Chapter 2

Introduction to Quantum Mechanics

1 Linear Algebra

We consider vectors $|\psi\rangle \in V = \mathbb{C}^n$.

1.1 Bases and linear independence

- A *spanning set* for V is a set of vectors $\{|v_1\rangle, \ldots, |v_n\rangle\}$ such that $|v\rangle = \sum_i a_i |v_i\rangle$ for any $|v\rangle \in V$.

  V may have many different spanning sets. E.g., for $\mathbb{C}^2$

  $$|v_1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |v_2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  and

  $$|v_1\rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} |v_2\rangle \equiv \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

  are both spanning sets.

- A set of non-zero vectors $\{|v_1\rangle, \ldots, |v_n\rangle\}$ are *linearly independent* if

  $$\sum_i a_i |v_i\rangle = 0 \implies a_i = 0 \ \forall i$$

- A *basis* for V is a linearly independent spanning set.

- A basis always exists and any two bases for V have the same number of elements. The number of elements in any basis for V is called the *dimension* of V. We will primarily be interested in finite-dimensional vector spaces.

1.2 Linear operators and matrices

- A *linear operator* is defined by

  $$A : V \to W$$
  $$A(\sum_i a_i |v_i\rangle) = \sum_i a_i A |v_i\rangle$$

It is clear that once the action of A on a basis is specified, the action of A is completely determined on all vectors.

- If $A : V \to W$ and $B : W \to X$, then the *composition* of B with A is defined by

$$\begin{aligned} BA &: V \to X \\ (BA)|v\rangle &= B(A|v\rangle) \end{aligned}$$

- We can view a linear operator as a matrix $A : V \to W$ by choosing a basis for both V and W. This is called the *matrix representation* of V. Let $|v_i\rangle$ be a basis for V and $|w_i\rangle$ be a basis for W. Then the matrix elements A_{ij} are given by

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle$$

1.3 The Pauli matrices

$$\sigma_0 \equiv I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1.4 Inner products

- An *inner product* is a map

$$\begin{aligned} (\cdot,\cdot) &: V \times V \to \mathbb{C} \\ (|v\rangle, |w\rangle) &\to \langle v|w\rangle \end{aligned}$$

such that

(1) $(\cdot,\cdot)$ is linear in the second argument,

$$(|v\rangle, \sum_i \lambda_i |w_i\rangle) = \sum_i \lambda_i \langle v|w_i\rangle$$

(2) $\langle v|w\rangle = \langle w|v\rangle^*$,

(3) $\langle v|v\rangle \geq 0$ with equality iff $|v\rangle = 0$.

Ex: $\mathbb{C}^n$ has an inner product defined by

$$(y, z) = \sum_i y_i^* z_i$$

- A vector space equipped with an inner product is called an *inner product space*.