Quantitative Spatial Economics 1

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Quantiative Spatial Economics (QSE)

- Quantitative spatial economic (QSE) model is a model that we can estimate/calibrate using the real-world data.
  - More classic spatial models were hard to take to data because they typically assume simple geography (symmetric locations, circular city etc).
- We can use this model for counterfactual analysis in a general-equilibrium framework
  - How does a historical event (e.g., disasters) affect the spatial economy?
  - What happens when mobility cost of goods or people gets lower, such as due to transportation improvement?
  - What is the impact of taxes on the spatial economy?
- Recommended introductory articles for QSE models:
  - Redding and Rossi-Hansberg (2017, Annual Review of Economics), Allen and Arkolakis (2023, Journal of Economic Perspective), Redding (2025, Handbook of Regional and Urban Economics).
  - Redding has a bunch of MATLAB codes for replication in his website.
  - If you speak Japanese and are interested in Redding and Rossi-Hansberg, Keisuke Takano's note is helpful for coding in R: `https://rpubs.com/k_takano/r_de_qse`

Inter-regional QSE model

- In this lecture, I introduce an inter-regional QSE model following Redding (2016, JIE)
  - This paper closely relates to Allen and Arkolakis (2014 QJE), which predates Redding (2016).
  - Both papers can be applicable to similar settings, but I chose Redding's paper as (1) its logit-like structure and discrete geography make algebra simpler and (2) it has detailed online appendix and MATLAB codes
- QSE papers are somewhat hard to read for beginners, especially when you are not very familiar with the trade models
  - The tradition of the "New Economic Geography (NEG)" literature, which starts from Krugman (1991 JPE) and culminates in Fujita, Krugman, Venable (1999, book), is influencing the formulation of QSE models.
    - This literature in turn builds on Krugman's (1980 AER) "New Trade Theory (NTT)" paper, so again an influence from trade literature.
  - The QSE literature also borrows techniques of quantitative trade models, especially Eaton and Kortum (2002 ECMA) and Dekle, Eaton, Kortum (2007, AER P&P).
- As such, I go through Redding (2016), our first exposure to the QSE literature, relatively slowly.

Redding (2016) model: Overview

- There are continuum of mobile workers, whose total mass is normalized to 1 ($\bar{L} = 1$).
- There are N locations, indexed by $i, n \in [1, 2, ..., N]$. When clear, I denote by N the set of available locations $[1, 2, ..., N]$.
- Locations can differ from one another in terms of land supply, productivity, amenities and geographical location.
- Geographic location is modeled as the *iceberg bilateral trade cost*: $d_{ni} > 1$ units of goods must be exported to deliver one unit of goods from i to n.
  - $d_{nn} = 1$ (no internal trade cost)
  - Maybe looks a bit weird at first sight, but this assumption is extremely popular in trade literature.
- Workers are mobile across locations, and have idiosyncratic tastes for them.
- This is an extension of the Rosen-Roback model. Note that unlike the Rosen-Roback model:
  - General equilibrium (no "outside option" that is exogenously given)
  - Bilateral trade under trade cost (geography)
  - Idiosyncratic shock to locations

  These features make the hedonic approach complicated to apply.

Consumers

- Preference for worker ω in location n:

$$U_n(\omega) = b_n(\omega) \left(\frac{C_n(\omega)}{\alpha} \right)^{\alpha} \left(\frac{H_n(\omega)}{1-\alpha} \right)^{1-\alpha}.$$

- Goods consumption index $C_n = \left[\sum_{i \in N} \int_0^{M_i} c_{ni}(j)^{\rho} dj \right]^{\frac{1}{\rho}}$. j is the name of a good $j \in [0, M_i]$ and ρ determines the elasticity of substitution between goods j and k.
  - M_i is the number of goods produced in location i. Due to trade, goods in all locations are available for consumers.
  - $c_{ni}(j)$ is the consumption of goods j, produced in location i, in location n.

- H_n is the housing consumption

- $b_n(\omega)$ is the idiosyncratic amenity shocks, capturing the idea that workers have heterogeneous preferences for living in each location
  - NOT a choice variable, so consumers take this value as given when choosing C and H.

Consumers

- Price index of the consumption index C_n is defined as follows:

$$P_n = \left[\sum_{i \in N} \int_0^{M_i} p_{ni}(j)^{1-\sigma} dj \right]^{\frac{1}{1-\sigma}},$$

  where $p_{ni}(j)$ is the price of each good j, consumed in location n and produced in location i. Also, $\sigma \equiv 1/(1-\rho)$.

- Why do we define the price index this way? To see this, consider the utility maximization in two steps:
  1. Consumers allocate the income ν_n to goods ν_{nC} and housing ν_{nH} to maximize U_n.
  2. Given ν_{nC}, consumers buy each variety of goods to maximize C_n.

- In the second problem, the Lagrangian is
  $\mathcal{L} = \left[\sum_{i \in N} \int_0^{M_i} c_{ni}(j)^{\rho} dj \right]^{\frac{1}{\rho}} - \lambda [\sum_{i \in N} \int_0^{M_i} p_{ni}(j) c_{ni}(j) dj - \nu_{nC}],$
  - Essentially, this is just a simple CES utility maximization problem (Dixit and Stiglitz 1977 AER). Looks complicated, but you should get used to it if you want to work on trade and spatial models.

Consumers

- The first order condition (FOC) with respect to $c_{ni}(j)$: $C_n^{1/\rho-1} c_{ni}(j)^{\rho-1} - \lambda p_{ni}(j) = 0$.
- Noting that we have the same FOC for good k produced in location x, we have $\left(\frac{c_{ni}(j)}{c_{nx}(k)}\right)^{\rho-1} = \frac{p_{ni}(j)}{p_{nx}(k)}$, representing the relationship between the consumption ratio and the price ratio of two goods j and k.
- Therefore, we have $c_{nx}(k) = c_{ni}(j)\left(\frac{p_{nx}(k)}{p_{ni}(j)}\right)^{1/(\rho-1)}$. Substituting this into the budget constraint $\sum_{x\in N}\int_0^{M_x} p_{nx}(k)c_{nx}(k)dk = \nu_n C$,

$$c_{ni}(j)\left(\frac{1}{p_{ni}(j)}\right)^{\sigma} \underbrace{\left[\sum_{x\in N}\int_0^{M_x} p_{nx}(k)^{1-\sigma} dj\right]}_{P_n^{1-\sigma}} = \nu_n C$$

- Therefore, the optimal consumption of the good j is written as $c_{ni}(j) = p_{ni}(j)^{-\sigma} P_n^{\sigma-1} \nu_n C$
  - The demand is decreasing in the own price $p_{ni}(j)$.
  - The demand also depends on P_n, which summarizes the "price level" of all goods available in location n.

Consumers

- Substituting $c_{ni}(j) = p_{ni}(j)^{-\sigma} P_n^{\sigma-1} \nu_{nC}$ into $C_n = \left[\sum_{i \in N} \int_0^{M_i} c_{ni}(j)^{\rho} dj\right]^{\frac{1}{\rho}}$, we have $C_n = P_n^{-1} \nu_{nC}$
  - Hence, P_n is the "price" of the consumption index C_n!
- Going back to the first step, consumers maximize U_n subject to the budget constraint $P_n C_n + r_n H_n = \nu_n$, where r_n is the land rent.
- The indirect utility is

$$V_n(\omega) = \frac{b_n(\omega)\nu_n}{P_n^{\alpha} r_n^{1-\alpha}}$$

- Except that the amenity is idiosyncratic and we have the goods price index P_n, the indirect utility looks the same as the Rosen-Roback model before.
  - We now begin talking about how to deal with the idiosyncratic shock.

Idiosyncratic shock for location choice

- Each workers chooses the location that offers the highest utility, after taking into account the idiosyncratic amenity $b_n(\omega)$.

- We assume that $b_n(\omega)$ follows the Frechet distribution, which has the following distribution and density functions:
$$G_n(b) = e^{-B_n b^{-\epsilon}}, \quad g_b(b) = B_n \epsilon b^{-\epsilon-1} e^{-B_n b^{-\epsilon}},$$
where B_n corresponds to the "level" and ϵ corresponds to the "dispersion."

- Intuitively, B_n captures the average amenity level of location n and ϵ governs the variance of the idiosyncratic taste shock.

- Useful result: the expected value of b is written as[1]
$$E(b) = \Gamma(\frac{\epsilon - 1}{\epsilon}) B_n^{1/\epsilon},$$
where $\Gamma()$ is the gamma function: $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$.

[1] For derivation, see `https://www.dropbox.com/scl/fi/bb8eltxzepahv0dbef5wa/FrechetAverage_derivation.pdf?rlkey=1gt26fln5a04j7sney70b3ui2&st=jlu1bu8t&dl=0`.

Idiosyncratic shock for location choice

- What is the distribution of $V_n(\omega) = \frac{b_n(\omega)\nu_n}{P_n^\alpha r_n^{1-\alpha}}$?
- Since V_n is written as a Frechet variable $(b_n(\omega))$ times a constant $(\frac{\nu_n}{P_n^\alpha r_n^{1-\alpha}})$, the probability of $V_n < \bar{V}_n$ is the same as the probability of $b_n < \bar{V}_n \frac{P_n^\alpha r_n^{1-\alpha}}{\nu_n}$. We already know that this probability is $G_n(b = \bar{V}_n \frac{P_n^\alpha r_n^{1-\alpha}}{\nu_n})$.
- Therefore, the distribution function of V_n is written as follows:

$$G_n(V) = e^{-\psi_n V^{-\epsilon}}, \text{where} \quad \psi_n \equiv B_n\left(\frac{\nu_n}{P_n^\alpha r_n^{1-\alpha}}\right)^\epsilon,$$

  which is also a Frechet distribution (with a different level parameter ψ_n)!
- ψ_n summarizes the attractiveness of living in n.

Location choice probability under the Frechet shock

- What is the probability of choosing to live in location n, L_n?
- Note that $L_n = Pr(V_n \geq \max_{j \neq n} V_j)$.
- Conditional on V_n taking the value V_n', the probability of this event is

$$\prod_{j \neq n} G_j(V_n') = e^{-\psi_{-n}(V_n')^{-\epsilon}},$$

where $\psi_{-n} = \sum_{j \neq n} \psi_j$. For later use, we also define $\psi = \sum_j \psi_j$.

- To get an unconditional probability, we integrate this over V_n:

$$\int_0^\infty e^{-\psi_{-n}(V_n)^{-\epsilon}} \underbrace{\psi_n \epsilon (V_n)^{-\epsilon-1} e^{-\psi_n (V_n)^{-\epsilon}}}_{\text{Density of } V_n} dV_n \quad = \left(\frac{\psi_n}{\psi}\right) \underbrace{\int_0^\infty \epsilon \psi (V_n)^{-\epsilon-1} e^{-\psi V_n^{-\epsilon}} dV_n}_{1} = \frac{\psi_n}{\psi},$$

where the last integral is one because the integrand is the Frechet density with level parameter ψ and the scale parameter ϵ.

Location choice probability under the Frechet shock

- Therefore, we have the location choice probability

$$L_n = \frac{B_n \left(\frac{\nu_n}{P_n^{\alpha} r_n^{1-\alpha}} \right)^{\epsilon}}{\sum_{k \in N} B_k \left(\frac{\nu_k}{P_k^{\alpha} r_k^{1-\alpha}} \right)^{\epsilon}}$$

- Note that ϵ is the "migration elasticity": elasticity of the location choice probability with respect to the utility changes
  - Taking log, we have $\ln L_n = \epsilon \times \ln(\text{The indirect utility } V_n(\omega) \text{ that excludes idiosyncratic amenity } b_n(\omega)) + \text{remaining things}$
  - In a free-mobility model, $\epsilon = \infty$

Aside: Frechet vs logit

- The choice probability may look familiar: Looks similar to the logit choice probability.
- Not a coincidence! To see this, Consider the indirect utility (inclusive of average amenity level) $V_n = B_n \left(\nu_n / P_n^{\alpha} r_n^{1-\alpha}\right)^{\epsilon}$. Taking the log and consider the additive logit error ϵ_{logit}, $\ln V_n + \epsilon_{\text{logit}}$, the choice probability is

$$\frac{\exp(\ln V_n)}{\sum_k \exp(\ln V_k)} = \frac{B_n \left(\frac{\nu_n}{P_n^{\alpha} r_n^{1-\alpha}}\right)^{\epsilon}}{\sum_{k \in N} B_k \left(\frac{\nu_k}{P_k^{\alpha} r_k^{1-\alpha}}\right)^{\epsilon}},$$

which is exactly the same as the Frechet choice probability!

Aside: Frechet vs logit

- It turns out that the following two approaches induce the same choice probability:
  1. Work on the "raw" indirect utility and use the Frechet idiosyncratic shock.
  2. Work on the "log-transformed" indirect utility and use the logit idiosyncratic shock
- Both are very similar, and it is not clear which approach is better. Both are used in the literature[2]
- My opinion:
  - When your model is simpler in multiplicative form, which is often the case in trade or spatial models, use the Frechet.[3]
  - When your model is simpler in additive form, use the logit.
  - If you are not sure, follow the tradition of your field:
    - Use Frechet if your paper belong to the tradition of trade or spatial economics.
    - Otherwise, use Logit.

[2] See also Matthew Turner's lecture note: `https://matthewturner.org/ec2410/lectures/5_Discrete_v4.pdf`

[3] The most notable example is the Eaton-Kortum model. See Eaton and Kortum (2002 ECMA) and Redding (2016 JIE).

(Ex-ante) expected utility

- As in the logit model, we can simply express the expected utility (before the realization of $b_n(\omega)$):

$$\Gamma(\frac{\epsilon-1}{\epsilon})\left[\sum_{k\in N} B_k\left(\frac{\nu_k}{P_k^{\alpha} r_k^{1-\alpha}}\right)^{\epsilon}\right]^{1/\epsilon}$$

- Why? Since consumers maximize utility, the utility is $\max_{n=1,\dots,N} V_n \equiv V^*$.
- Since V^* is smaller than V when $G_n(V) < V$ for all n, the distirbution function of V^* is

$$G(V) = \prod_n G_n(V) = e^{-(\sum_n \psi_n) V^{-\epsilon}},$$

  where the second equality uses the iid assumption of $b_n(\omega)$.
- This is the Frechet distribution with the level parameter $\sum_n \psi_n$ and the dispersion parameter ϵ. Hence, using the expected value formula for Frechet distribution, we get the above.

Production

- So far we have solved for consumers' behavior given the prices.
- For production, we assume the monopolistic competition with trade, as in Krugman (1980 AER).
- There are many varieties of goods in each location.
- To produce a variety in location i, firms must incur, in unit of labor, a fixed cost F and a constant per-unit cost $1/A_i$.
  - The fixed cost F implies increasing returns to scale, as the average cost decreases with production size.
- Therefore, the total amount of labor $l_i(j)$ to produce $x_i(j)$ units of variety j in location i is

$$l_i(j) = F + \frac{x_i(j)}{A_i}.$$

Production

- The demand function for good j in location n is $c_{ni}(j) = p_{ni}(j)^{-\sigma} P_n^{\sigma-1} \alpha \nu_n$.
- The firm sets the price $p_{ni}(j)$ to maximize the profit $p_{ni}(j)c_{ni}(j) - w_i(F + c_{ni}(j)d_{ni}/A_i)$.
  - The impact of $p_{ni}(j)$ on price indices P is zero as each firm is infinitesimally small
- The first order condition yields the constant mark-up over marginal cost:

$$p_{ni}(j) = \frac{\sigma}{\sigma - 1} \frac{d_{ni} w_i}{A_i}$$

- Note: constant mark-up is a consequence of assuming the CES utility of consumers. Very convenient, but somewhat weird because price is independent of the competitiveness of the market (e.g., the number of firms)
  - There are ways to have a variable mark-up. For instance, see Zhelobodko, Kokovin, Parenti, Thisse 2012 ECMA).

Production

- From the zero-profit condition in equilibrium, we have

$$\sum_n \underbrace{\left(\frac{\sigma}{\sigma-1}-1\right)}_{\text{Profit per unit}} \underbrace{\frac{d_{ni}w_i}{A_i}}_{\text{Unit cost to sell in } n} \underbrace{\frac{x_{ni}(j)}{d_{ni}}}_{\text{Sales amount in location } n} = \underbrace{w_i F}_{\text{Fixed cost of operation}},$$

  where $x_{ni}(j)$ is the output of good j that is sold to location n (prior to incurring the iceberg trade cost).

- The LHS is the total profit from all markets. The RHS is the fixed cost.

- From this, we have $\sum_n x_{ni}(j) \equiv x_i(j) = A_i(\sigma-1)F$.

- Since the total amount of labor employed by firm j is $l_i(j) = x_i(j)/A_i + F$, we have $l_i(j) = \sigma F$.
    - Note that the firm size is determined only by the preference parameter σ and the fixed cost F.

- From the labor market clearing condition, the number of firms (equivalently, the number of varieties) M_i is proportional to population size L_i:

$$M_i = \frac{L_i}{\sigma F}$$

Trade flow

- The share of location n's expenditure on goods produced in location i is written as follows:

$$\pi_{ni} = \frac{M_i c_{ni} p_{ni}}{\sum_k M_k c_{nk} p_{nk}} = \frac{M_i p_{ni}^{1-\sigma}}{\sum_k M_k p_{nk}^{1-\sigma}} = \frac{L_i \left(\frac{d_{ni} w_i}{A_i}\right)^{1-\sigma}}{\sum_k L_k \left(\frac{d_{nk} w_k}{A_k}\right)^{1-\sigma}}$$

- This is a *gravity equation* of trade flows
  - If you take the log, $\ln \pi_{ni} = -(\sigma - 1) \ln d_{ni} +$ origin fixed effect $+$ destination fixed effect.
  - See Anderson and Van Wincoop (2003 AER) for more discussions on trade gravity equations.
- The elasticity of substitution $(\sigma - 1)$ determines the trade elasticity with respect to trade costs.
- The expenditure share depends on the population size L_i
  - Larger market implies more varieties, and hence exports more.

Price index

- Recall that the price index is defined as $P_n = \left[\sum_{i \in N} \int_0^{M_i} p_{ni}(j)^{1-\sigma} dj\right]^{\frac{1}{1-\sigma}}$.
- Given that we have already derived expressions for M_i and p_{ni}, this is rewritten as follows:

$$P_n = \frac{\sigma}{\sigma - 1} \left(\frac{1}{\sigma F}\right)^{1/(1-\sigma)} \left[\sum_{i \in N} L_i \left(\frac{d_{ni} w_i}{A_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}},$$

- Agglomeration force: Price index P_n is decreasing in population size L_n.
  - Love of variety and more goods availability: The decrease is larger when σ is smaller (i.e., goods are less substitutable), so that larger M_n matters more.
- Moreover, using the own trade share π_{nn} and $d_{nn} = 1$, this can be further simplified as follows:

$$P_n^{1-\sigma} = \frac{\frac{L_n}{\sigma F} \left(\frac{\sigma}{\sigma-1} \frac{w_n}{A_n}\right)^{1-\sigma}}{\pi_{nn}}$$

- Note: In trade and spatial models, things can sometimes be simplified using the own trade share. See, for instance, Arkolakis, Costinot, Rodoriguez-Clare (2012 AER).

Land market clearing

- We assume that the land rent is redistributed to local residents, so that

$$\underbrace{\nu_n L_n}_{\text{Total local income}} = \underbrace{w_n L_n}_{\text{Labor income}} + \underbrace{(1-\alpha)\nu_n}_{\text{Land spending per capita}} \underbrace{L_n}_{\text{Population}} = \frac{w_n L_n}{\alpha},$$

where the second equality follows because the first equality implies $\nu_n = w_n/\alpha$

- Land supply H_n is fixed.
- From land market clearing, the equilibrium land rent is

$$r_n = \frac{(1-\alpha)\nu_n L_n}{H_n} = \frac{1-\alpha}{\alpha}\frac{w_n L_n}{H_n}$$

Equilibrium conditions

Summarizing the arguments so far, the following equilibrium conditions determine the endogenous variables (L_n, π_{ni}, w_n):

1. Trade balance (inclusive of within-location consumption):

$$w_i L_i = \sum_n \pi_{ni} w_n L_n$$

2. Trade share:

$$\pi_{ni} = \frac{L_i \left(\frac{d_{ni} w_i}{A_i}\right)^{1-\sigma}}{\sum_k L_k \left(\frac{d_{nk} w_k}{A_k}\right)^{1-\sigma}}$$

3. Residential choice probability:

$$L_n = \frac{B_n \left(\frac{\nu_n}{P_n^{\alpha} r_n^{1-\alpha}}\right)^{\epsilon}}{\sum_{k \in N} B_k \left(\frac{\nu_k}{P_k^{\alpha} r_k^{1-\alpha}}\right)^{\epsilon}} = \frac{B_n A_n^{\alpha\epsilon} H_n^{\epsilon(1-\alpha)} \pi_{nn}^{-\alpha\epsilon/(\sigma-1)} L_n^{-(\epsilon(1-\alpha)-\alpha\epsilon/(\sigma-1))}}{\sum_k B_k A_k^{\alpha\epsilon} H_k^{\epsilon(1-\alpha)} \pi_{kk}^{-\alpha\epsilon/(\sigma-1)} L_k^{-(\epsilon(1-\alpha)-\alpha\epsilon/(\sigma-1))}}$$

Existence and uniqueness of equilibrium

Proposition

Assume $\sigma(1-\bar{\alpha}) > 1$, where $\bar{\alpha} \equiv \frac{\alpha}{1+1/\epsilon}$. Take as given the land area, productivity, and amenity parameters (H_n, A_n, B_n) and quasi-symmetric bilateral trade frictions (i.e., d_{ni} is such that $d_{ni} = D_n D_i D_{ni}$ and $D_{ni} = D_{in}$). Then, there exist unique equilibrium population (L_n^*), trade shares (π_{ni}^*), and wages (w_n^*)

- I omit the proof as it is technical. See Redding (2016).
- In numerical analysis, Redding assumes $\sigma(1-\bar{\alpha}) > 1$. But what does $\sigma(1-\bar{\alpha}) > 1$ mean?
- It means "dispersion forces" $>$ "agglomeration forces"
  - It is more likely satisfied when σ, the elasticity of substitution parameter, increases.
    - Larger σ means weaker "agglomeration forces" because variety is evaluated less.
  - $1-\bar{\alpha}$ is increasing in the spending share for land (1-α), and decreasing in the dispersion parameter of the Frechet shock ϵ.
    - They imply stronger "dispersion forces" that lead to more dispersed population distribution.

Existence and uniqueness of equilibrium

- Why do we need "dispersion forces" $>$ "agglomeration forces" condition for the uniqueness of equilibrium?
- When agglomeration forces are strong, there can be multiple equilibria
- To understand why strong agglomeration forces lead to multiple equilibria, we consider a very simple location-choice game.

Agglomeration forces and multiple equilibria

	Location A	Location B
Location A	$f+A, f+A$	$f, 0$
Location B	$0, f$	A, A

- Consider a binary location choice game
- $f \geq 0$ is the relative locational advantage of location A
  - Capturing exogenous amenity or productivity differences (A_n, B_n)
- $A \geq 0$ is the agglomeration forces.
  - You are happier by living closer to others.
  - In Redding's (2016) model, this comes from more goods varieties in a larger market.

Agglomeration forces and multiple equilibria

	Location A	Location B
Location A	$f+A, f+A$	$f, 0$
Location B	$0, f$	A, A

- When $f < A$ so that agglomeration forces are important, then there are two equilibria.
  - When $f > A$, only one pure strategy equilibrium.
- Intuitively, as long as you live close to others, you do not care about where you are.
  - This leads to multiple equilibria, which differ in where the agglomeration arises
- See Krugman (1991) and Fujita, Krugman, Venables (1999 book) for these situations
  - Multiple equilibria are important in the "New Economic Geography" literature, which the QSE literature builds on.

Multiple equilibria in QSE

- Going back to Redding (2016), when $\sigma(1-\bar{\alpha}) > 1$ is not satisfied, there can be multiple equilibria because agglomeration forces are "too strong"
  - Essentially, we are assuming that agglomeration forces are not too strong. Since this may or may not be true in your empirical context, this can be a problematic assumption.

- My take 1: multiple equilibria can pose difficulty in the analysis, so assuming the unique equilibrium may help your analysis when you are not so interested in the multiplicity
  - Counterfactual analysis can be tricky under multiple equilibria:
    - In a counterfactual world, we do not know which equilibrium of the model realizes.
  - It can also be difficult to estimate model parameters
    - The model parameter values may change, depending on which equilibrium is assumed to be describing the reality.

- My take 2: But potential multiplicity of equilibria is a feature of the spatial economy (not a bug!). And a carefully-designed QSE models can potentially address/circumvent these difficulties.
  - Solve for the nearest equilibrium to the observed equilibrium (Ahlfeldt, Redding, Sturm, Wolf 2015 ECMA)
  - Multiple steady states, but unique equilibrium when history is given (Allen and Donaldson 2022 JPE R&R)
  - Agent-based approach for equilibrium selection (Ahlfeldt, Albers, Behrens 2022 wp)
  - Expectations may select equilibria: Takeda and Yamagishi (2024 JPE R&R).

Model calibration and counterfactual analysis: Step 1

- So, how can we calibrate the parameter values of the model to conduct a counterfactual analysis?
- First, you should determine several exogenous parameters of the model.
  - Spending share for goods (α)
  - Elasticity of substitution (σ)
  - Frechet dispersion parameter (ϵ)
  - Trade costs (d_{ni})
  - Land supply (H_n)
- You can use the observed values in the data, some parameter values estimated in other papers, or match the moments (e.g., mean, variance) of the data.
  - There is no single way to determine these values. Depends on the purpose and structure of your model.
  - Look at Redding (2016) or other prior studies (e.g., those mentioned later in the "applications" section of this lecture) for how to determine these values.

Model calibration and counterfactual analysis: Step 2

- So far, we have solved for endogenous variables of the model (population, wages), and checked the existence/uniqueness of equilibrium.
  - $(A_n, B_n) \mapsto (L_n, W_n)$
- But we have assumed that amenities and productivity of each location (A_n, B_n) are known and observed. In the data, we do not have these values!
  - In the data, we observe population and wages.
- It turns out that by assuming what we observe in the data is the equilibrium outcome of the model, we can calibrate (A_n, B_n) from the observed population and wages
  - $(L_n, W_n) \mapsto (A_n, B_n)$
  - This process is also called "model inversion" (Redding and Rossi-hansberg 2017).
- After calibrating (A_n, B_n), we can again solve the model to obtain counterfactual population and wages.
  - For instance, you can change trade costs to analyze infrastructure improvement. You can use $(A_n, B_n) \mapsto (L_n, W_n)$ in the counterfactual scenario to obtain predictions about population and wages.

Model calibration and counterfactual analysis: Step 2

- More specifically, how can we calculate (A_n, B_n) from (L_n, w_n)?
- From the "trade balance" and "trade share" equilibrium conditions, we can define an function of A:

$$D_i(A) = w_i L_i - \sum_n \frac{L_i(d_{ni} w_i / A_i)^{1-\sigma}}{\sum_k L_k (d_{nk} w_k / A_k)^{1-\sigma}} w_n L_n,$$

  and we can define a similar thing for B from the residential choice probability.
- We can show that given (L_i, w_i), this has the unique solution of A_i
  - See Redding (2016) for the proof.
- In numerical calculation, we can "gradually update" A to approach the solution:
  1. Make an guess of A_1, where 1 represents this is the 1st guess.
  2. To get A_2 (the second guess), we can calculate the following:

$$f_i(A_1) = \frac{1}{\sum_i A_{i1} D_i^+(A_1)} (A_{i1} + D^+(A_1)), \text{where} D_i^+(A) = \max\{D_i(A), 0\}.$$

     Intuitively, we upwardly adjust A_{i1} when $D_i(A) > 0$ and vice versa.
  3. Let $A_{i2} = \vartheta f_i(A_1) + (1 - \vartheta) A_{i1}$, where ϑ is the updating step size. Repeat until convergence.

Model calibration and counterfactual analysis: Step 3

- After calculating (A_n, B_n), we can calculate the equilibrium of the model when we change some parameter values
  - For instance, change d_{ni} to explore the effect of transportation improvement.
- We can numerically solve the following equilibrium conditions for (L_n, w_n, π_{ni}) under the new values $d_{ni} = d'_{ni}$:
  1. Trade balance (inclusive of within-location consumption):
$$w_i L_i = \sum_n \pi_{ni} w_n L_n$$
  2. Trade share:
$$\pi_{ni} = \frac{L_i \left(\frac{d_{ni} w_i}{A_i}\right)^{1-\sigma}}{\sum_k L_k \left(\frac{d_{nk} w_k}{A_k}\right)^{1-\sigma}}$$
  3. Residential choice probability:
$$L_n = \frac{B_n A_n^{\alpha\epsilon} H_n^{\epsilon(1-\alpha)} \pi_{nn}^{-\alpha\epsilon/(\sigma-1)} L_n^{-(\epsilon(1-\alpha)-\alpha\epsilon/(\sigma-1))}}{\sum_k B_k A_k^{\alpha\epsilon} H_k^{\epsilon(1-\alpha)} \pi_{kk}^{-\alpha\epsilon/(\sigma-1)} L_k^{-(\epsilon(1-\alpha)-\alpha\epsilon/(\sigma-1))}}$$

Model calibration and counterfactual analysis: Step 3

- Sometimes, the "exact hat approach" of Dekle, Eaton, Kortum (2007) simplifies the calculation.
- Let's express the variables in the counterfactual world by prime (ex., d'_{ni}), and $\hat{x} = x'/x$ denote the "percentage change" of the variable x.
- By dividing the "new" equilibrium condition (with prime) by the "old" equilibrium condition (without prime), we have the following:
  1. Trade balance (exact hat):
  $$\hat{w}_i \hat{\lambda}_i Y_i = \sum_n \pi'_{ni} \hat{w}_n \hat{\lambda}_n Y_n$$
  2. Trade share:
  $$\hat{\pi}_{ni} \pi_{ni} = \frac{\pi_{ni} \hat{L}_i \left(\frac{\hat{d}_{ni} \hat{w}_i}{\hat{A}_i} \right)^{1-\sigma}}{\sum_k \pi_{nk} \hat{L}_k \left(\frac{\hat{d}_{nk} \hat{w}_k}{\hat{A}_k} \right)^{1-\sigma}}$$
  3. Residential choice probability:
  $$\hat{\lambda}_n \lambda_n = \frac{\hat{B}_n \hat{A}_n^{\alpha\epsilon} \hat{H}_n^{\epsilon(1-\alpha)} \hat{\pi}_{nn}^{-\alpha\epsilon/(\sigma-1)} \hat{L}_n^{-(\epsilon(1-\alpha)-\alpha\epsilon/(\sigma-1))} \lambda_n}{\sum_k \hat{B}_k \hat{A}_k^{\alpha\epsilon} \hat{H}_k^{\epsilon(1-\alpha)} \hat{\pi}_{kk}^{-\alpha\epsilon/(\sigma-1)} \hat{L}_k^{-(\epsilon(1-\alpha)-\alpha\epsilon/(\sigma-1))} \lambda_k}$$

where $Y_i = w_i L_i$ and $\lambda_n = L_n/\bar{L} = L_n$.

Model calibration and counterfactual analysis: Step 3

- We can then solve these equations for $(\hat{\lambda}_n, \hat{w}_n, \hat{\pi}_{ni})$, taken as given the initial equilibrium values $(\lambda_n, w_n, \pi_{ni})$, which are assumed to be observed in the data.
- The benefit of the exact-hat approach is that we can actually "ignore" parameters that are not of our interest.
- For instance, when we are interested in changes of trade costs only, then $\hat{d}_{ni} \neq 1$ but we can assume $\hat{A}_n = 1$ and $\hat{B}_n = 1$
  - *Even without knowing the exact value of* (A_n, B_n), we can assume so!
- Thus, we can actually dispense with Step 2 to obtain the counterfactual predictions about population and wages.
  - This can be helpful when the model is computationally hard to solve
- But knowing the values of (A_n, B_n) is informative about the attractiveness of each location, and for other purposes (e.g., welfare level) having these values can be essential.
  - Moreover, the exact hat approach does not always simplify the analysis.

Applications

- We now introduce several applications of the QSE models to illustrate what kind of questions has been analyzed
- The models in these applications are similar to Redding (2016) but not exactly the same. However, the basic structure of the analysis in Redding (2016) is portable to these applications
  - See "the menu of quantitative spatial models" in Redding and Rossi-Hansberg (2017) for variations of the QSE models. See also Redding (2024) for a more recent survey.
- Applications to cover:
  - Market access and geopolitical change (Redding and Sturm 2008 AER; Nakajima 2008 JJIE)
  - Migration restriction (Tombe and Zhu 2019 AER)
  - Fiscal transfers (Henkel, Seidel, Suedekum 2021 AEJ Policy)
  - Global warming and transportation infrastructure (Balboni 2025 AER)

Redding and Sturm (2008, AER)

- Western German cities near the border of East Germany suddenly lost nearby trade partners after 1945.
- Reduced-form evidence that their economic activities (population etc) indeed stagnate after 1945

MAP 1. THE DIVISION OF GERMANY AFTER THE SECOND WORLD WAR

Notes: The map shows Germany in its borders prior to the Second World War (usually referred to as the 1937 borders) and the division of Germany into West Germany, East Germany, areas that became part of Poland, and an area that became part of Russia. The West German cities in our sample which were within 75 kilometers of the East-West German border are denoted by squares, all other cities with a population greater than 20,000 in 1919 by circles.

Redding and Sturm (2008, AER)

- By suitably calibrating the parameters of a QSE model, the model can predict the decline of such border cities after German division
- Evidence of the importance of market access

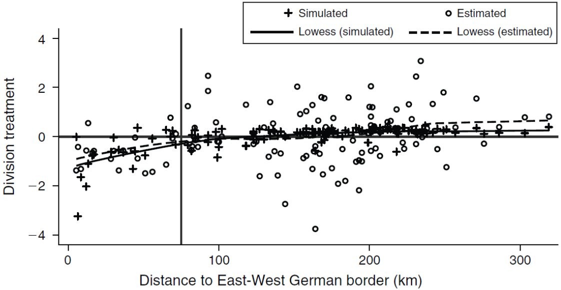


FIGURE 6. SIMULATED AND ESTIMATED DIVISION TREATMENTS

Notes: Simulated division treatments based on the parameter configuration that minimizes the sum of squared deviations between the simulated and estimated division treatments for small and large cities. Lowess is a locally weighted linear least squares regression of the division treatment against distance to the East-West German border (bandwidth 0.8).

Nakajima (2008, JJIE)

- Similar to Redding and Sturm (2008), but in a Japanese context
- Japan lost Korea after WW2. Combined with the Korean war, it substantially reduced the trade volume with Korea.
- As a result, Japanese cities on the Sea of Japan lost market access to Korea
- Nakajima finds population decline of the cities close to Korea.

Fig. 1. Map of Japan and Korea.

Tombe and Zhu (2019 AER)

- What is the cause of recent remarkable growth of China?
- Tombe and Zhu investigate the role of reduced costs in trade and migration
- Relative to Redding's model, Tombe and Zhu's model is richer in migration costs:
  - Individuals differ in their *hukou* status (location, agriculture vs non-agriculture)
  - If one does not follow the hukou, then one must incur the utility cost (e.g., lost access to public service) and they also lose income from land
- Otherwise, the model is similar to Redding (2016)
  - The trade structure is the version of Eaton-Kortum rather than the differentiated goods version we have covered (appearing Section 3 of Redding 2016). See Section 2 of Redding (2016) for the Eaton-Kortum version.

Tombe and Zhu (2019 AER)

- "Growth decomposition" of China using the QSE model
  - To isolate the contribution of internal trade cost in the growth 2000-2007, keep the internal trade cost as the initial level (in 2000) while letting other parameters to change.
  - Compare the growth in this counterfactual equilibrium and the data

- Internal trade cost > (internal) migration costs > external trade cost
  - Highlights the role of domestic factors in Chinese growth, unlike some other anecdotes that highlight export-led growth of China (e.g., the importance of WTO participation)

Table 9—Decomposing China's Aggregate Labor Productivity Growth

	Marginal effects		
	Real GDP per worker growth (%)	Share of growth	Standard deviation (%)
Overall (all changes)	57.1	–	–
Productivity changes	36.9	0.64	1.3
Internal trade cost changes	10.2	0.18	0.3
External trade cost changes	4.5	0.08	0.7
Migration cost changes	5.6	0.10	0.9
Of the migration cost changes			
Between-province, within-non-agriculture	0.9	0.02	0.4
Between-province, within-agriculture	0.0	0.00	0.0
Between-province, agriculture-non-agriculture	3.2	0.06	0.9
Within-province, agriculture-non-agriculture	1.5	0.03	0.3

Notes: Decomposes the change in real GDP into contributions from productivity, internal trade cost changes, external trade cost changes, and migration cost changes. The bottom panel decomposes the change due to migration cost changes into various different types of migration. To attribute contributions from each component, we report the marginal contribution to aggregate growth of each component across all permutations. In the last column, we report the standard deviation of those growth rates across permutations. Shares may not sum to 1 due to rounding. The growth rates are continuously compounded rates.

Henkel, Seidel, Suedekum (2021 AEJ Policy)

- Fiscal transfers across different local governmetns are prevalent
  - What is the effect of fiscal transfers on production and welfare?
- Preferences are modified in the following way:

$$U_n(\omega) = b_n(\omega) \left(\frac{G_n}{L_n^\eta}\right)^\gamma \left(\frac{C_n(\omega)}{\alpha}\right)^\alpha \left(\frac{H_n(\omega)}{1-\alpha}\right)^{1-\alpha},$$

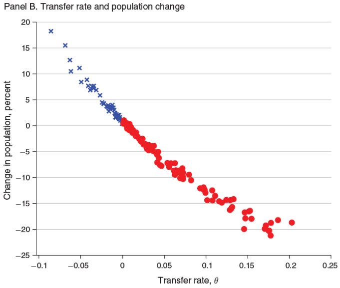
  where G_n is the local public spending. $\eta > 0$ is the degree of congestion of public goods.
- The government budget constraint:

$$G_n = (t_n + \theta_n) w_n L_n / P_n,$$

  where t_n is the local tax rate, θ_n is the fiscal transfers, and P_n is the local price level (, which is also the price of public goods). The transfer must satisfy $\sum_n \theta_n w_n L_n = 0$.

Henkel, Seidel, Suedekum (2021 AEJ Policy)

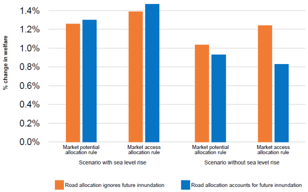
- Take the model to the actual data (with fiscal transfers), then compute the counterfactual equilibrium that hypothetically abandon the fiscal transfers.
- The massive population increase in Western Germany, which is the net payer of fiscal transfers
- But the welfare effect of fiscal transfers is slightly positive
  - Fiscal transfers reduce the congestion of public goods in Western Germany
- Endogenizing local tax determination (but without fiscal transfers): Ferrari and Ossa (2023 JPUBE), Borck, Oshiro, Sato (2023 JUE R&R)



Balboni (2025 AER)

- Cities are often on the coastal area, but they face high risk of flooding due to sea level rise.
- Should we invest in coastal areas, in which population is large, or inland areas that are safer?
- Using a QSE model, Balboni compares two situations in Vietnam:
  - The actual investment pattern on road that favors coastal areas
  - Hypothetical road investment that favors more inland areas
- There can be welfare improvement by more favoring inland areas than the status quo
  - But taking into account future inundation risk may not bring a small benefit

Figure 8: Welfare gains from counterfactual road investments relative to status quo road investments



Notes: The figure shows aggregate welfare gains versus the status quo road investments. The percentage change in welfare is measured in terms of consumption equivalent variation. Scenario with sea level rise incorporates a 1 meter rise in the sea level realized gradually over 100 years. Scenario without sea level rise assumes that sea levels will remain at current levels. Blue (orange) bars correspond to counterfactual road allocations that account for (ignore) future sea level rise.

Taking stock

- We have seen how QSE models work, using Redding (2016) as an example.
  - We have also covered applications of QSE models that look relatively similar to Redding (2016).
- The model is more general than the Rosen-Roback model and allows us to evaluate the effects of various things
  - But as we have seen, the analysis is substantially more complicated than the hedonic approach.
  - This is also a fully parametric approach with specific functional forms. The hedonic approach imposes some strong assumptions, but often we can dispense with several functional form assumptions.
  - You should carefully compare two approaches to determine how you answer your question
- Redding's (2016) model does not incorporate commuting, and hence it is not well-suited for small gepgraphical area (e.g., the distribution of economic activities within Tokyo).
  - We next cover a QSE model with commuting.