Stationarity & introductory functions

FISH 550 – Applied Time Series Analysis

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30 March 2023

Topics for today

Characteristics of time series

- Expectation, mean & variance
- Covariance & correlation
- Stationarity
- Autocovariance & autocorrelation
- Correlograms

White noise

Random walks

Backshift & difference operators

Expectation & the mean

The expectation (E) of a variable is its mean value in the population

$\mathrm{E}(x) \equiv$ mean of $x = \mu$

We can estimate μ from a sample as

$$m = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Variance

$\mathrm{E}([x-\mu]^2) \equiv$ expected deviations of x about μ

$\mathrm{E}([x-\mu]^2) \equiv$ variance of $x = \sigma^2$

We can estimate σ^2 from a sample as

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - m)^2$$

Covariance

If we have two variables, x and y, we can generalize variance

$$\sigma^2 = \mathrm{E}([x_i - \mu][x_i - \mu])$$

into *covariance*

$$\gamma_{x,y} = \mathrm{E}([x_i - \mu_x][y_i - \mu_y])$$

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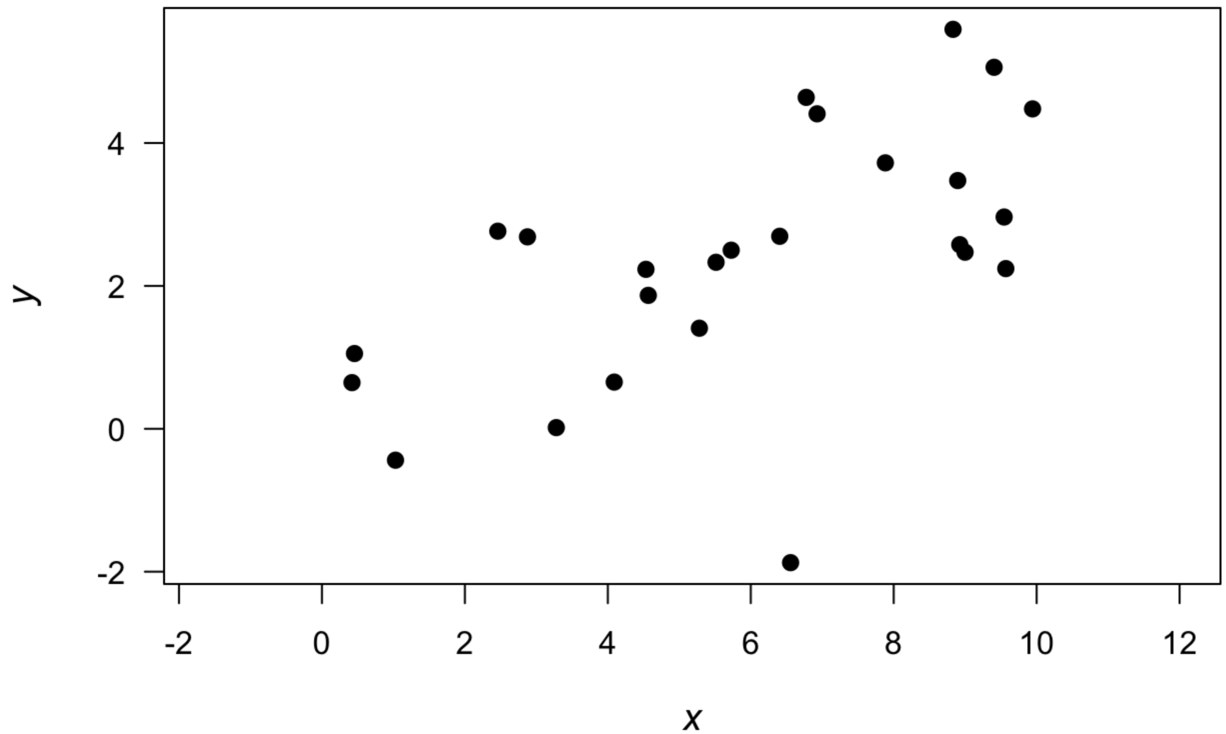
We can estimate $\gamma_{x,y}$ from a sample as

$$\mathrm{Cov}(x, y) = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - m_x)(y_i - m_y)$$

Graphical example of covariance



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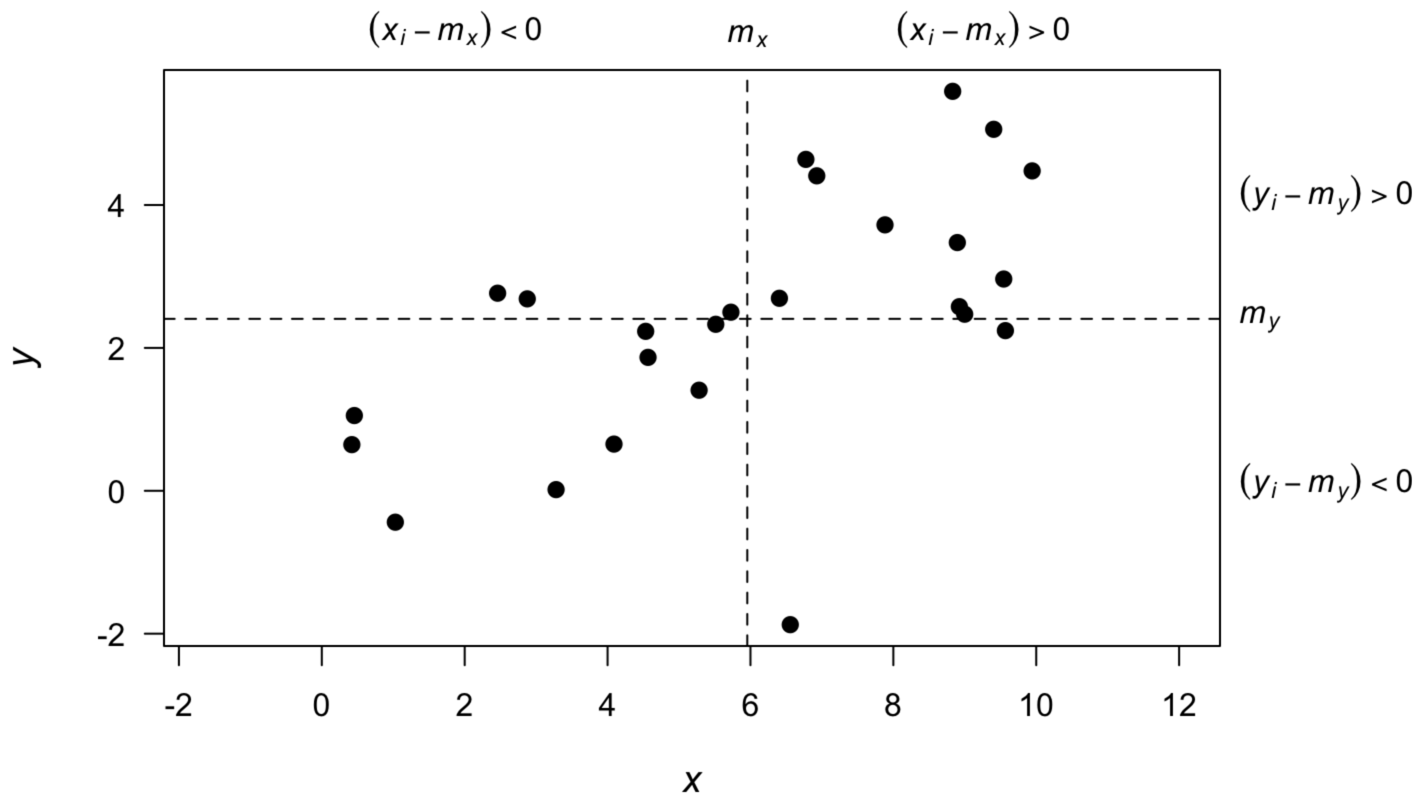

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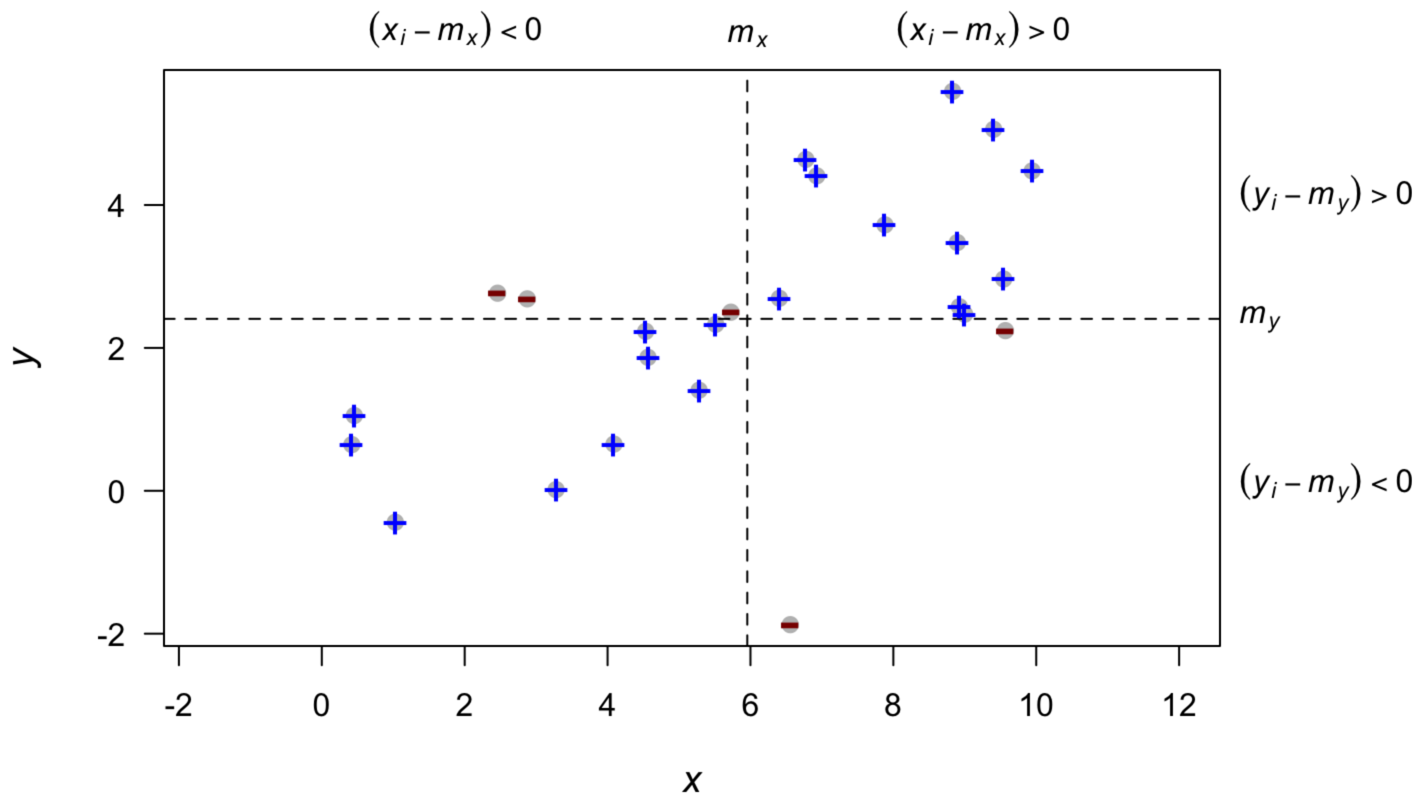

Correlation

Correlation is a dimensionless measure of the linear association between 2 variables, x & y

It is simply the covariance standardized by the standard deviations

$$\rho_{x,y} = \frac{\gamma_{x,y}}{\sigma_x \sigma_y}$$

$$-1 < \rho_{x,y} < 1$$

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Stationarity & the mean

Consider a single value, x_t

Stationarity & the mean

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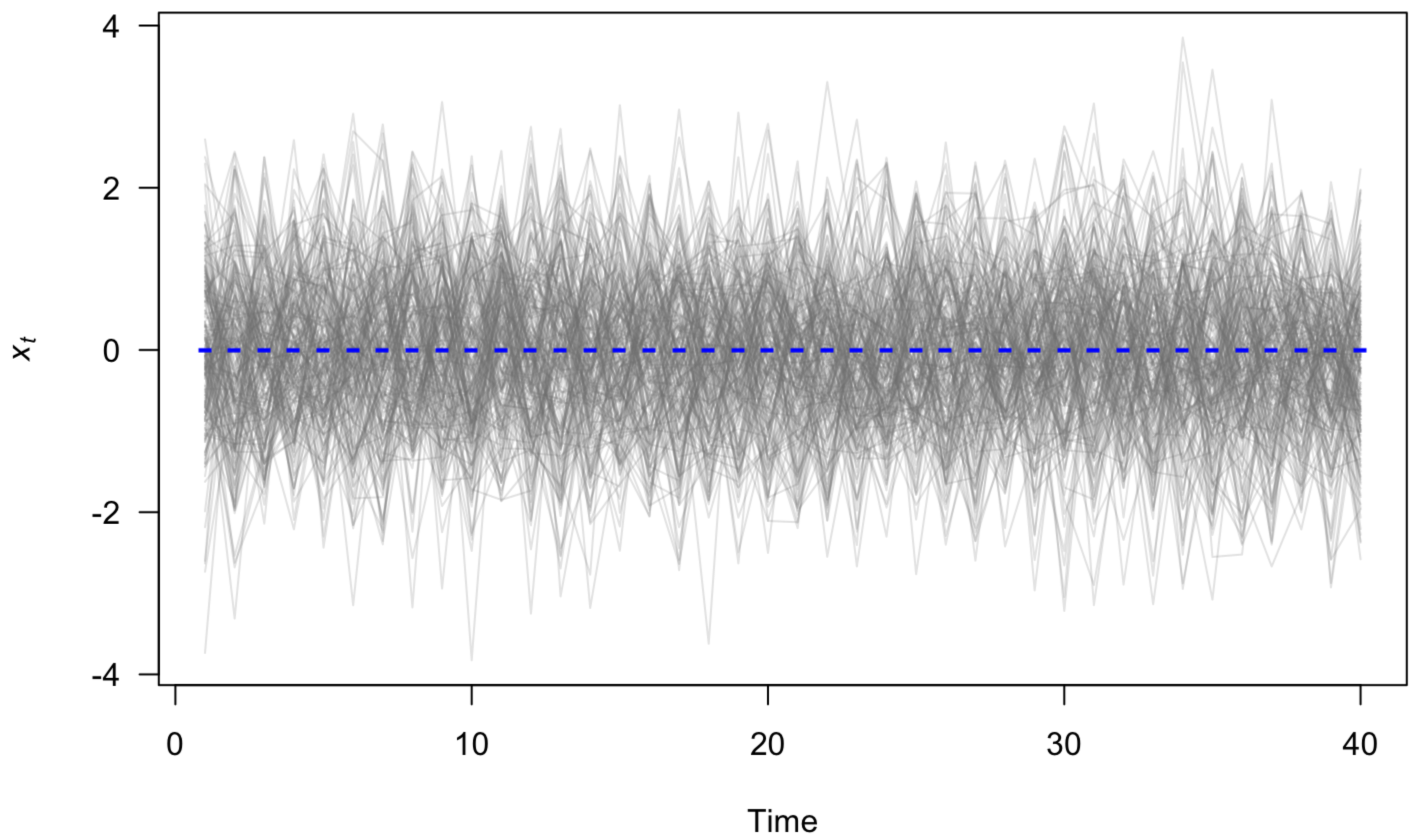
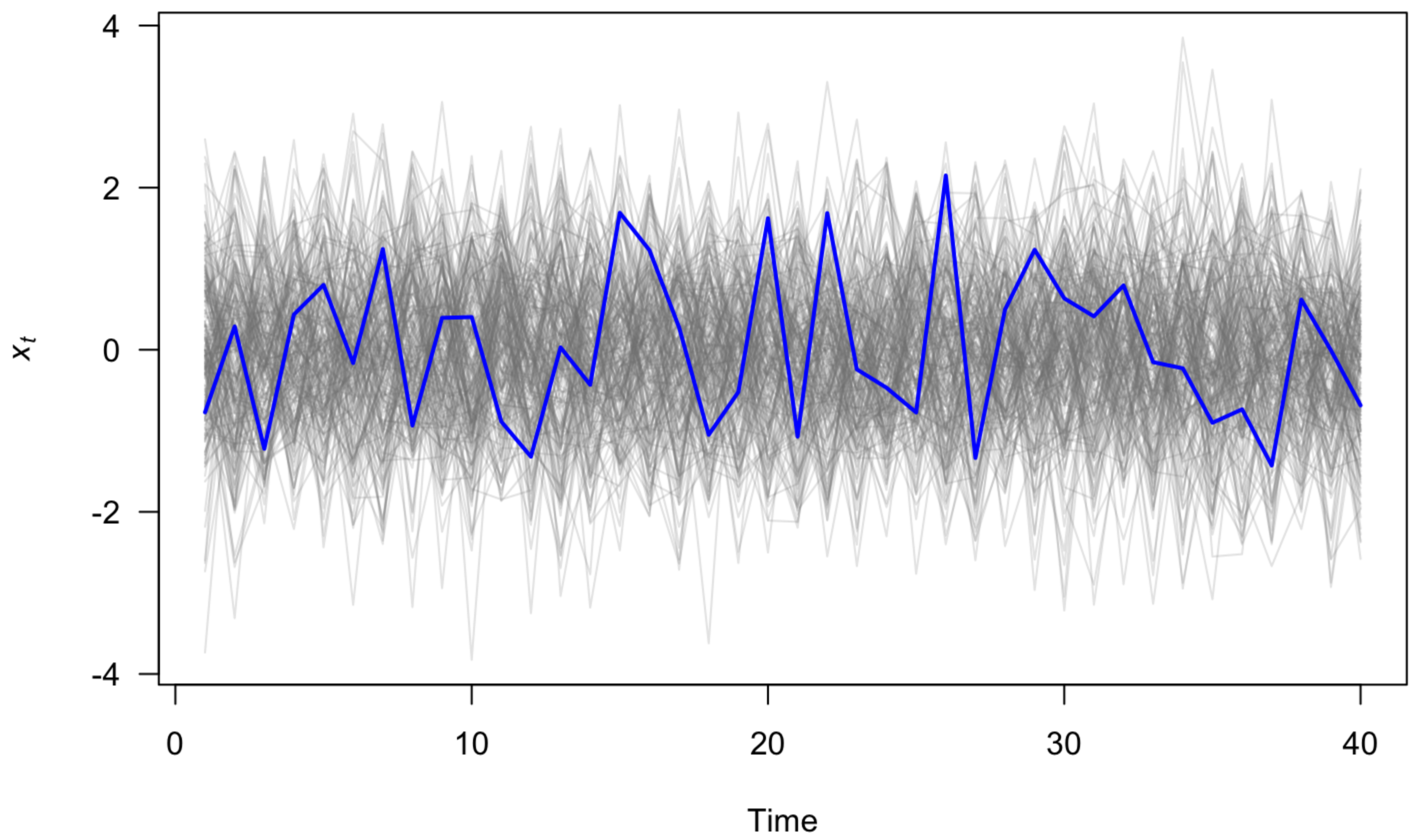
$\mathrm{E}(x_t)$ is taken across an ensemble of *all* possible time series

Stationarity & the mean



Stationarity & the mean



Our single realization is our estimate!

Stationarity & the mean

If $\text{E}(x_t)$ is constant across time, we say the time series is *stationary* in the mean

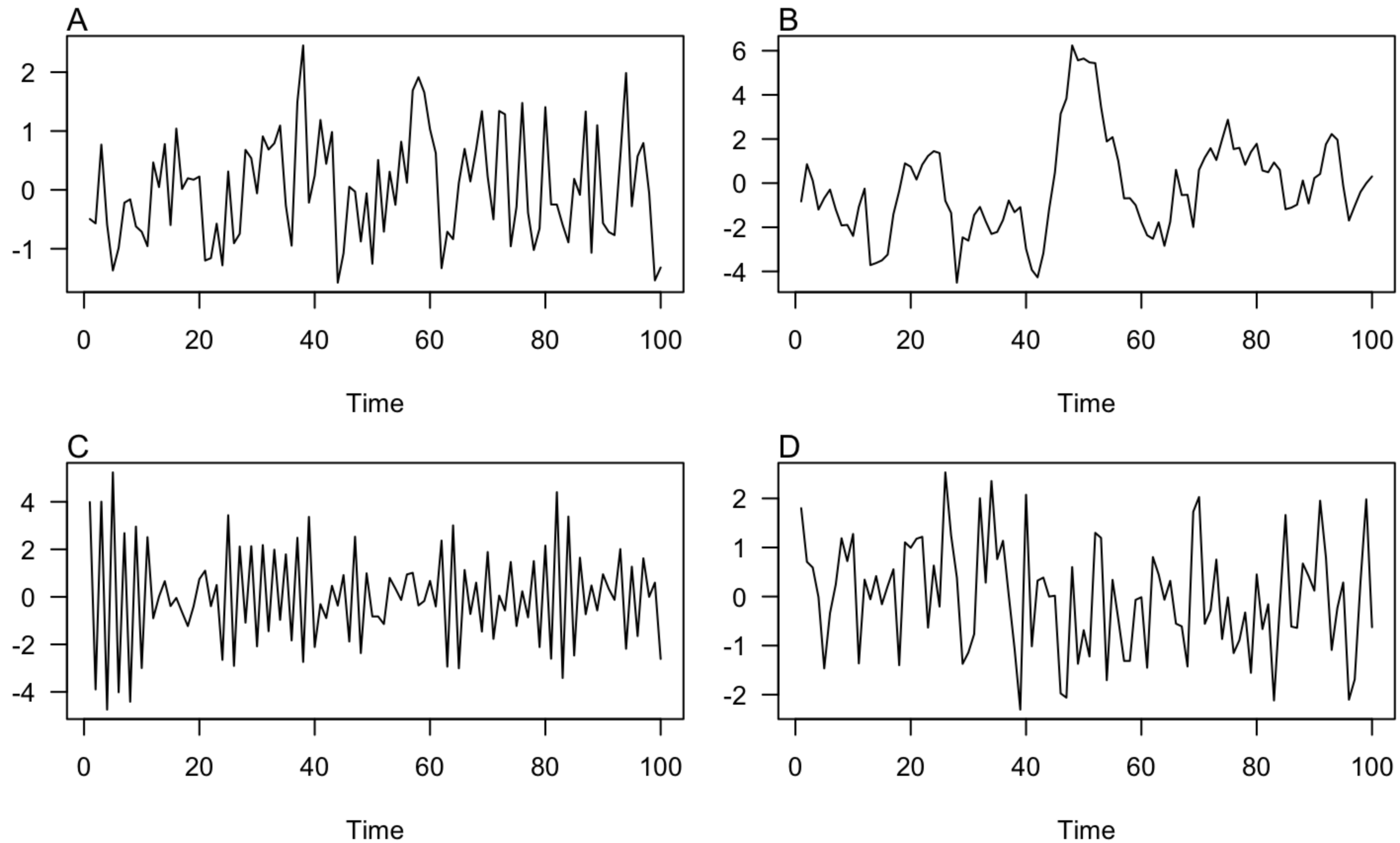
Stationarity of time series

Stationarity is a convenient assumption that allows us to describe the statistical properties of a time series.

In general, a time series is said to be stationary if there is

1. no systematic change in the mean or variance
2. no systematic trend
3. no periodic variations or seasonality

Identifying stationarity



Identifying stationarity

Our eyes are really bad at identifying stationarity, so we will learn some tools to help us

Autocovariance function (ACVF)

For stationary ts, we define the *autocovariance function* (γ_k) as

$$\gamma_k = \mathrm{E}([x_t - \mu][x_{t+k} - \mu])$$

which means that

$$\gamma_0 = \mathrm{E}([x_t - \mu][x_t - \mu]) = \sigma^2$$

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"Smooth" time series have large ACVF for large k

"Choppy" time series have ACVF near 0 for small k

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We can estimate γ_k from a sample as

$$c_k = \frac{1}{N} \sum_{t=1}^{N-k} (x_t - m)(x_{t+k} - m)$$

Autocorrelation function (ACF)

The *autocorrelation function* (ACF) is simply the ACVF normalized by the variance

$$\rho_k = \frac{\gamma_k}{\sigma^2} = \frac{\gamma_k}{\gamma_0}$$

The ACF measures the correlation of a time series against a time-shifted version of itself

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The ACF measures the correlation of a time series against a time-shifted version of itself

We can estimate ACF from a sample as

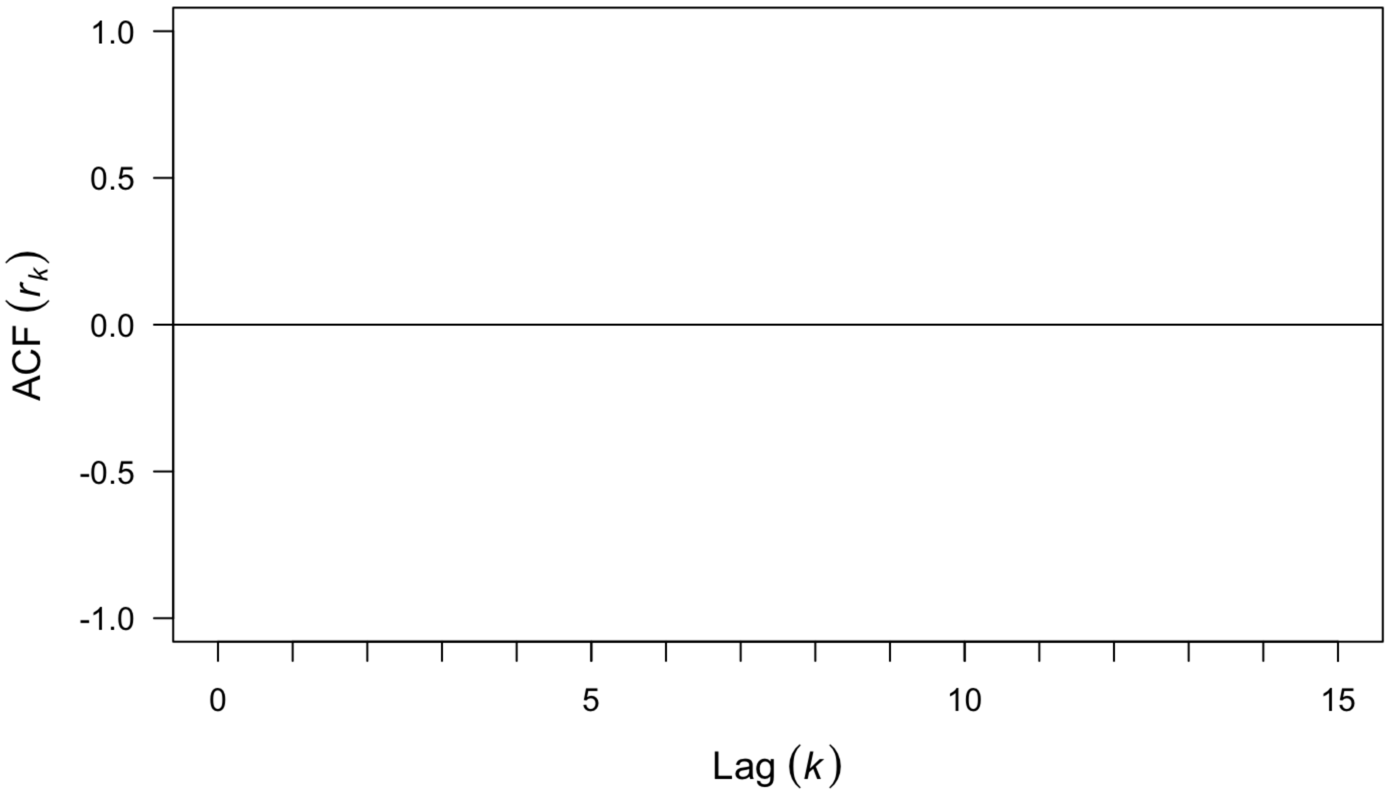
$$r_k = \frac{c_k}{c_0}$$

Properties of the ACF

The ACF has several important properties:

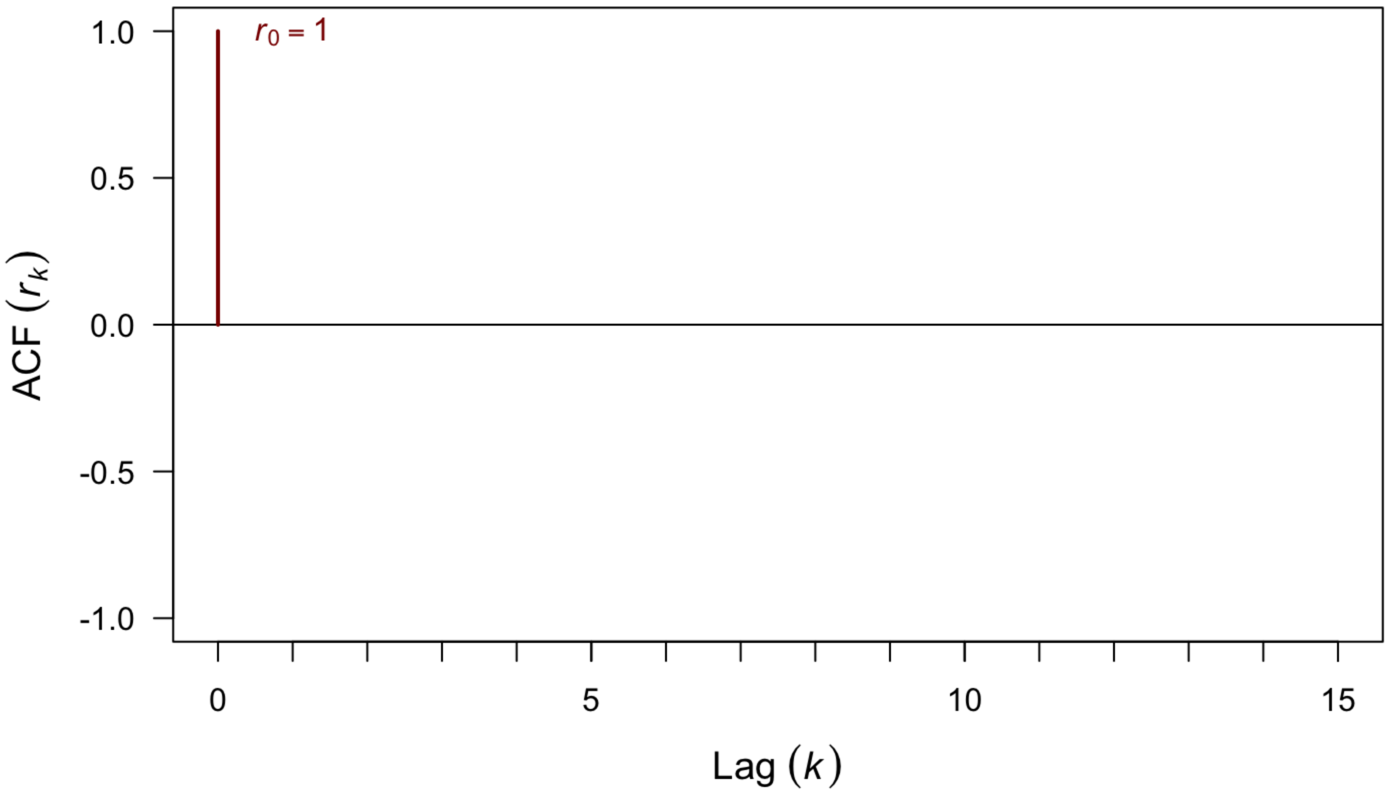
- $-1 \leq r_k \leq 1$
- $r_k = r_{-k}$
- r_k of a periodic function is itself periodic
- r_k for the sum of 2 independent variables is the sum of r_k for each of them

The correlogram



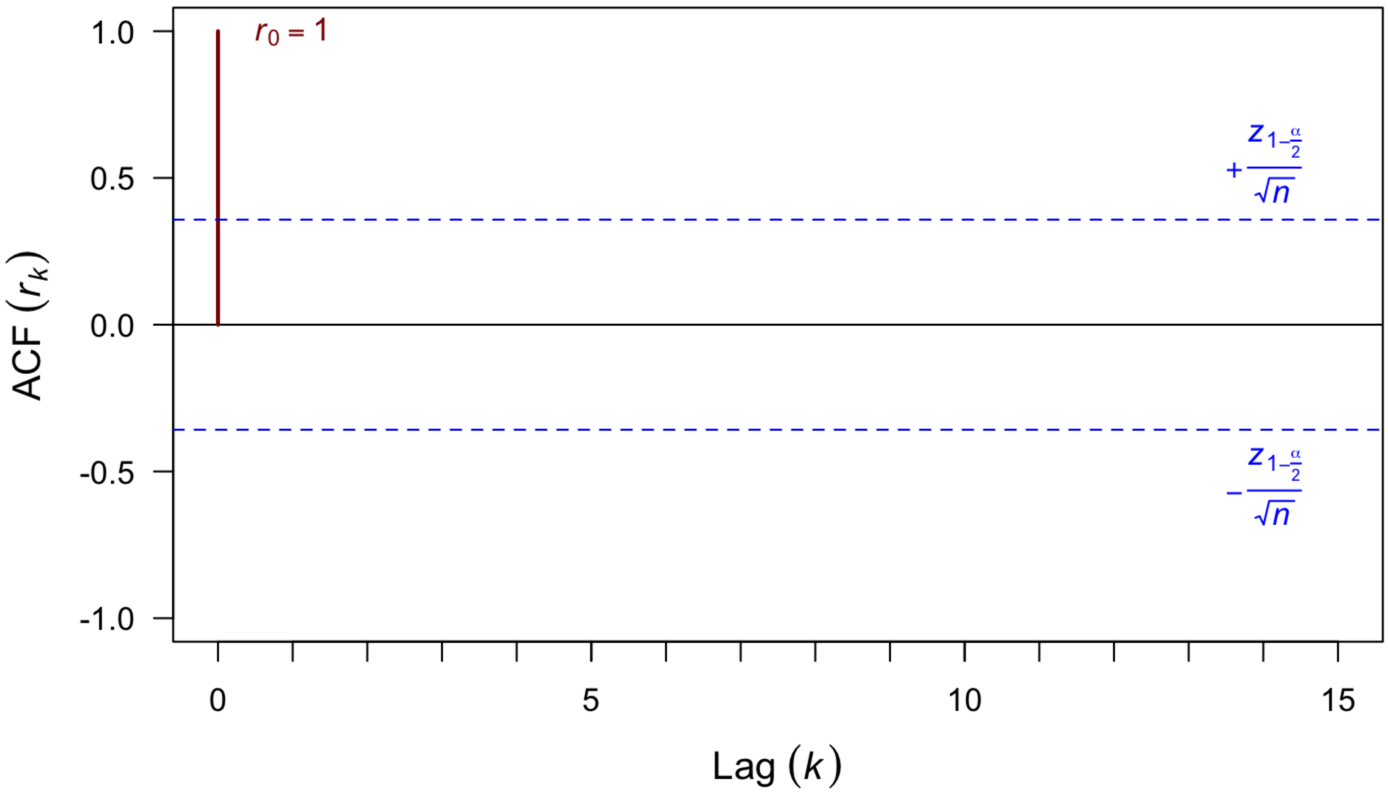
Graphical output for the ACF

The correlogram



The ACF at lag = 0 is always 1

The correlogram

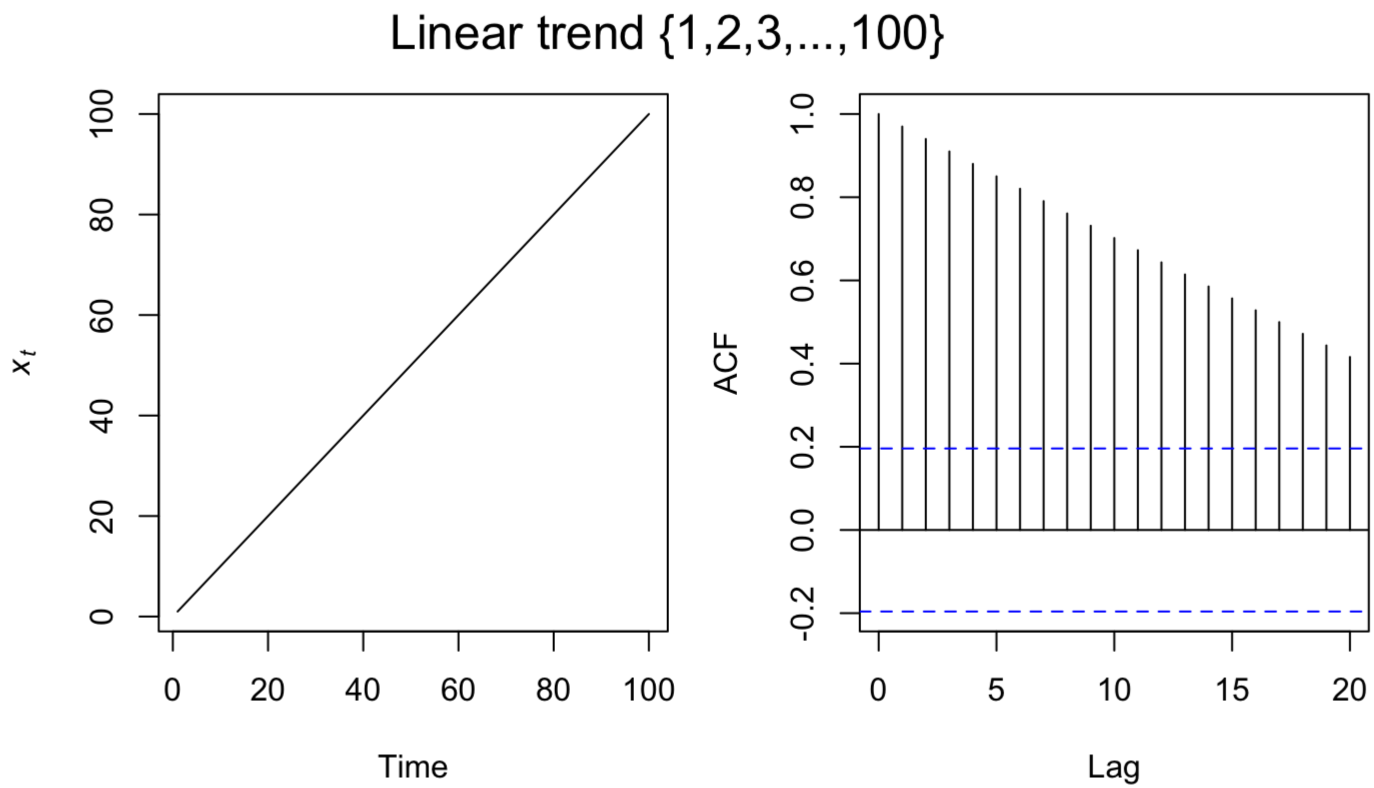


Approximate confidence intervals

Estimating the ACF in R

```
acf(ts_object)
```

ACF for deterministic forms



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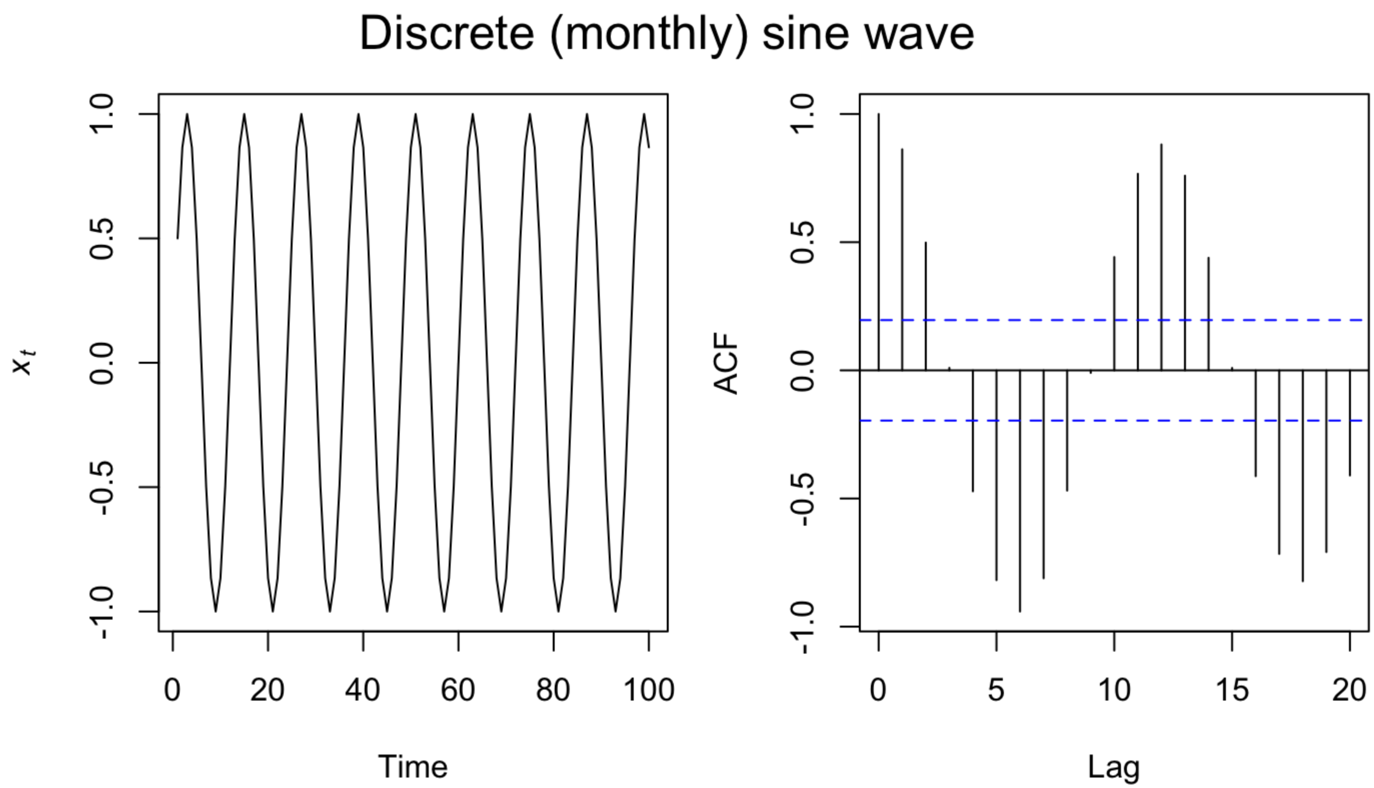

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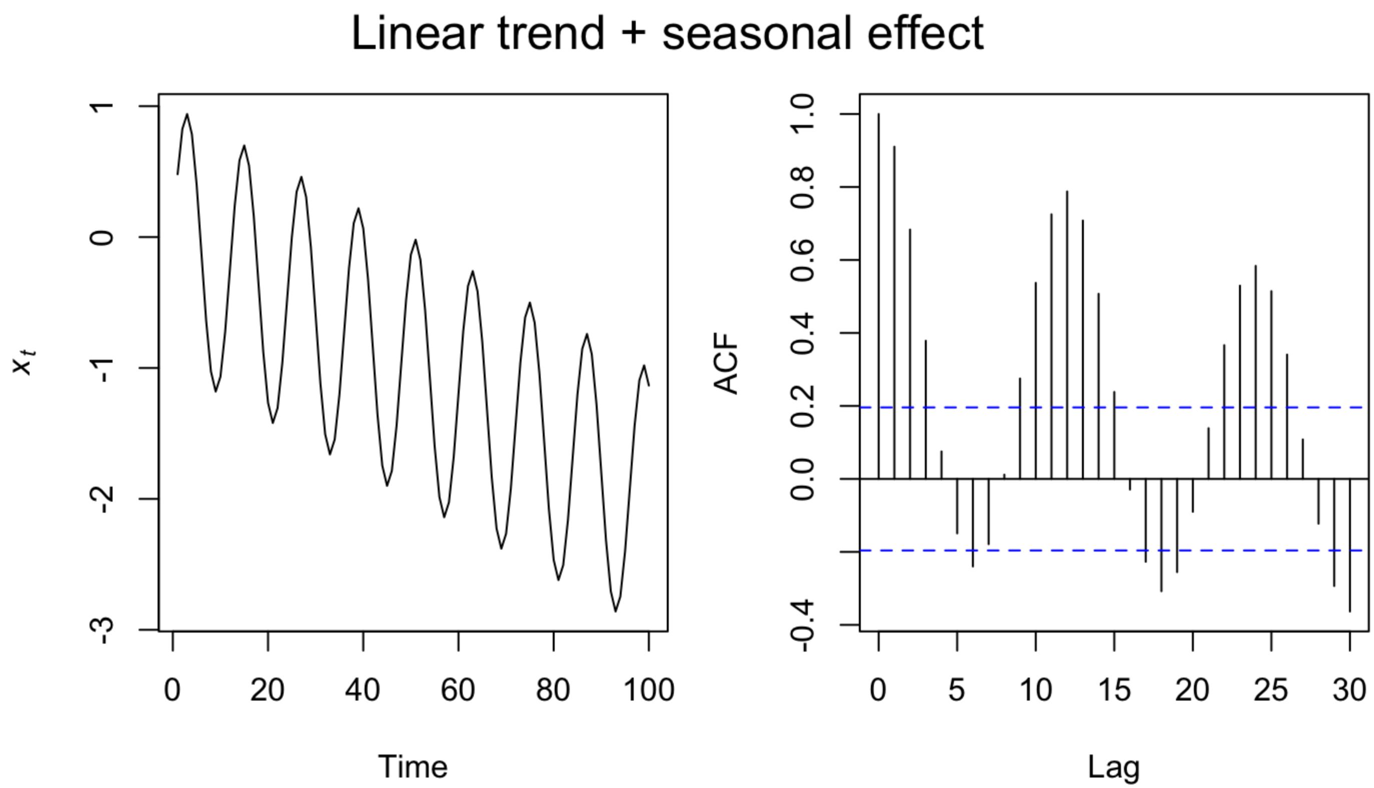
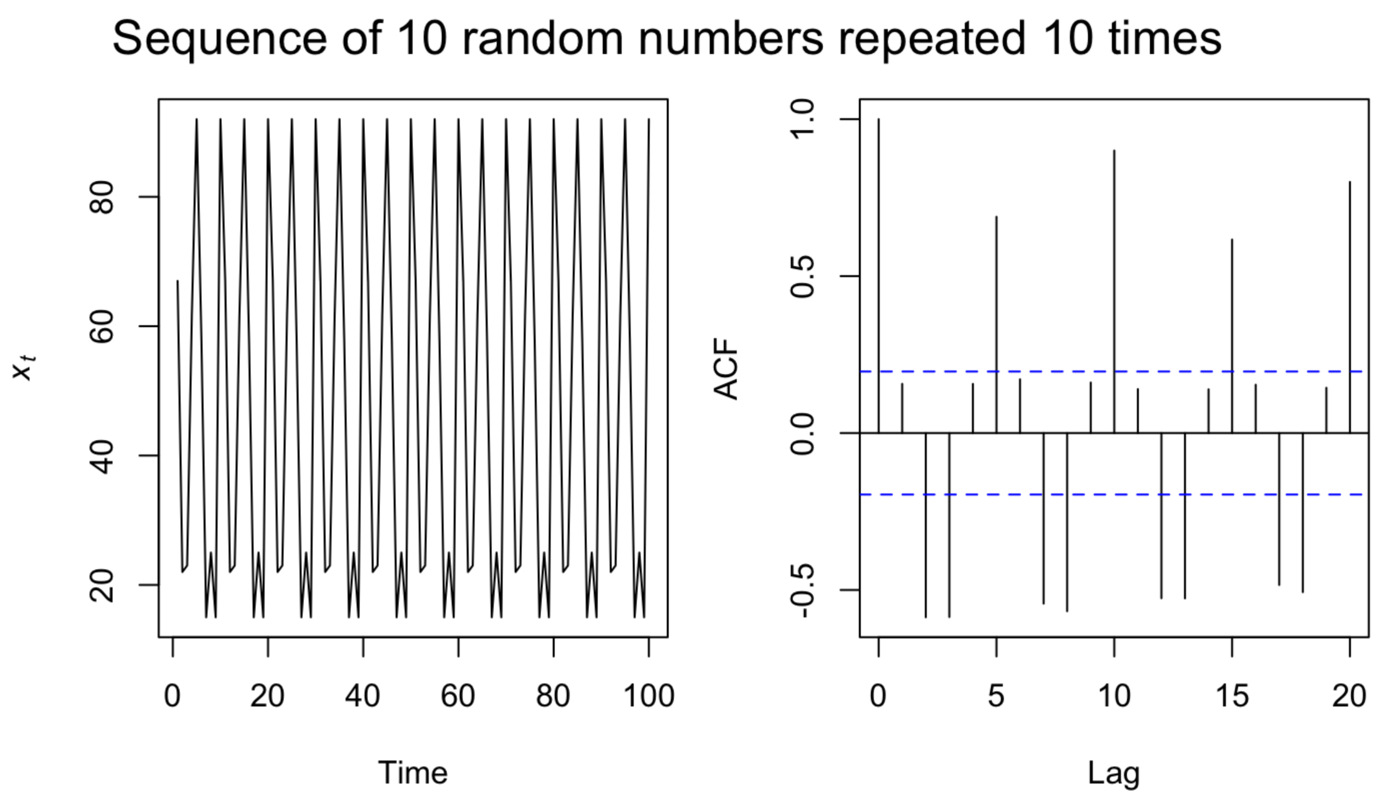


ACF for deterministic forms



Induced autocorrelation

Recall the transitive property, whereby

If $A = B$ and $B = C$, then $A = C$

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which suggests that

If $x \propto y$ and $y \propto z$, then $x \propto z$

and thus

If $x_t \propto x_{t+1}$ and $x_{t+1} \propto x_{t+2}$, then $x_t \propto x_{t+2}$

Partial autocorrelation funcion (PACF)

The *partial autocorrelation function* (ϕ_k) measures the correlation between a series x_t and x_{t+k} with the linear dependence of $\{x_{t-1}, x_{t-2}, \ldots, x_{t-k-1}\}$ removed

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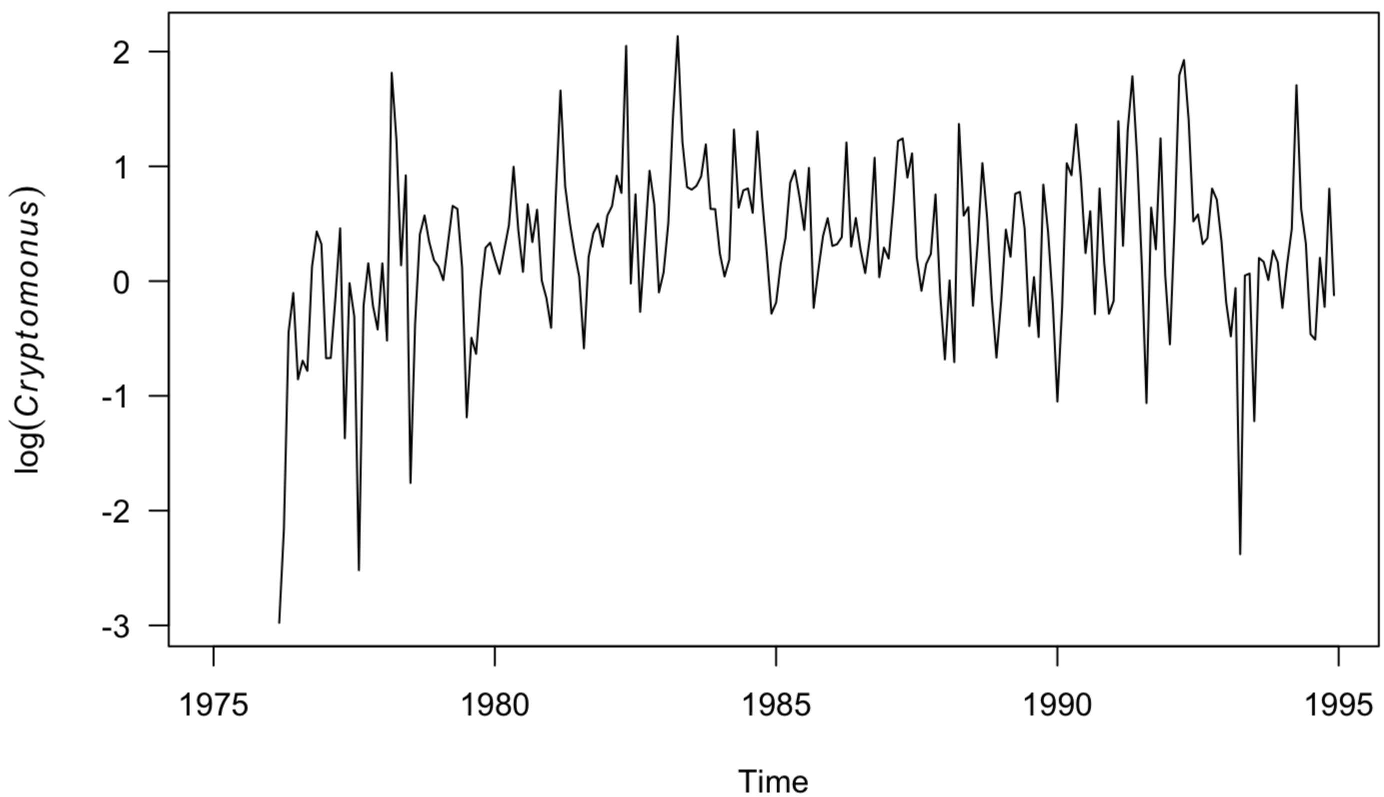
We can estimate ϕ_k from a sample as

$$
\phi_k = \begin{cases} \text{Cor}(x_1, x_0) = \rho_1 & \text{if } k = 1 \\ \text{Cor}(x_k - x_k^{k-1}, x_0 - x_0^{k-1}) & \text{if } k \geq 2 \end{cases}
$$

$$
x_k^{k-1} = \beta_1 x_{k-1} + \beta_2 x_{k-2} + \cdots + \beta_{k-1} x_1
$$

$$
x_0^{k-1} = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{k-1} x_{k-1}
$$

Lake Washington phytoplankton



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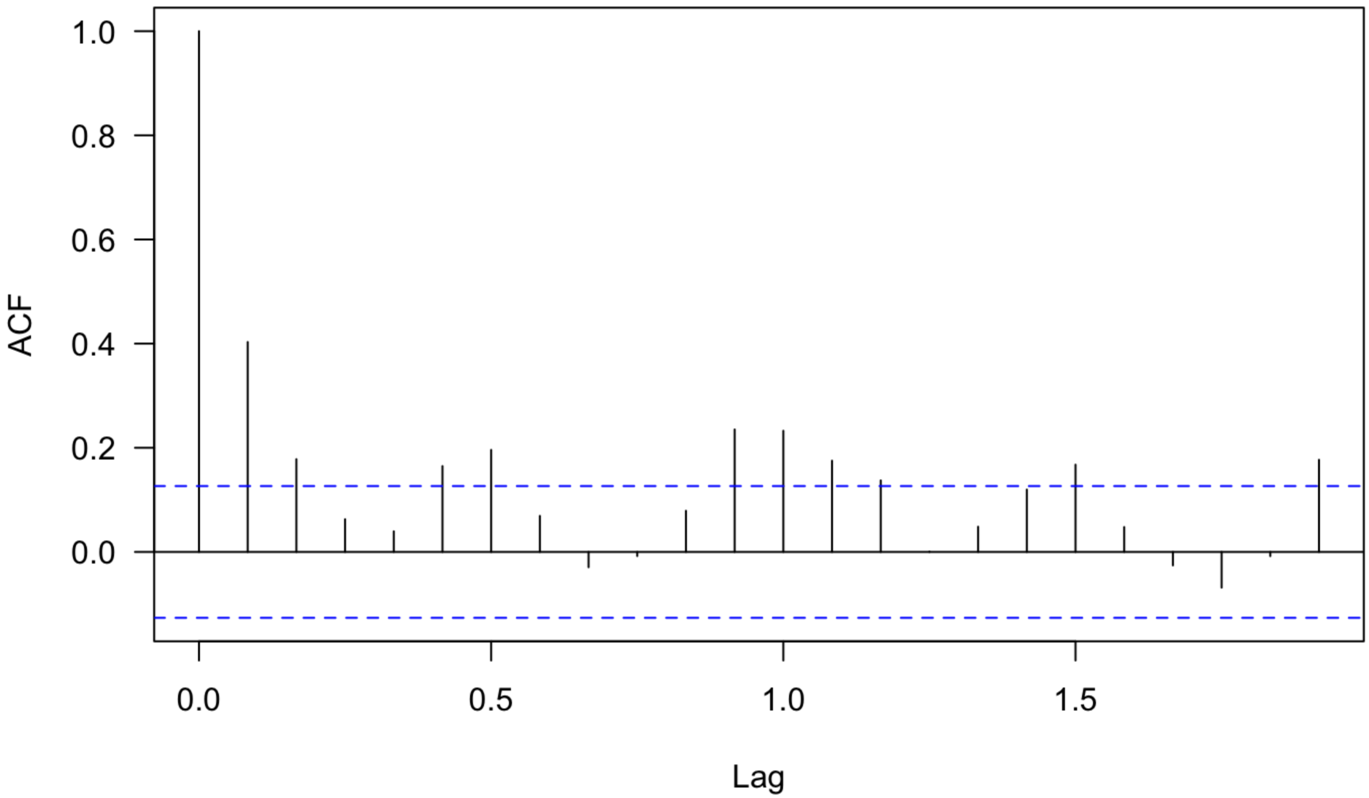

Autocorrelation

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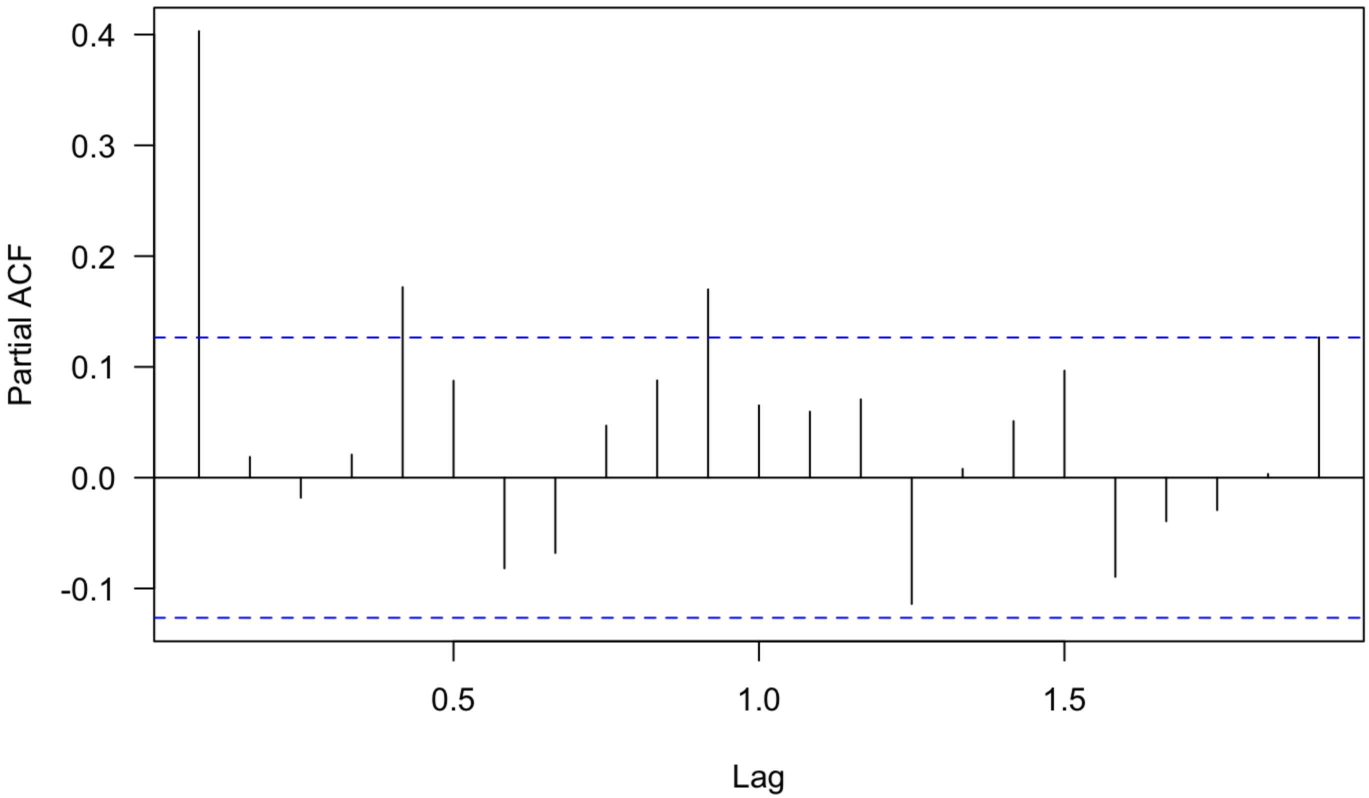

Partial autocorrelation

ACF & PACF in model selection

We will see that the ACF & PACF are *very* useful for identifying the orders of ARMA models

Cross-covariance function (CCVF)

Often we want to look for relationships between 2 different time series

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We can estimate the CCVF ($g_k^{x,y}$) from a sample as

$$g_k^{x,y} = \frac{1}{N} \sum_{t=1}^{N-k} (x_t - m_x)(y_{t+k} - m_y)$$

Cross-correlation function (CCF)

The cross-correlation function is the CCVF normalized by the standard deviations of x & y

$$r_k^{x,y} = \frac{g_k^{x,y}}{s_x s_y}$$

Just as with other measures of correlation

$$-1 \leq r_k^{x,y} \leq 1$$

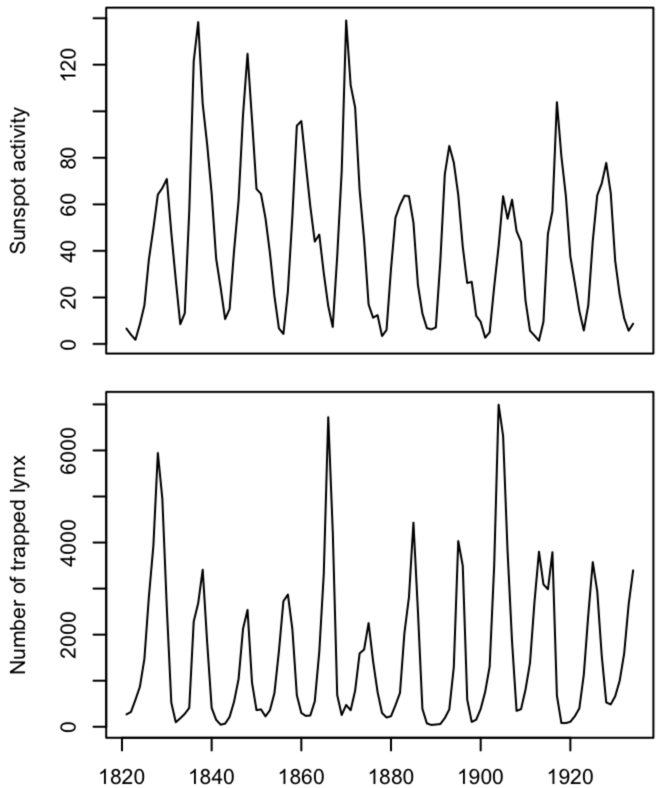
Estimating the CCF in R

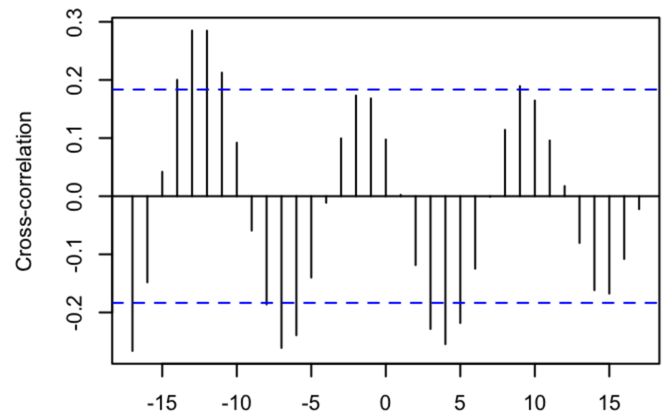
```
ccf(x, y)
```

Note: the lag `k` value returned by `ccf(x, y)` is the correlation between `x[t+k]` and `y[t]`

In an explanatory context, we often think of $y = f(x)$, so it's helpful to use `ccf(y, x)` and only consider positive lags

Example of cross-correlation





SOME SIMPLE MODELS

White noise (WN)

A time series $\{w_t\}$ is discrete white noise if its values are

1. independent

2. identically distributed with a mean of zero

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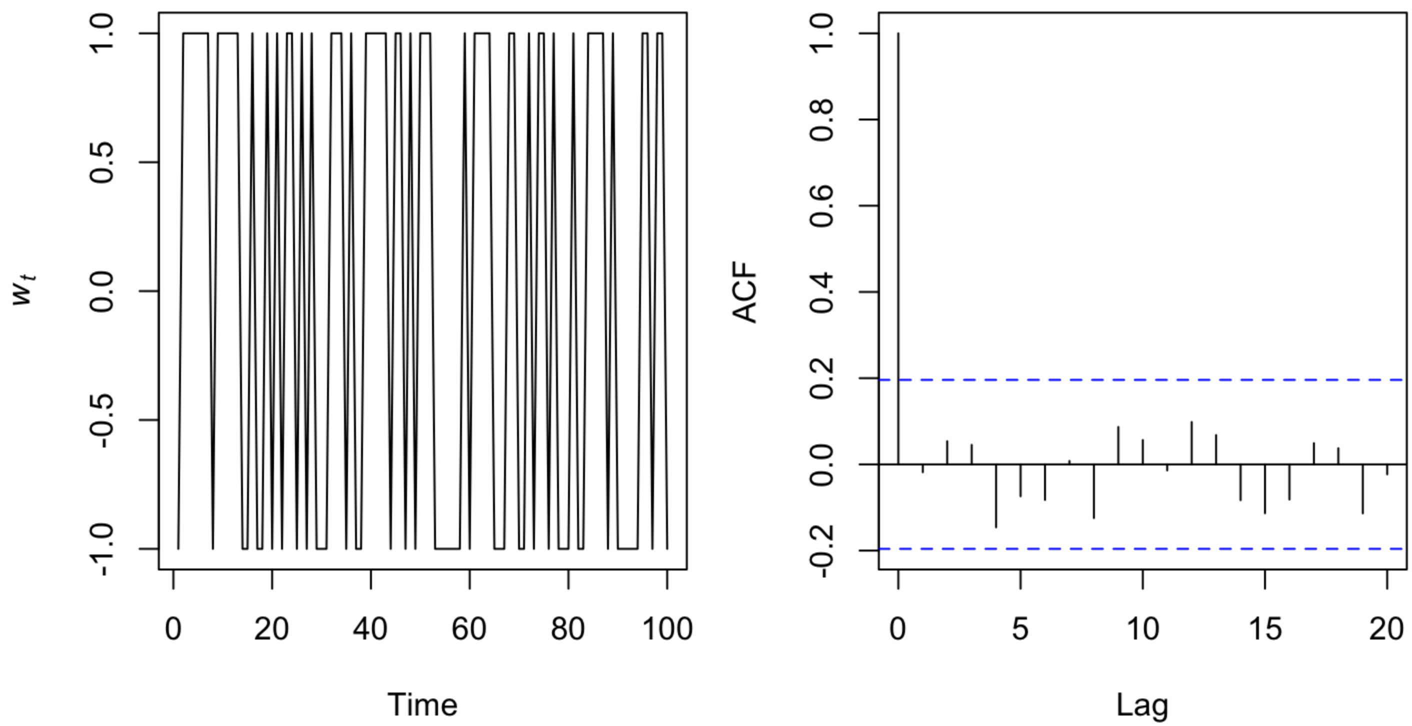
Note that distributional form for $\{w_t\}$ is flexible

White noise (WN)



$w_t = 2e_t - 1; e_t \sim \text{Bernoulli}(0.5)$

Gaussian white noise

We often assume so-called *Gaussian white noise*, whereby

$$w_t \sim \mathrm{N}(0, \sigma^2)$$

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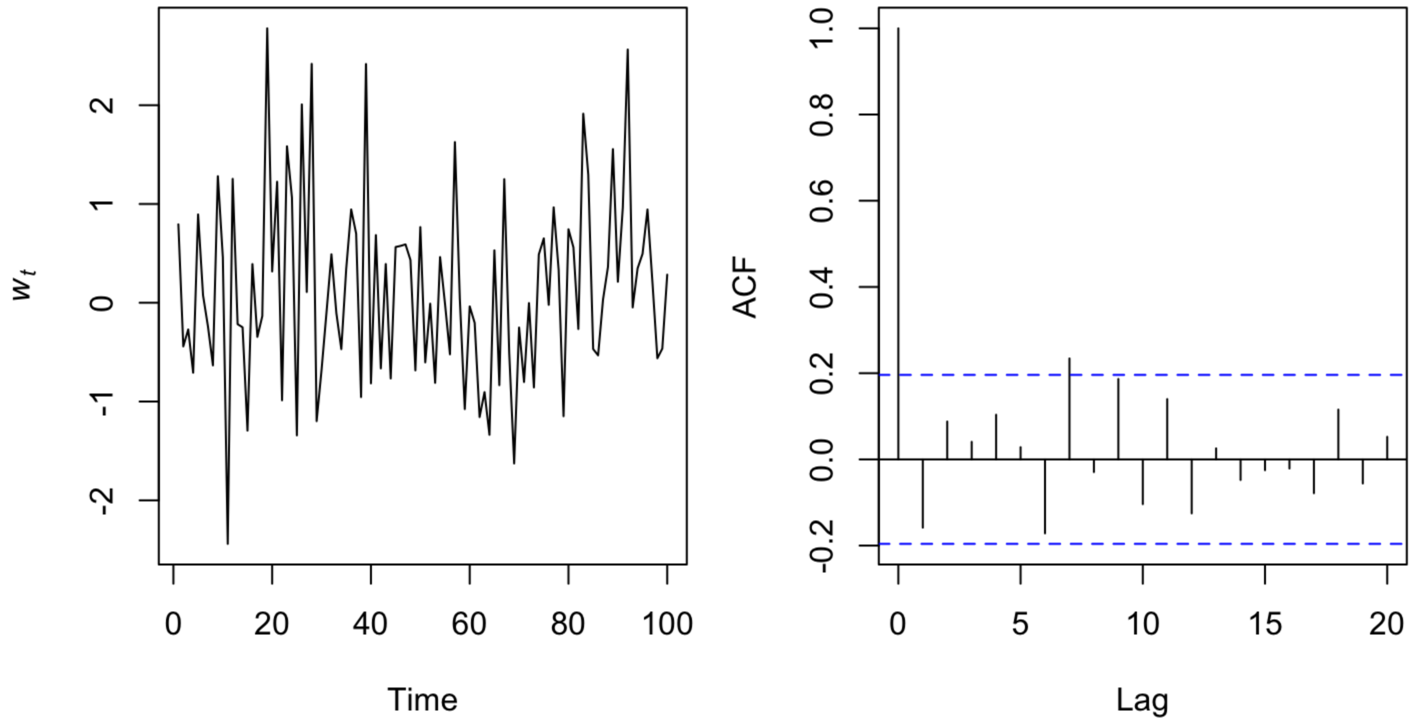
$$w_t \sim \mathrm{N}(0, \sigma^2)$$

and the following apply as well

autocovariance: $\gamma_k = \begin{cases} \sigma^2 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$

autocorrelation: $\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$

Gaussian white noise



$w_t \sim \mathrm{N}(0, 1)$

Random walk (RW)

A time series $\{x_t\}$ is a random walk if

1. $x_t = x_{t-1} + w_t$

2. w_t is white noise

Random walk (RW)

The following apply to random walks

mean: $\mu_x = 0$

autocovariance: $\gamma_k(t) = t\sigma^2$

autocorrelation: $\rho_k(t) = \frac{t\sigma^2}{\sqrt{t\sigma^2(t+k)\sigma^2}}$

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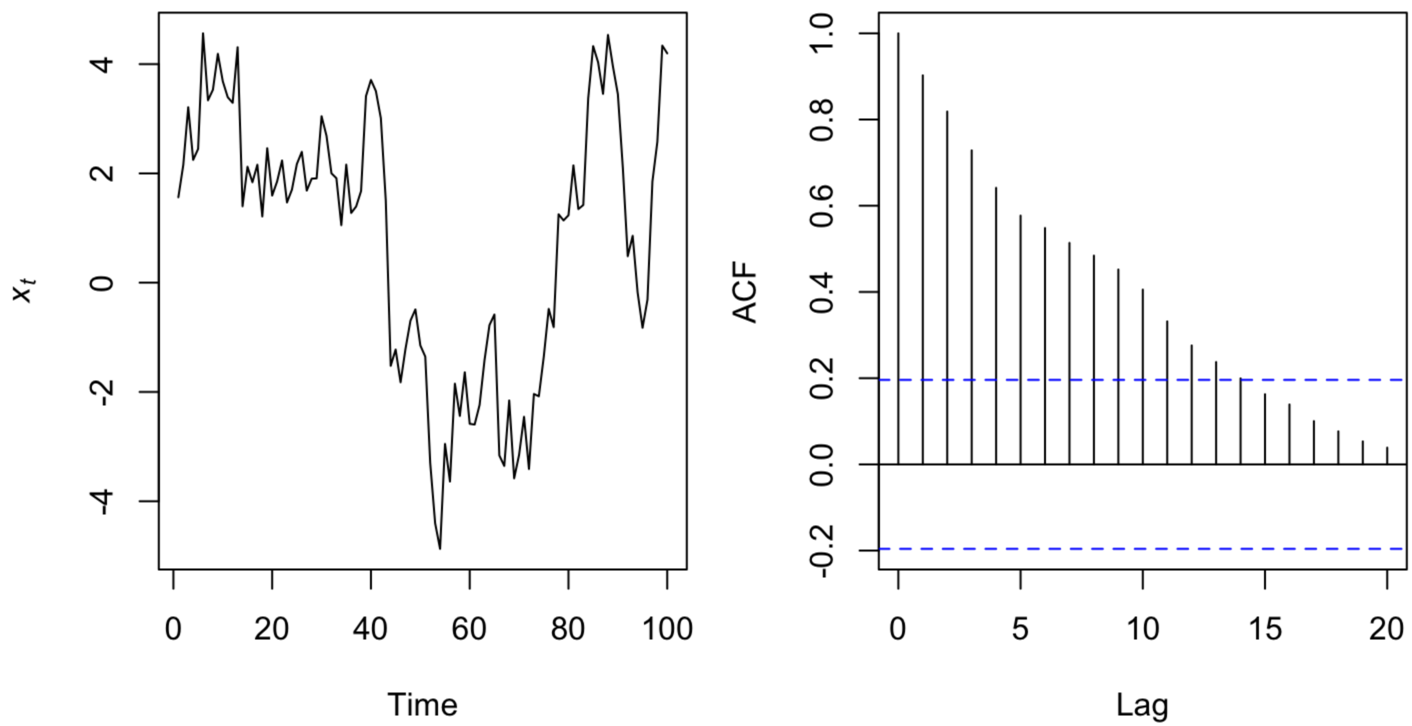
Note: Random walks are not stationary

Random walk (RW)



$$x_t = x_{t-1} + w_t; w_t \sim \mathrm{N}(0, 1)$$

SOME IMPORTANT OPERATORS

The backshift shift operator

The *backshift shift operator* ($\mathbf{B}$) is an important function in time series analysis, which we define as

$$\mathbf{B}x_t = x_{t-1}$$

or more generally as

$$\mathbf{B}^k x_t = x_{t-k}$$

The backshift shift operator

For example, a random walk with

$$x_t = x_{t-1} + w_t$$

can be written as

$$\begin{aligned} x_t &= \mathbf{B}x_t + w_t \\ x_t - \mathbf{B}x_t &= w_t \\ (1 - \mathbf{B})x_t &= w_t \\ x_t &= (1 - \mathbf{B})^{-1}w_t \end{aligned}$$

The difference operator

The *difference operator* (∇) is another important function in time series analysis, which we define as

$$\nabla x_t = x_t - x_{t-1}$$

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$$\nabla x_t = x_t - x_{t-1}$$

For example, first-differencing a random walk yields white noise

$$\begin{aligned}
\nabla x_t &= x_{t-1} + w_t \\
x_t - x_{t-1} &= x_{t-1} + w_t - x_{t-1} \\
x_t - x_{t-1} &= w_t
\end{aligned}$$

The difference operator

The difference operator and the backshift operator are related

$$\nabla^k = (1 - \mathbf{B})^k$$

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For example

$$\begin{aligned} \nabla x_t &= (1 - \mathbf{B}) x_t \\ x_t - x_{t-1} &= x_t - \mathbf{B} x_t \\ x_t - x_{t-1} &= x_t - x_{t-1} \end{aligned}$$

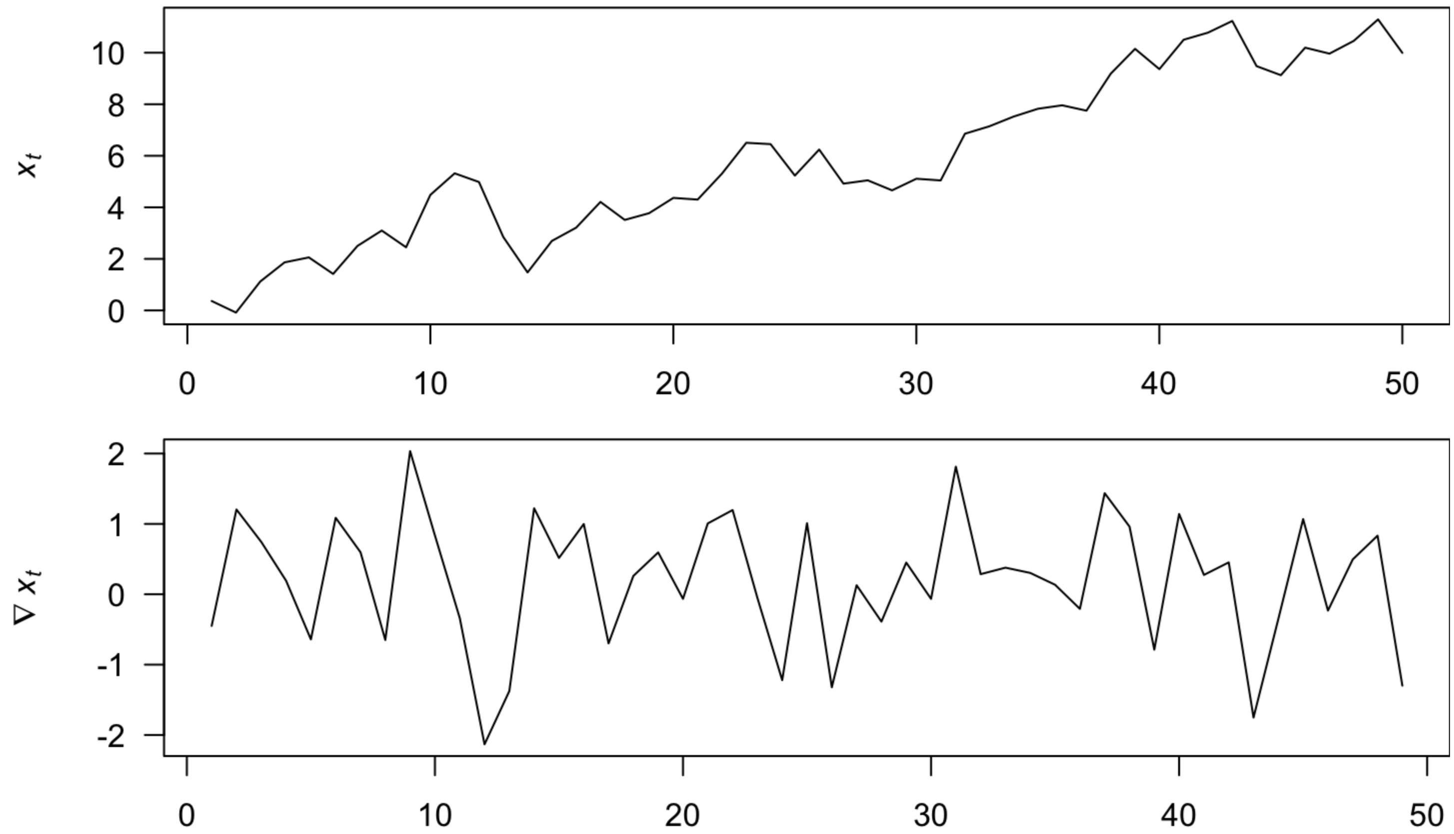
Differencing to remove a trend

Differencing is a simple means for removing a trend

The 1st-difference removes a linear trend

A 2nd-difference will remove a quadratic trend

Differencing to remove a trend

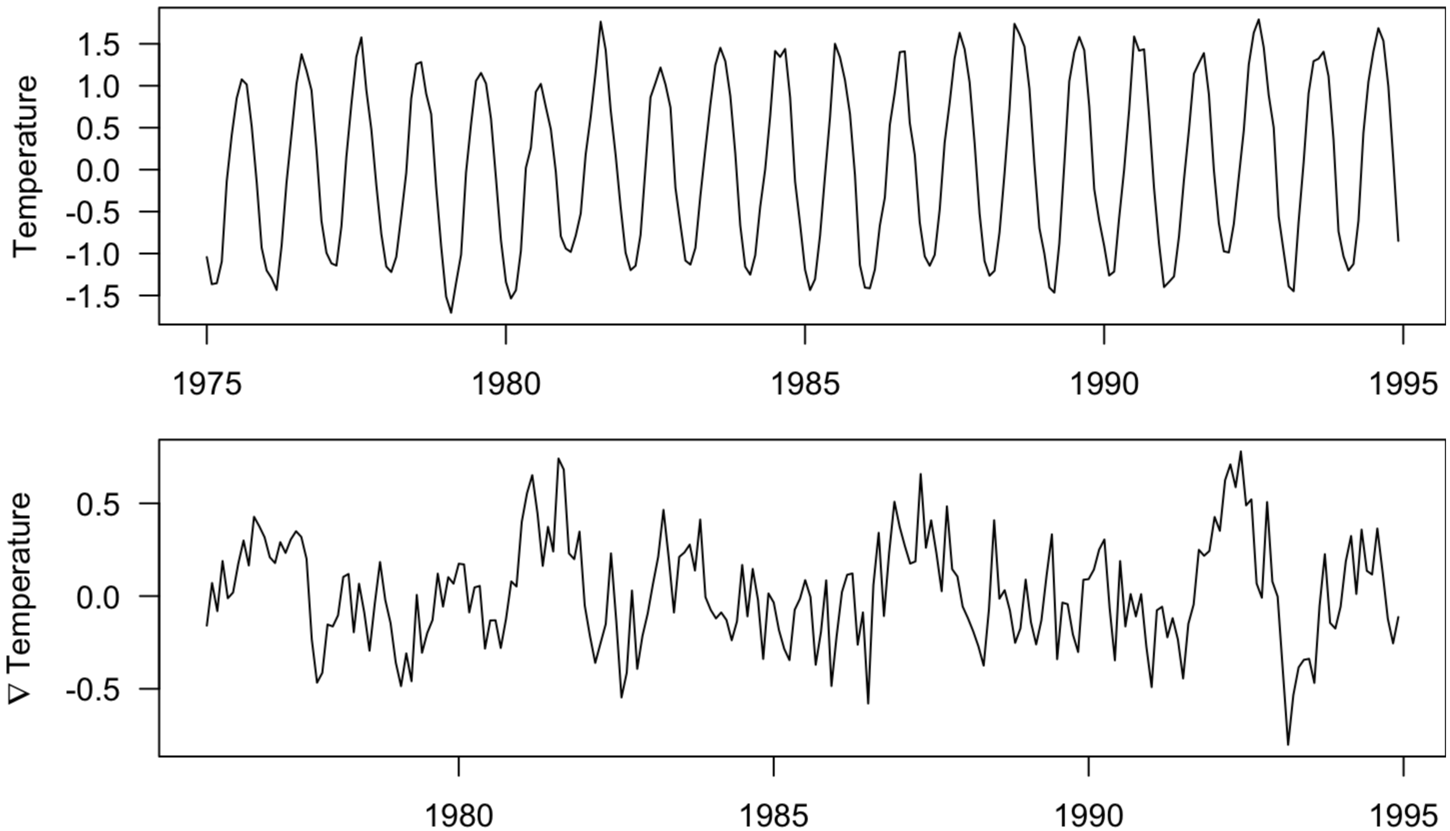


Differencing to remove seasonality

Differencing is a simple means for removing a seasonal effect

Using a 1st-difference with $k = period$ removes both trend & seasonal effects

Differencing to remove seasonality



Differencing to remove a trend in R

We can use `diff()` to easily compute differences

```
diff(x,
     lag,
     differences
     )
```

Differencing to remove a trend in R

```
diff(x,
     lag,
     differences
     )
```

`lag` (h) specifies $t - h$

`lag = 1` (default) is for non-seasonal data

`lag = 4` would work for quarterly data or

`lag = 12` for monthly data

Differencing to remove a trend in R

```
diff(x,
     lag,
     differences
     )
```

`differences` is the number of differencing operations

`differences = 1` (default) is for a linear trend

`differences = 2` is for a quadratic trend

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