User's Guide for Stage 2 Disk Evolution Code

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1 Overview

This document describes the structure, numerical scheme, and physical implementation of the Stage 2 version of the one–dimensional viscous disk evolution code. Stage 2 corresponds to the first fully time–dependent implementation of the disk energy equation coupled with irradiation heating including a finite time delay.

The code is designed for robustness, extensibility, and physical transparency, and is intended for studies of accretion and decretion disks in high–energy binary systems.

2 General Structure of the Code

The time integration is organized around three nested layers:

1. **Output–level time stepping** with a fixed interval Δt_{out}.
2. **Adaptive substepping** to reach the next output time.
3. **Local TRY/COMMIT evolution** for each substep.

The main program advances the system by repeatedly calling `evolve_try_to_target`, which ensures that the system evolves from time t_n to $t_n + \Delta t_{\mathrm{out}}$ using as many internal substeps as required for stability.

3 Main Modules

3.1 mod_global

This module stores global grid definitions and state arrays.

Key quantities include:

- Radial grid: `r(i)`, `nr`
- Surface density: `sigma_cur(i)`
- Viscosity: `nu_cur(i)`
- Thermal quantities: `Tmid_cur`, `H_cur`, `rho_cur`, `kappa_cur`, `tau_cur`
- Heating and cooling rates: `Qvis_cur`, `Qrad_cur`, `Qirr_cur`

The suffix `_cur` indicates the current committed state inside a substep.

3.2 evolve_substep_mod

This module controls the evolution over a single adaptive substep.
The central routine is:

```
substep_try_and_commit()
```

It performs:

1. Construction of the irradiation profile at the substep start.
2. A TRY evolution using `evolve_physics_one_substep_try`.
3. Acceptance or rejection of the step.
4. Commitment of the result to the `_cur` arrays.

No retry loop exists inside this routine; retries are handled at a higher level.

3.3 evolve_try_mod

This module contains the physics solver for a single substep.
The main routine is:

```
evolve_physics_one_substep_try()
```

It advances:

- Surface density Σ
- Temperature T
- Viscosity ν
- Disk structure and heating/cooling terms

for a given substep size Δt.

3.4 evolve_to_target (or evolve_try_to_target)

This routine advances the system from t_n to $t_n + \Delta t_{\rm out}$ using adaptive substeps.
Algorithm:

1. Initialize `t_local = t_nd`.
2. While remaining time > 0:
    - Attempt a substep of size $\Delta t_{\rm try}$.
    - On failure, reduce Δt.
    - On success, commit and advance `t_local`.
3. Ensure that output times are uniformly spaced.

4 Governing Equations

4.1 Surface Density Evolution

The disk surface density evolves according to the standard viscous diffusion equation:

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(\nu\Sigma r^{1/2}\right)\right] + S(r,t), \tag{1}$$

where $S(r,t)$ is an external source or sink term.

Time integration uses the θ–method:

$$\Sigma^{n+1} = \Sigma^n + \Delta t\left[(1-\theta)F(\Sigma^n) + \theta F(\Sigma^{n+1})\right]. \tag{2}$$

4.2 Energy Equation

At each radius, the midplane temperature evolves according to:

$$\Sigma c_V \frac{dT}{dt} = Q_{\rm vis} + Q_{\rm irr} - Q_{\rm rad}. \tag{3}$$

An implicit Euler step is used:

$$\frac{T^{n+1} - T^n}{\Delta t} = \frac{Q_{\rm vis}(T^{n+1}) + Q_{\rm irr} - Q_{\rm rad}(T^{n+1})}{\Sigma c_V}. \tag{4}$$

This nonlinear equation is solved locally using a damped Newton method with bracketing safeguards.

5 Convergence and Acceptance Criteria

Each substep is accepted if:

- No numerical failure occurs (NaN, divergence).
- $\Sigma > 0$, $T > 0$, $\nu > 0$ in active cells.

The Newton iteration itself enforces local convergence of the energy equation, so no global thermal balance test is required at this stage.

6 Irradiation Model

6.1 Irradiation Heating Rate

The irradiation heating rate is computed using Eq. (16) of Lee, Okazaki, and Hayasaki (2024):

$$Q_{\rm irr}(\xi) = \frac{A_1 L_1}{2\pi r_{\rm in}^2}\frac{1}{\xi}\left[(1+Q_{12})\frac{dY}{d\xi} - \frac{\beta_1 + Q_{12}\beta_2}{\xi^2}\left(\frac{Y}{\xi} - \frac{1}{2}\frac{dY}{d\xi}\right)\right], \tag{5}$$

where $\xi = r/r_{\rm in}$ and $Y = H/r$.

At Stage 2:

- Shadowing is disabled.
- $Q_{\rm irr}(r)$ is computed once per substep using the geometry at the substep start.

6.2 Time Delay (Scheme C)

Irradiation is powered by accretion luminosity with a finite delay.

Procedure:

1. Measure instantaneous accretion rate $\dot{M}_{\rm in}(t)$ at the inner boundary.
2. Store $\dot{M}_{\rm in}$ in a physical–time history buffer.
3. Define a delay time:
$$t_{\rm delay} = f_{\rm delay} \frac{r_{\rm in}^2}{\nu(r_{\rm in})}. \tag{6}$$
4. Compute delayed accretion rate $\dot{M}_{\rm in}(t - t_{\rm delay})$ by linear interpolation.
5. Set irradiation luminosity:
$$L_{\rm irr} = \eta_{\rm acc} \dot{M}_{\rm in}(t - t_{\rm delay}) c^2, \tag{7}$$
optionally capped at the Eddington luminosity.

This scheme ensures causality and avoids instantaneous feedback.

7 Scope and Limitations of Stage 2

Stage 2 includes:

- Fully time–dependent energy equation.
- Adaptive substepping with fixed output times.
- Irradiation with time delay.

Not included at this stage:

- Shadowing effects.
- Implicit irradiation–thermal coupling.
- Stability branch tracking.
- Checkpointing of irradiation history buffers.

These are intended for later development stages.

8 Summary

The Stage 2 code provides a robust and physically consistent framework for studying irradiated viscous disks with delayed feedback. It forms a solid foundation for future extensions including shadowing, branch stability analysis, and global energy diagnostics.