Probability Theorems

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Sample Space: The sample space **S** is the collection of all possible outcomes of a random study
Event: An event is a subset of the sample space S
Probability: Probability is a real-valued set function **P** that assigns to each event **A** in the sample space **S** a number **P(A)** called the "probability of the event **A** such that:

1. The probability of any event **A** must be nonnegative, that is, $P(A) \geq 0$
2. The probability of the sample space is 1, that is, $P(S) = 1$
3. Given mutually exclusive events $A_1, A_2, A_3, \ldots$ that is, where $A_i \cap A_j \neq \emptyset$, for $i \neq j$
   a. The probability of a finite union of the events is the sum of the probability of the individual events, that is, $P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$
   b. The probability of a countably infinite union of the events is the sum of the probability of the individual events, that is, $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \cdots$

Five Theorems:

1. Prove: $P(A) = 1 - P(A')$

$$S = A \cap A'$$
$$P(S) = P(A \cap A')$$
$$1 = P(A) + P(A')$$
$$P(A) = 1 - P(A')$$

2. Prove: $P(\emptyset) = 0$

$$A = \emptyset$$
$$A' = S$$
$$\begin{aligned} P(\emptyset) &= P(A) \\ &= 1 - P(A') \\ &= 1 - P(S) \\ &= 1 - 1 = 0 \end{aligned}$$
$$P(\emptyset) = 0$$

3. If events A and B are such that $A \subseteq B$, then prove $P(A) \leq P(B)$

$$B = A \cup (A' \cap B)$$
$$\begin{aligned} P(B) &= P\big(A \cup (A' \cap B)\big) \\ &= P(A) + P(A' \cap B) \end{aligned}$$

Given $P(A' \cap B) \geq 0$, therefore $P(A) \leq P(B)$

4. Prove $P(A) \leq 1$

   Note that by definition, $A \subseteq S$, and $P(S) = 1$, therefore from theorem 3 we have $P(A) \leq 1$

5. For any two events A and B, prove $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   $A \cup B = A \cup (A' \cap B)$
   $\quad = A + (A' \cap B)$
   $P(A \cup B) = P(A) + P(A' \cap B)$

   $B = (B \cap A) + (B \cap A')$
   $P(B) = P(B \cap A) + P(B \cap A')$
   $P(B \cap A') = P(B) - P(B \cap A)$

   $P(A \cup B) = P(A) + P(A' \cap B)$
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

6. For any two events A and B, prove 1) $(A \cup B)' = (A' \cap B')$, and 2) $(A \cap B)' = (A' \cup B')$

   $x \in (A \cup B)'$
   $x \notin (A \cup B)$
   $x \notin A$ and $x \notin B$
   $x \in A'$ and $x \in B'$
   $x \in (A' \cap B')$

   And:
   $x \in (A \cap B)'$
   $x \notin (A \cap B)$
   $x \notin A$ or $x \notin B$
   $x \in A'$ or $x \in B'$
   $x \in (A' \cup B')$