# TOWARD WEIL–PETERSSON GEOMETRY OF KÄHLER MODULI SPACES

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Abstract. This article is an extended version of the author's talk at the Kinosaki Algebraic Geometry Symposium 2017. We discuss Weil–Petersson geometry of the Kähler moduli space of a Calabi–Yau manifold via the Bridgeland stability conditions.

## 1. Introduction

The purpose of the present article is to provide a step toward differential geometric study of Kähler moduli spaces via Bridgeland stability conditions. The motivation of our work comes from mirror symmetry. Mirror symmetry is duality between complex geometry of a Calabi–Yau manifold $X$ and symplectic (Kähler) geometry of a mirror Calabi–Yau manifold $Y$. In light of the fact that there exists a canonical Kähler metric, called the Weil–Petersson metric, on the complex moduli space of a Calabi–Yau manifold, we would like to propose the following question:

*What is the canonical metric on the Kähler moduli space of a Calabi–Yau manifold, which is mirror to the classical Weil–Petersson metric?*

Our approach taken in the article [7] is to use the stability conditions on a triangulated category introduced by Bridgeland [3]. Let $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^bCoh}(X))$ be the space of numerical stability conditions of the derived category $\mathrm{D^bCoh}(X)$ of a Calabi–Yau manifold $X$. Then it is conjectured that there exists an embedding of the Kähler moduli space $\mathfrak{M}_{\mathrm{Kah}}(X)$ into

$$\mathrm{Aut}(\mathrm{D^bCoh}(X))\backslash\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^bCoh}(X))/\mathbb{C}.$$

On careful comparison of the two sides of Kontsevich's homological mirror symmetry conjecture [10]: $\mathrm{D^bCoh}(X) \cong \mathrm{D^bFuk}(Y)$, we will give a provisional definition of Weil–Petersson geometry of the above double quotient space (not the Kähler moduli space $\mathfrak{M}_{\mathrm{Kah}}(X)$). We warn the reader that, since we do not directly work with $\mathfrak{M}_{\mathrm{Kah}}(X)$, our metric is in general degenerate. Nevertheless, we will see that the degeneracy turns out to be useful in a sense (Conjecture 4.1).

We will provide some supporting evidence of our proposal by computing a few basic examples. It will be observed that our Weil–Petersson metrics coincide with classically well-knwon metrics in some cases. The most important example is the following (where the above conjectural embedding is really an isomorphism).

**Theorem 1.1** (Theorem 3.1)**.** *Let $A$ be the self-product $E_\tau \times E_\tau$ of an elliptic curve $E_\tau$. Then there is an identification*

$$\overline{\mathrm{Aut}}_{\mathrm{CY}}(\mathrm{D^bCoh}(A))\backslash \mathrm{Stab}^+_{\mathcal{N}}(\mathrm{D^bCoh}(A))/\mathbb{C}^\times \cong \mathrm{Sp}(4,\mathbb{Z})\backslash \mathfrak{H}_2.$$

*Moreover, the Weil–Petersson metric on the LHS is identified with the Bergman metric on the Siegel modular variety* $\mathrm{Sp}(4,\mathbb{Z})\backslash \mathfrak{H}_2$.

This result is compatible with the mirror duality between $A$ and the principally polarized abelian surface since the complex moduli space of the latter is the Siegel modular variety $\mathrm{Sp}(4,\mathbb{Z})\backslash \mathfrak{H}_2$.

It is worth mentioning Wilson and Trenner's relevant work [21, 19]. They studied the so-called *asymptotic Weil–Petersson metrics* on the complexified Kähler cones, which are considered as approximations of the mirror Weil–Petersson metrics near large volume limits. An advantage of our approach is the fact that our Weil–Petersson metric is inherently global and makes perfect sense away from large volume limits, in contrast to Wilson and Trenner's local study. As a matter of fact, the global aspects of the moduli space are of special importance in recent study of mirror symmetry. A very little is known about global aspects of the Kähler moduli spaces and we hope our work will provide a step forward in this research direction.

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## 2. Bridgeland stability conditions

2.1. **Bridgeland stability conditions.** The notion of stability conditions on a triangulated category was introduced by Bridgeland [3], inspired by ideas of the Π-stabilities of D-branes [6]. In this article, a triangulated category $\mathcal{D}$ is essentially small, linear over the complex numbers $\mathbb{C}$, and of finite type. We define the Euler form $\chi$ on the Grothendieck group $K(\mathcal{D})$ by the formula

$$\chi(E,F) := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(E, F[i]).$$

The numerical Grothendieck group $\mathcal{N}(\mathcal{D}) := K(\mathcal{D})/K(\mathcal{D})^{\perp_\chi}$ is the quotient of $K(\mathcal{D})$ by the null space $K(\mathcal{D})^{\perp_\chi}$ of $\chi$. We always assume that $\mathcal{N}(\mathcal{D})$ is of finite rank.

**Definition 2.1** ([3, 11])**.** *A numerical stability condition $\sigma = (\mathcal{Z}, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of*

- *a group homomorphism $\mathcal{Z} : \mathcal{N}(\mathcal{D}) \to \mathbb{C}$ (central charge),*
- *a collection of full additive subcategories $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}$ of $\mathcal{D}$ (semistable objects)*

*such that:*

(1) *If* $0 \neq E \in \mathcal{P}(\phi)$, *then* $\mathcal{Z}(E) \in \mathbb{R}_{>0} \cdot e^{\sqrt{-1}\pi\phi}$.
(2) $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$.
(3) *If* $\phi_1 > \phi_2$ *and* $A_i \in \mathcal{P}(\phi_i)$, *then* $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$.
(4) *For every* $0 \neq E \in \mathcal{D}$, *there exists a collection of exact triangles*

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{k-1} \arrow[r] & E \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & & A_k \arrow[ul, dashed]
\end{tikzcd}$$

*such that* $A_i \in \mathcal{P}(\phi_i)$ *and* $\phi_1 > \phi_2 > \cdots > \phi_k$.
(5) *There exist a constant* $C > 0$ *and a norm* $||*||$ *on* $\mathcal{N}(\mathcal{D})_{\mathbb{R}}$ *such that* $||E|| \leq C|\mathcal{Z}(E)|$ *for any semistable object* $E$.

We denote by $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ the space of numerical stability conditions on $\mathcal{D}$. Bridgeland defined a nice topology on it such that the forgetful map

$$\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}) \longrightarrow \mathrm{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C}), \quad \sigma = (\mathcal{Z}, \mathcal{P}) \mapsto \mathcal{Z}$$

is a local homeomorphism [3, 11]. Thereby $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ is naturally a complex manifold, which is locally modelled on the $\mathbb{C}$-vector space $\mathrm{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C})$.

Moreover, $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ naturally carries a right action of the group $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, the universal cover of the group $\mathrm{GL}^+(2, \mathbb{R})$ of orientation-preserving linear automorphism of $\mathbb{R}^2$, as well as a left action of the group $\mathrm{Aut}(\mathcal{D})$ of autoequivalences of $\mathcal{D}$. The $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action is given by post-composition on the central charge $\mathcal{Z} : \mathcal{N}(\mathcal{D}) \to \mathbb{C} \cong \mathbb{R}^2$ and a suitable relabelling of the phases. We often restrict this action to the subgroup $\mathbb{C} \subset \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ which acts freely.

2.2. **Central charge via twisted Mukai pairing.** Let $X$ be a smooth projective variety. Inspired by work of Mukai in the case of K3 surfaces [13], Căldăraru defined the the Mukai pairing on $H^*(X; \mathbb{C})$ [5]:

$$\langle v, v' \rangle_{\mathrm{Muk}} := \int_X e^{c_1(X)/2} \cdot v^\vee \cdot v'.$$

Here $v = \sum_j v_j \in \oplus_j H^j(X; \mathbb{C})$ and its Mukai dual $v^\vee = \sum_j \sqrt{-1}^j v_j$. This Mukai paring differs from Mukai's original definition [13] for K3 surfaces by the sign. We define a twisted Mukai vector of $E \in \mathcal{D}_X = \mathrm{D^b Coh}(X)$ by

$$v_\Lambda(E) := \mathrm{ch}(E)\sqrt{\mathrm{Td}_X}\exp(\sqrt{-1}\Lambda)$$

for any $\Lambda \in H^*(X; \mathbb{C})$ such that $\Lambda^\vee = -\Lambda$. By the Hirzebruch–Riemann–Roch theorem, we observe that a twisted Mukai pairing is compatible with the Euler pairing;

$$\chi(E, F) = \int_X \mathrm{ch}(E^\vee)\mathrm{ch}(F)\mathrm{Td}_X = \langle v_\Lambda(E), v_\Lambda(F) \rangle_{\mathrm{Muk}}.$$

A geometric twisting $\Lambda$ compatible with the integral structure on the quantum cohomology was first introduced by Iritani [8] and Katzarkov–Kontsevich–Pantev [9]. A familiar identity from complex analysis reads

$$\frac{z}{1-e^{-z}} = e^{z/2}\frac{z/2}{\sinh(z/2)} = e^{z/2}\Gamma(1+\frac{z}{2\pi\sqrt{-1}})\Gamma(1-\frac{z}{2\pi\sqrt{-1}}),$$

where $\Gamma(z)$ is the Gamma function. The power series in the LHS induces the Todd class $\mathrm{Td}_X$. Then we consider a square root of the Todd class by writing

$$\sqrt{\frac{z}{1-z}}\exp(\sqrt{-1}\Lambda(z)) = e^{z/4}\Gamma(1+\frac{z}{2\pi\sqrt{-1}}),$$

and solve it for $\Lambda(z)$, where $z$ is a real variable, as

$$\begin{aligned}\Lambda(z) &= \Im(\log\Gamma(1+\frac{z}{2\pi\sqrt{-1}})) \\ &= \frac{\gamma z}{2\pi} + \sum_{j\geq 1}(-1)^j\frac{\zeta(2j+1)}{2j+1}\left(\frac{z}{2\pi}\right)^{2j}\end{aligned}$$

where $\gamma$ is Euler's constant. Since the constant term of $\Lambda(z)$ is zero, we may use it to define an additive characteristic class $\Lambda_X$, called the *log Gamma class*. In the Calabi–Yau case, we can explicitly write it as

$$\Lambda_X = -\frac{\zeta(3)}{(2\pi)^3}c_3(X) + \frac{\zeta(5)}{(2\pi)^5}(c_5(X) - c_2(X)c_3(X)) + \dots$$

For K3 and abelian surfaces, there is no effect of twisting as $\Lambda_X = 0$. For Calabi–Yau 3-folds, the modification is given by the leading term, which is familiar in period computations in the B-model side. We define $v_X(E)$ to be the twisted Muaki vector of an object $E \in \mathcal{D}_X$ associated to the log Gamma class $\Lambda_X$:

$$v_X(E) := \mathrm{ch}(E)\sqrt{\mathrm{Td}_X}\exp(\sqrt{-1}\Lambda_X)$$

Now let $X$ be a projective Calabi–Yau manifold equipped with a complexified Kähler parameter

$$\omega = B + \sqrt{-1}\kappa \in H^2(X;\mathbb{C}),$$

where $\kappa$ is a Kähler class. Let also $q = \exp(2\pi\sqrt{-1}\omega)$. We define the quantum exponential $\exp_*(\omega)$ by

$$\exp_*(\omega) := 1 + \omega + \frac{1}{2!}\omega*\omega + \frac{1}{3!}\omega*\omega*\omega + \cdots.$$

where $*$ denotes the quantum product, which implicitly depends on $\omega$. It is conjectured that near the large volume limit, which means that $\int_C \Im(\omega) \gg 0$ for all effective curve $C \subset X$, there exists a Bridgeland stability condition on $\mathcal{D}_X$ with central charge of the form

$$\mathcal{Z}(E) = -\left\langle \exp_*(\omega), v_X(E)\right\rangle_{\mathrm{Muk}}. \tag{1}$$

The asymptotic behavior of the above central charge near the large volume limit is given by

$$\mathcal{Z}(E) \sim -\int_X e^{-\omega} v_X(E) + O(q). \tag{2}$$

The existence of a Bridgeland stability condition with the asymptotic central charge given by the leading term of the above expression has been proved for various important examples including K3 and abelian surfaces [4], as well as abelian threefolds [2, 12].

2.3. **Weil–Petersson geometry on** $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$**.** We shall propose a formulation of Weil–Petersson geometry on a suitable quotient of the space of Bridgeland stability conditions on a Calabi–Yau triangulated category $\mathcal{D}$ of dimension $n \in \mathbb{N}$, i.e. for every pair of objects $E$ and $F$, there is a natural isomorphism

$$\mathrm{Hom}^*_{\mathcal{D}}(E, F) \cong \mathrm{Hom}^*_{\mathcal{D}}(F, E[n])^\vee.$$

A prototypical example is the derived category of a Calabi–Yau $n$-fold. An important consequence is that the Euler form on $\mathcal{N}(\mathcal{D})$ is (skew-)symmetric if $n$ is even (odd).

For a basis $\{E_i\}$ of $\mathcal{N}(\mathcal{D})$, we define a bilinear form $\mathfrak{b} : \mathrm{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C})^{\otimes 2} \to \mathbb{C}$ by

$$\mathcal{Z}_1 \otimes \mathcal{Z}_2 \mapsto \mathfrak{b}(\mathcal{Z}_1, \mathcal{Z}_2) := \sum_{i,j} \chi^{i,j} \mathcal{Z}_1(E_i)\mathcal{Z}_2(E_j),$$

where $(\chi^{i,j}) := (\chi(E_i, E_j))^{-1}$. It is an easy exercise to check that the bilinear form $\mathfrak{b}$ is independent of the choice of a basis.

Then we define the subset $\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D}) \subset \mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ by

$$\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D}) := \{\sigma = (\mathcal{Z}, \mathcal{P}) \mid \mathfrak{b}(\mathcal{Z}, \mathcal{Z}) = 0,\ (\sqrt{-1})^{-n}\mathfrak{b}(\mathcal{Z}, \overline{\mathcal{Z}}) > 0\}.$$

The first condition is vacuous when $n$ is odd as the bilinear form $\mathfrak{b}$ is skew-symmetric. The natural free $\mathbb{C}$-action on $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ preserves the subset $\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D})$.

**Definition 2.2.** *Let* $s = (\mathcal{Z}_{\bar{\sigma}}, \mathcal{P}_{\bar{\sigma}})$ *be a local holomorphic section of the* $\mathbb{C}$*-torsor* $\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D}) \to \mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D})/\mathbb{C}$*, then*

$$K_{\mathrm{WP}}(\bar{\sigma}) := -\log\left((\sqrt{-1})^{-n}\mathfrak{b}(\mathcal{Z}_{\bar{\sigma}}, \overline{\mathcal{Z}_{\bar{\sigma}}})\right)$$

*defines a local smooth function on* $\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D})/\mathbb{C}$*.*

We call $K_{\mathrm{WP}}$ the Weil–Petersson potential on $\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D})/\mathbb{C}$. It is an analogue of the Weil–Petersson potential of the complex moduli space of a Calabi–Yau manifold (Hodge theoretic description due to Tian [17]).

**Proposition 2.3** (Fan–K–Yau [7])**.** *The complex Hessian* $\frac{\sqrt{-1}}{2}\partial\overline{\partial}K_{\mathrm{WP}}$ *of the Weil–Petersson potential* $K_{\mathrm{WP}}$ *does not depend on the choice of a local section* $s$*. Moreover, it descends to the double quotient space*

$$\mathrm{Aut}(\mathcal{D})\backslash\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D})/\mathbb{C}$$

*away from singular loci.*

We would like to propose the above (possibly degenerate) metric $\frac{\sqrt{-1}}{2}\partial\overline{\partial}K_{\mathrm{WP}}$ as the Weil–Petersson metric on the Kähler moduli space, which is mirror to the classical Weil–Petersson metric on the complex moduli space under the mirror identification.

## 3. Computation

The purpose of this section is to back up our proposal by computing a few basic examples. We will observe that our Weil–Petersson metrics coincide with the classical important metrics in these cases.

3.1. **Elliptic curve.** Let us consider $\mathcal{D}_X = \mathrm{D^b Coh}(X)$ for an elliptic curve $X$. Since the action of $\widetilde{\mathrm{GL}^+(2,\mathbb{R})}$ on $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_X)$ is free and transitive [3, Theorem 9.1], we have

$$\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D}_X) = \mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_X) \cong \widetilde{\mathrm{GL}^+(2,\mathbb{R})} \cong \mathbb{C} \times \mathbb{H},$$

as a complex manifold. Moreover the double quotient becomes

$$\mathrm{Aut}(\mathcal{D}_X)\backslash \mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D}_X)/\mathbb{C} \cong \mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}.$$

This is indeed the Kähler moduli space of the elliptic curve $X$. The normalized central charge at $\tau \in \mathbb{H}$ is given by

$$\mathcal{Z}(E) = -\deg(E) + \tau \cdot \mathrm{rank}(E).$$

Hence the Weil–Petersson potential is, via a symplectic basis $\{\mathcal{O}_X, \mathcal{O}_p\}$ of $\mathcal{N}(\mathcal{D}_X)$,

$$\begin{aligned} K_{\mathrm{WP}}(\tau) &= -\log\Big((\sqrt{-1})^{-1}(\mathcal{Z}(\mathcal{O}_p)\overline{\mathcal{Z}}(\mathcal{O}_X) - \mathcal{Z}(\mathcal{O}_X)\overline{\mathcal{Z}}(\mathcal{O}_p))\Big) \\ &= -\log(\Im(\tau)) - \log 2. \end{aligned}$$

This is the Poincaré potential on $\mathbb{H}$ and it descends to the Kähler moduli space $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$.

3.2. **Self-product of elliptic curve.** We consider the self-product $A := E_\tau \times E_\tau$ of an elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for $\tau \in \mathbb{H}$. We denote by $\mathrm{NS}(A) = H^2(A,\mathbb{Z}) \cap H^{1,1}(A)$ the Néron–Severi lattice of $A$.

A result of Orlov [14, Proposition 3.5] shows that every autoequivalence of $\mathcal{D}_A = \mathrm{D^b Coh}(A)$ induces a Hodge isometry of the lattice $H^*(A;\mathbb{Z})$ equipped with the Mukai pairing. Hence there is a group homomorphism

$$\delta : \mathrm{Aut}(\mathcal{D}_A) \longrightarrow \mathrm{Aut}H^*(A;\mathbb{Z}).$$

The kernel of the homomorphism will be denoted by $\mathrm{Aut}^0(\mathcal{D}_A)$.

Let $\Omega \in H^2(A;\mathbb{C})$ be the class of a holomorphic volume form on $A$. The sublattice

$$\mathcal{N}(A) := H^*(A;\mathbb{Z}) \cap \Omega^\perp \subset H^*(A;\mathbb{C})$$

can be identified with $\mathcal{N}(\mathcal{D}_A) = H^0(A;\mathbb{Z}) \oplus \mathrm{NS}(A) \oplus H^4(A;\mathbb{Z})$. In fact, if the complex moduli $\tau \in \mathbb{H}$ is generic, $\mathcal{N}(\mathcal{D}_A) \cong U^{\oplus 2} \oplus \langle 2 \rangle$. Here $U$ is the hyperbolic lattice, and $\langle 2\rangle$ denotes an integral lattice of rank 1 with the Gram matrix $(2)$.

We now recall the definition of *Calabi–Yau autoequivalences* following the work of Bayer and Bridgeland [1]. Define

$$\mathrm{Aut}^+_{\mathrm{CY}} H^*(A) \subset \mathrm{Aut}^+ H^*(A)$$

to be the subgroup of Hodge isometries which preserve the class of holomorphic 2-form $[\Omega] \in \mathbb{P}H^*(A;\mathbb{C})$. Any such isometry restricts to give an isometry of $\mathcal{N}(\mathcal{D}_A)$. In fact,

$$\mathrm{Aut}^+_{\mathrm{CY}} H^*(A) \subset \mathrm{Aut}\mathcal{N}(\mathcal{D}_A)$$

is the subgroup of index 2 which do not exchange the two components of $\mathcal{P}(A) \subset \mathcal{N}(\mathcal{D}_A)_{\mathbb{C}}$, the subset consisting of vectors $\mho \in \mathcal{N}(\mathcal{D}_A)_{\mathbb{C}}$ such that $\mathbb{R}\Re(\mho) \oplus \mathbb{R}\Im(\mho)$ is a negative definite 2-plane in $\mathcal{N}(\mathcal{D}_A)_{\mathbb{R}}$.

An autoequivalence $\Phi \in \mathrm{Aut}(\mathcal{D}_A)$ is said to be *Calabi–Yau* if the induced Hodge isometry $\delta(\Phi)$ lies in $\mathrm{Aut}^+_{\mathrm{CY}} H^*(A)$. We denote $\mathrm{Aut}_{\mathrm{CY}}(\mathcal{D}_A) \subset \mathrm{Aut}(\mathcal{D}_A)$ the group of Calabi–Yau autoequivalences. There exists a short exact sequence

$$1 \longrightarrow \mathrm{Aut}^0(\mathcal{D}_A) \longrightarrow \mathrm{Aut}_{\mathrm{CY}}(\mathcal{D}_A) \longrightarrow \mathrm{Aut}^+_{\mathrm{CY}} H^*(A) \longrightarrow 1.$$

We write $\mathrm{Aut}^0_{\mathrm{tri}}(\mathcal{D}_A) \subset \mathrm{Aut}^0(\mathcal{D}_A)$ for the subgroup generated by twists by elements of $\mathrm{Pic}^0(A)$ and pullbacks by automorphisms of $A$ acting trivially on $H^*(A;\mathbb{Z})$. Since $\mathrm{Aut}^0_{\mathrm{tri}}(\mathcal{D}_A)$ acts trivially on $\mathrm{Stab}^\dagger(\mathcal{D}_A)$, it is useful to define

$$\overline{\mathrm{Aut}}_{\mathrm{CY}}(\mathcal{D}_A) := \mathrm{Aut}_{\mathrm{CY}}(\mathcal{D}_A)/\mathrm{Aut}^0_{\mathrm{tri}}(\mathcal{D}_A).$$

Then $\overline{\mathrm{Aut}}_{\mathrm{CY}}(\mathcal{D}_A)$ acts on $\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D}_A)$, and there is a short exact sequence

$$1 \longrightarrow \langle [2]\rangle \longrightarrow \overline{\mathrm{Aut}}_{\mathrm{CY}}(\mathcal{D}_A) \longrightarrow \mathrm{Aut}^+_{\mathrm{CY}} H^*(A) \longrightarrow 1.$$

The following is a main result of our article [7].

**Theorem 3.1** (Fan–K–Yau [7]). *There exists a canonical isomorphism*

$$\overline{\mathrm{Aut}}_{\mathrm{CY}}(\mathcal{D}_A)\backslash \mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D}_A)/\mathbb{C}^\times \cong \mathrm{Sp}(4,\mathbb{Z})\backslash \mathfrak{H}_2$$

We shall call the LHS the Kähler moduli space of $A$.

Now let us take a close look at the differential geometric structures of the above two space. First, there exists a canonical metric on the Siegel modular variety $\mathrm{Sp}(4,\mathbb{Z})\backslash\mathfrak{H}_2$, namely the Bergman metric. It is known to be a complete Kähler–Einstein metric.

**Proposition 3.2** ([16]). *The Bergman kernel $K_{\mathrm{Ber}} : \mathfrak{H}_g \times \mathfrak{H}_g \to \mathbb{C}$ of the Siegel upper half-space $\mathfrak{H}_g$ of degree $g$ is given by*

$$K_{\mathrm{Ber}}(M,N) = -\mathrm{tr}(\log(-\sqrt{-1}(M-\overline{N}))).$$

*The Bergman metric is the complex Hessian of the Bergman potential*

$$K_{\mathrm{Ber}}(M) := K_{\mathrm{Ber}}(M,M) = -\mathrm{tr}(\log(2\Im(M))).$$

An important observation of the work [7] is the fact that the Bergman metric coincides with our Weil–Petersson metric.

**Theorem 3.3** (Fan–K–Yau [7])**.** *The Weil–Petersson potential on* $\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D}_A)/\mathbb{C}$ *coincides with the Bergman potential of the Siegel upper half-plane* $\mathfrak{H}_2$ *up to a constant.*

Therefore the Weil–Petersson metric on the Kähler moduli space is identified with the Bergman metric on $\mathrm{Sp}(4,\mathbb{Z})\backslash\mathfrak{H}_2$ via the isomorphism in Theorem 3.1.

3.3. **Split abelian surfaces.** Let us next examine a variant of the above example. Let $A$ be a split abelian surface, that is, $A \cong E_{\tau_1} \times E_{\tau_2}$ for elliptic curves $E_{\tau_1}$ and $E_{\tau_2}$. Such a splitting is unique provided that $E_{\tau_1}$ and $E_{\tau_2}$ are generic, or equivalently $\mathrm{NS}(A) \cong U$. We have

$$\mathrm{Aut}^+_{\mathrm{CY}} H^*(A) \cong O^+(U^{\oplus 2}) \cong \mathrm{P}(\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})) \rtimes \mathbb{Z}_2,$$

where $\mathrm{P}(\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}))$ represents the quotient group of $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ by the involution $(A,B) \mapsto (-A,-B)$ and the semi-direct product structure is given by the generator of $\mathbb{Z}_2$ acting on $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ by exchanging the two factors.

**Theorem 3.4** (Fan–K–Yau [7])**.** *There exists a canonical identification*

$$\overline{\mathrm{Aut}}_{\mathrm{CY}}(\mathcal{D}_A)\backslash\mathrm{Stab}^+_{\mathcal{N}}(\mathcal{D}_A)/\mathbb{C}^\times \cong \mathrm{P}(\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})) \rtimes \mathbb{Z}_2\backslash(\mathbb{H} \times \mathbb{H})$$

*Moreover, the Weil–Petersson metric on the LHS is identified with the Bergman metric on the RHS.*

It is worth mentioning that the computation in the above two examples are compatible with mirror symmetry. The mirror of the self-product of an elliptic curve is a principally polarized abelian surface, or equivalently $\langle 2\rangle$-polarized abelian surface. A split abelian surface is self-mirror symmetric. A lattice polarized version of the global Torelli Theorem asserts that the complex moduli space of such surfaces are given by the above Siegel modular varieties.

3.4. **Abelian variety.** Let $X$ be an abelian variety of dimension $n$. Since there is no quantum corrections and the Chern classes are trivial, the expected central charge at the complexified Kähler moduli $\omega \in H^2(X;\mathbb{C})$ is given by

$$\mathcal{Z}_{\exp(\omega)}(E) = -\langle \exp(\omega), v_X(E)\rangle_{\mathrm{Muk}} = -\int_X e^{-\omega}\mathrm{ch}(E).$$

The existence of Bridgeland stability condition with this central charge is known for $n \leq 3$. Then the Weil–Petersson potential is

$$K_{\mathrm{WP}}(\omega) = -\log(\Im(\omega)^n) - \log\frac{2^n}{n!}.$$

Fix a polarization $H$. We think of $\omega = \tau H$ for $\tau \in \mathbb{H}$ as a slice of the Kähler moduli space $\mathfrak{M}_{\mathrm{Kah}}(X)$. Then the Weil–Petersson metric on $\mathbb{H}$ is essentially the Poincaré metric. This observation is compatible with Wang's mirror result [20], which asserts that in the case of infinite distance, the Weil–Petersson metric is asymptotic to a scaling of the Poincaré metric.

3.5. **Quintic threefold.** Although the existence of a Bridgeland stability condition for a quintic threefold $X \subset \mathbb{P}^4$ has not yet been confirmed, we can still compute the Weil–Petersson potential using the central charge in Equation (1) near the large volume limit.

Let $\tau H \in H^2(X;\mathbb{C})$ be the complexified Kähler class, where $H$ is the hyperplane class and $\tau \in \mathbb{H}$. Let $N_d^X$ denote the genus 0 Gromov–Witten invariant of $X$ of degree $d$. We observe that

$$\exp_*(\tau H) = 1 + \tau H + \frac{\tau^2}{2}(1 + \frac{1}{5}\sum_{d\geq 1} N_d d^3 q^d)H^2 + \frac{\tau^3}{6}(1 + \frac{1}{5}\sum_{d\geq 1} N_d d^3 q^d)H^3,$$

where $q = e^{2\pi\sqrt{-1}\tau}$ and we use the quantum product

$$H * H = \Phi(q)H^2 = \frac{1}{5}(5 + \sum_{d\geq 1} N_d^X q^d d^3)H^2.$$

Then the central charge computes to be

$$\begin{aligned}\mathcal{Z}(E) &= -\left\langle \exp_*(\tau H), v_X(E)\right\rangle_{\mathrm{Muk}} \\ &= -\int_X e^{-\tau H} v_X(E) + \frac{\zeta(3)\chi(X)}{(2\pi)^3}(\frac{\tau^2}{10}H^2\mathrm{ch}_1(E) - \frac{\tau^3}{6}\mathrm{ch}_0(E))\sum_{d\geq 1} N_d^X d^3 q^d,\end{aligned}$$

where $\chi(X)$ is the topological Euler number of $X$. Near the large volume limit, the Weil–Petersson potential is given by

$$\begin{aligned}K_{\mathrm{WP}}(\tau) &= -\log\Big(H^3(\overline{\Phi(q)}(\frac{\bar\tau^3}{6} + \frac{\tau\bar\tau^2}{2}) - \Phi(q)(\frac{\tau^3}{6} + \frac{\tau^2\bar\tau}{2})\Big) - 2\log\Big(\frac{\zeta(3)\chi(X)}{(2\pi)^3}\Big) \\ &\sim -\log(\frac{4}{3}H^3\Im(\tau)^3) - 2\log\Big(\frac{\zeta(3)\chi(X)}{(2\pi)^3}\Big) + O(q).\end{aligned}$$

Therefore the Weil–Petersson metric is a quantum deformation of the Poincaré metric on $\mathbb{H}$. In particular, for sufficiently small $q$, it is non-degenerate and the Weil–Petersson distance to the large volume limit is infinite. We remark that if there is no B-field, the correction term $O(q)$ is precisely given by $\log(\Phi(q))$.

It is known that the complex moduli space $\mathfrak{M}_{\mathrm{Cpx}}(Y)$ of the mirror quintic 3-fold $Y$ is the 3-punctured $\mathbb{P}^1$. The points corresponds to the large complex structure limit, the conifold point and the Gepner point. Since our Weil–Petersson metric is inherently global, an alluring research direction is to examine the Weil–Petersson metric away from the large volume limit, where central charges are not of the form (1). For instance, the Weil–Petersson metric around the Gepner point may be studied via matrix factorization categories via the Orlov equivalence [15]

$$\mathrm{D^b Coh}(X) \cong \mathrm{HMF}(W),$$

where HMF($W$) is the homotopy category of a graded matrix factorization of the defining equation $W$ of the quintic 3-fold $X$. Toda studied stability conditions, called the Gepner type stability conditions, conjecturally corresponding to the Gepner point [18].

## 4. Application

Lastly we would like to discuss potential application of our work. Let $X$ be a projective Calabi–Yau $n$-fold. Then $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_X)$ is considered as an extension of the Kähler moduli space $\mathfrak{M}_{\mathrm{Kah}}(X)$. It is akin to the big quantum cohomology rather than the small quantum cohomology in the sense that the tangent space of $\mathfrak{M}_{\mathrm{Kah}}(X)$ is $H^{1,1}(X)$ while that of $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_X)$ is $\oplus_p H^{p,p}(X)$.

Motivated by mirror symmetry and classical Weil–Petersson geometry, especially the fact that Weil–Petersson metric is non-degenerate on $\mathfrak{M}_{\mathrm{Cpx}}(X)$, we can now propose the following [7], which slightly refines Bridgeland conjecture.

**Conjecture 4.1** (Fan–K–Yau [7]). *There exists an embedding of the Kähler moduli space*

$$\iota : \mathfrak{M}_{\mathrm{Kah}}(X) \hookrightarrow \mathrm{Aut}(\mathcal{D}_X)\backslash \mathrm{Stab}^{+}_{\mathcal{N}}(\mathcal{D}_X)/\mathbb{C}.$$

*The complex Hessian of the pullback $\iota^* K_{\mathrm{WP}}$ of the Weil–Petersson potential $K_{\mathrm{WP}}$ defines a Kähler metric on $\mathfrak{M}_{\mathrm{Kah}}(X)$, i.e. non-degenerate. Moreover, it is identified with the Weil–Petersson metric on the complex moduli space $\mathfrak{M}_{\mathrm{Cpx}}(Y)$ of a mirror manifold $Y$ under the mirror map*

$$\mathfrak{M}_{\mathrm{Kah}}(X) \cong \mathfrak{M}_{\mathrm{Cpx}}(Y).$$

It is worth noting that the real difficulty lies in providing a mathematical definition of the Kähler moduli space $\mathfrak{M}_{\mathrm{Kah}}(X)$. An implication of Conjecture 4.1 is the potential of making use of the non-degeneracy condition on the Weil–Petersson metric to characterize $\mathfrak{M}_{\mathrm{Kah}}(X)$.

## References


[1] A. Bayer and T. Bridgeland, *Derived automorphism groups of K3 surfaces of Picard rank 1.* Duke Math. J. 166 (2017), no. 1, 75-124.

[2] A. Bayer, E. Macri and P. Stellari, *The space of stability conditions on abelian threefolds, and on some Calabi–Yau threefolds.* Invent. Math. 206 (2016), no. 3, 869-933.

[3] T. Bridgeland, *Stability conditions on triangulated categories.* Ann. of Math. (2) 166 (2007), no. 2, 317-345.

[4] T. Bridgeland, *Stability conditions on K3 surfaces.* Duke Math. J. 141 (2008), no. 2, 241-291. Proc. Sympos. Pure Math., 80, Part 1, Amer. Math. Soc., Providence, RI, 2009.

[5] A. Căldăraru, *The Mukai pairing II: the Hochschild–Kostant–Rosenberg isomorphism.* Adv. Math. 194 (2005), 34-66.

[6] M. Douglas, *Dirichlet branes, homological mirror symmetry, and stability.* Proc. of ICM, Vol. III (Beijing, 2002), 395-408.

[7] Y.-W. Fan, A. Kanazawa, and S.-T. Yau, *Weil-Petersson geometry on the space of Bridgeland stability conditions.* arXiv:1708.02161.

[8] H. Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*. Adv. Math. 222 (2009), 1016-1079.

[9] L. Katzarkov, M. Kontsevich and T. Pantev, *Hodge theoretic aspects of mirror symmetry*. From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008. 87-174.

[10] M. Kontsevich, *Homological algebra of mirror symmetry*. Proc. of ICM (Zürich, 1994), 120-139.

[11] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*. arXiv: 0811.2435.

[12] A. Maciocia and D. Piyaratne, *Fourier–Mukai transforms and Bridgeland stability conditions on abelian threefolds*. Algebr. Geom. 2 (2015), no. 3, 270-297.

[13] S. Mukai, *On the moduli space of bundles on K3 surfaces*. Vector bundles on algebraic varieties (Bombay, 1984), 341-413, Tata Inst. Fund. Res. Stud. Math., 11, Tata Inst. Fund. Res., Bombay, 1987.

[14] D. Orlov, *Equivalences of derived categories and K3 surfaces*. J. Math. Sci. (New York) 84 (1997), no. 5, 1361-1381.

[15] D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*. Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 503-531, Progr. Math., 270, Birkhauser, Boston, Inc., 2009.

[16] G. Schumacher, *On the geometry of moduli spaces*. Manuscripta Math. 50 (1985), 229-267.

[17] G. Tian, *Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Petersson–Weil metric*. Mathematical aspects of string theory (San Diego, Calif., 1986), 629-646, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.

[18] Y. Toda, *Gepner type stability conditions on graded matrix factorizations*. Algebr. Geom. 1 (2014), no. 5, 613-665.

[19] T. Trenner and P. M. H. Wilson. *Asymptotic curvature of moduli spaces for Calabi–Yau threefolds*. J. Geom. Anal. 21 (2011), no. 2, 409-428.

[20] C.-L. Wang, *On the incompleteness of the Weil–Petersson metric along degenerations of Calabi–Yau manifolds*. Math. Res. Lett. 4 (1997), no. 1, 157-171.

[21] P. M. H. Wilson. *Sectional curvatures of Kähler moduli*. Math. Ann. 330 (2004), no. 4, 631-664.

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