Let $p_0 = -1$, $p_1 = 3$, and define p_i for $i \geq 2$ with the recurrence:

$$p_i = -2(2n-1)p_{i-1} + p_{i-2} \tag{1}$$

Similarly, let $q_0 = 1$, $q_1 = -1$, and define q_i for $i \geq 2$ with the same recurrence:

$$q_i = -2(2n-1)q_{i-1} + q_{i-2} \tag{2}$$

Lastly, let

$$F(x) = (e-1)x + 1 + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n \left(p_n x - \frac{p_n - p_{n-1}}{2} + e\left(q_n x - \frac{q_n - q_{n-1}}{2} \right) \right) \tag{3}$$

Theorem 1. $F(x) = e^x$

Proof. Define

$$f(x,n) = p_n x - \frac{p_n - p_{n-1}}{2} + e\left(q_n x - \frac{q_n - q_{n-1}}{2} \right) \tag{4}$$

We can express F more simply as

$$F(x) = (e-1)x + 1 + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n f(x,n)$$

Firstly, we will take the derivative of $F(x)$,

$$F'(x) = e - 1 + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1}(x-1)^{n-1}(2x-1) f(x,n) + \frac{1}{n!} x^n (x-1)^n (p_n + e q_n)$$

Then, we can split this into two sums, and re-index the first one

$$F'(x) = e - 1 + \sum_{n=0}^{\infty} \frac{1}{n!} x^n (x-1)^n (2x-1) f(x,n+1) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n (p_n + e q_n)$$

Now we will look at $(2x-1)f(x,n+1)$. We will first split $(2x-1)f(x,n)$ into two functions $g_1(x,n)$ and $g_2(x,n)$.

$$g_1(x,n) = -f(x,n) - 2x\left(\frac{p_n - p_{n-1}}{2} - p_n + e\left(\frac{q_n - q_{n-1}}{2} - q_n \right) \right) \tag{5}$$

$$g_2(x,n) = 2x f(x,n) + 2x\left(\frac{p_n - p_{n-1}}{2} - p_n + e\left(\frac{q_n - q_{n-1}}{2} - q_n \right) \right) \tag{6}$$

Clearly $g_1(x,n) + g_2(x,n) = (2x-1)f(x,n)$. Looking at $g_2(x,n)$ first, we see that

$$\begin{aligned} g_2(x,n) &= 2x\left(p_n x - \frac{p_n - p_{n-1}}{2} + e\left(q_n x - \frac{q_n - q_{n-1}}{2} \right) \right) + 2x\left(\frac{p_n - p_{n-1}}{2} - p_n + e\left(\frac{q_n - q_{n-1}}{2} - q_n \right) \right) \\ &= 2x(p_n x - p_n + e\,(q_n x - q_n) \\ &= 2x(x-1)(p_n + e q_n) \end{aligned}$$

Next we will look at $g_1(x,n)$

$$\begin{aligned}
g_1(x,n) &= \frac{p_n - p_{n-1}}{2} - p_n x + e\frac{q_n - q_{n-1}}{2} - eq_n x - 2x\left(\frac{p_n - p_{n-1}}{2} - p_n + e\left(\frac{q_n - q_{n-1}}{2} - q_n\right)\right) \\
&= \frac{p_n - p_{n-1}}{2} - p_n x + e\frac{q_n - q_{n-1}}{2} - eq_n x + x\left(p_{n-1} + p_n + eq_{n-1} + eq_n\right) \\
&= \frac{p_n - p_{n-1}}{2} + e\frac{q_n - q_{n-1}}{2} + x\left(p_{n-1} + eq_{n-1}\right)
\end{aligned}$$

Then we can use equations (1) and (2) for p_n and q_n:

$$\begin{aligned}
g_1(x,n) &= \frac{p_n - p_{n-1}}{2} + e\frac{q_n - q_{n-1}}{2} + x\left(p_{n-1} + eq_{n-1}\right) \\
&= \frac{p_{n-2} - 2(2n-1)p_{n-1} - p_{n-1}}{2} + e\frac{q_{n-2} - 2(2n-1)q_{n-1} - q_{n-1}}{2} + x\left(p_{n-1} + eq_{n-1}\right) \\
&= xp_{n-1} + \frac{p_{n-2} - p_{n-1}}{2} + eq_{n-1}x + e\frac{q_{n-2} - q_{n-1}}{2} - (2n-1)p_{n-1} - (2n-1)eq_{n-1}
\end{aligned}$$

But we can see that this first part is just the formula for $f(x, n-1)$, so we have

$$g_1(x,n) = f(x,n-1) - (2n-1)(p_{n-1} + eq_{n-1})$$

Picking up where we left off earlier, we have

$$F'(x) = e - 1 + \sum_{n=0}^{\infty} \frac{1}{n!} x^n (x-1)^n (g_1(x,n+1) + g_2(x,n+1)) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n (p_n + eq_n)$$

We can split the first sum into two by distributing over $g_1(x,n+1) + g_2(x,n+1)$

$$F'(x) = e-1+\sum_{n=0}^{\infty} \frac{1}{n!} x^n (x-1)^n g_1(x,n+1) + \sum_{n=0}^{\infty} \frac{1}{n!} x^n (x-1)^n g_2(x,n+1) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n (p_n+eq_n)$$

Then we can apply the work we did to g_2 to get

$$F'(x) = e-1+\sum_{n=0}^{\infty} \frac{1}{n!} x^n (x-1)^n g_1(x,n+1) + \sum_{n=0}^{\infty} \frac{1}{n!} x^n (x-1)^n 2x(x-1)(p_{n+1}+eq_{n+1}) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n (p_n+eq_n)$$

Then we can re-index the middle sum, to get

$$F'(x) = e-1+\sum_{n=0}^{\infty} \frac{1}{n!} x^n (x-1)^n g_1(x,n+1) + \sum_{n=1}^{\infty} \frac{n}{n!} x^n (x-1)^n (2p_n+2eq_n) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n (p_n+eq_n)$$

At this point we can combine the second and third sums back together

$$F'(x) = e - 1 + \sum_{n=0}^{\infty} \frac{1}{n!} x^n (x-1)^n g_1(x,n+1) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n (2n+1)(p_n + eq_n)$$

We can pull of the $n = 0$ term of the first sum, and then combine the sums again to get

$$F'(x) = e - 1 + g_1(x,1) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n (g_1(x,n+1) + (2n+1)(p_n + eq_n))$$

Now we can apply the work we did to $g_1(x,n)$ to get

$$F'(x) = e - 1 + g_1(x,1) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n (f(x,n) - (2n+1)(p_n + eq_n) + (2n+1)(p_n + eq_n))$$

Lastly, we can evaluate $g_1(x,1)$ to get

$$F'(x) = e - 1 + 2 - e - x + ex + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n f(x,n) = (e-1)x + 1 + \sum_{n=1}^{\infty} \frac{1}{n!} x^n (x-1)^n f(x,n)$$

Thus we have shown that $F'(x) = F(x)$. That means that $F(x)$ is of the form ce^x. When we plug in 0 for x, almost everything disappears, as each term in the sum has a factor of x. We have $F(0) = 1$. Therefore $c = 1$, and $F(x) = e^x$ $\square$