

# Dynamic Factor Analysis

FISH 550 – Applied Time Series Analysis

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# Topics for today

Deterministic vs stochastic elements

Regression with autocorrelated errors

Regression with temporal random effects

Dynamic Factor Analysis (DFA)

- Forms of covariance matrix
- Constraints for model fitting
- Interpretation of results

# Code for today

You can find the R code for these lecture notes and other related exercises [here](#).

# A very simple model

Consider this simple model, consisting of a mean  $\mu$  plus error

$$y_i = \mu + e_i \text{ with } e_i \sim N(0, \sigma^2)$$

# A very simple model

The right-hand side of the equation is composed of *deterministic* and *stochastic* pieces

$$y_i = \underbrace{\mu}_{\text{deterministic}} + \underbrace{e_i}_{\text{stochastic}}$$

# A very simple model

Sometime these pieces are referred to as *fixed* and *random*

$$y_i = \underbrace{\mu}_{\text{fixed}} + \underbrace{e_i}_{\text{random}}$$

# A very simple model

This can also be seen by rewriting the model

$$y_i = \mu + e_i \text{ with } e_i \sim N(0, \sigma^2)$$

as

$$y_i \sim N(\mu, \sigma^2)$$

# Simple linear regression

We can expand the deterministic part of the model, as with linear regression

$$y_i = \underbrace{\alpha + \beta x_i}_{\text{mean}} + e_i \text{ with } e_i \sim \text{N}(0, \sigma^2)$$

so

$$y_i \sim \text{N}(\alpha + \beta x_i, \sigma^2)$$



# A simple time series model

Consider a simple model with a mean  $\mu$  plus white noise

$$y_t = \mu + e_t \text{ with } e_t \sim N(0, \sigma^2)$$

# Time series model with covariates

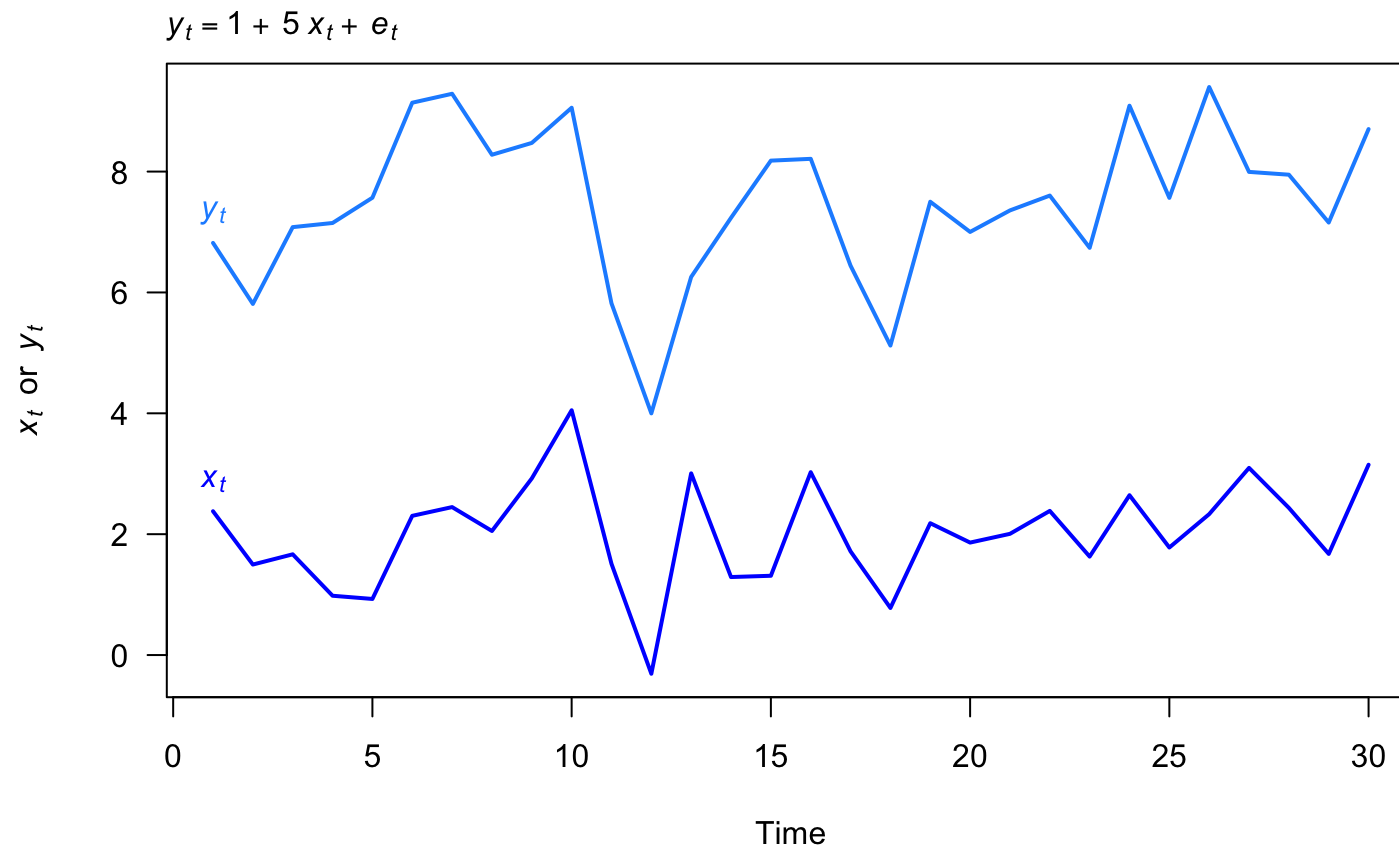
We can expand the deterministic part of the model, as before with linear regression

$$y_t = \underbrace{\alpha + \beta x_t}_{\text{mean}} + e_t \text{ with } e_t \sim N(0, \sigma^2)$$

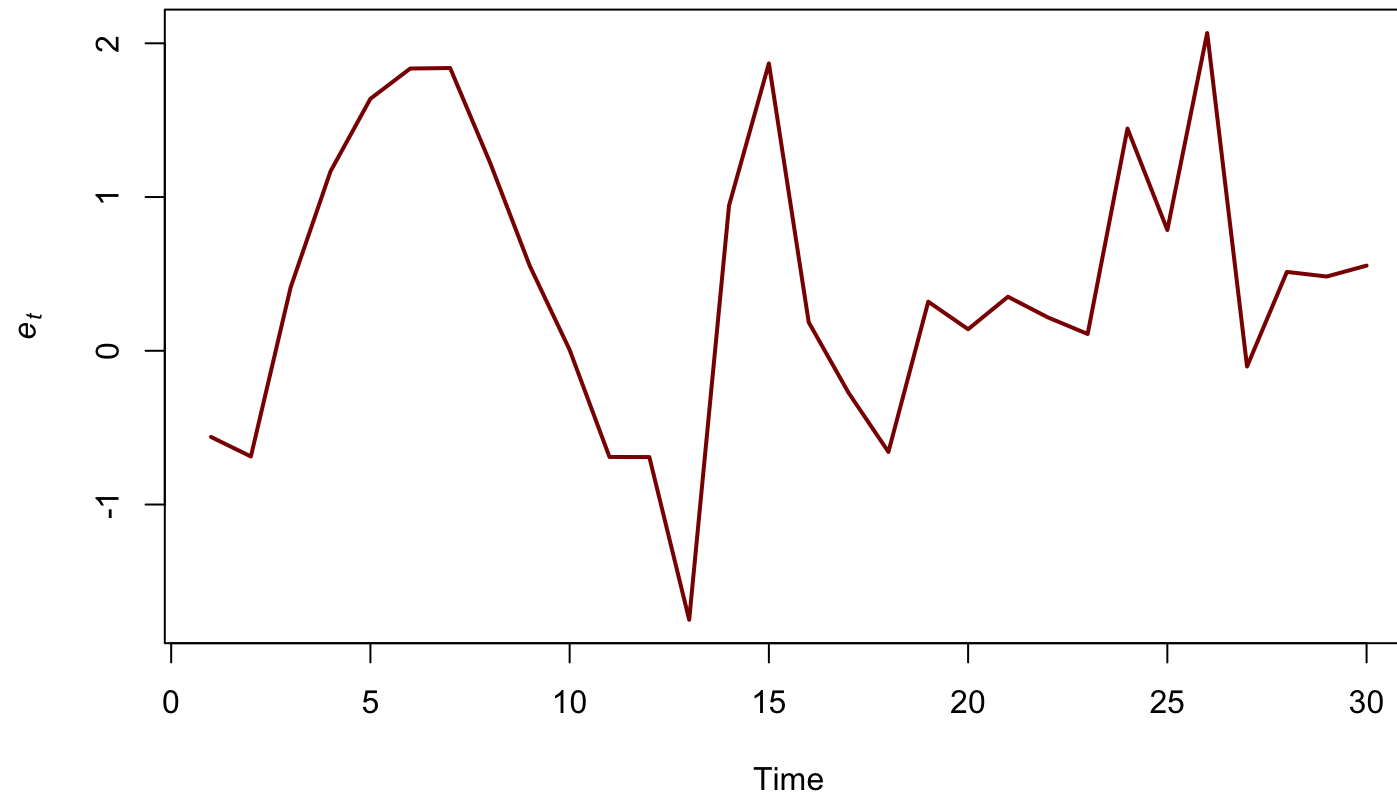
so

$$y_t \sim N(\alpha + \beta x_t, \sigma^2)$$

# Example of linear model

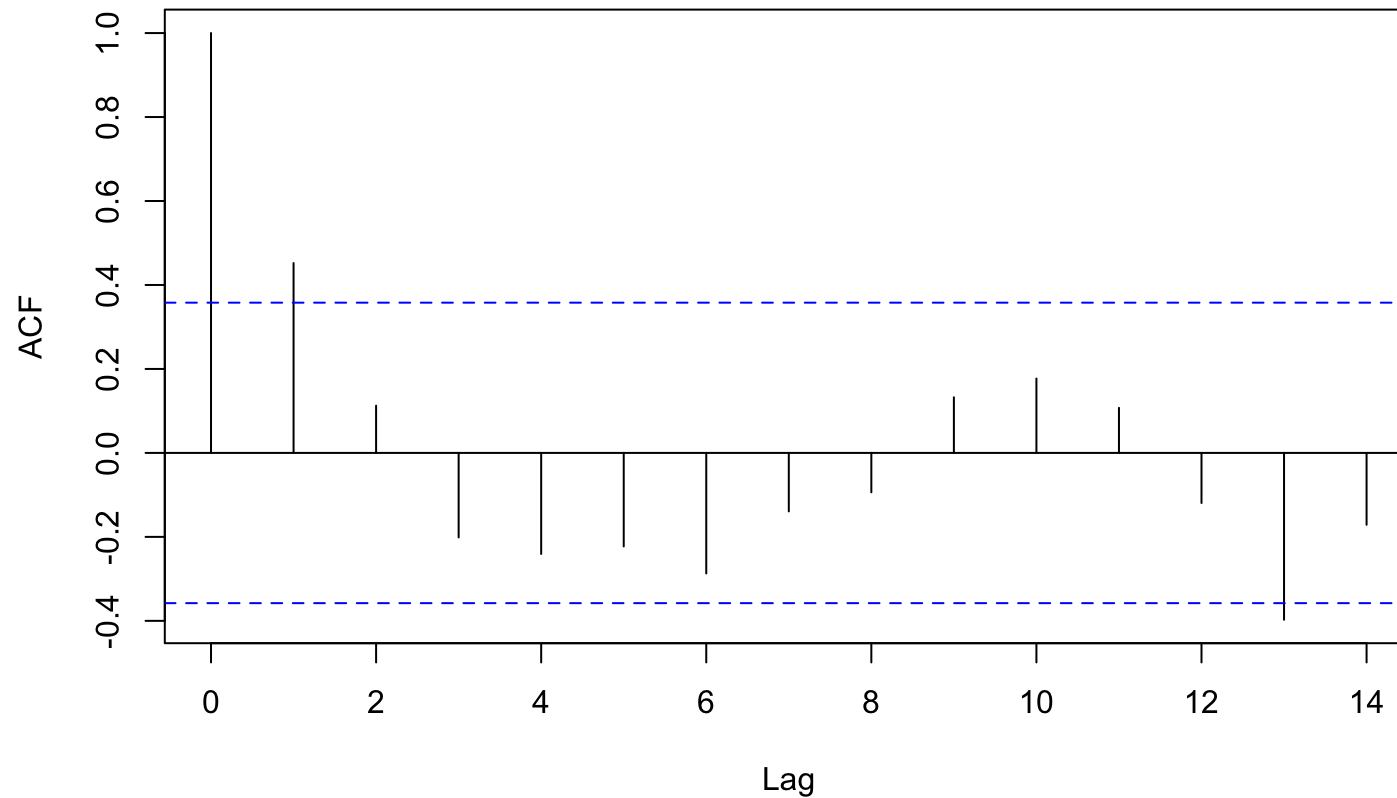


# Model residuals



These do *not* look like white noise!

# ACF of model residuals



There is significant autocorrelation at lag = 1

# Model with autocorrelated errors

We can expand the stochastic part of the model to have autocorrelated errors

$$y_t = \alpha + \beta x_t + e_t$$

$$e_t = \phi e_{t-1} + w_t$$

with  $w_t \sim N(0, \sigma^2)$

# Model with autocorrelated errors

We can expand the stochastic part of the model to have autocorrelated errors

$$y_t = \alpha + \beta x_t + e_t$$
$$e_t = \phi e_{t-1} + w_t$$

with  $w_t \sim \text{N}(0, \sigma^2)$

We can write this model as our standard state-space model

# State-space model

Observation equation

$$\begin{aligned}y_t &= \alpha + \beta x_t + e_t \\ &= e_t + \alpha + \beta x_t\end{aligned}$$

$\Downarrow$

$$y_t = x_t + a + Dd_t + v_t$$

with

$$x_t = e_t, a = \alpha, D = \beta, d_t = x_t, v_t = 0$$



# State-space model

State equation

$$e_t = \phi e_{t-1} + w_t$$

$\Downarrow$

$$x_t = Bx_t + w_t$$

with

$$x_t = e_t \text{ and } B = \phi$$

# State-space model

Full form

$$y_t = \alpha + \beta x_t + e_t$$

$$e_t = \phi e_{t-1} + w_t$$

$\Downarrow$

$$y_t = a + Dd_t + x_t$$

$$x_t = Bx_t + w_t$$

# State-space model

Observation model in `MARSS()`

$$y_t = a + Dd_t + x_t$$

$\Downarrow$

$$y_t = Zx_t + a + Dd_t + v_t$$

```
y = data          ## [1 x T] matrix of data
a = matrix("a")  ## intercept
D = matrix("D")   ## slope
d = covariate     ## [1 x T] matrix of measured covariate
Z = matrix(1)     ## no multiplier on x
R = matrix(0)     ## v_t ~ N(0,R); want v_t = 0 for all t
```

# State-space model

State model in MARSS ( )

$$x_t = Bx_t + w_t$$

$\Downarrow$

$$x_t = Bx_t + u + Cc_t + w_t$$

```
B = matrix("b")  ## AR(1) coefficient for model errors
Q = matrix("q")  ## w_t ~ N(0,Q); var for model errors
u = matrix(0)     ## u = 0
C = matrix(0)     ## C = 0
c = matrix(0)     ## c_t = 0 for all t
```

# MORE RANDOM EFFECTS

# Expanding the random effect

Recall our simple model

$$y_t = \underbrace{\mu}_{\text{fixed}} + \underbrace{e_t}_{\text{random}}$$

# Expanding the random effect

We can expand the random portion

$$y_t = \underbrace{\mu}_{\text{fixed}} + \underbrace{f_t + e_t}_{\text{random}}$$

$$e_t \sim N(0, \sigma)$$

$$f_t \sim N(f_{t-1}, \gamma)$$

# Expanding the random effect

We can expand the random portion

$$y_t = \underbrace{\mu}_{\text{fixed}} + \underbrace{f_t + e_t}_{\text{random}}$$

$$e_t \sim N(0, \sigma)$$

$$f_t \sim N(f_{t-1}, \gamma)$$

This is simply a random walk observed with error



# Random walk observed with error

$$y_t = \mu + f_t + e_t \text{ with } e_t \sim \text{N}(0, \sigma)$$

$$f_t = f_{t-1} + w_t \text{ with } w_t \sim \text{N}(0, \gamma)$$

$\Downarrow$

$$y_t = a + x_t + v_t \text{ with } v_t \sim \text{N}(0, R)$$

$$x_t = x_{t-1} + w_t \text{ with } w_t \sim \text{N}(0, Q)$$

# Expanding fixed & random effects

We can expand the fixed portion

$$y_t = \underbrace{\alpha + \beta x_t}_{\text{fixed}} + \underbrace{f_t + e_t}_{\text{random}}$$

$$e_t \sim N(0, \sigma)$$

$$f_t \sim N(f_{t-1}, \gamma)$$

# Fixed & random effects

In familiar state-space form

$$y_t = \alpha + \beta x_t + f_t + e_t \text{ with } e_t \sim \text{N}(0, \sigma)$$

$$f_t = f_{t-1} + w_t \text{ with } w_t \sim \text{N}(0, \gamma)$$

$\Downarrow$

$$y_t = a + Dd_t + x_t + v_t \text{ with } v_t \sim \text{N}(0, R)$$

$$x_t = x_{t-1} + w_t \text{ with } w_t \sim \text{N}(0, Q)$$

# MULTIPLE TIME SERIES

# Simple model for 2+ time series

Random walk observed with error

$$y_{i,t} = x_{i,t} + a_i + v_{i,t}$$

$$x_{i,t} = x_{i,t-1} + w_{i,t}$$

with

$$v_{i,t} \sim \text{N}(0, R)$$

$$w_{i,t} \sim \text{N}(0, Q)$$

# Random walk observed with error

$$y_{1,t} = x_{1,t} + a_1 + v_{1,t}$$

$$y_{2,t} = x_{2,t} + a_2 + v_{2,t}$$

$$\vdots$$

$$y_{n,t} = x_{n,t} + a_2 + v_{n,t}$$

$$x_{1,t} = x_{1,t-1} + w_{1,t}$$

$$x_{2,t} = x_{2,t-1} + w_{2,t}$$

$$\vdots$$

$$x_{n,t} = x_{n,t-1} + w_{n,t}$$

# Random walk observed with error

In matrix form

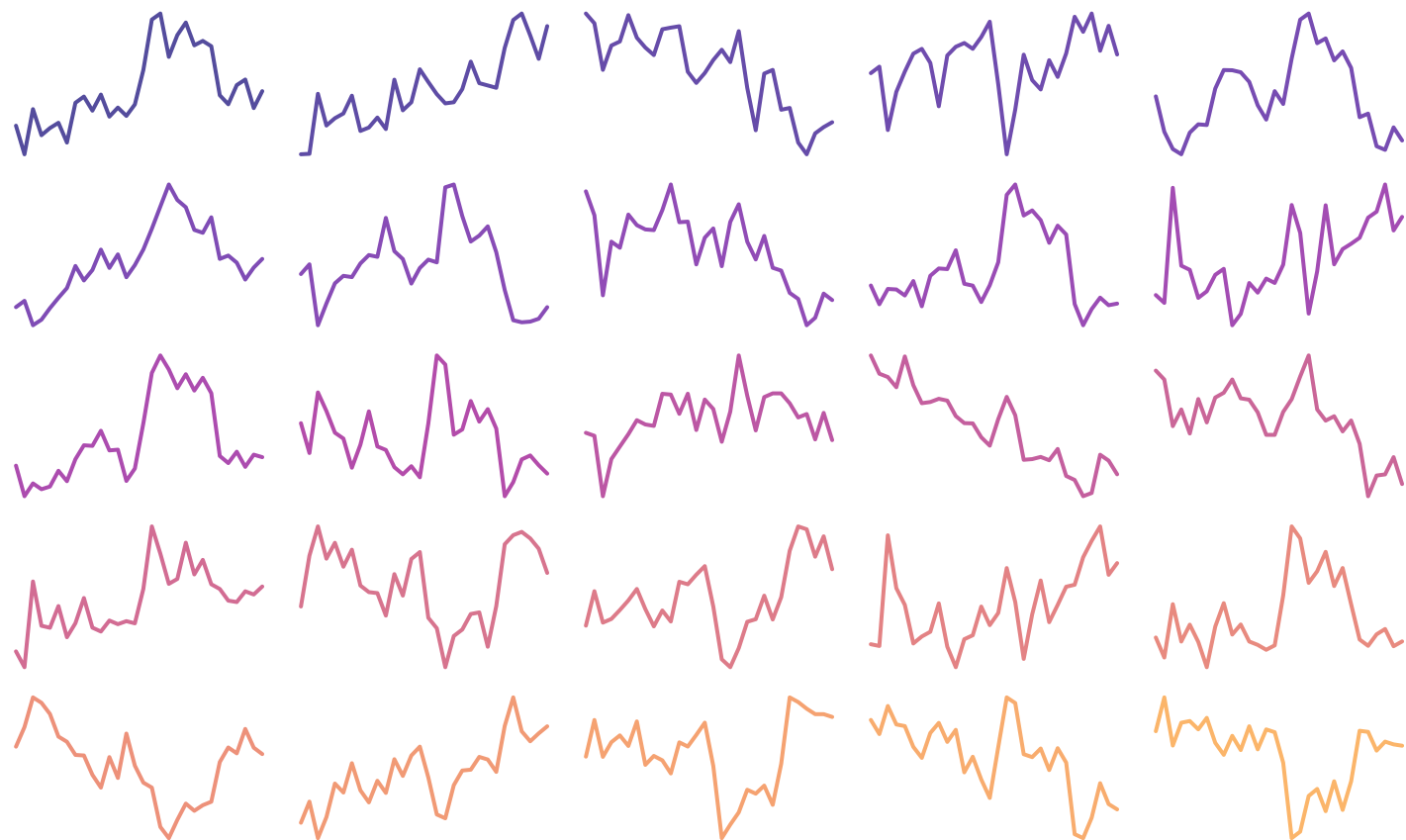
$$\mathbf{y}_t = \mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

with

$$\mathbf{v}_t \sim \text{MVN}(\mathbf{0}, \mathbf{R})$$

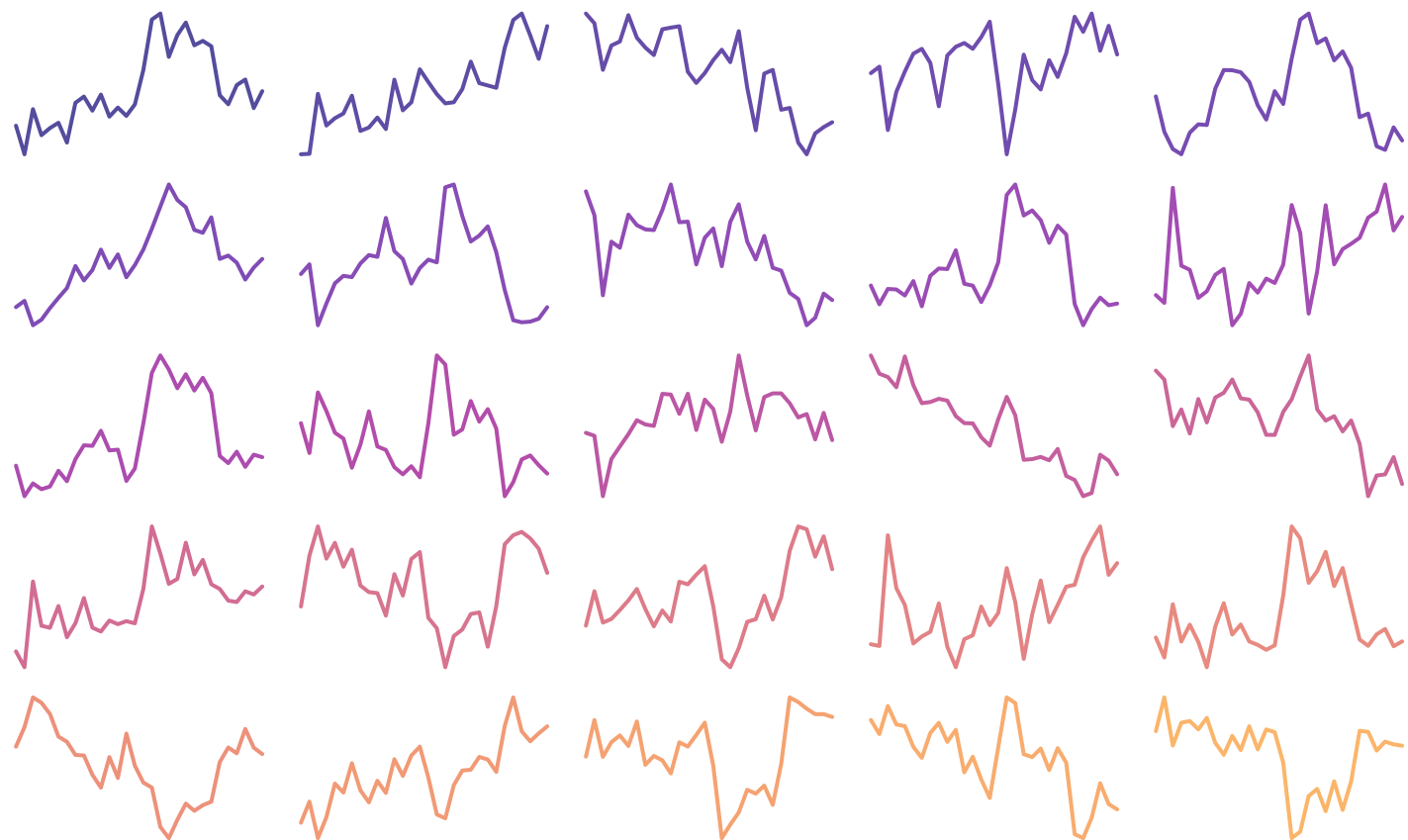
$$\mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$$





# Environmental time series

We often observe covariance among environmental time series, especially for those collected close to one another in space



Are there some common patterns here?

# Common patterns in time series



# State-space model

Ex: population structure

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

We can make (test) assumptions by specifying different forms for  $\mathbf{Z}$

# State-space model

Ex: Harbor seal population structure

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_{JF} \\ x_N \\ x_S \end{bmatrix}_t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}_t$$

$$\begin{bmatrix} x_{JF} \\ x_N \\ x_S \end{bmatrix}_t = \begin{bmatrix} x_{JF} \\ x_N \\ x_S \end{bmatrix}_{t-1} + \begin{bmatrix} w_{JF} \\ w_N \\ w_S \end{bmatrix}_t$$

# Finding common patterns

What if our observations were instead a mixture of 2+ states?

For example, we sampled haul-outs located between several breeding sites

# Mixtures of states

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}_t = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0.9 & 0.1 \\ 0 & 0.3 & 0.7 \\ 0 & 0.1 & 0.9 \end{bmatrix} \times \begin{bmatrix} x_{JF} \\ x_N \\ x_S \end{bmatrix}_t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}_t$$

$$\begin{bmatrix} x_{JF} \\ x_N \\ x_S \end{bmatrix}_t = \begin{bmatrix} x_{JF} \\ x_N \\ x_S \end{bmatrix}_{t-1} + \begin{bmatrix} w_{JF} \\ w_N \\ w_S \end{bmatrix}_t$$

# Finding common patterns

What if our observations were a mixture of states, but we didn't know how many or the weightings?

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$
$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

What are the dimensions of  $\mathbf{Z}$ ?

What are the elements within  $\mathbf{Z}$ ?



# Dynamic Factor Analysis (DFA)

DFA is a *dimension reduction* technique, which models  $n$  observed time series as a function of  $m$  hidden states (patterns), where  $n \gg m$

# Dynamic Factor Analysis (DFA)

State-space form

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

data:  $\mathbf{y}_t$  is  $n \times 1$

loadings:  $\mathbf{Z}$  is  $n \times m$  with  $n > m$

states:  $\mathbf{x}_t$  is  $m \times 1$

# Dimension reduction

## Principal Components Analysis (PCA)

Goal is to reduce some large number of correlated variates into a few uncorrelated factors

# Principal Components Analysis (PCA)

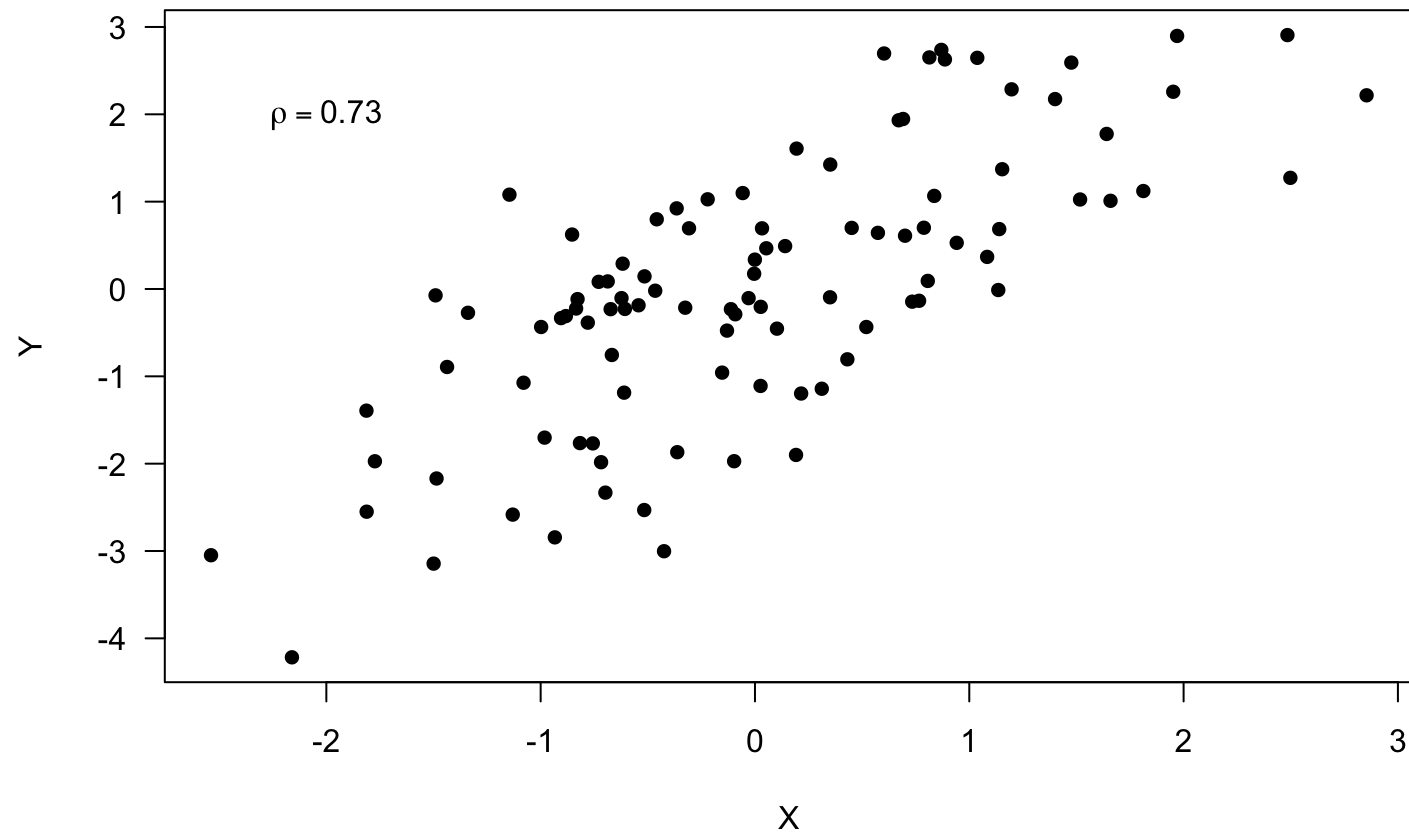
Calculating the principal components requires us to estimate the covariance of the data

$$\text{PC} = \text{eigenvectors}(\text{cov}(\mathbf{y}))$$

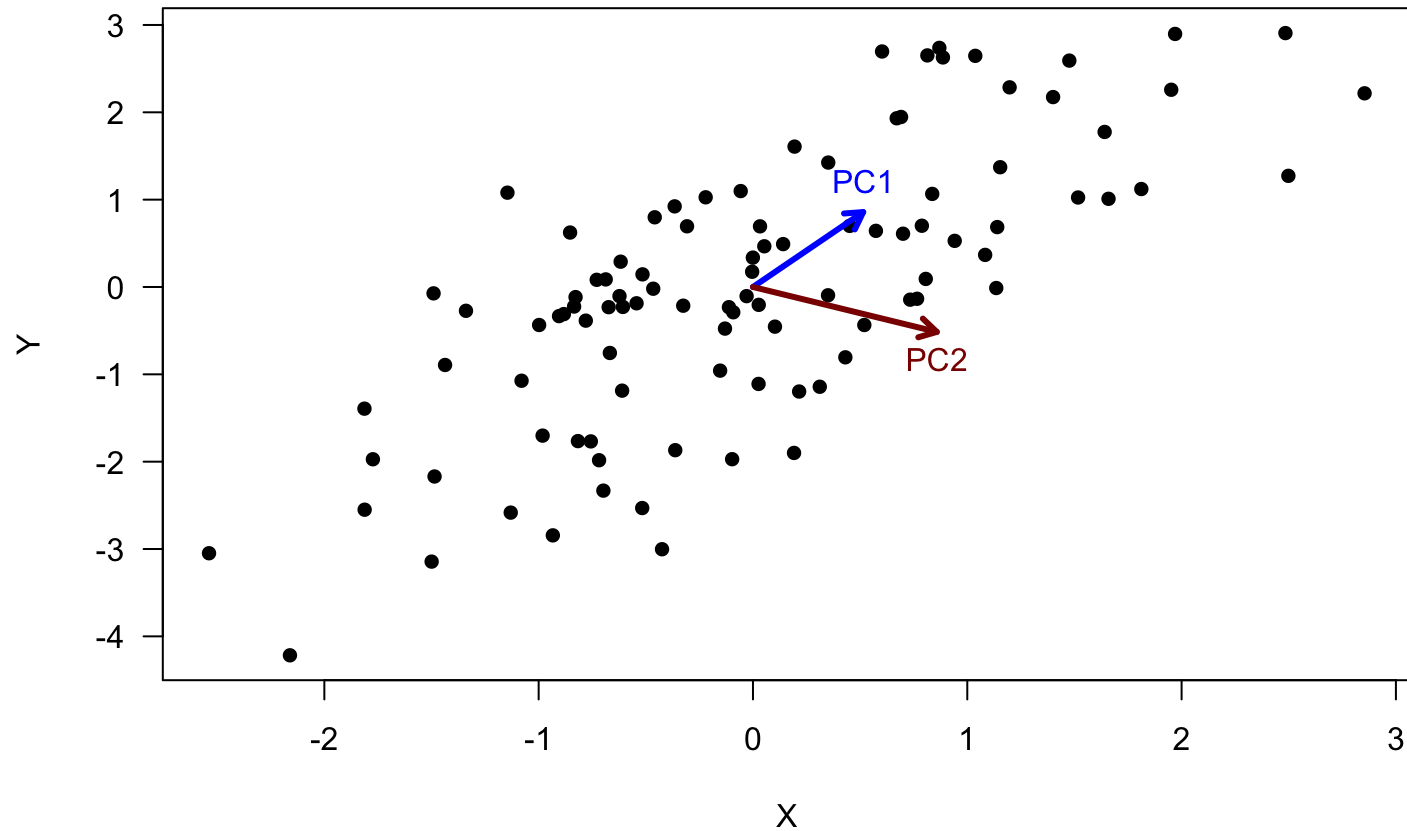
There will be  $n$  principal components (eigenvectors) for an  $n \times T$  matrix  $\mathbf{y}$

We reduce the dimension by selecting a subset of the components that explain much of the variance (eg, the first 2)

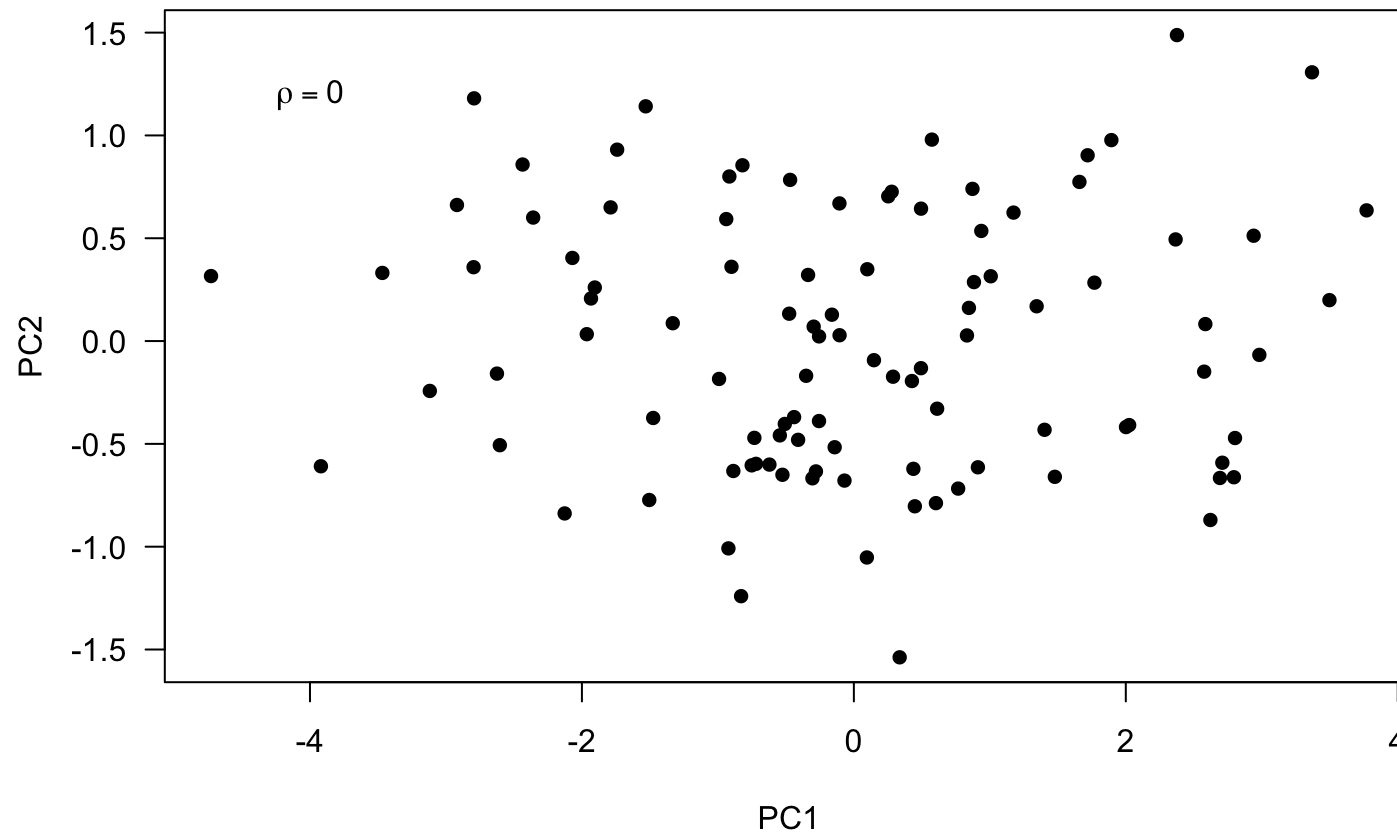
# Principal Components Analysis (PCA)



# Principal Components Analysis (PCA)



# Principal Components Analysis (PCA)



# Relationship between PCA & DFA

We need to estimate the covariance in the data  $\mathbf{y}$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t, \text{ with } \mathbf{v}_t \sim \text{MVN}(\mathbf{0}, \mathbf{R})$$

so

$$\text{cov}(\mathbf{y}_t) = \mathbf{Z}\text{cov}(\mathbf{x}_t)\mathbf{Z}^\top + \mathbf{R}$$

In PCA, we require  $\mathbf{R}$  to be diagonal, but not so in DFA



# Principal Components Analysis (PCA)

Forms for  $\mathbf{R}$  with  $n = 4$

$$\mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix} \text{ or } \mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \end{bmatrix}$$

# Dynamic Factor Analysis (DFA)

Forms for  $\mathbf{R}$  with  $n = 4$

$$\mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix} \text{ or } \mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \end{bmatrix}$$

$$\mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma & \gamma & \gamma & \gamma \\ \gamma & \sigma & \gamma & \gamma \\ \gamma & \gamma & \sigma & \gamma \\ \gamma & \gamma & \gamma & \sigma \end{bmatrix} \text{ or } \mathbf{R} \stackrel{?}{=} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & \gamma_{2,4} \\ 0 & 0 & \sigma_3 & 0 \\ 0 & \gamma_{2,4} & 0 & \sigma_4 \end{bmatrix}$$

# Dynamic Factor Analysis (DFA)

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

What form should we use for  $\mathbf{Z}$ ?

$$\mathbf{Z} \stackrel{?}{=} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \quad \text{or} \quad \mathbf{Z} \stackrel{?}{=} \begin{bmatrix} z_{1,1} & z_{2,1} \\ z_{1,2} & z_{2,2} \\ z_{1,3} & z_{2,3} \\ z_{1,4} & z_{2,4} \\ z_{1,5} & z_{2,5} \end{bmatrix} \quad \text{or} \quad \mathbf{Z} \stackrel{?}{=} \begin{bmatrix} z_{1,1} & z_{2,1} & z_{3,1} \\ z_{1,2} & z_{2,2} & z_{3,2} \\ z_{1,3} & z_{2,3} & z_{3,3} \\ z_{1,4} & z_{2,4} & z_{3,4} \\ z_{1,5} & z_{2,5} & z_{3,5} \end{bmatrix}$$

# Dynamic Factor Analysis (DFA)

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$
$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

What form should we use for  $\mathbf{Z}$ ?

$$\mathbf{Z} \stackrel{?}{=} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_5 \end{bmatrix} \quad \text{or} \quad \mathbf{Z} \stackrel{?}{=} \begin{bmatrix} z_{1,1} & z_{2,1} \\ z_{1,2} & z_{2,2} \\ z_{1,3} & z_{2,3} \\ \vdots & \vdots \\ z_{1,n} & z_{2,n} \end{bmatrix} \quad \text{or} \quad \mathbf{Z} \stackrel{?}{=} \begin{bmatrix} z_{1,1} & z_{2,1} & z_{3,1} \\ z_{1,2} & z_{2,2} & z_{3,2} \\ z_{1,3} & z_{2,3} & z_{3,3} \\ \vdots & \vdots & \vdots \\ z_{1,n} & z_{2,n} & z_{3,n} \end{bmatrix}$$

We'll use model selection criteria to choose (eg, AICc)

# Fitting DFA models

Unless  $\mathbf{Z}$  is unconstrained in some manner, there are an infinite number of combinations of  $\mathbf{Z}$  and  $\mathbf{x}$  that will equal  $\mathbf{y}$

Therefore we need to impose some constraints on the model

# Constraints on DFA models

## 1) The offset $\mathbf{a}$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

# Constraints on DFA models

## 1) The offset $\mathbf{a}$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

We will set the first  $m$  elements of  $\mathbf{a}$  to 0

# Constraints on DFA models

## 1) The offset **a**

For example, if  $n = 5$  and  $m = 2$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \Rightarrow \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$



# Constraints on DFA models

## 1) The offset $\mathbf{a}$

For example, if  $n = 5$  and  $m = 2$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \Rightarrow \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \Rightarrow \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note, however, that this causes problems for the EM algorithm so we will often de-mean the data and set  $a_i = 0$  for all  $i$

# Constraints on DFA models

## 2) The loadings $\mathbf{Z}$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

$$\mathbf{Z} = \begin{bmatrix} z_{1,1} & z_{2,1} & \dots & z_{m,1} \\ z_{1,2} & z_{2,2} & \dots & z_{m,2} \\ z_{1,3} & z_{2,3} & \dots & z_{m,3} \\ \vdots & \vdots & \ddots & z_{m,4} \\ z_{1,n} & z_{2,n} & \dots & z_{m,n} \end{bmatrix}$$

# Constraints on DFA models

## 2) The loadings $\mathbf{Z}$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

$$\mathbf{Z} = \begin{bmatrix} z_{1,1} & z_{2,1} & \dots & z_{m,1} \\ z_{1,2} & z_{2,2} & \dots & z_{m,2} \\ z_{1,3} & z_{2,3} & \dots & z_{m,3} \\ \vdots & \vdots & \ddots & z_{m,4} \\ z_{1,n} & z_{2,n} & \dots & z_{m,n} \end{bmatrix}$$

We will set the upper right triangle of  $\mathbf{Z}$  to 0

# Constraints on DFA models

## 2) The loadings $\mathbf{Z}$

For example, if  $n = 5$  and  $m = 3$

$$\mathbf{Z} = \begin{bmatrix} z_{1,1} & 0 & 0 \\ z_{1,2} & z_{2,2} & 0 \\ z_{1,3} & z_{2,3} & z_{3,3} \\ z_{1,4} & z_{2,3} & z_{3,4} \\ z_{1,5} & z_{2,5} & z_{3,5} \end{bmatrix}$$

For the first  $m - 1$  rows of  $\mathbf{Z}$ ,  $z_{i,j} = 0$  if  $j > i$

# Constraints on DFA models

## 2) The loadings $\mathbf{Z}$

An additional constraint is necessary in a Bayesian context

$$\mathbf{Z} = \begin{bmatrix} \underline{z_{1,1}} & 0 & 0 \\ z_{1,2} & \underline{z_{2,2}} & 0 \\ z_{1,3} & z_{2,3} & \underline{z_{3,3}} \\ z_{1,4} & z_{2,3} & z_{3,4} \\ z_{1,5} & z_{2,5} & z_{3,5} \end{bmatrix}$$

Diagonal of  $\mathbf{Z}$  is positive:  $z_{i,j} > 0$  if  $i = j$

# Constraints on DFA models

## 3) The state variance $\mathbf{Q}$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

$$\mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$$

# Constraints on DFA models

## 3) The state variance $\mathbf{Q}$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

$$\mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$$

We will set  $\mathbf{Q}$  equal to the Identity matrix  $\mathbf{I}$

# Constraints on DFA models

## 3) The state variance $\mathbf{Q}$

For example, if  $m = 4$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This allows our random walks to have a *lot* of flexibility



# Dynamic Factor Analysis (DFA)

Including  $p$  covariates

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \underline{\mathbf{D}\mathbf{d}_t} + \mathbf{v}_t$$
$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

$\mathbf{d}_t$  is a  $p \times 1$  vector of covariates at time  $t$

$\mathbf{D}$  is an  $n \times p$  matrix of covariate effects

# Dynamic Factor Analysis (DFA)

Form for **D**

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \underline{\mathbf{D}}\mathbf{d}_t + \mathbf{v}_t$$
$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

Careful thought must be given *a priori* as to the form for **D**

Should the effect(s) vary by site, species, etc?

# Dynamic Factor Analysis (DFA)

Form for **D**

For example, given 2 covariates, Temp and Salinity

$$\mathbf{D} = \underbrace{\begin{bmatrix} d_{\text{Temp}} & d_{\text{Salinity}} \\ d_{\text{Temp}} & d_{\text{Salinity}} \\ \vdots & \vdots \\ d_{\text{Temp}} & d_{\text{Salinity}} \end{bmatrix}}_{\text{effects same by site/species}} \quad \text{or} \quad \mathbf{D} = \underbrace{\begin{bmatrix} d_{\text{Temp},1} & d_{\text{Salinity},1} \\ d_{\text{Temp},2} & d_{\text{Salinity},2} \\ \vdots & \vdots \\ d_{\text{Temp},n} & d_{\text{Salinity},n} \end{bmatrix}}_{\text{effects differ by site/species}}$$

# A note on model selection

Earlier we saw that we could use model selection criteria to help us choose among the different forms for  $\mathbf{Z}$

However, caution must be given when comparing models with and without covariates, and varying numbers of states

# A note on model selection

Think about the DFA model form

$$\mathbf{y}_t = \mathbf{Z}\underline{\mathbf{x}_t} + \mathbf{a} + \mathbf{D}\underline{\mathbf{d}_t} + \mathbf{v}_t$$

$\mathbf{x}_t$  are *undetermined* random walks

$\mathbf{d}_t$  are *predetermined* covariates

# An example with 3 times series

Model 1 has 2 trends and no covariates

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_t = \begin{bmatrix} z_{1,1} & z_{2,1} \\ z_{1,2} & z_{2,2} \\ z_{1,3} & z_{2,3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_t$$

Model 2 has 1 trend and 1 covariate

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_t = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} [x]_t + \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} [d]_t + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_t$$

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Unless  $\mathbf{d}$  is *highly correlated* with  $\mathbf{y}$ , Model 1 will be favored

# A note on model selection

For models with covariates

- fit the *most complex model you can envision* based on all of your possible covariates and random factors (states)
- keep the covariates fixed and choose the number of trends (states) using AICc
- keep the covariates & states fixed and choose the form for **R**
- sort out the covariates while keeping the states & **R** fixed



# Interpreting DFA results

Recall that we had to constrain the form of  $\mathbf{Z}$  to fit the model

$$\mathbf{Z} = \begin{bmatrix} z_{1,1} & 0 & \dots & 0 \\ z_{1,2} & z_{2,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & z_{m,m} \\ \vdots & \vdots & \vdots & \vdots \\ z_{1,n} & z_{2,n} & z_{3,n} & z_{m,n} \end{bmatrix}$$

So, the 1st common factor is determined by the 1st variate, the 2nd common factor by the first two variates, etc.

# Interpreting DFA results

To help with this, we can use a *basis rotation* to maximize the loadings on a few factors

If  $\mathbf{H}$  is an  $m \times m$  non-singular matrix, these 2 DFA models are equivalent

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{D}\mathbf{d}_t + \mathbf{v}_t$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{H}^{-1}\mathbf{x}_t + \mathbf{a} + \mathbf{D}\mathbf{d}_t + \mathbf{v}_t$$

$$\mathbf{H}\mathbf{x}_t = \mathbf{H}\mathbf{x}_{t-1} + \mathbf{H}\mathbf{w}_t$$

How should we choose  $\mathbf{H}$ ?

# Basis rotation

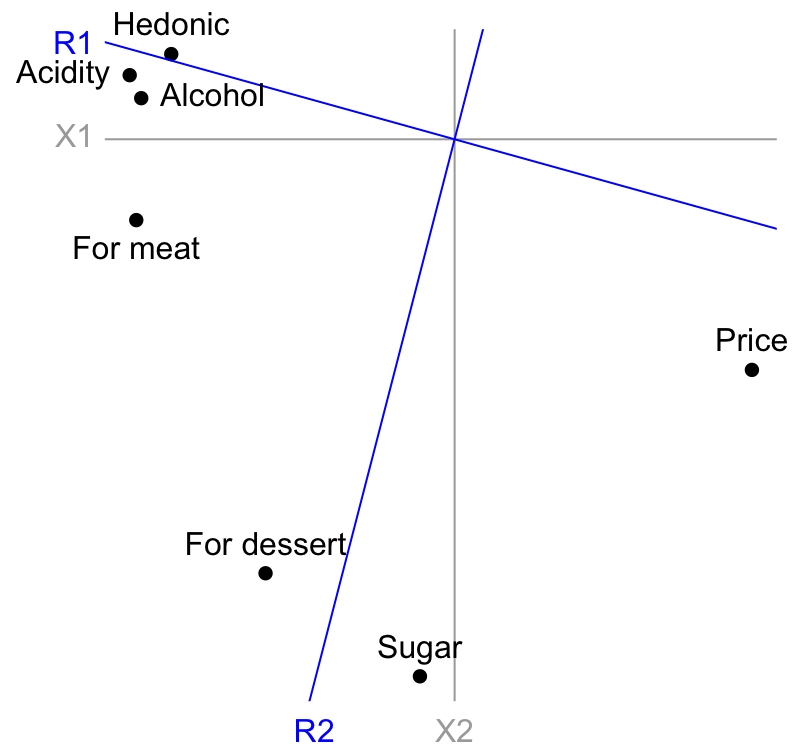
## Varimax

A *varimax* rotation will maximize the variance of the loadings in **Z** along a few of the factors

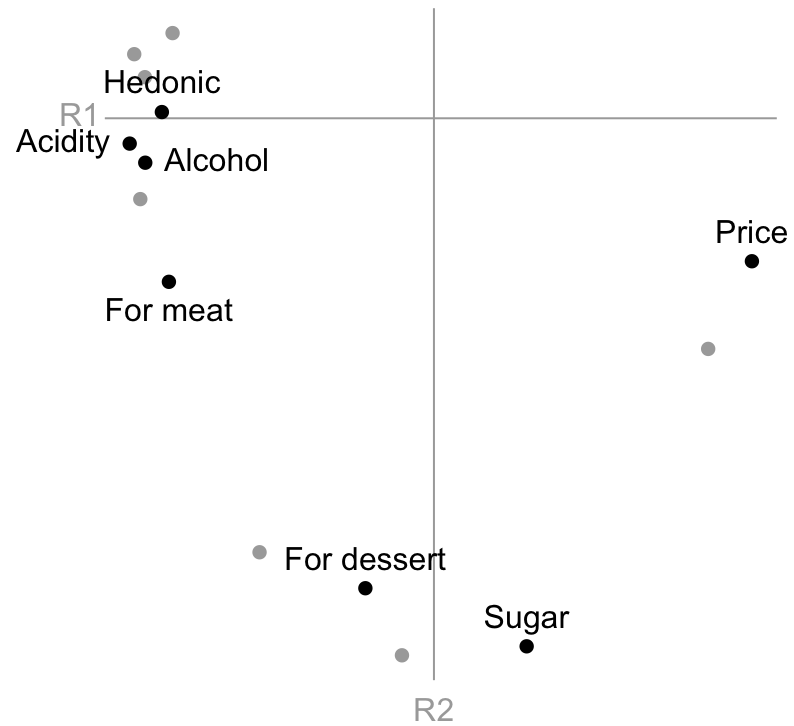
# PCA of 5 wines with 8 attributes



# Rotated loadings



# Rotated loadings



# Topics for today

Deterministic vs stochastic elements

Regression with autocorrelated errors

Regression with temporal random effects

Dynamic Factor Analysis (DFA)

- Forms of covariance matrix
- Constraints for model fitting
- Interpretation of results