair of vectors there is a complex number (Φ, Ψ), such that

$$\begin{aligned}(\Phi, \Psi) &= (\Psi, \Phi)^* \\ (\Phi, \xi_1\Psi_1 + \xi_2\Psi_2) &= \xi_1(\Phi, \Psi_1) + \xi_2(\Phi, \Psi_2) \\ (\eta_1\Phi_1 + \eta_2\Phi_2, \Psi) &= \eta_1^*(\Phi_1, \Psi) + \eta_2(\Phi_2, \Psi) \\ (\Phi, \Phi) &\geq 0\ (= 0 \text{ iff } \Psi = 0)\end{aligned}$$

- A *ray* is a set of normalized vectors (i.e., $(\Phi, \Phi) = 1$) with Ψ and Ψ' belonging to the same ray if $\Psi' = \xi\Psi$ with ξ a complex number with $|\xi| = 1$.

2. Observables are represented by Hermitian operators.

    - These are mappings $\Psi \to A\Psi$ of the Hilbert space into itself, linear in the sense that

    $$A(\xi\Psi + \eta\Phi) = \xi A\Psi + \eta A\Phi \tag{9.33}$$

    and satisfying the reality condition $A^\dagger = A$, where for any linear operator A the adjoint $A^\dagger$ is defined by

    $$(\Phi, A^\dagger\Psi) \equiv (A\Phi, \Psi) = (\Psi, A\Phi)^* \tag{9.34}$$

    - A state represented by a ray $\mathscr{R}$ has a definite value α for the observable represented by an operator A if vectors Ψ belonging to this ray are eigenvectors of A with eigenvalue α:

    $$A\Psi = \alpha\Psi \text{ fot } \Psi \text{ in } \mathscr{R} \tag{9.35}$$

    - If also $A\Psi' = a'\Psi'$, then because A is Hermitian we have

    $$a(\Psi', \Psi) = (\Psi', A\Psi) = (A^\dagger\Psi', \Psi) = (A\Psi', \Psi) = a^{'*}(\Psi', \Psi). \tag{9.36}$$

    If $\Psi = \Psi' \neq 0$, then $a' = a$ and therefore $a^* = a$ (Hermitian operators have real eigenvalues). If $a \neq a'$, then $(\Psi', \Psi) = 0$ (eigenvectors corresponding to different eigenvalues are orthogonal).

    - Hermitian operators representing observables have eigenvectors that form complete sets, which can be taken to be orthonormal [66]. They therefore form a basis for the Hilbert space. Finding such a basis is referred to as *diagonalizing* the operator A.

    - If we have two or more Hermitian operators, we can find a basis in which they are all diagonal if and only if they commute.

3. If a system is in a state represented by a ray $\mathscr{R}$, and an experiment is done to test whether it is in any one of the different states represented by mutually orthogonal rays $\mathscr{R}_1, \mathscr{R}_2, \ldots$ (for instance, by measuring one or more observables) then the probability of finding it in the state represented by $\mathscr{R}_n$ is

    $$P(\mathscr{R} \to \mathscr{R}_n) = |(\Psi, \Psi_n)|^2, \tag{9.37}$$

    where Ψ and Ψ_n are any vectors belonging to rays $\mathscr{R}$ and $\mathscr{R}_n$, respectively. (A pair of rays is said to be orthogonal if the state-vectors from the two rays have vanishing scalar products.) Another elementary theorem gives a total probability unity:

    $$\sum_n P(\mathscr{R} \to \mathscr{R}_n) = 1 \tag{9.38}$$

    if the state vectors Ψ_n form a complete set.

[66]This is always the case for finite-dimensional Hilbert spaces, and we sill assume it is also true in the infinite-dimensional case

A Partially Ordered Sets

Definition A.1 (Partially ordered set). A *partially ordered set* is an ordered pair $(P, \leq)$ where P is a nonempty set and $\leq$ is a relation on P called a *partial order* (read "less than or equal to") with the following properties:

1. (Reflexivity) For all $a \in P$,
$$a \leq a.$$
2. (Antisymmetry) For all $a, b \in P$,
$$a \leq b \text{ and } b \leq a \implies a = b.$$
3. (Transitivity) For all $a, b, c \in P$,
$$a \leq b \text{ and } b \leq c \implies a \leq c.$$

Partially ordered sets are also called *posets*. When $\leq$ is understood, we say that "P is a partially ordered set".

Definition A.2 (Max,min elements, upper bounds). Let P be a partially ordered set.

1. The *maximum* (*greatest*, *largest*, etc.) element of P, should it exist, is an element $M \in P$ with the property that all elements of P are less than or equal to M, that is,
$$p \leq M \text{ for all } p \in P.$$
Similarly, the *minimum* (*least*, *smallest*, etc.) element of P, should it exist, is an element $N \in P$ with the property that all elements of P are greater than or equal to N, that is,
$$N \leq p \text{ for all } p \in P.$$
2. A *maximal element* is an element $m \in P$ with the property that there is no larger element in P, that is,
$$p \in P, m \leq p \implies m = p.$$
Similarly, a *minimal element* is an element $n \in P$ with the property that there is no smaller element in P, that is,
$$p \in P, p \leq n \implies p = n.$$
3. If $S \subseteq P$, then $u \in P$ is an *upper bound* for S if
$$s \leq u \text{ for all } s \in S.$$
The minimum element of the set of all upper bounds of S, if it exists, is called the *least upper bound* or *supremum* of S, and is denoted by $\sup S$ (read "soup S") or $\operatorname{lub} S$.

4. If $S \subseteq P$, then $\ell \in P$ is a *lower bound* for S if

$$\ell \leq s \text{ for all } s \in S.$$

The maximum element of the set of all lower bounds of S, if it exists, is called the *greatest lower bound* or *infemum* of S, and is denoted by $\inf S$ or glb$S$.

Example A.3. If M is a maximum element for a poset P, then M is unique. To see this, suppose M' is another maximum element for P. Since $M' \in P$ and

$$p \leq M \text{ for all } p \in P,$$

we have

$$M' \leq M.$$

Similarly, since $M \in P$ and

$$p \leq M' \text{ for all } p \in P,$$

we have

$$M \leq M'.$$

But $M' \leq M$ and $M \leq M'$ implies $M = M'$ by antisymmetry, so M is unique.

Exercise A.1. Prove that if N is a minimum element for a poset P, then N is unique.

It follows immediately that the supremum and infimum for a subset S of a poset P, if they exist, are unique.

Note that in a partially ordered set, it is possible that not all elements are comparable. In other words, it is possible to have $x, y \in P$ with the property that $x \not\leq y$ and $y \not\leq x$.

Definition A.4 (Totally ordered set). A partially ordered set in which every pair of elements is comparable is called a *totally ordered set*, or a *linearly ordered set*. Any totally ordered subset of a partially ordered set P is called a *chain* in P.

Example A.5. (a) The set $\mathbb{R}$ of real numbers, with the usual binary relation $\leq$ is a partially ordered set. It is also a totally ordered set. It has no maximal elements.

(b) The set $\{0, 1, 2, \dots\}$ of nonnegative integers, together with the binary relation $n < m$ if $n \mid m$ (that is, if n divides m), is a partially ordered set. The subset S of $\mathcal{N}$ consisting of all powers of 2 is a totally ordered subset of $\mathcal{N}$, that is, it is a chain in $\mathcal{N}$. The set $P = \{2, 4, 8, 3, 9, 27\}$ is a partially ordered subset. It has two maximal elements, namely 8 and 27. The subset $Q = \{2, 3, 5, 7, 11\}$ is a partially ordered set in which every element is both maximal and minimal.

(c) Let S be any set and let $\mathscr{P}(S)$ be the power set of S, that is, the set of all subsets of S. Then $\mathscr{P}(S)$, together with the subset relation $\subseteq$, is a partially ordered set.

Exercise A.2. Verify all the claims in the examples above.

The following gives a condition under which a partially ordered set has a maximal element.

Lemma A.6 (Zorn's lemma). If P is a poset in which every chain has an upper bound, then P has a maximal element.

Zorn's lemma is equivalent to the *axiom of choice*, which is an independent axiom of set theory. As such, it is not subject to proof from the other axioms of ordinary (Zermelo-Fraenkel) set theory. Zorn's lemma has many important equivalences, one of which is the well-ordering principle.

Definition A.7 (Well ordering). A *well ordering* on a nonempty subset X is a total order on X with the property that every nonempty subset of X has a least element.

Theorem A.8 (Well-ordering principle). Every nonempty set has a well ordering.

The usual ordering on the natural numbers $\mathbb{N}$ ($n < m$ if $m - n > 0$) is a well ordering. Remarkably, the well-ordering principle implies that even $\mathbb{R}$ can be well ordered! Of course the usual ordering on $\mathbb{R}$ is not a well ordering, and no one has ever given an explicit well ordering on $\mathbb{R}$, but the well-ordering principle proves that one exists.

B Complex Numbers

B.1 Extending the Real Number System

In a previous course, you studied quadratic equations:

$$ax^2 + bx + c = 0$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$. You found that equations with $b^2 - 4ac < 0$ do not have solutions in $\mathbb{R}$. In particular, the equation

$$x^2 + 1 = 0$$

has no solution in $\mathbb{R}$. Indeed, since $x^2 \geq 0$ for every real number x, the quantity $x^2 + 1$ is always positive, and therefore cannot equal zero for any value of x. Visually, the graph is a parabola $y = x^2$ translated up one unit:

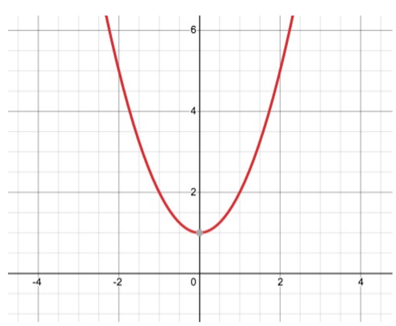


Figure 29: Graph of $y = x^2 + 1$.

A general quadratic function $y = ax^2 + bx + c$ can be put in vertex form by completing the square:

$$y = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}.$$

Since $4a^2 > 0$, if $b^2 - 4ac < 0$ we see that the second term is *positive*, so y is the sum of a nonnegative term and a positive term, and is therefore positive for all values of x. Thus, if $b^2 - 4ac < 0$, $ax^2 + bx + c = 0$ cannot have a solution in $\mathbb{R}$.

As we have done many times now, rather than giving up, we will attempt to invent a new system of numbers which extend $\mathbb{R}$. We will see that this is indeed possible, and denote this new set of numbers by $\mathbb{C}$. We require $\mathbb{C}$ to have the following properties:

1. We must be able to define operations on $\mathbb{C}$ which satisfy the same algebraic properties as those of $\mathbb{R}$.
2. $\mathbb{C}$ must contain a subset which is a copy of $\mathbb{R}$; that is, elements of this subset must be in 1-1 correspondence with elements of $\mathbb{R}$, and the operations of $\mathbb{C}$ must agree with the usual ones on $\mathbb{R}$ when restricted to this special subset.
3. Every quadratic equation must have a solution in $\mathbb{C}$.

For reference, let us recall the algebraic properties of $\mathbb{R}$ (which are exactly the same as those of $\mathbb{Q}$). In algebra, any set F together with operations $+, \times$ of addition and multiplication which satisfy the nine properties listed below is said to be a *field*. When the operations are clear from context, a field $(F, +, \times)$ is simply denoted by F. Thus, $\mathbb{Q}$ and $\mathbb{R}$ are fields, but $\mathbb{N}$ and $\mathbb{Z}$ are not.

Field Properties

(i) Addition is commutative: $a + b = b + a$ for all $a, b \in F$;

(ii) Addition is associative: $a + (b + c) = (a + b) + c$ for all $a, b, c \in F$;

(iii) Existence of an additive identity: There exists an element $0 \in F$ such that $0 + a = a$ for all $a \in F$;

(iv) Existence of additive inverses: For every $a \in F$, there exists $-a \in F$ such that $a + (-a) = 0$ (note that this allows us to define subtraction as addition of the inverse);

(v) Multiplication is commutative: $ab = ba$ for all $a, b \in F$;

(vi) Multiplication is associative: $a(bc) = (ab)c$ for all $a, b, c \in F$;

(vii) Existence of a multiplicative identity: there exists an element $1 \in F$ such that $1a = a$ for all $a \in F$;

(viii) Existence of multiplicative inverses: every *nonzero* element of F has a multiplicative inverse, that is, for every $a \in F$ there exists a number a^{-1} such that $aa^{-1} = 1$. This allows us to define division by a nonzero real number as multiplication by the inverse ($\frac{a}{b} = ab^{-1}$). We cannot define a multiplicative inverse for 0 without contradicting the other properties, so we leave this undefined.

(ix) Distributive property: $a(b + c) = ab + ac$ for all $a, b, c \in F$.

We now *construct* $\mathbb{C}$ from $\mathbb{R}$. That is, we will define the elements of the set $\mathbb{C}$, as well as operations of addition and multiplication, out of those of $\mathbb{R}$. We will then show that $\mathbb{C}$ satisfies the three requirements set out above.

Definition B.1.

(a) Let $\mathbb{C}$ denote the collection of all *ordered pairs* of real numbers; that is

$$\mathbb{C} = \{(a,b) : a,b \in \mathbb{R}\}.$$

Elements of $\mathbb{C}$ are called *complex numbers*.

(b) Given $(a,b) \in \mathbb{C}$, the real number a is called the *real part* of (a,b) and the real number b is called the *imaginary part* (a,b).

(c) Two complex numbers (a_1,b_2) and (a_2,b_2) are *equal* if $a_1 = b_1$ and $a_2 = b_2$. [67]

(d) We define addition and multiplication of complex numbers as follows: [68]

$$\begin{aligned}(a_1,b_1) + (a_2,b_2) &:= (a_1 + a_2, b_1 + b_2)\\ (a_1,b_1) \cdot (a_2,b_2) &:= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2)\end{aligned}$$

Example B.2. Consider $(1,3), (-2,4) \in \mathbb{C}$. Using the definitions above, their sum and product are given by

$$\begin{aligned}(1,3) + (-2,4) &= (1-2, 3+4) = (-1,7)\\ (1,3)\cdot(-2,4) &= (1(-2) - (3)(4), 1(4) + 3(-2))\\ &= (-2-12, 4-6)\\ &= (-14,-2).\end{aligned}$$

Since a complex number (a,b) is an ordered pair of real numbers, we can visualize a complex number geometrically as a point in the plane, with x-coordinate a and y-coordinate b. In this context, the plane is called the *complex plane*. The x-axis is called the *real axis* and the y-axis is called the *imaginary axis*.

We now show that $\mathbb{C}$ has exactly the same algebraic properties as $\mathbb{R}$; that is, we will show that $\mathbb{C}$ is a field.

Theorem B.3. $\mathbb{C}$ is a field.

Proof. (i) We first show that addition is commutative in $\mathbb{C}$. Let $(a_1,b_1),(a_2,b_2) \in \mathbb{C}$. Then, by definition of addition in $\mathbb{C}$,

$$(a_1,b_1) + (a_2,b_2) = (a_1 + a_2, b_1 + b_2).$$

Since $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and addition is commutative in $\mathbb{R}$,

$$a_1 + a_2 = a_2 + a_1 \text{ and } b_1 + b_2 = b_2 + b_1.$$

Therefore,

$$\begin{aligned}(a_1,b_1) + (a_2,b_2) &= (a_1 + a_2, b_1 + b_2)\\ &= (a_2 + a_1, b_2 + b_1).\end{aligned}$$

[67] This statement is nontrivial. Recall that in $\mathbb{Q}$, $(m_1,n_1) = (m_2,n_2)$ if $m_1n_2 = n_1m_2$.

[68] We do not yet have the tools to explain this choice for the definition of multiplication. Sometime in the spring we will be able to show that it is, in fact, forced upon us, but for now we will just postulate it and see that it works.

But $(a_2 + a_1, b_2 + b_1) = (a_2, b_2) + (a_1, b_1)$ by definition of addition in $\mathbb{C}$, so we have shown that

$$(a_1, b_1) + (a_2, b_2) = (a_2, b_2) + (a_1, b_1)$$

for all $(a_1, b_1), (a_2, b_2) \in \mathbb{C}$. Thus, *addition is commutative in* $\mathbb{C}$.

(ii) We now show that addition is associative in $\mathbb{C}$. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \mathbb{C}$. Then

$$\begin{aligned}((a_1, b_1) + (a_2, b_2)) + (a_3, b_3) &= (a_1 + a_2, b_1 + b_2) + (a_3, b_3) \text{ (by def of addition in } \mathbb{C})\\ &= ((a_1 + a_2) + a_3, (b_1 + b_2) + b_3) \text{ (by def of addition in } \mathbb{C})\\ &= (a_1 + (a_2 + a_3), b_1 + (b_2 + b_3)) \text{ (since addition is associative in } \mathbb{R})\\ &= (a_1, b_1) + (a_2 + a_3, b_2 + b_3) \text{ (by def of addition in } \mathbb{C})\\ &= (a_1, b_1) + ((a_2, b_2) + (a_3, b_3)) \text{ (by def of addition in } \mathbb{C})\end{aligned}$$

Hence, *addition is associative in* $\mathbb{C}$.

(iii) Let us try to find an additive identity in $\mathbb{C}$. We are looking for a fixed complex number (a', b') such that $(a, b) + (a', b') = (a + a', b + b') = (a, b)$ for all $(a, b) \in \mathbb{C}$. Since two complex numbers are equal if and only if their corresponding entries are equal, the equation

$$(a + a', b + b') = (a, b)$$

implies that

$$a + a' = a \text{ and } b + b' = b.$$

Both of these equations involve real numbers only, and we easily solve them to obtain $a' = b' = 0$. Thus, $\mathbb{C}$ has a unique additive identity, $(0, 0)$. [69]

(iv) Let us now show that every element of $\mathbb{C}$ has an additive inverse. For $(a, b) \in \mathbb{C}$, an additive inverse is a complex number (a', b') such that $(a, b) + (a', b') = (a + a', b + b') = (0, 0)$. Again, this implies the two *real* equations

$$a + a' = 0 \text{ and } b + b' = 0$$

from which we obtain $(a', b') = (-a, -b)$. Hence, every complex number (a, b) has a unique additive inverse $(-a, -b)$. [70]

(v) We now show that multiplication is commutative in $\mathbb{C}$. Let $(a_1, b_1), (a_2, b_2) \in \mathbb{C}$. Then

$$\begin{aligned}(a_1, b_1) \cdot (a_2, b_2) &= (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1) \text{ (by def of multiplication in } \mathbb{C})\\ &= (a_2 a_1 - b_2 b_1, a_2 b_1 + b_2 a_1) \text{ (since multiplication is commutative in } \mathbb{R})\\ &= (a_2, b_2) \cdot (a_1, b_1) \text{ (by def of multiplication in } \mathbb{C}).\end{aligned}$$

Hence, multiplication is commutative in $\mathbb{C}$.

[69] Actually, it is easy to show that if an additive identity exists in any set with a commutative operation of addition, then it must be unique. To see this, suppose that we have two numbers 0 and $0'$ such that $0 + a = a$ for all a and $0' + a = a$ for all a. Then we must have $0' = 0' + 0 = 0$ (since $0' = 0' + 0$ and $0' + 0 = 0$), so the additive identity is indeed unique.

[70] It is also easy to show that if an inverse exists for a, then it follows from commutativity and associativity that it is necessarily unique. For if $b + a = 0$ and $b' + a = 0$, then $b = b + 0 = b + (b' + a) = b + (a + b') = (b + a) + b' = 0 + b' = b'$.

(vi) We now show that multiplication in $\mathbb{C}$ is associative. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \mathbb{C}$. Then

$$\begin{aligned}
&((a_1, b_1) \cdot (a_2, b_2)) \cdot (a_3, b_3) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) \cdot (a_3, b_3) \text{ (def of multiplication in } \mathbb{C})\\
&= ((a_1a_2 - b_1b_2)a_3 - (a_1b_2 + a_2b_1)b_3, (a_1a_2 - b_1b_2)b_3 + (a_1b_2 + a_2b_1)a_3) \text{ (def of multiplication in } \mathbb{C})\\
&= (a_1a_2a_3 - b_1b_2a_3 - a_1b_2b_3 - a_2b_1b_3, a_1a_2b_3 - b_1b_2b_3 + a_1b_2a_3 + a_2b_1a_3) \text{ (distributive property in } \mathbb{R})\\
&= (a_1a_2a_3 - a_1b_2b_3 - b_1b_2a_3 - a_2b_1b_3, a_1a_2b_3 + a_1b_2a_3 - b_1b_2b_3 + a_2b_1a_3) \text{ (addition is commutative in } \mathbb{R})\\
&= (a_1a_2a_3 - a_1b_2b_3 - b_1b_2a_3 - b_1a_2b_3, a_1a_2b_3 + a_1b_2a_3 - b_1b_2b_3 + b_1a_2a_3) \text{ (multiplication is commutative in } \mathbb{R})\\
&= (a_1(a_2a_3 - b_2b_3) - b_1(b_2a_3 + a_2b_3), a_1(a_2b_3 + b_2a_3) - b_1(b_2b_3 - a_2a_3)) \text{ (distributive property in } \mathbb{R})\\
&= (a_1, b_1) \cdot (a_2a_3 - b_2b_3, b_2a_3 + a_2b_3) \text{ (def of multiplication in } \mathbb{C})\\
&= (a_1, b_1) \cdot ((a_2, b_2) \cdot (a_3, b_3)) \text{ (def of multiplication in } \mathbb{C}).
\end{aligned}$$

Hence, *multiplication is associative in* $\mathbb{C}$.

(vii) Let us try to find a multiplicative identity in $\mathbb{C}$. This is a complex number (a', b') such that

$$(a', b') \cdot (a, b) = (a'a - b'b, a'b + b'a) = (a, b)$$

for all $(a, b) \in \mathbb{C}$. Similar to before, we must solve the pair of *real* equations

$$a'a - b'b = a \tag{B.1}$$
$$a'b + b'a = b \tag{B.2}$$

Multiplying Equation (B.1) by b and Equation (B.2) by a, we obtain

$$\begin{aligned} a'ab - b'b^2 &= ab\\ a'ab + b'a^2 &= ab. \end{aligned}$$

Now, subtracting Equation (B.1) from Equation (B.2), we obtain

$$b'a^2 + b'b^2 = 0$$

or

$$b'(a^2 + b^2) = 0. \tag{B.3}$$

To satisfy Equation (B.3) for all $a, b \in \mathbb{R}$, we must have $b' = 0$. Equations (B.1) and (B.2) then simplify to

$$\begin{aligned} a'a &= a\\ a'b &= b. \end{aligned}$$

These are satisfied for all $a, b \in \mathbb{R}$ if and only if $a' = 1$. Therefore, we find a unique multiplicative identity in $\mathbb{C}$, $(1, 0)$. [71]

(viii) We now show that every nonzero complex number has a multiplicative inverse. Let $(a, b) \in \mathbb{C}$. A multiplicative inverse for (a, b) is a complex number (a', b') such that

$$(a, b) \cdot (a', b') = (aa' - bb', ab' + ba') = (1, 0).$$

[71] As before, if a multiplicative identity exists then it is necessarily unique, since if we have numbers 1 and $1'$ such that $1 \cdot a = a$ and $1' \cdot a = a$ for all a, then $1 = 1 \cdot 1' = 1'$.

This implies the two real equations for a', b':

$$aa' - bb' = 1, \tag{B.4}$$
$$ab' + ba' = 0 \tag{B.5}$$

Equation (B.5) implies that $ba' = -ab'$. Multiplying Equation (B.4) by b:

$$aba' - b^2b' = b$$

and substituting $ba' = -ab'$ then gives

$$\begin{aligned} a(-ab') - b^2b' &= b \\ -a^2b' - b^2b' &= b \\ -(a^2+b^2)b' &= b. \end{aligned}$$

Since $(a,b) \neq (0,0)$ by assumption, $a^2 + b^2 > 0$ so we can divide both sides of this equation by $a^2 + b^2$ to obtain

$$b' = \frac{-b}{a^2+b^2}.$$

Substituting this expression for b' into $ba' = -ab'$ then gives

$$\begin{aligned} ba' &= -a\left(\frac{-b}{a^2+b^2}\right) \\ ba' &= \frac{ab}{a^2+b^2} \end{aligned}$$

and therefore

$$\begin{aligned} ba' - \frac{ab}{a^2+b^2} &= 0 \\ b\left(a' - \frac{a}{a^2+b^2}\right) &= 0. \end{aligned}$$

For this equation to hold for all $b \in \mathbb{R}$, we must therefore have that

$$a' - \frac{a}{a^2+b^2} = 0$$

and therefore

$$a' = \frac{a}{a^2+b^2}.$$

Thus, we have found that every nonzero complex number (a,b) has a unique multiplicative inverse, given by $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$.

(ix) Finally, we check that the distributive property holds. Let $(a_1,b_1), (a_2,b_2)$, and $(a_3,b_3) \in \mathbb{C}$. Then

$$\begin{aligned} &(a_1,b_1)\cdot((a_2,b_2)+(a_3,b_3)) = (a_1,b_1)\cdot(a_2+a_3, b_2+b_3) \text{ (def of addition in } \mathbb{C}) \\ &= (a_1(a_2+a_3) - b_1(b_2+b_3), a_1(b_2+b_3) + b_1(a_2+a_3)) \text{ (def of multiplication in } \mathbb{C}) \\ &= (a_1a_2 + a_1a_3 - b_1b_2 - b_1b_3, a_1b_2 + a_1b_3 + b_1a_2 + b_1a_3) \text{ (distributive property in } \mathbb{R}) \\ &= (a_1a_2 - b_1b_2 + a_1a_3 - b_1b_3, a_1b_2 + b_1a_2 + a_1b_3 + b_1a_3) \text{ (commutativity of addition in } \mathbb{R}) \\ &= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) + (a_1a_3 - b_1b_3, a_1b_3 + b_1a_3) \text{ (def of addition in } \mathbb{C}) \\ &= (a_1,b_1)\cdot(a_2,b_2) + (a_1,b_1)\cdot(a_3,b_3) \text{ (def of multiplication in } \mathbb{C}). \end{aligned}$$

Hence, *the distributive property holds in* $\mathbb{C}$, completing the proof that $\mathbb{C}$ is a field.

$\square$

Note that the set of complex numbers of the form $(a, 0)$ are in 1-1 correspondence with elements of $\mathbb{R}$:

$$(a, 0) \leftrightarrow a \text{ for all } a \in \mathbb{R}.$$

Moreover,

$$\begin{aligned}(a, 0) + (b, 0) &= (a + b, 0)\\ (a, 0) \cdot (b, 0) &= (ab - 0 \cdot 0, a \cdot 0 + 0 \cdot b) = (ab, 0)\end{aligned}$$

which shows that the operations of $\mathbb{C}$ reduce to those of $\mathbb{R}$ when restricted to this special subset. Thus, $\mathbb{C}$ contains a copy of $\mathbb{R}$. On the other hand, complex numbers of the form $(0, b)$ with $b \neq 0$ are said to be *pure imaginary*.

Proposition B.4. The complex number $(0, 1)$ satisfies

$$(0, 1) \cdot (0, 1) = (-1, 0).$$

Proof. By direct computation,

$$(0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0).$$

$\square$

This shows that -1, when viewed as the element $(-1, 0) \in \mathbb{C}$, has a square root in $\mathbb{C}$. In fact,

Proposition B.5. Every negative real number has two square roots in $\mathbb{C}$.

Proof. Let $b > 0$. Then $-b < 0$. We can view $-b$ as the complex number $(-b, 0)$. A square root of $(-b, 0)$ is a complex number (a', b') such that

$$(a', b')^2 = ((a')^2 - (b')^2, 2a'b') = (-b, 0).$$

This implies the two real equations

$$\begin{aligned}(a')^2 - (b')^2 &= -b\\ 2a'b' &= 0\end{aligned}$$

which have solutions $(a', b') = (0, \pm\sqrt{b})$. $\square$

By Theorem B.15 in Section B.2, there is no way to distinguish between these square roots since there is no notion of a "positive" or "negative" complex number. Thus, we will use the notation $\sqrt{-b}$, when $b > 0$, to stand for the set of square roots of $-b$; i.e., $\sqrt{-b} = \{(0, \sqrt{b}), (0, -\sqrt{b})\}$.

Using the 1-1 correspondence $a \leftrightarrow (a, 0)$ we can view the equation

$$x^2 + 1 = 0$$

as an equation in $\mathbb{C}$ by writing

$$x^2 + (1, 0) = (0, 0). \tag{B.6}$$

By Proposition B.4, we see that $(0, 1)$ is a solution to Equation (B.6), since

$$(0, 1)^2 + (1, 0) = (-1, 0) + (1, 0) = (-1 + 1, 0) = (0, 0).$$

Exercise B.1. Show that $(0, -1)$ is a also a solution to Equation (B.6).

Before we show that all quadratic equations have a root in $\mathbb{C}$, we will first show that we can rewrite complex numbers using a more convenient notation.

Proposition B.6. Let $(a, b) \in \mathbb{C}$. Then

$$(a, b) = (a, 0) + (0, 1)(b, 0).$$

Proof. Since

$$(0, 1)(b, 0) = (0 \cdot b - 1 \cdot 0, 1 \cdot b + 0 \cdot 0) = (0, b),$$

we have

$$(a, 0) + (0, 1)(b, 0) = (a, 0) + (0, b) = (a, b)$$

for all $(a, b) \in \mathbb{C}$. $\square$

Let us denote $i := (0, 1)$. By the 1-1 correspondence $a \leftrightarrow (a, 0)$ and Proposition B.6, we can write

$$(a, b) = (a, 0) + (0, 1)(b, 0) = a + ib.$$

Proposition B.7. Writing complex numbers (a_1, b_1) and (a_2, b_2) as $a_1 + ib_1$ and $a_2 + ib_2$, we see that the rules for addition and multiplication of complex numbers in Definition B.1 are exactly what one would obtain by simply treating i as a real number and making the replacement $i^2 = -1$ wherever it appears.

Proof. By direct computation,

$$\begin{aligned}
(a_1, b_1) + (a_2, b_2) &= (a_1 + ib_1) + (a_2 + ib_2) \\
&= a_1 + ib_1 + a_2 + ib_2 \\
&= a_1 + a_2 + ib_1 + ib_2 \\
&= a_1 + a_2 + i(b_1 + b_2) = (a_1 + a_2, b_1 + b_2),
\end{aligned}$$

and

$$\begin{aligned}
(a_1, b_1) \cdot (a_2, b_2) &= (a_1 + ib_1)(a_2 + ib_2) \\
&= a_1a_2 + a_1(ib_2) + (ib_1)a_2 + (ib_1)(ib_2) \\
&= a_1a_2 + ia_1b_2 + ib_1a_2 + i^2b_1b_2 \\
&= a_1a_2 + i(a_1b_2 + b_1a_2) + (-1)b_1b_2 \\
&= a_1a_2 - b_1b_2 + i(a_1b_2 + b_1a_2) \\
&= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2).
\end{aligned}$$

$\square$

By Proposition B.7 we see that we can write each complex number (a, b) as $a + ib$ $(= a + bi)$, and simply manipulate every expression by treating i as a real number and replacing every occurrence of i^2 by -1. We will do this from now on, rather than continuing to write complex numbers as ordered pairs.

In manipulating complicated expressions, it is helpful to know that we can easily simplify larger powers of i by noting that since $i^2 = -1$, $i^4 = (-1)^2 = 1$. Consequently, for any integer n, i^n has only 4 possible values: $1, i, -1$, or $-i$.

Example B.8. To simplify i^{117}, we note that $117 = 29(4) + 1$. Thus

$$i^{117} = i^{29(4)+1} = i^{29(4)}i = (i^4)^{29}i = 1^{29}i = 1i = i.$$

Thus, in computing $i^{117} = i \cdot i \cdot i \cdots i$, we see that in the first 116 multiplications we cycle through the 4 values 29 times, taking us back to 1, with the final multiplication giving $1i = i$. We can therefore quickly simplify any power i^n just by finding the remainder when n is divided by 4. The 4 possible values of the remainder (0,1,2,3) correspond, respectively, to the simplified power of i ($i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$).

Exercise B.2. Simplify i^{325}.

Solution. $325 \div 4$ has remainder 1, so $i^{325} = i$. □

We note also that in deriving the rules for complex addition and multiplication in the proof of Proposition B.7, we used only the fact that $i^2 = -1$. Since $(-i)^2 = i^2 = -1$, everything goes through unchanged if we replace i everywhere by $-i$. The operation of changing i to $-i$ is called **complex conjugation**, and two complex numbers $z = a + ib$ and $w = a - ib$, which differ only by replacing i by $-i$, are called **complex conjugates.** Given a complex number $z = a + ib$, we denote its complex conjugate by $\bar{z} = a - ib$. We note that, geometrically, a complex number and its complex conjugate are related by reflection across the x-axis, as shown in the figure below.

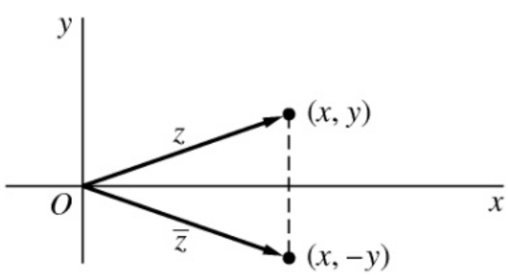


Figure 30: Complex conjugates are related by reflection across the real axis.

Finally, we prove that every quadratic equation has a root in $\mathbb{C}$.

Theorem B.9. Every quadratic equation has a root in $\mathbb{C}$.

Proof. Suppose

$$ax^2 + bx + c = 0,$$

with $a, b, c \in \mathbb{R}$ and $a \neq 0$. Dividing by a and completing the square, we obtain

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

This is the celebrated *quadratic formula* for the roots of the quadratic equation $ax^2 + bx + c = 0$. The quantity $\Delta := b^2 - 4ac$ is called the *discriminant* of the quadratic equation. We see that the discriminant controls the nature of the roots:

$$\begin{aligned}
\Delta > 0 &\implies \text{distinct real solutions,} \\
\Delta = 0 &\implies \text{single real solution with multiplicity two (i.e., a repeated real solution),} \\
\Delta < 0 &\implies \text{complex conjugate solutions.}
\end{aligned}$$

This covers all possibilities, so we see that every quadratic equation has a root in $\mathbb{C}$. $\square$

Example B.10. The quadratic equation $3x^2 - 4x + 5 = 0$ has discriminant $\Delta = (-4)^2 - 4(3)(5) = 16 - 60 = -44 < 0$, so the roots are complex conjugates.

B.2 $\mathbb{C}$ is not an ordered field

We discuss here an important difference between $\mathbb{R}$ and $\mathbb{C}$, which has to do with ordering these sets. While we can define an order on $\mathbb{R}$ in a way which is consistent with the algebraic operations, this turns out not to be the case in $\mathbb{C}$.

Definition B.11. A *order* on a set S is a relation, denoted by $\prec$, with the following properties: [72]

(i) (Trichotomy) If $x, y \in S$, then one and only one of the statements

$$x \prec y,\ x = y,\ y \prec x$$

is true.

(ii) (Transitivity) If $x, y, z \in S$, and if $x \prec y$ and $y \prec z$, then $x \prec z$.

[72] We use the symbol $\prec$ rather than $<$ to emphasize that we may consider any order relation, not just the usual one on $\mathbb{R}$.

If $x \prec y$, we also write $y \succ x$ with the same meaning.

Definition B.12. If the set S on which we have defined $\prec$ is a *field*, then it is an *ordered field* if it also satisfies the following two properties for all $x, y, z \in S$:

(a) If $y \prec z$, then $x + y \prec x + z$.

(b) If $x \succ 0$ and $y \succ 0$, then $xy \succ 0$.

It is important to note that an ordered field is *more* than just an ordered set that is also a field, as it is not guaranteed that the two conditions in Definition B.12 will be satisfied.

Theorem B.13. $\mathbb{R}$ is an ordered field.

Proof. College-level, so we omit the proof. □

The next theorem shows that, by extending $\mathbb{R}$ to $\mathbb{C}$, we lose the ordered field property. We will need the following lemma:

Lemma B.14. In any ordered field, if $x \neq 0$, then $x^2 \succ 0$.

Proof. By trichotomy, if $x \neq 0$, then either $x \succ 0$ or $x \prec 0$. If $x \succ 0$, then by part (b) of Definition B.12, $x \cdot x = x^2 \succ 0$. If $x \prec 0$, then $(-x) \succ 0$, so again by by part (b) of Definition B.12, $(-x)(-x) = x^2 \succ 0$. [73] □

Theorem B.15. There is no order relation $\prec$ on $\mathbb{C}$ making $\mathbb{C}$ into an ordered field.

Proof. (By contradiction.) Suppose $(\mathbb{C}, \prec)$ is an ordered field. By Lemma B.14, since $1 \neq 0$, $1^2 = 1 \succ 0$. By Lemma B.14 again, since $i \neq 0$, $i^2 = -1 \succ 0$. By part (b) of Definition B.12, since

$$-1 \succ 0$$

we have

$$1 - 1 \succ 1 + 0$$

which implies

$$0 \succ 1,$$

contradicting the fact that $1 \succ 0$. Thus, we conclude that there is no order relation $\prec$ on $\mathbb{C}$ making $\mathbb{C}$ into an ordered field. □

Theorem B.15 does *not* say that complex numbers cannot be ordered. For instance, we might put $\mathbb{C}$ in *dictionary order*, where we define $(a_1, b_1) > (a_2, b_2)$ if $a_1 > a_2$ or if $a_1 = a_2$ and $b_1 > b_2$. What Theorem B.15 says is that while this makes $\mathbb{C}$ into an ordered *set*, it does *not* make $\mathbb{C}$ into an ordered *field*; that is, the order (or any order) is not consistent with the operations of addition and multiplication, which leads to contradictions.

Since $\mathbb{C}$ is not an ordered field, all inequalities must therefore be between *real* numbers. For example, we define the *absolute value* (or *modulus*) of a complex number $z = a + ib$ to be the *real* number $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$. [74] We see that $|z| \geq 0$. (This is also geometrically evident, since $|z| = \sqrt{a^2 + b^2}$ is the distance of the point (a, b) from the origin.) However, writing $z > 0$ makes no sense.

[73] Note that $(-x)(-x) = x^2$ follows from the field properties.

[74] Note that this definition reduces to the definition of absolute value of a real number when z is real, since if $b = 0$ we have $|z| = \sqrt{a^2} = |a|$.