Phase retrieval algorithms

Definition 1 (Phase retrieval projection operators)**.** For $A \in L^2(\mathbb{R}^d, \mathbb{R}_{\geqslant 0})$, define the positivity and modulus projection operators $P_P,\ P_M :\ : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ as follows: for any $g \in L^2(\mathbb{R}^d)$ and L^2- almost all $x,\ k$,

$$P_P g(x) = \frac{1}{2}\big(g(x) - |g(x)|\big) = \begin{cases} g(x), \text{ if } g(x) \geqslant 0; \\ 0 \text{ otherwise}; \end{cases} \tag{1}$$

$$P_M g(x) = \mathcal{F}^{-1}\left(A \cdot e^{i\cdot\arg(\widehat{g})}\right) = \mathcal{F}^{-1}\left(A \cdot \frac{\widehat{g}}{|\widehat{g}|}\right)(x), \tag{2}$$

where $\frac{\widehat{g}(k)}{|\widehat{g}(k)|}$ is understood to be 1 for $|\widehat{g}(k)| = 0$.

Definition 2 (Phase retrieval algorithms)**.** Recall the following definitions of the phase retrieval algorithms:

$$\text{(ER)} \qquad \psi_{n+1} = P_P \circ P_M \psi_n, \tag{3}$$

$$\text{(BIO)} \qquad \psi_{n+1} = (1 - \beta P_M + \beta P_P \circ P_M)\psi_n, \tag{4}$$

$$\text{(HIO)} \qquad \psi_{n+1} = (1 - P_P - \beta P_M + (1+\beta) P_P \circ P_M)\psi_n, \tag{5}$$

$$\text{(DM)} \qquad \psi_{n+1} = (1 + \beta D)[\psi_n] = (1 - P_P - P_M - (1+\beta) P_P \circ P_M + (1-\beta) P_M \circ P_P)[\psi_n] \tag{6}$$

with the parameter $\beta > 0$ and a random $\psi_0 \in L^2(\mathbb{R}^d)$. These iteration methods are known as error reduction (ER), basic input-output (BIO), hybrid input-output algorithms (HIO) [1], and optimal difference map algorithm (DM) [2]. Note that [1] uses the support projection instead of the positivity projection; the results below remain valid if the positivity projection is replaced by the support projection.

References

[1] H. H. Bauschke, P. L. Combettes, and D. R. Luke. Phase retrieval, error reduction algorithm, and fienup variants: A view from convex optimization. *J. Opt. Soc. Am. A*, 19, 2002.

[2] V. Elser. Phase retrieval by iterated projections. *J. Opt. Soc. Am. A*, 20:40–52, 2003.