1 SPH w/ nuclear reaction

Suppose we know acceleration, time derviative of energy, coefficient viscosity, and the next timestep at the initial time. We integrate each particle as follows:

1. Calculate the position, and predict the velocity and energy:

$$\boldsymbol{r}_i = \boldsymbol{r}_i^{(0)} + \boldsymbol{v}_i^{(0)}\Delta t + \frac{1}{2}\boldsymbol{a}_i^{(0)}\Delta t^2 \tag{1}$$

$$\boldsymbol{v}_i^{(1/2)} = \boldsymbol{v}_i^{(0)} + \frac{1}{2}\boldsymbol{a}^{(0)}\Delta t \tag{2}$$

$$u_i^{(1/2)} = u_i^{(0)} + \frac{1}{2}\dot{u}^{(0)}\Delta t + \epsilon_{\mathrm{nuc},i}(\Delta t/2) \tag{3}$$

$$\tilde{\boldsymbol{v}}_i = \boldsymbol{v}_i^{(0)} + \boldsymbol{a}_i^{(0)}\Delta t \tag{4}$$

$$\tilde{\boldsymbol{u}}_i = \boldsymbol{u}_i^{(0)} + \dot{\boldsymbol{u}}_i^{(0)}\Delta t + \epsilon_{\mathrm{nuc},i}(\Delta t) \tag{5}$$

$$\tilde{\alpha}_i = \alpha_i^{(0)} + \dot{\alpha}_i^{(0)}\Delta t, \tag{6}$$

   where ($\epsilon_{\mathrm{nuc},i}$) is energy generated through nuclear reaction. When the reaction is exothermic, the density and temperature are fixed at this time. On the other hand, when the reaction is endothermic, only the density is fixed at this time.

2. Calculate the density, kernel-support length and grad-h term of each particle by an iterative method:

   (a) Estimate the density by the following expression:

$$\rho_i = \sum_j m_j W_i \tag{7}$$

$$\Omega_i = 1 + \frac{1}{D}\frac{H_i}{\rho_i}\sum_j m_j \frac{\partial W_i}{\partial H_i} \tag{8}$$

   where $W_i = W(r_{ij}, H_i)$, $r_{ij} = |\boldsymbol{r}_{ij}|$, and $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$. The kernel function is expressed as

$$W(r, H) = H^{-D} w(r/H), \tag{9}$$

$$\frac{\partial W(r,H)}{\partial H} = -H^{-(D+1)}\left[Dw(q) + q\frac{\partial w}{\partial q}\right]. \tag{10}$$

   The formula of $w(q)$ and $\partial w/\partial q$ are described in Appendix A.

   (b) calculate the kernel length as follow:

$$H_i = (H/h) \times \eta \left(\frac{m_i}{\rho_i}\right)^{1/D}, \tag{11}$$

   where h is the kernel length. The values of η and H/h are described in Appendix A.

(c) Return to step (2a) unless this is the 3rd time.

(d) Calculate divergence and rotation of $\boldsymbol{v}$:

$$\nabla \cdot \tilde{\boldsymbol{v}}_i = -\frac{1}{\Omega_i \rho_i} \sum \tilde{\boldsymbol{v}}_{ij} \cdot \left[m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{12}$$

$$\nabla \times \tilde{\boldsymbol{v}}_i = \frac{1}{\Omega_i \rho_i} \sum \tilde{\boldsymbol{v}}_{ij} \times \left[m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right]. \tag{13}$$

The derivative of the kernel function is as follows:

$$\frac{\partial W(r,H)}{\partial r} = H^{-(D+1)} \frac{\partial w(r/H)}{\partial (r/H)}. \tag{14}$$

3. Calculate the pressure ($\tilde{P}_i$) and sound speed ($\tilde{c}_{\mathrm{s},i}$) as follows:

$$P_i = (\gamma - 1)\rho_i \tilde{u}_i \tag{15}$$

$$c_{\mathrm{s},i} = \left(\gamma \frac{P_i}{\rho_i} \right)^{1/2}, \tag{16}$$

or Helmholtz EOS, in which case the temperature ($\tilde{T}_i$) is also obtained.

4. Calculate the Balsara switch:

$$f_i = \frac{|\nabla \cdot \tilde{\boldsymbol{v}}_i|}{|\nabla \cdot \tilde{\boldsymbol{v}}_i| + |\nabla \times \tilde{\boldsymbol{v}}_i| + 0.0001 \tilde{c}_{\mathrm{s},i}/h_i}. \tag{17}$$

5. Calculate the acceleration and the time derivative of the energy:

$$\boldsymbol{a}_i = -\sum \left(\frac{\tilde{P}_i}{\Omega_i \rho_i^2} + \frac{\tilde{P}_j}{\Omega_j \rho_j^2} + f_{ij} \Pi_{ij} \right) \left[\frac{m_j}{2} \left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}} \right) \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{18}$$

$$\dot{u}_i = \sum \left(\frac{\tilde{P}_i}{\Omega_i \rho_i^2} + \frac{f_{ij} \Pi_{ij}}{2} \right) \left[\frac{m_j}{2} \left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}} \right) \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right] \cdot \tilde{\boldsymbol{v}}_{ij}, \tag{19}$$

where $f_{ij} = (f_i + f_j)/2$, and Π_{ij} is an artificial viscosity. The artificial viscosity is expressed as follows:

$$\Pi_{ij} = -\frac{\tilde{\alpha}_i}{2} \frac{v_{ij}^{\mathrm{sig}} w_{ij}}{\rho_{ij}} \tag{20}$$

$$v_{ij}^{\mathrm{sig}} = c_{\mathrm{s},i} + c_{\mathrm{s},j} - 3 w_{ij} \tag{21}$$

$$w_{ij} = \begin{cases} \dfrac{\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij}}{|\boldsymbol{r}_{ij}|} & (\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij} < 0) \\ 0 & (\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij} \geq 0) \end{cases} \tag{22}$$

where $\rho_{ij} = (\rho_i + \rho_j)/2$.

6. Calculate $\dot{\alpha}$:

$$\dot{\alpha}_i = -\frac{\tilde{\alpha}_i - \alpha_{\rm min}}{h_i/(0.25\tilde{c}_{{\rm s},i})} + S_i \tag{23}$$
$$S_i = \max\left[-(\nabla \cdot \tilde{\boldsymbol{v}}_i)(\alpha_{\rm max} - \tilde{\alpha}_i), 0\right] \tag{24}$$

7. Correct the velocity and energy:

$$\boldsymbol{v}_i = \boldsymbol{v}_i^{(1/2)} + \frac{1}{2}\boldsymbol{a}_i \Delta t \tag{25}$$
$$u_i = u_i^{(1/2)} + \frac{1}{2}\dot{u}_i \Delta t + \epsilon_{{\rm nuc},i}(\Delta t/2) \tag{26}$$
$$\alpha_i = \alpha_i^{(1/2)} + \frac{1}{2}\dot{\alpha}_i \Delta t. \tag{27}$$

When the reaction is exothermic, the density and temperature are fixed at this time. On the other hand, when the reaction is endothermic, only the density is fixed at this time.

8. Calculate the next timestep:

$$\Delta t_{\rm next} = C \min_i \left[\frac{2H_i}{\max_j\left(v_{ij}^{\rm sig}\right)}, \Delta t \left|\frac{u_i^{(0)}}{u_i - u_i^{(0)}}\right|\right]. \tag{28}$$

9. Return to step 1.

2 Helmholtz EOS

Coorporate Timmes's EOS. The compositions are 100 % carbon, 50 % carbon and 50 % oxygen, and will be 100 % helium.

A Kernels

A.1 Cubic spline kernel

We describe the cubic spline kernel:

$$w(r/H) = 2^D \sigma \times \begin{cases} \left(1 - 6q^2 + 6q^3\right) & (0 \le q < 1/2) \\ \left[2(1-q)^3\right] & (1/2 \le q < 1) \\ 0 & (q \ge 1) \end{cases} \tag{29}$$

$$\frac{\partial w(q)}{\partial q} = (-6)2^D \sigma \times \begin{cases} 1/3 & (0 \le q < 1/3) \\ \left(2q - 3q^2\right) & (1/3 \le q < 1/2) \\ (1-q)^2 & (1/2 \le q < 1) \\ 0 & (q \ge 1) \end{cases}, \tag{30}$$

where $\sigma = 2/3$ (1D), $10/7\pi$ (2D), and $1/\pi$ (3D).

For this kernel, $\eta = 1.2$, and $H/h = 2$.

The equations (29) and (30) can be rewritten as

$$w(r/H) = 2^D \sigma \times 2 \left[(1-q)_+^3 - 4(1/2-q)_+^3 \right], \tag{31}$$

$$\frac{\partial w(q)}{\partial q} = (-6) 2^D \sigma \times \left[(1-q)_+^2 - 4(1/2-q)_+^2 + 3(1/3-q)_+^2) \right], \tag{32}$$

where $(x)_+ \equiv \max(0, x)$.

A.2 Wendland C^2 kernel

We describe Wendland C^2 kernel: In the case of $D = 1$,

$$w(q) = C_{\mathrm{W}} (1-q)_+^3 (1+3q) \tag{33}$$

$$\frac{\partial w(q)}{\partial q} = 3C_{\mathrm{W}} \left[(1-q)_+^3 - (1-q)_+^2 (1+3q) \right], \tag{34}$$

and in the case of $D = 2, 3$,

$$w(q) = C_{\mathrm{W}} (1-q)_+^4 (1+4q) \tag{35}$$

$$\frac{\partial w(q)}{\partial q} = 4C_{\mathrm{W}} \left[(1-q)_+^4 - (1-q)_+^3 (1+4q) \right], \tag{36}$$

where $C_{\mathrm{W}} = 5/4$ ($D = 1$), $7/\pi$ ($D = 2$), and $21/(2\pi)$ ($D = 3$). For this kernel, $\eta = 1.6$, and $H/h = 1.620185(D = 1)$, $1.897367(D = 2)$, and $1.93492(D = 3)$.

A.3 Wendland C^4 kernel

We describe Wendland C^4 kernel: In the case of $D = 1$,

$$w(q) = C_{\mathrm{W}} (1-q)_+^5 (1+5q+8q^2) \tag{37}$$

$$\frac{\partial w(q)}{\partial q} = C_{\mathrm{W}} (1-q)_+^4 \left[-5(1+5q+8q^2) + (1-q)_+ (5+16q) \right] \tag{38}$$

and in the case of $D = 2, 3$,

$$w(q) = C_{\mathrm{W}} (1-q)_+^6 \left[1+6q+(35/3)q^2 \right] \tag{39}$$

$$\frac{\partial w(q)}{\partial q} = C_{\mathrm{W}} (1-q)_+^5 \left\{ -6 \left[1+6q+(35/3)q^2 \right] + (1-q)_+ \left[6+(70/3)q \right] \right\} \tag{40}$$

where $C_{\mathrm{W}} = 3/2$ ($D = 1$), $9/\pi$ ($D = 2$), and $495/(32\pi)$ ($D = 3$). For this kernel, $\eta = 1.6$, and $H/h = 1.936492(D = 1)$, $2.171239(D = 2)$, and $2.207940(D = 3)$.

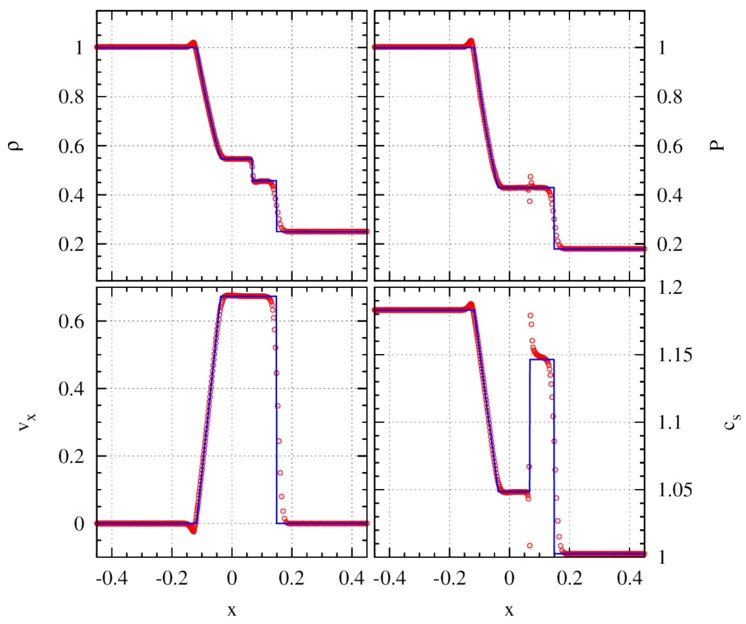

Figure 1: 1D shock tube (cubic spline, $\gamma = 1.4$, $\alpha = 1.0$).

B SPH tests

1. 1D shock tube (CUBICSPLINE, USE_AT1D)
2. 3D shock tube (CUBICSPLINE)
3. Strong shock (CUBICSPLINE, USE_AT1D)
4. Point like explosion (CUBICSPLINE)
5. Evrard test (CUBICSPLINE, USE_AT3D, GRAVITY)

C EOS tests

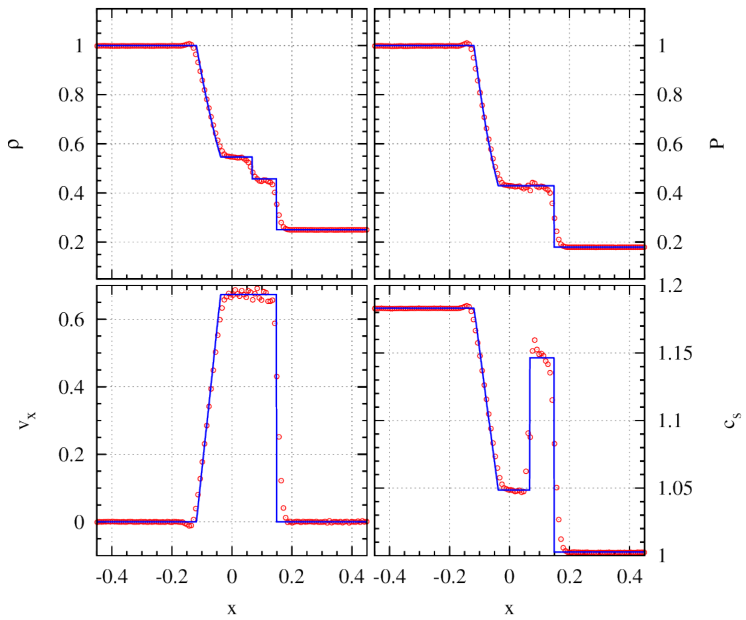

Figure 2: 3D shock tube (cubic spline, $\gamma = 1.4$, $\alpha = 1.0$).

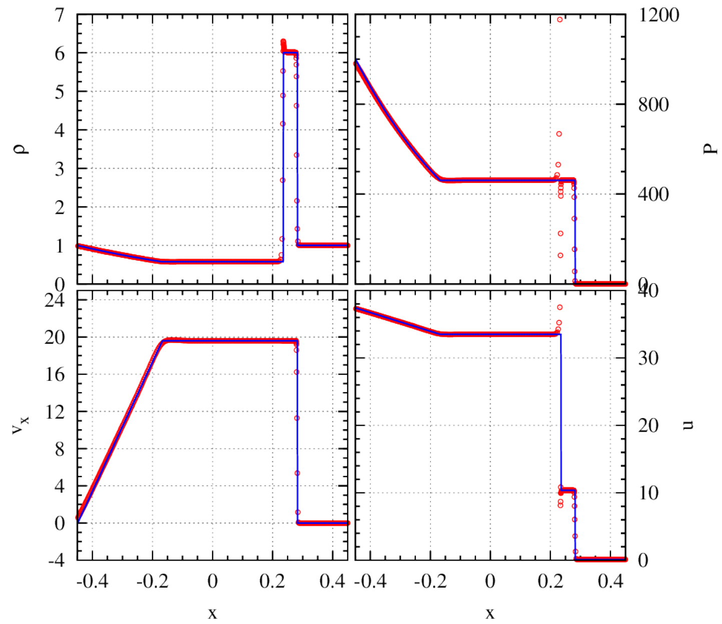

Figure 3: Strong shock (cubic spline, $\gamma = 1.4$, $\alpha = 1.0$).

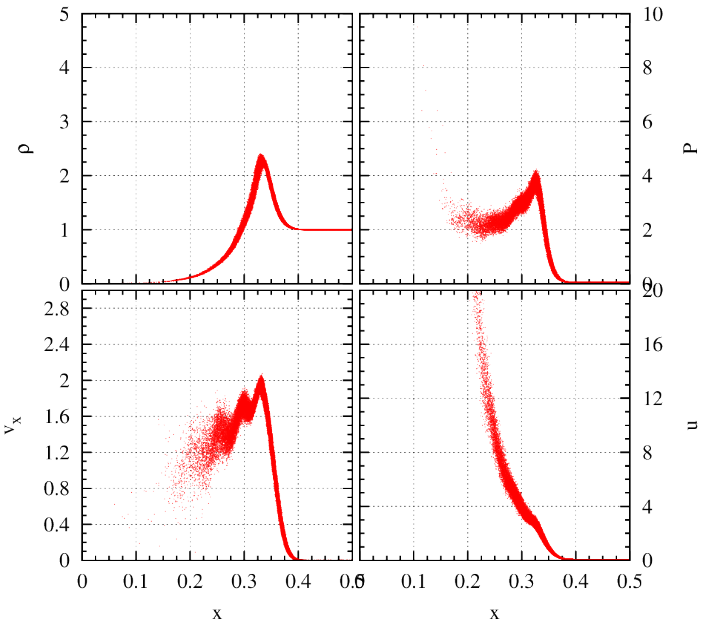

Figure 4: Point like explosion (cubic spline, $\gamma = 5/3$, $\alpha = 3.0$).

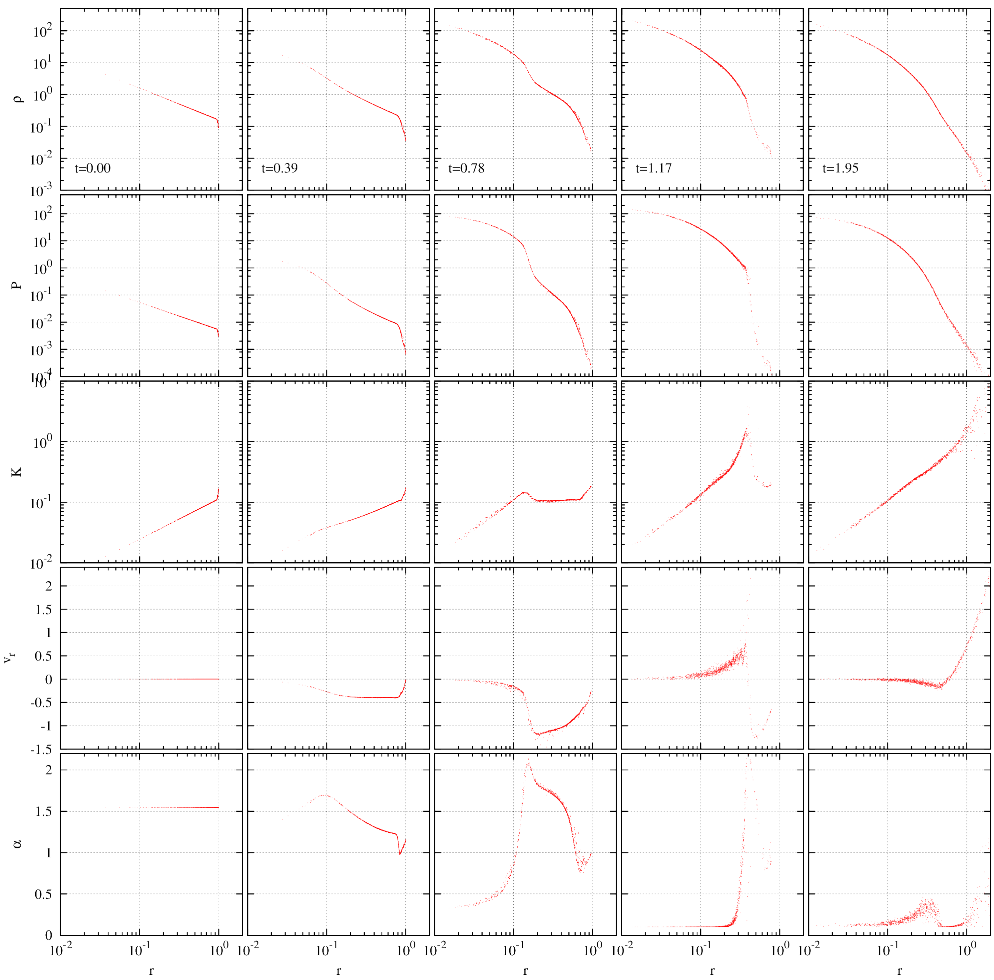

Figure 5: Evrard test (cubic spline, $\gamma = 5/3$, $\alpha_{\min} = 0.1$, $\alpha_{\max} = 3.0$).

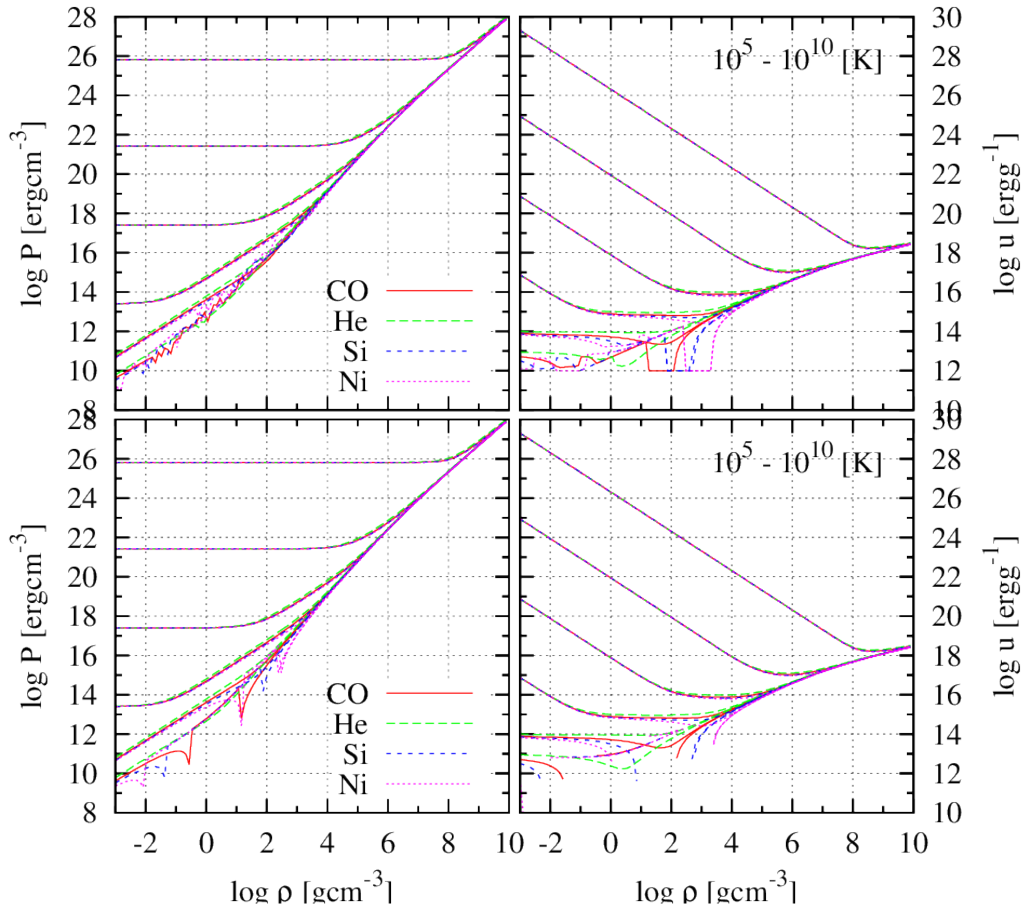

Figure 6: Helmholtz EOS w/ lookup table (top) and w/o (bottom).

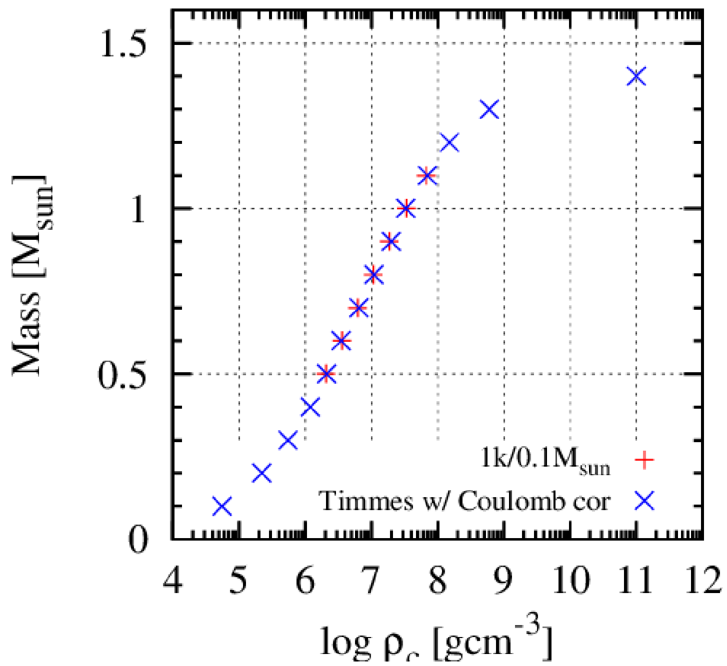

Figure 7: Single COWD equilibrium ($1k/0.1M_{\odot}$) w/ new damping. The new damping mode is the addtion of a sort of cooling during 100 s, such that $u_{\text{new},i} = (u_{\text{old},i} - u_{\min}(\rho_i))\exp(-0.1\Delta t) + u_{\min}(\rho_i)$.

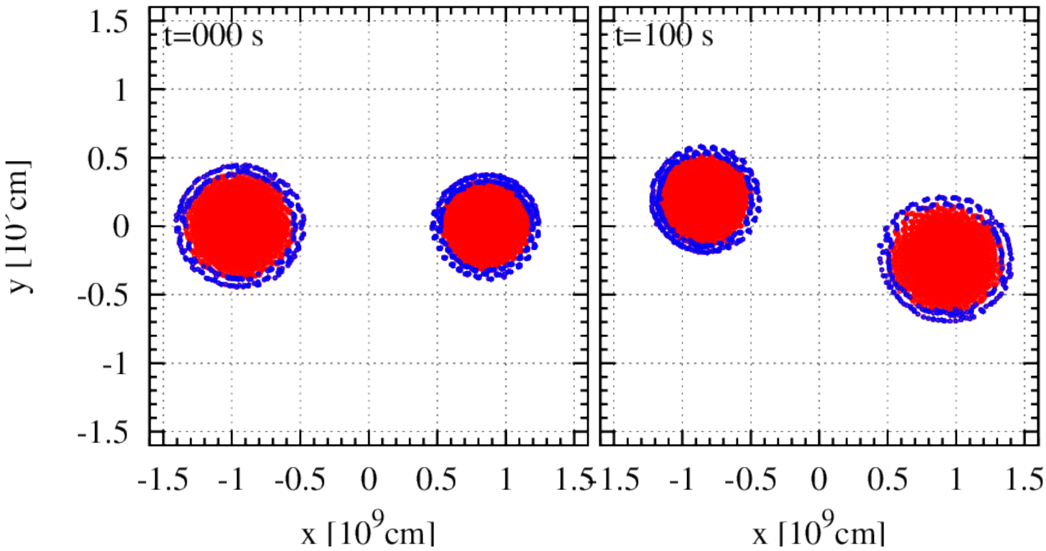

Figure 8: Density and temperature of $1.1M_{\odot}$ and $1.0M_{\odot}$ COWDs ($1k/0.1M_{\odot}$) from 1.8×10^9cm. Blue points indicate helium particles ($f_{\rm He} = 0.1$).

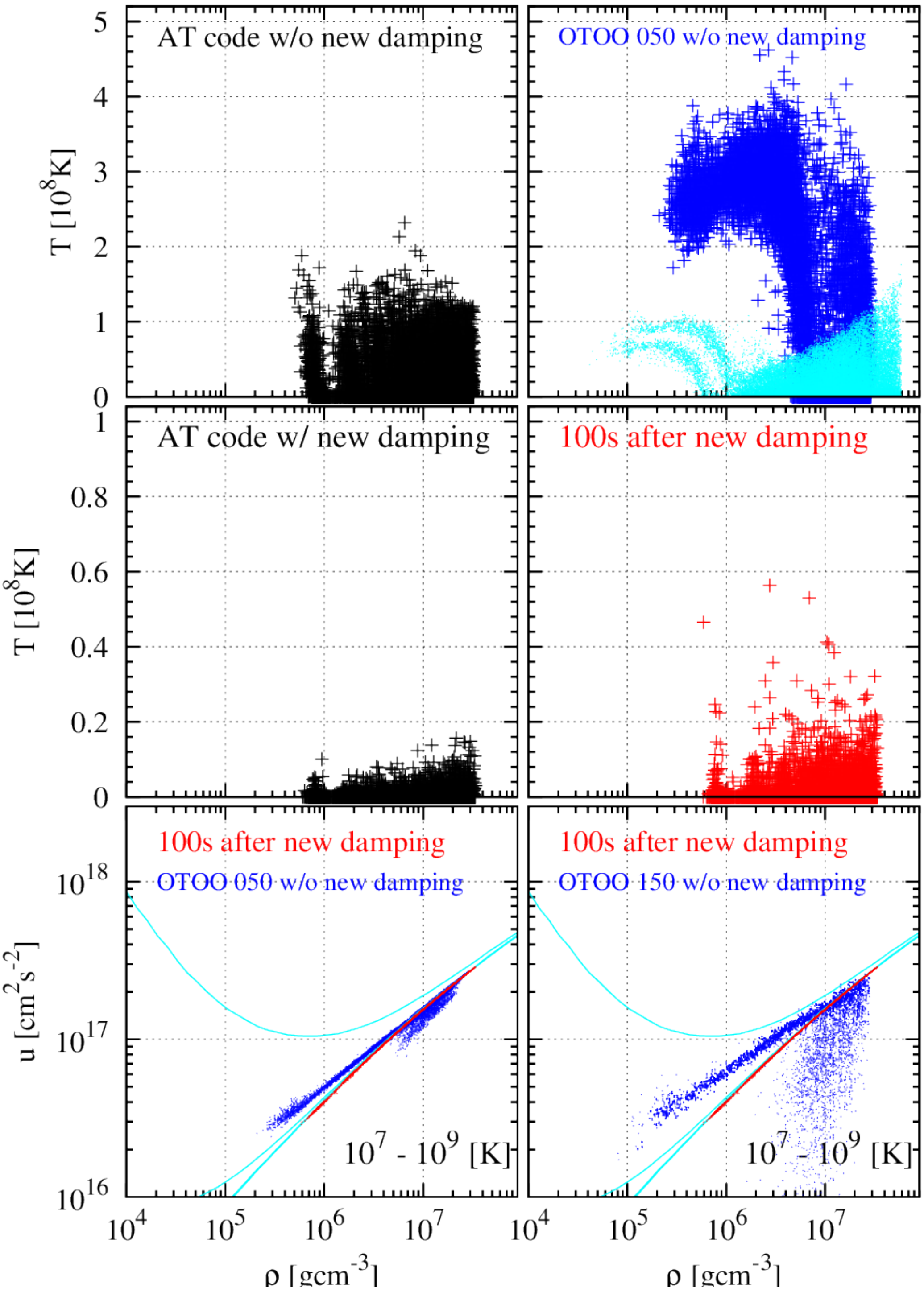

Figure 9: $1M_\odot$ COWD ($1k/0.1M_\odot$).

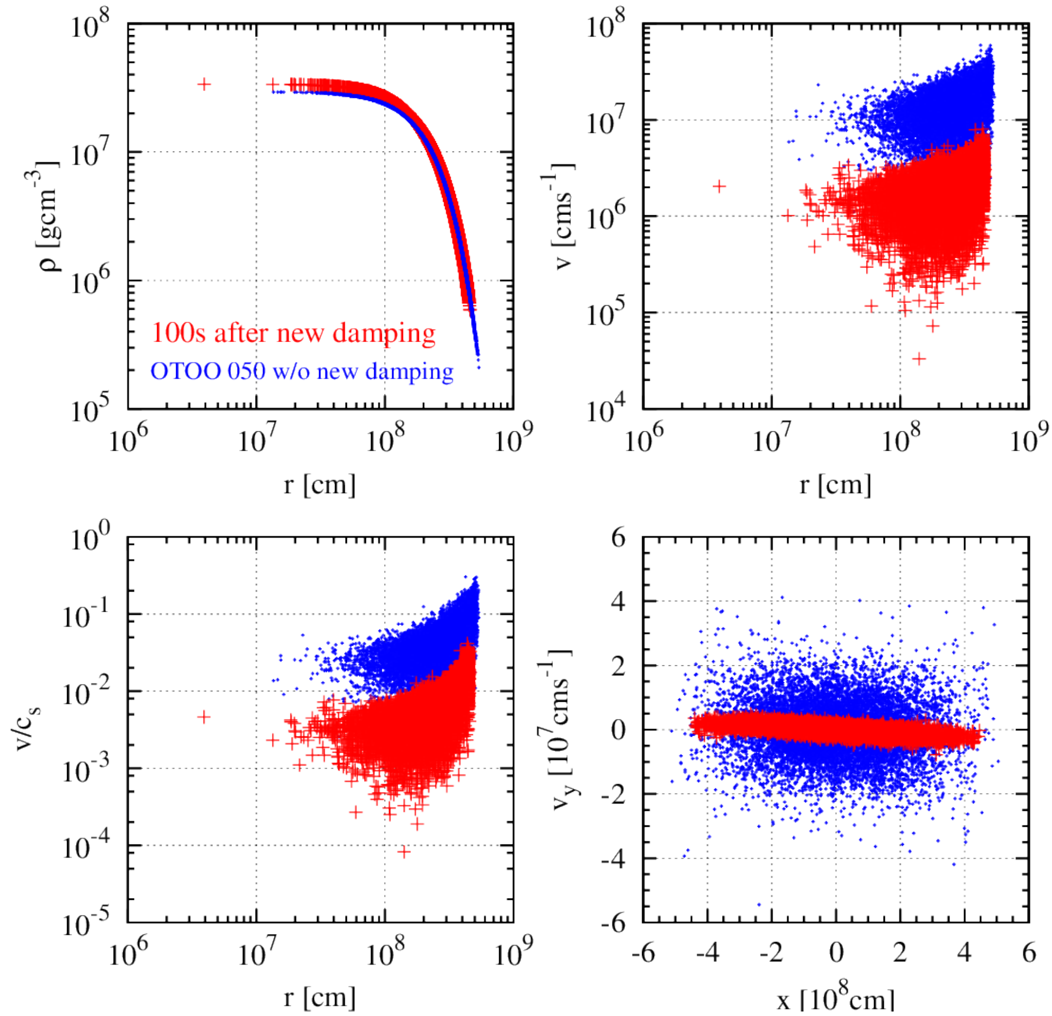

Figure 10: $1M_{\odot}$ COWD ($1k/0.1M_{\odot}$).

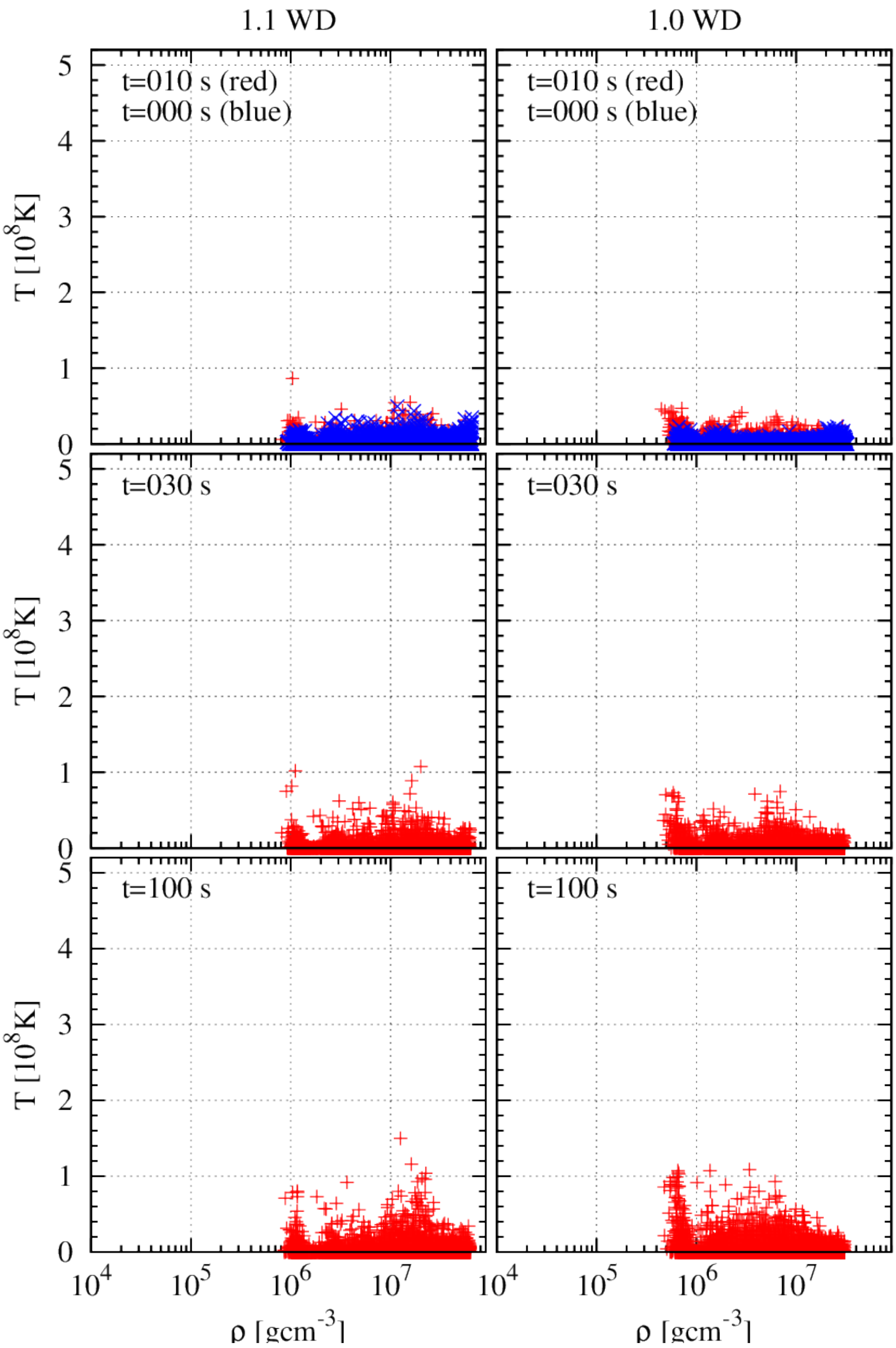

Figure 11: Density and temperature of $1.1M_\odot$ and $1.0M_\odot$ COWDs ($1k/0.1M_\odot$) from 1.8×10^9cm.

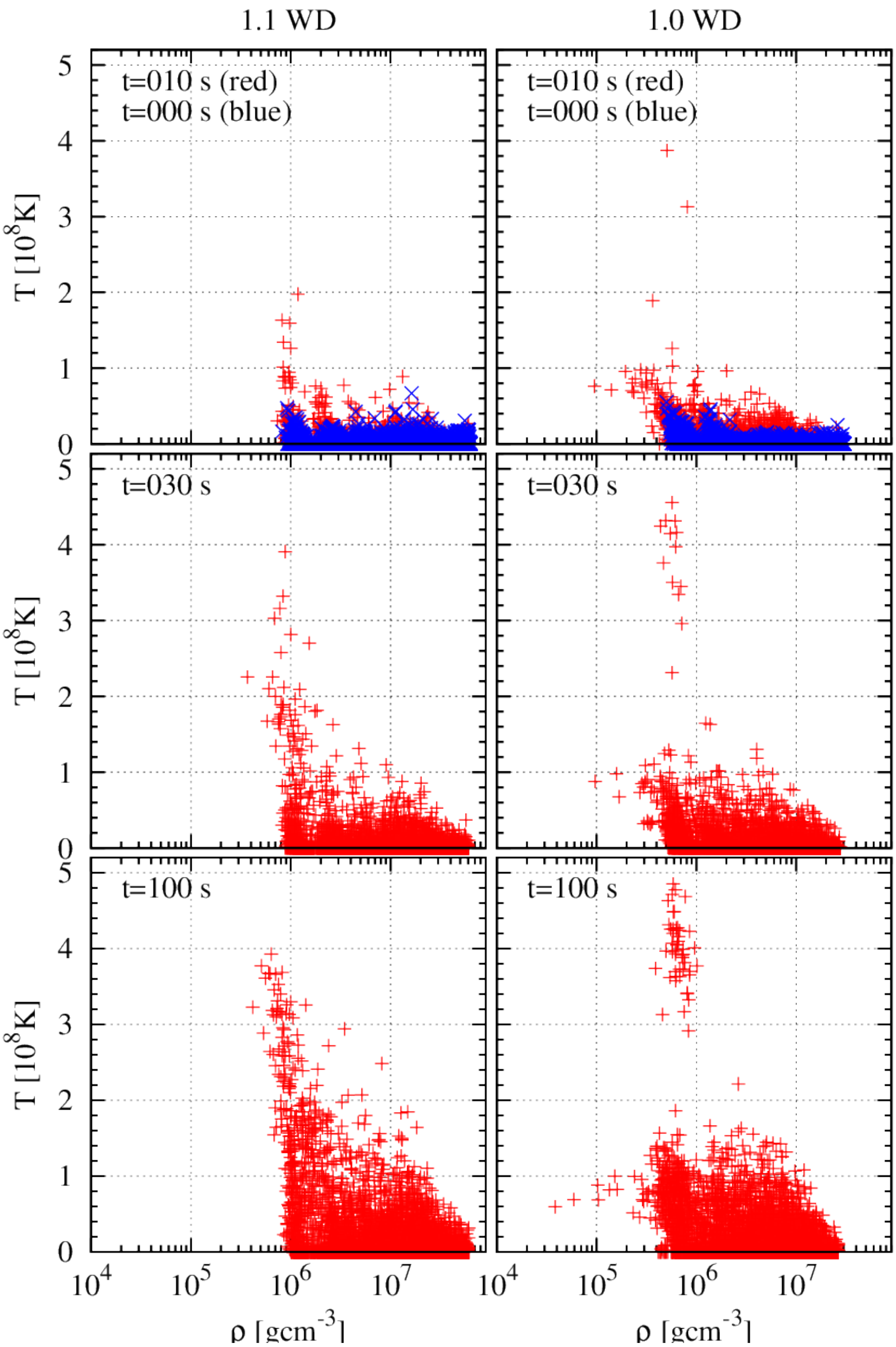

Figure 12: Density and temperature of $1.1M_{\odot}$ and $1.0M_{\odot}$ COWDs ($1k/0.1M_{\odot}$) from 1.5×10^9cm.