Ch3 : Differentiation Rules

Sec 3.1 Derivatives of Polynomials and Exponential Functions

Rule : Derivative of a constant function

$$\frac{d}{dx}(c) = 0 \quad , \quad c \text{ is a constant}$$

The power rule (General)

If n is any real number, then $\frac{d}{dx}(x^n) = n x^{n-1}$

Ex1:

a) If $f(x) = x^6$ then $f'(x) = 6x^5$

b) If $y = x^{100}$ then $\frac{dy}{dx} = 100x^{99}$

c) If $y = t^4$ then $\frac{dy}{dt} = 4t^3$

d) $\frac{d}{dr}(r^3) = 3r^2$

Ex2: Differentiate

a) $f(x) = \frac{1}{x^2}$

$f(x) = x^{-2}$

$\Rightarrow \frac{d}{dx}(x^{-2}) = -2x^{-3} = \frac{-2}{x^3}$

b) $y = \sqrt[3]{x^2}$

$y = x^{2/3}$

$\Rightarrow \frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{-1/3} = \frac{2}{3} \cdot \frac{1}{x^{1/3}}$

Q4 / 181 If $f(x) = \sqrt{30}$, find $f'(x)$

$f'(x) = 0$ since $\sqrt{30}$ is constant.

The constant Multiple Rule

If c is a constant and f is a differentiable function then $\frac{d}{dx}(c f(x)) = c \frac{d}{dx} f(x)$

* The Sum Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x) + g(x) \right] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

- Using prime notation: $(f+g)' = f' + g'$

* The difference Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x) - g(x) \right] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

Exs Differentiate $y = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$

$$\frac{dy}{dx} = 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6 + 0$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

Q83/181 Find the equation of the tangent line to the curve $y = x - \sqrt{x}$ at $(1,0)$

$$y = x - x^{1/2}$$

$$\Rightarrow y' = 1 - \frac{1}{2} x^{-1/2}$$

$$m = y'\Big|_{x=1} = 1 - \frac{1}{2} = \frac{1}{2}$$

So $m = \frac{1}{2}$, we have the point $(1,0)$

Equation of the tangent line:

$$y - 0 = \frac{1}{2}(x-1)$$

$$y = \frac{1}{2}x - \frac{1}{2}$$

Q22/121 $y = \sqrt{x}\,(x-1)$ find $\frac{dy}{dx}$

$\boxed{\text{Simplify first}}$

$$y = x^{3/2} - x^{1/2}$$

$$y' = \frac{3}{2} x^{1/2} - \frac{1}{2} x^{-1/2}$$

Ex6/178 Find the points on the curve of $y = x^4 - 6x^2 + 4$ where the tangent is Horizontal

$\boxed{\text{idea: Horizontal Tangent occurs when } y' = 0}$

$$y' = 4x^3 - 12x$$

Now set $y' = 0$

$$y' = 0 \Rightarrow 4x^3 - 12x = 0$$
$$4x(x^2 - 3) = 0$$
$$4x(x - \sqrt{3})(x + \sqrt{3}) = 0$$
$$x = 0 , \quad x = \sqrt{3} , \quad x = -\sqrt{3}$$

the points are $(0, 4)$, $(\sqrt{3}, -5)$ and $(-\sqrt{3}, -5)$

Exponential Functions

Def: e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

Derivative of the natural exponential function

$$\frac{d}{dx}(e^x) = e^x$$

Q32/181 Differentiate $y = e^{x+1} + 1$

$$y = e \cdot e^x + 1$$ simplify first
$$y' = e \cdot e^x + 0$$
$$\Rightarrow y' = e^{x+1}$$

Q56/181 At what point on $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$

idea! Parallel lines have equal slopes

- $3x - y = 5 \Rightarrow$ the line $y = 3x - 5$ has the slope $m = 3$
- For $y = 1 + 2e^x - 3x$
  $y' = 2e^x - 3$

Now solve $2e^x - 3 = 3$
$$2e^x = 6 \Rightarrow e^x = 3 \Rightarrow x = \ln 3$$

Find y by subsituting on $y = 1 + 2e^x - 3x$
$$y = 1 + 2e^{\ln 3} - 3\ln 3$$
$$= 1 + 2(3) - 3\ln 3$$
$$= 7 - 3\ln 3$$

the point is $(\ln 3, 7 - 3\ln 3)$

$\frac{Q69}{182}$ Find $f'(x)$ where $f(x) = |x^2-9|$

rewrite the function by

$x^2-9 \geq 0 \Rightarrow x^2 \geq 9 \Rightarrow |x| \geq 3 \Rightarrow x \geq 3$ or $x \leq -3$

thas mean: $|x^2-9| = x^2-9$ if $x \geq 3$ or $x \leq -3$

$$f(x) = \begin{cases} x^2-9 & \text{if } x \leq -3 \\ 9-x^2 & \text{if } -3 \leq x \leq 3 \\ x^2-9 & \text{if } x \geq 3 \end{cases}$$

f	x^2-9	$9-x^2$	x^2-9
	-3	3	

$$f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases}$$

Note that we remove the equality because we will investigate them alone

$\boxed{\text{at } x = -3}$

$f'_-(-3) = 2(-3) = -6$

$f'_+(-3) = -2(-3) = 6$

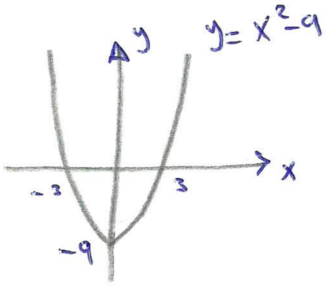
$\Rightarrow f'(-3)$ DNE since $f'_-(-3) \neq f'_+(-3)$ "corner"

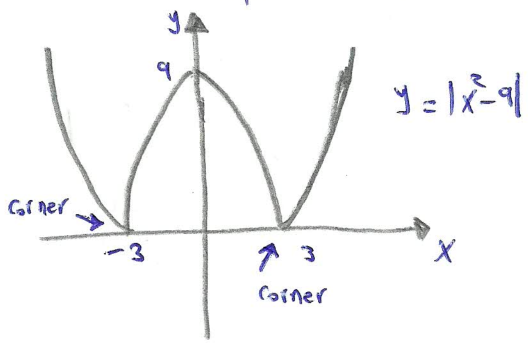
$\boxed{\text{at } x = 3}$

$f'_+(3) = 2(3) = 6$

$f'_-(3) = -2(3) = -6$

$\Rightarrow f'(3)$ DNE since $f'_+(3) \neq f'_-(3)$ "corner"





Q 75 Let $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ mx+b & \text{if } x > 2 \end{cases}$

Find the values of m and b that make f differentiable every where.

- f must be diff at $x=2$

$$f'(x) = \begin{cases} 2x & \text{if } x<2 \\ m & \text{if } x>2 \end{cases}$$

$$\left.\begin{array}{l} f'_-(2) = 4 \\ f'_+(2) = m \end{array}\right\} \Rightarrow \boxed{m=4} \quad \text{since } f \text{ is diff at } x=2$$

- f is diff at $x=2 \Rightarrow f$ is cont at $x=2$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$$

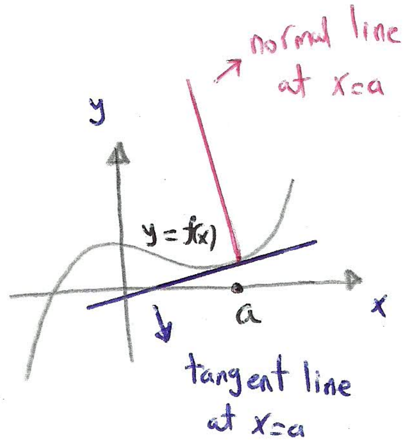
$$2m+b = 4 = 4$$

Now: $2m+b=4 \Rightarrow 8+b=4 \Rightarrow \boxed{b=-4}$

Remarks:

1) the slope of the tangent line to $y = f(x)$ at $x = a$ is $f'(a)$

2) the slope of the normal line to $y = f(x)$ at $x = a$ is $\frac{-1}{f'(a)}$



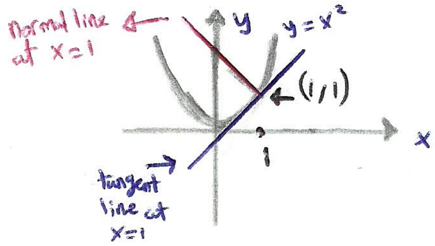
3) the point $(a, f(a))$ is common in $y = f(x)$, the tangent line at $x=a$ and the normal line at $x = a$.

Q: Find the equation of the tangent line to $y = x^2$ at $x = 1$

$y' = 2x$

$m_t = 2(1) = 2$, Point $(1,1)$

$\Rightarrow$ Equation: $y - 1 = 2(x-1)$



Q: Find the equation of the normal line to $y = x^2$ at $x = 1$

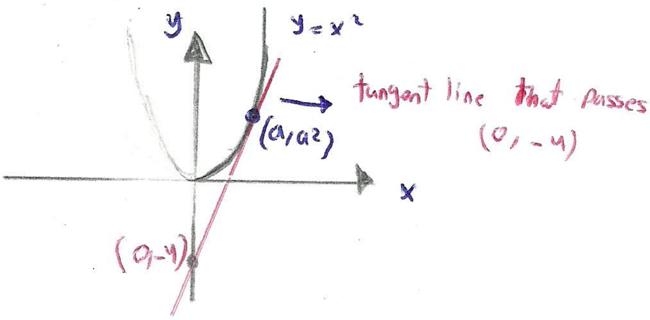
$y' = 2x \Rightarrow m_t = 2(1) = 2$

$\Rightarrow m_N = -\frac{1}{2}$, Point $(1,1)$

Equation: $y - 1 = -\frac{1}{2}(x-1)$

Q: Find the equation of the tangent line to $y = x^2$, $x \geq 0$ that passes through the point $(0,-4)$

$$\boxed{\begin{array}{l}\text{idea: Solve } \frac{\Delta y}{\Delta x} = y'\big|_{x=a} \text{ for } a. \\ \text{one Point is given, the other one is } (a, f(a))\end{array}}$$



$y' = 2x$, $y'\big|_{x=a} = 2a$

$\frac{\Delta y}{\Delta x} = \frac{a^2 - (-4)}{a - 0} = \frac{a^2+4}{a}$

$\Rightarrow \frac{a^2+4}{a} = 2a \Rightarrow a^2 + 4 = 2a^2 \Rightarrow a^2 = 4 \Rightarrow a = 2, a = -2$

rejected

So the needed line is the tangent line to $y = x^2$ at $x = 2$

$y' = 2x$

$m = 4$, Point $(2, 4)$

Equation: $y - 4 = 4.(x-2) \Rightarrow \boxed{y = 4x - 4}$

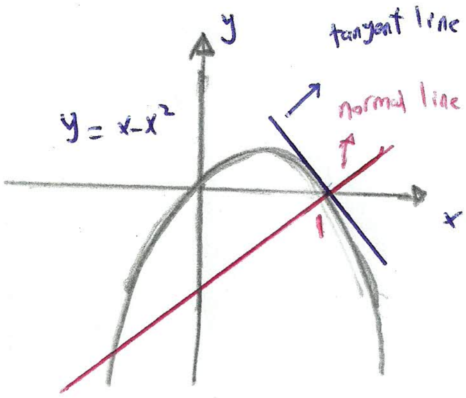
Or simply find the equation of line through $(0,-4)$ and $(2,4)$

$m = \frac{4-(-4)}{2-0} = \frac{8}{2} = 4$

Equation: $y + 4 = 4(x-0) \Rightarrow \boxed{y = 4x - 4}$

Solved Exercises

$\frac{Q58}{182}$ Where does the normal line to the parabola $y = x - x^2$ at $(1,0)$ intersect the parabola a second time.



$y' = 1 - 2x$

$m_t = 1 - 2(1) = -1 \Rightarrow m_N = \frac{-1}{-1} = 1$

Equation of the normal line: $y - 0 = 1(x-1) \Rightarrow y = x - 1$

Now to find the intersection point : solve

$$x - x^2 = x - 1 \Rightarrow -x^2 = -1$$

$$x^2 = 1 \Rightarrow x = \pm 1$$

The other point is $(-1, -2)$

Note: you can sub $x = -1$ in $y = x - x^2$ or in $y = x - 1$ because the point is common

at $x = 1$, the point $(+1, 0)$ is the first point of intersection

Q: Find the equation of both → tangent lines to $y = x^2$ that pass through the point $(0, -9)$ (Ans $y + 9 = 6x$, $y + 9 = -6x$)

Q For what values of a and b is the line $2x + y = b$ tangent to $y = ax^2$ when $x = 2$ (Ans $a = -\frac{1}{2}$, $b = 2$)