# ON THE MINIMAL VOLUME OF SIMPLICES ENCLOSING A CONVEX BODY

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ABSTRACT. Let $C \subset \mathbb{R}^n$ be a compact convex body. We prove that there exists an $n$-simplex $S \subset \mathbb{R}^n$ enclosing $C$ such that $\mathrm{Vol}(S) \leq n^{n-1}\mathrm{Vol}(C)$.

## 1. THEOREM

Throughout this article, an $n$-simplex is the convex hull of $(n+1)$ points of $\mathbb{R}^n$ in general position, and a convex body in $\mathbb{R}^n$ is a compact convex set with non-empty interior. For a bounded subset $B \subset \mathbb{R}^n$, we denote by $\mathrm{Vol}(B)$ the volume of $B$. The objective of this article is to prove the following theorem:

**Theorem 1.** *Let $C \subset \mathbb{R}^n$ be a convex body and $S$ an $n$-simplex with minimum volume among all $n$-simplices enclosing $C$. Then*

$$\mathrm{Vol}(S) \leq n^{n-1}\mathrm{Vol}(C).$$

It is easy to see that the constant $n^{n-1}$ is optimal for $n \leq 2$ but we do not know whether or not it gives a sharp bound for $n \geq 3$ at this point. In fact, the case for $n = 2$ is a classical result due to Gross [2], which was later generalized by Kuperberg [3]. However, our approach is different from theirs and we deal with the higher dimensional case as well.

Theorem 1 can be seen as dual to the classical result of Blaschke [1][1] and its higher dimensional analogue due to Macbeath [5], which approximate a given convex body by inscribed simplices.

It has been an open problem to find a higher dimensional analogue of Gross's result [2] and we believe that Theorem 1 presents the first result in this direction. We hope that our simple formula is useful in both pure and applied mathematics.

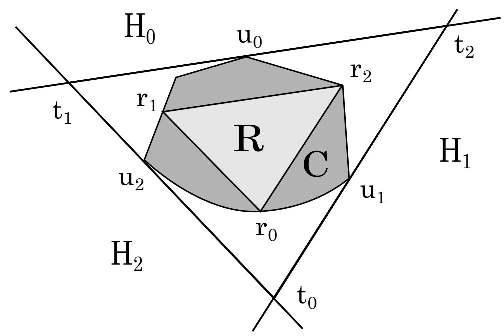
## 2. PROOF

We first prepare two lemmas. In what follows, $\mathrm{Conv}(D)$ denotes the convex hull of a set $D \subset \mathbb{R}^n$. We also set $\mathrm{Conv}(D, p) := \mathrm{Conv}(D \cup \{p\})$ for

[1]Let $C \subset \mathbb{R}^2$ be a convex body and $T$ a triangle with maximum area among all triangles inscribed in $C$. Then $\mathrm{Vol}(T) \geq \frac{3^{3/2}}{4\pi}\mathrm{Vol}(C)$ with equality if and only if $C$ is an ellipse.

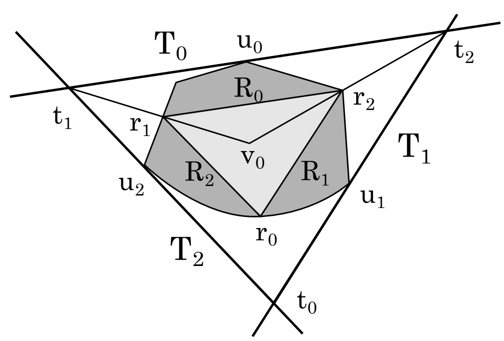
$D \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$.

Let $C \subset \mathbb{R}^n$ be a convex body and $R$ an $n$-simplex inscribed in $C$. We denote the vertices of $R$ by $r_0, \ldots, r_n \in \partial C$. Let $H_i$ be the hyperplane which is parallel to $\mathrm{Conv}(\{r_0, \ldots, r_n\} \setminus \{r_i\})$ and touches $C$ at the opposite side to $r_i$. Let $u_i$ be a point in $H_i \cap C$. The region $T_{C,R}$ bounded by the hyperplanes $H_0, \ldots, H_n$ is an $n$-simplex similar to $R$. The vertices of $T_{C,R}$ are given by $t_i := \cap_{j \neq i} H_j$ for $0 \leq i \leq n$. Let $r$ be the homothetic ratio of $T_{C,R}$ and $R$, that is, $\mathrm{Vol}(T_{C,R}) = r^n \mathrm{Vol}(R)$.




**Lemma 1.** $\mathrm{Vol}(C) \geq r\mathrm{Vol}(R)$.

*Proof.* Set $R_i := \mathrm{Conv}(\{r_0, \ldots, r_n\} \backslash \{r_i\})$ and $T_i := \mathrm{Conv}(\{t_0, \ldots, t_n\} \backslash \{t_i\})$, which are the faces of $R$ and $T_{C,R}$ respectively. Let $I := \{i \mid R_i \not\subset T_i\}$. For $i \in I$, there exists a unique $v_i \in \mathbb{R}^n$ such that $\mathrm{Conv}(R_i, v_i)$ and $\mathrm{Conv}(T_i, v_i)$ are similar and $\mathrm{Conv}(R_i, v_i) \subset \mathrm{Conv}(T_i, v_i)$.




We first consider $B_i := \mathrm{Conv}(R_i \cup T_i) \setminus \mathrm{Conv}(R_i, u_i)$ for $i \in I$. Using the relation

$$\mathrm{Vol}(\mathrm{Conv}(R_i, u_i)) = (r - 1)\mathrm{Vol}(\mathrm{Conv}(R_i, v_i)),$$

we get

$$\begin{aligned}
\mathrm{Vol}(B_i) &= \mathrm{Vol}(\mathrm{Conv}(R_i \cup T_i)) - \mathrm{Vol}(\mathrm{Conv}(R_i, u_i)) \\
&= \mathrm{Vol}(\mathrm{Conv}(T_i, v_i)) - \mathrm{Vol}(\mathrm{Conv}(R_i, v_i)) - (r-1)\mathrm{Vol}(\mathrm{Conv}(R_i, v_i)) \\
&= r^n \mathrm{Vol}(\mathrm{Conv}(R_i, v_i)) - r\mathrm{Vol}(\mathrm{Conv}(R_i, v_i)) \\
&= (r^n - r)\mathrm{Vol}(\mathrm{Conv}(R_i, v_i)) \\
&= (\sum_{j=1}^{n-1} r^j)\mathrm{Vol}(\mathrm{Conv}(R_i, u_i)).
\end{aligned}$$

Since

$$\mathrm{Vol}(R) + \sum_{i \in I} [\mathrm{Vol}(\mathrm{Conv}(R_i, u_i)) + \mathrm{Vol}(B_i)] = \mathrm{Vol}(T_{C,R}) = r^n \mathrm{Vol}(R),$$

we have

$$\mathrm{Vol}(R) + \sum_{i \in I} (\sum_{j=0}^{n-1} r^j)\mathrm{Vol}(\mathrm{Conv}(R_i, u_i)) = r^n \mathrm{Vol}(R).$$

It thus follows that

$$\sum_{i \in I} \mathrm{Vol}(\mathrm{Conv}(R_i, u_i)) = (r-1)\mathrm{Vol}(R).$$

Finally we obtain, by the convexity of $C$,

$$\mathrm{Vol}(C) \geq \mathrm{Vol}(R) + \sum_{i \in I} \mathrm{Vol}(\mathrm{Conv}(R_i, u_i)) = r\mathrm{Vol}(R).$$

Note that the argument above is valid even if $H_i \cap C$ is not a point. □

Let $R \subset \mathbb{R}^n$ be a simplex with vertices $r_0, \ldots, r_n$. Define $I_i \subset \mathbb{R}^n$ to be the hyperplane which passes though $r_i$ and parallel to $\mathrm{Conv}(\{r_0, \ldots, r_n\} \setminus \{r_i\})$. The region $U_R$ bounded by the hyperplanes $I_0, \ldots, I_n$ is an $n$-simplex similar to $R$.

**Lemma 2.** *$\mathrm{Vol}(U_R) = n^n \mathrm{Vol}(R)$.*

*Proof.* Moving $r_i$ on the hyperplane $I_i$ does not change $\mathrm{Vol}(U_R)$ nor $\mathrm{Vol}(R)$. Hence, after rescaling, we may assume that $r_1 - r_0, \ldots, r_n - r_0$ form an orthonormal basis of $\mathbb{R}^n$. The assertion then follows from a direct computation. □

We now start the proof of Theorem 1.

*Proof of Theorem 1.* Let $R$ be an $n$-simplex with maximum volume $m := \mathrm{Vol}(R)$ among all $n$-simplices inscribed in $C$. We apply Lemma 1 to the pair $C$ and $R$ to obtain

$$\mathrm{Vol}(C) \geq rm.$$

If $n > \mathrm{Vol}(C)/m$ holds,

$$\mathrm{Vol}(T_{C,R}) = r^n m \leq (\mathrm{Vol}(C)/m)^n m < n^{n-1}\mathrm{Vol}(C).$$

Thus the assertion holds in this case.

So suppose that $n \leq \mathrm{Vol}(C)/m$ holds. Let $r_0, \ldots, r_n$ be the vertices of $R$. By the maximality of the volume of $R$, the hyperplane $I_i$ is tangent to $C$ at $r_i$, and therefore we see that $C \subset U_R$. By Lemma 2 we obtain

$$\mathrm{Vol}(U_R) = n^n m \leq n^{n-1}\mathrm{Vol}(C).$$

This proves the assertion. $\square$

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