

## 1 Variable kernel w/ grad-h

Suppose we know acceleration, time derviative of energy, coefficient viscosity, and the next timestep at the initial time. We integrate each particle as follows:

1. Calculate the position, and predict the velocity and energy:

$$\mathbf{r}_i = \mathbf{r}_i^{(0)} + \mathbf{v}_i^{(0)} \Delta t + \frac{1}{2} \mathbf{a}_i^{(0)} \Delta t^2 \quad (1)$$

$$\mathbf{v}_i^{(1/2)} = \mathbf{v}_i^{(0)} + \frac{1}{2} \mathbf{a}_i^{(0)} \Delta t \quad (2)$$

$$u_i^{(1/2)} = u_i^{(0)} + \frac{1}{2} \dot{u}_i^{(0)} \Delta t \quad (3)$$

$$\tilde{\mathbf{v}}_i = \mathbf{v}_i^{(0)} + \mathbf{a}_i^{(0)} \Delta t \quad (4)$$

$$\tilde{\mathbf{u}}_i = \mathbf{u}_i^{(0)} + \dot{\mathbf{u}}_i^{(0)} \Delta t \quad (5)$$

$$\tilde{\alpha}_i = \alpha_i^{(0)} + \dot{\alpha}_i^{(0)} \Delta t. \quad (6)$$

2. Calculate the density, kernel-support length and grad-h term of each particle by an iterative method:

- (a) Estimate the density by the following expression:

$$\rho_i = \sum_j m_j W_i \quad (7)$$

$$\Omega_i = 1 + \frac{1}{D} \frac{H_i}{\rho_i} \sum_j m_j \frac{\partial W_i}{\partial H_i} \quad (8)$$

where  $W_i = W(r_{ij}, H_i)$ ,  $r_{ij} = |\mathbf{r}_{ij}|$ , and  $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ . The kernel function is expressed as

$$W(r, H) = H^{-D} w(r/H), \quad (9)$$

$$\frac{\partial W(r, H)}{\partial H} = -H^{-(D+1)} \left[ Dw(q) + q \frac{\partial w}{\partial q} \right]. \quad (10)$$

The formula of  $w(q)$  and  $\partial w / \partial q$  are described in Appendix A.

- (b) calculate the kernel length as follow:

$$H_i = (H/h) \times \eta \left( \frac{m_i}{\rho_i} \right)^{1/D}, \quad (11)$$

where  $h$  is the kernel length. The values of  $\eta$  and  $H/h$  are described in Appendix A.

- (c) Return to step (2a) if this is the first time.

(d) Calculate divergence and rotation of  $\mathbf{v}$ :

$$\nabla \cdot \tilde{\mathbf{v}}_i = -\frac{1}{\Omega_i \rho_i} \sum \tilde{\mathbf{v}}_{ij} \cdot \left[ m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\mathbf{r}_{ij}}{r_{ij}} \right], \quad (12)$$

$$\nabla \times \tilde{\mathbf{v}}_i = \frac{1}{\Omega_i \rho_i} \sum \tilde{\mathbf{v}}_{ij} \times \left[ m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\mathbf{r}_{ij}}{r_{ij}} \right]. \quad (13)$$

The derivative of the kernel function is as follows:

$$\frac{\partial W(r, H)}{\partial r} = H^{-(D+1)} \frac{\partial w(r/H)}{\partial (r/H)}. \quad (14)$$

3. Calculate the pressure ( $\tilde{P}_i$ ) and sound speed ( $\tilde{c}_{s,i}$ ) as follows:

$$P_i = (\gamma - 1) \rho_i \tilde{u}_i \quad (15)$$

$$c_{s,i} = \left( \gamma \frac{P_i}{\rho_i} \right)^{1/2}. \quad (16)$$

4. Calculate the Balsara switch:

$$f_i = \frac{|\nabla \cdot \tilde{\mathbf{v}}_i|}{|\nabla \cdot \tilde{\mathbf{v}}_i| + |\nabla \times \tilde{\mathbf{v}}_i| + 0.0001 \tilde{c}_{s,i}/h_i}. \quad (17)$$

5. Calculate the acceleration and the time derivative of the energy:

$$\mathbf{a}_i = -\sum \left( \frac{\tilde{P}_i}{\Omega_i \rho_i^2} + \frac{\tilde{P}_j}{\Omega_j \rho_j^2} + f_{ij} \Pi_{ij} \right) \left[ \frac{m_j}{2} \left( \frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}} \right) \frac{\mathbf{r}_{ij}}{r_{ij}} \right], \quad (18)$$

$$\dot{u}_i = \sum \left( \frac{\tilde{P}_i}{\Omega_i \rho_i^2} + \frac{f_{ij} \Pi_{ij}}{2} \right) \left[ \frac{m_j}{2} \left( \frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}} \right) \frac{\mathbf{r}_{ij}}{r_{ij}} \right] \cdot \tilde{\mathbf{v}}_{ij}, \quad (19)$$

where  $f_{ij} = (f_i + f_j)/2$ , and  $\Pi_{ij}$  is an artificial viscosity. The artificial viscosity is expressed as follows:

$$\Pi_{ij} = -\frac{\tilde{\alpha}_i}{2} \frac{v_{ij}^{\text{sig}} w_{ij}}{\rho_{ij}} \quad (20)$$

$$v_{ij}^{\text{sig}} = c_{s,i} + c_{s,j} - 3w_{ij} \quad (21)$$

$$w_{ij} = \begin{cases} \frac{\mathbf{r}_{ij} \cdot \tilde{\mathbf{v}}_{ij}}{|\mathbf{r}_{ij}|} & (\mathbf{r}_{ij} \cdot \tilde{\mathbf{v}}_{ij} < 0) \\ 0 & (\mathbf{r}_{ij} \cdot \tilde{\mathbf{v}}_{ij} \geq 0) \end{cases} \quad (22)$$

where  $\rho_{ij} = (\rho_i + \rho_j)/2$ .

6. Calculate  $\dot{\alpha}$ :

$$\dot{\alpha}_i = -\frac{\tilde{\alpha}_i - \alpha_{\min}}{h_i/(0.25 \tilde{c}_{s,i})} + S_i \quad (23)$$

$$S_i = \max [-(\nabla \cdot \tilde{\mathbf{v}}_i)(\alpha_{\max} - \tilde{\alpha}_i), 0] \quad (24)$$

7. Correct the velocity and energy:

$$\mathbf{v}_i = \mathbf{v}_i^{(1/2)} + \frac{1}{2}\mathbf{a}_i\Delta t \quad (25)$$

$$u_i = u_i^{(1/2)} + \frac{1}{2}\dot{u}_i\Delta t \quad (26)$$

$$\alpha_i = \alpha_i^{(1/2)} + \frac{1}{2}\dot{\alpha}_i\Delta t. \quad (27)$$

8. Calculate the next timestep:

$$\Delta t = C \min_i \left[ \min \left( \frac{2H_i}{\max_j (v_{ij}^{\text{sig}})}, \frac{u_i}{|\dot{u}_i|} \right) \right]. \quad (28)$$

9. Return to step 1.

## 2 Helmholtz EOS

Cooperate Timmes's EOS. The compositions are 100 % carbon, 50 % carbon and 50 % oxygen, and 100 % helium.

## A Kernels

### A.1 Cubic spline kernel

We describe the cubic spline kernel:

$$w(r/H) = 2^D \sigma \times \begin{cases} (1 - 6q^2 + 6q^3) & (0 \leq q < 1/2) \\ [2(1 - q)^3] & (1/2 \leq q < 1) \\ 0 & (q \geq 1) \end{cases} \quad (29)$$

$$\frac{\partial w(q)}{\partial q} = (-6)2^D \sigma \times \begin{cases} 1/3 & (0 \leq q < 1/3) \\ (2q - 3q^2) & (1/3 \leq q < 1/2) \\ (1 - q)^2 & (1/2 \leq q < 1) \\ 0 & (q \geq 1) \end{cases}, \quad (30)$$

$$(31)$$

where  $\sigma = 2/3$  (1D),  $10/7\pi$  (2D), and  $1/\pi$  (3D).

For this kernel,  $\eta = 1.2$ , and  $H/h = 2$ .

### A.2 Wendland C<sup>2</sup> kernel

We describe Wendland C<sup>2</sup> kernel: In the case of  $D = 1$ ,

$$w(q) = C_W(1 - q)_+^3(1 + 3q) \quad (32)$$

$$\frac{\partial w(q)}{\partial q} = 3C_W [(1 - q)_+^3 - (1 - q)_+^2(1 + 3q)], \quad (33)$$

and in the case of  $D = 2, 3$ ,

$$w(q) = C_W(1 - q)_+^4(1 + 4q) \quad (34)$$

$$\frac{\partial w(q)}{\partial q} = 4C_W [(1 - q)_+^4 - (1 - q)_+^3(1 + 4q)], \quad (35)$$

where  $C_W = 5/4$  ( $D = 1$ ),  $7/\pi$  ( $D = 2$ ), and  $21/(2\pi)$  ( $D = 3$ ). For this kernel,  $\eta = 1.6$ , and  $H/h = 1.620185$  ( $D = 1$ ),  $1.897367$  ( $D = 2$ ), and  $1.93492$  ( $D = 3$ ).

### A.3 Wendland $C^4$ kernel

We describe Wendland  $C^4$  kernel: In the case of  $D = 1$ ,

$$w(q) = C_W(1 - q)_+^5(1 + 5q + 8q^2) \quad (36)$$

$$\frac{\partial w(q)}{\partial q} = C_W(1 - q)_+^4 [-5(1 + 5q + 8q^2) + (1 - q)_+(5 + 16q)] \quad (37)$$

and in the case of  $D = 2, 3$ ,

$$w(q) = C_W(1 - q)_+^6 [1 + 6q + (35/3)q^2] \quad (38)$$

$$\frac{\partial w(q)}{\partial q} = C_W(1 - q)_+^5 \{-6 [1 + 6q + (35/3)q^2] + (1 - q)_+[6 + (70/3)q]\} \quad (39)$$

where  $C_W = 3/2$  ( $D = 1$ ),  $9/\pi$  ( $D = 2$ ), and  $495/(32\pi)$  ( $D = 3$ ). For this kernel,  $\eta = 1.6$ , and  $H/h = 1.936492$  ( $D = 1$ ),  $2.171239$  ( $D = 2$ ), and  $2.207940$  ( $D = 3$ ).

## B SPH tests

1. 1D shock tube (CUBICSPLINE, USE\_AT1D)
2. 3D shock tube (CUBICSPLINE)
3. Strong shock (CUBICSPLINE, USE\_AT1D)
4. Point like explosion (CUBICSPLINE)
5. Evrard test (CUBICSPLINE, USE\_AT3D, GRAVITY)

## C EOS tests

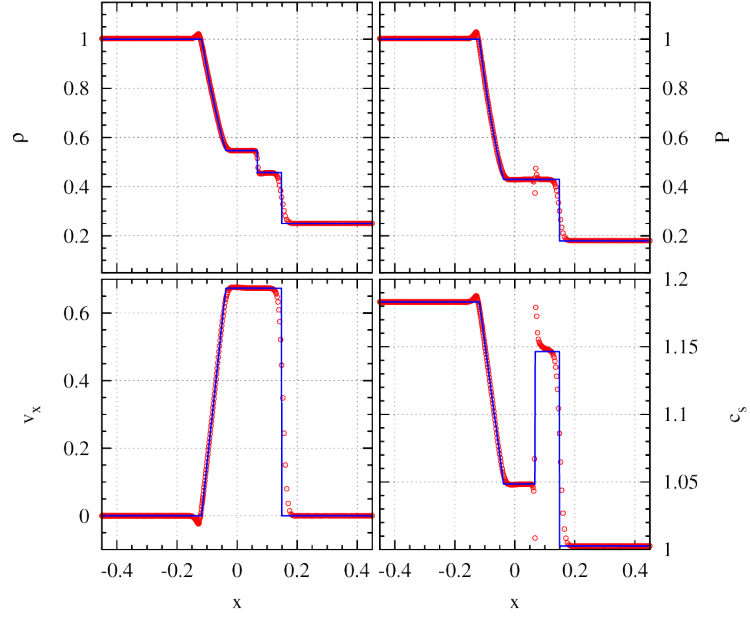


Figure 1: 1D shock tube (cubic spline,  $\gamma = 1.4$ ,  $\alpha = 1.0$ ).

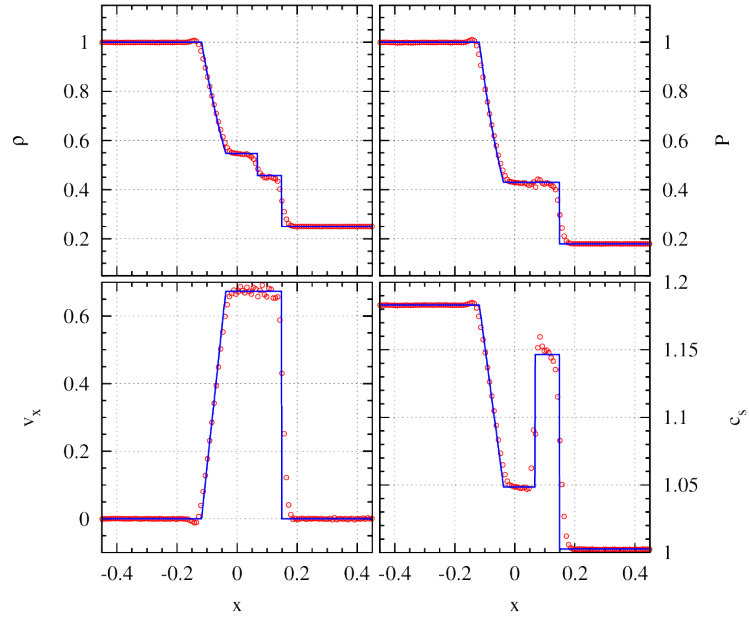


Figure 2: 3D shock tube (cubic spline,  $\gamma = 1.4$ ,  $\alpha = 1.0$ ).

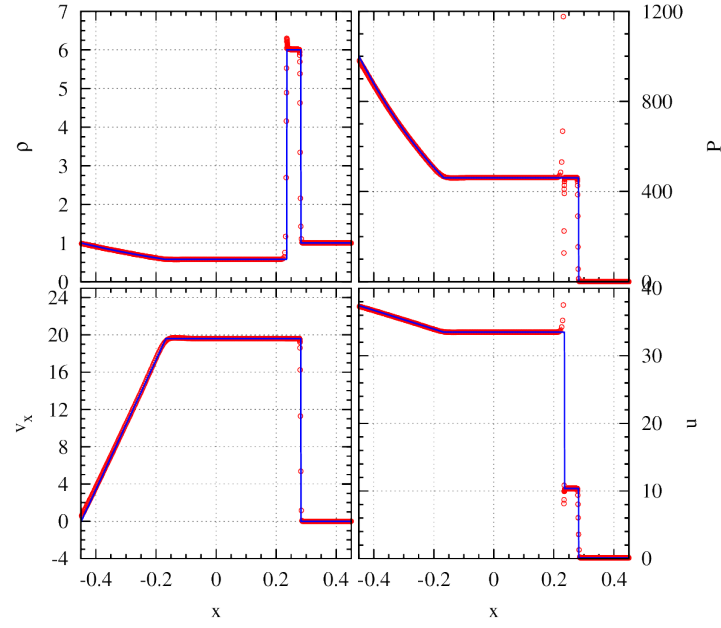


Figure 3: Strong shock (cubic spline,  $\gamma = 1.4$ ,  $\alpha = 1.0$ ).

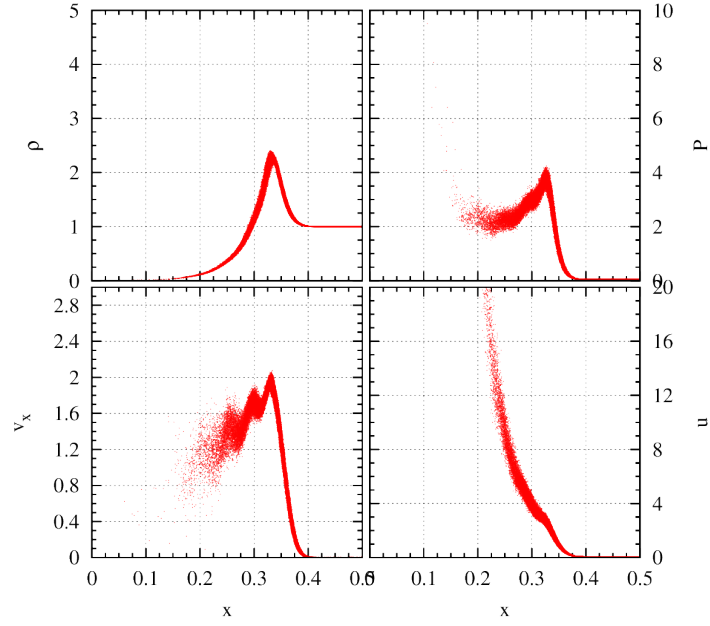


Figure 4: Point like explosion (cubic spline,  $\gamma = 5/3$ ,  $\alpha = 3.0$ ).

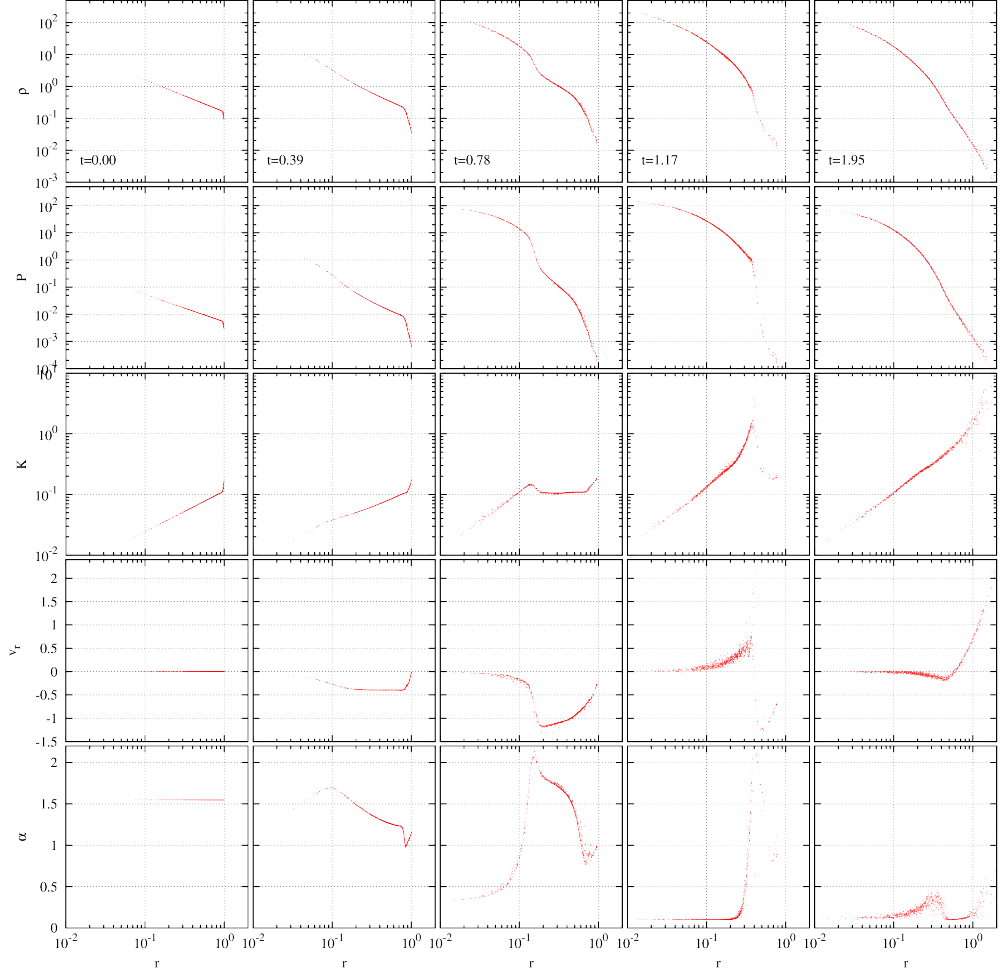


Figure 5: Evrard test (cubic spline,  $\gamma = 5/3$ ,  $\alpha_{\min} = 0.1$ ,  $\alpha_{\max} = 3.0$ ).

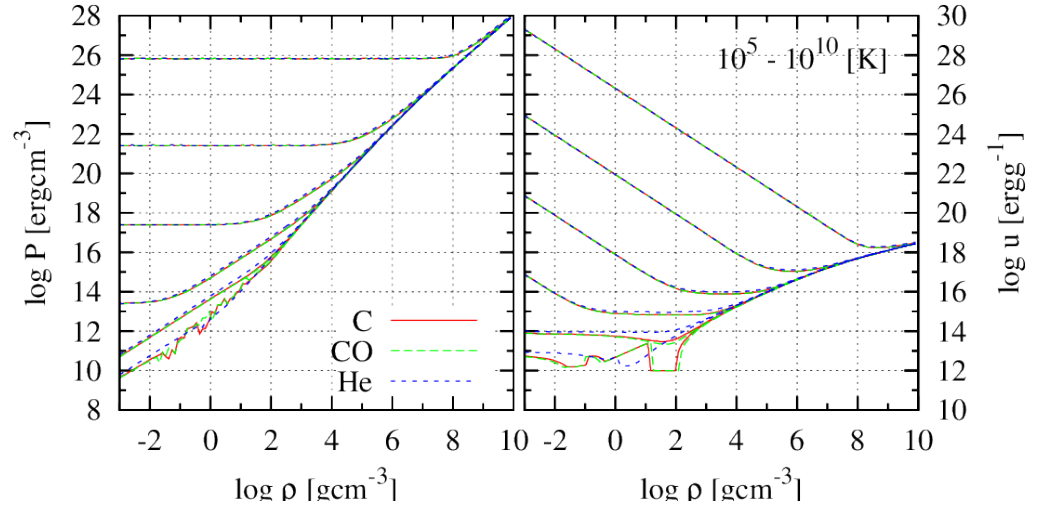


Figure 6: Helmholtz EOS.



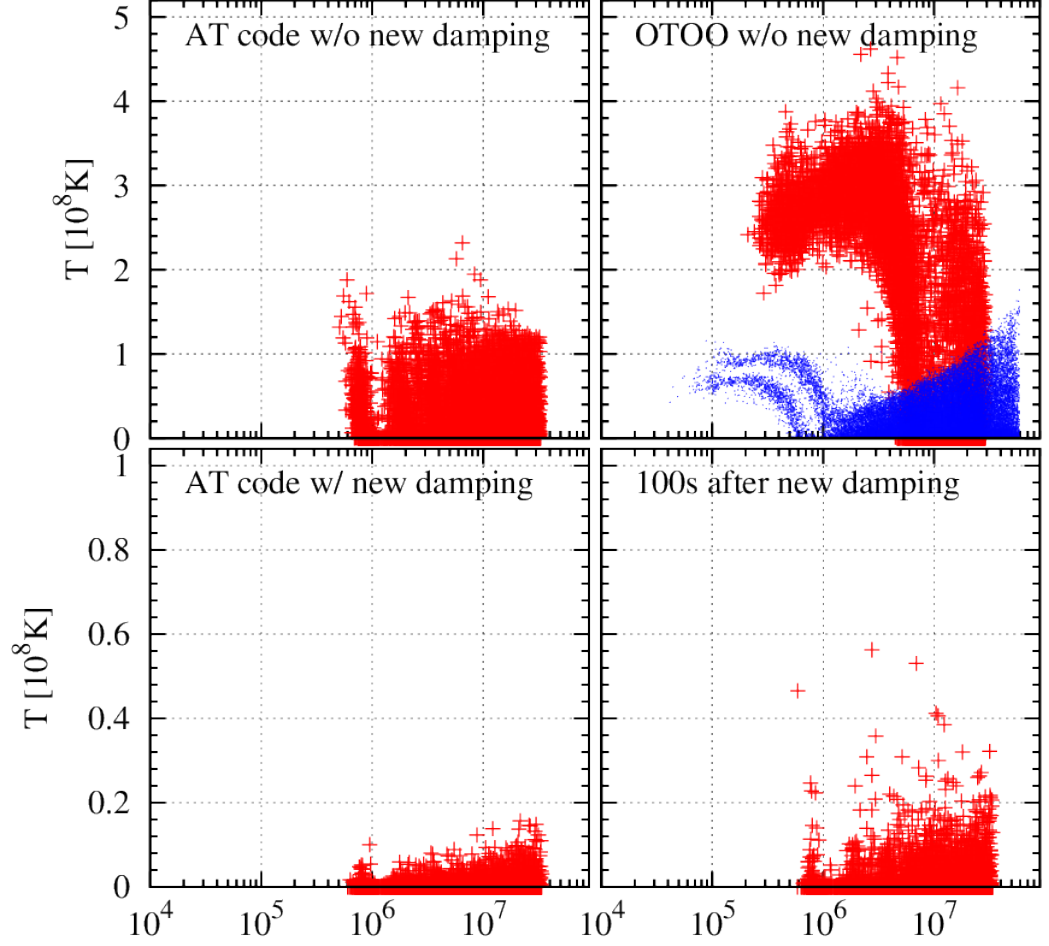


Figure 7: Density and temperature of  $1M_{\odot}$  COWD ( $1k/0.1M_{\odot}$ ). A new damping mode with cooling during 100 s, such that  $u_{\text{new},i} = (u_{\text{old},i} - u_{\text{min}}(\rho_i)) \exp(-0.1\Delta t) + u_{\text{min}}(\rho_i)$ .

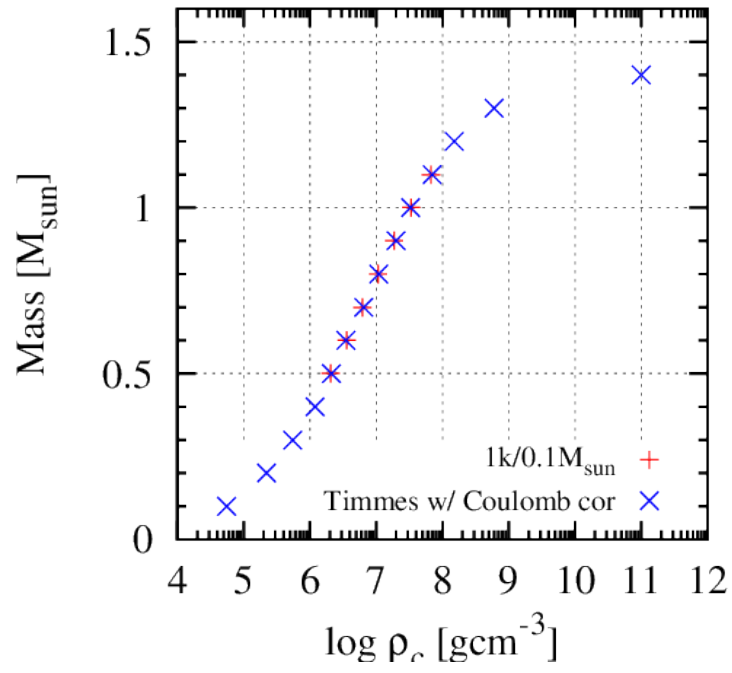


Figure 8: Single COWD equilibrium ( $1k/0.1M_{\odot}$  w/ new damping).

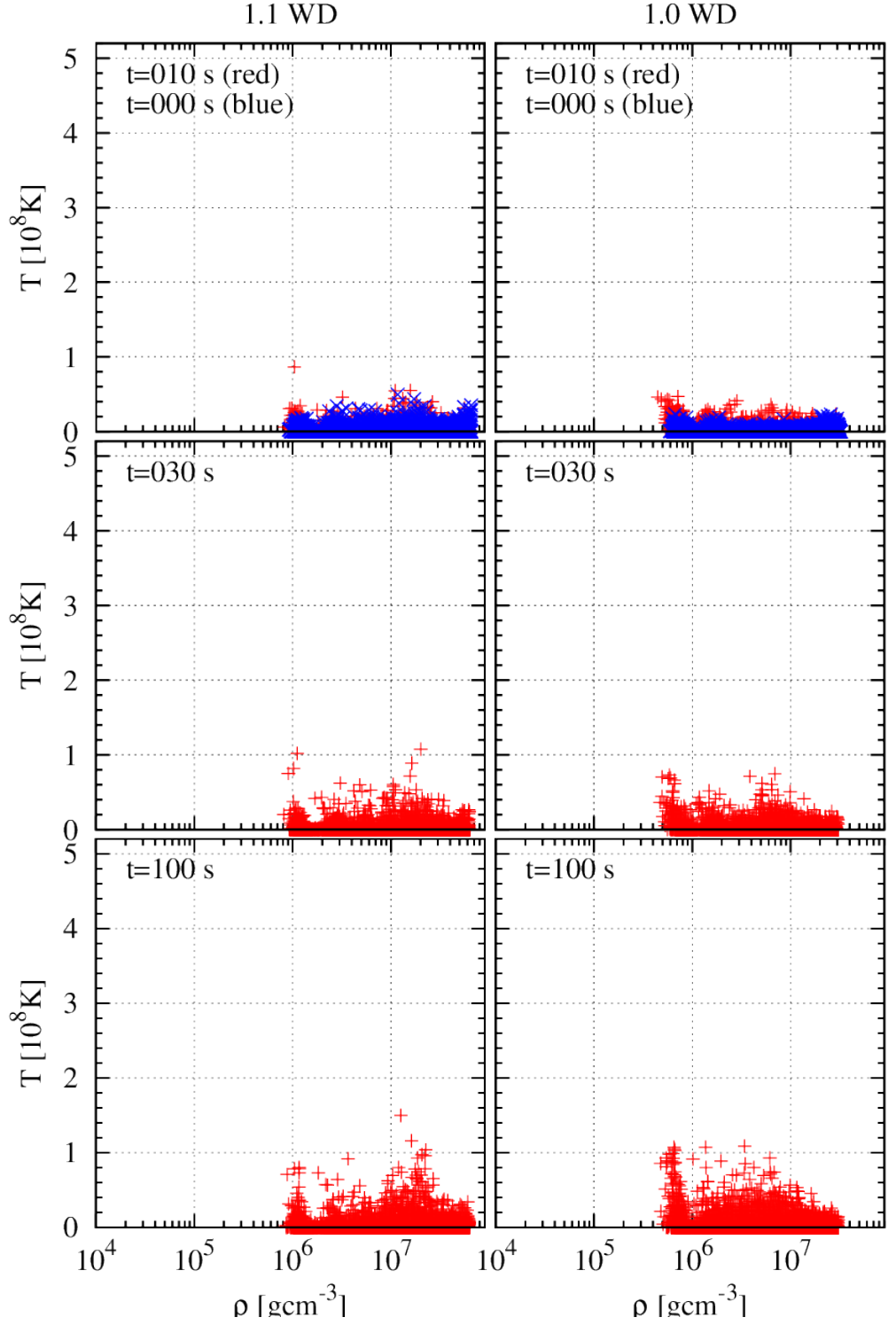


Figure 9: Density and temperature of  $1.1M_{\odot}$  and  $1.0M_{\odot}$  COWDs ( $1k/0.1M_{\odot}$ ) from  $1.8 \times 10^9 \text{cm}$ .

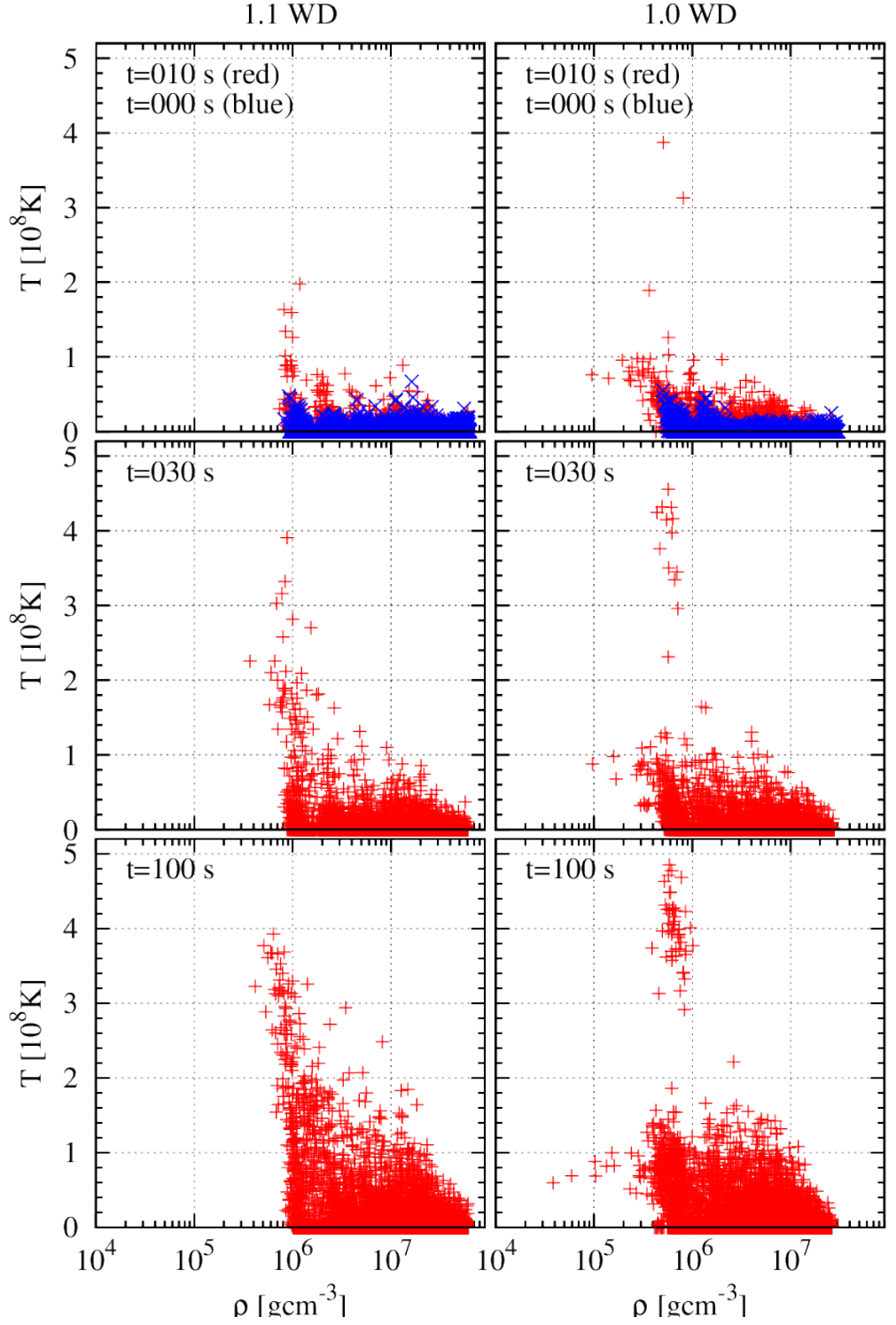


Figure 10: Density and temperature of  $1.1M_{\odot}$  and  $1.0M_{\odot}$  COWDs ( $1k/0.1M_{\odot}$ ) from  $1.5 \times 10^9$ cm.

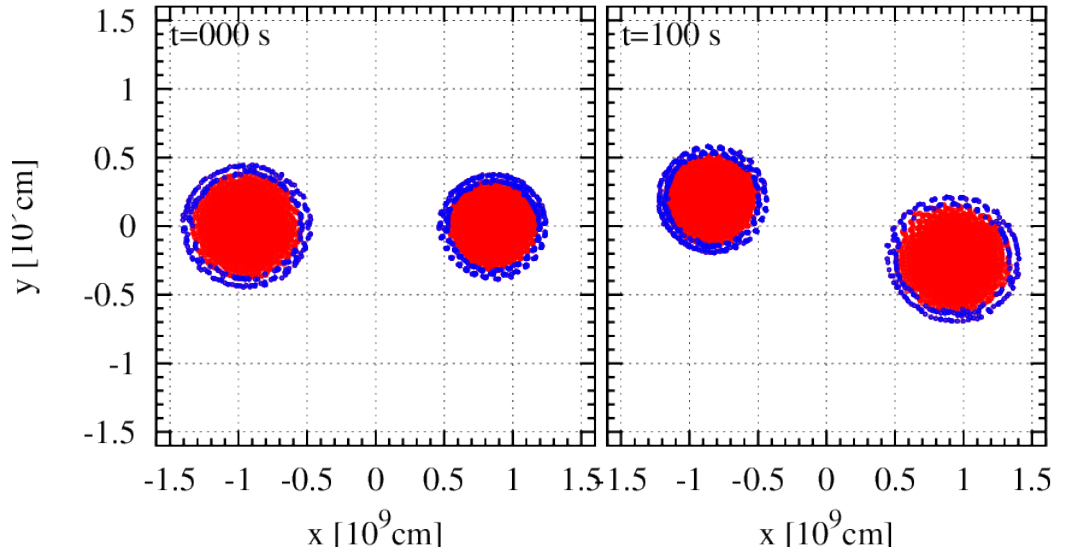


Figure 11: Density and temperature of  $1.1M_{\odot}$  and  $1.0M_{\odot}$  COWDs ( $1k/0.1M_{\odot}$ ) from  $1.8 \times 10^9$  cm. Blue points indicate helium particles ( $f_{\text{He}} = 0.1$ ).