**Disturbance Rejection Control Based on State-Reconstruction and Persistence Disturbance Estimation**

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**Abstract**

In this note, disturbance rejection control (DRC) based on unknown input observation (UIO), and disturbance-observer based control (DOBC) methods are revisited for a class of MIMO systems with mismatch disturbance conditions. In both of these methods, the estimated disturbance is considered to be in the feedback channel. The disturbance term could represent either unknown mismatched signals penetrating the states, or unknown dynamics not captured in the modeling process, or physical parameter variations not accounted for in the mathematical model of the plant. Unlike the high-gain approaches and variable structure methods, a systematic synthesis of the state/disturbance observer-based controller is carried out. For this purpose, first, using a series of singular value decompositions, the linearized plant is transformed into disturbance-free and disturbance-dependent subsystems. Then, functional state reconstruction based on generalized detectability concept is proposed for the disturbance-free part. Then, a DRC based on quadratic stability theorem is employed to guarantee the performance of the closed-loop system. An important contribution offered in this article is the independence of the estimated disturbance from the control input which seem to be missing in the literature for disturbance decoupling problems. In the second method, DOBC is reconsidered with the aim of achieving a high level of robustness against modeling uncertainties and matched/mismatched disturbances, while at the same time retaining performance. Accordingly, unlike the first method, DRC, full information state observation is developed independent of the disturbance estimation. An advantage of such a combination is that disturbance estimation does not involve output derivatives. Finally, the case of systems with matched disturbances is presented as a corollary of the main results.

**Keywords:** Unknown input observation, Disturbance rejection control, Disturbance observer-based control, noise decoupling.

1. **Introduction**

Research in the area of state observation and disturbance estimation in linear multivariable systems has reached a mature level, and various existence and sufficient conditions have been proposed in the last few decades (Guan and Saif [1]; Hou and Muller [2]). In the case of measurable disturbances, feedforward and feedback/feedforward strategies have been used to prevent performance degradation through optimization of some user-defined performance indices. This concept is also referred to as state reconstruction, as in Kurek [3], where unknown exogenous input signals are the source of erroneous system data. For such systems, a simple and well-stablished approach is the sliding mode observer (SMO), which allows for robust reconstruction of the states under unknown inputs (Darouach et al. [4], Mondal et al. [5], and Oveisi and Nestorović [6]). Considering the disturbance as a bounded unknown mismatched exogenous input instead of opting for the classical stochastic approach based on statistical distribution of the signal affecting the plant seems to be more practical in control applications and system theory. The mismatched unknown inputs are those signals that penetrate the system states through channels that are independent of the control input signals. Additionally, due to the lack of information about the external disturbances, intuitively high gain controllers are designed in the early development of disturbance rejection control (DRC). It is wellknown however that high gain controllers may lead to actuator windup problem and peaking phenomenon (Sussmann and Kokotovic [7]).

To a large extent, although categorized under separate control synthesis approach, disturbances (and modeling uncertainties) may be handled by classical (robust) techniques. However, often conflicting performance and stability constraints on the closed-loop system require more advanced alternative approaches in estimating and then rejecting the disturbance. Recently, linear and nonlinear disturbance and uncertainty estimation and attenuation (LDUEA/NDUEA) methods have attracted considerable interest. Disturbance-observer based control (DOBC) method as one candidate of DUEA, in general, delivers high robustness with respect to modeling uncertainties and matched/mismatched disturbances without performance loss (Shim and Jo [8]). Additionally, more rigorous stability analysis and synthesis can be obtained through the use of DOBC (Chen et al. [9]) compared to the methods of extended state observer (ESO), disturbance accommodation control (DAC) (Gao et al. [10]), and uncertainty and disturbance estimator (UDE) (Zhong et al. [11]). The unknown input excitations acting on the system also known as the lumped disturbance represent a generalized form of external stimuli that also takes into account the unmodeled dynamics of high-order nature and parameter variations. The case of static matched/mismatched disturbances can be considered as special cases of the control scheme proposed in this paper [12,13].

In summary, the following contributions are offered in this paper:

1. Two Lyapunov-based methods are proposed to reconstruct the system states in the presence of unknown and unmeasurable input signals. In this regards, the disturbance decoupling problem is revisited, and, based on the concept of unknown input observation (UIO), a reduced order dynamic system is developed to simultaneously estimate the disturbance and observe the states. In order to transform DRC into a convex feasibility (optimization) problem, the controller synthesis is separated from that of UIO. Compared to most recent researches, one distinct novelty in the proposed combination is the independence of the dynamics of disturbance estimation mechanism from the control input. Consequently, the non-convexity of a systematical DRC synthesis is addressed in this paper.
2. The second contribution, compared to Luenburger state observer, is that the stability of the designed reduced order UIO for disturbance decoupled subsystem is guaranteed which is missing in the literature. The stability guarantee of the decoupled UIO for systems with modeling uncertainties is a highly advantageous property which may relax the observability condition constraints required for the transformed system. The observability condition, as discussed in this paper, is a binding constraint in the previous methods due to the uniqueness of the singular value decomposition.

In the second part of the paper, by aiming at suppressing the time-derivative of the measurement output (inherently polluted by noise) in the existing disturbance estimation mechanism that are based on the concept of DOBC, the UIO synthesis process is broken into two separate sequential phases. In the first, a revised form of the strong functional observer is proposed. This is followed by the second phase where a disturbance estimation and rejection is carried out. Two salient contributions are therefore made: (i) The state observer-based DOBC method, which is missing in literature, and (ii) The problem formulation accounts for the most general case, namely mismatch disturbance. Additionally, for the first time, the composite controller gain for the estimated disturbance (and of course the observed states) is calculated based on convex optimization. Moreover, the limitations on transforming the design procedure into eigenvalue problem (EVP) in semi-definite programming (SDP) framework are carefully investigated. Finally, the effect of the measurement noise in output is examined in disturbance/noise decoupling framework, based on available techniques of fault detection.

In the rest of the paper, represents the identity matrix with appropriate dimension, superscript signifies the transpose of a matrix or a vector, and stands for the set of real numbers. Additionally, , , and represent the space of time-dependent vectors, matrices, and positive definite matrices with dimensions of , , and for , respectively. Finally, in linear matrix inequality conditions, represent a zero matrix with appropriate dimension.

1. **System representation and linearization**

Consider a multivariable nonlinear dynamic system described in state space form as

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|  | (1) |

where, ,, and are the state, input, square-integrable disturbance (including un-modeled dynamics of high-order nature and parameter variations in the system elements), and output vectors at time , respectively. In addition, in order to express the plant as a nonlinear Lipschitz system, it is assumed that , , and are sufficiently smooth and differentiable around the equilibrium points of Eq. (1). Then, the necessary and sufficient conditions on the asymptotic stability of the observer-based control system can be ascertained from the eigenvalues of the linearized system (Rajamani [14]). For causal plant, with globally Lipschitz continuous dynamics, assuming and , Eq. (1) can be linearized in the multi-input multi-output (MIMO) form of Eq. (2). Moreover, the class of systems studied in this paper satisfies the following assumptions: *i*) the linearized plant is asymptotically stable, *ii*) and (only for results in section 3.1), and , *iii*) matrices and have full column rank.

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|  | (2) |

where , , and . In Eq. (2) and throughout the rest of the paper, to simplify the notation, the dependence of the state space variables on time is omitted. It should be pointed out that the feedthrough term in the linearized output equation is assumed to be zero (), since without this assumption the feasibility problem of disturbance estimation may become non-convex. Additionally, the disturbance term may appear in the output equation () in which case, as long as the condition (*ii*) is satisfied, the system can be transformed into Eq. (2) (Trinh et al. [15]). The form of Eq. (2) as a linearization of (1) around the equilibrium point of the plant, where the nonlinear system usually operates, opens the possibility of using global approximation methods such as Takagi-Sugeno-Kang method of fuzzy inference which may generalize the applicability of the proposed techniques. Alternatively, large departures from the equilibrium could be modeled as uncertainties, and appropriate robust controllers may be synthesized to address these mismatching uncertainties. However, this problem is out of the scope of this paper.

1. **Problem formulation and main results**

We start this section with a brief summary of its content, followed by detailed analysis of existing estimation and disturbance rejection techniques. Then, the main results of this paper will be presented.

In subsection 3.1, the design problem of UIO-based DRC for the disturbance decoupled system is investigated. The effect of output measurement noise is treated as a generalization of Theorem 1a. However, in the case that ( representing the dimension of noise on output) multiple UIO (MUIO) should be designed. The case of matched disturbances is given as Corollary 1a to Theorem 1b.

In subsection 3.2, DOBC is investigated. Although DOBC with matching conditions has been addressed in the existing literature, (for example Kempf and Kobayashi [16], Chen [17], Thum et al. [18], Chen et al. [19], and Yang et al. [20]), the mismatch case for nonlinear systems which satisfy the Lipschitz condition has not yet received much attention, with some exceptions (for example Barmish and Leitmann [21] and Guo and Chen [22]). Most recently, two notable results are reported by Wei and Guo [23, 24] to address the problems of variable structure compensator and controller for mismatch DOBC. Since, controller design is based on the worst case scenario, the performance of the closed-loop system is often compromised to preserve robustness (Chen et al. [25], Chen [26], Yang et al. [20]).

In this paper, a systematic Lyapunov-based approach is proposed to reconstruct the states using a full-order filter on system outputs and then, the input disturbance is simultaneously estimated and rejected in the plant output.

* 1. **DRC based on disturbance decoupled UIO**

Following the UIO method in Chang [27] and Koenig and Mammar [28], the state equation can be decoupled by applying singular value decomposition (SVD) on the disturbance matrix (. This operation decouples the system into two subsystems, one of which is independent of the unknown inputs. Accordingly, the SVD of can be written as , where ,, , and. Then, mapping the state equation by and defining and , the decoupled state-space model can be written as in Eq. (3). This representation, in the terminology of fault observation/detection, is referred to as fault-dependent/fault-free decoupling:

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|  | (3) |

where,, ,, , and . Similar decomposition is applied on output equation (). Then, assuming the non-singular mapping with and being respectively the arbitrary matrix that guarantees the non-singularity of (as a result: the non-singularity of the transformation) and pseudo-inverse of, the following two conditions can be stated: (*a*), (*b*) . Using condition (), the fault-free equation (3) can be decoupled from the fault-dependent states () as where and . In the same fashion, one can obtain with and . It should be noted that, based on assumption *ii*) and , it is trivial to show the existence of for the non-singular mapping where . The aim is then to formulate some conditions for existence of a stable UIO that has more flexible structure compared to the classical Luenberger’s observer or Kalman filter in determining the state estimation error dynamics. Since one of the ultimate goals of this paper is to include an estimation of the unknown mismatch disturbance input in feedback control law, the flexibility of the UIO is a key requirement. Consequently, the structure of UIO, and therefore the estimation error dynamics, defined in terms of some constraints under which designing the feedback gain for the estimated disturbance signal in DRC is transformable into convex feasibility problem (see Eq. (7) compared to Eq. (6)). Therefore, the standard solutions such as interior point method can be employed. Accordingly, by selecting the dynamics of fault-free functional state-observer as and defining the decoupled state observation residual as , one can derive Eq. (4) similar to [29].

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|  | (4) |

Using Eq. (4), the estimation of the disturbance can be calculated simultaneously as with . Then, in contrast to conventional disturbance decoupling methods such as Hou and Müller [2], Trinh et al. [15], and Mondal et al. [5], the linear matrix equality (LME) constraints are defined as and ) .

**Definition 1.** The extended state reconstruction system is asymptotically stable if: *a*) The pair () is observable. *b*) The Lyapunov stability of the estimator is satisfied and *c*) if the matrices , , and can be calculated such that .

**Theorem 1a.** *Decoupled state reconstruction system based on the concept of UIO defined as is asymptotically stable in terms of Definition 1 and rejects the unknown time-dependent input in state observation residuals as well as the decoupled fault-free/dependent states if (1) assumptions i, ii, and iii are satisfied, (2) the pair* () *is observable*, *and (3) there exist positive definite matrices , matrices , , and with appropriate dimensions, and scalar such that the linear matrix inequality (LMI)/LME conditions stated in Eq. (5a) are satisfied*

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|  | (5a) |

*where*  *is a symmetric matrix and are zero matrices with appropriate dimension except for the elements in Eq. (5b)*

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|  | (5b) |

*Proof*. Consider a Lyapunov equation in the form of ( and ). Then with will lead to LMI 1 in Eq. (5a).

Note that the Lyapunov function above is proposed on the grounds of the non-expansive nature of the system (also known as a contractive system) expressed in Eq. (2). In other words, the -norm of the linear system in Eq. (2) can be formulated in an LMI framework using Bounded Real Lemma (BRL). The rest is direct parameter specifications defined in the UIO system construction. It should be noted that there is no guarantee that with the decomposition in Eq. (2) is Hurwitz and therefore the observability condition of the theorem cannot be relaxed in conventional state observation techniques. Additionally, the proof for existence of a solution is along similar lines as in Chang [27], and thus is omitted in this paper. This completes the proof.

■

It should be pointed out that the Lyapunov equation in the proof of Theorem 1a is chosen in the least conservative form. This is because of the fact that the design parameters of UIO are independent of those in DRC (Theorem 1b). In other words, in most conservative configuration, the quadratic function with should be selected as a Lyapunov candidate in which is a symmetric positive definite matrix with appropriate dimension. However, due to the coupling introduced by off-diagonal block matrices in , the problem of designing observer and controller parameters simultaneously becomes non-convex (Projection Lemma is inapplicable). In literature, either the off-diagonal block matrices are simply set to zero (decoupling the controller and observer design) (Remark 3 and 4 of [30]) or additional linear matrix equalities are introduced [6, 31]. However, for UIO in Theorem 1a such LMEs do not exist, and additionally Congruence transformation is not applicable. Therefore, the decoupled version of Lyapunov stability theorem is implemented separately for UIO and DRC (similar to the idea in [17]). In such a setting, since the UIO design algorithm is prior to DRC, the term of the control law introduces zero diagonal elements in () after employing Schur complement Lemma. This is the scenario for which the negative definiteness of is replaced with . For a Hurwitz nominal system with the assumptions in Theorem 1a, the Lyapunov stability theorem guarantees that: For the UIO remains stable for any arbitrary bounded initial condition (). In addition, for , the trajectories of the state estimation error dynamics remain bounded as long as is an bounded function. LMI (5) is a non-strict, which can be converted to a strict feasible LMI through eradicating the existing LME constraints and then reducing the resulted LMI by eliminating any constant null space (page 19 of [32]). Moreover, non-strict LMI constraints do not reduce the generality of the method, since satisfying strict inequality constraints in applied optimization interfaces e.g. YALMIP toolbox package is infeasible.

The functional observer designed based on Theorem 1a *simultaneously* (unlike Theorem 2a) observes the system states and detects the disturbance. This concept is intimately related to UIO in which the observation problem of number of systems is reduced to the design problem of estimators. In view of LME conditions (5a), the formulation for the estimated disturbance can be simplified as . It should be pointed out that an advantage of the proposed method in comparison to Gao and Wang [33], and Aldeen and Sharma [34] is that the dynamics of the disturbance estimation mechanism (see Eq. (7)) are independent of the control input. As a result, in order to bridge this gap in literature, the non-convexity of a systematical DRC synthesis by minimizing the induced norm of the transfer function from disturbance to output can be overcome. One should note that the nominal plant model in Aldeen and Sharma [34] includes a stable Lipschitz nonlinear term in the state equation that accounts for un-modeled dynamics and therefore is distinguished from the current paper. In previous publications, such as in Chang [27], due to this issue, the feedback law in terms of the estimated disturbance is reported as while the state-observer-based feedback gain is calculated by solving an Algebraic Riccati Equation (ARE). In another approach, as an alternative view, Trinh et al. [15] used parametric solution of generalized Sylvester equation of Duan [35] to design a reduced order disturbance decoupled observer. Once they reconstructed the state vector, by reformulating the system in Eq. (2), the fault detection filter in their approach is obtained as expressed in Eq. (6) [27].

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|  | (6) |

where is the observer gain to be designed. Considering the control law as , where , will bring to the right hand side (RHS) of Eq. (6), and lead to a nonlinear feasibility problem in terms of designing . This problem is addressed in this paper. Moreover, in the method proposed in the present paper, compared to the standard Luenberger’s observer, which is designed on the fault-free counterpart of the mapped states, it should be pointed out that although in, under the assumption of observability of the pair (), can set the eigenvalues of the decoupled state estimation error dynamics in LHS of complex plain, the lack of information on and uniqueness of the SVD may destabilize the observer for relatively small mismatch uncertainty in modeling the system. However, in the proposed method of this paper, this problem is addressed by using full information observer (). This trend of observation and rejection of unknown disturbance, which acts on system states and outputs, is referred to as fault detection and isolation (FDI) [9].

**Remark 1.** In condition (I) of disturbance decoupling, , and in defining the mapping , an arbitrary matrix is required to guarantee the non-singularity of the transformation. It follows from Theorem 1a that in order to have an LMI constraint (for possibly a convex optimization problem with ), matrix should be determined prior to the feasibility/optimization phase. The assumption that is fixed is a conservative way to design the observer. It should be pointed out that defining a new variable postpones the calculation of while keeping Eq. (5) as a convex constraint. This would reduce the conservatism of the obtained solution, however, the lack of appropriate algorithm in decomposing back into individual variables and is the main drawback of this alternative. As a more practical approach in reducing the conservativeness of the UIO design problem, is to construct a bilinear matrix inequality (BMI) that can be defined over the design parameters and . Convex optimization problem such as Eq. (5) is classified as P-hard problem in which P refers to a family of optimization problems (independent of the solution algorithm) that requires a bounded *polynomial time* in reaching a feasible/optimal solution [36, 37]. In contrast, defining and in BMI framework is classified as NP-hard which suffers from having a computing *polynomial time*. Therefore, a global solver is required for convergence of non-convex feasibility/optimization subjected to BMI constraints [38]. A practical tool to solve Eq. (5) with the revised block matrices (replacing , , and ) is to use the well-known branch and bound algorithm using linear programming relaxations (LPR) and convex envelope approximations (BMIBNB [39]). An implementation of this algorithm is available in YALMIP[[2]](#footnote-2) (see sdpsettings) [40].

In addition, in an analogy to the disturbance observation technique based on the geometrical approach for the systems without uncertainty, the state reconstruction and disturbance rejection has a solution if where is the span of the transformation, represents the subspace summation of and , and finally is the supremum subspace that is -invariant (iff for matrices in Eq. (2)). However, the DRC in this paper, unlike the geometrical approach reduces to quadratic stability problem instead of setting the transfer function from disturbance to output equal to null (Rubio et al. [41, 42]). Koenig and Mammar [28] generalized the conditions in linear UIO to nonlinear systems and developed a reduced-order observer for this purpose (FDI) using disturbance decoupling technique. Sharma and Aldeen [34, 43] followed the same technique to reconstruct the states as well as estimating the disturbance for nonlinear systems. Accordingly, in this paper, selecting the DRC law as for the estimation mechanism in Theorem 1a, Eq. (7) can be obtained

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|  | (7) |

Next, the synthesis of the persistent DRC is reduced to multi-dimensional convex minimization in terms of the system of LMIs/LMEs.

**Theorem 1b.** *The system of Eq. (2) with the feedback control law as , where and are the observed state vector and the estimated disturbance vector of Theorem 1a is asymptotically stable and rejects the disturbance in system states quadratically if 1) is of full column rank and 2) there exist symmetric matrices , , , and, matrices ,, and , and positive scalar such that the system of LMI/LME in conditions (8a) are satisfied*

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|  | (8a) |

*where* *is a symmetric matrix and are zero matrices with appropriate dimensions except for the elements in Eq. (8b)*

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|  | (8b) |

*Then, the control gains can be obtained as and .*

*Proof*. Let’s propose the Lyapunov candidate as and differentiate the Lyapunov function along the system trajectory in Eq. (2). Due to the independent design variables of and , the resulting equation will be nonlinear, and the well-known Congruence transformation cannot be used due to the term in the obtained equation. The LME constraint in Eq. (8a) is therefore introduced [44]. Following comparable procedure as in the proof of Theorem 1a, with will lead to the LMI 1 expressed in (8a). Existence of solution for LME constraint is guaranteed by satisfying the condition (1) of Theorem 1b. This completes the proof.

■

The two-step method proposed in Theorems 1a and b solves the issues related to the high-gain controllers such as the windup problem in actuator and peaking phenomenon in states and control law. Despite the SMO-based controller mentioned before, the transient response of the control law in DRC is not subjected to peaking problem. However, the proposed disturbance estimator originally requires the calculation of , which includes the derivative of the output. As a result for the real-time implementation, an appropriate low-pass analog filter on the output is required to reject or attenuate noise. Alternatively the results of Theorem 2a and b can be used. Another advantage of the results stated in Theorems 1a and b is the order of observer, where for a plant of order whole states can be estimated by a functional UIO of order in comparison to the strong functional observer in Theorem 2a where the total order is for the auxiliary observer states.

Before proceeding to the strong functional observer, for the sake of completeness, the case in which the output is polluted by noise and the state equation is under the effect of input disturbance () is considered. Assuming then the output equation is formulated as where is a known matrix which can be decomposed as using SVD in which ,, , and. Defining , the noise-dependent and noise-free decoupled output counterparts can be respectively stated as and , where and . Accordingly, after decoupling the output equation, becomes noise-independent. Unlike the system in Eq. (2) and the fault-free functional state-observer in Eq. (4) with penalizing term (), in the case of polluted output measurement, only the noise-free counterpart () is included in UIO dynamics. Henceforth, following similar approach as in Theorem 1a, first the state equation is decoupled using the SVD on the disturbance matrix (. Then, the noise-free output equation is formulated in terms of disturbance-free decoupled states only. Now using superscript as the associated variable of Theorem 1a in the case that the output is polluted by noise (), the arbitrary matrix and matrix in previous case change to and , respectively. Then, and where . Likewise, with . The design procedure of the observer and the controller synthesis according to the control law are almost identical to Theorems 1a and b in which the design parameters should be resized. For the sake of brevity regenerating Theorems 1a/b is omitted in this paper.

Next, the matched disturbance case is presented in corollary 1a (without proof). One should note that in case that the system is under the effect of matched and mismatched disturbances simultaneously () combination of Corollary 1a and Theorems 1a/b can be used to create a composite control law () for separate estimation and rejection of and , respectively.

**Corollary 1a.** (matched DRC/UIO)*For matched disturbance in Eq. (2), , a decoupled state estimator is asymptotically stable in terms of Definition 1 and rejects the unknown time-dependent input quadratically in state observation residual, output, and the decoupled fault-free/-dependent states with control law if assumptions (i), (ii) (without ), and (iii) are satisfied, if the pair* () *is observable*, *and if there exist , and scalar such that LMI/LME condition (9a) are satisfied*

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|  | (9a) |

*where*  *is symmetric matrix and are zero matrices with appropriate dimensions except for the elements in Eq. (9b)*

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|  | (9b) |

*Using SVD, we write input matrix The control law of the DRC can then be obtained from B together with the transformed decoupled state matrix and an arbitrary matrix as*

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|  | (9c) |

It is now obvious that as long as the quadratic stability based on Eq. (9) is fulfilled, the two-step DRC (Theorems 1a and 1b) reduces to a single step procedure as stated in Corollary 1a. However, asymptotic stability cannot be fulfilled because . In the matched case, and since , one expects to solve this issue for . But, because in is a design variable, the obtained matrix inequality obtained after substituting (9c) and using Schur complement Lemma is nonlinear. As an alternative approach to full information UIO design based on disturbance/noise decoupling technique for DRC, DOBC schemes for perturbed systems are investigated in Theorems 2a/b. This is presented in the next section.

* 1. **DOBC based on full information state reconstruction**

Instead of UIO in Theorem 1a, and assuming that the transfer matrix between the unknown input and system output has a relative degree of one, i.e. , let us propose the dynamics of the states observer to be , in which is the vector of the observer states. In order to simplify the design process, it is assumed that in this paper. Additionally, , , , and are unknown matrices to be determined such that the conditions in Definition 1 are satisfied for the related full-order system matrices (instead of decoupled system in section 3.1). Defining the observation residual as , the dynamics of the state estimation error can be determined as . Additionally the following conditions should be satisfied ) ) and ) , in which . Then, based on (), with , , and is an unknown matrix to guarantee the satisfaction of () (assumption *ii*). Then, using the calculated and following (), it is easy to obtain: where , and . Assuming , one can obtain which changes the state estimation error dynamics to . Next, using Theorem 2a, the stability of the state estimation error dynamics will be guaranteed by calculating and , properly. The UIO technique in the generalized framework of (strong) -detectable observer is addressed to avoid the appearance of in disturbance estimation (). In view of observability/detectability, an observer is called -detectable if for an arbitrary bounded initial condition and and it is strong -detectable if . It is proven that there exist an UIO for system (2) *iff* it is strongly -detectable (Hautus, [45]: Theorem 3.2; Moreno [46]).

**Theorem 2a.** (Strong functional observer (SFO))*The UIO of the second scheme with dynamics of , asymptotically observes the system states of Eq. (2) while keeping the transfer function from disturbance and control input in state estimation error dynamics equal to null if is full rank and if for a symmetric positive definite matrix and , the constraint of Eq. (10a) has a solution for and*

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|  | (10a) |

*Then, and can be calculated.*

*Proof.* The proof is trivial and is suppressed. The necessary condition can be proven by following the similar approach to Kurek [3].

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**Remark 2.** The process of determining should be carried out prior to Theorem 2a. In other words, is the variable that guarantees the satisfaction of some additional LMEs which are omitted in Theorem 2a and otherwise leads to infeasible convex problem. If , there exists a unique solution for the intermediate matrix that guarantees the satisfaction of (). If , then a solution for may be calculated by means of Least Square (LS) algorithm. Not surprisingly, in contrast to the results in section 3.1, condition (*ii*), , can be relieved. This is because of the fact that in DOBC based on SFO (and revised SFO), no disturbance decoupling is included. In other words, in section 3.1 requires . It is obvious that for in Theorem 2a, there exist more than one feasible solution for matrix that satisfies condition and subsequently . If calculating is not possible, then, even for a bounded , the state estimation error dynamics change to and due to the optimization problem becomes non-convex. In order to address this issue for systems with , the structure of SFO is revised to , . Assuming that , in order to make the SFO synthesis algorithm less conservative, LMEs are released and instead Theorem 3a (without proof) returns , , and . In this case, the transfer function from control input and disturbance to state estimation error dynamics is minimized in the sense of induced norm, in contrast to Theorem 2a in which, the transfer functions are set to null.

**Theorem 3a.** (Revised SFO)*SFO with revised dynamics as , observes the system states of Eq. (2) in the sense of Definition 1 while minimizing the induced -norm of the transfer functions from control input and unknown bounded disturbance signal () to state estimation error dynamics if is full rank, if the plant is asymptotically stable, and if for symmetric positive definite matrix , matrices ,*, and  *with appropriate dimensions, and positive scalars and , the constraint of Eq. (10b) has a feasible solution (for user defined positive weighting scalars and )*

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|  | (10b) |

The result in Theorem 2a in contrast to full-order Luenberger’s observer is a stronger form of the functional state reconstruction mechanism and in the case of unknown input excitation is referred to SFO (Moreno [46]). Unlike the reduced-order observer that simultaneously calculates the gain matrices for unknown input estimation and states reconstruction, in this method, an individual mechanism is assigned for disturbance estimation. Accordingly, in view of proposed method in Chen et al. [25], Chen [26], and Yang et al. [20], the auxiliary state is defined to recast the UIO problem in the framework of a second observation mechanism in terms of the disturbance and states. This mechanism is defined as while where . Contrary to the result of Theorem 1a, the estimation order is equal to . By defining the dynamics of auxiliary disturbance estimation residuals as and derivation with respect to time and defining , Eq. (11) can be obtained

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|  | (11) |

Next, the disturbance observer-based compensation gain is redesigned by including an estimated measure of the bounded unknown input in the feedback channel to create a composite controller that counteracts the mismatched lumped disturbances in the input channel. Accordingly, persistence disturbance rejection is carried out by means of static observer-based state/disturbance feedback.

**Theorem 2b.** (Static observer-based control) *For the state-space representation in Eq. (2), the control law defined as a feedback of the reconstructed states and estimated disturbance ( with and being the unknown controller gains to be calculated) can reject the disturbance in output quadratically and simultaneously design the estimation gain if both the disturbance () and its derivative () are bounded, if plant is controllable, and also if there exist symmetric positive definite matrices and , matrices , , and and scalars and such that the LMI/LME conditions (12) are satisfied.*

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|  | (12a) |

*where* *is a symmetric matrix and are zero matrices with appropriate dimensions except for the elements in Eq. (12b)*

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|  | (12b) |

*Then, , , and , .*

*Proof.* Define the quadratic Lyapunov equation in terms of disturbance estimation residuals and real system states as and then transforming the scalar in the form of with will lead to LMI 1 in (12a). Existence of solution for LME constraint is guaranteed by satisfying condition (1) of Theorem 1b. The bounded-input bounded-output (BIBO) stability with effective design of Theorem 2b is achieved for system of Eq. (2) under the assumptions of (*i*), (*ii*), and (*iii*). Therefore, the proof is complete.

■

**Remark 3.** The LMI constraints defined in Theorems 1a/b, Theorems 2a/b, Theorem 3a, and Corollary 1a include BRL as the basis of robust process control in an LMI framework. BRL as the worst case performance measure is utilized to calculate the norm of the transfer function , () in which is the maximum singular value of , under the assumption that . Therefore, based on the concept of SDP and EVP, a minimizing problem on maximum singular value of can be defined as (Similar to results in Eq. (10a)) [pages 1-25 of 47, 48]. represent Eqs. (5, 8, 9, and 12).

It should be mentioned that in Wei and Gao [23] compared to [22] a more general DOBC framework is proposed that includes both matched and mismatched disturbance in the nominal state space model (in analogy to Eq. (2): ). In such a representation, the matched disturbance is assumed to be generated by an external system of known dimension and dynamics. Then, the composite DOBC and -controller are used to form a control law as . Accordingly, the idea of internal model control (IMC) is employed in creating a counteracting control effort with the matched disturbance. However, the mismatch case still remains to be handled by -controller. In contrast to this idea, in this paper both of the proposed methods are based on including the mismatch disturbance estimation in feedback channel () which is more general. Additionally, two important contributions in this paper are: 1) Ref. [23] assumes that the states of the system are available in constructing the control law (). However, in this paper, careful attention is paid to reconstruct and observe the states based on the concept of UIO and disturbance decoupling problem. Accordingly, it is assumed that only measured output signal is available and more generally, the effect of noise is incorporated in derivation of generalized solution in section 3.1. 2) In the state reconstruction based on UIO and disturbance/noise decoupling, the matched disturbance case is investigated as Corollary 1a without additional assumptions on dynamics of the exogenous system generating . Such a view covers the modeling parametric uncertainties as well as the common harmonic disturbances. Moreover, this paper similar to Chen [17], is proposed in a two-stage design procedure. However, in Ref. [17], in contrast to this paper, first the nonlinear controller is designed based on measurable disturbance assumption and satisfying performance requirements and then, based on DOBC concept the disturbance is estimated and replaced in control law. A limiting assumption, that is relieved in this paper, is then the nature of unknown signal which is assumed to be harmonic in [17] ( ).

**Remark 4.** It should be noted that the asymptotic stability of the closed-loop system based on Theorem 2b is only guaranteed if the steady state value of satisfies: . However, as it is reported in Chen et al. [49], the proposed approach has a robust performance in tracking disturbances with fast dynamics. Moreover, even though the high frequency inputs may be estimated in this method, in most of applications due to bandwidth limitation of the actuators, it is challenging to reject the unknown input using the estimation of disturbance in the feedback channel.

**Remark 5.** It should be pointed out that, since the input matrix () and disturbance matrix () are distinct in mismatch case, in general, the control law in Eqs. (12b) or (8b) cannot guarantee the complete elimination of the unknown input in system output. In other words, the null-ness of transfer function from disturbance to output is reduced to quadratic rejection problem. Additional constraints based on subspace theory will lead to conservative results.

**Remark 6.** Both of the methods preserve the nominal performance of the controller ( in Eq. (8a) and in Eq. (12a)) if the unknown input stops to drive the system since the inner loop for estimation of disturbance () will automatically become zero.

1. **Numerical Example**

The system we consider here as the numerical example is an abstract sixth order plant with three control inputs, one disturbance channel, and three output measurement signals. The state space matrices of the plant in the form of Eq. (2) are presented in Eq. (13)

|  |  |
| --- | --- |
|  | (13) |

Then, using Theorem 1a, the unknown input observer matrices are obtained as Eq. (14). It should be pointed out that the auxiliary intermediate variable is calculated based on metaheuristic method such that conditions and in LME constraints of Theorem 1a are satisfied.

|  |  |
| --- | --- |
|  | (14) |

After observing the decoupled disturbance-free states () and using to observe the faulted (disturbance-dependent) states, unknown input disturbance is estimated and the system states are reconstructed based on the SVD of disturbance matrix () as: . Then, using Theorem 1b appropriate controller matrices are calculated ( and ) and the results are shown in Eq. (15)

|  |  |
| --- | --- |
|  | (15) |

Now based on the concept of SFO, to obtain a full-order unknown input state observer, LME condition (10) in Theorem 2a is solved using MATLAB LMI toolbox and the resulted matrices are shown in Eq. (16)

|  |  |
| --- | --- |
|  | (16) |

Then, using the obtained UIO of Theorem 2a with matrices in Eq. (16) and employing Theorem 2b, the disturbance estimation mechanism can be calculated simultaneously with the static feedback gain control law (). The associated matrices , , and are calculated and presented in Eq. (17)

|  |  |
| --- | --- |
|  | (17) |

In order to evaluate the performance of UIO of decoupled system (Theorem 1a) in disturbance estimation and compare the estimation quality with two-step feasibility problem of Theorem 2b, the plant is excited through the disturbance channel with a non-stationary signal. Figure 1 represents the disturbance estimation comparison of the two methods. As it can be seen, both of the methods are estimating the unknown input signal which consists of four parts: 1) Chirp signal with damped amplitude (swept frequency in 6 sec). 2) Sudden jump disturbance between 5 and 6 sec. 3) saw-tooth harmonic signal with frequency 0.5 Hz. 4) Realization of white Gaussian noise signal between 12 to 15 sec. It is observed that disturbance estimation based on independent dynamics in Eq. (11) and Theorem 2b has a slightly better performance than simultaneous disturbance estimation mechanism based on disturbance/noise decoupling problem. This performance difference can be quantitatively measured by integrating over the absolute value of estimation error. As it can be seen in Fig. 2, DOBC based on Theorem 2b stays in lower values for non-stationary unknown signal compared to UIO/DRC of Theorem 1a/b. Moreover, in the range of sec, in which the realization of white Gaussian signal with sampling time of 1 msec is active, the estimation error based on UIO/DRC increases significantly. However, it should be noted that for large systems with various source of erroneous signals, the disturbance estimation based on decoupling method is more computationally efficient.



Fig. 1. Comparison of the performance of disturbance estimation mechanism



Fig 2. Comparison of disturbance estimation error by means of integral of absolute value

Next, in order to assess the performance of disturbance rejection in both methods, the closed-loop systems after the implementation of the DRC (Theorem 1b) and DOBC (Theorem 2b), with given matrices in Eqs. (15) and (17), are excited with a sinusoidal signal of frequency 1.59 Hz in time window of ec. The three output signals of the three system are compared in Fig. 3: 1) open loop system; 2) closed-loop system based on UIO/DRC of decoupling problem (Theorems 1a/b); 3) SFO/DOBC based on Theorems 2a and 2b.



Fig. 3. Comparison of the performance of the disturbance rejection

1. **Conclusion**

In this paper, considering the limitations for the systematic control synthesis in disturbance rejection control (DRC) based on estimated disturbance, the disturbance/noise decoupling problem is revisited. On one hand, the disturbance/noise decoupling reduces the size of the state observer dynamics, but on the other hand, the design problem of estimated disturbance gain matrix in DRC becomes non-convex. Using the concept of unknown input observation (UIO) this issue is addressed in this paper. Two open problems for this combination are: (i) the design of the intermediate variable which guarantees the satisfaction of the additional linear matrix equality constraints defined to reject the effect of disturbance and suppress the direct effect of control input in state estimation error dynamics (instead of bilinear matrix inequality constraint), and (ii) The appearance of the first derivative of the output measurement in disturbance estimation. For the class of systems with equal number of outputs and unknown input signals (i) can be addressed (as of this paper). To address (ii), a possible solution, as followed in the second part of this paper, is to separate the disturbance estimation dynamics from UIO. This approach separates the observation problem into two sub-problems and an obvious disadvantage is that after designing the strong functional state observer (SFO), the feasibility of second sub-problem (disturbance estimation) is not guaranteed. Then again, if plant is under the effect of multiple erroneous signals, the order of the observer/estimator will increase. Moreover, if the general governing dynamics of the system has multiple equilibrium points in the framework of Rajamani (1998) [14] satisfaction of linear matrix inequality conditions of all linearized systems is a challenging task. It should be noted that since the general view of DRC in this paper is for mismatch case, feedforward methods are neglected as an alternative to inclusion of estimated disturbance in feedback channel which is mostly undesirable in practice. The implementation of the main results in the methodology section on two vibrating mechanical smart structures in real-time is an ongoing research [6, 50]. Moreover, systematic sensitivity analyses on the closed-loop performance in terms of the tuning parameters based on one-at-a-time (OAT) technique is proposed as a potential future research topic. Additionally, considering the current investigations in nonlinear system identification [51], the semi-analytical modeling of geometrically nonlinear systems [52], and the nonlinear polynomial state space method (PNLSS) [53], the adaptation/generalization of the main results of this paper is an interesting topic. In contrast to the robust techniques based on worst-case scenarios, such a view is less conservative because the nonlinearity of the plant is only effecting the control design when they appear in system output.

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