1 SPH w/ artificial conductivity

Suppose we know acceleration, time derviative of energy, coefficient viscosity, and the next timestep at the initial time. We integrate each particle as follows:

1. Calculate the position, and predict the velocity and energy:

$$\boldsymbol{r}_i = \boldsymbol{r}_i^{(0)} + \boldsymbol{v}_i^{(0)}\Delta t + \frac{1}{2}\boldsymbol{a}_i^{(0)}\Delta t^2, \tag{1}$$

$$\boldsymbol{v}_i^{(1/2)} = \boldsymbol{v}_i^{(0)} + \frac{1}{2}\boldsymbol{a}^{(0)}\Delta t, \tag{2}$$

$$u_i^{(1/2)} = u_i^{(0)} + \frac{1}{2}\dot{u}^{(0)}\Delta t \tag{3}$$

$$\alpha_i^{(1/2)} = \alpha_i^{(0)} + \frac{1}{2}\dot{\alpha}^{(0)}\Delta t, \tag{4}$$

$$\alpha_i^{\mathrm{u},(1/2)} = \alpha_i^{\mathrm{u},(0)} + \frac{1}{2}\dot{\alpha}^{\mathrm{u},(0)}\Delta t, \tag{5}$$

$$\tilde{\boldsymbol{v}}_i = \boldsymbol{v}_i^{(0)} + \boldsymbol{a}_i^{(0)}\Delta t, \tag{6}$$

$$\tilde{\boldsymbol{u}}_i = \boldsymbol{u}_i^{(0)} + \dot{\boldsymbol{u}}_i^{(0)}\Delta t + \epsilon_{\mathrm{nuc},i}(\Delta t), \tag{7}$$

$$\tilde{\alpha}_i = \alpha_i^{(0)} + \dot{\alpha}_i^{(0)}\Delta t \tag{8}$$

$$\tilde{\alpha}_i^{\mathrm{u}} = \alpha_i^{\mathrm{u},(0)} + \dot{\alpha}_i^{\mathrm{u},(0)}\Delta t, \tag{9}$$

   where ($\epsilon_{\mathrm{nuc},i}$) is energy generated through nuclear reaction. (When the reaction is exothermic, the density and temperature are fixed at this time. On the other hand, when the reaction is endothermic, only the density is fixed at this time.)

2. Calculate the density, kernel-support length and grad-h term of each particle by an iterative method:

   (a) Estimate the density by the following expression:

$$\rho_i = \sum_j m_j W_i \tag{10}$$

   where $W_i = W(r_{ij}, H_i)$, $r_{ij} = |\boldsymbol{r}_{ij}|$, and $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$. The kernel function is expressed as

$$W(r, H) = H^{-D} w(r/H). \tag{11}$$

   The formula of $w(q)$ is described in Appendix A.

   (b) Calculate the kernel length as follow:

$$H_i = \max\left[C_{H/h} \times \eta \left(\frac{m_i}{\rho_i}\right)^{1/D}, H_{\max}\right], \tag{12}$$

   where h is the kernel length. Here, $H_{\max}$ is the initial distance of the binary separation, or the maximum `double` in the case of the single WD. The values of η and $C_{H/h}$ are described in Appendix A.

(c) Return to step (2a) unless this is the 3rd time.

(d) Calculate divergence and rotation of $\boldsymbol{v}$, grad-h term, and gravity correction term:

$$\nabla \cdot \tilde{\boldsymbol{v}}_i = -\frac{1}{\Omega_i \rho_i} \sum \tilde{\boldsymbol{v}}_{ij} \cdot \left[m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{13}$$

$$\nabla \times \tilde{\boldsymbol{v}}_i = \frac{1}{\Omega_i \rho_i} \sum \tilde{\boldsymbol{v}}_{ij} \times \left[m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{14}$$

$$\Omega_i = 1 + \frac{1}{D} \frac{H_i}{\rho_i} \sum_j m_j \frac{\partial W_i}{\partial H_i}, \tag{15}$$

The derivatives of the kernel function are as follows:

$$\frac{\partial W(r,H)}{\partial r} = H^{-(D+1)} \frac{\partial w(r/H)}{\partial (r/H)} \tag{16}$$

$$\frac{\partial W(r,H)}{\partial H} = -H^{-(D+1)} \left[D w(q) + q \frac{\partial w}{\partial q} \right]. \tag{17}$$

The formula of $\partial w / \partial q$ is described in Appendix A.

3. Calculate $(\bar{A}_i)$ and $(\bar{Z}_i)$ as follows:

$$\bar{A}_i = \frac{1}{\sum_k X_{i,k}/A_k} \tag{18}$$

$$\bar{Z}_i = \bar{A}_i \sum_k \left(\frac{Z_k}{A_k} X_{i,k} \right), \tag{19}$$

where $A_k = (4, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56)$, and $Z_k = (2, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28)$.

4. Calculate the pressure $(\tilde{P}_i)$ and sound speed $(\tilde{c}_{\mathrm{s},i})$ as follows:

$$P_i = (\gamma - 1)\rho_i \tilde{u}_i \tag{20}$$

$$c_{\mathrm{s},i} = \left(\gamma \frac{P_i}{\rho_i} \right)^{1/2}, \tag{21}$$

or Helmholtz EOS, in which case the temperature $(\tilde{T}_i)$ is also obtained.

5. Calculate the Balsara switch:

$$f_i = \frac{|\nabla \cdot \tilde{\boldsymbol{v}}_i|}{|\nabla \cdot \tilde{\boldsymbol{v}}_i| + |\nabla \times \tilde{\boldsymbol{v}}_i| + 0.0001 \tilde{c}_{\mathrm{s},i}/h_i}. \tag{22}$$

6. Calculate the gravity and its correction term:

$$\boldsymbol{g}_{2,i} = -\frac{G}{2}\sum_j m_j \frac{\boldsymbol{r}_{ij}}{r_{ij}} \left[\frac{\partial\phi(r_{ij},H_i)}{\partial r_{ij}} + \frac{\partial\phi(r_{ij},H_j)}{\partial r_{ij}}\right], \tag{23}$$

$$\phi_i = \frac{G}{2}\sum_j m_j \left[\phi(r_{ij},H_i) + \phi(r_{ij},H_j)\right] \tag{24}$$

$$\eta_i = \frac{H_i}{D\rho_i}\frac{1}{\Omega_i}\sum_j m_j \frac{\partial\phi(r_{ij},H_i)}{\partial H_i}, \tag{25}$$

where the potential is expressed as:

$$\phi(r,H) = -\frac{1}{\left(r^2 + H^2 C_{H/h}^{-2}\right)^{1/2}} \tag{26}$$

$$\frac{\partial\phi(r,H)}{\partial r} = \frac{r}{\left(r^2 + H^2 C_{H/h}^{-2}\right)^{3/2}}, \tag{27}$$

$$\frac{\partial\phi(r,H)}{\partial H} = \frac{H C_{H/h}^{-2}}{\left(r^2 + H^2 C_{H/h}^{-2}\right)^{3/2}}. \tag{28}$$

7. Calculate the hydro acceleration, the time derivative of the energy, and one term of the gravity:

$$\boldsymbol{a}_i = -\frac{1}{2}\sum_j \left(\frac{\tilde{P}_i}{\Omega_i\rho_i^2} + \frac{\tilde{P}_j}{\Omega_j\rho_j^2} + f_{ij}\Pi_{ij}\right)\left[m_j\left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}}\right)\frac{\boldsymbol{r}_{ij}}{r_{ij}}\right], \tag{29}$$

$$\dot{u}_i = \frac{1}{2}\sum_j \left(\frac{\tilde{P}_i}{\Omega_i\rho_i^2} + \frac{f_{ij}\Pi_{ij}}{2}\right)\left[m_j\left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}}\right)\frac{\boldsymbol{r}_{ij}}{r_{ij}}\right]\cdot\tilde{\boldsymbol{v}}_{ij}, \tag{30}$$

$$\boldsymbol{g}_{1,i} = \frac{G}{2}\sum_j m_j \left[\left(\eta_i\frac{\partial W_i}{\partial r_{ij}} + \eta_j\frac{\partial W_j}{\partial r_{ij}}\right)\frac{\boldsymbol{r}_{ij}}{r_{ij}}\right], \tag{31}$$

$$\left(\nabla^2 u\right)_i = \sum_j \frac{u_i - u_j}{\rho_{ij}}\left[m_j\left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}}\right)\frac{1}{r_{ij}}\right], \tag{32}$$

where $f_{ij} = (f_i + f_j)/2$, and Π_{ij} is an artificial viscosity. The artificial viscosity is expressed as follows:

$$\Pi_{ij} = -\frac{\tilde{\alpha}_{ij}}{2}\frac{v_{ij}^{\mathrm{sig}} w_{ij}^{0}}{\rho_{ij}} \tag{33}$$

$$v_{ij}^{\mathrm{sig}} = c_{\mathrm{s},i} + c_{\mathrm{s},j} - 3w_{ij}^{0} \tag{34}$$

$$w_{ij}^{0} = \min\left(w_{ij}, 0\right) \tag{35}$$

$$w_{ij} = \frac{\boldsymbol{r}_{ij}\cdot\tilde{\boldsymbol{v}}_{ij}}{|\boldsymbol{r}_{ij}|} \tag{36}$$

where $\tilde{\alpha}_{ij} = (\tilde{\alpha}_i + \tilde{\alpha}_j)/2$, and $\rho_{ij} = (\rho_i + \rho_j)/2$. If we introduce Price's thermal conductivity, equation (30) can be rewritten as:

$$\begin{aligned}\dot{u}_i =& \frac{1}{2}\sum_j m_j \left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}}\right) \\ &\times \left\{\frac{\tilde{P}_i}{\Omega_i \rho_i^2} w_{ij} - \frac{1}{\rho_{ij}}\left[\frac{1}{4} f_{ij}\tilde{\alpha}_{ij} v_{ij}^{\mathrm{sig}} (w_{ij}^{0})^2 - \tilde{\alpha}_{ij}^{\mathrm{u}} v_{ij}^{\mathrm{u,sig}}\left(u_i - u_j\right)\right]\right\},\end{aligned} \tag{37}$$

where

$$v_{ij}^{\mathrm{u,sig}} = \left(\frac{|\tilde{P}_i - \tilde{P}_j|}{\rho_{ij}}\right)^{1/2} \tag{38}$$

8. Calculate $\dot{\alpha}$ and $\dot{\alpha}^{\mathrm{u}}$:

$$\dot{\alpha}_i = -\frac{\tilde{\alpha}_i - \alpha_{\mathrm{min}}}{h_i/(0.25\tilde{c}_{\mathrm{s},i})} + \max\left[-(\nabla\cdot\tilde{\boldsymbol{v}}_i)(\alpha_{\mathrm{max}} - \tilde{\alpha}_i), 0\right] \tag{39}$$

$$\dot{\alpha}_i^{\mathrm{u}} = -\frac{\tilde{\alpha}_i^{\mathrm{u}} - \alpha_{\mathrm{min}}^{\mathrm{u}}}{h_i/(0.25\tilde{c}_{\mathrm{s},i})} + \frac{h_i|\nabla^2 u|_i}{(u_i + \epsilon^{\mathrm{u}})^{1/2}}\left(\alpha_{\mathrm{max}} - \tilde{\alpha}_i^{\mathrm{u}}\right), \tag{40}$$

where we set $\epsilon^{\mathrm{u}} = 0.0001 u_{0,\mathrm{min}}$.

9. Correct the velocity and energy:

$$\boldsymbol{v}_i = \boldsymbol{v}_i^{(1/2)} + \frac{1}{2}\boldsymbol{a}_i\Delta t \tag{41}$$

$$u_i = u_i^{(1/2)} + \frac{1}{2}\dot{u}_i\Delta t + \epsilon_{\mathrm{nuc},i}(\Delta t) \tag{42}$$

$$\alpha_i = \alpha_i^{(1/2)} + \frac{1}{2}\dot{\alpha}_i\Delta t, \tag{43}$$

$$\alpha_i^{\mathrm{u}} = \alpha_i^{\mathrm{u},(1/2)} + \frac{1}{2}\dot{\alpha}_i^{\mathrm{u}}\Delta t, \tag{44}$$

where $\epsilon_{\mathrm{nuc},i}(\Delta t)$ has been already obtained in step 1.

10. Calculate the next timestep:

$$\Delta t_{\rm next} = C \min_i \left[\frac{C_{H/h}^{-1} H_i}{\max_j \left(v_{ij}^{\rm sig} \right)}, \left| \frac{u_i^{(0)}}{\dot{u}_i^{(0)}} \right| \right]. \tag{45}$$

When nuclear reaction is considered, we should see Raskin et al.

11. Return to step 1.

2 SPH w/ IAD

Between step 6 and 7, we calculate a matrix $\mathcal{C}_i$. The size of the matrix is $D \times D$. The matrix is expressed as

$$\mathcal{C}_i = \{c_{kl,i}\} = \{\tau_{kl,i}\}^{-1}, \tag{46}$$

The component of the inverse matrix, τ_{ij}, is written as

$$\tau_{kl,i} = \sum_j \frac{m_j}{\rho_j} x_{k,ij} x_{l,ij} W_i, \tag{47}$$

where $\boldsymbol{r}_i j = (x_{1,ij}, x_{2,ij}, x_{3,ij})$ in 3D.

Furthermore, we replace eqs (29) and (30) (or (37)) with the following equations:

$$\boldsymbol{a}_i = -\sum_j m_j \left(\frac{\tilde{P}_i}{\Omega_i \rho_i^2} \boldsymbol{A}_{ij} - \frac{\tilde{P}_j}{\Omega_j \rho_j^2} \boldsymbol{A}_{ji} + f_{ij} \Pi_{ij} \bar{\boldsymbol{A}}_{ij} \right) \tag{48}$$

$$\dot{u}_i = \sum_j m_j \boldsymbol{v}_{ij} \cdot \left(\frac{\tilde{P}_i}{\Omega_i \rho_i^2} \boldsymbol{A}_{ij} + \frac{f_{ij} \Pi_{ij}}{2} \bar{\boldsymbol{A}}_{ij} \right), \tag{49}$$

where $\boldsymbol{A}_{ij} = -\left(\mathcal{C}_i \boldsymbol{r}_{ij}\right) W_i$, and $\bar{\boldsymbol{A}}_{ij} = \left(\boldsymbol{A}_{ij} - \boldsymbol{A}_{ji}\right)/2$.

3 DISPH

As the initial condition, we have m_i, $\boldsymbol{r}_i$, $\boldsymbol{v}_i$, u_i, α_i, and H_i. We integrate each particle in the following way.

1. We take step (1a) at the initial time, and step (1b) at the other times.

   (a) We calculate a tentative density ρ_i, such that

   $$\tilde{\rho}_i = \sum_j m_j W_i. \tag{50}$$

   Then, we obtain $\tilde{P}_i$ from ρ_i and u_i, and $\tilde{Y}_i = m_i \tilde{P}_i^k / \rho_i$, where $k = 0.05$.

(b) We predict physical quantities at the next time:

$$\boldsymbol{r}_i = \boldsymbol{r}_i^{(0)} + \boldsymbol{v}_i^{(0)}\Delta t + \frac{1}{2}\boldsymbol{a}_i^{(0)}\Delta t^2, \tag{51}$$

$$\boldsymbol{v}_i^{(1/2)} = \boldsymbol{v}_i^{(0)} + \frac{1}{2}\boldsymbol{a}^{(0)}\Delta t, \tag{52}$$

$$u_i^{(1/2)} = u_i^{(0)} + \frac{1}{2}\dot{u}^{(0)}\Delta t, \tag{53}$$

$$Y_i^{(1/2)} = Y_i^{(0)} + \frac{1}{2}\dot{Y}^{(0)}\Delta t, \tag{54}$$

$$\alpha_i^{(1/2)} = \alpha_i^{(0)} + \frac{1}{2}\dot{\alpha}^{(0)}\Delta t, \tag{55}$$

$$\tilde{\boldsymbol{v}}_i = \boldsymbol{v}_i^{(0)} + \boldsymbol{a}_i^{(0)}\Delta t, \tag{56}$$

$$\tilde{\boldsymbol{u}}_i = \boldsymbol{u}_i^{(0)} + \dot{\boldsymbol{u}}_i^{(0)}\Delta t + \epsilon_{\mathrm{nuc},i}(\Delta t), \tag{57}$$

$$\tilde{Y}_i = Y_i^{(0)} + \dot{Y}^{(0)}\Delta t, \tag{58}$$

$$\tilde{\alpha}_i = \alpha_i^{(0)} + \dot{\alpha}_i^{(0)}\Delta t, \tag{59}$$

$$\tilde{P}_i = P_i^{(0)} + \dot{P}_i\Delta t, \tag{60}$$

$$\tilde{H}_i = H_i^{(0)} + \dot{H}_i\Delta t. \tag{61}$$

We obtain the time derivatives $\dot{P}_i$ and $\dot{H}_i$ as follows:

$$\dot{P}_i = -\frac{\partial P_i}{\partial \rho}\rho_i(\nabla\cdot\boldsymbol{v}_i) + \frac{\partial P_i}{\partial u}\dot{u}_i, \tag{62}$$

$$\dot{H}_i = \frac{H_i}{D}(\nabla\cdot\boldsymbol{v}_i). \tag{63}$$

Then, we calculate $\tilde{P}_i^k$.

2. We derive a volume element iteratively.

(a) We obtain the density, such that $\rho_i = m_i\tilde{P}_i^k/\tilde{Y}_i$.

(b) We obtain the non-smoothed pressure $\hat{P}_i$ from ρ_i and $\tilde{u}_i$.

(c) We update $\tilde{Y}_i$, such that $\tilde{Y}_i = m_i\hat{P}^k/\rho_i$.

(d) We calculate $\tilde{P}_i^k$ as follows:

$$\tilde{P}_i^k = \sum_j \tilde{Y}_j W_i, \tag{64}$$

where the kernel function is

$$W(r, H) = H^{-D}w(r/H). \tag{65}$$

(e) We obtain kernel support radius: $H_i = C_{H/h}\eta(\tilde{Y}_i/\tilde{P}_i)^{1/D}$.

(f) We return step 2a unless this is the 2nd time.

(g) We calculate density, such that $\rho_i = m_i \tilde{P}_i^k / \tilde{Y}_i$.

3. We calculate some auxiliary variables as follows:

$$\nabla \cdot \tilde{\boldsymbol{v}}_i = -\frac{1}{\Omega_i \tilde{P}_i^k} \sum \tilde{Y}_j \tilde{\boldsymbol{v}}_{ij} \cdot \left[\frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{66}$$

$$\nabla \times \tilde{\boldsymbol{v}}_i = -\frac{1}{\Omega_i \tilde{P}_i^k} \sum \tilde{Y}_j \tilde{\boldsymbol{v}}_{ij} \times \left[\frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{67}$$

$$\Omega_i = 1 + \frac{1}{D} \frac{H_i}{\tilde{P}_i^k} \sum_j \tilde{Y}_j \frac{\partial W_i}{\partial H_i}, \tag{68}$$

where the derivative of the kernel function is

$$\frac{\partial W(r, H)}{\partial r} = H^{-(D+1)} \frac{\partial w(r/H)}{\partial (r/H)}, \tag{69}$$

$$\frac{\partial W(r, H)}{\partial H} = -H^{-(D+1)} \left[D w(q) + q \frac{\partial w}{\partial q} \right]. \tag{70}$$

4. We calculate the sound of speed ($\tilde{c}_{\mathrm{s},i}$) and the Balsara switch:

$$f_i = \frac{|\nabla \cdot \tilde{\boldsymbol{v}}_i|}{|\nabla \cdot \tilde{\boldsymbol{v}}_i| + |\nabla \times \tilde{\boldsymbol{v}}_i| + 0.0001 C_{H/h} \tilde{c}_{\mathrm{s},i} / \tilde{H}_i}. \tag{71}$$

5. We calculate the gravity and its correction term:

$$\boldsymbol{g}_{2,i} = -\frac{G}{2} \sum_j m_j \frac{\boldsymbol{r}_{ij}}{r_{ij}} \left[\frac{\partial \phi(r_{ij}, \tilde{H}_i)}{\partial r_{ij}} + \frac{\partial \phi(r_{ij}, \tilde{H}_j)}{\partial r_{ij}} \right], \tag{72}$$

$$\phi_i = \frac{G}{2} \sum_j m_j \left[\phi(r_{ij}, \tilde{H}_i) + \phi(r_{ij}, \tilde{H}_j) \right] \tag{73}$$

$$\eta_i = \frac{\tilde{H}_i}{D \rho_i} \frac{1}{\Omega_i} \sum_j m_j \frac{\partial \phi(r_{ij}, \tilde{H}_i)}{\partial \tilde{H}_i}, \tag{74}$$

where the potential is expressed as:

$$\phi(r, H) = -\frac{1}{\left(r^2 + H^2 C_{H/h}^{-2} \right)^{1/2}} \tag{75}$$

$$\frac{\partial \phi(r, H)}{\partial r} = \frac{r}{\left(r^2 + H^2 C_{H/h}^{-2} \right)^{3/2}}, \tag{76}$$

$$\frac{\partial \phi(r, H)}{\partial H} = \frac{H C_{H/h}^{-2}}{\left(r^2 + H^2 C_{H/h}^{-2} \right)^{3/2}}. \tag{77}$$

6. We calculate the hydro acceleration, the time derivative of the energy, and one term of the gravity:

$$\boldsymbol{a}_i = -\frac{1}{2m_i}\sum_j \left[\frac{\tilde{P}_i\tilde{Y}_i}{\tilde{P}_i^{2k}\Omega_i}\tilde{Y}_j + \frac{\tilde{P}_j\tilde{Y}_j}{\tilde{P}_j^{2k}\Omega_j}\tilde{Y}_i + m_i m_j f_{ij}\Pi_{ij}\right]\left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}}\right)\frac{\boldsymbol{r}_{ij}}{r_{ij}}, \tag{78}$$

$$\dot{u}_i = \frac{1}{2m_i}\sum_j \left[\frac{\tilde{P}_i\tilde{Y}_i}{\tilde{P}_i^{2k}\Omega_i}\tilde{Y}_j + \frac{m_i m_j f_{ij}\Pi_{ij}}{2}\right]\tilde{\boldsymbol{v}}_{ij}\cdot\left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}}\right)\frac{\boldsymbol{r}_{ij}}{r_{ij}}, \tag{79}$$

$$\dot{Y}_i = \frac{(\gamma_i - 1)}{2}\sum_j \frac{\tilde{P}_i\tilde{Y}_i}{\tilde{P}_i^{2k}\Omega_i}\tilde{Y}_j\tilde{\boldsymbol{v}}_{ij}\cdot\left(\frac{\partial W_i}{\partial r_{ij}} + \frac{\partial W_j}{\partial r_{ij}}\right)\frac{\boldsymbol{r}_{ij}}{r_{ij}}, \tag{80}$$

$$\boldsymbol{g}_{1,i} = \frac{G}{2}\sum_j m_j\left[\left(\eta_i\frac{\partial W_i}{\partial r_{ij}} + \eta_j\frac{\partial W_j}{\partial r_{ij}}\right)\frac{\boldsymbol{r}_{ij}}{r_{ij}}\right], \tag{81}$$

where $\boldsymbol{A}_{ij} = -\left(\mathcal{C}_i\boldsymbol{r}_{ij}\right)W_i$, and $\bar{\boldsymbol{A}}_{ij} = \left(\boldsymbol{A}_{ij} - \boldsymbol{A}_{ji}\right)/2$. The Balsara switch is $f_{ij} = (f_i + f_j)/2$. The artificial viscosity is as follows:

$$\Pi_{ij} = -\frac{\tilde{\alpha}_{ij}}{2}\frac{v_{ij}^{\mathrm{sig}}w_{ij}^0}{\tilde{\rho}_{ij}}, \tag{82}$$

$$v_{ij}^{\mathrm{sig}} = \tilde{c}_{\mathrm{s},i} + \tilde{c}_{\mathrm{s},j} - 3w_{ij}^0, \tag{83}$$

$$w_{ij}^0 = \min\left(w_{ij}, 0\right), \tag{84}$$

$$w_{ij} = \frac{\boldsymbol{r}_{ij}\cdot\tilde{\boldsymbol{v}}_{ij}}{|\boldsymbol{r}_{ij}|}, \tag{85}$$

where $\tilde{\alpha}_{ij} = (\tilde{\alpha}_i + \tilde{\alpha}_j)/2$, and $\tilde{\rho}_{ij} = (\tilde{\rho}_i + \tilde{\rho}_j)/2$.

7. We calculate $\dot{\alpha}$:

$$\dot{\alpha}_i = -\frac{\tilde{\alpha}_i - \alpha_{\min}}{H_i/(0.25C_{H/h}\tilde{c}_{\mathrm{s},i})} + \max\left[-(\nabla\cdot\tilde{\boldsymbol{v}}_i)(\alpha_{\max} - \tilde{\alpha}_i), 0\right]. \tag{86}$$

8. Correct the velocity and energy:

$$\boldsymbol{v}_i = \boldsymbol{v}_i^{(1/2)} + \frac{1}{2}\boldsymbol{a}_i\Delta t \tag{87}$$

$$u_i = u_i^{(1/2)} + \frac{1}{2}\dot{u}_i\Delta t + \epsilon_{\mathrm{nuc},i}(\Delta t) \tag{88}$$

$$Y_i = Y_i^{(1/2)} + \frac{1}{2}\dot{Y}_i\Delta t \tag{89}$$

$$\alpha_i = \alpha_i^{(1/2)} + \frac{1}{2}\dot{\alpha}_i\Delta t, \tag{90}$$

where $\epsilon_{\mathrm{nuc},i}(\Delta t)$ has been already obtained in step 1b.

9. We calculate the next timestep:

$$\Delta t_{\rm next} = C \min_i \left[\frac{C_{H/h}^{-1} H_i}{\max_j \left(v_{ij}^{\rm sig} \right)}, \left| \frac{u_i^{(0)}}{\dot{u}_i^{(0)}} \right| \right]. \tag{91}$$

When nuclear reaction is considered, we should see Raskin et al.

10. We return to step 1b.

4 DISPH (not work)

As the initial condition, we have m_i, $\boldsymbol{r}_i$, $\boldsymbol{v}_i$, u_i, α_i, and H_i. We integrate each particle in the following way.

1. We take step (1a) at the initial time, and step (1b) at the other times.

   (a) We calculate a tentative density $\tilde{\rho}_i$, such that

   $$\tilde{\rho}_i = \sum_j m_j W_i. \tag{92}$$

   Then, we obtain $\tilde{P}_i$ from $\tilde{\rho}_i$ and u_i.

   (b) We predict physical quantities at the next time:

   $$\begin{aligned}
   \boldsymbol{r}_i &= \boldsymbol{r}_i^{(0)} + \boldsymbol{v}_i^{(0)} \Delta t + \frac{1}{2} \boldsymbol{a}_i^{(0)} \Delta t^2, & (93)\\
   \boldsymbol{v}_i^{(1/2)} &= \boldsymbol{v}_i^{(0)} + \frac{1}{2} \boldsymbol{a}^{(0)} \Delta t, & (94)\\
   u_i^{(1/2)} &= u_i^{(0)} + \frac{1}{2} \dot{u}^{(0)} \Delta t, & (95)\\
   \alpha_i^{(1/2)} &= \alpha_i^{(0)} + \frac{1}{2} \dot{\alpha}^{(0)} \Delta t, & (96)\\
   \tilde{\boldsymbol{v}}_i &= \boldsymbol{v}_i^{(0)} + \boldsymbol{a}_i^{(0)} \Delta t, & (97)\\
   \tilde{\boldsymbol{u}}_i &= \boldsymbol{u}_i^{(0)} + \dot{\boldsymbol{u}}_i^{(0)} \Delta t + \epsilon_{{\rm nuc},i}(\Delta t), & (98)\\
   \tilde{\alpha}_i &= \alpha_i^{(0)} + \dot{\alpha}_i^{(0)} \Delta t, & (99)\\
   \tilde{P}_i &= P_i^{(0)} + \dot{P}_i \Delta t, & (100)\\
   \tilde{H}_i &= H_i^{(0)} + \dot{H}_i \Delta t. & (101)
   \end{aligned}$$

   We obtain the time derivatives $\dot{P}_i$ and $\dot{H}_i$ as follows:

   $$\begin{aligned}
   \dot{P}_i &= -\frac{\partial P_i}{\partial \rho} \rho_i (\nabla \cdot \boldsymbol{v}_i) + \frac{\partial P_i}{\partial u} \dot{u}_i, & (102)\\
   \dot{H}_i &= \frac{H_i}{D} (\nabla \cdot \boldsymbol{v}_i). & (103)
   \end{aligned}$$

2. We derive a volume element iteratively.

   (a) We tentatively obtain a volume element as follows:

$$V_i = \frac{1}{\sum_j (\tilde{P}_j/\tilde{P}_i)^k W_i}, \tag{104}$$

   where we adopt $k = 0.05$. Here, the kernel function is

$$W(r, H) = H^{-D} w(r/H). \tag{105}$$

   (b) We obtain $\tilde{H}_i$ and $\tilde{\rho}_i$ as follows:

$$\tilde{H}_i = C_{H/h} \eta V_i^{1/D}, \tag{106}$$

$$\tilde{\rho}_i = \frac{m_i}{V_i}, \tag{107}$$

   where the values of $C_{H/h}$ and η are seen in Appendix A.

   (c) We calculate $\tilde{P}_i$ from $\tilde{\rho}_i$ and $\tilde{u}_i$.

   (d) We return step (2a) unless this is the 3rd time.

3. We calculate some auxiliary variables as follows:

$$\nabla \cdot \tilde{\boldsymbol{v}}_i = -V_i \sum \left(\frac{\tilde{P}_j}{\tilde{P}_i} \right)^k \tilde{\boldsymbol{v}}_{ij} \cdot \left[\frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{108}$$

$$\nabla \times \tilde{\boldsymbol{v}}_i = V_i \sum \left(\frac{\tilde{P}_j}{\tilde{P}_i} \right)^k \tilde{\boldsymbol{v}}_{ij} \times \left[\frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{109}$$

$$\tau_{kl,i} = \sum_j V_j x_{k,ij} x_{l,ij} W_i, \tag{110}$$

   where the derivative of the kernel function is

$$\frac{\partial W(r, H)}{\partial r} = H^{-(D+1)} \frac{\partial w(r/H)}{\partial (r/H)}. \tag{111}$$

   Then, we obtain a matrix $\mathcal{C}_i (= \{c_{kl,i}\})$, which is the inverse matrix of $\{\tau_{kl,i}\}$.

4. We calculate the sound of speed ($\tilde{c}_{\mathrm{s},i}$) and the Balsara switch:

$$f_i = \frac{|\nabla \cdot \tilde{\boldsymbol{v}}_i|}{|\nabla \cdot \tilde{\boldsymbol{v}}_i| + |\nabla \times \tilde{\boldsymbol{v}}_i| + 0.0001 C_{H/h} \tilde{c}_{\mathrm{s},i}/\tilde{H}_i}. \tag{112}$$

5. We calculate the gravity and its correction term:

$$\boldsymbol{g}_{2,i} = -\frac{G}{2}\sum_j m_j \frac{\boldsymbol{r}_{ij}}{r_{ij}} \left[\frac{\partial \phi(r_{ij}, \tilde{H}_i)}{\partial r_{ij}} + \frac{\partial \phi(r_{ij}, \tilde{H}_j)}{\partial r_{ij}}\right], \tag{113}$$

$$\phi_i = \frac{G}{2}\sum_j m_j \left[\phi(r_{ij}, \tilde{H}_i) + \phi(r_{ij}, \tilde{H}_j)\right] \tag{114}$$

$$\eta_i = \frac{\tilde{H}_i}{D\rho_i}\frac{1}{\Omega_i}\sum_j m_j \frac{\partial \phi(r_{ij}, \tilde{H}_i)}{\partial \tilde{H}_i}, \tag{115}$$

where the potential is expressed as:

$$\phi(r, H) = -\frac{1}{\left(r^2 + H^2 C_{H/h}^{-2}\right)^{1/2}} \tag{116}$$

$$\frac{\partial \phi(r, H)}{\partial r} = \frac{r}{\left(r^2 + H^2 C_{H/h}^{-2}\right)^{3/2}}, \tag{117}$$

$$\frac{\partial \phi(r, H)}{\partial H} = \frac{H C_{H/h}^{-2}}{\left(r^2 + H^2 C_{H/h}^{-2}\right)^{3/2}}. \tag{118}$$

6. We calculate the hydro acceleration, the time derivative of the energy, and one term of the gravity:

$$\boldsymbol{a}_i = -\frac{1}{m_i}\sum_j \left[\tilde{P}_i V_i^2 \left(\frac{\tilde{P}_j}{\tilde{P}_i}\right)^k \boldsymbol{A}_{ij} - \tilde{P}_j V_j^2 \left(\frac{\tilde{P}_i}{\tilde{P}_j}\right)^k \boldsymbol{A}_{ji} + m_i m_j f_{ij} \Pi_{ij} \bar{\boldsymbol{A}}_{ij}\right], \tag{119}$$

$$\dot{u}_i = \frac{1}{m_i}\sum_j \tilde{\boldsymbol{v}}_{ij} \cdot \left[\tilde{P}_i V_i^2 \left(\frac{\tilde{P}_j}{\tilde{P}_i}\right)^k \boldsymbol{A}_{ij} + \frac{m_i m_j f_{ij} \Pi_{ij}}{2} \bar{\boldsymbol{A}}_{ij}\right], \tag{120}$$

$$\boldsymbol{g}_{1,i} = \frac{G}{2}\sum_j m_j \left[\left(\eta_i \frac{\partial W_i}{\partial r_{ij}} + \eta_j \frac{\partial W_j}{\partial r_{ij}}\right) \frac{\boldsymbol{r}_{ij}}{r_{ij}}\right], \tag{121}$$

where $\boldsymbol{A}_{ij} = -\left(\mathcal{C}_i \boldsymbol{r}_{ij}\right) W_i$, and $\bar{\boldsymbol{A}}_{ij} = \left(\boldsymbol{A}_{ij} - \boldsymbol{A}_{ji}\right)/2$. The Balsara switch is $f_{ij} = (f_i + f_j)/2$. The artificial viscosity is as follows:

$$\Pi_{ij} = -\frac{\tilde{\alpha}_{ij}}{2}\frac{v_{ij}^{\mathrm{sig}} w_{ij}^0}{\tilde{\rho}_{ij}}, \tag{122}$$

$$v_{ij}^{\mathrm{sig}} = \tilde{c}_{\mathrm{s},i} + \tilde{c}_{\mathrm{s},j} - 3w_{ij}^0, \tag{123}$$

$$w_{ij}^0 = \min\left(w_{ij}, 0\right), \tag{124}$$

$$w_{ij} = \frac{\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij}}{|\boldsymbol{r}_{ij}|}, \tag{125}$$

where $\tilde{\alpha}_{ij} = (\tilde{\alpha}_i + \tilde{\alpha}_j)/2$, and $\tilde{\rho}_{ij} = (\tilde{\rho}_i + \tilde{\rho}_j)/2$.

7. We calculate $\dot{\alpha}$:

$$\dot{\alpha}_i = -\frac{\tilde{\alpha}_i - \alpha_{\rm min}}{H_i/(0.25 C_{H/h} \tilde{c}_{{\rm s},i})} + \max\left[-(\nabla\cdot\tilde{\boldsymbol{v}}_i)(\alpha_{\rm max} - \tilde{\alpha}_i), 0\right]. \tag{126}$$

8. Correct the velocity and energy:

$$\boldsymbol{v}_i = \boldsymbol{v}_i^{(1/2)} + \frac{1}{2}\boldsymbol{a}_i \Delta t \tag{127}$$

$$u_i = u_i^{(1/2)} + \frac{1}{2}\dot{u}_i \Delta t + \epsilon_{{\rm nuc},i}(\Delta t) \tag{128}$$

$$\alpha_i = \alpha_i^{(1/2)} + \frac{1}{2}\dot{\alpha}_i \Delta t, \tag{129}$$

where $\epsilon_{{\rm nuc},i}(\Delta t)$ has been already obtained in step 1b.

9. We calculate the next timestep:

$$\Delta t_{\rm next} = C \min_i \left[\frac{C_{H/h}^{-1} H_i}{\max_j \left(v_{ij}^{\rm sig}\right)}, \left|\frac{u_i^{(0)}}{\dot{u}_i^{(0)}}\right|\right]. \tag{130}$$

When nuclear reaction is considered, we should see Raskin et al.

10. We return to step 1b.

5 Helmholtz EOS

Coorporate Timmes's EOS. The compositions are 100 % carbon, 50 % carbon and 50 % oxygen, and will be 100 % helium.

A Kernels

A.1 Cubic spline kernel

We describe the cubic spline kernel:

$$w(r/H) = 2^D \sigma \times \begin{cases} \left(1 - 6q^2 + 6q^3\right) & (0 \le q < 1/2) \\ \left[2(1-q)^3\right] & (1/2 \le q < 1) \\ 0 & (q \ge 1) \end{cases} \tag{131}$$

$$\frac{\partial w(q)}{\partial q} = (-6) 2^D \sigma \times \begin{cases} 1/3 & (0 \le q < 1/3) \\ \left(2q - 3q^2\right) & (1/3 \le q < 1/2) \\ (1-q)^2 & (1/2 \le q < 1) \\ 0 & (q \ge 1) \end{cases}, \tag{132}$$

where $\sigma = 2/3$ (1D), $10/7\pi$ (2D), and $1/\pi$ (3D).

For this kernel, $\eta = 1.2$, and $C_{H/h} = 2$.

The equations (131) and (132) can be rewritten as

$$w(r/H) = 2^D\sigma \times 2\left[(1-q)_+^3 - 4(1/2-q)_+^3\right], \tag{133}$$

$$\frac{\partial w(q)}{\partial q} = (-6)2^D\sigma \times \left[(1-q)_+^2 - 4(1/2-q)_+^2 + 3(1/3-q)_+^2)\right], \tag{134}$$

where $(x)_+ \equiv \max(0,x)$.

A.2 Wendland C^2 kernel

We describe Wendland C^2 kernel: In the case of $D=1$,

$$w(q) = C_{\rm W}(1-q)_+^3(1+3q) \tag{135}$$

$$\frac{\partial w(q)}{\partial q} = 3C_{\rm W}\left[(1-q)_+^3 - (1-q)_+^2(1+3q)\right], \tag{136}$$

and in the case of $D=2,3$,

$$w(q) = C_{\rm W}(1-q)_+^4(1+4q) \tag{137}$$

$$\frac{\partial w(q)}{\partial q} = 4C_{\rm W}\left[(1-q)_+^4 - (1-q)_+^3(1+4q)\right], \tag{138}$$

where $C_{\rm W} = 5/4$ $(D=1)$, $7/\pi$ $(D=2)$, and $21/(2\pi)$ $(D=3)$. For this kernel, $\eta = 1.6$, and $C_{H/h} = 1.620185(D=1)$, $1.897367(D=2)$, and $1.93492(D=3)$.

A.3 Wendland C^4 kernel

We describe Wendland C^4 kernel: In the case of $D=1$,

$$w(q) = C_{\rm W}(1-q)_+^5(1+5q+8q^2) \tag{139}$$

$$\frac{\partial w(q)}{\partial q} = C_{\rm W}(1-q)_+^4\left[-5(1+5q+8q^2) + (1-q)_+(5+16q)\right] \tag{140}$$

and in the case of $D=2,3$,

$$w(q) = C_{\rm W}(1-q)_+^6\left[1+6q+(35/3)q^2\right] \tag{141}$$

$$\frac{\partial w(q)}{\partial q} = C_{\rm W}(1-q)_+^5\left\{-6\left[1+6q+(35/3)q^2\right] + (1-q)_+\left[6+(70/3)q\right]\right\} \tag{142}$$

where $C_{\rm W} = 3/2$ $(D=1)$, $9/\pi$ $(D=2)$, and $495/(32\pi)$ $(D=3)$. For this kernel, $\eta = 1.6$, and $C_{H/h} = 1.936492(D=1)$, $2.171239(D=2)$, and $2.207940(D=3)$.

B SPH tests

1. 1D shock tube (CUBICSPLINE, USE_AT1D)

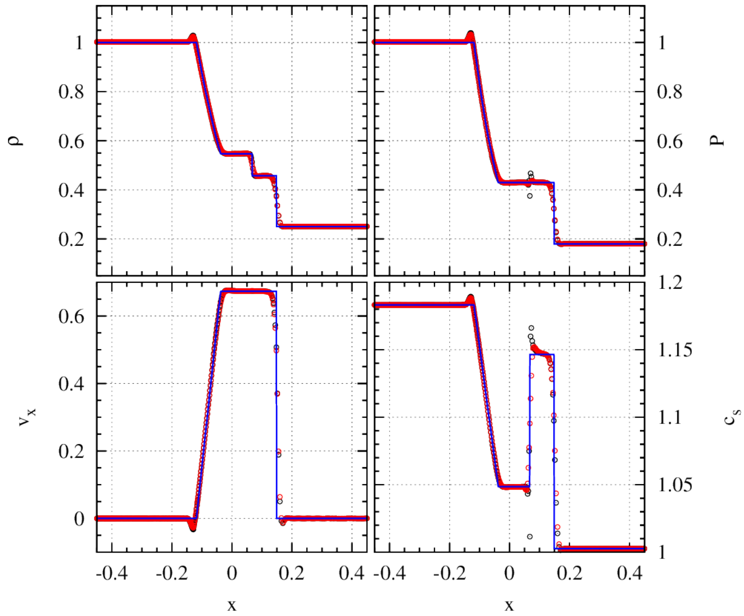

Figure 1: 1D shock tube (W2, $\gamma = 1.4$, $\alpha = \alpha^{\mathrm{u}} = 1.0$).

2. 3D shock tube (CUBICSPLINE)
3. Strong shock (CUBICSPLINE, USE_AT1D)
4. Point like explosion (CUBICSPLINE)
5. Evrard test (CUBICSPLINE, USE_AT3D, GRAVITY)

C EOS tests

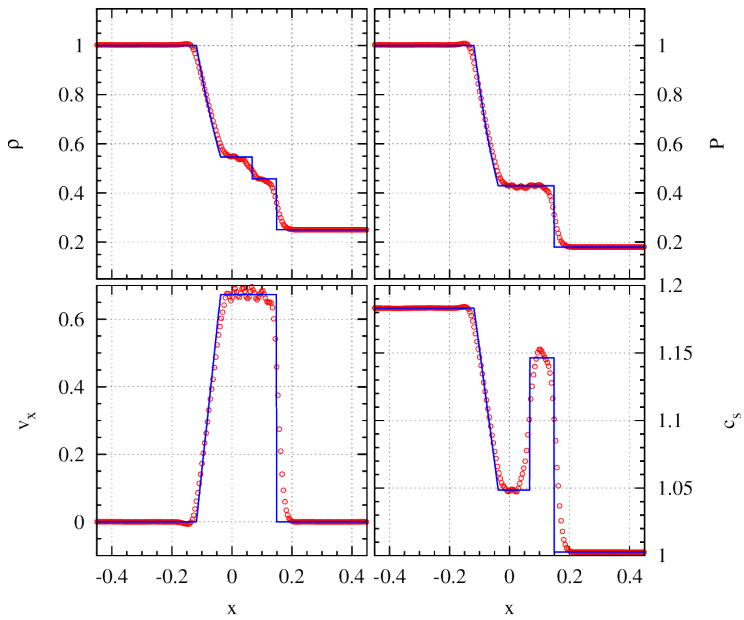

Figure 2: 3D shock tube (W2, $\gamma = 1.4$, $\alpha = \alpha^{\mathrm{u}} = 1.0$).

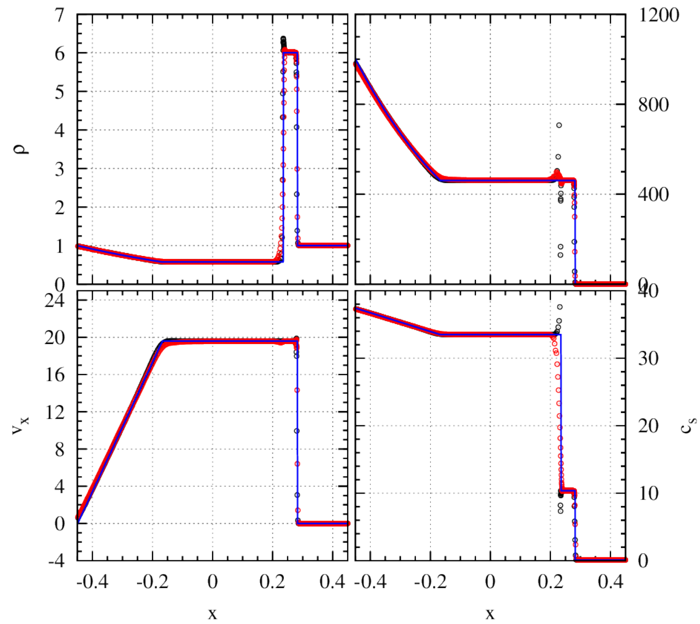

Figure 3: Strong shock (W2, $\gamma = 1.4$, $\alpha = \alpha^{\mathrm{u}} = 1.0$).

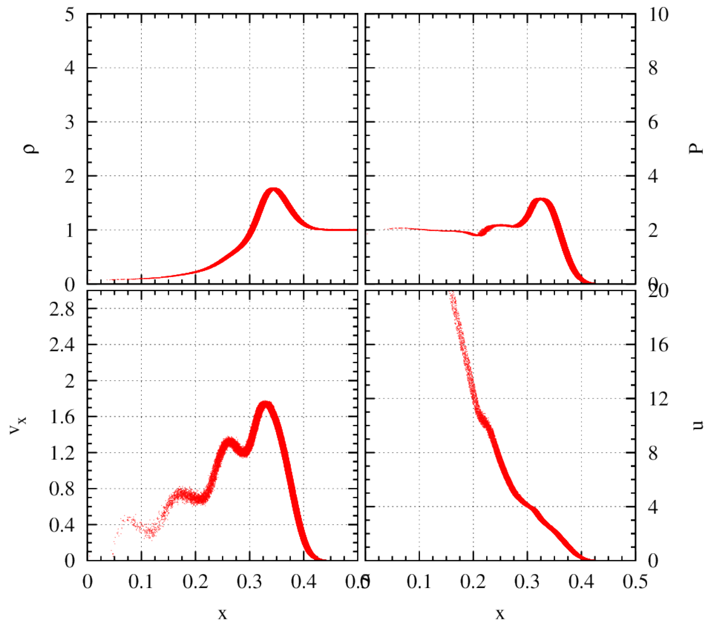

Figure 4: Point like explosion (W2, $\gamma = 5/3$, $\alpha = \alpha^{\rm u} = 3.0$).

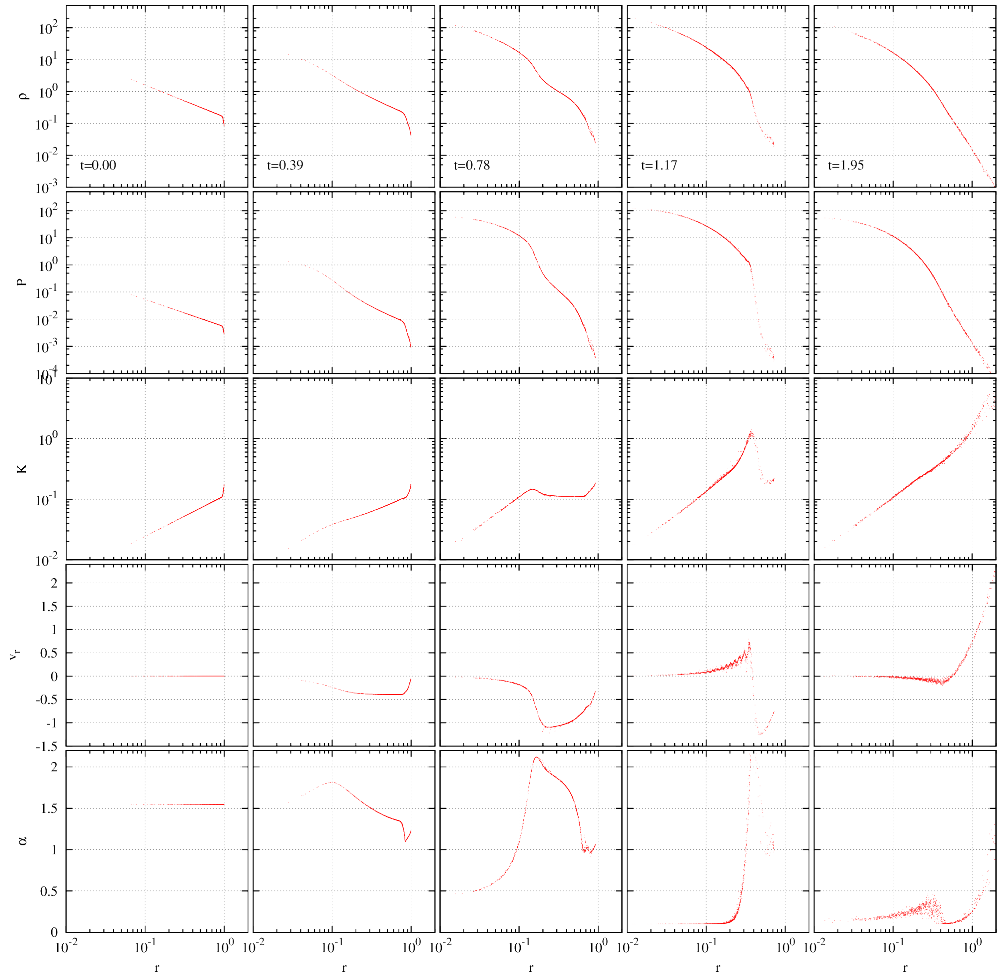

Figure 5: Evrard test (W2, $\gamma = 5/3$, $\alpha_{\min} = \alpha^{\mathrm{u}}_{\min} = 0.1$, $\alpha_{\max} = \alpha^{\mathrm{u}}_{\max} = 3.0$).

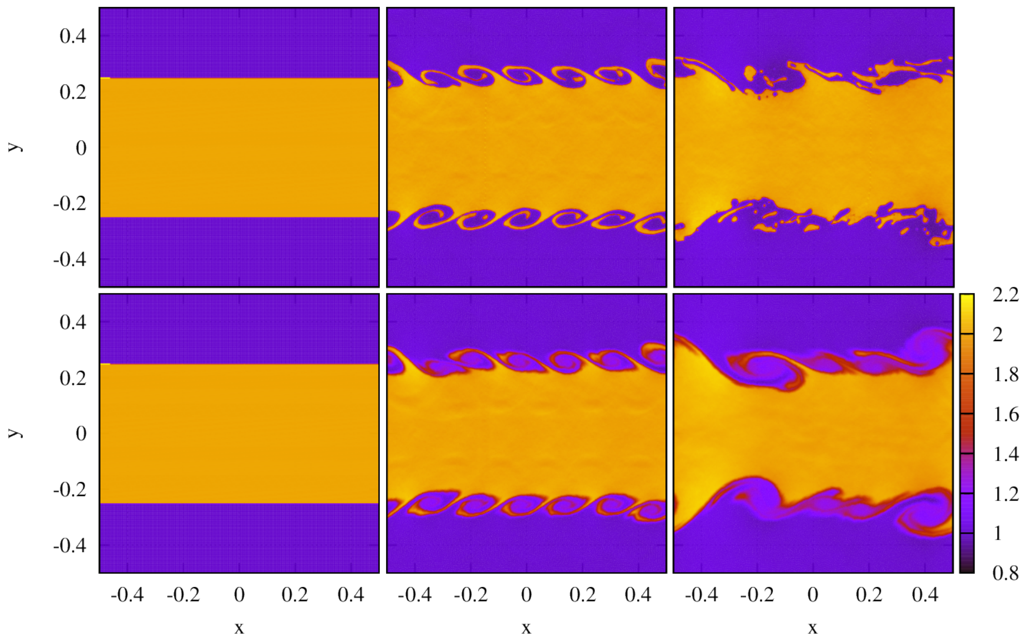

Figure 6: KH test. Top: W2, $\gamma = 5/3$, $\alpha = 3, \alpha^{\mathrm{u}} = 0$. Bottom: W2, $\gamma = 5/3$, $\alpha = \alpha^{\mathrm{u}} = 3$. From left to right, $t = 0$, $t = 2\tau_{\mathrm{KH}}$, $t = 4\tau_{\mathrm{KH}}$.

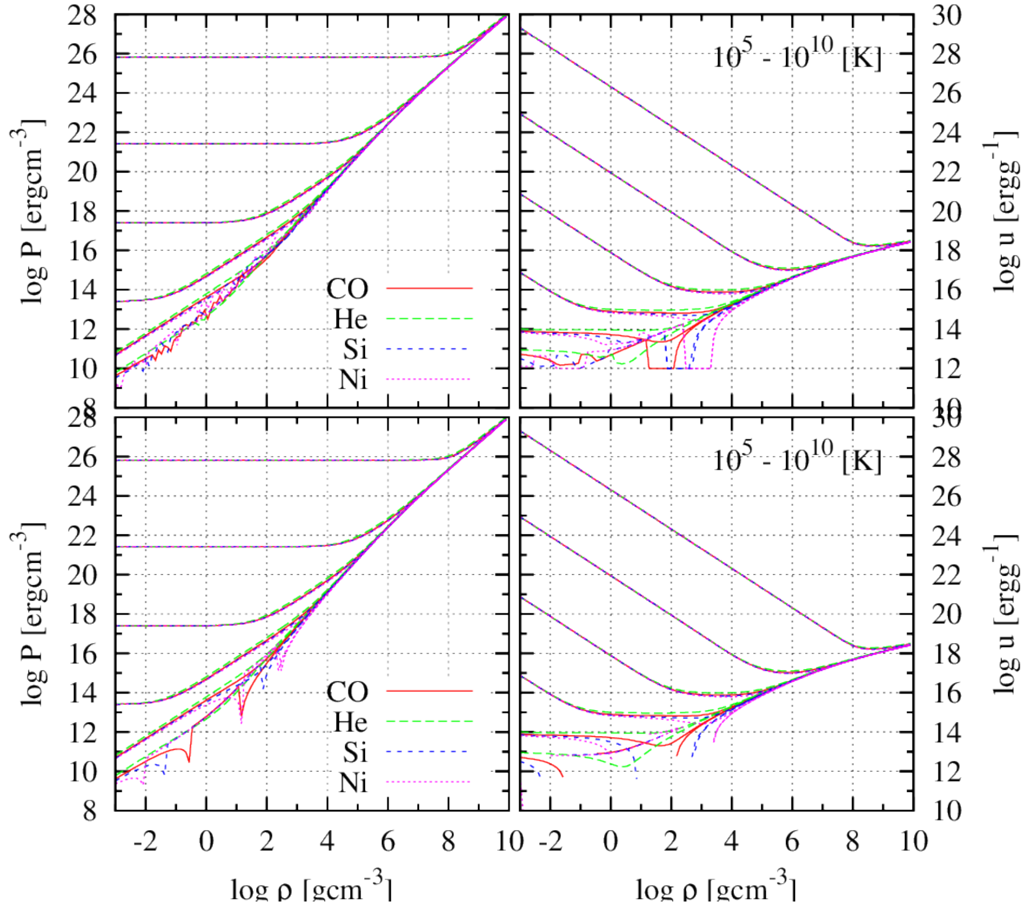

Figure 7: Helmholtz EOS w/ lookup table (top) and w/o (bottom).

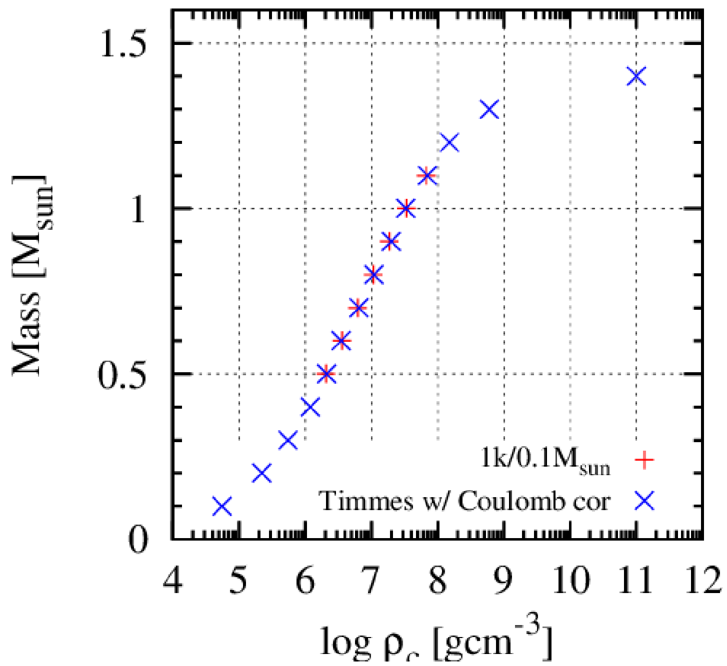

Figure 8: Single COWD equilibrium ($1k/0.1M_{\odot}$) w/ new damping. The new damping mode is the addtion of a sort of cooling during 100 s, such that $u_{\mathrm{new},i} = (u_{\mathrm{old},i} - u_{\mathrm{min}}(\rho_i))\exp(-0.1\Delta t) + u_{\mathrm{min}}(\rho_i)$.

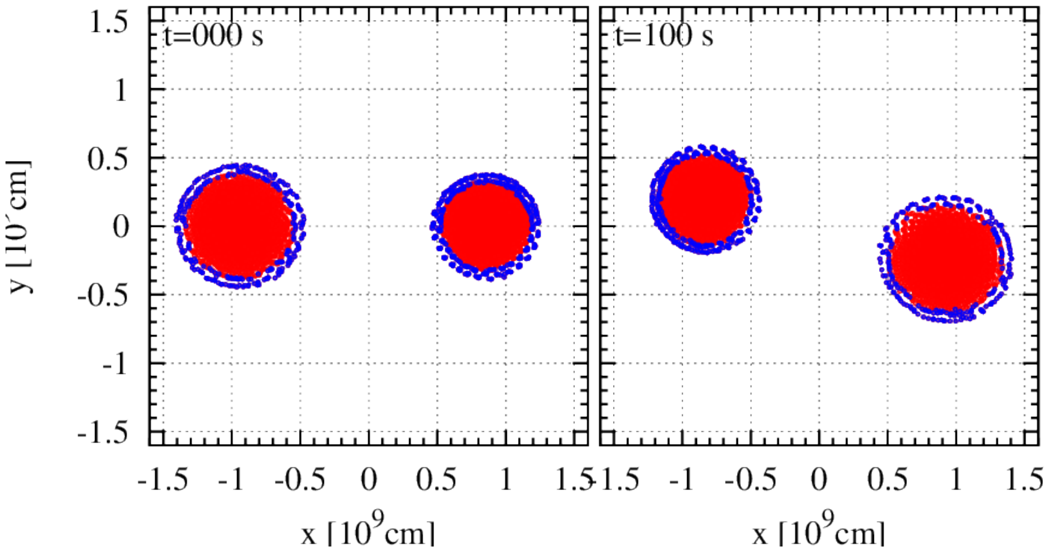

Figure 9: Density and temperature of $1.1M_{\odot}$ and $1.0M_{\odot}$ COWDs ($1k/0.1M_{\odot}$) from 1.8×10^{9}cm. Blue points indicate helium particles ($f_{\rm He}=0.1$).

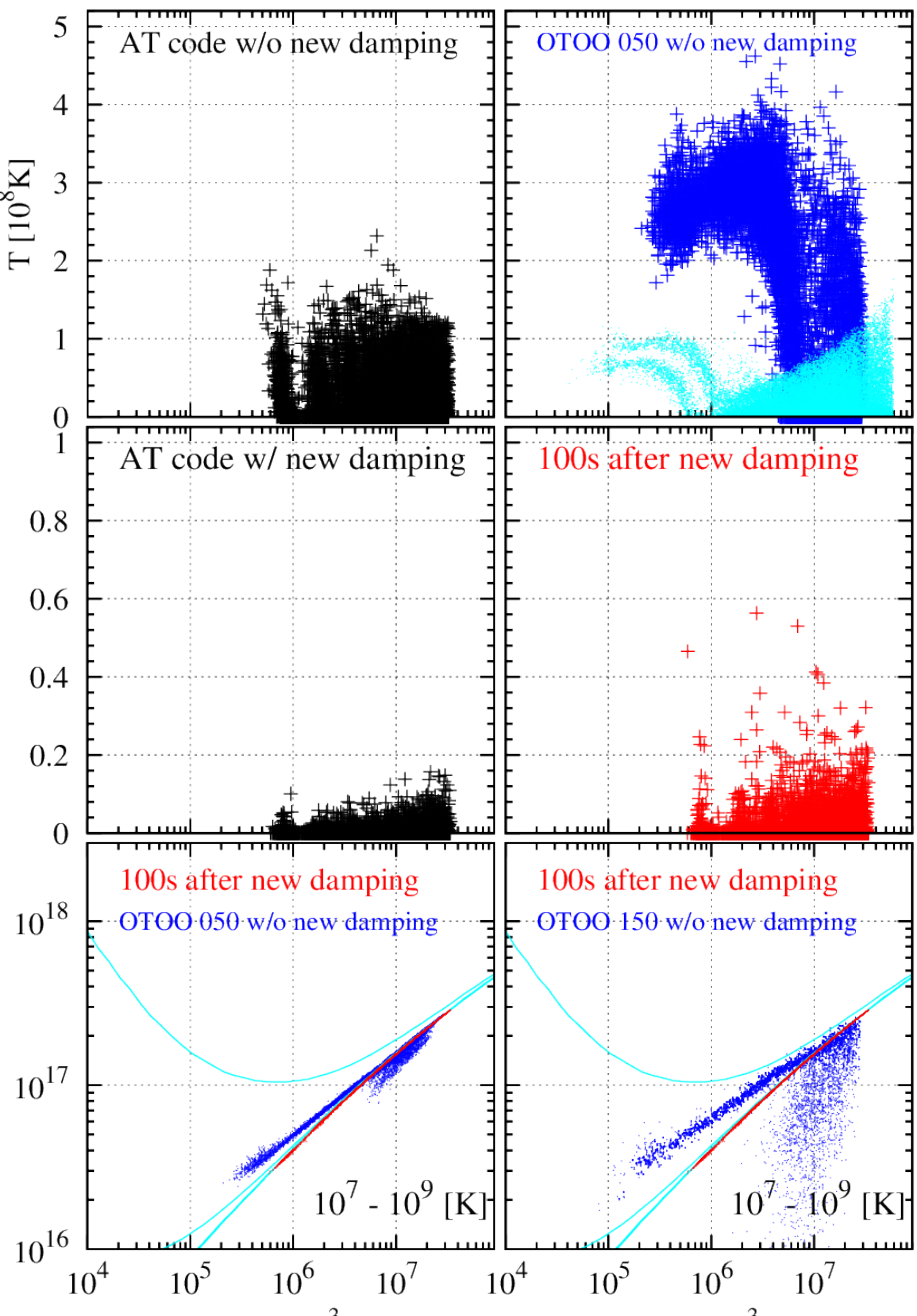

Figure 10: $1M_\odot$ COWD ($1k/0.1M_\odot$).

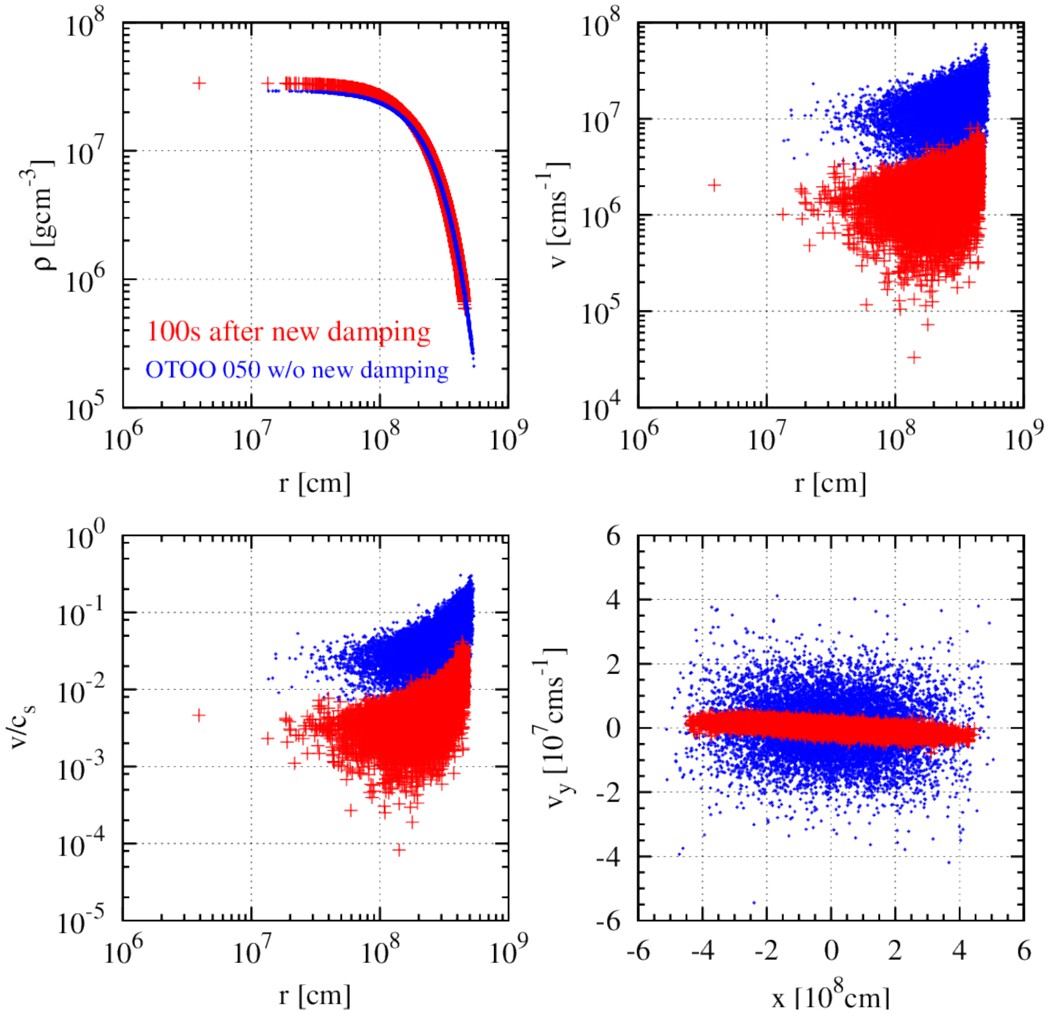

Figure 11: $1M_{\odot}$ COWD $(1k/0.1M_{\odot})$.

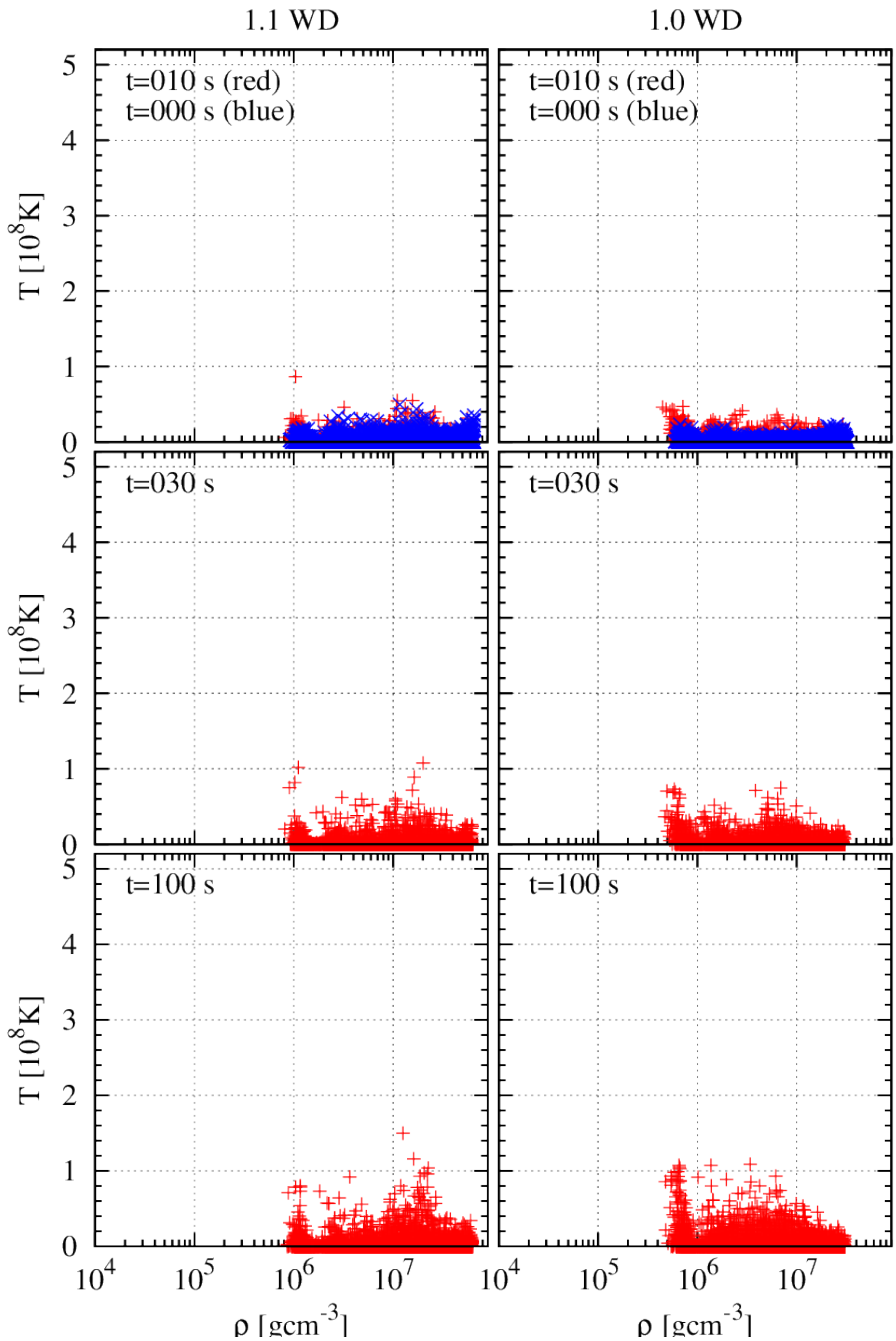

Figure 12: Density and temperature of $1.1M_\odot$ and $1.0M_\odot$ COWDs ($1k/0.1M_\odot$) from 1.8×10^9cm.

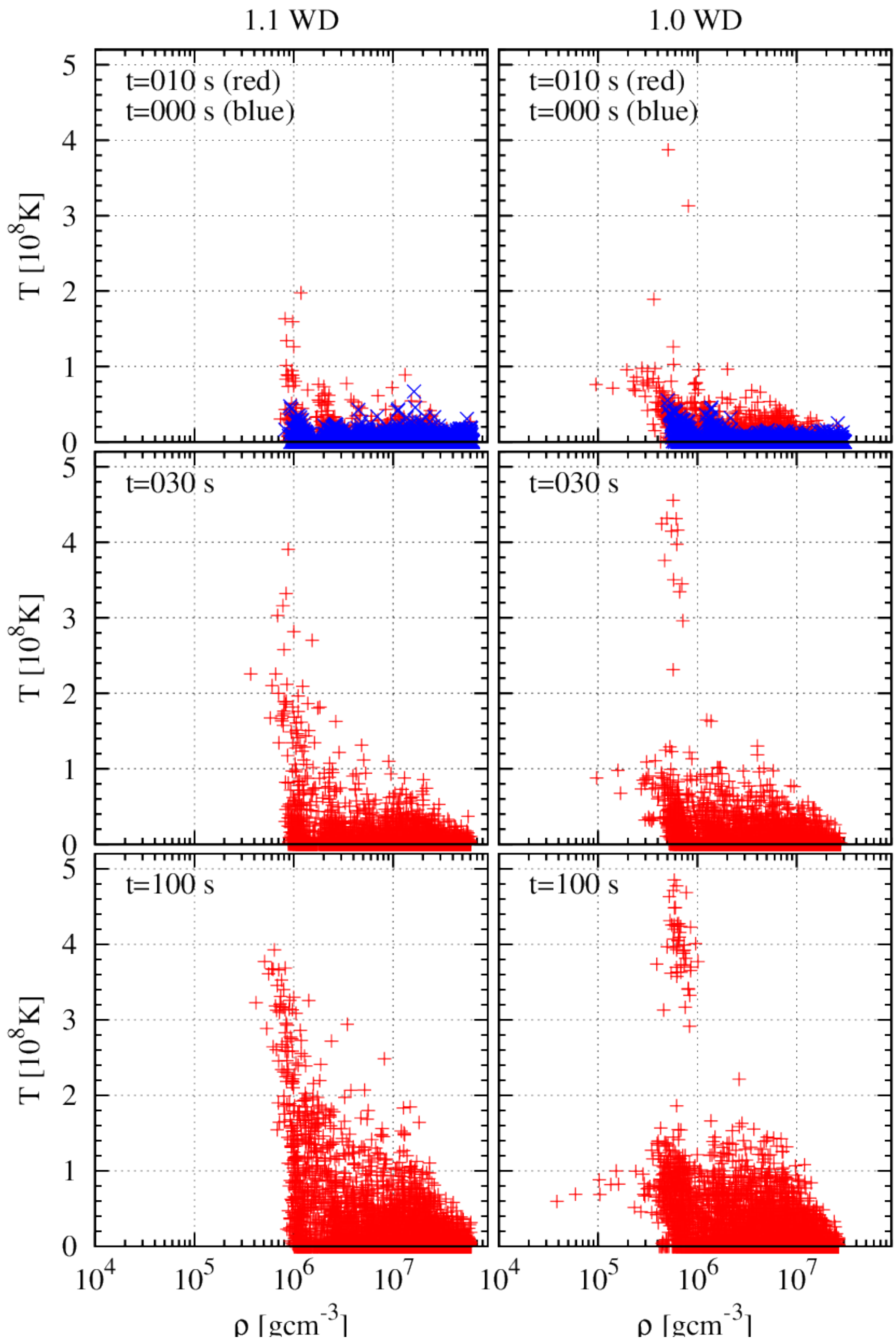

Figure 13: Density and temperature of $1.1M_{\odot}$ and $1.0M_{\odot}$ COWDs ($1k/0.1M_{\odot}$) from 1.5×10^9cm.

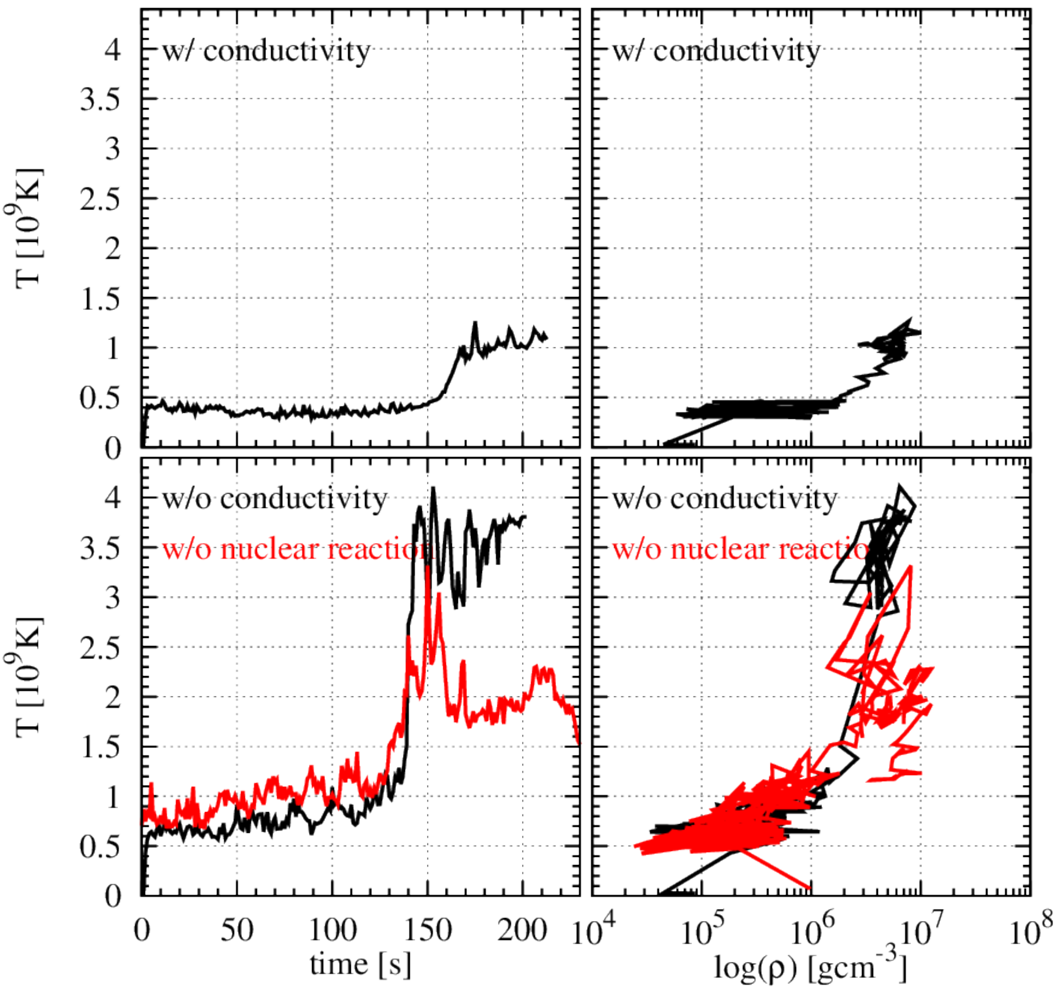

Figure 14: Temperature evolution with mass resolution $64k/0.1M_{\odot}$ in the cases of w/ conductivity (top) and w/o conductivity (bottom).

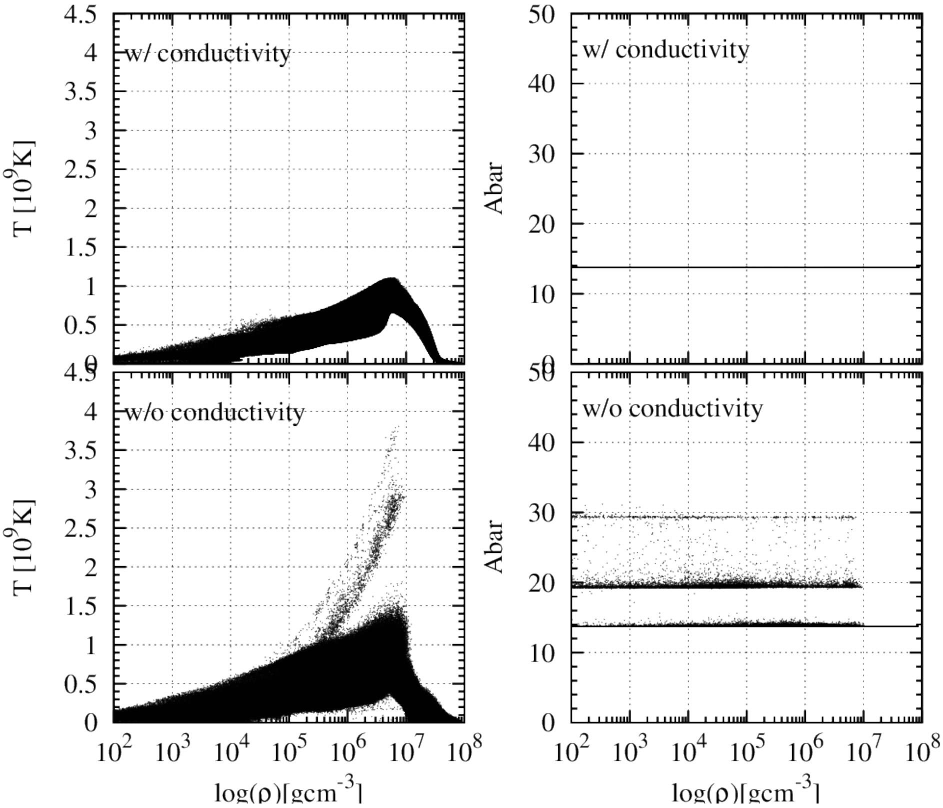

Figure 15: ρ-T and ρ-$\bar{A}$ plains with mass resolution $64k/0.1M_{\odot}$ in the cases of w/ conductivity (top) and w/o conductivity (bottom).