# Introduction to multivariate state-space models

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*FISH 507 – Applied Time Series Analysis*

21 January 2021

# Topics

## Lecture

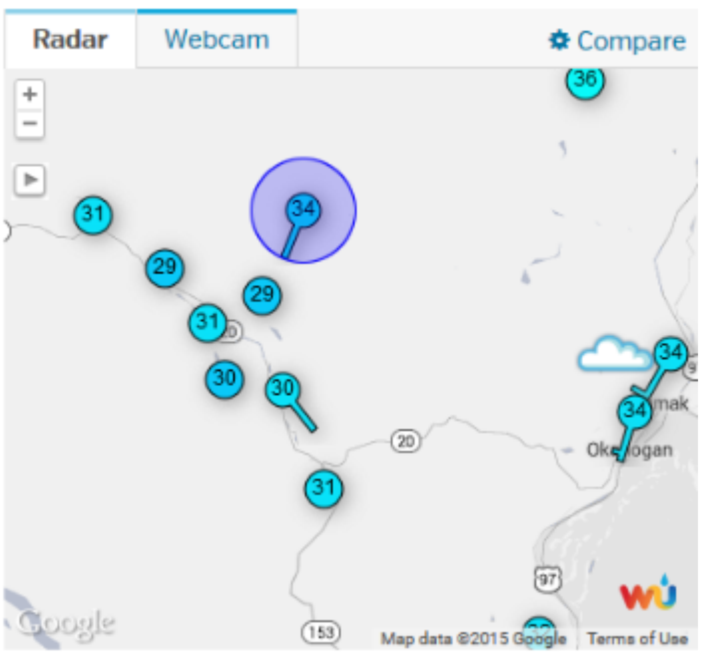
- Short example of multivariate observations
- Examples of multivariate structure in population data
- How to express these structures mathematically
- Adding a multivariate observation process

## Computer Labs

- Analysis of population structure using multi-site data
- Combining diverse data sources to estimate an underlying model

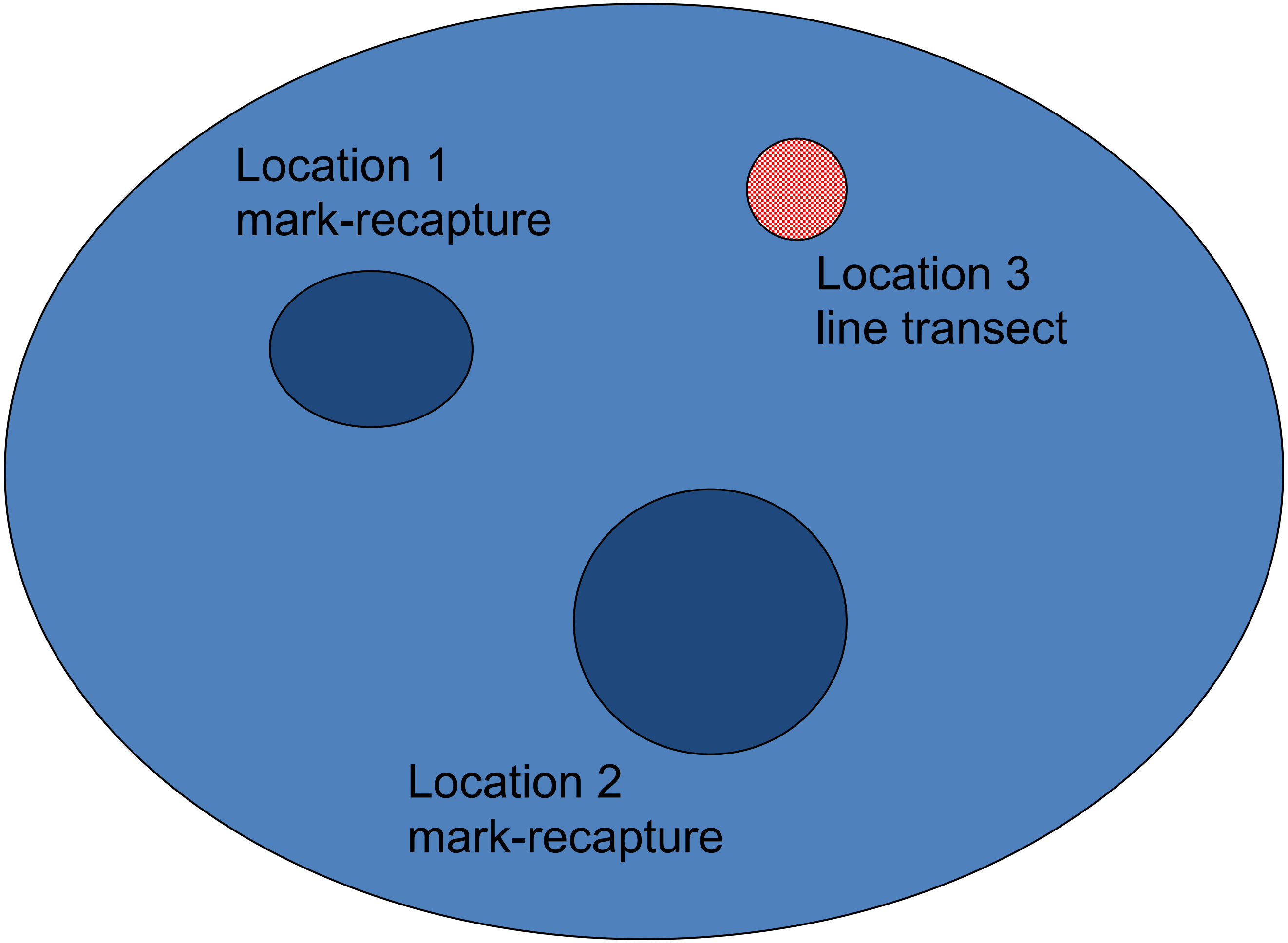
# Other examples

Combine multiple station data into a single metric




Multiple individuals measured over time

# Imagine we have 3 sampling locations for a population



# Mathematically we can write

$$x_t = x_{t-1} + u + w_t, w_t \sim N(0, q)$$

$$y_{1,t} = x_t + v_{1,t}, v_{1,t} \sim N(a_1, r_1)$$

$$y_{2,t} = x_t + v_{2,t}, v_{2,t} \sim N(a_2, r_2)$$

$$y_{3,t} = x_t + v_{3,t}, v_{3,t} \sim N(a_3, r_3)$$

observations

population size

noise

# The observation part can be rewritten

We need to fix one of the a's.
Traditionally we fix to the first to 0.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_t = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_t$$

observations Z matrix population size bias noise

# The model with one a fixed to zero

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_t = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_t$$

observations Z matrix population size bias noise

# The observation errors are multivariate. For now, let's assume Normality

The variance-covariance matrix tells you how the observation errors are related. Are they independent? Or do they covary? Do have the same variance or difference variances?

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_t \sim MVN\left( 0, \begin{bmatrix} \eta_1^2 & \eta_{1,2} & \eta_{1,3} \\ \eta_{1,2} & \eta_2^2 & \eta_{3,2} \\ \eta_{1,3} & \eta_{3,2} & \eta_3^2 \end{bmatrix} \right)$$

# Example observation error var-cov matrices

$$\begin{bmatrix} \eta_1^2 & \eta_{1,2} & \eta_{1,3} \\ \eta_{1,2} & \eta_2^2 & \eta_{3,2} \\ \eta_{1,3} & \eta_{3,2} & \eta_3^2 \end{bmatrix}$$

unconstrained

$$\begin{bmatrix} \eta^2 & \alpha & \alpha \\ \alpha & \eta^2 & \alpha \\ \alpha & \alpha & \eta^2 \end{bmatrix}$$

"equal varcov"

diagonal

$$\begin{bmatrix} \eta_1^2 & 0 & 0 \\ 0 & \eta_2^2 & 0 \\ 0 & 0 & \eta_3^2 \end{bmatrix}$$

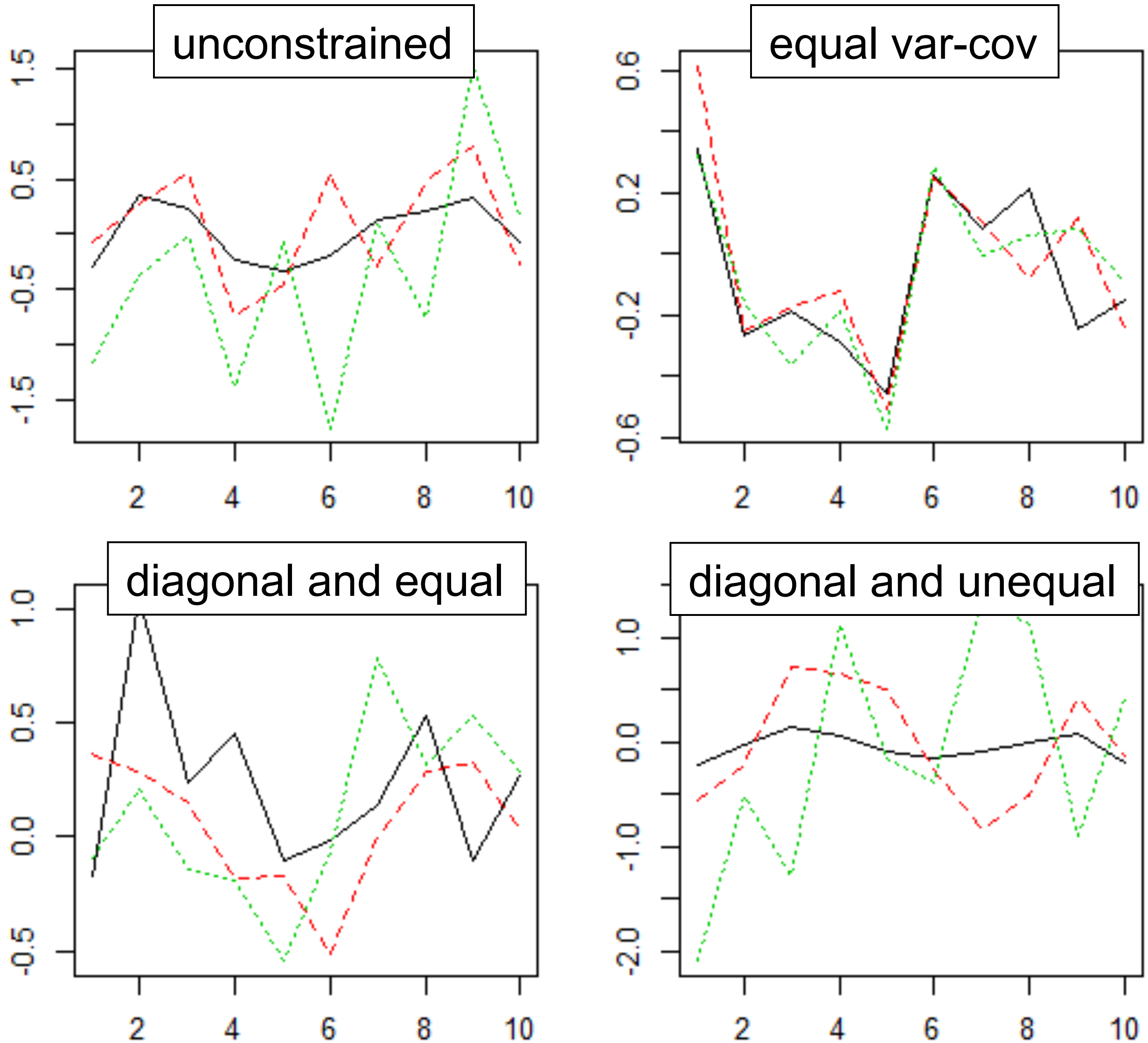
unique variances and uncorrelated errors

$$\begin{bmatrix} \eta^2 & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & \eta^2 \end{bmatrix}$$

identical variances and uncorrelated errors

# Example of errors coming from these variance-covariance matrices

error = how much the pop. growth rate is above or below average




```
mvrnorm(10,rep(0,3),diag(.1,3))
```

# Fitting MARSS models using the MARSS R Package

- Fits MARSS models
- *Model specification is 1-to-1 with the equation for the model*
- General, fits any MARSS model with Gaussian errors.
- **BUT**
- Maximum likelihood
- Slow for large data sets.  Huge speed improvements are possible by coding their models in TMB (or ADMB or greta). Mark will talk about this.

# MARSS R Package Model Syntax

$$X(t) = \mathbf{B}\, X(t-1) + \mathbf{U} + w(t),\ w(t) \sim N(0, \mathbf{Q})$$

$$Y(t) = \mathbf{Z}\, X(t) + \mathbf{A} + v(t),\ v(t) \sim N(0,\mathbf{R})$$

- **`fit2=MARSS(y,model=mod.list)`**

- **`y`** is data; **`model`** tells MARSS what the parameters are
- The parameters are MATRICES
- You write matrices just like they appear in your model on paper. Matrices must be MATRICES (not scalars, not vectors)
- You pass **`model`** to MARSS as a list

X(t) = **B** X(t-1) + **U** + w(t), w(t) ~ N(0, **Q**)

Y(t) = **Z** X(t) + **A** + v(t), v(t) ~ N(0,**R**)

```
mod.list=list(
 U=matrix("u"),
 x0=matrix(0),
 B=matrix(1),
 Q=matrix(0.1),
 Z=matrix(1),
 A=matrix(0),
 R=matrix("r"),
 tinitx=0)
```

Let's say we want to fit this model:

$$x_t = x_{t-1} + u + w_t, w_t \sim N(0, 0.1)$$
$$y_t = x_t + v_t, v_t \sim N(0, r)$$
$$x_0 = 0$$

Write in matrix form:

$$[x]_t = [1][x]_{t-1} + [u] + [w]_t, [w]_t \sim MVN(0, [0.1])$$
$$[y]_t = [1][x]_t + [v]_t, [v]_t \sim MVN(0, [r])$$
$$x_0 = [0]$$

```
mod.list=list(
 Q=matrix(0.1)
)
```

X(t) = **B** X(t-1) + **U** + w(t), w(t) ~ N(0, **Q**)

Y(t) = **Z** X(t) + **A** + v(t), v(t) ~ N(0,**R**)

```
mod.list=list(
 U=matrix("u"),
 x0=matrix(0),
 B=matrix(1),
 Q=matrix(0.1),
 Z=matrix(1,2,1),
 A= matrix(list(0,"a2"),2,1),
 R= matrix(list("r",0,0,"r"),
 tinitx=0)
```

Let's say we want to fit a model where two sites are sampling temperature x in a lake:

Our temperature model:

$$[x]_t = [1][x]_{t-1} + [u] + [w]_t, [w]_t \sim MVN(0, [0.1])$$
$$[x]_0 = [0]$$

Our two temperature sensors:

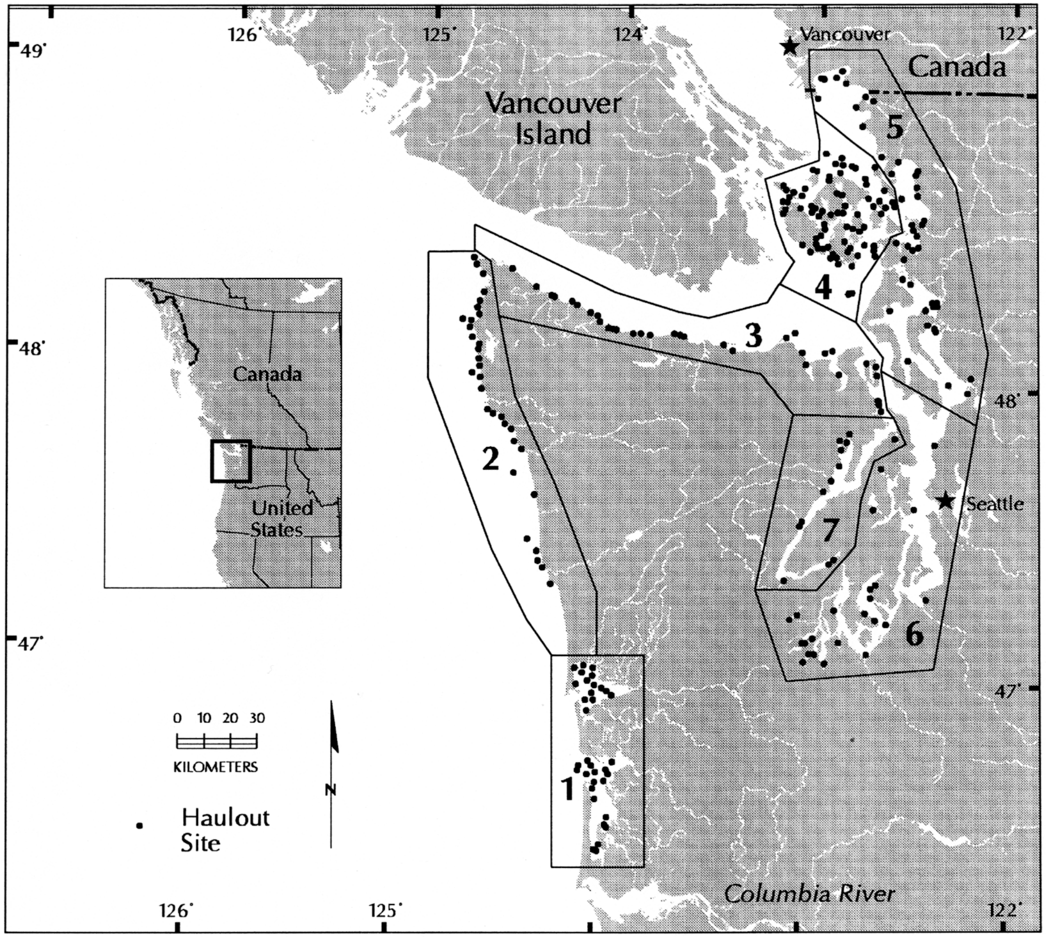
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_t = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [x]_t + \begin{bmatrix} 0 \\ a_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_t$$
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_t \sim MVN\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \right)$$

# Some short examples

- marss_example_1.R
- marss_example_2.R
- marss_example_3.R

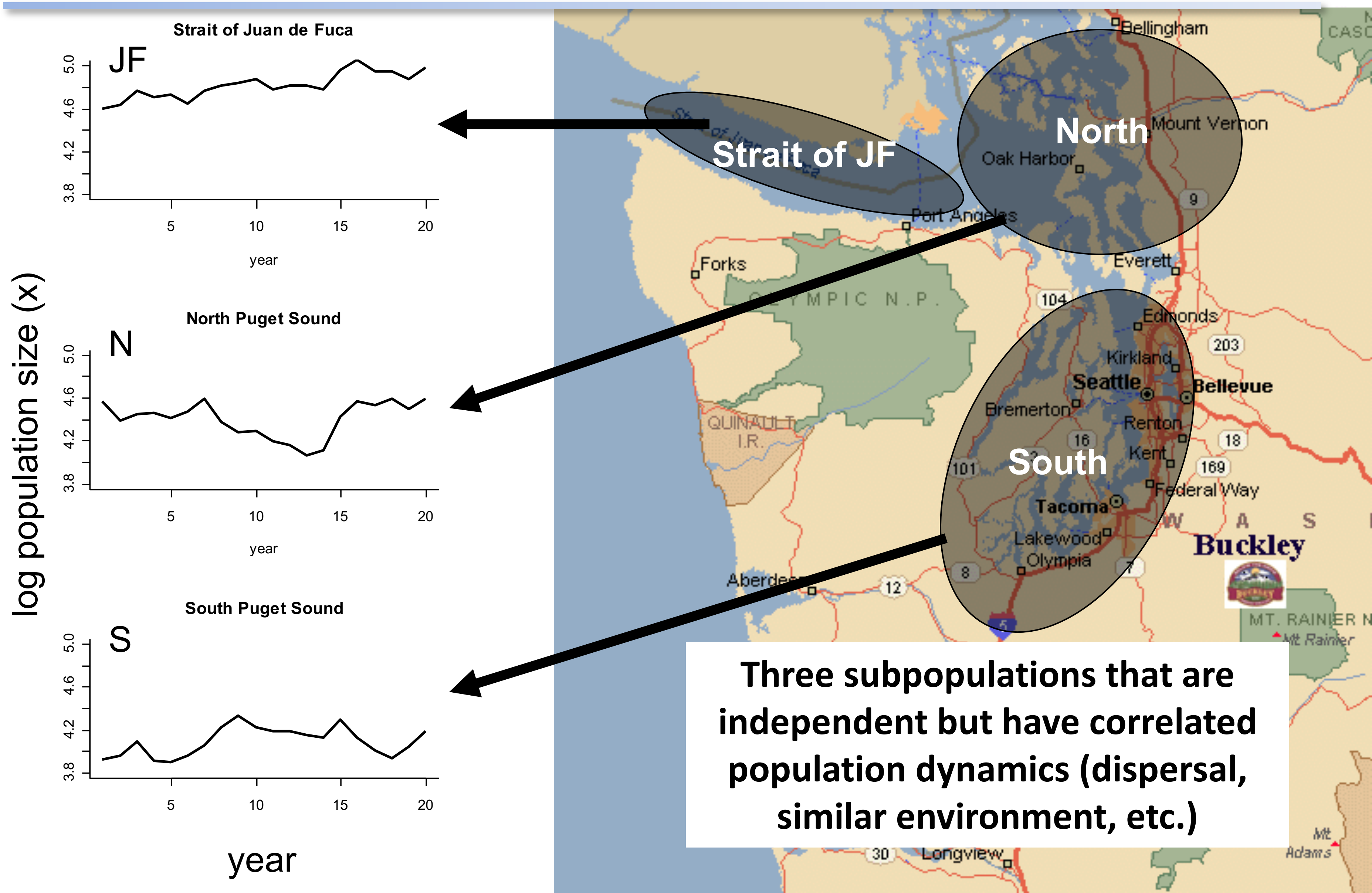
# An example: modeling the population dynamics of harbor seals in Puget Sound, WA

# Multi-site data (Pacific harbor seals)




Jeffries et al. 2003. Trends and Status of Harbor Seals in WA State: 1978-1999. J of Wildlife Management 67: 208-219.

# Let's hypothesize (and model) that the population has 3 subpopulations




**Three subpopulations that are independent but have correlated population dynamics (dispersal, similar environment, etc.)**

# A multivariate model for the population (not the observations but the actual population)

Multivariate stochastic exponential growth

$$\begin{bmatrix} x_{JF,t} \\ x_{N,t} \\ x_{S,t} \end{bmatrix} = \begin{bmatrix} x_{JF,t-1} \\ x_{N,t-1} \\ x_{S,t-1} \end{bmatrix} + \begin{bmatrix} u_{JF} \\ u_{N} \\ u_{S} \end{bmatrix} + \begin{bmatrix} w_{JF,t} \\ w_{N,t} \\ w_{S,t} \end{bmatrix}$$

3 different x's, one for each subpopulation

3 mean population growth rate terms

3 different process errors

$\boldsymbol{e} \sim \mathrm{MVN}(0,\mathbf{Q})$

# The population model in matrix form

Exponential population growth with drift (tendency to increase or decline)

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{u} + \mathbf{w}_t$$

$$\mathbf{w}_t \sim \text{MVN}(0,\mathbf{Q})$$

Each parameter has "structure". Different structures imply different population structure.

# The mean population growth rates (u) can have spatial structure

$$\begin{bmatrix} u_{JF} \\ u_{N} \\ u_{S} \end{bmatrix}$$

unconstrained (all different)

$$\begin{bmatrix} u \\ u \\ u \end{bmatrix}$$

all the same

$$\begin{bmatrix} u_{JF} \\ u_{N\&S} \\ u_{N\&S} \end{bmatrix}$$

Strait of Juan de Fuca different
North and South same

# The process error var-cov matrix can have structure: $\mathbf{w}_t \sim \mathrm{MVN}(0,\mathbf{Q})$

$$\begin{bmatrix} \sigma^2_{JF} & \sigma_{JF,N} & \sigma_{JF,S} \\ \sigma_{JF,N} & \sigma^2_{N} & \sigma_{N,S} \\ \sigma_{JF,S} & \sigma_{N,S} & \sigma^2_{S} \end{bmatrix}$$

unconstrained

variances all different and year-to-year population changes covary

$$\begin{bmatrix} \sigma^2_{JF} & 0 & 0 \\ 0 & \sigma^2_{N} & 0 \\ 0 & 0 & \sigma^2_{S} \end{bmatrix}$$

diagonal

unique variances and year-to-year population changes are uncorrelated

$$\begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$

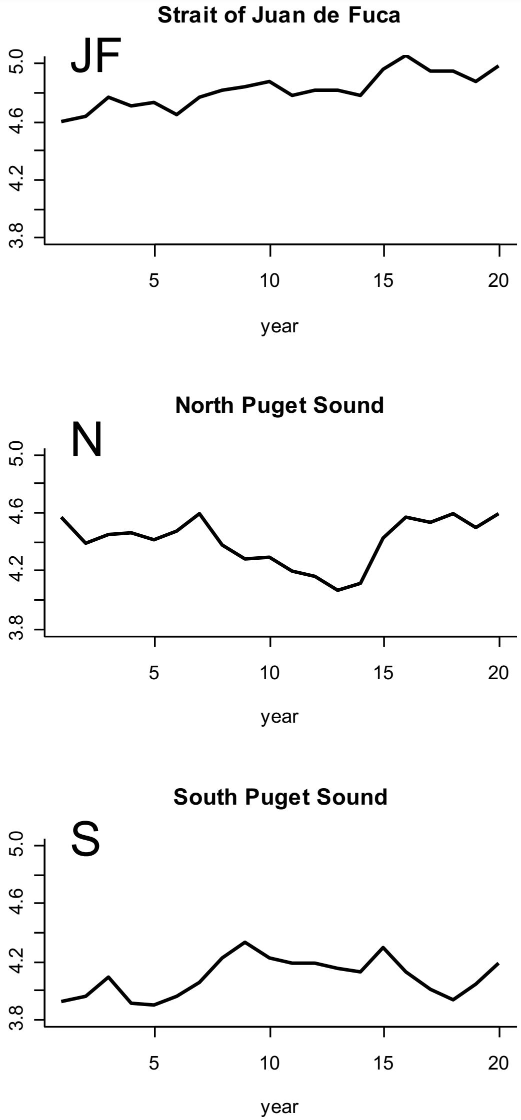
diagonal

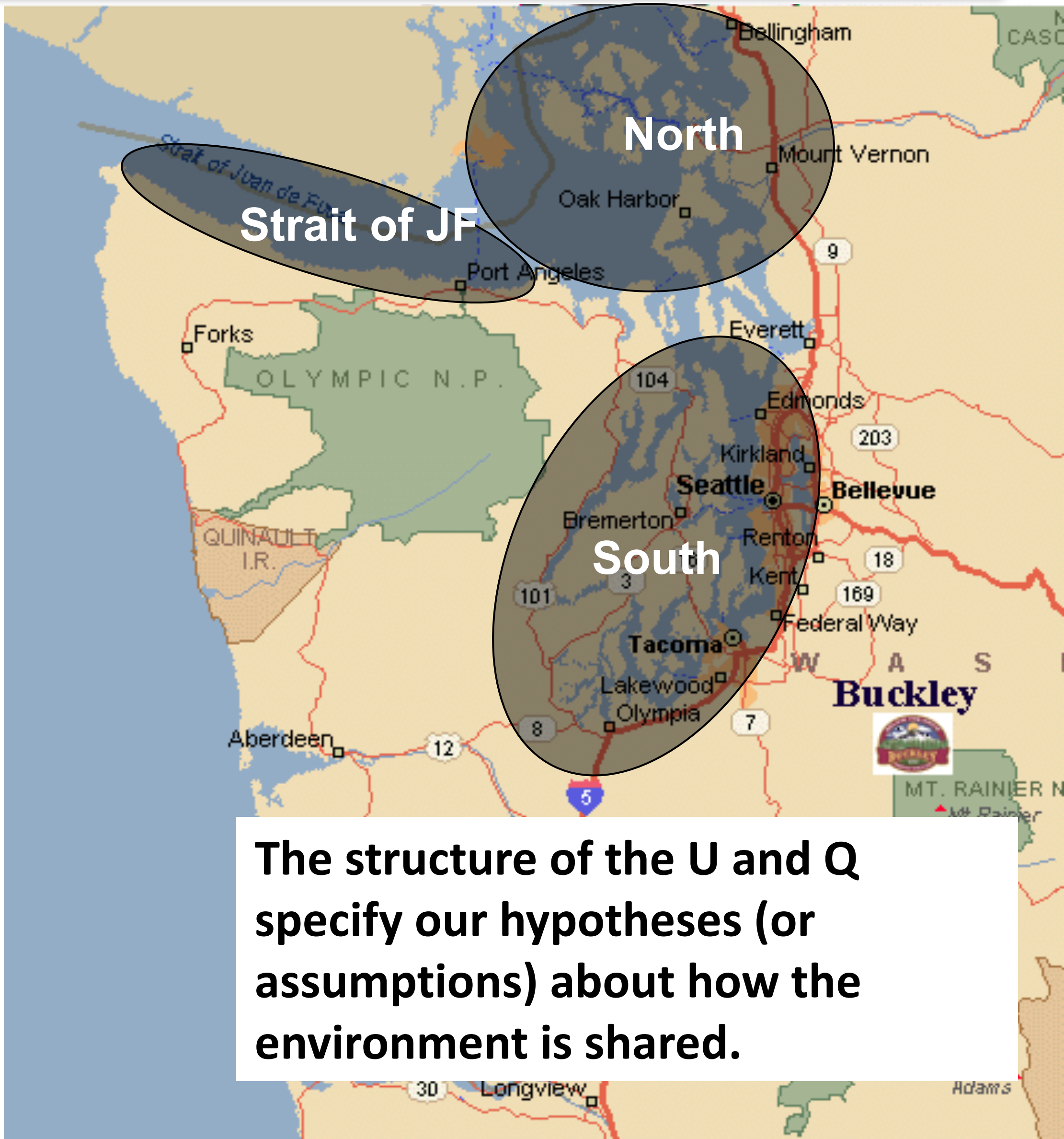
same variances and year-to-year population changes are uncorrelated

$$\begin{bmatrix} \sigma^2 & \alpha & \alpha \\ \alpha & \sigma^2 & \alpha \\ \alpha & \alpha & \sigma^2 \end{bmatrix}$$

JF has unique variance;
N & S share the same variance
yr-to-yr changes have equal covariance

$$x_t = x_{t-1} + u + w_t$$



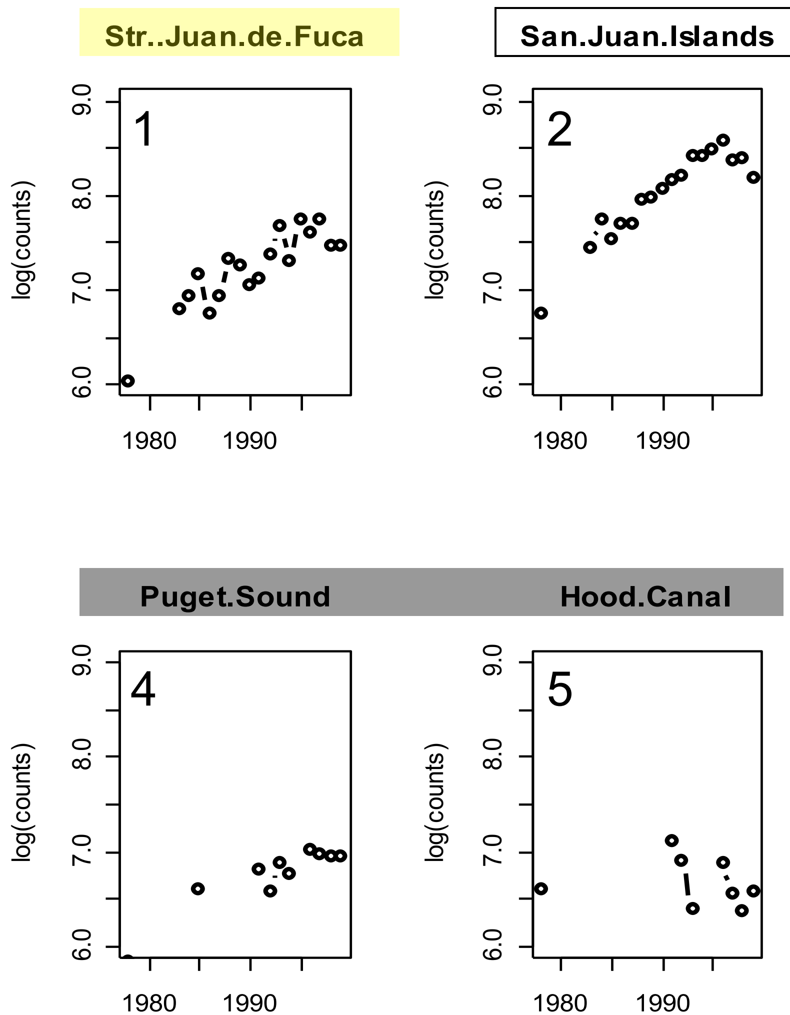

**The structure of the U and Q specify our hypotheses (or assumptions) about how the environment is shared.**

# We observe our subpopulations and those observations have error

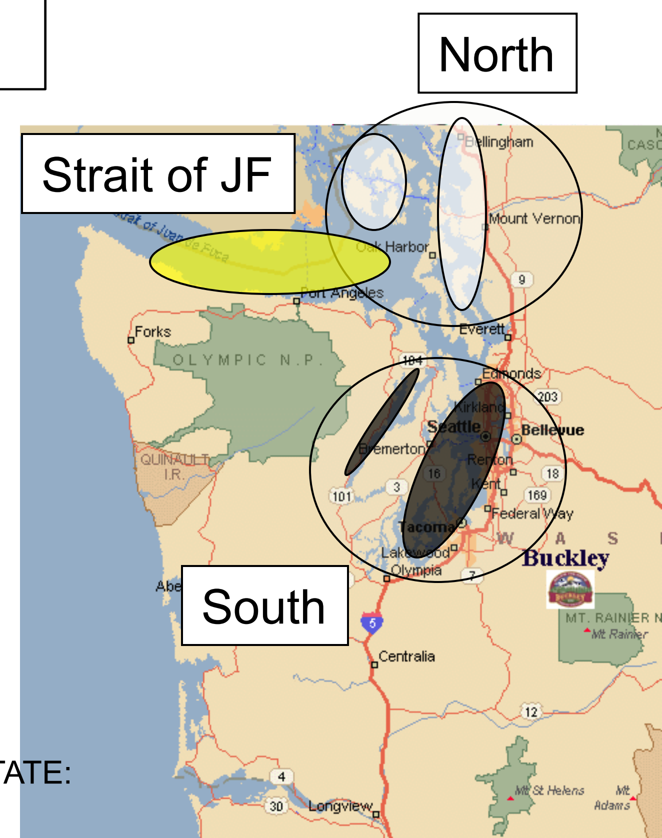
For example, some surveys are from boats. Counting is not perfect and some animals are in the water.


NEFSC/NOAA

# The obs. err. model specifies how the observed time series are related to the true subpopulation sizes




5 sampling locations




Jeffries et al. 2003. TRENDS AND STATUS OF HARBOR SEALS IN WASHINGTON STATE: 1978–1999. J. Wildl. Manage. 67(1):208-219

# The observation model

true population "hidden"

Log of counts

$$
\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \\ y_{4,t} \\ y_{5,t} \end{bmatrix}
=
\begin{array}{ccc} JF & N & S \\ \end{array}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\times
\begin{bmatrix} x_{JF,t} \\ x_{N,t} \\ x_{S,t} \end{bmatrix}
+
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}
+
\begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \\ \eta_{3,t} \\ \eta_{4,t} \\ \eta_{5,t} \end{bmatrix}
$$

**Z** matrix

relates each observation time series to a different state process

observations

observation biases

measurement errors

# The observation errors have a var-cov matrix

$$\begin{bmatrix} \eta_1^2 & \eta_{1,2} & \eta_{1,3} & \eta_{1,4} & \eta_{1,5} \\ \eta_{1,2} & \eta_2^2 & \eta_{3,2} & \eta_{2,4} & \eta_{2,5} \\ \eta_{1,3} & \eta_{3,2} & \eta_3^2 & \eta_{3,4} & \eta_{3,5} \\ \eta_{1,4} & \eta_{2,4} & \eta_{3,4} & \eta_4^2 & \eta_{4,5} \\ \eta_{1,5} & \eta_{2,5} & \eta_{3,5} & \eta_{4,5} & \eta_5^2 \end{bmatrix}$$

unconstrained

$$\begin{bmatrix} \eta_1^2 & 0 & 0 & 0 & 0 \\ 0 & \eta_2^2 & 0 & 0 & 0 \\ 0 & 0 & \eta_3^2 & 0 & 0 \\ 0 & 0 & 0 & \eta_4^2 & 0 \\ 0 & 0 & 0 & 0 & \eta_5^2 \end{bmatrix}$$

unique variances and uncorrelated errors

diagonal

$$\begin{bmatrix} \eta^2 & 0 & 0 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \end{bmatrix}$$

identical variances and uncorrelated errors

diagonal

# The harbor seal multivariate state-space model in matrix form

identity

3x1 vectors

3x3 matrix

$$\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \mathbf{u} + \mathbf{w}_t \quad \text{where } \mathbf{w}_t \sim MVN(0, \mathbf{Q})$$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t \quad \text{where } \mathbf{v}_t \sim MVN(0, \mathbf{R})$$

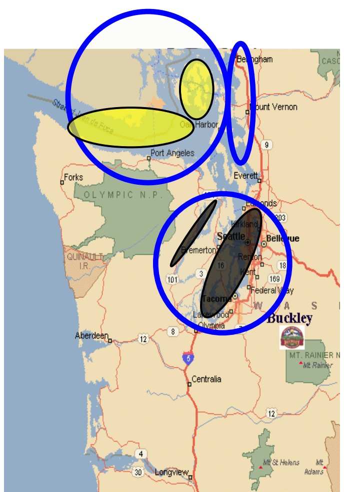
5x5 matrix

5x1 vectors

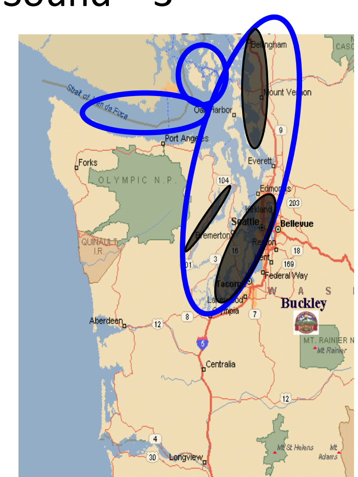
# Instead of N, S, Str. J subpopulations, we could have other combinations and numbers of subpopulations

$$\begin{array}{r} \text{Str. JF} \\ \text{San Isl.} \\ \text{E. Bays} \\ \text{P.S.} \\ \text{Hood C.} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Str of Juan de Fuca & San Juan Is sites = 1st subpop
Eastern bays = $2^{nd}$
Hood C. & S. Puget S. = $3^{rd}$




Strait of Juan de Fuca = $1^{st}$ sub pop
San Juan Is sites = 2nd
Eastern bays, Hood Canal & S. Puget Sound = $3^{rd}$



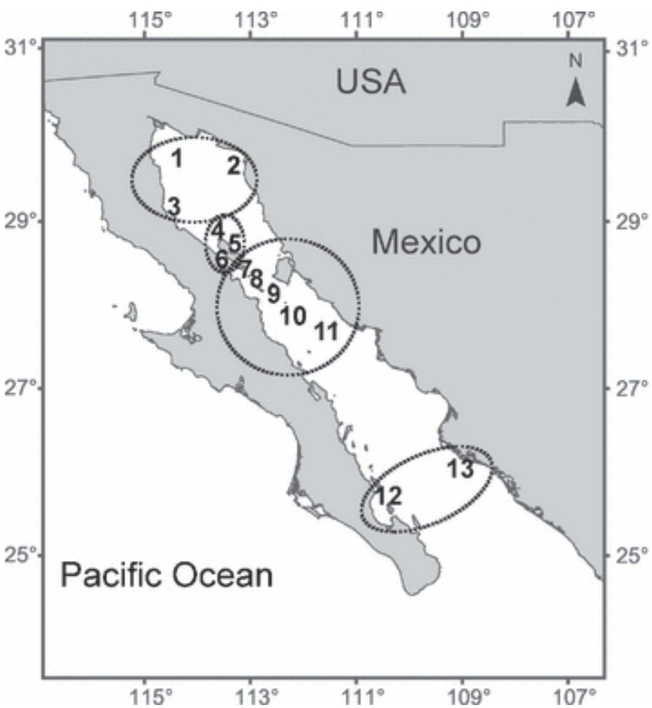

One Puget Sound population and all sites are sampling it
One population

# The same model can capture many different underlying population structures and observation structures

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{u} + \mathbf{w}_t \quad \text{where } \mathbf{w}_t \sim MVN(0, \mathbf{Q})$$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t \quad \text{where } \mathbf{v}_t \sim MVN(0, \mathbf{R})$$

***Inferring spatial structure from time-series data: using multivariate state-space models to detect metapopulation structure of California sea lions in the Gulf of California, Mexico***




(NOAA, Channel Is)

Hypotheses about the population structure:
Diet, Disease, DNA, Distance
(2 null models: no structure and fully structured)

**Journal of Applied Ecology**
Volume 47, Issue 1, pages 47-56, 15 DEC 2009 DOI: 10.1111/j.1365-2664.2009.01745.x
http://onlinelibrary.wiley.com/doi/10.1111/j.1365-2664.2009.01745.x/full#f1

**Table 1.** **Model performance, given by Akaike's Information Criterion (AIC) *b*-value, across the six hypotheses for the subpopulation configuration**

| Parameters | | | Hypotheses ($m$ = no. subpopulations) | | | | | |
|---|---|---|---|---|---|---|---|---|
| **u** | **Q** | **R** | **Panmictic ($m = 1$)** | **Diet ($m = 4$)** | **Disease ($m = 4$)** | **Distance ($m = 4$)** | **DNA ($m = 2$)** | **Independent ($m = 11$)** |
| Same | Same | Same | 68·2 | 48·4 | 49·8 | 26·8 | 38·9 | 22·2 |
| Unique | Same | Same | | 63·9 | 72·8 | 46·9 | 46·6 | 25·5 |
| Same | Unique | Same | | 55·8 | 57·6 | 26·6 | 34·2 | 64·4 |
| Same | Same | Unique | 97·3 | 74·4 | 73·2 | 68·4 | 67·3 | 32·5 |
| Unique | Same | Unique | | 87·1 | 91·3 | 71·8 | 69·3 | 65·6 |
| Unique | Unique | Same | | 61·4 | 84·1 | 39·8 | 38·3 | 50·0 |
| Same | Unique | Unique | | 102·8 | 103·8 | 202·1 | 82·7 | 114·7 |
| Unique | Unique | Unique | | 111·8 | 133·8 | 167·8 | 77·5 | 169·2 |
| Same | Correlated | Same | | 40·3 | 63·1 | 37·0 | 38·3 | 4804·7 |
| **Unique** | **Correlated** | **Same** | | 44·9 | 87·2 | **13·7** | 39·6 | 989·4 |
| Same | Correlated | Unique | | 110·3 | 163·8 | 321·4 | 102·2 | NA |
| Unique | Correlated | Unique | | 116·3 | 176·5 | 467·9 | 94·5 | NA |

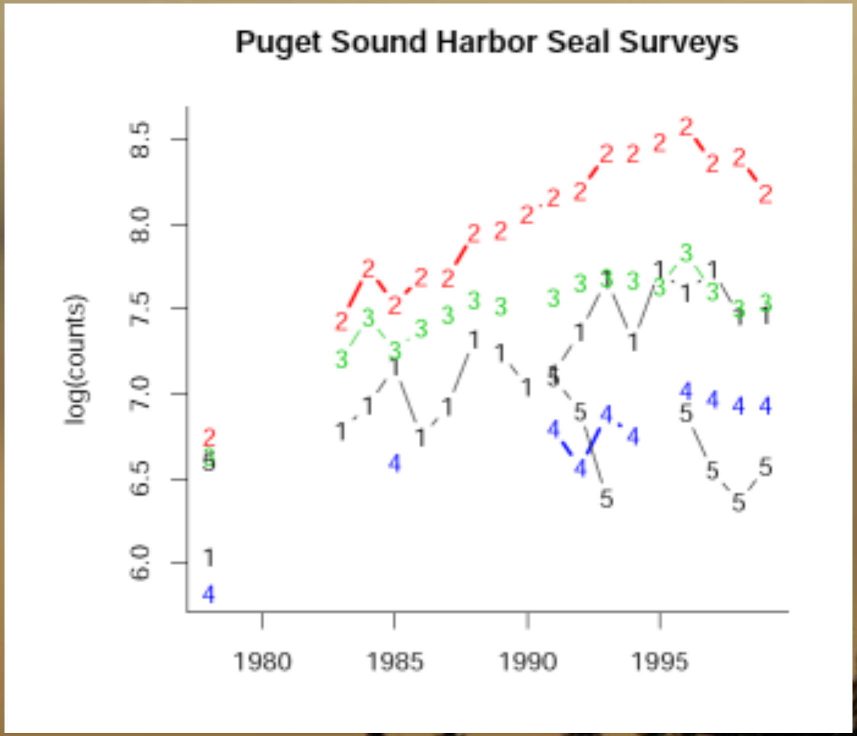
Process errors (**Q**) may be independent (a diagonal matrix) with variances that are the same magnitude across subpopulations (same), independent with unequal variances across subpopulations (unique) or may be temporally correlated, meaning an unconstrained **Q** matrix (correlated). The growth rate (**u**) and observation error matrix (**R**) parameters may also be equal (same) or unique across subpopulations. The model best supported by the data is shown in bold; complex models that did not fully converge are not applicable.

# The MARSS manual has two chapters using MARSS models to analyze spatial count data

harbor seal count data from the west coast of the USA


# Chapter 7 in HWS 2014

Chapter 7: Combining multi-site data to estimate regional population trends




# Chapter 8 in HWS 2014

Identifying spatial structure and covariance in harbor seals on the west coast of the USA

2000km

# Shortcut for the Z matrix

$$\text{count} \quad \begin{array}{r} \texttt{Coastal Estuaries} \\ \texttt{Olympic Peninsula} \\ \texttt{Str. Juan de Fuca} \\ \texttt{San Juan Islands} \\ \texttt{Eastern Bays} \\ \texttt{Puget Sound} \\ \texttt{CA.Mainland} \\ \texttt{CA.ChannelIslands} \\ \texttt{OR North Coast} \\ \texttt{OR South Coast} \\ \texttt{Georgia Strait} \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{wa.or,t} \\ x_{ps,t} \\ x_{ca,t} \end{bmatrix} + a + v$$

Z matrix

factor(c(“or.wa”,”or.wa,”ps”,”ps”,”ps”,”ps”,”ca”,”ca”,”or.wa”,”or.wa”,”ps”))