Expectation, Variance, Binomial, and Poisson proofs

by Andrew Rothman

Expectation:

1) Prove $\boldsymbol{E[Y+X]=E[X]+E[Y]}$

$$E[Y+X] = \sum_x \sum_y (x+y)P(X=x, Y=y)$$

$$= \sum_x \sum_y xP(X=x, Y=y) + \sum_x \sum_y yP(X=x, Y=y)$$

$$= \sum_x xP(X=x) + \sum_y yP(Y=y)$$

$$= E[X] + E[Y]$$

2) Prove $\boldsymbol{E[a+X]=a+E[X]}$

$$E[a+X] = \sum_x (a+x)P(X=x)$$

$$= \sum_x aP(X=x) + \sum_x xP(X=x)$$

$$= (a*1) + \sum_x xP(X=x)$$

$$= a + E[X]$$

3) Prove $\boldsymbol{E[aX]=aE[X]}$

$$E[aX] = \sum_x axP(X=x)$$

$$= a\sum_x xP(X=x)$$

$$= aE[X]$$

Variance Proof:

1) **Prove $\boldsymbol{Var[X] = E[X^2] - (E[X])^2}$**

$$
\begin{aligned}
Var[X] &= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + (E[X])^2] \\
&= E[X^2] - 2E[X]E[X] + (E[X])^2 \\
&= E[X^2] - 2(E[X])^2 + (E[X])^2 \\
&= E[X^2] - (E[X])^2
\end{aligned}
$$

2) **Prove $\boldsymbol{Var[a + X] = Var[X]}$**

$$
\begin{aligned}
Var[a + X] &= E[(a + X - E[a + X])^2] \\
&= E[(a + X - a - E[X])^2] \\
&= E[(X - E[X])^2] \\
&= Var[X]
\end{aligned}
$$

3) **Prove $\boldsymbol{Var[aX] = a^2 Var[X]}$**

$$
\begin{aligned}
Var[aX] &= E[(aX - E[aX])^2] \\
&= E[(aX - aE[X])^2] \\
&= E[a^2(X - E[X])^2] \\
&= a^2 E[(X - E[X])^2] \\
&= a^2 Var[X]
\end{aligned}
$$

4) **Prove $\boldsymbol{Var[X + Y] = Var[X] + Var[Y] + 2Cov(X, Y)}$**

$$
\begin{aligned}
Var[X + Y] &= E[(X + Y - E[X + Y])^2] \\
&= E[(X + Y - E[X] - E[Y])^2] \\
&= E\big[XX + XY - XE[X] - XE[Y] + YX + YY - YE[X] - YE[Y] - XE[X] - \\
&\quad YE[X] + E[X]E[X] + E[X]E[Y] - XE[Y] - YE[Y] + E[Y]E[X] + E[Y]E[Y]\big] \\
&= (E[X^2 - 2XE[X] + E[X]^2]) + (E[Y^2 - 2XE[Y] + E[X]^2]) + \\
&\quad 2E\big[XY - XE[Y] - YE[X] + E[X]E[Y]\big] \\
&= E[(X - E[X])^2] + E[(Y - E[Y])^2] + 2E[(X - E[X])(Y - E[Y])] \\
&= Var[X] + Var[Y] + 2Cov(X, Y)
\end{aligned}
$$

Covariance Proof:

1) **Prove $\boldsymbol{Cov(X,Y) = E[XY] - E[X]E[Y]}$**

$$\begin{aligned} Covar(X,Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E\big[XY - XE[Y] - E[X]Y + E[X]E[Y]\big] \\ &= E[XY] - 2E[X]E[Y] + [X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

2) **Prove $\boldsymbol{Cov(X+Y,Z) = Cov(X,Z) + Cov(Y,Z)}$**

$$\begin{aligned} Covar(X+Y,Z) &= E[(X + Y - E[X+Y])(Z - E[Z])] \\ &= E\big[XZ + YZ - E[X]Z - E[Y]Z - XE[Z] - YE[Z] + E[X]E[Z] + E[Y]E[Z]\big] \\ &= E[XZ] + E[YZ] - 2E[X]E[Z] - 2E[Y]E[Z] + E[X]E[Z] + E[Y]E[Z] \\ &= E[XZ] - E[X]E[Z] + E[YZ] - E[Y]E[Z] \\ &= Cov(X,Z) + Cov(Y,Z) \end{aligned}$$

Bernoulli Distribution:

PMF: $f(k;p) = p^k(1-p)^{1-k}$

Parameters: $p \in [0,1]$

Support: $k \in \{0,1\}$

1) $X \sim Bern(p)$, prove $E[X] = p$

$$
\begin{aligned}
E[X] &= \sum_{x=0}^{1} x\, p^x (1-p)^{1-x} \\
&= (0)p^0(1-p)^1 + (1)p^1(1-p)^0 \\
&= 0 + p \\
&= p
\end{aligned}
$$

2) $X \sim Bern(p)$, prove $Var[X] = p(1-p)$

$$Var[X] = E[X^2] - (E[X])^2$$

$$
\begin{aligned}
E[X^2] &= \sum_{x=0}^{1} x^2\, p^x (1-p)^{1-x} \\
&= (0)p^0(1-p)^1 + (1)p^1(1-p)^0 \\
&= 0 + p \\
&= p
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= p - (p)^2 \\
&= p(1-p)
\end{aligned}
$$

Binomial Distribution:

PMF: $f(k;n,p) = \binom{n}{k} p^k (1-p)^{n-k}$

Parameters: $p \in [0,1]$

$n \in \{0, 1, 2, \dots, \infty\}$

Support: $k \in \{0,1,2,\dots,n\}$

1) $X \sim Binom(n,p)$, prove $E[X] = np$

Approach #1:

$$
\begin{aligned}
E[X] &= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} \binom{n}{x-1} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{(n-1)-(x-1)} \\
&= np \sum_{j=0}^{m} \binom{m}{j} p^j (1-p)^{m-j} = np * (1) \\
&= np
\end{aligned}
$$

Approach #2:

Note that with $Y \sim Binom(n,p)$, $Y = \sum_{i=1}^{n} X$, where we have “n” iid $X \sim Bern(p)$

$$Y = \sum_{i=1}^{n} X$$

$$E[Y] = E\left[\sum_{i=1}^{n} X\right] = \sum_{i=1}^{n} E[X] = \sum_{i=1}^{n} p = np$$

2) $X \sim Binom(n, p)$, prove $Var[X] = np(1-p)$

<u>Approach #1</u>:

$Var[X] = E[X^2] - (E[X])^2$

Note that:

$$
\begin{aligned}
E[X^2] &= E[X^2] - E[X] + E[X] \\
&= E[X^2 - X] + E[X] \\
&= E[X(X-1)] + E[X]
\end{aligned}
$$

$$
\begin{aligned}
E[X(X-1)] &= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} \binom{n}{x-2} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} n(n-1)\binom{n-2}{x-2} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} n(n-1)p^2\binom{n-2}{x-2} p^{x-2} (1-p)^{(n-2)-(x-2)} \\
&= n(n-1)p^2 \sum_{j=0}^{m} \binom{m}{j} p^j (1-p)^{m-j} = n(n-1)p^2
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= E[X(X-1)] + E[X] \\
&= n(n-1)p^2 + np
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= n(n-1)p^2 + np - n^2p^2 \\
&= n^2p^2 - np^2 + np - n^2p^2 \\
&= np - np^2 \\
&= np(1-p)
\end{aligned}
$$

<u>Approach #2</u>:

Note that with $Y \sim Binom(n, p)$, $Y = \sum_{i=1}^{n} X$, where we have “n” iid $X \sim Bern(p)$

$$Y = \sum_{i=1}^{n} X$$

$$Var[Y] = Var\left[\sum_{i=1}^{n} X\right] = \sum_{i=1}^{n} Var[X] = \sum_{i=1}^{n} p(1-p) = np(1-p)$$

Poisson Distribution:

PMF: $f(k;\lambda) = \frac{\lambda^k e^\lambda}{k!}$

Parameters: $\lambda \in (0, \infty)$

Support: $k \in \{0,1,2,\dots\}$

1) $X \sim Poisson(\lambda)$, prove $E[X] = \lambda$

$$
\begin{aligned}
E[X] &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^\lambda}{x!} \\
&= \sum_{x=1}^{\infty} x \frac{\lambda^x e^\lambda}{x!} \\
&= \sum_{x=1}^{\infty} \frac{\lambda^x e^\lambda}{(x-1)!} \\
&= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^\lambda}{(x-1)!} \\
&= \lambda \sum_{y=0}^{\infty} \frac{\lambda^y e^\lambda}{y!} \\
&= \lambda
\end{aligned}
$$

2) $X \sim Poisson(\lambda)$, prove $Var[X] = \lambda$

Note that:

$$
\begin{aligned}
E[X^2] &= E[X^2] - E[X] + E[X] \\
&= E[X^2 - X] + E[X] \\
&= E[X(X-1)] + E[X]
\end{aligned}
$$

$$
\begin{aligned}
E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^\lambda}{x!} \\
&= \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x e^\lambda}{x!} \\
&= \sum_{x=2}^{\infty} \frac{\lambda^x e^\lambda}{(x-2)!} \\
&= \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2} e^\lambda}{(x-2)!} \\
&= \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y e^\lambda}{y!} \\
&= \lambda^2
\end{aligned}
$$

$$Var[X] = E[X^2] - (E[X])^2 = E[X(X-1)] + E[X] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Sample mean Proof:

$x_1, x_2, \dots, x_n \sim X$, with $E[X] = \mu$

1) Prove that the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is an unbiased estimate of μ

$$
\begin{aligned}
E[\bar{x}] &= E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] \\
&= \frac{1}{n}E[(x_1 + x_2 + \cdots + x_n)] \\
&= \frac{1}{n}(E[x_1] + E[x_2] + \cdots + E[x_n]) \\
&= \frac{1}{n}(\mu + \mu + \cdots + \mu) \\
&= \frac{n\mu}{n} = \mu
\end{aligned}
$$

Sample variance Proof:

$x_1, x_2, \dots, x_n \sim X$, with $\mathrm{Var}[X] = \sigma^2$

1) Prove that $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$ is an unbiased estimate of σ^2 when μ is known:

$$
\begin{aligned}
E[s^2] &= E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2\right] \\
&= \frac{1}{n}E[(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_n - \mu)^2] \\
&= \frac{1}{n}(E[(x_1 - \mu)^2] + E[(x_2 - \mu)^2] + \cdots + E[(x_n - \mu)^2]) \\
&= \frac{1}{n}(\sigma^2 + \sigma^2 \dots + \sigma^2) \\
&= \frac{n\sigma^2}{n} = \sigma^2
\end{aligned}
$$

2) Prove that $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is an unbiased estimate of σ^2 when μ is unknown:

$$E[s^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$$

$$= \frac{1}{n-1}E\left[\sum_{i=1}^{n} {x_i}^2 - 2x_i\bar{x} + \bar{x}^2\right]$$

$$= \frac{1}{n-1}E\left[\left(\sum_{i=1}^{n} {x_i}^2\right) - \left(\sum_{i=1}^{n} 2\bar{x}x_i\right) + \left(\sum_{i=1}^{n} \bar{x}^2\right)\right]$$

$$= \frac{1}{n-1}E\left[\left(\sum_{i=1}^{n} {x_i}^2\right) - \left(2\bar{x}\sum_{i=1}^{n} x_i\right) + n\bar{x}^2\right]$$

$$= \frac{1}{n-1}E\left[\left(\sum_{i=1}^{n} {x_i}^2\right) - 2n\bar{x}^2 + n\bar{x}^2\right]$$

$$= \frac{1}{n-1}E\left[\left(\sum_{i=1}^{n} {x_i}^2\right) - n\bar{x}^2\right]$$

$$= \frac{1}{n-1}\left(\left(\sum_{i=1}^{n} E[{x_i}^2]\right) - nE[\bar{x}^2]\right)$$

$$= \frac{1}{n-1}\left((E[{x_1}^2] + E[{x_2}^2] + \cdots + E[{x_n}^2]) - nE[\bar{x}^2]\right)$$

$$= \frac{1}{n-1}(n(\sigma^2 + \mu^2) - nE[\bar{x}^2])$$

$$= \frac{n}{n-1}(\sigma^2 + \mu^2 - E[\bar{x}^2])$$

** note that: $Var[\bar{x}] = E[\bar{x}^2] - E[\bar{x}]^2$

$$E[\bar{x}^2] = Var[\bar{x}] + E[\bar{x}]^2$$

$$= Var\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] + \mu^2$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} Var[x_i]\right) + \mu^2$$

$$= \frac{n\sigma^2}{n^2} + \mu^2 = \frac{\sigma^2}{n} + \mu^2$$

Therefore:

$$E[s^2] = \frac{n}{n-1}(\sigma^2 + \mu^2 - E[\bar{x}^2])$$

$$= \frac{n}{n-1}\left(\sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2\right)$$

$$= \frac{n}{n-1}\left(\sigma^2 - \frac{\sigma^2}{n}\right)$$

$$= \frac{n}{n-1}\left(\frac{n\sigma^2 - \sigma^2}{n}\right)$$

$$= \frac{n}{n-1}\left(\frac{(n-1)\sigma^2}{n}\right) = \sigma^2$$