

CALABI–YAU THREEFOLDS OF TYPE K (I): CLASSIFICATION

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ABSTRACT. Any Calabi–Yau threefold X with infinite fundamental group admits an étale Galois covering either by an abelian threefold or by the product of a K3 surface and an elliptic curve. We call X of type A in the former case and of type K in the latter case. In this paper, we provide the full classification of Calabi–Yau threefolds of type K, based on Oguiso and Sakurai’s work [26]. Together with a refinement of their result on Calabi–Yau threefolds of type A, we finally complete the classification of Calabi–Yau threefolds with infinite fundamental group.

1. INTRODUCTION

The present paper is concerned with the Calabi–Yau threefolds with infinite fundamental group. Throughout the paper, a Calabi–Yau threefold is a smooth complex projective threefold X with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$. Let X be a Calabi–Yau threefold with infinite fundamental group. Then the Bogomolov decomposition theorem [3] implies that X admits an étale Galois covering either by an abelian threefold or by the product of a K3 surface and an elliptic curve. We call X of type A in the former case and of type K in the latter case. Among many candidates for such coverings, we can always find a unique smallest one, up to isomorphism as a covering [4]. We call the smallest covering the minimal splitting covering of X . The main result of this paper is the following:

Theorem 1.1 (Theorem 3.1). *There exist exactly eight Calabi–Yau threefolds of type K, up to deformation equivalence. The equivalence class is uniquely determined by the Galois group G of the minimal splitting covering. Moreover, the Galois group is isomorphic to one of the following combinations of cyclic and dihedral groups*

$$C_2, \ C_2 \times C_2, \ C_2 \times C_2 \times C_2, \ D_6, \ D_8, \ D_{10}, \ D_{12}, \ \text{or} \ C_2 \times D_8.$$

Most Calabi–Yau threefolds we know have finite fundamental groups: for example, complete intersection Calabi–Yau threefolds in toric varieties or homogeneous spaces, and (resolutions of singularities of) finite quotients thereof. Calabi–Yau threefolds with infinite fundamental group were only partially explored before the pioneering work of Oguiso and Sakurai [26]. In their paper, they made a list of possible Galois groups for type K but it was not settled whether they really occur or not. In this paper, we complement

their work by providing the full classification (Theorem 1.1) and also give an explicit presentation for the deformation classes of the eight Calabi–Yau threefolds of type K.

The results described in this paper represent the first step in our program which is aimed at more detailed understanding of Calabi–Yau threefolds of type K. Calabi–Yau threefolds of type K are relatively simple yet rich enough to display the essential complexities of Calabi–Yau geometries, and we expect that they will provide good testing-grounds for general theories and conjectures. Indeed, the simplest example, known as the Enriques Calabi–Yau threefold, has been one of the most tractable compact Calabi–Yau threefolds both in string theory and mathematics (see for example [10, 1, 17, 15, 22]). A particularly nice property of Calabi–Yau threefolds of type K is their fibration structure; they all admit a K3 fibration, an abelian surface fibration, and an elliptic fibration. This rich structure suggests that they play an important role in dualities among various string theories. In the sequel paper [13], we discuss mirror symmetry of Calabi–Yau threefolds of type K.

We will also provide the full classification of Calabi–Yau threefolds of type A, again based on Oguiso and Sakurai’s work [26]. In contrast to Calabi–Yau threefolds of type K, Calabi–Yau threefolds of type A are classified not by the Galois groups of the minimal splitting coverings, but by the *minimal totally splitting coverings*, where abelian threefolds that cover Calabi–Yau threefolds of type A split into the product of three elliptic curves (Theorem 6.2). Together with the classification of Calabi–Yau threefolds of type K, we finally complete the full classification of Calabi–Yau threefolds with infinite fundamental group:

Theorem 1.2 (Theorem 6.4). *There exist exactly fourteen deformation classes of Calabi–Yau threefolds with infinite fundamental group. More precisely, six of them are of type A, and eight of them are of type K.*

It is remarkable that we can study Calabi–Yau threefolds very concretely by simply assuming that their fundamental groups are infinite. Recall that a fundamental gap in the classification of algebraic threefolds is the lack of understanding of Calabi–Yau threefolds. We hope that our results unveil an interesting class of Calabi–Yau threefolds and shed some light on the further investigation of general compact Calabi–Yau threefolds.

Structure of Paper. In Section 2 we recall some basics on lattices and K3 surfaces. Lattice theory will be useful when we study finite automorphism groups on K3 surfaces in later sections. Section 3 is the main part of this paper. It begins with a review of Oguiso and Sakurai’s fundamental work [26], which essentially reduces the study of Calabi–Yau threefolds of type K to that of K3 surfaces equipped with Calabi–Yau actions (Definition 3.6). It then provides the full classification of Calabi–Yau threefolds of type K, presenting all the deformation classes. Section 4 is devoted to the proof of a key technical result, Lemma 3.14 (Key Lemma), which plays a crucial

role in Section 3. Section 5 addresses some basic properties of Calabi–Yau threefolds of type K. Section 6 improves Oguiso and Sakurai’s work on Calabi–Yau threefolds of type A. It finally completes the classification of Calabi–Yau threefolds with infinite fundamental group.

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2. LATTICES AND K3 SURFACES

We begin with a brief summary of the basics of lattices and K3 surfaces. This will also serve to set conventions and notations. Standard references are [2, 24].

2.1. Lattices. A lattice is a free \mathbb{Z} -module L of finite rank together with a symmetric bilinear form $\langle *, ** \rangle: L \times L \rightarrow \mathbb{Z}$. By an abuse of notation, we denote a lattice simply by L . With respect to a choice of basis, the bilinear form is represented by a Gram matrix and the discriminant $\text{disc}(L)$ is the determinant of the Gram matrix. We denote by $O(L)$ the group of automorphisms of L . We define $L(n)$ to be the lattice obtained by multiplying the bilinear form L by an integer n . For $a \in \mathbb{Q}$ we denote by $\langle a \rangle$ the lattice of rank 1 generated by x with $x^2 := \langle x, x \rangle = a$. A lattice L is called even if $x^2 \in 2\mathbb{Z}$ for all $x \in L$. L is non-degenerate if $\text{disc}(L) \neq 0$ and unimodular if $\text{disc}(L) = \pm 1$. If L is a non-degenerate lattice, the signature of L is the pair (t_+, t_-) where t_+ and t_- respectively denote the dimensions of the positive and negative eigenvalues of $L \otimes \mathbb{R}$. We define $\text{sign } L := t_+ - t_-$.

A sublattice M of a lattice L is a submodule of L with the bilinear form of L restricted to M . A sublattice M of L is called primitive if L/M is torsion free. For a sublattice M of L , we denote the orthogonal complement of M in L by M_L^\perp (or simply M^\perp). An action of a group G on a lattice L preserves the bilinear form unless otherwise stated and we define the invariant part L^G and the coinvariant part L_G of L by

$$L^G := \{x \in L \mid g \cdot x = x \ (\forall g \in G)\}, \quad L_G := (L^G)_L^\perp.$$

We simply denote $L^{\langle g \rangle}$ and $L_{\langle g \rangle}$ by L^g and L_g respectively for $g \in G$. If another group H acts on L , we denote $L^G \cap L_H$ by L_H^G .

The hyperbolic lattice U is the rank 2 lattice whose Gram matrix is given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the corresponding basis e, f is called the standard basis.

Let A_m , D_n , E_l , ($m \geq 1$, $n \geq 4$, $l = 6, 7, 8$) be the lattices defined by the corresponding Cartan matrices. Every indefinite even unimodular lattice can

be realized as an orthogonal sum of copies of U and $E_8(\pm 1)$ in an essentially unique way, the only relation being $E_8 \oplus E_8(-1) \cong U^{\oplus 8}$. Thus an even unimodular lattice of signature $(3, 19)$ is isomorphic to $\Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, which is called the K3 lattice.

Let L be a non-degenerate even lattice. The bilinear form determined a canonical embedding $L \hookrightarrow L^\vee := \text{Hom}(L, \mathbb{Z})$. The discriminant group $A(L) := L^\vee/L$ is an abelian group of order $|\text{disc}(L)|$ equipped with a quadratic map $q(L): A(L) \rightarrow \mathbb{Q}/2\mathbb{Z}$ by sending $x + L \mapsto x^2 + 2\mathbb{Z}$. Two even lattices L and L' have isomorphic discriminant form if and only if they are stably equivalent, that is, $L \oplus K \cong L' \oplus K'$ for some even unimodular lattices K and K' . Since the rank of an even unimodular is divisible by 8, $\text{sign } q(L) := \text{sign } L \bmod 8$ is well-defined. Let $M \hookrightarrow L$ be a primitive embedding of non-degenerate even lattices and suppose that L is unimodular, then there is a natural isomorphism $(A(M), q(M)) \cong (A(M^\perp), -q(M^\perp))$. The genus of L is defined as the set of isomorphism classes of lattices L' such that the signature of L' is the same as that of L and $(A(L), q(L)) \cong (A(L'), q(L'))$.

Theorem 2.1 ([24, 28]). *Let L be a non-degenerate even lattice with $\text{rank } L \geq 3$. If $L \cong U(n) \oplus L'$ for a positive integer n and a lattice L' , then the genus of L consists of only one class.*

Let L be a lattice and M a module such that $L \subset M \subset L^\vee$. We say that M equipped with the induced bilinear form $\langle *, ** \rangle$ is an overlattice of L if $\langle *, ** \rangle$ takes integer values on M . Any lattice which includes L as a sublattice of finite index is considered as an overlattice of L .

Proposition 2.2 ([24]). *Let L be a non-degenerate even lattice and M a submodule of L^\vee such that $L \subset M$. Then M is an even overlattice of L if and only if the image of M in $A(L)$ is an isotropic subgroup, that is, the restriction of $q(L)$ to M/L is zero. Moreover, there is a natural one-to-one correspondence between the set of even overlattices of L and the set of isotropic subgroups of $A(L)$.*

Proposition 2.3 ([24]). *Let K and L be non-degenerate even lattices. Then there exists a primitive embedding of K into an even unimodular lattice Γ such that $K^\perp \cong L$, if and only if $(A(K), q(K)) \cong (A(L), -q(L))$. More precisely, any such Γ is of the form $\Gamma_\lambda \subset K^\vee \oplus L^\vee$ for some isomorphism*

$$\lambda: (A(K), q(K)) \rightarrow (A(L), -q(L)),$$

where Γ_λ is the lattice corresponding to the isotropic subgroup

$$\{(x, \lambda(x)) \in A(K) \oplus A(L) \mid x \in A(K)\} \subset A(K) \oplus A(L).$$

Lemma 2.4. *Let L be a non-degenerate lattice and $\iota \in \text{O}(L)$ an involution. Then $L/(L^\iota \oplus L_\iota) \cong (\mathbb{Z}/2\mathbb{Z})^n$ for some $n \leq \min\{\text{rank } L^\iota, \text{rank } L_\iota\}$.*

Proof. For any $x \in L$, we have a decomposition $x = x_+ + x_-$ with $x_+ \in L^\iota \otimes \mathbb{Q}$ and $x_- \in L_\iota \otimes \mathbb{Q}$. We have $2x_+ = x + \iota(x) \in L$. We define $\phi(x \bmod L^\iota \oplus L_\iota) = 2x_+ \bmod 2L^\iota$. We can easily see that $\phi: L/(L^\iota \oplus L_\iota) \rightarrow$

$L^\iota/2L^\iota$ is a well-defined injection. Hence we have $L/(L^\iota \oplus L_\iota) \cong (\mathbb{Z}/2\mathbb{Z})^n$ with $n \leq \text{rank } L^\iota$. Similarly, we have $n \leq \text{rank } L_\iota$. \square

2.2. K3 Surfaces. A K3 surface S is a simply-connected compact complex surface with trivial canonical bundle. Then $H^2(S, \mathbb{Z})$ with its intersection form is isomorphic to the K3 lattice $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. It is endowed with a weight-two Hodge structure

$$H^2(S, \mathbb{C}) = H^2(S, \mathbb{Z}) \otimes \mathbb{C} = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S).$$

Let ω_S be a nowhere vanishing holomorphic 2-form on S . The space $H^{2,0}(S) \cong \mathbb{C}$ is generated by the class of ω_S , which we denote by the same ω_S . The Néron–Severi lattice $\text{NS}(S)$ is given by

$$(2.1) \quad \text{NS}(S) := \{x \in H^2(S, \mathbb{Z}) \mid \langle x, \omega_S \rangle = 0\}.$$

Here we extend the bilinear form $\langle *, * \rangle$ on $H^2(S, \mathbb{Z})$ to that on $H^2(S, \mathbb{C})$ linearly. The open subset $\mathcal{K}_S \subset H^{1,1}(S, \mathbb{R})$ of Kähler classes is called the Kähler cone of S . It is known that \mathcal{K}_S is a connected component of

$$\{x \in H^{1,1}(S, \mathbb{R}) \mid x^2 > 0, \langle x, \delta \rangle \neq 0 \ (\forall \delta \in \Delta_S)\}, \quad \Delta_S := \{\delta \in \text{NS}(S) \mid \delta^2 = -2\}.$$

The study of K3 surfaces reduces to lattice theory by the following two theorems.

Theorem 2.5 (Global Torelli Theorem [27, 6]). *Let S and T be K3 surfaces. Let $\phi: H^2(S, \mathbb{Z}) \rightarrow H^2(T, \mathbb{Z})$ be an isomorphism of lattices satisfying the following conditions.*

- (1) $(\phi \otimes \mathbb{C})(H^{2,0}(S)) = H^{2,0}(T)$.
- (2) *There exists an element $\kappa \in \mathcal{K}_S$ such that $(\phi \otimes \mathbb{R})(\kappa) \in \mathcal{K}_T$.*

Then there exists a unique isomorphism $f: T \rightarrow S$ such that $f^ = \phi$.*

Theorem 2.6 (Surjectivity of the period map [30]). *Assume that vectors $\omega \in \Lambda \otimes \mathbb{C}$ and $\kappa \in \Lambda \otimes \mathbb{R}$ satisfy the following conditions:*

- (1) $\langle \omega, \omega \rangle = 0$, $\langle \omega, \bar{\omega} \rangle > 0$, $\langle \kappa, \kappa \rangle > 0$ and $\langle \kappa, \omega \rangle = 0$.
- (2) $\langle \kappa, x \rangle \neq 0$ for any $x \in (\omega)_\Lambda^\perp$ such that $\langle x, x \rangle = -2$.

Then there exist a K3 surface S and an isomorphism $\alpha: H^2(S, \mathbb{Z}) \rightarrow \Lambda$ of lattices such that $\mathbb{C}\omega = (\alpha \otimes \mathbb{C})(H^{2,0}(S))$ and $\kappa \in (\alpha \otimes \mathbb{R})(\mathcal{K}_S)$.

An action of a group G on a K3 surface S induces a (left) G -action on $H^2(S, \mathbb{Z})$ by

$$g \cdot x := (g^{-1})^* x, \quad g \in G, \ x \in H^2(S, \mathbb{Z}).$$

The following lemma is useful to study finite group actions on a K3 surface.

Lemma 2.7 ([26, Lemma 1.7]). *Let S be a K3 surface with an action of a finite group G and let x be an element in $\text{NS}(S)^G \otimes \mathbb{R}$ with $x^2 > 0$. Suppose that $\langle x, \delta \rangle \neq 0$ for any $\delta \in \text{NS}(S)$ with $\delta^2 = -2$. Then there exists $\gamma \in \text{O}(H^2(S, \mathbb{Z}))$ such that $\gamma(H^{2,0}(S)) = H^{2,0}(S)$, $\gamma(x) \in \mathcal{K}_S$, and γ commutes with G .*

An automorphism g of a K3 surface S is said to be symplectic if $g^*\omega_S = \omega_S$, and anti-symplectic if $g^*\omega_S = -\omega_S$.

Theorem 2.8 ([23]). *Let g be a symplectic automorphism of S of finite order, then $\text{ord}(g) \leq 8$ and the number of fixed points depends only on $\text{ord}(g)$ as given in the following table.*

$\text{ord}(g)$	2	3	4	5	6	7	8
$ S^g $	8	6	4	4	2	3	2

A fixed point free involution ι of a K3 surface (necessarily anti-symplectic) is called an Enriques involution. The quotient surface $S/\langle\iota\rangle$ is called an Enriques surface.

Theorem 2.9 ([2, Section VIII.19]). *An involution ι of a K3 surface S is an Enriques involution if and only if*

$$H^2(S, \mathbb{Z})^\iota \cong U(2) \oplus E_8(-2), \quad H^2(S, \mathbb{Z})_\iota \cong U \oplus U(2) \oplus E_8(-2).$$

An example of a K3 surface with an Enriques involution we should keep in mind is:

Example 2.10 (Horikawa model [2, Section V.23]). *The double covering $\pi: S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ branching along a smooth curve B of bidegree $(4, 4)$ is a K3 surface. We denote by θ the covering involution on S . Assume that B is invariant under the involution ι of $\mathbb{P}^1 \times \mathbb{P}^1$ given by $(x, y) \mapsto (-x, -y)$, where x and y are the inhomogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$. The involution ι lifts to a symplectic involution of the K3 surface S . Then $\theta \circ \iota$ is an involution of S without fixed points unless B passes through one of the four fixed points of ι on $\mathbb{P}^1 \times \mathbb{P}^1$. The quotient surface $T = S/\langle\theta \circ \iota\rangle$ is therefore an Enriques surface.*

$$\begin{array}{ccc} S & \xrightarrow{\text{id}} & S \\ \downarrow / \langle \theta \rangle & & \downarrow / \langle \theta \circ \iota \rangle \\ \mathbb{P}^1 \times \mathbb{P}^1 & & T \end{array}$$

Proposition 2.11 ([2, Proposition XIII.18.1]). *Any generic K3 surface with an Enriques involution is realized as a Horikawa model defined above.*

3. CLASSIFICATION

The Bogomolov decomposition theorem [3] implies that a Calabi–Yau threefold X with infinite fundamental group admits an étale Galois covering either by an abelian threefold or by the product of a K3 surface and an elliptic curve. We call X of type A in the former case and of type K in the latter case. Among many candidates of such coverings, we can always find a unique smallest one, up to isomorphism as a covering [4]. We call the smallest covering the minimal splitting covering of X . The goal of this section is to provide the following classification theorem of Calabi–Yau threefolds of type K:

Theorem 3.1. *There exist exactly eight Calabi–Yau threefolds of type K, up to deformation equivalence. The equivalence class is uniquely determined by the Galois group G of the minimal splitting covering. Moreover, the Galois group is isomorphic to one of the following combinations of cyclic and dihedral groups*

$$C_2, C_2 \times C_2, C_2 \times C_2 \times C_2, D_6, D_8, D_{10}, D_{12}, \text{ or } C_2 \times D_8.$$

We will also give an explicit presentation for the eight Calabi–Yau threefolds. For the reader’s convenience, we outline the proof of Theorem 3.1. Firstly, by the work of Oguiso and Sakurai [26], the classification of Calabi–Yau threefolds of type K essentially reduces to that of K3 surfaces S equipped with a Calabi–Yau action (Definition 3.6) of a finite group of the form $G = H \rtimes \langle \iota \rangle$. Here the action of H on S is symplectic and ι is an Enriques involution. A sketch of the proof of the classification of such K3 surfaces is the following:

- (1) We make a list of examples of K3 surfaces S with a Calabi–Yau G -action. They are given as double coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ (H -equivariant Horikawa models).
- (2) For a K3 surface S with a Calabi–Yau G -action, it is proven that there exists an element $v \in \text{NS}(S)^G$ such that $v^2 = 4$ (Key Lemma).
- (3) It is shown that S has a projective model of degree 4 and admits a G -equivariant double covering of a quadric hypersurface in \mathbb{P}^3 isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if it is smooth. Therefore, S is generically realized as an H -equivariant Horikawa model constructed in Step 1.
- (4) We classify the deformation equivalence classes of S on a case-by-case basis and also exclude an unrealizable Galois group.

It is worth noting that a realization of a K3 surface S with a Calabi–Yau G -action as a Horikawa model is in general not unique (Propositions 3.11 and 3.13).

3.1. Work of Oguiso and Sakurai. We begin with a brief review of Oguiso and Sakurai’s work [26]. Let X be a Calabi–Yau threefold of type K. Then the minimal splitting covering $\pi: S \times E \rightarrow X$ is obtained by imposing the condition that the Galois group of the covering π does not contain any elements of the form $(\text{id}_S, \text{non-zero translation of } E)$.

Definition 3.2. *We call a finite group G a Calabi–Yau group if there exist a K3 surface S , an elliptic curve E and a faithful G -action on $S \times E$ such that the following conditions hold.*

- (1) G contains no elements of the form $(\text{id}_S, \text{non-zero translation of } E)$.
- (2) The G -action on $H^{3,0}(S \times E) \cong \mathbb{C}$ is trivial.
- (3) The G -action is free, that is, $(S \times E)^g = \emptyset$ for all $g \in G$, $g \neq 1$.
- (4) G does not preserve any holomorphic 1-form, that is, $H^0(S \times E, \Omega_{S \times E})^G = 0$.

We call $S \times E$ a target threefold of G .

The Galois group G of the minimal splitting covering $S \times E \rightarrow X$ of a Calabi–Yau threefold X of type K is a Calabi–Yau group. Conversely, if G is a Calabi–Yau group with a target space $S \times E$ of G , then the quotient $(S \times E)/G$ is a Calabi–Yau threefold of type K.

Let G be a Calabi–Yau group and $S \times E$ a target threefold of G . Thanks to a result of Beauville [4], we have a canonical isomorphism $\text{Aut}(S \times E) \cong \text{Aut}(S) \times \text{Aut}(E)$. The images of $G \subset \text{Aut}(S \times E)$ under the two projections to $\text{Aut}(S)$ and $\text{Aut}(E)$ are denoted by G_S and G_E respectively. It can be proven that $G_S \cong G \cong G_E$ via the natural projections:

$$\begin{array}{ccccc} \text{Aut}(S) & \xleftarrow{p_1} & \text{Aut}(S \times E) & \xrightarrow{p_2} & \text{Aut}(E) \\ \cup & & \cup & & \cup \\ G_S & \xleftarrow[p_1|_G]{\cong} & G & \xrightarrow[p_2|_G]{\cong} & G_E. \end{array}$$

We denote by g_S and g_E the elements in G_S and G_E respectively corresponding to $g \in G$, that is, $p_1(g) = g_S$, $p_2(g) = g_E$.

Proposition 3.3 ([26, Lemma 2.28]). *Let G be a Calabi–Yau group and $S \times E$ a target threefold of G . Define $H := \text{Ker}(G \rightarrow \text{GL}(H^{2,0}(S)))$ and take any $\iota \in G \setminus H$. Then the following hold.*

- (1) $\text{ord}(\iota) = 2$ and $G = H \rtimes \langle \iota \rangle$, where the semi-direct product structure is given by $\iota h \iota = h^{-1}$ for all $h \in H$.
- (2) g_S is an Enriques involution for any $g \in G \setminus H$.
- (3) $\iota_E = -\text{id}_E$ and $H_E = \langle t_a \rangle \times \langle t_b \rangle \cong C_n \times C_m$ under an appropriate origin of E , where t_a and t_b are translations of order n and m respectively such that $n|m$. Moreover we have $(n, m) \in \{(1, k) \mid 1 \leq k \leq 6\}, (2, 2), (2, 4), (3, 3)\}$.

Although the case $(n, m) = (2, 4)$ is eliminated from the list of possible Calabi–Yau groups in [26], there is an error in the proof of Lemma 2.29 in [26], which is used to prove the proposition above¹. In fact, there exists a Calabi–Yau group of the form $(C_2 \times C_4) \rtimes C_2 \cong C_2 \times D_8$ (Proposition 3.11). For the sake of completeness, here we settle the proof of Lemma 2.29 in [26]. We do not repeat the whole argument but give a proof of the non-trivial part: (n, m) cannot be $(1, 7), (1, 8), (2, 6)$ nor $(4, 4)$. The reader can skip this part, assuming Proposition 3.3.

Proof of $(n, m) \neq (1, 7), (1, 8), (2, 6), (4, 4)$.

- (1) $(1, 7)$: Let $H_S = \langle g \rangle \cong C_7$. Since $\iota g \iota = g^{-1}$, $\langle \iota \rangle \cong C_2$ acts on S^g , which has cardinality 3 and thus has a fixed point. But this contradicts with the fact that $S^f = \emptyset$ for any $f \in G_S \setminus H_S$.
- (2) $(1, 8)$: Let $H_S = \langle g \rangle \cong C_8$. Note that $\langle g, \iota \rangle / \langle g^2 \rangle \cong C_2 \times C_2$ and acts on $S^{g^2} \setminus S^g$ which has cardinality $4 - 2 = 2$. Then this action induces a

¹The error in [26, Lemma 2.29] is that, with their notation, the group $\langle \alpha, h_S^2 \rangle$ is not necessarily isomorphic to either $C_2 \times C_2$ or $C_2 \times C_4$, but may be isomorphic to C_4 .

homomorphism $\phi: C_2 \times C_2 \rightarrow S_2$. Since $S^f = \emptyset$ for all $f \in G_S \setminus H_S$, $\text{Ker}(\phi) (\neq 1) \subset \langle g \rangle / \langle g^2 \rangle \cong C_2$ and thus $\text{Ker}(\phi) \cong \langle g \rangle / \langle g^2 \rangle$. This contradicts with our subtracting S^g from S^{g^2} .

- (3) (2, 6) : Let $H_S = \langle g \rangle \times \langle h \rangle \cong C_2 \times C_6$. $|S^h| = 2$ and thus there is a homomorphism $\phi: \langle g, \iota \rangle \cong C_2 \times C_2 \rightarrow S_2$. Since $S^f = \emptyset$ for all $f \in G_S \setminus H_S$, $\text{Ker}(\phi) = \langle g \rangle \cong C_2$. Let $p \in S^g$ be one of the fixed points. Then we have a faithful representation $H_S = C_2 \times C_6 \rightarrow \text{SL}(T_p S) \cong \text{SL}(2, \mathbb{C})$. This contradicts with the classification of finite subgroups in $\text{SL}(2, \mathbb{C})$.
- (4) (4, 4) : Let $H_S = \langle g \rangle \times \langle h \rangle \cong C_4 \times C_4$. Note that $\langle g, h, \iota \rangle / \langle g^2, h^2 \rangle \cong C_2 \times C_2 \times C_2$ and acts on S^{h^2} . Note also that $|S^{h^2} \setminus S^h| = 8 - 4 = 4$. Then this induces a homomorphism $\phi: C_2 \times C_2 \times C_2 \rightarrow S_4$. Since S_4 does not contain $C_2 \times C_2 \times C_2$, $\text{Ker}(\phi) \subset \langle g, h \rangle / \langle g^2, h^2 \rangle$ is not trivial. Let α be a lift of a non-trivial element of $\text{Ker}(\phi)$ and take a fixed point $p \in S^{h^2} \setminus S^h$. Then we have a natural injection $\langle \alpha, h^2 \rangle \rightarrow \text{SL}(T_p S) \cong \text{SL}(2, \mathbb{C})$. In addition using $h \notin \text{Ker}(\phi)$, we obtain $\langle \alpha, h^2 \rangle \cong C_4 \times C_2$, which contradicts with the classification of finite subgroups in $\text{SL}(2, \mathbb{C})$.

□

Now we can state a main result of [26] with a slight correction.

Theorem 3.4 ([26, Theorem 2.23]). *Let X be a Calabi–Yau threefold of type K. Let $S \times E \rightarrow X$ be the minimal splitting covering and G its Galois group. Then the following hold.*

- (1) *G is isomorphic to one of the following:*

C_2 , $C_2 \times C_2$, $C_2 \times C_2 \times C_2$, D_6 , D_8 , D_{10} , D_{12} , $C_2 \times D_8$, or $(C_3 \times C_3) \rtimes C_2$.

- (2) *In each case the Picard number $\rho(X)$ of X is uniquely determined by G and is calculated as $\rho(X) = 11, 7, 5, 5, 4, 3, 3, 3, 3$ respectively.*

- (3) *The cases $G \cong C_2$, $C_2 \times C_2$, $C_2 \times C_2 \times C_2$ really occur.*

It has not been settled yet whether or not there exist Calabi–Yau threefolds of type K with Galois group $G \cong D_{2n}$ ($3 \leq n \leq 6$), $C_2 \times D_8$ or $(C_3 \times C_3) \rtimes C_2$. Note that the example of a Calabi–Yau threefold of type K with $G \cong D_8$ presented in Proposition 2.33 in [26] is incorrect². We will settle this classification problem of Calabi–Yau threefolds of type K and also give an explicit presentation of the deformation classes.

Example 3.5 (Enriques Calabi–Yau threefold). *Let S be a K3 surface with an Enriques involution ι and E an elliptic curve with the negation -1_E . The free quotient*

$$X := (S \times E) / \langle (\iota, -1_E) \rangle$$

is the simplest Calabi–Yau threefold of type K, known as the Enriques Calabi–Yau threefold.

²The error in [26, Proposition 2.33] is that, with their notation, $S^{ab} \neq \emptyset$.

3.2. Construction. The goal of this section is to make a list of concrete examples of Calabi–Yau threefolds of type K. We will later show that the list covers all the generic Calabi–Yau threefolds of type K. We begin with the definition of *Calabi–Yau actions*, which is based on Proposition 3.3.

Definition 3.6. *Let G be a finite group. We say that an action of G on a K3 surface S is a Calabi–Yau action if the following hold.*

- (1) $G = H \rtimes \langle \iota \rangle$ for some $H \cong C_n \times C_m$ with $(n, m) \in \{(1, k) \mid 1 \leq k \leq 6\}, (2, 2), (2, 4), (3, 3)\}$, and ι with $\text{ord}(\iota) = 2$. The semi-direct product structure is given by $\iota h = h^{-1}$ for all $h \in H$.
- (2) H acts on S symplectically and any $g \in G \setminus H$ acts on S as an Enriques involution.

Recall that a generic K3 surface with the simplest Calabi–Yau action, namely an Enriques involution, is realized as a Horikawa model (Proposition 2.11). We will see that a K3 surface equipped with a Calabi–Yau G -action is realized as an H -equivariant Horikawa model (Proposition 3.11).

We can construct a Calabi–Yau G -action on a K3 surface as follows. Let u, x, y, z, w be affine coordinates of $\mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^2$. Define L by

$$L := (\mathbb{C} \times (\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})) / (\mathbb{C}^\times)^2,$$

where the action of $(\mu_1, \mu_2) \in (\mathbb{C}^\times)^2$ is given by

$$(u, x, y, z, w) \mapsto (\mu_1^2 \mu_2^2 u, \mu_1 x, \mu_1 y, \mu_2 z, \mu_2 w).$$

The projection $\mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}^2 \times \mathbb{C}^2$ descends to the map

$$\pi: L \rightarrow Z := ((\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})) / (\mathbb{C}^\times)^2 \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Note that L is naturally identified with the total space of $\mathcal{O}_Z(2, 2)$. Let $F = F(x, y, z, w) \in H^0(\mathcal{O}_Z(4, 4))$ be a homogeneous polynomial of bidegree $(4, 4)$. Assume that the curve $B \subset Z$ defined by $F = 0$ has at most ADE-singularities. We define S_0 by

$$S_0 := \{u^2 = F\} \subset L.$$

In other words, S_0 is a double covering of Z branching along B . The minimal resolution S of S_0 is a K3 surface (see the proof of Lemma 3.9 below). The group $\Gamma := \text{GL}(2, \mathbb{C}) \times \text{GL}(2, \mathbb{C})$ acts on L by, for $\gamma = M_1 \times M_2 \in \Gamma$,

$$\gamma(u, x, y, z, w) = (u, x', y', z', w'), \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = M_1 \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} z' \\ w' \end{bmatrix} = M_2 \begin{bmatrix} z \\ w \end{bmatrix}.$$

If F is invariant under the action of $\gamma \in \Gamma$, then γ naturally acts on S_0 as well. We denote by γ^+ the induced action of γ on S . The covering transformation of $S_0 \rightarrow Z$, which is defined by $(u, x, y, z, w) \mapsto (-u, x, y, z, w)$, induces the involution $j := (\sqrt{-1} I_2 \times I_2)^+$ on S . Note that there are two lifts of the action of γ on Z : γ^+ and $\gamma^- := \gamma^+ j$. The K3 surface S with the polarization $\mathcal{L} := \pi^* \mathcal{O}_Z(1, 1)$ is a polarized K3 surface of degree 4. Let $\text{Aut}(S, \mathcal{L})$ denote the automorphism group of the polarized K3 surface (S, \mathcal{L}) .

Definition 3.7. We denote by (S, \mathcal{L}, G) a triplet consisting of a polarized K3 surface (S, \mathcal{L}) defined above and a finite subgroup $G \subset \text{Aut}(S, \mathcal{L})$.

We define $\text{Aut}(S, \mathcal{L})^+$ to be the subgroup of $\text{Aut}(S, \mathcal{L})$ preserving each ruling $Z \rightarrow \mathbb{P}^1$. Then we have

$$(3.1) \quad \text{Aut}(S, \mathcal{L})^+ = \{\gamma \in \Gamma \mid \gamma^*F = F\} / \{\lambda_1 I_2 \times \lambda_2 I_2 \mid \lambda_1^2 \lambda_2^2 = 1\}.$$

Remark 3.8. Since the Picard group of Z is isomorphic to $\mathbb{Z}^{\oplus 2}$ (hence torsion free), a line bundle \mathcal{M} on Z such that $\mathcal{M}^{\otimes 2} \cong \mathcal{O}_Z(4, 4)$ is isomorphic to $\mathcal{O}_Z(2, 2)$. Therefore, a double covering of Z branching along B is unique and isomorphic to S_0 .

Lemma 3.9. Let ω_S be a nowhere vanishing holomorphic 2-form on S . If F is invariant under the action of $\gamma = M_1 \times M_2 \in \Gamma$, we have

$$(\gamma^\pm)^* \omega_S = \pm \det(M_1) \det(M_2) \omega_S.$$

Proof. Since $(xdy - ydx) \wedge (zdw - wdz)/u$ gives a nowhere vanishing holomorphic 2-form on S , the equality in the assumption holds. \square

Lemma 3.10. Let $\phi: Y \rightarrow Y_0$ be the minimal resolution of a surface Y_0 with at most ADE-singularities. Then an automorphism g of Y_0 has a fixed point if and only if the induced action of g on Y has a fixed point.

Proof. The assertion follows from the fact that any automorphism of a connected ADE-configuration has a fixed point. \square

Proposition 3.11. Let (S, \mathcal{L}, G) be a triplet defined in Definition 3.7. Assume that the action of G on S is a Calabi-Yau action. Then such triplets (S, \mathcal{L}, G) are classified into the types in the following table up to isomorphism. Here two triplets (S, \mathcal{L}, G) and (S', \mathcal{L}', G') are isomorphic if there exists an isomorphism $f: S \rightarrow S'$ such that $f^* \mathcal{L}' = \mathcal{L}$ and $f^{-1} \circ G' \circ f = G$.

H	Ξ	basis of $H^0(\mathcal{O}_Z(4, 4))^G$
C_1	\emptyset	$x^i y^{4-i} z^j w^{4-j} + x^{4-i} y^i z^{4-j} w^j$
C_2	$\{(1/4, 1/4)\}$	$x^i y^{4-i} z^j w^{4-j} + x^{4-i} y^i z^{4-j} w^j \ (i \equiv j \pmod{2})$
C_2	$\{(1/4, 0)\}$	$x^i y^{4-i} z^j w^{4-j} + x^{4-i} y^i z^{4-j} w^j \ (i \equiv 0 \pmod{2})$
C_3	$\{(1/3, 1/3)\}$	$x^i y^{4-i} z^j w^{4-j} + x^{4-i} y^i z^{4-j} w^j \ (i + j \equiv 1 \pmod{3})$
C_4	$\{(1/8, 1/8)\}$	$x^4 z^4 + y^4 w^4, x^4 w^4 + y^4 z^4,$ $x^3 y z w^3 + x y^3 z^3 w, x^2 y^2 z^2 w^2$
C_4	$\{(1/8, 1/4)\}$	$x^4 z^3 w + y^4 z w^3, x^4 z w^3 + y^4 z^3 w,$ $x^2 y^2 z^4 + x^2 y^2 w^4, x^2 y^2 z^2 w^2$
C_5	$\{(1/5, 2/5)\}$	$x^4 z w^3 + y^4 z^3 w, x^3 y z^4 + x y^3 w^4, x^2 y^2 z^2 w^2$
C_6	$\{(1/12, 1/6)\}$	$x^4 z^4 + y^4 w^4, x^4 z w^3 + y^4 z^3 w, x^2 y^2 z^2 w^2$
$C_2 \times C_2$	$\{(1/4, 0), (0, 1/4)\}$	$x^i y^{4-i} z^j w^{4-j} + x^{4-i} y^i z^{4-j} w^j \ (i \equiv j \equiv 0 \pmod{2})$
$C_2 \times C_4$	$\{(1/8, 1/8), (0, 1/4)\}$	$x^4 z^4 + y^4 w^4, x^4 w^4 + y^4 z^4, x^2 y^2 z^2 w^2$

A triplet (S, \mathcal{L}, G) for each type is defined as follows.

- (1) F is invariant under the action of $M(a) \times M(b)$ for all $(a, b) \in \Xi$ and ι_1 , where

$$M(a) := \begin{bmatrix} \exp(2\pi ia) & 0 \\ 0 & \exp(-2\pi ia) \end{bmatrix}, \iota_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

- (2) $H := \langle (M(a) \times M(b))^+ \mid (a, b) \in \Xi \rangle$.

- (3) $G := H \rtimes \langle \iota \rangle$, $\iota := \iota_1^- = \left(\sqrt{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^+$.

- (4) for any $g \in G \setminus H$, the action of g on B has no fixed point.

Furthermore, for a generic $F \in H^0(\mathcal{O}_Z(4, 4))^G$, the surface S_0 is smooth and the condition (4) is satisfied, and thus the action of G on $S_0 (= S)$ is a Calabi–Yau action.

Remark 3.12. In Proposition 3.11, the group $G \subset \text{Aut}(S, \mathcal{L})$ for each type acts on $H^0(\mathcal{O}_Z(4, 4))$ in a natural way by using the generator matrices. Hence we can define the G -invariant space $H^0(\mathcal{O}_Z(4, 4))^G$. We will use a similar convention in Proposition 3.13.

Proof. Let (S, \mathcal{L}, G) be a triplet such that the action of $G = H \rtimes \langle \iota \rangle$ on S is a Calabi–Yau action. By Definition 3.6, H is isomorphic to one in the table or $C_3 \times C_3$. For $g \in G \setminus H$, the action of g preserves each ruling $Z \rightarrow \mathbb{P}^1$; otherwise, Z^g is 1-dimensional and $S^g \neq \emptyset$. Since G is generated by $G \setminus H$, we may assume that any element in G is of the form γ^\pm for $\gamma \in \Gamma$ with $\gamma^*F = F$ by (3.1). By Lemma 3.10, the condition (4) is satisfied.

We begin with the case $H = C_1$. Since $S^\iota = \emptyset$, it follows that Z^ι is (at most) 0-dimensional. Hence we may assume that

$$\iota = \left(\lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^+$$

after changing the coordinates of Z . Since the action of ι on S is anti-symplectic by Theorem 2.9, we have $\lambda = \pm\sqrt{-1}$ by Lemma 3.9, thus $\iota = \iota_1^-$.

Let us next consider the case $H = C_n$ ($1 \leq n \leq 6$). Let σ be a generator of H . By the argument above and the relation $\iota\sigma\iota = \sigma^{-1}$, we may assume that $\iota = \iota_1^-$ and $\sigma = (\lambda M(a) \times M(b))^+$ for some $a, b \in \mathbb{Q}$ after changing the coordinates of Z . Since the action of σ on S is symplectic, we have $\lambda = \pm 1$ by Lemma 3.9, thus $\sigma = (M(a) \times M(b))^+$. If $ka \notin \frac{1}{4}\mathbb{Z}$ and $kb \in \frac{1}{2}\mathbb{Z}$ for some $k \in \mathbb{Z}$, then F is divisible by x^2y^2 . Hence this case is excluded. We can see that (a, b) , (b, a) , $(-a, b)$, $(a + 1/2, b)$, and (ka, kb) with $\text{GCD}(k, n) = 1$ give isomorphic triplets. Therefore, we may assume that (a, b) is one of the following:

$$(1/4, 1/4), (1/4, 0), (1/3, 1/3), (1/8, 1/8), (1/8, 1/4), \\ (1/5, 1/5), (1/5, 2/5), (1/12, 1/12), (1/12, 1/6).$$

Here we have $n = \min\{k \in \mathbb{Z}_{>0} \mid ka \in \frac{1}{2}\mathbb{Z}, kb \in \frac{1}{2}\mathbb{Z}\}$. Suppose that $(a, b) = (1/5, 1/5)$ or $(1/12, 1/12)$. Then F is a linear combination of

$$x^4w^4 + y^4z^4, x^3yzw^3 + xy^3z^3w, x^2y^2z^2w^2,$$

and hence B has a singular point of multiplicity 4 at, for instance, $[1 : 0] \times [1 : 0]$. This contradicts to the assumption that B admits only ADE-singularities. Therefore the cases $(a, b) = (1/5, 1/5)$ and $(1/12, 1/12)$ are excluded.

Lastly, let us consider the cases $H = C_2 \times C_2$, $C_2 \times C_4$, and $C_3 \times C_3$. Let σ, τ be generators of H such that $\text{ord}(\sigma)$ is divisible by $\text{ord}(\tau)$. By a similar argument for $H = C_n$, $n = \text{ord}(\sigma)$, we may assume that

$$\sigma = (M(a) \times M(b))^+, \tau = \left(\lambda M(a') \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^e \times M(b') \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^f \right)^+, \iota = \iota_1^-,$$

where $e, f \in \{0, 1\}$, $\lambda^2(-1)^{e+f} = 1$ and $a, b, a', b' \in \mathbb{Q}$. We can chose (a, b) as is given in the table for $H = C_n$. Moreover, we may assume that $a' = 0$ after replacing τ by $\sigma^k \tau$ for some $k \in \mathbb{Z}$. By the argument for $H = C_1$, it follows that Z^{h_u} is 0-dimensional for $h \in H$. This implies that $e = 0$, and $f = 0$ if $(a, b) \neq (1/4, 0)$. We may assume that one of the following cases occurs.

	(a, b)	(a', b')	(e, f)
(i)	$(1/4, 0)$	$(0, 1/4)$	$(0, 0)$
(ii)	$(1/4, 0)$	$(0, 1/4)$	$(0, 1)$
(iii)	$(1/8, 1/8)$	$(0, 1/4)$	$(0, 0)$
(iv)	$(1/8, 1/4)$	$(0, 1/4)$	$(0, 0)$
(v)	$(1/3, 1/3)$	$(0, 1/3)$	$(0, 0)$

We can check that the cases (i) and (ii) for $H = C_2 \times C_2$ give isomorphic triplets. Since F is divisible by x^2y^2 in the cases (iv) and (v), these cases are excluded.

Conversely, we check that the action of G on S is a Calabi–Yau action for (S, \mathcal{L}, G) of each type. By the argument above, the action of H on S is symplectic. Let $g \in G \setminus H$. Then $B^g = \emptyset$ and Z^g is 0-dimensional, thus S^g is either empty or 0-dimensional. Recall that the fixed locus of an anti-symplectic involution of a K3 surface is 1-dimensional if it is not empty (see Section 2.2). This implies that $S^g = \emptyset$. Therefore, the action of G on S is a Calabi–Yau action.

To show the smoothness of S_0 for a generic F , it suffices, by Bertini’s theorem, to show that B is smooth on the base locus of the linear system defined by $H^0(\mathcal{O}_Z(4, 4))^G$. We can check this directly. We also find that the action of any $g \in G \setminus H$ on B has no fixed point for a generic F . \square

For $H = C_2$ or C_4 , we obtain two families of K3 surfaces with a Calabi–Yau action of $G = H \rtimes \langle \iota \rangle$ by Proposition 3.11. As we will see in Proposition 3.13 below, if we forget the polarizations, they are essentially and generically

the same families of K3 surfaces with a Calabi–Yau G -action. Let

$$Q = Q(s_1, t_1, s_2, t_2, s_3, t_3) \in H^0(\mathcal{O}_{\mathbb{P}}(2, 2, 2)), \quad \mathbb{P} := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

be a homogeneous polynomial of tridegree $(2, 2, 2)$, where $[s_i : t_i]$ is homogeneous coordinates of the i -th \mathbb{P}^1 . Assume that the surface

$$S_0 := \{Q = 0\} \subset \mathbb{P}$$

has at most ADE-singularities. Then the minimal resolution S of S_0 is a K3 surface. Let $p_i: \mathbb{P} \rightarrow \mathbb{P}^1$ denote the i -th projection. With this notation, we have the following.

Proposition 3.13. *Let $G \subset \mathrm{PGL}(2, \mathbb{C})^3$ be the group defined by Ξ' in the following table, in a similar way to Proposition 3.11.*

H	Ξ'	basis of $H^0(\mathcal{O}_{\mathbb{P}}(2, 2, 2))^G$
C_1	\emptyset	$s_1^i t_1^{2-i} s_2^j t_2^{2-j} s_3^k t_3^{2-k} + s_1^{2-i} t_1^i s_2^{2-j} t_2^j s_3^{2-k} t_3^k$
C_2	$\{(1/4, 1/4, 0)\}$	$s_1^i t_1^{2-i} s_2^j t_2^{2-j} s_3^k t_3^{2-k} + s_1^{2-i} t_1^i s_2^{2-j} t_2^j s_3^{2-k} t_3^k$ ($i + j \equiv 0 \pmod{2}$)
C_3	$\{(1/3, 1/3, 1/3)\}$	$s_1^i t_1^{2-i} s_2^j t_2^{2-j} s_3^k t_3^{2-k} + s_1^{2-i} t_1^i s_2^{2-j} t_2^j s_3^{2-k} t_3^k$ ($i + j + k \equiv 0 \pmod{3}$)
C_4	$\{(1/8, 1/8, 1/4)\}$	$s_1^i t_1^{2-i} s_2^j t_2^{2-j} s_3^k t_3^{2-k} + s_1^{2-i} t_1^i s_2^{2-j} t_2^j s_3^{2-k} t_3^k$ ($i + j + 2k \equiv 0 \pmod{4}$)
$C_2 \times C_2$	$\{(1/4, 0, 1/4), (0, 1/4, 1/4)\}$	$s_1^i t_1^{2-i} s_2^j t_2^{2-j} s_3^k t_3^{2-k} + s_1^{2-i} t_1^i s_2^{2-j} t_2^j s_3^{2-k} t_3^k$ ($i + k \equiv j + k \equiv 0 \pmod{2}$)

More precisely, H is generated by $M(a) \times M(b) \times M(c)$ for $(a, b, c) \in \Xi'$, and $G := H \rtimes \langle \iota \rangle$, where

$$\iota = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For a generic $Q \in H^0(\mathcal{O}_{\mathbb{P}}(2, 2, 2))^G$, the surface S_0 is smooth and the action of G on $S (= S_0)$ is a Calabi–Yau action. Moreover, a generic triplet for $H = C_n$ ($1 \leq n \leq 4$) or $C_2 \times C_2$ constructed in Proposition 3.11 is of the form (S, \mathcal{L}, G) , where

$$\mathcal{L} = ((p_\alpha \times p_\beta)^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))|_S$$

with $\alpha, \beta \in \{1, 2, 3\}$, $\alpha \neq \beta$.

Proof. In each case, we can check, for a generic $Q \in H^0(\mathcal{O}_{\mathbb{P}}(2, 2, 2))^G$, that S_0 is smooth and that the action of any $g \in G \setminus H$ on $S (= S_0)$ has no fixed point by a direct computation (see the proof of Proposition 3.11). Since the action of H on S is symplectic (Lemma 3.9), the action of G on S is a Calabi–Yau action. This proves the first assertion.

Let us next show the second assertion. We assume that $(\alpha, \beta) = (1, 2)$ as the other cases are similar. Define the map

$$\phi: V := H^0(\mathcal{O}_{\mathbb{P}}(2, 2, 2))^G \rightarrow W := H^0(\mathcal{O}_{\mathbb{P}}(4, 4, 0))^G$$

by

$$Q = As_3^2 + Bs_3t_3 + Ct_3^2 \mapsto F = \det \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}.$$

The branching locus B of the double covering $p_1 \times p_2: S_0 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is defined by $F = 0$. The map ϕ gives a correspondence between $(2, 2, 2)$ -hypersurfaces in \mathbb{P} with a Calabi–Yau G -action and double coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ branching along a $(4, 4)$ -curve with a Calabi–Yau G -action. Hence it suffices to show that ϕ is dominant. We show this by comparing the dimensions of $\phi^{-1}(F)$, V and W . By the argument above and generic smoothness ([12, Corollary III.10.7]), we may assume that S_0 and $\phi^{-1}(F)$ is smooth (hence $S = S_0$) by taking a generic $Q \in V$. Let Δ be a contractible open neighborhood of Q in $\phi^{-1}(F)$. We construct a natural family of embeddings $S \rightarrow \mathbb{P}$ parametrized by Δ as follows. Since the branching locus of the double covering

$(p_1 \times p_2) \times \text{id}_\Delta: \bar{S} \rightarrow (\mathbb{P}^1 \times \mathbb{P}^1) \times \Delta$, $\bar{S} := \{(x, Q') \mid Q'(x) = 0\} \subset \mathbb{P} \times \Delta$, is $B \times \Delta$, we have a natural commutative diagram

$$\begin{array}{ccc} S \times \Delta & \xrightarrow[\cong]{f} & \bar{S} \\ & \searrow (p_1 \times p_2) \times \text{id}_\Delta & \downarrow (p_1 \times p_2) \times \text{id}_\Delta \\ & & (\mathbb{P}^1 \times \mathbb{P}^1) \times \Delta, \end{array}$$

where

$$f_{Q'} := f(*, Q'): S \rightarrow \{Q' = 0\}, \quad Q' \in \Delta,$$

is a G -equivariant isomorphism and $f_Q = \text{id}_S$. Since Δ is connected and the Picard group of S is discrete, the map $f_{Q'}$ is represented by $\gamma \in \text{GL}(2, \mathbb{C})^3$ such that γ commutes with G in $\text{PGL}(2, \mathbb{C})^3$. Furthermore, since $(p_1 \times p_2) \circ f_{Q'} = p_1 \times p_2$, we may assume that $\gamma = I_2 \times I_2 \times M$. Then $\phi(\gamma^*Q) = \det(M)^2 F$. Therefore, we have $\Delta \subset \Gamma^*Q$, where Γ is a subgroup of $\text{GL}(2, \mathbb{C})^3$ defined by

$$\Gamma := \{\gamma = I_2 \times I_2 \times M \mid \gamma \text{ commutes with } G \text{ in } \text{PGL}(2, \mathbb{C})^3, \det(M) = \pm 1\}.$$

Since $\dim \Delta \leq \dim \Gamma$, ϕ is dominant if $\dim \Gamma \leq \dim V - \dim W$. This can be checked directly, as indicated in the following table.

H	Ξ	$\dim \Gamma$	$\dim V$	$\dim W$
C_1	\emptyset	1	14	13
C_2	$\{(1/4, 1/4)\}$	1	8	7
C_2	$\{(1/4, 0)\}$	0	8	8
C_3	$\{(1/3, 1/3)\}$	0	5	5
C_4	$\{(1/8, 1/8)\}$	0	4	4
C_4	$\{(1/8, 1/4)\}$	0	4	4
$C_2 \times C_2$	$\{(1/4, 0), (0, 1/4)\}$	0	5	5

□

3.3. Uniqueness. In this section, we will prove the uniqueness theorem of Calabi–Yau actions (Theorem 3.19). We will also show the non-existence of a K3 surface with a Calabi–Yau G -action for $G \cong (C_3 \times C_3) \rtimes C_2$ (Theorem 3.20). Henceforth we fix a semi-direct product decomposition $G = H \rtimes \langle \iota \rangle$ of a Calabi–Yau group G as in Proposition 3.3. The key to proving the uniqueness is the following lemma, whose proof will be given in Section 4.

Lemma 3.14 (Key Lemma). *Let S be a K3 surface with a Calabi–Yau G -action. Then there exists an element $v \in \mathrm{NS}(S)^G$ such that $v^2 = 4$.*

First, we consider the (coarse) moduli space of K3 surfaces S with a Calabi–Yau G -action. Let Ψ_G denote the set of actions $\psi: G \rightarrow \mathrm{O}(\Lambda)$ of G on $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ such that there exist a K3 surface S with a Calabi–Yau G -action and a G -equivariant isomorphism $H^2(S, \mathbb{Z}) \rightarrow \Lambda_\psi$. Here we denote Λ with a G -action ψ by Λ_ψ . The group $\mathrm{O}(\Lambda)$ acts on Ψ_G by conjugation:

$$(\gamma \cdot \psi)(g) = \gamma \psi(g) \gamma^{-1}, \quad \gamma \in \mathrm{O}(\Lambda), \quad \psi \in \Psi_G, \quad g \in G.$$

Define the period domain $\tilde{\mathcal{D}}^G$ by

$$\tilde{\mathcal{D}}^G := \bigsqcup_{\psi \in \Psi'_G} \tilde{\mathcal{D}}^{G, \psi}, \quad \tilde{\mathcal{D}}^{G, \psi} := \{\mathbb{C}\omega \in \mathbb{P}((\Lambda_\psi)_\ell^H \otimes \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\},$$

where Ψ'_G is a complete representative system of the quotient $\Psi_\Lambda / \mathrm{O}(\Lambda)$. For any K3 surface S with a Calabi–Yau G -action, there exist a unique $\psi \in \Psi'_G$ and a G -equivariant isomorphism $\alpha: H^2(S, \mathbb{Z}) \rightarrow \Lambda_\psi$. Under the period map, S with the G -action corresponds to the period point

$$(\alpha \otimes \mathbb{C})(H^{2,0}(S)) \in \tilde{\mathcal{D}}^{G, \psi} \subset \tilde{\mathcal{D}}^G.$$

Lemma 3.15. *Let S be a K3 surface with a G -action. If the induced action $\psi: G \rightarrow \mathrm{O}(\Lambda)$ (which is defined modulo the conjugate action of $\mathrm{O}(\Lambda)$) is an element in Ψ_G , the G -action on S is a Calabi–Yau action.*

Proof. In general, a symplectic automorphism g of a K3 surface S of finite order is characterized as an automorphism such that $H^2(S, \mathbb{Z})_g$ is negative definite [23, Theorem 3.1]. Also, an Enriques involution is characterized by Lemma 2.9. \square

Proposition 3.16. *For the moduli space \mathcal{M}^G of K3 surfaces S with a Calabi–Yau G -action, the period map defined above induces an isomorphism*

$$\tau: \mathcal{M}^G \rightarrow \bigsqcup_{\psi \in \Psi'_G} (\mathcal{D}^{G, \psi} / \mathrm{O}(\Lambda, \psi)).$$

Here

$$\begin{aligned} \mathcal{D}^{G, \psi} &:= \{\mathbb{C}\omega \in \tilde{\mathcal{D}}^{G, \psi} \mid \langle \omega, \delta \rangle \neq 0 \ (\forall \delta \in \Delta_\psi)\}, \\ \Delta_\psi &:= \{\delta \in (\Lambda_\psi)_G \mid \delta^2 = -2\}, \\ \mathrm{O}(\Lambda, \psi) &:= \{\gamma \in \mathrm{O}(\Lambda) \mid \gamma \psi(g) = \psi(g) \gamma \ (\forall g \in G)\}. \end{aligned}$$

Proof. Let $\mathbb{C}\omega = (\alpha \otimes \mathbb{C})(H^{2,0}(S)) \in \tilde{\mathcal{D}}^{G,\psi}$ be the period point of a K3 surface S with a Calabi–Yau G -action. We can check that $\mathbb{C}\omega$ modulo the action of $O(\Lambda, \psi)$ is independent of the choice of α . Since G is finite, there exists a G -invariant Kähler class κ_S of S . We have $\kappa := (\alpha \otimes \mathbb{R})(\kappa_S) \in (\Lambda_\psi)^G \otimes \mathbb{R}$. For any $\delta \in \Delta_\psi$, we have $\langle \kappa, \delta \rangle = 0$, and thus $\langle \omega, \delta \rangle \neq 0$ (see Section 2.2). Therefore we see that $\mathbb{C}\omega \in \mathcal{D}^{G,\psi}$. Assume that a K3 surface S' with a Calabi–Yau G -action is mapped to the same point as S by τ . Then there exists a G -equivariant isomorphism $\phi: H^2(S, \mathbb{Z}) \rightarrow H^2(S', \mathbb{Z})$ such that $(\phi \otimes \mathbb{C})(H^{2,0}(S)) = H^{2,0}(S')$. By Lemma 2.7, we may assume that $(\phi \otimes \mathbb{R})(\kappa_S)$ is a Kähler class of S' . By Theorem 2.5, ϕ induces a G -equivariant isomorphism between S and S' . Therefore, τ is injective. Let $\mathbb{C}\omega_1 \in \mathcal{D}^{G,\psi}$. By the definition of $\mathcal{D}^{G,\psi}$, for any $\delta \in \Lambda_\psi$ with $\delta^2 = -2$ and $\langle \omega_1, \delta \rangle = 0$, we have $\langle \delta, (\Lambda_\psi)^G \rangle \supsetneq \{0\}$. Hence there exists $\kappa_1 \in (\Lambda_\psi)^G \otimes \mathbb{R}$ such that ω_1 and κ_1 satisfy the conditions (1) and (2) in Theorem 2.6. Therefore, by Theorem 2.6, there exist a K3 surface S_1 and an isomorphism $\alpha_1: H^2(S_1, \mathbb{Z}) \rightarrow \Lambda_\psi$ such that $(\alpha_1^{-1} \otimes \mathbb{C})(\mathbb{C}\omega_1) = H^{2,0}(S_1)$ and $(\alpha_1^{-1} \otimes \mathbb{R})(\kappa_1)$ is a Kähler class of S_1 . By Theorem 2.5, the G -action on Λ_ψ induces a G -action on S_1 such that α_1 is G -equivariant, which is a Calabi–Yau action by Lemma 3.15. This implies the surjectivity of τ . \square

Next let us consider projective models of the K3 surfaces with a Calabi–Yau action.

Lemma 3.17. *Let S be a K3 surface with a Calabi–Yau G -action. Assume that there exists an element $v \in \text{NS}(S)^G$ such that $v^2 = 4$. Then there exists a G -invariant line bundle \mathcal{L} on S satisfying the following conditions.*

- (1) $\mathcal{L}^2 = 4$ and $h^0(\mathcal{L}) = 4$.
- (2) The linear system $|\mathcal{L}|$ defined by \mathcal{L} is base-point free and defines a map $\phi_{\mathcal{L}}: S \rightarrow \mathbb{P}^3$.
- (3) $\dim \phi_{\mathcal{L}}(S) = 2$.
- (4) The degree $\deg \phi_{\mathcal{L}}$ of the map $\phi_{\mathcal{L}}: S \rightarrow \phi_{\mathcal{L}}(S)$ is 2, and $\phi_{\mathcal{L}}(S)$ is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or a cone (i.e. a nodal quadric surface).

Proof. Note that the closure $\overline{\mathcal{K}}_S$ of \mathcal{K}_S is the nef cone of S . We may assume that v is nef by Lemma 2.7. Let \mathcal{L} be a line bundle on S representing v . By [29, Sections 4 and 8], we have $h^0(\mathcal{L}) = 4$ and either of the following occurs.

- (a) \mathcal{L} is base-point free, $\dim \phi_{\mathcal{L}}(S) = 2$, and $\deg \phi_{\mathcal{L}} = 1$ or 2. Any connected component of $\phi_{\mathcal{L}}^{-1}(p)$ for any p is either a point or an ADE-configuration.
- (b) $\mathcal{L} \cong \mathcal{O}_S(3E + \Gamma)$ and $|\mathcal{L}| = \{D_1 + D_2 + D_3 + \Gamma \mid D_i \sim E\}$, where E and $\Gamma \cong \mathbb{P}^1$ are irreducible divisors such that $E^2 = 0$, $\Gamma^2 = -2$ and $\langle E, \Gamma \rangle = 1$.

In Case (b), the base locus $\Gamma \cong \mathbb{P}^1$ of $|\mathcal{L}|$ is stable under the action of ι and thus ι has a fixed point in Γ , which is a contradiction. Hence Case (a) occurs. Since the fixed locus of any (projective) involution of \mathbb{P}^3 is at least 1-dimensional, there exists a fixed point p of the action of ι on $\phi_{\mathcal{L}}(S)$. If

$\deg \phi_{\mathcal{L}} = 1$, then $S^i \neq \emptyset$ by Lemma 3.10, which is a contradiction. Hence $\deg \phi_{\mathcal{L}} = 2$, and $\phi_{\mathcal{L}}(S)$ is an irreducible quadric surface in \mathbb{P}^3 , which is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a cone. \square

Proposition 3.18. *For a generic point in \mathcal{M}^G , the corresponding K3 surface S with a Calabi–Yau action $\chi: G \rightarrow \text{Aut}(S)$ has a projective model as in Proposition 3.11. More precisely, S and χ are realized as the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branching along a $(4, 4)$ -curve and a projective G -action on S .*

Proof. Let S be a K3 surface with a Calabi–Yau G -action. By Lemmas 3.14 and 3.17, there exists a G -invariant line bundle \mathcal{L} satisfying the conditions (1)–(4) in Lemma 3.17. Let $\phi_{\mathcal{L}} = u \circ \theta$ be the Stein factorization of $\phi_{\mathcal{L}}$. Then $\theta(S)$ is a normal surface possibly with ADE-singularities, and u is a finite map of degree 2. Assume that $\phi_{\mathcal{L}}(S)$ is a cone with the singular point p . By Lemma 3.10, $u^{-1}(p)$ consists of two points \bar{p}_1, \bar{p}_2 , which are interchanged by any $g \in G \setminus H$. Hence each $\theta^{-1}(\bar{p}_i)$ is a (-2) -curve C_i on S and $C_1 - C_2 \in H^2(S, \mathbb{Z})_{\iota}^H$ with $(C_1 - C_2)^2 = -4$. In particular, the Picard number of S is greater than the generic Picard number $= 22 - \text{rank } H^2(S, \mathbb{Z})_{\iota}^H$. Therefore, if the period point of (S, χ) is contained in

$$\bigsqcup_{\psi \in \Psi'_G} \{ \mathbb{C}\omega \in \mathcal{D}^{G, \psi} \mid \langle \omega, \delta \rangle \neq 0 \ (\forall \delta \in \Delta'_{\psi}) \}, \quad \Delta'_{\psi} := \{ \delta \in (\Lambda_{\psi})_{\iota}^H \mid \delta^2 = -4 \},$$

then $\phi_{\mathcal{L}}(S) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the branching curve B of u has at most ADE-singularities. Moreover, since the canonical bundle of S is trivial, the bidegree of B is $(4, 4)$. \square

Let S be a K3 surface with a Calabi–Yau action $\chi: G \rightarrow \text{Aut}(S)$. We will prove (Theorem 3.19) that the pair $(S, \chi(G))$ is unique up to equivariant deformation. A pair (S, χ) represents a K3 surface S with a Calabi–Yau G -action with a fixed group G , while a pair $(S, \chi(G))$ represents a K3 surface S with a subgroup of $\text{Aut}(S)$ which gives a Calabi–Yau G -action. The difference is whether or not we keep track of a way to identify G with a subgroup of $\text{Aut}(S)$. Let $\text{Aut}_H(G)$ denote the subgroup of $\text{Aut}(G)$ consisting of elements which preserve H . The group $\text{Aut}_H(G)$ acts on the moduli space \mathcal{M}^G of pairs (S, χ) from the right by

$$(S, \chi) \cdot \sigma = (S, \chi \circ \sigma), \quad \sigma \in \text{Aut}_H(G).$$

Note that $\text{Aut}(G)$ does not necessarily act on \mathcal{M}^G because the action of H on S is symplectic by definition. The orbit of (S, χ) under the action of $\text{Aut}_H(G)$ is identified with $(S, \chi(G))$, and $\mathcal{M}^G / \text{Aut}_H(G)$ is considered as the moduli space of pairs $(S, \chi(G))$.

Theorem 3.19. *Let $H = C_n$ ($1 \leq n \leq 6$), $C_2 \times C_2$ or $C_2 \times C_4$. Then there exists a unique subgroup of $\text{O}(\Lambda)$ induced by a Calabi–Yau G -action up to conjugation in $\text{O}(\Lambda)$, that is,*

$$(3.2) \quad \Psi_G = \text{O}(\Lambda) \cdot \psi \cdot \text{Aut}_H(G), \quad \psi \in \Psi_G.$$

Moreover, we have

$$(3.3) \quad \mathcal{M}^G / \text{Aut}_H(G) \cong \mathcal{D}^{G,\psi} / \Gamma_\psi, \quad \Gamma_\psi := \{\gamma \in \text{O}(\Lambda) \mid \gamma\psi(G)\gamma^{-1} = \psi(G)\}.$$

In particular, the moduli space $\mathcal{M}^G / \text{Aut}_H(G)$ of pairs $(S, \chi(G))$ as above is irreducible, and a pair $(S, \chi(G))$ exists uniquely up to equivariant deformation.

Proof. By Proposition 3.18, a generic pair $(S, \chi(G))$ has a projective model as in Proposition 3.11. Hence the existence of Calabi–Yau G -actions follows from Proposition 3.11. Also, the connectedness of $\mathcal{M}^G / \text{Aut}_H(G)$ follows from Propositions 3.11 and 3.13. The stabilizer subgroup Σ of $\mathcal{D}^{G,\psi} / \text{O}(\Lambda, \psi)$ in $\text{Aut}_H(G)$ is given by

$$\Sigma = \{\sigma \in \text{Aut}_H(G) \mid \psi \circ \sigma = \gamma \cdot \psi \ (\exists \gamma \in \text{O}(\Lambda))\}.$$

Since Σ is naturally isomorphic to $\Gamma_\psi / \text{O}(\Lambda, \psi)$, we can check (3.2) and (3.3) by Proposition 3.16. \square

Theorem 3.20. *For $G \cong (C_3 \times C_3) \rtimes C_2$, there does not exist a K3 surface with a Calabi–Yau G -action, that is, $\mathcal{M}^G = \emptyset$.*

Proof. As in the proof of Theorem 3.19, a generic pair $(S, \chi(G))$ admits a projective model as in Proposition 3.11. However, there is no such a projective model. \square

3.4. Moduli Spaces of Complex Structures. In this section, we describe the complex moduli spaces of Calabi–Yau threefolds of type K. We in particular show the irreducibility of the moduli space with a prescribed Galois group G . Throughout this section, we fix a semi-direct product decomposition $G = H \rtimes \langle \iota \rangle$ as in Proposition 3.3. In Section 3.3, we studied the moduli space of K3 surfaces S with a Calabi–Yau G -action, which is denoted by \mathcal{M}_S^G instead of \mathcal{M}^G in this section.

Let us consider the moduli problem of elliptic curves with a G -action given in Proposition 3.3, which we also call a Calabi–Yau G -action. Let \mathcal{M}_E^G denote the moduli space of elliptic curves with a Calabi–Yau G -action. An element in \mathcal{M}_E^G is the isomorphism class of an elliptic curve with a faithful translation action of H . A faithful translation action of $C_2 \times C_2$ on an elliptic curve E is given by a level 2 structure on E . Therefore \mathcal{M}_E^G for $H = C_2 \times C_2$ is identified with the (non-compact) modular curve $Y(2) := \mathbb{H} / \Gamma(2)$. In the same manner, \mathcal{M}_E^G for $H = C_n$ is identified with the modular curve $Y_1(n) := \mathbb{H} / \Gamma_1(n)$. For $H = C_2 \times C_4$, we want the moduli space of elliptic curves with linearly independent 2- and 4-torsion points. It is not difficult to see that it is identified with the modular curve $Y(2 \mid 4) := \mathbb{H} / \Gamma(2 \mid 4)$, where

$$\Gamma(2 \mid 4) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{Z}) \mid a - 1 \equiv c \equiv 0 \pmod{2}, \ b \equiv d - 1 \equiv 0 \pmod{4} \right\}.$$

We summarize the argument above in the following lemma.

Lemma 3.21. *Let $1 \leq n \leq 6$. The moduli space \mathcal{M}_E^G of elliptic curves with a Calabi–Yau G -action is irreducible and given by the following.*

G	$C_2 \times C_2 \times C_2$	$C_n \rtimes C_2$	$C_2 \times D_8$
\mathcal{M}_E^G	$Y(2)$	$Y_1(n)$	$Y(2 \mid 4)$

Theorem 3.22. *Let $\text{Aut}_H(G)$ denote the subgroup of $\text{Aut}(G)$ consisting of elements which preserve H . The quotient space*

$$(\mathcal{M}_S^G \times \mathcal{M}_E^G) / \text{Aut}_H(G)$$

is the coarse moduli space of Calabi–Yau threefolds of type K whose minimal splitting covering has the Galois group isomorphic to G . The moduli space is in particular irreducible.

Proof. Two Calabi–Yau threefolds X and Y of type K are isomorphic if and only if the corresponding minimal splitting coverings $S_X \times E_X$ and $S_Y \times E_Y$ are isomorphic as Galois coverings. Suppose that the Galois group is isomorphic to G . The condition is equivalent to the existence of an isomorphism $f: S_X \times E_X \rightarrow S_Y \times E_Y$ and an automorphism $\phi \in \text{Aut}(G)$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ \downarrow & \circlearrowleft & \downarrow \\ \text{Aut}(S_X \times E_X) & \longrightarrow & \text{Aut}(S_Y \times E_Y), \end{array}$$

that is, $f(g \cdot x) = \phi(g) \cdot f(x)$ for any $g \in G$ and any $x \in S_X \times E_X$. Note that we have $\phi \in \text{Aut}_H(G)$ because we fix a subgroup H as in Proposition 3.3. Since a Calabi–Yau G -action on $S_X \times E_X$ induces that on each S_X and E_X , $S_X \times E_X$ is represented by a point in $\mathcal{M}_S^G \times \mathcal{M}_E^G$. The quotient space $(\mathcal{M}_S^G \times \mathcal{M}_E^G) / \text{Aut}_H(G)$ is then the coarse moduli space of the isomorphism classes of Calabi–Yau threefolds of type K with Galois group isomorphic to G . The moduli space is irreducible because the action of $\text{Aut}_H(G)$ on the set of connected components of \mathcal{M}_S^G is transitive and \mathcal{M}_E^G is irreducible by Theorem 3.19 and Lemma 3.21. \square

Combining Proposition 3.11, Theorems 3.19, 3.20 and 3.22, we complete the proof of the main theorem (Theorem 3.1) of the present section.

4. KEY LEMMA

Let S be a K3 surface with a Calabi–Yau G -action. We fix a semi-direct product decomposition $G = H \rtimes \langle \iota \rangle$ as in Proposition 3.3. This section is devoted to the proof of the Key Lemma:

Key Lemma (Lemma 3.14). *There exists an element $v \in \text{NS}(S)^G$ such that $v^2 = 4$.*

4.1. Preparation.

Lemma 4.1. *Set $r := \text{rank } H^2(S, \mathbb{Z})^H$. We then have*

$$\text{rank } H^2(S, \mathbb{Z})^G = \frac{r}{2} - 1, \quad \text{rank } H^2(S, \mathbb{Z})_l^H = \frac{r}{2} + 1.$$

Proof. Let X denote a Calabi-Yau threefold $(S \times E)/G$ of type K. Since a holomorphic 2-form ω_S on S is contained in $H^2(S, \mathbb{C})_l^H$, we see that $H^2(S, \mathbb{C})^G \subset H^{1,1}(S)$. We therefore have

$$H^{1,1}(X) \cong H^{1,1}(S \times E, \mathbb{C})^G \cong (H^2(S, \mathbb{C})^G \otimes H^0(E, \mathbb{C})) \oplus (H^0(S, \mathbb{C}) \otimes H^2(E, \mathbb{C})).$$

as \mathbb{C} -vector spaces and conclude that $h^{1,1}(X) = \text{rank } H^2(S, \mathbb{Z})^G + 1$. On the other hand we have canonical isomorphisms of \mathbb{C} -vector spaces:

$$H^{2,1}(X) \cong H^{2,1}(S \times E, \mathbb{C})^G \cong (H^{1,1}(S)_l^H \otimes H^{1,0}(E)) \oplus (H^{2,0}(S) \otimes H^{0,1}(E)).$$

By the decomposition

$$H^2(S, \mathbb{C})_l^H \cong H^{2,0}(S) \oplus H^{1,1}(S)_l^H \oplus H^{0,2}(S),$$

we conclude that $h^{2,1}(X) = \text{rank } H^2(S, \mathbb{Z})_l^H - 1$. Since the Euler characteristic $e(X) = e(S \times E)/|G| = 0$, we have $h^{1,1}(X) = h^{2,1}(X)$. Then the claim readily follows. \square

Recall that the action of H on S is symplectic. Hence the quotient surface S/H has at most ADE-singularities and the minimal resolution \tilde{S} of S/H is again a $K3$ surface. Let $\tilde{\iota}$ denote the involution of \tilde{S} induced by ι .

Lemma 4.2. *The involution of S/H induced by ι has no fixed point. In particular, $\tilde{\iota}$ is an Enriques involution of \tilde{S} .*

Proof. If the involution of S/H induced by ι has a fixed point, then the action of $h\iota$ on S has a fixed point for some $h \in H$, which is a contradiction. \square

Each irreducible curve M_i which contracts under the resolution $\tilde{S} \rightarrow S/H$ is a (-2) -curve. We denote by M the negative definite lattice generated by $\{M_i\}_i$ and set $K := M_{H^2(\tilde{S}, \mathbb{Z})}^\perp$.

Lemma 4.3. *If $H = C_n$ ($3 \leq n \leq 6$), then there is no nef class $v \in K^{\tilde{\iota}}$ such that $v^2 = 4$.*

Proof. We assume that a nef class $v \in K^{\tilde{\iota}}$ satisfies $v^2 = 4$ and derive a contradiction. By Lemma 4.2, the (induced) action of ι on S/H has no fixed point. By the same argument as in the proof of Proposition 3.17, the class v induces a morphism $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^3$ such that $\tilde{f}(\tilde{S})$ is a quadric surface and the degree of \tilde{f} is 2. Since we have $v \perp M$ by the assumption, the morphism \tilde{f} induces a morphism $f: S/H \rightarrow \mathbb{P}^3$. By the proof of Lemma 3.18, we may assume that $f(S/H) \cong \mathbb{P}^1 \times \mathbb{P}^1$ by taking a generic S . The action of ι on S/H is of the form $\sigma\tau$, where σ is induced by a symplectic involution of \tilde{S} and τ is the covering transformation of f . Let $\bar{\tau} \in \text{Aut}(S)$ be a lift of τ . Note that $\bar{\tau}$ normalizes H . Since f induces a generically one-to-one morphism

$S/\langle H, \bar{\tau} \rangle \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, it follows that $S/\langle H, \bar{\tau} \rangle$ is smooth and that the action of τ fixes each singular point of S/H . Hence the actions of a generator of H and $\bar{\tau}$ are represented by the matrices $\begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ respectively, in local coordinates around a point in S^H , where $\zeta_n := \exp(2\pi i/n)$. Therefore we have $\bar{\tau}h\bar{\tau} = h^{-1}$ for any $h \in H$ (\star).

We checked that τ fixes each point in $\text{Sing}(S/H)$. Hence the action of σ has 8 fixed points $q_i \notin \text{Sing}(S/H)$ ($1 \leq i \leq 8$) by Theorem 2.8. Let $Q_i \subset S$ denote the inverse image of q_i , which consists of $|H|$ points. Take a point $p \in Q_i$. Since H acts on Q_i transitively, we can take a lift $\bar{\sigma} \in \text{Aut}(S)$ of σ such that $\bar{\sigma} \cdot p = p$. The action of $\bar{\sigma}$ around p is locally identified with that of σ around q_i . Therefore $\text{ord}(\bar{\sigma}) = 2$. Since $\bar{\sigma}\bar{\tau} \in H$, the condition (\star) implies that $\bar{\sigma}$ commutes with H . Hence the action of $\bar{\sigma}$ on each Q_i is trivial or free. If $n = 3, 5$ or 6 , this contradicts to the fact $|S^{\bar{\sigma}}| = 8$. Let us next consider the case $n = 4$. Let $h \in H$ be a generator of H . By a similar argument, for each Q_i , we can check that the action of either $\bar{\sigma}$ or $\bar{\sigma}h^2$ on Q_i is trivial. Therefore we have $\cup_{i=1}^8 Q_i = S^{\bar{\sigma}} \cup S^{\bar{\sigma}h^2}$. On the other hand, Theorem 2.8 implies that $|\cup_{i=1}^8 Q_i| = 8 \cdot |H| = 32$ and $|S^{\bar{\sigma}} \cup S^{\bar{\sigma}h^2}| = 2 \cdot 8 = 16$. This is a contradiction. \square

4.2. Proof of the Key Lemma. In the following, we write $L_R := L \otimes_{\mathbb{Z}} R$ for a lattice L and a \mathbb{Z} -module R . The bilinear form on L naturally extends to that on L_R which takes values in R . We denote by \mathbb{Z}_p the p -adic integers. Lattices over \mathbb{Z}_p , and their discriminant groups and forms are defined in a similar way to lattices (over \mathbb{Z}). Note that a lattice over \mathbb{Z}_2 is not necessarily even. Assume that L is non-degenerate and even. Then $A(L_{\mathbb{Z}_p})$ and $q(L_{\mathbb{Z}_p})$ are the p -parts of $A(L)$ and $q(L)$ respectively (see [24] for details). In particular, if $|\text{disc}(L)|$ is a power of p , then we have $(A(L), q(L)) \cong (A(L_{\mathbb{Z}_p}), q(L_{\mathbb{Z}_p}))$.

Some remarks are in order before the proof. We fix an identification $H^2(S, \mathbb{Z}) = \Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Since $H^{2,0}(S)$ is contained in $(\Lambda_{\mathbb{C}}^H)_{\mathbb{C}}$, we have $\text{NS}(S)^G = \Lambda^G$ by (2.1). By [24], the H -invariant lattice Λ^H is non-degenerate, and the rank of Λ^H , which depends only on the group H , is given in the following table.

H	C_1	C_2	C_3	C_4	C_5	C_6	$C_2 \times C_2$	$C_2 \times C_4$	$C_3 \times C_3$
$\text{rank } \Lambda^H$	22	14	10	8	6	6	10	6	6

Since $\Lambda^G(1/2)$ is contained in $\Lambda^{\iota}(1/2)$, which is isomorphic to $U \oplus E_8(-1)$ by Theorem 2.9, it follows that $\Lambda^G(1/2)$ is even. Similarly, $K^{\tilde{\iota}}(1/2)$ is even by Lemma 4.2. Since G is finite, there exists a G -invariant Kähler class of S . Therefore Λ^G has signature $(1, \text{rank } \Lambda^G - 1)$. Set

$$S' := S \setminus \{p \in S \mid h \cdot p = p \ (\exists h \in H, h \neq 1)\},$$

and let $\pi: S' \rightarrow \tilde{S}$ be the natural map. Since $S \setminus S'$ is a finite set, the pushforward π_* and Poincaré duality induce a natural map

$$f: \Lambda = H^2(S, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \rightarrow H_2(\tilde{S}, \mathbb{Z}) \cong H^2(\tilde{S}, \mathbb{Z}).$$

For any $x, y \in \Lambda^H$, we have $\langle f(x), f(y) \rangle = |H| \langle x, y \rangle$. The map f decomposes as

$$f: \Lambda \rightarrow (\Lambda^H)^\vee \rightarrow H^2(\tilde{S}, \mathbb{Z}),$$

where the first map is the restriction of the first projection of the decomposition $\Lambda_{\mathbb{Q}} = (\Lambda^H)_{\mathbb{Q}} \oplus (\Lambda_H)_{\mathbb{Q}}$ and the second map is the natural injection. Since $\Lambda^H / (\Lambda^G \oplus \Lambda_t^H) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus l}$ for some l by Lemma 2.4, we have $2(\Lambda^G)^\vee \subset (\Lambda^H)^\vee$. Hence we find that

$$f(2(\Lambda^G)^\vee) \subset f((\Lambda^G)_{\mathbb{Q}} \cap (\Lambda^H)^\vee) \subset K^{\tilde{t}}.$$

Set $L := \Lambda^G(1/2)$. Then we have $2(\Lambda^G)^\vee \cong 2L^\vee(1/2) \cong L^\vee(2)$. Thus we have

$$L^\vee(|H|) \cong f(2(\Lambda^G)^\vee)(1/2) \subset K^{\tilde{t}}(1/2).$$

Therefore L satisfies the following conditions.

- (1) L and $L^\vee(|H|)$ are even.
- (2) If $H = C_n$ ($3 \leq n \leq 6$), then $v^2 \neq 2/n$ for any $v \in L^\vee$.

Here (2) is a conclusion of Lemmas 2.7 and 4.3. These conditions are derived from geometry of K3 surfaces. On the other hand, the argument below is essentially lattice theoretic.

Proof of Key Lemma. If Λ^G contains $U(2)$, we see that the assertion of the Key Lemma holds.

Case $H = C_1$. We have $\Lambda^G \cong U(2) \oplus E_8(-2)$ by Theorem 2.9.

Case $H = C_2$. This case has been studied by Ito and Ohashi (No. 13 in their paper [14]). They showed that $\Lambda^G \cong U(2) \oplus D_4(-2)$. Here we give a proof of this fact for the sake of completeness. By Lemma 4.1, the signature of Λ^G is $(1, 5)$. For each prime p , the lattice $L_{\mathbb{Z}_p}$ over the local ring \mathbb{Z}_p admits an orthogonal decomposition $L_{\mathbb{Z}_p} \cong \bigoplus_{i \geq 0} L_i^{(p)}(p^i)$, where $L_i^{(p)}$'s are unimodular lattices (see [24] for details). By (1), we have $L_i^{(p)} = 0$ for any $p \neq 2$ and any $i \geq 1$. Thus $|\text{disc}(L)|$ is a power of 2. Again, by (1), $L_0^{(2)}$ and $L_1^{(2)}$ are even, and we have $L_i^{(2)} = 0$ for any $i \geq 2$. Let V denote the lattice over \mathbb{Z}_2 defined by the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. In general, a lattice over \mathbb{Z}_2 is expressed as an orthogonal sum of the lattices in the following table [24,

Propositions 1.8.1 and 1.11.2].

N	$\langle 2^k a \rangle$	$U(2^k) = \begin{bmatrix} 0 & 2^k \\ 2^k & 0 \end{bmatrix}$	$V(2^k) = \begin{bmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{bmatrix}$
$A(N)$	$\mathbb{Z}/2^k\mathbb{Z}$	$(\mathbb{Z}/2^k\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2^k\mathbb{Z})^{\oplus 2}$
$q(N)$	$\langle a/2^k \rangle$	$u(2^k) := \begin{bmatrix} 0 & 1/2^k \\ 1/2^k & 0 \end{bmatrix}$	$v(2^k) := \begin{bmatrix} 1/2^{k-1} & 1/2^k \\ 1/2^k & 1/2^{k-1} \end{bmatrix}$
$\text{sign } q(N)$	$a + k(a^2 - 1)/2$	0	$4k$

Here $k \geq 0$ and $a = \pm 1, \pm 3$. (Note that $\langle \pm 1/2 \rangle \cong \langle \mp 3/2 \rangle$.) Since $L_0^{(2)}$ and $L_1^{(2)}$ are even, $L_{\mathbb{Z}_2}$ has an orthogonal decomposition

$$L_{\mathbb{Z}_2} \cong U^{\oplus \nu} \oplus V^{\oplus \mu} \oplus U(2)^{\oplus \nu'} \oplus V(2)^{\oplus \mu'}.$$

Then we have

$$A(L) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2(\nu' + \mu')}, \quad \text{sign } q(L) \equiv 4\mu' \pmod{8}.$$

Since we have $\Lambda^t(1/2) \cong U \oplus E_8(-1)$ and $\Lambda^G = (\Lambda^t)^H$, it follows that $\nu' + \mu' \leq 2$ by Proposition 2.3 and Lemma 2.4. The fact that $\text{sign } \Lambda^G \equiv \text{sign } q(L) \pmod{8}$ implies that $\mu' = 1$. Hence either of the following cases occurs.

	$A(L)$	$q(L)$	L
(a)	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$v(2)$	$U \oplus D_4(-1)$
(b)	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$	$u(2) \oplus v(2)$	$U(2) \oplus D_4(-1)$

Here, in each case, L is uniquely determined by $q(L)$ by Theorem 2.1. In Case (b), we have $q(\Lambda_H^t(1/2)) \cong u(2) \oplus v(2)$ by Proposition 2.3. Since $q(\Lambda_H^t(1/2))$ takes values in $\mathbb{Z}/2\mathbb{Z}$, it follows that $\Lambda_H^t(1/4)$ is an even unimodular lattice of rank 4, which contradicts to the fact that any even unimodular lattice has rank divisible by 8. Hence Case (a) occurs: $\Lambda^G = L(2) \cong U(2) \oplus D_4(-2)$.

Case $H = C_3$. The signature of Λ^G is $(1, 3)$. By a similar argument, the condition (1) implies that $A(L) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus l}$ for some $0 \leq l \leq 4$. Due to the relations $\langle 2/3 \rangle^{\oplus 2} \cong \langle -2/3 \rangle^{\oplus 2}$ and $\text{sign} \langle \pm 2/3 \rangle \equiv \pm 2 \pmod{8}$ (see [24] for details), we conclude that

$$q(L) \cong \langle 2/3 \rangle^{\oplus l-1} \oplus \langle \pm 2/3 \rangle, \quad \text{sign } q(L) \equiv 2(l-1) \pm 2 \pmod{8}.$$

We can check that either of the following cases occurs.

	$A(L)$	$q(L)$	L
(a)	$\mathbb{Z}/3\mathbb{Z}$	$\langle -2/3 \rangle$	$U \oplus A_2(-1)$
(b)	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 3}$	$\langle 2/3 \rangle^{\oplus 3}$	$U(3) \oplus A_2(-1)$

Case (b) cannot occur by (2). Therefore Case (a) occurs: $\Lambda^G \cong U(2) \oplus A_2(-2)$.

Case $H = C_4$. The signature of Λ^G is $(1, 2)$. Similarly, we find that $|\text{disc}(L)|$ is a power of 2. Moreover, $L_0^{(2)}$ and $L_2^{(2)}$ are even, and we have $L_i^{(2)} = 0$ for any $i \geq 3$. If $L_0^{(2)} = L_2^{(2)} = 0$, then $L(1/2) \cong U \oplus \langle -1 \rangle$ by the

uniqueness of indefinite odd unimodular lattices. Otherwise, $q(L) \cong \langle -1/2 \rangle$ or $u(4) \oplus \langle -1/2 \rangle$ because of the relation $\langle a/2^k \rangle \oplus v(2^{k+1}) \cong \langle 5a/2^k \rangle \oplus u(2^{k+1})$ for any a with $a \equiv 1 \pmod{2}$ (see [24] for more details). Therefore we conclude that $L \cong U(2^k) \oplus \langle -2 \rangle$ for $k = 0, 1$ or 2 . By (2), it follows that $k \neq 1, 2$. Thus $\Lambda^G \cong U(2) \oplus \langle -4 \rangle$.

Case $H = C_5$. In a similar way, we can check that L is an indefinite even lattice of rank 2 such that $A(L) \cong (\mathbb{Z}/5\mathbb{Z})^{\oplus l}$ for some $0 \leq l \leq 2$. By [8, Table 15.2], we see that

$$L \cong U, \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \text{ or } U(5).$$

The second and third cases cannot occur by (2). Hence we conclude that $\Lambda^G \cong U(2)$.

Case $H = C_6$. The signature of Λ^G is $(1, 1)$. We make use of the argument in Case $H = C_3$. Let h be a generator of H . We define $N := \Lambda^{\langle h^2, \iota \rangle}(1/2) \cong U \oplus A_2(-1)$. Then h acts on N as an involution and we have $N^h = L$. By Lemma 2.4, we can check that $A(N^h)$ and $A(N_h)$ are of the form $(\mathbb{Z}/2\mathbb{Z})^{\oplus l} \oplus (\mathbb{Z}/3\mathbb{Z})^{\oplus m}$ for some $0 \leq l \leq 2$ and $0 \leq m \leq 1$. Therefore, according to [8, Tables 15.1 and 15.2], we have

$$\begin{aligned} N^h &\cong U, U(2) \text{ or } \pm \begin{bmatrix} 2 & 0 \\ 0 & -6 \end{bmatrix}; \\ N_h &\cong A_2(-1), \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix} \text{ or } A_2(-2). \end{aligned}$$

By (1), we have $N^h \cong U$ or $U(2)$. Note that $N^h \oplus N_h$ is a sublattice of N of finite index and that N^h and N_h are primitive sublattices of N . Hence we have $N^h \cong U$ and $N_h \cong A_2(-1)$ (see Proposition 2.2). Thus $\Lambda^G = N^h(2) \cong U(2)$.

Case $H = C_2 \times C_2$. The signature of Λ^G is $(1, 3)$. An almost identical argument to that in Case $H = C_4$ shows that $L \cong U(2^k) \oplus \langle -2 \rangle^{\oplus 2}$ for $k = 0, 1$ or 2 . In order to show $k \neq 2$, we make use of the argument in Case $H = C_2$. Let $H' \cong C_2$ be a subgroup of H . Then we know that $N := \Lambda^{\langle H', \iota \rangle}(1/2) \cong U \oplus D_4(-1)$. Since H/H' acts on N as an involution and we have $N^{H/H'} = L$, it follows that $|\text{disc}(L)|$ divides 2^4 by Lemma 2.4. Therefore we have $\Lambda^G \cong U(2^{k+1}) \oplus \langle -4 \rangle^{\oplus 2}$ for $k = 0$ or 1 , and the assertion of the Key Lemma holds.

Case $H = C_2 \times C_4$. The signature of Λ^G is $(1, 1)$. Let $H' \cong C_4$ be a subgroup of H . We then have $N := \Lambda^{\langle H', \iota \rangle}(1/2) \cong U \oplus \langle -2 \rangle$ by the argument in Case $H = C_4$. Similarly, we find that $|\text{disc}(N^{H/H'})|$ and $|\text{disc}(N_{H/H'})|$ divide 2^2 . By [8, Table 15.2], we have

$$N^{H/H'} \cong U, U(2) \text{ or } \langle 2 \rangle \oplus \langle -2 \rangle; \quad N_{H/H'} \cong \langle -2 \rangle \text{ or } \langle -4 \rangle.$$

Therefore, by Proposition 2.2, we have $N^{H/H'} \cong U$ or $\langle 2 \rangle \oplus \langle -2 \rangle$, and $N_{H/H'} \cong \langle -2 \rangle$. Thus $\Lambda^G \cong U(2)$ or $\langle 4 \rangle \oplus \langle -4 \rangle$. Hence the assertion of the Key Lemma holds.

Case $H = C_3 \times C_3$. The signature of Λ^G is $(1, 1)$. We make use of the argument in Case $H = C_3$. Let $H', H'' \cong C_3$ be subgroups of H such that $H = H' \times H''$. Then $N := \Lambda^{\langle H', \iota \rangle}(1/2) \cong U \oplus A_2(-1)$. By a similar argument to the proof of Lemma 2.4, we have $\Lambda/(\Lambda^{H''} \oplus \Lambda_{H''}) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus l}$ for some l . Hence $N/(L \oplus L_N^\perp) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus m}$ for some $0 \leq m \leq 2$, and $|\text{disc}(L)|$ and $|\text{disc}(L_N^\perp)|$ divide 3^3 . Therefore, by [8, Tables 15.1 and 15.2], we have

$$L \cong U, U(3) \text{ or } \pm \begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}; \quad L_N^\perp \cong A_2(-1), A_2(-3) \text{ or } \begin{bmatrix} 2 & 1 \\ 1 & 14 \end{bmatrix}.$$

By Proposition 2.2, we can check that $L \cong U$ or $U(3)$, and that $L_N^\perp \cong A_2(-1)$. Assume that $L \cong U(3)$. Note that $N' := L_N^\perp = \Lambda^{\langle H'', \iota \rangle}(1/2)$. Hence, by interchanging H' and H'' , we have

$$N'' := \Lambda^{\langle H'', \iota \rangle}(1/2) \cong A_2(-1), \quad L \oplus N' \oplus N'' \subset \Lambda^\iota(1/2).$$

Since we have $N/(L \oplus N') \cong \mathbb{Z}/3\mathbb{Z}$, there exist elements $v \in L^\vee$ and $w \in (N')^\vee$ such that $v^2 = 2/3$, $(w')^2 = -2/3$ and $v + w' \in N$. Similarly, there exists an element $w'' \in (N'')^\vee$ such that $(w'')^2 = -2/3$ and $v + w'' \in \Lambda^{\langle H'', \iota \rangle}(1/2)$. This implies that $w' - w'' \in \Lambda^\iota(1/2)$ and that $(w' - w'')^2 = -4/3$. Since $\Lambda^\iota(1/2)$ is integral, this is a contradiction. Therefore we conclude that $\Lambda^G \cong U(2)$. \square

Proposition 4.4. *The G -invariant lattice $H^2(S, \mathbb{Z})^G$ is given by the following table³.*

G	$H^2(S, \mathbb{Z})^G$
C_2	$U(2) \oplus E_8(-2)$
$C_2 \times C_2$	$U(2) \oplus D_4(-2)$
$C_2 \times C_2 \times C_2$	$U(2) \oplus \langle -4 \rangle^{\oplus 2}$
D_6	$U(2) \oplus A_2(-2)$
D_8	$U(2) \oplus \langle -4 \rangle$
D_{10}	$U(2)$
D_{12}	$U(2)$
$C_2 \times D_8$	$U(2)$

Proof. It is worth noting that $H^2(S, \mathbb{Z})^G$ does not depend on the choice of S . By the proof of the Key Lemma, it suffices to show the assertion for $G = C_2 \times C_2$ and $C_2 \times D_8$. Note that a generic K3 surface S with a Calabi–Yau G -action is realized as a Horikawa model (Proposition 3.11) and thus $H^2(S, \mathbb{Z})^G$ contains $U(2)$, which is the pullback of the Néron–Severi lattice of $\mathbb{P}^1 \times \mathbb{P}^1$. For $G = C_2 \times C_2$, we have $H^2(S, \mathbb{Z})^G \cong U(2^{k+1}) \oplus \langle -4 \rangle^{\oplus 2}$ for $k = 0$ or 1 by the proof of the Key Lemma. Hence we have $H^2(S, \mathbb{Z})^G \cong$

³In the proof of Proposition 3.11, we already checked that Case $H = C_3 \times C_3$ does not occur.

$U(2) \oplus \langle -4 \rangle^{\oplus 2}$. Similarly, for $G = C_2 \times D_8$, we have $H^2(S, \mathbb{Z})^G \cong U(2)$ or $\langle 4 \rangle \oplus \langle -4 \rangle$. We thus conclude that $H^2(S, \mathbb{Z})^G \cong U(2)$. \square

5. PROPERTIES

In this section, we will investigate some basic properties of Calabi–Yau threefolds of type K. The explicit description obtained in the preceding section plays a central role in our study. Throughout this section, X is a Calabi–Yau threefold of type K and $\pi: S \times E \rightarrow X$ is the minimal splitting covering with Galois group G . We also fix a semi-direct decomposition $G = H \rtimes \langle \iota \rangle$.

There exist G -equivariant Ricci-flat Kähler metrics g_S and g_E on S and E respectively [31]. Then the product metric $g_S \times g_E$ on $S \times E$ descends to a Ricci-flat Kähler metric g' and g on the quotients $(S \times E)/H$ and X respectively. Let $T := S/\langle \iota \rangle$ be the Enriques surface with the metric g_T induced by g_S . We denote by $\text{Hol}_h(Y)$ the holonomy group of a manifold Y with respect to a metric h (we do not refer to a based point).

Proposition 5.1. (1) $\text{Hol}_{g_T}(T) \cong \{A \in \text{U}(2) \mid \det A = \pm 1\} \subset \text{U}(2)$.
 (2) $\text{Hol}_g(X) \cong \text{S}(\text{U}(2) \times C_2) \subset \text{SU}(3)$.

Proof. Since the holonomy group $\text{Hol}_{g_T}(T)$ cannot be $\text{SU}(2)$, it must be a C_2 -extension of $\text{Hol}_{g_S}(S) \cong \text{SU}(2)$ in $\text{U}(2)$. Such an extension is unique and this proves the first assertion. In order to prove the second assertion, we first consider the quotient $(S \times E)/H$, which admits a smooth isotrivial K3 fibration $(S \times E)/H \rightarrow E/H$. Since the action of H on S is symplectic, we see that $\text{Hol}_{g'}((S \times E)/H) \cong \text{SU}(2)$. Therefore the holonomy group $\text{Hol}_g(X)$ is an extension of $\text{SU}(2)$ in $\text{SU}(3)$ of index at most 2. Since X contains an Enriques surface, we conclude that $\text{Hol}_g(X) \cong \text{S}(\text{U}(2) \times C_2) \subset \text{SU}(3)$. \square

Proposition 5.2. *The following hold.*

- (1) $\pi_1(X) = (\mathbb{Z} \times \mathbb{Z}) \rtimes G$, where the G -action on $\mathbb{Z} \times \mathbb{Z}$ is identified with that on $\pi_1(E)$.
- (2) $H_1(X, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^n$, where the exponent n is given by the following table.

G	C_2	$C_2 \times C_2$	$C_2 \times C_2 \times C_2$	D_6	D_8	D_{10}	D_{12}	$C_2 \times D_8$
n	3	4	5	3	4	3	4	5

Proof. The first assertion readily follows from the exact sequence $0 \rightarrow \pi_1(S \times E) \rightarrow \pi_1(X) \rightarrow G \rightarrow 0$, and the Calabi–Yau G -action on E . The second follows from the fact that $H_1(X, \mathbb{Z}) \cong \pi_1(X)^{\text{Ab}}$, or the Cartan–Leray spectral sequence associated to the étale map $S \times E \rightarrow X$. \square

Proposition 5.3. *There exists no isolated (smooth) rational curve on X . Here we say a curve is isolated if it is not a member of any non-trivial family.*

Proof. Suppose that there exists an isolated rational curve $C \subset X$. Since π is étale, the pullback $\pi^{-1}(C)$ consists of $|G|$ isolated rational curves. On the

other hand, there is no isolated rational curve on the product $S \times E$ as any morphism $\mathbb{P}^1 \rightarrow E$ is constant and any smooth rational curve on any K3 surface has self-intersection number -2 . This leads us to a contradiction. \square

All rational curves show up in families (parametrized by the elliptic curve E). It is shown that they do not contribute to Gromov–Witten invariants but the higher genus quantum corrections are present at least for the Enriques Calabi–Yau threefold [22].

Proposition 5.4. $\text{Aut}(X) = \text{Bir}(X)$.

Proof. Any birational morphism between minimal models is decomposed into finitely many flops up to automorphisms [16]. Hence it is enough to prove that there exists no flop of X . In the case of threefolds, the exceptional locus of any flopping contraction must be a tree of isolated rational curves [18, Theorems 1.3 and 3.7]. The previous proposition therefore shows that there exists no flop of X . \square

Proposition 5.5. *The following hold.*

- (1) *If $G \cong D_{10}$, D_{12} or $C_2 \times D_8$, we have $|\text{Aut}(X)| < \infty$.*
- (2) *If $G \cong C_2$, $C_2 \times C_2$, $C_2 \times C_2 \times C_2$, D_6 or D_8 , and X is generic in the moduli space, we have $|\text{Aut}(X)| = \infty$.*

Proof. For $G \cong D_{10}$, D_{12} or $C_2 \times D_8$, we have $\rho(X) = 3$ and the intersection form on $H^2(X, \mathbb{Z})$ splits into the product of three linear forms. Hence the assertion (1) follows from the result of [20].

For $G \cong C_2$, $C_2 \times C_2$, $C_2 \times C_2 \times C_2$, D_6 or D_8 , the K3 surface S is realized as a $(2, 2, 2)$ -hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (Proposition 3.13). Hence, by [7], $\text{Aut}(S)$ contains $C_2 * C_2 * C_2$, which commutes with G . Therefore the assertion (2) follows. \square

Remark 5.6. *In Proposition 5.5 (2), the genericity assumption is essential at least in the case $G = C_2$. In fact, if $G = C_2$, it follows that $\text{Aut}(X)$ is infinite if and only if the automorphism group of the Enriques surface $S/\langle \iota \rangle$ is infinite. Although the automorphism group of a generic Enriques surface is infinite, there exist Enriques surfaces with finite automorphism group (see [19] for the classification of such Enriques surfaces).*

It is known that the automorphism group of a Calabi–Yau threefold with $\rho = 1, 2$ is finite [25]. On the other hand, it is expected that there is a Calabi–Yau threefold with infinite automorphism group for each $\rho \geq 4$ (see for example [5, 11]). Proposition 5.5 provides a supporting evidence for this folklore conjecture, giving examples for small and new ρ . It is an open problem whether or not a Calabi–Yau threefold with $\rho = 3$ admits infinite automorphism group [20].

6. CALABI–YAU THREEFOLDS OF TYPE A

In this final section, we slightly change the topic and probe Calabi–Yau threefolds of type A. Recall that a Calabi–Yau threefold is called of type A if it is an étale quotient of an abelian threefold. By refining Oguiso and Sakurai’s fundamental work [26] on Calabi–Yau threefolds of type A, we will finally settle the full classification of Calabi–Yau threefolds with infinite fundamental group (Theorem 6.4).

Let $A := \mathbb{C}^d/\Lambda$ be a d -dimensional complex torus. There is a natural semi-direct decomposition $\text{Aut}(A) = A \rtimes \text{Aut}_{\text{Lie}}(A)$, where the first factor is the translation group of A and $\text{Aut}_{\text{Lie}}(A)$ consists of elements that fix the origin of A . We call the second factor of $g \in \text{Aut}(A)$ the Lie part of g and denote it by g_0 . The fundamental result in the theory of Calabi–Yau threefolds of type A is the following.

Theorem 6.1 (Oguiso–Sakurai [26, Theorem 0.1]). *Let X be a Calabi–Yau threefold of type A. Then the following hold.*

- (1) $X = A/G$, where A is an abelian threefold and G is a finite group acting freely on A in such a way that either of the following is satisfied:

- (a) $G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2$ and

$$a_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad b_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

- (b) $G = \langle a, b \mid a^4 = b^2 = abab = 1 \rangle \cong D_8$ and

$$a_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad b_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where a_0 and b_0 are the Lie part of a and b respectively and the matrix representation is the one given by an appropriate realization of A as \mathbb{C}^3/Λ .

- (2) In the first case, $\rho(X) = 3$ and in the second case $\rho(X) = 2$.
- (3) Both cases really occur.

Theorem 6.1 provides a classification of the Lie part of the Galois groups of the minimal splitting coverings, where the Galois groups do not contain any translation element. We will see that, in contrast to Calabi–Yau threefolds of type K, Calabi–Yau threefolds of type A are not classified by the Galois groups of the minimal splitting coverings. That is, a choice of Galois group does not determine the deformation family of a Calabi–Yau threefold of type A. We improve Theorem 6.1 by allowing non-minimal splitting coverings as follows.

Proposition 6.2. *Let X be a Calabi–Yau threefold of type A. Then X is isomorphic to the étale quotient A/G of an abelian threefold A by an action of a finite group G , where A and G are given by the following.*

- (1) $A = A'/T$, where A' is the direct product of three elliptic curves E_1, E_2 and E_3 :

$$A' := E_1 \times E_2 \times E_3, \quad E_i := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau_i), \quad \tau_i \in \mathbb{H}$$

and T is one of the subgroups of A' in the following table, which consists of 2-torsion points of A' .

T_1	T_2	T_3	T_4
0	$\langle (0, 1/2, 1/2)_{A'} \rangle$	$\langle (1/2, 1/2, 0)_{A'}, (1/2, 0, 1/2)_{A'} \rangle$	$\langle (1/2, 1/2, 1/2)_{A'} \rangle$

Here $(z_1, z_2, z_3)_{A'}$ denotes the image of $(z_1, z_2, z_3) \in \mathbb{C}^3$ in A' .

- (2) $G \cong C_2 \times C_2$ or D_8 .

(a) If $G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2$, then G is generated by

$$a: (z_1, z_2, z_3)_A \mapsto (z_1 + \tau_1/2, -z_2, -z_3)_A,$$

$$b: (z_1, z_2, z_3)_A \mapsto (-z_1, z_2 + \tau_2/2, -z_3 + \tau_3/2)_A.$$

(b) If $G = \langle a, b \mid a^4 = b^2 = abab = 1 \rangle \cong D_8$, then $\tau_2 = \tau_3 =: \tau$, $T = T_2$ or T_3 , and G is generated by

$$a: (z_1, z_2, z_3)_A \mapsto (z_1 + \tau_1/4, -z_3, z_2)_A,$$

$$b: (z_1, z_2, z_3)_A \mapsto (-z_1, z_2 + \tau/2, -z_3 + (1 + \tau)/2)_A.$$

Moreover, each case really occurs.

Proof. By Theorem 6.1, X is of the form A/G with G isomorphic to either $C_2 \times C_2$ or D_8 . Let \mathbb{C}^3/Λ be a realization of A as a complex torus. In the case $G \cong C_2 \times C_2$, we may assume that G is generated by

$$a: (z_1, z_2, z_3)_A \mapsto (z_1 + u_1, -z_2, -z_3)_A,$$

$$b: (z_1, z_2, z_3)_A \mapsto (-z_1, z_2 + u_2, -z_3 + u_3)_A,$$

after changing the origin of A if necessary. Hence Λ is stable under the following actions:

$$a_0: (z_1, z_2, z_3) \mapsto (z_1, -z_2, -z_3),$$

$$b_0: (z_1, z_2, z_3) \mapsto (-z_1, z_2, -z_3).$$

From this, we see that there exist lattices $\Lambda_i \subset \mathbb{C}$ for $i = 1, 2, 3$ such that

$$2\Lambda \subset \Lambda_1 \times \Lambda_2 \times \Lambda_3 \subset \Lambda.$$

Let e_1, e_2, e_3 be the standard basis of \mathbb{C}^3 . Set

$$\Lambda' := \Lambda'_1 \times \Lambda'_2 \times \Lambda'_3 \subset \Lambda, \quad \Lambda'_i := \{z \in \mathbb{C} \mid ze_i \in \Lambda\}.$$

Then Λ/Λ' is a 2-elementary group, that is, $\Lambda/\Lambda' \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus n}$ for some n . Since $a^2 = b^2 = (ab)^2 = \text{id}_A$ and the G -action is free, we have $u_i \notin \Lambda'_i$ but $2u_i \in \Lambda'_i$. Let $v = (v_1, v_2, v_3) \in \Lambda$. Suppose $v_1 \equiv u_1 \pmod{\Lambda'_1}$, then $(z_1, v_2/2, v_3/2) \in A^a$. Hence we conclude that $v_1 \not\equiv u_1 \pmod{\Lambda'_1}$. Similarly, $v_i \not\equiv u_i \pmod{\Lambda'_i}$ for $i = 2, 3$. Therefore, we may assume that there exist $\tau_i \in \mathbb{H}$ for $i = 1, 2, 3$ such that

- (1) $\Lambda'_i = \mathbb{Z} \oplus \mathbb{Z}\tau_i$,

- (2) $u_i \equiv \tau_i/2 \pmod{\Lambda'_i}$,
- (3) $v_i \equiv 0 \text{ or } 1/2 \pmod{\Lambda'_i}$ for all $v \in \Lambda$,

after changing each coordinate z_i if necessary. Now that we can check the assertion of the theorem in this case by a direct computation. In particular, $T = \Lambda/\Lambda'$ coincides with one in the table up to permutation of the coordinates.

Similarly, in the case $G \cong D_8$, we may assume that G is generated by

$$\begin{aligned} a: (z_1, z_2, z_3)_A &\mapsto (z_1 + u_1, -z_3, z_2)_A, \\ b: (z_1, z_2, z_3)_A &\mapsto (-z_1, z_2 + u_2, -z_3 + u_3)_A. \end{aligned}$$

We use the same notation as above. It follows that $\Lambda'_2 = \Lambda'_3$, $4u_1 \in \Lambda'_1$, $2u_1 \notin \Lambda'_1$, $2u_i \in \Lambda'_i$, $u_i \notin \Lambda'_i$ for $i = 2, 3$. We have

$$\begin{aligned} ab: (z_1, z_2, z_3)_A &\mapsto (-z_1 + u_1, z_3 - u_3, z_2 + u_2)_A \\ (ab)^2: (z_1, z_2, z_3)_A &\mapsto (z_1, z_2 + u_2 - u_3, z_3 + u_2 - u_3)_A. \end{aligned}$$

By $(ab)^2 = 1$ and $A^{ab} = \emptyset$, it follows that $(0, u_2 - u_3, u_2 - u_3) \in \Lambda$ and $u_2 - u_3 \notin \Lambda'_2$. Since the action of $SL(2, \mathbb{Z})$ on the set of level 2 structures on an elliptic curve is transitive, we may assume that $\tau_2 = \tau_3 =: \tau$, $u_2 = \tau_2/2$, $u_3 = (1 + \tau_3)/2$. By a similar argument to the case $G \cong C_2 \times C_2$, we can check that $v_i \equiv 0 \text{ or } 1/2 \pmod{\Lambda'_i}$ for any $v = (v_1, v_2, v_3) \in \Lambda$. In particular, we have $(0, 1/2, 1/2) \in \Lambda$. Since $T = T_4$ implies that $(1/2, 0, 0) \in \Lambda'_1$, which is a contradiction, it follows that T is either T_2 or T_3 . Moreover, we can check that the action of G has no fixed point for $T = T_2, T_3$. \square

Remark 6.3. *The above four cases for $G \cong C_2 \times C_2$ have previously been studied by Donagi and Wendland [9].*

As was mentioned earlier, in contrast to Calabi–Yau threefolds of type K, Calabi–Yau threefolds of type A are not classified by the Galois groups of the minimal splitting coverings. They are classified by the minimal totally splitting coverings, where abelian threefolds A which cover X split into the product of three elliptic curves.

Together with Theorem 3.1, Proposition 6.2 finally completes the full classification of Calabi–Yau threefolds with infinite fundamental group:

Theorem 6.4. *There exist precisely fourteen Calabi–Yau threefolds with infinite fundamental group, up to deformation equivalence. To be more precise, six of them are of type A and eight of them are of type K.*

REFERENCES

- [1] P. Aspinwall, An $N = 2$ Dual Pair and a Phase Transition, Nucl. Phys. B 460 (1996), 57–76.
- [2] W. P. Barth, K. Hulek, C. A. M. Peters and A. van de Ven, Compact complex surfaces, Second edition, Springer-Verlag, Berlin, 2004.
- [3] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Diff. Geom. 18 (1983), no. 4, 755–782.

- [4] A. Beauville, Some remarks on Kähler manifolds with $c_1 = 0$, in *Classification of Algebraic and Analytic Manifolds*, K. Ueno, ed., Progress Math. 39 (1983), 1–26.
- [5] C. Borcea, On desingularized Horrocks–Mumford quintics, *J. Reine Angew. Math.* 421 (1991), 23–41.
- [6] D. Burns and M. Rapoport, On the Torelli problem for Kählerian $K3$ surfaces, *Ann. Sci. École Norm. Sup. (4)* 8 (1975), no. 2, 235–273.
- [7] S. Cantat and K. Oguiso, Birational automorphism groups and the movable cone theorem for Calabi–Yau manifolds of Wehler type via universal Coxeter groups, *Amer. J. Math.* 137 (2015), no. 4, 1013–1044.
- [8] J. H. Conway and N. J. Sloane, *Sphere packings, lattices and groups*, Third edition, Springer-Verlag, New York, 1999.
- [9] R. Donagi and K. Wendland, On orbifolds and fermion constructions, *J. Geom. Phys.* 59 (2009), no. 7, 942–968.
- [10] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, Second quantized mirror symmetry, *Phys. Lett. B* 361 (1995), 59–65.
- [11] A. Grassi and D. Morrison, Automorphisms and the Kähler cone of certain Calabi–Yau manifolds, *Duke Math. J.* 71 (1993), no. 3, 831–838.
- [12] R. Hartshorne, *Algebraic geometry*, GTM 52, Springer-Verlag, New York-Heidelberg, 1977.
- [13] K. Hashimoto and A. Kanazawa, Calabi–Yau threefolds of type K (II) Mirror Symmetry, *Comm. in Number Theory and Physics*. 2016;10 (2) 157–192.
- [14] H. Ito and H. Ohashi, Classification of involutions on Enriques surfaces, *arXiv:1302.2316*.
- [15] A. Kanazawa and P. H. M. Wilson, Trilinear forms and Chern classes of Calabi–Yau threefolds, *Osaka J. Math.* Vol 51. No 1, 203–213.
- [16] Y. Kawamata, Flops connect minimal models, *Publ. Res. Inst. Math. Sci.* 44 (2008), 419–423.
- [17] A. Klemm and M. Mariño, Counting BPS states on the Enriques Calabi–Yau, *Comm. Math. Phys.* 280 (2008), no. 1, 27–76.
- [18] J. Kollár and S. Mori, *Birational geometry of algebraic varieties* (English translation), Cambridge Tracts in Mathematics, 134, 1998.
- [19] S. Kondō, Enriques surfaces with finite automorphism groups, *Japan. J. Math.* 12 (1986), 191–282.
- [20] V. Lazić, K. Oguiso and T. Peternell, Automorphisms of Calabi–Yau threefolds with Picard number three, *arXiv:1310.8151*.
- [21] M. Mariño, Gromov–Witten invariants and topological strings: a progress report, *International Congress of Mathematicians Vol. III*, 409–419, Eur. Math. Soc., Zürich, 2006.
- [22] D. Maulik and R. Pandharipande, New calculations in Gromov–Witten theory, *PAMQ* 4 (2008), 469–500.
- [23] V. V. Nikulin, Finite groups of automorphisms of Kählerian surfaces of Type $K3$, *Trudy Moskov. Mat. Obshch.* 38 (1979), 75–137.
- [24] V. V. Nikulin, Integer symmetric bilinear forms and some of their geometric applications, *Izv. Akad. Nauk SSSR Ser. Mat.* 43 (1979), no. 1, 111–177.
- [25] K. Oguiso, Automorphism groups of Calabi–Yau manifolds of Picard number two, to appear in *J. Alg. Geom.*
- [26] K. Oguiso and J. Sakurai, Calabi–Yau threefolds of quotient type, *Asian J. Math.* 5 (2001), no.1, 43–77.
- [27] I. I. Pjateckiĭ–Šapiro and I. R. Šafarevič, Torelli’s theorem for algebraic surfaces of type $K3$ (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* 35 (1971), 530–572.
- [28] O. T. O’Meara, *Introduction to quadratic forms*, Springer-Verlag, Berlin, 1973.
- [29] B. Saint-Donat, Projective models of $K3$ surfaces, *Math. Z.* 189 (1985), 1083–1119.

- [30] A. Todorov, Applications of the Kähler–Einstein–Calabi–Yau metric to moduli of K3 surfaces, *Invent. Math.*, 61 (1980), 251–265.
- [31] S. -T. Yau, On the Ricci curvature of compact Kähler manifold and the complex Monge–Ampere equation I, *Comm. Pure Appl. Math* 31 (1978), no. 3, 339–411.

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