ASSIGNMENT 2

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1) Let's say there exists a strategy profile ~~$\{(p,1-p)\times(q,1-q)\}$~~
$\sigma^* = ([p, 1-p], [q, 1-q])$

Where $\sigma_1^* = [p, 1-p]$
$\sigma_2^* = [q, 1-q]$

The game is given by

P1/P2	C	NC
C	2,2	0,3
NC	3,0	1,1

Necessary and Sufficient Conditions for MSNE:

$u_1(C, \sigma_2^*) = u_1(NC, \sigma_2^*)$ — (1)

$u_2(C, \sigma_1^*) = u_2(NC, \sigma_1^*)$ — (2)

For (1),

LHS: $u_1(C, \sigma_2^*) = 2q + 0(1-q) = 2q$

RHS: $u_1(NC, \sigma_2^*) = 3q + 1 - q = 2q + 1$

Clearly, $u_1(NC, \sigma_2^*) > u_1(C, \sigma_2^*) \ \forall q \in (0,1)$

~~$\therefore$ NC is a strongly dominant strategy~~

For (2)

LHS: $u_2(C, \sigma_1^*) = 2p + 0(1-p) = 2p$

RHS: $u_2(NC, \sigma_1^*) = 3p + 1 - p = 2p + 1$

$u_2(NC, \sigma_1^*) > u_1(C, \sigma_1^*) \ \forall p \in (0,1)$

$\therefore$ NC is a strongly dominant strategy.

$\therefore$ The degenerate strategy equilibrium $\boxed{[0,1]\times[0,1]}$ is the only MSNE

This is the same as the pure strategy equilibrium

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2A) Consider the profile : $\sigma^* = ([p, 1-p], [q, 1-q])$

So $\sigma_1^* = [p, 1-p]$

$\sigma_2^* = [q, 1-q]$

Now, necessary and sufficient conditions,

$u_1(H, \sigma_2^*) = u_1(T, \sigma_2^*)$ — ①

$u_2(H, \sigma_1^*) = u_2(T, \sigma_1^*)$ — ②

For ①

$u_1(H, \sigma_2^*) = -q + 2(1-q) = -3q + 2$

$u_1(T, \sigma_2^*) = +2q + q - 1 = 3q - 1$

For equality : $-3q + 2 = 3q - 1 \Rightarrow q = 0.5$

For ②

$u_2(H, \sigma_1^*) = p + p - 1 = 2p - 1$

$u_2(T, \sigma_1^*) = -p + p + 1 = 1 - 2p$

For equality : $1 - 2p = 2p - 1 \Rightarrow p = 0.5$

$\therefore ([\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}])$ is an MSNE

Checking for pure strategy equilibrium,

(H, H) : P1 can play T to improve his payoff. Eliminated

(H, T) : P2 can play H to improve her payoff. Eliminated

(T, H) : P2 can play T to improve her payoff. Eliminated

(T, T) : P1 can play H to improve his payoff. Eliminated

$\therefore$ No pure stratergy NE exists.

$\therefore$ The only MSNE of this game is $([\frac{1}{2}, \frac{1}{2}] \times [\frac{1}{2}, \frac{1}{2}])$

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Q2c) ~~Case 1: All three~~

Case 1: Consider the profile: $([t, m, 1-t-m], [l, c, 1-l-c]) = \sigma^*$

Apply necessary and sufficient conditions:

1) $u_1(T, \sigma_2^*) = u_2(M, \sigma_2^*) = u_3(B, \sigma_2^*)$

$u_1(T, \sigma_2^*) = l - 4c$

$u_1(M, \sigma_2^*) = -4l + c$

$u_1(B, \sigma_2^*) = 1 - l - c$

$\Rightarrow 5l = 5c \Rightarrow l = c$

$\Rightarrow -3l = 1 - 2l \Rightarrow l = -1$

Not possible

$\therefore$ No MSNE can exist here as we get negative probability

Also, we can conclude that for all MNSE, $l \neq c$, $t = m$

Case 2: $\sigma^* = ([t, 1-t, 0] \times [l, 1-l, 0])$

Applying conditions,

1) $u_1(T, \sigma_2^*) = l - 4 + 4l = 5l - 4$

$u_1(M, \sigma_2^*) = -4l + 1 - l = -5l + 1$

$\Rightarrow u_1(T, \sigma_2^*) = u_1(M, \sigma_2^*)$

$\Rightarrow 5l - 4 = -5l + 1$

$\Rightarrow l = 0.5$

2) $u_2(L, \sigma_1^*) = -4t + 1 - t = -5t + 1$

$u_2(C, \sigma_1^*) = t - 4 + 4t = 5t - 4$

$u_2(L, \sigma_1^*) = u_2(C, \sigma_1^*)$

$\Rightarrow -5t + 1 = 5t - 4$

$\Rightarrow t = 0.5$

$\therefore$ We get an MSNE: $([\frac{1}{2}, \frac{1}{2}, 0] \times [\frac{1}{2}, \frac{1}{2}, 0])$

Case 3: $\sigma^*([t, 0, 1-t] \times [l, 0, 1-l])$

Applying conditions

1) $u_1(T, \sigma_2^*) = l$

$u_1(B, \sigma_2^*) = 1 - l$

$u_1(T, \sigma_2^*) = u_1(B, \sigma_2^*)$

$\Rightarrow l = 1 - l \Rightarrow l = 0.5$

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Case 3: $\sigma^* = ([0,0,1] \times [0,0,1])$

This is degenerate to pure strategy $[B \times R]$

~~However P1 can play~~

P1 cannot improve payoff by playing T/M

P2 cannot improve payoff by playing L/C

$\therefore$ ~~B and R are their~~ b.

$\therefore$ $(B \times R)$ ~~forms~~ or $([0,0,1] \times [0,0,1])$ forms a purestrategy equilibrium.

$\therefore$ MSNE: $\sigma([\frac{1}{2}, \frac{1}{2}, 0] \times [\frac{1}{2}, \frac{1}{2}, 0])$

and $\sigma([0,0,1] \times [0,0,1])$

2B) Say $\sigma^*([p, 1-p], [q, 1-q])$ is an MSNE

$\therefore$ $u_1(L, \sigma_2^*) = u_1(R, \sigma_2^*)$

$\Rightarrow$ $\cancel{2q}\ 0 = 2q + 0$

$\Rightarrow$ $q = 0$

and $u_2(L, \sigma_1^*) = u_1(R, \sigma_1^*)$

$\Rightarrow$ $p + 2 - 2p = 2p + 1 - p$

$\Rightarrow$ $1 = 2p$

$\Rightarrow$ $p = 0.5$

$\therefore$ $\sigma([\frac{1}{2}, \frac{1}{2}] \times [0, 1])$

forms an MSNE

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3) Let this be the BOS game

P_1 / P_2	S	B
S	a_1, b_1	0,0
B	0,0	a_2, b_2

Given: $\sigma = \left(\left[\frac{3}{5}, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{3}{5}\right] \right)$ is an MSNE

$\therefore \quad u_1(S, \sigma_2) = u_1(B, \sigma_2)$

$\Rightarrow \quad \frac{2}{5} a_1 = \frac{3}{5} a_2$

$\Rightarrow \quad \frac{a_1}{a_2} = \frac{3}{2}$

And $\quad u_2(S, \sigma_1) = u_2(B, \sigma_1)$

$\Rightarrow \quad \frac{3b_1}{5} = \frac{2b_2}{5} \Rightarrow \frac{b_1}{b_2} = \frac{2}{3}$

$\therefore$ Take $a_1 = 3, a_2 = 2, b_1 = 2, b_2 = 3$

$\therefore$ Required BOS game,

P1 / P2	S	B
S	3,2	0,0
B	0,0	2,3

4) Notice, the symmetry of this situation. All the witnesses are equivalent. Hence, we only need to check for MSNE in profiles where everyone has the same probability

Consider a profile ~~[illegible]~~ σ with $\sigma_i = p \ \forall i \in \{1, 2, \dots, N\}$ where p is the probability that witness i does NOT report the crime.

Now payoffs,

$$u_i = \begin{cases} v & , \text{if } i \text{ does not report, but someone does} \\ v-c & , \text{if } i \text{ reports} \\ 0 & , \text{if nobody reports} \end{cases}$$

and $v > c > 0$

Say σ is an NE,

$\therefore u_i(1-p, \sigma^*_{-i}) = v - c$

$u_i(p, \sigma^*_{-i}) = 0 \times (P(\text{no one reports})) + v \times (P(\text{someone reports}))$

$= v(1 - p^{N-1})$

If $v-c > v(1-p^{n-1})$: i must report

$\therefore p = 0$ does not make an NE

If $v-c < v(1-p^{n-1})$: i should not report

$\therefore p = 1$ does not make an NE

If $v - c = v(1-p^{n-1})$: MSNE

$\Rightarrow c = vp^{n-1}$

$\Rightarrow p = \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$ is the probability of a witness not reporting

Now $v > c > 0 \Rightarrow 0 < \frac{c}{v} < 1$ and $n > 1 \Rightarrow n \geq 2$

$\therefore$ As $n \uparrow$, $\frac{1}{n-1} \downarrow$ and $p \uparrow$

$\therefore$ If the number of witnesses increase, each is less likely to report.

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Probability that no one reports,

$$= p^N = \left(\frac{c}{v}\right)^{\frac{N}{N-1}}$$

$\therefore$ As $n \uparrow$, $p^N \uparrow$ towards $\frac{c}{v}$.

$\therefore$ The probability of no-one reporting the crime increases with increasing N and is given by

$$\boxed{P = \left(\frac{c}{v}\right)^{\frac{N}{N-1}}}$$

5) i) 2 Bidders : i.i.d Unif $[0,1]$

$\therefore F(s) = s$ [Probability of a ~~bid~~ signal $< s$]

$f(s) = 1$ [Probability of any specific ~~bid~~ signal s]

Now, Probability that ~~a winner~~ bidder with signal s wins :

$G(s) = P_r[\, b(Y) \leq b(s)\,]$ [Probability of other bidder getting a smaller signal]

$= s^{2-1} = s$

[Here we assume that $b(i)$ is an increasing function]

Probability density $= 1$

Expected payment of bidder 1 with signal s.

$$m(s) = \int_0^s y\, g(y)\, dy = \int_0^s y\, dy = \frac{s^2}{2}$$

Expected payment of bidder 1

$$m = \int_0^1 m(s)\, f(s)\, ds = \int_0^1 \frac{y^2}{2} \cdot 1 = \left[\frac{y^3}{6}\right]_0^1 = \frac{1}{6}$$

Expected revenue

$$ER = Nm = 2 \times \frac{1}{6} = \boxed{\frac{1}{3}}$$

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5 ii) 2 Bidders : (θ_1, θ_2) iid Unif $[0,1]$

$\therefore F(s) = s$

$f(s) = 1$

~~G(s) =~~ Let Y be the second highest bid.

$$G(s) = P_r[\, Y \leq s\,] = F^{2-1}(s) = s$$

$g(s) = 1$

$$m(s) = \int_0^s y\, g(y)\, dy = \int_0^s y\, dy = \frac{s^2}{2}$$

$$m = \int_0^1 \frac{s^2}{2} \cdot 1\, ds = \left[\frac{s^3}{6}\right]_0^1 = \frac{1}{6}$$

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$ER = Nm = 2 \times \frac{1}{6} = \boxed{\frac{1}{3}}$

$\therefore$ Expected revenue is $\frac{1}{3}$

$\therefore$ Revenue in I price Auction = Revenue in II price Auction

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6>

Answer: First Price Auction is ~~not~~ NOT incentive-compatible

Second Price Auction is DSIC

Explanation:

i> Second Price Auction

Let's assume player i's valuation is v_i,
his bid is b_i
and the highest bid from all other players is h.

~~Case 1:~~ To show: $b_i = v_i$ is a weakly dominant strategy

Case 1: $h > v_i$

Here, one ~~or~~ pays nothing by bidding $b_i = v_i$ or $b_i < v_i$, in either case ~~it~~, the payoff is zero.
However overbidding: $b_i > h$ gives a negative payoff $(v_i - h)$. Hence bidding ~~th~~ truthfully is ~~one of the best options~~ the dominant option

Case 2: $v_i > h$

Here, one wins the object by bidding $b_i = v_i$ or $b_i > v_i$ and in both cases there is a positive payoff $(v_i - h)$.
But underbidding: $b_i < h$ leads to zero payoff which is a worse outcome.
Hence bidding truthfully is the better option.

Case 3: $v_i = h$

~~Here overbidding gives nega~~ Zero probability event is ignored.

In all cases, bidding truthfully ~~gives the best possible payoff.~~ is atleast not worse than all other options.
Hence truthfulness is a weakly dominant strategy.

Hence II price auction is DSIC

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ii) First price Auction

We saw already that in a first price auction, the BNE is formed when the n players who have valuations v_i bid

$$b_i = \frac{N-1}{N} v_i < v_i$$

Hence the bayesian nash equilibrium is formed by bidding less than your true valuation.

Also note how, if your true valuation is higher than the highest bid of everyone else (h), then you maximise your payoff by ~~biddingh~~ bidding h and not v_i. Hence bidding truthfully is dominated by underbidding in this case.

Thus as truth-telling does not form a BNE and is not a weakly dominant strategy,

First price Auction is not incentive compliant

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8) $v(i)$ iid Unif $[0,1]$

Set $v(i) = s_i$ for simplicity

~~$n\,v(i)\,2\,\&\ldots$~~

Now $F(s) = s$, $f(s) = 1$ $\because$ s_i is iid Unif$[0,1]$

~~Payoff for bidd~~ From the symmetry, it is obvious that the equilibrium scenario will have everyone use the same optimal bid

ie $b(i) = b(\,)$

But say this is not true and bidder i uses the function b_i instead.

$\therefore$ Payoff,

$$U_i(b_i, b_{-i}, s_i) = (s_i - b_i)\,F^{n-1}(b^{-1}(b_i))$$
$$= (s_i - b_i)\,(b^{-1}(b_i))^{n-1}$$

Now, as b_i must be the optimal bid, U_i must be maximised at b_i

ie $b_i = \text{argmax}\ (s_i - b_i)(b^{-1}(b_i))^{n-1}$

$\therefore$ To find b_i, $\dfrac{\partial U_i}{\partial b_i} = 0$

$$\therefore \frac{\partial U_i}{\partial b_i} = (s_i - b_i)(n-1)(b^{-1}(b_i))^{n-2} \cdot \frac{1}{b'(b^{-1}(b_i))} - (b^{-1}(b_i))^{n-1} = 0$$

$$\Rightarrow \frac{(s_i - b_i)(n-1)}{b'(b^{-1}(b_i))} = b^{-1}(b_i)$$

$$\Rightarrow b'(b^{-1}(b_i)) = \frac{(s_i - b_i)(n-1)}{b^{-1}(b_i)}$$

But for a symmetric equilibrium,

set $s_i = s$, $b_i = b(i) = b(s)$

$$\therefore b'(b^{-1}(b(s))) = \frac{(s - b(s))(n-1)}{s}$$

$$\Rightarrow b'(s) = \left[1 - \frac{b(s)}{s}\right](n-1)$$

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$$\Rightarrow \frac{d}{dt_s}\left[b(t_s) \cdot s^{n-1}\right] = (n-1)\, s^{n-1}$$

$$\Rightarrow \frac{d}{dt}\left[b(t) \cdot t^{n-1}\right] = (n-1)\, t^{n-1}$$

$$\Rightarrow b(t) \cdot t^{n-1} = \int (n-1)\, t^{n-1}$$

$$\Rightarrow b(t) \cdot t^{n-1} = \frac{(n-1)\, t^{n}}{n}$$

$$\Rightarrow b(t) = \frac{s(n-1)}{n}$$

But $s = V(i)$

$\cancel{\Rightarrow b(V(i)) =}$

$$\Rightarrow b(i) = V(i) \cdot \left(\frac{n-1}{n}\right)$$

b) They form a Nash Equilibrium. This strategy is not dominant always.

c) i> Say that player i has valuation $V(i)$

Say that he bids $b(i)$

and the highest bid from everyone else is h.

Case 1 : $h > V(i)$

Here, $U_i = \begin{cases} 0, & b(i) < h \\ -h + V(i), & b(i) > h \end{cases}$

$\because h > V(i)$, $V(i) - h < 0$

$\therefore$ The better response is for $b(i) < h$ if $h > V(i)$

$\therefore \underbrace{b(i) \leq V(i)}_{\text{Truthful bidding \& Underbidding.}}$ or $\underbrace{h < b(i) < V(i)}_{\text{Slight overbidding}}$

Clearly overbidding is a worse strategy

Case 2: $h < V(i)$

Here $U_i = \begin{cases} V(i) - h, & b_i > h \\ 0, & b_i < h \end{cases}$ (>0)

Here, it is better to overbid or bid truthfully ($b_i \geq V(i)$) as underbidding gives worse payoff.

Now, taking union of cases, we can see that truthful bidding is weakly dominant strategy and always gives the best payoff.

$\therefore$ The Dominant Strategy Nash Equilibrium will be the one where everyone bids their valuation.

$\therefore B(i) = V(i)$

This is a Dominant Strategy Equilibrium

9) Since $V(i)$ is iid Unif $[0,1]$ $\boxed{\text{First Price Auction}}$

$F(s) = s$, $f(s) = 1$

$G(s) = Pr[V(i) > V_{j \neq i}] = [V(i)]^{n-1}$

Set $V(i) = s$ for convinience.

$\therefore G(s) = s^{n-1}$, $g(s) = (n-1) s^{n-2}$

$\therefore$ Expected payment for signal s,

$m(s) = \int_0^s y\, g(y)\, dy = \int_0^s y (n-1) y^{n-2} dy = \frac{(n-1)}{n} s^n$

Expected payment, $m = \int_0^1 m(s) f(s) ds$

$m = \int_0^1 \frac{n-1}{n} s^n . 1 ds = \frac{n-1}{n} \cdot \frac{1}{n+1} [s^{n+1}]_0^1 = \frac{n-1}{n(n+1)}$

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Expected revenue $= \cancel{\;} nm = \boxed{\dfrac{n-1}{n+1}}$

Second Price Auction

Say: $V(i) = s$ is iid $\mathrm{Unif}[0,1]$

$F(s) = s, \; f(s) = 1$

Let Y denote second highest bid

$G(s) = Pr[Y \leq s] = s^{n-1}$

$g(s) = (n-1)s^{n-2}$

$$m(s) = \int_0^s y\, g(y)\, dy = \int_0^s y\,(n-1)\, y^{n-2}\, dy = \frac{n-1}{n} s^n$$

$$m = \int_0^1 m(s) f(s) ds = \int_0^1 \frac{n-1}{n} s^n \cdot 1\, ds = \frac{n-1}{n(n+1)}$$

Expected revenue $= nm = \boxed{\dfrac{n-1}{n+1}}$
