Counting Suborders

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Abstract

In this paper, we show that any order has at least as many suborder types as elements.

1 Notation

Throughout, T will denote a linear order, letters κ, λ, μ will denote cardinals, and α, β, γ will denote ordinals. For any order T, we will use T^* to denote the order dual of T. Often we will be considering cardinals κ as orders. As such, κ^* means the order dual of κ.

For multiple sections, we will need to be careful about subtly different notions of intervals. We will use the following conventions:

- An **open interval** (resp. **closed interval**) of T is any set of the form $(a, b)_T = \{x \in T \mid a < x < b\}$ (resp. $[a, b]_T = \{x \in T \mid a \le x \le b\}$.)

- A set $X \subseteq T$ is called **convex** if whenever $x, y \in X$ and $x \le a \le y$, we have $a \in X$. Equivalently, if $[x, y]_X = [x, y]_T$ for all $x, y \in X$.

- An **open left ray** of T is a set of the form $(-\infty, a)_T = \{x \in T \mid x < a\}$. Analogous definitions and notations can be made exchanging open/closed or left/right.

- A set $X \subseteq T$ is called an **initial segment** (resp. **final segment**) if whenever $x \in X$ and $a \le x$ (resp. $a \ge x$), we have $a \in X$.

We note that left/right rays are always initial/final segments, and that intervals and initial/final segments are always convex. The converses to these statements are all false in general. Additionally, singletons are always convex somwhat trivially. Hence, we call a convex set **nontrivial** if it contains at least 2 points.

2 Balanced Orders

Definition 1. A linear order T is κ-**balanced** if $|T| \ge \kappa$, and for every subset $X \subseteq T$ with $|X| \ge \kappa$, there exists both a left ray and a right ray each of size at least κ.

Lemma 2. Let T be a linear order and let $\kappa \le |T|$ be regular. If every left ray of T has size less than κ, then κ embeds as an order into T.

Proof. We will define an increasing sequence $(t_\alpha)_{\alpha < \kappa}$ in T recursively. Suppose for some $\alpha < \kappa$ that we have defined t_β for all $\beta < \alpha$. Consider the set

$$X = \bigcup_{\beta < \alpha} (-\infty, t_\beta]_T.$$

By assumption, each ray $(-\infty, t_\beta]$ has size less than κ. Then by the regularity of κ, we have

$$|X| = \sum_{\beta < \alpha} |(-\infty, t_\beta]_T| < \kappa.$$

Since $|T| \ge \kappa$, it follows that $T \setminus X \ne \emptyset$. Then pick t_α to be any element of $T \setminus X$. By construction, $t_\beta < t_\alpha$ for all $\beta < \alpha$. This completes the construction of the sequence, hence κ embeds into T. $\quad\square$

Corollary 2.1. If κ is regular, then an order T is not κ-balanced iff it embeds either κ or κ^*

Proof. If T is not κ-balanced then there exists an $X \subseteq T$ with $|X| \geq \kappa$ such that either all of X's left rays have size less than *kappa*, or all of X's right rays have size less than κ. Lemma 2 tells us that in the former case, κ embeds into X, and in the latter case, κ embeds into X^*. Either way, one of κ and κ^* embeds into X, and therefore into T.

Conversely, κ and κ^* are not κ-balanced. Thus, any order that embeds κ or κ^* is not κ-balanced either. $\qquad\square$

Definition 3. If T is an order, then an equivalence relation $\sim$ on T is called **convex** if all of its classes are convex.

Definition 4. Given a linear order T and an infinite cardinal κ, define the κ-close relation $\approx_\kappa$ by $x \approx_\kappa y$ if there are fewer than κ many elements between x and y in T.

Proposition 5. The κ-close relation is a convex equivalence relation.

Proof. It is clearly reflexive and symmetric. Let

$$I_{x,y} = [\min\{x,y\}, \max\{x,y\}]_T.$$

We have that $x \approx_\kappa y$ if and only if $|I_{x,y}| < \kappa$. Towards showing the transitivity of $\approx_\kappa$, suppose $x \approx_\kappa y$ and $z \approx_\kappa y$. It is easy to check by casework that

$$I_{x,z} \subseteq I_{x,y} \cup I_{y,z}.$$

Therefore, we have

$$|I_{x,z}| \leq |I_{x,y}| + |I_{y,z}| < \kappa + \kappa = \kappa.$$

Thus, $x \approx_\kappa z$.

Finally, let $x \leq y \leq z$ and $x \approx_\kappa z$. In this case, $I_{x,y} \subseteq I_{x,z}$. Hence, $|I_{x,z}| < kappa$ impies $|I_{x,y}| < \kappa$. Thus, $\approx_\kappa$ is convex. $\qquad\square$

Lemma 6. For any infinite κ, if T is κ-balanced, then each equivalence class of $\approx_\kappa$ has size smaller than κ.

Proof. Fix an equivalence class C, and pick a point $a \in C$. Define $L = (-\infty, a]_C$ and $R = [a, \infty)_C$. Clearly $C = L \cup R$. Note that for any $b \in L$, we have that $[b, \infty)_L = [b, a]_C$. Since $\approx_\kappa$ is convex, $[b, a]_C = [b, a]_T$. Then since a, b are in the same $\approx_\kappa$ equivalence class, we have $|[b, a]_T| < \kappa$. Tracing this back, we have shown that the size of every right ray of L is smaller than κ. By the κ-balancedness of T, it follows that $|L| < \kappa$. We may use an analogous argument to show $|R| < \kappa$. Hence, $|C| = |L| + |R| < \kappa$. $\qquad\square$

Corollary 6.1. If κ is regular, and T is a κ-balanced linear order then $|T/\approx_\kappa| \geq \kappa$.

Proof. Each equivalence class of $\approx_\kappa$ has size less than κ, and their union is T — a set of size at least κ. By the regularity of κ, there must be at least κ many equivalence classes. $\qquad\square$

Lemma 7. If κ is regular and T is κ-balanced then T contains a κ-dense subset.

Proof. Choose some set $Q \subseteq T$ of representatives of $\approx_\kappa$. By corollary 6.1, $|Q| \geq \kappa$. Consider $a, b \in Q$ with $a < b$. We wish to show that $|[a,b]_Q| \geq \kappa$. Firstly, since $a \neq b$, we know that $|(a,b)_T| \geq \kappa$. Now, take any $x \in T$ with $a < x < b$. Since Q is a choice of representatives for $\approx_\kappa$, there is some $\tilde{x} \in Q$ with $x \approx_\kappa x$. We claim that from the convexity of $\approx_\kappa$, we must have $a \leq \tilde{x} \leq b$. Otherwise we would have either $\tilde{x} < a < x$ or $x < b < \tilde{x}$. In the former case, the convexity of $\approx_\kappa$ gives $a \approx_\kappa \tilde{x}$ and in the latter case $b \approx_\kappa \tilde{x}$. But since $a, b, \tilde{x}$ are representatives of equivalence classes, it would follow that $a = \tilde{x}$ or $b = \tilde{x}$ respectively.

We have just shown that if $a < x < b$, then for the $\tilde{x} \in Q$ with $x \approx_\kappa \tilde{x}$, we must have $a \leq \tilde{x} \leq b$. Thus, we have

$$(a,b)_T \subseteq \bigcup_{\tilde{x} \in [a,b]_Q} [\tilde{x}]_{\approx_\kappa}.$$

By lemma 6, each class $[\tilde{x}]_{\approx_\kappa}$ has size less than κ. Since $|(a,b)_T| \geq \kappa$ and κ is regular, the index set $[a,b]_Q$ must have size at least κ. $\qquad\square$

Corollary 7.1. For any regular κ, every total order of size at least κ either:

- embeds κ

- embeds κ^*

- contains a κ-dense subset

Proof. If the order is κ-balanced then it contains a κ-dense subset by lemma 7. Otherwise, it embeds either κ or κ^* by corollary 2.1. $\qquad\square$

3 Combinatorics

Lemma 8. For any infinite cardinal κ, we have

$$(2^\kappa)^+ \to (\kappa^+)_\kappa^2.$$

Proof. Let $\mu = (2^\kappa)^+$. Let $F : [\mu]^2 \to \kappa$ be any function. Our goal is to find a subset $H \subseteq \mu$ of size κ^+ such that F is constant on $[H]^2$. For each $a \in \kappa$, let $F_a : \kappa \setminus \{a\} \to \kappa$ take x to $F(\{a, x\})$. First we will prove the following claim:

   (1) There exists a subset $A \subseteq \mu$ of size 2^κ such that for any $C \subseteq A$ with $|C| \leq \kappa$, and any $u \in \kappa \setminus C$, there exists a $v \in A \setminus C$ such that F_v agrees with F_u on C.

We will construct an ascending inclusion sequence $\{A_\alpha\}_{\alpha < \kappa^+}$ of subsets of μ of size 2^κ. First let A_0 be any subset of μ of size 2^κ. Suppose A_α is defined for some $\alpha < \kappa^+$. For any $C \subseteq A_\alpha$ of size at most κ, the number of functions from C to ω is

$$(|C| \cdot |C|)^\kappa \leq \kappa^\kappa \leq (2^\kappa)^\kappa = 2^{\kappa \times \kappa} = 2^\kappa.$$

The number of subsets $C \subseteq A_\alpha$ of size at most κ is $(2^\kappa)^\kappa = 2^\kappa$. Thus, there are only 2^κ many functions from any $C \subseteq A_\alpha$ of size at most κ to κ. In particular,

$$\{F_u \restriction C \mid C \subseteq A_\alpha, |C| \leq \kappa, u \in \mu \setminus C\}$$

has size at most 2^κ. Choosing some representative v for each function in the above set, we obtain a new set $A_{\alpha+1} \supseteq A_\alpha$ of size 2^κ. For any limit $\alpha < \kappa^+$, let $A_\alpha = \cup_{\beta < \alpha} A_\beta$. We have $|A_\alpha| = |\alpha| \cdot 2^\kappa \leq \kappa \cdot 2^\kappa = 2^\kappa$.

Let $A = \cup_{\alpha < \kappa^+} A_\alpha$. Let $C \subseteq A$ have size at most κ, and let $u \in \mu \setminus C$. Each $c \in C$ is in some A_{α_c}. Let $\alpha = \sup\{\alpha_c \mid c \in C\}$. Since $|C| \leq \kappa$, and each $\alpha_c < \kappa^+$, we have $\alpha < \kappa^+$. Then $C \subseteq A_\alpha$. By construction, there is some $v \in A_{\alpha+1} \setminus C$ such that F_u agrees with F_v on C. Furthermore, that v is also in A, therefore the claim is proven.

Fix some $b \in \mu \setminus A$. We will construct a sequence $\{x_\alpha\}_{\alpha < \kappa^+}$ in A. First, let x_0 be arbitrary. Given that x_β is defined for all $\beta < \alpha$ for some $\alpha < \kappa^+$, let $C = \{x_\beta \mid \beta < \alpha\}$. Choose some $x_\alpha \in A$ using (1) such that F_{x_α} agrees with F_b on C. Let $X = \{x_\alpha \mid \alpha < \kappa^+\}$. Define $G : X \to \kappa$ by $G(x) = F_b(x)$. For any $\alpha < \beta < \kappa$, we have $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha)$. By construction, F_{x_β} agrees with F_b on $\{x_\gamma \mid \gamma < \beta\}$, so $F_{x_\beta}(x_\alpha) = F_b(x_\alpha) = G(x_\alpha)$.

Since the codomain of G is κ and its domain has size κ^+, there must be some $y \in \kappa$ whose preimage under G has κ^+ many elements. In other words, there is an $H \subseteq X$ of size κ^+ that G is constant on. For all $x_\alpha, x_\beta \in H$ with $\alpha < \beta$, we have $F(\{x_\alpha, x_\beta\}) = G(x_\beta) = y$. Thus F is constant on $[H]^2$ (with value y). $\square$

Proposition 9. For any regular λ, we have

$$(2^{<\lambda})^+ \to (\lambda)_2^2$$

Proof. We use as a special case of 1.1 in [1] that if κ is some strongly compact cardinal, $\lambda > \kappa$ is regular, and $\zeta, \theta < \kappa$ then

$$(2^{<\lambda})^+ \to (\lambda + \zeta)_\theta^2.$$

As noted in the paper, the result is valid in the case $\kappa = \omega$. For our purposes, we only need the case $\zeta = 0$ and $\theta = 2$. Thus, we have the simplified statement that if $\lambda > \omega$ is regular then $(2^{<\lambda})^+ \to (\lambda)_2^2$. We note that even in the case $\lambda = \omega$, the statement becomes $\omega \to (\omega)_2^2$, which is also true. $\square$

4 Order Types

We introduce some notation which ease the proof of the next lemma. Suppose Ψ is some class of linear orders. Then we say Φ **holds convex hereditarily of** T if Ψ holds for every nontrivial convex subset of T. We notate the class of such T by $\mathcal{CH}[\Psi]$.

Definition 10. For a cardinal κ, let $\mathrm{Emb}(\kappa)$ be the class of orders that embed κ. Equivalently, the class of orders that contain a well order of size κ.

Lemma 11. If κ is a singular cardinal and $T \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cup \mathrm{Emb}(\lambda^*)]$ for all $\lambda < \kappa$, then $T \in \mathrm{Emb}(\kappa) \cup \mathrm{Emb}(\kappa^*)$.

Proof. First we will try to find a nontrivial convex $C \subseteq T$ so that either C is in $\mathcal{CH}[\mathrm{Emb}(\lambda)]$ for all $\lambda < \kappa$, or C^* is. Firstly, suppose $T \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cap \mathrm{Emb}(\lambda^*)]$ for all $\lambda < \kappa$. Then clearly $C = T$ is satisfactory. Otherwise, let λ_0 be the minimal cardinal less than κ so that $T \notin \mathcal{CH}[\mathrm{Emb}(\lambda_0) \cap \mathrm{Emb}(\lambda_0^*)]$. That means there is some nontrivial, convex $C \subseteq T$ so that $C \notin \mathrm{Emb}(\lambda_0)$ or $C \notin \mathrm{Emb}(\lambda_0^*)$. This is the same as saying either C^* or C is not in $\mathrm{Emb}(\lambda_0^*)$.

We may assume that $C \notin \mathrm{Emb}(\lambda_0^*)$, as we may use an analogous argument to the following in the case that $C^* \notin \mathrm{Emb}(\lambda_0)$. We will show $C \in \mathcal{CH}(\lambda)$ for all $\lambda < \kappa$. Fix a nontrivial convex $D \subseteq C$, and some cardinal $\lambda < \kappa$. If $\lambda < \lambda_0$ then by the minimality of λ_0 we have $T \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cap \mathrm{Emb}(\lambda^*)]$. In particular, D is a nontrivial convex subset of T, so $D \in \mathrm{Emb}(\lambda)$. On the other hand, suppose $\lambda \geq \lambda_0$. Since C cannot embed λ_0^* and $\lambda \geq \lambda_0$, clearly a subset $D \subseteq C$ cannot embed λ^*. That is, $D \notin \mathrm{Emb}(\lambda^*)$. However, by assumption $T \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cup \mathrm{Emb}(\lambda^*)]$. Then since D is a nontrivial convex subset of D, we must have $D \in \mathrm{Emb}(\lambda)$. Thus, the claim is shown.

Finally, we will use the fact that $C \in \mathcal{CH}[\mathrm{Emb}(\lambda)$ for all $\lambda < \kappa$ to build an embedding of κ into C. If on the other hand $C^* \in \mathcal{CH}[\mathrm{Emb}(\lambda)]$ for all $\lambda < \kappa$, then we would end up building an embedding of κ into C^*, and hence an embedding of κ^* into C. Either case achieves an embedding of κ or κ^* into T, which is what we are trying to show.

As κ is singular, we may take a cofinal sequence of cardinals $(\lambda_\alpha)_{\alpha < \mu}$ where $\mu = \mathrm{cf}(\kappa) < \kappa$. Since $\mu < \kappa$, we have $C \in \mathrm{Emb}(\mu)$. Hence, we may choose an embedding $f : \mu \to C$. Next, for each $\alpha < \mu$, we have that $[f(\alpha \cdot 2), f(\alpha \cdot 2 + 1)]_C$ is a nontrivial convex subset of C. Hence, $[f(\alpha \cdot 2), f(\alpha \cdot 2 + 1)]_C \in \mathrm{Emb}(\lambda_\alpha)$. Then we may choose some well ordered subset $W_\alpha \subseteq [f(\alpha \cdot 2), f(\alpha \cdot 2 + 1)]_C$ such that the order type of C_α is λ_α. Now, consider the set

$$W = \bigcup_{\alpha < \mu} W_\alpha.$$

Firstly, it is well ordered. Take any nonempty $X \subseteq W$. There is a least $\alpha < \mu$ so that $X \cap W_\alpha \neq \emptyset$. Since W_α is well ordered, there is a least element x of $X \cap W_\alpha$. Now since f is order preserving and $W_\alpha \subseteq [f(\alpha \cdot 2), f(\alpha \cdot 2 + 1)]_C$, we have that x is the least element of all of X. Secondly, we have

$$|W| = \sum_{\alpha < \mu} |W_\alpha| = \sum_{\alpha < \mu} \lambda_\alpha = \kappa.$$

Hence, C contains a well order of size κ. Thus, $C \in \mathrm{Emb}(\kappa)$. $\qquad \square$

Lemma 12. If $\kappa \to (\lambda)_2^2$, then every order of size at least κ embeds either λ or λ^*.

Proof. Assume without loss of generallity that the underlying set of T is κ, ordered by some relation $\prec$. Let $F : [\kappa]^2 \to \{0, 1\}$ send $\{a, b\}$ to 0 if either $a < b$ and $a \prec b$ or $a > b$ and $a \succ b$ and to 1 otherwise. In other words, $F(\{a, b\})$ is 0 if $<$ and $\prec$ agree and is 1 if they disagree. Using the
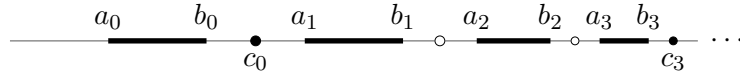
above, there is an $H \subseteq \kappa$ of size λ such that F is constant on $[H]^2$. Since H is a subset of κ, and $|H| \geq \lambda$ it has $<$-order type some ordinal at least λ. Choose some subset $X \subseteq H$ with $<$-order type exactly λ. If $F[H] = \{0\}$ then $\prec$ agrees with $<$ everywhere on X, and so X also has $\prec$-order type λ. In other words, X is an ascending λ sequence in $\prec$. If $F[H] = \{1\}$ then $\prec$ disagrees with $<$ everywhere on X. Then the $\prec$-order type of X will be the dual of the $<$-order type of X. Thus, X is a descending λ sequence in $\prec$. $\qquad \square$

Lemma 13. If λ embeds into a dense order T then T contains at least 2^λ suborder types.

Proof. Choose some embedding $\iota : \lambda \to T$. Using the density of T, choose points $a_\alpha, b_\alpha, c_\alpha$ for $\alpha < \lambda$ such that $\iota(\alpha) = a_\alpha < b_\alpha < c_\alpha < \iota(\alpha + 1)$. Given a subset $X \subseteq \lambda$, define the set $S(X) \subseteq T$ by

$$S(X) = \bigcup_{\alpha < \lambda} [a_\alpha, b_\alpha]_T \cup \{c_\alpha \mid \alpha \in X\}.$$

The following is a depiction of $S(X)$ for some X that contains 0 and 3, but not 1 or 2:

We aim to show that if $X \neq Y$ then $S(X)$ and $S(Y)$ are not isomorphic. It follows that there are at least $|\mathcal{P}(\lambda)| = 2^\lambda$ subordertypes of T.

Suppose there is an order isomorphism $f : S(X) \to S(Y)$. By inspection, we see that the set of b points is a definable subset of $S(X)$ given by the first order formula

$$\varphi_b(x) = (\exists y < x \text{ s.t. the interval } [y, x] \text{ is dense}) \wedge (x \text{ has a successor}).$$

Since f is an order isomorphism, it must preserve definable subsets. In particular, x is a b point iff $f(x)$ is a b point. Since the set of b points is well ordered, it follows by transfinite induction that $f(b_\alpha) = b_\alpha$ for all $\alpha < \lambda$. Similarly, we see that the set of a points is defined by the formula

$$\varphi_a(x) = \exists y > x([\text{the interval } [x, y] \text{ is dense}] \wedge [\forall x' < x \text{ the interval } [x', y] \text{ is not dense}]),$$

and so $f(a_\alpha) = a_\alpha$ for all $\alpha < \lambda$.

Let $\alpha \in X$. There is exactly one point in $S(X)$ between b_α and $a_{\alpha+1}$, namely c_α. Therefore, there must be exactly one point between $f(b_\alpha) = b_\alpha$ and $f(a_{\alpha+1}) = a_{\alpha+1}$. This is only possible if $\alpha \in Y$. By the symmetric argument, if $\alpha \in Y$ then $\alpha \in X$. Thus, we have shown $X = Y$. $\qquad \square$

Theorem 14. Any order T has as at least as many suborder types as elements

Proof. Suppose not. Let κ be the least cardinality so that there is some order T of size κ that has fewer than κ many suborder types. Fix such an order T. From our minimality assumptions, for each $\lambda < \kappa$ there are at least λ many suborder types of T. It follows that there are at least $\sup\{\lambda \mid \lambda < \kappa\}$ suborder types of T. If κ were a limit cardinal then $\sup\{\lambda \mid \lambda < \kappa\} = \kappa$, so κ must be a successor cardinal. In particular, κ is regular.

Next, if T embeds either κ or κ^* then T embeds each ordinal less than κ, or the dual of each ordinal less than κ, giving at least κ many suborder types. Thus, neither can be the case. By corollary 7.1, T must contain a κ-dense subset S. It suffices to show that S contains κ many suborder types.

Let μ be the least cardinal such that $2^\mu \geq \kappa$. By lemma 13, it suffices to show that S has embeds either μ or μ^*. We now argue in two cases:

Case 1: μ is regular. By proposition 9, we have

$$(2^{<\mu})^+ \to (\mu)_2^2.$$

Since μ is minimal, for each $\lambda < \mu$ we have $2^\lambda < \kappa$. Then, since κ is a successor cardinal, we have $2^{<\mu} < \kappa$. It follows that $(2^{<\mu})^+ \leq \kappa$, so $\kappa \to (\mu)_2^2$. By lemma 12, either μ or μ^* embeds into S.

Case 2: μ is singular. By the minimality of μ, for any $\lambda < \mu$, we have $2^\lambda < \kappa$. It follows that $(2^\lambda)^+ \leq \kappa$ for each $\lambda < \mu$, so by lemma 8, we have $(2^\lambda)^+ \to (\lambda)_2^2$. By construction, S is κ-dense. In particular, if C is a nontrivial convex subset of S, then $|C|$ has size at least κ. Then lemma 12 gives us that $|C|$ embeds either λ or λ^* for each $\lambda < \mu$. Hence, $S \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cup \mathrm{Emb}(\lambda^*)]$ for every $\lambda < \mu$. By lemma 11, we are done. $\square$

References

[1] Saharon Shelah. A partition relation using strongly compact cardinals. *Proc. Amer. Math. Soc.*, 131(8):2585–2592, 2003. arXiv: math/0103155.