Expectation, Variance, Binomial, and Poisson proofs

by Andrew Rothman

Expectation:

1) **Prove $\boldsymbol{E[Y+X]=E[X]+E[Y]}$**

$$E[Y+X] = \sum_x \sum_y (x+y)P(X=x, Y=y)$$

$$= \sum_x \sum_y xP(X=x, Y=y) + \sum_x \sum_y yP(X=x, Y=y)$$

$$= \sum_x xP(X=x) + \sum_y yP(Y=y)$$

$$= E[X] + E[Y]$$

2) **Prove $\boldsymbol{E[a+X]=a+E[X]}$**

$$E[a+X] = \sum_x (a+x)P(X=x)$$

$$= \sum_x aP(X=x) + \sum_x xP(X=x)$$

$$= (a * 1) + \sum_x xP(X=x)$$

$$= a + E[X]$$

3) **Prove $\boldsymbol{E[aX]=aE[X]}$**

$$E[aX] = \sum_x axP(X=x)$$

$$= a\sum_x xP(X=x)$$

$$= aE[X]$$

Variance Proof:

1) **Prove $\boldsymbol{Var[X] = E[X^2] - (E[X])^2}$**

$$
\begin{aligned}
Var[X] &= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + (E[X])^2] \\
&= E[X^2] - 2E[X]E[X] + (E[X])^2 \\
&= E[X^2] - 2(E[X])^2 + (E[X])^2 \\
&= E[X^2] - (E[X])^2
\end{aligned}
$$

2) **Prove $\boldsymbol{Var[a + X] = Var[X]}$**

$$
\begin{aligned}
Var[a + X] &= E[(a + X - E[a + X])^2] \\
&= E[(a + X - a - E[X])^2] \\
&= E[(X - E[X])^2] \\
&= Var[X]
\end{aligned}
$$

3) **Prove $\boldsymbol{Var[aX] = a^2 Var[X]}$**

$$
\begin{aligned}
Var[aX] &= E[(aX - E[aX])^2] \\
&= E[(aX - aE[X])^2] \\
&= E[a^2(X - E[X])^2] \\
&= a^2E[(X - E[X])^2] \\
&= a^2Var[X]
\end{aligned}
$$

Bernoulli Distribution:

PMF: $f(k;p) = \ p^k(1-p)^{1-k}$

Parameters: $p \in [0,1]$

Support: $k \in \{0,1\}$

1) $X \sim Bern(p)$, prove $E[X] = p$

$$\begin{aligned} E[X] &= \sum_{x=0}^{1} x\,p^x(1-p)^{1-x} \\ &= (0)p^0(1-p)^1 + (1)p^1(1-p)^0 \\ &= 0 + p \\ &= p \end{aligned}$$

2) $X \sim Bern(p)$, prove $Var[X] = p(1-p)$

$$Var[X] = E[X^2] - (E[X])^2$$

$$\begin{aligned} E[X^2] &= \sum_{x=0}^{1} x^2\,p^x(1-p)^{1-x} \\ &= (0)p^0(1-p)^1 + (1)p^1(1-p)^0 \\ &= 0 + p \\ &= p \end{aligned}$$

$$\begin{aligned} Var[X] &= p - (p)^2 \\ &= p(1-p) \end{aligned}$$

Binomial Distribution:

PMF: $f(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$

Parameters: $p \in [0,1]$

$n \in \{0, 1, 2, \dots, \infty\}$

Support: $k \in \{0,1,2, \dots, n\}$

1) $X \sim Binom(n, p)$, prove $E[X] = np$

Approach #1:

$$
\begin{aligned}
E[X] &= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} \binom{n}{x-1} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{(n-1)-(x-1)} \\
&= np \sum_{j=0}^{m} \binom{m}{j} p^j (1-p)^{m-j} = np * (1) \\
&= np
\end{aligned}
$$

Approach #2:

Note that with $Y \sim Binom(n, p)$, $Y = \sum_{i=1}^{n} X$, where we have "n" iid $X \sim Bern(p)$

$$Y = \sum_{i=1}^{n} X$$

$$E[Y] = E\left[\sum_{i=1}^{n} X\right] = \sum_{i=1}^{n} E[X] = \sum_{i=1}^{n} p = np$$

2) $X \sim Binom(n,p)$, prove $Var[X] = np(1-p)$

<u>Approach #1</u>:
$Var[X] = E[X^2] - (E[X])^2$

Note that:

$$\begin{aligned} E[X^2] &= E[X^2] - E[X] + E[X] \\ &= E[X^2 - X] + E[X] \\ &= E[X(X-1)] + E[X] \end{aligned}$$

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=2}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=2}^{n} \binom{n}{x-2} p^x (1-p)^{n-x} \\ &= \sum_{x=2}^{n} n(n-1)\binom{n-2}{x-2} p^x (1-p)^{n-x} \\ &= \sum_{x=2}^{n} n(n-1)p^2\binom{n-2}{x-2} p^{x-2} (1-p)^{(n-2)-(x-2)} \\ &= n(n-1)p^2 \sum_{j=0}^{m} \binom{m}{j} p^j (1-p)^{m-j} = n(n-1)p^2 \end{aligned}$$

$$\begin{aligned} E[X^2] &= E[X(X-1)] + E[X] \\ &= n(n-1)p^2 + np \end{aligned}$$

$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= n^2p^2 - np^2 + np - n^2p^2 \\ &= np - np^2 \\ &= np(1-p) \end{aligned}$$

<u>Approach #2</u>:

Note that with $Y \sim Binom(n,p)$, $Y = \sum_{i=1}^{n} X$, where we have “n” iid $X \sim Bern(p)$

$$Y = \sum_{i=1}^{n} X$$

$$Var[Y] = Var\left[\sum_{i=1}^{n} X\right] = \sum_{i=1}^{n} Var[X] = \sum_{i=1}^{n} p(1-p) = np(1-p)$$

Poisson Distribution:

PMF: $f(k;\lambda) = \frac{\lambda^k e^\lambda}{k!}$

Parameters: $\lambda \in (0, \infty)$

Support: $k \in \{0,1,2,\dots\}$

1) $X \sim Poisson(\lambda)$, prove $E[X] = \lambda$

$$\begin{aligned} E[X] &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^\lambda}{x!} \\ &= \sum_{x=1}^{\infty} x \frac{\lambda^x e^\lambda}{x!} \\ &= \sum_{x=1}^{\infty} \frac{\lambda^x e^\lambda}{(x-1)!} \\ &= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^\lambda}{(x-1)!} \\ &= \lambda \sum_{y=0}^{\infty} \frac{\lambda^y e^\lambda}{y!} \\ &= \lambda \end{aligned}$$

2) $X \sim Poisson(\lambda)$, prove $Var[X] = \lambda$

Note that:

$$\begin{aligned} E[X^2] &= E[X^2] - E[X] + E[X] \\ &= E[X^2 - X] + E[X] \\ &= E[X(X-1)] + E[X] \end{aligned}$$

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^\lambda}{x!} \\ &= \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x e^\lambda}{x!} \\ &= \sum_{x=2}^{\infty} \frac{\lambda^x e^\lambda}{(x-2)!} \\ &= \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2} e^\lambda}{(x-2)!} \\ &= \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y e^\lambda}{y!} \\ &= \lambda^2 \end{aligned}$$

$$Var[X] = E[X^2] - (E[X])^2 = E[X(X-1)] + E[X] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$