L^p-CONVERGENCE OF THE LAPLACE–BELTRAMI EIGENFUNCTION EXPANSIONS

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Abstract. We provide a simple sufficient condition for the L^p-convergence of the Laplace–Beltrami eigenfunction expansions of functions on a compact Riemannian manifold with a Dirichlet boundary condition.
Nous fournissons une condition suffisante simple pour la convergence L^p de Laplace–Beltrami expansions de fonctions sur une variété riemannienne compacte avec une condition aux limites de Dirichlet.

1. Introduction

Let M be a smooth compact Riemannian manifold possibly with boundary and Δ the Laplace–Beltrami operator associated to the Riemannian metric [6]. We denote by $C_0^k(M)$ the space of continuous functions on M which are C^k on the interior M^o with the Dirichlet condition $f|_{\partial M} = 0$[1]. We define the Hilbert space $L^2(M)$ as the completion of $C_0^\infty(M)$ with respect to the inner product $\langle f, g\rangle := \int_M f(x)g(x)dx$, where we simply write $dx := \mathrm{dvol}(x)$. The Dirichlet eigenfunctions $\{\omega_i\}_i$ of the operator Δ form an orthonormal basis of $L^2(M)$. Then for a function $f \in L^2(M)$ we have the eigenfunction expansion

$$f(x) = \sum_{i=1}^{\infty} f_i \omega_i(x), \quad f_i := \langle f, \omega_i\rangle.$$

In light of the Fourier analysis, we would like to propose the following question: *when does the right-hand side L^p-converges to the function $f(x)$?*. It is a classical fact that the Fourier series expansion of a C^1-function on the circle S^1 absolutely and uniformly converges [8, Theorem 3.10.2]. Moreover, a C^1-function on the 2-sphere S^2 admits a uniformly convergent series expansion by the spherical harmonics [4, Chap.10.4]. We refer the reader to the articles [5, 2] for results on the uniform convergence of the expansion of a function in terms of spherical harmonics in higher dimensions.

The objective of this short article is to record the following simple theorem that works for a smooth compact Riemannian manifold M of dimension $m \geq 2$.

2010 *Mathematics Subject Classification.* 58J05, 35P10.
Key words and phrases. Laplace–Beltrami operator, eigenfunction expansion, L^p-convergence, Dirichlet boundary condition, Green's function.

[1]No condition is imposed when M has no boundary.

Theorem 1.1. *Let k be the minimal integer greater than $\frac{m}{4}$. Then the Laplace–Beltrami eigenfunction expansion of a function $f \in L^2(M)$ $L^{\frac{2k}{k-l}}$-converges[2] if $f \in C_0^{2l}(M)$ for an integer $1 \leq l \leq k$.*

At this point, we do not know whether or not the above estimate provides a sharp bound for the Riemannian manifolds of dimension $m \geq 2$ (see Section 3 for a discussion). The only result in this direction known to the author is the fact that if M has no boundary and a function f is in the Sobolev space $H^k(M)$, then the convergence of the Laplace–Beltrami eigenfunction expansion is also in $H^k(M)$ [3, proof of Theorem 3.1]. It is also worth mentioning that the Riesz mean version of the L^p-convergence problem is discussed in [7]. We hope that our simple result is useful in geometric analysis of Riemannian manifolds.

2. Proof of Theorem

We denote by $L^p(M)$ the space of L^p-integrable functions on M where the norm is defined by $\|f\|_p := \left(\int_M |f(x)|^p dx\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|f\|_\infty := \max_{x \in M}(|f(x)|)$ for $p = \infty$. The Hölder conjugate of p is denoted by p'. The essential ingredients of the proof of Theorem 1.1 are the invertible elliptic operator $\Delta - I$, where I denotes the identity operator, and its Green's function $G(x,y) = G_x(y)$. We define an operator Gr by

$$\mathrm{Gr}[f](x) := \int_M G_x(y) f(y) dy,$$

where the domain of the operator Gr will be specified in Lemma 2.1 below.

We assume that $m \geq 3$ until otherwise stated. Then we observe that $G_x(y) \in L^p(M)$ for $p < \frac{m}{m-2}$ because for sufficiently close x and y in M the Green's function behaves as $|G_x(y)| \leq C \mathrm{d}(x,y)^{2-m}$ for some constant $C > 0$. Here $\mathrm{d}(*, **)$ denotes the distance defined by the Riemannian metric on M. The following lemma is an analogue of Young's inequality for convolutions [1].

Lemma 2.1. *For $1 \leq p < \frac{m}{m-2}$, $q \geq 1$, $r \geq 1$ and $\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r} = 1$. Then for $f \in L^q(M)$ we have $\mathrm{Gr}[f] \in L^r$ and*

$$\|\mathrm{Gr}[f]\|_r \leq K_p \|f\|_q,$$

where $K_p := \sup_{x \in M}\left(\|G_x\|_p\right) < \infty$.

Proof. The proof is almost identical to that of Young's inequality, but we include it here for the sake of completeness. First the generalized Hölder

[2] Throughout the article, a fraction with 0 in the denominator is always understood as ∞.

inequality shows that

$$\begin{aligned}
\big|\operatorname{Gr}[f](x)\big| &\le \int_M |G_x(y)|^{\frac{p}{r}}|G_x(y)|^{1-\frac{p}{r}}|f(y)|^{\frac{q}{r}}|f(y)|^{1-\frac{q}{r}}dy \\
&\le \Big(\int_M |G_x(y)|^p|f(y)|^q dy\Big)^{\frac{1}{r}}\Big(\int_M |G_x(y)|^{q'(1-\frac{p}{r})}dy\Big)^{\frac{1}{q'}}\Big(\int_M |f(y)|^{p'(1-\frac{q}{r})}dy\Big)^{\frac{1}{p'}} \\
&= \Big(\int_M |G_x(y)|^p|f(y)|^q dy\Big)^{\frac{1}{r}}\big\|G_x\big\|_p^{1-\frac{p}{r}}\big\|f\big\|_q^{1-\frac{q}{r}},
\end{aligned}$$

where we used relations $q'(1-\frac{r}{p}) = p$ and $p'(1-\frac{r}{q'}) = q$. Therefore it follows that

$$\begin{aligned}
\int_M \big|\operatorname{Gr}[f](x)\big|^r dx &\le \big\|f\big\|_q^{r-q}\int_M \big\|G_x\big\|_p^{r-p}\Big(\int_M |G_x(y)|^p|f(y)|^q dy\Big)dx \\
&\le K_p^{r-p}\big\|f\big\|_q^{r-q}\int_M \Big(\int_M |G_y(x)|^p dx\Big)|f(y)|^q dy \\
&\le K_p^{r-p}\big\|f\big\|_q^{r-q}\int_M \big\|G_y\big\|_p^p|f(y)|^q dy \\
&\le K_p^r\big\|f\big\|_q^r.
\end{aligned}$$

Moreover $K_p < \infty$ because M is compact and $1 \le p < \frac{m}{m-2}$. □

Lemma 2.2. *For an integer $k > \frac{m}{4}$ and a function $f \in L^2(M)$, we have $\operatorname{Gr}^{l-1}[f] \in L^{\frac{2k}{k-l+1}}(M)$ and*

$$\big\|\operatorname{Gr}^l[f]\big\|_{\frac{2k}{k-l}} \le K_{\frac{2k}{2k-1}}\big\|\operatorname{Gr}^{l-1}[f]\big\|_{\frac{2k}{k-l+1}}.$$

for $1 \le l \le k$. Here Gr^l denotes l-th iteration of Gr.

Proof. We can prove the assertion by induction on l, by setting

$$p = \frac{2k}{2k-1}, \quad q = \frac{2k}{k-l+1}, \quad r = \frac{2k}{k-l}$$

in Lemma 2.1. Here the assumption $k > \frac{m}{4}$ guarantees that $p < \frac{m}{m-2}$. We leave it to the reader to check the above p, q and r satisfy the assumptions of Lemma 2.1 for $1 \le l \le k$. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let k and l be integers such that $k > \frac{m}{4}$ and $1 \le l \le k$. Let $f(x) = \sum_{i=1}^{\infty} f_i\omega_i(x)$ be the eigenfunction expansion of a function $f \in C_0^{2l}(M)$. A repeated use of integration by parts shows that

$$\operatorname{Gr}^l[(\Delta - I)^l f] = (-1)^l f$$

because there is no boundary contribution by the Dirichlet condition. Let λ_i be the eigenvalue of the eigenfunction $\omega_i(x)$. Then it follows that for an arbitrary $n \in \mathbb{N}$

$$\begin{aligned}\Big\| f - \sum_{i=1}^{n} f_i \omega_i \Big\|_{\frac{2k}{k-l}} &= \Big\| \mathrm{Gr}^l \left[(I-\Delta)^l f - \sum_{i=1}^{n} (1-\lambda_i)^l f_i \omega_i \right] \Big\|_{\frac{2k}{k-l}} \\ &\leq K^l_{\frac{2k}{2k-1}} \Big\| (I-\Delta)^l f - \sum_{i=1}^{n} (1-\lambda_i)^l f_i \omega_i \Big\|_2 \end{aligned} \tag{1}$$

where we repeatedly used Lemma 2.2 in the second line. On the other hand, the eigenfunction expansion of $(I-\Delta)^l f(x) \in L^2(M)$ is given by

$$(I-\Delta)^l f(x) = \sum_{i=1}^{\infty} (1-\lambda_i)^l f_i \omega_i(x)$$

by the self-adjointness of $I - \Delta$. Therefore we conclude that the right-hand side of Eq.(1) converges to zero as n increases. This proves the $L^{\frac{2k}{k-l}}$-convergence of the eigenfunction expansion of $f \in C_0^{2l}(M)$ for $m \geq 3$.

We finally deal with the case where $m = 2$. With the same notation as before, we have $K_2 = \sup_{x \in M} \left(\|G_x\|_2 \right) < \infty$ because for sufficiently close x and y in M the Green's function behaves as $|G_x(y)| \leq -C \log(\mathrm{d}(x,y))$ for some constant $C > 0$. Then, for $p = 2$, $k = 1$ and $l = 1$, an almost identical argument works for Lemma 2.1 and Lemma 2.2 to obtain the estimate $\| \mathrm{Gr}[f] \|_\infty \leq K_2 \|f\|_2$. The rest follows from the same argument as the last paragraph. □

3. Optimality

Lastly we make an observation about the optimality of Theorem 1.1. The best result known for the sphere is the following:

Theorem 3.1 (Ragozin [5] and Kalf [2]). *Assume that $m \geq 2$. Then a function $f \in C^{[\frac{m}{2}]}(S^m)$ admits a uniform convergent expansion in terms of spherical harmonics.*

Moreover, the regularity assumption in Theorem 3.1 is known to be optimal [2, Remark 2]. Although our general result does not recover the above optimal estimate, the defect is at worst twice differentiability in the spheres case. As was mentioned earlier, we do not know at this point whether or not our estimate provides a sharp bound for $m \geq 2$ and $1 \leq p \leq \infty$, and we hope to come back to this optimality problem in future work.

Acknowledgement. The problem discussed in this article stemmed from Professor Y.-H. Kim's class on PDE in 2012 Fall. The author is very grateful to C. Behan, J. Ma and Professor Y.-H. Kim for useful discussion on the subject.

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