1 Notation and conventions

An **interval** in T will always be some set of the form

$$(a, b) = \{x \in T \mid a < x < b\},$$

in which case a and b are called the endpoints of the interval. A **convex subset** of T is a set C so that for all $a, b \in C$, we have $(a, b) \subseteq C$. This definition includes all intervals, but also things such as intial/final segments, closed intervals. For any order T, we will use T^{op} to denote the order dual of T. So as not to confuse elements of T and T^{op} with each other, we will mark elements of T^{op} with a dot above them like $\dot{t}$. When we have two orders S, T, we will let $S + T$ denote the order on $S \cup T$ such that every element of S is less than every element of T. As we will frequently be dealing with ordinals, it is important to note that ordinal addition and this sum of ordinals fortunately coincide. On the other hand, we will always take the lexicographic order on $T \times S$, whereas the ordinal product $\alpha \cdot \beta$ comes from the antilexicographic order on $\alpha \times \beta$. Hence, we will always use $\times$ to denote the lexicographic product, and $\cdot$ to denote the product of ordinals.

2 Embedding rank

Definition 2.1. Let T be a total order. The *embedding rank* of T, denoted $\mathrm{emrk}(T)$, is the least ordinal α so that α does not embed as an order into T. In other words, the following are equivalent for any ordinal α:

- $\alpha < \mathrm{emrk}(T)$
- There is an order preserving injection $f : \alpha \hookrightarrow T$
- There is a well ordered subset $A \subseteq T$ of order type α.

Example 2.2. For any ordinal α, it is clear that $\mathrm{emrk}(\alpha) = \alpha + 1$. Moreover, if α is infinite, we have $\mathrm{emrk}(\alpha^{\mathrm{op}}) = \omega$.

Example 2.3. One can show that $\mathrm{emrk}(\mathbb{Q}) = \mathrm{emrk}(\mathbb{R}) = \omega_1$. For $\mathbb{Q}$, this follows from the fact that $\mathbb{Q}$ is a saturated model of the theory of total orders, hence it embeds any countable ordinal. Yet, it cannot embed ω_1, as $\mathbb{Q}$ is countable. For $\mathbb{R}$, it is clear that $\mathrm{emrk}(\mathbb{R}) \geq \omega_1$ as $\mathbb{R}$ contains $\mathbb{Q}$. Morover, if $\mathbb{R}$ could embed ω_1, then between any r_α and $r_{\alpha+1}$, we could find some rational number, thereby finding $\aleph_1$ many rationals.

We may use the previous example as inspiration for the following fact:

Proposition 2.4. If T is a total order, and S is a dense subset of T which contains T's endpoints, if T has any, then $\mathrm{emrk}(T) = \mathrm{emrk}(S)$.

We start by working towards the following theorem:

Theorem 2.5. For any ordinal $\alpha > 0$ there is an order T with $\mathrm{emrk}(T) = \alpha$.

We will construct T in two parts using the following lemma:

Lemma 2.6. For orders T, S, the embedding rank on their sum follows the following formula:

$$\text{emrk}(T+S) = \sup_{\alpha<emrk(T)} (\alpha + \text{emrk}(S))$$

With this in mind, we define the following operation on ordinals:

$$\alpha \oplus \beta = \sup_{\gamma<\alpha} (\gamma + \beta).$$

Proof. Firstly suppose $W \subseteq T + S$ is well ordered. We may split W as $W_0 + W_1$ where $W_0 \subseteq T$ and $W_1 \subseteq S$. By definition we have $\text{ot}(W_0) < \text{emrk}(T)$ and $\text{ot}(W_1) < \text{emrk}(S)$. Then we have

$$\text{ot}(W) = \text{ot}(W_0) + \text{ot}(W_1) < \text{ot}(W_0) + \text{emrk}(S) \leq \sup_{\alpha<\text{emrk}(T)} \alpha + \text{emrk}(S).$$

On the other hand, suppose $\gamma < \sup_{\alpha<emrk(T)} \alpha + \text{emrk}(S)$. Then there is an ordinal $\alpha < \text{emrk}(T)$ so that $\gamma < \alpha + \text{emrk}(S)$. Hence, there is a $\beta < \text{emrk}(S)$ such that $\gamma \leq \alpha + \beta$. We may well ordered subsets $A \subseteq T$ and $B \subseteq S$ so that $\text{ot}(A) = \alpha$ and $\text{ot}(B) = \beta$. Then $A + B \subseteq T + S$ is a well ordered subset with $\gamma \leq \text{ot}(A + B)$. Hence, $\gamma < \text{emrk}(T + S)$. □

Proposition 2.7. We have the following properties of $\oplus$:

(i) $\alpha \oplus \beta \leq \alpha + \beta$.

(ii) $(\alpha + 1) \oplus \beta = \alpha + \beta$.

(iii) If $\alpha_1 \leq \alpha_2$ then $\alpha_1 \oplus \beta \leq \alpha_2 \oplus \beta$ and $\beta \oplus \alpha_1 \leq \beta \oplus \alpha_2$

(iv) $(\alpha_1 \oplus \alpha_2) \oplus \alpha_3 = \alpha_1 \oplus (\alpha_2 \oplus \alpha_3)$

(v) $(\alpha_1 + \alpha_2) \oplus \alpha_3 = \alpha_1 + (\alpha_2 \oplus \alpha_3)$

(vi) If $\alpha \geq \omega^\gamma \geq \beta$ then $\beta \oplus \alpha = \alpha$.

(vii) If $\alpha \oplus \beta_1 = \beta_1$ and $\beta_2 \geq \beta_2$ then $\alpha \oplus \beta_2 = \beta_2$

Proof. The proofs of (i)-(iii) are clear. The proof of (iv) comes down to the fact that for any function $f : X \times Y \to \text{Ord}$, we have

$$\sup_{x\in X}(\sup_{y\in Y} f(x, y)) = \sup_{y\in Y}(\sup_{x\in X} f(x, y)).$$

In particular, let $f(x, y) = x + y + \alpha_3$ and let $X = \alpha_1$, and $Y = \alpha_2$. For (v), we see that

$$\begin{aligned} LHS &= \sup_{\beta<\alpha_1+\alpha_2} (\beta + \alpha_3) \\ &= \sup_{\beta'<\alpha_2} (\alpha_1 + \beta' + \alpha_3) \\ &= \sup_{\beta''<\alpha_2\oplus\alpha_3} (\alpha_1 + \beta'') \\ &= RHS \end{aligned}$$

For (vi), we verify that whenever $\delta < \omega^\gamma \leq \alpha$, we have $\delta + \alpha = \alpha$. Hence,

$$\sup_{\delta<\beta} \delta + \alpha = \alpha.$$

For (vii), we see that if $\alpha \oplus \beta_1 = \beta_1$, then $\gamma + \beta_1 = \beta_1$ for all $\gamma < \alpha$. Hence $\gamma + \beta_2 = \beta_2$ for all $\gamma < \alpha$, and so $\alpha \oplus \beta_2 = \beta_2$. □

Corollary 2.7.1. For any total order T and any ordinal α, we have $\mathrm{emrk}(\alpha + T) = \alpha + \mathrm{emrk}(T)$

Proof. This follows from (ii), as $\mathrm{emrk}(\alpha) = \alpha + 1$. $\square$

Now, using Cantor normal form, any $\alpha > 0$ may be written as $\beta + \omega^\gamma$ For some ordinals β and γ. Hence, using corollary 2.7.1 it suffices to prove theorem 2.5 for ordinals of the form ω^γ.

Lemma 2.8. For any ordinal α, there is a total order T with $\mathrm{emrk}(T) = \omega^\alpha$.

Proof. Firstly, take the successor case $\alpha = \beta + 1$. Define

$$T = \omega^{\mathrm{op}} \times \omega^\beta.$$

We see that if $W \subseteq T$, then there must be some finite n so that $W \subseteq n \times \omega^\beta$. Hence, we have

$$\mathrm{emrk}(T) = \sup_{n\in\omega} \mathrm{emrk}(n \times \omega^\beta) = \sup_{n\in\omega}(\omega^\beta \cdot n + 1) = \omega^{\beta+1}.$$

Now suppose α is a limit. Define

$$T = \sum_{\dot\beta\in\gamma^{\mathrm{op}}} \omega^\beta.$$

We have that for any well ordered $W \subseteq T$, there must be some finite subset $A \subseteq \gamma^{\mathrm{op}}$ such that

$$W \subseteq \sum_{\dot\beta\in A} \omega^\beta.$$

In particular, sinec α is a limit, there is some β_0 such that $\beta_0 > \beta$ for all $\dot\beta \in A$. Hence,

$$\mathrm{ot}(W) \leq \sum_{\dot\beta\in A} \omega^{\beta_0} = \omega^{\beta_0}\dot{|}A| < \omega^\alpha.$$

Therefore, $\mathrm{emrk}(T) \leq \omega^\alpha$. On the other hand, ω^β embeds into T for each $\beta < \alpha$. Hence,

$$\mathrm{emrk}(T) \geq \sup_{\beta<\alpha} \mathrm{emrk}(\omega^\beta) = \omega^\alpha.$$

$\square$

Proof of 2.5. Using Cantor normal form, α can be expressed as $\beta + \omega^\gamma$. By lemma 2.8 there exists an order T with $\mathrm{emrk}(T) = \omega^\gamma$. Then by corollary 2.7.1, $\mathrm{emrk}(\beta + T) = \beta + \omega^\gamma = \alpha$. $\square$

3 Basis for Orders of Uncountable Embedding Rank

For this section, we will be interested primarily in orders of uncountable embedding rank. In other words, orders which embed every countable ordinal. Two very natural orders of uncountable embedding rank are $\mathbb{Q}$ and ω_1, and it is clear that neither embed in the other. However, if we follow the construction from the previous section to construct a linear order of embedding rank $\omega^{\omega_1} = \omega_1$, we obtain a linear order which does not embed ω_1 or $\mathbb{Q}$. For this section, we will give this order the name μ.

Definition 3.1.

$$\mu = \sum_{\dot\beta\in\omega_1^{\mathrm{op}}} \beta.$$

The goal of the section is to show that in fact $\{\mathbb{Q}, \omega_1, \mu\}$, is a basis for orders of uncountable embedding rank.

Theorem 3.2. If T is an order with uncountable embedding rank, then T contains a subset isomorphic to one of $\mathbb{Q}$, ω_1, and μ.

The following proposition shows that having countable embedding rank is a nice notion of smallness, in that it is closed under interleavings.

Lemma 3.3. Let T be a total order and let X_0 and X_1 be subsets of T. If X_0 and X_1 have countable embedding rank then so does $X_0 \cup X_1$.

Proof. Let $\alpha_i = \mathrm{emrk}(X_i)$ for $i = 0, 1$. By assumption, both α_i are countable. We show that if $C \subseteq X_0 \cup X_1$ is well ordered, then $\mathrm{ot}(C) \leq \alpha_0 \cdot \alpha_1$. Hence $\mathrm{emrk}(X \cup Y) \leq \alpha_0 \cdot \alpha_1 + 1 < \omega_1$.

Suppose $C \subseteq X_0 \cup X_1$ is well ordered. Then we may enumerate $C \cap X_1$ in order as $\{c_\beta\}_{\beta<\gamma}$ where $\gamma = \mathrm{ot}(C \cap X_1)$. Now, we may split C into segments C_β defined as:

$$x \in C_\beta \Leftrightarrow [\beta = \gamma \wedge X_1 \leq x] \vee [\beta \text{ is minimal so that } x < c_\beta].$$

Abusing notation by comparing simultaneously all elements of a set to a single element, we have the following (if $\gamma \geq \omega$, for example):

$$C_0 < c_0 \leq C_1 < c_1 \leq \cdots \leq C_\omega < c_\omega \leq \cdots \leq C_\gamma$$

In other words,

$$C = \sum_{\beta \leq \gamma} C_\beta. \tag{1}$$

As C is well ordered, each piece C_β is well ordered. So

$$\mathrm{ot}(C) = \sum_{\beta \leq \gamma} \mathrm{ot}(C_\beta).$$

Fix some $\beta \leq \gamma$. Suppose $x \in C_\beta \cap X_1$. Then $x = c_{\beta'}$ for some $\beta' < \gamma$. It follows from the definition of C_β that β' must be the predecessor of β. Furthermore, $c_{\beta'}$ will be the least element of C_β, as any $y < c_{\beta'}$ must be in some $C_{\beta''}$ for some $\beta'' \leq \beta'$. What we have shown is that if $C_\beta \cap X_1$ is nonempty, then it is the singleton containing the least element of C_β. Therefore, we have

$$C_\beta = (C_\beta \cap X_1) + (C_\beta \setminus X_1).$$

As $C \subseteq X_0 \cup X_1$, the remainder $C_\beta \setminus X_1$ is contained in X_0. Then we have the bound $\mathrm{ot}(C_\beta \setminus X_1) < \alpha_0$. Moreover, $|C_\beta \cap X_1| \leq 1$. Thus,

$$\mathrm{ot}(C_\beta) \leq 1 + \mathrm{ot}(C_\beta \setminus X_1) \leq \alpha_0.$$

Now, using this bound, and equation (1), we get the bound

$$\mathrm{ot}(C) \leq \alpha_0 \cdot (\gamma + 1).$$

Since $\gamma = \mathrm{ot}(C \cap X_1)$, we have $\gamma < \alpha_1$. Thus,

$$\mathrm{ot}(C) \leq \alpha_0 \cdot \alpha_1.$$

□

Definition 3.4. Let T be a total order, and suppose $\alpha \leq \mathrm{emrk}(T)$. We say a total order T is *α-left-heavy* if the only final segment of T whose embedding rank is at least α is T. Similarly, a total order T is *α-right-heavy* if the only initial segment T with embedding rank at least α is T.

We caution that the notions of left-heaviness and right-heaviness are not dual. That is, if T is α-left-heavy, then T^{op} is not necessarily α-right-heavy. This is due to the fact that embedding rank is not symmetric. In general $\mathrm{emrk}(T)$ is not equal to $\mathrm{emrk}(T^{\mathrm{op}})$.

Lemma 3.5. Any countable subset of a ω_1-left-heavy set T is bounded below in T.

Proof. First we show that T cannot have a least element. Suppose to the contrary that T has a least element ℓ. Taking $T' = T \setminus \{\ell\}$, we have $T = \{\ell\} \cup T'$. Since T is ω_1-left-heavy, and T is a final segment, T has countable embedding rank. But clearly so does $\{\ell\}$. Then by lemma 3.3, T has a countable embedding rank. This is a contradiction.

Now, let X be a countable subset of T. For any $x \in X$, as we just showed, $[x, \infty)$ cannot be all of T. Then $[x, \infty)$ is a proper final segment of T, so its embedding rank must countable. Define

$$\alpha = \sup_{x \in X} \{\mathrm{emrk}([x, \infty))\}.$$

Since X is countable and $\mathrm{emrk}([x, \infty))$ is countable for each $x \in X$, it follows that α is countable. Hence, there is some $A \subseteq T$ of order type α. Let a_0 be the least element of A. Fix an $x \in X$. If $x \leq a_0$, then $A \subseteq [x, \infty)$. But then $\mathrm{emrk}([x, \infty)) > \alpha$, which is impossible from the definition of α. Thus, $a_0 < x$ for all $x \in X$. □

Theorem 3.6. Any left heavy T embeds μ.

Proof. We will build a decreasing sequence $\{t_\alpha\}_{\alpha < \omega_1}$ in T so that $\alpha \hookrightarrow [t_{\alpha+1}, t_\alpha)$ for each $\alpha < \omega_1$.

We build this sequence by transfinite recursion. In the limit case of defining t_α, we simply use lemma 3.5 and let t_α be a lower bound for the set $\{t_\beta \mid \beta < \alpha\}$. Now, for a successor stage, suppose we have defined t_α. We aim to find some $s < t_\alpha$ so that $\alpha \hookrightarrow [s, t_\alpha)$.

As a special case of lemma 3.5 taking $X = \{t_\alpha\}$ (and as proved directly in the proof of lemma 3.5), t_α cannot be the least element of T. Hence, $[x, infty)$ is a proper final segment, and so it has countable embedding rank. By lemma 3.3, that must mean (∞, x) has uncountable embedding rank. Then there is some $A \subseteq (\infty, x)$ of order type α. Taking $t_{\alpha+1}$ to be the least element of A, we have that $A \subseteq [t_{\alpha+1}, t_\alpha)$. This completes the recursion.

Finally, for each $\alpha < \omega_1$, choose some $A_\alpha \subseteq [t_{\alpha+1}, t_\alpha)$ of order type α. Then we have

$$A = \bigcup_{\alpha \in \omega_1} A_\alpha = \sum_{\dot{\alpha} \in \omega_1^{\mathrm{op}}} A_\alpha$$

is a subset of T isomorphic to μ. □

Lemma 3.7. Any countable subset of a right heavy set T has a strict upper bound in T.

Proof. Let X be a countable subset of T. For each $x \in X$, the inital segment $(-\infty, x)$ is proper, so it has countable embedding rank. Let

$$\alpha = \sup_{x \in X} \mathrm{emrk}((-\infty, x))$$

Since X is countable, and each $\mathrm{emrk}((-\infty, x))$ is countable, α is countable. Then there is some subset $A \subseteq T$ of order type $\alpha + 1$. Let a be the greatest element of A, and let $A' = A \setminus \{a\}$. For any $x \in X$, if $a \leq x$, then $A' \subseteq (-\infty, x)$. But then $\mathrm{emrk}((-\infty, x)) > \mathrm{ot}(A') = \alpha$. This is impossible from the definition of α. Hence, $x < a$ for all $x \in X$. □

Theorem 3.8. Any ω_1-right-heavy T embeds ω_1.

Proof. This is the same as the proof of 3.6, except easier as we only need to find an increasing sequence $\{t_\alpha\}_{\alpha<\omega_1}$. Simply define t_α given that t_β are defined for $\beta < \alpha$ by applying lemma 3.7 to $\{t_\beta \mid \beta < \alpha\}$. □

In fact, theorems 3.6 and 3.8 can be seen as exactly characterizing when an order of uncountable cofinality embeds μ or ω_1 respectively. Since μ is ω_1-left-heavy, we have that an order T of uncountable embedding rank embeds μ if and only if T contains a subset which is ω_1-left-heavy. Similarly, an order of uncountable embedding rank embeds ω_1 if and only if it is ω_1-right-heavy. Hence, throwing out all orders which do not embed μ or ω_1, we are left with the following orders:

Definition 3.9. A total order T is *distributed* if every subset of T is neither ω_1-left-heavy nor ω_1-right-heavy.

The following theorem completes the proof of theorem 3.2. Moreover, since $\mathbb{Q}$ is distributed, it characterizes exactly the orders of uncountable embedding rank which embed $\mathbb{Q}$.

Theorem 3.10. If an order T has uncountable embedding rank, and is distributed, then T embeds $\mathbb{Q}$.

Proof. For any elements x, y, let $[\{x, y\}]$ denote the closed interval $[\min\{x, y\}, \max\{x, y\}]$. Define the relation $\sim$ on T by

$$x \sim y \Leftrightarrow \mathrm{emrk}[\{x, y\}] < \omega_1$$

In words, $x \sim y$ if the closed interval between them has countable embedding rank. Considering countable embedding rank as a smallness notion, $\sim$ should be thought of as a closeness notion for T. It is clearly symmetric and reflexive. Moreover, since $[\{x, z\}] \subseteq [\{x, y\}] \cup [\{y, z\}]$, it follows from lemma 3.3 that $\sim$ is transitive. Finally, it is convex, since if $z \in [\{x, y\}]$, then $[\{x, z\}] \subseteq [\{x, y\}]$. Hence, $x \sim y$ implies $x \sim z$. This is enough to justify taking the quotient $Q = T/\sim$, and equipping Q with the order inherited from T.

We claim that each $\sim$ class has countable embedding rank. Suppose to the contrary that some class $C = [c]$ has uncountable embedding rank. Consider $X = C \cap [c, \infty)$ and $Y = C \cap (-\infty, c)$. Since $C = X \cap Y$, and $\mathrm{emrk}(C) \geq \omega_1$, it follows from lemma 3.3 that at least one of X and Y must have uncountable embedding rank. We will assume $\mathrm{emrk}(Y) \geq \omega_1$; the argument in the other case is symmetric. Since T is balanced, Y cannot be ω_1-right-heavy. Hence, it has some proper initial segment I with uncountable embedding rank. In particular, I has some upper bound $y \in Y$. Then $I \subseteq [c, y]$. This is a problem though, as then $\mathrm{emrk}([c, y]) \geq \omega_1$, even though c and y are in the same $\sim$ class C. Hence, we have a contradiction, and the claim is shown.

Finally, we claim Q is dense and not a singleton. It follows that Q embeds $\mathbb{Q}$. Moreover, by choosing a representative from each class in Q, we can embed Q into T. Hence, T embeds $\mathbb{Q}$. Since each $\sim$ class has countable embedding rank, lemma 3.3 implies Q is not a single element, as $\mathrm{emrk}(T) \geq \omega_1$. If $C = [c]$ and $D = [d]$ are distinct elements of Q, then $\mathrm{emrk}([c, d]) \geq \omega_1$. By lemma 3.3, $\mathrm{emrk}(C \cup D) < \omega_1$. Hence, $[c, d] \not\subseteq C \cup D$, and so there must be some $b \in [c, d]$ with $[c] < [b] < [d]$. □

4 Embedding Ranks of Symmetric Orders

In this section, we will try to control the embedding ranks of symmetric orders, i.e. orders T which are isomorphic to their duals. Such orders can always be decomposed as either $S^{\mathrm{op}} + S$, or

$S^{\mathrm{op}} + \{*\} + S$, for some order S. We will stick with constructing the former type of symmetric order, as they are slightly easier to work with, and suffice for finding orders of a given embedding rank α for any α except finite even numbers.

Proposition 4.1. Suppose T is an order and α is an infinite cardinal. Let $S = \alpha + T$. Then

$$\operatorname{emrk}(S^{\mathrm{op}} + S) = \operatorname{emrk}(T^{\mathrm{op}}) \oplus (\alpha + \operatorname{emrk}(T)). \tag{i}$$

If in addition we have $\operatorname{emrk}(T) = \operatorname{emrk}(T^{\mathrm{op}} + T)$, then

$$\operatorname{emrk}(S^{\mathrm{op}} + S) = \operatorname{emrk}(S) = \alpha + \operatorname{emrk}(T). \tag{ii}$$

Proof. From lemma 2.6, we have

$$\begin{aligned} \operatorname{emrk}(S^{\mathrm{op}} + S) &= \operatorname{emrk}(T^{\mathrm{op}}) \oplus \operatorname{emrk}(\alpha^{\mathrm{op}}) \oplus \operatorname{emrk}(\alpha) \oplus \operatorname{emrk}(T) \\ &= \operatorname{emrk}(T^{\mathrm{op}}) \oplus \omega \oplus (\alpha + 1) \oplus \operatorname{emrk}(T). \end{aligned}$$

Using proposition 2.7 (ii), (vi), and the assumption that $\alpha \geq \omega$, we get

$$\operatorname{emrk}(T^{\mathrm{op}}) \oplus \omega \oplus (\alpha + 1) \oplus \beta = \operatorname{emrk}(T^{\mathrm{op}}) \oplus (\alpha + \operatorname{emrk}(T)).$$

This proves (i). Now, assume $\operatorname{emrk}(T) = \operatorname{emrk}(T^{\mathrm{op}} + T)$. Then $\operatorname{emrk}(T) = \operatorname{emrk}(T^{\mathrm{op}}) \oplus \operatorname{emrk}(T)$. By proposition 2.7,

$$\operatorname{emrk}(T^{\mathrm{op}}) \oplus (\alpha + \beta) = \alpha + \beta.$$

□

We will need to make use of an order we construted in the first section again.

Definition 4.2. If α is an ordinal, define

$$T_\alpha = \sum_{\dot{\beta} \in \alpha^{\mathrm{op}}} \beta.$$

We are in particular interested in the case when α is a regular cardinal. For our sakes, we will consider ω a regular cardinal.

Proposition 4.3. If κ is a regular cardinal, then

- $\operatorname{emrk}(T_\kappa) = \kappa$
- $\operatorname{emrk}(T_\kappa^{\mathrm{op}}) = \kappa + 1$.

Proof. The first item was shown in lemma 2.8. For the latter, we see that

$$(T_\kappa)^{\mathrm{op}} = \sum_{\alpha \in \kappa} \alpha^{\mathrm{op}}.$$

Then we may easily embed κ into T_κ^{op} by taking a point from each component. It remains to show that no well ordered subset of T_κ^{op} has order type $\kappa + 1$. Suppose to the contrary that there were such a well ordered subset W. Since $\operatorname{ot}(W) = \kappa + 1$, there is a greatest element of W, say m. Then we have $m \in \alpha^{\mathrm{op}}$ for some $\alpha < \kappa$. Moreover, since m is the greatest element of W we have

$$W \subseteq \sum_{\beta \leq \alpha} \beta^{\mathrm{op}}.$$

But since κ is a regular cardinal, the right hand side has size less than κ. However, $\operatorname{ot}(W) = \kappa + 1$ so $|W| = \kappa$. This is a contradiction, so the proposition is proven. □

Theorem 4.4. If $\alpha \geq 2$ is an ordinal which is not a weakly compact cardinal, then there exists an order T such that $\operatorname{emrk}(T) = \operatorname{emrk}(T^{\mathrm{op}} + T) = \omega^\alpha$.

Proof. We will break this up into three cases.

Case 1: (successor)

Suppose $\alpha = \beta + 1$. Take

$$T = \omega^{\mathrm{op}} \times \omega^\beta.$$

As seen in section 2, we have $\operatorname{emrk}(T) = \omega^\alpha$. We also claim $\operatorname{emrk}(T^{\mathrm{op}}) \leq \omega + 1$. We observe that $T^{\mathrm{op}} \cong \omega \times (\omega^\beta)^{\mathrm{op}}$. Then if $W \subseteq T^{\mathrm{op}}$ is well ordered, W can contain only finitely many points within each copy of $(\omega^\beta)^{\mathrm{op}}$. That means W is the ω-indexed sum of finite orders, and so $\operatorname{ot}(W) \leq \omega$. Thus, $\operatorname{emrk}(T^{\mathrm{op}}) \leq \omega + 1$. Now, since $\alpha \geq 2$, we have

$$\begin{aligned}
\operatorname{emrk}(T^{\mathrm{op}} + T) &= \operatorname{emrk}(T^{\mathrm{op}}) \oplus \operatorname{emrk}(T) \\
&\leq (\omega + 1) \oplus \omega^\alpha \\
&= \omega^\alpha.
\end{aligned}$$

Case 2: (α a limit ordinal and $\operatorname{cf}(\alpha) < \alpha$)

Suppose $\operatorname{cf}(\alpha) < \alpha$. Let $\lambda = \operatorname{cf}(\alpha)$, and let $(\beta_\nu)_{\nu<\lambda}$ be an increasing sequence of ordinals with supremum α. Take

$$T = \sum_{\dot{\nu} \in \lambda^{\mathrm{op}}} \omega^{\beta_\nu}.$$

First, note that T is a subset of the order we defined in lemma 2.8, and so $\operatorname{emrk}(T) \leq \omega^\alpha$. However, as the β_ν are cofinal in α, we still get $\operatorname{emrk}(T) \geq \omega^\alpha$. For the same reasoning as in case 1, we have $\operatorname{emrk}(T^{\mathrm{op}}) \leq \lambda + 1$. Now, we have $\lambda < \alpha \leq \omega^\alpha$. Hence $\lambda + \omega^\alpha = \omega^\alpha$, and so

$$\begin{aligned}
\operatorname{emrk}(T^{\mathrm{op}} + T) &= \operatorname{emrk}(T^{\mathrm{op}}) \oplus \operatorname{emrk}(T) \\
&\leq (\lambda + 1) \oplus \omega^\alpha \\
&= \omega^\alpha.
\end{aligned}$$

Case 3: (not weakly compact but regular) Let α be an regular cardinal κ that is not weakly compact. Then there exists an order T of size κ that embeds neither κ nor κ^{op}. That is, $\operatorname{emrk}(T)$ and $\operatorname{emrk}(T^{\mathrm{op}})$ are both at most κ. Since $|T| = \kappa$ we may index κ by elements of T. Fix such an indexing $(\alpha_t)_{t\in T}$. Define

$$S = \sum_{t \in T} \alpha_t.$$

It is clear that $\operatorname{emrk}(S) \geq \kappa$. Now, we show that $\operatorname{emrk}(S) \leq \kappa$. Suppose $W \subseteq S$ is a well order. Then we must have a well ordered subset $V \subseteq T$ so that

$$W \subseteq \sum_{t \in V} \alpha_t.$$

But by assumption, $\operatorname{ot}(V) < \operatorname{emrk}(T) \leq \kappa$. Hence, by the regularity of κ, the above sum is smaller than κ.

The same argument shows $\operatorname{emrk}(S^{\mathrm{op}}) \leq \kappa$. Thus, we have

$$\operatorname{emrk}(S^{\mathrm{op}} + S) \leq \kappa \oplus \kappa = \kappa.$$

$\square$

Theorem 4.5. For an ordinal $\alpha > 0$, there exists a symmetric order with embedding rank α if and only if α is not weakly compact.

Proof. Firstly, suppose α is a weakly compact cardinal κ. Then for any order T with $|T| \geq \kappa$, there either is an ascending or descending sequence $W \subseteq T$ of order type κ. In our language, if $|T| \geq \kappa$ then $\operatorname{emrk}(T) > \kappa$ or $\operatorname{emrk}(T^{\mathrm{op}}) > \kappa$. Now, if T is symmetric, then $\operatorname{emrk}(T) = \operatorname{emrk}(T^{\mathrm{op}})$. Hence, if $|T| \geq \kappa$ we can conclude $\operatorname{emrk}(T) > \kappa$. On the other hand, if $|T| < \kappa$, then certainly $\operatorname{emrk}(T) \leq |T|^+ < \kappa$. Hence, We can never have $\operatorname{emrk}(T) = \kappa$ if T is symmetric.

Now suppose α is not weakly compact. We may write

$$\alpha = \beta + \omega^\gamma \cdot n$$

for ordinals β, γ, and $n \in \omega$ such that either $\beta \geq \omega^{\gamma+1}$ or $\beta = 0$. We split into three cases:

Case 1: γ is a weakly compact cardinal κ and $\beta = 0$

In this case, $\alpha = \kappa \cdot n$. Since we assumed α is not weakly compact, we must have $n \geq 2$. Let $\delta = \kappa \cdot (n-2)$. Consider the order

$$T = \delta + T_\kappa.$$

From proposition 4.1 (i) and 4.3, we get

$$\operatorname{emrk}(T^{\mathrm{op}} + T) = (\kappa + 1) \oplus (\delta + \kappa).$$

Proposition 2.7 (ii) gives

$$(\kappa + 1) \oplus (\delta + \kappa) = \kappa + \delta + \kappa = \kappa \cdot n.$$

Case 2: γ is a weakly compact cardinal κ, and $\beta \geq \omega^{\gamma+1}$

In this case, let $\delta = \beta + \kappa \cdot (n-1)$, and let

$$T = \delta + T_\kappa.$$

We again have

$$\operatorname{emrk}(T^{\mathrm{op}} + T) = \kappa + \delta + \kappa.$$

Now, since $\delta \geq \beta \geq \gamma^{\gamma+1} > \kappa$, we get

$$\kappa + \delta + \kappa = \delta + \kappa = \alpha.$$

Case 3: γ is not weakly compact In this case, we may use lemma 4.4 to find an order T' so that $\operatorname{emrk} T' = \operatorname{emrk} T'^{\mathrm{op}} + T'$. Let

$$T = \beta + \omega^\gamma \cdot (n-1) + T'.$$

By propositon 4.1 (ii), we have

$$\operatorname{emrk}(T) = \operatorname{emrk}(T^{\mathrm{op}} + T) = \beta + \omega^\gamma \cdot (n-1) + \operatorname{emrk}(T') = \alpha.$$

□