Proc. Nat. Acad. Sci. USA
Vol. 72, No. 10, pp. 3789–3793, October 1975
Applied Mathematics

Adiabatic evolution of plasma equilibrium

(bifurcation/islation/weak solution/generalized differential equation/Tokamak)

H. GRAD, P. N. HU, AND D. C. STEVENS

Courant Institute of Mathematical Sciences, New York University, New York, N.Y. 10012

Contributed by H. Grad, July 21, 1975

ABSTRACT **A new theory of plasma equilibrium is introduced in which adiabatic constraints are specified. This leads to a mathematically nonstandard structure, as compared to the usual equilibrium theory, in which prescription of pressure and current profiles leads to an elliptic partial differential equation. Topologically complex configurations require further generalization of the concept of adiabaticity to allow irreversible mixing of plasma and magnetic flux among islands. Matching conditions across a boundary layer at the separatrix are obtained from appropriate conservation laws. Applications are made to configurations with planned islands (as in Doublet) and accidental islands (as in Tokamaks). Two-dimensional, axially symmetric, helically symmetric, and closed line equilibria are included.**

1. Introduction

The work described in this paper arose from a confluence of two distinct streams of thought. First was the recognition that time-dependent plasma diffusion is both physically subtle (1–3) and mathematically nonstandard (4). Second was the realization that deformation of equilibria under adiabatic constraints is equally subtle and nonstandard (5–8), and moreover involves similar mathematical and numerical difficulties as in the diffusion problem, but in a more transparent form.

Conventional equilibrium theory in two dimensions is governed by the nonlinear elliptic equation

$$\Delta\psi = -\frac{dp}{d\psi} \qquad [1.1]$$

where $p(\psi)$, the pressure profile in terms of the flux function, is presumed to be given. Consider first a domain as in Fig. 1 and solution $\psi(x,y)$ with simple flux contours. A solution will be determined (disregarding subtleties) by a boundary condition, say $\psi = 0$. The value, ψ_1, at the "center" is found in the course of solving the elliptic problem. Adiabatic deformation of the equilibrium implies conservation of flux; thus, under adiabatic constraints the value ψ_1 must be specified in addition to the elliptic boundary condition.

Consider next a more complex solution of Eq. **1.1**, as in Fig. 2. Adiabatic deformation conventionally implies that all critical flux values, ψ_1, ψ_2, and ψ_c (at the separatrix), are invariant. This is evidently incorrect. For example, further adiabatic constriction of the central waist in Fig. 2 would lead to large $\nabla\psi$ (magnetic field strength), destroying any pressure balance. The conventional adiabatic constraint must be violated, allowing plasma and flux to transfer among regions. This requires dissipation. We postulate that the dissipation is confined to a boundary layer near the separatrix (assuming high conductivity), leaving the interior domains classically adiabatic. Recalling the Hugoniot conditions for an ideal shock, we present jump conditions across the separatrix which allow the basically dissipative problem to be solved without overt regard to any specific dissipation mechanism. The jump conditions arise from appropriate conservation laws, valid through the boundary layer. Specifically, use is made of conservation of volume, mass, momentum, and flux—but energy is not always conserved, and the system is not always variational.

The adiabatic constraint on the pressure is

$$p = \mu(\psi)[\psi'(V)]^\gamma \qquad [1.2]$$

Here $V(\psi)$ represents the volume (area in two dimensions) within a flux contour, $\psi(V)$ the inverse function, γ the gas constant, and $\mu(\psi)$ [instead of $p(\psi)$] is given. For simplicity, taking $\gamma = 2$ and the special profile μ = constant = ½, eliminating p yields the prototype adiabatic equation

$$\Delta\psi = -\psi''(V) \qquad [1.3]$$

On the left is an elliptic operator on $\psi(x,y)$; on the right is an ordinary differential operator on $\psi(V)$. The combination is not a differential equation by the highly implicit definition of the independent variable, V, through $\psi(x,y)$ = constant. We call this combination of an ordinary differential equation (ODE) and partial differential equation (PDE) a *Generalized Differential Equation* (GDE)*. The general profile [1.2], as well as extensions to axial and helical equilibria and some diffusion problems, retain the form of a GDE, combining an elliptic second order PDE and ODE.

We remark that in the complex topology of Fig. 2, a time or parameter dependent adiabatic solution is path independent in the case of *mixing* (shrinking islands in Fig. 2), but not in the case of *splitting* (growing islands).

To see heuristically that [1.3] does allow proper boundary data, consider the (microcanonical) volume-weighted average on a flux surface.

$$\langle\phi\rangle = \oint \phi\, dS/|\nabla V|, \qquad \oint dS/|\nabla V| = 1 \qquad [1.4]$$

An elementary calculation yields

$$\langle\Delta\psi\rangle = (K\psi')', \quad K(V) = \langle|\nabla V|^2\rangle \qquad [1.5]$$

The coefficient $K(V)$ depends on geometry, i.e., on the contours ψ = constant, only. Since $K(0) = 0$, the ODE $(K\psi')' = 0$ does not allow a boundary condition to be given at $V = 0$ (this is a reflection of the fact that $V = 0$ is an interior point for the elliptic operator). However, the average of Eq. **1.3**,

Abbreviations: ODE, PDE, and GDE, ordinary, partial, and generalized differential equation, respectively.

* This was originally termed a QDE ("Queer" differential equation) in various lectures during 1973–1975.

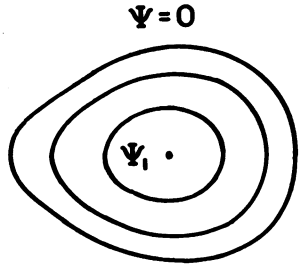

FIG. 1. Simple configuration.

$(K + 1)\psi'' + K'\psi' = 0$, is not singular at the origin, allowing ψ_1 to be specified.

This remark also indicates a plausible method of solving a GDE by iteration. The basic method consists of iterating by alternately determining the geometry (flux contours and K) and the flux profile $\psi(V)$, the former from the GDE treated as a PDE, the latter from the averaged ODE†. Insofar as solution of an ODE is held to be trivial, the method is an iteration on the geometrical shape of the flux contours. This method is very rapidly convergent, giving accurate answers in very distorted geometries and in highly compressed plasmas with strongly nonuniform current profiles.

We present in this paper a very brief outline of the beginning of a theory of the GDE; mentioned are a few examples in which mathematical proofs have been obtained, several formal solutions in limiting and special cases, and the empirical evidence of a large variety of successful numerical computations. The physical applications are very diverse, including reinterpretation of local (Mercier) stability in complex geometry, radical reinterpretation of resistive tearing instability, as well as the more evident specific application to adiabatic evolution of various types of noncircular Tokamaks and Belt Pinches, including overt island formation (as in Doublet) and unintentional island formation (as in Mirnov oscillation of circular Tokamaks); application of the GDE theory to explicitly dissipative systems is left to later publication.

2. Theory of the generalized differential equation in simple topology

There are only two evident cases in which a GDE is explicitly solvable: in one dimension, $\psi(x)$, and in a circular cylinder, $\psi(r)$. In each case the GDE reduces to an ODE. Small perturbations about each of these elementary configurations are separable into ODE and PDE components. The reason for this simplification is that in these configurations the operations Δ and $\langle\rangle$ commute.

Linearization about an arbitrarily given equilibrium does not simplify. The variational equation satisfied by $\partial\psi/\partial t$, where $\psi(x,y,t)$ is a one parameter family of solutions, is (for $\gamma = 2$ in two dimensions)

$$(\Delta + \ddot{p})\frac{\partial\psi}{\partial t} = -2(\mu\psi_t')' + 2\psi_t(\mu\psi'')'/\psi' \qquad [2.1]$$

$$(') = d/dV, \qquad (\dot{}) = d/d\psi$$

Note that $\psi_t = \partial\psi(V,t)/\partial t = \langle\partial\psi(x,y,t)/\partial t\rangle$. The left side is a linear self-adjoint elliptic operator on $\partial\psi/\partial t$; the right side is a self-adjoint operator on $\psi_t(V) = \langle\partial\psi/\partial t\rangle$ provided that two boundary values are given. This description may lead to a firm mathematical theory; but some effort is required since two different spaces are involved. A more concrete existence proof for this linearized equation can be carried out using the Green's function method used in ref. 4 for the diffusion problem.

† For the diffusion problem, this method was described in various lectures in 1970 and is in Oak Ridge National Laboratory memoranda dated April 22, 1970 and July 6, 1970.

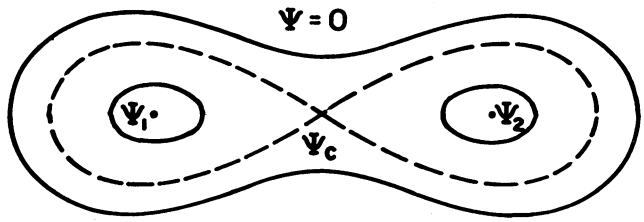

FIG. 2. Complex (doublet) configuration.

Two limiting cases are partially solvable in that the limiting profile is explicit and the limiting geometry, although not explicit, is decoupled from the profile, namely, when a large or small parameter multiplies $\Delta\psi$. In the case $(1/\epsilon)\Delta\psi$, the formal result, $\Delta\psi \sim 0$, is not valid near $V = 0$ where a boundary layer is found. The solution approximates a Green's function. Matching the inner and outer solutions leads to a compatibility condition that the net force be zero on the concentrated current; this serves to locate the point $V = 0$. This limit (e.g., of a highly compressed plasma) is identical to the classical fluid solution of a vortex that is at rest in a given bounded fluid domain (9). One can show that there exists at least one solution, and this is unique in a convex domain.

In the opposite limit, the profile is determined by $dp/d\psi \sim 0$ or $V(\psi) = \int (\mu/p_0)^{1/\gamma}d\psi$, but the geometry is not immediately evident. In this case it is convenient to consider an equation satisfied by $V(x,y)$ rather than $\psi(x,y)$. We mention, in passing, a family of related problems, $\Delta\psi = f(V)$ (f given), $\Delta V = f(V)$ (f open), [1.1] given $\psi(V)$ (p open), [1.1] given $p(V)$, etc.

The variational function for [1.1] as a PDE [$p(\psi)$ given] is

$$F_1 = \int \left[\frac{1}{2}\left|\nabla\psi\right|^2 - p(\psi)\right] dxdy \qquad [2.2]$$

and as a GDE [$\mu(\psi)$ given] it is

$$F_2 = \int\frac{1}{2}\left|\nabla\psi\right|^2 dxdy + \int\frac{1}{\gamma-1}\mu(\psi')^\gamma\, dV \qquad [2.3]$$

A steepest descent argument for [2.3] leads to an equation resembling a heat equation ($\dot{\mu} = d\mu/d\psi$),

$$\frac{\partial\psi}{\partial\tau} = \Delta\psi + \dot{\mu}(\psi')^\gamma + \gamma\mu(\psi')^{\gamma-2}\psi'' \qquad [2.4]$$

Numerical solution of this equation must face the formidable obstacle of the essential incompatibility of meshes (and stability criteria) in (x,y) and V. An evident attempt at numerical solution would be to alternate advancement in the (x,y) and V meshes; the iteration described in Section 1 can be interpreted as an extreme over-relaxation in which $t \to \infty$ at each step.

3. Complex topology

The first law that is established in a nonsimple equilibrium such as that in Fig. 2 is that the poloidal flux is conserved.

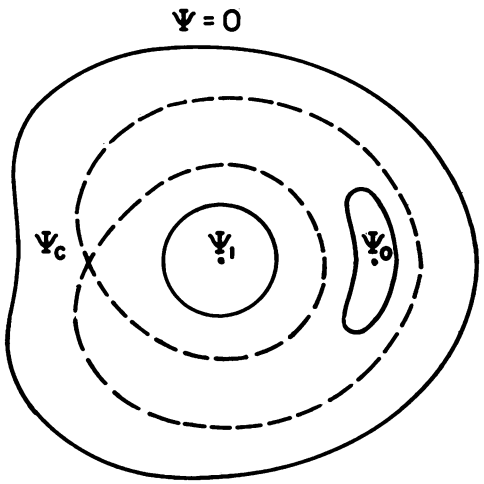

FIG. 3. Complex ("tearing") configuration.

This means that the difference in flux, $\delta\psi$, is fixed between two points, each moving with the fluid, neither of which is within the separatrix boundary layer. In particular, ψ_1 and ψ_2 are fixed, but nothing can be said about ψ_c. As ψ_c varies, both regions 1 and 2 either grow or shrink together, with respect to mass and flux (but not necessarily volume). If the islands shrink, then two thermodynamically different states mix to form one. If the islands grow, a single state is split into two not necessarily identical states. The two situations, *mixing* and *splitting*, evidently must be considered separately. Note that in the topology of Fig. 3, it is the intermediate zone labeled "0" that is distinguished and the outer and inner regions that move together.

All equilibria that have been computed using the elliptic equation have been assigned bounded current density, $\Delta\psi$, near the separatrix. From this alone follows $V \sim \psi \log \psi$ or $\psi'(V) = 0$ at the separatrix; either $p = 0$ or $\mu = \infty$ at the separatrix. If, however, an equilibrium has been obtained by adiabatic evolution (with mixing or splitting), the current is automatically singular near the separatrix, altering the contours so that ψ', p, and μ are finite and nonzero. To be precise, it turns out that $\Delta\psi \sim K' \sim V^{-1/3}$ on the narrow (acute angle) side, whereas on the other side, current density and K' are bounded. In other words, adiabatically processed and elliptic equilibria concern profiles that are, in practice, mutually exclusive. For example, local (Mercier) stability criteria are strongly modified.

Despite the singular profile and thermodynamic complications of mixing and splitting, the case of *symmetric* islands can be numerically solved exactly as the simple case, Fig. 1. We state the procedure (the justification is deferred). In region 1, double the value of V assigned to a contour, and multiply μ by 2^γ; it will follow that ψ', μ, and p, are continuous across the separatrix. The ODE can be solved directly from the center of region 1 to the outer boundary of the domain without taking notice of the separatrix. The iterated elliptic PDE is, of course, solved in the full domain. The value of ψ_c is found incidentally. Greater numerical accuracy will be obtained if the singular (but integrable) current is properly handled, but this is not a necessity.

In the general case of asymmetric islands, the justification for the jump conditions is quite complicated. It must be realized that any such (weak) extension of the theory is basically *ad hoc*. To derive the matching conditions from basic principles is not evident, even though there are many plausible procedures which are suggested by similar problems. For example, we list three essentially different cases: (*a*) two dimensions, $\gamma = 2$; (*b*) two dimensions, $\gamma \neq 2$; and (*c*) axial or helical symmetry. Case (*a*) is variational; correct jumps can be calculated as natural boundary conditions of a variational principle; p can be discontinuous. Case (*b*) is not variational, energy is not conserved in the separatrix, and p can be discontinuous. Case (*c*) can be variational, energy is conserved, and p is continuous. Both variational formulations, [2.2] and [2.3], are formally impeccable; but the former is never applicable in complex geometry, while the latter sometimes is.

We remark that, during iteration, some matching conditions will be automatically guaranteed by inversion of the elliptic operator; the remaining ones must be accommodated at the second step of iteration, by suitably joining solutions of the ODE in the three regions.

The two basic *local* matching conditions (described for axial symmetry, see Appendix for notation) are

$$[\nabla\psi] = 0, \quad \left[p + \tfrac{1}{2}B^2\right] = 0 \qquad [3.1]$$

The ODE needs averaged formulas. Without discussion, we list some of the most important:

$$1/\psi_0' = 1/\psi_1' + 1/\psi_2', \ \alpha \equiv \psi_0'/\psi_1' \qquad [3.2]$$

$$\langle 1/r^2\rangle_0 = \alpha\langle 1/r^2\rangle_1 + (1-\alpha)\langle 1/r^2\rangle_2 \qquad [3.3]$$

$$K_0 = K_1/\alpha + K_2/(1-\alpha) \qquad [3.4]$$

$$p_0 = p_1 = p_2, \qquad f_0 = f_1 = f_2 \qquad [3.5]$$

$$\nu_0 = \nu_1 + \nu_2, \qquad \mu_0^{1/\gamma} = \mu_1^{1/\gamma} + \mu_2^{1/\gamma} \qquad [3.6]$$

$$\left.\begin{aligned} \nu_1 &= \alpha\nu_0\langle 1/r^2\rangle_0/\langle 1/r^2\rangle_1, \\ \nu_2 &= (1-\alpha)\nu_0\langle 1/r^2\rangle_0/\langle 1/r^2\rangle_2 \end{aligned}\right\} \qquad [3.7]$$

$$\mu_1 = \mu_0\alpha^\gamma, \quad \mu_2 = \mu_0(1-\alpha)^\gamma \qquad [3.8]$$

Equations [3.6] apply to mixing, [3.7] and [3.8] to splitting. A process that involves only mixing is path independent; the profiles $\mu_0(\psi)$, $\nu_0(\psi)$ in region 0 can be calculated *a priori* in terms of those in regions 1 and 2. But a process that involves splitting creates values μ_1, μ_2, ν_1, ν_2 which depend on the geometry at the instant of splitting.

We remark that adiabatic evolution of closed line equilibria (with reflection symmetry) (6) can also be handled by roughly similar methods. But it must be kept in mind that flux surfaces are not adiabatically invariant, and, in complex geometry a plasma element can cross a separatrix without recourse to dissipation.

4. Islating and bifurcating equilibria

The linearized magneto-fluid dynamic equations of motion can be written in the form $\xi_{tt} = L\xi$, where ξ is the vector plasma displacement and L is a certain second order differential operator. Stability is sometimes analyzed in terms of the sign of $\delta W = -(\xi, L\xi)$. The spectrum, defined by $L\xi = \lambda\xi$, $\lambda = -\omega^2$ gives normal modes and also a number of continua (10). The linearized displacement ξ from one equilibrium to a neighboring one caused by a change in external constraint is governed by $L\xi = f$ (or $L\xi = 0$ with an inhomogeneous boundary condition representing the change in constraint). There is a unique neighboring equilibrium only if L is invertible; i.e., provided that there is no eigenvalue at the

origin and the continuum does not extend to the origin. The operator L is very closely related to the *linearized* GDE, [2.1].

The case of a point eigenvalue passing through the origin is classical. The correct implication is that linearization has become invalid. With regard to equilibrium, there is a bifurcation in parameter space with crossing of distinct families of equilibria. Each branch must be tested for stability. If any branch is stable, the system will presumably follow that branch, and the linearized conclusion, that there is a transition from stability to instability when ω^2 changes sign, is incorrect.

In the case of a continuum reaching the origin, the correct nonlinear interpretation is a change in the topology of the solution (involving violation of the strict adiabatic constraint) (5, 6). *Unique* continuation past the point of *islation* follows by introducing the weak extension of the concept of adiabaticity involving mixing and splitting (7, 8).

We describe two specific applications, one analytic (a boundary layer analysis), the second numerical. In a problem with a flux function, the criterion for islation is a higher order isolated critical point, $\nabla\psi = 0$, or $\nabla\psi = 0$ on a curve rather than only at isolated points. In a one-dimensional problem with $\psi = \psi(x)$ and $\psi'(x_0) = 0$, a small perturbation leads to an *islated* boundary layer near $x = x_0$. Matching to the outer solution (which is found by conventional means from the linearized operator L) shows that the dominant (most singular) component of the outer solution must vanish at x_0; it also shows that the thickness of the boundary layer is $0(\epsilon)$ (rather than the expected $\epsilon^{1/2}$), where ϵ represents the magnitude of the external perturbation.

The same boundary layer analysis shows the impossibility of matching any island region to a perfectly conducting, linearized solution on the exterior in the case of the presumed appearance of a resistive tearing instability. This implies that on a long time scale the resistive instability is not an instability at all but represents an initially thickened resistive layer which ultimately diffuses classically, without instability. The resistive instability calculation merely gives the short time growth in thickness of the layer.

Next consider the configuration of Fig. 2 which has arisen by gradual squeezing from an initially elliptic shape. The first effect is a simple distortion of the elliptical belt pinch. Beyond a certain bifurcation point, the equilibrium is discovered to be asymmetric (but still with a simple topology). At a second critical value, the observed asymmetric equilibrium is found to islate. There is evidently an additional, unstable, symmetric equilibrium beyond the point of bifurcation which the numerical iteration does not find. It is a trivial matter to *impose* symmetry on the numerical scheme; in this way the symmetric solution is found beyond the point of bifurcation. The symmetric solution also eventually islates, at a parameter value beyond the one at which islation first occurs in the asymmetric equilibrium.

5. Conclusion

We have presented the main features of a theory of adiabatic compression of plasma. In a simple configuration this involves development of the theory of a new and unusual mathematical structure, the Generalized Differential Equation. In a complex topological configuration this involves, in addition, introduction of an extension of the concept of adiabaticity to allow an irreversible boundary layer at a separatrix. We have listed some analytical and numerical procedures and properties and have mentioned a few of the most important physical applications. Details will appear elsewhere.

Appendix

Average Pressure Balance in General Geometry. By assigning proper orientation to toroidal and poloidal closed curves and surface cuts, the equilibrium identity

$$-p' = I_1'\psi_1' + I_2'\psi_2' \qquad [A.1]$$

can be derived; this form, as an inner product rather than a cross product in indices, is obtained by associating the magneto-motive force $I = \oint B\cdot dx$ with a curve rather than with a surface as in $I = \int J\cdot dS$.

On a pressure surface, the magnetic field, B, satisfies an elliptic equation and is uniquely determined by two periods, either I_i or ψ_i', $(i = 1,2)$ (11). The periods are related by a 2×2 surface inductance matrix

$$\psi_i' = L_{ij}I_j, \quad I_i = \Lambda_{ij}\psi_j' \qquad [A.2]$$

where $L_{ij}(V)$ depends on the geometry of the flux surfaces only (12). We retain $\psi_1 = \psi$ and $I_2 = f$ in [A.1], eliminating ψ_2 and I_1 by use of [A.2]

$$-\dot{p} = (\psi'/L_{11})' - (L_{12}/L_{11})'f + f\dot{f}/\Lambda_{22} \qquad [A.3]$$

$$f = \psi'(\Lambda_{21} + \nu\Lambda_{22}), \qquad \nu = \psi_2'/\psi_1' \qquad [A.4]$$

[A.4] is used to eliminate f in terms of the adiabatic invariant $\nu(\psi)$, and as always, $p = \mu(\psi')^\gamma$. In two dimensions, $L_{11} = \langle|\nabla V|^2\rangle^{-1}$, $L_{12} = 0$, $\Lambda_{22} = 1$. In axial symmetry, $L_{11} = \langle|\nabla V|^2/r^2\rangle^{-1}$, $L_{12} = 0$, $\Lambda_{22} = \langle 1/r^2\rangle^{-1}$. In helical symmetry, with helical angle $\phi = \theta - kz$, introducing

$$\left.\begin{aligned} \sigma^2 &= 1 + k^2r^2, \quad \lambda = \langle 1/\sigma^2\rangle^{-1} \\ \kappa &= 2k\int (1/\sigma^4)dV, \quad K = \langle|\nabla V|^2/\sigma^2\rangle \end{aligned}\right\} \qquad [A.5]$$

one can show

$$\left.\begin{aligned} &\Lambda_{11} = K + \lambda\kappa^2, \quad \Lambda_{12} = -\lambda\kappa, \quad \Lambda_{22} = \lambda \\ &L_{11} = 1/K, \quad L_{12} = \kappa/K, \quad L_{22} = 1/\lambda + \kappa^2/K \end{aligned}\right\} \qquad [A.6]$$

The elliptic equation, of which [A.3] is the average, is

$$\mathrm{div}(\nabla\psi/r^2) = -\dot{p} - f\dot{f}/r^2, \quad f = rB_\theta \qquad [A.7]$$

$$\mathrm{div}(\nabla\psi/\sigma^2) - 2kf/\sigma^4 = -\dot{p} - f\dot{f}/\sigma^2 \qquad [A.8]$$

in axial and helical symmetry, respectively.

This work was supported by U.S. Energy Research and Development Administration under Contract E(11-1)-3077.

1. Grad, H. & Hogan, J. (1970) *Phys. Rev. Lett.* **24**, 1337–1340.
2. Grad, H, (1971) *Courant Institute Report MF-69.*
3. Grad, H. (1974) in *Advances in Plasma Physics,* (John Wiley & Sons, Inc., New York), Vol. 5, pp. 103–119.
4. Grad, H. (1970) in *"Proceedings of Congress of Mathematicians"* (Nice) (Gauthier-Villars, Paris), Vol. 3, pp. 105–113.
5. Grad, H. (1969) *Phys. Today* **22**, no. 12, 34–44.

6. Grad, H. (1971) in *Plasma Physics and Controlled Nuclear Fusion Research* (Int. Atomic Energy Agency, Vienna), Vol. III, pp. 229–239.
7. Grad, H. (1974) in *Proceedings of Conference on Theoretical Controlled Thermonuclear Research* (Lawrence Livermore Laboratory), p. 31.
8. Grad, H., Hu, P. N. & Stevens, D. C. (1974) *Bull. Am. Phys. Soc.* **19**, 865.
9. Milne-Thomson, L. M. (1938) *Theoretical Hydrodynamics* (Macmillan, New York), chap. XIII.
10. Grad, H. (1973) *Proc. Nat. Acad. Sci. USA* **70**, 3277–3281.
11. Grad, H. & Rubin, H. (1958) in *Proceedings of Second International Conference on the Peaceful Uses of Atomic Energy* (Geneva), Vol. 31, pp. 190–197.
12. Bateman, G. (1973) *Nucl. Fusion* **13**, 227–238.