Intro to ARMA models

FISH 550 – Applied Time Series Analysis

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Topics for today

Review

- White noise
- Random walks

Autoregressive (AR) models

Moving average (MA) models

Autoregressive moving average (ARMA) models

Using ACF & PACF for model ID

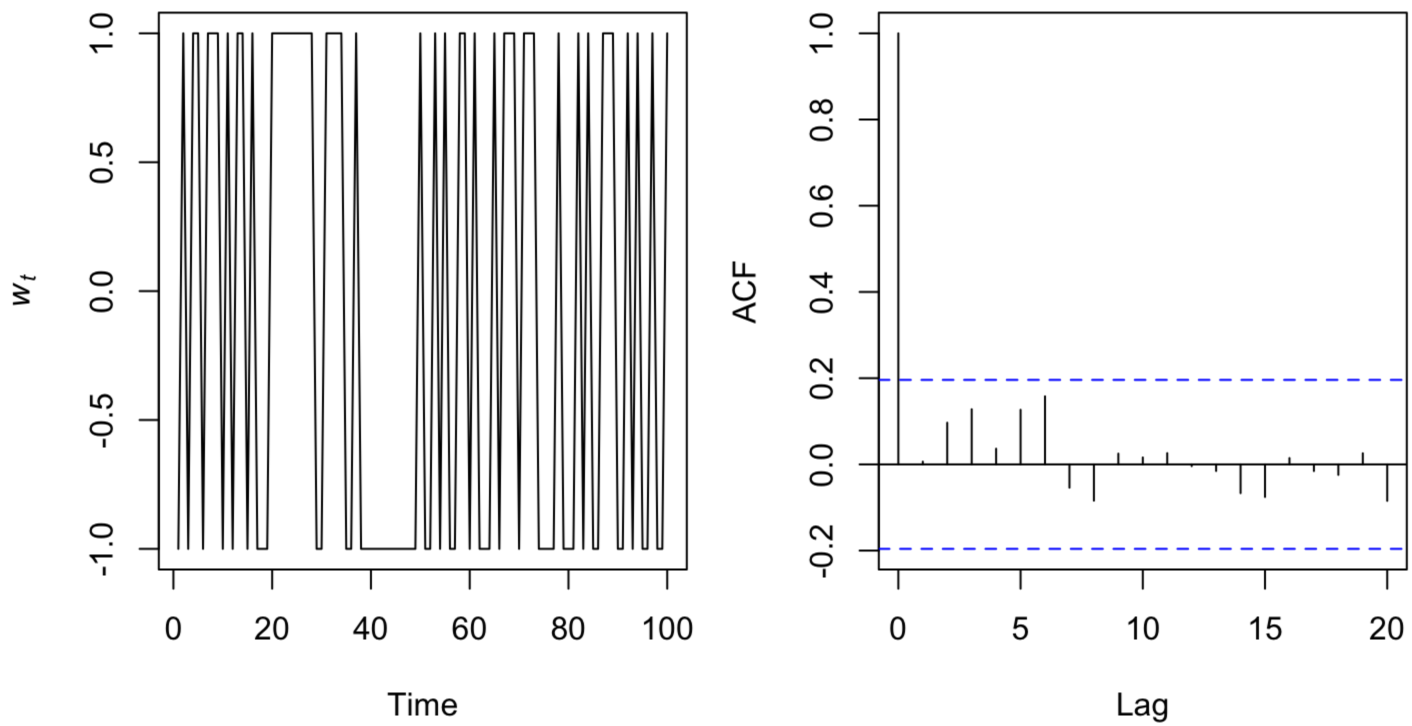
White noise (WN)

A time series $\{w_t\}$ is discrete white noise if its values are

1. independent
2. identically distributed with a mean of zero

The distributional form for $\{w_t\}$ is flexible

White noise (WN)



$$w_t = 2e_t - 1; e_t \sim \text{Bernoulli}(0.5)$$

Gaussian white noise

We often assume so-called *Gaussian white noise*, whereby

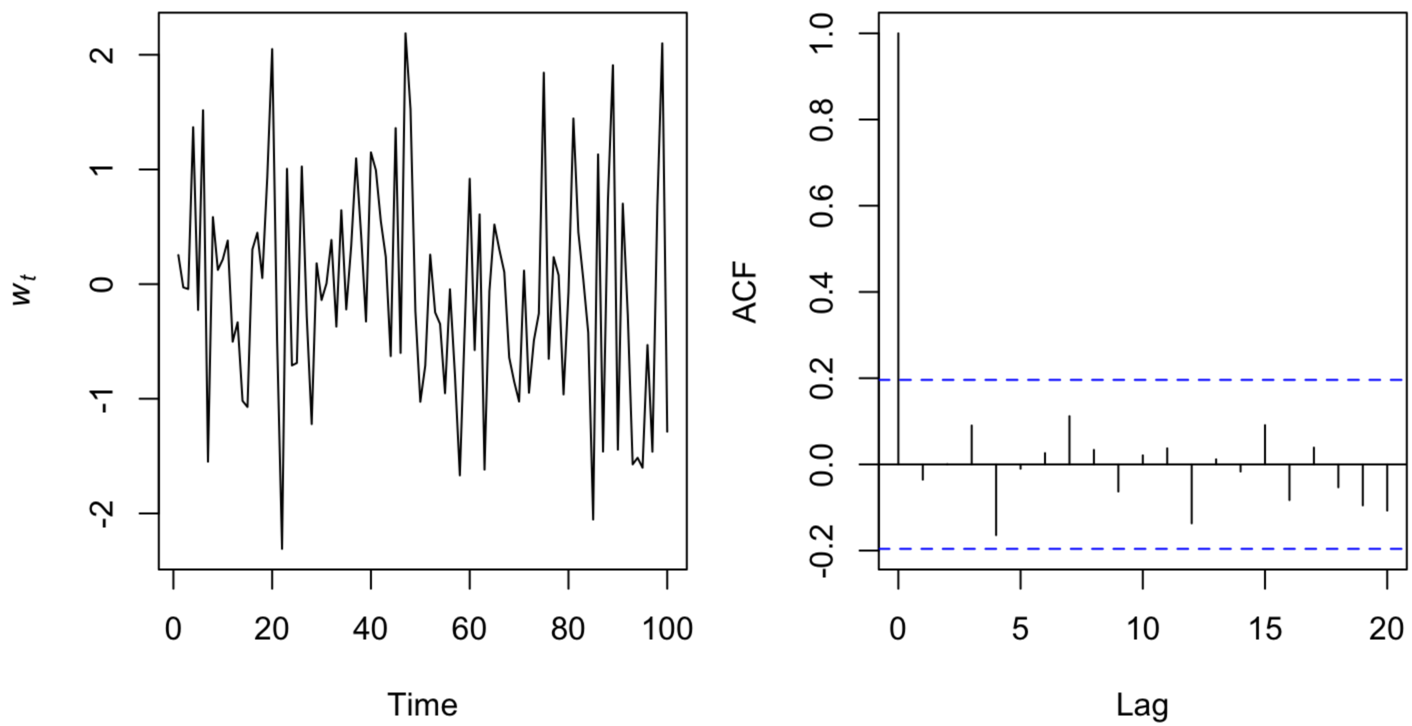
$$w_t \sim \mathrm{N}(0, \sigma^2)$$

and the following apply as well

autocovariance: $\gamma_k = \begin{cases} \sigma^2 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$

autocorrelation: $\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$

Gaussian white noise



$$w_t \sim \mathrm{N}(0, 1)$$

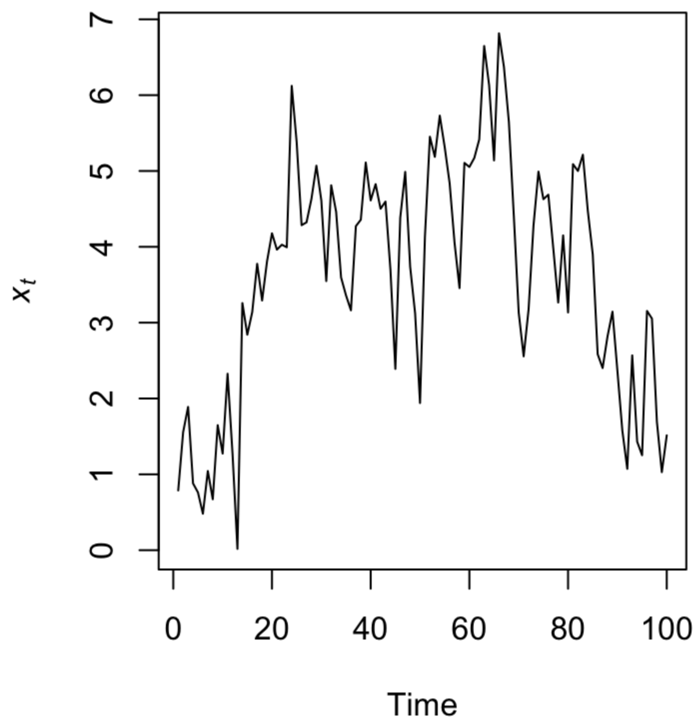
Random walk (RW)

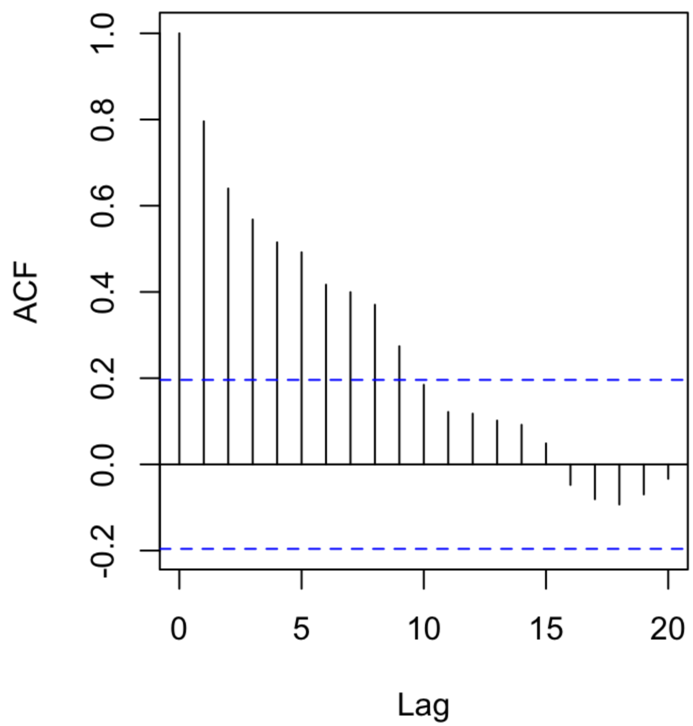
A time series $\{x_t\}$ is a random walk if

1. $x_t = x_{t-1} + w_t$

2. w_t is white noise

Random walk (RW)





$$x_t = x_{t-1} + w_t; w_t \sim \mathrm{N}(0, 1)$$

Random walk (RW)

Of note: Random walks are extremely flexible models and can be fit to many kinds of time series

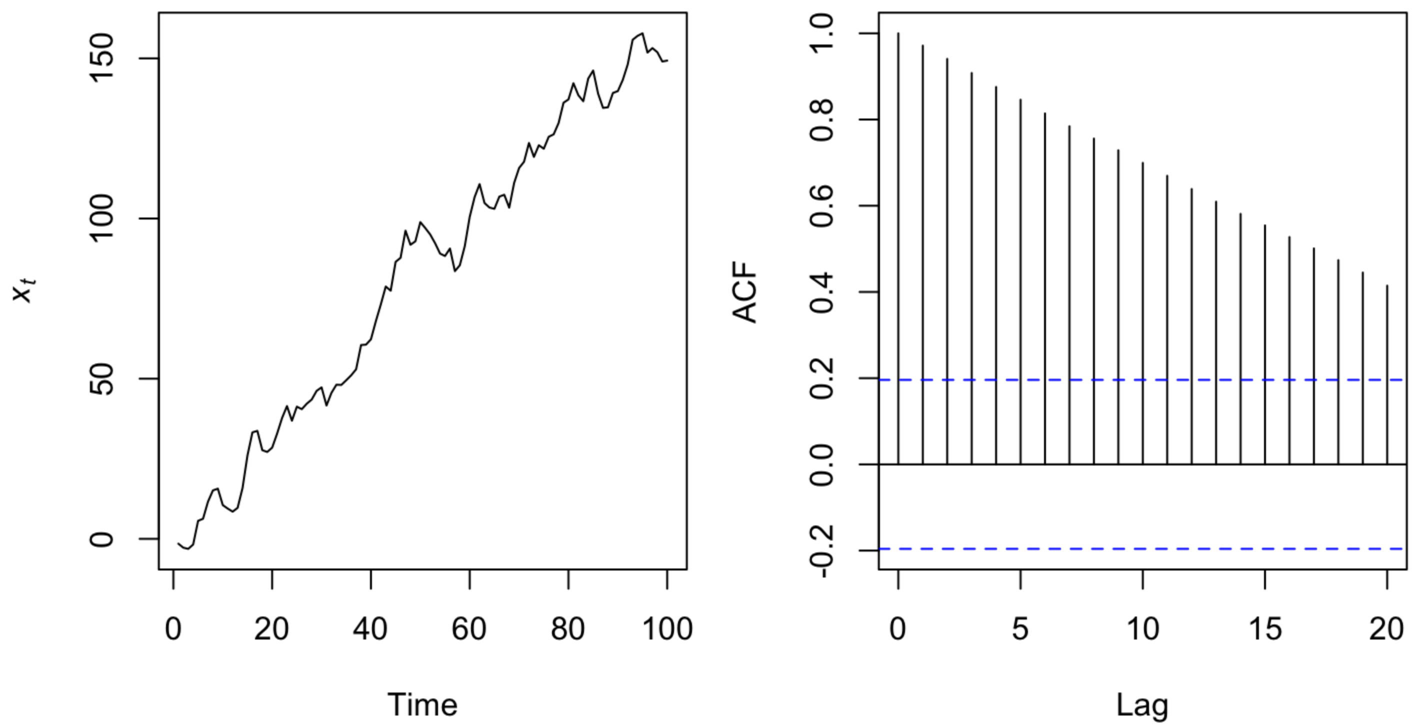
Biased random walk

A *biased random walk* (or *random walk with drift*) is written as

$$x_t = x_{t-1} + u + w_t$$

where u is the bias (drift) per time step and w_t is white noise

Biased random walk



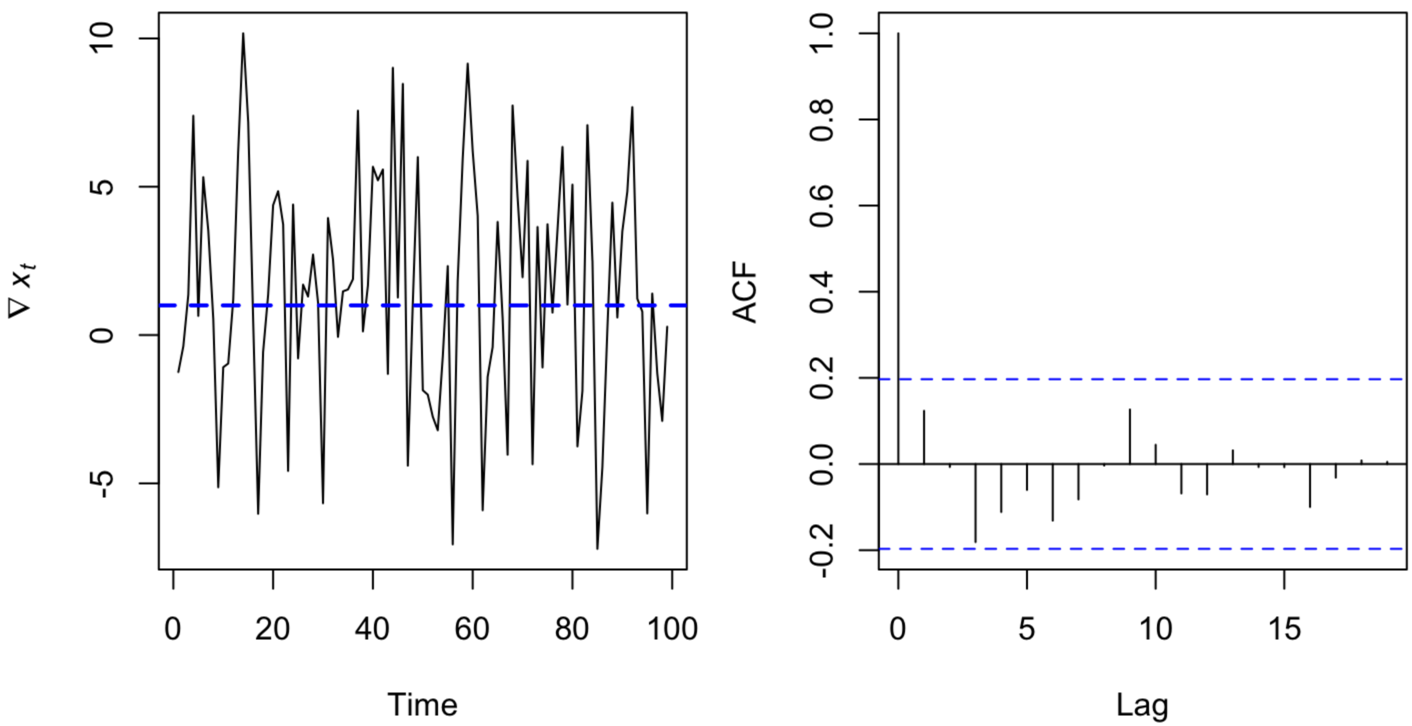
$$x_t = x_{t-1} + 1 + w_t; w_t \sim \mathrm{N}(0, 4)$$

Differencing a biased random walk

First-differencing a biased random walk yields a constant mean (level) u plus white noise

$$
\begin{aligned}
x_t &= x_{t-1} + u + w_t \\
&\Downarrow \\
\nabla(x_t &= x_{t-1} + u + w_t) \\
x_t - x_{t-1} &= x_{t-1} + u + w_t - x_{t-1} \\
x_t - x_{t-1} &= u + w_t
\end{aligned}
$$

Differencing a biased random walk



$$x_t - x_{t-1} = 1 + w_t; w_t \sim \mathrm{N}(0, 1)$$

Linear stationary models

Linear stationary models

We saw last week that linear filters are a useful way of modeling time series

Here we extend those ideas to a general class of models call *autoregressive moving average* (ARMA) models

Autoregressive (AR) models

Autoregressive models are widely used in ecology to treat a current state of nature as a function its past state(s)

Autoregressive (AR) models

An *autoregressive* model of order *p*, or AR(*p*), is defined as

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t$$

where we assume

1. w_t is white noise
2. $\phi_p \neq 0$ for an order-*p* process

Examples of AR(p) models

AR(1)

$$x_t = 0.5x_{t-1} + w_t$$

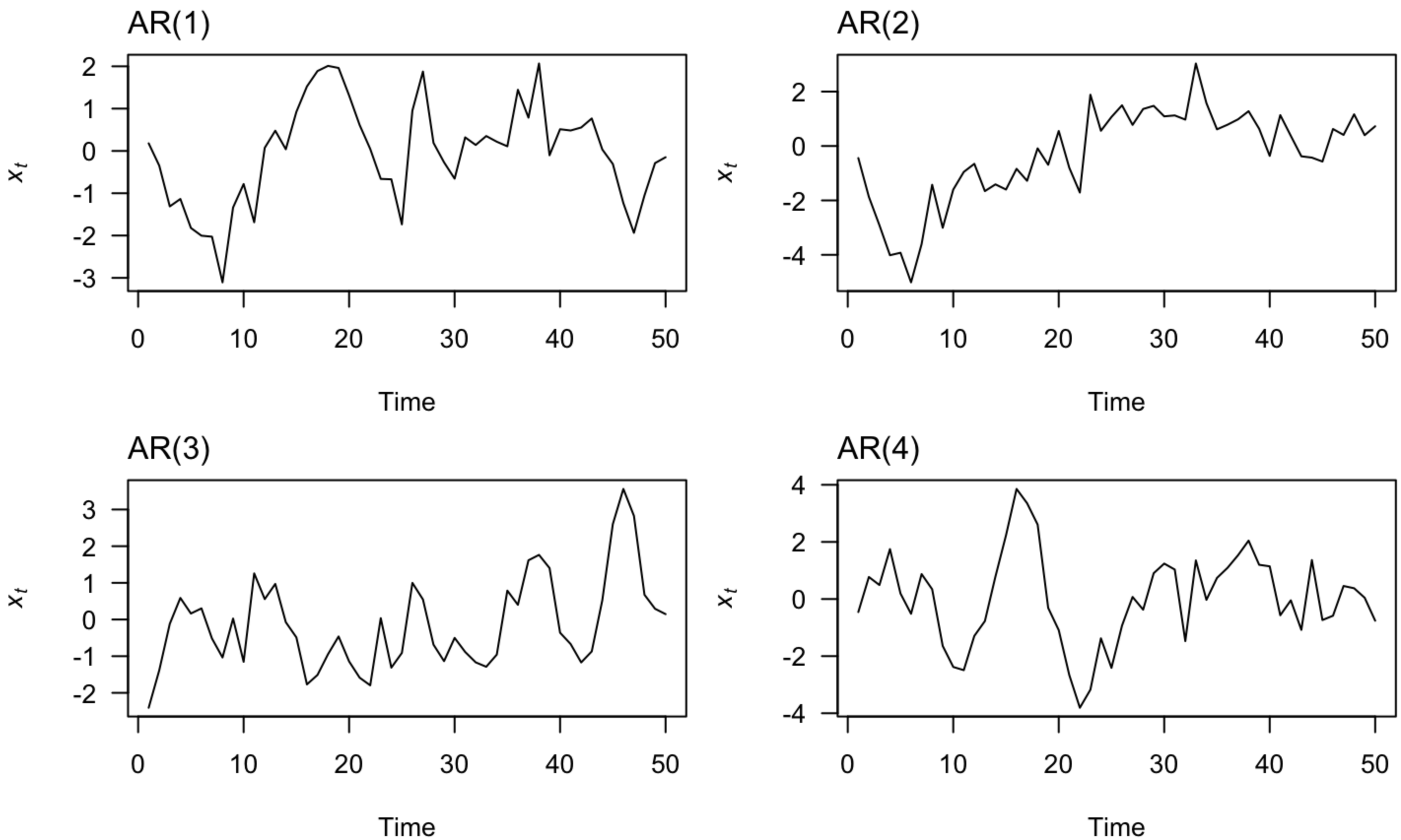
AR(1) with $\phi_1 = 1$ (random walk)

$$x_t = x_{t-1} + w_t$$

AR(2)

$$x_t = -0.2x_{t-1} + 0.4x_{t-2} + w_t$$

Examples of AR(p) models



Stationary AR(p) models

Recall that *stationary* processes have the following properties

1. no systematic change in the mean or variance
2. no systematic trend
3. no periodic variations or seasonality

We seek a means for identifying whether our AR(p) models are also stationary

Stationary AR(p) models

We can write out an AR(p) model using the backshift operator

$$
\begin{aligned}
x_t &= \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + w_t \\
&\Downarrow \\
x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - \dots - \phi_p x_{t-p} &= w_t \\
(1 - \phi_1 \mathbf{B} - \phi_2 \mathbf{B}^2 - \dots - \phi_p \mathbf{B}^p) x_t &= w_t \\
\phi_p (\mathbf{B}^p) x_t &= w_t
\end{aligned}
$$

Stationary AR(p) models

If we treat $\mathbf{B}$ as a number (or numbers), we can out write the *characteristic equation* as

$$\phi_p(\mathbf{B})x_t = w_t$$
$$\Downarrow$$
$$\phi_p(\mathbf{B}^p) = 0$$

To be stationary, **all roots** of the characteristic equation **must exceed 1 in absolute value**

Stationary AR(p) models

For example, consider this AR(1) model from earlier

$$x_t = 0.5x_{t-1} + w_t$$

Stationary AR(p) models

For example, consider this AR(1) model from earlier

$$
\begin{gathered}
x_t = 0.5 x_{t-1} + w_t \\
\Downarrow \\
x_t - 0.5 x_{t-1} = w_t \\
x_t - 0.5 \mathbf{B} x_t = w_t \\
(1 - 0.5 \mathbf{B}) x_t = w_t
\end{gathered}
$$

Stationary AR(p) models

For example, consider this AR(1) model from earlier

$$
\begin{aligned}
(1 - 0.5\mathbf{B})x_t &= w_t \\
&\Downarrow \\
1 - 0.5\mathbf{B} &= 0 \\
-0.5\mathbf{B} &= -1 \\
\mathbf{B} &= 2
\end{aligned}
$$

This model is indeed stationary because $\mathbf{B} > 1$

Stationary AR(p) models

What about this AR(2) model from earlier?

$$x_t = -0.2x_{t-1} + 0.4x_{t-2} + w_t$$

Stationary AR(p) models

What about this AR(2) model from earlier?

$$x_t = -0.2x_{t-1} + 0.4x_{t-2} + w_t$$
$$\Downarrow$$
$$\begin{aligned} x_t + 0.2x_{t-1} - 0.4x_{t-2} &= w_t \\ x_t + 0.2\mathbf{B}x_t - 0.4\mathbf{B}^2x_t &= w_t \\ (1 + 0.2\mathbf{B} - 0.4\mathbf{B}^2)x_t &= w_t \end{aligned}$$

Stationary AR(p) models

What about this AR(2) model from earlier?

$$(1 + 0.2\mathbf{B} - 0.4\mathbf{B}^2)x_t = w_t$$
$$\Downarrow$$
$$1 + 0.2\mathbf{B} - 0.4\mathbf{B}^2 = 0$$
$$\Downarrow$$
$$\mathbf{B}_1 \approx -1.35 \text{ and } \mathbf{B}_2 \approx 1.85$$

This model is *not* stationary because only $\mathbf{B}_2 > 1$

What about random walks?

Consider our random walk model

$$x_t = x_{t-1} + w_t$$

What about random walks?

Consider our random walk model

$$x_t = x_{t-1} + w_t$$
$$\Downarrow$$
$$\begin{aligned} x_t - x_{t-1} &= w_t \\ x_t - 1\mathbf{B}x_t &= w_t \\ (1 - 1\mathbf{B})x_t &= w_t \end{aligned}$$

What about random walks?

Consider our random walk model

$$\begin{aligned}
x_t - x_{t-1} &= w_t \\
x_t - 1\mathbf{B}x_t &= w_t \\
(1 - 1\mathbf{B})x_t &= w_t \\
&\Downarrow \\
1 - 1\mathbf{B} &= 0 \\
-1\mathbf{B} &= -1 \\
\mathbf{B} &= 1
\end{aligned}$$

Random walks are **not** stationary because $\mathbf{B} = 1 \ngtr 1$

Stationary AR(1) models

We can define a parameter space over which all AR(1) models are stationary

$$x_t = \phi x_{t-1} + w_t$$

Stationary AR(1) models

We can define a parameter space over which all AR(1) models are stationary

$$
\begin{aligned}
x_t &= \phi x_{t-1} + w_t \\
&\Downarrow \\
x_t - \phi x_{t-1} &= w_t \\
x_t - \phi \mathbf{B} x_t &= w_t \\
(1 - \phi \mathbf{B}) x_t &= w_t
\end{aligned}
$$

Stationary AR(1) models

For $x_t = \phi x_{t-1} + w_t$, we have

$$
\begin{aligned}
(1 - \phi \mathbf{B}) x_t &= w_t \\
&\Downarrow \\
1 - \phi \mathbf{B} &= 0 \\
-\phi \mathbf{B} &= -1 \\
\mathbf{B} &= \frac{1}{\phi} \\
&\Downarrow \\
\mathbf{B} = \frac{1}{\phi} &> 1 \text{ iff } 0 < \phi < 1
\end{aligned}
$$

Stationary AR(1) models

What if ϕ is negative, such that $x_t = -\phi x_{t-1} + w_t$?

$$x_t = -\phi x_{t-1} + w_t$$

$$\Downarrow$$

$$\begin{aligned} x_t + \phi x_{t-1} &= w_t \\ x_t + \phi \mathbf{B} x_t &= w_t \\ (1 + \phi \mathbf{B}) x_t &= w_t \end{aligned}$$

Stationary AR(1) models

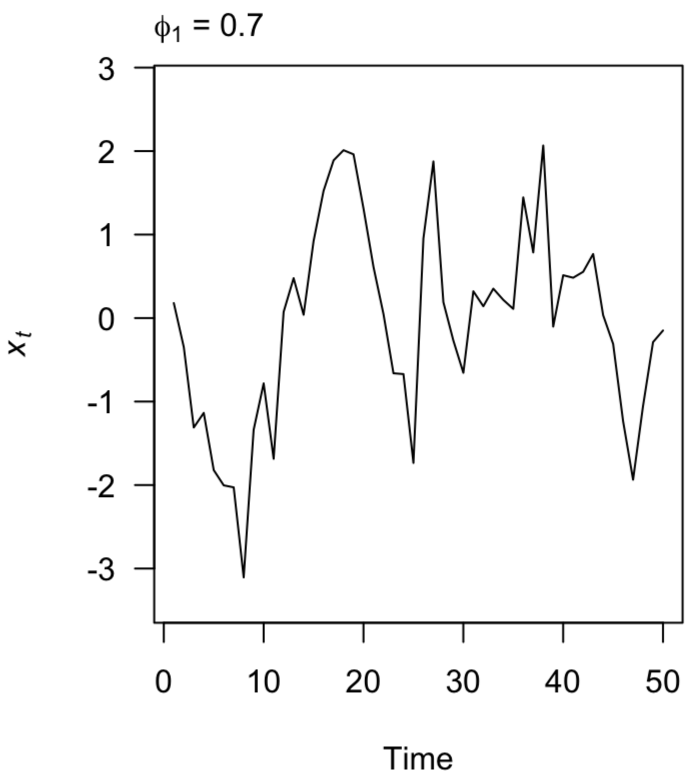
For $x_t = -\phi x_{t-1} + w_t$, we have

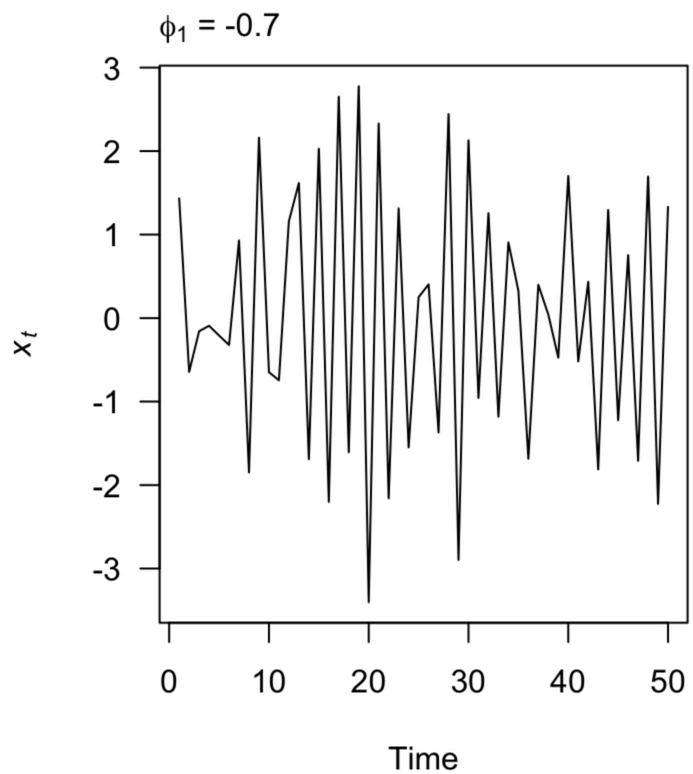
$$
\begin{gathered}
(1 + \phi \mathbf{B}) x_t = w_t \\
\Downarrow \\
\begin{aligned}
1 + \phi \mathbf{B} &= 0 \\
\phi \mathbf{B} &= -1 \\
\mathbf{B} &= -\frac{1}{\phi}
\end{aligned} \\
\Downarrow \\
\mathbf{B} = -\frac{1}{\phi} > 1 \text{ iff } -1 < \phi < 0
\end{gathered}
$$

Stationary AR(1) models

Thus, AR(1) models are stationary if and only if $|\phi| < 1$

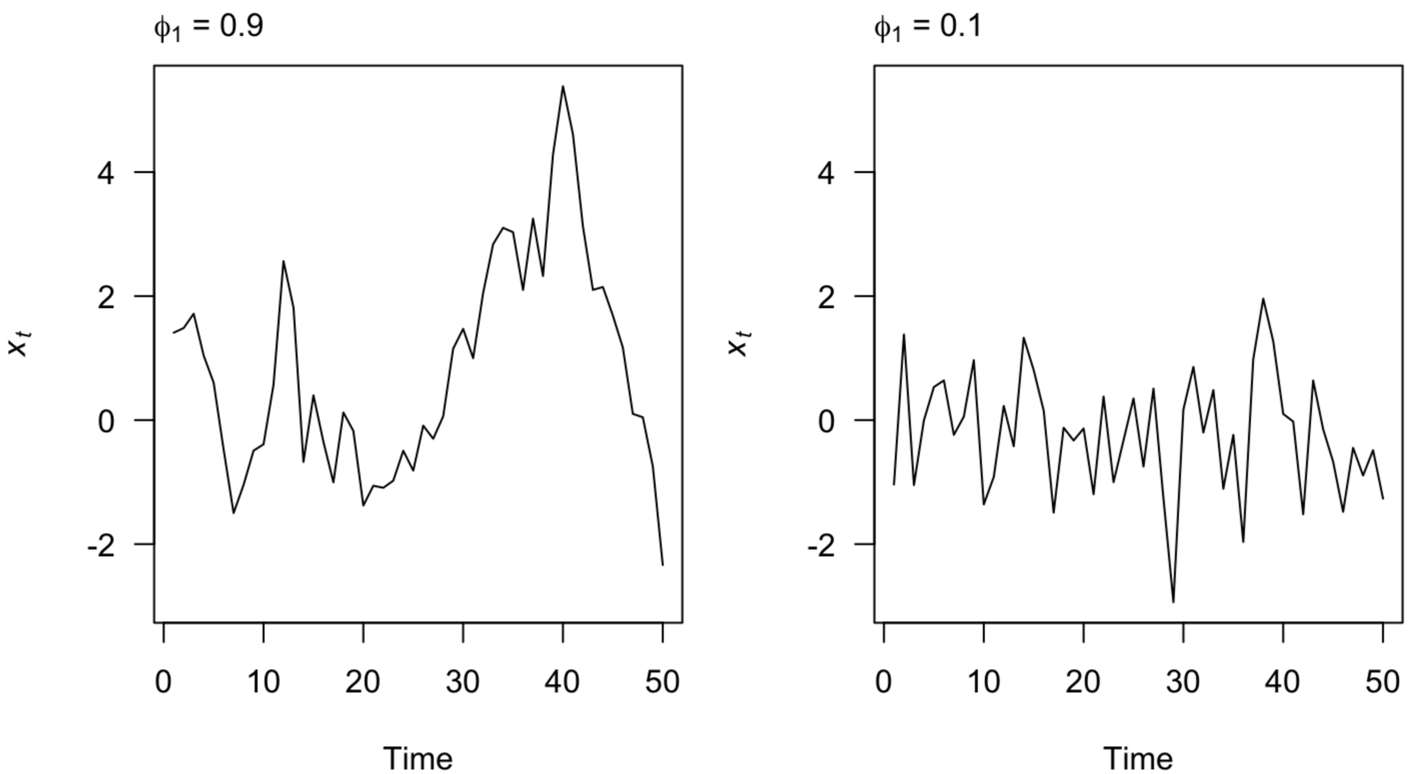
Coefficients of AR(1) models





Same value, but different sign

Coefficients of AR(1) models

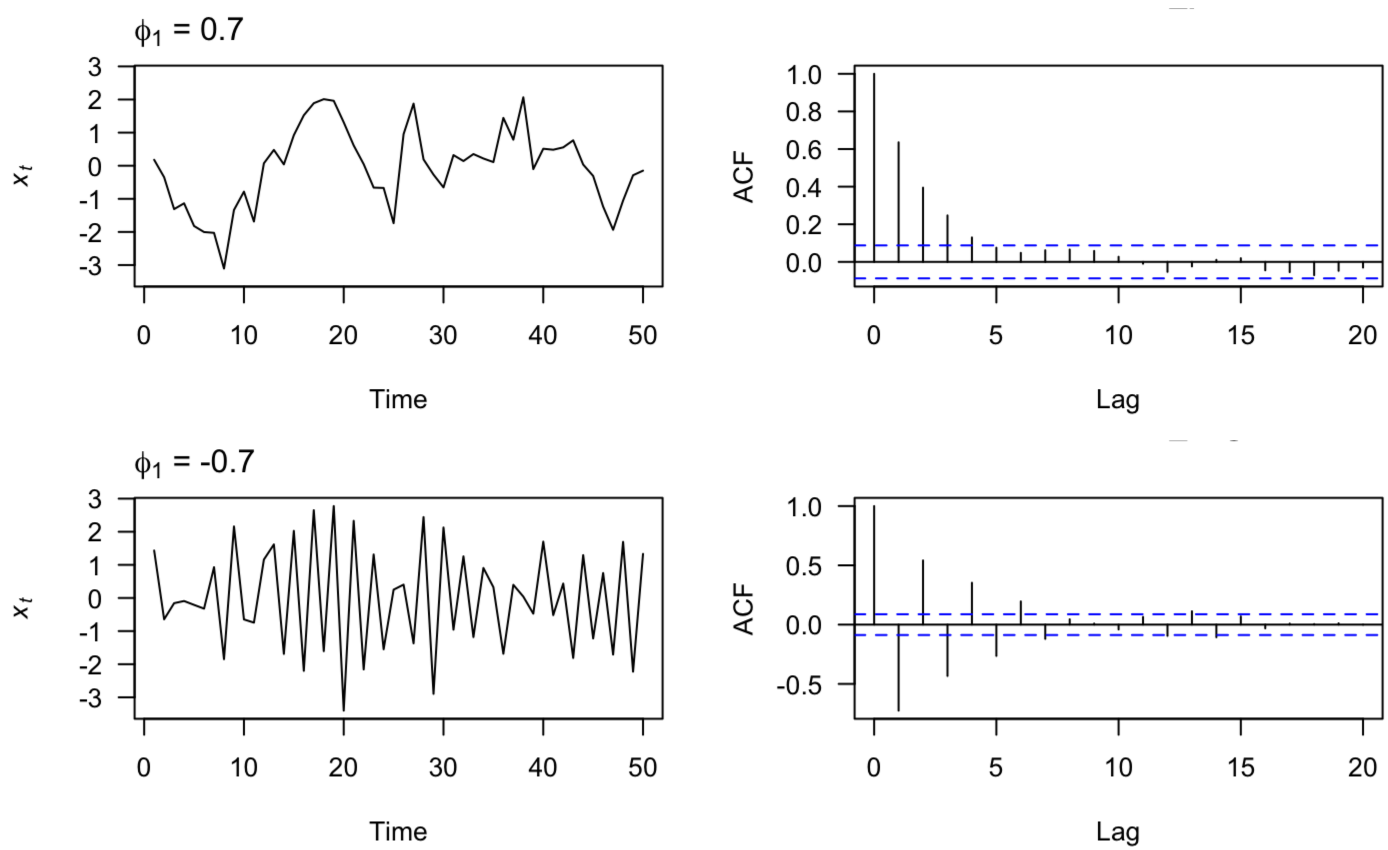


Both positive, but different magnitude

Autocorrelation function (ACF)

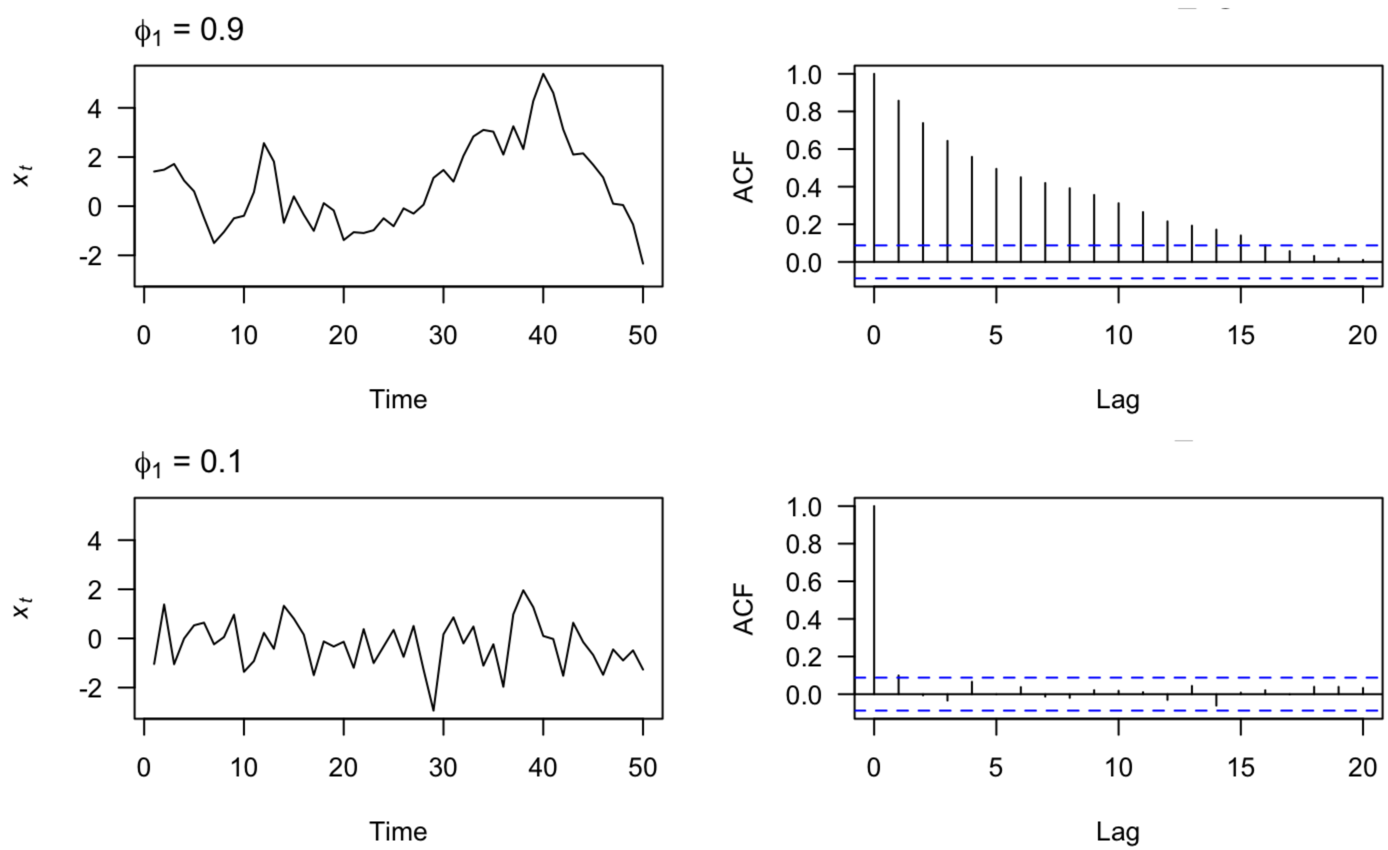
Recall that the *autocorrelation function* (ρ_k) measures the correlation between $\{x_t\}$ and a shifted version of itself $\{x_{t+k}\}$

ACF for AR(1) models



ACF oscillates for model with $-\phi$

ACF for AR(1) models

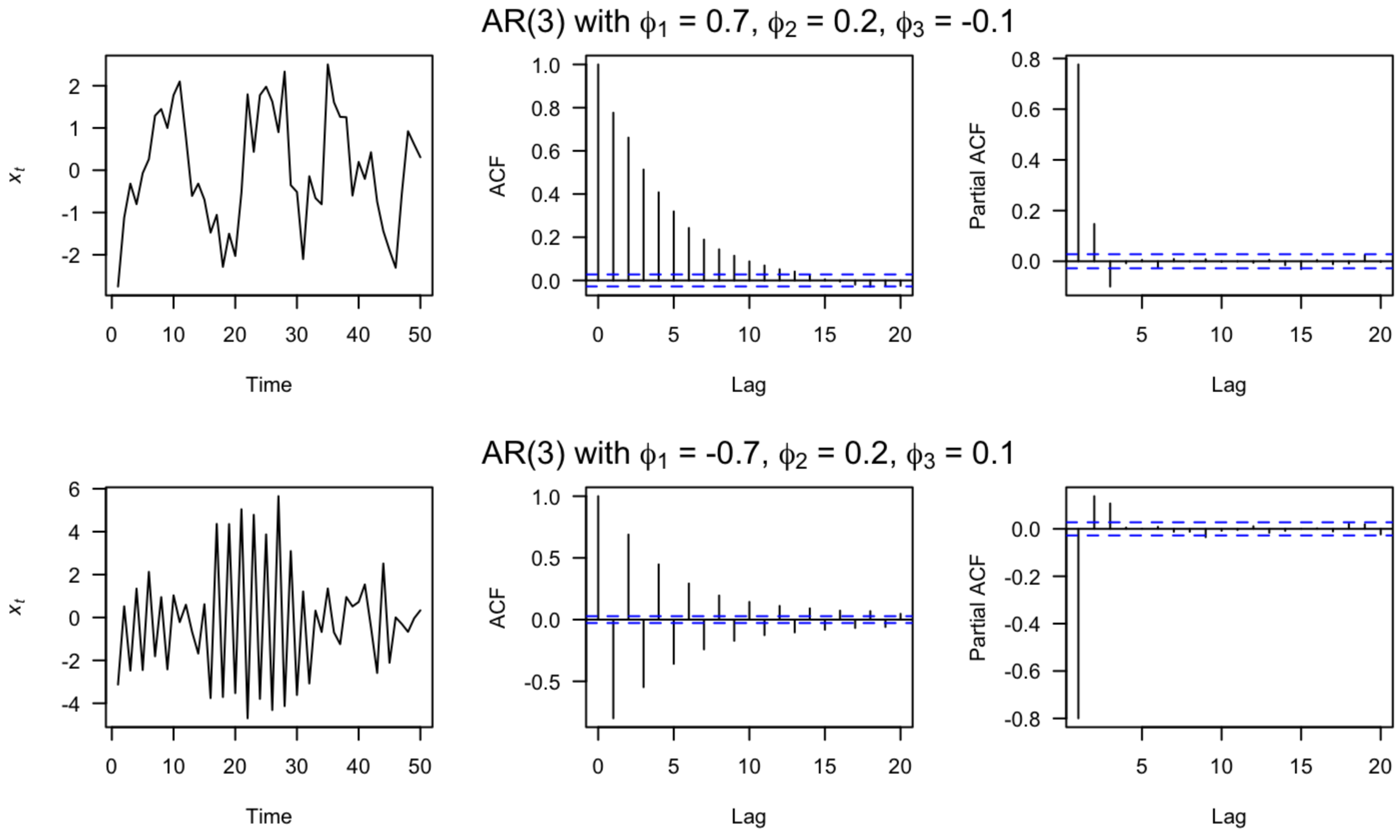


For model with large ϕ, ACF has longer tail

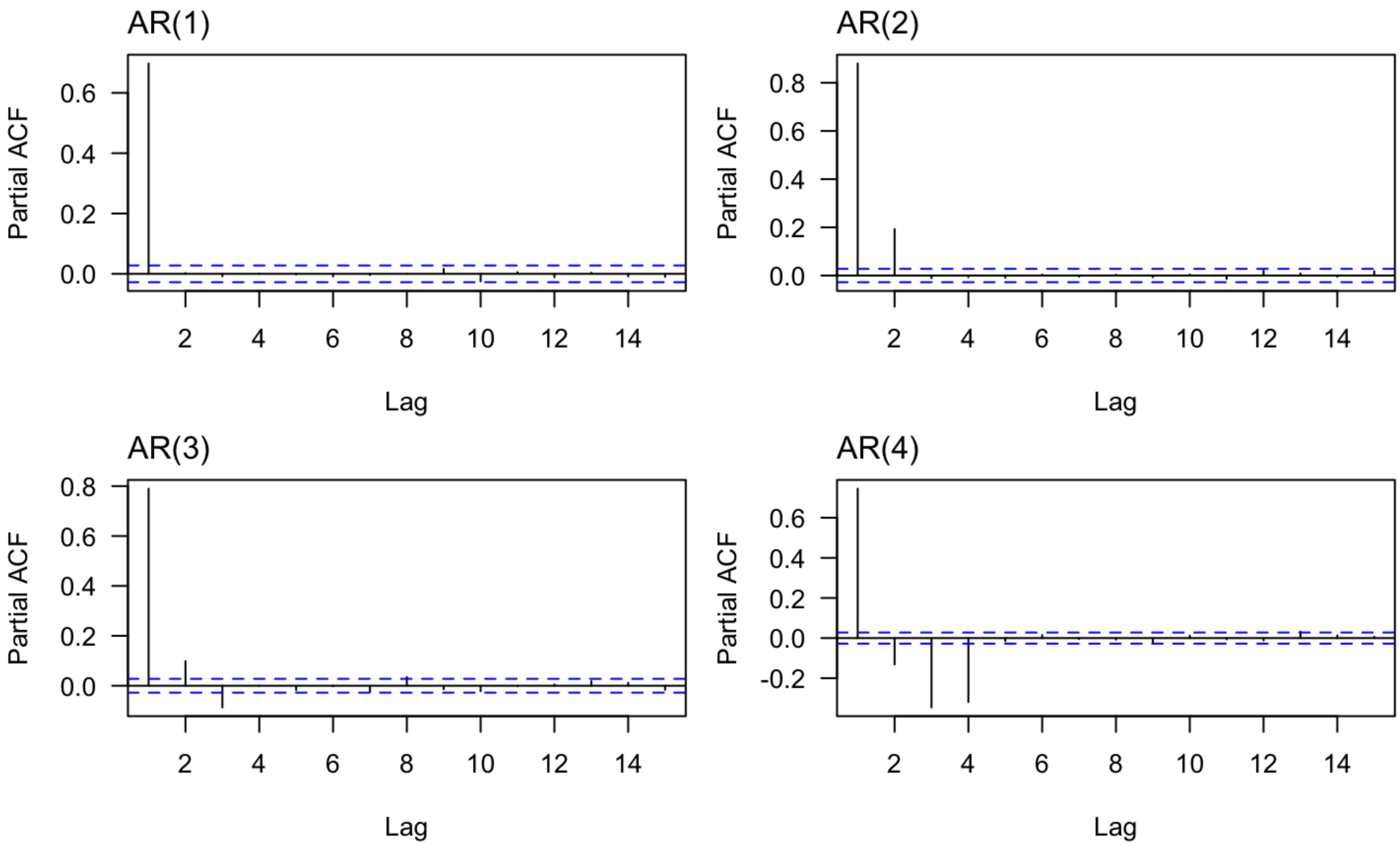
Partial autocorrelation funcion (PACF)

Recall that the *partial autocorrelation function* (ϕ_k) measures the correlation between $\{x_t\}$ and a shifted version of itself $\{x_{t+k}\}$, with the linear dependence of $\{x_{t-1}, x_{t-2}, \dots, x_{t-k-1}\}$ removed

ACF & PACF for AR(p) models



PACF for AR(p) models



Do you see the link between the order p and lag k?

Using ACF & PACF for model ID

Model	ACF	PACF
AR(*p*)	Tails off slowly	Cuts off after lag *p*

Moving average (MA) models

Moving average models are most commonly used for forecasting a future state

Moving average (MA) models

A moving average model of order q, or MA(q), is defined as

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

where w_t is white noise

Each of the x_t is a sum of the most recent error terms

Moving average (MA) models

A moving average model of order q, or MA(q), is defined as

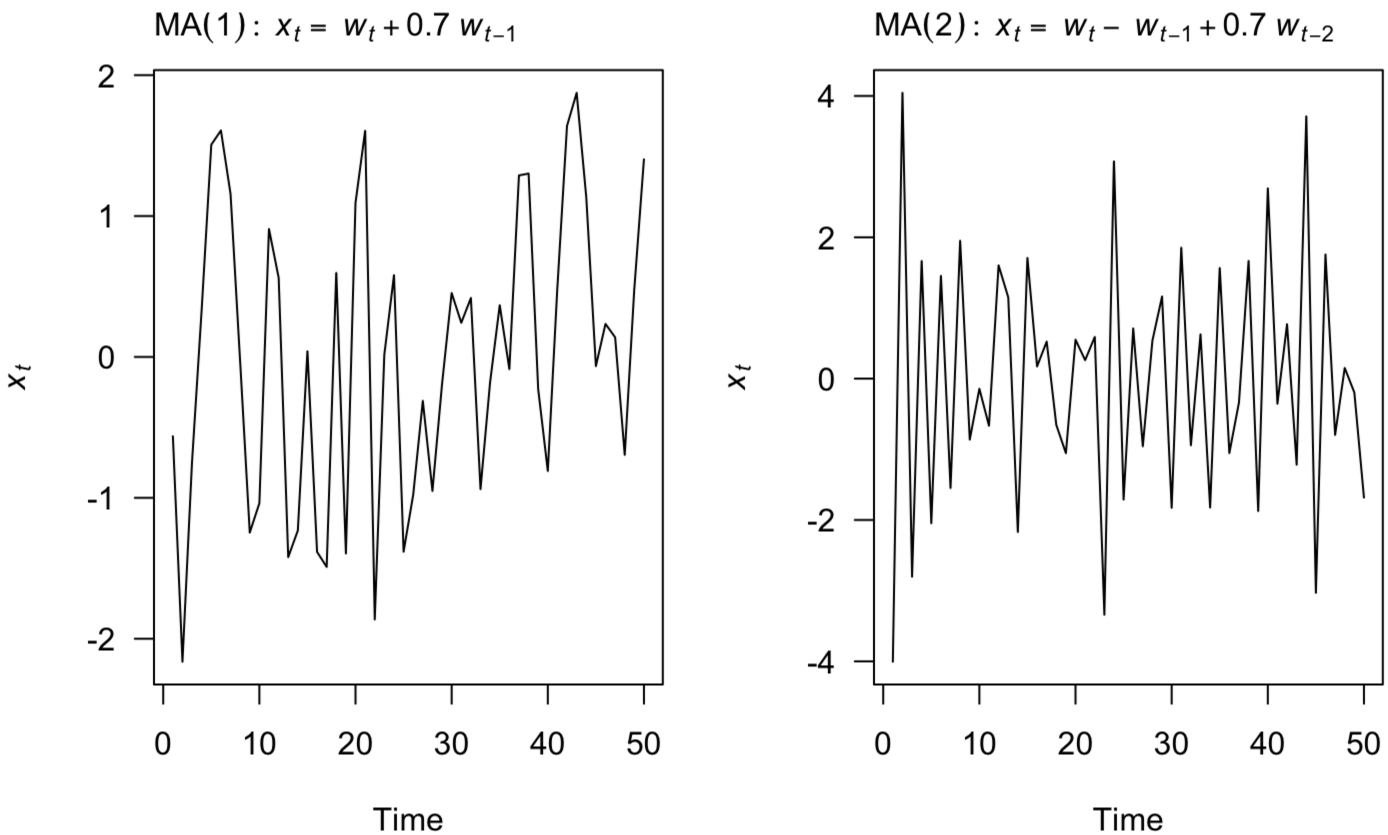
$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

where w_t is white noise

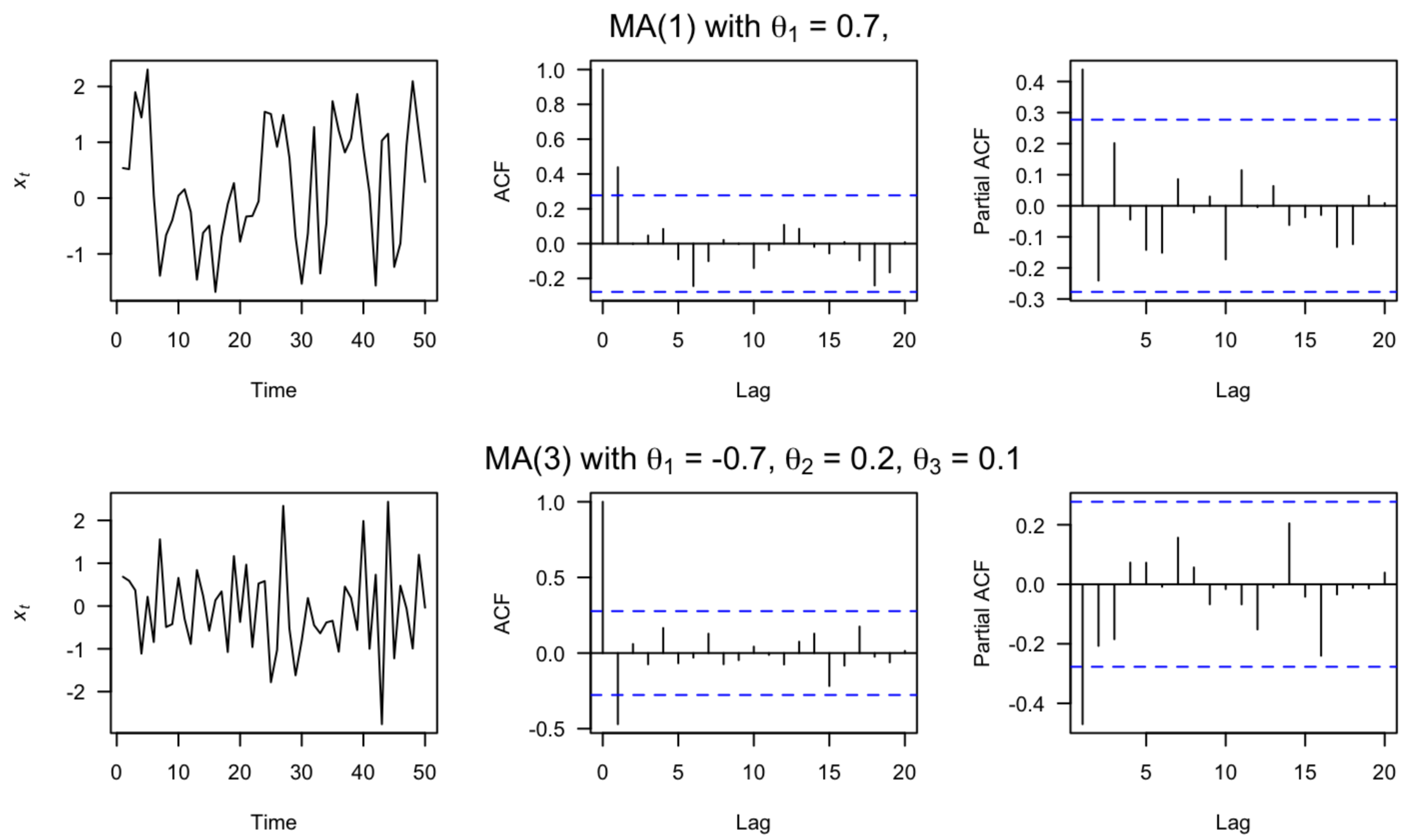
Each of the x_t is a sum of the most recent error terms

Thus, *all* MA processes are stationary because they are finite sums of stationary WN processes

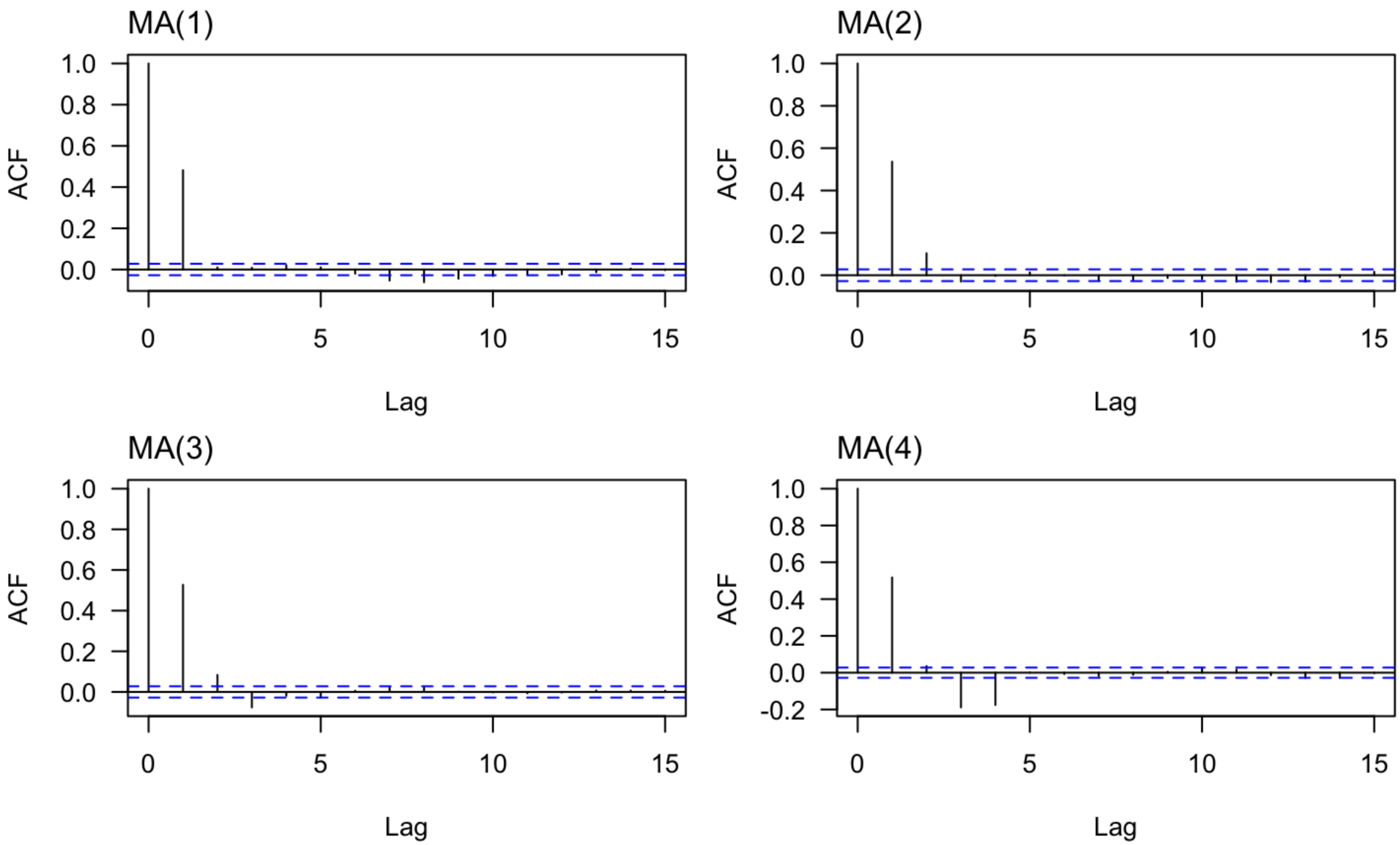
Examples of MA(q) models



ACF & PACF for MA(q) models



ACF for MA(q) models



Do you see the link between the order q and lag k?

Using ACF & PACF for model ID

Model	ACF	PACF
AR(p)	Tails off slowly	Cuts off after lag p
MA(q)	Cuts off after lag q	Tails off slowly

AR(p) model as an MA(∞) model

It is possible to write an AR(p) model as an MA(∞) model

AR(1) model as an MA(∞) model

For example, consider an AR(1) model

$$x_t = \phi x_{t-1} + w_t$$

AR(1) model as an MA(∞) model

For example, consider an AR(1) model

$$
\begin{gathered}
x_t = \phi x_{t-1} + w_t \\
\Downarrow \\
x_{t-1} = \phi x_{t-2} + w_{t-1} \\
\Downarrow \\
x_{t-2} = \phi x_{t-3} + w_{t-2} \\
\Downarrow \\
x_{t-3} = \phi x_{t-4} + w_{t-3}
\end{gathered}
$$

AR(1) model as an MA(∞) model

Substituting in the expression for x_{t-1} into that for x_t

$$x_t = \phi x_{t-1} + w_t$$
$$\Downarrow$$
$$x_{t-1} = \phi x_{t-2} + w_{t-1}$$
$$\Downarrow$$
$$x_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t$$
$$x_t = \phi^2 x_{t-2} + \phi w_{t-1} + w_t$$

AR(1) model as an MA(∞) model

And repeated substitutions yields

$$
\begin{aligned}
x_t &= \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\
&\Downarrow \\
x_t &= \phi^3 x_{t-3} + \phi^2 w_{t-2} + \phi w_{t-1} + w_t \\
&\Downarrow \\
x_t &= \phi^4 x_{t-4} + \phi^3 w_{t-3} + \phi^2 w_{t-2} + \phi w_{t-1} + w_t \\
&\Downarrow \\
x_t &= w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots + \phi^k w_{t-k} + \phi^{k+1} x_{t-k-1}
\end{aligned}
$$

AR(1) model as an MA(∞) model

If our AR(1) model is stationary, then

$$|\phi| < 1$$

which then implies that

$$\lim_{k \to \infty} \phi^{k+1} = 0$$

AR(1) model as an MA(∞) model

If our AR(1) model is stationary, then

$$|\phi| < 1$$

which then implies that

$$\lim_{k \to \infty} \phi^{k+1} = 0$$

and hence

$$x_t = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots + \phi^k w_{t-k} + \phi^{k+1} x_{t-k-1}$$
$$\Downarrow$$
$$x_t = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots + \phi^k w_{t-k}$$

Invertible MA(q) models

An MA(q) process is *invertible* if it can be written as a stationary autoregressive process of infinite order without an error term

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

$$\Downarrow ?$$

$$w_t = x_t + \sum_{k=1}^{\infty} (-\theta)^k x_{t-k}$$

Invertible MA(q) models

Q: Why do we care if an MA(q) model is invertible?

A: It helps us identify the model's parameters

Invertible MA(q) models

For example, these MA(1) models are equivalent

$$x_t = w_t + \frac{1}{5} w_{t-1} \text{ with } w_t \sim \text{ N}(0, 25)$$

$$\Updownarrow$$

$$x_t = w_t + 5 w_{t-1} \text{ with } w_t \sim \text{ N}(0, 1)$$

Variance of an MA(1) model

The variance of x_t is given by

$$x_t = w_t + \frac{1}{5} w_{t-1} \text{ with } w_t \sim \text{ N}(0, 25)$$

$$\Downarrow$$

$$\begin{aligned} \text{Var}(x_t) &= \text{Var}(w_t) + \left(\frac{1}{25} \right) \text{Var}(w_{t-1}) \\ &= 25 + \left(\frac{1}{25} \right) 25 \\ &= 25 + 1 \\ &= 26 \end{aligned}$$

Variance of an MA(1) model

The variance of x_t is given by

$$x_t = w_t + 5w_{t-1} \text{ with } w_t \sim \text{N}(0, 1)$$

$$\Downarrow$$

$$\begin{aligned}
\text{Var}(x_t) &= \text{Var}(w_t) + (25)\text{Var}(w_{t-1}) \\
&= 1 + (25)1 \\
&= 1 + 25 \\
&= 26
\end{aligned}$$

Rewriting an MA(1) model

We can rewrite an MA(1) model in terms of x

$$
\begin{gathered}
x_t = w_t + \theta w_{t-1} \\
\Downarrow \\
w_t = x_t - \theta w_{t-1}
\end{gathered}
$$

Rewriting an MA(1) model

And now we can substitute in previous expressions for w_t

$$
\begin{aligned}
w_t &= x_t - \theta w_{t-1} \\
&\Downarrow \\
w_{t-1} &= x_{t-1} - \theta w_{t-2} \\
&\Downarrow \\
w_t &= x_t - \theta(x_{t-1} - \theta w_{t-2}) \\
w_t &= x_t - \theta x_{t-1} - \theta^2 w_{t-2} \\
&\vdots \\
w_t &= x_t - \theta x_{t-1} - \cdots - \theta^k x_{t-k} - \theta^{k+1} w_{t-k-1}
\end{aligned}
$$

Invertible MA(1) model

If we constrain $|\theta| < 1$, then

$$\lim_{k\to\infty} (-\theta)^{k+1} w_{t-k-1} = 0$$

and

$$\begin{aligned} w_t &= x_t - \theta x_{t-1} - \cdots - \theta^k x_{t-k} - \theta^{k+1} w_{t-k-1} \\ &\Downarrow \\ w_t &= x_t - \theta x_{t-1} - \cdots - \theta^k x_{t-k} \\ w_t &= x_t + \sum_{k=1}^{\infty} (-\theta)^k x_{t-k} \end{aligned}$$

Autoregressive moving average models

An autoregressive moving average, or ARMA(p,q), model is written as

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}$$

Autoregressive moving average models

We can write an ARMA(p,q) model using the backshift operator

$$\phi_p(\mathbf{B}^p)x_t = \theta_q(\mathbf{B}^q)w_t$$

Autoregressive moving average models

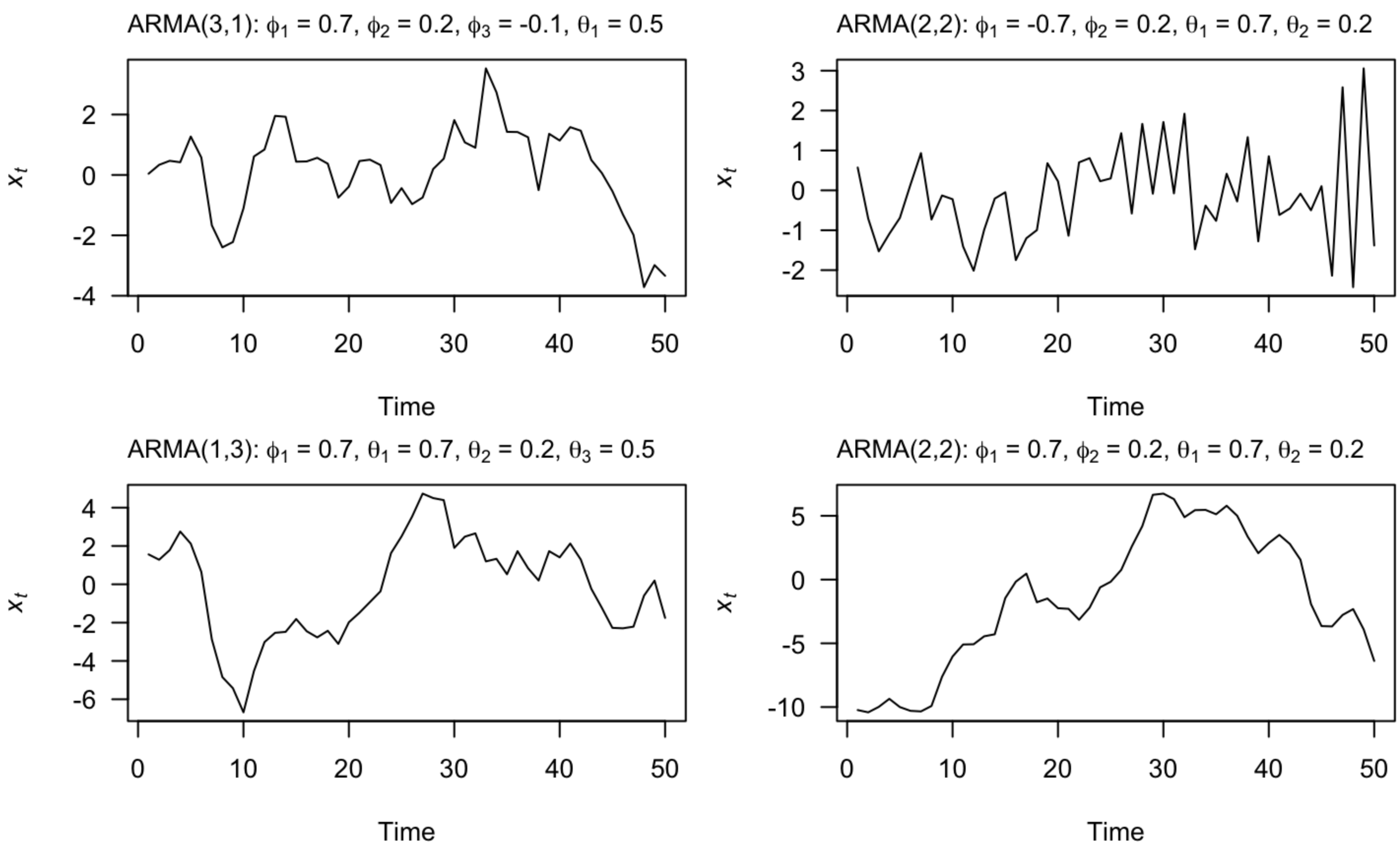
We can write an ARMA(*p*,*q*) model using the backshift operator

$$\phi_p(\mathbf{B}^p)x_t = \theta_q(\mathbf{B}^q)w_t$$

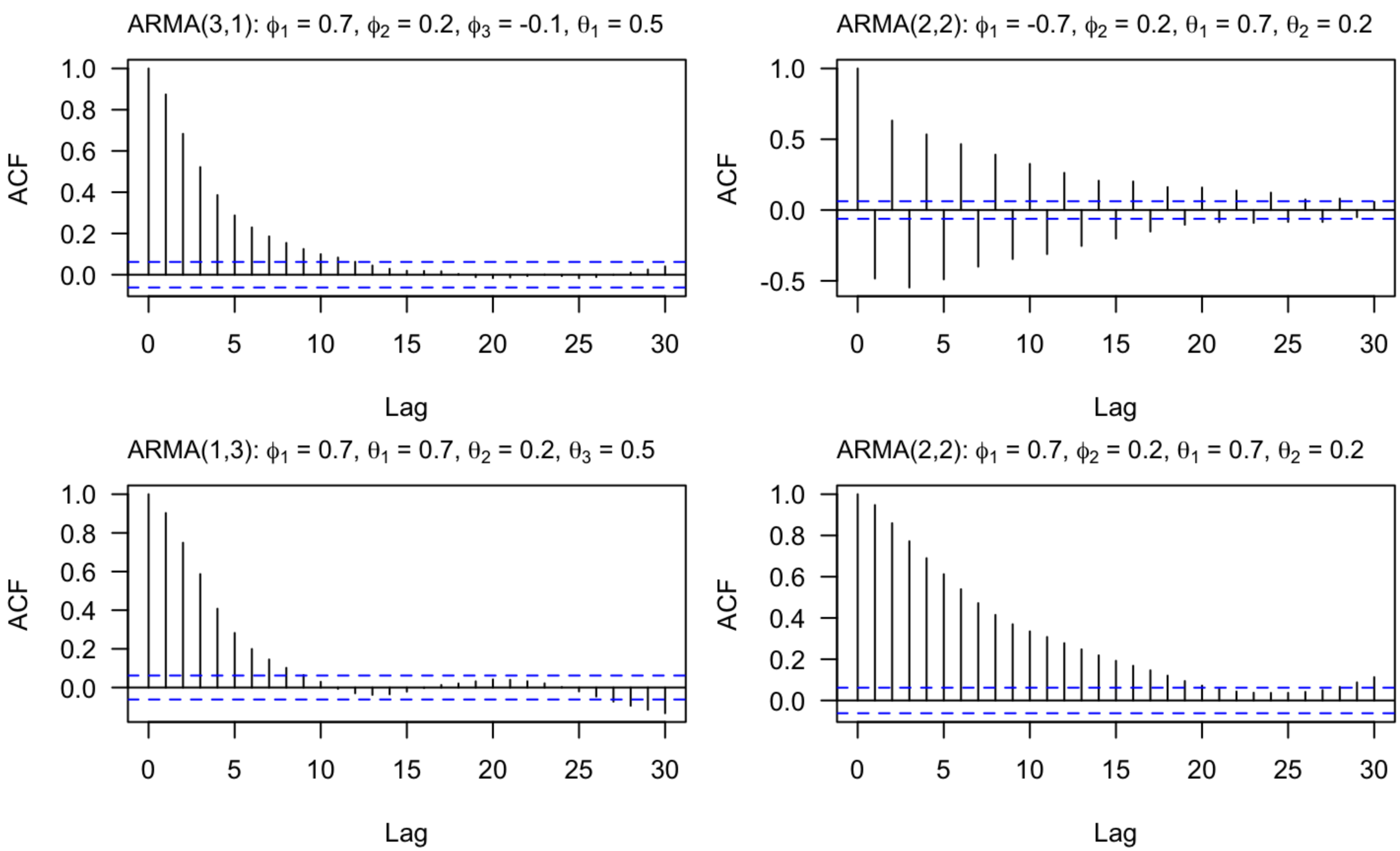
ARMA models are *stationary* if all roots of $\phi_p(\mathbf{B}) > 1$

ARMA models are *invertible* if all roots of $\theta_q(\mathbf{B}) > 1$

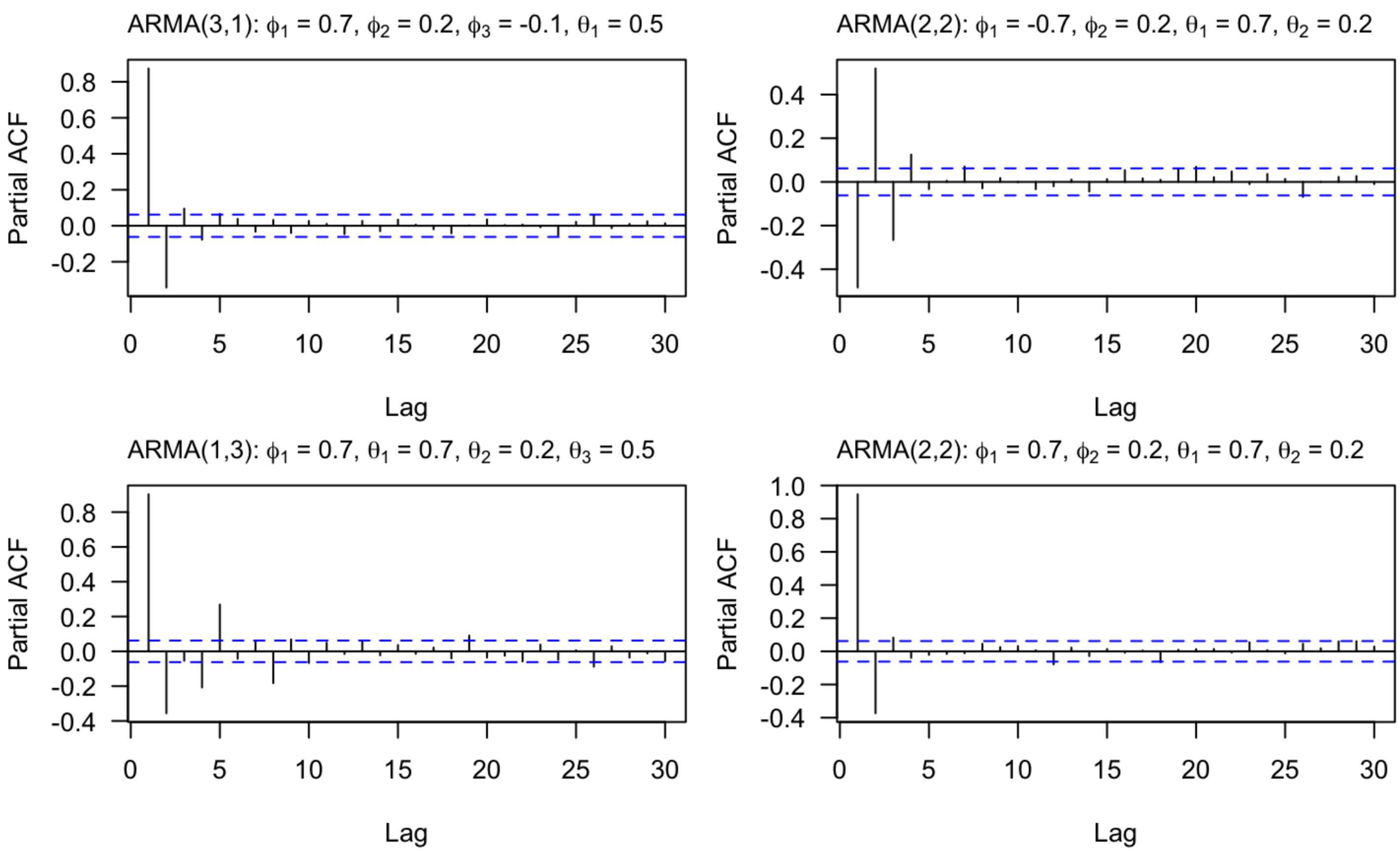
Examples of ARMA(p,q) models



ACF for ARMA(p,q) models



PACF for ARMA(p,q) models



Using ACF & PACF for model ID

Model	ACF	PACF
AR(*p*)	Tails off slowly	Cuts off after lag *p*
MA(*q*)	Cuts off after lag *q*	Tails off slowly
ARMA(*p*,*q*)	Tails off slowly	Tails off slowly

NONSTATIONARY MODELS

Autoregressive integrated moving average (ARIMA) models

If the data do not appear stationary, differencing can help

This leads to the class of *autoregressive integrated moving average* (ARIMA) models

ARIMA models are indexed with orders (p,d,q) where d indicates the order of differencing

ARIMA(p,d,q) models

Definition

$\{x_t\}$ follows an ARIMA(p,d,q) process if $(1 - \mathbf{B})^d x_t$ is an ARMA(p,q) process

ARIMA(p,d,q) models

An example

Consider an ARMA(1,0) = AR(1) process where

$$x_t = (1 + \phi) x_{t-1} + w_t$$

ARIMA(*p*,*d*,*q*) models

An example

Consider an ARMA(1,0) = AR(1) process where

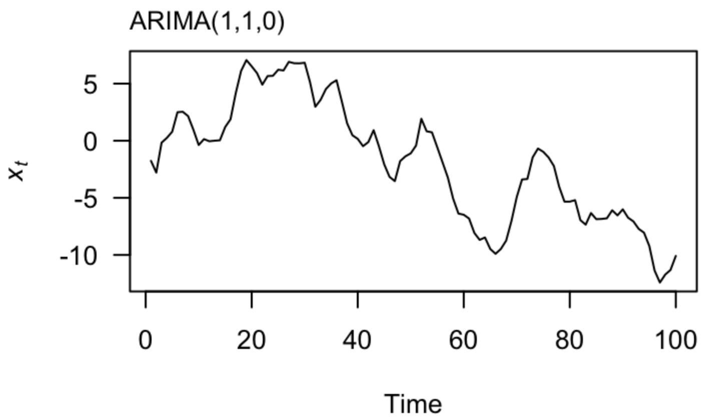
$$x_t = (1 + \phi)x_{t-1} + w_t$$

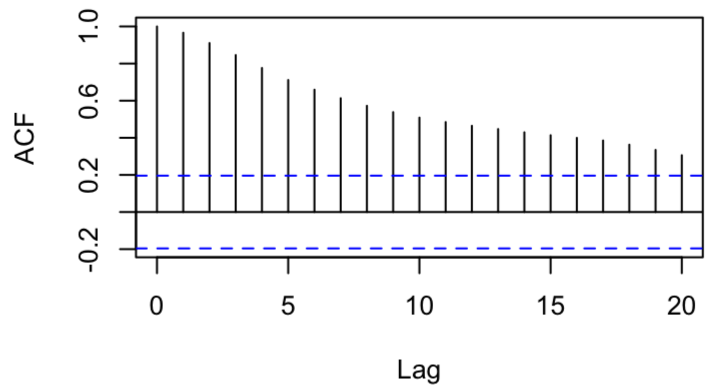
$$\Downarrow$$

$$\begin{aligned} x_t &= x_{t-1} + \phi x_{t-1} + w_t \\ x_t - x_{t-1} &= \phi x_{t-1} + w_t \\ (1 - \mathbf{B})x_t &= \phi x_{t-1} + w_t \end{aligned}$$

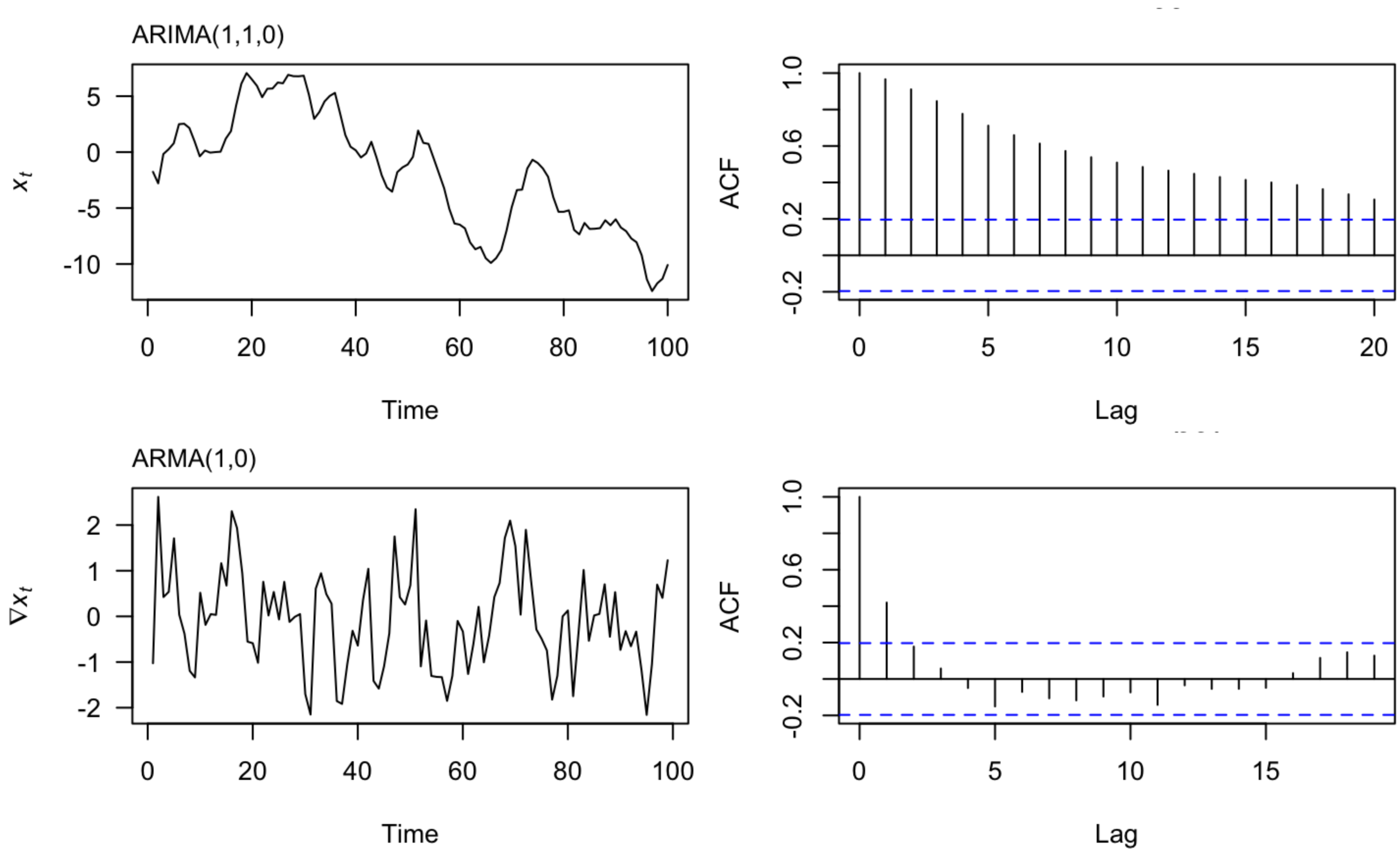
So x_t is indeed an ARIMA(1,1,0) process

ARIMA(*p*,*d*,*q*) models





ARIMA(*p*,*d*,*q*) models



Topics for today

Review

- White noise
- Random walks

Autoregressive (AR) models

Moving average (MA) models

Autoregressive moving average (ARMA) models

Using ACF & PACF for model ID