Chapter 2 Solutions

1 Linear Algebra

1.1 Bases and linear independence

1. We see that
$$a_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
has solution $a_1 = a_2 = -a_3 \neq 0$. Therefore the vectors are linearly dependent.

2. Choosing the basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the matrix representation of A with respect to this basis is
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Taking instead the basis $|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, the matrix representation of A with respect to this basis is
$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. Using equation (2.12) repeatedly, we have
$$\begin{aligned} BA|v_i\rangle &= B(\sum_j A_{ji}|w_j\rangle) \\ &= \sum_j A_{ji}(B|w_j\rangle) \\ &= \sum_j A_{ji} \sum_k B_{kj}|x_k\rangle \\ &= \sum_k (\sum_j B_{kj}A_{ji})|x_k\rangle \end{aligned}$$
Also by (2.12), we have
$$BA|v_i\rangle = \sum_k (BA)_{ki}|x_k\rangle$$

Comparing the two expressions, we find

$$(BA)_{ki} = \sum_j B_{kj} A_{ji},$$

which is the matrix product of the matrix representations for B and A.

4. Introducing the basis $|v_i\rangle$ for V, equation (2.12) gives

$$\begin{aligned} I_V |v_j\rangle &= \sum_i (I_V)_{ij} |v_i\rangle \\ &= |v_j\rangle \end{aligned}$$

which implies $(I_V)_{ij} = \delta_{ij}$.

5. We need to check the three conditions:

(1)

$$\begin{aligned} (y, \sum_k \lambda_k z^{(k)}) &= \sum_i y_i^* \sum_k \lambda_k z_i^{(k)} \\ &= \sum_k \lambda_k \sum_i y_i^* z_i^{(k)} \\ &= \sum_k \lambda_k (y, z^{(k)}) \end{aligned}$$

(2)

$$(z, y)^* = (\sum_i z_i^* y_i)^* = \sum_i z_i y_i^* = (y, z)$$

(3)

$$(y, y) = \sum_i y_i^* y_i = \sum_i |y_i|^2 \geq 0, = 0 \text{ iff } y = 0.$$

Therefore, this is an inner product on $\mathbb{C}^n$.

6. Using property (2) followed by (1), we have

$$\begin{aligned} (\sum_i \lambda_i |w_i\rangle, |v\rangle) &= (|v\rangle, \sum_i \lambda_i |w_i\rangle)^* \\ &= (\sum_i \lambda_i \langle v|w_i\rangle)^* \\ &= \sum_i \lambda_i^* \langle v|w_i\rangle^* \\ &= \sum_i \lambda_i^* \langle w_i|v\rangle. \end{aligned}$$

7. Taking the inner product,

$$\begin{aligned}\langle w|v\rangle &= \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}\\ &= 1-1\\ &= 0.\end{aligned}$$

Thus, the vectors are orthogonal. The norm of each vector is $\sqrt{2}$, so the normalized form of the vectors are given by

$$|w\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad |v\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$