COMPLEX LINEAR REPRESENTATIONS

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Abstract. In this expository paper, we develop necessary definitions and explain the basic theory of finite dimensional complex representation theory of compact and especially finite groups. We conclude with a detailed exploration of the irreducible representations of the symmetric group.

Contents

1. Introduction to Representation Theory

1.1. **Some Context.** Morally, groups are sets of symmetries of objects. Groups act on these objects by manipulating them according to the symmetry. For example the cube group Cu is the symmetries of a cube. There is a standard action of the cube group on the cube itself which rotates the cube in all possible orientation preserving ways to map the cube back to itself. In this way, a group can be used to learn about the space it acts on, and a space with a group action can be used to learn about the group itself. This is useful when one domain or the other is easier to work with or more understood.

Date: August 2024.

Representation theory studies the actions of groups on vector spaces. By restricting the actions to be linear, this allows well developed knowledge of linear algebra to be applied to the study of groups. We will consider the case of finite dimensional vector spaces over $\mathbb{C}$. The finite dimensional restriction allows us to break down large representations into irreducible pieces in a result known as *complete reducibility*, which is discussed in the first section. As an algebraically closed field, $\mathbb{C}$ is a natural easy case to start with, but many of the results generalize to other fields. Complex representations are also of interest for physically motivated problems in quantum mechanics, as will be discussed in Section 7 on compact groups.

In this expository paper, we develop the necessary definitions and theorems to gain an understanding of basic representation theory in the finite dimensional complex case. This will lay the groundwork for future study in representation theory.

1.2. **What Are Representations?** A representation is a special type of group action. A general group action $G \circlearrowright X$ is equivalent to a homomorphism $\psi : G \to S_X$ to the symmetric group on X. When we restrict our attention to linear actions on vector spaces, we instead consider group homomorphisms to the general linear group.

Definition 1.1. For a group G, an *F-linear representation* is a pair (V, ρ), where V is a vector space over the field F, and ρ is a homomorphism $\rho : G \to GL(V)$.

Convention 1.1. For clarity, we will introduce some shorthand. We may refer to either ρ or V individually as a "representation" of G. It is convenient to work in an algebraically closed field, in order to take advantage of theorems regarding diagonalization, so we use "representation" as shorthand for "finite-dimensional complex representation". We write ρ_g or just g for $\rho(g)$ when the context is clear.

There is a notion of when two representations are the same. We call two representations isomorphic if there exists a change of basis taking one to the other.

Definition 1.2. For representations $(\rho^1, V_1), (\rho^2, V_2)$, a *morphism* of representations between ρ_1 and ρ_2 is a linear map $\varphi : V_1 \to V_2$ such that the following diagram commutes.

$$\begin{tikzcd} V_1 \arrow[r, "\varphi"] \arrow[d, "\rho_1"'] & V_2 \arrow[d, "\rho_2"] \\ V_1 \arrow[r, "\varphi"'] & V_2 \end{tikzcd}$$

Definition 1.3. An *isomorphism* of representations is an invertible morphism.

In the above diagram, when φ is invertible, there is an equivalent formulation.

$$\rho_2 = \varphi \circ \rho_1 \circ \varphi^{-1}.$$

In this sense ρ_2 can be thought of as ρ_1 under a change of basis.

1.3. **Subrepresentations and Irreducibility.** In order to understand large and complicated representations, one useful strategy is to break them down into smaller pieces. One would like the way in which the group acts on each piece to be independent in some sense from how it acts on the others so that they may be considered in isolation an then pieced together to create a larger whole. This is precisely what subrepresentations do for us.

Definition 1.4. A *subrepresentation* of a representation (ρ, V) is a subspace $W \subset V$ that is invariant under the action of G.

We might desire to break representations down as far as possible until we reach a representation that cannot be reduced further. We call these representations *irreducible.*

Definition 1.5. A representation is *irreducible* if it has no proper subrepresentations.

There are certain representations which are then built up from irreducible representations.

Definition 1.6. A representation V is *completely reducible* if it is isomorphic to a direct sum of irreducible representations.

$$V \cong W_1 \oplus ... \oplus W_k$$

Rather than building representations from irreducible parts, we may take another perspective and, given a representation known to be completely reducible, find its irreducible factors. It turns out a large class of representations are completely reducible, and so in order to understand these representations, we need only understand their irreducible representations. We will work toward this result in our main theorem, Theorem 1.16.

One pitfall one can imagine in factoring a representation into irreducibles is that when a representation V is factored as $V \cong W \oplus U$ and W is a subrepresentation, U may *not* be a subrepresentation! Here we find a class of representations for which this pitfall doesn't happen.

Definition 1.7. A representation is *unitarizable* if it can be equipped with a hermitian, positive definite, inner product $\langle , \rangle$ with respect to which G acts unitarily:

$$\langle v, w\rangle = \langle \rho_g(v), \rho_g(w)\rangle, \ \ \forall g \in G, \ v, w \in V.$$

Lemma 1.8. *Let V be a unitarizable representation equipped with an inner product with respect to which G acts unitarily, and let W be a subrepresentation. Then the orthogonal complement $W^\perp$ is also a subrepresentation.*

Proof. $W^\perp$ is the set $\{v : \langle v, w\rangle = 0 \ \forall w \in W\}$. Since W is fixed by G, we have for any $g \in G$ and $w \in W$, $w = \rho_g(w')$, where $w' = \rho_{g^{-1}}(w) \in W$. Then for any $v \in W^\perp$, $\langle w, \rho_g(v)\rangle = \langle \rho_g(w'), \rho_g(v)\rangle = \langle w', v\rangle = 0$. The second to last equality follows from the unitarity of ρ_g. Therefore $\rho_g(v) \in W^\perp$. It follows that $W^\perp$ is a subrepresentation of V. □

Theorem 1.9. ***(Weyl)*** *Finite dimensional representations of finite groups are unitarizable.*

Proof. Let (ρ, V) be a finite dimensional representation of a finite group G. Given a hermitian, positive definite inner product $\langle \cdot, \cdot \rangle$ on V, average over the group to obtain a new inner product.

$$\langle v, w \rangle' := \frac{1}{|G|} \sum_{g \in G} \langle \rho_g v, \rho_g w \rangle$$

The new inner product inherits hermiticity and positive definiteness from the original product by linearity. To see invariance, multiply by an arbitrary element $h \in G$.

$$\langle \rho_h v, \rho_h w \rangle' = \frac{1}{|G|} \sum_{g \in G} \langle \rho_g \rho_h v, \rho_g \rho_h w \rangle$$

The sum is invariant under G by the Sudoku Lemma, since multiplication by ρ_h just permutes the elements of G. Therefore $\langle , \rangle'$ is fixed by G, and so ρ is unitarizable. $\square$

In fact, there is a larger class of representations which are unitarizable and includes finite group representations.

Definition 1.10. A *topological group* is a group and also a topological space for which the group action and inverses are continuous maps. A *compact group* is a topological group which is compact.

Example 1.11. Any finite group can be given the discrete topology to make it a topological group. Compact groups thus include finite groups with discrete topology.

Example 1.12. For the classical groups $SU2, SO3, U_N$ we consider later, we give them the subspace topology of the usual topology on $GL_n(\mathbb{C})$.

In order to show unitarizability for compact groups, we require the following result of Haar, which we cite without proof.

Theorem 1.13. ***(Haar)***[6] *For any compact Hausdorff topological group G, there exists a unique normalized regular Borel measure on G that is invariant under left and right multiplication by elements of G. This measure is called the Haar Measure.*

In the examples of compact groups we explore later, we will construct this measure explicitly.

Lemma 1.14. *Finite dimensional representations of compact groups are unitarizable.*

Proof. Let $\int_G dg$ denote integration with respect to the Haar measure of G. Then, following the same logic as the proof of Theorem 1.9, given any inner product $\langle , \rangle$ on G, a unitary inner product can be constructed as

$$\langle v, w \rangle' := \frac{1}{N} \int_G \langle \rho_g v, \rho_g w \rangle dg, \tag{1.15}$$

where $N = \int_G 1 dg$ is the volume of the group G. By Theorem 1.13, $\langle , \rangle'$ is translation invariant, making G unitarizable. $\square$

We are ready to state the main theorem of the section.

Theorem 1.16. *Finite dimensional unitarizable representations are completely reducible.*

Proof. Suppose V is a finite dimensional unitarizable representation of a group G. If there are no nontrivial subrepresentations, then V is irreducible. Otherwise, let W be a subrepresentation, and let $W^\perp$ be its orthogonal compliment. Thus $V \cong W \oplus W^\perp$. It follows from Lemma 1.8 that $W^\perp$ is a subrepresentation of V. Noting that one dimensional representations are irreducible, it follows from induction on the number of dimensions that V is completely reducible. □

Thus we have that representations of compact groups, including finite groups, are completely reducible: they factor into direct sums of irreducible representations as in Definition 1.6. All representations of compact groups can be constructed by taking direct sums of irreducible representations.

One goal of representation theory is to discover all the irreducible representations of a given group. To this end, in the next section we present a key tool in identifying irreducible representations: *Schur's Lemma*.

1.4. **Schur's Lemma.** Schur's Lemma is a cornerstone of basic representation theory. It formulates a condition for irreducibility in terms of maps from a representation to itself. We will make continued use of Schur's Lemma throughout the paper. First a quick but important definition.

Definition 1.17. For representations $(\rho^1, V_1), (\rho^2, V_2)$, an *intertwiner* is a linear map $\varphi : V_1 \to V_2$ that commutes with the action of G:

$$\rho^2_g \circ \varphi = \varphi \circ \rho^1_g$$

It makes this diagram commute:

$$\begin{tikzcd}
V_1 \arrow[r, "\varphi"] \arrow[d, "\rho_1"'] & V_2 \arrow[d, "\rho_2"] \\
V_1 \arrow[r, "\varphi"'] & V_2
\end{tikzcd}$$

The space of intertwiners is called $Hom_G(V_1, V_2)$. We define $End_G(V) = Hom_G(V, V)$.

Notice that in this language, isomorphisms of representations are isomorphisms of vector spaces that are intertwiners; equivalently, bijective intertwiners. Since intertwiners aren't necessarily invertible, we don't view them as conjugates like we do isomorphisms.

Schur's lemma gives a powerful restriction on itertwiners of irreducible representations.

Theorem 1.18. ***(Schur's Lemma)*** *Intertwiners $\varphi : V \to V$ on irreducible representations are scalar multiples of the identity map.*

Proof. Let ρ be irreducible, and let φ be an intertwiner of ρ. Then, since φ is a linear map between complex vector spaces, it has at least one nonzero eigenspace E, say with eigenvalue λ. Then for any $g \in G$ the eigenspace E is fixed by ρ_g.

$$(\varphi \circ \rho_g)(v) = (\rho_g \circ \varphi)(v) = \rho_g(\lambda v) = \lambda \rho_g(v) \ \forall v \in E$$

It follows that eigenspaces of φ are subrepresentations. Since V is irreducible, subrepresentations must be trivial, so there is a single eigenspace which is all of V. □

The following corollary to Schur's Lemma allows one to determine whether irreducible representations are isomorphic to each other by the dimension of the space of isomorphisms between them.

Corollary 1.19. *For irreducible representations V, W, the space $Hom_G(V,W)$ is 1 dimensional and comprised of isomorphisms if $V \cong W$ (as representations) and is $\{0\}$ otherwise.*

Proof. The kernel and image of $\varphi \in Hom_G(V,W)$ are subrepresentations of the irreducible representations V,W, so they must be trivial. Therefore $\varphi = 0$ or φ is an isomorphism. If φ_1, φ_2 are two nontrivial interwiners, then $\varphi_1 \circ \varphi_2$ is an intertwiner on V and so is a scalar by Schur's Lemma. Thus elements of $Hom_G(V,W)$ differ only by a scalar and so $Hom_G(V,W)$ is one dimensional. □

This provides a converse to Schur's Lemma for completely reducible representations: since each pair of isomorphic copies of irreducible representations have a one dimensional space of intertwiners between them, representations V that are not irreducible (that is, contain multiple irreducible representations) will have elements of $End_G(V)$ that are not simply scalar multiples of the identity. In fact, $End_G(V)$ will in general turn out to be a direct sum of matrix algebras, where the individual matrix entries give the scalars that define scalar maps required by Schur's Lemma between each pair of isomorphic representations. This will be fleshed out in the next section.

Let's review where we've been this section. We started out by introducing the idea of a representation as a linear group action on a vector space. We then found that compact groups and in particular finite groups factor (complete reducibility) into a direct sum of spaces which are fixed by the group action (irreducible representations). Finally, we found a criterion for irreducibility and a restriction on maps between irreducible representations (Schur's Lemma).

The space of maps between representations can be further understood through a module-theoretic formulation, culminating in the Artin-Wedderburn Theorem, which we explore next.

2. Module Theory

We next approach the subject from the point of view of module theory, culminating in the main theorem which characterizes the structure of complex representations.

Definition 2.1. For a finite group G, the *group algebra* $\mathbb{C}[G]$ consists of $\mathbb{C}$-linear combinations of the group elements of G.

By extending linearly in $\mathbb{C}$, a representation $\rho : G \to GL(V)$ defines an algebra homomorphism $\rho' : \mathbb{C}[G] \to End(V)$. Restricting to $G \subset \mathbb{C}[G]$ gives the inverse. Since V is an $End(V)$-module, the extended homomorphism makes V a $\mathbb{C}[G]$-module. This gives a bijection between $\mathbb{C}[G]$-modules and representations of G. We therefore want to understand all $\mathbb{C}[G]$-modules.

In this correspondence, irreducible representations correspond to simple $\mathbb{C}[G]$-modules. In this section, we develop a theorem on the structure of a class of modules which includes $\mathbb{C}[G]$, which will help to understand the structure of $\mathbb{C}[G]$ for finite groups, and therefore the structure of representations of finite groups. We give an abbreviated version of some of the module-theoretic proofs and refer reader to [3] for a more thorough treatment.

In the language of module theory, Schur's Lemma says the following:

Lemma 2.2. ***(Schur's Lemma, module version)*** *A module homomorphism ψ between simple modules is either an isomorphism or the zero map.*

Proof. The kernel and image of a homomorphism are both submodules. □

The analogue of creating completely reducible representations from irreducible representations is creating "semisimple modules" from simple modules. We will see later that for a finite group G, the group algebra $\mathbb{C}[G]$ is a semisimple module. In order to understand the structure of $\mathbb{C}[G]$ then it is worthwhle to understand semisimple modules.

Definition 2.3. A *semisimple* module is a direct sum of simple modules.

The next lemma is the first step in understanding the structure of semisimple modules.

Lemma 2.4. *The image of a finite dimensional semisimple module under homomorphism is semisimple.*

Proof. Suppose $S = S_1 \oplus S_2 \oplus ... \oplus S_n$ is a semisimple module and $\varphi : S \to M$ is a homomorphism. Then $\varphi|_{S_i}$ is an isomorphism or 0 by Schur's Lemma. Since simple modules have trivial intersections, $\varphi(S) \cong \bigoplus_{i=1}^{n} \varphi(S_i)$ where each $\varphi|_{S_i}$ is 0 or an isomorphism. □

There is a notion of when an algebra is semisimple, and it is heavily related to the module case.

Definition 2.5. We say a finite dimensional $\mathbb{C}-$algebra A is *semisimple* if all $A-$modules are semisimple.

Proposition 2.6. *A finite dimensional $\mathbb{C}$-algebra A is semisimple iff it is semisimple as an A-module.*

Proof. If A is semisimple, then all modules over A are semisimple, so A would be as well. If A is semisimple as an A-module, then for a generating set $m_1, ..m_r$ for any finitely generated $A-$module M, the result follows from Lemma 2.4 and the module homomorphism $A^r \to M$ given by $(a_1, ..a_r) \to a_1 m_1 + .. + a_r m_r$. □

Again, later we will see that for finite G, $\mathbb{C}[G]$ is a semisimple $\mathbb{C}$-algebra, and so we are working towards a structure theorem for semisimple algebras in order to understand the structure of $\mathbb{C}[G]$, which tells us the structure of representations of G.

We can use the constraints on maps between simple modules to understand the structure of semisimple algebras.

Proposition 2.7. *If $A \cong S_1 \oplus .. \oplus S_n$ where S_i is a simple module, then any simple A-module is isomorphic to one of the S_i.*

Proof. For S a simple A module, given $v \in S$, define the homomorphism $\varphi : A \to S$ by $\varphi(a) = av$. Then $\varphi(A) = S$ must be isomorphic to one of the S_i by Schur's lemma and the simplicity of S. □

Lemma 2.8. *As $M_n(\mathbb{C})$-modules,*

$$M_n(\mathbb{C}) \cong n\mathbb{C}^n. \tag{2.9}$$

Proof. As a sketch of the proof, consider each of the columns of a given matrix $M \in M_n(\mathbb{C})$ as an element of one of the copies of $\mathbb{C}^n$. One can check this is a module isomorphism. □

Lemma 2.10. *If A is a finite dimensional $\mathbb{C}$-algebra and S a simple A-module, then $End_A(S) \cong \mathbb{C}$.*

Proof. This is a restatement of Schur's Lemma. □

Definition 2.11. For an algebra A, the *opposite algebra* A^{op} is the alegbra on the same set as A but with multiplication operation $\cdot$ defined as $a \cdot b := ba$ where the second multiplication is carried out in A.

Lemma 2.12. $A^{op} \cong End_A(A)$

Proof. Define the map $\varphi : A^{op} \to End_A(A)$ by $\varphi(a)(1) = a$. Injectivity is immediate from the definition. Any map $\psi \in End_A(A)$ is determined by $\psi(1)$ by $\psi(b) = b\psi(1)$. Therefore φ is surjective. Next, $\varphi(a \cdot b)(1) = \varphi(ba)(1) = ba = b\varphi(a)(1) = (\varphi(b) \circ \varphi(a))(1)$, so φ is a homomorphism. It follows that φ is an isomorphism. □

Lemma 2.13. *If S is simple, then $End_A(nS) \cong M_n(End_A(S))$.*

Proof sketch. For the full proof see Lemma 3.6 in [3]. Here we consider just the special case $S = \mathbb{C}$, $A = \mathbb{C}[G]$ which is all we need for representation theory. As A-modules, $End_A(S^{\oplus n}) := Hom_A(S^{\oplus n}, S^{\oplus n})$. Homomorphisms from direct sums factor as direct sums of homomorphisms, so $Hom_A(S^{\oplus n}, S^{\oplus n}) \cong \bigoplus_{i=1}^n Hom_A(S, S^{\oplus n})$. By Schur's Lemma, $Hom_A(S, S^{\oplus n}) \cong \mathbb{C}^n$. Thus $End_A(nS) = n\mathbb{C}^n$. By Lemma 2.8, we have $n\mathbb{C}^n \cong M_n(\mathbb{C})$. Letting $S = \mathbb{C}, A = \mathbb{C}[G]$, we have $End_{\mathbb{C}}[G](\mathbb{C}^n) = \mathbb{C}$, so $End_A(nS) \cong M_n(End_A(S))$ for this case. □

We are prepared to state the main theorem of this section.

Theorem 2.14. ***(Artin - Wedderburn)*** [3] *Let A be a finite dimensional $\mathbb{C}$-algebra. Then A is semisimple iff A is isomorphic to a finite direct sum of matrix algebras over $\mathbb{C}$.*

Proof. ($\Leftarrow$)Suppose $A \cong \bigoplus_{i=1}^r M_{n_i}(\mathbb{C})$. Then by Lemma 2.8, we have $A \cong \bigoplus_{i=1}^r n_i\mathbb{C}^{n_i}$. By restricting the action of A to a single summand, it follows that simplicity of $n_i\mathbb{C}^{n_i}$ as an $M_{n_i}(\mathbb{C})$-module implies simplicity as an A-module. Therefore A is a direct sum of simple modules and so semisimple.

($\Rightarrow$) Let $A \cong n_iS_i \oplus ... \oplus n_rS_r$ be a semisimple $\mathbb{C}$-algebra with $S_i \ncong S_j$ if $i \neq j$. Then by Lemma 2.12, $A^{op} \cong End_A(A) \cong End_A(n_iS_i \oplus ... \oplus n_rS_r)$. Since S_i, S_j are pairwise nonisomorphic and simple, we have $End_A(n_iS_i \oplus ... \oplus n_rS_r) \cong \bigoplus_{i=1}^r End_A(n_iS_i)$. By Lemma 2.13, $End_A(n_iS_i) \cong M_{n_i}(End_A(S_i))$. By Schur's Lemma, $End_A(S_i) \cong \mathbb{C}$. Thus $A^{op} \cong \bigoplus_{i=1}^r M_{n_i}(\mathbb{C})$. Then, $A \cong (\bigoplus_{i=1}^r M_{n_i}(\mathbb{C}))^{op} \cong \bigoplus_{i=1}^r (M_{n_i}(\mathbb{C}))^{op}$. Then, the transpose map, sending a matrix A to A^T satisfies the relation that $A^TB^T = (BA)^T$, which gives an isomorphism from $(M_{n_i}(\mathbb{C}))^{op}$ to $M_{n_i}(\mathbb{C})$. Therefore $A \cong \bigoplus_{i=1}^r (M_{n_i}(\mathbb{C}))^{op} \cong \bigoplus_{i=1}^r M_{n_i}(\mathbb{C})$ as desired. □

The connection to representation theory is given by *Maschke's Theorem*.

Theorem 2.15. ***(Maschke)*** *For a finite group G, the group ring $\mathbb{C}[G]$ is a finite dimensional semisimple $\mathbb{C}$-algebra.*

Proof. From the discussion following Definition 2.1, we identify $\mathbb{C}[G]-$modules as representations of G. In particular, $\mathbb{C}[G]$ is a module over itself. With this in mind, the decomposition of $\mathbb{C}[G]$ into simple modules follows from Complete Reducibility. □

Putting the previous two theorems together, we arrive at a classification of the structure of $\mathbb{C}[G]$.

Corollary 2.16. *As algebras,*

$$\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \oplus ... M_{n_r}(\mathbb{C}) \cong \bigoplus_{V \in Irr(G)} End_{\mathbb{C}} V.$$

In the isomorphism, an element $g \in G$ is mapped to the linear map $\rho(g)$ for the corresponding irreducible representation (ρ, V) in each factor.

Proof sketch. The first isomorphism follows from Theorem 2.15 and Theorem 2.14. The second isomorphism follows from Lemma 2.8. The second isomorphism follows from the identification of $\{\mathbb{C}^{n_i}\}$ as the unique up to isomorphism simple $M_{n_i}(\mathbb{C})$ modules and then $M_{n_i}(\mathbb{C})$ as $End_{\mathbb{C}} V$ where $V \cong \mathbb{C}^{n_i}$. The last statement follows from viewing V as a $\mathbb{C}[G]$-module. □

We see that there are r isomorphism classes of simple modules (irreducible representations) with dimensions $\{n_i\}$. For the next result we need a quick definition which will continue to be of importance in following sections.

Definition 2.17. For a group G, let Cl be the set of conjugacy classes of G. A *class function* on G is a function $f : Cl \to \mathbb{C}$. Based on context, functions $h : G \to \mathbb{C}$ that are constant on conjugacy classes may also be called class functions.

Proposition 2.18. *From Corollary 2.16 we can deduce three important results in finite dimensional complex representation theory. Let G be a finite group.*

1. *The regular representation of G decomposes as the direct sum of irreducible representations with multiplicity equal to their dimension.*
2. $\sum_{V \in Irr(G)} dim(V)^2 = |G|$
3. *The number of irreducible representations of G is the number of conjugacy classes in G*

Proof. To see (1), note that the regular representation is isomorphic to $\mathbb{C}[G]$ itself, which by Corollary 2.16 is a direct sum over irreducible representations V of $End_{\mathbb{C}} V$. We have $End_{\mathbb{C}}(V) \cong M_n(\mathbb{C}) \cong n\mathbb{C}^n$ by Lemma 2.8. This gives n copies of irreducible representation V where $n = dim(V)$.

Item (2) follows directly from a dimension count.

For (3), an irreducible representation of G is a simple module of $\mathbb{C}[G]$. Considering $\mathbb{C}[G]$ as the space of complex valued functions on G, for a function $\varphi : G \to \mathbb{C}$, $\varphi \in Z(\mathbb{C}[G])$ iff $\forall x \in G$, $x\varphi(g) = \varphi x(g)$. Thus $\varphi(g) = x^{-1}\varphi x(g) = \varphi(x^{-1}gx)$. It follows that the center $Z(\mathbb{C}[G])$ consists precisely of class functions. A basis of class functions is given by $\{\varphi_i\}$, where φ_i is unity on elements of a chosen conjugacy class and zero elsewhere. Thus the dimension of the space of class functions and thus $Z(\mathbb{C}[G])$ is the number of conjugacy classes of G.

By Theorem 2.14, the group algebra decomposes as a direct sum of r matrix algebras.

$$\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \oplus ... \oplus M_{n_r}(\mathbb{C}).$$

Then

$$Z\mathbb{C}[G] = Id_{n_1} \oplus .. \oplus Id_{n_r}.$$

Therefore $Dim(Z(\mathbb{C}[G]) = r$. Since each factor $M_{n_i}(\mathbb{C})$ acts on simple modules $\mathbb{C}^{n_i}$, there are exactly r isomorphism classes of irreducible representations.

It follows that the number of isomorphism classes of irreducible representations is the number of conjugacy classes of G. □

We can use this result to understand irreducible representations.

Example 2.19. Let G be a finite abelian group, and let (ρ, V) be a representation of G. Then elements of $\rho(G)$ must commute, and are diagonalizable. Diagonal matrices act on direct sums of one dimensional spaces, so it follows that they are simultaneously diagonalizable. Therefore by Definition 1.6, V decomposes as a direct sum of one dimensional irreducible representations. Seen from the point of view of Corollary 2.16, the only matrix algebras that are commutative are one dimensional, so that $\mathbb{C}[G] \cong \mathbb{C} \oplus .. \oplus \mathbb{C}$ for which all modules are one dimensional.

Let's recap where we've been. First, we noted that representations of a group G could be identified with $\mathbb{C}[G]$-modules. We therefore sought to understand the structure of the group algebra $\mathbb{C}[G]$. Maschke's Theorem classifies $\mathbb{C}[G]$ as what is called a semisimple algebra. We therefore developed some module theory to work towards the main theorem, the Artin-Wedderburn Theorem which classifies the structure of semisimple algebras as direct sums of matrix algebras. We used this in Corollary 2.16 and Proposition 2.18 to deduce three foundational results in representation theory of finite groups.

In the next section we will prove these same results from a different angle, and go further to understand more about the individual irreducible representations of a group though a lens called *character theory*.

3. Character Theory

In general, group homomorphisms are complicated and messy to keep track of. It turns out that for representations (ρ, V) which are completely reducible, the essential information to list and differentiate the irreducible representations is contained in just the trace of all the linear maps $\rho(g) : g \in G$. This is called the *character* of the representation. In this section, we will use characters to identify and deduce important properties of irreducible representations of compact groups. We will develop a toolbox which can be used to catalogue information about completely reducible representations. We will use these tools on some specific groups in Section 5 after a brief foray into tensor products in Section 4.

Definition 3.1. For a representation (ρ, V), the *character* of (ρ, V) is the function $\chi : G \to \mathbb{C}$ given by taking the trace.

$$\chi_V(g) = tr_V \rho(g)$$

By extending linearly this gives a map $\mathbb{C}[G] \to \mathbb{C}$.

Next we will deduce a few properties of the character function that make calculations easier.

Proposition 3.2. *For a representation (ρ, V) and compact group G and $g \in G$,*
(1) $\chi(g^{-1}) = \overline{\chi(g)}$.
(2) $\chi(e) = Dim(V)$

Proof. (1); Recall that by Lemma 1.14, (ρ, V) is unitarizable. It follows that

$$\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^{\dagger}.$$

Then $tr(\rho(g)) = \overline{tr(\rho(g)^{\dagger})}$ and the proposition (1) follows.

(2) Since ρ is a homomorphism we have $\rho(e) = Id$ and $tr(Id) = Dim(V)$. □

This gives an easy way of determining the dimension of a representation. Now, since $tr(A \oplus B) = tr(A) + tr(B)$, the character of a completely reducible representation is determined by the character of its irreducible factors. Let's introduce some lingo.

Definition 3.3. An *irreducible character* is the character of an irreducible representation.

So we know that the decomposition of a representation into irreducible representations determines its character via irreducible characters. One may ask whether the converse is true: whether given the character of a representation we may determine its decomposition into irreducibles. The answer is 'yes' for an interesting reason: It will turn out that the irreducible characters form an orthonormal basis of class functions with respect to an inner product which averages over the group. Let's define that inner product.

Definition 3.4. For a finite group G define an inner product on characters of G by the following:

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)} \cdot \chi_2(g) \tag{3.5}$$

For a compact group G, define an inner product using the Haar Measure.

$$\langle \chi_1, \chi_2 \rangle \cong \frac{1}{N} \int_G \overline{\chi_1(g)} \chi_2(g) dg \tag{3.6}$$

Proposition 3.7. *For a compact or finite group G, the irreducible characters of G are orthonormal with respect to the inner product (3.5), (3.6) respectively.*

Proof. [1] Let (ρ^1, V), (ρ^2, W) be two irreducible representations of G and let χ^1, χ^2 be their characters. Choose $G-$invariant inner products on V, W and bases $\{v_i\}, \{w_j\}$ for V, W, letting the bases agree if $V \cong W$. Then for finite G we have

$$\langle \chi^1, \chi^2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^1(g)} \chi^2(g) = \frac{1}{|G|} \sum_{i,j} \sum_G \langle v_i, \rho^1(g)^{-1} v_i \rangle \langle w_j, \rho^2(g) w_j \rangle. \tag{3.8}$$

The last expression can be interpreted as summing over matrix elements of a linear map $f : V \to W$. The map f restricted to the span of w_j and then projected onto the span of v_i is a map $f_{ij} = |\rho^1(g)^{-1} v_i \rangle \langle w_j | \rho^2(g)$. If $V \cong W$, then by Schur's lemma, f is a scalar multiple of the identity. Therefore $f_{ij} = \frac{Tr(f_{ij})}{Dim(V)} = \frac{\delta_{ij}}{Dim(V)}$. Otherwise, $f = 0$. Summing over matrix elements and group elements cancels the factors of $1/Dim(V)$ and $1/|G|$ to attain:

$$\langle \chi^1, \chi^2 \rangle = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W. \end{cases} \tag{3.9}$$

If G is a compact group, define

$$\langle \chi^1, \chi^2 \rangle = \frac{1}{N} \int_G \overline{\chi^1(g)} \chi^2(g) dg = \frac{1}{N} \sum_{i,j} \int_G \langle v_i, \rho^1(g)^{-1} v_i \rangle \langle w_j, \rho^2(g) w_j \rangle dg,$$

where N is the volume of the compact group G. The same logic as in the finite case follows here, with the only difference being that integrating over the elements of G cancels out the factor of $1/N$ [1]. $\square$

Theorem 3.10. *The irreducible characters form an orthonormal basis of the space of class functions on G with respect to the inner product (3.5), (3.6).*

Proof. The characters are class functions because the trace is. The fact that irreducible characters form a spanning set of class functions follows from Proposition 2.18, while orthonormality follows from Proposition 3.7. $\square$

Let's review. For a given representation (ρ, V) we take the trace of each map $\rho(g) : g \in G$ to get what is called the character. The character is a class function on G. Think of the characters now as being vectors in the space of class functions with coordinates indexed by the conjugacy classes of G. The terms in the inner product Definition 3.4 are constant on conjugacy classes so we may think of the sum as ranging over the conjugacy classes of G, rather than the elements. Thus working in the space of class functions we found an inner product with respect to which the irreducible characters form an orthonormal basis. As we will see next, this space contains all the information needed to determine whether characters are the same and whether they are irreducible.

Proposition 3.11. *Two representations are isomorphic iff their characters are the same.*

Proof. The preservation of trace under conjugation implies the forward direction. The reverse follows from linear independence of irreducible characters in Theorem 3.10. $\square$

Proposition 3.12. *A representation χ is irreducible iff $\langle \chi, \chi \rangle = 1$.*

Proof. Complete reducibility implies representations decompose as integer combinations of irreducible representations. Therefore the inner product between representations takes on integral values. By orthonormality, the multiplicity of an irreducible representation χ_i in a representation χ is given by $\langle \chi_i, \chi \rangle$. $\square$

This gives an easy way to detect irreducible characters.

Example 3.13. One very useful representation is given by the action of G on $\mathbb{C}[G]$ given by $g \cdot \lambda h = \lambda gh$, and extending linearly. This representation is called the *regular representation* and has a number of nice properties.

First notice that the character $\chi_{reg}(g)$ is given by the number of elements fixed by g. The Sudoku Lemma then implies

$$\chi_{reg}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

From this it follows that for any representation χ, we have

$$\langle \chi_{reg}, \chi \rangle = Dim(\chi)$$

From this we can extract a nice identity:

$$|G| = \langle \chi_{reg}, \chi_{reg} \rangle = Dim(\chi_{reg}) = \sum_i \langle \chi_{reg}, \chi_i \rangle \cdot Dim(\chi_i) = \sum Dim(\chi_i)^2 \tag{3.14}$$

Remark 3.15. The results of the previous example also follow from Corollary 2.16. To see this, note that the regular representation is isomorphic to $\mathbb{C}[G]$ itself as a $\mathbb{C}[G]$-module. From the decomposition

$$\mathbb{C}[G] = n_1\mathbb{C}^{n_1} \oplus ... \oplus n_r\mathbb{C}^{n_r},$$

we see that the multiplicity of each irreducible representation $V \cong C^{n_i}$ is $n_i = Dim(V)$ and that

$$|G| = Dim(\mathbb{C}[G]) = n_1 Dim(\mathbb{C}^{n_1}) + .. + n_r Dim(\mathbb{C}^{n_r}) = \sum n_i^2.$$

The regular representation is an example of what is called a permutation representation.

Definition 3.16. For an action $G \circlearrowright X$ on a finite set X, define a representation of G on the complex vector space $\mathbb{C}[X]$ with basis $\{x_i\}$ labeled by elements of X. The action is given by linearly extending the group action. This representation is called a *permutation representation*.

Proposition 3.17. *The vector* $(1, 1, 1...)$ *spans a trivial subrepresentation in any permutation representation.*

Characters can also be used to find the normal subgroups of a group.

Definition 3.18. For a character χ, let the *kernel* of χ be the set

$$ker\chi := \{g : \chi(g) = \chi(1)\}.$$

Proposition 3.19. *Normal subgroups of G are the arbitrary intersections of kernels of irreducible characters of G.*

Proof. From the regular representation restricted to the cosets G/N, we find $\chi(h) = 0$ if $h \notin N$ and $\chi(h) = \chi(1)$ if $h \in N$. Thus $N = ker\chi := \{g : \chi(g) = \chi(1)\}$. The converse holds, so that any normal subgroup of G can be written as the kernel of some representation. Next, note that $\ker\rho = \ker\chi$ because the only way to have $\chi(g) = dim(V)$ with diagonal entries being roots of unity is to have them all be identically 1. Finally, for any character $\chi = \sum \chi_i$, with χ_i irreducible, $ker\chi = \bigcap ker\chi_i$ follows from the same statement on ρ. It follows that normal subgroups are the intersection of arbitrary combinations of the kernels $ker\chi_i$ of irreducible characters. □

Let's review what we've seen in this section. We defined *characters*, which are vastly simpler than representations, requiring only knowledge of the traces of $\{\rho(g) : g \in G\}$. Characters encode the dimension of a representation and can be decomposed into irreducible characters. These irreducible characters, with a cleverly chosen inner product, form an orthonormal basis for the space of class functions. Working in this space, we can use the characters to determine which irreducible representations are factors in a given representation, whether two representations are isomorphic, whether a representation is irreducible, what the normal subgroups of the group are, and the dimensions and number of the irreducible representations. We will use these tools to find all irreducible representations of specific groups in Section 5 after some preliminaries about tensor products.

4. Tensor Products

One strategy of constructing irreducible representations takes advantage of an algebraic construction called a *tensor product*. In this section, we introduce tensor products and use them to construct the *symmetric* and *alternating* powers of a space. Tensor products will continue to show up throughout the remainder of the paper and into more advanced representation theory.

Definition 4.1. Let R be a ring and let M be a right R-module, and N a left R-module. Define the *tensor product* $M \otimes_R N$ of M and N over R to be the quotient group of the free group on the symbols $m \otimes n \in M \times N$ by the following relations:

$$(m + m') \otimes n = m \otimes n + m' \otimes n,$$

$$m \otimes (n + n') = m \otimes n + m \otimes n'$$

$$mr \otimes n = m \otimes rn$$

Thus tensor products are linear in each component and transfer factors of $r \in R$ across components.

Remark 4.2. If R is commutative, then $M \otimes_R N$ is an R-module, where R acts on the right on M or equivilently on the left on N.

Next we see how to define operators on tensor products. They have a particularly nice matrix form.

Proposition 4.3. [1] *Let $A \in End(V)$ and $B \in End(W)$. Define the operator $A \otimes B \in End(V \otimes W)$ by $A \otimes B(v \otimes w) = Av \otimes Bw$. For bases $\{e_i\}, \{f_k\}$ of V, W respectively, there exists a basis $\{e_i \otimes f_k\}$ of $V \otimes W$ in which the matrix of $A \otimes B$ has the following simple form:*

$$(A \otimes B)_{(i,k)(j,l)} = A_{ij} \cdot B_{kl} \tag{4.4}$$

Naturally we can define representations that are tensor products of other tensor products.

Definition 4.5. Given representations (ρ, V) and (σ, W) of a group G, define the representation $(\rho \otimes \sigma, V \otimes W)$ to be the representation defined by the following condition:

$$(\rho \otimes \sigma)_g (v \otimes w) = \rho_g v \otimes \sigma_g w.$$

Working with characters of tensor product representations is particularly nice because of the following proposition, which allows characters to easily be calculated from the decomposition of representations into irreducible representations and tensors.

Proposition 4.6. *Let V and W be representations of a finite group G.*

(1) $\chi_{V \oplus W} = \chi_V + \chi_W$
(2) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

Proof. This follows directly from the basis decomposition on $V \oplus W$ and $V \otimes W$. $\square$

There is an intuitive representation of the symmetric group on tensor products of spaces that is given by permuting the factors in the tensor product. This representation is related to other important representations so we take some time to illumniate some of its properties here.

Proposition 4.7. *For a representation (ρ, V), the representation $\rho^{\otimes n}$ on $V^{\otimes n}$ given by $\rho_g^{\otimes n}(v_1 \otimes ... \otimes v_n) = \rho_g v_1 \otimes .. \otimes \rho_g v_n$ commutes with the representation of S_n on $V^{\otimes n}$ which permutes the factors.*

Proof. Since ρ_g is applied to each factor, the action of G is not affected by the action of S_n. □

It can be useful to think of all the isomorphic copies of an irreducible representation that appear in a given representation as a single block.

Definition 4.8. For a completely reducible representation (ρ, V). V decomposes as a direct sum $V \cong \bigoplus_{i=1}^{r} V_i^{\oplus n_i}$. The blocks $V_i^{\oplus n_i}$ are called *isotypical components.*

Proposition 4.9. *Let G, H be compact groups and let (ρ, V) be a representation of G and (φ, V) a representation of H. If for all $h \in H$, $\varphi(h)$ is an intertwiner of ρ, then all isotypical components of ρ are subrepresentations of φ.*

Proof. Let W^n be an isotypical component of ρ where W is an irreducible representations of G. For any $h \in H, g \in G$, φ_h commutes with ρ_g, so by Schur's Lemma, the restriction of ρ_g to a single summand is 0 on nonisomorphic irreducible representations. It follows that φ_h fixes isotypical components of ρ. □

The following corollary to the proposition allows one to find representations of an arbitrary group G given representations of S_n.

Corollary 4.10. [1] *Every S_n-isotypical component of $V^{\otimes n}$ is a subrepresentation of G with the action given in Proposition 4.7.*

Proof. This follows directly from Proposition 4.9, since the representations of G and S_n commute by Proposition 4.7. □

There are two isotypical components of S_n-representations that are of special interest, called the *symmetric* and *alternating powers* respectively.

Definition 4.11. Given a vector space V, define the *n-th symmetric power of V*, *$Sym^n(V)$* to be the subspace of $V^{\otimes n}$ that is symmetric under interchange of factors. Define the *n-th alternrating power of V*, $\Lambda^n V$, to be the subspace of $V^{\otimes n}$ that is antisymmetric under interchange of factors.

Proposition 4.12. [1] *Given a basis $\{v_1, ..., v_n\}$ of V, then a basis for $Sym^n V$ is given by*

$$\left\{ \frac{1}{n!} \sum_{\sigma \in S_n} v_{k_{\sigma(1)}} \otimes ... \otimes v_{k_{\sigma(n)}} \right\} \text{ as } 1 \leqslant k_1 \leqslant ... \leqslant k_n \leqslant m,$$

and a basis for $\Lambda^n V$ is given by

$$\left\{ \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) v_{k_{\sigma(1)}} \otimes ... \otimes v_{k_{\sigma(n)}} \right\} \text{ as } 1 \leqslant k_1 < ... < k_n \leqslant m,$$

where $\varepsilon(\sigma)$ is the sign of σ.

Proof. For the proof see [1]. □

Under the action of S_n permuting the components of $V^{\otimes n}$, the construction of the bases in Proposition 4.12 illucidates the fact that the elements of $Sym^n V$ are fixed by this action while the elements of $\Lambda^n V$ are negated by transpositions. In fact, the space $Sym^n V$ is the isotypical component of $V^{\otimes n}$ associated with the trivial representation, while $\Lambda^n V$ is the isotypical component associated with the sign representation. This allows us to use Corollary 4.10 to attain representations of any group G on $V^{\otimes n}$.

Let's review what we covered in this section. We defined the notion of *tensor product*, which is a way of building up a larger space out of other spaces by letting scalars transfer between the factors. We defined representations on tensor products and found that calculating characters of tensor product spaces to be straightforward (multiplicative). Next we covered isotypical components, subspaces composed of all the copies of a given irreducible representation that appear in a completely reducible representation. We exhibited a basis for two particularly important issotypical components of S_n: $Sym^n V$ and $\Lambda^n V$. These spaces and tensor products more broadly will appear as irreducible representations in specific groups studied later. We turn next to finding the irreducible characters of a few specific groups.

5. Applying Character Theory

One useful way of displaying information about irreducible representations of a group is by constructing its character table. A a character table includes much of the relevant information about the irreducible representations of a given group, including their dimensions and their characters. In this table, columns are indexed by representatives of conjugacy classes (since characters are class functions), with the size of the conjugacy class given in parentheses. Rows are indexed by irreducible representation. Every group has the trivial representation χ_1 which always appears as the first row.

Rep	e	a ($\lvert C_a \rvert$)	b ($\lvert C_b \rvert$)	c ($\lvert C_c \rvert$)
χ_1	1	1	1	1
χ_2	$\mathrm{Dim}(\chi_2)$	$\chi_2(a)$	$\chi_2(b)$	$\chi_2(c)$
χ_3	$\mathrm{Dim}(\chi_3)$	$\chi_3(a)$	$\chi_3(b)$	$\chi_3(c)$
χ_4	$\mathrm{Dim}(\chi_4)$	$\chi_4(a)$	$\chi_4(b)$	$\chi_4(c)$

Remark 5.1. When the columns are weighted by $\sqrt{|C|/|G|}$, where $|C|$ is the conjugacy class size, to account for the size of the conjugacy classes, the orthogonality relations of characters implies the rows are orthonormal [1]. It follows that the columns are also orthonormal and therefore the matrix formed by the weighted entries of the table is unitary. This is a useful tool in piecing together the character table.

5.1. **Character Table of A_5.** We will construct the character table of A_5 - the even permutations of 5 items - using some of the previously developed tools. We start by listing the conjugacy classes of A_5, of which there are five, represented by elements $e, (12)(34), (123), (12345), (12354)$. By Proposition 2.18, we are thus looking for five irreducible representations. We get the trivial representation for free, with character $\chi_1 = (1, 1, 1, 1, 1)$. We don't get the sign representation, since all elements of A_5 have even sign, so it is the same as the trivial representation.

We will now proceed by examining the action of A_5 on a few different spaces. Let's first consider the permutation representation of A_5 on $\mathbb{C}^5$ given by permuting the coordinates. The trace of a permutation representation is the number of elements fixed, so it follows that $\chi_{\mathbb{C}^5} = (5, 1, 2, 0, 0)$.

Noting that the terms in the inner product (3.5) are the same on conjugacy classes since characters are class functions, for R a system of representatives of conjugacy classes of G, with sizes $|C_g|$, we can rewrite (3.5) as

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in R} |C_g| \cdot \chi^{-1}(g)\chi'(g). \tag{5.2}$$

We may thus evaluate inner products just on representatives, with the appropriate weighting. The sizes of the conjugacy classes of A_5 are $1, 15, 20, 12$, and 12.

Thus we can evaluate $\langle \chi_{\mathbb{C}^5}, \chi_{\mathbb{C}^5} \rangle$ given by the inner product (5.2), as

$$\langle \chi_{\mathbb{C}^5}, \chi_{\mathbb{C}^5} \rangle = \frac{1}{60}\left(1 \cdot 5^2 + 15 \cdot 1^2 + 20 \cdot 2^2 + 12 \cdot 0^2 + 12 \cdot 0^2\right) = 2.$$

From Proposition 3.12 and Theorem 1.16, it follows that $\chi_{\mathbb{C}^5}$ is the sum of two irreducible characters. By Proposition 3.17, one of these is the trivial character χ_1, so subtracting this we have $\chi_{\mathbb{C}^5} - \chi_1 = (5, 1, 2, 0, 0) - (1, 1, 1, 1, 1) = (4, 0, 1, -1, -1) := \chi_2$. We call V the subspace on which χ_2 acts.

We next consider the action of A_5 on the tensor square $V^{\otimes 2}$.

Lemma 5.3. *The character of Sym^2V can be calculated from that of V in the following way:*

$$\chi_{Sym^2V}(g) = \frac{\chi_V^2(g) + \chi_V(g^2)}{2}. \tag{5.4}$$

The character of $\Lambda^2 V$ is the following:

$$\chi_{\Lambda^2 V}(g) = \frac{\chi_V^2(g) - \chi_V(g^2)}{2} \tag{5.5}$$

Proof. This follows from counting over the bases defined in Proposition 4.12 [1]. $\square$

On the subspace Sym^2V therefore we have $\chi_{Sym^2V} = (10, 2, 1, 0, 0)$ and on $\Lambda^2 V$ we have $\chi_{\Lambda^2 V} = (6, -2, 0, 1, 1)$. We calculate $\langle \chi_{Sym^2V}, \chi_{Sym^2V} \rangle = 3$, so χ_{Sym^2V} is the sum of three irreducible characters, since this is the only way to have a modulus of 3 when components take integer coefficients. We have $\langle \chi_{Sym^2V}, \chi_1 \rangle = 1$ and $\langle \chi_{Sym^2V}, \chi_2 \rangle = 1$, so χ_{Sym^2V} includes a copy of χ_2 and χ_1, which can then be subtracted out. The difference $\chi_{Sym^2V} - \chi_1 - \chi_2 = (5, 1, -1, 0, 0) := \chi_3$ is irreducible.

Now, we have $\langle \chi_{\Lambda^2 V}, \chi_{\Lambda^2 V} \rangle = 2$, so $\chi_{\Lambda^2 V}$ is composed of two irreducible representations. We have $\langle \chi_{\Lambda^2 V}, \chi_1 \rangle = \langle \chi_{\Lambda^2 V}, \chi_2 \rangle = \langle \chi_{\Lambda^2 V}, \chi_3 \rangle = 0$, so the two irreducible representations summing to $\chi_{\Lambda^2 V}$ are nonisomorphic to the previously found irreducible representations, and so are the last two we need to find.

Now, given a representation ρ, conjugation by $\tau \in S_n$ defines a new representation ρ^τ that acts as $\rho^\tau(g) = \rho(\tau g \tau^{-1})$. Conjugation by τ preserves the conjugacy classes of A_5 that were already conjugacy classes of S_5. However, (12345) and (12354) are conjugate in S_5 but not A_5. Let τ then be the element of S_5 by which they are conjugate: $\tau(12345)\tau^{-1} = (12354)$. It follows that

$$\rho^{\tau}(12345) = \rho(\tau(12345)\tau^{-1}) = \rho(12354),$$

and vice versa. Letting ρ be one of the two irreducible representations that makes up $\Lambda^2 V$, it follows that ρ^{τ} is also an irreducible representation. Since it is unaccounted for elsewhere it must be the other summand of $\chi_{\Lambda^2 V}$.

We therefore know the following things about ρ and ρ^{τ}. First, by (3.14 we have $|G| = 60 = \sum d_i^2 = 1^2 + 4^2 + 5^2 + Dim(\rho)^2 + Dim(\rho^{\tau})^2$. The only integer solutions to this are $Dim(\rho^{\tau}) = Dim(\rho) = 3$. By the definitions of ρ, ρ^{τ}, we know that $\chi_{\rho}, \chi_{\rho^{\tau}}$ are the same on conjugacy classes $e, (12)(34), (123)$ which are shared conjugacy classes of S_5 and A_5 and have swapped values on conjugacy classes $(12345), (12354)$ which split a class of S_5. By construction we also know that $\chi_{\rho^{\tau}} + \chi_{\rho} = \chi_{\Lambda^2 V}$. Finally, by 3.10, $\chi_{\rho^{\tau}}, \chi_{\rho}$ are orthogonal to each other.

All of these conditions combined are enough, with some algebra, to nail down the final two irreducible representations as $\chi_{\rho^{\tau}} = (3, -1, 0, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$ and $\chi_{\rho} = (3, -1, 0, \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$. We can thus present the completed character table of A_5 :

Rep	e	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_{ρ}	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$\chi_{\rho^{\tau}}$	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$

Thus by drawing on the properties of characters developed in previous sections and considering the action of A^5 on specific spaces, we were able to find all irreducible characters of A^5. This shows the power of character theory in finding the irreducible representations of finite groups. Before considering additional specific groups in the last two sections, we next introduce some more abstract results in representation theory on the topic of how the representations of a group relate to the representations of its subgroups.

6. Induced and Restricted Representations

In this section we explore a way to generate representations of a group given representations of a subgroup, or vice versa. These representations are called the *induced* and *restricted* representations respectively. We introduce these representations and prove some theorems relating the two.

First, suppose we have a representation of a group G. This is also a representation of each subgroup $H \subset G$. When we restrict to the subgroup H, we call it the *restricted representation.* The same notion works for any homomorphism into G, so we give the broader definition below.

Definition 6.1. Let (ρ, V) be a representation of a group G, and let $\varphi : H \to G$ be a homomorphism of groups. Define the *restricted representation* of G to H to be the representation $(Res_G^H V, V)$ of H given by the composition

$$Res_G^H V = \rho \circ \varphi.$$

The composition is illustrated in the following diagram.

$$\begin{tikzcd} H \arrow[r, "\varphi"] \arrow[dr, "Res_G^H"'] & G \arrow[d, "\rho"] \\ & V \end{tikzcd}$$

This allows us to produce a new representation of a group H given a representation of G and a group homomorphism from H to G. This is particularly useful when H is a subgroup of G and the map φ is the inclusion of H into G. In this case the restricted representation reduces to $Res_G^H(h) = \rho_h$.

Given a representation of a subgroup $H \subset G$, a new representation of G and adjoint functor of Res_G^H is given by the *induced representation*.

Definition 6.2. Given a representation (ρ, W) of a subgroup $H \subset G$, define the *induced representation* $Ind_H^G W$ to be the extension of scalars of the $\mathbb{C}[H]$-module W to the group ring $\mathbb{C}[G]$.

$$Ind_H^G W := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \tag{6.3}$$

This defines a representation of G with the action given by right multiplication on the factor $\mathbb{C}[G]$.

Example 6.4. $Ind_{\{1\}}^G \mathbf{1}$ is the regular representation of G.
This can be seen by considering $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W = \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbf{1} \cong \mathbb{C}[\mathbf{G}]$. The left action of G on $\mathbb{C}[G]$ is precisely the regular representation of G.

This may at first appear quite abstract, but in the following few propositions, we will get a better idea of how Ind_H^G behaves. First, the following proposition will be useful in proving later theorems, though for what I believe to be the best intuitive picture of the induced representation I direct the reader to Propositon 6.9.

The following lemma gives a useful isomorphism between spaces of functions and tensor products. We will use it in the proof of the next proposition and later.

Lemma 6.5. *For finite dimensional vector spaces U, V, there is an isomorphism of vector spaces,*

$$Hom(U, V) \cong U^* \otimes V$$

which preserves $G-$actions.

Proof. Given bases $\{u_i\}, \{v_i\}$ of U, V respectively, identify the linear map L_{ij} satisfying $L_{ij}(u_i) = v_j$ with $u_i^* \otimes v_j$ and extend linearly in $\mathbb{C}$ and to sums of simple tensors. To see that G-actions are preserved, see that $g \in G$ acts on L_{ij} by $gL_{ij}(u) = L_{ij}(\rho_g u)$. Thus $gL_{ij}(\rho_g^{-1} u_i) = v_j$. Similarly, $g(u_i^* \otimes v_j) := \rho_g^{-1} u_i^* \otimes v_j$. □

Proposition 6.6. *For a representation (ρ, W) of $H \subset G$, as G-representations, $Ind_H^G W$ is isomorphic to the space $Hom_H(G, W)$ of linear maps $\varphi : G \to W$, which are H-invariant under the simultaneous right action $\varphi(g) \cdot h = \varphi(gh)$ and left action $h \cdot \varphi(g) = \rho(h) \cdot \varphi(g)$.*

Morally this is because $\mathbb{C}[G]$ can be thought of as the space of maps $\varphi : G \to \mathbb{C}$. In general for a tensor product $A \otimes_R B$, factors of $r \in R$ transfer across the factors as $ar \otimes_R b = a \otimes rb$. Therefore we should have $ar^{-1} \otimes_R rb = a \otimes b$. That is, simultaneous action by r^{-1} on one factor and r on the other cancels out. Here the action of h^{-1} on $\varphi(g)$ is $\varphi(g) \cdot h^{-1} = \varphi(gh)$.

Proof. Following Lemma 6.5, the space $Hom_{\mathbb{C}}(G, W)$ is isomorphic to the tensor product $\mathbb{C}[G] \otimes W$ by the map

$$\varphi \to \sum_{g \in G} e_g \otimes w_g,$$

where $\varphi(g) = w_g$, $e_g(g) = 1$, and $e_g(h) = 0$ for $h \neq g$. Here we are identifying $\mathbb{C}[G]$ with the space of linear maps from G to $\mathbb{C}$.

The left action of H maps $e_g \to e_{gh^{-1}}$ while the right action of H maps w to $\rho_h w$. Thus the H-invariance condition reads as

$$\sum_{g \in G} e_g \otimes w_g = \sum_{g \in G} e_{gh^{-1}} \otimes \rho_h w_g \; \forall h \in H. \tag{6.7}$$

By extending linearly, we see this is precisely an element of $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$, agreeing with the tensor product structure of invariance under acting on one factor of the product with an element g and acting by the inverse g^{-1} on he other factor. To check $G-$invariance, we let $f : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to Hom_H(G, W)$ be the map taking $g \otimes w$ to $e_g \otimes w$. Then we see $f(g \cdot \sum g_i \otimes w_i) = f(\sum g_i g^{-1} \otimes w_i) = \sum e_{g_i g^{-1}} \otimes w_i$, while $g \cdot f(\cdot \sum g_i \otimes w_i) = g \cdot \sum e_{g_i} \otimes w_i = \sum e_{g_i g^{-1}} \otimes w_i$. ☐

Example 6.8. For $(\rho^1, W_1), (\rho^2, W_2)$ representations of $H \subset G$, $Ind_H^G(W_1 \oplus W_2) \cong Ind_H^G W_1 \oplus Ind_H^G W_2$.

Proof. Using the construction in Proposition 6.6, $Hom(G, W_1 \oplus W_2) \cong Hom(G, W_1) \oplus Hom(G, W_2)$, and the linear operator $\rho_g = \rho_g^1 \oplus \rho_g^2$ acts on the factors W_1, W_2 separately, so that H-invariance is preserved by the isomorphism. ☐

Though the preceding proposition is correct and useful, and the original definition is satisfyingly concise, the induced representation $Ind_H^G W$ is perhaps most intuitively thought of in the following way: for each coset in G/H create a copy of W. Choose a representative g_r for each coset and call the associated copy W_r. Then an element $g \in G$ acts both by permuting the cosets (multiplying $g \cdot g_r$ and checking what coset it is now in) as well as acting on the copy of W by the factor belonging to H that is left over. As we would desire, action by elements of H agrees with the representation of H on the copy of W associated to the identity coset. The following proposition gives the details.

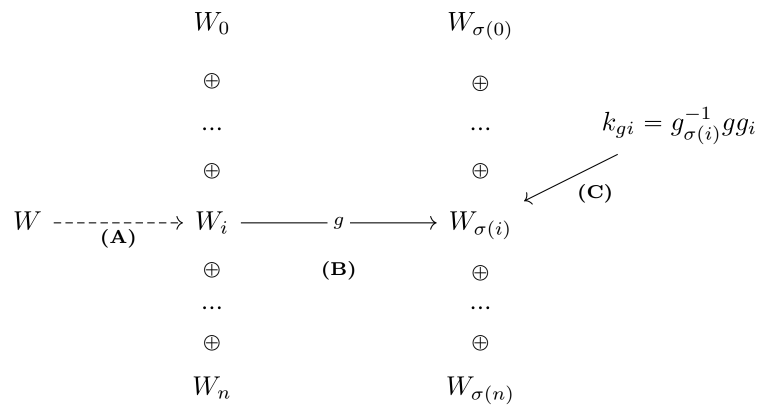

FIGURE 1. The action of G on the induced representation $Ind_H^G W$. **(A):** For every coset $i \in G/H$, create an isomorphic copy of the representation W indexed by the cosets as W_i. Their direct sum is the representation $Ind_H^G W$. **(B):** The action of G on $Ind_H^G W$ is to permute the factors of W_i as well as to linearly transform each individual W_i. For an element $g \in G$, denote the permutation of $\{W_i\}$ given by the action of g as σ. Then $g|_{W_i}(W_i) = W_{\sigma(i)}$. **(C):** Let $\{g_i\}$ be a system of representatives of the cosets G/H and denote $W_i = g_i W$. Then $g g_i W_i = g_{\sigma(i)} k_{gi} W$ where $k_{gi} = g^{-1}_{\sigma(i)} g g_i$. Thus k_{gi} acts within the factor $W_{\sigma(i)}$.

Proposition 6.9. *Let (ρ, W) be a representation of a subgroup H of a finite group G. Then for a system of represenatatives $R = \{g_r\}$ of the cosets G/H,*

$$Ind_H^G W \cong \bigoplus_{r \in R} W_r.$$

Each W_r is isomorphic to W and for any $g \in G$ and $g_r \in R$ we have

$$g g_r = g_{r'} h_r$$

for $g_{r'} \in R$ and $h_r \in H$, so the action of G is well defined as

$$g \cdot \bigoplus_{r \in R} W_r = \bigoplus_{r \in R} \rho(h_r) W_{r'}. \tag{6.10}$$

Proof. We will show this form agrees with the form $Hom_H(G, W)$ from Proposition 6.6. From the $H-$ invariance condition (6.7) it follows that we have $\rho_h w_g = w_{gh^{-1}}\ \forall h, g \in H, G$. Therefore given the image $Ind_H^G W(g)$ of any element g of the coset gH, the invariance condition specifies the map on the entire coset. Let W_r be the space of maps in $Hom_H(G, W)$ that vanish on all but the single coset containing r. We then have an isomorphism $f : W \to W_r$ given by

$$f(w) = \sum_{h \in H} e_{gh} \otimes \rho_h^{-1}(w). \tag{6.11}$$

This satisfies the $H-$invariance condition (6.7) because $w = w_r$ and so $\rho_h^{-1}(w_r) = w_{rh}$. In particular, we can define the $\mathbb{C}[G]$-module homomorphism $\varphi : Hom_H(W, G) \to$

$\bigoplus_{r\in R} W_r$ by

$$\varphi(\sum_{g\in G} e_g \otimes w_g) = \sum_{r\in R}\sum_{h\in H} e_{rh} \otimes \rho_h^{-1} w_r.$$

Bijectivity is given by the fact that cosets partition a group, so φ is an isomorphism. □

Having constructed the induced representation and having found an intuitive way to think of it, we turn next to the question of computing the characters of induced representations. For this a preliminary definition will be useful.

Definition 6.12. Let ψ be a class function on G, and let R be a system of representatives of cosets G/H. The *induced function* $Ind_H^G\psi$ is given by

$$Ind_H^G\psi(g) = \sum_{r\in R} \varepsilon \circ \psi(r^{-1}gr).$$

Here ε is an indicator function which is 0 on $G - H$ and 1 on H.

Proposition 6.13. *Let $H \subset G$ and let (ρ, V) be a representation of H with character χ. Then the character of $Ind_H^G V$ is the induced function $Ind_H^G\chi$.*

Proof. By the decomposition in Proposition 6.9, the character χ_g will have nonzero contribution from the blocks W_r which are fixed by the action of g. We have $W_r = W_{r'}$ iff $gg_r = g_r k$ for $k \in H$. Thus $g = g_r k g_r^{-1}$. Following (6.11, for those blocks W_r which are fixed, g acts as

$$g \cdot \sum_{h\in H} e_{g_r h} \otimes \rho_h^{-1} w = \sum_{h\in H} e_{g_r h} \otimes \rho_h^{-1} k w.$$

It follows that if $g_r^{-1} g g_r = k \in H$, the action of g on W_r corresponds to the action of $k = g_r^{-1} g g_r$ on W. If $g_r^{-1} g r \notin H$, the block W_r contributes zero to the trace. The proposition follows from summing over blocks W_r which are fixed. Summing over only fixed blocks mirrors the role of the indicator function ε in Definition 6.12. □

Lemma 6.14. *For completely reducible representations $(\rho, V)$and (ρ', U) of a group G, write $\langle V, U\rangle_G$ for the inner product $\langle \chi_V, \chi_U\rangle$ over G. Then $\langle V, U\rangle_G$ is the dimension of $Hom^G(V, U)$, the space of G-invariant linear maps, or intertwiners, from V to U.*

Proof. We will prove this in three steps.

First, we show that the statement holds when U, V are irreducible. Second, we show that both $\langle V, U\rangle_G$ and $Hom^G(V, U)$ are linear in each argument. Finally, we recall complete reducibility. The third step allows us to write

$$\langle V, U\rangle_G = \langle V_1 \oplus ... \oplus V_n, U_1 \oplus ... \oplus U_m\rangle_G,$$

where U_i, V_i are irreducible. Then the second step gives

$$\langle V_1 \oplus ... \oplus V_n, U_1 \oplus ... \oplus U_m\rangle_G = \langle V_1, U_1\rangle_G + \langle V_1, U_2\rangle_G + ... + \langle V_n, U_m\rangle_G.$$

Then step 1 gives

$$\langle V_1, U_1\rangle_G + \langle V_1, U_2\rangle_G + ... + \langle V_n, U_m\rangle_G = dim(Hom^G(V_1, U_1)) + ... + dim(Hom^G(V_n, U_m)).$$

Then finally step 2 again gives

$$dim(Hom^G(V_1, U_1)) + ... + dim(Hom^G(V_n, U_m)) = dim(Hom^G(V, U)),$$

and the proposition follows.

So lets prove these three things. First, suppose U, V irreducible. Then $\langle V, U\rangle_G = 1$ if $V \cong U$ and 0 otherwise. Schur's Lemma gives the same values for $dim(Hom^G(V, U))$. For the second step, let U_1, U_2, V be irreducible representations. Then

$$\langle U_1 \oplus U_2, V\rangle_G = \langle \chi_{U_1} + \chi_{U_2}, \chi_V\rangle,$$

and then we have

$$\langle \chi_{U_1} + \chi_{U_2}, \chi_V\rangle = \langle \chi_{U_1}, \chi_V\rangle + \langle \chi_{U_2}, \chi_V\rangle$$

by linearity of the character inner product per Proposition 4.6. By the vector space isomorphism

$$Hom(A \oplus B, C) \cong Hom(A, C) \oplus Hom(B, C),$$

$Hom(U_1 \oplus U_2, V)$ decomposes as $Hom(U_1, V) \oplus Hom(U_2, V)$ and a vector is G-invariant in $Hom(U_1 \oplus U_2, V)$ iff its image is so in $Hom(U_1, V) \oplus Hom(U_2, V)$. Therefore we can write

$$Hom(U_1 \oplus U_2, V) \cong Hom(U_1, V) \oplus Hom(U_2, V).$$

The claim on dimension follows. This same argument works for the right hand factor as well so we have both Hom^G and $\langle, \rangle_G$ are linear in each argument as desired. The third step is just to use the definition of complete reducibility. □

Induced representations and restricted representations are connected via the following theorem.

Theorem 6.15 (Frobenius Reciprocity)**.**

$$Hom^G(Ind_H^G W, V) \cong Hom^H(W, Res_G^H V)$$

Proof. By the Definition 6.2 of induced representation, we have

$$Hom^G(V, Ind_H^G W) = Hom^G(V, \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W).$$

By Lemma 6.5, we have

$$Hom^G(V, \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W) \cong (V^* \otimes \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W)^G,$$

where the superscript G denotes vectors invariant under the action of G on V^* and $\mathbb{C}[G]$. We further have

$$(V^* \otimes \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W)^G \cong (V^* \otimes \mathbb{C}[G] \otimes W)^{G,H}$$

where the superscript H denotes vectors invariant under the simultaneous right multiplication of H on $\mathbb{C}[G]$ and left action on W. With G, H acting the same, we permute the order of the tensor product to have

$$(V^* \otimes \mathbb{C}[G] \otimes W)^{G,H} \cong (\mathbb{C}[G] \otimes V^* \otimes W)^{G,H}.$$

The G-invariance restriction determines all invariant vectors in $\mathbb{C}[G] \otimes V^* \otimes W$ given its restriction to $e \otimes V^* \otimes W$ by the condition

$$g \cdot e \otimes v^* \otimes w = e \otimes \rho_g^{-1} v^* \otimes w$$

for a G-invariant vector. Therefore the dimension of $(\mathbb{C}[G] \otimes V^* \otimes W)^{G,H}$ is the same as the dimension of $(e \otimes V^* \otimes W)^H$, which is the same as $(V^* \otimes W)^H$ given that the notion of H-invariance is well defined on $V^* \otimes W$.

Indeed, in the map $f : (\mathbb{C}[G] \otimes V^* \otimes W)^G \to V^* \otimes W$, a vector $\varphi \in (\mathbb{C}[G] \otimes V^* \otimes W)^G$ is H invariant when for any simple tensor $q = e \otimes v^* \otimes w$, q is invariant under the action of H as $hq = h \otimes v^* \otimes \rho_h w$. The G-invariance condition gives

$hq = e \otimes (h^{-1}v)^* \otimes hw$. Thus H invariance implies invariance under the action of H on V^* and on W, and thus passes to the desired notion of H-invariance in $V^* \otimes W$. By Lemma 6.5 we can then identify $(V^* \otimes W)^H \cong Hom^H(Res_G^H V, W)$. □

Remark 6.16. Frobenius Reciprocity is the statement that the functors $Ind_H^G : Rep_H \to Rep_G$ and $Res_G^H : Rep_G \to Rep_H$ are *adjoint functors*. Here Rep_G is the category of G-representations where morphisms are G-intertwiners, and Rep_H is the analogue for the subgroup H. The functor Res_G^H maps a representation (ρ, V) of G to a representation $(\rho|_H, V)$ of H. It acts as the identity on morphisms since G-invariance implies H-invariance. The functor Ind_H^G maps a representation W to $Ind_H^G W$.

Corollary 6.17. *By counting dimensions in the above, we have*

$$\langle V, Ind_H^G W\rangle_G = \langle Res_G^H V, W\rangle_H.$$

6.1. **Mackey Theory.** We will develop a tool to determine if a representation is irreducible through a relation between its induced and restricted representations.

Theorem 6.18 (Mackey's Restriction Formula)**.** *For H a subgroup of G, let $S \subset G$ be a set of representatives for the double coset space $H\backslash G/H$. For each $s \in S$, define*

$$H_s := sHs^{-1} \cap H.$$

For a representation (ρ, W) of H, a representation (ρ^s, W_s) of H_s is defined by $\rho^s(x) = \rho(s^{-1}xs)$, where $W_s = W$. Then,

$$Res_G^H Ind_H^G(W) \cong \bigoplus_{s\in S} Ind_{H_s}^H(W_s).$$

Proof sketch. For the full proof see [4] (p. 58-59). We give an instructive and abridged version here.

The space $V := Ind_H^G(W)$ is a direct sum of the spaces $g_r W$ where $g_r \in R_1$ for R_1 a system of representatives for G/H. It is a fact that double cosets are partitioned by cosets, so we may group together all the cosets that sit inside the same double cosets to make the space $V(s)$. The space $V(s)$ is formally defined as the subspace of V generated by the images $g_r W$ for $g_r \in HsH$.

The approach is to show that $V(s)$ is isomorphic as an H-representation to the summand $Ind_{H_s}^H(W_s)$. To do so, we need only check the characters are the same.

Let R_2 be a system of representatives for H/H_s. Then by Proposition 6.13 the character of $Ind_{H_s}^H(W_s)$ is given by

$$\chi(Ind_{H_s}^H(W_s)) = \sum_{h\in H}\sum_{r\in R_2} \varepsilon(h \in rH_s r^{-1})\chi^s(r^{-1}hr),$$

where $\varepsilon(h \in rH_s r^{-1}) = 1$ when $h \in rH_s r^{-1}$ and 0 otherwise. Recalling the definition of ρ^s, we have $\chi^s(h) = (s^{-1}hs)$. An element of $rH_s r^{-1}$ can be written as $rsh's^{-1}r^{-1}$ for $h' \in H$. When this element is in H, it is included in the sum, so we may write

$$\sum_{h\in H}\sum_{r\in R_2} \varepsilon(h \in rH_s r^{-1})\chi^s(r^{-1}hr) = \sum_{h'\in H}\sum_{r\in R_2} \varepsilon(h' \in rH_s r^{-1})\chi(s^{-1}r^{-1}h'rs).$$

On the other hand, the character of $Res_G^H Ind_H^G(W)$ is given as

$$\sum_{h\in H}\sum_{a\in R_3} \varepsilon(h \in aHa^{-1})\chi(a^{-1}ha),$$

where R_3 is the subset of R_1 that is contained in the double coset represented by s. The trick is the realize that elements of R_3 are in bijection with elements of R_1 by mapping r to rs. To see this, consider $r(sHs^{-1}) \in R_3$. Multiplying by s gives $r(sHs^{-1})s = rsH$. This is an element of the double coset HsH. It also represents a unique coset in G/H because $rs \in G$. We can thus write

$$\sum_{h\in H}\sum_{a\in R_3\cap H}\varepsilon(h\in aHa^{-1})\chi(a^{-1}ha) = \sum_{h\in H}\sum_{r\in R_2}\varepsilon(h\in rsHs^{-1}r^{-1})\chi(s^{-1}r^{-1}hrs).$$

Since this matches our equation for the character of $Ind_{H_s}^H(W_s)$ above we conclude

$$\chi(Res_G^H Ind_H^G(W)) = \chi(Ind_{H_s}^H(W_s)).$$

This implies the two are isomorphic, and since this follows for every $s \in R_1$, we have the entire isomorphism as desired.

□

Theorem 6.19 (Mackey's Irreducibility Criterion)**.** *$Ind_H^G$is irreducible iff W is irreducible and $\forall s \in S - H$, the representations W_s and $Res_H^{H_s}W$ are disjoint.*

Proof. From a direct application of Mackey's Restriction Formula and Frobenius reciprocity we have

$$\langle Ind_H^G W, Ind_H^G W\rangle_G = \langle Res_G^H Ind_H^G W, W\rangle_H = \sum_s\langle Ind_{H_s}^H W_s, W\rangle_H = \sum_s\langle W, Ind_{H_s}^H W_s\rangle_H = \sum_s\langle Res_H^{H_s}W, W_s\rangle_{H_s}. \tag{6.20}$$

Since inner products of characters take integral values, $Ind_H^G W$ is irreducible iff all terms but one in the last sum are zero, and the one nonzero term has the value 1. Indeed, there is a single coset for which $H_s = H$ ($s \in H$), which contributes the term $\langle Res_H^H W, W\rangle_H = \langle W, W\rangle_H$ to the sum. If W is irreducible, this takes value 1, while if all other pairs are disjoint, their inner products vanish as desired. □

In this chapter, we defined the induced and restricted representations and showed a few ways in which they can be related. We turn next to the representation theory of compact groups, exemplified by the *classical groups.*

7. Classical Groups

In this section we show how to use representation theory and character theory when the groups in question are not finite. We consider a few important compact groups known as *classical groups* for their historic use as easy cases. In the representation theory of compact groups G, we restrict our attention to continuous group homomorphisms $\rho : G \to GL(V)$.

7.1. U(1).

Perhaps the simplest compact group is $U(1)$, the group of unitary 1×1 complex matrices: that is $x \in \mathbb{C}$ s.t. $x\bar{x} = 1$. It can be identified with the unit circle S^1 in the complex plane.

Proposition 7.1. *A continuous irreducible representation of $U(1)$ has the form $\rho(z) = z^n$ for integer n.*

proof sketch. We will exhibit all one dimensional, and thus irreducible, representations of $U(1)$.

A one dimensional representation of $U(1)$ is a continuous group homomorphism $\rho : U(1) \to \mathbb{C}^{\times}$. Consider the kernel of ρ.

In general, the kernel of a continuous group homomorphism is a closed subgroup. The closed proper subgroups of $U(1)$ are cyclic subgroups C_n of nth roots of unity. Therefore

$$Ker\,\rho = C_n$$

for some $n \in \mathbb{N}$. It follows

$$\rho(C_n) = 1.$$

Next let $a \in C_{mn}$ be an mnth root of unity. Then $(a^m)^n = 1$, so $a^m \in C_n$. Since ρ is a homomorphism, $\rho(a)^m = \rho(a^m) = 1$, so $\rho(a^m) \in C_n$. Thus $\rho(C_{nm}) \subset C_m$, so ρ maps mn-th roots of unity to m-th roots of unity $\forall m \in \mathbb{Z}_+$ in an order preserving way. This condition, along with continuity, is enough to imply that $\rho(z) = z^n$ precisely. For the detailed proof see [1]. ☐

Though the Haar Measure implies the existence of a G-invariant inner product on all compact groups with respect to which irreducible characters are orthonormal, in general it is hard to find this product from first principles. Instead, in this as well as future examples, we will exhibit an inner product derived from the geometric intuition present in the structure of the groups, and then check orthonormality of characters and G-invariance.

Proposition 7.2. *There is a normalized hermitian inner product on characters of $U(1)$ given by*

$$\langle \chi, \chi' \rangle = \frac{1}{2\pi} \int_0^{2\pi} \overline{\chi(\theta)} \chi'(\theta) d\theta, \tag{7.3}$$

where $\chi_n(\theta)$ is interpreted as $e^{in\theta}$. With respect to this inner product, the irreducible characters $\chi_n := z^n$ are orthonormal.

In the finite case, we found the characters to be a basis for class functions. We obtain a similar result here. Since $U(1)$ is abelian, each element is its own conjugacy class. Since (7.3) is only well defined when it converges, our statement about irreducible characters being a basis on class functions must take this into account.

Theorem 7.4 (Fourier)**.** *The functions z^n form an orthonormal basis for $L_2(U(1))$, the Hilbert space of square integrable complex function on $U(1)$.*

Proof. It is a famous theorem of Fourier [7] that any $f \in L_2(S^1)$ has a *fourier series* expansion as

$$f(\theta) = \sum_{-\infty}^{+\infty} f_n \cdot e^{in\theta}. \tag{7.5}$$

The fourier coefficients f_n are given by the inner product of characters using the inner product (7.3).

$$f_n = \langle z^n, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(\theta) d\theta$$

□

Definition 7.6. *Laurent polynomials* are polynomials in z and z^{-1}.

7.2. $SU(2)$.

Definition 7.7. Let the *quaternion algebra* $\mathcal{H}$ be the algebra generated by elements i, j, k with the relations

$$ij = -ji = k$$
$$jk = -kj = i$$
$$ki = -ik = j$$
$$i^2 = j^2 = k^2 = -1$$

The quaternion algebra $\mathcal{H}$ is naturally embedded in $M_2(\mathbb{C})$ via the following map $\varphi : \mathcal{H} \to M_2(\mathbb{C})$:

$$\psi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} := I$$
$$\psi(i) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$
$$\psi(j) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$\psi(k) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Physicists will recognize the quaternions as the *Pauli matrices*, up to a factor of i. These matrices are used to analyze the group $SU(2)$ which is the state space for a Spin-1/2 particle.

The image $\psi(\mathcal{H}) \subset M_2(\mathbb{C})$ generates the subspace of unit determinant unitary matrices, called $SU(2)$.

Definition 7.8. The group $SU(2)$ is the group of determinant 1 unitary 2×2 complex matrices.

$$SU(2) := \left\{ P = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \; ; \; a, b \in \mathbb{C} \; ; \; det(P) = 1 \right\}$$

A matrix $P \in SU(2)$ can be given in coordinates (x_0, x_1, x_2, x_3) by defining $P = x_0 I + x_1 i + x_2 j + x_3 k$ with the restriction that $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. Geometrically $SU(2)$ can be identified with the unit sphere S^3 in R^4.

Proposition 7.9. *The following facts will prove useful in analyzing $SU(2)$. Let $P, P' \in SU(2)$.*

(1) *The eigenvalues of P are $z, \bar{z} : z\bar{z} = 1$.*
(2) *$P = cos\theta I + sin\theta A$ where $trA = 0$.*
(3) *P, P' are conjugate iff $tr(P) = tr(p')$.*

Proof. Fact (1) follows from the diagonalizability of complex matrices and the requirement that $det(P) = 1$. Fact (2) folows from the decomposition $x_1 i + x_2 j + x_3 k := A$ and the normalization condition. Fact (3) follows directly from fact (1). For a detailed proof see [2]. □

Following the geometric intuition granted by fact (3), the conjugacy classes are called *latitudes* and are indexed by the value of θ. In particular, the set of traceless elements is called the equator E. For a fixed traceless element $A \in E$, the set of elements $\{cos\theta I + sin\theta A\}$ is called a *longitude*. A quick computation $(A')(cI + sA)(A'^{-1}) = cI + sA'AA'^{-1}$, where $c^2 + s^2 = 1$, shows that longitudes are conjugate subgroups of $SU(2)$, since all traceless matrices are conjugate by fact (3).

Proposition 7.10. *The map $f : SU(2) \to S^1$ given by $f(cos\theta I + sin\theta A) = e^{i\theta}$ gives a homeomorphism from latitudes in $SU(2)$ to the circle S^1. The latitudes are given the quotient topology in $SU(2)$. The map is invariant under the interchange $\theta \to -\theta$.*

Therefore any continuous class function on $SU(2)$ is a function on the eigenvalue $z = e^{i\theta}$ that is invariant under the interchange $z \to z^{-1}$. Since a representation of $SU(2)$ restricts to a representation of $U(1)$ by the homomorphism mapping $cos\theta I + sin\theta A$ to $e^{i\theta}$, it follows that characters of $SU(2)$ are Laurent polynomials.

By viewing $SU(2)$ as topologically the sphere $S^3 \in \mathbb{R}^4$, one can arrive at an invariant integration measure.

Proposition 7.11 (Weyl integration Formula)**.** *The following is a $SU(2)$-invariant inner product on continuous class functions on $SU(2)$ with regard to which the irreducible characters are orthonormal.*

$$\langle \varphi, \psi \rangle = \frac{1}{\pi} \int_0^{2\pi} \overline{\varphi(\theta)} \psi(\theta) \, sin^2\theta \, d\theta$$

Proof. For the proof see [8]. □

Theorem 7.12. *The irreducible characters of $SU(2)$ are given by $\chi_n(z) = z^n + z^{n-2} + ... + z^{-n}$ where $z = e^{i\theta}$ parametrizes the conjugacy classes of $SU(2)$.*

Proof. For the full proof see [1]. Here we check orthogonality using Proposition 7.11.

The conjugacy classes of $SU(2)$ are latitudes parametrized by angle θ. Thus let $z = e^{i\theta}$. Let

$$\chi_n = z^n + z^{n-2} + ... + z^{-n}$$

and let

$$\chi_m = z^m + z^{m-2} + ... + z^{-m}.$$

This simplifies to

$$\chi_n(\theta) = \frac{sin((m+1)\theta)}{sin\theta}$$
$$\chi_n(\theta) = \frac{sin((n+1)\theta)}{sin\theta}.$$

Carrying out the integration we have

$$\langle \chi_n, \chi_m \rangle = \frac{1}{\pi} \int_0^{2\pi} \frac{sin((m+1)\theta)}{sin\theta} \cdot \frac{sin((n+1)\theta)}{sin\theta} \cdot sin^2(\theta) \, d\theta$$
$$= \frac{1}{\pi} \int_0^{2\pi} sin((m+1)\theta) \cdot sin((n+1)\theta) \, d\theta$$
$$= \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2} \left(cos((m-n)\theta) - cos((m+n+2)\theta)\right) = \delta_{mn}.$$

It follows that the characters χ_n form an orthonormal set with repect to the inner product in Proposition 7.11. $\square$

Remark 7.13. Notice that in keeping with Proposition 7.10, the irreducible representations of $SU(2)$ are Laurent polynomials that are preserved under the interchange $z \to -z$.

Proposition 7.14. *As $SU(2)$-representations, $Sym^n(\mathbb{C}^2) \cong P_n$, the space of homogeneous polynomials of degree n in two complex variables.*

Proof. The isomorphism is constructed my mapping $e_1^i e_2^j$ to the symmetrization of $e_1^{\otimes i} \otimes e_2^{\otimes j}$. This commutes with the action of $SU(2)$ as $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} ae_1 + be_2 \\ -\bar{b}e_1 + \bar{a}e_2 \end{bmatrix}$. $\square$

In this section we found the irreducible characters for the groups $U(1)$ and $SU(2)$, which are simple cases of infinite compact groups. We found the precise form of the inner product (3.5) for these groups by starting with the geometric interpretation of the groups as a circle and a sphere respectively. In the next section, we turn to the irreducible characters of the symmetric group.

8. S_n

We will now use the tools we have developed to find the irreducible representations of the symmetric group S_n. The following result is of interest in its own right, and also has connections to the representations of $U(1)$ through a notion called *Schur-Weyl Duality*, which is discussed in [1].

Two immediate representations of S_n for any n are the trivial representation and the *sign representation*, which maps σ to $sign(\sigma) \cdot Id$. However there are in general many more. They are indexed by *partitions of* n and can be listed and studied by *Young diagrams*, which make apparent the essential features of a partition.

Definition 8.1. A *partition* λ of $[1, n] := \{1, 2, ..n\}$ is a set $\lambda_1 \geqslant ... \geqslant \lambda_k$ s.t. $\sum_i \lambda_i = n$. The set of partitions of n is denoted $\mathcal{P}_n$.

A partition λ divides $[1, n]$ into blocks $I_i = [\lambda_1 + .. + \lambda_{i-1}, \lambda_1 + ... + \lambda_i]$. We define a special polynomial Δ_λ in the following way:

$$\Delta_\lambda = \prod_{i=1}^{k} \prod_{l<j \in I_i} (x_l - x_j) \tag{8.2}$$

Thus Δ_λ contains all antisymmetric factors $(x_i - x_j)$ for pairs i, j in the same block I. Define $S_\lambda \subset S_n$ to be permutations which preserve the blocks I_i. Then we will show that functions that are antisymmetric under S_λ necessarily contain a factor of Δ_λ. First, a short proposition.

Proposition 8.3. *Δ_λ consists of homogeneous polynomials of degree $d_\lambda = \sum_{i=1}^{k} \frac{\lambda_i(\lambda_i - 1)}{2}$.*

We will need this later.

Let $\mathbb{C}[x_1, ..x_n] := A$ be complex polynomials in n variables. Then S_n acts on A by permuting the variables. Let $A^+ \subset A$ denote S_λ-invariant polynomials. Let $A^- \subset A$ denote S_λ-anti-invariant polynomials.

Theorem 8.4. *As an A^+-module, $A^- = A^+ \cdot \Delta_\lambda$.*

Proof. Let $f \in A^-$. Then for any transposition $(i,j) \in S_\lambda : i \neq j$, we have $(i,j)f = -f$. It follows that $f|_{x_i=x_j} = 0$. Thus f contains a factor of $(x_i - x_j)$. Since this holds for all distinct pairs i,j, which are coprime, f contains a factor of Δ_λ. So let $f = h \cdot \Delta_\lambda$. Then $(i,j)f = -f = (i,j)h \cdot (i,j)\Delta_\lambda = (i,j)h \cdot -\Delta_\lambda$. It follows that $h \in A^+$. Thus any $f \in A^-$ is of the form $f = h \cdot \Delta_\lambda$ for $h \in A^+$. □

Next for our main result.

Definition 8.5. For $\lambda \in \mathcal{P}_n$, let $V(\lambda)$ be the $\mathbb{C}$-span of the set $\{s\Delta_\lambda : s \in S_n\}$.

Remark 8.6. The space $V(\lambda)$ is S_n stable.

Theorem 8.7. *The spaces $V(\lambda)$ as λ ranges over all partitions $\mathcal{P}_n$, are all the irreducible representations of S_n.*

Proof. To prove this theorem we need to show three things.

(1) $V(\lambda)$ are irreducible.
(2) $V(\lambda)$ are distinct for distinct partitions.
(3) Any irreducble representation is one of the $V(\lambda)$.

Given (1) and (2), item (3) follows immediately from Proposition 2.18, considering that both the representations $V(\lambda)$ and conjugacy classes of S_n are indexed by partitions of n, in the latter case to determine cycle type. Thus it remains to show (1) and then (2). Item (1) will be given by Corollary 8.9 below and item (2) will be given by Proposition 8.13 below. □

Lemma 8.8. *Let $F : V(\lambda) \to V(\lambda)$ be an intertwiner of S_n-representations. Then F is a scalar multiple of the identity.*

Proof. For $s \in S_n$, by assumption we have $F(s(\Delta_\lambda)) = s(F(\Delta_\lambda))$. Let $F(\Delta_\lambda) := D$. Then $Im(F)$ is the $\mathbb{C}$-span of $\{sD : s \in S_n\}$. Furthermore, since $\Delta_\lambda \in A^-$, for $s \in S^\lambda$, we have

$$sD = F(s\Delta_\lambda) = F(\varepsilon_s\Delta_\lambda) = \varepsilon_s D,$$

where ε_s is the sign of s. It follows that $D \in A^-$. Therefore by Theorem 8.4, D contains a factor of Δ_λ or $F = 0$. Since Δ_λ is a degree d_λ polynomial, the degree of D requires that D is a scalar multiple $c\Delta_\lambda$ of Δ_λ. It follows that for $s \in S^n$,

$$F(s\Delta_\lambda) = sD = sc\Delta_\lambda = c(s\Delta_\lambda),$$

so that F is a scalar function. □

Corollary 8.9. *$V(\lambda)$ is irreducible for any $\lambda \in \mathcal{P}_n$*

Proof. By Schur's Lemma and the discussion following its corrolary, a completely reducible representation V is irreducible iff all G-intertwiners are scalar multiples of the identity. This condition is precisely what Lemma 8.8 showed. □

We have thus shown item (1). It remains to show (2). Before doing so, we introduce a new way of visualizing partitions which will be used in the proof.

To every partition λ there is assigned a *Young diagram* $D(\lambda)$.

Definition 8.10. For $\lambda \in \mathcal{P}_n$, the *Young diagram* $D(\lambda)$ of λ is an array with λ_i cells in column i.

A Young diagram is a nice visual way to understand a partition. It also offers a geometric way of constructing a new, related partition from any given partition. This partition is called the *transposed partition.*

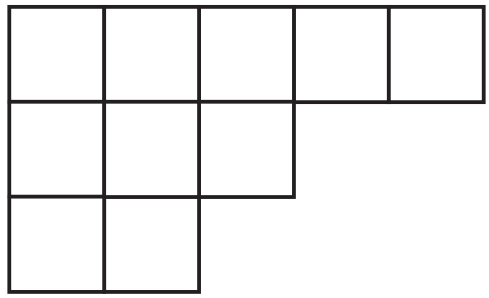

FIGURE 2. The Young diagram associated with the partition $(5, 3, 2)$.

Definition 8.11. For a partition $\lambda \in \mathcal{P}_n$ and its associated Young diagram $D(\lambda)$, define a new Young diagram $D(\lambda^T)$ by reflecting $D(\lambda)$ over the diagonal. Then $D(\lambda^T)$ defines a partition λ^T.

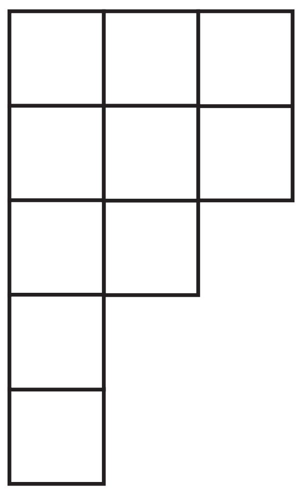

FIGURE 3. The transposed Young diagram $D(\lambda^T)$ associated with the partition $(5, 3, 2)^T = (3, 3, 2, 1, 1)$.

Proposition 8.12. *Notice that the number of cells in row i of $D(\lambda^T)$, or equivalently λ_i^T, is the number of rows of $D(\lambda)$ that have at least i cells. The definition also shows that for $\lambda, \mu \in \mathcal{P}_n$, $\lambda^T = \mu^T \Rightarrow \lambda = \mu$.*

Now we are prepared to prove (2).

Proposition 8.13. *The representations $V(\lambda) : \lambda \in \mathcal{P}_n$, are pairwise nonisomorphic.*

Proof. We wish to show that

$$V(\lambda) = V(\mu) \Rightarrow \mu = \lambda.$$

To do this, we introduce new notation. Define $v(n)$ to be the tuple $(1, 2, ...n)$. Similarly, for $\lambda = (\lambda_1, ..\lambda_k) \in \mathcal{P}_n$, let $v(\lambda) = (v(\lambda_1), .., v(\lambda_k)) = (1, 2, ...\lambda_1, 1, 2, ..., \lambda_k)$, and denote $v(\lambda)_j$ the j-th entry. Then let $x^{v(\lambda)} := x_1^{v(\lambda)_1} \cdot \ldots \cdot x_n^{v(\lambda)_n}$. For an n-tuple v and a permutation $s \in S_n$, let $x^{sv} = x_1^{v_{s(1)}} \cdot .. \cdot s_n^{v_{s(n)}}$.

Now, the Leibniz formula for the Vandermonde determinant says

$$\Delta_n = \frac{1}{x_1 \cdot .. \cdot x_n} \sum_{s \in S_n} \varepsilon(s) \cdot x^{sv(n)}.$$

A generalization of this formula shows that

$$\Delta_\lambda = \frac{1}{x_1 \cdot .. \cdot x_n} \sum_{s \in S_\lambda} \varepsilon(s) \cdot x^{sv(\lambda)}. \tag{8.14}$$

For a tuple v, let $m_j(v)$ be the number of times j appears in v. Then from Proposition 8.12, we see $m_j(v(\lambda)) = \lambda_j^t$. Now suppose $V(\lambda) = V(\mu)$. It follows that $\Delta_\lambda \in V(\mu)$, and so the monomial $x^{v(\lambda)}$ which appears in the formula for Δ_λ in (8.14 must appear in as some monomial in $s\Delta(\mu)$ for some $s \in S_n$. It follows that $x^{v(\lambda)} = sx^{s'v(\mu)}$ for some $s' \in S_n$. Therefore $v(\lambda) = ss'v(\mu)$.

Now, since $m_j(v)$ counts the instances of j, ignoring order, it is permutation invariant, and so

$$m_j(v(\lambda)) = m_j(ss'v(\mu)) = m_j(v(\mu)).$$

From this it follows that $\lambda_j^t = \mu_j^t \ \forall j > 0$ and thus by Proposition 8.12, $\lambda = \mu$ as desired. □

Young diagrams give us a useful way to visualize partitions and they also are useful in understanding the representations $V(\lambda)$, as the following Theorem shows.

Definition 8.15. Given a Young diagram $D(\lambda)$, a *standard* Young diagram of λ is a way of assigning numbers to the cells of $D(\lambda)$ such that any cell has a higher value than those to the left or above it.

Visually, the standard Young diagrams index the different ways a Young diagram could be constructed by placing one cell at a time which was forced into the top left corner of the array.

1	2
3	4

1	3
2	4

FIGURE 4. The two standard Young diagrams of the partition $(2, 2)$.

Theorem 8.16. [9] *For an irreducible representation $V(\lambda)$ of S_n, the dimension of $V(\lambda)$ is given by the number of standard Young diagrams of λ.*

For any n, there exists the partition $\lambda_1 = (n)$, which results in a totally antisymmetric polynomial Δ_{λ_1}, which yields the sign representation. There also exists the partition $\lambda_2 = (1, 1, 1, ...1)$ which has $\Delta_{\lambda_2} = 1$, which yields the trivial representation. The Young tableaus associated to these representations can be seen in both Figure 4 and Figure 5.

Example 8.17 (S_3)**.** For the case $n = 3$, there is one other partition which is $\lambda_3 = (2, 1)$. By Theorem 8.16, the dimension of $V(\lambda_3)$ is 2. Using x, y, z for the three variables, by inspection we see that $\Delta_\lambda = (z-y)$. In general polynomials in n variables are hard to visualize, but in this case where they are of degree 1 they can be understood as coordinates in $\mathbb{C}^3$. Under the action of S_3, $V(\lambda_3)$ is then defined to be the $\mathbb{C}$-span of the polynomials $\{\lambda_3 = (z-y), (x-z), (x-y), (y-x), (y-z), (z-x)\}$, which span the plane $x+y+z = 0$. In this plane S_3 acts by permuting the coordinate axes, and so $V(\lambda_3)$ can be seen to be the regular representation of S_3 which acts on a triangle in the plane.

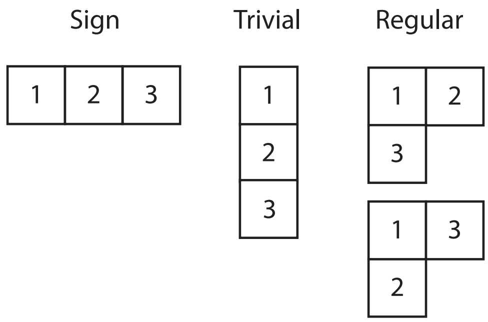

FIGURE 5. Standard Young diagrams associated to irreducible representations of S_3.

Example 8.18 (S_4)**.** The standard Young diagrams associated with the irreducible representations of S_4 are given below. There are the usual sign and trivial representations. Identifying S_4 with the symmetries of the tetrahedron, the irreducible representations also include the standard representation as well as the tensor product of the standard representation with the sign representation. Finally there is a 2-dimensional representation, which corresponds to the regular representation of the copy of S_3 in S_4. Visualizing S_4 as the symmetries of a cube, the three pairs of opposite faces are permuted, giving a copy of S_3 in S_4.

ACKNOWLEDGMENTS

I'd like to thank my mentor Dr. Victor Ginzburg for guiding me this summer, teaching me, and generously giving me his time. Thank you to Dr. Peter May for orchestrating an excellent REU experience for me this summer. Finally, I'd like

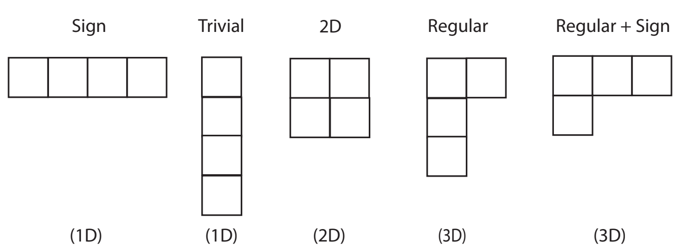

FIGURE 6. Young diagrams associated with the irreducible representations of S_4. The dimensions are listed along the bottom.

to thank my amazing girlfriend Corinne Motl for her constant support and talking about math with me!

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