

## 1 Equations in $\rho$ coordinates

$$0 = \frac{1}{V'} \frac{\partial}{\partial \rho} \left[ V' \langle |\nabla \rho|^2 \rangle \frac{\partial}{\partial \rho} \Phi \right] + \frac{1}{\epsilon_0} (-en_e + Z_i en_i + Z_b en_b + Z_b e g n_b^{\text{rp}}) \quad (1.1)$$

$$\begin{aligned} \frac{1}{c^2} \frac{\partial}{\partial t} \dot{\psi}_t &= V' \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle \frac{\partial}{\partial \rho} \left[ \frac{1}{V' \langle R^{-2} \rangle} \frac{\partial}{\partial \rho} \psi_t \right] \\ &+ \frac{\mu_0}{\langle B^\theta \rangle} \left[ -en_e \left( \langle Bu_{e\parallel} \rangle - I \left\langle \frac{u_{e\zeta}}{R} \right\rangle \right) + Z_i en_i \left( \langle Bu_{i\parallel} \rangle - I \left\langle \frac{u_{i\zeta}}{R} \right\rangle \right) \right. \\ &\left. + Z_b en_b \left( \langle Bu_{b\parallel} \rangle - I \left\langle \frac{u_{b\zeta}}{R} \right\rangle \right) \right] \end{aligned} \quad (1.2)$$

$$\begin{aligned} \frac{1}{c^2} \frac{\partial}{\partial t} \dot{\psi} &= \frac{1}{V' \langle R^{-2} \rangle} \frac{\partial}{\partial \rho} \left[ V' \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle \frac{\partial}{\partial \rho} \psi \right] \\ &- \frac{\mu_0}{\langle R^{-2} \rangle} \left( -en_e \left\langle \frac{u_{e\zeta}}{R} \right\rangle + Z_i en_i \left\langle \frac{u_{i\zeta}}{R} \right\rangle + Z_b en_b \left\langle \frac{u_{b\zeta}}{R} \right\rangle \right) \end{aligned} \quad (1.3)$$

$$\frac{\partial}{\partial t} \psi = \dot{\psi} \quad (1.4)$$

$$\frac{\partial}{\partial t} \psi_t = \dot{\psi}_t \quad (1.5)$$

$$\begin{aligned} \frac{1}{V'} \frac{\partial}{\partial t} (V' n_e) &= -\frac{1}{V'} \frac{\partial}{\partial \rho} [V' n_e (u_e^\rho - u_g^\rho)] + \nu_{\text{ionize}} \frac{n_e}{n_{01} + n_{02} + n_{03}} (n_{01} + n_{02} + n_{03}) \\ &- \nu_L (n_e - n_{\text{ediv}}) + (1 - f_{\text{CX}}) \frac{P_b}{E_b} \end{aligned} \quad (1.6)$$

$$\begin{aligned} 0 &= -(\langle B^2 \rangle \langle R^2 \rangle - I^2) \frac{\partial}{\partial \rho} n_e T_e + en_e (\langle B^2 \rangle \langle R^2 \rangle - I^2) \frac{\partial}{\partial \rho} \Phi \\ &- e \frac{\partial \psi}{\partial \rho} I n_e \langle Bu_{e\parallel} \rangle + e \frac{\partial \psi}{\partial \rho} \langle B^2 \rangle n_e \langle Ru_{e\zeta} \rangle \end{aligned} \quad (1.7)$$

$$\begin{aligned} m_e n_e \frac{\partial}{\partial t} \langle Bu_{e\parallel} \rangle &= -\hat{\mu}_1^e \langle Bu_{e\parallel} \rangle - \hat{\mu}_2^e \langle B \hat{q}_{e\parallel} \rangle + \hat{\mu}_1^e B V_{1e} + \hat{\mu}_2^e B V_{2e} \\ &+ \ell_{11}^{\text{ee}} \langle Bu_{e\parallel} \rangle + \ell_{11}^{\text{ei}} \langle Bu_{i\parallel} \rangle + \ell_{11}^{\text{eb}} \langle Bu_{b\parallel} \rangle - \ell_{12}^{\text{ee}} \langle B \hat{q}_{e\parallel} \rangle - \ell_{12}^{\text{ei}} \langle B \hat{q}_{i\parallel} \rangle \\ &- en_e \langle BE_{\parallel} \rangle - \nu_L m_e n_e \langle Bu_{e\parallel} \rangle - \nu_{0e} m_e n_e \langle Bu_{e\parallel} \rangle \\ &- \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho| \rangle v_{e\zeta} n_e \langle Bu_{e\parallel} \rangle + \langle |\nabla \rho| \rangle v_{e\zeta} \frac{\partial}{\partial \rho} \left( \frac{\langle B^2 \rangle}{I} \right) n_e \langle Bu_{e\parallel} \rangle \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho|^2 \rangle \chi_{e\zeta} n_e \frac{\partial}{\partial \rho} \langle Bu_{e\parallel} \rangle - \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho|^2 \rangle \chi_{e\zeta} n_e \frac{\partial}{\partial \rho} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \langle Bu_{e\parallel} \rangle \\
& - \langle |\nabla \rho|^2 \rangle \chi_{e\zeta} n_e \frac{\partial}{\partial \rho} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \frac{\partial}{\partial \rho} \langle Bu_{e\parallel} \rangle + \langle |\nabla \rho|^2 \rangle \chi_{e\zeta} n_e \left[ \frac{\partial}{\partial \rho} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \right]^2 \langle Bu_{e\parallel} \rangle \quad (1.8)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{V'} \frac{\partial}{\partial t} (V' m_e n_e \langle Ru_{e\zeta} \rangle) & = - \frac{1}{V'} \frac{\partial}{\partial \rho} V' \left[ \langle |\nabla \rho| \rangle v_{e\zeta} m_e n_e \langle Ru_{e\zeta} \rangle + (u_e^\rho - u_g^\rho) m_e n_e \langle Ru_{e\zeta} \rangle \right. \\
& \quad \left. - \langle |\nabla \rho|^2 \rangle \chi_{e\zeta} m_e n_e \frac{\partial}{\partial \rho} \langle Ru_{e\zeta} \rangle + \langle \Pi_e^{\text{res}} \rangle \right] \\
& + \ell_{11}^{\text{ee}} \langle Ru_{e\zeta} \rangle + \ell_{11}^{\text{ei}} \langle Ru_{i\zeta} \rangle + \ell_{11}^{\text{eb}} \langle Ru_{b\zeta} \rangle - \ell_{12}^{\text{ee}} \frac{I}{\langle B^2 \rangle} \langle B \hat{q}_{e\parallel} \rangle - \ell_{12}^{\text{ei}} \frac{I}{\langle B^2 \rangle} \langle B \hat{q}_{i\parallel} \rangle \\
& - e n_e \langle RE_\zeta \rangle - e \frac{\partial \psi}{\partial \rho} n_e u_e^\rho \\
& - \frac{e^2 \langle |\nabla \rho|^2 \rangle D_e}{T_e} \frac{\partial \psi}{\partial \rho} n_e \left[ \frac{\partial \psi}{\partial \rho} \frac{\langle B^2 \rangle}{\langle B^2 \rangle \langle R^2 \rangle - I^2} \langle Ru_{e\zeta} \rangle - \frac{\partial \psi}{\partial \rho} \frac{I}{\langle B^2 \rangle \langle R^2 \rangle - I^2} \langle Bu_{e\parallel} \rangle \right. \\
& \left. + \frac{C_T - 1}{e} \frac{\partial}{\partial \rho} T_e + \frac{\partial}{\partial \rho} \Phi + \frac{T_e}{e} \frac{\partial \psi}{\partial \rho} C_p \right] - \nu_L m_e n_e \langle Ru_{e\zeta} \rangle - \nu_{0e} m_e n_e \langle Ru_{e\zeta} \rangle \quad (1.9)
\end{aligned}$$

$$\begin{aligned}
\frac{3}{2} \frac{1}{V'^{5/3}} \frac{\partial}{\partial t} (V'^{5/3} n_e T_e) & = - \frac{1}{V'} \frac{\partial}{\partial \rho} V' \left[ - \chi_e \langle |\nabla \rho|^2 \rangle n_e \frac{\partial}{\partial \rho} T_e + V_e^{\text{hp}} \langle |\nabla \rho| \rangle n_e T_e + \frac{5}{2} n_e T_e (u_e^\rho - u_g^\rho) \right] \\
& - u_g^\rho \frac{\partial}{\partial \rho} n_e T_e - u_i^\rho \frac{\partial}{\partial \rho} n_i T_i - \hat{u}_{i\theta} \langle \mathbf{B} \cdot \nabla \cdot \vec{\pi}_i \rangle - \frac{3}{2} n_i \frac{T_e - T_i}{\tau_{ie}} \\
& - \dot{\psi}_t \langle B^\theta \rangle (-e n_e \hat{u}_{e\theta} + Z_i e n_i \hat{u}_{i\theta}) + \dot{\psi} \left( -e n_e \left\langle \frac{u_{e\zeta}}{R} \right\rangle + Z_i e n_i \left\langle \frac{u_{i\zeta}}{R} \right\rangle \right) \\
& - \nu_L T_e (n_e - n_{\text{ediv}}) - \frac{3}{2} \nu_{LT_e} n_e (T_e - T_{\text{ediv}}) \\
& - E_H \nu_{\text{ionize}} \frac{n_e}{n_{01} + n_{02} + n_{03}} (n_{01} + n_{02} + n_{03}) \\
& + f_{\text{ei}} P_b + P_{\alpha e} + P_{\text{RFe}} - P_{\text{Br}} \quad (1.10)
\end{aligned}$$

$$\begin{aligned}
\frac{5}{2} m_e n_e \frac{\partial}{\partial t} \langle B \hat{q}_{e\parallel} \rangle & = - \hat{\mu}_2^e \langle Bu_{e\parallel} \rangle - \hat{\mu}_3^e \langle B \hat{q}_{e\parallel} \rangle + \hat{\mu}_2^e B V_{1e} + \hat{\mu}_3^e B V_{2e} \\
& - \ell_{21}^{\text{ee}} \langle Bu_{e\parallel} \rangle - \ell_{21}^{\text{ei}} \langle Bu_{i\parallel} \rangle - \ell_{21}^{\text{eb}} \langle Bu_{b\parallel} \rangle + \ell_{22}^{\text{ee}} \langle B \hat{q}_{e\parallel} \rangle + \ell_{22}^{\text{ei}} \langle B \hat{q}_{i\parallel} \rangle \\
& - \nu_L m_e n_e \langle B \hat{q}_{e\parallel} \rangle - \nu_{0e} m_e n_e \langle B \hat{q}_{e\parallel} \rangle \quad (1.11)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{V'} \frac{\partial}{\partial t} (V' n_i) & = - \frac{1}{V'} \frac{\partial}{\partial \rho} V' [V' n_i (u_i^\rho - u_g^\rho)] + \nu_{\text{ionize}} \frac{n_i}{n_{01} + n_{02} + n_{03}} (n_{01} + n_{02} + n_{03}) \\
& - \nu_L (n_i - n_{\text{idiv}}) + \nu_b n_b + \nu_b g n_b^{\text{rp}} - f_{\text{CX}} \frac{P_b}{E_b} + \nu_{bL} n_b + S_{\text{LC}} - \nu_{\text{OL}} n_i \quad (1.12)
\end{aligned}$$

$$\begin{aligned}
0 = & -(\langle B^2 \rangle \langle R^2 \rangle - I^2) \frac{\partial}{\partial \rho} n_i T_i - Z_i e n_i (\langle B^2 \rangle \langle R^2 \rangle - I^2) \frac{\partial}{\partial \rho} \Phi \\
& + Z_i e n_i \frac{\partial \psi}{\partial \rho} I \langle Bu_{i\parallel} \rangle - Z_i e n_i \frac{\partial \psi}{\partial \rho} \langle B^2 \rangle \langle Ru_{i\zeta} \rangle
\end{aligned} \tag{1.13}$$

$$\begin{aligned}
m_i n_i \frac{\partial}{\partial t} \langle Bu_{i\parallel} \rangle = & -\hat{\mu}_1^i \langle Bu_{i\parallel} \rangle - \hat{\mu}_2^i \langle Bq_{i\parallel} \rangle + \hat{\mu}_1^i B V_{1i} + \hat{\mu}_2^i B V_{2i} \\
& + \ell_{11}^{ie} \langle Bu_{e\parallel} \rangle + \ell_{11}^{ii} \langle Bu_{i\parallel} \rangle + \ell_{11}^{ib} \langle Bu_{b\parallel} \rangle - \ell_{12}^{ie} \langle B\hat{q}_{e\parallel} \rangle - \ell_{12}^{ii} \langle B\hat{q}_{i\parallel} \rangle \\
& + Z_i e n_i \langle BE_{\parallel} \rangle - \nu_L m_i n_i \langle Bu_{i\parallel} \rangle + \nu_b m_b n_b \langle Bu_{b\parallel} \rangle - \nu_{0i} m_i n_i \langle Bu_{i\parallel} \rangle \\
& - \nu_{CX} m_i n_i \langle Bu_{i\parallel} \rangle - \nu_{OL} m_i n_i \langle Bu_{i\parallel} \rangle \\
& - \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho| \rangle v_{i\zeta} n_i \langle Bu_{i\parallel} \rangle + \langle |\nabla \rho| \rangle v_{i\zeta} \frac{\partial}{\partial \rho} \left( \frac{\langle B^2 \rangle}{I} \right) n_i \langle Bu_{i\parallel} \rangle \\
& + \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho|^2 \rangle \chi_{i\zeta} n_i \frac{\partial}{\partial \rho} \langle Bu_{i\parallel} \rangle - \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho|^2 \rangle \chi_{i\zeta} n_i \frac{\partial}{\partial \rho} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \langle Bu_{i\parallel} \rangle \\
& - \langle |\nabla \rho|^2 \rangle \chi_{i\zeta} n_i \frac{\partial}{\partial \rho} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \frac{\partial}{\partial \rho} \langle Bu_{i\parallel} \rangle + \langle |\nabla \rho|^2 \rangle \chi_{i\zeta} n_i \left[ \frac{\partial}{\partial \rho} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \right]^2 \langle Bu_{i\parallel} \rangle
\end{aligned} \tag{1.14}$$

$$\begin{aligned}
\frac{1}{V'} \frac{\partial}{\partial t} (V' m_i n_i \langle Ru_{i\zeta} \rangle) = & -\frac{1}{V'} \frac{\partial}{\partial \rho} V' \left[ \langle |\nabla \rho| \rangle v_{i\zeta} m_i n_i \langle Ru_{i\zeta} \rangle + (u_1^\rho - u_g^\rho) m_i n_i \langle Ru_{i\zeta} \rangle \right. \\
& \left. - \langle |\nabla \rho|^2 \rangle \chi_{i\zeta} m_i n_i \frac{\partial}{\partial \rho} \langle Ru_{i\zeta} \rangle + \langle \Pi_i^{\text{res}} \rangle \right] \\
& + \ell_{11}^{ie} \langle Ru_{e\zeta} \rangle + \ell_{11}^{ii} \langle Ru_{i\zeta} \rangle + \ell_{11}^{ib} \langle Ru_{b\zeta} \rangle - \ell_{12}^{ie} \frac{I}{\langle B^2 \rangle} \langle B\hat{q}_{e\parallel} \rangle - \ell_{12}^{ii} \frac{I}{\langle B^2 \rangle} \langle B\hat{q}_{i\parallel} \rangle \\
& + Z_i e n_i \langle RE_\zeta \rangle + Z_i e \frac{\partial \psi}{\partial \rho} n_i u_i^\rho \\
& + Z_i \frac{e^2 \langle |\nabla \rho|^2 \rangle D_e}{T_e} \frac{\partial \psi}{\partial \rho} n_e \left[ \frac{\partial \psi}{\partial \rho} \frac{\langle B^2 \rangle}{\langle B^2 \rangle \langle R^2 \rangle - I^2} \langle Ru_{e\zeta} \rangle - \frac{\partial \psi}{\partial \rho} \frac{I}{\langle B^2 \rangle \langle R^2 \rangle - I^2} \langle Bu_{e\parallel} \rangle \right. \\
& \left. + \frac{C_T - 1}{e} \frac{\partial}{\partial \rho} T_e + \frac{\partial}{\partial \rho} \Phi + \frac{T_e}{e} \frac{\partial \psi}{\partial \rho} C_p \right] \\
& - \nu_L m_i n_i \langle Ru_{i\zeta} \rangle + \nu_b m_b n_b \langle Ru_{b\zeta} \rangle - \nu_{0i} m_i n_i \langle Ru_{i\zeta} \rangle - \nu_{CX} m_i n_i \langle Ru_{i\zeta} \rangle \\
& - \nu_{OL} m_i n_i \langle Ru_{i\zeta} \rangle
\end{aligned} \tag{1.15}$$

$$\begin{aligned}
\frac{3}{2} \frac{1}{V'^{5/3}} \frac{\partial}{\partial t} (V'^{5/3} n_i T_i) = & -\frac{1}{V'} \frac{\partial}{\partial \rho} V' \left[ -\chi_i \langle |\nabla \rho|^2 \rangle n_i \frac{\partial}{\partial \rho} T_i + V_i^{\text{hp}} \langle |\nabla \rho| \rangle n_i T_i + \frac{5}{2} n_i T_i (u_1^\rho - u_g^\rho) \right] \\
& + (u_1^\rho - u_g^\rho) \frac{\partial}{\partial \rho} n_i T_i + \hat{u}_{i\theta} \langle \mathbf{B} \cdot \nabla \cdot \overleftrightarrow{\pi}_i \rangle + \frac{3}{2} n_i \frac{T_e - T_i}{\tau_{ie}} \\
& - \nu_L T_i (n_i - n_{\text{div}}) - \frac{3}{2} \nu_{LT_i} n_i (T_i - T_{\text{div}})
\end{aligned}$$

$$\begin{aligned}
& + \frac{3}{2} \nu_{\text{ionize}} \frac{n_i}{n_{01} + n_{02} + n_{03}} (n_{01} T_{01} + n_{02} T_{02} + n_{03} T_{03}) \\
& - \frac{3}{2} \nu_{\text{CX}} n_i (T_i - T_{01}) + (1 - f_{\text{ei}}) P_{\text{b}} + m_{\text{b}} \frac{P_{\text{b}}}{E_{\text{b}}} \frac{\langle Bu_{\text{b}\parallel} \rangle}{\langle B^2 \rangle} + P_{\alpha\text{i}} + P_{\text{RFi}} \quad (1.16)
\end{aligned}$$

$$\begin{aligned}
\frac{5}{2} m_i n_i \frac{\partial}{\partial t} \langle B \hat{q}_{\text{i}\parallel} \rangle & = -\hat{\mu}_2^{\text{i}} \langle Bu_{\text{i}\parallel} \rangle - \hat{\mu}_3^{\text{i}} \langle B \hat{q}_{\text{i}\parallel} \rangle + \hat{\mu}_2^{\text{i}} B V_{1\text{i}} + \hat{\mu}_3^{\text{i}} B V_{2\text{i}} \\
& - \ell_{21}^{\text{ie}} \langle Bu_{\text{e}\parallel} \rangle - \ell_{21}^{\text{ii}} \langle Bu_{\text{i}\parallel} \rangle - \ell_{21}^{\text{ib}} \langle Bu_{\text{b}\parallel} \rangle + \ell_{22}^{\text{ie}} \langle B \hat{q}_{\text{e}\parallel} \rangle + \ell_{22}^{\text{ii}} \langle B \hat{q}_{\text{i}\parallel} \rangle \\
& - \nu_{\text{L}} m_i n_i \langle B \hat{q}_{\text{i}\parallel} \rangle - \nu_{0\text{i}} m_i n_i \langle B \hat{q}_{\text{i}\parallel} \rangle \quad (1.17)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{V'} \frac{\partial}{\partial t} (V' n_{\text{b}}) & = -\frac{1}{V'} \frac{\partial}{\partial \rho} [V' n_{\text{b}} (u_{\text{b}}^{\rho} - u_{\text{g}}^{\rho})] + \frac{P_{\text{b}}}{E_{\text{b}}} - g \frac{P_{\text{b}\perp}}{E_{\text{b}}} - \nu_{\text{b}} n_{\text{b}} - \nu_{\text{bL}} n_{\text{b}} - \nu_{\text{eff}} g (\sqrt{\delta} n_{\text{b}} - n_{\text{b}}^{\text{rp}}) \\
& + \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho|^2 \rangle f_t D_{\text{b}} \frac{\partial}{\partial \rho} n_{\text{b}} \quad (1.18)
\end{aligned}$$

$$\begin{aligned}
0 & = -(\langle B^2 \rangle \langle R^2 \rangle - I^2) \frac{\partial}{\partial \rho} n_{\text{b}} T_{\text{b}} - Z_{\text{b}} e n_{\text{b}} (\langle B^2 \rangle \langle R^2 \rangle - I^2) \frac{\partial}{\partial \rho} \Phi \\
& + Z_{\text{b}} e n_{\text{b}} \frac{\partial \psi}{\partial \rho} I \langle Bu_{\text{b}\parallel} \rangle - Z_{\text{b}} e n_{\text{b}} \frac{\partial \psi}{\partial \rho} \langle B^2 \rangle \langle Ru_{\text{b}\zeta} \rangle \quad (1.19)
\end{aligned}$$

$$\begin{aligned}
m_{\text{b}} \frac{\partial}{\partial t} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle & = -(\hat{\mu}_1^{\text{b}} - \ell_{11}^{\text{bb}}) \langle Bu_{\text{b}\parallel} \rangle + \sum_{a \neq \text{b}} \ell_{11}^{\text{ba}} \langle Bu_{a\parallel} \rangle + \hat{\mu}_1^{\text{b}} B V_{1\text{b}} + Z_{\text{b}} e n_{\text{b}} \langle B E_{\parallel}^A \rangle \\
& + \langle B S_{\text{m}\parallel\text{b}} \rangle - \nu_{\text{b}} m_{\text{b}} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle - \nu_{0\text{b}} m_{\text{b}} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle - \nu_{\text{CX}} m_{\text{b}} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle - \nu_{\text{bL}} m_{\text{b}} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle \\
& - \nu_{\text{eff}} g \sqrt{\delta} m_{\text{b}} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle + \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho|^2 \rangle m_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle f_t D_{\text{b}} \frac{\partial}{\partial \rho} n_{\text{b}} \quad (1.20)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{V'} \frac{\partial}{\partial t} (V' m_{\text{b}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle) & = -\frac{1}{V'} \frac{\partial}{\partial \rho} [V' (u_{\text{b}}^{\rho} - u_{\text{g}}^{\rho}) m_{\text{b}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle] \\
& + \ell_{11}^{\text{be}} \langle Ru_{\text{e}\zeta} \rangle + \ell_{11}^{\text{bi}} \langle Ru_{\text{i}\zeta} \rangle + \ell_{11}^{\text{bb}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle \\
& + Z_{\text{b}} e n_{\text{b}} \dot{\psi} + Z_{\text{b}} e \frac{\partial \psi}{\partial \rho} n_{\text{b}} u_{\text{b}}^{\rho} - \nu_{\text{L}} m_{\text{b}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle - \nu_{\text{b}} m_{\text{b}} n_{\text{b}} \langle Bu_{\text{b}\zeta} \rangle \\
& - \nu_{0\text{i}} m_{\text{b}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle - \nu_{\text{CX}} m_{\text{b}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle + \frac{I}{\langle B^2 \rangle} \langle B S_{\text{m}\parallel\text{b}} \rangle \quad (1.21)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{V'} \frac{\partial}{\partial t} (V' n_{01}) & = \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho|^2 \rangle D_{01} \frac{\partial}{\partial \rho} n_{01} - \nu_{\text{ionize}} \frac{n_{\text{i}}}{n_{01} + n_{02} + n_{03}} n_{01} \\
& - \nu_{\text{CX}} \frac{n_{\text{i}}}{n_{01} + n_{02}} n_{01} + \gamma_0 \nu_{\text{L}} (n_{\text{i}} - n_{\text{idiv}}) + \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle \mathbf{\Gamma}^{\text{puff}} \cdot \nabla \rho \rangle \Big|_{\text{b}} \quad (1.22)
\end{aligned}$$

$$\frac{1}{V'} \frac{\partial}{\partial t} (V' n_{02}) = \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho|^2 \rangle D_{02} \frac{\partial}{\partial \rho} n_{02} - \nu_{\text{ionize}} \frac{n_{\text{i}}}{n_{01} + n_{02} + n_{03}} n_{02} + \nu_{\text{CX}} \frac{n_{\text{i}}}{n_{01} + n_{02}} n_{01} \quad (1.23)$$

$$\frac{1}{V'} \frac{\partial}{\partial t} (V' n_{03}) = \frac{1}{V'} \frac{\partial}{\partial \rho} V' \langle |\nabla \rho|^2 \rangle D_{03} \frac{\partial}{\partial \rho} n_{03} - \nu_{\text{ionize}} \frac{n_i}{n_{01} + n_{02} + n_{03}} n_{03} + f_{\text{CX}} \frac{P_b}{E_b} \quad (1.24)$$

$$\frac{1}{V'} \frac{\partial}{\partial t} (V' n_b^{\text{rp}}) = \frac{P_{b\perp}}{E_b} + \nu_{\text{eff}} (\sqrt{\delta} n_b - n_b^{\text{rp}}) - \nu_b n_b^{\text{rp}} - \frac{1}{V'} \frac{\partial}{\partial \rho} V' \left( \langle |\nabla \rho| \rangle u_b^{\text{rp}} n_b^{\text{rp}} - \langle |\nabla \rho|^2 \rangle D_b^{\text{rp}} \frac{\partial}{\partial \rho} n_b^{\text{rp}} \right) \quad (1.25)$$

## 2 Additional important relationship

$$B_p = \frac{|\nabla \rho|}{R} \frac{\partial \psi}{\partial \rho}, \quad \langle B_p^2 \rangle = \left\langle \frac{|\nabla V|^2}{R^2} \right\rangle \left( \frac{\partial \psi}{\partial V} \right)^2 \quad (2.1)$$

$$B_t = \frac{I}{R} = \frac{1}{R} \frac{4\pi^2}{\langle R^{-2} \rangle} \frac{\partial \psi_t}{\partial V}, \quad \langle B_t^2 \rangle = \frac{16\pi^4}{\langle R^{-2} \rangle} \left( \frac{\partial \psi_t}{\partial V} \right)^2 \quad (2.2)$$

$$\langle B^2 \rangle = \langle B_t^2 \rangle + \langle B_p^2 \rangle = \frac{16\pi^4}{\langle R^{-2} \rangle} \left( \frac{\partial \psi_t}{\partial V} \right)^2 + \left\langle \frac{|\nabla V|^2}{R^2} \right\rangle \left( \frac{\partial \psi}{\partial V} \right)^2 \quad (2.3)$$

$$\langle B^\theta \rangle = \frac{4\pi^2}{V'} \frac{\partial \psi}{\partial \rho} = 4\pi^2 \frac{\partial \psi}{\partial V} = \frac{I \langle R^{-2} \rangle}{q} = 2\pi \left( \oint \frac{dl_p}{B_p} \right)^{-1} \quad (2.4)$$

$$\hat{q}^2 \equiv \frac{I^2}{2\langle B_p^2 \rangle} \left( \left\langle \frac{1}{R^2} \right\rangle - \frac{1}{\langle R^2 \rangle} \right) \quad (2.5)$$

$$\langle B^2 \rangle \langle R^2 \rangle - I^2 = \langle R^2 \rangle \left\langle \frac{|\nabla V|^2}{R^2} \right\rangle \left( \frac{\partial \psi}{\partial V} \right)^2 (1 + 2\hat{q}^2) \quad (2.6)$$

$$\langle B \hat{q}_{s\parallel} \rangle \equiv \frac{2\langle B q_{s\parallel} \rangle}{5p_s} \quad (2.7)$$

$$\hat{u}_{s\theta} = \frac{\langle B u_{s\parallel} \rangle - B V_{1s}}{\langle B^2 \rangle} \quad (2.8)$$

$$\begin{aligned} \left\langle \frac{u_{s\zeta}}{R} \right\rangle &= \frac{1}{1 + 2\hat{q}^2} \frac{1}{\langle R^2 \rangle} \langle R u_{s\zeta} \rangle + \frac{2\hat{q}^2}{1 + 2\hat{q}^2} \frac{1}{I} \langle B u_{s\parallel} \rangle \\ &= \frac{\langle R u_{s\zeta} \rangle}{\langle R^2 \rangle} + \frac{I}{\langle B^2 \rangle} \left( \left\langle \frac{1}{R^2} \right\rangle - \frac{1}{\langle R^2 \rangle} \right) (\langle B u_{s\parallel} \rangle - B V_{1s}) \\ &= \frac{\langle R u_{s\zeta} \rangle}{\langle R^2 \rangle} + I \left( \left\langle \frac{1}{R^2} \right\rangle - \frac{1}{\langle R^2 \rangle} \right) \hat{u}_{s\theta} \end{aligned} \quad (2.9)$$

$$B V_{1s} = -I \left( \frac{1}{e_s n_s} \frac{\partial n_s T_s}{\partial \psi} + \frac{\partial \Phi}{\partial \psi} \right), \quad B V_{2s} = -\frac{I}{e_s} \frac{\partial T_s}{\partial \psi} \quad (2.10)$$

$$u^\rho \equiv \langle \mathbf{u} \cdot \nabla \rho \rangle, \quad u^V = \langle \mathbf{u} \cdot \nabla V \rangle \quad (2.11)$$

$$u_g^\rho = -\frac{4\pi^2}{IV'\langle R^{-2}\rangle}\dot{\psi}_t, \quad u_g^V = -\frac{4\pi^2}{I\langle R^{-2}\rangle}\dot{\psi}_t = -\frac{V'}{2\rho\dot{\psi}_{ta}}\dot{\psi}_t \quad (2.12)$$

$$\frac{1}{V'}\frac{\partial V'}{\partial t} = \frac{\partial}{\partial V}\langle \mathbf{u}_g \cdot \nabla V \rangle = \frac{\partial u_g^V}{\partial V} \quad (2.13)$$

$$\langle E_r \rangle = -\langle |\nabla \rho|^2 \rangle \frac{\partial r}{\partial \rho} \frac{\partial}{\partial \rho} \Phi \quad (2.14)$$

$$\langle BE_{\parallel} \rangle = I\langle R^{-2} \rangle \left( -\frac{\dot{\psi}_t}{q} + \dot{\psi} \right) \quad (2.15)$$

$$RE_\zeta = \dot{\psi} = \frac{V_{\text{loop}}}{2\pi} \quad (2.16)$$

$$\langle \mathbf{B} \cdot \nabla \cdot \overleftrightarrow{\pi}_i \rangle = \hat{\mu}_1^i (\langle Bu_{i\parallel} \rangle - BV_{1i}) + \hat{\mu}_2^i \left( \frac{2\langle Bq_{i\parallel} \rangle}{5p_i} - BV_{2i} \right) \quad (2.17)$$

$$\hat{\ell}_{ij}^{ab} \equiv \frac{\ell_{ij}^{ab}}{m_a n_a}, \quad \hat{\ell}_{ij}^{bb} \equiv \frac{\ell_{ij}^{bb}}{m_e n_e} \quad (2.18)$$

$$\Gamma^{\text{puff}} \simeq \frac{\langle \mathbf{\Gamma}^{\text{puff}} \cdot \nabla \rho \rangle}{\langle |\nabla \rho| \rangle} \quad (2.19)$$

### 3 Equations in the coordinates suitable for implementation

$$0 = \frac{\partial}{\partial V} \left[ \langle |\nabla V|^2 \rangle \frac{\partial}{\partial V} \Phi \right] + \frac{1}{\epsilon_0} (-e n_e + Z_i e n_i + Z_b e n_b + Z_b e g n_b^{\text{rp}}) \quad (3.1)$$

$$\begin{aligned} \frac{1}{c^2} \frac{\partial}{\partial t} \dot{\psi}_t &= \frac{\partial}{\partial V} \left[ \left\langle \frac{|\nabla V|^2}{R^2} \right\rangle \frac{1}{\langle R^{-2} \rangle} \frac{\partial}{\partial V} \psi_t \right] - \frac{1}{\langle R^{-2} \rangle} \frac{\partial}{\partial V} \left[ \left\langle \frac{|\nabla V|^2}{R^2} \right\rangle \right] \frac{\partial}{\partial V} \psi_t \\ &+ \frac{\mu_0}{\langle B^\theta \rangle} \left[ -e \left( n_e \langle Bu_{e\parallel} \rangle - I n_e \left\langle \frac{u_{e\zeta}}{R} \right\rangle \right) + Z_i e \left( n_i \langle Bu_{i\parallel} \rangle - I n_i \left\langle \frac{u_{i\zeta}}{R} \right\rangle \right) \right. \\ &\left. + Z_b e \left( n_b \langle Bu_{b\parallel} \rangle - I n_b \left\langle \frac{u_{b\zeta}}{R} \right\rangle \right) \right] \end{aligned} \quad (3.2)$$

$$\begin{aligned} \frac{1}{c^2} \left\langle \frac{1}{R^2} \right\rangle \frac{\partial}{\partial t} \dot{\psi} &= \frac{\partial}{\partial V} \left[ \left\langle \frac{|\nabla V|^2}{R^2} \right\rangle \frac{\partial}{\partial V} \psi \right] \\ &- \mu_0 \left( -e n_e \left\langle \frac{u_{e\zeta}}{R} \right\rangle + Z_i e n_i \left\langle \frac{u_{i\zeta}}{R} \right\rangle + Z_b e n_b \left\langle \frac{u_{b\zeta}}{R} \right\rangle \right) \end{aligned} \quad (3.3)$$

$$\frac{\partial}{\partial t} \psi = \dot{\psi} \quad (3.4)$$

$$\frac{\partial}{\partial t}\psi_t = \dot{\psi}_t \quad (3.5)$$

$$\begin{aligned} \frac{\partial}{\partial t}n_e = & -\frac{\partial}{\partial V}n_e u_e^V + u_g^V \frac{\partial}{\partial V}n_e + \nu_{\text{ionize}} \frac{n_e}{n_{01} + n_{02} + n_{03}} (n_{01} + n_{02} + n_{03}) \\ & - \nu_L(n_e - n_{\text{ediv}}) + (1 - f_{\text{CX}}) \frac{P_b}{E_b} \end{aligned} \quad (3.6)$$

$$\begin{aligned} 0 = & -\left(\langle B^2 \rangle \langle R^2 \rangle - I^2\right) \frac{1}{m_e} \frac{\partial}{\partial V} n_e T_e + \frac{e}{m_e} n_e \left(\langle B^2 \rangle \langle R^2 \rangle - I^2\right) \frac{\partial}{\partial V} \Phi \\ & - \frac{e}{m_e} \frac{\partial \psi}{\partial V} I n_e \langle B u_{e\parallel} \rangle + \frac{e}{m_e} \frac{\partial \psi}{\partial V} \langle B^2 \rangle n_e \langle R u_{e\zeta} \rangle \end{aligned} \quad (3.7)$$

$$\begin{aligned} n_e \frac{\partial}{\partial t} \langle B u_{e\parallel} \rangle = & -\frac{1}{m_e} \hat{\mu}_1^e \langle B u_{e\parallel} \rangle - \frac{1}{m_e} \hat{\mu}_2^e \langle B \hat{q}_{e\parallel} \rangle + \frac{1}{em_e} \hat{\mu}_1^e I \frac{\partial V}{\partial \psi} \frac{1}{n_e} \frac{\partial}{\partial V} n_e T_e - \frac{1}{m_e} \hat{\mu}_1^e I \frac{\partial V}{\partial \psi} \frac{\partial}{\partial V} \Phi \\ & + \frac{1}{em_e} \hat{\mu}_2^e I \frac{\partial V}{\partial \psi} \frac{1}{n_e} \frac{\partial}{\partial V} n_e T_e - \frac{1}{em_e} \hat{\mu}_2^e I \frac{\partial V}{\partial \psi} \frac{T_e}{n_e} \frac{\partial}{\partial V} n_e \\ & + \hat{\ell}_{11}^{\text{ee}} n_e \langle B u_{e\parallel} \rangle + \hat{\ell}_{11}^{\text{ei}} n_e \langle B u_{i\parallel} \rangle + \hat{\ell}_{11}^{\text{eb}} \frac{n_e}{n_b} n_b \langle B u_{b\parallel} \rangle - \hat{\ell}_{12}^{\text{ee}} n_e \langle B \hat{q}_{e\parallel} \rangle - \hat{\ell}_{12}^{\text{ei}} n_e \langle B \hat{q}_{i\parallel} \rangle \\ & - e \frac{n_e}{m_e} I \langle R^{-2} \rangle \left( -\frac{\dot{\psi}_t}{q} + \dot{\psi} \right) - \nu_L n_e \langle B u_{e\parallel} \rangle - \nu_{0e} n_e \langle B u_{e\parallel} \rangle \\ & - \frac{\partial}{\partial V} \langle |\nabla V| \rangle v_{e\zeta} n_e \langle B u_{e\parallel} \rangle + \langle |\nabla V| \rangle v_{e\zeta} \frac{\partial}{\partial V} \ln \left( \frac{\langle B^2 \rangle}{I} \right) n_e \langle B u_{e\parallel} \rangle \\ & + \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle \chi_{e\zeta} n_e \frac{\partial}{\partial V} \langle B u_{e\parallel} \rangle - \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle \chi_{e\zeta} n_e \frac{\partial}{\partial V} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \langle B u_{e\parallel} \rangle \\ & - \langle |\nabla V|^2 \rangle \chi_{e\zeta} n_e \frac{\partial}{\partial V} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \frac{\partial}{\partial V} \langle B u_{e\parallel} \rangle + \langle |\nabla V|^2 \rangle \chi_{e\zeta} n_e \left[ \frac{\partial}{\partial V} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \right]^2 \langle B u_{e\parallel} \rangle \end{aligned} \quad (3.8)$$

$$\begin{aligned} \frac{\partial}{\partial t} n_e \langle R u_{e\zeta} \rangle = & -(u_e^V - u_g^V) \frac{\partial}{\partial V} n_e \langle R u_{e\zeta} \rangle - \frac{\partial u_e^V}{\partial V} n_e \langle R u_{e\zeta} \rangle - \frac{\partial}{\partial V} [\langle |\nabla V| \rangle v_{e\zeta} n_e \langle R u_{e\zeta} \rangle] \\ & + \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle \chi_{e\zeta} \frac{\partial}{\partial V} n_e \langle R u_{e\zeta} \rangle - \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle \chi_{e\zeta} \langle R u_{e\zeta} \rangle \frac{\partial}{\partial V} n_e - \frac{1}{m_e} \frac{\partial}{\partial V} \langle \Pi_e^{\text{res}} \rangle \\ & + \hat{\ell}_{11}^{\text{ee}} n_e \langle R u_{e\zeta} \rangle + \hat{\ell}_{11}^{\text{ei}} \frac{n_e}{n_i} n_i \langle R u_{i\zeta} \rangle + \hat{\ell}_{11}^{\text{eb}} \frac{n_e}{n_b} n_b \langle R u_{b\zeta} \rangle - \hat{\ell}_{12}^{\text{ee}} \frac{I}{\langle B^2 \rangle} n_e \langle B \hat{q}_{e\parallel} \rangle - \hat{\ell}_{12}^{\text{ei}} \frac{I}{\langle B^2 \rangle} n_e \langle B \hat{q}_{i\parallel} \rangle \\ & - e \frac{n_e}{m_e} \dot{\psi} - e \frac{1}{m_e} \frac{\partial \psi}{\partial V} n_e u_e^V \\ & - \frac{e^2 \langle |\nabla V|^2 \rangle D_e}{m_e T_e} \frac{\partial \psi}{\partial V} \left[ \frac{\partial \psi}{\partial V} \frac{\langle B^2 \rangle}{\langle B^2 \rangle \langle R^2 \rangle - I^2} n_e \langle R u_{e\zeta} \rangle - \frac{\partial \psi}{\partial V} \frac{I}{\langle B^2 \rangle \langle R^2 \rangle - I^2} n_e \langle B u_{e\parallel} \rangle \right] \\ & + \frac{C_T - 1}{e} \left( \frac{\partial}{\partial V} n_e T_e - T_e \frac{\partial}{\partial V} n_e \right) + n_e \frac{\partial}{\partial V} \Phi + \frac{T_e}{e} \frac{\partial \psi}{\partial V} C_p n_e \Big] - \nu_L n_e \langle R u_{e\zeta} \rangle - \nu_{0e} n_e \langle R u_{e\zeta} \rangle \end{aligned} \quad (3.9)$$

$$\begin{aligned}
\frac{3}{2} \frac{\partial}{\partial t} n_e T_e &= -\frac{5}{2} (u_e^V - u_g^V) \frac{\partial}{\partial V} n_e T_e - \frac{5}{2} \frac{\partial u_e^V}{\partial V} n_e T_e - \frac{\partial}{\partial V} [V_e^{\text{hp}} \langle |\nabla V| \rangle n_e T_e] \\
&+ \frac{\partial}{\partial V} \chi_e \langle |\nabla V|^2 \rangle \frac{\partial}{\partial V} n_e T_e - \frac{\partial}{\partial V} \chi_e \langle |\nabla V|^2 \rangle T_e \frac{\partial}{\partial V} n_e \\
&- u_g^V \frac{\partial}{\partial V} n_e T_e - u_i^V \frac{\partial}{\partial V} n_i T_i - \hat{u}_{i\theta} \langle \mathbf{B} \cdot \nabla \cdot \hat{\pi}_i \rangle - \frac{3}{2} \nu_{\text{Tei}} \left( n_e T_e - \frac{n_e}{n_i} n_i T_i \right) \\
&- 4\pi^2 \frac{\partial \psi}{\partial V} \dot{\psi}_t (-en_e \hat{u}_{e\theta} + Z_i en_i \hat{u}_{i\theta}) + \dot{\psi} \left( -en_e \left\langle \frac{u_{e\zeta}}{R} \right\rangle + Z_i en_i \left\langle \frac{u_{i\zeta}}{R} \right\rangle \right) \\
&- \nu_L (n_e T_e - n_{\text{ediv}} T_e) - \frac{3}{2} \nu_{LT_e} (n_e T_e - n_e T_{\text{ediv}}) - E_H \nu_{\text{ionize}} \frac{n_e}{n_{01} + n_{02} + n_{03}} (n_{01} + n_{02} + n_{03}) \\
&+ f_{\text{ei}} P_b + P_{\alpha e} + P_{\text{RFe}} - P_{\text{Br}}
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\frac{5}{2} n_e \frac{\partial}{\partial t} \langle B \hat{q}_{e\parallel} \rangle &= -\frac{1}{m_e} \hat{\mu}_2^e \langle B u_{e\parallel} \rangle - \frac{1}{m_e} \hat{\mu}_3^e \langle B \hat{q}_{e\parallel} \rangle + \frac{1}{em_e} \hat{\mu}_2^e I \frac{\partial V}{\partial \psi} \frac{1}{n_e} \frac{\partial}{\partial V} n_e T_e - \frac{1}{m_e} \hat{\mu}_2^e I \frac{\partial V}{\partial \psi} \frac{\partial}{\partial V} \Phi \\
&+ \frac{1}{em_e} \hat{\mu}_3^e I \frac{\partial V}{\partial \psi} \frac{1}{n_e} \frac{\partial}{\partial V} n_e T_e - \frac{1}{em_e} \hat{\mu}_3^e I \frac{\partial V}{\partial \psi} \frac{T_e}{n_e} \frac{\partial}{\partial V} n_e \\
&- \hat{\ell}_{21}^{\text{ee}} n_e \langle B u_{e\parallel} \rangle - \hat{\ell}_{21}^{\text{ei}} n_e \langle B u_{i\parallel} \rangle - \hat{\ell}_{21}^{\text{eb}} \frac{n_e}{n_b} n_b \langle B u_{b\parallel} \rangle + \hat{\ell}_{22}^{\text{ee}} n_e \langle B \hat{q}_{e\parallel} \rangle + \hat{\ell}_{22}^{\text{ei}} n_e \langle B \hat{q}_{i\parallel} \rangle \\
&- \nu_L n_e \langle B \hat{q}_{e\parallel} \rangle - \nu_{0e} n_e \langle B \hat{q}_{e\parallel} \rangle
\end{aligned} \tag{3.11}$$

$$n_e \left\langle \frac{u_{e\zeta}}{R} \right\rangle = \frac{1}{\langle R^2 \rangle} n_e \langle R u_{e\zeta} \rangle - \frac{2\hat{q}^2}{1 + 2\hat{q}^2} \frac{1}{\langle R^2 \rangle} n_e \langle R u_{e\zeta} \rangle + \frac{2\hat{q}^2}{1 + 2\hat{q}^2} \frac{1}{I} n_e \langle B u_{e\parallel} \rangle \tag{3.12}$$



$$\begin{aligned}
\frac{\partial}{\partial t} n_i &= -\frac{\partial}{\partial V} n_i u_i^V + u_g^V \frac{\partial}{\partial V} n_i + \nu_{\text{ionize}} \frac{n_i}{n_{01} + n_{02} + n_{03}} (n_{01} + n_{02} + n_{03}) - \nu_L (n_i - n_{\text{idiv}}) \\
&+ \nu_b n_b + \nu_b g n_b^{\text{rp}} - f_{\text{CX}} \frac{P_b}{E_b} + \nu_{bL} n_b + S_{\text{LC}} - \nu_{\text{OL}} n_i
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
0 &= -\left(\langle B^2 \rangle \langle R^2 \rangle - I^2\right) \frac{1}{m_i} \frac{\partial}{\partial V} n_i T_i - Z_i \frac{e}{m_i} n_i \left(\langle B^2 \rangle \langle R^2 \rangle - I^2\right) \frac{\partial}{\partial V} \Phi \\
&+ Z_i \frac{e}{m_i} \frac{\partial \psi}{\partial V} I n_i \langle B u_{i\parallel} \rangle - Z_i \frac{e}{m_i} \frac{\partial \psi}{\partial V} \langle B^2 \rangle n_i \langle R u_{i\zeta} \rangle
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
n_i \frac{\partial}{\partial t} \langle B u_{i\parallel} \rangle &= -\frac{1}{m_i} \hat{\mu}_1^i \langle B u_{i\parallel} \rangle - \frac{1}{m_i} \hat{\mu}_2^i \langle B \hat{q}_{i\parallel} \rangle - \frac{1}{Z_i e m_i} \hat{\mu}_1^i I \frac{\partial V}{\partial \psi} \frac{1}{n_i} \frac{\partial}{\partial V} n_i T_i - \frac{1}{m_i} \hat{\mu}_1^i I \frac{\partial V}{\partial \psi} \frac{\partial}{\partial V} \Phi \\
&- \frac{1}{Z_i e m_i} \hat{\mu}_2^i I \frac{\partial V}{\partial \psi} \frac{1}{n_i} \frac{\partial}{\partial V} n_i T_i + \frac{1}{Z_i e m_i} \hat{\mu}_2^i I \frac{\partial V}{\partial \psi} \frac{T_i}{n_i} \frac{\partial}{\partial V} n_i \\
&+ \hat{\ell}_{11}^{\text{ie}} n_i \langle B u_{e\parallel} \rangle + \hat{\ell}_{11}^{\text{ii}} n_i \langle B u_{i\parallel} \rangle + \hat{\ell}_{11}^{\text{ib}} \frac{n_i}{n_b} n_b \langle B u_{b\parallel} \rangle - \hat{\ell}_{12}^{\text{ie}} n_i \langle B \hat{q}_{e\parallel} \rangle - \hat{\ell}_{12}^{\text{ii}} n_i \langle B \hat{q}_{i\parallel} \rangle \\
&+ Z_i e \frac{n_i}{m_i} I \langle R^{-2} \rangle \left( -\frac{\dot{\psi}_t}{q} + \dot{\psi} \right) \\
&- \nu_L n_i \langle B u_{i\parallel} \rangle + \nu_b \frac{m_b}{m_i} n_b \langle B u_{b\parallel} \rangle - \nu_{0i} n_i \langle B u_{i\parallel} \rangle - \nu_{\text{CX}} n_i \langle B u_{i\parallel} \rangle - \nu_{\text{OL}} n_i \langle B u_{i\parallel} \rangle \\
&- \frac{\partial}{\partial V} \langle |\nabla V| \rangle v_{i\zeta} n_i \langle B u_{i\parallel} \rangle + \langle |\nabla V| \rangle v_{i\zeta} \frac{\partial}{\partial V} \ln \left( \frac{\langle B^2 \rangle}{I} \right) n_i \langle B u_{i\parallel} \rangle \\
&+ \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle \chi_{i\zeta} n_i \frac{\partial}{\partial V} \langle B u_{i\parallel} \rangle - \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle \chi_{i\zeta} n_i \frac{\partial}{\partial V} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \langle B u_{i\parallel} \rangle \\
&- \langle |\nabla V|^2 \rangle \chi_{i\zeta} n_i \frac{\partial}{\partial V} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \frac{\partial}{\partial V} \langle B u_{i\parallel} \rangle + \langle |\nabla V|^2 \rangle \chi_{i\zeta} n_i \left[ \frac{\partial}{\partial V} \ln \left( \frac{\langle B^2 \rangle}{I} \right) \right]^2 \langle B u_{i\parallel} \rangle
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\frac{\partial}{\partial t} n_i \langle R u_{i\zeta} \rangle &= -(u_i^V - u_g^V) \frac{\partial}{\partial V} n_i \langle R u_{i\zeta} \rangle - \frac{\partial u_i^V}{\partial V} n_i \langle R u_{i\zeta} \rangle - \frac{\partial}{\partial V} \langle |\nabla V| \rangle v_{i\zeta} n_i \langle R u_{i\zeta} \rangle \\
&+ \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle \chi_{i\zeta} \frac{\partial}{\partial V} n_i \langle R u_{i\zeta} \rangle - \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle \chi_{i\zeta} \langle R u_{i\zeta} \rangle \frac{\partial}{\partial V} n_i - \frac{1}{m_i} \frac{\partial}{\partial V} \langle \Pi_i^{\text{res}} \rangle \\
&+ \hat{\ell}_{11}^{\text{ie}} \frac{n_i}{n_e} n_e \langle R u_{e\zeta} \rangle + \hat{\ell}_{11}^{\text{ii}} n_i \langle R u_{i\zeta} \rangle + \hat{\ell}_{11}^{\text{ib}} \frac{n_i}{n_b} n_b \langle R u_{b\zeta} \rangle - \hat{\ell}_{12}^{\text{ie}} \frac{I}{\langle B^2 \rangle} n_i \langle B \hat{q}_{e\parallel} \rangle - \hat{\ell}_{12}^{\text{ii}} \frac{I}{\langle B^2 \rangle} n_i \langle B \hat{q}_{i\parallel} \rangle \\
&+ Z_i \frac{e}{m_i} n_i \dot{\psi} + Z_i \frac{e}{m_i} \frac{\partial \psi}{\partial V} n_i u_i^V \\
&+ Z_i \frac{e^2 \langle |\nabla V|^2 \rangle D_e}{m_i T_e} \frac{\partial \psi}{\partial V} \left[ \frac{\partial \psi}{\partial V} \frac{\langle B^2 \rangle}{\langle B^2 \rangle \langle R^2 \rangle - I^2} n_e \langle R u_{e\zeta} \rangle - \frac{\partial \psi}{\partial V} \frac{I}{\langle B^2 \rangle \langle R^2 \rangle - I^2} n_e \langle B u_{e\parallel} \rangle \right] \\
&+ \frac{C_T - 1}{e} \left( \frac{\partial}{\partial V} n_e T_e - T_e \frac{\partial}{\partial V} n_e \right) + n_e \frac{\partial}{\partial V} \Phi + \frac{T_e}{e} \frac{\partial \psi}{\partial V} C_p n_e
\end{aligned}$$

$$- \nu_L n_i \langle Ru_{i\zeta} \rangle + \nu_b \frac{m_b}{m_i} n_b \langle Bu_{b\zeta} \rangle - \nu_{0i} n_i \langle Ru_{i\zeta} \rangle - \nu_{CX} n_i \langle Ru_{i\zeta} \rangle - \nu_{OL} n_i \langle Ru_{i\zeta} \rangle \quad (3.16)$$

$$\begin{aligned} \frac{3}{2} \frac{\partial}{\partial t} n_i T_i &= -\frac{5}{2} (u_i^V - u_g^V) \frac{\partial}{\partial V} n_i T_i - \frac{5}{2} \frac{\partial u_i^V}{\partial V} n_i T_i - \frac{\partial}{\partial V} V_i^{\text{hp}} \langle |\nabla V| \rangle n_i T_i \\ &+ \frac{\partial}{\partial V} \chi_i \langle |\nabla V|^2 \rangle \frac{\partial}{\partial V} n_i T_i - \frac{\partial}{\partial V} \chi_i \langle |\nabla V|^2 \rangle T_i \frac{\partial}{\partial V} n_i \\ &+ (u_i^V - u_g^V) \frac{\partial}{\partial V} n_i T_i + \hat{u}_{i\theta} \langle \mathbf{B} \cdot \nabla \cdot \vec{\pi}_i \rangle - \frac{3}{2} \nu_{\text{Tei}} \left( \frac{n_e}{n_i} n_i T_i - n_e T_e \right) \\ &- \nu_L (n_i T_i - n_{\text{idiv}} T_i) - \frac{3}{2} \nu_{LT_i} (n_i T_i - n_i T_{\text{idiv}}) \\ &+ \frac{3}{2} \nu_{\text{ionize}} \frac{n_i}{n_{01} + n_{02} + n_{03}} (n_{01} T_{01} + n_{02} T_{02} + n_{03} T_{03}) \\ &- \frac{3}{2} \nu_{CX} (n_i T_i - n_i T_{01}) + (1 - f_{\text{ei}}) P_b + m_b \frac{1}{n_b} \frac{P_b}{E_b} \frac{\langle Bu_{i\parallel} \rangle}{\langle B^2 \rangle} n_b \langle Bu_{b\parallel} \rangle + P_{\text{ai}} + P_{\text{RFi}} \end{aligned} \quad (3.17)$$

$$\begin{aligned} \frac{5}{2} n_i \frac{\partial}{\partial t} \langle B \hat{q}_{i\parallel} \rangle &= -\frac{1}{m_i} \hat{\mu}_2^i \langle Bu_{i\parallel} \rangle - \frac{1}{m_i} \hat{\mu}_3^i \langle B \hat{q}_{i\parallel} \rangle - \frac{1}{Z_i e m_i} \hat{\mu}_2^i I \frac{\partial V}{\partial \psi} \frac{1}{n_i} \frac{\partial}{\partial V} n_i T_i - \frac{1}{m_i} \hat{\mu}_2^i I \frac{\partial V}{\partial \psi} \frac{\partial}{\partial V} \Phi \\ &- \frac{1}{Z_i e m_i} \hat{\mu}_3^i I \frac{\partial V}{\partial \psi} \frac{1}{n_i} \frac{\partial}{\partial V} n_i T_i + \frac{1}{Z_i e m_i} \hat{\mu}_3^i I \frac{\partial V}{\partial \psi} \frac{T_i}{n_i} \frac{\partial}{\partial V} n_i \\ &- \hat{\ell}_{21}^{\text{ie}} n_i \langle Bu_{e\parallel} \rangle - \hat{\ell}_{21}^{\text{ii}} n_i \langle Bu_{i\parallel} \rangle - \hat{\ell}_{21}^{\text{ib}} \frac{n_i}{n_b} n_b \langle Bu_{b\parallel} \rangle + \hat{\ell}_{22}^{\text{ie}} n_i \langle B \hat{q}_{e\parallel} \rangle + \hat{\ell}_{22}^{\text{ii}} n_i \langle B \hat{q}_{i\parallel} \rangle \\ &- \nu_L n_i \langle B \hat{q}_{i\parallel} \rangle - \nu_{0i} n_i \langle B \hat{q}_{i\parallel} \rangle \end{aligned} \quad (3.18)$$

$$n_i \left\langle \frac{u_{i\zeta}}{R} \right\rangle = \frac{1}{\langle R^2 \rangle} n_i \langle Ru_{i\zeta} \rangle - \frac{2\hat{q}^2}{1 + 2\hat{q}^2} \frac{1}{\langle R^2 \rangle} n_i \langle Ru_{i\zeta} \rangle + \frac{2\hat{q}^2}{1 + 2\hat{q}^2} \frac{1}{I} n_i \langle Bu_{i\parallel} \rangle \quad (3.19)$$

$$\begin{aligned} \frac{\partial}{\partial t} n_b &= -\frac{\partial}{\partial V} n_b u_b^V + u_g^V \frac{\partial}{\partial V} n_b + \frac{P_b}{E_b} - g \frac{P_{b\perp}}{E_b} - \nu_b n_b - \nu_{bL} n_b - \nu_{\text{eff}} g (\sqrt{\delta} n_b - n_b^{\text{rp}}) \\ &+ \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle f_t D_b \frac{\partial}{\partial V} n_b \end{aligned} \quad (3.20)$$

$$\begin{aligned} 0 &= -\left( \langle B^2 \rangle \langle R^2 \rangle - I^2 \right) \frac{1}{m_b} T_b \frac{\partial}{\partial V} n_b - \left( \langle B^2 \rangle \langle R^2 \rangle - I^2 \right) \frac{1}{m_b} \frac{\partial T_b}{\partial V} n_b \\ &- Z_b \frac{e}{m_b} n_b \left( \langle B^2 \rangle \langle R^2 \rangle - I^2 \right) \frac{\partial}{\partial V} \Phi + Z_b \frac{e}{m_b} \frac{\partial \psi}{\partial V} I n_b \langle Bu_{b\parallel} \rangle - Z_b \frac{e}{m_b} \frac{\partial \psi}{\partial V} \langle B^2 \rangle n_b \langle Ru_{b\zeta} \rangle \end{aligned} \quad (3.21)$$

$$\begin{aligned} \frac{\partial}{\partial t} n_b \langle Bu_{b\parallel} \rangle &= -\frac{1}{m_b n_b} \hat{\mu}_1^b n_b \langle Bu_{b\parallel} \rangle - \frac{1}{Z_b e m_b} \hat{\mu}_1^b I \frac{\partial V}{\partial \psi} \frac{T_b}{n_b} \frac{\partial}{\partial V} n_b - \frac{1}{Z_b e m_b} \hat{\mu}_1^b I \frac{\partial V}{\partial \psi} \frac{\partial T_b}{\partial V} - \frac{1}{m_b} \hat{\mu}_1^b I \frac{\partial V}{\partial \psi} \frac{\partial}{\partial V} \Phi \\ &+ \hat{\ell}_{11}^{\text{bb}} \frac{m_e n_e}{m_b n_b} n_b \langle Bu_{b\parallel} \rangle + \sum_{a \neq b} \hat{\ell}_{11}^{\text{ba}} \frac{m_e n_e}{m_b} \langle Bu_{a\parallel} \rangle \\ &+ Z_b \frac{e}{m_b} n_b I \langle R^{-2} \rangle \left( -\frac{\dot{\psi}_t}{q} + \dot{\psi} \right) + \frac{1}{m_b} \langle BS_{\text{m}\parallel b} \rangle - \nu_b n_b \langle Bu_{b\parallel} \rangle - \nu_{0b} n_b \langle Bu_{b\parallel} \rangle \end{aligned}$$

$$- \nu_{\text{CX}} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle - \nu_{\text{bL}} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle - \nu_{\text{eff}} g \sqrt{\delta} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle + \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle \langle Bu_{\text{b}\parallel} \rangle f_t D_{\text{b}} \frac{\partial}{\partial V} n_{\text{b}} \quad (3.22)$$

$$\begin{aligned} \frac{\partial}{\partial t} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle &= u_g^V \frac{\partial}{\partial V} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle - \frac{\partial}{\partial V} \langle Ru_{\text{b}\zeta} \rangle n_{\text{b}} u_{\text{b}}^V \\ &+ \frac{1}{m_{\text{b}}} \ell_{11}^{\text{be}} \frac{1}{n_{\text{e}}} n_{\text{e}} \langle Ru_{\text{e}\zeta} \rangle + \frac{1}{m_{\text{b}}} \ell_{11}^{\text{bi}} \frac{1}{n_{\text{i}}} n_{\text{i}} \langle Ru_{\text{i}\zeta} \rangle + \frac{1}{m_{\text{b}}} \ell_{11}^{\text{bb}} \frac{1}{n_{\text{b}}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle \\ &+ Z_{\text{b}} \frac{e}{m_{\text{b}}} n_{\text{b}} \dot{\psi} + Z_{\text{b}} \frac{e}{m_{\text{b}}} \frac{\partial \psi}{\partial V} n_{\text{b}} u_{\text{b}}^V \\ &- \nu_{\text{L}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle - \nu_{\text{b}} n_{\text{b}} \langle Bu_{\text{b}\zeta} \rangle - \nu_{0\text{i}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle - \nu_{\text{CX}} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle + \frac{1}{m_{\text{b}}} \frac{I}{\langle B^2 \rangle} \langle BS_{\text{m}\parallel\text{b}} \rangle \end{aligned} \quad (3.23)$$

$$n_{\text{b}} \left\langle \frac{u_{\text{b}\zeta}}{R} \right\rangle = \frac{1}{\langle R^2 \rangle} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle - \frac{2\hat{q}^2}{1 + 2\hat{q}^2} \frac{1}{\langle R^2 \rangle} n_{\text{b}} \langle Ru_{\text{b}\zeta} \rangle + \frac{2\hat{q}^2}{1 + 2\hat{q}^2} \frac{1}{I} n_{\text{b}} \langle Bu_{\text{b}\parallel} \rangle \quad (3.24)$$

$$\begin{aligned} \frac{\partial}{\partial t} n_{01} &= - \frac{\partial u_g^V}{\partial V} n_{01} + \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle D_{01} \frac{\partial}{\partial V} n_{01} - \nu_{\text{ionize}} \frac{n_{\text{i}}}{n_{01} + n_{02} + n_{03}} n_{01} \\ &- \nu_{\text{CX}} \frac{n_{\text{i}}}{n_{01} + n_{02}} n_{01} + \gamma_0 \nu_{\text{L}} (n_{\text{i}} - n_{\text{div}}) + \frac{\partial}{\partial V} V' \Gamma^{\text{puff}} \Big|_{\text{b}} \end{aligned} \quad (3.25)$$

$$\frac{\partial}{\partial t} n_{02} = - \frac{\partial u_g^V}{\partial V} n_{02} + \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle D_{02} \frac{\partial}{\partial V} n_{02} - \nu_{\text{ionize}} \frac{n_{\text{i}}}{n_{01} + n_{02} + n_{03}} n_{02} + \nu_{\text{CX}} \frac{n_{\text{i}}}{n_{01} + n_{02}} n_{01} \quad (3.26)$$

$$\frac{\partial}{\partial t} n_{03} = - \frac{\partial u_g^V}{\partial V} n_{03} + \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle D_{03} \frac{\partial}{\partial V} n_{03} - \nu_{\text{ionize}} \frac{n_{\text{i}}}{n_{01} + n_{02} + n_{03}} n_{03} + f_{\text{CX}} \frac{P_{\text{b}}}{E_{\text{b}}} \quad (3.27)$$

$$\begin{aligned} \frac{\partial}{\partial t} n_{\text{b}}^{\text{rp}} &= - \frac{\partial u_g^V}{\partial V} n_{\text{b}}^{\text{rp}} \frac{P_{\text{b}\perp}}{E_{\text{b}}} + \nu_{\text{eff}} (\sqrt{\delta} n_{\text{b}} - n_{\text{b}}^{\text{rp}}) - \nu_{\text{b}} n_{\text{b}}^{\text{rp}} \\ &- \frac{\partial}{\partial V} \langle |\nabla V| \rangle u_{\text{b}}^{\text{rp}} n_{\text{b}}^{\text{rp}} + \frac{\partial}{\partial V} \langle |\nabla V|^2 \rangle D_{\text{b}}^{\text{rp}} \frac{\partial}{\partial V} n_{\text{b}}^{\text{rp}} \end{aligned} \quad (3.28)$$

## 4 Boundary Conditions

num.	name	variable	highest order derivative	center	edge
1	LQm1	$\Phi$	2	N	0
2	LQm2	$\dot{\psi}_t$	2	0	$\langle  \nabla V ^2 / R^2 \rangle _{V_b} I_{\text{vac}} / (4\pi^2)$
3	LQm3	$\dot{\psi}$	2	N	$2\pi\mu_0 I_p$
4	LQm4	$\psi$	0	×	×
5	LQm5	$\psi_t / \mu_0$	0	×	×
6	LQe1	$n_e$	2	N	N
7	LQe2	$n_e u_e^V$	1	0	×
8	LQe3	$\langle Bu_{e\parallel} \rangle$	2(0)	N(×)	N(×)
9	LQe4	$n_e \langle Ru_{e\zeta} \rangle$	2	N	N
10	LQe5	$n_e T_e$	2	N	N
11	LQe6	$\langle B\hat{q}_{e\parallel} \rangle$	0	×	×
12	LQe7	$n_e \langle u_{e\zeta} / R \rangle$	0	×	×
13	LQi1	$n_i$	2	N	N
14	LQi2	$n_i u_i^V$	1	0	×
15	LQi3	$\langle Bu_{i\parallel} \rangle$	2(0)	N(×)	N(×)
16	LQi4	$n_i \langle Ru_{i\zeta} \rangle$	2	N	N
17	LQi5	$n_i T_i$	2	N	N
18	LQi6	$\langle B\hat{q}_{i\parallel} \rangle$	0	×	×
19	LQi7	$n_i \langle u_{i\zeta} / R \rangle$	0	×	×
20	LQb1	$n_b$	0(2)	×(N)	×(N)
21	LQb2	$n_b u_b^V$	1	0	×
22	LQb3	$n_b \langle Bu_{b\parallel} \rangle$	0	×	×
23	LQb4	$n_b \langle Ru_{b\zeta} \rangle$	0	×	×
24	LQb7	$n_b \langle u_{b\zeta} / R \rangle$	0	×	×
25	LQn1	$n_{01}$	2	N	$\langle  \nabla V  \rangle S_{\text{gas}} _b$
26	LQn2	$n_{02}$	2	N	0
27	LQn3	$n_{03}$	2	N	0
28	LQr1	$n_b^{\text{rp}}$	0(2)	×(N)	×(N)

0: Dirichlet condition ( $u = 0$ )

N: Neumann condition ( $u' = 0$ )

value: Neumann condition ( $u' = \text{value}$ )

×: no condition

NOTE: The dependent variable of LQm4 turns to be  $\psi / \mu_0$  when NB is applied. Subsequently, the boundary condition of LQm3 is also changed to  $2\pi I_p$ .

NOTE: In the table the column labeled by the highest order derivative means the order of the highest order derivative of a dependent variable, not an equation, in the corresponding row. Please note that the term not integrated by parts is never counted. For example, the first row shows the highest order of derivatives is equal to 2 for  $\Phi$  because the LQm1 equation has the diffusion term having the second derivative of  $\Phi$  with respect to  $V$ . In addition, the LQe2 and LQi2 equations also have the first derivative of  $\Phi$ , which is not a dependent

variable of these equations. As a whole, the highest order derivative of  $\Phi$  in this equation system is 2 that comes from the LQm1 equation. When we look at  $n_e$ , we find that the highest order derivative of  $n_e$  is 2 shown in the LQe5 equation, even though no derivative appears in the LQe1 equation. The Neumann conditions for  $n_e$  come from the LQe5 equation. When it comes to  $n_e u_e^V$ , the highest derivative is found as order 1 in the LQe1 equation, but this term is not integrated by parts. It means that any boundary condition is not naturally imposed on  $n_e u_e^V$ . The Dirichlet boundary condition should be imposed at axis.