The tau inclusive hadronic decay from lattice QCD

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This is the review of the inclusive calculation to the tau decay by lattice QCD.

1 Feynman diagram calculation of the tau decay

In this chapter, we examined the R ratio calculation and the decay rate of tau inclusive hadronic decay $\Gamma(\tau^- \to X + \nu_\tau)$. Additionally, I explored the hadronic structure of the W boson. To help understand these observables, I used the two-currents lattice correlator $\langle C_\mu C_\nu \rangle_E$.

1.1 The leptonic decay of tau particle

In the simplest scenario, we consider the leptonic decay of the tau particle, which results in two possible final states. The only difference between these two final states is the mass of the lepton in the final state. For our analysis, we assume that the mass of the final state lepton is denoted as "m". Our objective is to determine the decay rate of the tau particle. The kinetic variables used in this analysis are measured in the rest frame of the tau particle. The kinetic variables in the rest frame of τ are:

$$\begin{aligned} p_{\tau^-} &= p_1 = (m_\tau, 0, 0, 0), \\ p_{\nu_\tau} &= p_2 = (|\vec{p_2}|, \vec{p_2}), \\ p_{\ell^-} &= p_3 = (|\vec{p_3}|, \vec{p_3}), \\ p_{\nu_\ell} &= p_4 = (|\vec{p_4}|, \vec{p_4}), \end{aligned}$$

To obtain the differential decay rate, we have to consider the phase space of each particles [**?**]:

$$\frac{d^2\Gamma}{dq^2} = \frac{1}{2m_\tau} \int d\text{LIPS} |M^2| \delta((p_3 + p_4 - q)^2) \tag{1}$$

the lorentz invariant phase space integral of 3 bodies decay is:

$$d\text{LIPS} = \frac{dp_2^3}{(2\pi)^3 2p_2^0} \frac{dp_3^3}{(2\pi)^3 2p_3^0} \frac{dp_4^3}{(2\pi)^3 2p_4^0} (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - p_4),$$

In the leading order, the matrix in terms of the trace is:

$$|M^2| = \frac{G_F^2}{4} Tr\left[\gamma^\mu (1-\gamma_5) \not{p_2} \gamma^\nu (1-\gamma_5) \left(\not{p_1} + m_\tau\right)\right] Tr\left[\gamma_\mu (1-\gamma_5) \not{p_4} \gamma_\nu (1-\gamma_5) \left(\not{p_3} + m\right)\right] \tag{2}$$

Inserting above matrix production into equation.(1), we can obtain the tau leptonic decay rate at leading order. Here we extend the trace of the leptonic part and define it as leptonic tensor $L^{\mu\nu}$. Remember that $\gamma_5^2 = 1$

$$L^{\mu\nu}/8 = Tr\left[\gamma^\mu (1-\gamma_5) \not{p_2} \gamma^\nu (1-\gamma_5) \left(\not{p_1} + m_\tau\right)\right]/8 = p_1^\mu p_2^\nu - (p_1 \cdot p_2) g^{\mu\nu} + p_1^\nu p_2^\mu - i p_{1\alpha} p_{2\beta} \epsilon^{\beta\nu\alpha\mu} \tag{3}$$

After integrated out the phase space, we can obtain the total decay rate of :

$$\Gamma^0 \left(\tau \to e^- + \bar{\nu_\tau} + \nu_\tau\right) = \frac{m_\tau^5 G_F^2}{192\pi^3} = \frac{m_\tau^5 g^4}{6144 m_W^4 \pi^3} \tag{4}$$

For the one-loop result, please check ref [**?**] for detail.

1.2 The Decay of the top quark

We will now examine the decay of a heavy quark into a lighter quark via the emission of a W boson. The phase space for this process is simpler than the previous scenario as we have one less final state particle. The matrix element for this process is given by:

$$\frac{1}{2}|M^2| = \frac{g^2}{16} Tr\left[\gamma^\mu (1-\gamma_5) \not{p_2} \gamma^\nu (1-\gamma_5) \left(\not{p_1} + m_t\right)\right] \sum_{\text{polarization}} \epsilon_\mu^*(Q) \epsilon_\mu(Q) \tag{5}$$

where the sum over the physical W boson polarization vector gives $\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_W^2}\right)$ the final state quark is regarded as massless particle. Once we have set the kinematics using a delta function to ensure that the incoming and outgoing particles are on-shell, we can insert the sum and scalar products into the matrix element.

$$\begin{aligned}
&2p_1\cdot p_2 = -\left(p_1-p_2\right)^2+p_1^2+p_2^2 = -\left(q\right)^2+m_t^2 = m_t^2-m_W^2,\\
&2p_1\cdot q = -\left(p_1-q\right)^2+p_1^2+q^2 = m_t^2+m_W^2,\\
&2p_2\cdot q = -\left(p_2-q\right)^2+q^2+p_2^2 = q^2-\left(p_1-2q\right)^2 = 2\left(m_t^2+m_W^2\right)-4m_W^2+m_W^2-m_t^2 = m_t^2-m_W^2
\end{aligned}\tag{6}$$

we obtain the matrix square as:

$$\begin{aligned}
\frac{1}{2}|M^2| &= \frac{g^2}{2}\left[\gamma^\mu\left(1-\gamma_5\right)\not{p}_2\gamma^\nu\left(1-\gamma_5\right)\left(\not{p}_1+m_t\right)\right]\left[-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_W^2}\right] = \frac{g^2}{2}\left[\left(p_1\cdot p_2\right)+2\frac{\left(p_1\cdot q\right)\left(p_2\cdot q\right)}{m_W^2}\right],\\
&= \frac{g^2}{4}\frac{m_t^4}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)\left(1+2\frac{m_W^2}{m_t^2}\right)
\end{aligned}\tag{7}$$

The phase space integral is:

$$\begin{aligned}
\int d\text{LIPS} &= \int\frac{d\vec{p_2}^3}{(2\pi)^3 2p_2^0}\frac{d\vec{q}^3}{(2\pi)^3 2q^0}(2\pi)^4\delta^4(p_1-p_2-q),\\
&= \frac{1}{(2\pi)^2}\int\frac{d\vec{p_2}^3}{2p_2^0}\frac{d\vec{q}^3}{2q^0}\delta(m_t-p_2^0-q^0)\delta^3(-\vec{p_2}-\vec{q}),\\
&= \frac{1}{4(2\pi)^2}\int\frac{d\vec{q}^3}{q^0|\vec{q}|}\delta(m_t-|\vec{q}|-\sqrt{m_W^2+|\vec{q}|^2}),\\
&= \frac{1}{4(2\pi)^2}\int\frac{|\vec{q}|^2 d\vec{q}d\text{Cos}(\theta)d\phi}{|\vec{q}|\sqrt{m_W^2+|\vec{q}|^2}}\delta(m_t-|\vec{q}|-\sqrt{m_W^2+|\vec{q}|^2}),\\
&= \frac{1}{8\pi}\int\frac{|\vec{q}|d\vec{q}}{\sqrt{m_W^2+|\vec{q}|^2}}\delta(m_t-|\vec{q}|-\sqrt{m_W^2+|\vec{q}|^2})
\end{aligned}\tag{8}$$

the square root inside the delta function is:

$$\begin{aligned}
&\delta(m_t-|\vec{q}|-\sqrt{m_W^2+|\vec{q}|^2}) = \left|\left(-1-\frac{|\vec{q}|}{\sqrt{|\vec{q}|^2+m_W^2}}\right)\right|^{-1}\delta\left(|\vec{q}|-\left(\frac{m_t^2-m_W^2}{2m_t}\right)\right),\\
&\sqrt{m_W^2+|\vec{q}|^2}\rightarrow\frac{m_t^2+m_W^2}{2m_t}
\end{aligned}\tag{9}$$

inserting into the phase integral, we obtain that:

$$\int d\text{LIPS} = \frac{m_t^2-m_W^2}{8\pi m_t^2}\tag{10}$$

therefore the decay rate of the top quark at leading order is:

$$\begin{aligned}
\Gamma(t\rightarrow W+b) &= \frac{g^2}{64\pi}\frac{m_t\left(m_t^2-m_W^2\right)}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)\left(1+2\frac{m_W^2}{m_t^2}\right),\\
&= \frac{g^2 m_t^3}{64\pi m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)^2\left(1+2\frac{m_W^2}{m_t^2}\right),\\
&= \frac{G_F m_t^3}{8\pi\sqrt{2}}\left(1-\frac{m_W^2}{m_t^2}\right)^2\left(1+2\frac{m_W^2}{m_t^2}\right)
\end{aligned}\tag{11}$$

2 The current correlator

Our objective is to calculate the semi-inclusive hadronic tau decay rate. The inclusive process is represented by the current correlator.:

$$\Pi^{\mu\nu}_{i,j,V-A}(q) = i\int d^4x e^{iqx}\langle 0|\,T\left\{J^\mu_{ij,V-A}(x)J^\nu_{ij,V-A}(0)^\dagger\right\}|0\rangle = -(g^{\mu\nu}Q^2-q^\mu q^\nu)\Pi^{(J=1)}_{V-A}(q^2)+(q^\mu q^\nu)\,\Pi^{(J=0)}_{V-A}(q^2)\tag{12}$$

for flavour changing current the dagger is essential to keep it. For instance, the $J=u\bar{d}\neq\bar{u}d=J^\dagger$. The imaginary part of the correlator is [**?**]:

$$\text{Im}\Pi^{\mu\nu}_{i,j}(q) = \frac{1}{2i}\left[\Pi^{\mu\nu}_{i,j}(q)-\Pi^{\mu\nu}_{i,j}(q)^\dagger\right] = \frac{1}{2}(2\pi)^4\sum_n\delta^4(q-p_n)\langle 0|\,J_\mu(0)\,|n\rangle\langle n|\,J_\nu(0)^\dagger\,|0\rangle\tag{13}$$

In the physical timelike regime, it's not possible to calculate the HVP perturbatively. However, we can extract the metric structure of the HVP using the ward identity, which gives us the equation $\Pi^{\mu\nu}_{i,j}(q) = -(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi_{i,j}(q)$. The QCD contributions are captured by the scalar function $\Pi_{i,j}$. A reference[?] shows that the lorentz structure of the HVP is related to the angular momentum J of the produced hadron in the final state.

$$\begin{aligned}\Pi^{\mu\nu}_{i,j}(q,Q^2) &= -\left(g^{\mu\nu}Q^2 - q^\mu q^\nu\right)\Pi^{(J=1)}_{i,j}(Q^2) + q^\mu q^\nu \Pi^{(J=0)}_{i,j}(Q^2),\\ &= -\left(g^{\mu\nu}Q^2 - q^\mu q^\nu\right)\Pi^{(0+1)}_{i,j}(Q^2) - g^{\mu\nu}Q^2\Pi^{(0)}_{i,j}(Q^2),\\ &= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}\right)Q^2\Pi^{(0+1)}_{i,j}(Q^2) - g^{\mu\nu}Q^2\Pi^{(0)}_{i,j}(Q^2)\end{aligned} \tag{14}$$

where we define that:

$$\Pi^{(0+1)}_{i,j}(Q^2) \equiv \Pi^{(1)}_{i,j}(Q^2) + \Pi^{(0)}_{i,j}(Q^2) \tag{15}$$

The imaginary part of a two-current operator can be used to expand the differential decay rate. Using the cutting rule, we find that equation (1) is equal to equation (4) when we square the matrix element. In the following sections, we will use the imaginary part of the HVP to replace the on-shell W-boson, and we will substitute the W-boson mass m_W with the virtuality of the off-shell W-boson, denoted as Q^2. The virtuality is limited by the hardware performance, as mentioned in [?]. Since the virtuality is typically around a few GeV^2, we can approximate the W-boson propagator. To start, we will calculate the matrix element square for J=1.

$$\begin{aligned}\frac{1}{2}|M^2|^{(J=1)} &= \frac{1}{2M_W^4}\frac{g^4|V_{ij}|^2}{(2\sqrt{2})^4}\mathrm{Tr}\left[\gamma^\mu(1-\gamma_5)\not{p}_2\gamma^\nu(1-\gamma_5)(\not{p}_1+m_\tau)\right]\left[-g_{\mu\nu}+\frac{q_\mu q_\nu}{Q^2}\right]Q^2\mathrm{Im}\Pi^{(J=1)}_{i,j}(Q^2),\\ &= \frac{g^4|V_{ij}|^2Q^2}{32M_W^4}\left[(p_1\cdot p_2) + 2\frac{(p_1\cdot q)(p_2\cdot q)}{Q^2}\right]\mathrm{Im}\Pi^{(J=1)}_{i,j}(Q^2),\\ &= \frac{g^4|V_{ij}|^2m_\tau^4}{32M_W^4}\left(1-\frac{Q^2}{m_\tau^2}\right)\left(1+2\frac{Q^2}{m_\tau^2}\right)\mathrm{Im}\Pi^{(J=1)}_{i,j}(Q^2)\end{aligned} \tag{16}$$

and so on:

$$\begin{aligned}\frac{1}{2}|M^2|^{(J=0)} &= \frac{Q^2}{2M_W^4}\frac{g^4|V_{ij}|^2}{(2\sqrt{2})^4}\mathrm{Tr}\left[\gamma^\mu(1-\gamma_5)\not{p}_2\gamma^\nu(1-\gamma_5)(\not{p}_1+m_\tau)\right]g_{\mu\nu}\mathrm{Im}\Pi^{(J=0)}_{i,j}(Q^2),\\ &= \frac{g^4|V_{ij}|^2Q^2}{32M_W^4}(p_1\cdot p_2)\mathrm{Im}\Pi^{(J=0)}_{i,j}(Q^2),\\ &= \frac{g^4|V_{ij}|^2m_\tau^4}{32M_W^4}\left(1-\frac{Q^2}{m_\tau^2}\right)\mathrm{Im}\Pi^{(J=0)}_{i,j}(Q^2)\end{aligned} \tag{17}$$

the phase space integral is similiar to the top decay despite that the w boson is not on it's mass shell. And we have to obtain the relation between the decay rate and the virtuality Q^2:

$$\begin{aligned}\int d\mathrm{LIPS} &= \int\left(\frac{d\vec{p_2}^3}{(2\pi)^3 2p_2^0}\right)\left(\frac{d^4q}{(2\pi)^4}\right)(2\pi)^4\delta^4(p_1-p_2-q),\\ &= (2\pi)^{-3}\int\left(\frac{d\vec{p_2}^3}{2p_2^0}\right)d^4q\,\delta(m_t-p_2^0-q^0)\delta^3(-\vec{p_2}-\vec{q}),\\ &= \left(\frac{(2\pi)^{-3}}{2}\right)\int d^4q\frac{1}{|\vec{q}|}\delta(m_t-|\vec{q}|-q^0),\\ &= \frac{1}{16\pi^3}\int\frac{d^4q}{|\vec{q}|}\delta(m_t-|\vec{q}|-q^0),\\ &= \frac{1}{16\pi^3}\int\frac{d^4q}{\sqrt{Q^2-q^{02}}}\delta(m_t-q^0-\sqrt{Q^2-q^{02}}),\\ &= \frac{1}{8\pi^2}\int dq^0dQ^2\delta(m_t-q^0-\sqrt{Q^2-q^{02}}),\\ &= \frac{1}{8\pi^2}\int\frac{|\vec{q}|d\vec{q}dQ^2}{\sqrt{q^2+|\vec{q}|^2}}\delta(m_t-|\vec{q}|-\sqrt{Q^2+|\vec{q}|^2}),\end{aligned} \tag{18}$$

now we want to change the variable from d^4q into $d^3\vec{q}dQ^2$, yielding the Jacobain determiant:

$$\begin{aligned}d^4q = dq^0d^3\vec{q} = 2\pi|\vec{q}|^2dq^0d\vec{q} = 2\pi\frac{|\vec{q}|^2dq^0dQ^2}{\vec{q}} &= 2\pi|\vec{q}|dq^0dQ^2\\ = 2\pi\frac{|\vec{q}|^2}{q^0}d\vec{q}dQ^2 = \frac{2\pi|\vec{q}|^2}{\sqrt{q^2+|\vec{q}|^2}}d\vec{q}dQ^2 &\end{aligned} \tag{19}$$

the square root inside the delta function is also like top decay:

$$\begin{aligned}
&\delta(m_t - |\vec{q}| - \sqrt{Q^2 + |\vec{q}|^2}) = \left|\left(1 + \frac{|\vec{q}|}{\sqrt{|\vec{q}|^2 + Q^2}}\right)\right|^{-1} \delta\left(|\vec{q}| - \left(\frac{m_\tau^2 - Q^2}{2m_\tau}\right)\right), \\
&\sqrt{Q^2 + |\vec{q}|^2} \to \frac{m_\tau^2 + Q^2}{2m_\tau}, \\
&\frac{|\vec{q}|}{\sqrt{Q^2 + |\vec{q}|^2}} \to \frac{m_\tau^2 - Q^2}{m_\tau^2 + Q^2}, \\
&1 + \frac{|\vec{q}|}{\sqrt{Q^2 + |\vec{q}|^2}} \to \frac{2m_\tau^2}{m_\tau^2 + Q^2}
\end{aligned} \tag{20}$$

inserting into the phase integral, we obtain that:

$$\begin{aligned}
\int d\text{LIPS} &= \frac{1}{8\pi^2} \int \frac{|\vec{q}| d\vec{q} dQ^2}{\sqrt{q^2 + |\vec{q}|^2}} \delta(m_t - |\vec{q}| - \sqrt{Q^2 + |\vec{q}|^2}), \\
&= \frac{1}{8\pi^2} \int dQ^2 \left(\frac{m_\tau^2 - Q^2}{m_\tau^2 + Q^2}\right) \left(\frac{2m_\tau^2}{m_\tau^2 + Q^2}\right)^{-1}, \\
&= \frac{1}{16\pi^2} \int dQ^2 \left(\frac{m_\tau^2 - Q^2}{m_\tau^2}\right)
\end{aligned} \tag{21}$$

yielding that:

$$\begin{aligned}
\frac{d\Gamma}{dQ^2} &= \frac{1}{2m_\tau} \left(\int d\text{LIPS}\right) \left(\frac{1}{2}|M^2|^{(J=1)} + \frac{1}{2}|M^2|^{(J=0)}\right), \\
&= \frac{g^4 |V_{ij}|^2 m_\tau^3}{512\pi^2 M_W^4} \left(\left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \left(1 + 2\frac{Q^2}{m_\tau^2}\right) \text{Im}\Pi_{i,j}^{(J=1)}(Q^2) + \left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \text{Im}\Pi_{i,j}^{(J=0)}(Q^2)\right) \\
&= \frac{G_F^2 |V_{ij}|^2 m_\tau^3}{16\pi^2} \left(\left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \left(1 + 2\frac{Q^2}{m_\tau^2}\right) \text{Im}\Pi_{i,j}^{(J=1)}(Q^2) + \left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \text{Im}\Pi_{i,j}^{(J=0)}(Q^2)\right)
\end{aligned} \tag{22}$$

where the Fermi constant is given by

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \tag{23}$$

Now we can obtain the decay width:

$$\Gamma = \frac{G_F^2 |V_{ij}|^2 m_\tau^3}{16\pi^2} \int dQ^2 \left[\left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \left(1 + 2\frac{Q^2}{m_\tau^2}\right) \text{Im}\Pi_{i,j}^{(J=1)}(Q^2) + \left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \text{Im}\Pi_{i,j}^{(J=0)}(Q^2)\right] \tag{24}$$

Considering the exclusive processes, we can separate the contribution from kaon and remaining states by a delta function $\pi f_K^2 \delta\left(Q^2 - M_K^2\right)$ and a contiune spetral:

$$\text{Im}\Pi_{i,j}\left(Q^2\right) = \pi f_K^2 \delta\left(Q^2 - M_K^2\right) + \text{Contiune spetral} \tag{25}$$

if we ignored all other modes, inserting above exclusive decompoisiton into decay rate, we can obtain:

$$\Gamma(\tau^- \to K^- + \nu_\tau) = \frac{G_F^2 f_K^2 m_\tau^3}{16\pi} \left(1 - \frac{M_K^2}{m_\tau^2}\right)^2 |V_{us}|^2, \tag{26}$$

$$\Gamma^1(\tau^- \to K^* + \nu_\tau) = \frac{G_F^2 f_{K^*}^2 m_\tau^3}{16\pi} \left(1 - \frac{M_{K^*}^2}{m_\tau^2}\right)^2 \left(1 + 2\frac{M_{K^*}^2}{m_\tau^2}\right) |V_{us}|^2. \tag{27}$$

dividing by the total decay rate $\Gamma^0\left(\tau \to e^- + \bar{\nu_\tau} + \nu_\tau\right) = \frac{m_\tau^5 G_F^2}{192\pi^3}$ to obtain the R-ratio, we have:

$$\begin{aligned}
R(Q^2) = \int_0^{m_\tau^2} dQ^2 \left(\frac{\frac{d^2\Gamma}{dQ^2}}{\frac{m_\tau^5 G_F^2}{192\pi^3}}\right) &= 12\pi |V_{ij}|^2 \int_0^{m_\tau^2} \frac{dQ^2}{m_\tau^2} \left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{Q^2}{m_\tau^2}\right) \text{Im}^{(J=1)}\Pi_{i,j}\left(Q^2\right) + \text{Im}^{(J=0)}\Pi_{i,j}\left(Q^2\right)\right], \\
&= 12\pi |V_{ij}|^2 \int_0^{m_\tau^2} \frac{dQ^2}{m_\tau^2} \left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{Q^2}{m_\tau^2}\right) \text{Im}^{(0+1)}\Pi_{i,j}\left(Q^2\right) - 2\left(\frac{Q^2}{m_\tau^2}\right) \text{Im}^{(0)}\Pi_{i,j}\left(Q^2\right)\right], \\
&= 12\pi^2 |V_{ij}|^2 \int_0^{m_\tau^2} \frac{dQ^2}{m_\tau^2} \left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{Q^2}{m_\tau^2}\right) \rho_{i,j}^{(0+1)}\left(Q^2\right) - 2\left(\frac{Q^2}{m_\tau^2}\right) \rho_{i,j}^{(0)}\left(Q^2\right)\right], \\
&= 12\pi^2 |V_{ij}|^2 \int_0^{m_\tau^2} \frac{dQ^2}{m_\tau^2} \left(1 - \frac{Q^2}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{Q^2}{m_\tau^2}\right) \rho_{i,j}^{(1)}\left(Q^2\right) + \rho_{i,j}^{(0)}\left(Q^2\right)\right]
\end{aligned} \tag{28}$$

We should also include the perturbative expansion of the EW effect S_{EW}. The one-loop result is discuss at [?].

3 Chebshev Approximation

The relation of 2point correlation function calculated from lattice and the spectral density gives:

$$C(t)=\int_0^{\infty}dQ\ Q^2\rho(Q^2)\mathrm{e}^{-Q\cdot t} \tag{29}$$

this relation indicates that the relation between the lattice data and the spectral density function. It's an inverse problem since the opposite operation can't be easily calculated. The spectral density $\rho(Q^2)$ is still unknown due to the numerically difficulty to extract it from above equation. Here we follow [?] to avoid this numerical ill problem. At first the smeared spectral density $\bar{\rho}\left(Q^2\right)$ is defined:

$$\bar{\rho}\left(Q\right)=\frac{\langle\Psi|\,\delta(\hat{H}-Q)\,|\Psi\rangle}{\langle\Psi|\Psi\rangle} \tag{30}$$

which indicates the number of states for each energy Q.We assume that this function is positive and normalized. The original spectral function and the smeared spectral function differ in their normalization. The states are the eigenvectors of the current operator. In [?], a new correlator is defined based on the lattice correlator C(t):

$$\begin{aligned}\bar{C}(t)\equiv\int_0^{\infty}dQ\bar{\rho}(Q)\exp^{-Q\cdot t}&=\frac{\langle\Psi|\,\mathrm{e}^{-\hat{H}t}\,|\Psi\rangle}{\langle\Psi|\Psi\rangle},\\&=\frac{\sum_{x,y}\langle 0|\,J_z(0,x)e^{-\hat{H}(t+2t_0)}J_z(0,y)\,|0\rangle}{\sum_{x,y}\langle 0|\,J_z(0,x)e^{-2\hat{H}t_0}J_z(0,y)\,|0\rangle},\\&=\frac{\sum_{x,y}\langle 0|\,J_z(t+2t_0,x)J_z(0,y)\,|0\rangle}{\sum_{x,y}\langle 0|\,J_z(2t_0,x)J_z(0,y)\,|0\rangle},\\&=\frac{C(t+2t_0)}{C(2t_0)}.\end{aligned} \tag{31}$$

The dimension of $\bar{C}(t)$ is massless, which is necessary to satisfy the definition of the translation operator. The total modification to the spectral function is:

$$\bar{\rho}(Q)=\frac{1}{C(2t_0)}Q^2\exp(-2Qt_0)\rho(Q) \tag{32}$$

The of the $\bar{\rho}(Q)$ is -1 in mass dimension. The decay width is expressed as:

$$\begin{aligned}\Gamma&=\frac{G_F^2|V_{ij}|^2m_\tau^3}{16\pi^2}\int dQ^2\left[\left(1-\frac{Q^2}{m_\tau^2}\right)^2\left(1+2\frac{Q^2}{m_\tau^2}\right)\mathrm{Im}\Pi_{i,j}^{(J=1)}\left(Q^2\right)+\left(1-\frac{Q^2}{m_\tau^2}\right)^2\mathrm{Im}\Pi_{i,j}^{(J=0)}\left(Q^2\right)\right],\\&=\frac{G_F^2|V_{ij}|^2m_\tau^3}{16\pi}\int dQ^2\left[\left(1-\frac{Q^2}{m_\tau^2}\right)^2\left(1+2\frac{Q^2}{m_\tau^2}\right)\rho^{(J=1)}(Q^2)+\left(1-\frac{Q^2}{m_\tau^2}\right)^2\rho^{(J=0)}(Q^2)\right],\\&=\frac{G_F^2|V_{ij}|^2m_\tau^3}{16\pi}\int dQ(2Q)\left[\left(1-\frac{Q^2}{m_\tau^2}\right)^2\left(1+2\frac{Q^2}{m_\tau^2}\right)\rho^{(J=1)}(Q^2)+\left(1-\frac{Q^2}{m_\tau^2}\right)^2\rho^{(J=0)}(Q^2)\right],\\&\equiv\frac{G_F^2|V_{ij}|^2m_\tau^3}{16\pi}\int dQ\left[K^{(J=1)}(Q)\rho^{(J=1)}(Q)+K^{(J=0)}\rho^{(J=0)}(Q^2)\right],\end{aligned} \tag{33}$$

Now we want to express the decay width in terms of $\bar{\rho}(Q)$.

$$\begin{aligned}\Gamma&=\frac{G_F^2|V_{ij}|^2m_\tau^3}{16\pi}\int dQ\left(\frac{1}{C(2t_0)}Q^2e(-2Qt_0)\right)^{-1}\left[K^{(J=1)}(Q)\bar{\rho}^{(J=1)}(Q)+K^{(J=0)}\bar{\rho}^{(J=0)}(Q^2)\right],\\&=\frac{G_F^2|V_{ij}|^2m_\tau^3}{16\pi}\int dQ\left[\bar{K}^{(J=1)}(Q)\bar{\rho}^{(J=1)}(Q)+\bar{K}^{(J=0)}\bar{\rho}^{(J=0)}(Q^2)\right],\end{aligned} \tag{34}$$

in the beginning, the Kernel functions are mass dimension 1. After we redefine the expression, the approximated Kernel functions $\bar{K}$ are dimension 2. The approximation theory is applied to solve the energy integral by using the transfer matrix element $e^{-\hat{H}}$ to construct the time evolution operator inside the two-point function.

$$\begin{aligned}&\bar{K}(Q)\approx\frac{c_0^j}{2}+\sum_{i=1}^{N_t}c_i^j\cdot\bar{T}_i^j(e^{-Q}),\\&\text{with,}\\&c_i^j=\frac{2}{\pi}\int_0^{\pi}d\theta\bar{K}^j\left(-\mathrm{Log}\left(\frac{1+\mathrm{Cos}\,(\theta)}{2}\right)\right)\mathrm{Cos}\,(i\theta)\end{aligned} \tag{35}$$

where the N_t present the truncation of the polynomial. The term of the polynomial is defined as $\bar{T}_1(x) = 2x - 1, \bar{T}_2(x) = 8x^2 - 8x + 1, ...$ Now we put it into the decay width without the prefactor:

$$
\begin{aligned}
\Gamma_{\text{app}} &\equiv \sum_j^{1,0} \int dQ \bar{K}_j(Q)\, \bar{\rho}_j(Q), \\
&\approx \sum_j^{1,0} \int dQ \left(\frac{c_0^j}{2} + \sum_{i=1}^{N_t} c_i^j \cdot \bar{T}_i^j\left(e^{-Q}\right) \right) \left(\frac{\langle \Psi | \delta(\hat{H} - Q) | \Psi \rangle}{\langle \Psi | \Psi \rangle} \right), \\
&= \sum_j^{1,0} \left[\frac{c_0^j}{2} + \sum_{i=1}^{N_t} c_i^j \left\langle \bar{T}_i^j\left(e^{-\hat{H}}\right) \right\rangle \right], \\
&\text{where,} \\
&\left\langle \bar{T}_i^j(\hat{H}) \right\rangle \equiv \frac{\langle \Psi | \bar{T}_i^j(\hat{H}) | \Psi \rangle}{\langle \Psi | \Psi \rangle},
\end{aligned}
\tag{36}
$$

Now expanding the exponential term, we have

$$
\begin{aligned}
\left\langle \bar{T}_1\left(e^{-\hat{H}}\right) \right\rangle &= 2 \left(\frac{\langle \Psi | \mathrm{e}^{-\hat{H}} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \right) - 1 = 2\bar{C}(1) - 1, \\
\left\langle \bar{T}_2\left(e^{-\hat{H}}\right) \right\rangle &= 8 \left(\frac{\langle \Psi | \mathrm{e}^{-2\hat{H}} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \right) - 8 \left(\frac{\langle \Psi | \mathrm{e}^{-\hat{H}} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \right) + 1 = 8\bar{C}(2) - 8\bar{C}(1) + 1, \\
&\vdots \\
\left\langle \bar{T}_i\left(e^{-\hat{H}}\right) \right\rangle &= \sum a_i \bar{C}(i)
\end{aligned}
\tag{37}
$$

where the a_i is presented as following table:

	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
$\bar{T}_1$	-1	2									
$\bar{T}_2$	1	-8	8								
$\bar{T}_3$	-1	18	-48	32							
$\bar{T}_4$	1	-32	160	-256	128						
$\bar{T}_5$	-1	50	-400	1120	-1280	512					
$\bar{T}_6$	1	-72	840	-3584	6912	-6144	2048				
$\bar{T}_7$	-1	98	-1568	9408	-26880	39424	-28672	8192			
$\bar{T}_8$	1	-128	2688	-21504	84480	-180224	212992	-131072	32768		
$\bar{T}_9$	-1	162	-4320	44352	-228096	658944	-1118208	1105920	-589824	131072	
$\bar{T}_{10}$	1	-200	6600	-84480	549120	-2050048	4659200	-6553600	5570560	-2621440	524288

Table 1: Coefficients of shifted chebshev polynoimals of the second linds

3.1 Error propagation

The discrepancy between the approximation and the actual physics encompasses both statistical and systematic errors. The systematic error falls beyond the scope of this paper. Considering the correlation within the data, if the data is truly Gaussian distributed, the statistical error (denoted as σ) of the Chebyshev method can be extracted from the covariance matrix of $\bar{C}$, which is denoted as $\overline{\text{Cov}}$.

$$
\sigma\left(\left\langle \bar{T}_i\left(e^{-\hat{H}}\right) \right\rangle \right) = \sqrt{ \sum_{(j,k) \le i} \left(a_j \overline{\text{Cov}}(j,k)\, a_k \right) }
\tag{38}
$$

The updated covariance matrix, denoted as $\overline{\text{Cov}}(t_i, t_f)$, is generated by applying the outer product to the arrays $\sigma\left(\bar{C}(t_i)\right)$ and $\sigma\left(\bar{C}(t_f)\right)$. Following the acquisition of this new covariance matrix, we are then able to compute the statistical error of $\left\langle \bar{T}_i\left(e^{-\hat{H}}\right) \right\rangle$ while taking into account correlated data.

$$
\begin{aligned}
\bar{C}(t) &= \frac{C(t + 2t_0)}{C(2t_0)}, \\
\frac{\sigma\left(\bar{C}(t)\right)}{\bar{C}(t)} &= \sqrt{ \left(\frac{\sigma(C(t + 2t_0))}{C(t + 2t_0)} \right)^2 + \left(\frac{\sigma(C(2t_0))}{C(2t_0)} \right)^2 - 2 \left(\frac{\text{Cov}\,(t + 2t_0, 2t_0)}{C(t + 2t_0) C(2t_0)} \right) }
\end{aligned}
\tag{39}
$$

the new covariance matrix $\overline{\text{Cov}}(t_i, t_f)$ can be constructed by the outer product of array $\sigma\left(\bar{C}(t_i)\right) \times \sigma\left(\bar{C}(t_f)\right)$. After obtained the new covariance matrix, we can compute error of the $\left\langle \bar{T}_i\left(e^{-\hat{H}}\right) \right\rangle$ with correlated data. The first five error and mean calculation to the RBC/UK QCD(2018), 64I is listed in the appendix.

3.2 Regularized kernel function

After we introduced the approximation, extra factors now comes into the kernel functions. The factor of Q^{-2} makes our kernel function become diverged in the infrared region:

$$
\begin{aligned}
\Gamma_{\text{app}} \equiv& \int_0^\infty dQ^2 \frac{C(2t_0)\exp(2Qt_0)}{Q^2}\left(1-\frac{Q^2}{m_\tau^2}\right)^2\left(1+2\frac{Q^2}{m_\tau^2}\right)H(m_\tau^2-Q^2)\bar{\rho}^{(1)}(Q^2)+\frac{C(2t_0)\exp(2Qt_0)}{Q^2}\left(1-\frac{Q^2}{m_\tau^2}\right)^2 H(m_\tau^2-Q^2)\bar{\rho}^{(0)}(Q^2) \\
=& \int_0^\infty dQC(2t_0)\exp(2Qt_0)\left(\frac{4Q^5}{m^6}-\frac{6Q^3}{m^4}+\frac{2}{Q}\right)H(m_\tau^2-Q^2)\bar{\rho}^{(1)}(Q)+C(2t_0)\exp(2Qt_0)\left(\frac{2Q^3}{m^4}-\frac{4Q}{m^2}+\frac{2}{Q}\right)H(m_\tau^2-Q^2)\bar{\rho}^{(0)}(Q) \\
=& \int_0^\infty dQ\bar{K}^1_{\text{org}}(Q)\bar{\rho}^{(1)}(Q)+\bar{K}^0_{\text{org}}(Q)\bar{\rho}^{(0)}(Q)
\end{aligned}
\tag{40}
$$

These extra dependence makes our kernel functions diverge in low energy region. We introduce a regulator to regularized this divergence. Unlike the renormalization, this regularization has no physical meaning. After introduced the regulator, the kernel is defined as:

$$
\begin{aligned}
K^1_{\text{Regularized}} \equiv& \tanh\left(\left(\frac{Q}{Q_0}\right)^N\right)\times\bar{K}^1_{\text{org}}, \\
K^0_{\text{Regularized}} \equiv& \tanh\left(\left(\frac{Q}{Q_0}\right)^N\right)\times\bar{K}^0_{\text{org}}
\end{aligned}
\tag{41}
$$

The value of the Q_0 should make smeared kernel close to the original function when $Q > M_{\text{Kaon}}$. Our method provides a powerful advantage of the lattice method. For instance, the quark-hadron duality only hold when Q is larger than 1 GeV. In our method, the lower bound can be reach to the Kaon mass, which significantly reduces the theoretical uncertainty. The total factors we used in the numerical calculation is:

$$
K_{\text{Regularized}} = C(2t_0)\times\exp(2Qt_0)\times\tanh\left(\left(\frac{Q}{Q_0^J}\right)^N\right)\times\text{Polynoimals},
\tag{42}
$$

In Ishikawa's paper, the t_0 is selected to one in order to prevent the loss of excited states. In previous case, we choose t_0 equal to 0.5. The approximation of the regularized kernel functions are: for each angular momentum:

$$
\begin{aligned}
c_i^1 =& \frac{2}{\pi}\int_0^\pi d\theta K^1\left(-\text{Log}\left(\frac{1+\text{Cos}(\theta)}{2}\right)\right)\text{Cos}(i\theta) \\
=& \frac{2}{\pi}\int_0^\pi d\theta\text{Cos}(i\theta)\left(-\frac{4}{m_\tau^6}\text{Log}^5\left(\frac{1+\text{Cos}(\theta)}{2}\right)+\frac{6}{m_\tau^4}\text{Log}^3\left(\frac{1+\text{Cos}(\theta)}{2}\right)-2\text{Log}^{-1}\left(\frac{1+\text{Cos}(\theta)}{2}\right)\right)\times \\
&\exp\left(2\left(-\text{Log}\left(\frac{1+\text{Cos}(\theta)}{2}\right)\right)t_0\right)\tanh\left(\left(\frac{\left(-\text{Log}\left(\frac{1+\text{Cos}(\theta)}{2}\right)\right)}{Q_0^J}\right)^N\right), \\
c_i^0 =& \frac{2}{\pi}\int_0^\pi d\theta K^0\left(-\text{Log}\left(\frac{1+\text{Cos}(\theta)}{2}\right)\right)\text{Cos}(i\theta) \\
=& \frac{2}{\pi}\int_0^\pi d\theta\text{Cos}(i\theta)\left(-\frac{2}{m_\tau^4}\text{Log}^3\left(\frac{1+\text{Cos}(\theta)}{2}\right)+\frac{4}{m_\tau^2}\text{Log}\left(\frac{1+\text{Cos}(\theta)}{2}\right)-2\text{Log}^{-1}\left(\frac{1+\text{Cos}(\theta)}{2}\right)\right)\times \\
&\exp\left(2\left(-\text{Log}\left(\frac{1+\text{Cos}(\theta)}{2}\right)\right)t_0\right)\tanh\left(\left(\frac{\left(-\text{Log}\left(\frac{1+\text{Cos}(\theta)}{2}\right)\right)}{Q_0^J}\right)^N\right)
\end{aligned}
\tag{43}
$$

In the latter chapter, the N will fixed at 1 to minimized the difference.

3.3 Extraction of the CKM matrix element

In our approach, the CKM matrix element can be extracted by comparing the experiment data and the result calculated by the Chebshev approximation:

$$
V_{us} = \sqrt{\frac{\Gamma_{\text{Exp}}}{\left(\frac{G_F^2 m_\tau^3}{16\pi}\right)\times\Gamma_{\text{app}}}}
\tag{44}
$$

4 Inclusive calculation

The codeset is listed here.

4.1 Mock data testing

To test this method work or not and the precision of this method. It's important to test by creating the mock data. To get the parameters, we use the following equation to fit the amplitude and phase:

$$C(t) = 2A \cdot \exp^{\left(-M \cdot \frac{T}{2}\right)} \mathrm{Cosh}\left(\left(\frac{T}{2} - t\right) M\right) \tag{45}$$

since we don't need the correlator from other direction, only decay mode is calculated. For the only data we have, the RBCQCD data. They call them 48I and 64I respectively, which stands for their volume. Our data is given by different gamma matrices. Here I select the $\gamma_5\gamma_4$ channel to apply above fitting euqation and obtain following table:

The inverse a is obtained by taking ratio between kaon mass and M_0. The lattice spacing is:

$$a_{48}^{-1} = \frac{M_k}{M_{0,48}} = 1.7295(38) \tag{46}$$

$$a_{64}^{-1} = \frac{M_k}{M_{0,64}} = 2.3586(7) \tag{47}$$

To check the consistency, next step is to calculate the kaon decay constant f_K. I do not have permission to access the 3-point functions, so I can only directly use the ZA,ZV mentioned in the article from RBCQCD. The renormalized decay constant is:

$$f_K = Z_V \sqrt{\frac{2 \cdot A}{M_K V}} \tag{48}$$

Applying above equation, we can obtain the table:

Table 2: Lattice ensembles of the data for Kaon

	48I	64I
Configurations	40	88
Measurements	48	64
Time slicing	96	128
Spatial volume	48^3	96^3
A_0	256.2 (1.7)	226.2 (1.4)
M_0	0.28816 (44)	0.21539 (43)
a^{-1} (GeV)	1.7295(38)	2.3586(7)
f_k	0.15587(55)	0.15684(28)
$Z_A = Z_V$	0.71076	0.74293

Table 3: Lattice ensembles of the data for Kstar

	48I	64I
Configurations	40	88
Measurements	48	64
Time slicing	96	128
Spatial volume	48^3	96^3
A_0	858 (48)	728 (67)
M_0	0.5170	0.3812 (51)
a^{-1} (GeV)	1.7295(38)	2.3586(7)
f_k	0.2129(52)	0.2114(84)
$Z_A = Z_V$	0.71076	0.74293

Therefore the f_K is obtained by $a^{-1} \times af_k \approx 0.1576$. For Kstar, we use the same lattice spacing extracted from kaon, the Kstar mass from RBC data is:

$$M_{48}^{k*} = a_{48}^{-1} M_{0,48} = 0.8942(67) \tag{49}$$

$$M_{64}^{k*} = a_{64}^{-1} M_{0,64} = 0.899(12) \tag{50}$$

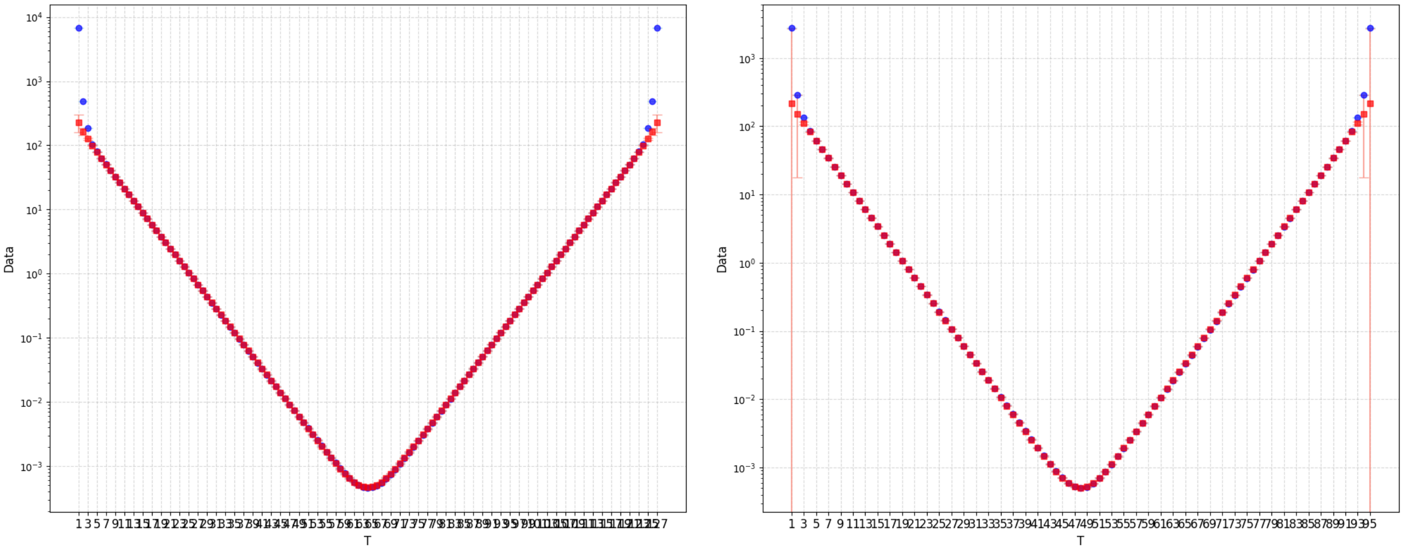


Figure 1: Orignal correlator vs Reconstructed Mock data

Our goal is to compare the exclusive decay width to by assuiming a delta function in the kernel or experimental value. I used mock data which constructed by:

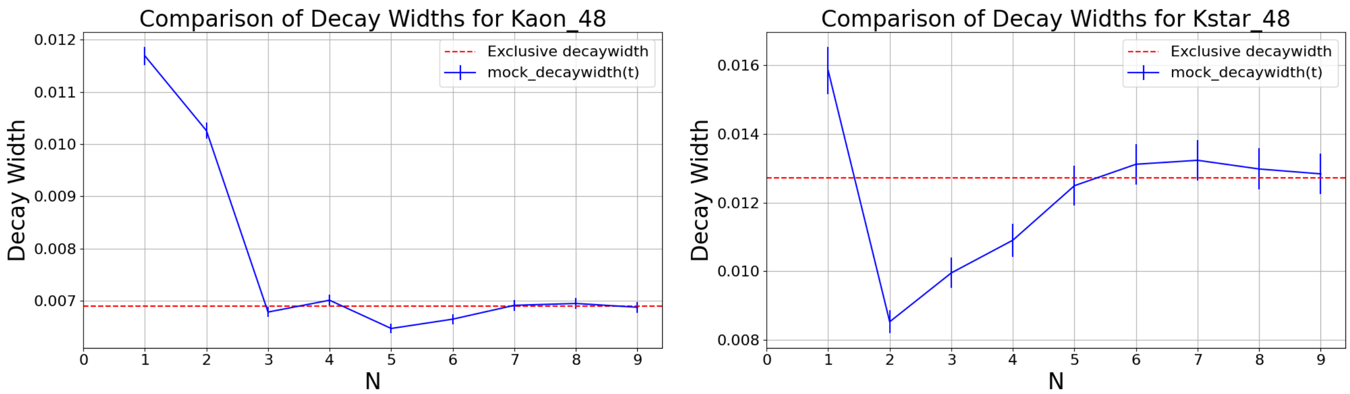
$$C(t) = \frac{M_K \cdot f_k^2}{2} e^{-M_K t} \tag{51}$$

Therefore the decay width is :

$$\Gamma_{\text{app}}(\tau^- \to K^- + \nu_\tau) = \frac{G_F^2 f_K^2 m_\tau^3}{16\pi} \left(1 - \frac{M_K^2}{m_\tau^2}\right)^2, \tag{52}$$

$$\Gamma_{\text{app}}(\tau^- \to K^* + \nu_\tau) = \frac{G_F^2 f_{K^*}^2 m_\tau^3}{16\pi} \left(1 - \frac{M_{K^*}^2}{m_\tau^2}\right)^2 \left(1 + 2\frac{M_{K^*}^2}{m_\tau^2}\right). \tag{53}$$

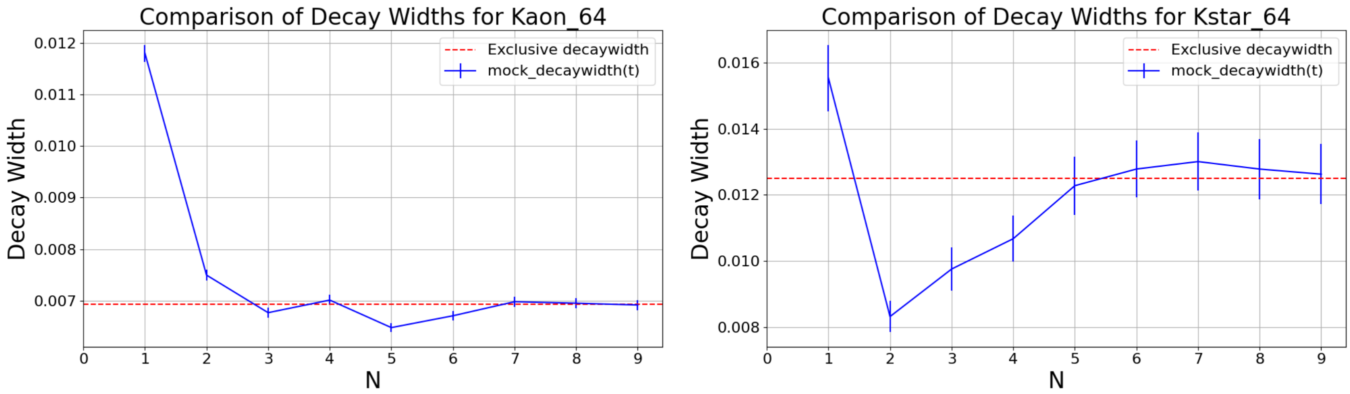
inserting Γ_{app} into equation.44 then the $V_{u,s}$ could be obtained.



(a) J = 0.

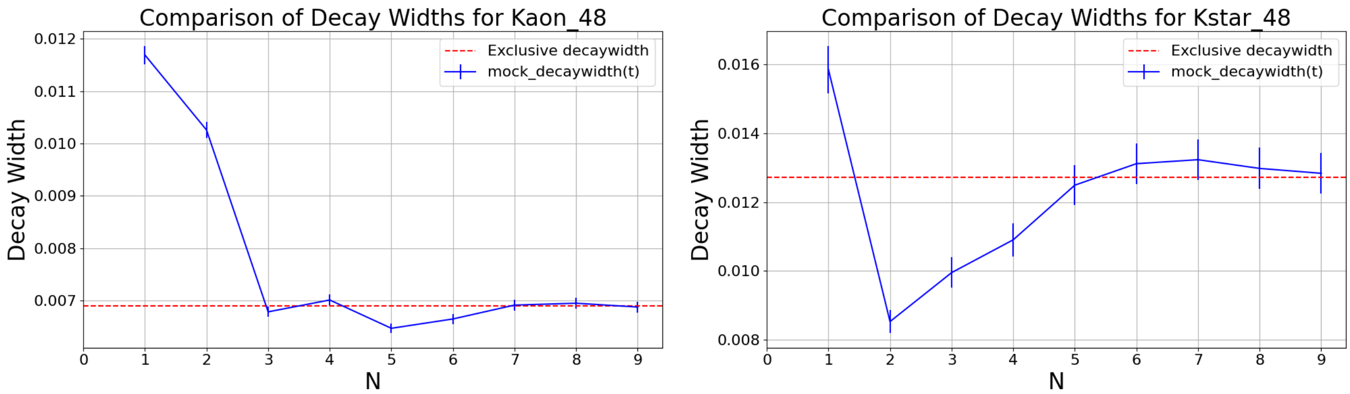
(b) J = 1.

Figure 2: Mock decaywidth for 48.



(a) J = 0.　　(b) J = 1.

Figure 3: Mock decaywidth for 64.



(a) J = 0.　　(b) J = 1.

Figure 4: Mock decaywidth for 48.

4.2 Precondition of lattice data

4.2.1 Angular momentum decomposition

It is important for us to understand the relationship between the angular momentum and our lattice data, which is separated by gamma matrices. For example, the data that we obtained looks like:

$$\begin{aligned} C_V^{t,t}(t) &= \int d^3x \, \langle 0| \, J_V^4(x,t) J_V^4(0)^\dagger \, |0\rangle \, e^{-ip\cdot x}, \\ C_V^{r,r}(t) &= \int d^3x \, \langle 0| \, J_V^r(x,t) J_V^r(0)^\dagger \, |0\rangle \, e^{-ip\cdot x}, \\ C_A^{t,t}(t) &= \int d^3x \, \langle 0| \, J_A^4(x,t) J_A^4(0)^\dagger \, |0\rangle \, e^{-ip\cdot x}, \\ C_A^{r,r}(t) &= \int d^3x \, \langle 0| \, J_A^r(x,t) J_A^r(0)^\dagger \, |0\rangle \, e^{-ip\cdot x}. \end{aligned} \tag{54}$$

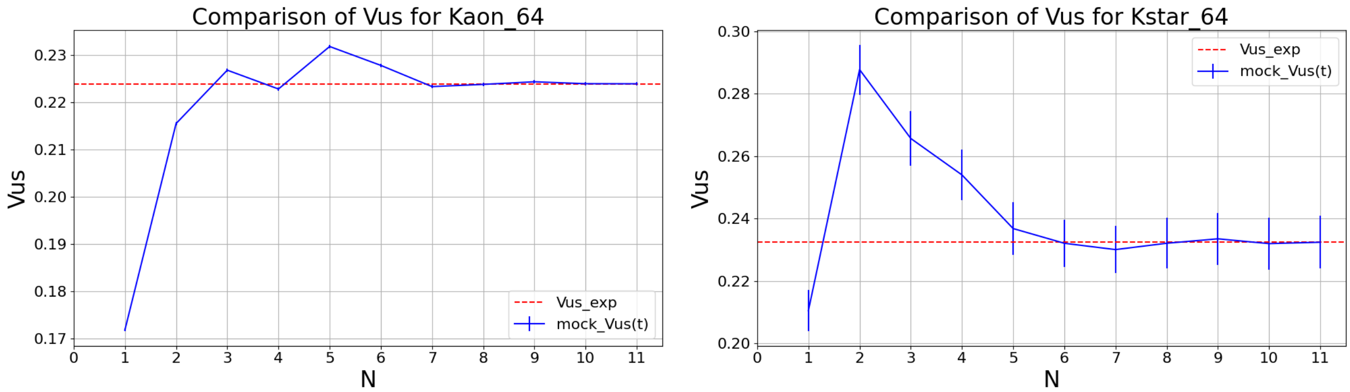
where r stands for the spatial direction. It is worth mentioning that the Fourier transform in the correlation function results in an additional factor that is proportional to the spatial volume of the lattice. We rearrange our data:

$$\begin{aligned} C_{V/A}^{(J=1)}(t) &= \frac{1}{3} \sum_r^{x,y,z} C_{V/A}^{r,r}(t) \\ C_{V/A}^{(J=0)}(t) &= C_{V/A}^{t,t}(t) \end{aligned} \tag{55}$$

4.2.2 Fitting the Chebshev matrix element

We expect the accuracy of the result will improve as we include more terms in the approximation. By combining with the $\bar{C}(t)$, we can use the lattice correlator to reconstruct the R ratio. However, if we plug the data directly into the Chebyshev polynomial to obtain $\bar{C}(t)$, the error becomes unacceptably large. To address this issue, we use the recursive relation of the second kind Chebyshev polynomial:

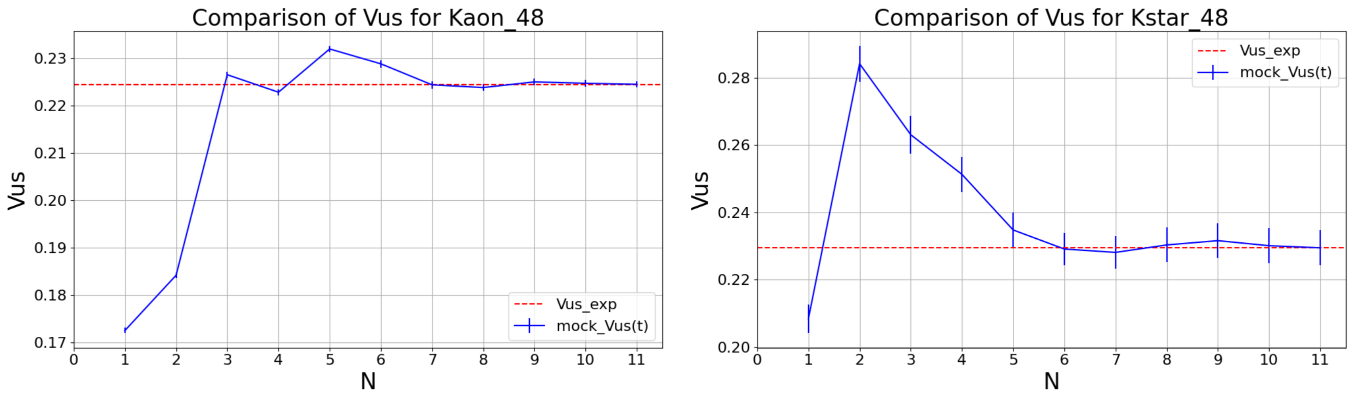
$$\bar{C}(t) = 2^{1-2t} \left[\frac{1}{2} \binom{2t}{t} + \sum_{j=1}^{t} \binom{2t}{t-j} \langle T_j^* \rangle \right], \tag{56}$$



(a) J = 0.

(b) J = 1.

Figure 5: Mock decaywidth for 64



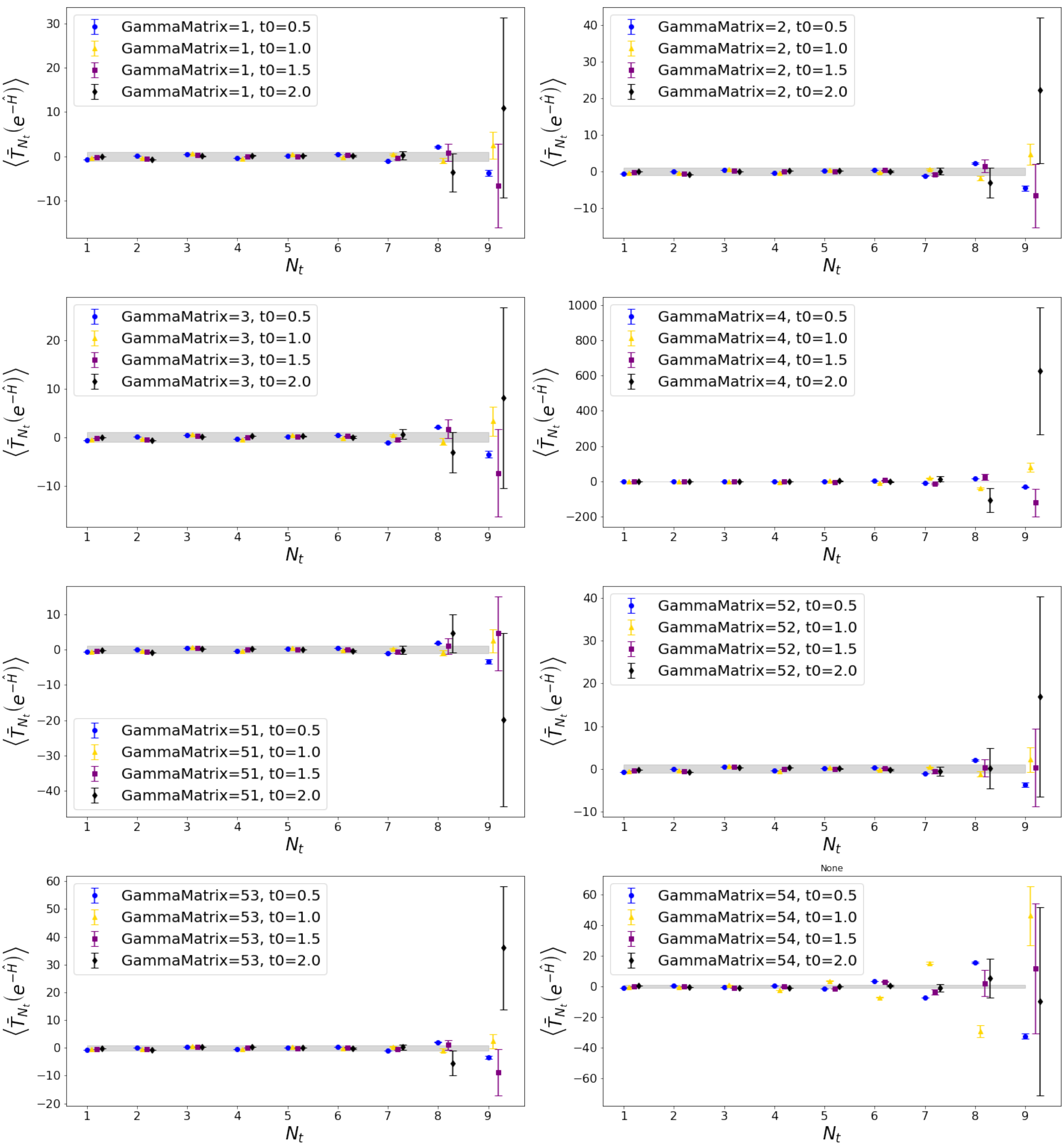
(a) J = 0.

(b) J = 1.

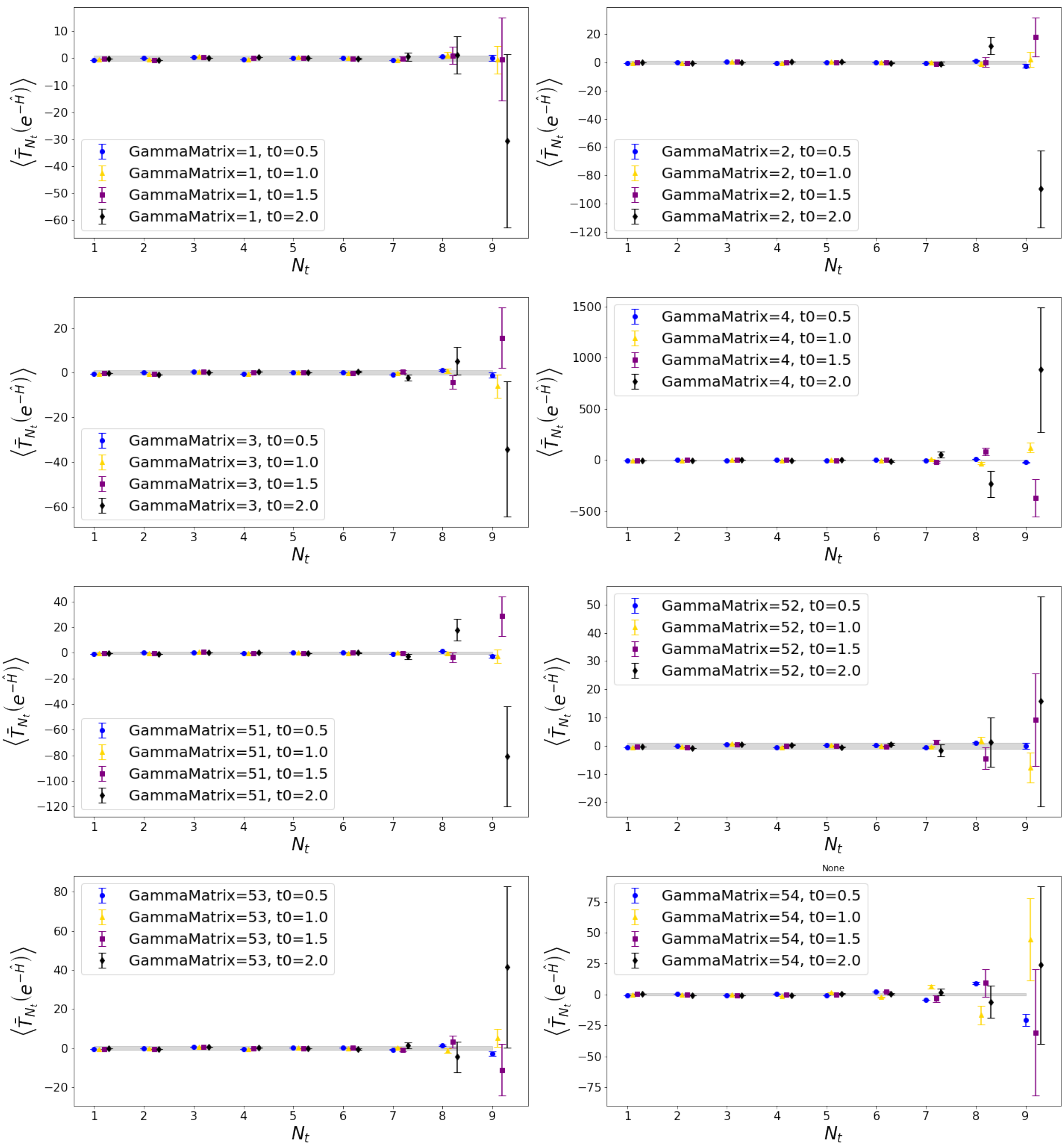
Figure 6: mock vus from 48.

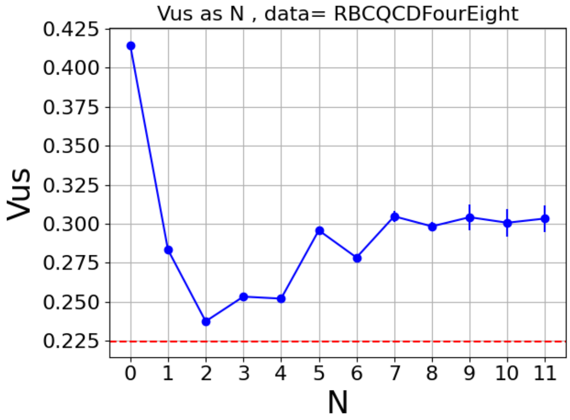
where time separation t_0 is introduced to avoid the OPE-divergence point near zero in the time direction. By using the fsqfit provided at python, we can fit the Chebyshev elements $\langle T_j^* \rangle$.

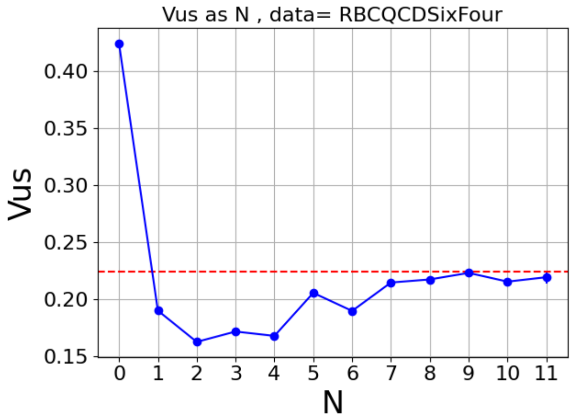
DirectSum of Chebeshev from RBC/UKQCD(2018) Data set, L=64



DirectSum of Chebeshev from RBC/UKQCD(2018) Data set, L=48







A Experiment Data Analysis

In order to check the behaviour of our approximation, it's important to check the function form which can presents the shape of the data. If only the delta pick is used, the error due to the non-zero width will be ignored. In this chapter, we analysis the data from Belle [**?**]. In the Belle, they define:

$$\begin{aligned}\frac{dR_{us;V+A}}{ds} &= \frac{1}{B_e}\frac{dB_{us;V+A}}{ds} \\ &= \frac{12\pi^2 S_{EW}|V_{us}|^2}{m_\tau^2}(1-y_\tau)^2 \cdot [(1+2y_\tau)\rho^{0+1}_{us;V+A}(s) - 2y_\tau \rho^0_{us;V+A}(s)] \\ \text{where } \frac{dB}{ds} &= B_{K^-\pi^0+\overline{K^0}\pi^-}\left(\frac{1}{N}\frac{dN}{ds}\right) \text{ and } y_\tau = \frac{s}{m_\tau^2}\end{aligned} \tag{57}$$

then

$$\begin{aligned}\mathrm{LC}(Q^2) &= m_\tau^2\left(\frac{dB_{V+A}}{ds}\right)\left(12\pi^2 S_{\mathrm{EW}}(1-y^2)B_e\right)^{-1}, \\ &= |V_{us}|^2\left[(1+2y)\,\rho^{(0+1)} - 2y\rho^{(0)}\right], \\ &= |V_{us}|^2\left[(1+2y)\,\rho^{(1)} + \rho^{(0)}\right]\end{aligned} \tag{58}$$

where $y = \frac{Q^2}{m_\tau^2}$.

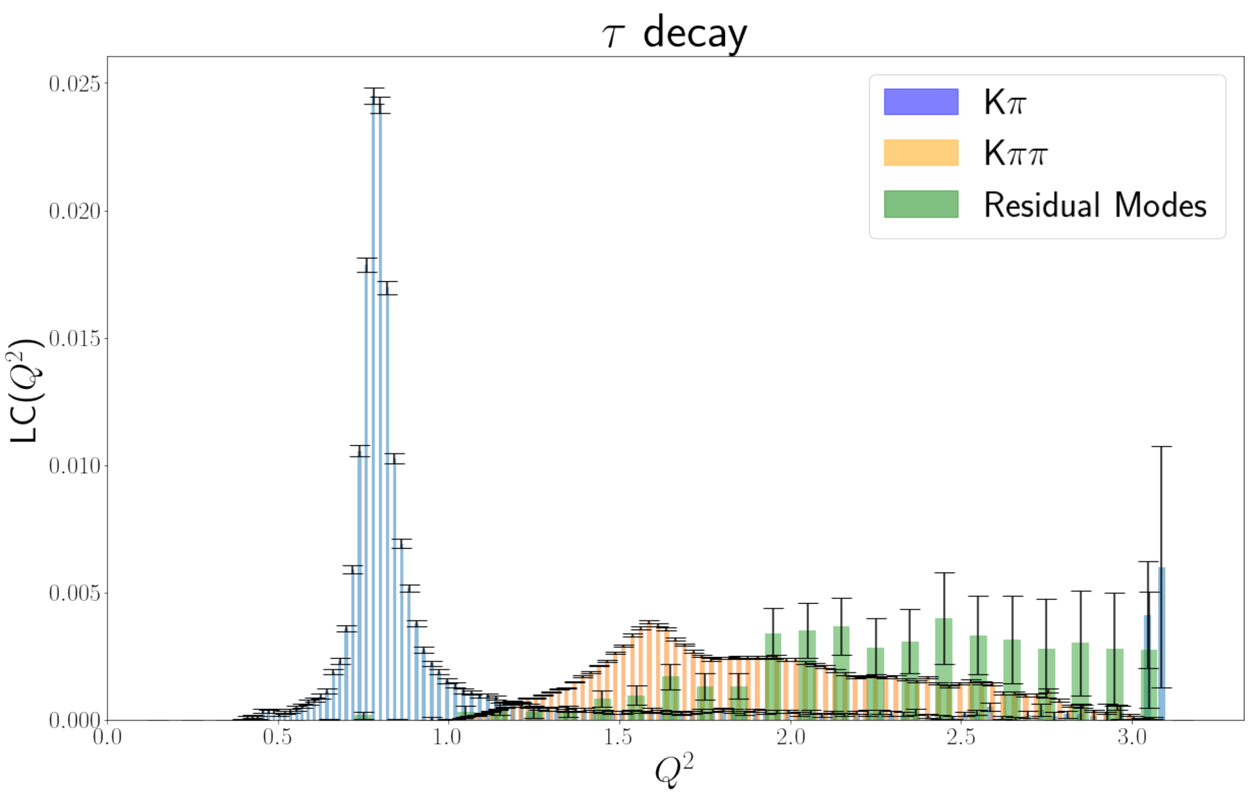


Figure 8: Histogram of Belle result present as LC(Q^2).

A.1 Lattice Configuration and the fitting formalism

In this chapter, I review how did I fit our lattice data and corresponded result. In begining, I list some proprety of the data. Basically wo data are used in our researching. I call them 48 and 64 respectively, which stands for their volume. Our data is given by different gamma matrices. For instance, in axial current, we have $\gamma_5\gamma_i$ and $\gamma_5\gamma_4$. The lattice data [?], some propreties are listed in following table:

Table 4: Lattice ensembles of the data

	48c	64c
Configurations	40	88
Measurements	48	64
Time slicing	96	128
a^{-1} (GeV)	1.7295(38)	2.3586(7)
am_π	0.08049(13)	0.05903(13)
am_k	0.28853(14)	0.21531(17)
af_k	0.090396(86)	0.21531(17)
$Z_A = Z_V$	0.71076	0.74293
Spatial Volume(L^3)	48^3	64^3
β	2.13	2.25

according to this table, we can average over the configurations on the raw data. Firstly we have to average over the measurement then average over configurations. We implement the Jackknife resampling method to reduce the error. The averaged data with different lorentz indices and different spatial volume are plotted logarithmic. In this chapter, we present the fitting result and check the consistence with the previous table. We want to indicate that the pion mass and renormalization constant cannot be checked by only the two-point function, since both of them needs the three point function to obtain the other bosons.

A.1.1 Hadron Mass

The hadron mass could be fitted from the correlation function. The spatial data are contaminated by excited states. In the contrary, the time component gives a very clean data, which we can assume that the ground state particle dominates this channel. We apply thre listed formalism to lsqfit package:

$$T_{\text{Fit}}(t) = A \times \exp^{(-M_K \cdot t)} \tag{59}$$

The fitting detail is presented as following table:

N	48	64
Fit range	(20,40)	(10,50)
Mass	0.28845 (10)	0.21555 (13)
χ^2/d.o.f	1.52	1.7

the fitting results of 48 and 64 data consist to the reference.

A.1.2 Decay constant

The fitting formalism is given from the reference (data). For instance, the decay constant could be extracted from the correlation function when the excited states are ignored. Inserting to the spectrum integral, we find that the result is nothing but with an additional factor to the equation.(34). The ground state only model on the correlator gives:

$$C(t) = 2A\exp^{\left(-M\cdot\frac{T}{2}\right)}\text{Cosh}\left(\left(\frac{T}{2} - t\right)M\right) \tag{60}$$

According to the reference, we also should obtain the renormalization factor Z_A, which in charge to cancel the divergence in the non-chiral region. The actually decay constant should be fitted from the following equation:

$$f_K = Z_V\sqrt{\frac{2\cdot A}{M_k \cdot V}} \tag{61}$$

where A is the amplitude fitted from the axial channel and the V is the spatial volume of the data.

N	48	64
Fit range	8,24	8,48
Amplitude	258.22(86)	229.46(71)
f_K	0.09024(15)	R0.06672(13)
χ^2/d.o.f	0.98	1.8

A.1.3 Two particles fits

Generally, we expect that the excited state are exist on our data. For instance, the slope of the logarithmic plot changes in the small time slicing region indicated the excited states might exist. To identify the excited state, we must extend our formalism to two particles. After we introduced the 2 particles to the fit. The formalism becomes:

$$\mathrm{T}_{\mathrm{CoshFit}}\left(t\right)=\sum_{i}^{2}A_i\cdot\exp\left(\frac{\mathrm{Nt}\cdot\mathrm{M}_{ki}}{2}\right)\cdot\mathrm{Cosh}\left(\mathrm{M}_{ki}\cdot\left(t-\frac{\mathrm{Nt}}{2}\right)\right) \tag{62}$$

where i presents the number of particles. In following results, the amplitude a.A is defined as:

$$a\cdot A\equiv A_i\cdot\exp\left(\frac{\mathrm{Nt}\cdot\mathrm{M}_{ki}}{2}\right). \tag{63}$$

The addition factor inside the amplitude is from the overlapped of two waves with opposite directions.

The following two tables present the result which fitted by Flavien Callet by using the lsqfit:

Table 5: Lattice Data, 48c, (5,92)

	$V, J=0$	$A, J=0$	$V, J=1$	$A, J=1$
$a\cdot A_0$	0.000237(36)	0.0023400(63)	0.00968(14)	0.0155(12)
$a\cdot m_0$	0.471(21)	0.28883(24)	0.5317(16)	0.7589(93)
$a\cdot f_0$	0.0225(12)	0.09047(11)	0.13562(77)	0.1438(49)
$a\cdot A_1$	0.0092(52)	0.0151(82)	0.1158(42)	0.199(27)
$a\cdot m_1$	1.41(14)	1.55(12)	1.245(11)	1.461(45)
$a\cdot f_1$	0.081(19)	0.099(23)	0.3065(42)	0.371(19)
chi2/dof	1.0915118467	1.1251417864	1.0850871593	1.1037699606

Table 6: Lattice Data, 64c, (7,121)

	$V, J=0$	$A, J=0$	$V, J=1$	$A, J=1$
$a\cdot A_0$	0.000109(32)	0.0008654(31)	0.003446(75)	0.00494(64)
$a\cdot m_0$	0.374(27)	0.21549(24)	0.3927(17)	0.552(11)
$a\cdot f_0$	0.0179(20)	0.06658(11)	0.09841(86)	0.0994(55)
$a\cdot A_1$	0.00030(38)	0.00040(34)	0.0285(16)	0.0411(76)
$a\cdot m_1$	0.82(31)	0.84(15)	0.881(13)	1.011(49)
$a\cdot f_1$	0.0201(89)	0.0229(78)	0.1890(41)	0.212(14)
chi2/dof	0.7735837421	1.1701979602	1.0024449565	1.0926596074

where the time slicing region of the 48 is selected between (10,85) and the 64 is selected betweeb (10,110). Comparing to the single particle fit, we found that the proprety of the ground state didn't change to much after we introduced the second particles. Some excited state might be a problem to our calculation, the first excited state of the A,J=0 channel contains a heavy particle which the mass is heavier than the tau, which is non-allowed in the phase space.

A.1.4 Q_0 selection algorithm

The selection of Q_0 should be as small as possible, to make sure the difference between the smeared and original kernel function. But if the Q_0 is chosen in very small value, the summation of approximation needs more terms to converge, which makes our approximation bad. Now we want to test the impact of the Q_0 selection. An simple algorithm is developed to find the value of Q_0. Defining the j_{CRI} which makes $|c(i)| < 0.5$ for any $i > j_{\mathrm{CRI}}$. For given list of Q_0(from 0.05 to 0.4 with a step of 0.00005), comparing different j_{CRI} and find the Q_0 which minized the j_{CRI}. We have:

$$Q_0^{\mathrm{J=0}} = 0.12\mathrm{GeV}, Q_0^{\mathrm{J=1}} = 0.185\mathrm{GeV}, \tag{64}$$

which makes the length of coefficients after $j_{\mathrm{CRI}} = 2, 3$ are all smaller than 0.5.

Comparing this to the experiment data provided from, we found the difference are dominated at small-Q^2, which is not physically important.

B Backup in many years ago

C Figures

C.1 Lattice correlator

In this section, I follow Ishikawa's method [**?**] to derive the relationship between correlator and the spectral function. Firstly, we define the di-current correlation function:

$$C(t) = \int d^3x < J_\mu(t,x) J_\nu(0,0)^\dagger > \tag{65}$$

and the HVP function :

$$\Pi^{\mu\nu}\left(q, Q^2\right) = -(g^{\mu\nu}Q^2 - q^\mu q^\nu)\Pi(Q^2) = \int d^4x < J_\mu(t,x) J_\nu(0)^\dagger > e^{iq\cdot x} \tag{66}$$

the relationship between the HVP and the spectral function ρ is given:

$$\begin{aligned} \Pi(Q^2) - \Pi(0) &= -\frac{Q^2}{\pi}\int_0^\infty ds \frac{\mathrm{Im}\left[\Pi(s)\right]}{s(s+Q^2)} \\ &= -\frac{Q^2}{\pi}\int_0^\infty ds \frac{\rho(s)}{s(s+Q^2)} \end{aligned} \tag{67}$$

where $\mathrm{Im}\left[\Pi(s)\right] = \frac{R(s)}{12\pi}$. Assuming that $Q = (\omega, 0)$, we can rewrite the HVP as:

$$-\omega^2\Pi(\omega^2) = \int_{-\infty}^{\infty} dt C(t) e^{i\omega\cdot t} \tag{68}$$

using the inverse FT, we can obtain the correlator from spectral function ρ:

$$C(t) = \int_0^\infty d\omega \omega^2 \rho(\omega) e^{-\omega t} \tag{69}$$

The imaginary part of the spectral function should be related to the lattice correlator:

$$\mathrm{Im}\Pi_{i,j}^{(J)}(q) = \frac{1}{2i}\left[\Pi_{i,j}^{(J)}(q) - \Pi_{i,j}^{(J)}(q)^\dagger\right] \tag{70}$$

In the lattice part, we generate the di-current correlator as [**?**]:

$$C_{V/A}^{\mu\nu}(t) = \sum_{\vec{x}} C_{V/A}^{\mu\nu}(\vec{x}, t) = \sum_{\vec{x}} \langle 0| J_{V/A}^{E,\nu}(\vec{x}, t) J_{V/A}^{E,\mu}(0,0)^\dagger |0\rangle \tag{71}$$

the relation of the lattice correlator to the HVP is given in[**?**]:

$$\begin{aligned} \Pi^E\left(K^2\right) F^{\mu\nu}(K) &= \int dt e^{iK_1 t} \int d^3\vec{x} e^{i\vec{k}\vec{x}} \langle 0| J_{V/A}^{E,\nu}(\vec{x}, t) J_{V/A}^{E,\mu}(0,0)^\dagger |0\rangle, \\ &= \int d^4x e^{-iKx} C_{V/A}^{\mu\nu}(\vec{x}, t) \end{aligned} \tag{72}$$

where $F^{\mu\nu}(K) = K_\mu K_\nu - \delta_{\mu\nu}K^2$ is the lorentz factor on the lattice and the momentum K^2 is a spacelike vector. We want to obtain the lattice correlator as different angular momentum J. Using equation.(14), the lattice correlator can rewrite as:

$$C_{V/A}^{\mu\nu}(t) = -(g^{\mu\nu}Q^2 - q^\mu q^\nu)\Pi_{i,j}^{(0+1)}\left(Q^2\right) + g^{\mu\nu}Q^2\Pi_{i,j}^{(0)}\left(Q^2\right) \tag{73}$$

References

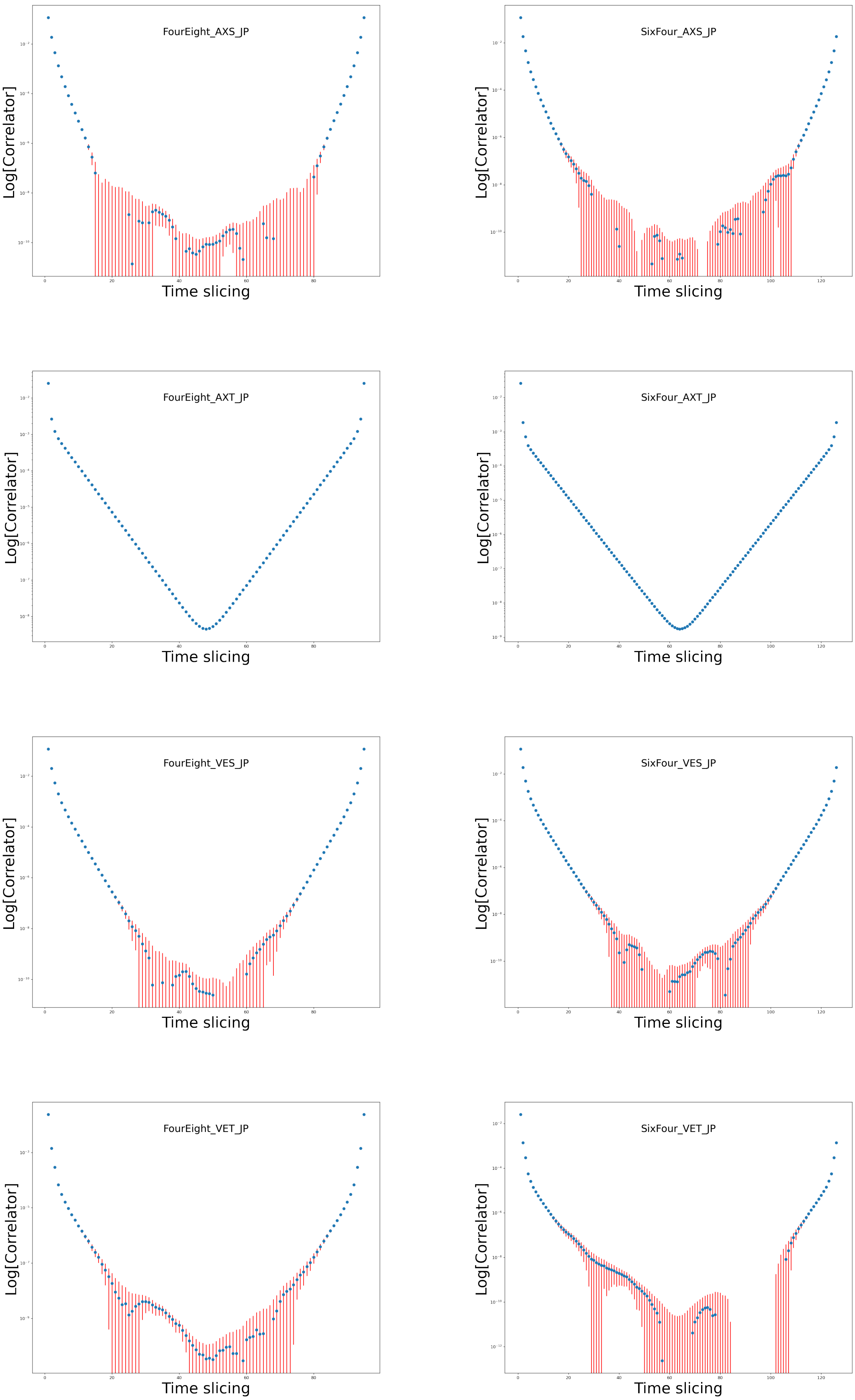


Figure 9: 2pt function with loargithemic plot. Jackknife-Averaged over the configurations