Intro to ARMA models

FISH 507 – Applied Time Series Analysis

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Topics for today

Review

- White noise
- Random walks

Autoregressive (AR) models

Moving average (MA) models

Autoregressive moving average (ARMA) models

Using ACF & PACF for model ID

White noise (WN)

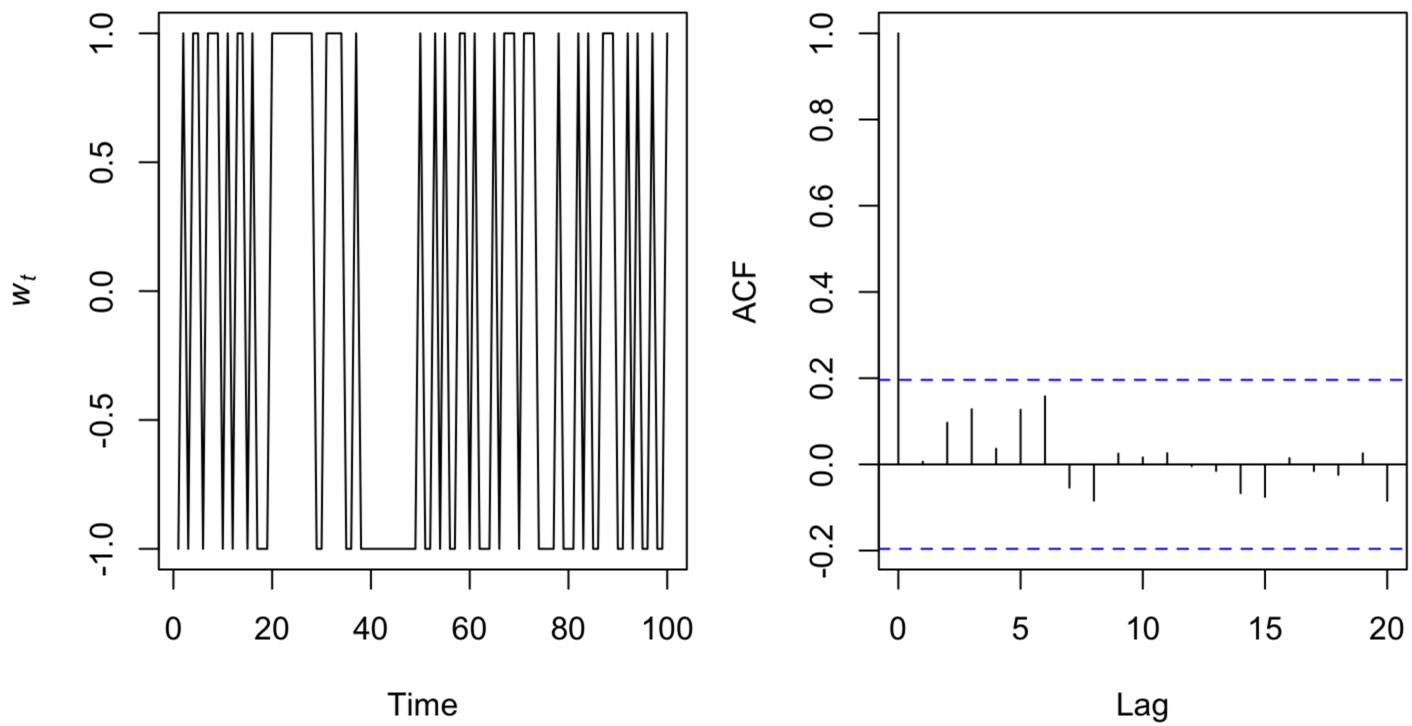
A time series $\{w_t\}$ is discrete white noise if its values are

1. independent

2. identically distributed with a mean of zero

The distributional form for $\{w_t\}$ is flexible

White noise (WN)



$$w_t = 2e_t - 1; e_t \sim \text{Bernoulli}(0.5)$$

Gaussian white noise

We often assume so-called *Gaussian white noise*, whereby

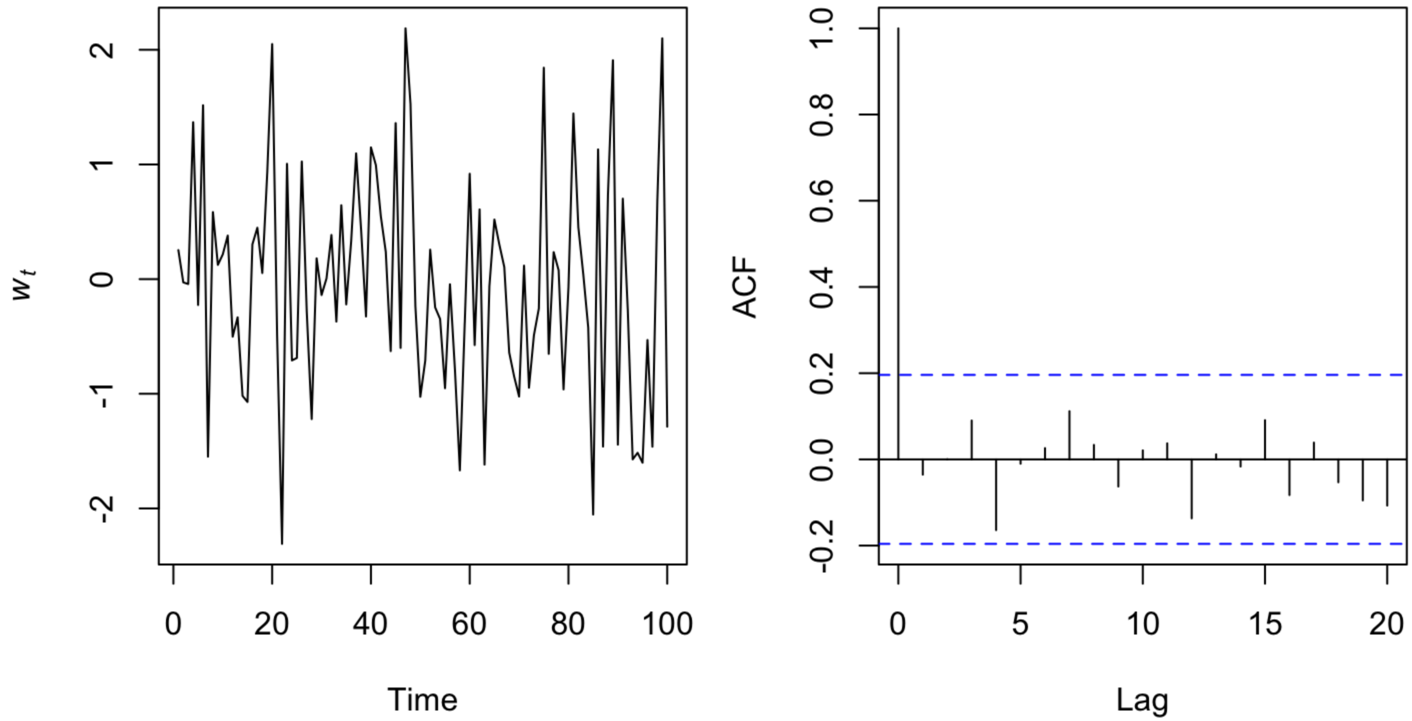
$$w_t \sim \text{N}(0, \sigma^2)$$

and the following apply as well

autocovariance: $\gamma_k = \begin{cases} \sigma^2 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$

autocorrelation: $\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$

Gaussian white noise



$w_t \sim \mathrm{N}(0, 1)$

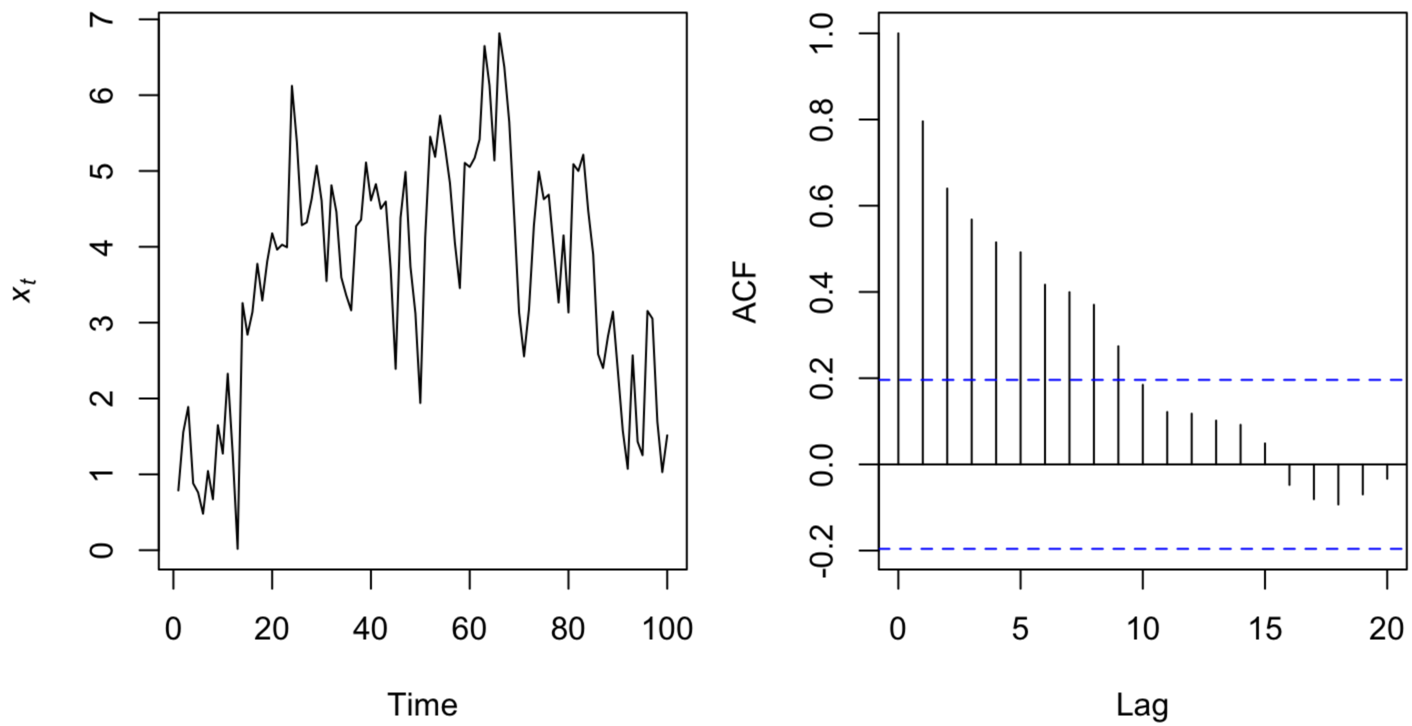
Random walk (RW)

A time series $\{x_t\}$ is a random walk if

1. $x_t = x_{t-1} + w_t$

2. w_t is white noise

Random walk (RW)



$$x_t = x_{t-1} + w_t; w_t \sim \mathrm{N}(0, 1)$$

Random walk (RW)

Of note: Random walks are extremely flexible models and can be fit to many kinds of time series

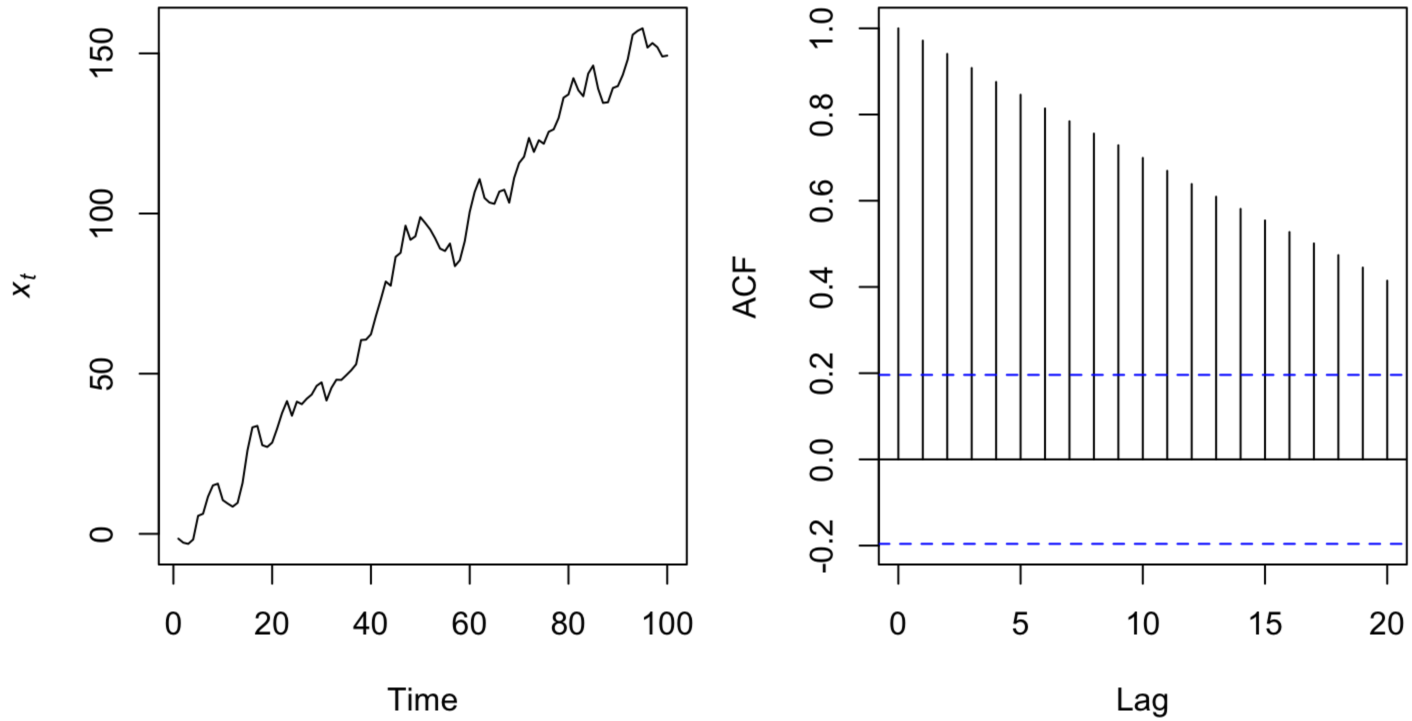
Biased random walk

A *biased random walk* (or *random walk with drift*) is written as

$$x_t = x_{t-1} + u + w_t$$

where u is the bias (drift) per time step and w_t is white noise

Biased random walk



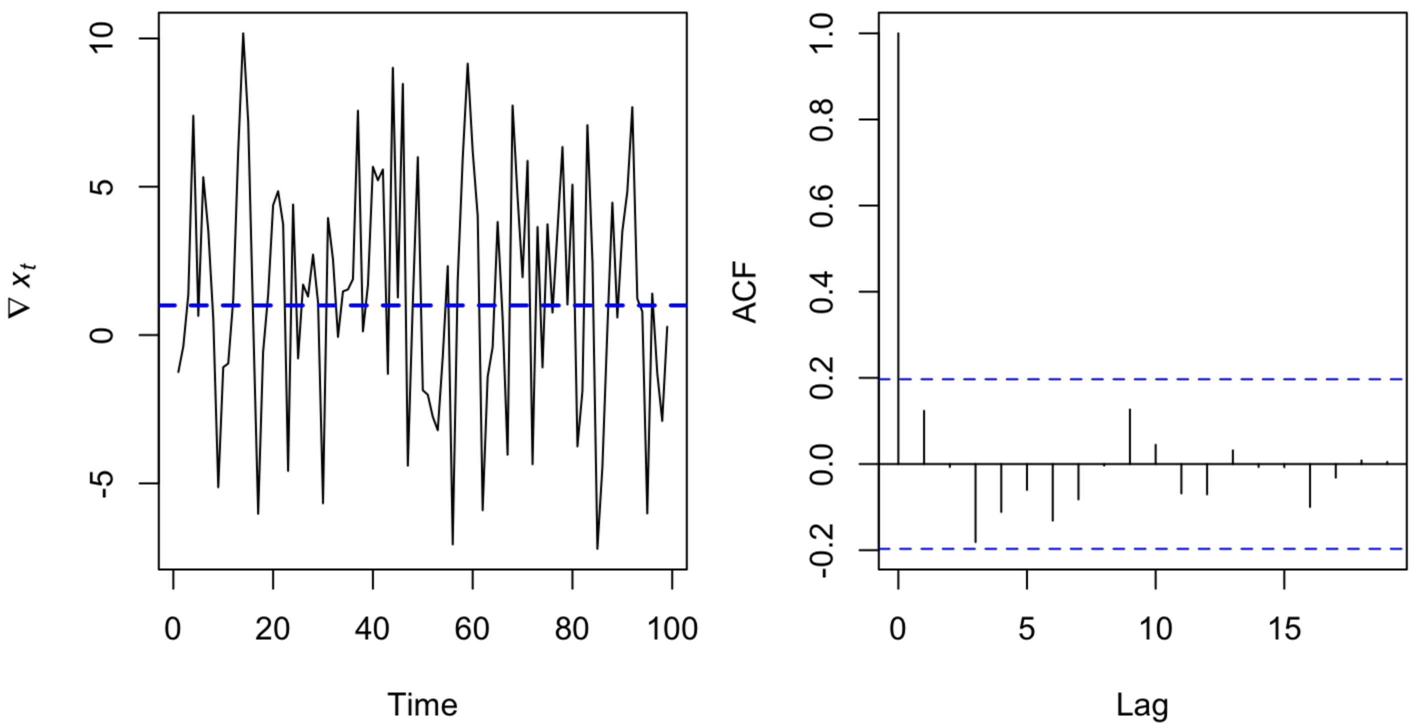
$$x_t = x_{t-1} + 1 + w_t; w_t \sim \mathrm{N}(0, 4)$$

Differencing a biased random walk

First-differencing a biased random walk yields a constant mean (level) u plus white noise

$$
\begin{aligned}
x_t &= x_{t-1} + u + w_t \\
&\Downarrow \\
\nabla(x_t &= x_{t-1} + u + w_t) \\
x_t - x_{t-1} &= x_{t-1} + u + w_t - x_{t-1} \\
x_t - x_{t-1} &= u + w_t
\end{aligned}
$$

Differencing a biased random walk



$$x_t - x_{t-1} = 1 + w_t; w_t \sim \mathrm{N}(0, 1)$$

Linear stationary models

Linear stationary models

We saw last week that linear filters are a useful way of modeling time series

Here we extend those ideas to a general class of models call *autoregressive moving average* (ARMA) models

Autoregressive (AR) models

Autoregressive models are widely used in ecology to treat a current state of nature as a function its past state(s)

Autoregressive (AR) models

An *autoregressive* model of order p, or AR(p), is defined as

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + w_t$$

where we assume

1. w_t is white noise

2. $\phi_p \neq 0$ for an order-p process

Examples of AR(p) models

AR(1)

$$x_t = 0.5x_{t-1} + w_t$$

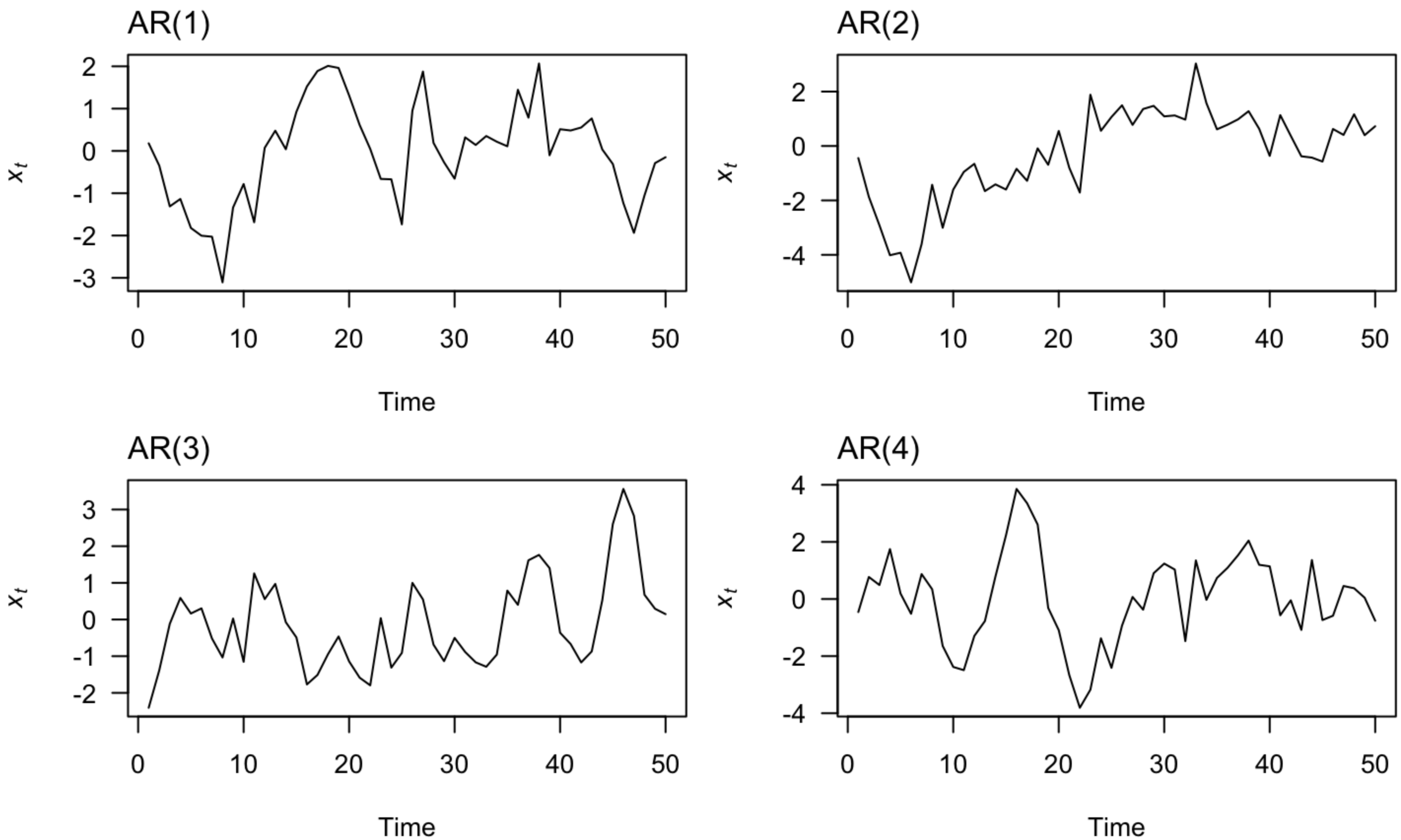
AR(1) with $\phi_1 = 1$ (random walk)

$$x_t = x_{t-1} + w_t$$

AR(2)

$$x_t = -0.2x_{t-1} + 0.4x_{t-2} + w_t$$

Examples of AR(p) models



Stationary AR(p) models

Recall that *stationary* processes have the following properties

1. no systematic change in the mean or variance
2. no systematic trend
3. no periodic variations or seasonality

We seek a means for identifying whether our AR(p) models are also stationary

Stationary AR(p) models

We can write out an AR(p) model using the backshift operator

$$
\begin{aligned}
x_t &= \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + w_t \\
&\Downarrow \\
x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - \dots - \phi_p x_{t-p} &= w_t \\
(1 - \phi_1 \mathbf{B} - \phi_2 \mathbf{B}^2 - \dots - \phi_p \mathbf{B}^p) x_t &= w_t \\
\phi_p(\mathbf{B}) x_t &= w_t
\end{aligned}
$$

Stationary AR(p) models

If we treat $\mathbf{B}$ as a number (or numbers), we can out write the *characteristic equation* as

$$
\begin{gathered}
\phi_p(\mathbf{B}) x_t = w_t \\
\Downarrow \\
\phi_p(\mathbf{B}) = 0
\end{gathered}
$$

To be stationary, **all roots** of the characteristic equation **must exceed 1 in absolute value**

Stationary AR(p) models

For example, consider this AR(1) model from earlier

$$x_t = 0.5x_{t-1} + w_t$$

Stationary AR(p) models

For example, consider this AR(1) model from earlier

$$x_t = 0.5x_{t-1} + w_t$$
$$\Downarrow$$
$$\begin{aligned}
x_t - 0.5x_{t-1} &= w_t \\
x_t - 0.5\mathbf{B}x_t &= w_t \\
(1 - 0.5\mathbf{B})x_t &= w_t
\end{aligned}$$

Stationary AR(p) models

For example, consider this AR(1) model from earlier

$$
\begin{aligned}
(1 - 0.5\mathbf{B})x_t &= w_t \\
&\Downarrow \\
1 - 0.5\mathbf{B} &= 0 \\
-0.5\mathbf{B} &= -1 \\
\mathbf{B} &= 2
\end{aligned}
$$

This model is indeed stationary because $\mathbf{B} > 1$

Stationary AR(p) models

What about this AR(2) model from earlier?

$$x_t = -0.2x_{t-1} + 0.4x_{t-2} + w_t$$

Stationary AR(p) models

What about this AR(2) model from earlier?

$$x_t = -0.2x_{t-1} + 0.4x_{t-2} + w_t$$
$$\Downarrow$$
$$x_t + 0.2x_{t-1} - 0.4x_{t-2} = w_t$$
$$x_t + 0.2\mathbf{B}x_t - 0.4\mathbf{B}^2x_t = w_t$$
$$(1 + 0.2\mathbf{B} - 0.4\mathbf{B}^2)x_t = w_t$$

Stationary AR(p) models

What about this AR(2) model from earlier?

$$(1 + 0.2\mathbf{B} - 0.4\mathbf{B}^2)x_t = w_t$$
$$\Downarrow$$
$$1 + 0.2\mathbf{B} - 0.4\mathbf{B}^2 = 0$$
$$\Downarrow$$
$$\mathbf{B}_1 \approx -1.35 \text{ and } \mathbf{B}_2 \approx 1.85$$

This model is *not* stationary because only $\mathbf{B}_2 > 1$

What about random walks?

Consider our random walk model

$$x_t = x_{t-1} + w_t$$

What about random walks?

Consider our random walk model

$$x_t = x_{t-1} + w_t$$

$$\Downarrow$$

$$\begin{aligned} x_t - x_{t-1} &= w_t \\ x_t - 1\mathbf{B}x_t &= w_t \\ (1 - 1\mathbf{B})x_t &= w_t \end{aligned}$$

What about random walks?

Consider our random walk model

$$
\begin{aligned}
x_t - x_{t-1} &= w_t \\
x_t - 1\mathbf{B}x_t &= w_t \\
(1 - 1\mathbf{B})x_t &= w_t \\
&\Downarrow \\
1 - 1\mathbf{B} &= 0 \\
-1\mathbf{B} &= -1 \\
\mathbf{B} &= 1
\end{aligned}
$$

Random walks are **not** stationary because $\mathbf{B} = 1 \ngtr 1$

Stationary AR(1) models

We can define a parameter space over which all AR(1) models are stationary

$$x_t = \phi x_{t-1} + w_t$$

Stationary AR(1) models

We can define a parameter space over which all AR(1) models are stationary

$$
\begin{aligned}
x_t &= \phi x_{t-1} + w_t \\
&\Downarrow \\
x_t - \phi x_{t-1} &= w_t \\
x_t - \phi \mathbf{B} x_t &= w_t \\
(1 - \phi \mathbf{B}) x_t &= w_t
\end{aligned}
$$

Stationary AR(1) models

For $x_t = \phi x_{t-1} + w_t$, we have

$$
\begin{aligned}
(1 - \phi \mathbf{B}) x_t &= w_t \\
&\Downarrow \\
1 - \phi \mathbf{B} &= 0 \\
-\phi \mathbf{B} &= -1 \\
\mathbf{B} &= \frac{1}{\phi} \\
&\Downarrow \\
\mathbf{B} = \frac{1}{\phi} &> 1 \text{ iff } 0 < \phi < 1
\end{aligned}
$$

Stationary AR(1) models

What if ϕ is negative, such that $x_t = -\phi x_{t-1} + w_t$?

$$
\begin{gathered}
x_t = -\phi x_{t-1} + w_t \\
\Downarrow \\
\begin{aligned}
x_t + \phi x_{t-1} &= w_t \\
x_t + \phi \mathbf{B} x_t &= w_t \\
(1 + \phi \mathbf{B}) x_t &= w_t
\end{aligned}
\end{gathered}
$$

Stationary AR(1) models

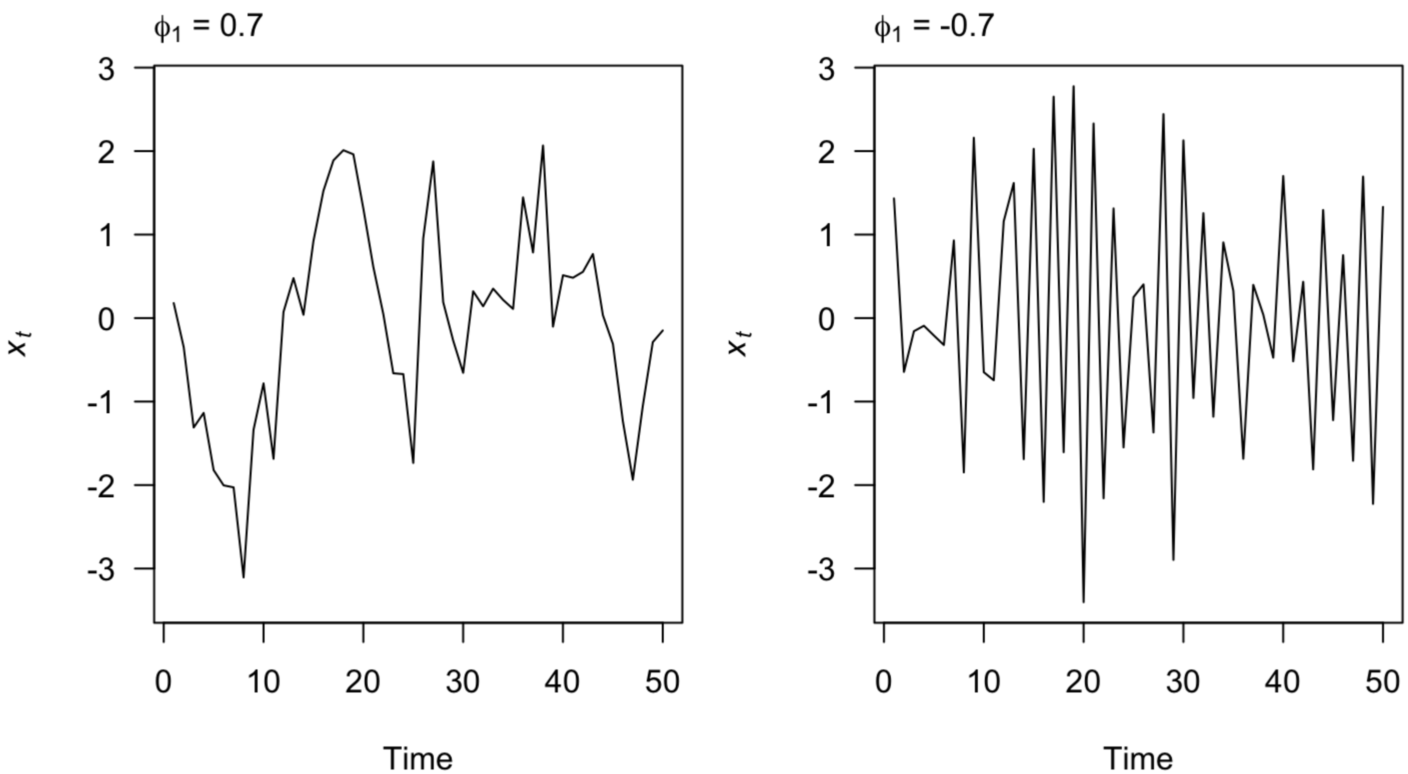
For $x_t = -\phi x_{t-1} + w_t$, we have

$$
\begin{aligned}
(1 + \phi \mathbf{B}) x_t &= w_t \\
&\Downarrow \\
1 + \phi \mathbf{B} &= 0 \\
\phi \mathbf{B} &= -1 \\
\mathbf{B} &= -\frac{1}{\phi} \\
&\Downarrow \\
\mathbf{B} = -\frac{1}{\phi} &> 1 \text{ iff } -1 < \phi < 0
\end{aligned}
$$

Stationary AR(1) models

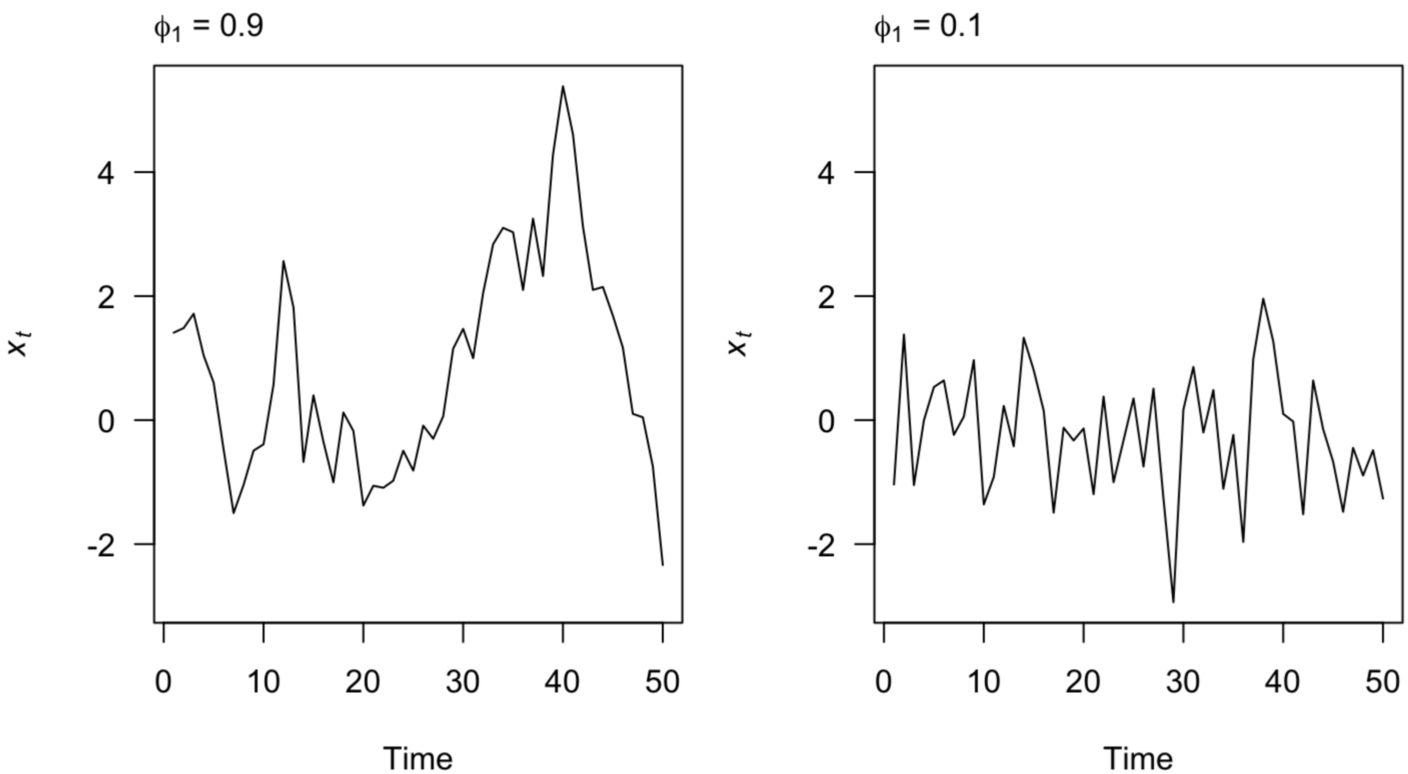
Thus, AR(1) models are stationary if and only if $|\phi| < 1$

Coefficients of AR(1) models



Same value, but different sign

Coefficients of AR(1) models

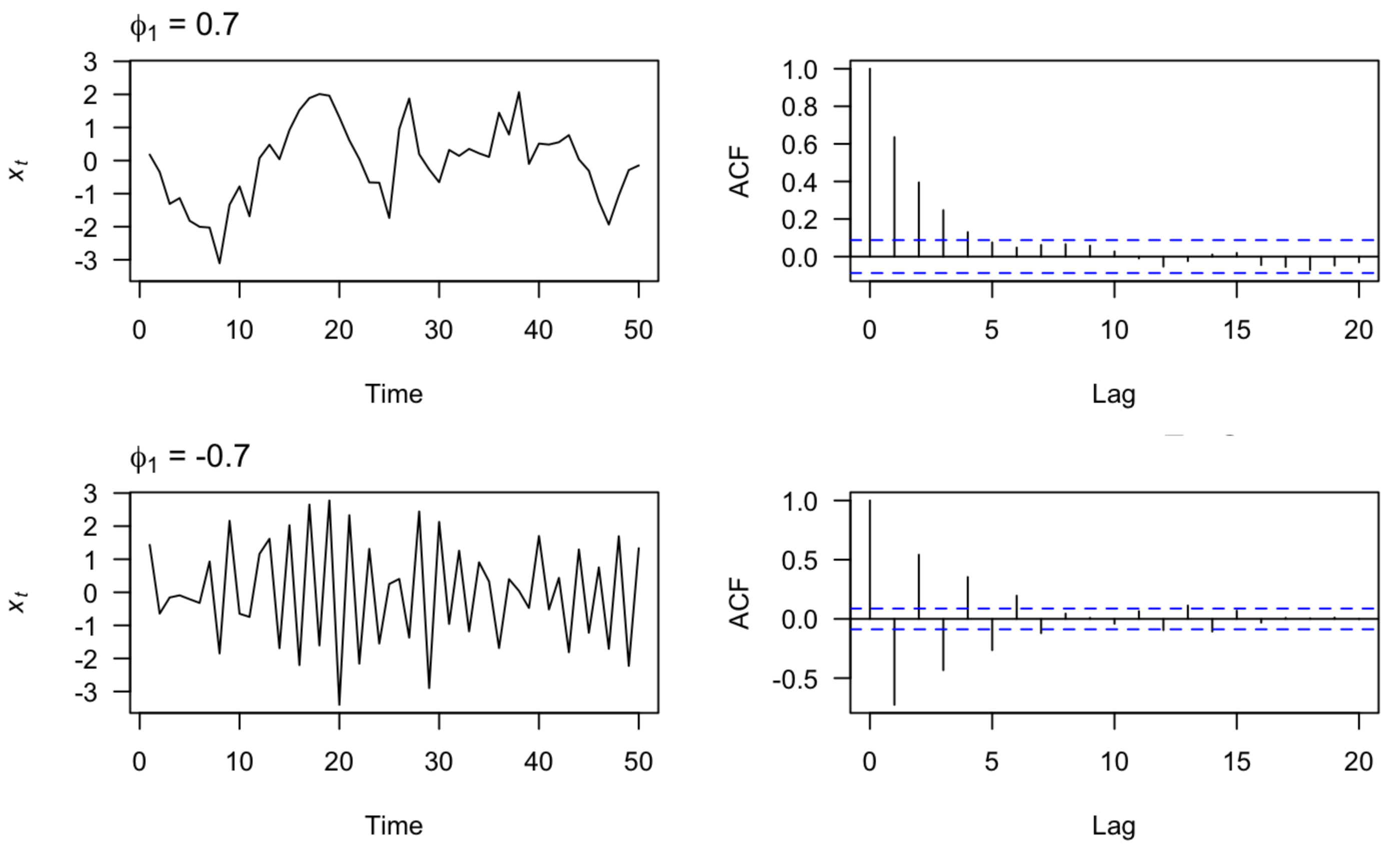


Both positive, but different magnitude

Autocorrelation function (ACF)

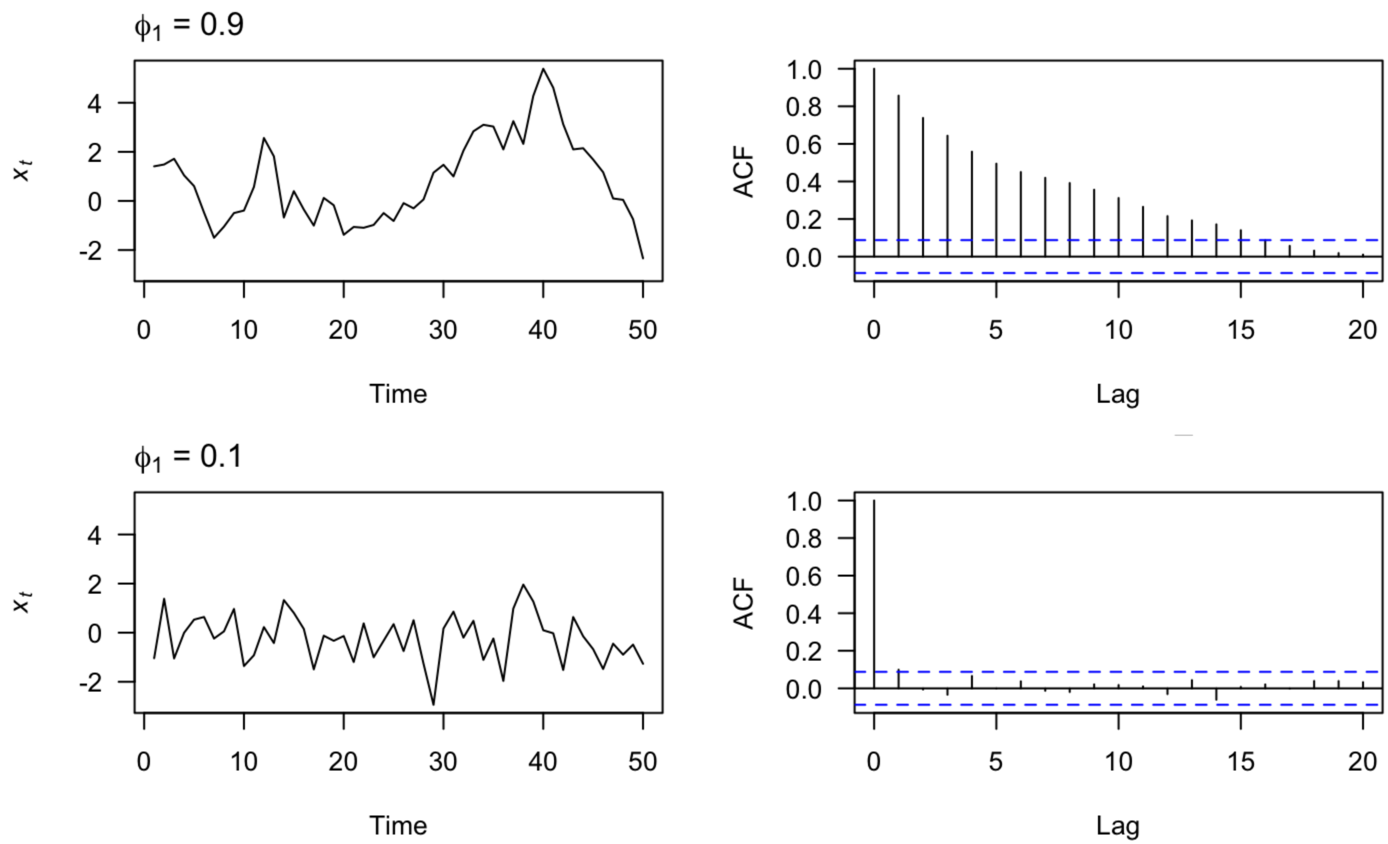
Recall that the *autocorrelation function* (ρ_k) measures the correlation between $\{x_t\}$ and a shifted version of itself $\{x_{t+k}\}$

ACF for AR(1) models



ACF oscillates for model with $-\phi$

ACF for AR(1) models

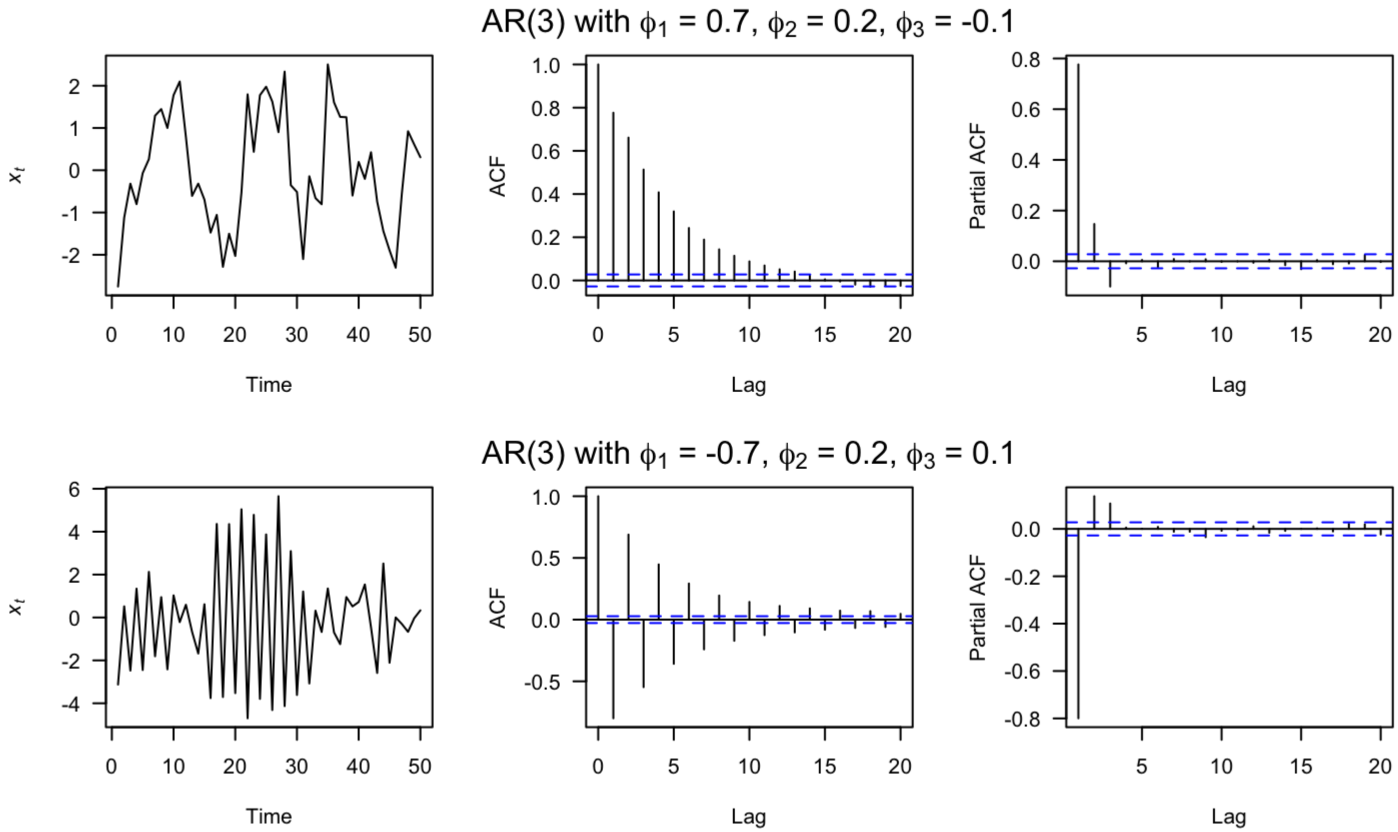


For model with large ϕ, ACF has longer tail

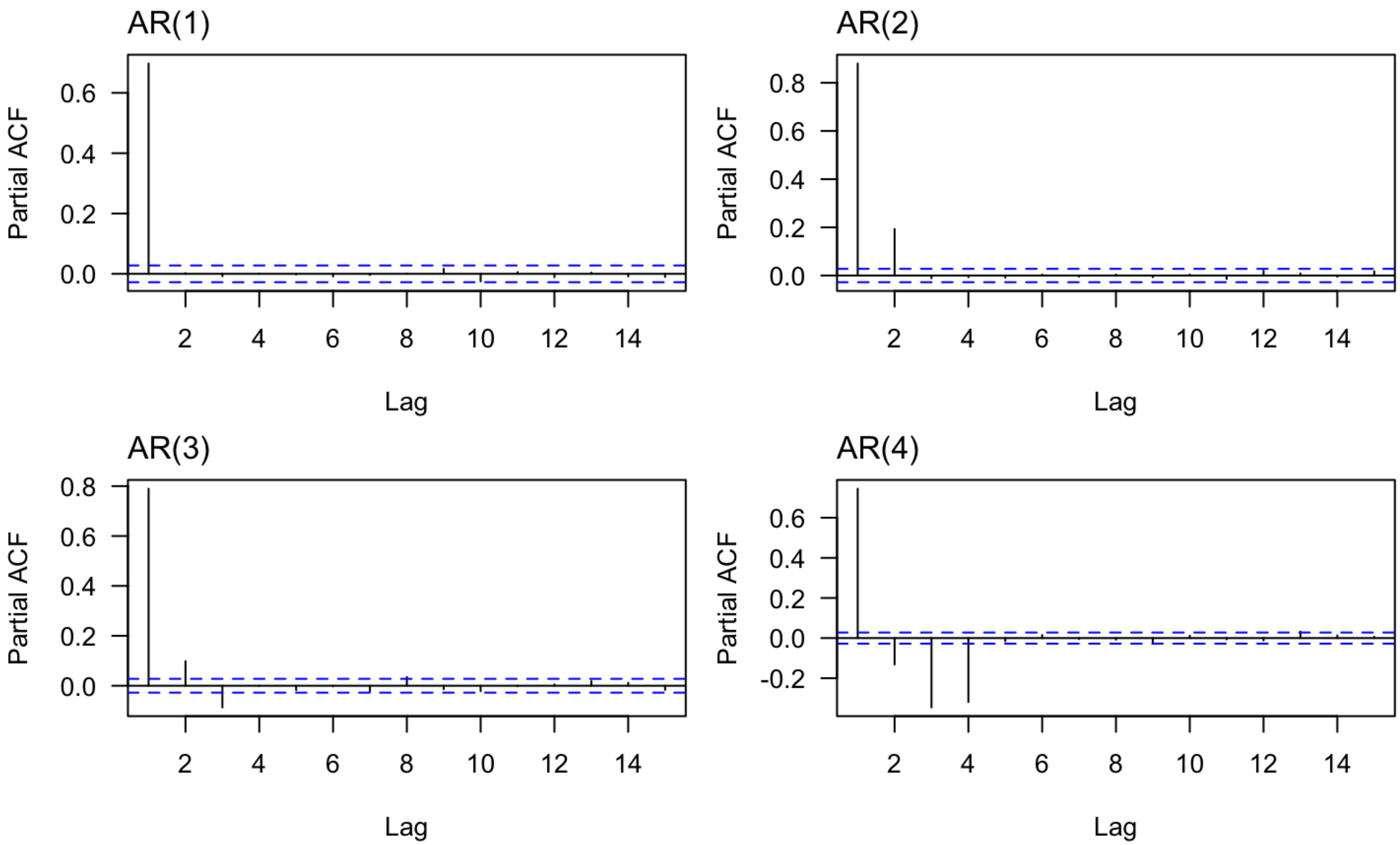
Partial autocorrelation funcion (PACF)

Recall that the *partial autocorrelation function* (ϕ_k) measures the correlation between $\{x_t\}$ and a shifted version of itself $\{x_{t+k}\}$, with the linear dependence of $\{x_{t-1}, x_{t-2}, \dots, x_{t-k-1}\}$ removed

ACF & PACF for AR(p) models



PACF for AR(p) models



Do you see the link between the order p and lag k?

Using ACF & PACF for model ID

Model	ACF	PACF
AR(p)	Tails off slowly	Cuts off after lag p

Moving average (MA) models

Moving average models are most commonly used for forecasting a future state

Moving average (MA) models

A moving average model of order q, or MA(q), is defined as

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

where w_t is white noise

Each of the x_t is a sum of the most recent error terms

Moving average (MA) models

A moving average model of order q, or MA(q), is defined as

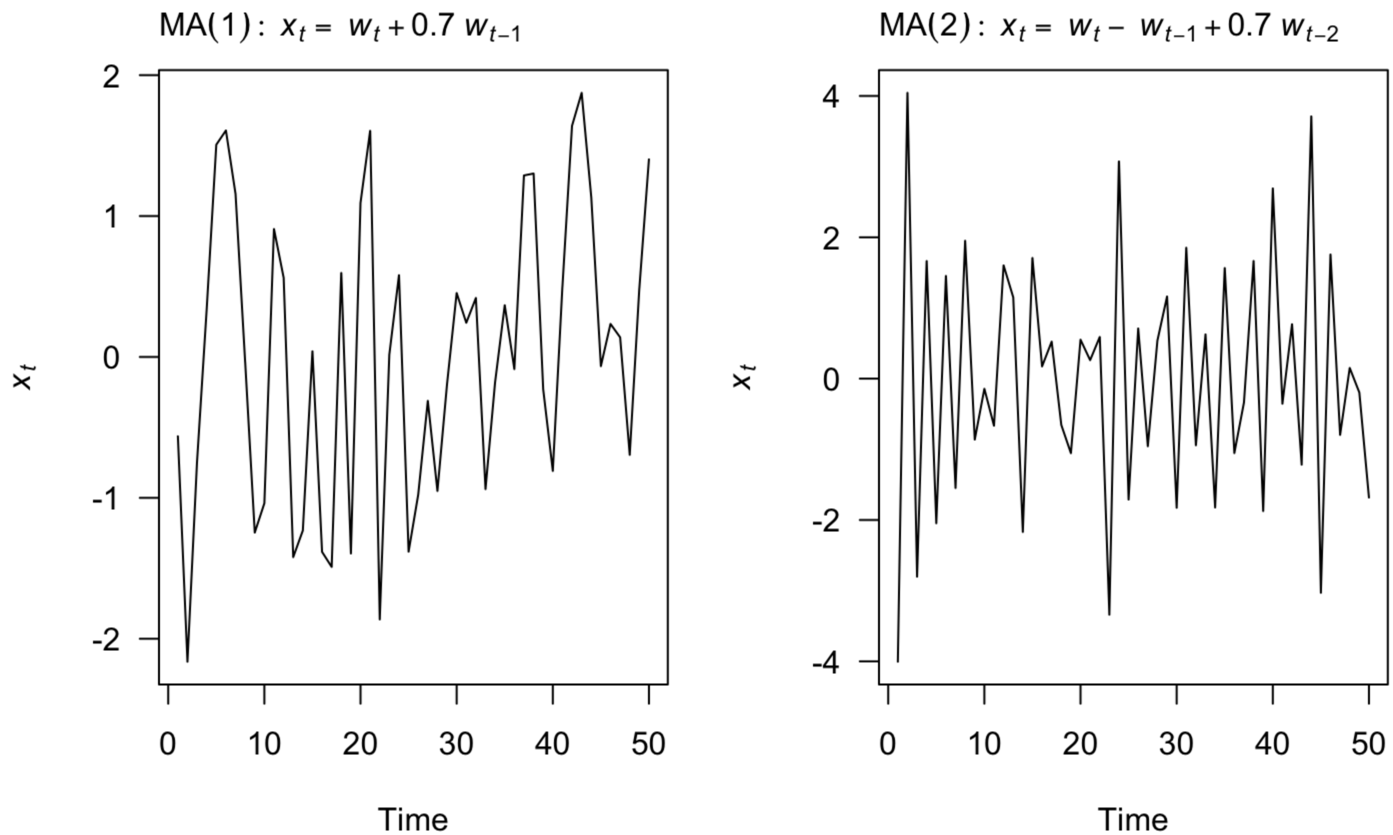
$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

where w_t is white noise

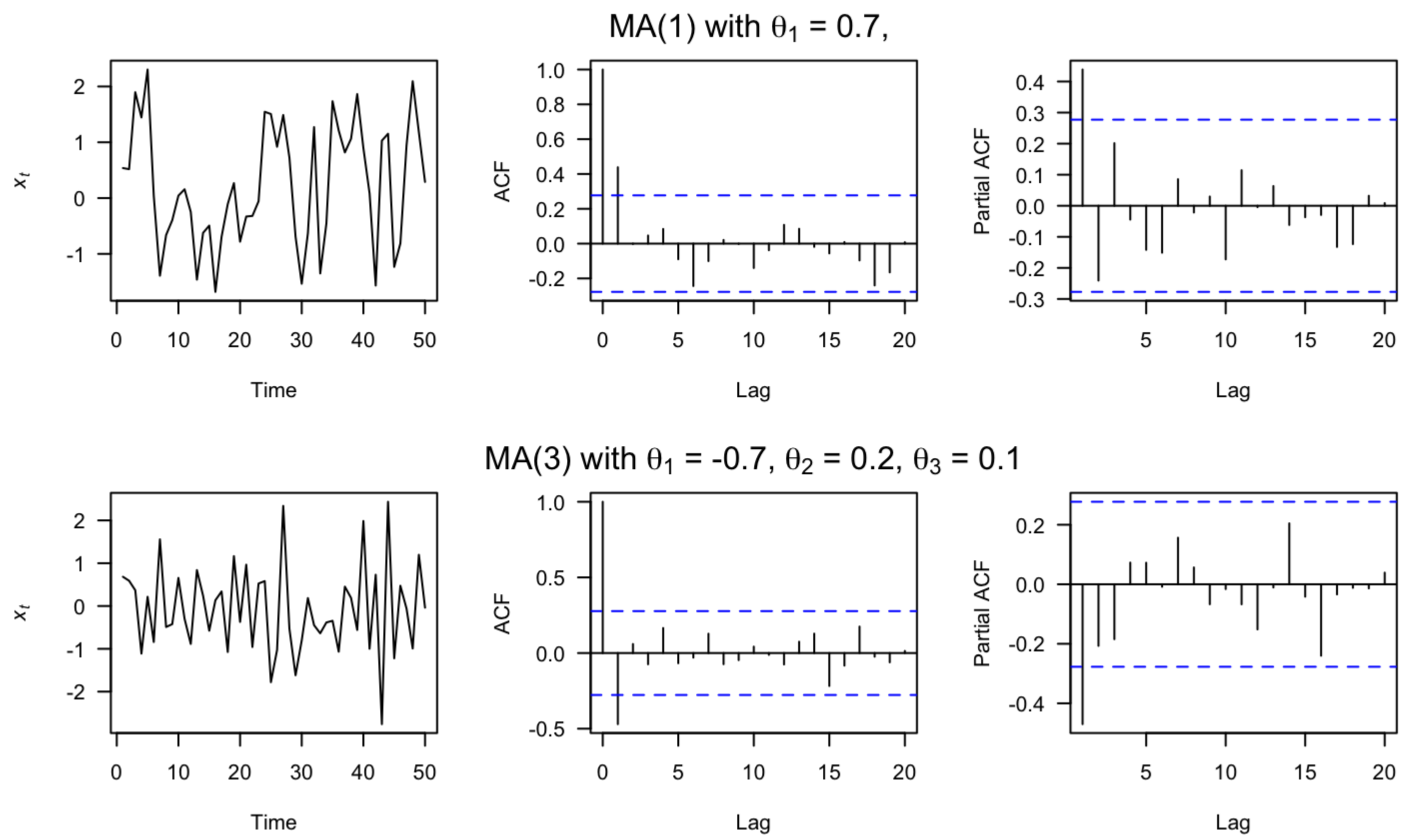
Each of the x_t is a sum of the most recent error terms

Thus, *all* MA processes are stationary because they are finite sums of stationary WN processes

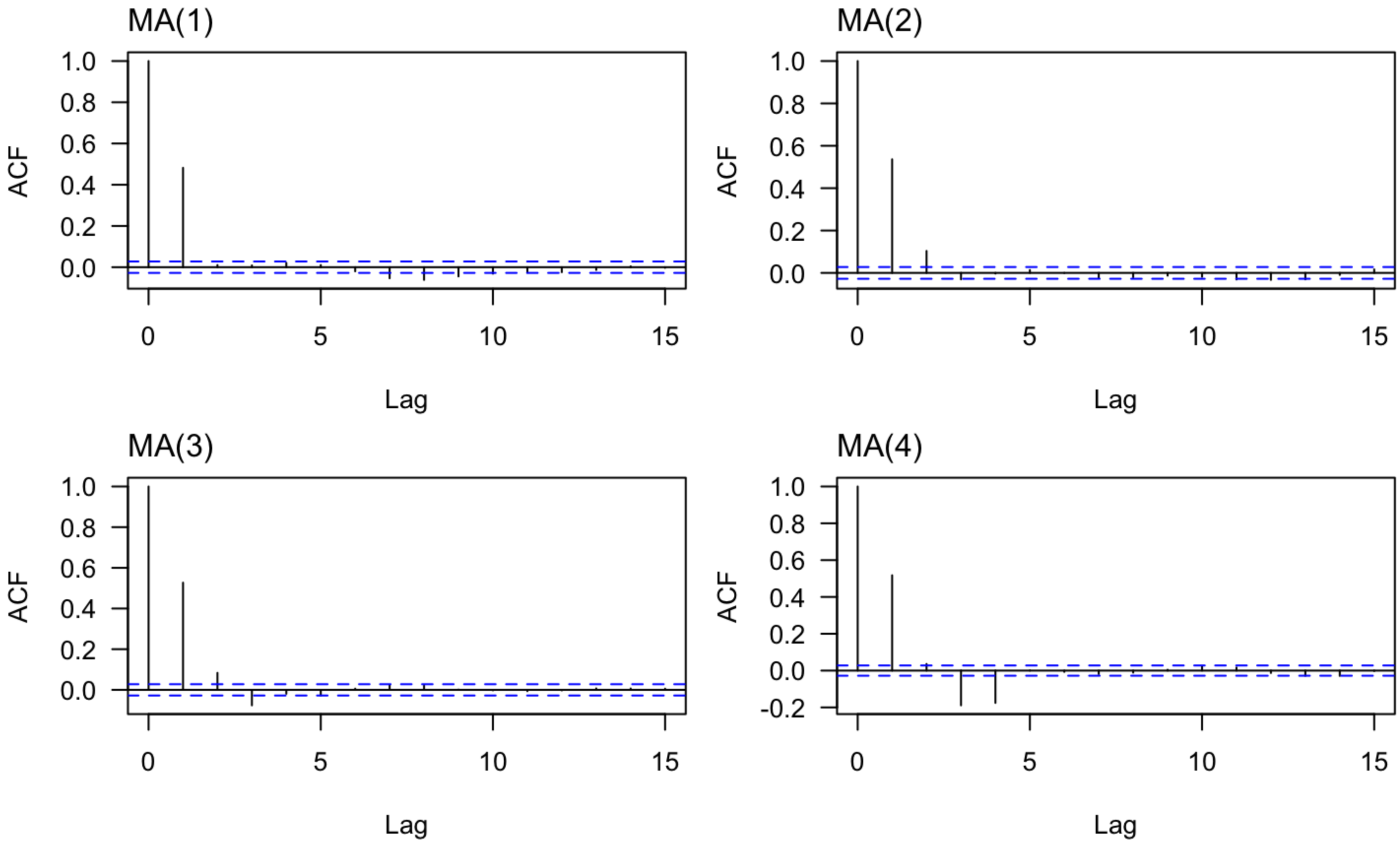
Examples of MA(q) models



ACF & PACF for MA(q) models



ACF for MA(q) models



Do you see the link between the order q and lag k?

Using ACF & PACF for model ID

Model	ACF	PACF
AR(p)	Tails off slowly	Cuts off after lag p
MA(q)	Cuts off after lag q	Tails off slowly

AR(p) model as an MA(∞) model

It is possible to write an AR(p) model as an MA(∞) model

AR(1) model as an MA(∞) model

For example, consider an AR(1) model

$$x_t = \phi x_{t-1} + w_t$$

AR(1) model as an MA(∞) model

For example, consider an AR(1) model

$$
\begin{gathered}
x_t = \phi x_{t-1} + w_t \\
\Downarrow \\
x_{t-1} = \phi x_{t-2} + w_{t-1} \\
\Downarrow \\
x_{t-2} = \phi x_{t-3} + w_{t-2} \\
\Downarrow \\
x_{t-3} = \phi x_{t-4} + w_{t-3}
\end{gathered}
$$

AR(1) model as an MA(∞) model

Substituting in the expression for x_{t-1} into that for x_t

$$
\begin{gathered}
x_t = \phi x_{t-1} + w_t \\
\Downarrow \\
x_{t-1} = \phi x_{t-2} + w_{t-1} \\
\Downarrow \\
x_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t \\
x_t = \phi^2 x_{t-2} + \phi w_{t-1} + w_t
\end{gathered}
$$

AR(1) model as an MA(∞) model

And repeated substitutions yields

$$
\begin{aligned}
x_t &= \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\
&\Downarrow \\
x_t &= \phi^3 x_{t-3} + \phi^2 w_{t-2} + \phi w_{t-1} + w_t \\
&\Downarrow \\
x_t &= \phi^4 x_{t-4} + \phi^3 w_{t-3} + \phi^2 w_{t-2} + \phi w_{t-1} + w_t \\
&\Downarrow \\
x_t &= w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots + \phi^k w_{t-k} + \phi^{k+1} x_{t-k-1}
\end{aligned}
$$

AR(1) model as an MA(∞) model

If our AR(1) model is stationary, then

$$|\phi| < 1$$

which then implies that

$$\lim_{k \to \infty} \phi^{k+1} = 0$$

AR(1) model as an MA(∞) model

If our AR(1) model is stationary, then

$$|\phi| < 1$$

which then implies that

$$\lim_{k \to \infty} \phi^{k+1} = 0$$

and hence

$$\begin{aligned}
x_t &= w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots + \phi^k w_{t-k} + \phi^{k+1} x_{t-k-1} \\
&\Downarrow \\
x_t &= w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots + \phi^k w_{t-k}
\end{aligned}$$

Invertible MA(q) models

An MA(q) process is *invertible* if it can be written as a stationary autoregressive process of infinite order without an error term

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

$$\Downarrow ?$$

$$w_t = x_t + \sum_{k=1}^{\infty} (-\theta)^k x_{t-k}$$

Invertible MA(q) models

Q: Why do we care if an MA(q) model is invertible?

A: It helps us identify the model's parameters

Invertible MA(q) models

For example, these MA(1) models are equivalent

$$x_t = w_t + \frac{1}{5} w_{t-1} \text{ with } w_t \sim \text{N}(0, 25)$$

$$\Updownarrow$$

$$x_t = w_t + 5 w_{t-1} \text{ with } w_t \sim \text{N}(0, 1)$$

Variance of an MA(1) model

The variance of x_t is given by

$$x_t = w_t + \frac{1}{5} w_{t-1} \text{ with } w_t \sim \text{ N}(0, 25)$$

$$\Downarrow$$

$$\begin{aligned}
\text{Var}(x_t) &= \text{Var}(w_t) + \left(\frac{1}{25} \right) \text{Var}(w_{t-1}) \\
&= 25 + \left(\frac{1}{25} \right) 25 \\
&= 25 + 1 \\
&= 26
\end{aligned}$$

Variance of an MA(1) model

The variance of x_t is given by

$$x_t = w_t + 5w_{t-1} \text{ with } w_t \sim \text{ N}(0, 1)$$

$$\Downarrow$$

$$\begin{aligned}
\text{Var}(x_t) &= \text{Var}(w_t) + (25)\text{Var}(w_{t-1}) \\
&= 1 + (25)1 \\
&= 1 + 25 \\
&= 26
\end{aligned}$$

Rewriting an MA(1) model

We can rewrite an MA(1) model in terms of x

$$
\begin{gathered}
x_t = w_t + \theta w_{t-1} \\
\Downarrow \\
w_t = x_t - \theta w_{t-1}
\end{gathered}
$$

Rewriting an MA(1) model

And now we can substitute in previous expressions for w_t

$$
\begin{aligned}
w_t &= x_t - \theta w_{t-1} \\
&\Downarrow \\
w_{t-1} &= x_{t-1} - \theta w_{t-2} \\
&\Downarrow \\
w_t &= x_t - \theta(x_{t-1} - \theta w_{t-2}) \\
w_t &= x_t - \theta x_{t-1} - \theta^2 w_{t-2} \\
&\vdots \\
w_t &= x_t - \theta x_{t-1} - \cdots - \theta^k x_{t-k} - \theta^{k+1} w_{t-k-1}
\end{aligned}
$$

Invertible MA(1) model

If we constrain $|\theta| < 1$, then

$$\lim_{k\to\infty} (-\theta)^{k+1} w_{t-k-1} = 0$$

and

$$\begin{aligned}
w_t &= x_t - \theta x_{t-1} - \cdots - \theta^k x_{t-k} - \theta^{k+1} w_{t-k-1} \\
&\Downarrow \\
w_t &= x_t - \theta x_{t-1} - \cdots - \theta^k x_{t-k} \\
w_t &= x_t + \sum_{k=1}^{\infty} (-\theta)^k x_{t-k}
\end{aligned}$$

Autoregressive moving average models

An autoregressive moving average, or ARMA(p,q), model is written as

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}$$

Autoregressive moving average models

We can write an ARMA(p,q) model using the backshift operator

$$\phi_p(\mathbf{B})x_t = \theta_q(\mathbf{B})w_t$$

Autoregressive moving average models

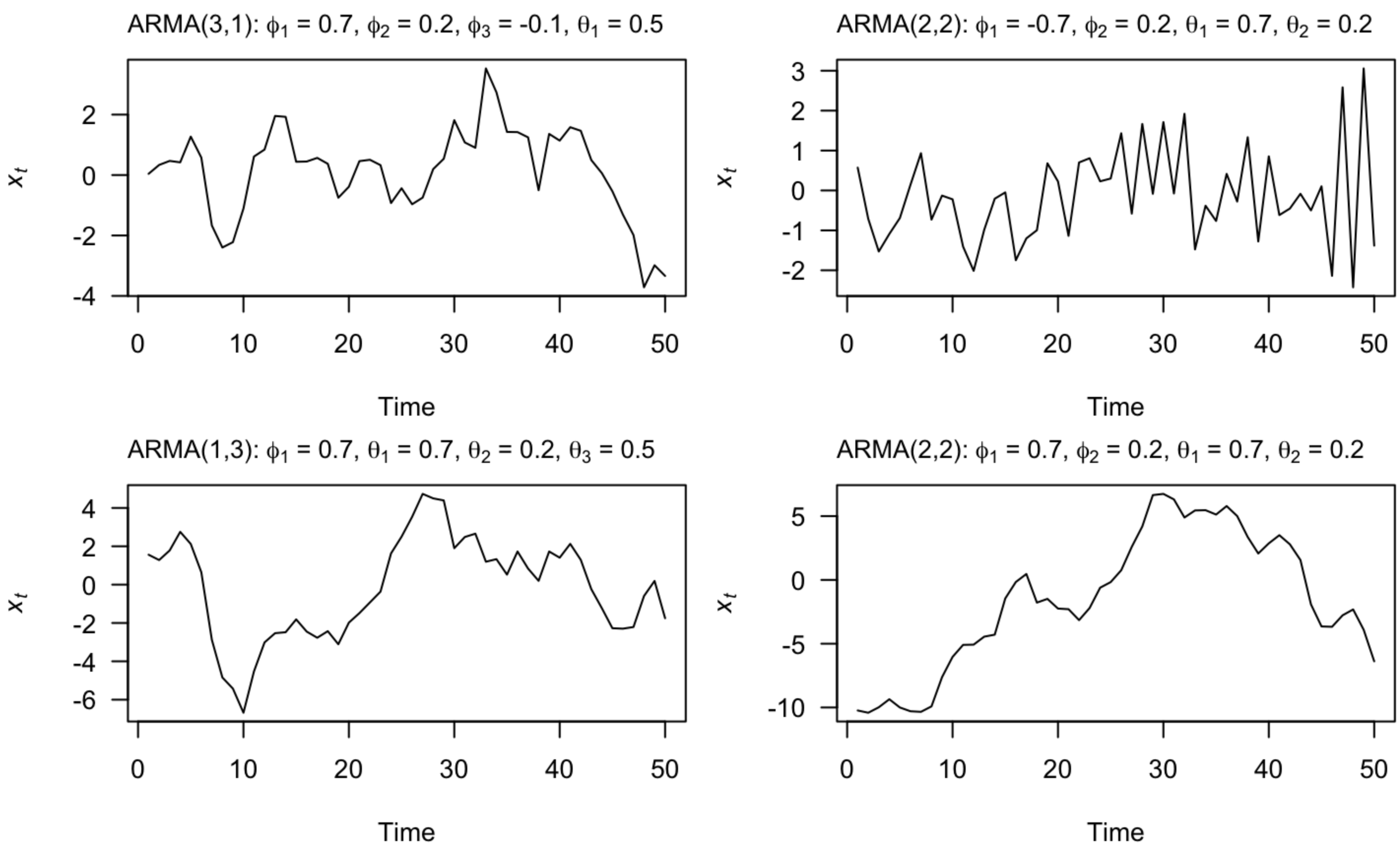
We can write an ARMA(p,q) model using the backshift operator

$$\phi_p(\mathbf{B}) x_t = \theta_q(\mathbf{B}) w_t$$

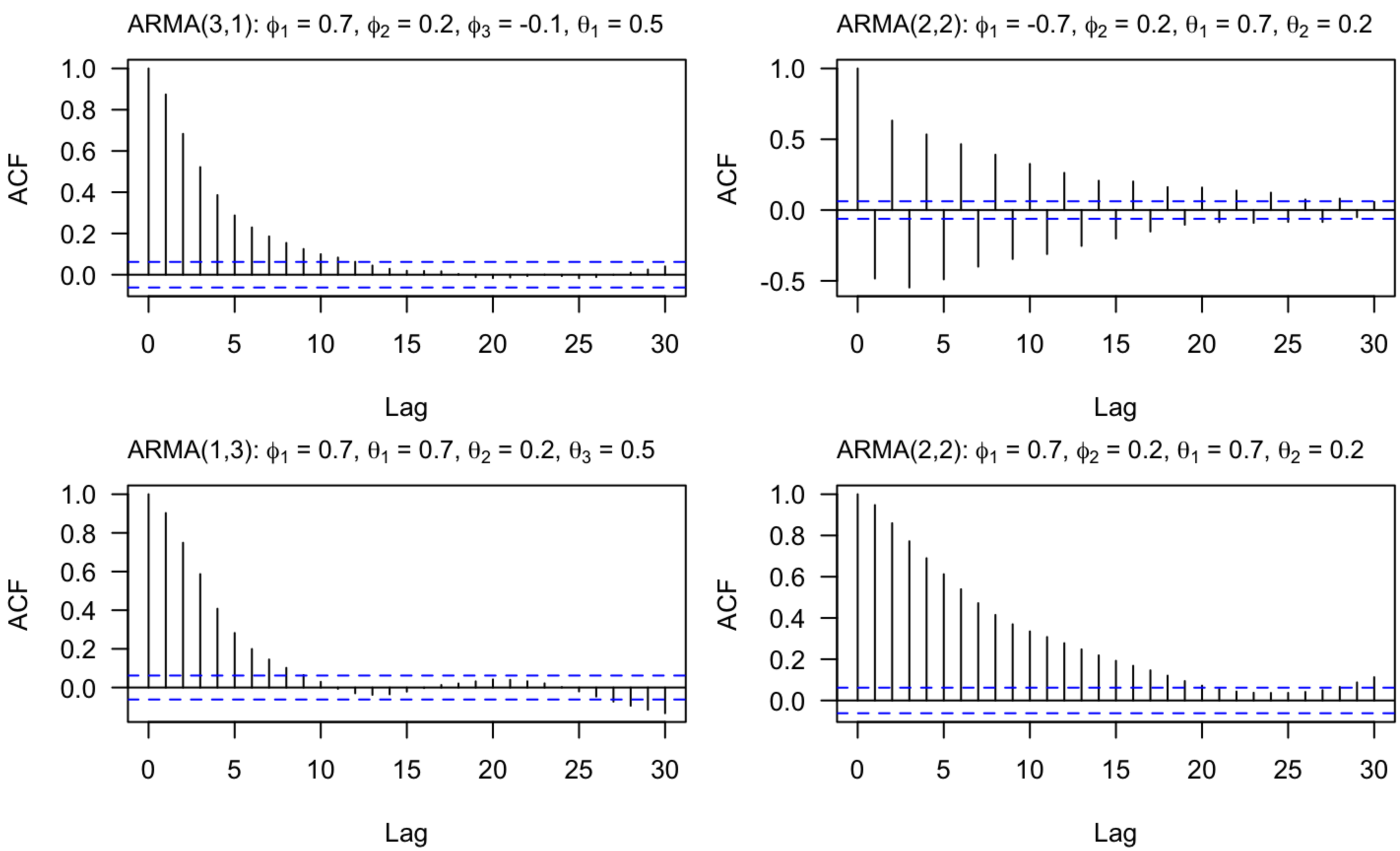
ARMA models are *stationary* if all roots of $\phi_p(\mathbf{B}) > 1$

ARMA models are *invertible* if all roots of $\theta_q(\mathbf{B}) > 1$

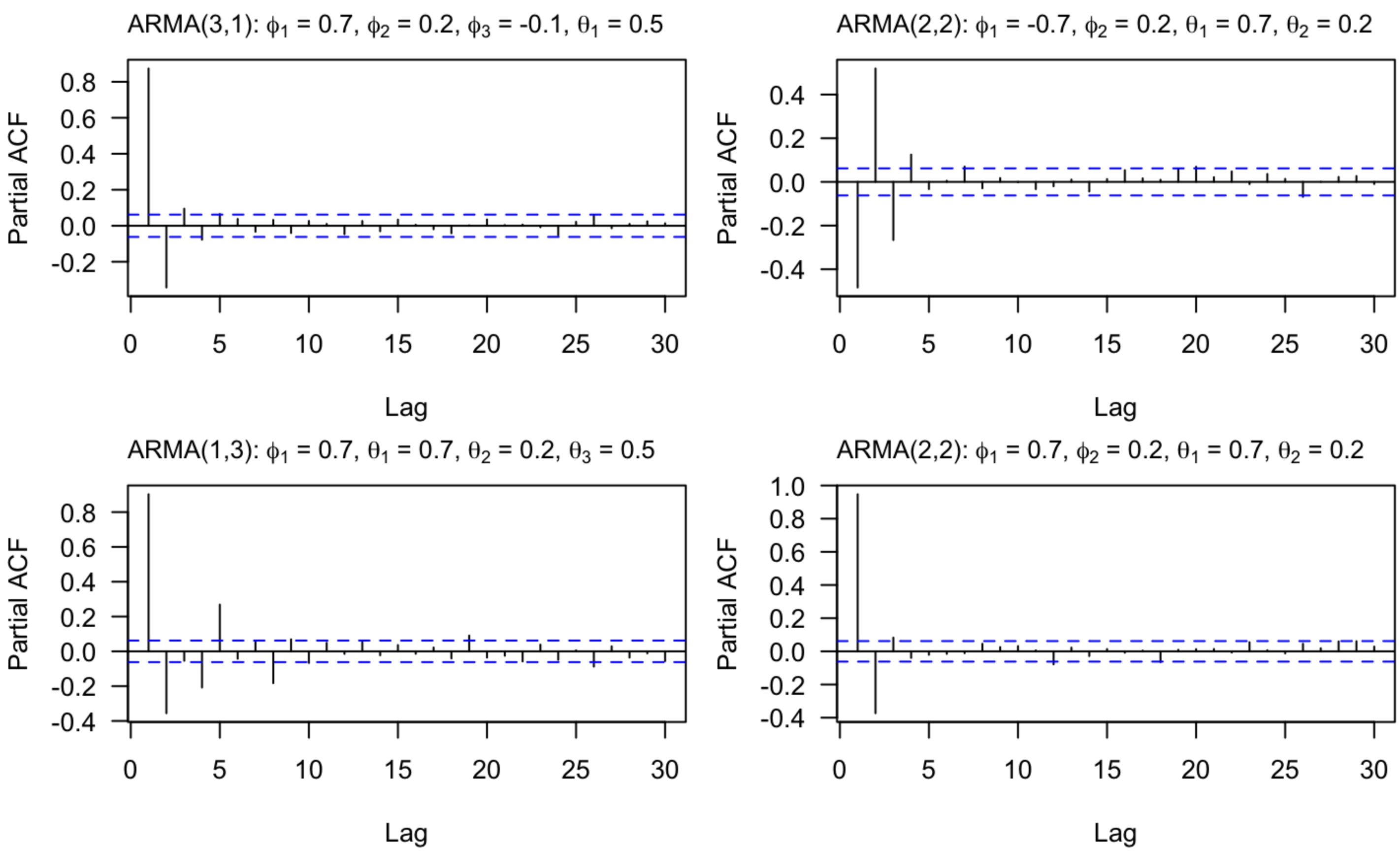
Examples of ARMA(*p*,*q*) models



ACF for ARMA(p,q) models



PACF for ARMA(p,q) models



Using ACF & PACF for model ID

Model	ACF	PACF
AR(p)	Tails off slowly	Cuts off after lag p
MA(q)	Cuts off after lag q	Tails off slowly
ARMA(p,q)	Tails off slowly	Tails off slowly

NONSTATIONARY MODELS

Autoregressive integrated moving average (ARIMA) models

If the data do not appear stationary, differencing can help

This leads to the class of *autoregressive integrated moving average* (ARIMA) models

ARIMA models are indexed with orders (p,d,q) where d indicates the order of differencing

ARIMA(*p*,*d*,*q*) models

Definition

$\{x_t\}$ follows an ARIMA(*p*,*d*,*q*) process if $(1 - \mathbf{B})^d x_t$ is an ARMA(*p*,*q*) process

ARIMA(p,d,q) models

An example

Consider an ARMA(1,0) = AR(1) process where

$$x_t = (1 + \phi)x_{t-1} + w_t$$

ARIMA(*p*,*d*,*q*) models

An example

Consider an ARMA(1,0) = AR(1) process where

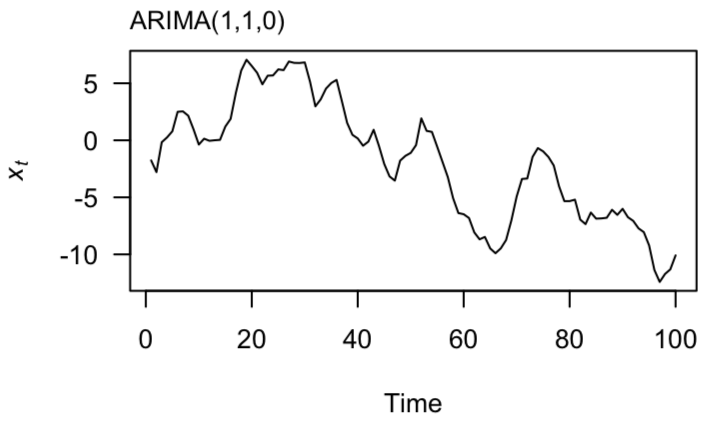
$$x_t = (1 + \phi)x_{t-1} + w_t$$

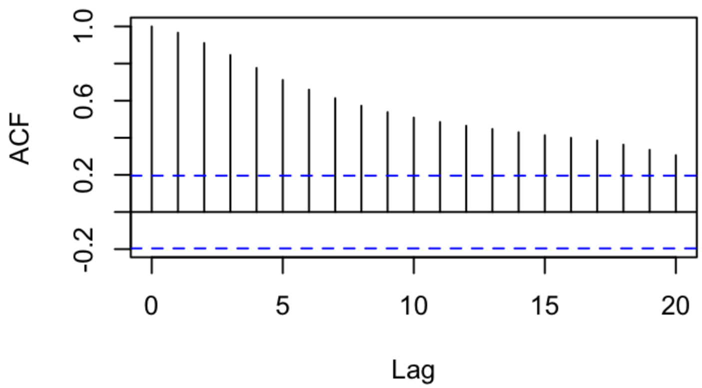
$$\Downarrow$$

$$\begin{aligned} x_t &= x_{t-1} + \phi x_{t-1} + w_t \\ x_t - x_{t-1} &= \phi x_{t-1} + w_t \\ (1 - \mathbf{B})x_t &= \phi x_{t-1} + w_t \end{aligned}$$

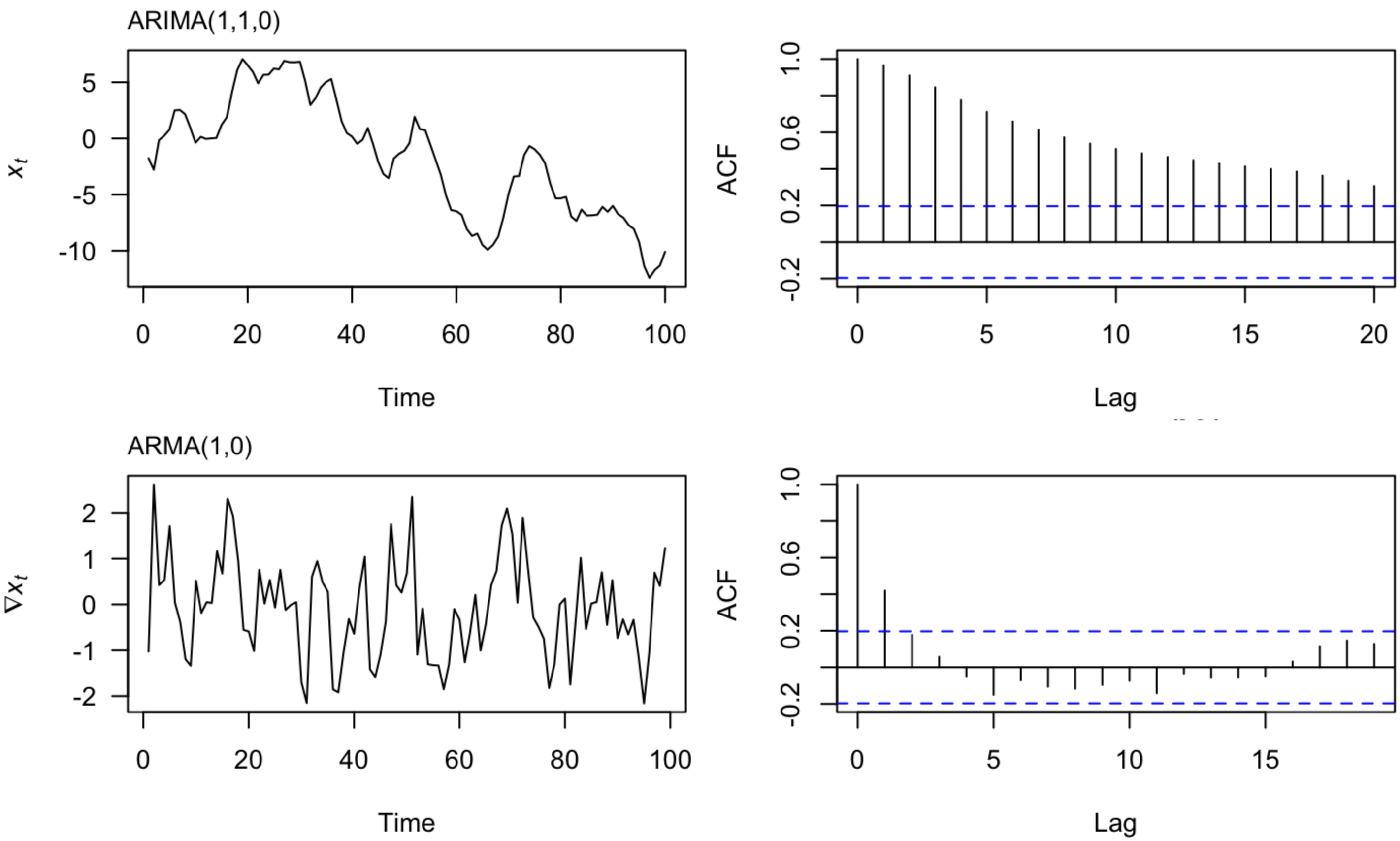
So x_t is indeed an ARIMA(1,1,0) process

ARIMA(p,d,q) models





ARIMA(p,d,q) models



Topics for today

Review

- White noise
- Random walks

Autoregressive (AR) models

Moving average (MA) models

Autoregressive moving average (ARMA) models

Using ACF & PACF for model ID