1 Variable kernel w/ grad-h

Suppose we know acceleration, time derviative of energy, coefficient viscosity, and the next timestep at the initial time. We integrate each particle as follows:

1. Calculate the position, and predict the velocity and energy:

$$\boldsymbol{r}_i = \boldsymbol{r}_i^{(0)} + \boldsymbol{v}_i^{(0)}\Delta t + \frac{1}{2}\boldsymbol{a}_i^{(0)}\Delta t^2 \tag{1}$$

$$\boldsymbol{v}_i^{(1/2)} = \boldsymbol{v}_i^{(0)} + \frac{1}{2}\boldsymbol{a}^{(0)}\Delta t \tag{2}$$

$$u_i^{(1/2)} = u_i^{(0)} + \frac{1}{2}\dot{u}^{(0)}\Delta t \tag{3}$$

$$\tilde{\boldsymbol{v}}_i = \boldsymbol{v}_i^{(0)} + \boldsymbol{a}_i^{(0)}\Delta t \tag{4}$$

$$\tilde{\boldsymbol{u}}_i = \boldsymbol{u}_i^{(0)} + \dot{\boldsymbol{u}}_i^{(0)}\Delta t \tag{5}$$

$$\tilde{\alpha}_i = \alpha_i^{(0)} + \dot{\alpha}_i^{(0)}\Delta t. \tag{6}$$

2. Calculate the density, kernel-support length and grad-h term of each particle by an iterative method:

   (a) Estimate the density by the following expression:

$$\rho_i = \sum_j m_j W_i \tag{7}$$

$$\Omega_i = 1 + \frac{1}{D}\frac{H_i}{\rho_i}\sum_j m_j \frac{\partial W_i}{\partial H_i} \tag{8}$$

   where $W_i = W(r_{ij}, H_i)$, $r_{ij} = |\boldsymbol{r}_{ij}|$, and $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$. The kernel function is expressed as

$$W(r, H) = H^{-D} w(r/H), \tag{9}$$

$$\frac{\partial W(r, H)}{\partial H} = -H^{-(D+1)}\left[D w(q) + q\frac{\partial w}{\partial q}\right]. \tag{10}$$

   In the case of $D = 1$,

$$w(q) = \frac{5}{4}(1-q)_+^3(1+3q) \tag{11}$$

$$\frac{\partial w(q)}{\partial q} = \frac{15}{4}\left[(1-q)_+^3 - (1-q)_+^2(1+3q)\right], \tag{12}$$

   and in the case of $D = 2, 3$,

$$w(q) = C_{\mathrm{W}}(1-q)_+^4(1+4q) \tag{13}$$

$$\frac{\partial w(q)}{\partial q} = 4C_{\mathrm{W}}\left[(1-q)_+^4 - (1-q)_+^3(1+4q)\right], \tag{14}$$

   where $C_{\mathrm{W}} = 7/\pi$ $(D = 2)$, and $C_{\mathrm{W}} = 21/(2\pi)$ $(D = 3)$.

(b) calculate the kernel length as follow:

$$H_i = (H/h) \times \eta \left(\frac{m_i}{\rho_i}\right)^{1/D}, \tag{15}$$

where h is the kernel length, $\eta = 1.6$, and $H/h = 1.620185(D = 1)$, $1.897367(D = 2)$, and $1.93492(D = 3)$.

(c) Return to step (2a) if this is the first time.

(d) Calculate divergence and rotation of $\boldsymbol{v}$:

$$\nabla \cdot \tilde{\boldsymbol{v}}_i = \frac{1}{\Omega_i \rho_i} \sum \tilde{\boldsymbol{v}}_{ij} \cdot \left[m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{16}$$

$$\nabla \times \tilde{\boldsymbol{v}}_i = \frac{1}{\Omega_i \rho_i} \sum \tilde{\boldsymbol{v}}_{ij} \times \left[m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right]. \tag{17}$$

The derivative of the kernel function is as follows:

$$\frac{\partial W(r, H)}{\partial r} = H^{-(D+1)} \frac{\partial w(r/H)}{\partial (r/H)}. \tag{18}$$

3. Calculate the pressure ($\tilde{P}_i$) and sound speed ($\tilde{c}_{\mathrm{s},i}$) as follows:

$$P_i = (\gamma - 1) \rho_i \tilde{u}_i \tag{19}$$

$$c_{\mathrm{s},i} = \left(\gamma \frac{P_i}{\rho_i}\right)^{1/2}. \tag{20}$$

4. Calculate the Balsara switch:

$$f_i = \frac{|\nabla \cdot \tilde{\boldsymbol{v}}_i|}{|\nabla \cdot \tilde{\boldsymbol{v}}_i| + |\nabla \times \tilde{\boldsymbol{v}}_i| + 0.0001 \tilde{c}_{\mathrm{s},i}/H_i}. \tag{21}$$

5. Calculate the acceleration and the time derivative of the energy:

$$\boldsymbol{a}_i = -\sum \left(\frac{\tilde{P}_i}{\Omega_i \rho_i^2} + \frac{\tilde{P}_j}{\Omega_j \rho_j^2} + f_{ij} \Pi_{ij} \right) \left[\frac{m_j}{2} \left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}} \right) \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{22}$$

$$\dot{u}_i = \sum \left(\frac{\tilde{P}_i}{\Omega_i \rho_i^2} + \frac{f_{ij} \Pi_{ij}}{2} \right) \left[\frac{m_j}{2} \left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}} \right) \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right] \cdot \tilde{\boldsymbol{v}}_{ij}, \tag{23}$$

where $f_{ij} = (f_i + f_j)/2$, and Π_{ij} is an artificial viscosity. The artificial viscosity is expressed as follows:

$$\Pi_{ij} = -\frac{\alpha}{2} \frac{v_{ij}^{\mathrm{sig}} w_{ij}}{\rho_{ij}} \tag{24}$$

$$v_{ij}^{\mathrm{sig}} = c_{\mathrm{s},i} + c_{\mathrm{s},j} - 3 w_{ij} \tag{25}$$

$$w_{ij} = \begin{cases} \dfrac{\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij}}{|\boldsymbol{r}_{ij}|} & (\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij} < 0) \\ 0 & (\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij} \geq 0) \end{cases} \tag{26}$$

where $\alpha = 1.0$ and $\rho_{ij} = (\rho_i + \rho_j)/2$.

6. Calculate $\dot{\alpha}$:

$$\dot{\alpha}_i = -\frac{\tilde{\alpha}_i - \alpha_{\min}}{h_i/(0.25\tilde{c}_{\mathrm{s},i})} + S_i \tag{27}$$

$$S_i = \max\left[-(\nabla \cdot \tilde{\boldsymbol{v}}_i)(\alpha_{\max} - \tilde{\alpha}_i), 0\right] \tag{28}$$

7. Correct the velocity and energy:

$$\boldsymbol{v}_i = \boldsymbol{v}_i^{(1/2)} + \frac{1}{2}\boldsymbol{a}_i \Delta t \tag{29}$$

$$u_i = u_i^{(1/2)} + \frac{1}{2}\dot{u}_i \Delta t \tag{30}$$

$$\alpha_i = \alpha_i^{(1/2)} + \frac{1}{2}\dot{\alpha}_i \Delta t. \tag{31}$$

8. Calculate the next timestep:

$$\Delta t = C \min_i \left[\min\left(\frac{2H_i}{\max_j \left(v_{ij}^{\mathrm{sig}}\right)}, \frac{u_i}{|\dot{u}_i|}\right)\right], \tag{32}$$

   where $C = 0.15$.

9. Return to step 1.

2 Test

1. Shock tube
2. Strong shock
3. Point like explosion
4. Kelvin-Helmholtz instability

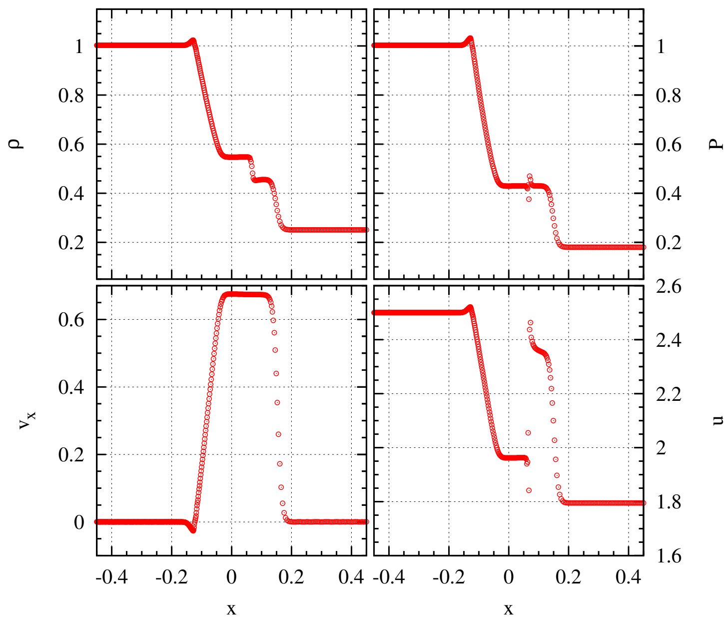

Figure 1: Shock tube ($\gamma = 1.4$, $\alpha = 2.0$, $\eta = 1.6$).

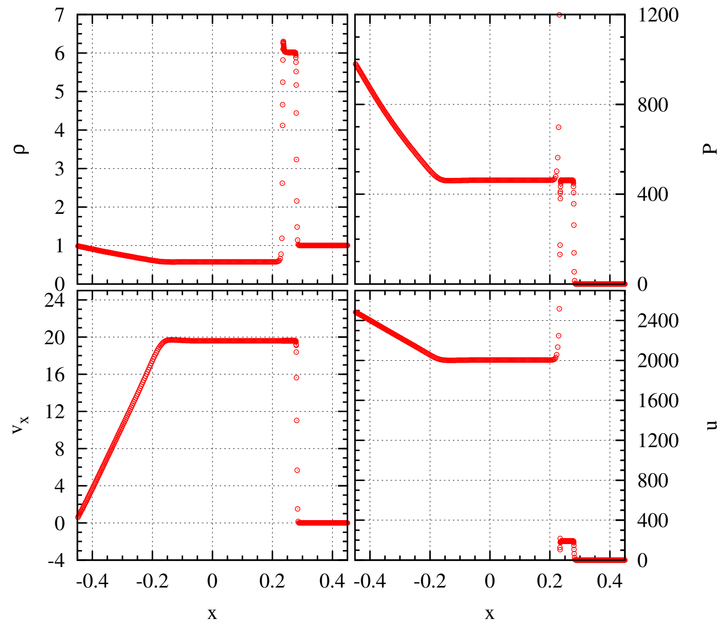

Figure 2: Strong shock ($\gamma = 1.4$, $\alpha = 2.0$, $\eta = 1.6$).

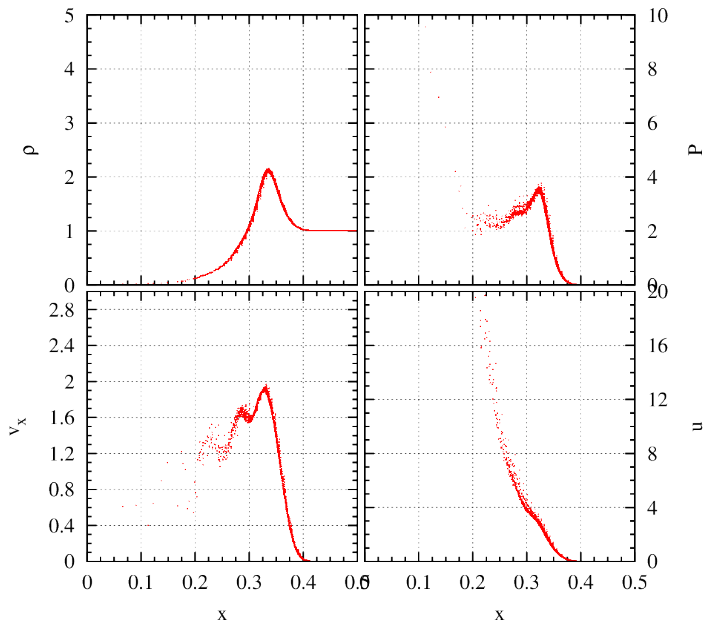

Figure 3: Point like explosion ($\gamma = 5/3$, $\alpha = 3.0$, $\eta = 1.6$).

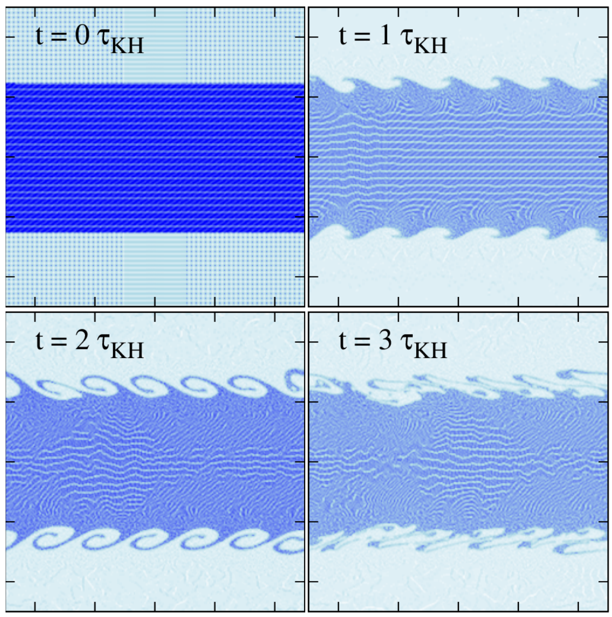

Figure 4: KH instability ($\gamma = 5/3$, $\alpha = 3.0$, $\eta = 1.6$).

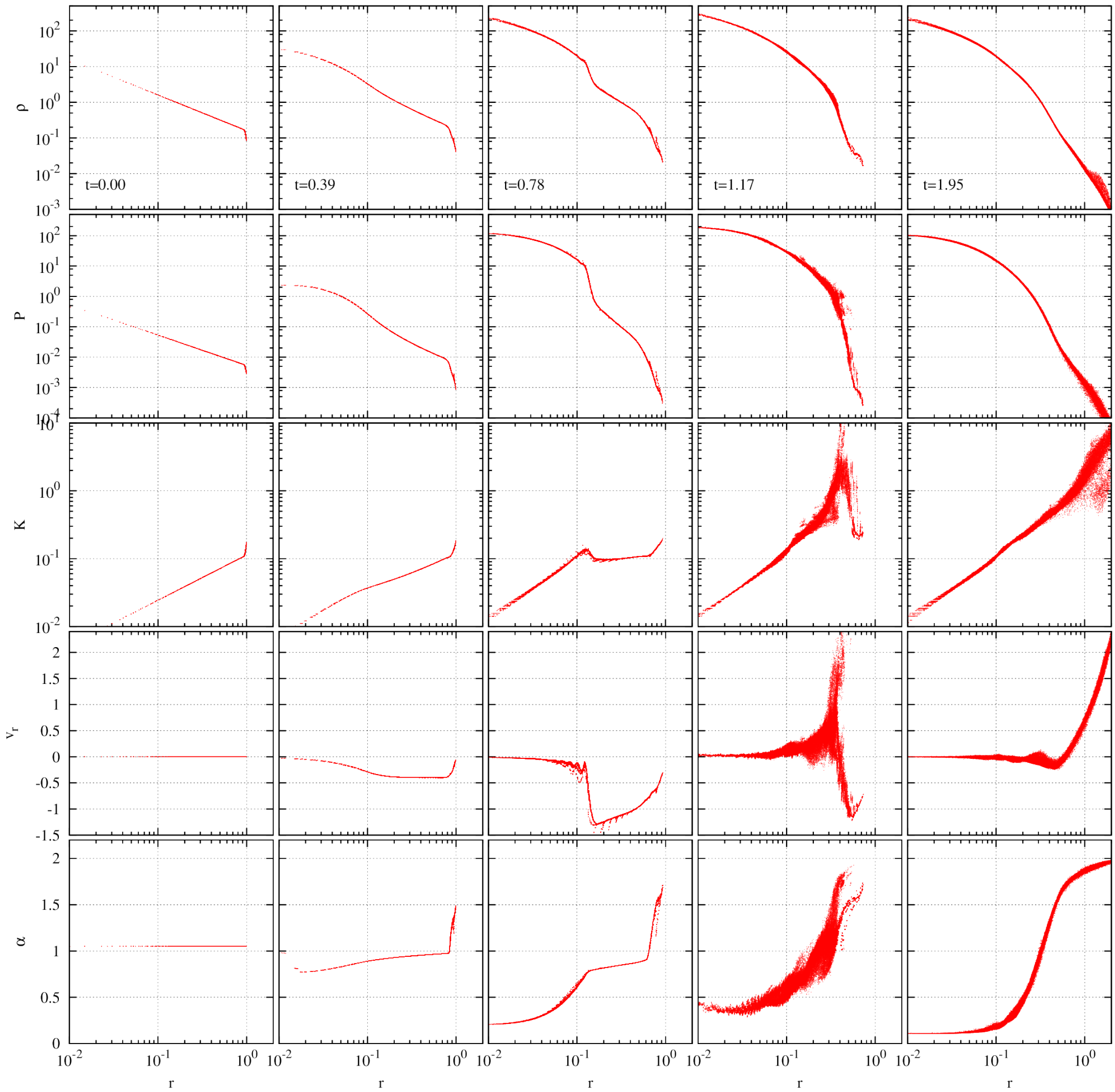

Figure 5: Evrard test.