# 1 Variable kernel w/ grad-h

Suppose we know acceleration, time derviative of energy, coefficient viscosity, and the next timestep at the initial time. We integrate each particle as follows:

1. Calculate the position, and predict the velocity and energy:

$$r_i = r_i^{(0)} + v_i^{(0)} \Delta t + \frac{1}{2} a_i^{(0)} \Delta t^2$$
 (1)

$$\boldsymbol{v}_{i}^{(1/2)} = \boldsymbol{v}_{i}^{(0)} + \frac{1}{2}\boldsymbol{a}^{(0)}\Delta t \tag{2}$$

$$u_i^{(1/2)} = u_i^{(0)} + \frac{1}{2}\dot{u}^{(0)}\Delta t \tag{3}$$

$$\tilde{\boldsymbol{v}}_i = \boldsymbol{v}_i^{(0)} + \boldsymbol{a}_i^{(0)} \Delta t \tag{4}$$

$$\tilde{\boldsymbol{u}}_i = \boldsymbol{u}_i^{(0)} + \dot{\boldsymbol{u}}_i^{(0)} \Delta t \tag{5}$$

$$\tilde{\alpha}_i = \alpha_i^{(0)} + \dot{\alpha}_i^{(0)} \Delta t. \tag{6}$$

- 2. Calculate the density, kernel-support length and grad-h term of each particle by an iterative method:
  - (a) Estimate the density by the following expression:

$$\rho_i = \sum_j m_j W_i \tag{7}$$

$$\Omega_i = 1 + \frac{1}{D} \frac{H_i}{\rho_i} \sum_j m_j \frac{\partial W_i}{\partial H_i}$$
 (8)

where  $W_i = W(r_{ij}, H_i)$ ,  $r_{ij} = |\mathbf{r}_{ij}|$ , and  $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ . The kernel function is expressed as

$$W(r,H) = H^{-D}w(r/H), \tag{9}$$

$$\frac{\partial W(r,H)}{\partial H} = -H^{-(D+1)} \left[ Dw(q) + q \frac{\partial w}{\partial q} \right]. \tag{10}$$

The formula of w(q) and  $\partial w/\partial q$  are described in Appendix A.

(b) calculate the kernel length as follow:

$$H_i = (H/h) \times \eta \left(\frac{m_i}{\rho_i}\right)^{1/D},\tag{11}$$

where h is the kernel length. The values of  $\eta$  and H/h are described in Appendix A.

(c) Return to step (2a) if this is the first time.

(d) Calculate divergence and rotation of v:

$$\nabla \cdot \tilde{\boldsymbol{v}}_i = \frac{1}{\Omega_i \rho_i} \sum \tilde{\boldsymbol{v}}_{ij} \cdot \left[ m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right], \tag{12}$$

$$\nabla \times \tilde{\boldsymbol{v}}_i = \frac{1}{\Omega_i \rho_i} \sum \tilde{\boldsymbol{v}}_{ij} \times \left[ m_j \frac{\partial W_i}{\partial r_{ij}} \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right]. \tag{13}$$

The derivative of the kernel function is as follows:

$$\frac{\partial W(r,H)}{\partial r} = H^{-(D+1)} \frac{\partial w(r/H)}{\partial (r/H)}.$$
 (14)

3. Calculate the pressure  $(\tilde{P}_i)$  and sound speed  $(\tilde{c}_{s,i})$  as follows:

$$P_i = (\gamma - 1)\rho_i \tilde{u}_i \tag{15}$$

$$c_{\mathrm{s},i} = \left(\gamma \frac{P_i}{\rho_i}\right)^{1/2}.\tag{16}$$

4. Calculate the Balsara switch:

$$f_i = \frac{|\nabla \cdot \tilde{\boldsymbol{v}}_i|}{|\nabla \cdot \tilde{\boldsymbol{v}}_i| + |\nabla \times \tilde{\boldsymbol{v}}_i| + 0.0001\tilde{c}_{\mathrm{s},i}/h_i}.$$
(17)

5. Calculate the acceleration and the time derivative of the energy:

$$\boldsymbol{a}_{i} = -\sum \left(\frac{\tilde{P}_{i}}{\Omega_{i}\rho_{i}^{2}} + \frac{\tilde{P}_{j}}{\Omega_{j}\rho_{j}^{2}} + f_{ij}\Pi_{ij}\right) \left[\frac{m_{j}}{2} \left(\frac{\partial W_{i}}{\partial r_{ij}} + \frac{\partial W_{j}}{\partial r_{ij}}\right) \frac{\boldsymbol{r}_{ij}}{r_{ij}}\right],$$
(18)

$$\dot{u}_{i} = \sum \left( \frac{\tilde{P}_{i}}{\Omega_{i} \rho_{i}^{2}} + \frac{f_{ij} \Pi_{ij}}{2} \right) \left[ \frac{m_{j}}{2} \left( \frac{\partial W_{i}}{\partial r_{ij}} + \frac{\partial W_{j}}{\partial r_{ij}} \right) \frac{\boldsymbol{r}_{ij}}{r_{ij}} \right] \cdot \tilde{\boldsymbol{v}}_{ij}, \quad (19)$$

where  $f_{ij} = (f_i + f_j)/2$ , and  $\Pi_{ij}$  is an artificial viscosity. The artificial viscosity is expressed as follows:

$$\Pi_{ij} = -\frac{\alpha}{2} \frac{v_{ij}^{\text{sig}} w_{ij}}{\rho_{ij}} \tag{20}$$

$$v_{ij}^{\text{sig}} = c_{\text{s},i} + c_{\text{s},j} - 3w_{ij} \tag{21}$$

$$w_{ij} = \begin{cases} \frac{\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij}}{|\boldsymbol{r}_{ij}|} & (\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij} < 0) \\ 0 & (\boldsymbol{r}_{ij} \cdot \tilde{\boldsymbol{v}}_{ij} \ge 0) \end{cases}$$

$$(22)$$

where  $\alpha = 1.0$  and  $\rho_{ij} = (\rho_i + \rho_j)/2$ .

6. Calculate  $\dot{\alpha}$ :

$$\dot{\alpha}_i = -\frac{\tilde{\alpha}_i - \alpha_{\min}}{h_i/(0.25\tilde{c}_{s,i})} + S_i \tag{23}$$

$$S_i = \max\left[-(\nabla \cdot \tilde{\boldsymbol{v}}_i)(\alpha_{\max} - \tilde{\alpha}_i), 0\right]$$
(24)

7. Correct the velocity and energy:

$$\mathbf{v}_i = \mathbf{v}_i^{(1/2)} + \frac{1}{2} \mathbf{a}_i \Delta t \tag{25}$$

$$u_i = u_i^{(1/2)} + \frac{1}{2}\dot{u}_i \Delta t \tag{26}$$

$$\alpha_i = \alpha_i^{(1/2)} + \frac{1}{2}\dot{\alpha}_i \Delta t. \tag{27}$$

8. Calculate the next timestep:

$$\Delta t = C \min_{i} \left[ \min \left( \frac{2H_i}{\max_{j} \left( v_{ij}^{\text{sig}} \right)}, \frac{u_i}{|\dot{u}_i|} \right) \right]. \tag{28}$$

9. Return to step 1.

#### Kernels $\mathbf{A}$

### Cubic spline kernel

We describe the cubic spline kernel:

$$w(r/H) = 2^{D} \sigma \times \begin{cases} (1 - 6q^{2} + 6q^{3}) & (0 \le q < 1/2) \\ [2(1 - q)^{3}] & (1/2 \le q < 1) \\ 0 & (q \ge 1) \end{cases}$$
 (29)

$$w(r/H) = 2^{D}\sigma \times \begin{cases} (1 - 6q^{2} + 6q^{3}) & (0 \le q < 1/2) \\ [2(1 - q)^{3}] & (1/2 \le q < 1) \\ 0 & (q \ge 1) \end{cases}$$

$$\frac{\partial w(q)}{\partial q} = (-6)2^{D}\sigma \times \begin{cases} 1/3 & (0 \le q < 1/3) \\ (2q - 3q^{2}) & (1/3 \le q < 1/2) \\ (1 - q)^{2} & (1/2 \le q < 1) \\ 0 & (q \ge 1) \end{cases} ,$$

$$(30)$$

(31)

where  $\sigma = 2/3$  (1D),  $10/7\pi$  (2D), and  $1/\pi$  (3D). For this kernel,  $\eta = 1.2$ , and H/h = 2.

#### Wendland $C^2$ kernel

We describe Wendland  $C^2$  kernel: In the case of D=1,

$$w(q) = C_{W}(1-q)_{+}^{3}(1+3q)$$
(32)

$$\frac{\partial w(q)}{\partial q} = 3C_{\rm W} \left[ (1-q)_+^3 - (1-q)_+^2 (1+3q) \right], \tag{33}$$

and in the case of D = 2, 3,

$$w(q) = C_{W}(1-q)_{+}^{4}(1+4q)$$
(34)

$$\frac{\partial w(q)}{\partial q} = 4C_{\rm W} \left[ (1-q)_+^4 - (1-q)_+^3 (1+4q) \right], \tag{35}$$

where  $C_{\rm W} = 5/4$  (D=1),  $7/\pi$  (D=2), and  $21/(2\pi)$  (D=3). For this kernel,  $\eta = 1.6$ , and H/h = 1.620185(D=1), 1.897367(D=2), and 1.93492(D=3).

## A.3 Wendland C<sup>4</sup> kernel

We describe Wendland  $C^4$  kernel: In the case of D=1,

$$w(q) = C_{\rm W}(1-q)_+^5(1+5q+8q^2)$$
(36)

$$\frac{\partial w(q)}{\partial q} = C_{W}(1-q)_{+}^{4} \left[ -5(1+5q+8q^{2}) + (1-q)_{+}(5+16q) \right]$$
 (37)

and in the case of D=2,3,

$$w(q) = C_{W}(1 - q)_{+}^{6} \left[ 1 + 6q + (35/3)q^{2} \right]$$
(38)

$$\frac{\partial w(q)}{\partial q} = C_{W}(1-q)_{+}^{5} \left\{ -6 \left[ 1 + 6q + (35/3)q^{2} \right] + (1-q)_{+} \left[ 6 + (70/3)q \right] \right\}$$
(39)

where  $C_{\rm W}=3/2$  (D=1),  $9/\pi$  (D=2), and  $495/(32\pi)$  (D=3). For this kernel,  $\eta=1.6$ , and H/h=1.936492(D=1), 2.171239(D=2), and 2.207940(D=3).

#### B Test

- 1. Shock tube
- 2. Strong shock
- 3. Point like explosion
- 4. Evrard test

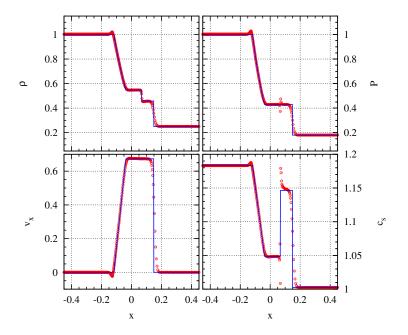


Figure 1: Shock tube (cubic spline,  $\gamma=1.4,\,\alpha=1.0$ ).

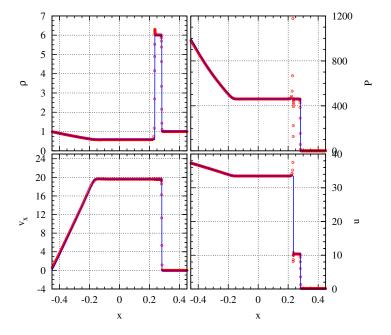


Figure 2: Strong shock (cubic spline,  $\gamma=1.4,\,\alpha=1.0).$ 

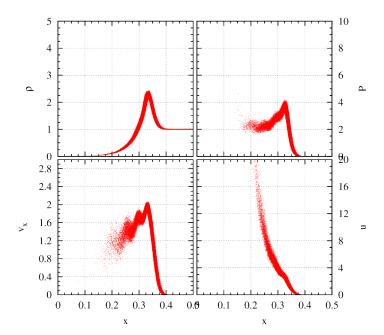


Figure 3: Point like explosion (cubic spline,  $\gamma=5/3,\,\alpha=3.0).$ 

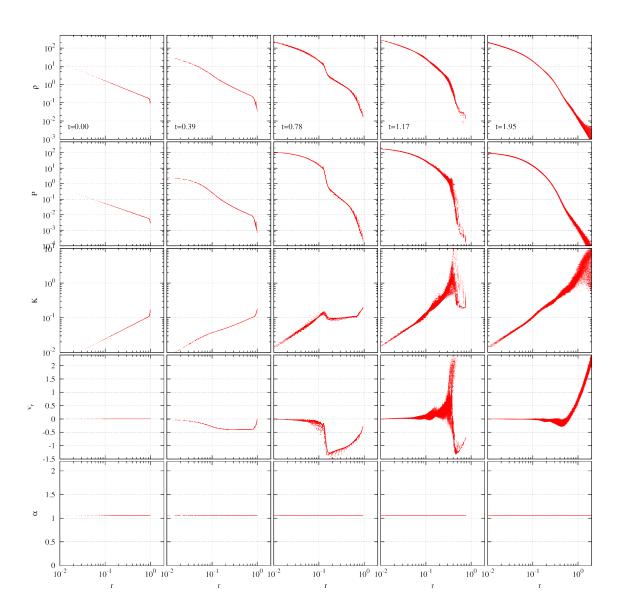


Figure 4: Evrard test (cubic spline,  $\gamma = 5/3, \, \alpha = 1.05.$