Chapter 3: DIFFERENTIATION RULES

In the previous chapter we saw how to apply the definition of the derivative to several functions. In this chapter we establish some differentiation rules that will help us find the derivative without going back to the definition.

3.1 Derivative of Polynomial & Exponential

The Goals:

After completing this section, you will be able to

1. To introduce the rules of differentiation.
2. To consider the derivative of the exponential functions

We start this section by few theorems.

Rule 1: Derivative of a constant function.

If $\boldsymbol{y = f(x) = c}$ where $\boldsymbol{c}$ any constant, then

$$\boldsymbol{y' = f'(x) = \frac{dy}{dx} = 0}$$

Rule 2: Derivative of a power function (power rule).

For any positive integer n

$$\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1}}$$

Rule 3: The General Power Rule

For any real number n

$$\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1}}$$

Example: Find the derivative for the following functions

1. $f(x) = -15$
2. $f(x) = x^3$
3. $f(x) = x^{104}$
4. $f(x) = \sqrt{15}$
5. $f(x) = x^{-\frac{4}{5}}$
6. $f(x) = \sqrt[3]{x}$
7. $f(r) = \pi r^2$
8. $f(x) = \frac{1}{x}$
9. $f(x) = \frac{1}{x^2}$
10. $f(x) = x^{\pi}$

Example: Find the tangent equation to the curve of $\mathrm{y} = x\sqrt{x}$ at the point $(4, 8)$

Solution:

$$\mathrm{y} = x\sqrt{x} = x\, x^{\frac{1}{2}} = x^{\frac{3}{2}}$$

The slope of the tangent line is

$$\boldsymbol{y' = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x} = \frac{3}{2}\sqrt{4} = 3}$$

The equation of the tangent line is

$$\mathrm{y} - \mathrm{y}_0 = m(x - x_0)$$
$$\mathrm{y} - 8 = 3(x - 4)$$

Rule 4: The Derivative of a function multiplied by constant

Let c is constant and $f(x)$ is differentiable, then

$$\frac{d}{dx} c\, f(x) = c \frac{d}{dx} f(x)$$

Rule 5: The Derivative of sum (difference) of functions

If f and g are differentiable, then

1. $\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) = f'(x) + g'(x)$

2. $\frac{d}{dx}\big(f(x) - g(x)\big) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x) = f'(x) - g'(x)$

In general:

$$\frac{d}{dx}\big(f(x) \pm g(x) \pm h(x)\big) = f'(x) \pm g'(x) \pm h'(x)$$

Example: differentiate each of the following functions

1. $f(x) = x^4 - 2x$
2. $f(t) = \sqrt{t} - \frac{1}{\sqrt{t}}$
3. $f(x) = \sqrt{x}\big(x^4 - 2\sqrt{x}\big)$
4. $y = \big(t - \sqrt{t}\big)^2$
5. $u = \sqrt[3]{t^2} - 2\sqrt{t^3}$
6. $y = \frac{x^2 - x}{\sqrt{x}}$

Example: Find the tangent equation to the curve of $y = \sqrt{x}\left(x^4 - 2\sqrt{x}\right)$ at $x = 1$

Solution:

$$y = \sqrt{x}\left(x^4 - 2\sqrt{x}\right) = x^4\sqrt{x} - 2\sqrt{x}\sqrt{x} = x^{\frac{9}{2}} - 2x$$

The slope of the tangent line is

$$\boldsymbol{y'(1) = \frac{9}{2}x^{\frac{7}{2}} - 2} = \frac{7}{2}$$

The equation of the tangent line is

$$y - y_0 = m(x - x_0)$$
$$y + 1 = \frac{7}{2}(x - 1)$$

Example Find all points on the curve $y = x^4 - 2x^2$ where the tangent line to the curve is horizontal.

Solution

The slope of the tangent line is

$$\boldsymbol{y' = 4x^3 - 4x = 0}$$

Since the tangent line is horizontal, we need to find all points $\left(a, f(a)\right)$ on the graph where the slope of the tangent line is zero.

$$\boldsymbol{y'|_{x=a} = 2a^3 - 4}a = 4a(a^2 - 1) = 0$$

$$a = 0,\ \ a = 1,\ \ a = -1$$

The points are
|

$$(0, 0), \qquad (1, -1) \qquad (-1, -1)$$

Example consider the function $f(x) = |x^2 - 9|$

1. Find $f'(x)$
2. For which values $f(x)$ is not differentiable?

Solution:

1. $f(x) = |x^2 - 9| = \begin{cases} x^2 - 9 & x \le -3 \\ 9 - x^2 & -2 < x < 3 \\ x^2 - 9 & x \ge 3 \end{cases}$

$$f'(x) = \begin{cases} 2x & x < -3 \\ -2x & -2 < x < 3 \\ 2x & x > 3 \end{cases}$$

2. When $x = -3$

$$f'_-(-3) = -6 \quad and \quad f'_+(-3) = 6 \quad \rightarrow \quad f'(-3) \ DNE$$

When $x = 3$

$$f'_-(3) = -6 \quad and \quad f'_+(3) = 6 \quad \rightarrow \quad f'(3) \ DNE$$

$f(x)$ is not differentiable at $x = -3$ & $x = 3$

Exponential Functions

Consider the exponential function $f(x) = a^x$. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h\to 0} \frac{(a^h - 1)}{h}$$

Note that the limit is the value of the derivative of f at 0, that is

$$\lim_{h\to 0} \frac{(a^h - 1)}{h} = f'(0)$$

Therefore, the exponential function $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere and

$$f'(x) = f'(0)a^x$$

So we need to find $f'(0)$. Using Calculator, we find that

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

Numerical evidence for the existence of $f'(0)$ is given in the table above for the cases $a = 2$ and $a = 3$. (Values are stated correct to four decimal places.) It appears that the limits exist and

$$a = 2, \qquad f'(0) = \lim_{h\to 0} \frac{(2^h - 1)}{h} = 0.69$$

$$a = 3, \qquad f'(0) = \lim_{h\to 0} \frac{(3^h - 1)}{h} = 1.1$$

Thus,

$$\frac{d}{dx}(2^x) = f'(0)2^x = (0.69)2^x \qquad and \qquad \frac{d}{dx}(3^x) = f'(0)3^x = (1.1)3^x$$

In view of the estimates of $f'(0)$ for $a = 2$ and $a = 3$, it seems reasonable that there is a number a between 2 and 3 for which $f'(0) = 1$.

It is traditional to denote this value by the letter e. Thus we have the following definition

Definition of the Number e

$$e \text{ is the number such that } \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Note that using calculator, $e \cong 2.71828$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Example Find the derivative of each of the following functions.

1. $f(x) = 3e^x - x^4 + 2$
2. $y = ae^x - \frac{b}{x} + \frac{c}{x^2}$
3. $y = e^{x+2} + 7x$

Example: Find all points on the curve $y = f(x) = 3e^x - 3x + 2$ where the tangent line to the curve is parallel to the line $y - 3x - 1 = 0$.

Solution
Since the tangent line is parallel to the line $y - 3x - 1 = 0$, it must have the same slope; i.e.; the slope of the tangent line is $m = 3$.
But the slope is equal to $f'(x) = 3e^x - 3$
Thus

$$3e^x - 3 = 3 \quad \rightarrow \quad e^x = 2 \quad \rightarrow x = \ln 2$$

That is the tangent line at the point $(\ln 2, f(\ln 2))$is parallel to $y - 3x - 1 = 0$.
Where

$$f(\ln 2) = 3e^{\ln 2} - 3\ln 2 + 2 = 8 - 3\ln 2$$

The point:

$$(\ln 2, 8 - 3\ln 2)$$

3.2 The Product and Quotient Rules

The Goal:

After completing this section, you will be able to

1. Introduce the rules of differentiation for the product and quotient

Let $f(x)$ and $g(x)$ be differentiable functions.

1. We define the product of two functions as:

$$f.g(x) = f(x)g(x)$$

2. and the quotient as

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

Theorems

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Proof:

1. $(f.g)'(x) = \lim\limits_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$

For the numerator add and subtract $f(x+h)g(x)$ and re arrange the terms

$$= \lim_{h\to 0} f(x+h)\left(\frac{g(x+h)-g(x)}{h}\right) + g(x)\left(\frac{f(x+h)-f(x)}{h}\right) = f(x)g'(x) + g(x)f'(x)$$

2. If we put $F = \frac{f}{g}$, then $f = Fg$. Differentiating both side and using (1) above,

$$f' = Fg' + gF'$$

and so

$$F' = \frac{f' - Fg'}{g} = \frac{f' - \frac{f}{g}g'}{g} = \frac{f'g - fg'}{g^2}$$

Example: Find y' for each of the following functions.

1. $y = (x^2 + 2)(3x - x^3)$
2. $y = \frac{x^2+1}{3x-x^3}$
3. $y = (1 - x)(x^2 + 1)(x + x^3)$
4. $y = \frac{1}{2-\frac{1}{x+3}}$
5. $y = \frac{(x^2-x)}{\sqrt{x}}$
6. $y = \sqrt{x}\, e^x$

Example: Find the slope of tangent line to the curve $y = \frac{(1+x)(2-x^2)}{2x+1}$ at $x = 0$

Solution:

The slope of the tangent equal to $m = y'$

$$y'|_{x=0} = \frac{(2x+1)[(1+x)(-2x)+(2-x^2)] - 2(1+x)(2-x^2)}{(2x+1)^2} = -\frac{2-4}{1} = -2$$

Example: If $\boldsymbol{h(2) = 4}$, $\boldsymbol{h'(2) = -3}$, find

1. $\boldsymbol{\frac{d}{dx}\left(\frac{h(x)}{x}\right)\Big|_{x=2}}$
2. $\boldsymbol{\frac{d}{dx}\left(\sqrt{x}h(x)\right)\Big|_{x=2}}$

Solution:

1. $\boldsymbol{\frac{d}{dx}\left(\frac{h(x)}{x}\right) = \frac{xh'(x)-h(x)}{x^2} = \frac{2h'(2)-h(2)}{2^2} = \frac{2(-3)-4}{2^2} = -\frac{10}{4} = -\frac{5}{2}}$
2. $\boldsymbol{\frac{d}{dx}\left(\sqrt{x}h(x)\right) = \sqrt{x}h'(x) + \frac{h(x)}{2\sqrt{x}} = -3\sqrt{2} + \frac{4}{2\sqrt{2}}}$

3.3 Derivatives of Trigonometric Functions

The Goal:

After completing this section, you will be able to Introduce the derivatives of trigonometric functions

In this section we shall look at the derivatives of the trigonometric functions. There are several useful trigonometric limits that are necessary for evaluating the derivatives of trigonometric functions. Let's start by stating some (hopefully) obvious limits:

$$\lim_{x\to 0} \sin x = 0$$
$$\lim_{x\to 0} \cos x = 1$$
$$\lim_{x\to 0} \tan x = 0$$

Since each of the above functions is continuous at $x = 0$, the value of the limit at $x = 0$ is the value of the function at $x = 0$; this follows from the definition of limits.

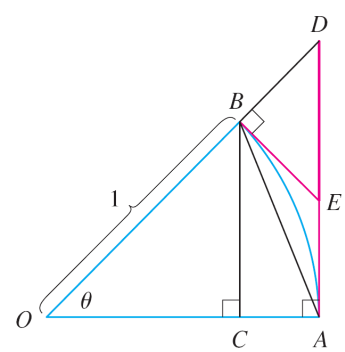
In order to evaluate the derivatives of sine and cosine we need to the following theorem.

Theorem:

$$\boxed{\lim_{\theta\to 0} \frac{\sin\,\theta}{\theta} = 1}$$

Proof:

We now use a geometric argument to prove the theorem. Assume first that θ lies between 0 and π /2. Figure shows a sector of a circle with center O, central angle θ, and radius 1.

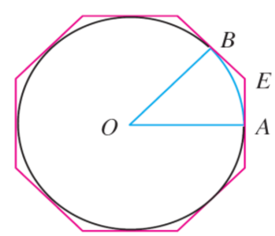


BC is drawn perpendicular to OA. By the definition of radian measure, we have arc AB = θ.
Also | BC | = | OB | sin θ = sin θ.

From the diagram we see that

$$|BC| < |AB| < \text{arc } AB$$

Therefore $\sin\theta < \theta$ so $\frac{\sin\theta}{\theta} < 1$



Let the tangent lines at A and B intersect at E. You can see from

Figure that the circumference of a circle is smaller than the length of a circumscribed polygon,
and so arc AB < | AE | + | EB |.

Thus

$$\begin{aligned}\theta = \text{arc } AB &< |AE| + |EB| \\ &< |AE| + |ED| \\ &= |AD| = |OA| \tan\theta \\ &= \tan\theta\end{aligned}$$

Therefore we have

$$\theta < \frac{\sin\theta}{\cos\theta}$$

so

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

We know that $\lim_{\theta\to 0} 1 = 1$ and $\lim_{\theta\to 0} \cos\theta = 1$, so by the Squeeze Theorem, we have

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$$

But the function $\frac{\sin\theta}{\theta}$ is an even function, so its right and left limits must be equal. Hence, we have

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Corollary

$$\boxed{\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0}$$

Theorem.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Proof of (1)

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h\ -\sin x}{h}$$
$$= \lim_{h\to 0}\frac{\sin x\,(\cos h - 1) + \cos x\sin h}{h}$$
$$= \lim_{h\to 0}\left(\frac{\sin x\,(\cos h-1)}{h} + \frac{\cos x\sin h}{h}\right)$$
$$= \lim_{h\to 0}\frac{\sin x\,(\cos h-1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h}$$
$$= \sin x\lim_{h\to 0}\frac{(\cos h-1)}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h} = 0 + \cos x = \cos x$$

Proof of (3)

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\cos x - \sin x\,(-\sin x)}{\cos^2 x}$$
$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Proof of (5)

$$\frac{d}{dx}(csc\, x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\sin x\,(0) - (cos\, x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$
$$= -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = -\csc x\cot x$$

Example. Find the derivative of the functions

1. $y = x\sin x + \cos x$
2. $y = x\cot x$
3. $y = e^x \sin x$
4. $y = \frac{\sin x}{1+\cos x}$
5. $y = 2\csc\ x + 5\cos x$
6. $y = \frac{tan\, x-1}{\sec x}$

Solution.

1. $y' = x\cos x + \sin x - \sin x = x\cos x$
2. $y' = x(-\csc^2 x) + \cot x$
3. $y' = e^x(\cos x) + sin\, x\,(e^x) = e^x(\cos x + \sin x)$
4. $y' = \frac{(1+\cos x)\cos x + \sin x(0-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x - \sin^2 x}{(1+\cos x)^2}$
5. $y' = -2\csc x\cot x - 5\sin x$
6. $y' = \frac{\sec x\,(\sec^2 x) - (\tan x - 1)(\sec x\tan x)}{\sec^2 x} = \frac{\sec^3 x - \sec x\tan^2 x + \sec x\tan x}{\sec^2 x}$

$$= \frac{\sec x\,(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{(\sec^2 x - \tan^2 x + \tan x)}{\sec x}$$

But $1 + \tan^2 x = \sec^2 x \quad\quad \to 1 = \sec^2 x - \tan^2 x$

$$y' = \frac{1 + \tan x}{\sec x}$$

Example. Find the equation of the tangent line to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$

Solution:

The slope of the tangent line

$$m = y'|_{x=\frac{\pi}{2}} = \sec^2 x = \sec^2 \frac{\pi}{4} = 2$$

The equation given by

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

Theorem:

$$\boldsymbol{\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad and \qquad \lim_{x\to 0}\frac{x}{\sin x}=1}$$

In general

$$\boldsymbol{\lim_{x\to 0}\frac{\sin ax}{bx}=\frac{a}{b} \qquad and \qquad \lim_{x\to 0}\frac{\sin ax}{\sin bx}=\frac{a}{b}}$$

Example. Find the limits

1. $\lim_{x\to 0}\frac{\sin 4x}{x}=$
2. $\lim_{x\to 0}\frac{\sin 5x}{\sin 6x}=$
3. $\lim_{x\to 0}x\cot x=$
4. $\lim_{x\to 0}\frac{\cos x-1}{\sin x}=$
5. $\lim_{x\to \pi}\frac{\sin 2x}{\sin x}=$
6. $\lim_{x\to 0}\frac{2x-\sin x}{x}=$
7. $\lim_{x\to \frac{\pi}{4}}\frac{\sin\left(x+\frac{\pi}{4}\right)-1}{x-\frac{\pi}{4}}=$

Solution:

1. $\lim_{x\to 0}\frac{\sin 4x}{x}=\lim_{x\to 0}\frac{\sin 4x}{x}\left(\frac{4}{4}\right)=4\lim_{x\to 0}\frac{\sin 4x}{4x}=4(1)=4$
2. $\lim_{x\to 0}\frac{\sin 5x}{\sin 6x}=\lim_{x\to 0}\frac{\sin 5x}{\sin 6x}\left(\frac{x}{x}\frac{5}{5}\frac{6}{6}\right)=\frac{5}{6}\lim_{x\to 0}\frac{\frac{\sin 5x}{5x}}{\frac{\sin 6x}{6x}}=\frac{5}{6}$
3. $\lim_{x\to 0}x\cot x=\lim_{x\to 0}\frac{x\cos x}{\sin x}=\lim_{x\to 0}\frac{\cos x}{\frac{\sin x}{x}}=\frac{1}{1}=1$
4. $\lim_{x\to 0}\frac{\cos x-1}{\sin x}=\lim_{x\to 0}\left(\frac{\cos x-1}{\sin x}\right)\left(\frac{\cos x+1}{\cos x+1}\right)=\lim_{x\to 0}\frac{\cos^2 x-1}{\sin x(\cos x+1)}=\lim_{x\to 0}\frac{-\sin^2 x}{\sin x(\cos x+1)}=$
   $\lim_{x\to 0}\frac{-\sin x}{(\cos x+1)}=\frac{0}{1+1}=0$
5. $\lim_{x\to \pi}\frac{\sin 2x}{\sin x}=\lim_{x\to \pi}\frac{2\sin x\cos x}{\sin x}=\lim_{x\to \pi}2\cos x=2(-1)=-2$
6. $\lim_{x\to 0}\frac{2x-\sin x}{x}=\lim_{x\to 0}\left(2-\frac{\sin x}{x}\right)=\lim_{x\to 0}2-\lim_{x\to 0}\frac{\sin x}{x}=2-1=1$
7. $\lim_{x\to \pi}\frac{\sin\left(x+\frac{\pi}{4}\right)-1}{x-\frac{\pi}{4}}=$ Let $y=x-\frac{\pi}{4}$ $\quad as \quad x\to\frac{\pi}{4}, \quad y\to 0$

$$\lim_{y\to 0}\frac{\sin\left(y+\frac{\pi}{4}+\frac{\pi}{4}\right)-1}{y}=\lim_{y\to 0}\frac{\sin\left(y+\frac{\pi}{2}\right)-1}{y} \qquad \sin\left(\frac{\pi}{2}-y\right)=\cos y \qquad \sin\left(\frac{\pi}{2}-(-y)\right)=\cos(-y)=\cos y \quad even$$

$$\lim_{y\to 0}\frac{\sin\left(y+\frac{\pi}{2}\right)-1}{y}=\lim_{y\to 0}\frac{\cos y-1}{y}=0$$

3.4 The Chain Rule

The Goal:

After completing this section, you will be able to

1. Introduce the chain rule
2. Introduce the power rule combined with chain rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F(x)$.

Observe that F is a composite function. In fact, if we let

$$y = f(x) = \sqrt{u} \quad and \quad g(x) = x^2 + 1$$

then we can write $y = F(x) = f\big(g(x)\big)$ *that is* $F = fog$.

We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = fog$ in terms of the derivatives of f and g.

It turns out that the derivative of the composite function fog is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the ***Chain Rule***

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Proof

Note first that if Δx is the change in x then $\Delta u = g(x + \Delta x) - g(x)$ and $\Delta y = f(u + \Delta u) - f(u)$. Furthermore, when $\Delta x \to 0$ then $\Delta u \to 0$. Now

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\frac{\Delta u}{\Delta u} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{dy}{du}\frac{du}{dx}$$

Example. Find $F'(x)$ if $y = F(x) = (x^3 + 1)^{100}$.

Solution:
Let $= x^3 + 1$, than $y = F(u) = u^{100}$

$$\frac{dy}{du} = 100\, u^{99} \quad and \quad \frac{du}{dx} = 3x^2$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 100\, u^{99}(3x^2) = 100\, (x^3 + 1)^{99}(3x^2) = 300x^2(x^3 + 1)^{99}$$

Remark: *Do not forget to substitute back and have the answer in terms of as given in the problem.*

Example . Find $\frac{dy}{dx}$ (write the answer in the variable *x*).

1. $y = u^2 + \sqrt{u} \quad and \quad u = x + 2$
2. $y = \frac{1}{1+u} \quad and \quad u = x^2 + x - 1$

Solution:

1. $\frac{dy}{du} = 2u - \frac{1}{2\sqrt{u}} \quad and \quad \frac{du}{dx} = 1$

   Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u - \frac{1}{2\sqrt{u}} = 2(x + 2) - \frac{1}{2\sqrt{x + 2}}$$

2. $\frac{dy}{du} = \frac{-1}{(1+u)^2} \quad and \quad \frac{du}{dx} = 2x + 1$

   Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{-(2x + 1)}{(x^2 + x)^2}$$

Example. Find the derivative for the following functions

1. $y = \tan 4x$
2. $y = \tan(\sin x)$
3. $y = \sqrt[3]{1 + \tan x}$

Solution:

1. Let $u = 4x$, than $y = \tan u$

$$\frac{dy}{du} = \sec^2 u \quad and \quad \frac{du}{dx} = 4$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4\sec^2 u = 4\sec^2 4x$$

2. Let $u = \sin x$, than $y = \tan u$

$$\frac{dy}{du} = \sec^2 u \quad and \quad \frac{du}{dx} = \cos x$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec^2 u)(\cos x) = (\sec^2(\sin x))(\cos x)$$

3. Let $u = 1 + \tan x$ than $y = \sqrt[3]{u} = u^{\frac{1}{3}}$

$$\frac{dy}{du} = \frac{1}{3}u^{-\frac{2}{3}} \quad and \quad \frac{du}{dx} = \sec^2 x$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{3}u^{-\frac{2}{3}}\sec^2 x = \frac{1}{3\sqrt[3]{u^2}}\sec^2 x = \frac{\sec^2 x}{3\sqrt[3]{(1 + \tan x)^2}}$$

$\boxed{4}$ **The Power Rule Combined with the Chain Rule** If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively, $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example 3. Find $\frac{dy}{dx}$ (write the answer in the variable x).

1. $f(x) = (1-3x^3)^{\frac{2}{3}}$
2. $g(x) = (x^2-1)^3(2x+3)^5$
3. $h(x) = \frac{1}{\sqrt{1-x+x^3}}$
4. $f(x) = \sqrt{\frac{1+x}{x}}$
5. Find the derivative at $x = 0$ of $y = \left(\frac{1-x^2}{1+x^2}\right)^{-3}$

Solution:

1. $f'(x) = \frac{2}{3}(1-3x^3)^{\frac{-1}{3}}(-6x^2) = -\frac{4x^2}{\sqrt[3]{1-3x^3}}$
2. $g'(x) = (x^2-1)^3[5(2x+3)^5(2)] + (2x+3)^5[3(x^2-1)^2(2x)]$
3. $h'(x) = \frac{-\frac{1}{3}(1-x+x^3)^{-\frac{2}{3}}(-1+3x^2)}{\left(\sqrt{1-x+x^3}\right)^2}$
4. $f'(x) = \frac{1}{2}\left(\frac{1+x}{x}\right)^{-\frac{1}{2}}\left[\frac{x-(1+x)}{x^2}\right]$
5. $y' = -3\left(\frac{1-x^2}{1+x^2}\right)^{-4}\left[\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}\right]$

$$y'|_{x=0} = 0$$

Remark. We may extend the chain rule for three or more functions. For example, if y is a differentiable function of u, u is a differentiable function of x, and x is a differentiable function of t then y is a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dt}$$

Example. Let $\boldsymbol{y = \frac{1}{1+u}}$, $\boldsymbol{u = \sqrt{x-2}}$, $\boldsymbol{x = t^2 - 2t + 3}$, find $\left.\frac{dy}{dt}\right|_{t=2}$

$$\frac{dy}{dt} = \frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dt} = -\frac{1}{(1+u)^2}\cdot\frac{1}{2}(x-2)^{-\frac{1}{2}}(2t-2)$$

$$= -\frac{1}{2}\frac{1}{\left(1+\sqrt{x-2}\right)^2}(x-2)^{-\frac{1}{2}}(2t-2)$$

$$= -\frac{1}{2}\frac{1}{\left(1+\sqrt{\boldsymbol{t^2-2t+1}}\right)^2}(\boldsymbol{t^2-2t+1})^{-\frac{1}{2}}(2t-2)$$

$$= \frac{-(2t-2)}{2\left(1+\sqrt{\boldsymbol{t^2-2t+1}}\right)^2\sqrt{\boldsymbol{t^2-2t+1}}}$$

$$\left.\frac{dy}{dt}\right|_{t=1} = \frac{-(4-2)}{2\left(1+\sqrt{4-\boldsymbol{4}+\boldsymbol{1}}\right)^2\sqrt{4-\boldsymbol{4}+\boldsymbol{1}}} = -\frac{2}{2(4)} = -\frac{1}{4}$$

Example. Find the slope of the line tangent to the curve of $f(x) = \left(x+\sqrt[3]{x}\right)^4$ at $x = 1$

Solution:
The slope of the tangent line

$$f'(x) = 4\left(x+\sqrt[3]{x}\right)^3\left[1+\frac{1}{3}x^{-\frac{2}{3}}\right]$$

$$m = f'(1) = 4\left(1+\sqrt[3]{1}\right)^3\left[1+\frac{1}{3}(1)^{-\frac{2}{3}}\right] = 32\left[\frac{4}{3}\right] = \frac{128}{3}$$

The equation given by

$$y - y_0 = 2(x - x_0)$$

To find y_0, when $x = 1$ $\quad y_0 = 16$

$$y - 16 = \frac{128}{3}(x-1)$$

Remark.

1. For any trigonometric function, say $y = \sin\big(g(x)\big)$, then

$$y' = \cos\big(g(x)\big)\ g'(x)$$

2. For any exponential function, say $y = e^{g(x)}$, where $g(x)$ is differentiable, then

$$y' = e^{g(x)}.g'(x)$$

3. We can use the Chain Rule to differentiate an exponential function with any base $a > 0$

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

Example. Differentiate the following

1. $y = e^{\sec(4x)}$
2. $y = e^{-5x\cos(3x)}$
3. $y = \tan^2(3x)$
4. $y = \sin^2(\cos^3(x^5))$
5. $y = e^{\sin(x)} + \cos(e^x)$
6. $y = 10^x$
7. $y = 3^{(1-x^2)}$

Solution

1. $y' = e^{\sec(4x)}(\sec(4x)\tan(4x)\,4) = 4e^{\sec(4x)}\sec(4x)\tan(4x)$
2. $y' = e^{-5x\cos(3x)}(5x\sin(3x)(3) - 5\cos(3x)) = e^{-5x\cos(3x)}(15x\sin(3x) - 5\cos(3x))$
3. $y' = 2\tan(3x)\sec^2(3x)\,(3) = 6\tan(3x)\sec^2(3x)$
4. $y' = 2\sin(\cos^3(x^5))\cos(\cos^3(x^5))\,[-3\cos^2(x^5)\sin(x^5)\,(5x^4)]$
5. $y' = \cos(x)\,e^{\sin(x)} - e^x\sin(e^x)$
6. $y' = 10^x \ln 10$
7. Let $u = 1 - x^2$, $y = 3^u$, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3^u \ln 3\ (-2x) = (-2x)3^{1-x^2}\ln 3$

Example. Find $f'(2)$, where

1. $f(x) = \frac{x}{\sqrt{1+x^3}} + e^{-3}$
2. $f(x) = (x+1)\sqrt{x^3 + x - 1}$

Example.

If $F(x) = f(g(x))$ and $g(3) = 6,\ \ g'(3) = 4,\ \ \ f'(3) = 2,\ \ \ f'(6) = 7$, find $F'(3)$

Solution:

$$F'(x) = f'(g(x))g'(x)$$

$$F'(3) = f'(g(3))g'(3) = f'(6)g'(3) = 7(4) = 28$$

Example. If $F(x) = f\left(3f(4f(x))\right)$ where $f(0) = 0$ *and* $f'(0) = 2$, find $F'(0)$

Solution

$$F'(x) = \frac{d}{dx} f\left(3f(4f(x))\right) . \frac{d}{dx}\left(3f(4f(x))\right) . \frac{d}{dx}\left(4f(x)\right)$$

$$= f'\left(3f(4f(x))\right) . 3f'(4f(x)). 4f'(x)$$

$$F'(0) = f'\left(3f(4f(0))\right) . 3f'(4f(0)). 4f'(0)$$
$$= f'(0). 3f'(0). 4f'(0) = 2.3.2.4.2 = 96$$

3.5 The Implicit Differentiation

The Goal:

After completing this section, you will be able to

1. To explain the method implicit differentiation.
2. To differentiate the inverse trigonometric functions.
3. To consider the orthogonal Trajectories

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example

$$y = \sqrt{x^2+1}, \quad or \quad y = x\sin x$$

Or, in general

$$y = f(x)$$

Some functions, however, are defined implicitly by relation between x & y such as

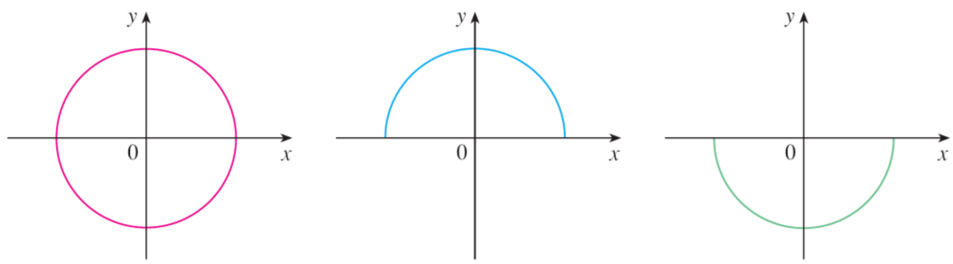
$$y^2 + x^2 = 25 \quad or \quad y^3 + x^3 = 6xy$$

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x.

The equation $y^2 + x^2 = 25$gives a circle which is not a function, we may solve for y to get

$$y = \pm\sqrt{25 - x^2}$$

The first part $y = +\sqrt{25 - x^2}$ represents the upper semicircle while the second part $y = -\sqrt{25 - x^2}$



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

To find the slope of the tangent line at the point $(1,1)$ we use the first function since the point on the upper semicircle, that is $y'|_{x=1} = \frac{1}{2\sqrt{2-x^2}}(-2x) = \frac{-x}{\sqrt{2-x^2}} = -1$

Sometimes it is hard or even impossible to solve an equation for y as we did above and we need to find y', so what we do?

Implicit Differentiation

To find $\frac{dy}{dx}$ from an equation in x & y, we differentiate both sides with respect to x and treat y as a function of x. This type of differentiation is called ***implicit differentiation***.

Example. Find $\frac{dy}{dx}$

1. $x^2 + y^2 = 1$
2. $x^2y^2 + 2y + 4x = 0$
3. $y^5 + x^2y^3 = 1 + ye^{x^2}$
4. $\sin(x^2y) + \cos(x) = 1$

Solution

1. $2x + 2y\frac{dy}{dx} = 0 \quad \rightarrow \frac{dy}{dx} = -\frac{2x}{2y}$
2. $x^2\, 2y\frac{dy}{dy} + 2xy^2 + 2\frac{dy}{dx} + 4 = 0 \quad \rightarrow \quad x^2\, 2y\frac{dy}{dy} + 2\frac{dy}{dx} = -4 - 2xy^2$
$$\frac{dy}{dx} = \frac{-4 - 2xy^2}{x^2\, 2y + 2}$$
3. $5y^4\frac{dy}{dx} + x^2 3y^2\frac{dy}{dx} + y^3 2x = ye^{x^2}2x + e^{x^2}\frac{dy}{dx}$
$$\frac{dy}{dx} = \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y^2 - e^{x^2}}$$
4. $\cos(x^2y)\left(x^2\frac{dy}{dx} + 2xy\right) - \sin(x) = 0 \quad \rightarrow \quad \frac{dy}{dx} = \frac{\sin(x) - 2xy\cos(x^2y)}{x^2\cos(x^2y)}$

Example.

a) Find the equation of the line tangent to the curve $y^2 = x^3 + 3x^2$ at $P(1, -2)$

b) At what points does this curve have a horizontal asymptotes

Solution:

a) Since the point on the curve, the slope equal to $m = \frac{dy}{dx}$

$$2y\frac{dy}{dx} = 3x^2 + 6x \qquad \rightarrow \quad \frac{dy}{dx} = \frac{3x^2 + 6x}{2y}$$

$$m = \left.\frac{dy}{dx}\right|_{(1,-2)} = \frac{3+6}{-4} = -\frac{9}{4}$$

The equation of the tangent line

$$y + 2 = -\frac{9}{4}(x - 1)$$

b) The curve has a horizontal asymptotes when the slope equal to 0

$$m = 0 = \frac{3x^2 + 6x}{2y} \qquad y \neq 0$$

$$3x^2 + 6x = 0$$

$$x(3x + 6) = 0$$

$$x = 0 \; and \; x = -2$$

When $x = 0, \quad y = 0$, so the point (0,0) but $y \neq 0$ rejected

When $x = -2, \quad y = \pm 2$, so the points $(-2, -2)$ & $(-2, 2)$

Example.

Find the points where the tangent line is horizontal for the curve $y^3 + x^3 = 3xy$

The tangent line is horizontal when $\frac{dy}{dx} = 0$

$$3y^2\frac{dy}{dx} + 3x^2 = 3x\frac{dy}{dx} + 3y \qquad \rightarrow \quad \frac{dy}{dx} = \frac{y - x^2}{y^2 - x} = 0 \qquad \rightarrow y = x^2$$

Substitute in the original equation, we get $x^6 - 2x^3 = 0 \qquad \rightarrow x = 0 \;\; or \; x = \sqrt[3]{2}$

Example. Find the equation of the line(s) tangent to the circle $(x-4)^2+y^2=4$ and passes through the origin.

Since the origin not on the circle, let the point of contact be when $x=a$, the cprresponding $y-value$ is

$$(a-4)^2+y^2=4$$

$$y=\pm\sqrt{4-(a-4)^2}$$

So the tangent passes through $\left(a,+\sqrt{4-(a-4)^2}\right)$ & $\left(a,-\sqrt{4-(a-4)^2}\right)$

The slope of the tangent line at the first point is $m=\frac{dy}{dx}$

$$2(x-4)+2y\frac{dy}{dx}=0 \qquad \rightarrow \; m=\frac{dy}{dx}=\frac{-(x-4)}{y}=\pm\frac{-(a-4)}{\sqrt{4-(a-4)^2}}$$

The equation of the tangent line is

$$y-\sqrt{4-(a-4)^2}=\pm\frac{-(a-4)}{\sqrt{4-(a-4)^2}}(x-a)$$

But since the tangent has to pass through the origin we must have

$$0-\sqrt{4-(a-4)^2}=\frac{-(a-4)}{\sqrt{4-(a-4)^2}}(0-a)$$

$$\sqrt{4-(a-4)^2}=\frac{a(a-4)}{\sqrt{4-(a-4)^2}}$$

$$-4+(a-4)^2=a(a-4)$$

$$-4+a^2-8a+16=a^2-4a \qquad a=3$$

The slope

$$m=\frac{dy}{dx}=\pm\frac{-(3-4)}{\sqrt{4-(3-4)^2}}=\pm\frac{1}{\sqrt{3}}$$

The equation of line passes through the origin given by

$$y=\pm\frac{x}{\sqrt{3}}$$

Orthogonal Trajectories.

Two curves are *orthogonal*, if at each point of intersection their tangent lines are perpendicular.

Two families of curves are said to be orthogonal trajectories of each other if each curve in the one family is the orthogonal to every curve in the other family.

Example. Show that the given families of curves are orthogonal trajectories of each other

$$xy = a \quad and \quad x^2 - y^2 = b$$

1. The slope of the first family at the point $(x, y)$is

$$x\frac{dy}{dx} + y = 0 \quad \rightarrow \quad m_1 = \frac{dy}{dx} = -\frac{y}{x}$$

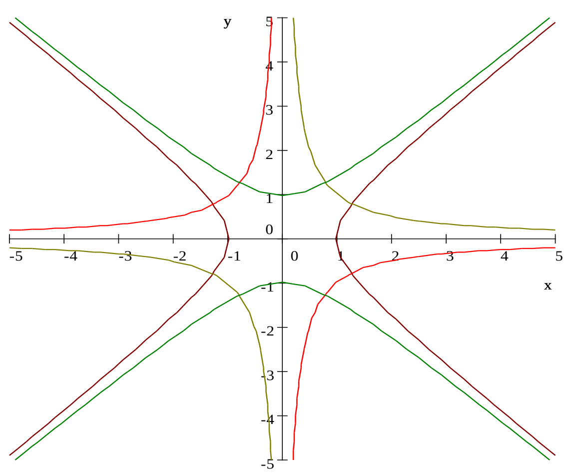
2. The slope of the first family at the point $(x, y)$is

$$2x - 2y\frac{dy}{dx} = 0 \quad \rightarrow \quad m_2 = \frac{dy}{dx} = \frac{x}{y}$$

3. The product of the slops

$$m_1\, m_2 = -\frac{y}{x}\frac{x}{y} = -1$$

4. It follows that the two families are orthogonal trajectory of each other



Example. Show that the given families of curves are orthogonal trajectories of each other

$$y = ax^3 \quad and \quad x^2 + 3y^2 = b$$

1. The slope of the first family at the point $(x, y)$is

$$\frac{dy}{dx} = 3ax^2 \quad \rightarrow \quad m_1 = \frac{dy}{dx} = 3ax^2$$

2. The slope of the first family at the point $(x, y)$is

$$2x + 6y\frac{dy}{dx} = 0 \quad \rightarrow \quad m_2 = \frac{dy}{dx} = \frac{-x}{3y}$$

3. The product of the slops

$$m_1\, m_2 = (3ax^2)\left(\frac{-x}{3y}\right)$$

But at the point of intersection $y = ax^3$

$$m_1\, m_2 = (3ax^2)\left(\frac{-x}{3y}\right) = (3ax^2)\left(\frac{-x}{3ax^3}\right) = -1 \quad where\ a \neq 0$$

4. It follows that the two families orthogonal trajectory of each other

Example. Show that the given families of curves are orthogonal trajectories of each other

$$x^2 - y^2 = 5 \quad and \quad 4x^2 + 9y^2 = 72$$

1. The slope of the first family at the point $(x, y)$is

$$2x - 2y\frac{dy}{dx} = 0 \quad \rightarrow \quad m_1 = \frac{dy}{dx} = \frac{x}{y}$$

2. The slope of the first family at the point $(x, y)$is

$$8x + 18y\frac{dy}{dx} = 0 \quad \rightarrow \quad m_2 = \frac{dy}{dx} = \frac{-4x}{9y}$$

3. The product of the slops

$$m_1\, m_2 = \left(\frac{x}{y}\right)\left(\frac{-4x}{9y}\right)$$

But at the point of intersection

$$y^2 = x^2 - 5 \quad and \quad y^2 = \frac{72 - 4x^2}{9}$$

$$x^2 - 5 = \frac{72 - 4x^2}{9} \quad \rightarrow \quad 9x^2 - 45 = 72 - 4x^2 \quad \rightarrow \quad x^2 = \frac{117}{13} = 9 \quad \rightarrow \quad x = \pm 3$$

When $x = -3,\ \ y = \pm 2 \qquad \rightarrow \qquad (-3, -2) \quad \& \, (-3, 2)$

When $x = 3,\ \ y = \pm 2 \qquad \rightarrow \qquad (3, -2) \quad \& \, (3, 2)$

At the point $(-3, -2)$, $m_1 = \frac{x}{y} = \frac{3}{2} \quad and \quad m_2 = \frac{-4x}{9y} = -\frac{2}{3} \quad then\ m_1\, m_2 = -1$

At the point $(-3, 2)$, $m_1 = \frac{x}{y} = \frac{-3}{2} \quad and \quad m_2 = \frac{-4x}{9y} = \frac{2}{3} \quad then\ m_1\, m_2 = -1$

At the point $(3, -2)$, $m_1 = \frac{x}{y} = \frac{-3}{2} \quad and \quad m_2 = \frac{-4x}{9y} = \frac{2}{3} \quad then\ m_1\, m_2 = -1$

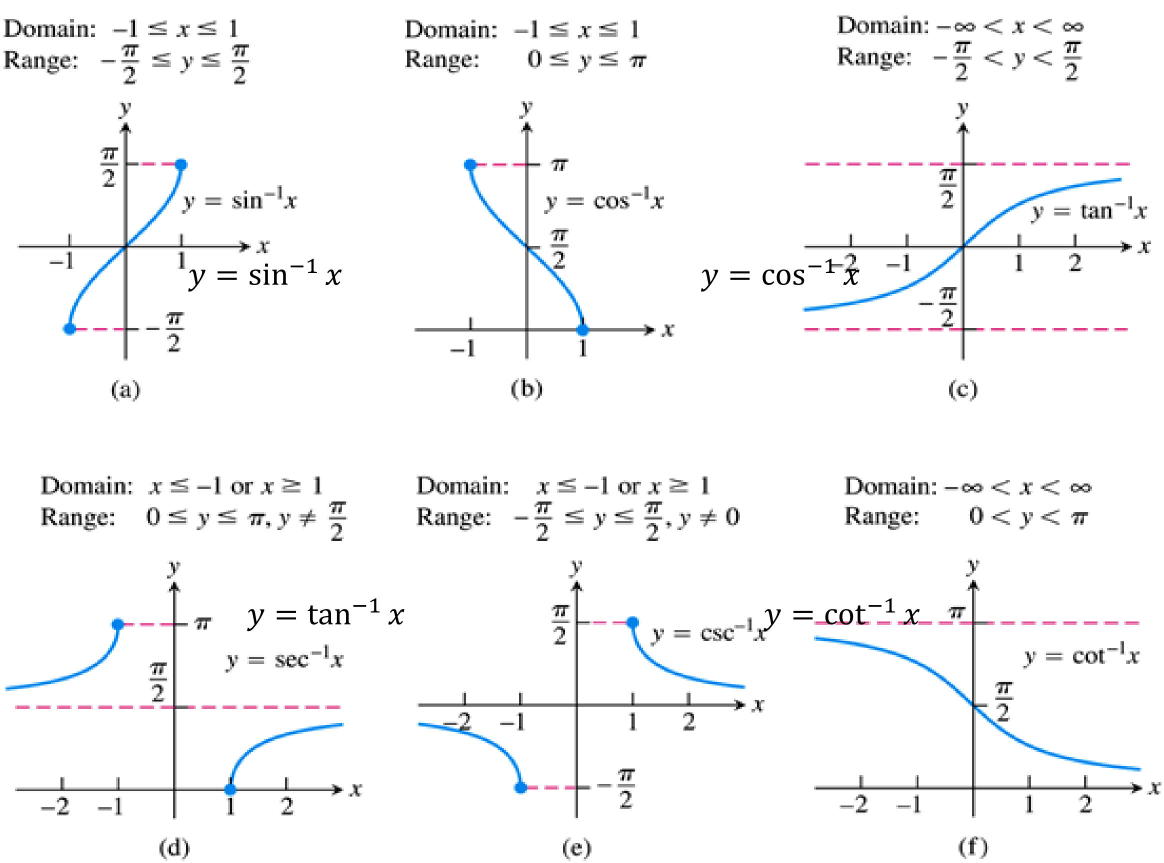
At the point $(-3, -2)$, $m_1 = \frac{x}{y} = \frac{3}{2} \quad and \quad m_2 = \frac{-4x}{9y} = -\frac{2}{3} \quad then\ m_1\, m_2 = -1$

4. It follows that the two families orthogonal trajectory of each other

Derivatives of Inverse Trigonometric Functions

Let us review the some topics regarding the inverse trigonometric functions.

Function	Inverse Function	Domain of Inv.	Range
1. $y=\sin x$	$y=\sin^{-1} x$	$-1\le x\le 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$
2. $y=\cos x$	$y=\cos^{-1} x$	$-1\le x\le 1$	$0\le y\le\pi$
3. $y=\tan x$	$y=\tan^{-1} x$	$(-\infty,\infty)$	$-\frac{\pi}{2}< y<\frac{\pi}{2}$
4. $y=\cot x$	$y=\cot^{-1} x$	$(-\infty,\infty)$	$0< y<\pi$
5. $y=\sec x$	$y=\sec^{-1} x$	$\lvert x\rvert\ge 1$	$0\le y<\frac{\pi}{2};\ \pi\le y<\frac{3\pi}{2}$
6. $y=\csc x$	$y=\csc^{-1} x$	$\lvert x\rvert\ge 1$	$0< y\le\frac{\pi}{2};\ \pi< y\le\frac{3\pi}{2}$



Graphs of the six basic inverse trigonometric functions.

Remarks

1. $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
2. $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$.
3. $\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$.
4. $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$.

Remarks

1. $\sin^{-1}(-x) = -\sin^{-1} x$
2. $\cos^{-1}(-x) = \pi - \cos^{-1} x$
3. $\tan^{-1}(-x) = -\tan^{-1} x$
4. $\cot^{-1}(-x) = \pi - \cot^{-1} x$

Now we can use implicit differentiation to find the derivatives of the inverse trigonometric functions.

Theorem

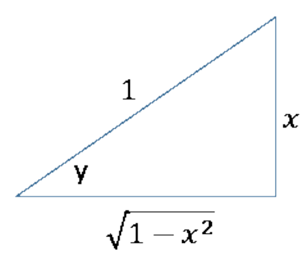
1. $D_x\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$
2. $D_x\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}}$
3. $D_x\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$
4. $D_x\left(\cot^{-1} x\right) = -\dfrac{1}{1+x^2}$
5. $D_x\left(\sec^{-1} x\right) = \dfrac{1}{x\sqrt{x^2-1}}$
6. $D_x\left(\csc^{-1} x\right) = -\dfrac{1}{x\sqrt{x^2-1}}$

Proof (1).

Let $y = \sin^{-1} x$, we need to find $\frac{dy}{dx}$

1. Write $x = \sin y$
2. Differentiate implicitly

$$1 = \cos y \frac{dy}{dx} \quad \rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$



And similarly for the others

Theorem Using the chain rule, if u is a functions of x, then

1. $D_x\left(\sin^{-1} u\right) = \dfrac{1}{\sqrt{1-u^2}} D_x u$
2. $D_x\left(\cos^{-1} u\right) = -\dfrac{1}{\sqrt{1-u^2}} D_x u$
3. $D_x\left(\tan^{-1} u\right) = \dfrac{1}{1+u^2} D_x u$
4. $D_x\left(\cot^{-1} u\right) = -\dfrac{1}{1+u^2} D_x u$
5. $D_x\left(\sec^{-1} u\right) = \dfrac{1}{u\sqrt{u^2-1}} D_x u$
6. $D_x\left(\csc^{-1} u\right) = -\dfrac{1}{u\sqrt{u^2-1}} D_x u$

Example. Differentiate

1. $y = \frac{1}{\sin^{-1} x}$
2. $y = x \arctan \sqrt{x}$
3. $y = \sin^{-1} e^x$
4. $y = \tan^{-1}(\sin x)$
5. $y = e^{\cos^{-1} x}$
6. $y = \sqrt{\tan^{-1} x}$
7. $f(x) = \sqrt{1 - x^2} \sin^{-1} x$

Solution:

1. $y' = \frac{\frac{1}{\sqrt{1-x^2}}}{(\sin^{-1} x)^2} = \frac{1}{(\sin^{-1} x)^2 \sqrt{1-x^2}}$
2. $y' = x\left(\frac{1}{1+\sqrt{x}^2}\right)\left(\frac{1}{2\sqrt{x}}\right) + \arctan \sqrt{x} = \frac{\sqrt{x}}{2(1+x)} + + \arctan \sqrt{x}$
3. $y' = \frac{1}{\sqrt{1+(e^x)^2}} e^x = \frac{e^x}{\sqrt{1+e^{2x}}}$
4. $y' = \frac{1}{1+(\sin x)^2} cos\, x$
5. $y' = -\frac{1}{\sqrt{1-x^2}} e^{\cos^{-1} x}$
6. $y' = \frac{1}{2}(\tan^{-1} x)^{-\frac{1}{2}}\left(\frac{1}{1+x^2}\right) = \frac{1}{2(1+x^2)\sqrt{\tan^{-1} x}}$
7. $f'(x) = \sqrt{1 - x^2}\left(\frac{1}{\sqrt{1-x^2}}\right) + \sin^{-1} x \left(\frac{-2x}{2\sqrt{1-x^2}}\right) = 1 - \frac{x \sin^{-1} x}{\sqrt{1-x^2}}$

Example.

a) Suppose that f is one – one differentiable function and its inverse function f^{-1} is also differentiable. Use the implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Provide that the denominator is not 0

b) If $f(4) = 5$ $f'(4) = \frac{2}{3}$, find $f^{-1\,\prime}(5)$

Solution

a) Let $y = f^{-1}(x)$ $\qquad x = f(y) = f\left((f^{-1})(x)\right)$

But

$$x = f(y) \quad \rightarrow \quad 1 = f'(y)\frac{dy}{dx} \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

b) Since $f(4) = 5 \quad \rightarrow \quad f^{-1}(5) = 4$

$$(f^{-1})'(5) = \frac{1}{f'\left(f^{-1}(5)\right)} = \frac{1}{f'(4)} = \frac{3}{2}$$

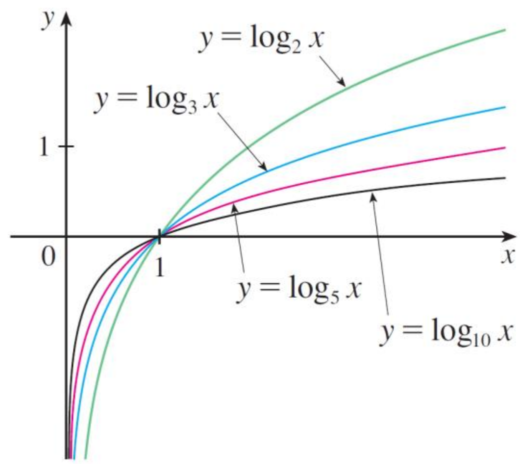
3.6 Derivative of Logarithmic Functions

The Goal:

After completing this section, you will be able to

1. To define the derivative of logarithmic functions
2. To use the logarithmic differentiation method
3. To define e as a limit

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_a x$ and, in particular, the natural logarithmic function $y = \ln x$. [It can be proved that logarithmic functions are differentiable; this is certainly plausible from their graphs



Review of logarithms

Definition. The inverse of the exponential function $y = a^x$ is the logarithmic function, denoted by $y = \log_a x$ with base $a > 0$ & $a \neq 1$. The graph is

Properties of logarithms:

1. $\log_a x\,y = \log_a x + \log_a y$
2. $\text{Log}_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^r = r \log_a x,$ for any positive real number r.

Remark. Two special cases of the logarithms.

1. If the base is $a = e$ we get the ***natural logarithm*** and is denoted by:
$$\ln x = \log_e x$$
2. If the base is $a = 10$, we get the ***common logarithm*** and is denoted by
$$\log x = \log_{10} x$$

Changing the base:

for any positive number b different from 1, we have

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular,

$$\log_a x = \frac{\ln x}{\ln a}$$

Special logarithms:

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\ln e = 1$

Note:

1. $y = a^x$ is equivalent to $x = \log_a y$ for $y > 0$ and all x similarly
2. $y = e^x$ is equivalent to $x = \ln y$

Derivative of logarithmic function

Theorem:

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Proof:

Since $y = \ln x$ is equivalent to $e^y = x$. Differentiate implicitly

$$y'e^y = 1$$

$$y' = \frac{1}{e^y} = \frac{1}{x}$$

Note also that if a is a positive real number not equal to 1, we have $y = \log_a x = \frac{\ln x}{\ln a}$, so

Theorem:

Let $y = \log_a x$, then

$$y' = \frac{1}{x \ln a}$$

Theorem:
if $u > 0$ is a differentiable function of x, we can use the chain rule to show that:

1. $\frac{d}{dx}(\ln u) = \frac{1}{u}\ \frac{du}{dx}$ $\quad or \quad$ $\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$
2. $\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a}\ \frac{du}{dx}$ $\quad or \quad$ $\frac{d}{dx}(\log_a g(x)) = \frac{g'(x)}{g(x)\ln a}$

Examples. Find y' for each of the following.

1. $y = \log_3(x^2 - 1)$
2. $y = \log_2(\sin x + \cos x)$
3. $y = \sqrt{\log_3(x^2 + 2x + 2)}$
4. $y = \ln(1 + \tan x)$
5. $y = x \ln(1 + e^x)$
6. $y = \frac{\ln x}{x}$
7. $y = \ln^3 x^2$
8. $y = \ln|x|$
9. $y = \ln|x^2 - 5|$
10. $y = \ln \ln x$

Solution

1. $y' = \frac{1}{\ln 3}\frac{2x}{x^2-1} = \frac{2x}{\ln 3(x^2-1)}$
2. $y' = \frac{1}{\ln 2}\frac{\cos x - \sin x}{\sin x + \cos x}$
3. $y' = \frac{1}{2\ln 3}\ \frac{2x+2}{(x^2+2x+2)}$
4. $y' = \frac{\sec^2 x}{1+\tan x}$
5. $y' = \frac{xe^x}{1+e^x} + \ln(1 + e^x)$
6. $y' = \frac{x\frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$
7. $y' = 3(\ln^2 x^2)\frac{2x}{x^2} = 6x\ \ln^2 x^2$
8. Let $f(x) = \ln|x| = \begin{cases} \ln x & x > 0 \\ \ln -x & x < 0 \end{cases}$

   $y' = \begin{cases} 1/x & x > 0 \\ -1/-x & x < 0 \end{cases}$ $\quad y' = \frac{1}{x}, x \neq 0$
9. $y' = \frac{2x}{x^2-5}$
10. $y' = \frac{1}{\ln x}\frac{1}{x} = \frac{1}{x\ \ln x}$

Example: Find the equation of the line tangent to the curve $y = e^x$ and passes through the origin. (Hint: you need to find the point of tangency).

Since the origin not on the curve, let the point of contact be when $x = a$, the cprresponding $y - value$ is

$$y = e^a$$

So the tangent passes through (a, e^a)

The slope of the tangent line at the first point is $m = \frac{dy}{dx}$

$$\left.\frac{dy}{dx}\right|_{x=a} = e^x = e^a$$

The equation of the tangent line is

$$y - e^a = e^x(x - a)$$

But since the tangent has to pass through the origin we must have

$$0 - e^a = e^a(0 - a)$$

$$-e^a = -ae^a \qquad \rightarrow \quad a = 1$$

The slope

$$m = \frac{dy}{dx} = e^a = e^1 = e$$

The equation of line passes through the origin given by

$$y = ex$$

Remark.

Since $y = f^{-1}(x) = a^x$ is the inverse of $y = f(x) = \log_a x$, we have

$$\frac{d}{dx}(a^x) = \frac{1}{(\log_a f^{-1}(x))'} = \frac{1}{\frac{1}{f^{-1}(x)\ln a}} = f^{-1}(x)\ln a = a^x \ln a$$

Note that if $a = e$ in the above formula we get:

$$\frac{d}{dx}(e^x) = e^x \ln e = e^x$$

Theorem:

If $u > 0$ is a differentiable function of x, we can use the chain rule to show that:

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$$

Example: Find y':

1. $y = a^{2x^3+5x}$
2. $y = 3^{\sin x}$

Solution:

1. $y' = a^{2x^3+5x} \ln a \ (6x+5)$
2. $y' = 3^{\sin x} \ln 3 \ (\cos x)$

Sometimes it is easier to use the properties of the logarithm to simplify the expression before we differentiae.

Examples: Find $y'$for each of the following

1. $y = \ln(x+1)^3(x-1)^4$
2. $y = \ln\sqrt[3]{\frac{x+5}{x^2+x+1}}$

Solution:

1. $y = \ln(x+1)^3(x-1)^4 = \ln(x+1)^3 + \ln(x-1)^4 = 3\ln(x+1) + 4\ln(x-1)$

$$y' = \frac{3}{x+1} + \frac{4}{x-1}$$

2. $y = \ln\sqrt[3]{\frac{x+5}{x^2+x+1}} = \ln\left(\frac{x+5}{x^2+x+1}\right)^{\frac{1}{3}} = \frac{1}{3}\ln\left(\frac{x+5}{x^2+x+1}\right) = \frac{1}{3}[\ln(x+5) - \ln(x^2+x+1)]$

$$y' = \frac{1}{3}\left[\frac{1}{x+5} - \frac{2x+1}{x^2+x+1}\right]$$

Logarithmic differentiation:

Derivatives of expressions involving products, quotients and powers may be simplified by taking the logarithm first and then differentiating. This method is called ***logarithmic differentiation***.

To differentiate $y = f(x)$ using logarithmic differentiation we do the following:

Step 1: take the ln of both sides.
Step 2: use properties of the ln to simplify the expression.
Step 3: differentiate both sides implicitly with respect to x.
Step 4: solve for y'.

Note: To differentiate $y = [f(x)]^{g(x)}$ using logarithmic differentiation, we use the same steps as above.

Example. If $y = \frac{(x+1)^2(x^2-x-1)^5}{\sqrt[3]{x+x^4}}$ find y'

Solution.

1. Take the *ln* of both sides

$$\ln y = \ln \frac{(x+1)^2(x^2-x-1)^5}{\sqrt[3]{x+x^4}}$$

2. Now we use the properties of the ln to simplify the expression

$$\ln y = \ln \frac{(x+1)^2(x^2-x-1)^5}{\sqrt[3]{x+x^4}} = \ln(x+1)^2 + \ln(x^2-x-1)^5 - \ln \sqrt[3]{x+x^4}$$
$$\ln y = 2\ln(x+1) + 5\ln(x^2-x-1) - \frac{1}{3}\ln(x+x^4)$$

3. Now we differentiate both sides.

$$\frac{y'}{y} = \frac{2}{x+1} + \frac{5(2x-1)}{x^2-x-1} - \frac{(1+4x^3)}{3(x+x^4)}$$

4. Solving for y':

$$y' = y\left(\frac{2}{x+1} + \frac{5(2x-1)}{x^2-x-1} - \frac{(1+4x^3)}{3(x+x^4)}\right)$$
$$= \left(\frac{(x+1)^2(x^2-x-1)^5}{\sqrt[3]{x+x^4}}\right)\left(\frac{2}{x+1} + \frac{5(2x-1)}{x^2-x-1} - \frac{(1+4x^3)}{3(x+x^4)}\right)$$

Example. Use log differentiation to find y'

1. $y = x^x$
2. $y = x^{\ln x}$
3. $y = (1 + e^x)^x$
4. $y = (1 + \ln x)^x$
5. $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$

Solution:

1. $\ln y = \ln x^x = x \ln x$

$$\frac{y'}{y} = x\frac{1}{x} + \ln x = 1 + \ln x$$

$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

2. $\ln y = \ln x^{\ln x} = \ln x \ln x = (\ln x)^2$

$$\frac{y'}{y} = \frac{2}{x}\ln x \qquad \rightarrow \qquad y' = y\left(\frac{2}{x}\ln x\right) = x^{\ln x}\left(\frac{2}{x}\ln x\right)$$

3. $\ln y = \ln(1 + e^x)^x = x \ln(1 + e^x)$

$$\frac{y'}{y} = x\left(\frac{e^x}{1 + e^x}\right) + \ln(1 + e^x)$$

$$y' = y\left(x\left(\frac{e^x}{1 + e^x}\right) + \ln(1 + e^x)\right) = (1 + e^x)^x\left(x\left(\frac{e^x}{1 + e^x}\right) + \ln(1 + e^x)\right)$$

4. $\ln y = \ln(1 + \ln x)^x = x \ln(1 + \ln x)$

$$\frac{y'}{y} = x\left(\frac{\frac{1}{x}}{1 + \ln x}\right) + \ln(1 + \ln x)$$

$$y' = (1 + \ln x)^x\left(\left(\frac{1}{1 + \ln x}\right) + \ln(1 + \ln x)\right)$$

5. $\ln y = \ln\left(\sqrt{x}\, e^{x^2}(x^2 + 1)^{10}\right) = \ln \sqrt{x} + \ln e^{x^2} + \ln(x^2 + 1)^{10}$

$$\ln y = \frac{1}{2}\ln x + x^2 + 10 \ln(x^2 + 1)$$

$$\frac{y'}{y} = \frac{1}{2x} + 2x + \frac{20x}{x^2 + 1}$$

$$y' = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}\left(\frac{1}{2x} + 2x + \frac{20x}{x^2 + 1}\right)$$

Example. Find the equation of the tangent line to the curve $y = (1 + x)^x$ at $(1, 2)$.

The number e as a limit:

The base of the natural log is the number *e*. and we defined it earlier to be

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

.

To see this, we note that if $f(x) = \ln x$ then $f'(x) = \frac{1}{x}$and so $f'(1) = 1$. But the definition of the derivative implies that

$$1 = f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\ln(1+h) = \lim_{h\to 0}\ln(1+h)^{\frac{1}{h}} = \ln\lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

By the continuity of the exponential function we have

$$e^1 = e^{\ln\lim_{h\to 0}(1+h)^{\frac{1}{h}}} = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

That is

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

This number e can be approximated to as $e \approx 2.7182818$.

Remark: If we put $n = \frac{1}{x}$ in the formula of e we get an alternate definition of e as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

.

Example. Show that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for any $x > 0$

Solution

Let $\frac{1}{m} = \frac{x}{n} \quad \to \quad m = \frac{n}{x} \quad$ *and as* $\quad n \to \infty,\ m \to \infty$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{xm} = \lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^m\right)^x = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x = e^x$$

Note:

1. $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{3n}=e$

2. $\lim_{n\to\infty}\left(1+\frac{1}{5n+4}\right)^{(5n+4)}=e$

3. $\lim_{n\to\infty}\left(1+\frac{1}{2n^2}\right)^{2n^2}=e$

4. $\lim_{n\to\infty}\left(1+\frac{1}{\frac{2n+5}{7}}\right)^{\left(\frac{2n+5}{7}\right)}=e$

Example.
Find the following

1. $\lim_{n\to\infty}\left(1+\frac{1}{4n}\right)^{n}=$
2. $\lim_{n\to\infty}\left(\frac{n+2}{n-3}\right)^{n}=$
3. $\lim_{n\to\infty}\left(\frac{n+2}{n-3}\right)^{\frac{2n-1}{5}}$

Solution:

1. $\lim_{n\to\infty}\left(1+\frac{1}{4n}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{1}{4n}\right)^{n\left(\frac{4}{4}\right)}=\left(\lim_{n\to\infty}\left(1+\frac{1}{4n}\right)^{4n}\right)^{\frac{1}{4}}=e^{\frac{1}{4}}$

2. $\lim_{n\to\infty}\left(\frac{n+2}{n-3}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{5}{n-3}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{\frac{5}{5}}{\frac{n-3}{5}}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{1}{\frac{n-3}{5}}\right)^{n\left(\frac{n-3}{n-5}\right)\frac{5}{5}}$

$$=\left(\lim_{n\to\infty}\left(1+\frac{1}{\frac{n-3}{5}}\right)^{\left(\frac{n-3}{5}\right)}\right)^{\frac{5n}{n-3}}=e^{5}$$

3. $\lim_{n\to\infty}\left(\frac{n+2}{n-3}\right)^{\frac{2n-1}{5}}=\lim_{n\to\infty}\left(1+\frac{5}{n-3}\right)^{\frac{2n-1}{5}}=\lim_{n\to\infty}\left(1+\frac{\frac{5}{5}}{\frac{n-3}{5}}\right)^{\frac{2n-1}{5}}$

$$=\lim_{n\to\infty}\left(1+\frac{1}{\frac{n-3}{5}}\right)^{\frac{2n-1}{5}\left(\frac{n-3}{5}\right)\left(\frac{5}{n-3}\right)}=\left(\lim_{n\to\infty}\left(1+\frac{1}{\frac{n-3}{5}}\right)^{\left(\frac{n-3}{5}\right)}\right)^{\frac{5}{5}\left(\frac{2n-1}{n-3}\right)}=e^{2}$$

Higher Derivatives

We have seen that the derivative of a function $y = f(x)$ is again a function with domain consisting of all points where the function is differentiable. The derivative of the derivative is called the ***second derivative*** of the function. Using Leibniz Notation we can write the second derivative of the function $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

There are many other notations for the second derivative of the function $y = f(x)$. Some of these notations are:

$$y'' = f''(x) = D^2y = D^2f(x)$$

The main benefit of these notation is that they are easy to write and understand, while the first is harder to understand but emphasizes which is the independent and which is the dependent variable.

We may differentiate the second derivative to get the third derivative

$$\frac{d^3y}{dx^3} = y''' = f'''(x) = D^3y = D^3f(x)$$

This process may continue to find the derivative of the derivative. For $n \geq 4$, the n^{th} derivative of $y = f(x)$is denoted by:

$$\frac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x) = D^ny = D^nf(x)$$

Example: Find the first four derivatives and the n^{th} derivative for each of the following functions.

1. $y = f(x) = e^x$
2. $y = f(x) = \frac{1}{x}$

Example: Find $f'''(2)$ when $f(x) = x^4 - 3x^2 + 9$

$$y' = \mathbf{4x^3 - 6x} \qquad y''_{x=2} = \mathbf{12x^2 - 6 = 12(2)^2 - 6 = 42}$$

Example: Find y'' if

1. $f(x) = \frac{2}{3x-1}$
2. $f(x) = x^2 e^x$
3. $f(x) = x \sin x$
4. $f(x) = \tan^{-1} x^2$
5. $x^4 + y^4 = 16$

Solution:

1. $y = f(x) = \frac{2}{3x-1} = 2(3x-1)^{-1}$

$$y' = -6(3x-1)^{-2}$$

$$y'' = -36(3x-1)^{-3}$$

2. $y = f(x) = x^2 e^x$

$$y' = x^2 e^x + 2xe^x = e^x(x^2 + 2x)$$
$$y'' = e^x(2x+2) + e^x(x^2+2x) = e^x(x^2 + 4x + 2)$$

3. $y = f(x) = x \sin x$

$$y' = x \cos x + \sin x$$

$$y'' = -x \sin x + \cos x + \cos x = 2\cos x - x \sin x$$

4. $y = f(x) = \tan^{-1} x^2$

$$y' = \frac{2x}{1 + (x^2)^2} = \frac{2x}{1 + x^4}$$
$$y'' = \frac{2(1+x^4) - 2x(4x^3)}{(1+x^4)^2} = \frac{2 + 2x^4 - 8x^4}{(1+x^4)^2} = \frac{2 - 6x^4}{(1+x^4)^2}$$

5. $x^4 + y^4 = 16$

$$4x^3 + 4y^3 y' = 0$$

$$y' = \frac{-x^3}{y^3}$$

$$y'' = -\left(\frac{y^3 3x^2 - x^3 3y^2 y'}{(y^3)^2}\right) = -\left(\frac{3y^3x^2 - 3x^3y^2\left(\frac{-x^3}{y^3}\right)}{(y^3)^2}\right) = -\left(\frac{3y^3x^2 + 3x^3\left(\frac{x^3}{y}\right)}{(y^3)^2}\right)$$
$$= -\left(\frac{3y^4x^2 + 3x^6}{y^6}\right)\frac{1}{y} = -\left(\frac{3y^4x^2 + 3x^6}{y^7}\right) = -3x^2\left(\frac{y^4 + x^4}{y^7}\right) = -\frac{38x^2}{y^7}$$

Example: Find a formula for $f^{(n)}(x)$ if

1. $f(x) = \ln(x-1)$
2. $f(x) = e^{2x}$
3. $f(x) = \sqrt{x}$

Solution:

1.

$$f'(x) = \frac{1}{x-1} = (x-1)^{-1}$$
$$f''(x) = -1(x-1)^{-2}$$
$$f'''(x) = 2(x-1)^{-3}$$
$$f^{(4)}(x) = -2(3)(x-1)^{-4}$$
$$f^{(5)}(x) = 2(3)(4)(x-1)^{-5}$$

$$f^{(n)}(x) = (-1)^{n-1}2(3)(4)\ldots.(n-1)(x-1)^{-n} = \frac{(-1)^{n-1}(n-1)!}{(x-1)^n}$$

2.

$$f'(x) = 2e^{2x}$$
$$f''(x) = 2(2)e^{2x} = 2^2e^{2x}$$
$$f'''(x) = 2(2)(2)e^{2x} = 2^3e^{2x}$$
$$f^{(4)}(x) = 2(2)(2)(2)e^{2x} = 2^4e^{2x}$$

$$f^{(n)}(x) = \underbrace{2(2)(2)(2)\ldots(2)}_{n\ times}e^{2x} = 2^ne^{2x}$$

3.

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$
$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$$
$$f^{(4)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-\frac{7}{2}}$$
$$f^{(5)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)x^{-\frac{9}{2}}$$

$$f^{(n)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\ldots\left(\frac{1}{2}-n+1\right)x^{-\frac{2n-1}{2}}$$
$$f^{(n)}(x) = (-1)^{n+1}\frac{1.3.5.7\ldots(2n-3)}{2^n}x^{-\frac{2n-1}{2}}$$

Example: Find the 50th derivative of $y = f(x) = \cos 2x$
Solution

$$y' = -2\sin 2x$$
$$y'' = -2.2\cos 2x = -2^2\cos 2x$$
$$y''' = -2.2.(-2)\sin 2x = 2^3\sin 2x$$
$$y^{(4)} = -2.2.(-2)(2)\cos 2x = 2^4\cos 2x \qquad \textit{original function}$$

It occurs in a cycle of length 4 and since $50 = 4(12) + 2$

$$y^{(50)} = 2^{48}y''(\cos 2x) = 2^{48}.(-2^2)\cos 2x = -2^{50}\cos 2x$$

Example: Find the 1000th derivative of $y = f(x) = xe^{-x}$
Solution

$$y' = -xe^{-x} + e^{-x} = (1-x)e^{-x}$$
$$y'' = -(1-x)e^{-x} - e^{-x} = (x-2)e^{-x}$$
$$y''' = -(x-2)e^{-x} - e^{-x} = (3-x)e^{-x}$$
$$y^{(4)} = -(3-x)e^{-x} - e^{-x} = (x-4)e^{-x}$$

$$y^{(n)} = \begin{cases} (n-x)e^{-x} & n \ odd \\ (x-n)e^{-x} & n \ even \end{cases}$$

So

$$y^{(1000)} = (x-1000)e^{-x}$$

Example: For which value of r does the function $y = f(x) = e^{rx}$ satisfies the equation $y'' - 4y' - 5y = 0$

Solution

$$y' = re^{rx} \quad and \quad y'' = r^2e^{rx}$$

Substituting both the equation

$$y'' - 4y' + y = 0$$
$$r^2e^{rx} - 4re^{rx} + e^{rx} = 0$$
$$e^{rx}(r^2 - 4r + 1) = 0$$

Since $e^{rx} \neq 0$, we must have

$$r^2 - 4r + 1 = 0 \qquad \rightarrow \qquad r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

3.7 Rate of Change

The Goal:

After completing this section, you will be able to

To solve velocity problems using differential rules

Let $y = f(x)$ be a function. If x changes from x_1 to x_2 then the change in x is $\Delta x = x_2 - x_1$ and the change in y is

$$\Delta y = f(x_2) - f(x_1)$$

Recall that the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ is defined to be the quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The (**instantaneous**) **rate of change of *y* with respect to *x*** is defined to be

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

In physics, if $s = f(t)$ is the position function of a particle moving in a straight line, then

1. $\frac{\Delta s}{\Delta t}$ represents the ***average velocity*** over a time period Δt,
2. $\frac{ds}{dt}$ represents the (instantaneous) ***velocity*.**
3. $\left|\frac{ds}{dt}\right|$ represents the ***speed*** of the particle
4. The ***acceleration*** of the particle is $a(t) = v'(t) = \frac{d^2s}{dt^2}$

Example: The position function of an object is given by $s = f(t) = t^3 - 6t^2 + 9t$, where s is in meters and t in seconds.

1. Find the velocity at time t
2. What is the velocity after 2 seconds? 4 seconds?
3. When the particle is at rest? (i.e. the velocity is 0)
4. When the particle is moving forward (in positive direction)?
5. Find the total distance traveled by the particle during the first five seconds.
6. Find the acceleration after 4 seconds.
7. Draw a diagram to represent the motion of the particle.
8. When is the particle speeding up? When is it slowing down?

Solution:

1. The velocity

$$v(t) = f'(t) = 3t^2 - 12t + 9$$

2. The velocity after 2 and 4 seconds

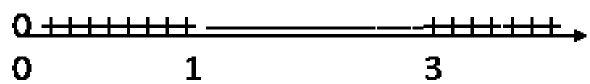
$$v(2) = 3(2^2) - 12(2) + 9 = -3\ m/s$$
$$v(4) = 3(4^2) - 12(4) + 9 = 9\ m/s$$

3. The particle at rest when the velocity $v(t) = 0$

$$3t^2 - 12t + 9 = 0 \qquad \rightarrow \quad t^2 - 4t + 3 = 0$$
$$(t-1)(t-3) = 0 \qquad t = 1 \ \ and \ t = 3$$

4. The particle moving forward when

$$v(t) > 0$$
$$(t-1)(t-3) > 0$$



It moves forward in the intervals $(0,1) \ \ and \ (3,\infty)$

5. We need to compute the distance in the intervals $[0,1] \ \ and \ [1,3] \ and \ [3,5]$

$$|f(1) - f(0)| = |4 - 0| = 4\ m$$
$$|f(3) - f(1)| = |0 - 4| = 4\ m$$
$$|f(5) - f(3)| = |20 - 0| = 20\ m$$

The total distance

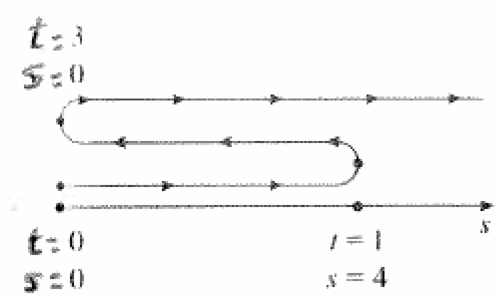
$$4 + 4 + 20 = 28\ m$$

6. The acceleration

$$a(t) = f''(t) = 6t - 12$$

$a(4) = 12\ m/s^2$

7. A diagram representing the motion of the particle



8. Note that the particle is speeding up when both the velocity and acceleration have the same sign (both positive or both negative). Also it is slowing down when the velocity and acceleration have different signs

3.9 Related Rates

The Goal:

After completing this section, you will be able to solve an application on differentiation using Chain rule

If we have an equation with two or more variables, each of which is a function of another variable (say time *t*), then we may differentiate with respect to *t* to get another equation relating the different rates with respect to *t*. If one rate is missing, and the others are known, we may find the missing rate using the others. Usually the problems on related rates are word problems.

To solve them do the following:

1. Define the variables and draw a picture (or the figure) if possible.
2. Identify the given rates and the one you want to find.
3. Find the equation relating the different variables (***find the relationship between the variables***)
4. Differentiate the equation (***with respect to time***) and solve for the missing rate of change.
5. Substitute the given values to find the missing rate of change.

Example. The radius of a metal sphere is increasing at a rate of 1 mm/sec. How fast its volume changing when its radius is 30 mm?

Solution.

Let the radius of the sphere be r, the volume be V , and the time be t. We are given the rate of change in the radius: $\frac{dr}{dt} = 1\ mm/sec$ and we need to find the rate of change of the volume $\left.\frac{dV}{dt}\right|_{r=30}$

We know that the volume of the sphere is given by

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to *t:*

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substituting $r = 30\ and \frac{dr}{dt} = 1$

we find:

$$\left.\frac{dV}{dt}\right|_{r=30} = 4\pi(30)^2(1) = 3600\pi\ mm^3/sec$$

Example. Gas is being pumped into spherical balloon at a rate of $5\, ft^3/min$. Find the rate at which the radius is changing when the diameter is 18 in.

Solution.
Let the radius of the sphere be r, the volume be V , and the time be t. We are given the rate of change in the volume $\frac{dV}{dt} = 5\, ft^3/min$ and we need to find the rate of change of the radius $\frac{dr}{dt}$ when $r = \frac{18}{2}\ in = 9\left(\frac{1}{12} ft\right) = \frac{3}{4}\ ft$

We know that the volume of the sphere is given by

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

.

Substituting

$$r = \frac{3}{4}\ \ and \frac{dV}{dt} = 5$$

We find

$$5 = 4\pi \left(\frac{3}{4}\right)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{20}{9\pi}\ ft/min$$

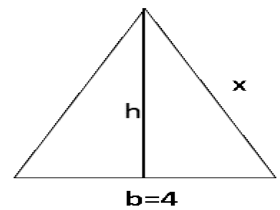
Example. The base of an isosceles triangle is 4 cm. If the side is changing at a rate of 2 cm/sec, how fast is the area changing when its side is 4 cm?

Solution.

Let the height be h, the side be x and the area be A. We are given that $\frac{dx}{dt} = 2\ cm/sec$, we need to find $\frac{dA}{dt}$ when $x = 4\ cm$.

We know that the area of the triangle is

$$A = \frac{1}{2}(4)h = 2h$$



We know that $h^2 + 2^2 = x^2 \qquad \rightarrow \quad h = \sqrt{x^2 - 4}$

So

$$A = 2h = 2\sqrt{x^2 - 4}$$

Differentiate both sides with respect to *t*:

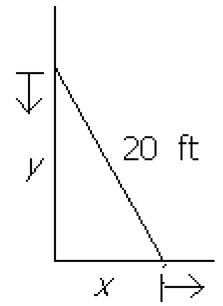
$$\frac{dA}{dt} = 2\frac{2x}{2\sqrt{x^2 - 4}}\frac{dx}{dt} = \frac{2x}{\sqrt{x^2 - 4}}\frac{dx}{dt}$$

Substituting $\frac{dx}{dt} = 2\ and\ x = 4$. we get

$$\frac{dA}{dt} = \frac{2x}{\sqrt{x^2 - 4}}\frac{dx}{dt} = \frac{2(4)}{\sqrt{(4)^2 - 4}}(2) = \frac{16}{\sqrt{12}} = \frac{8}{\sqrt{3}}\ cm/sec$$

Example. A ladder 20 ft long is leaning against a wall. If the bottom of the ladder slides away from the wall at a rate of 3ft/sec, how fast is the top slides down when the top of the ladder is 8 ft. from the ground?

Solution.
Let the top of the ladder be at distance y from the ground, and let the bottom be at distance x from the wall. We are given that
$\frac{dx}{dt} = 3\ ft/sec$ and we need to find $\frac{dy}{dt}$when $y = 8\ ft$



Since

$$x^2 + y^2 = 20^2$$

When $y = 8\ ft \quad x = \sqrt{400 - y^2} = \sqrt{400 - 8^2} = \sqrt{336}$

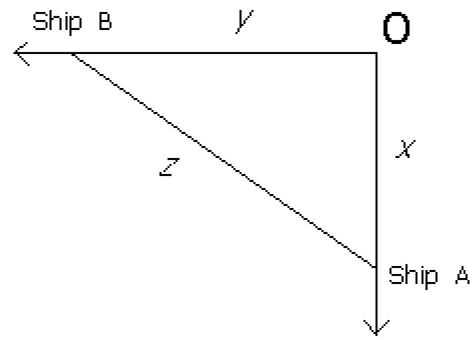
Now differentiating $x^2 + y^2 = 20^2$ with respect to t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Solving for $\frac{dy}{dt}$:

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{\sqrt{336}}{8}(3)\frac{ft}{sec} = -\frac{3\sqrt{336}}{8}\ ft/sec$$

Example. Two ships A and B meet in the ocean. Ship A is steaming south at 15 mi/h and ship B is steaming west at 20 mi/h. At what rate is the distance between the ships changing after two hours?



Solution.
Assume that that two ships met at the point O; let the distance of ship A from O be *x* and the distance of ship B from O be *y*; assume also that the distance between the two ships is *z*.

Given: $\frac{dx}{dt} = 15\ mi/h$ and $\frac{dy}{dt} = 20\ mi/h$, Find $\left.\frac{dz}{dt}\right|_{t=2}$

After two hours ($t = 2$):

$$x = 2(15) = 30\ mi \qquad and \quad y = 2(20) = 40\ mi \quad and \quad z = \sqrt{30^2 + 40^2} = 50\ mi$$

Since

$$x^2 + y^2 = z^2$$

differentiating with respect to *t* we get:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

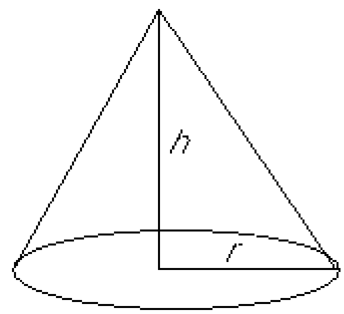
Substituting we get:

$$2(50)\frac{dz}{dt} = (2)(30)(15) + (2)(40)(20)$$

$$\frac{dz}{dt} = 25\ mi/h$$

Example. Wheat is leaking through a hole to form a conical pile whose altitude is always the same as its radius. If the height of the pile is increasing at a rate of 6 *in/min*. find the rate at which the wheat is leaking when the height is 10 in.

Solution.



Let r be the radius of the base and let h be the height.
Since the altitude is equal to the base we have $h = r$.
The volume V is given by

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3$$

We are given that

$$\frac{dh}{dt} = \frac{dr}{dt} = 6\ in/min$$

To find $\frac{dV}{dt}$ when $h = r = 10\ in$

Differentiate the equation $V$with respect to t:

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt} = \pi(10)^2(6) = 600\ in^3/min$$

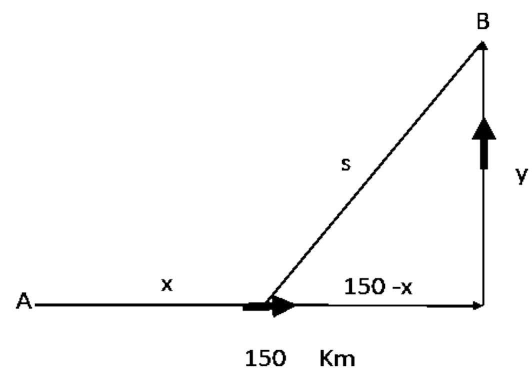
Example: At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast the distance between the ships changing at 4:00 pm?

Solution

Let x be the distance moved by ship A

Let y be the distance moved by ship B

Let $s$be the distance between the two ships



We are given that

$$\frac{dx}{dt} = 35\ km/h \qquad \frac{dy}{dt} = 25\ km/h$$

Since

$$(150 - x)^2 + y^2 = s^2$$

At 4:00 pm ($t = 4$):

$$x = 4(35) = 140\ km \qquad and \quad y = 4(25) = 100\ km \quad and$$

$$(150 - x)^2 + y^2 = s^2$$

$$s = \sqrt{(150 - x)^2 + y^2} = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10100} = 10\sqrt{101}$$

differentiating with respect to t we get:

$$s^2 = (150 - x)^2 + y^2$$

$$z\frac{ds}{dt} = -(150 - x)\frac{dx}{dt} + y\frac{dy}{dt} =$$

$$10\sqrt{101}\frac{ds}{dt} = -(150 - 140)(35) + (100)(25) = 2150$$

$$\frac{ds}{dt} = \frac{2150}{10\sqrt{101}} = \frac{215}{\sqrt{101}}\ km/h$$

3.10 Linear Approximations and Differentials

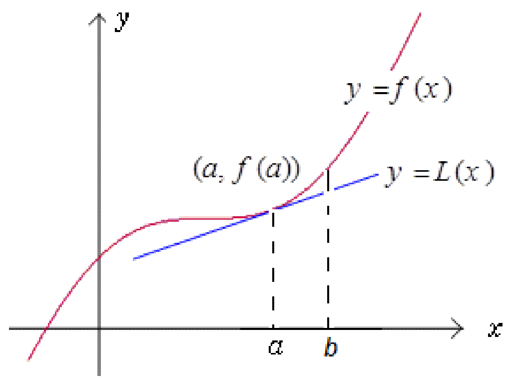
The Goal:

After completing this section, you will be able

1. to define the linear approximation and solve problems
2. to define the differentials and solve problems

Linear approximation

Let $y = f(x)$ be a differentiable function and assume that the tangent line at the point $(a, f(a))$ is very close to the graph of $f(x)$. We may use the tangent line to approximate the values of $f(x)$ near the point a. (Say at b, See the Figure)



The slope of the tangent line at the point $(a, f(a))$ is $f'(a)$. Thus if (x, y) is a point on the tangent line then the equation of the tangent line is

$$\frac{y - f(a)}{x - a} = f'(a)$$

Or

$$y = f(a) + f'(a)(x - a)$$

And the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a

The linear function whose graph is this tangent line, that is

$$L(x) = f(a) + f'(a)(x - a)$$

Is called the **linearization** of f at a

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Example. Find the linearization of $y = f(x) = \sqrt{x+1}$ at $a = 0$.

Solution. Note first that

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

So

$$f'(0) = \frac{1}{2\sqrt{0+1}} = \frac{1}{2}$$

Since

$$f(0) = \sqrt{0+1} = 1$$

The linearization of $f(x) = \sqrt{x+1}$ is

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= f(0) + f'(0)(x-0) \\ &= 1 + \frac{1}{2}(x) \\ &= \frac{1}{2}x + 1 \end{aligned}$$

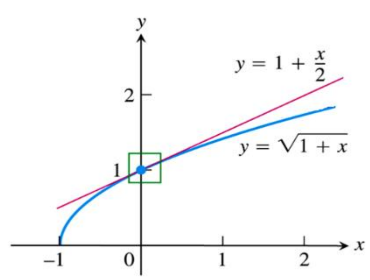
Note that the above linearization may help us in approximating some roots. We have

$$\sqrt{x+1} \approx L(x) = \frac{1}{2}x + 1$$

Use this linearization to approximate $\sqrt{1.2}$, $\sqrt{1.05}$, and $\sqrt{1.005}$.

From the following table we see that

Approximation	True value	|True value − approximation|
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$



The graph of $y = \sqrt{1+x}$ and its linearizations at $x = 0$

Example. Find the linearization of $y = f(x) = \sin x$ at $a = 60^0$ and use it to approximate $\sin 61$.

Solution.
We use radians instead of degrees (because when writing the equation we deal with numbers).

Note first that

$$f'(x) = \cos x$$

So

$$f'(60) = \cos 60 = \cos\frac{\pi}{3} = \frac{1}{2}$$

Since

$$f(60) = \sin 60 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

The linearization of $f(x) = \sin x$ is

$$\begin{aligned} L(x) &= f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) \end{aligned}$$

To approximate $\sin 61$

$$\begin{aligned} \sin 61 = \sin\frac{61\,\pi}{180} = L(61) &= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{61\,\pi}{180} - \frac{\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{61\,\pi}{180} - \frac{60\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{360} \end{aligned}$$

Remark. The linear approximation of a function depends on the point at which it is evaluated. For example the linear approximation of $y = f(x) = \sin x$ at $x = 0$ is $\sin x \approx \sin(0) + \cos(0)(x - 0)$. That is $\sin x \approx x$. This demonstrate the fact that near zero we have $\frac{\sin x}{x} \approx 1$.

Example. Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1

$$\ln(1+x) \approx x \qquad a = 0$$

Solution

Let

$$f(x) = \ln(1+x) \ \rightarrow \ f'(x) = \frac{1}{1+x}$$

So

$$f(0) = \ln(1) = 0 \qquad f'(0) = 1$$

Thus

$$f(x) \approx f(0) + f'(x)(x-0) = 0 + 1(x-0) = x$$

The accuracy is

$$|f(x) - L(x)| = |ln(1+x) - x| < \mathbf{0.1}$$

$$-0.1 < ln(1+x) - x < \mathbf{0.1}$$
$$ln(1+x) - 0.1 < x < ln(1+x) + \mathbf{0.1}$$

Differentials

Definition:
Let $y = f(x)$ be differentiable function. The change in x is denoted by Δx.The corresponding change in y is denoted by Δy and is defined as $\Delta y = f(x + \Delta x) - f(x)$

The differential of x denoted by dx is defined to be $dx = \Delta x$.
The differential of y denoted by dy, is defined as $dy = f'(x)\, dx$.

The geometric meaning of the differentials is shown in the figure

Let
$P\big(x, f(x)\big)$ & $Q\big(x + \Delta x\,, f(x + \Delta x)\big)$
Points on the graph of f
Let

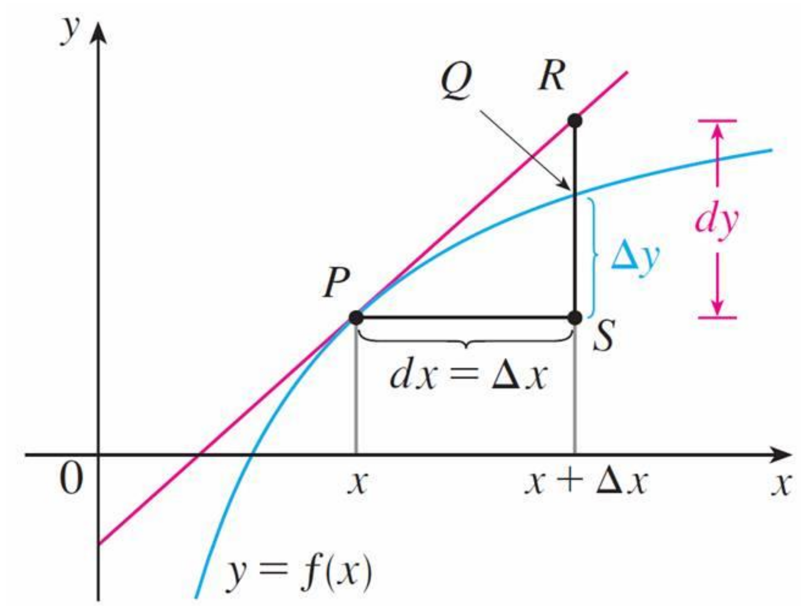
$$dx = \Delta x$$

Then, the corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative of $f(x)$
Thus, the directed distance from S to R is

$$f'(x)\, dx = dy$$



Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Remark. Since $dy = f'(x)\, dx$, then for any non zero value of dx we may divide by dx to get $\frac{dy}{dx} = f'(x)$. This gives us another way of looking at the expression $\frac{dy}{dx}$ other than a name for the derivative; we may regard it as the ***quotient of two differentials***.

Example. Consider the function $y = f(x) = 3x^2 - 4x + 5$

1. Find a formulas for Δy and dy.
2. Find Δy and dy at $x = 3$ and $\Delta x = 0.01$.

Solution

1. Using the definition of Δy

$$\begin{aligned}\Delta y = f(x + \Delta x) - f(x) &= 3(x + \Delta x)^2 - 4(x + \Delta x) + 5 - (3x^2 - 4x + 5\ ) \\ &= (6x - 4)\Delta x + 3(\Delta x)^2\end{aligned}$$

and

$$dy = f'(x)dx = (6x - 4)dx = (6x - 4)\Delta x$$

2. Substituting $= 3$ and $\Delta x = 0.01$ in Δy and dy we get:

$$\Delta y = \mathbf{0.1403} \quad \boldsymbol{and} \quad \boldsymbol{dy} = \mathbf{0.14}$$

Example. Compare Δy and dy where $\Delta x = dx$ for

$y = f(x) = \frac{16}{x}$ when $x = 4$ and $\Delta x = \mathbf{-1}$

Solution

1. $\Delta y = (x + \Delta x) - f(x) = \frac{16}{x+\Delta x} - \frac{16}{x} = \frac{16}{3} - \frac{16}{4} = \frac{4}{3}$

2. $dy = f'(x)dx = -\frac{16}{x^2}\ dx = -\frac{16}{4^2}(-1) = 1$

Remark. Since

$$\Delta y = f(x + \Delta x) - f(x)$$

Then

$$f(x + \Delta x) = f(x) + \Delta y \approx f(x) + dy$$

The differential approximation of $f(x)$ is given by:

$$f(x + dx) \approx f(x) + dy, \qquad dx = \Delta x$$

Note that if we put $dy = f'(x)dx$ in the above formula, we get the linear approximation

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

Example. Use differentiate to approximate $\sqrt{80}$.

Solution.
Let

$$f(x) = \sqrt{x} \qquad \rightarrow \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

we need to find $f(80) = \sqrt{80}$
We can write $f(80) = f\big(81 + (-1)\big)$
Using the formula in the remark above with $x = 80$ and $\Delta x = -1$
We get

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

$$\sqrt{80} = f(80) = f\big(81 + (-1)\big)$$

$$\approx f(81) + f'(81)\Delta x$$

$$\approx \sqrt{81} + \frac{1}{2\sqrt{81}}(-1) = 9 - \frac{1}{18} = 8.9444$$

Error in differential approximation

The error in using differentials to approximate the change in f is given by

$$\Delta y - dy = \big(f(a+\Delta x) - f(a)\big) - f'(a)dx$$
$$= \frac{\big(f(a+\Delta x) - f(a)\big)}{\Delta x}\,\Delta x - f'(a)\Delta x$$
$$= \left(\frac{\big(f(a+\Delta x) - f(a)\big)}{\Delta x} - f'(a)\right)\Delta x = \varepsilon\,\Delta x$$

So

$$\Delta y - dy = \varepsilon\,\Delta x$$

$$\Delta y = dy + \varepsilon\,\Delta x = f'(a)\Delta x + \varepsilon\,\Delta x$$

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

Definitions. Let $y = f(x)$ be differentiable function and assume that y changes from y_0 to y_1 as x changes x_0 to x_1.

If Δx represents the error in approximating x_0; i.e. $\Delta x = x_1 - x_0$ then

The *absolute change* (*error*) in y is $\Delta y = y_1 - y_0$ can be approximated by $dy = f'(a)\Delta x$

The *relative change* (*error*) in y is $\frac{\Delta y}{y_0}$ can be approximated by $\frac{dy}{y_0}$.

The *percentage change* (*error*) in y is $\frac{\Delta y}{y_0} \times 100\%$ is can be approximated by $\frac{dy}{y_0} \times 100\%$

	True	**Estimated**
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$

Example. The diameter of a sphere is determined to be 16 cm with maximum error of 0.001 cm.

a) Use differentials to determine an estimate for the maximum error in computing the volume.
b) Find the relative error in the volume.
c) Find the percent error in the volume

Solution: Let r be the radius, D be the diameter, and V be the volume of the sphere. Then

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{1}{6}\pi D^3$$

We are given that $dD = \Delta D \le 0.001$when $D = 16$ cm.

a) The error in the volume is given by:

$$\Delta V \approx dV = V'dD$$
$$= \frac{3}{6}\pi D^2\ \Delta D \le \frac{1}{2}\pi 16^2(0.001) = 0.4\ cm^3$$

That is the maximum error in the volume is $0.4\ cm^3$.

b) The relative error in the volume is

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{\frac{1}{2}\pi D^2\ dD}{\frac{1}{6}\pi D^3} = \frac{3dD}{D} = \frac{3(0.001)}{16} = 0.0001875$$

c) The percentage error in V is

$$\frac{\Delta V}{V} \times 100\% = 0.01875\%$$

Example. The radius of a sphere is found to be 9 cm. If this measurement is accurate to within 0.01 cm. what is the maximum error in the volume?

Solution:
Given that

$$|\Delta r| \leq 0.01 \text{ when } r = 9\ cm$$

Since the volume is

$$V = \frac{4}{3}\pi r^3$$

We have

$$|\Delta V| \approx |dV| = |4\pi r^2\ dr| \leq 4\pi r^2\ (0.01) = 3.24\ cm^3$$

Example. Approximate $\sqrt[3]{7}$ using differentials

Solution:
Let

$$f(x) = \sqrt[3]{x} \qquad f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

, we need to find $f(7)$

$$\begin{aligned} f(7) &\approx f\big(8 + (-1)\big) \\ &\approx f(8) + f'(8)dx = \sqrt[3]{8} + \frac{1}{3\sqrt[3]{8^2}}(-1) \\ &= 2 - \frac{1}{3(4)} = 2 - \frac{1}{12} = 1.917 \end{aligned}$$

Example. Approximate $\sqrt{123}$ using differentials
Example. Approximate $\ln 1.1$ using differentials

3.11 Hyperbolic Functions

The Goal:

After completing this section, you will be able

1. To define Hyperbolic functions
2. To consider Hyperbolic identities.
3. To define Hyperbolic derivatives.
4. To consider inverse Hyperbolic functions and their derivatives.

There are many nice combinations of the natural exponential functions $e^x$and e^{-x}. These functions are called the ***hyperbolic functions*** and they enjoy properties similar to the trigonometric functions; their definitions, though, are much more straightforward:

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Here are their graphs

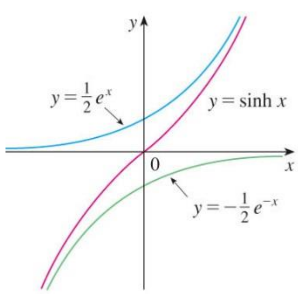


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

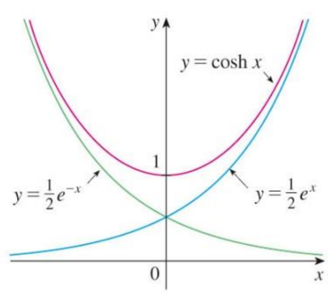


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

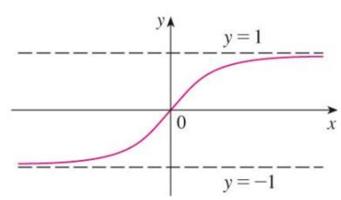


FIGURE 3
$y = \tanh x$

Note:

1. For $y = \sinh x$, $\quad$ *the doman*: $(-\infty, +\infty)$ $\quad$ *and the range* $(-\infty, +\infty)$
2. For $y = \cosh x$, $\quad$ *the doman*: $(-\infty, +\infty)$ $\quad$ *and the range* $[1, +\infty)$
3. For $y = \tanh x$, $\quad$ *the doman*: $(-\infty, +\infty)$ $\quad$ *and the range* $(-1,1)$

Hyperbolic identities

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

We list some of them here and leave most of the proofs to the exercises.

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

As their trigonometric counterparts, the ***$\cosh x$ function is even***, while the ***$\sinh x$ function is odd***. Their most important property is their version of the Pythagorean Theorem is

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

The verification is straightforward:

$$(\cosh x)^2 - (\sinh x)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}[(e^{2x} + 2e^x e^{-x} + e^{-2x}) - (e^{2x} - 2e^x e^{-x} + e^{-2x})]$$

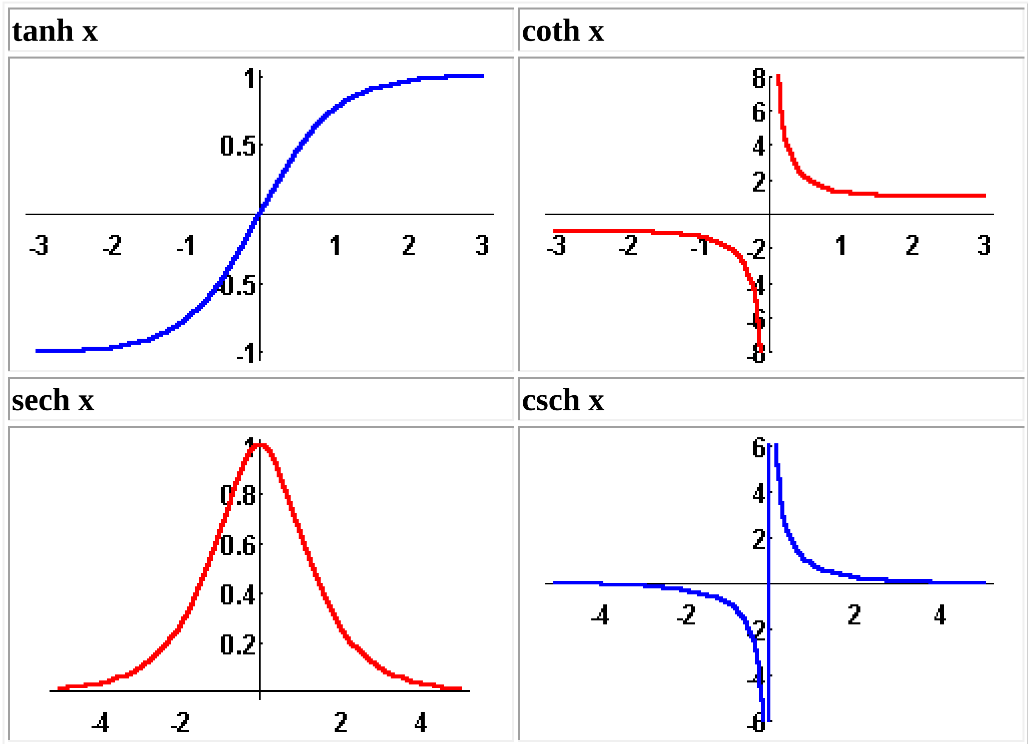
$$= \frac{1}{4}\, 4e^x e^{-x} = 1$$

The other hyperbolic functions are defined the same way as the rest of the trigonometric functions are defined:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

The graphs of these hyperbolic functions are



For every formula for the trigonometric functions, there is a similar (not necessary identical) formula for the hyperbolic functions:

1. $\sinh(-x) = -\sinh x$
2. $\cosh(-x) = \cosh x$
3. $\tanh(-x) = -\tanh x$
4. $\cosh x + \sinh x = e^x$
5. $\cosh x - \sinh x = e^{-x}$
6. $\cosh^2 x - \sinh^2 x = 1$
7. $1 - \tanh^2 x = \operatorname{sech}^2 x$
8. $\sinh 2x = 2 \sinh x \cosh x$
9. $\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$
10. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \ \sinh y$
11. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \ \sinh y$

Proof
The proof of these identities is straightforward:
Proof of (1):

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = -\frac{e^{x} - e^{-x}}{2} = -\sinh x$$

Proof of (4):

$$\cosh x + \sinh x = \frac{e^{x} + e^{-x}}{2} + \frac{e^{x} - e^{-x}}{2} = \frac{2e^{x}}{2} = e^{x}$$

Proof of (8):

$$2\sinh x \cosh x = 2\left(\frac{e^{x} + e^{-x}}{2}\right)\left(\frac{e^{x} - e^{-x}}{2}\right) = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$$

Proof of (11):
Start with the right side and multiply out:

$$\begin{aligned}
&\sinh x \cosh y + \cosh x \ \sinh y \\
&\qquad = \left(\frac{e^{x} + e^{-x}}{2}\right)\left(\frac{e^{y} + e^{-y}}{2}\right) + \left(\frac{e^{x} - e^{-x}}{2}\right)\left(\frac{e^{y} - e^{-y}}{2}\right) \\
&= \frac{e^{x+y} + e^{-(x+y)} + e^{x-y} + e^{-x+y}}{4} + \frac{e^{x+y} + e^{-(x+y)} - e^{x-y} - e^{-x+y}}{4} \\
&\qquad = \frac{2e^{x+y} + 2e^{-(x+y)}}{4} = \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x+y)
\end{aligned}$$

Example. If $\tanh x = \frac{4}{5}$, find the other Hyperbolic functions
Solution:
1. $\tanh x = \frac{4}{5} \qquad \rightarrow \qquad \coth x = \frac{1}{\tanh x} = \frac{5}{4}$

2. $1 - \tanh^2 x = \operatorname{sech}^2 x \ \rightarrow \ 1 - \frac{16}{25} = \operatorname{sech}^2 x \ \rightarrow \ \operatorname{sech}^2 x = \frac{9}{25} \rightarrow \ \operatorname{sech} x = \frac{3}{5} \quad x > 0$

3. $\cosh x = \frac{1}{\operatorname{sech} x} = \frac{5}{3}$

4. $\cosh^2 x - \sinh^2 x = 1 \rightarrow \ \sinh^2 x = 1 - \cosh^2 x = 1 - \frac{25}{9} = \frac{16}{9} \ \rightarrow \ \sinh x = \frac{4}{3}$

5. $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{3}{4}$

Derivatives of hyperbolic functions

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Note
We may also use the chain rule together with these derivatives such that

If

$$y = \sinh g(x)$$

Then

$$y' = \frac{d}{dx}(\sinh g(x)) = \cosh g(x) \,.\, g'(x)$$

Which is valid for the others

Proof
Proof of (1):

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

Proof of (3)

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x + \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

Example. Find $\frac{dy}{dx}$ for each of the following

1. $y = \tanh(4x)$
2. $y = x\ \cosh(\ln x)$
3. $y = \sinh(x^2 + x)$
4. $y = \sin(\cosh x)$
5. $y = \operatorname{sech}^2 x + \coth^2 x$
6. $y = \operatorname{sech}^2(x^2)$
7. $y = e^{\operatorname{sech} x}$
8. $y = e^x \operatorname{sech} x$
9. $f(t) = \ln(\sinh t)$

Solution
1. $y' = 4\operatorname{sech}^2(4x)$
2. $y' = x\sinh(\ln x)\frac{1}{x} + \cos(\ln x) = \sinh(\ln x) + \cos(\ln x)$
3. $y' = \cosh(x^2 + x)\ (2x + 1)$
4. $y' = \cos(\cosh x)\sinh x$
5. $y' = -2\operatorname{sech} x \operatorname{sech} x \tanh x + 2\coth x \operatorname{csch} x$
6. $y' = -2\operatorname{sech}(x^2)\operatorname{sech}(x^2)\tanh(x^2)\,(2x) = -4\operatorname{sech}^2(x^2)\tanh(x^2)$
7. $y' = e^{\operatorname{sech} x}(-\operatorname{sech} x \tanh x)$
8. $y' = e^x(-\operatorname{sech} x \tanh x) + \operatorname{sech} x\, e^x = e^x \operatorname{sech} x\,(1 - \tanh x)$
9. $f'(t) = \frac{\cosh t}{\sinh t} = \coth t$

Inverse Hyperbolic Functions

We now define the inverse hyperbolic function.

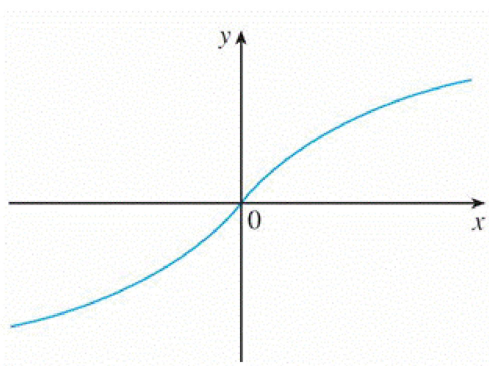
1. We start with the hyperbolic $\sinh x \quad and \quad \tanh x$ functions; Since both are one-to-one function, it has an inverse denoted by $\sinh^{-1} x \quad and \quad \tanh^{-1} x$ (also denoted by $\operatorname{arcsinh} x \quad and \quad \operatorname{arctanh} x$). Thus we say that

$$y = \sinh^{-1} x \qquad \Leftrightarrow \qquad x = \sinh y \qquad for\ all\ x\ \epsilon(-\infty, \infty)$$

and

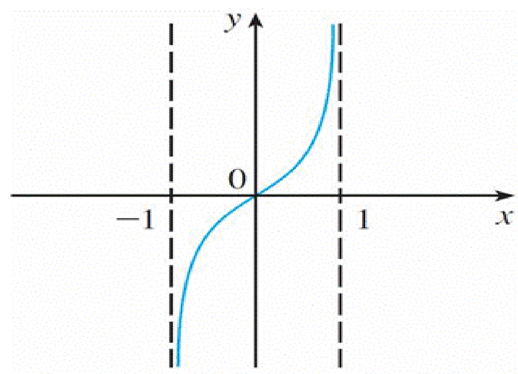
$$y = \tanh^{-1} x \qquad \Leftrightarrow \qquad x = \tanh y \qquad for\ all\ x\ \epsilon(-\infty, \infty)$$

As usual, we obtain the graph of the inverse hyperbolic sine or tan functions $\sinh^{-1} x$ or $\tanh^{-1} x$ by reflecting the graph of $\sinh x \quad or \quad \tanh x$about the line $y = x$:



$y = \sinh^{-1} x$

domain = $\mathbb{R}$ range = $\mathbb{R}$



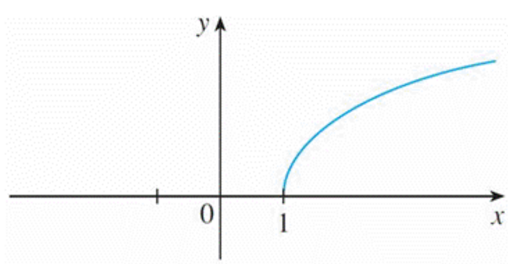
$y = \tanh^{-1} x$

domain = (−1, 1) range = $\mathbb{R}$

For the hyperbolic cosine function is not one-to one, so we need to restrict the domain to the interval $[0, \infty)$. Thus we say that

$$y = \cosh^{-1} x \qquad \Leftrightarrow \qquad x = \cosh y \qquad and\ x \geq 0$$

The graph of the $y = \cosh^{-1} x$ is given.



$y = \cosh^{-1} x$

domain = $[1, \infty)$ range = $[0, \infty)$

Similarly we define the other inverse hyperbolic functions

$$y = \coth^{-1} x \quad \Leftrightarrow \quad x = \coth y \quad for\ all\ \ x\ \epsilon\ (1, \infty)$$

$$y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad x = \operatorname{sech} y \quad for\ all\ \ x\ \epsilon\ (0,1)$$

$$y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad x = \operatorname{csch} y \quad for\ all\ \ x\ \epsilon\ (0, \infty)$$

Since the hyperbolic functions are defined in terms of the exponential functions; we may define the inverse hyperbolic functions in terms of logarithmic functions as follows.

1. $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right),$ $\qquad x \in (-\infty, \infty)$
2. $y = \cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right),$ $\qquad x \in [1, \infty)$
3. $y = \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right),$ $\qquad x \in (-1,1)$
4. $y = \coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right),$ $\qquad x \in (-1, \infty)$
5. $y = \operatorname{sech}^{-1} x = \ln\left(\frac{x+\sqrt{1-x^2}}{x}\right),$ $\qquad x \in (0,1)$
6. $y = \operatorname{csch}^{-1} x = \ln\left(\frac{x+\sqrt{x^2+1}}{x}\right),$ $\qquad x \in (0, \infty)$

Example. Show that $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

Solution.

Since

$$y = \sinh^{-1} x \quad \rightarrow \quad x = \sinh y = \frac{e^y - e^{-y}}{2}$$

So

$$2x = e^y - e^{-y} \qquad \rightarrow \quad e^y - 2x - e^{-y} = 0$$

Multiply be e^y:

$$e^{2y} - 2xe^y - 1 = 0$$

Solving for e^y:

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

But $e^y > 0$ and $x - \sqrt{x^2+1} < 0$ $(negative)$ which is impossible

Thus

$$e^y = x + \sqrt{x^2+1} \qquad \rightarrow \qquad y = \ln\left(x + \sqrt{x^2+1}\right)$$

Derivatives of inverse hyperbolic functions

1. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$

2. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

3. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

4. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

5. $\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\ \sqrt{1-x^2}}$

6. $\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\ \sqrt{x^2+1}}$

Proof
Proof of (2):

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{d}{dx}\left(x + \sqrt{x^2-1}\right)$$

$$= \frac{1 + \frac{2x}{2\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} = \frac{\frac{2\sqrt{x^2-1} + 2x}{2\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} = \frac{2\sqrt{x^2-1} + 2x}{\left(x + \sqrt{x^2-1}\right)\left(2\sqrt{x^2-1}\right)} = \frac{1}{\sqrt{x^2-1}}$$

Proof of (3)

$$\frac{d}{dx}\left(\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)\right) = \frac{d}{dx}\left(\frac{1}{2}(\ln(x+1) - \ln(x-1))\right)$$

$$= \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) = \frac{-2}{2(x^2-1)} = \frac{1}{1-x^2}$$

Remark. We may also use the chain rule together with these derivatives.
For example to differentiate $y = \cosh^{-1} x$ we write the equation in the form $x = \cosh y$.
Differentiating both sides implicitly with respect to *x* we get

$$1 = (\sinh y)y' \qquad \to \quad y' = \frac{1}{\sinh y}$$

But

$$(\cosh y)^2 - (\sinh y)^2 = 1 \qquad \to \qquad \sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$$

substituting we get

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{x^2-1}}$$

which is the same result we obtained in the first proof.

Similarly we can use the identities and implicit differentiation to find the other formulas.
Example. Find y'

1. $y = \sinh^{-1} x$
2. $y = \sinh^{-1}(2x)$
3. $y = x^2 \sinh^{-1}(2x)$
4. $y = x \coth^{-1} x$
5. $y = \coth^{-1}\left(\sqrt{x^2+1}\right)$
6. $y = \tanh^{-1}(\sin x)$
7. $y = \operatorname{sech}^{-1}(e^x)$

Solution

1. $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

$$y' = \frac{1 + \frac{2x}{2\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} = \frac{\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\left(\sqrt{x^2+1}\right)} = \frac{1}{\sqrt{x^2+1}}$$

2. $y = \sinh^{-1}(2x)$

$$y' = \frac{1}{\sqrt{(2x)^2+1}}(2) = \frac{2}{\sqrt{4x^2+1}}$$

3. $y = x^2 \sinh^{-1}(2x)$

$$y' = x^2\left(\frac{2}{\sqrt{4x^2+1}}\right) + (2x)\sinh^{-1}(2x) = \frac{2x^2}{\sqrt{4x^2+1}} + (2x)\sinh^{-1}(2x)$$

4. $y = x \coth^{-1} x$

$$y' = x\left(\frac{1}{1-x^2}\right) + \coth^{-1} x$$

5. $y = \coth^{-1}\left(\sqrt{x^2+1}\right)$

$$y' = \frac{1}{1-\left(\sqrt{x^2+1}\right)^2}\left(\frac{2x}{2\sqrt{x^2+1}}\right) = \frac{1}{1-x^2-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{-1}{x\sqrt{x^2+1}}$$

6. $y = \tanh^{-1}(\sin x)$

$$y' = \frac{1}{1-(\sin x)^2}(\cos x) = \frac{\cos x}{\cos^2 x} = \frac{1}{\cos x} = \sec x$$

7. $y = \operatorname{sech}^{-1}(e^x)$

$$y' = -\frac{1}{e^x\sqrt{1-(e^x)^2}}(e^x) = \frac{-1}{1-e^{2x}}$$