CALABI–YAU THREEFOLDS OF TYPE K (II): MIRROR SYMMETRY

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ABSTRACT. A Calabi–Yau threefold is called of type K if it admits an étale Galois covering by the product of a K3 surface and an elliptic curve. In our previous paper [16], based on Oguiso–Sakurai's fundamental work [24], we have provided the full classification of Calabi–Yau threefolds of type K and have studied some basic properties thereof. In the present paper, we continue the study, investigating them from the viewpoint of mirror symmetry. It is shown that mirror symmetry relies on duality of certain sublattices in the second cohomology of the K3 surface appearing in the minimal splitting covering. The duality may be thought of as a version of the lattice duality of the anti-symplectic involution on K3 surfaces discovered by Nikulin [23]. Based on the duality, we obtain several results parallel to what is known for Borcea–Voisin threefolds. Along the way, we also investigate the Brauer groups of Calabi–Yau threefolds of type K.

1. INTRODUCTION

The present paper studies mirror symmetry and some topological properties of Calabi–Yau threefolds of type K. A Calabi–Yau threefold[1] is a compact Kähler threefold X with trivial canonical bundle $\Omega_X^3 \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. In [24], Oguiso and Sakurai call X a Calabi–Yau threefold *of type K* if it admits an étale Galois covering by the product of a K3 surface and an elliptic curve. Among many candidates of such coverings, we can always find a unique smallest one, up to isomorphism as a covering, and we call it the minimal splitting covering. The importance of this class of Calabi–Yau threefolds comes from the fact that it is one of the two classes of Calabi–Yau threefolds with infinite fundamental group. In our previous work [16], based on Oguiso–Sakurai's fundamental work, we have given the full classification of Calabi–Yau threefolds of type K.

Theorem 1.1 ([24, 16])**.** *There exist exactly eight Calabi–Yau threefolds of type K, up to deformation equivalence. The equivalence class is uniquely determined by the Galois group of the minimal splitting covering, which is isomorphic to one of the following combinations of cyclic and dihedral*

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[1]We do not assume that X is simply-connected. One reason is that simply-connected Calabi–Yau threefolds are not closed under mirror symmetry; a mirror partner of a simply-connected Calabi–Yau threefold may not be simply-connected [11]

groups:

$$C_2, \ C_2 \times C_2, \ C_2 \times C_2 \times C_2, \ D_6, \ D_8, \ D_{10}, \ D_{12}, \ \text{and} \ C_2 \times D_8.$$

The Hodge numbers $h^{1,1} = h^{2,1}$ are respectively given by $11, 7, 5, 5, 4, 3, 3, 3$.

In [16], we also obtain explicit presentations of the eight Calabi–Yau three-folds of type K. Although Calabi–Yau threefolds of type K are very special, their explicit nature makes them an exceptionally good laboratory for general theories and conjectures. Indeed, the simplest example, known as the Enriques Calabi–Yau threefold, has been one of the most tractable compact Calabi–Yau threefolds (for example [10, 19, 18]). The objective of this paper is to investigate Calabi–Yau threefolds of type K with a view toward self-mirror symmetry.

Here is a brief summary of the paper. Let X be a Calabi–Yau threefold of type K and $\pi \colon S \times E \to X$ its minimal splitting covering, with a K3 surface S and an elliptic curve E. The geometry of X is equivalent to the G-equivariant geometry of the covering space $S \times E$, where $G := \mathrm{Gal}(\pi)$ is the Galois group of the covering π. Moreover, G turns out to be of the form $G = H \rtimes C_2$ and it acts on each factor, S and E. Let $M_G := H^2(S, \mathbb{Z})^G$ and $N_G := H^2(S, \mathbb{Z})_{C_2}^H$ (see Section 2.1 for the notation). The former represents the G-equivariant algebraic cycles and the latter the transcendental cycles. The first main result is an existence of duality between these lattices.

Theorem 1.2 (Theorem 3.9)**.** *There exists a lattice isomorphism $U \oplus M_G \cong N_G$ over the rational numbers $\mathbb{Q}$ (or some extension of $\mathbb{Z}$), where U is the hyperbolic lattice.*

Recall that Calabi–Yau threefolds of type K are close cousins of Borcea–Voisin threefolds [6, 29], whose mirror symmetry stems from the lattice duality of the anti-symplectic involution on K3 surfaces discovered by Nikulin [23]. Theorem 1.2 can be thought of as an H-equivariant version of Nikulin's duality, although it does not hold over $\mathbb{Z}$. It also indicates the fact that X is self-mirror symmetric[2]. Based on this fundamental duality, we obtain several results on the Yukawa couplings (Theorem 4.3) and special Lagrangian fibrations (Propositions 5.5 & 5.6) parallel to what is known for Borcea–Voisin threefolds (Voisin [29] and Gross–Wilson [13]). The mirror symmetry for Borcea–Voisin threefolds has a different flavour from that for the complete intersection Calabi–Yau threefolds in toric varieties and homogeneous spaces, and hence it has been a very important source of examples beyond the Batyrev–Borisov toric mirror symmetry. We hope that our work provides new examples of interesting mirror symmetry.

We also investigate Brauer groups, which are believed to play an important role in mirror symmetry but have not been much explored in the literature (for example [3, 5]). An importance of the Brauer group $\mathrm{Br}(X)$ of

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[2]It is important to note that the K3 surface S is in not self-mirror symmetry in the sense of Dolgachev [9] unless $G = C_2$, i.e. H is trivial.

a Calabi–Yau threefold X lies in the fact [1] that it is intimately related to another torsion group $H_1(X, \mathbb{Z})$ and the derived category $\mathrm{D^bCoh}(X)$. An explicit computation shows the second main result.

Theorem 1.3 (Theorem 3.16). *Let X be a Calabi–Yau threefold of type K, then $\mathrm{Br}(X) \cong \mathbb{Z}_2^{\oplus m}$, where m is given by the following.*

G	C_2	$C_2 \times C_2$	$C_2 \times C_2 \times C_2$	D_6	D_8	D_{10}	D_{12}	$C_2 \times D_8$
m	1	2	3	1	2	1	2	3

As an application, we show that any derived equivalent Calabi–Yau threefolds of type K have isomorphic Galois groups of the minimal splitting coverings (Corollary 3.17). It is also interesting to observe that $H_1(X, \mathbb{Z}) \cong \mathrm{Br}(X) \oplus \mathbb{Z}_2^{\oplus 2}$ holds in our case. The role of the Brauer group in mirror symmetry is tantalising, and deserves further explorations. For example, Gross discusses in [12] the subject in the context of SYZ mirror symmetry [28]. Unfortunately, we are not able to unveil the role of Brauer group in mirror symmetry of a Calabi–Yau threefolds of type K at this point. We hope that our explicit computation is useful for future investigation.

Structure of Paper. Section 2 sets conventions and recalls some basics of lattices and K3 surfaces. Section 3 begins with a brief review of the classification [16] of Calabi–Yau threefolds of type K and then investigates their topological properties, in particular their Brauer groups. Section 4 is devoted to the study of mirror symmetry, probing the A- and B-Yukawa couplings. Section 5 explores special Lagrangian fibrations inspired by the Strominger–Yau–Zaslow conjecture [28].

2. Lattices and K3 surfaces

In this section we summarize some basics of lattices and K3 surfaces, following [4, 22].

2.1. **Lattices.** A lattice is a free $\mathbb{Z}$-module L of finite rank together with a symmetric bilinear form $\langle *, ** \rangle \colon L \times L \to \mathbb{Z}$. By an abuse of notation, we often denote a lattice simply by L. Given a basis, the bilinear form is represented by a Gram matrix and the discriminant $\mathrm{disc}(L)$ is the determinant of the Gram matrix. We define $L(\lambda)$ to be the lattice obtained by multiplying the bilinear form by $\lambda \in \mathbb{Q}$. We denote by $\langle a \rangle$ the lattice of rank 1 generated by x with $x^2 := \langle x, x \rangle = a$. A lattice L is called even if $x^2 \in 2\mathbb{Z}$ for all $x \in L$. L is non-degenerate if $\mathrm{disc}(L) \neq 0$ and unimodular if $\mathrm{disc}(L) = \pm 1$. If L is a non-degenerate lattice, the signature of L is the pair (t_+, t_-) where t_+ and

t_- respectively denote the numbers of positive and negative eigenvalues of the Gram matrix.

A sublattice M of a lattice L is a submodule of L with the bilinear form of L restricted to M. It is called primitive if L/M is torsion free. We denote by $M_L^\perp$ (or simply $M^\perp$) the orthogonal complement of M in L. We always assume that an action of a group G on a lattice L preserves the bilinear form. Then the invariant part L^G and the coinvariant part L_G are defined as

$$L^G := \{x \in L \mid g \cdot x = x \ (\forall g \in G)\}, \quad L_G := (L^G)_L^\perp.$$

We simply denote $L^{\langle g \rangle}$ and $L_{\langle g \rangle}$ by L^g and L_g respectively for $g \in G$. If another group H acts on L, we denote $L^G \cap L_H$ by L_H^G.

The hyperbolic lattice U is the lattice given by the Gram matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The corresponding basis e, f is called the standard basis. Let A_m, D_n, E_l, $(m \geq 1, \ n \geq 4, \ l = 6, 7, 8)$ be the lattices defined by the corresponding Cartan matrices. Every indefinite even unimodular lattice is realized as an orthogonal sum of copies of U and $E_8(\pm 1)$ in an essentially unique way, the only relation being $E_8 \oplus E_8(-1) \cong U^{\oplus 8}$. Hence an even unimodular lattice of signature $(3, 19)$ is isomorphic to $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, which is called the K3 lattice.

Let L be a non-degenerate even lattice. We have a natural identification

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z} \ (\forall y \in L)\}.$$

The discriminant group $A(L) := L^\vee/L$ is a finite abelian group of order $|\mathrm{disc}(L)|$ equipped with a quadratic map $q(L) \colon A(L) \to \mathbb{Q}/2\mathbb{Z}$ given by $x + L \mapsto x^2 + 2\mathbb{Z}$. The genus of L is defined as the set of isomorphism classes of lattices L' such that the signature of L' is the same as that of L and $q(L) \cong q(L')$. (We sometimes write $q(L) \cong q(L')$ instead of $(A(L), q(L)) \cong (A(L'), q(L'))$.)

Theorem 2.1 ([22, 25])**.** *Let L be a non-degenerate even lattice. If $L \cong U(n) \oplus L'$ for a positive integer n and a lattice L', then the genus of L consists of only one class.*

Let L be an even lattice and M a module such that $L \subset M \subset L^\vee$. We say that M equipped with the induced bilinear form $\langle *, ** \rangle$ is an overlattice of L if $\langle *, ** \rangle$ takes integer values on M. Suppose that M is an overlattice of L. Then the subgroup $W := M/L \subset A(L)$ is isotropic, that is, the restriction of $q(L)$ to W is zero. There is a natural isomorphism $A(M) \cong W^\perp/W$, where

$$W^\perp = \{x + L \in A(L) \mid \langle x, y \rangle \equiv 0 \bmod \mathbb{Z} \ (\forall y \in M)\}.$$

Proposition 2.2 ([22])**.** *Let K and L be non-degenerate even lattices. There exists a primitive embedding of K into an even unimodular lattice Γ such that $K^\perp \cong L$, if and only if $(A(K), q(K)) \cong (A(L), -q(L))$. More precisely, any such Γ is of the form $\Gamma_\lambda \subset K^\vee \oplus L^\vee$ for some isomorphism*

$\lambda \colon (A(K), q(K)) \to (A(L), -q(L))$, *where* Γ_λ *is the overlattice of* $K \oplus L$ *corresponding to the isotropic subgroup*

$$\{(x, \lambda(x)) \in A(K) \oplus A(L) \mid x \in A(K)\} \subset A(K) \oplus A(L).$$

2.2. **K3 Surfaces.** A K3 surface S is a simply-connected compact complex surface S with trivial canonical bundle $\Omega_S^2 \cong \mathcal{O}_S$. Then $H^2(S, \mathbb{Z})$ with the cup product is isomorphic to the K3 lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. It is also endowed with a weight-two Hodge structure

$$H^2(S, \mathbb{C}) = H^2(S, \mathbb{Z}) \otimes \mathbb{C} = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S).$$

Let ω_S be a nowhere vanishing holomorphic 2-form on S. The space $H^{2,0}(S) \cong \mathbb{C}$ is generated by the class of ω_S, which we denote by the same ω_S. The algebraic lattice $NS(S)$ and the transcendental lattice $T(S)$ of S are primitive sublattices of $H^2(S, \mathbb{Z})$ defined by

$$NS(S) := \{x \in H^2(S, \mathbb{Z}) \mid \langle x, \omega_S \rangle = 0\}, \quad T(S) := NS(S)^{\perp}_{H^2(S,\mathbb{Z})}.$$

Here we extend the bilinear form $\langle *, ** \rangle$ on $H^2(S, \mathbb{Z})$ to that on $H^2(S, \mathbb{Z}) \otimes \mathbb{C}$ linearly. Note that $NS(S)$ is naturally isomorphic to the Picard group of S.

If a group G acts on a K3 surface S, the action of G induces a left action on $H^2(S, \mathbb{Z})$ by $g \cdot x := (g^{-1})^* x$ for $g \in G$ and $x \in H^2(S, \mathbb{Z})$. An automorphism g of S is called symplectic if $g^* \omega_S = \omega_S$. An Enriques surface is the quotient of a K3 surface S by a fixed point free involution ι, which we call an Enriques involution. Then we have $\iota^* \omega_S = -\omega_S$, i.e. an anti-symplectic involution.

Proposition 2.3 ([2, Section 2.3]). *Let S be a K3 surface. An involution $\iota \in \mathrm{Aut}(S)$ is an Enriques involution if and only if*

$$H^2(S, \mathbb{Z})^\iota \cong U(2) \oplus E_8(-2), \quad H^2(S, \mathbb{Z})_\iota \cong U \oplus U(2) \oplus E_8(-2).$$

3. Calabi–Yau Threefolds of Type K

3.1. **Classification and Construction.** We begin with a review of [24, 16]. By the Bogomolov decomposition theorem, a Calabi–Yau threefold X with infinite fundamental group admits an étale Galois covering either by an abelian threefold or by the product of a K3 surface and an elliptic curve. We call X of type A in the former case and of type K in the latter case. Among many candidates of such coverings, we can always find a unique smallest one, up to isomorphism as a covering, and we call it the minimal splitting covering. The full classification of Calabi–Yau threefolds with infinite fundamental group was completed in [16]. Here we focus on type K:

Theorem 3.1 ([16]). *There exist exactly eight Calabi–Yau threefolds of type K, up to deformation equivalence. The equivalence class is uniquely determined by the Galois group of the minimal splitting covering, which is isomorphic to one of the following combinations of cyclic and dihedral groups*

$$C_2, \ C_2 \times C_2, \ C_2 \times C_2 \times C_2, \ D_6, \ D_8, \ D_{10}, \ D_{12}, \ and \ C_2 \times D_8.$$

We now briefly summarize the construction in [16]. Let X be a Calabi–Yau threefold of type K and $\pi\colon S \times E \to X$ its minimal splitting covering with Galois group G. There exists a canonical isomorphism $\mathrm{Aut}(S \times E) \cong \mathrm{Aut}(S) \times \mathrm{Aut}(E)$, which induces a faithful G-action on each S and E:

$$\mathrm{Aut}(S) \xleftarrow{\ p_1\ } \mathrm{Aut}(S \times E) \xrightarrow{\ p_2\ } \mathrm{Aut}(E)$$

$$\cup \qquad\qquad\qquad \cup \qquad\qquad\qquad \cup$$

$$p_1(G) \xleftarrow[p_1|_G]{\cong} G \xrightarrow[p_2|_G]{\cong} p_2(G).$$

Proposition 3.2 ([24, 16])**.** *Let $H := \mathrm{Ker}(G \to \mathrm{GL}(H^{2,0}(S)))$ and take any $\iota \in G \setminus H$. Then the following hold:*

(1) $\mathrm{ord}(\iota) = 2$ *and* $G = H \rtimes \langle \iota \rangle$, *where the semi-direct product structure is given by* $\iota h \iota = h^{-1}$ *for all* $h \in H$;

(2) g *acts on S as an Enriques involution if* $g \in G \setminus H$;

(3) ι *acts on E as* -1_E *and H as translations of the form* $\langle t_a \rangle \times \langle t_b \rangle \cong C_n \times C_m$ *under an appropriate origin of E. Here t_a and t_b are translations of order n and m respectively for some* $(n,m) \in \{(1,k)(1 \leq k \leq 6),\ (2,2),\ (2,4)\}$.

Conversely, such a G-action yields a Calabi–Yau threefold $X := (S \times E)/G$ of type K. Proposition 3.2 provides us with a complete understanding of the G-action on E, and therefore the classification essentially reduces to that of K3 surfaces equipped with actions described in Proposition 3.2.

Definition 3.3. *Let G be a finite group. We say that an action of G on a K3 surface S is Calabi–Yau if the following hold:*

(1) $G = H \rtimes \langle \iota \rangle$ *with* $H \cong C_n \times C_m$ *for some* $(n,m) \in \{(1,k)\ (1 \leq k \leq 6),\ (2,2),\ (2,4)\}$, *and* $\mathrm{ord}(\iota) = 2$. *The semi-direct product structure is given by* $\iota h \iota = h^{-1}$ *for all* $h \in H$;

(2) H *acts on S symplectically, and any $g \in G \setminus H$ acts as an Enriques involution.*

In what follows, G is always one of the finite groups listed above. A basic example of a K3 surface which the reader could bear in mind is the following Horikawa model.

Proposition 3.4 (Horikawa model [4, Section V, 23])**.** *The double covering $S \to \mathbb{P}^1 \times \mathbb{P}^1$ branching along a bidegree $(4,4)$-divisor B is a K3 surface if it is smooth. We denote by θ the covering involution on S. Assume that B is invariant under the involution λ of $\mathbb{P}^1 \times \mathbb{P}^1$ given by $(x,y) \mapsto (-x,-y)$, where x and y are the inhomogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$. The involution λ lifts to a symplectic involution of S. Then $\theta \circ \lambda$ is an involution of S without fixed points unless B passed through one of the four fixed points of λ on $\mathbb{P}^1 \times \mathbb{P}^1$.*

The quotient surface $T = S/\langle \theta \circ \lambda \rangle$ is therefore an Enriques surface.

$$
\begin{array}{ccc}
S & \xrightarrow{\;\mathrm{id}\;} & S \\
{\scriptstyle /\langle\theta\rangle}\downarrow & & \downarrow{\scriptstyle /\langle\theta\circ\lambda\rangle} \\
\mathbb{P}^1 \times \mathbb{P}^1 & & T
\end{array}
$$

The classical theory of Enriques surfaces says that any generic K3 surface with an Enriques involution is realized as a Horikawa model ([4, Propositions 18.1, 18.2]).

Example 3.5 (Enriques Calabi–Yau threefold). *Let S be a K3 surface with an Enriques involution ι and E an elliptic curve with negation -1_E. The free quotient $X := (S \times E)/\langle (\iota, -1_E) \rangle$ is the simplest Calabi–Yau threefold of type K with Galois group $G = C_2$, known as the Enriques Calabi–Yau threefold.*

We saw that the Horikawa model gives rise to the Enriques Calabi–Yau threefold. In order to obtain other Calabi–Yau threefolds of type K, we consider special classes of Horikawa models as follows. We introduce a group $G_0 := H \rtimes C_2$ which is isomorphic to G in Definition 3.3 as an abstract group and we denote by λ the generator of the second factor C_2. Let $\rho_1, \rho_2 \colon G_0 \to \mathrm{PGL}(2, \mathbb{C})$ be 2-dimensional complex projective representations. We thus get a G_0-action $\rho_1 \times \rho_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that there exists a G_0-stable smooth curve B of bidegree $(4, 4)$. We then obtain a Horikawa K3 surface S as the double covering $\pi \colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ branching along B and the G_0-action on $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to S as a symplectic G_0-action. We further assume that the curve B does not pass through any of fixed points of $g \in G_0 \setminus H$. With the same notation as in Proposition 3.4, it can be checked that the symplectic G_0-action and the covering transformation θ commute.

$$
\begin{array}{ccccc}
& \xrightarrow{\;\exists symplectic\;} & S & \xrightarrow{\;\mathrm{id}\;} & S \\
& \nearrow & & & \\
& {\scriptstyle \rho_1\times\rho_2} & \downarrow{\scriptstyle /\langle\theta\rangle} & & \downarrow{\scriptstyle /\langle\theta\circ\lambda\rangle} \\
G_0 & \searrow & & & \\
& & \mathbb{P}^1 \times \mathbb{P}^1 & & T
\end{array}
$$

By twisting λ by θ, we obtain a G-action on S, i.e.

$$
\mathrm{Aut}(S) \supset G_0 \times \langle \theta \rangle \supset H \rtimes \langle \theta \circ \lambda \rangle = G.
$$

It is shown in [16] that the G-action on S is a Calabi–Yau action and a generic K3 surface equipped with a Calabi–Yau action is realized (not necessarily uniquely) in this way. To put it another way, there exist projective representations $\rho_1, \rho_2 \colon G_0 \to \mathrm{PGL}(2, \mathbb{C})$ which satisfy all the assumptions mentioned above. In this paper, we do not need explicit presentations of Calabi–Yau actions and thus close this section by just providing two examples.

Example 3.6. *Suppose that* $G = D_{12} = \langle a, b \mid a^6 = b^2 = baba = 1 \rangle$. *Let* x, y *(resp.* z, w*) be homogeneous coordinates of the first (resp. second)* $\mathbb{P}^1$. *For* $i = 1, 2$, *we define* $\rho_i \colon D_{12} \to \mathrm{PGL}(2, \mathbb{C})$ *by*

$$a \mapsto \begin{bmatrix} \zeta_{12}^i & 0 \\ 0 & \zeta_{12}^{12-i} \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where ζ_k *denotes a primitive k-th root of unity. A basis of the space of* D_{12}*-invariant polynomials of bidegree* $(4, 4)$ *are given by* $x^4 z^4 + y^4 w^4, x^4 z w^3 + y^4 z^3 w, x^2 y^2 z^2 w^2$. *A generic linear combination of these cuts out a smooth curve of bidegree* $(4, 4)$.

Example 3.7. *Suppose that* $G = D_8 \times C_2 = \langle a, b, c \mid a^4 = b^2 = baba = 1, ac = ca, bc = cb \rangle$. *For* $i = 1, 2$, *we define* $\rho_i \colon D_8 \times C_2 \to \mathrm{PGL}(2, \mathbb{C})$ *by*

$$a \mapsto \begin{bmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^7 \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} \sqrt{-1}^{i-1} & 0 \\ 0 & \sqrt{-1}^{1-i} \end{bmatrix}.$$

A basis of the space of $D_8 \times C_2$*-invariant polynomials of bidegree* $(4, 4)$ *are given by* $x^4 z^4 + y^4 w^4, x^4 w^4 + y^4 z^4, x^2 y^2 z^2 w^2$. *A generic linear combination of these cuts out a smooth curve of bidegree* $(4, 4)$.

3.2. **Lattices** $H^2(S, \mathbb{Z})^G$ **and** $H^2(S, \mathbb{Z})^H_{C_2}$. In what follows, we fix the decomposition $G = H \rtimes C_2$ with $C_2 = \langle \iota \rangle$ as in Proposition 3.2. We define two sublattices of $H^2(S, \mathbb{Z})$ by $M_G := H^2(S, \mathbb{Z})^G$ and $N_G := H^2(S, \mathbb{Z})^H_{C_2}$. The lattices M_G and N_G depend only on G (as an abstract group) and a generic K3 surface S with a Calabi–Yau G-action has transcendental lattice N_G (see [16] for details).

Proposition 3.8 (cf. [21, 15]). *Let* F *be a group listed in the table below. Suppose that* F *acts on a K3 surface* S *faithfully and symplectically. The isomorphism class of the invariant lattice* $H^2(S, \mathbb{Z})^F$ *is given by the following.*

F	$H^2(S, \mathbb{Z})^F$	rank $H^2(S, \mathbb{Z})^F$
C_2	$U^{\oplus 3} \oplus E_8(-2)$	14
$C_2 \times C_2$	$U \oplus U(2)^{\oplus 2} \oplus D_4(-2)$	10
$C_2 \times C_2 \times C_2$	$U(2)^{\oplus 3} \oplus \langle -4 \rangle^{\oplus 2}$	8
C_3	$U \oplus U(3)^{\oplus 2} \oplus A_2(-1)^{\oplus 2}$	10
C_4	$U \oplus U(4)^{\oplus 2} \oplus \langle 2 \rangle^{\oplus 2}$	8
D_6	$U(3) \oplus A_2(2) \oplus A_2(-1)^{\oplus 2}$	8
D_8	$U \oplus \langle 4 \rangle^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 3}$	7
C_5, D_{10}	$U \oplus U(5)^2$	6
C_6, D_{12}	$U \oplus U(6)^2$	6
$C_2 \times C_4, C_2 \times D_8$	$U(2) \oplus \langle 4 \rangle^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$	6

The main result in this section is the following classification of the lattices M_G and N_G.

Theorem 3.9. *The lattices M_G and N_G for each G are given by the following.*

G	H	M_G	N_G
C_2	C_1	$U(2) \oplus E_8(-2)$	$U \oplus U(2) \oplus E_8(-2)$
$C_2 \times C_2$	C_2	$U(2) \oplus D_4(-2)$	$U(2)^{\oplus 2} \oplus D_4(-2)$
$C_2 \times C_2 \times C_2$	$C_2 \times C_2$	$U(2) \oplus \langle -4 \rangle^{\oplus 2}$	$U(2)^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$
D_6	C_3	$U(2) \oplus A_2(-2)$	$U(3) \oplus U(6) \oplus A_2(-2)$
D_8	C_4	$U(2) \oplus \langle -4 \rangle$	$U(4)^{\oplus 2} \oplus \langle -4 \rangle$
D_{10}	C_5	$U(2)$	$U(5) \oplus U(10)$
D_{12}	C_6	$U(2)$	$U(6)^{\oplus 2}$
$C_2 \times D_8$	$C_2 \times C_4$	$U(2)$	$U(4) \oplus \langle 4 \rangle \oplus \langle -4 \rangle$

Remark 3.10. *From the table in Theorem 3.9, one can check that there exists an isomorphism $N_G \otimes \mathbb{Q} \cong (U \oplus M_G) \otimes \mathbb{Q}$ as quadratic spaces over $\mathbb{Q}$.*

Proof of Theorem 3.9. It suffices to determine N_G because the classification of M_G was completed in [16, Proposition 4.4]. We set $\Lambda := H^2(S, \mathbb{Z})$ and employ the lattice theory reviewed in Section 2. In [16, Proposition 3.11], we gave a projective model of a generic K3 surface S with a Calabi–Yau G-action as the double covering $S \to \mathbb{P}^1 \times \mathbb{P}^1$ branching along a smooth curve C of bidegree $(4, 4)$ (see Section 3.1). Let θ denote the covering transformation of $S \to \mathbb{P}^1 \times \mathbb{P}^1$. We define G' to be the group generated by H and $\iota\theta$. Then G' is isomorphic to G as an abstract group, the action of G' on S is symplectic, and we have $N_G = \Lambda_\iota^{G'}$. Let $L \subset \Lambda$ denote the pullback of $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$, which is isomorphic to $U(2)$. Let e, f denote the standard basis of L, that is, $e^2 = f^2 = 0$ and $\langle e, f \rangle = 2$. For our projective model, G acts on L trivially. Since $S/\langle\theta\rangle \cong \mathbb{P}^1 \times \mathbb{P}^1$, we have $L = \Lambda^\theta = \Lambda^{\langle G, \theta \rangle}$. Hence

$$N_G = \Lambda_\iota^{G'} = L_{\Lambda^{G'}}^\perp = (\Lambda_{G'} \oplus L)_\Lambda^\perp.$$

Recall that $\Lambda^{G'}$ is given in Proposition 3.8.

   Case $H = C_1$. By Proposition 2.3, we have $N_G = U \oplus U(2) \oplus E_8(-2)$.

   Case $H = C_2$. Set $\Gamma := U(2)^{\oplus 2} \oplus D_4(-2)$. We have $\Lambda^{G'} \cong U \oplus \Gamma$. For our projective model, the branching curve C is defined by a (generic) linear combination of polynomials of the form

$$x^i y^{4-i} z^j w^{4-j} + x^{4-i} y^i z^{4-j} w^j, \quad 0 \le i, j \le 4, \ i \equiv 0 \mod 2,$$

where x, y (resp. z, w) are homogeneous coordinates of the first (resp. second) $\mathbb{P}^1$. Note that the equation $z = 0$ defines a reducible curve on S, which has the two components D_1, D_2 such that $D_i^2 = -2$ and $\langle D_1, D_2 \rangle = 2$. Hence we may assume that the class of D_1 or $-D_1$ is given as $u := (v + e)/2 \in \Lambda$ for some $v \in \Lambda_{G'}$ by interchanging e and f if necessary. We then have the following inclusions:

$$K_0 := \Lambda_{G'} \oplus L \subset K_1 := K_0 + \mathbb{Z}u \subset (N_G)_\Lambda^\perp.$$

We show $K_1 = (N_G)_\Lambda^\perp$. Let $\overline{u}$ denote the class of u in $A(K_0)$. We have

$$A(K_1) = \overline{u}^\perp / \langle \overline{u} \rangle \cong B \cap \overline{u}^\perp, \quad B := \left((\Lambda_{G'})^\vee \oplus \mathbb{Z}(f/2) \right) / K_0 \subset A(K_0).$$

Hence the projection $B \to A(\Lambda_{G'})$ induces the isomorphism $q(K_1) \cong q(\Lambda_{G'})$. Hence

$$(1) \qquad\qquad q(K_1) \cong q(\Lambda_{G'}) \cong -q(\Lambda^{G'}) \cong -q(\Gamma)$$

by Proposition 2.2. Again, by Proposition 2.2, there exists a primitive embedding $\varphi \colon K_1 \to \Lambda$ such that $(\varphi(K_1))^\perp_\Lambda \cong \Gamma$. By Proposition 3.8, we have $(\Lambda_{G'})^\iota = \Lambda_{\iota\theta} \cong E_8(-2)$. Thus

$$(K_0)^\iota \cong E_8(-2) \oplus U(2) \subset (K_1)^\iota \subset ((N_G)^\perp_\Lambda)^\iota \cong E_8(-2) \oplus U(2)$$

by Proposition 2.3. Hence $(K_1)^\iota \cong E_8(-2) \oplus U(2)$. Since H acts on $A(K_1)$ trivially and $A((K_1)^\iota)$ is a 2-elementary group, the action of G on $K_1' := \varphi(K_1)$ (via φ) extends to that on Λ in such a way that H (resp. ι) acts on $(K_1')^\perp_\Lambda$ trivially (resp. as negation). This action of G on Λ gives a Calabi–Yau G-action on a K3 surface (see [16, Section 3.3] for details). By the uniqueness of a Calabi–Yau G-action [16, Theorem 3.19], we have $K_1 = (N_G)^\perp_\Lambda$. Hence $q(N_G) \cong -q(K_1) \cong q(\Lambda^{G'}) \cong q(\Gamma)$ by (1) and Proposition 2.2. Therefore $N_G \cong \Gamma$ by Theorem 2.1.

Case $H = C_2 \times C_2$. Since $\Lambda^{G'} \cong U(2)^{\oplus 3} \oplus \langle -4 \rangle^{\oplus 2}$, we have $N_G = L^\perp_{\Lambda^{G'}} \cong U(2)^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$ by Theorem 2.1.

Case $H = C_3, C_5$. By a similar argument to Case $H = C_2$, we conclude that $q(N_G) \cong q(\Lambda^{G'}) \oplus q(L)$. The lattice N_G is uniquely determined by Theorem 2.1.

Case $H = C_4, C_6$. For our projective model, the branching curve C is defined by a (generic) linear combination of the following polynomials:

$$x^4 z^3 w + y^4 z w^3, x^4 z w^3 + y^4 z^3 w, x^2 y^2 z^4 + x^2 y^2 w^4, x^2 y^2 z^2 w^2 \quad \text{if } H = C_4,$$

$$x^4 z^4 + y^4 w^4, x^4 z w^3 + y^4 z^3 w, x^2 y^2 z^2 w^2 \quad \text{if } H = C_6.$$

Note that the curve on S defined by $z = 0$ is reducible in each case. Similarly to Case $H = C_2$, we conclude that $(N_G)^\perp_\Lambda$ contains $\Lambda_{G'} \oplus L$ as a sublattice of index 2 and that $q(N_G) \cong q(\Lambda^{G'})$. The lattice N_G is uniquely determined by Theorem 2.1. For Case $H = C_4$, we use the isomorphism $\langle 1 \rangle \oplus \langle -1 \rangle^{\oplus 2} \cong U \oplus \langle -1 \rangle$.

Case $H = C_2 \times C_4$. Since $\Lambda^{G'} \cong U(2) \oplus \langle 4 \rangle^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$, we have $N_G = L^\perp_{\Lambda^{G'}} \cong \langle 4 \rangle^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2} \cong U(4) \oplus \langle 4 \rangle \oplus \langle -4 \rangle$ by Theorem 2.1. $\qquad\qquad \square$

3.3. Brauer Groups $\mathrm{Br}(X)$.

Throughout this section, X denotes a Calabi–Yau threefold of type K and $\pi \colon S \times E \to X$ its minimal splitting covering with Galois group $G = H \rtimes \langle \iota \rangle$.

Definition 3.11. *The Brauer group $\mathrm{Br}(Y)$ of a smooth projective variety Y is defined by $\mathrm{Br}(Y) := \mathrm{Tor}(H^2(Y, \mathcal{O}_Y^\times))$. Here $\mathrm{Tor}(-)$ denotes the torsion part.*

By the exact sequence $H^2(Y, \mathcal{O}_Y) \to H^2(Y, \mathcal{O}_Y^\times) \to H^3(Y, \mathbb{Z}) \to H^3(Y, \mathcal{O}_Y)$ and the universal coefficient theorem, for a Calabi–Yau threefold Y, we have

$$\mathrm{Br}(Y) \cong \mathrm{Tor}(H^3(Y, \mathbb{Z})) \cong \mathrm{Tor}(H_2(Y, \mathbb{Z})) \cong \mathrm{Tor}(H^4(Y, \mathbb{Z})).$$

Therefore the Brauer group of a Calabi–Yau threefold is topological, in contrast to that of a K3 surface, which is analytic.

The projection $E \times S \to E$ descends to the locally isotrivial K3-fibration $f \colon X \to B := E/G \cong \mathbb{P}^1$ with 4 singular fibers. Each singular fiber $f^{-1}(p_i)$ ($p_i \in B$, $1 \le i \le 4$) is a (doubled) Enriques surface. Let U_i be a neighborhood of p_i isomorphic to a disk such that $U_i \cap U_j = \emptyset$ for $i \ne j$. We use the following notations: $U_i^* = U_i \setminus \{p_i\}$, $\overline{U_i} = f^{-1}(U_i)$, $\overline{U_i^*} = f^{-1}(U_i^*)$, $B^* = B \setminus \{p_i\}_{i=1}^4$ and $\overline{B^*} = f^{-1}(B^*)$. Since $S/\langle \iota \rangle \hookrightarrow \overline{U_i}$ is a deformation retraction, we have[3]

$$H_2(\overline{U_i}, \mathbb{Z}) \cong H_2(S/\langle \iota \rangle, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 10} \oplus \mathbb{Z}_2.$$

Moreover, by considering monodromies, we obtain[4]

$$H_2(\overline{U_i^*}, \mathbb{Z}) \cong H_2(S, \mathbb{Z})/\{x - \iota x \mid x \in H_2(S, \mathbb{Z})\} \cong \mathbb{Z}^{\oplus 10} \oplus (\mathbb{Z}_2)^{\oplus 2}$$

and

$$H_2(\overline{B^*}, \mathbb{Z}) \cong H_2(S, \mathbb{Z})/\langle x - gx \mid x \in H_2(S, \mathbb{Z}), \ g \in G \rangle.$$

The Mayer–Vietoris sequence reads
(2)
$$\bigoplus_{i=1}^4 H_2(\overline{U_i^*}, \mathbb{Z}) \xrightarrow{\alpha} H_2(\overline{B^*}, \mathbb{Z}) \oplus \left(\bigoplus_{i=1}^4 H_2(\overline{U_i}, \mathbb{Z}) \right) \longrightarrow H_2(X, \mathbb{Z}) \longrightarrow \bigoplus_{i=1}^4 H_1(\overline{U_i^*}, \mathbb{Z}).$$

Since $H_1(\overline{U_i^*}, \mathbb{Z}) \cong \mathbb{Z}$, we conclude that $\mathrm{Br}(X) \cong \mathrm{Tor}(P)$, where

(3)
$$P = \frac{H_2(\overline{B^*}, \mathbb{Z}) \oplus \left(\bigoplus_{i=1}^4 H_2(\overline{U_i}, \mathbb{Z}) \right)}{\alpha(\bigoplus_{i=1}^4 H_2(\overline{U_i^*}, \mathbb{Z}))} \cong \frac{H_2(\overline{B^*}, \mathbb{Z})}{Q}.$$

Here Q denotes the image in $H_2(\overline{B^*}, \mathbb{Z})$ of the kernel of the natural map $H_2(\overline{U_i^*}, \mathbb{Z}) \to H_2(\overline{U_i}, \mathbb{Z})$, which is surjective [8, Section 9.3]. Note that Q does not depend on the index i and, by Lemma 3.12 below, we have $Q \cong \mathbb{Z}_2$. In what follows, we determine the torsion part of $H_2(\overline{B^*}, \mathbb{Z})$. We identify $H_2(S, \mathbb{Z})$ with $\Lambda := H^2(S, \mathbb{Z})$ by Poincaré duality.

Lemma 3.12. *In the equation (3), we have $Q \cong \mathbb{Z}_2$.*

Proof. As an example, we consider the case $H \cong C_2 \times C_4$. We can show the assertion for the other cases in a similar manner. We use the same notation as in Example 3.7. The branching curve B is defined by

$$c_1(x^4 z^4 + y^4 w^4) + c_2(x^4 w^4 + y^4 z^4) + c_3 x^2 y^2 z^2 w^2 = 0, \quad (c_1, c_2, c_3) \in \mathbb{C}^3.$$

We consider the degeneration of S to S_0, where S_0 is defined by generic (c_1, c_2, c_3) such that $p := (1:1) \times (1:1) \in B$, that is, $2c_1 + 2c_2 + c_3 = 0$. Note that S_0 has a singular point of type A_1 at the inverse image of p. Let γ denote the vanishing cycle corresponding to this singular point. Then we

---

[3]For the homology groups of Enriques surfaces, see e.g. [8, Section 9.3].

[4]For the proof of the isomorphism $H_2(S, \mathbb{Z})/\{x - \iota x \mid x \in H_2(S, \mathbb{Z})\} \cong \mathbb{Z}^{\oplus 10} \oplus (\mathbb{Z}_2)^{\oplus 2}$, see Lemma 3.15.

have $\iota\gamma = -\gamma$ and $\gamma^2 = -2$. Let $\widetilde{S}$ denote the minimal desingularization of S/H, which is a K3 surface. Since the induced action of ι on $\widetilde{S}$ is an Enriques involution, we have

$$\{x - \iota x \mid x \in H_2(\widetilde{S}, \mathbb{Z})\} \cong 2(H_2(\widetilde{S}, \mathbb{Z})_\iota)^\vee \cong U(4) \oplus U(2) \oplus E_8(-2)$$

by Proposition 2.3 (cf. [8, Section 4.4.4]). This implies that $(x - \iota x)^2 \equiv 0 \bmod 4$ for any $x \in H_2(\widetilde{S}, \mathbb{Z})$. Assume that

$$\gamma \in \langle x - gx \mid x \in H_2(S, \mathbb{Z}), \ g \in G \rangle =: D.$$

Then the pushforward of γ in $H_2(\widetilde{S}, \mathbb{Z})$ is of the form $x - \iota x$ with $x \in H_2(\widetilde{S}, \mathbb{Z})$, which contradicts to $\gamma^2 = -2$. Threfore $\gamma \notin D$ and Q is generated by $\gamma \bmod D$. $\qquad\square$

Lemma 3.13. *We have $\Lambda_H = \langle x - gx \mid x \in \Lambda, \ g \in H \rangle$.*

Proof. First assume that H is a non-trivial cyclic group generated by h. Since Λ is unimodular, any element $x \in \Lambda$ is written as $x' + x''$ with $x' \in (\Lambda_H)^\vee$ and $x'' \in (\Lambda^H)^\vee$ (Proposition 2.2). Hence the map γ defined by

$$\gamma := \mathrm{id} - h \colon (\Lambda_H)^\vee \to L := \langle x - gx \mid x \in \Lambda, \ g \in H \rangle$$

is an isomorphism. Note that we have the following inclusions: $L \subset \Lambda_H \subset (\Lambda_H)^\vee$. The eigenvalues of the action of h on Λ_H are given as in the following table (see Proposition 3.8).

H	Eigenvalues	$\det(\gamma)$
C_2	$(-1)^8$	2^8
C_3	$(\zeta_3)^6(\zeta_3^2)^6$	3^6
C_4	$(-1)^6(\zeta_4)^4(-\zeta_4)^4$	2^{10}
C_5	$(\zeta_5)^4(\zeta_5^2)^4(\zeta_5^3)^4(\zeta_5^4)^4$	5^4
C_6	$(-1)^4(\zeta_3)^4(\zeta_3^2)^4(-\zeta_3)^2(-\zeta_3^2)^2$	$2^4 \cdot 3^4$

In each case, we have $\det(\gamma) = |\operatorname{disc}(\Lambda_H)|$. Hence

$$|(\Lambda_H)^\vee/L| = \det(\gamma) = |\operatorname{disc}(\Lambda_H)| = |(\Lambda_H)^\vee/\Lambda_H|,$$

which implies $L = \Lambda_H$.

Next we consider the case $H = C_2 \times K$ with $K = C_2$ or C_4. We define E and F by

$$E := \Lambda_H / \left((\Lambda_H)_K \oplus (\Lambda_H)^K \right), \quad F := \Lambda_{C_2} / \left((\Lambda_{C_2})_K \oplus (\Lambda_{C_2})^K \right).$$

We have

$$(\Lambda_H)_K = \Lambda_K, \quad \Lambda_K \cap \Lambda_{C_2} = (\Lambda_{C_2})_K, \quad M := (\Lambda_H)^K = (\Lambda_{C_2})^K.$$

Hence F is considered as a subgroup of E. Note that Λ_K and Λ_{C_2} are contained in L by the argument above. Therefore, in order to show $\Lambda_H = L$, it is enough to show $E = F$. We have

$$|E|^2 \cdot |\operatorname{disc}(\Lambda_H)| = |\operatorname{disc}(\Lambda_K)| \cdot |\operatorname{disc}(M)|.$$

Since $\Lambda_{C_2}(1/2)$ is isomorphic to the unimodular lattice $E_8(-1)$ by Proposition 3.8, we obtain

$$|F| = |\operatorname{disc}(M(1/2))| = |\operatorname{disc}(M)| \cdot 2^{-r}, \quad r = \operatorname{rank} M = \operatorname{rank} \Lambda_H - \operatorname{rank} \Lambda_K,$$

by Propositions 2.2. We have the following table (see Proposition 3.8).

K	$\operatorname{disc}(\Lambda_H)$	$\operatorname{disc}(\Lambda_K)$	$\operatorname{rank} \Lambda_H$	$\operatorname{rank} \Lambda_K$	r
C_2	2^{10}	2^8	8	12	4
C_4	2^{10}	2^{10}	16	14	2

Hence, in each case, we have $|E|^2/|F| = 2^r \cdot \operatorname{disc}(\Lambda_K)/\operatorname{disc}(\Lambda_H) = 2^2$. Note that $M(1/2)$ is not unimodular, that is, $|F| > 1$, because the rank of any definite even unimodular lattice is divisible by 8 (see Section 2.1). Since $|E|/|F| = |E/F|$ is an integer, we have $|E| = |F| = 2^2$, which implies $E = F$. $\qquad\square$

Lemma 3.14. *The torsion part of*

(4) $\quad H_2(\overline{B^*}, \mathbb{Z}) \cong \Lambda/\langle x - gx \mid x \in \Lambda, \ g \in G \rangle \cong (\Lambda^H)^\vee/\{x - \iota x \mid x \in (\Lambda^H)^\vee\}$

is isomorphic to $\mathbb{Z}_2^{\oplus n}$. Here n is given by $n = \operatorname{rank} \Lambda_\iota^H - a$ for $\Lambda^H/(\Lambda^G \oplus \Lambda_\iota^H) \cong \mathbb{Z}_2^{\oplus a}$.

Proof. The projection $\Lambda \to (\Lambda^H)^\vee$ induces an isomorphism $\varphi \colon \Lambda/\Lambda_H \to (\Lambda^H)^\vee$ by Proposition 2.2. Note that the action of ι preserves Λ^H. By Lemma 3.13, the second isomorphism in (4) is derived from φ. The assertion of the lemma follows from Lemma 3.15 below. $\qquad\square$

Lemma 3.15. *Let Γ be a non-degenerate lattice. If an involution ι acts on Γ non-trivially, then*

$$\operatorname{Tor}(\Gamma^\vee/\{x - \iota x \mid x \in \Gamma^\vee\}) \cong \mathbb{Z}_2^{\oplus n}, \quad n = \operatorname{rank} \Gamma_\iota - a.$$

Here $a \in \mathbb{N}$ is determined by $\Gamma/(\Gamma_\iota \oplus \Gamma^\iota) \cong \mathbb{Z}_2^{\oplus a}$.

Proof. For a free $\mathbb{Z}$-module $W = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ of rank 2, we define an action of ι on W by $\iota(w_1) = w_2$ and $\iota(w_2) = w_1$. As a $\mathbb{Z}$-module with an ι-action, $\Gamma^\vee$ is decomposed into the direct sum $\Gamma^\vee \cong V_+ \oplus V_- \oplus W^{\oplus b}$, where ι acts as ± 1 on $V_\pm$ (see [7, §74]). One can check that there is a (non-canonical) isomorphism $\Gamma \cong \Gamma^\vee = \operatorname{Hom}_\mathbb{Z}(\Gamma, \mathbb{Z})$ of $\mathbb{Z}$-modules with an ι-action. Hence $a = b$. We have $\operatorname{Tor}(\Gamma^\vee/\{x - \iota x \mid x \in \Gamma^\vee\}) \cong V_-/2V_-$. Since $\operatorname{rank} \Gamma_\iota = \operatorname{rank} V_- + a$, the assertion holds. $\qquad\square$

Theorem 3.16. *We have $\operatorname{Br}(X) \cong \mathbb{Z}_2^{\oplus m}$, where m is given by the following.*

G	C_2	$C_2 \times C_2$	$C_2 \times C_2 \times C_2$	D_6	D_8	D_{10}	D_{12}	$C_2 \times D_8$
m	1	2	3	1	2	1	2	3

Proof. We use n and a as in Lemma 3.14: $n = \operatorname{rank} \Lambda_\iota^H - a$ and $\Lambda^H/(\Lambda^G \oplus \Lambda_\iota^H) \cong \mathbb{Z}_2^{\oplus a}$. Then

$$|\operatorname{disc}(\Lambda^G)| \cdot |\operatorname{disc}(\Lambda_\iota^H)|/|\operatorname{disc}(\Lambda^H)| = 2^{2a}.$$

Recall that we determined $M_G = \Lambda^G$ and $N_G = \Lambda^H_\iota$ in Theorem 3.9. Combined with Proposition 3.8, n is computed as in the following table.

G	$|\operatorname{disc}(\Lambda^H)|$	$|\operatorname{disc}(\Lambda^G)|$	$|\operatorname{disc}(\Lambda^H_\iota)|$	a	rank Λ^H_ι	n
C_2	1	2^{10}	2^{10}	10	12	2
$C_2 \times C_2$	2^8	2^8	2^{10}	5	8	3
$C_2 \times C_2 \times C_2$	2^{10}	2^6	2^8	2	6	4
D_6	3^6	$2^4 \cdot 3$	$2^4 \cdot 3^5$	4	6	2
D_8	2^{10}	2^4	2^{10}	2	5	3
D_{10}	5^4	2^2	$2^2 \cdot 5^4$	2	4	2
D_{12}	$2^4 \cdot 3^4$	2^2	$2^4 \cdot 3^4$	1	4	3
$C_2 \times D_8$	2^{10}	2^2	2^8	0	4	4

Since $\operatorname{Br}(X) \cong \operatorname{Tor}(P)$, where P is defined in (3), we have $\operatorname{Br}(X) \cong \mathbb{Z}_2^{\oplus n-1}$ by Lemma 3.14. $\qquad\square$

We observe that $H_1(X, \mathbb{Z}) \cong \operatorname{Br}(X) \oplus \mathbb{Z}_2^2$ holds for Calabi–Yau threefolds of type K by the result of [16]. It is interesting to investigate this isomorphism via self-mirror symmetry of a Calabi–Yau threefold of type K.

Corollary 3.17. *Let X_1 and X_2 be Calabi–Yau threefolds of type K whose Galois groups are G_1 and G_2 respectively. Then a derived equivalence $\mathrm{D}^b\mathrm{Coh}(X_1) \cong \mathrm{D}^b\mathrm{Coh}(X_2)$ implies an isomorphism $G_1 \cong G_2$.*

Proof. The work [1] of Addington shows that the finite abelian group $H_1(X, \mathbb{Z}) \oplus \operatorname{Br}(X)$ is a derived invariant of a Calabi–Yau threefold. It is also known that the Hodge numbers are derived invariants. These invariants completely determine the Galois group of a Calabi–Yau threefold of type K. We refers the reader to [16] for the Hodge numbers and $H_1(X, \mathbb{Z})$. $\qquad\square$

3.4. **Some Topological Properties.** In this section, we study some topological properties of Calabi–Yau threefolds of type K. We also refer the reader to [16, Section 5].

Definition 3.18 ([18])**.** *Let M be a free abelian group of finite rank and $\mu\colon M^{\otimes 3} \to \mathbb{Z}$ a symmetric trilinear form on M. Let L be a lattice with bilinear form $\langle *, ** \rangle_L \colon L \times L \to \mathbb{Z}$. We call μ of type L if the following hold:*

(1) *There is a decomposition $M \cong L \oplus N$ as an abelian group such that $N \cong \mathbb{Z}$.*

(2) *We have $\mu(\alpha, \beta, n) = n\langle \alpha, \beta \rangle_L$ $(\alpha, \beta \in L, \; n \in N \cong \mathbb{Z})$ and the remaining values are given by extending the form symmetrically and linearly and by setting other non-trivial values to be 0.*

Proposition 3.19. *If H is a cyclic group, the trilinear intersection form μ_X on $H^2(X, \mathbb{Z})$ modulo torsion defined by the cup product is of type $L = M_G(1/2)$.*

Proof. Let 0_E denote the origin of E and pick a point pt_S in S^H, which is not empty [21]. We define
(5)
$$\alpha := (\pi_*[S \times \{0_E\}])^{\mathrm{PD}} \in H^2(X, \mathbb{Z}), \quad \beta := (\pi_*[\{\mathrm{pt}_S\} \times E])^{\mathrm{PD}} \in H^4(X, \mathbb{Z}),$$

where the superscript PD means the Poincaré dual. Since π is 2-to-1 on $S \times \{0_E\}$, it follows that α is divisible by 2 in $H^2(X, \mathbb{Z})$. Similarly, β is divisible by $|H|$. Note that $\frac{1}{2}\alpha \cup \frac{1}{|H|}\beta = 1$. We have natural isomorphisms

$$H^2(X, \mathbb{Q}) \cong H^2(S \times E)^G \cong (M_G \otimes \mathbb{Q}) \otimes H^0(E, \mathbb{Q}) \oplus \mathbb{Q}\alpha \cong (M_G \otimes \mathbb{Q}) \oplus \mathbb{Q}\alpha$$

and

$$H^4(X, \mathbb{Q}) \cong H^4(S \times E)^G \cong (M_G \otimes \mathbb{Q}) \otimes H^2(E, \mathbb{Q}) \oplus \mathbb{Q}\beta \cong (M_G \otimes \mathbb{Q}) \oplus \mathbb{Q}\beta.$$

For a $\mathbb{Z}$-module E, we denote by E_f its free part: $E_f := E/\mathrm{Tor}(E)$. By the exact sequence (2), it follows that $H_2(\overline{B^*}, \mathbb{Z})_f$ is embedded into $H_2(X, \mathbb{Z})_f$ primitively. By Proposition 2.2, we can identify $H_2(\overline{B^*}, \mathbb{Z})_f \cong H_2(S, \mathbb{Z})/H_2(S, \mathbb{Z})_G$ with $(M_G)^\vee$. By the Poincaré duality, we have

$$H^2(X, \mathbb{Z})_f \cong \frac{1}{|G|}M_G \oplus \mathbb{Z}\frac{1}{2}\alpha, \quad H^4(X, \mathbb{Z})_f \cong (M_G)^\vee \oplus \mathbb{Z}\frac{1}{|H|}\beta.$$

Hence the trilinear form μ_X is of type L with $L = M_G(1/2)$ and $N = \mathbb{Z}\frac{1}{2}\alpha$. In fact, we have

$$\mu_X\big(\frac{1}{|G|}\gamma_1, \frac{1}{|G|}\gamma_2, n \cdot \frac{1}{2}\alpha\big) = |G|^2 \cdot \frac{n}{2}\langle \frac{1}{|G|}\gamma_1, \frac{1}{|G|}\gamma_2 \rangle_{M_G} = \frac{n}{2}\langle \gamma_1, \gamma_2 \rangle_{M_G}$$

for $\gamma_1, \gamma_2 \in M_G$ and $n \in \mathbb{Z}$. $\qquad\square$

Remark 3.20. *For $H = C_2 \times C_2$ and $C_2 \times C_4$, the cohomology group $H^4(X, \mathbb{Z})_f$ is generated by $(M_G)^\vee$ and $\frac{1}{|G|}\gamma$, where γ is defined as follows. We use the same notation as in Example 3.7. Let D denote the reducible curve in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $x^2w^2 + y^2z^2 = 0$. Then there is a G-equivariant continuous map $\varphi\colon E \to D$. The inverse image in $S \times E$ of the graph of φ is a 2-cycle. We define γ to be the Poincaré dual of the image of this 2-cycle in X, which is divisible by $|G|$.*

Proposition 3.21. *The second Chern class $c_2(X)$ is given by $c_2(X) = \frac{24}{|G|}\beta$, where β is defined in (5).*

Proof. Let β_S be a generator of $H^4(S, \mathbb{Z})$ with $\int_S \beta_S = 1$. By the universal property of characteristic classes, we have $\pi^*c_2(X) = c_2(S \times E) = 24p_1^*\beta_S$, where $p_1\colon S \times E \to S$ is the first projection. Applying the transfer map π_*, we obtain $|G|c_2(X) = 24\pi_*p_1^*\beta_S = 24\beta$. $\qquad\square$

4. Yukawa Couplings

We begin our discussion on mirror symmetry. Calabi–Yau threefolds X of type K are, by the construction, a topological self-mirror threefolds, that is, $h^{1,1}(X) = h^{1,2}(X)$. However, mirror symmetry should involve more than the

mere exchange of Hodge numbers. We will see that the mirror symmetry of Calabi–Yau threefolds of type K bears a resemblance to the mirror symmetry of Borcea–Voisin threefolds [29, 6]. The latter relies on the strange duality of certain involutions of K3 surfaces discovered by Nikulin [23]. Our study, on the other hand, will employ the classification of lattices M_G and N_G (Theorem 3.9).

As a warm-up, let us consider the case $G \cong C_2$. A generic K3 surface S with an Enriques involution is a self-mirror K3 surface in the sense of Dolgachev [9], that is,

$$(6) \qquad\qquad\qquad U \oplus NS(S) \cong T(S)$$

where $NS(S)$ and $T(S)$ are given as M_G and N_G respectively in this case. Thus it is no wonder that the corresponding Enriques Calabi–Yau threefold (Example 3.5) is a self-mirror threefold. For $G \not\cong C_2$, the corresponding K3 surface cannot be self-mirror symmetric as $\operatorname{rank} T(S) < 12$. Nevertheless, as we will see, the Calabi–Yau threefold $X = (E \times S)/G$ is self-mirror symmetric. In general, M_G and N_G do not manifest symmetry over $\mathbb{Z}$, but the duality

$$(7) \qquad\qquad\qquad U \oplus M_G \cong N_G$$

still holds over $\mathbb{Q}$ or an extension of $\mathbb{Z}$ (Remark 3.10). Note that M_G is not equal to $NS(S)$ but to $NS(S)^G$, while $T(S) = N_G$ generically holds. The duality (7) can be thought of as an H-equivariant version of the duality (6) since we have $M_G = (H^2(S, \mathbb{Z})^H)^\iota$ and $N_G = (H^2(S, \mathbb{Z})^H)_\iota$.

4.1. **Moduli Spaces and Mirror Maps.** Throughout this section, X is a Calabi–Yau threefold of type K and $S \times E \to X$ is its minimal splitting covering with Galois group G. We will analyze the moduli space of X equipped with a complexified Kähler class. Such a moduli space locally splits into the product of the complex moduli space and the complexified Kähler moduli space. We will also sketch the mirror map.

We begin with the moduli space of marked K3 surfaces S with a Calabi–Yau G-action (Definition 3.3). The period domain of such K3 surfaces is given by

$$\mathcal{D}_S^G := \{ \mathbb{C}\omega_S \in \mathbb{P}(N_G \otimes \mathbb{C}) \mid \langle \omega_S, \omega_S \rangle = 0, \ \langle \omega_S, \overline{\omega}_S \rangle > 0 \}.$$

The extended G-invariant complexified Kähler moduli space[5] is defined as the tube domain

$$\mathcal{K}_{S,\mathbb{C}}^G := \{ B_S + i\kappa_S \in M_G \otimes \mathbb{C} \mid \kappa_S \in \mathcal{K} \},$$

where $\mathcal{K}$ is the connected component of $\{ \kappa \in M_G \otimes \mathbb{R} \mid \langle \kappa, \kappa \rangle > 0 \}$ containing a Kähler class. The class B_S is called the B-field, and usually defined modulo $H^2(S, \mathbb{Z})$. This is where the Brauer group comes into the play, but for a moment we regard $\mathcal{K}_{S,\mathbb{C}}^G$ as a covering of the G-invariant complexified

---

[5]For simplicity, we use this definition, even though κ_S may not be a Kähler class.

Kähler moduli space $\mathcal{K}_{S,\mathbb{C}}^G/H^2(S,\mathbb{Z})$, analogous to $\mathcal{D}_S^G$ being a covering of the complex moduli space. We construct a mirror map at the level of theses coverings.

Proposition 4.1. *There exists a holomorphic isomorphism* $q_S^G\colon \mathcal{K}_{S,\mathbb{C}}^G \to \mathcal{D}_S^G$.

Proof. There exists an isomorphism $N_G \otimes \mathbb{Q} \cong (U \oplus M_G) \otimes \mathbb{Q}$ as quadratic spaces over $\mathbb{Q}$ (Remark 3.10). The holomorphic isomorphism $q_S^G\colon \mathcal{K}_{S,\mathbb{C}}^G \to \mathcal{D}_S^G$ is then given by the mapping

$$B_S + i\kappa_S \mapsto \mathbb{C}(e - \frac{1}{2}\langle B_S + i\kappa_S, B_S + i\kappa_S\rangle f + B_S + i\kappa_S),$$

where e and f are the standard basis of U. This is known as the tube domain realization [9, 29]. $\square$

Let us next recall mirror symmetry of the elliptic curves. Consider an elliptic curve $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau_E)$ with a Calabi–Yau G-action, by which we mean a G-action on E described in Proposition 3.2. We can continuously deform the G-action as we vary the moduli parameter. The period domain of such elliptic curves is thus given by

$$\mathcal{D}_E := \{\mathbb{C}\phi \in \mathbb{P}(H^1(E,\mathbb{Z}) \otimes \mathbb{C}) \mid i\int_E \phi \cup \overline{\phi} > 0\}.$$

Given a symplectic basis (α, β) of $H^1(E, \mathbb{Z})$, we may normalize $\phi = \alpha + \tau_E\beta$ for $\tau_E \in \mathbb{H}$. The extended complexified Kähler moduli space of E is identified, through integration over E, with the set

$$\mathcal{K}_{E,\mathbb{C}} := \{B_E + i\kappa_E \in H^{1,1}(E) \cong \mathbb{C} \mid \kappa_E > 0\}.$$

There is an isomorphism $q_E\colon \mathcal{K}_{E,\mathbb{C}} \to \mathcal{D}_E$ given by

$$(8) \qquad q_E(B_E + i\kappa_E) := \mathbb{C}(\alpha + (B_E + i\kappa_E)\beta).$$

We associate to a pair $(E_1, B_{E_1} + i\kappa_{E_1})$ the mirror pair (E_2, τ_{E_1}) with $\tau_{E_2} := B_{E_1} + i\kappa_{E_1}$.

Given a point $(B_S + i\kappa_S, B_E + i\kappa_E)$ in the extended complexified Kähler moduli space $\mathcal{K}_{X,\mathbb{C}} := \mathcal{K}_{S,\mathbb{C}}^G \times \mathcal{K}_{E,\mathbb{C}}$ of X, the point

$$(q_S^G(B_S + i\kappa_S), q_E(B_E + i\kappa_E)) \in \mathcal{D}_X := \mathcal{D}_S^G \times \mathcal{D}_E$$

determines a pair $(S^\vee, E^\vee)$ of a K3 surface and an elliptic curve, and hence another Calabi–Yau threefold $X^\vee := (S^\vee \times E^\vee)/G$ of type K[6]. In the same manner, a point $(\omega_S, \tau_E) \in \mathcal{D}_X$ determines an extended complexified Kähler structure $((q_S^G)^{-1}(\omega_S), q_E^{-1}(\tau_E)) \in \mathcal{K}_{X,\mathbb{C}}$ on the pair $(S^\vee, E^\vee)$, which subsequently determines an extended complexified Kähler structure on $X^\vee$. Therefore we can think of the map $q = (q_S^G, q_E)\colon \mathcal{K}_{X,\mathbb{C}} \to \mathcal{D}_X$ as the *self-mirror map* of the family of Calabi–Yau threefolds X.

---

[6]Precisely, the set of points in $\mathcal{D}_X$ corresponding to Calabi–Yau threefolds of type K is an open dense subset of $\mathcal{D}_X$ (see [16]).

4.2. **A- and B-Yukawa Couplings.** We now introduce the A-Yukawa coupling with a complexified Kähler class $B + i\kappa \in \mathcal{K}_{X,\mathbb{C}}/H^2(X,\mathbb{Z})$. The A-Yukawa coupling $Y_A^X \colon H^{1,1}(X)^{\otimes 3} \to \mathbb{C}[[\mathbf{q}]]$ is the symmetric trilinear form given by

$$
Y_A^X(H_1, H_2, H_3) = H_1 \cup H_2 \cup H_3 + \sum_{\beta \neq 0} n_\beta \frac{e^{2\pi i \int_\beta (B+i\kappa)}}{1 - e^{2\pi i \int_\beta (B+i\kappa)}} \prod_{i=1}^3 \int_\beta H_i,
$$

where the sum is over the effective classes $\beta \in H_2(X,\mathbb{Z})$ and $\mathbb{C}[[\mathbf{q}]]$ is the Novikov ring. The number n_β is naively the number of rational curves on X in the homology class β. A mathematical definition of n_β is carried out via the Gromov–Witten invariants of X. At the *large volume limit* (LVL) of $H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$, where $\int_\beta \kappa \to \infty$ for all effective curve classes $\beta \in H_2(X,\mathbb{Z})$, the A-Yukawa coupling Y_A asymptotically converges to the cup product as the quantum correction terms vanish[7].

It is instructive to briefly review a role of the Brauer group in mirror symmetry [3]. In the A-model topological string theory, the correlation functions depend on *instantons*, namely holomorphic maps $\sigma \colon \Sigma_g \to X$ from the world-sheet Σ_g to the target threefold X. The action μ is required to depend linearly on the homology class of the image of the map; $\mu \in \mathrm{Hom}(H_2(X,\mathbb{Z}), \mathbb{C}^\times) \cong H^2(X, \mathbb{C}^\times)$. Note that $\mathrm{Tor}(H^2(X, \mathbb{C}^\times))$ is isomorphic to $\mathrm{Br}(X)$ as an abstract group. By the exact sequence

$$
0 \to H^2(X,\mathbb{Z})/\mathrm{Tor}(H^2(X,\mathbb{Z})) \to H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C}^\times) \to \mathrm{Br}(X) \to 0
$$

we see that

$$
H^2(X,\mathbb{C}^\times) \cong \left(H^2(X,\mathbb{C})/H^2(X,\mathbb{Z}) \right) \oplus \mathrm{Br}(X).
$$

Assuming that the B-field represents a class in $H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$, we think of the complexified Kähler moduli space $\mathcal{K}_{X,\mathbb{C}}$ lying in the first summand. Then the above isomorphism shows that we need to incorporate in the A-model moduli space the contribution coming from $\mathrm{Br}(X)$. However, the choice of $\alpha \in \mathrm{Br}(X)$ does not matter at the LVL, and the components of the moduli space[8], each parametrized by $B + i\kappa \in \mathcal{K}_{X,\mathbb{C}}/H^2(X,\mathbb{Z})$ but having different values of α, are joined at the LVL (Figure 1). For example, it is interesting to study the role of Brauer group in the space of Bridgeland stability conditions. For the rest of this paper, we will ignore the issue of the Brauer group without harm, by assuming that such an α is chosen.

We next define the B-Yukawa coupling. First recall the isomorphism $H^{2,1}(X) \cong H^1(X, T_X)$. The differential of the period map on the Kuranishi space for X is identified with the interior product $H^1(X, T_X) \to$

---

[7]For Calabi–Yau threefolds of type K, no quantum correction in Y_A^X is expected because the moduli space of $g = 0$ stable maps probably involves a factor of E. This is also examined in Section 4.3.

[8]No flop is allowed [16, Proposition 5.5].
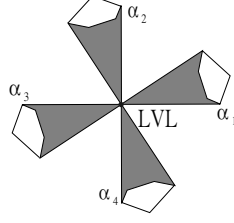
FIGURE 1. A-model moduli space for $\mathrm{Br}(X) \cong \mathbb{Z}_2^{\oplus 2}$

$\mathrm{Hom}(H^{k,l}(X), H^{k-1,l+1}(X))$ for $k, l \in \mathbb{N}$. By iterating these, we obtain a symmetric trilinear form, called the B-Yukawa coupling,

$$Y_B^X : H^1(X, T_X)^{\otimes 3} \longrightarrow \mathrm{Hom}(H^{3,0}(X), H^{0,3}(X)) \cong \mathbb{C},$$

where the last identification depends on a trivialization $H^{3,0}(X) \cong \mathbb{C}$. Recall that we have the natural isomorphism

$$H^{2,1}(X) \cong H^{1,1}(S)_{C_2}^H \otimes H^{1,0}(E) \oplus H^{2,0}(S) \otimes H^{0,1}(E).$$

Choose holomorphic volume forms dz_E and ω_S on E and S respectively. Using the isomorphism $\varphi \colon T_S \to \Omega_S, v \mapsto \iota_v \omega_S = \omega_S(v, *)$, we identify the following:

$$H^{1,1}(S)_{C_2}^H \otimes H^{1,0}(E) \cong H^1(S, T_S), \quad \eta \wedge \overline{\theta} \otimes dz_E \leftrightarrow \varphi^{-1}(\eta) \otimes \overline{\theta};$$

$$H^{2,0}(S) \otimes H^{0,1}(E) \cong H^1(E, T_E), \quad \omega_S \otimes d\overline{z}_E \leftrightarrow \frac{\partial}{\partial z_E} \otimes d\overline{z}_E.$$

Here η ($\overline{\theta}$) is the (anti-)holomorphic part of $\eta \wedge \overline{\theta} \in H^{1,1}(S)_{C_2}^H$.

Lemma 4.2. *Assume that the trivialization of $H^{3,0}(X) \cong H^{2,0}(S) \otimes H^{1,0}(E)$ is given by a nowhere-vanishing global section $\omega_S \otimes dz_E$. Then we have*

$$Y_B^X(\omega_S \otimes d\overline{z}_E, \eta_1 \wedge \overline{\theta}_1 \otimes dz_E, \eta_2 \wedge \overline{\theta}_2 \otimes dz_E) = \langle \eta_1 \wedge \overline{\theta}_1, \eta_2 \wedge \overline{\theta}_2 \rangle_{H^{1,1}(S)_{C_2}^H}$$

where $\langle *, ** \rangle_{H^{1,1}(S)_{C_2}^H}$ denotes the cup product restricted to $H^{1,1}(S)_{C_2}^H$. The B-Yukawa coupling Y_B^X is obtained by extending the above form trilinearly and symmetrically and by setting other non-trivial couplings to be 0.

Proof. Using the above identification of cohomology groups, we may show the assertion by a straightforward calculation. $\square$

Therefore the B-Yukawa coupling is of type $\langle *, ** \rangle_{H^{1,1}(S)_{C_2}^H}$ after $\mathbb{C}$-extension (see Definition 3.18). On the other hand, by Theorem 3.9, there always exists a decomposition

$$N_G = U(k) \oplus U(k)_{N_G}^{\perp}, \quad U(k)_{N_G}^{\perp} \otimes \mathbb{Q} \cong M_G \otimes \mathbb{Q},$$

for some $k \in \mathbb{N}$. The complex moduli space of S admits the Bailey–Borel compactfication and the primitive isotropic sublattices of $T(S) = N_G$ correspond to the cusps of the compactification [26]. Therefore, an isotropic

vector $e \in U(k)$ corresponds to a *large complex structure limit* (LCSL) in the context of mirror symmetry.

Let us now investigate the behavior of the B-Yukawa coupling Y_B^X when the moduli point of our K3 surface S approaches to the cusp. The Hodge $(1,1)$-part of the quadratic space $H^2(S,\mathbb{C})_{C_2}^H$ is identified with the quotient quadratic space

$$H^{1,1}(S)_{C_2}^H \cong (\omega_S)_{H^2(S,\mathbb{C})_{C_2}^H}^\perp / \mathbb{C}\omega_S.$$

Hence, at the limit as the period ω_S approaches to e, the quadratic space $H^{1,1}(S)_{C_2}^H$ becomes the $\mathbb{C}$-extension of $e^\perp/\mathbb{Q}e \cong M_G \otimes \mathbb{Q}$. Thereby, the quadratic factor of Y_B^X asymptotically converges to $M_G \otimes \mathbb{C}$ near the LCSL.

Given a pair $(\mathscr{X}, \mathscr{Y})$ of mirror families of Calabi–Yau threefolds. One feature of mirror symmetry is the identification of the Yukawa couplings

$$Y_A^X(H_1, H_2, H_3) = Y_B^Y(\theta_1, \theta_2, \theta_3)$$

after a transformation, called the *mirror map*, of local moduli parameters H_i around the LVL $\mathcal{K}_{X,\mathbb{C}}/H^2(X,\mathbb{Z})$ and θ_i around a LCSL of $\mathcal{M}_Y$. Here X is a general member of $\mathscr{X}$ near the LVL and Y is a general member of $\mathscr{Y}$ near the LCSL.

Theorem 4.3. *Let X be a Calabi–Yau threefold of type K. The asymptotic behavior of the A-Yukawa coupling Y_A^X around the LVL coincides, up to scalar multiplication, with that of the B-Yukawa coupling Y_B^X around a LCSL. The identification respects the rational structure of the trilinear forms.*

Proof. The assertion readily follows from the discussions in this section. As the complexified Kähler moduli $B + i\kappa \in \mathcal{K}_{X,\mathbb{C}}/H^2(X,\mathbb{Z})$ approaches to the LVL, the A-Yukawa coupling Y_A^X becomes the classical cup product on $H^{1,1}(X)$, which is the $\mathbb{C}$-extension of the trilinear form on $H^2(X,\mathbb{Z})$. By Theorem 3.19 and Remark 3.20, the trilinear form on $H^2(X,\mathbb{Z})$ is of type $L = M_G(1/2)$ over $\mathbb{Q}$. On the other hand, as the period ω_S of the K3 surface S approaches to the cusp of the period domain $\mathcal{D}_S^G$, the B-Yukawa coupling Y_B^X becomes a trilinear form, which has a linear factor and whose residual quadratic form is the $\mathbb{C}$-extension of $e^\perp/\mathbb{Q}e \cong M_G \otimes \mathbb{Q}$. $\qquad\square$

The identification in the proof of Theorem 4.3 respects the integral structure, that is, $e^\perp/\mathbb{Z}e \cong M_G$, for some cases including $G \cong C_2$. However, this is not true in general (see Theorem 3.9).

4.3. **B-Model Computation.** In this section, we will compute the B-models of Calabi–Yau threefolds of type K with 3- or 4-dimensional moduli spaces. The necessary integrals reduce to those of S and E, and thus there should be no instanton corrections. However it is interesting to see how they recover the lattice M_G. We will also find some modular properties in the computation, which may be useful in further investigations, such as the

BCOV theory. For $G \cong D_{12}$, with the same notation as in Example 3.6, the period integral of S is, for the cycle γ given by $|\frac{x}{y}| = |\frac{z}{w}| = \epsilon$ ($0 < \epsilon \ll 1$),

$$\Phi_0 := \int_\gamma \frac{(xdy - ydx) \wedge (zdw - wdz)}{\sqrt{Ax^2y^2z^2w^2 + B(x^4z^4 + y^4w^4) + C(x^4zw^3 + y^4z^3w)}},$$

where $[A : B : C] \in \mathbb{P}^2$ is the moduli parameter. On the open set $\{A \neq 0\} \cong \mathbb{C}^2$ with $z_1 := (B/2A)^2$ and $z_2 := (C/2A)^2$, the period integral reads

$$\Phi_0(z_1, z_2) = 1 + 12(z_1 + z_2) + 420(z_1^2 + 4z_1z_2 + z_2^2) + 18480(z_1 + z_2)(z_1^2 + 8z_1z_2 + z_2^2) + \cdots.$$

A numerical computation shows that the Picard–Fuchs system is generated by

$$\Theta_i^2 - 4z_i(4\Theta_1 + 4\Theta_2 + 3)(4\Theta_1 + 4\Theta_2 + 1)$$

for $i = 1, 2$, where $\Theta_i := z_i\frac{\partial}{\partial z_i}$ is the Euler differential. We observe that the point $(z_1, z_2) = (0, 0)$ is a LCSL of this family. The linear-logarithmic solutions are

$$\Phi_1(z_1, z_2) := \Phi_0 \log(z_1) + 40z_1 + 64z_2 + 1556z_1^2 + 7904z_1z_2 + 2816z_2^2 + \cdots,$$

$$\Phi_2(z_1, z_2) := \Phi_0 \log(z_2) + 64z_1 + 40z_2 + 2816z_1^2 + 7904z_1z_2 + 1556z_2^2 + \cdots.$$

Then the mirror map around the LCSL is given by

$$q_1 = \exp(\Phi_1/\Phi_0) = z_1 + 8z_1(5z_1 + 8z_2) + 4z_1(469z_1^2 + 2304z_1z_2 + 1024z_2^2) + \cdots,$$

$$q_2 = \exp(\Phi_2/\Phi_0) = z_2 + 8z_2(8z_1 + 5z_2) + 4z_2(1024z_1^2 + 2304z_1z_2 + 469z_2^2) + \cdots.$$

We write $q_i = e^{2\pi i t_i}$ ($i = 1, 2$) and we regard (t_1, t_2) as the G-invariant complexified Kähler parameter of S so that it descends to the complexified Kähler parameter of X. The techniques in [20, 19] shows that the inverse mirror maps read

$$z_1(q_1, q_2) = \frac{\vartheta_2^8(q_1)}{64(\vartheta_3^4(q_1) + \vartheta_4^4(q_1))^2}\left(1 - \frac{\vartheta_2^8(q_2)}{(\vartheta_3^4(q_2) + \vartheta_4^4(q_2))^2}\right),$$

$$z_2(q_1, q_2) = \frac{\vartheta_2^8(q_2)}{64(\vartheta_3^4(q_2) + \vartheta_4^4(q_2))^2}\left(1 - \frac{\vartheta_2^8(q_1)}{(\vartheta_3^4(q_1) + \vartheta_4^4(q_1))^2}\right),$$

where $\vartheta_2(q), \vartheta_3(q)$ and $\vartheta_4(q)$ are the Jacobi theta functions[9]. The period Φ_0 also reads, with respect to the mirror coordinates,

$$\Phi_0(q_1, q_2) = \frac{1}{2}\sqrt{(\vartheta_3^4(q_1) + \vartheta_4^4(q_1))(\vartheta_3^4(q_2) + \vartheta_4^4(q_2))}.$$

We denote by ∇_{z_i} the Gauss–Manin connection with respect to the local moduli parameter z_i and define $C_{z_i,\ldots,z_j} := \int_S \omega_S \wedge (\nabla_{z_i} \cdots \nabla_{z_j}\omega_S)$. Using the Griffith transversality relations

$$C_{z_i} = 0, \quad 3\partial_{z_1}C_{z_1,z_1} = 2C_{z_1,z_1,z_1}, \quad \partial_{z_2}C_{z_1,z_1} + 2\partial_{z_1}C_{z_1,z_2} = 2C_{z_1,z_1,z_2},$$

---

[9] $\vartheta_2(q) := \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}, \vartheta_3(q) := \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}}, \vartheta_4(q) := \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}}.$

we determine the B-Yukawa couplings, up to multiplication by a constant, as follows:

$$C_{z_i,z_i}(z_1, z_2) = \frac{1}{2^5 z_i (1 - 2^7(z_1 + z_2 + 2^6 z_1 z_2) + 2^{12}(z_1^2 + z_2^2))} \quad (i = 1, 2),$$

$$C_{z_1,z_2}(z_1, z_2) = \frac{1 - 64(z_1 + z_2)}{2^{12} z_1 z_2 (1 - 2^7(z_1 + z_2 + 2^6 z_1 z_2) + 2^{12}(z_1^2 + z_2^2))}.$$

Via the mirror map, we determine the A-Yukawa couplings

$$K_{t_i,t_j}(q_1, q_2) = \frac{1}{\Phi_0(z_1, z_2)^2} \sum_{k,l=1}^{2} C_{z_k,z_l}(z_1, z_2) \frac{\partial z_k}{\partial t_i} \frac{\partial z_l}{\partial t_j}.$$

There is no quantum correction for a K3 surface and we confirm that $K_{t_1,t_1} = K_{t_2,t_2} = 0$ and $K_{t_1,t_2} = 1$. This recovers the fact $H^2(S, \mathbb{Z})^{D_{12}} \cong U(2)$, up to multiplication by a constant. An identical argument works for $G \cong D_{10}$ and $C_2 \times D_8$, and the B-model calculations are compatible with the previous sections.

We turn to the elliptic curve and consider the following family of elliptic curves in $\mathbb{P}^2$ [27]

$$x_1 x_2 x_3 - z(x_1 + x_2)(x_2 + x_3)(x_3 + x_1) = 0.$$

It is equipped with a translation $(x_1 : x_2 : x_3) \mapsto (1/x_2 : 1/x_3 : 1/x_1)$, which generates the group $H \cong C_6$, and the parameter z is considered as a coordinate of the modular curve $X_1(6) \cong \mathbb{P}^1$. The Picard–Fuchs operator and its regular solution are given by

$$(8z - 1)(z + 1)\Theta^2 + z(16z + 7)\Theta + 2z(4z + 1), \quad \Theta := z\frac{d}{dz},$$

and

$$\Phi_0^{\text{ell}}(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^3 z^n = 1 + 2z + 10z^2 + 56z^3 + 346z^4 + 2252z^5 + 15184z^6 + \cdots.$$

The parameter z as a modular function on the upper half-plane $\mathbb{H}$ has the expansion

$$z(q) = \left(\frac{\eta(t)^3 \eta(t/6)}{\eta(t/2)^3 \eta(t/3)}\right)^3 = q^{1/6}(1 - 3q^{1/6} + 3q^{1/3} + 5q^{1/2} - 18q^{2/3} + 15q^{5/6} + \cdots)$$

around $z = 0$, where we write $q = e^{2\pi i t}$ ($t \in \mathbb{H}$) and $\eta(t)$ is the Dedekind eta function. As in the K3 surface case, the fundamental period is expressed as a modular form:

$$\Phi_0^{\text{ell}}(q) = \frac{\eta(t/2)^6 \eta(t/3)}{\eta(t)^3 \eta(t/6)^2} = \frac{1}{3} \sum_{(n,m)\in\mathbb{Z}^2} \left(q^{(n^2+nm+m^2)/6} + 2q^{(n^2+nm+m^2)/3}\right)$$

$$= 1 + 2q^{1/6} + 4q^{1/3} + 2q^{1/2} + 2q^{2/3} + 4q + \cdots.$$

Therefore, we observe that the period integral and the mirror maps of the threefold X are all written in terms of modular forms for $G \cong D_{12}$.

We may apply a similar argument for $G \cong D_8$. In this case, the period integral for the K3 surface is given by

$$\Phi_0(z_1, z_2, z_3) = 1+12(z_1+z_2+z_3)+420(z_1^2+z_2^2+z_3^2+4z_1z_2+4z_2z_3+4z_1z_3)+\cdots$$

and the Picard–Fuchs system is generated by, for $\mathbf{a} = [a_1, \ldots, a_6] \in \mathbb{C}^6$,

$$
\begin{aligned}
\mathcal{D}_{\mathbf{a}} =& (a_1 - 64a_1z_1 + 4(-16a_1 - 16a_2 + 3a_5)z_2 - 12a_4z_3)\Theta_1^2 \\
& + (a_2 - 12a_5z_1 - 64a_2z_2 + 4(-16a_2 - 16a_3 + 3a_6)z_3)\Theta_2^2 \\
& + (a_3 + 4(-16a_1 - 16a_3 + 3a_4)z_1 - 12a_6z_2 - 64a_3z_3)\Theta_3^2 \\
& - 128(a_1z_1 + a_2z_2 + a_3z_3)(\Theta_1\Theta_2 + \Theta_2\Theta_3 + \Theta_3\Theta_1) \\
& - 64(a_1z_1 + a_2z_2 + a_3z_3)(\Theta_1 + \Theta_2 + \Theta_3) - 12(a_1z_1 + a_2z_2 + a_3z_3).
\end{aligned}
$$

5. Special Lagrangian Fibrations

Let X be a Calabi–Yau manifold of dimension n with a Ricci-flat Kähler metric g. Let κ be the Kähler form associated to g. There exists a nowhere-vanishing holomorphic n-form Ω_X on X and we normalize Ω_X by requiring

$$(-1)^{\frac{n(n-1)}{2}}(\frac{i}{2})^n\Omega_X \wedge \overline{\Omega}_X = \frac{\kappa^n}{n!},$$

which uniquely determines Ω_X up to a phase $e^{i\theta} \in S^1$. A submanifold $L \subset X$ of real dimension n is called special Lagrangian if $\kappa|_L = 0$ and $\Re(\Omega_X)|_L = e^{i\theta_L}\mathrm{vol}_L$ for a constant $\theta_L \in \mathbb{R}$. Here vol_L denotes the Riemannian volume form on L induced by g. A surjective map $\pi\colon X \to B$ is called a special Lagrangian T^n-fibration if a generic fiber is a special Lagrangian n-torus T^n.

In [28] Strominger, Yau and Zaslow proposed that mirror symmetry should be what physicists call *T-duality*. In this so-called SYZ description, a Calabi–Yau manifold X admits a special Lagrangian T^n-fibration $\pi\colon X \to B$ near a LCSL and mirror Calabi–Yau manifold Y is obtained by fiberwise dualization of π, modified by instanton corrections. The importance of this conjecture lies in the fact that it leads to an intrinsic characterization of the mirror Calabi–Yau manifold Y; it is the moduli space of these special Lagrangian fibers $T^n \subset X$ decorated with flat U(1)-connection.

Example 5.1. *Let $E := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be an elliptic curve with $\tau \in \mathbb{H}$. Fix a holomorphic 1-form $dz := dx + idy$ and a Kähler form $\kappa := dx \wedge dy$. A special Lagrangian submanifold $L \subset X$ is a real curve such that $\kappa|_L = 0$ and $dy|_L = e^{i\theta_L}\mathrm{vol}_L$ for some $\theta_L \in \mathbb{R}$. The first condition is vacuous as L is of real dimension 1, and the second implies that L is a line. For example, the map*

$$\pi_E\colon E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \to S^1 = \mathbb{R}/\Im(\tau), \quad z \mapsto \Im(z)$$

is a special Lagrangian smooth T^1-fibration.

Given a Calabi–Yau manifold, finding a special Lagrangian fibration is an important and currently unsolved problem in high dimensions. Among

others, a well-known result is Gross and Wilson's work on Borcea–Voisin threefolds [13]. Recall that a Borcea–Voisin threefold is obtained from a K3 surface S with an anti-symplectic (non-Enriques) involution ι and an elliptic curve E by resolving singularities of the quotient $(S \times E)/\langle(\iota, -1_E)\rangle$ [6, 29]. There are two drawbacks in this case. Firstly, we need to allow a degenerate metric. Secondly, we need to work on a slice of the complex moduli space, where the threefold is realized as a blow-up of the orbifold $(S \times E)/\langle(\iota, -1_E)\rangle$.

In this section we will construct special Lagrangian fibrations of the Calabi–Yau threefolds of type K with smooth Ricci-flat metric. Such a fibration for the Enriques Calabi–Yau threefold was essentially constructed in [13] and our arguments below are its modifications. We state it here and do not repeat it each time in the sequel.

5.1. **K3 Surfaces as HyperKähler Manifolds.** Let S be a K3 surface with a Ricci-flat Kähler metric g. Then the holonomy group is $SU(2) \cong Sp(1)$ and the parallel transport defines complex structures I, J, K satisfying the quaternion relations: $I^2 = J^2 = K^2 = IJK = -1$ such that $S^2 = \{aI + bJ + cK \in \mathrm{End}(TS) \mid a^2 + b^2 + c^2 = 1\}$ is the possible complex structures for which g is a Kähler metric. The period of S in the complex structure I is given by the normalized holomorphic 2-form $\omega_I(*, **) := g(J*, **) + ig(K*, **)$, and the compatible Kähler form is given by $\kappa_I(*, **) := g(I*, **)$. We denote, for instance, by S_K the K3 surface S with the complex structure K.

Complex structure	Holomorphic 2-form	Kähler form
I	$\omega_I := \Re\omega_I + i\Im\omega_I$	κ_I
J	$\omega_J := \kappa_I + i\Re\omega_I$	$\kappa_J := \Im\omega_I$
K	$\omega_K := \Im\omega_I + i\kappa_I$	$\kappa_K := \Re\omega_I$

Recall the *hyperKähler trick*, which asserts that a special Lagrangian T^2-fibration with respect to the complex structure I is the same as an elliptic fibration with respect to the complex structure K, due to the following proposition.

Proposition 5.2 (Harvey–Lawson [14]). *A real smooth surface $L \subset S_I$ is a special Lagrangian submanifold if and only if $L \subset S_K$ is a complex submanifold.*

5.2. **Calabi–Yau Threefolds of Type K.** Let X be a Calabi–Yau threefold of type K and $S \times E \to X$ its minimal splitting covering with Galois group G. We equip S with a G-invariant Kähler class κ, which uniquely determines a G-invariant Ricci-flat metric on S [30]. We may assume that κ is generic in the sense that $\kappa^\perp \cap H^2(S, \mathbb{Z})^G = 0$. The product Ricci-flat metric on $S \times E$ is G-invariant and hence descends to a Ricci-flat metric on the quotient X, which we will always use in the following.

Proposition 5.3. *There exists $\omega_I \in \mathcal{D}_S^G$ such that its hyperKähler rotation S_K admits an elliptic fibration $\pi_S \colon S_K \to \mathbb{P}^1$ with a G-stable multiple-section. In other words, S_I admits a special Lagrangian T^2-fibration $\pi_S \colon S_I \to \mathbb{P}^1$ with a special Lagrangian multiple-section which is G-stable as a set.*

Proof. By Theorem 3.9, the transcendental lattice $N_G = H^2(S, \mathbb{Z})_{C_2}^H$ always contains $U(k)$ for some $k \in \mathbb{N}$. We denote by e, f the standard basis of $U(k)$. We choose a generic $\omega_I \in \mathcal{D}_S^G$ with $\Im \omega_I \in U(k)_{N_G}^\perp \otimes \mathbb{R}$. Then it follows that e, f are contained in $NS(S_K)$. By [24, Lemma 1.7], we may assume that f is nef after applying a sequence of reflections with respect to (-2)-curves in $NS(S_K)$, if necessary. Hence the linear system of f induces an elliptic fibration $\pi_S \colon S_K \to \mathbb{P}^1$. Let C be an irreducible curve C in S_K such that $\langle C, f \rangle > 0$. Then C is a multiple-section of the elliptic fibration with multiplicity $\langle C, f \rangle$. Thus $\cup_{g \in G} g(C)$ is a G-stable multiple-section. $\qquad\square$

It is possible to show the existence of a special Lagrangian fibration for a *generic* choice of complex structure of S with a *numerical* multiple-section (see [13, Proposition 1.3]). There is in general no canonical choice of such a fibration in contrast to the Borcea–Voisin case. Henceforth we denote simply by S the K3 surface S_I with the complex structure I obtained in Proposition 5.3.

Proposition 5.4. *The action of H on S_K is holomorphic and that of ι is anti-holomorphic, where K is the complex structure obtained in Proposition 5.3. Moreover the action of G on S descends to the base $\mathbb{P}^1$ in such a way that the fibration $\pi_S \colon S \to \mathbb{P}^1$ is G-equivariant.*

Proof. Recall that the Kähler form κ is G-invariant. The metric g is still Ricci-flat after a hyperKähler rotation and hence the G-action is isometry with respect to the metric g on S_K. The holomorphic 2-form ω_K is harmonic and so is $g^* \omega_K$ for any $g \in G$. For any $h \in H$, we hence have $h^* \omega_K = \omega_K$ as a 2-form. It then follows that the H-action is holomorphic on S_K. In the same manner, we can show that $\iota^* \omega_K = -\overline{\omega_K}$ as a 2-form and thus ι is anti-holomorphic. Since H leaves the fiber class of π_S invariant and ι simply changes its sign, the action of G on S_K descends to the base $\mathbb{P}^1$ in such a way that the fibration $\pi_S \colon S_K \to \mathbb{P}^1$ is G-equivariant. $\qquad\square$

Let $\pi_S \colon S \to \mathbb{P}^1 \cong S^2$ be the special Lagrangian fibration in Proposition 5.3. Combining the maps π_S and π_E, we obtain a special Lagrangian T^3-fibration $\pi_{S \times E} \colon S \times E \to S^2 \times S^1$ with respect to the Ricci-flat metric on $S \times E$. Moreover, by Proposition 5.4, $\pi_{S \times E}$ induces a map

$$\pi_X \colon X = (S \times E)/G \to B := (S^2 \times S^1)/G$$

with a multiple-section.

Proposition 5.5. *The map $\pi_X \colon X \to B$ is a special Lagrangian T^3-fibration such that each (possibly singular) fiber has Euler number zero.*

Proof. Let $p\colon S^2 \times S^1 \to B$ denote the quotient map. We define $S^{[H]}$ to be the set of points in S fixed by some $h \in H$. Similarly we define $(S^2)^{[G\backslash H]}$. For any generic $b \in B$ in the sense that the fiber $\pi_{S\times E}^{-1}(p^{-1}(b))$ is smooth and

$$b \notin p\big((\pi_S(S^{[H]}) \cup (S^2)^{[G\backslash H]}) \times S^1\big),$$

the fiber $\pi_{S\times E}^{-1}(p^{-1}(b))$ consists of some copies of T^3 and the fiber $\pi_X^{-1}(b) = \pi_{S\times E}^{-1}(p^{-1}(b))/G$ is again T^3. This shows that π_X is a special Lagrangian T^3-fibration because the special Lagrangian condition is clearly preserved by the G-action. Moreover, for any $c = (c_1, c_2) \in S^2 \times S^1$, we have the Euler number $\chi_{\mathrm{top}}(\pi_{S\times E}^{-1}(c)) = \chi_{\mathrm{top}}(\pi_S^{-1}(c_1) \times S^1) = 0$. It then follows that $\chi_{\mathrm{top}}(\pi_X^{-1}(b)) = \chi_{\mathrm{top}}\big(\pi_{S\times E}^{-1}(p^{-1}(b))\big)/|G| = 0$ for any $b \in B$. $\qquad\square$

Proposition 5.6. *The base space B is topologically identified either as the 3-sphere S^3 or an S^1-bundle over $\mathbb{RP}^2$.*

Proof. In the same manner as above, it is easy to see that the map $(S \times E)/H \to (S^2 \times S^1)/H$ is a special Lagrangian T^3-fibration. Since the H-actions on S^2 and S^1 are rotations, the base $(S^2 \times S^1)/H$ is topologically isomorphic to $S^2 \times S^1$. Let $\bar{\iota}$ be the generator of G/H. The action of $\bar{\iota}$ on the first factor $S^2 \cong \mathbb{P}^1$ is anti-holomorphic and hence is identified with either

$$z \mapsto \bar{z} \text{ with } (S^2)^{\bar{\iota}} = \mathbb{R} \cup \infty, \quad \text{or} \quad z \mapsto -\frac{1}{\bar{z}} \text{ with } (S^2)^{\bar{\iota}} = \emptyset.$$

The action of $\bar{\iota}$ on the second factor $S^1 = \{|z| = 1\} \subset \mathbb{C}$ is identified with the reflection along the real axis with $(S^1)^{\bar{\iota}} = S^0$. The case when $(S^2)^{\bar{\iota}} = \mathbb{R} \cup \infty$ was previously studied in [13, Section 3] and the base $B = (S^2 \times S^1)/\langle\bar{\iota}\rangle$ is topologically identified with S^3. On the other hand, when $(S^2)^{\bar{\iota}} = \emptyset$, the base $B = (S^2 \times S^1)/\langle\bar{\iota}\rangle$ is endowed with an S^1-bundle structure via the first projection:

$$B = (S^2 \times S^1)/\langle\bar{\iota}\rangle \to S^2/\langle\bar{\iota}\rangle \cong \mathbb{RP}^2.$$

$$\square$$

Note that B need not be a 3-sphere S^3 as X is not simply-connected, but we do not know whether or not the latter case really occurs.

For discussion on Brauer groups, dual fibrations and their sections, we refer the reader to [12, comment after Remark 3.12]. Another interesting problem is to relate the Hodge theoretic mirror symmetry with the SYZ description. Aspects of this have been studied for Borcea–Voisin threefolds in [13] (later more generally in [12]), where they recover the mirror map from the Leray spectral sequence associated to a special Lagrangian T^3-fibration. For a Borcea–Voisin threefold, this is the first description of the mirror map and conjecturally their method can be applied to a larger class of Calabi–Yau manifolds. The difficulty to tackle the problem in the present case is that the singular fibers may be reducible and the spectral sequence is more involved.

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