9/17/91

LP in 2-D

$$\min_{x_1, x_2} c_1 x_1 + c_2 x_2$$

$$\ni \; a_{i,1} x_1 + a_{i,2} x_2 \geq B_i \qquad i = 1, \ldots n.$$

$\Downarrow$

$$\min_{x,y} y$$

$$\ni \; y \geq a_i x + b_i \qquad i \in I_1$$

$$y \leq a_i x + b_i \qquad i \in I_2$$

$$a \leq x \leq b.$$

$|I_1| + |I_2| \leq n$

$-\infty \leq a, b \leq \infty$

$\Downarrow$

$$\min g(x)$$

$$\ni \; g(x) \leq h(x)$$

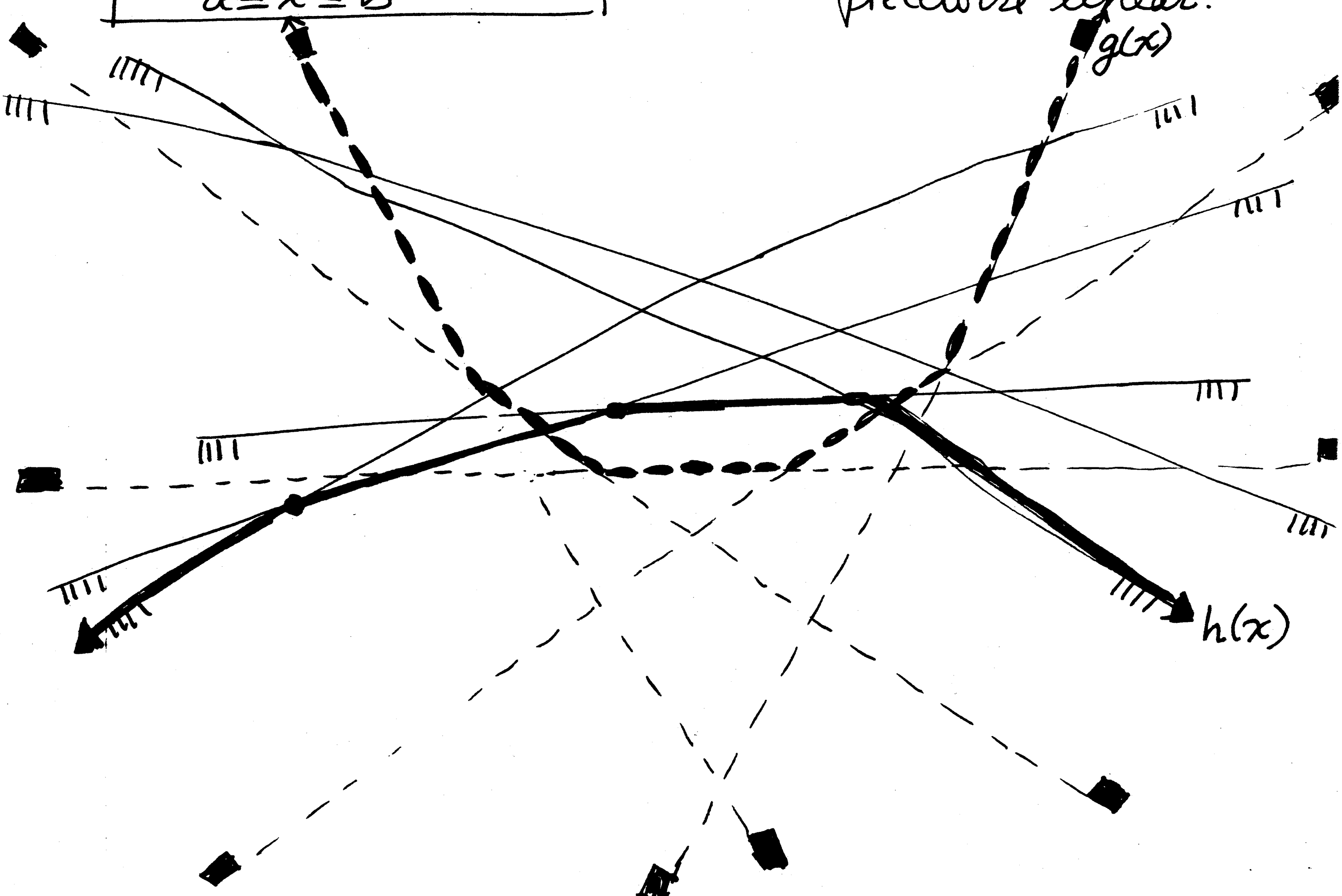
$$a \leq x \leq b$$

Define:

$g(x) = \max \{a_i x + b_i : i \in I_1\}$

$h(x) = \min \{a_i x + b_i : i \in I_2\}$

piecewise linear.



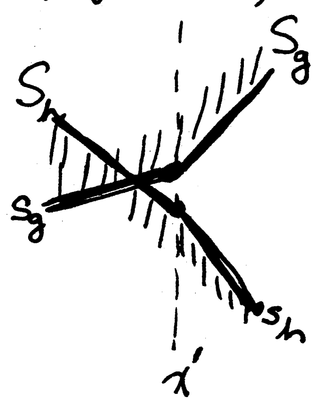
A given value x' of x, $a \le x' \le b$, is feasible iff $g(x') \le h(x')$. If x' is <u>not</u> feasible, any feasible values must lie to one side of x'.
If x' <u>is</u> feasible, we test whether x' is optimal and, if not, determine on which side the minimum lies. Let x^* denote optimal solution

Define $f(x) = g(x) - h(x)$ [For feasible x, $f(x) \le 0$]

<u>INFEASIBLE x'</u>: $f(x') > 0$ and $g(x') > h(x')$.

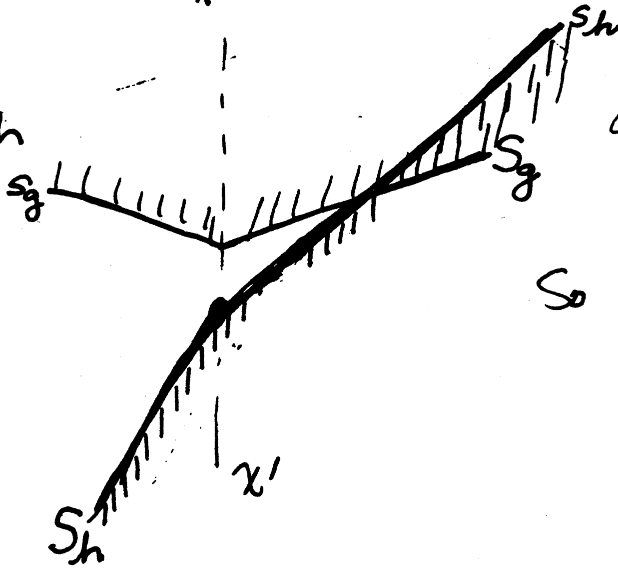
$s_g(x') = \min\{a_i : i \in I_1,\ a_i x' + b_i = g(x')\}$
$S_g(x') = \max\{a_i : i \in I_1,\ a_i x' + b_i = g(x')\}$
$s_h(x') = \min\{a_i : i \in I_2,\ a_i x' + b_i = h(x')\}$
$S_h(x') = \max\{a_i : i \in I_2,\ a_i x' + b_i = h(x')\}$

If $s_g > S_h$



all feasible x are <u>smaller</u> than x'
So $x^* < x'$, if $\exists x^*$

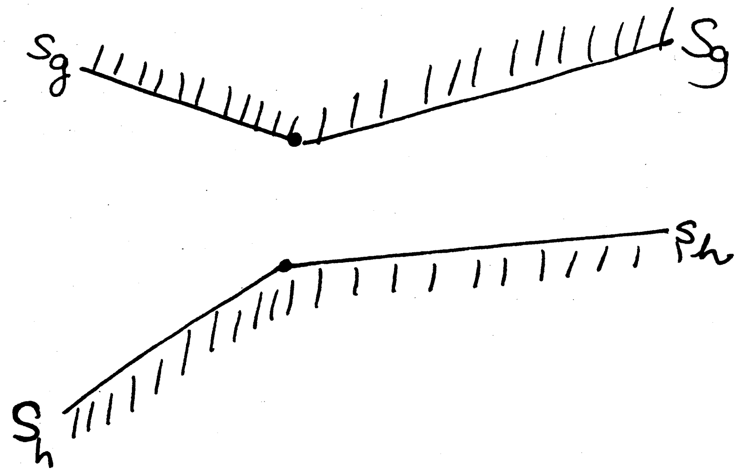
If $S_g < s_h$



all feasible x are <u>larger</u> than x'
So $x^* > x'$, if $\exists x^*$

If $s_g - S_h \le 0 \le S_g - s_h$

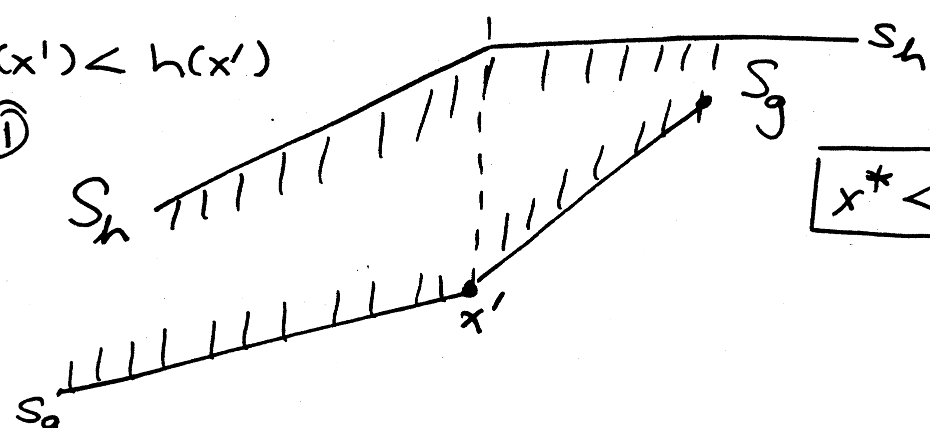
<u>NO</u> FEASIBLE X.



FEASIBLE x': $f(x') \le 0$ and $g(x') \le h(x')$

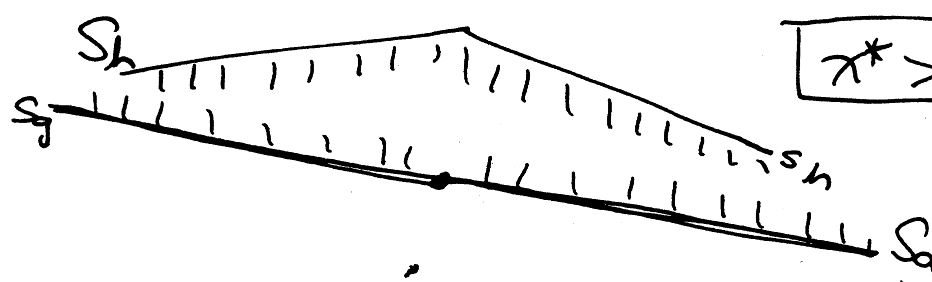
If $g(x') < h(x')$

1



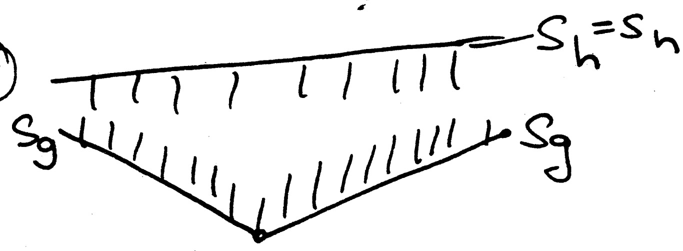
$\boxed{x^* < x'}$

2



$\boxed{x^* > x'}$

3

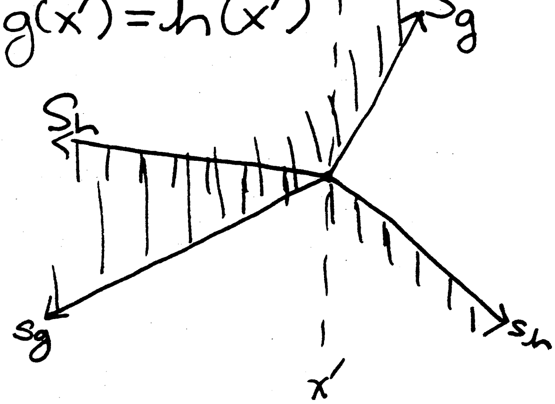


$s_g \le 0 \le S_g$

$x^* = x'$

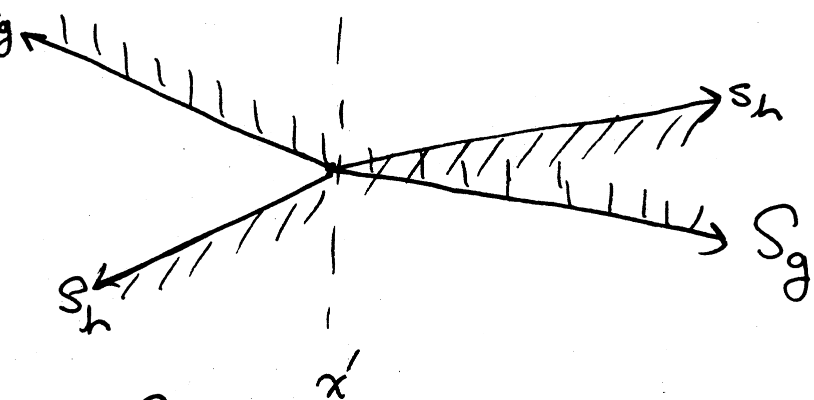
If $g(x') = h(x')$

(1)



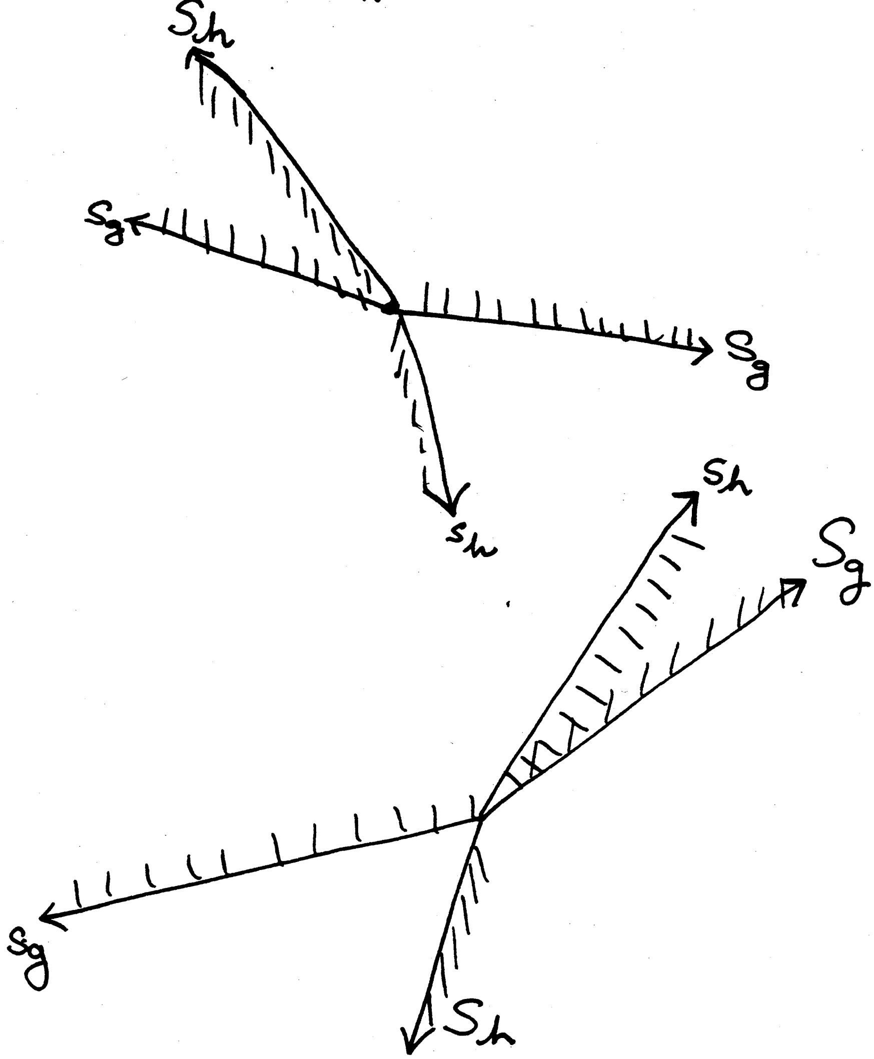
$x^* < x'$

(2)



$x^* > x'$

(3)

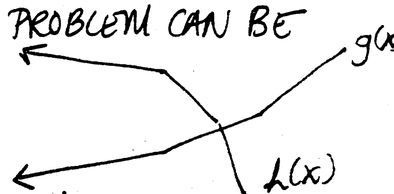


$x^* = x'$

2D LP continued

If $x' \in [a,b]$, then $O(n)$ time decides whether
ⓐ problem infeasible
ⓑ $x^* = x'$
ⓒ $x^* \in [a, x']$ if $\exists x^*$
ⓓ $x^* \in [x', b]$ if $\exists x^*$

[REMEMBER: PROBLEM CAN BE UNBOUNDED.



So, how shall we choose which value of x' to test?

Arrange elements of I_1 (resp I_2) in disjoint pairs
Either ① If $a_i = a_j$ in pair (i,j), drop redundant line
OR ② Compute $x_{ij} = \dfrac{b_i - b_j}{a_j - a_i}$, the intersection pt.
Assume $a_j > a_i$

If $x^* \in [a, x_{ij}]$ and $i,j \in I_1$, then line j is red.
If $x^* \in [x_{ij}, b]$ and $i,j \in I_1$, then line i is red.
If $x^* \in [a, x_{ij}]$, $i,j \in I_2$, line i is redundant
If $x^* \in [x_{ij}, b]$, $i,j \in I_2$, line j is redundant

Algorithm

Find median value of x_{ij}, x_m
Choose appropriate interval $[a, x_m]$ or $[x_m, b]$
That interval contains at most half of the x_{ij}.
For each of the remaining x_{ij}'s, one constraint can be removed.

$$T(n) \leq Cn + T\left(\tfrac{3n}{4}\right) = O(n).$$

LP in $\mathbb{R}^3$

$$\min_{x_1,x_2,x_3} \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 \ni a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \geq \beta_i \quad i=1,\dots n$$

$\Downarrow$ Transformation of coordinates

$$\min_{x,y,z} z \quad \text{such that}$$

$$z \geq a_i x + b_i y + c_i \quad i \in I_1$$

$$z \leq a_i x + b_i y + c_i \quad i \in I_2 \qquad |I_1|+|I_2|+|I_3| = n$$

$$a_i x + b_i y + c_i \leq 0 \quad i \in I_3$$

$\Downarrow$

$$\min_{x,y} g(x,y) \quad \text{such that} \quad f(x,y) \leq 0$$

$\Uparrow$

$$f(x,y) = \max\{g(x,y) - h(x,y), e(x,y)\}$$

DEFINE

$$g(x,y) = \max_{i \in I_1}\{a_i x + b_i y + c_i\}$$

$$h(x,y) = \min_{i \in I_2}\{a_i x + b_i y + c_i\}$$

$$e(x,y) = \max_{i \in I_3}\{a_i x + b_i y + c_i\}$$

<u>NOTE</u> 1) $e(x,y) = 0$ defines a convex polygon in the x-y plane which delimits the <u>domain</u> of the LP problem

2) for a point (x,y) in the <u>domain</u> to be <u>feasible</u>, then $g(x,y) \leq h(x,y)$, so:

$(x,y) \in \mathbb{R}^2$ is feasible iff $f(x,y) \leq 0$.

GOAL: Test a straight line in $\mathbb{R}^2$ and restrict further search to one half-plane or the other, or else STOP.

PROCEDURE: W.L.O.G, assume line is x-axis (transform coordinates).

Determine (a) problem infeasible; (b) global minumum has $y=0$; (c) problem is unbounded; (d) must search $\{(x,y): y>0\}$; (e) must search $\{x,y \mid y<0\}$ by first solving the 2D LP $\min (g(x,0)) \ni f(x,0) \leq 0$.

Either (a) problem unbounded $\Rightarrow$ 3D problem unbounded

OR (b) problem infeasible, with $(x^*,0)$ the min of $f(x,0)$.

OR (c) $(x^*,0)$ min of $g(x,0)$ for $f(x,0) \leq 0$.

If a), we are done

Assume $(x^*,0)$ is reported and WLOG assume $x^*=0$.

Further steps from here rely only on constraints which are tight at $(0,0)$. We isolate those constraints

$\boxed{\text{DEFINE}}$ $I_j^* \subset I_j$, $j \in \{1,2,3\}$

$i \in I_1 \Rightarrow i \in I_1^*$ if $c_i = g(0,0)$

$i \in I_2 \Rightarrow i \in I_2^*$ if $c_i = h(0,0)$ AND $f(0,0) = g(0,0) - h(0,0) \geq 0$

$i \in I_3 \Rightarrow i \in I_3^*$ if $c_i = e(0,0)$ AND $f(0,0) = e(0,0) \geq 0$

Using only those constraints which are tight at $(0,0)$, we can search locally at $(0,0)$ and decide in which half-space an optimal solution may lie. But we shall consider then a very small neighborhood of $(0,0)$ so as to be sure not to violate any other constraints.

We divide the procedure from here into two halves;

~~Both~~ first we assume $f(0,0) \leq 0$, i.e $x^*=0$ is feasible.

second we assume $f(0,0) > 0$, i.e. $x^*=0$ is infeasible

CASE I: $f(0,0) \le 0$, so $x^* = 0$ IS A FEASIBLE POINT

Prop 1 $\exists (x,y)$ with $y>0$, $g(x,y) < g(0,0)$ and $f(x,y) \le 0$

IFF $\exists \lambda$ such that

① $\max_{i\in I_1^*}\{a_i\lambda + b_i\} < 0$

AND

② $\max_{i\in I_1^*}\{a_i\lambda + b_i\} \le \min_{i\in I_2^*}\{a_i\lambda + b_i\}$

AND

③ $\max_{i\in I_3^*}\{a_i\lambda + b_i\} \le 0$

PROP 2 ... [redacted] $y<0$ ... IFF

① $\min_{i\in I_1^*}\{a_i\lambda + b_i\} > 0$

② $\min_{i\in I_1^*}\{a_i\lambda + b_i\} \ge \max_{i\in I_2^*}\{a_i\lambda + b_i$

③ $\min_{i\in I_3^*}\{a_i\lambda + b_i\} \ge 0$

Proof of Prop 1

$\Rightarrow$ Assume such an (x,y). Let $\lambda = \frac{x}{y}$. Recall that $y>0$.

① $\max_{i\in I_1^*}\{a_i\lambda y + b_i y + c_i\} \le g(\lambda y, y) < g(0,0) \equiv \max_{i\in I_1^*}\{c_i\}$

So $\max_{i\in I_1^*}\{a_i\lambda + b_i\} < 0$

② If $I_2^* = \emptyset$, trivial. Assume $I_2^* \neq \emptyset$. Then $f(0,0) = g(0,0) - h(0,0) =$

Therefore $\forall i \in I_1^*$, $\forall j \in I_2^*$, $c_i \equiv c_j = c$

$\max_{i\in I_1^*}\{a_i\lambda y + b_i y + c_i\} \le g(\lambda y, y) \le h(\lambda y, y) \le \min_{i\in I_2^*}\{a_i\lambda y + b_i y + c_i$

Subtracting $c_i = c$ and dividing by $y>0$ yields

$$\max_{i\in I_1^*}\{a_i\lambda + b_i\} \le \min_{i\in I_2^*}\{a_i\lambda + b_i\}$$

③ If $I_3^* = \emptyset$, trivial. Assume $I_3^* \neq \emptyset$. Then $f(0,0) = e(0,0) = c_j \ge$ for all $j \in I_3^*$. But $f(0,0) \le 0$ by the premise. So $c_j \equiv 0$.
AND $c_j = \max_{i\in I_3}\{c_i\}$.

So $\max_{i\in I_3^*}\{a_i\lambda y + b_i y + c_i\} \le \overset{c_i}{\overset{\shortparallel}{0}}$ and

$$\max_{i\in I_3^*}\{a_i\lambda + b_i\} \le 0$$

Proof of Prop 1 $\Leftarrow$ Assume $\exists \lambda$ which satisfies 1, 2, 3.
If $y>0$ is sufficiently small, then the constraints in I_1^*, I_2^*, and I_3^* are still the only ones which apply.
So $g(\lambda y, y) < g(0,0)$ since

$$\max_{i \in I_1^*} \{a_i \lambda y + b_i y + c_i\} \leq \max_{i \in I_1^*} \{a_i \lambda + b_i\} y + \max_{i \in I_1^*} \{c_i\}$$

$$\leq 0 y + g(0,0) \leq g(0,0)$$

Furthermore $f(\lambda y, y) \leq 0$ since $\forall i \in I_1^*, \forall j \in I_2^*; \ c_i \equiv c_j$
and $\max_{i \in I_1^*} \{a_i \lambda + b_i\} y + \max_{i \in I_1^*} \{c_i\} \leq \min_{i \in I_2^*} \{a_i \lambda + b_i\} y + \min_{i \in I} \{c_i$

The Test Given I_j^*, $j = 1, 2, 3$

First solve

$$\min_{\lambda, n} n$$
$$n \geq a_i \lambda + b_i \quad i \in I_1^*$$
$$n \leq a_i \lambda + b_i \quad i \in I_2^*$$
$$a_i \lambda + b_i \leq 0 \quad i \in I_3^*$$

If $n < 0$, choose $y > 0$

Second solve

$$\min_{\lambda, n} n$$
$$n \leq a_i \lambda + b_i \quad i \in I_1^*$$
$$n \geq a_i \lambda + b_i \quad i \in I_2^*$$
$$a_i \lambda + b_i \geq 0 \quad i \in I_3^*$$

If $n > 0$, choose $y < 0$

ELSE Keep $y = 0$, so $(0,0)$ is optimal.

CASE II: $f(0,0) > 0$ (INFEASIBLE POINT)

Prop 3 $\exists (x,y)$ with $y>0$, $f(x,y) < f(0,0)$ IFF $\exists \lambda$ such that

① $\max_{i \in I_1^*} \{a_i\lambda + b_i\} < \min_{i \in I_2^*} \{a_i\lambda + b_i\}$

AND

② $\max_{i \in I_3^*} \{a_i\lambda + b_i\} < 0$

Prop 4 ... $y<0$ ...

① $\min_{i \in I_1^*} \{a_i\lambda + b_i\} > \max_{i \in I_2^*} \{a_i\lambda + b_i\}$

AND

② $\min_{i \in I_3^*} \{a_i\lambda + b_i\} > 0$

The TEST

First $\min_{\lambda} \psi(\lambda)$ where $\psi(\lambda)$ represents the convex, piecewise linear

$$\psi(\lambda) = \max\{\max_{i \in I_1^*}\{a_i\lambda + b_i\} - \min_{i \in I_2^*}\{a_i\lambda + b_i\}, \max_{i \in I_3^*}\{a_i\lambda + b_i\}\}$$

[Use 2D LP where you pair indices only if $i,j \in I_k^*$ for the same $k \in \{1,2,3\}$]

If $\psi < 0$, then choose $y > 0$.

Second $\max_{\lambda} \psi(\lambda) = \min\{\min_{i \in I_1^*}\{a_i\lambda+b_i\} - \max_{i \in I_2^*}\{a_i\lambda+b_i\}, \min_{i \in I_3^*}\{a_i\lambda+b_i\}\}$

If $\psi > 0$, then choose $y < 0$.

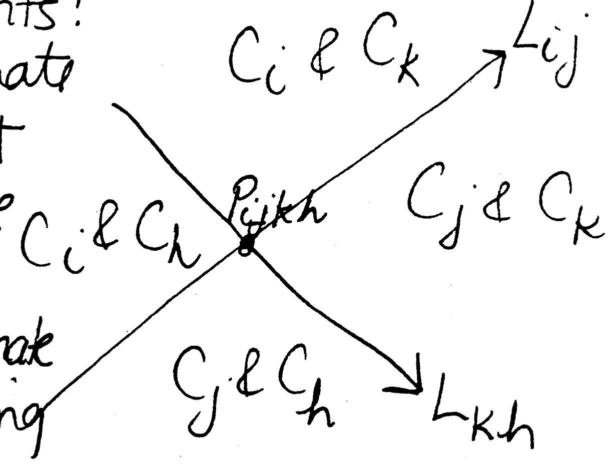
ELSE f attains its global minimum on the x-axis
$\Rightarrow$ problem is infeasible.

THE ALGORITHM

① Pair the constraints in I_1, I_2, and I_3 individually. In each pair (i,j), either discard one parallel constraint or compute the intersection line $L_{i,j}$ in the xy-plane. That line divides the xy-plane into 2 half-planes: in one, constraint i is redundant; in the other, j is.

② Once you have computed all the lines L_{i_k, j_k} for each disjoint pair $i_k, j_k \in I_k$, transform coordinates so that half the lines have positive slope and half have negative slope.

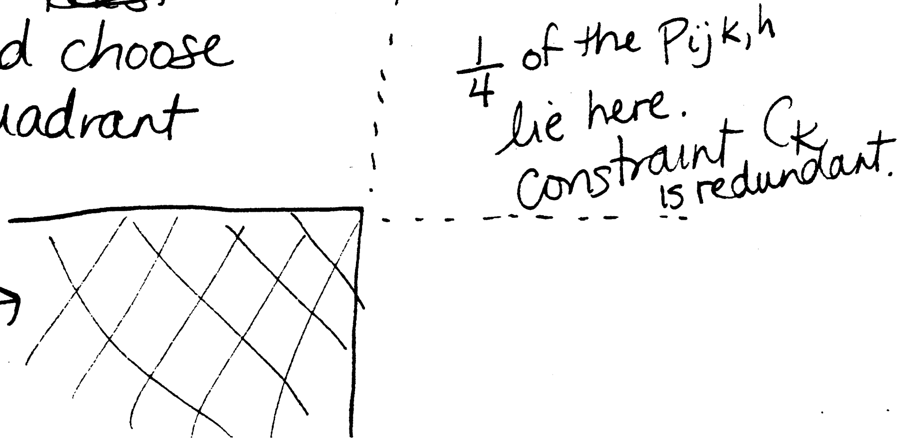
③ Pair one line L_{ij} with positive slope with one line L_{kh} of negative slope and compute the intersection point P_{ijkh}. The xy-plane is divided into four sectors each of which is associated with just two constraints:



④ Choose the median y-coordinate of all the P_{ijkh}. Test that line + choose the appropriate half-plane.

⑤ Choose the median x-coordinate of the P_{ijkh} NOT belonging to that half plane. ~~Test~~ Test that line and choose the appropriate quadrant

optimal sol'n is here, if it exists



(6) At least $\frac{1}{4}$ of the p_{ijkh} lie in the diagonally opposing quadrant. Since one of L_{ij} and L_{kh} has positive slope and one has negative slope, one of these lines <u>never</u> enters the chosen quadrant. Therefore one constraint is redundant. So $\frac{1}{16}$ of the constraints, at least, are eliminated.

$$LP(3,n) \leq Cn + 2(LP(2,n)) + LP(3, \tfrac{15n}{16})$$
$$= O(n)$$