Counting Suborders

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Abstract

In this paper, we show that any order has at least as many suborder types as elements.

1 Introduction and Notation

We show one main result in this paper, Theorem 14, which says that every linear order contains at least as many nonisomorphic suborders as elements. In section 2 we prove a useful lemma which will reduce the problem in the case that $|T| = \kappa$ is regular. In section 3 we discuss some infinitary combinatorics which we will use to embed sufficiently large ordinals into an order T. In section 4, we use these ordinal embeddings to build many distinct order types within T, and prove the main theorem.

Throughout, T will denote a linear order, letters κ, λ, μ will denote cardinals, and α, β, γ will denote ordinals. For any order T, we will use T^* to denote the order dual of T. Often we will be considering cardinals κ as orders. As such, κ^* means the order dual of κ.

For multiple sections, we will need to be careful about subtly different notions of intervals. We will use the following conventions:

- An **open interval** (resp. **closed interval**) of T is any set of the form $(a,b)_T = \{x \in T \mid a < x < b\}$ (resp. $[a,b]_T = \{x \in T \mid a \leq x \leq b\}$.)
- A set $X \subseteq T$ is called **convex** if whenever $x, y \in X$ and $x \leq a \leq y$, we have $a \in X$. Equivalently, if $[x,y]_X = [x,y]_T$ for all $x, y \in X$.
- An **open left ray** of T is a set of the form $(-\infty, a)_T = \{x \in T \mid x < a\}$. Analogous definitions and notations can be made exchanging open/closed or left/right.
- A set $X \subseteq T$ is called an **initial segment** (resp. **final segment**) if whenever $x \in X$ and $a \leq x$ (resp. $a \geq x$), we have $a \in X$.

We note that left/right rays are always initial/final segments, and that intervals and initial/final segments are always convex. The converses to these statements are all false in general. Additionally, singletons are always convex somwhat trivially. Hence, we call a convex set **nontrivial** if it contains at least 2 points.

2 Balanced Orders

Definition 1. A linear order T is κ-**balanced** if $|T| \geq \kappa$, and for every subset $X \subseteq T$ with $|X| \geq \kappa$, there exists both a left ray and a right ray each of size at least κ.

Lemma 2. Let T be a linear order and let $\kappa \leq |T|$ be regular. If every left ray of T has size less than κ, then κ embeds as an order into T.

Proof. We will define an increasing sequence $(t_\alpha)_{\alpha<\kappa}$ in T recursively. Suppose for some $\alpha < \kappa$ that we have defined t_β for all $\beta < \alpha$. Consider the set

$$X = \bigcup_{\beta<\alpha} (-\infty, t_\beta]_T.$$

By assumption, each ray $(-\infty, t_\beta]$ has size less than κ. Then by the regularity of κ, we have

$$|X| = \sum_{\beta<\alpha} |(-\infty, t_\beta]_T| < \kappa.$$

Since $|T| \geq \kappa$, it follows that $T \setminus X \neq \emptyset$. Then pick t_α to be any element of $T \setminus X$. By construction, $t_\beta < t_\alpha$ for all $\beta < \alpha$. This completes the construction of the sequence, hence κ embeds into T. $\square$

Corollary 2.1. If κ is regular, then an order T is not κ-balanced iff it embeds either κ or κ^*

Proof. If T is not κ-balanced then there exists an $X \subseteq T$ with $|X| \geq \kappa$ such that either all of X's left rays have size less than *kappa*, or all of X's right rays have size less than κ. Lemma 2 tells us that in the former case, κ embeds into X, and in the latter case, κ embeds into X^*. Either way, one of κ and κ^* embeds into X, and therefore into T.

Conversely, κ and κ^* are not κ-balanced. Thus, any order that embeds κ or κ^* is not κ-balanced either. $\square$

Definition 3. If T is an order, then an equivalence relation $\sim$ on T is called **convex** if all of its classes are convex.

Definition 4. Given a linear order T and an infinite cardinal κ, define the κ-close relation $\approx_\kappa$ by $x \approx_\kappa y$ if there are fewer than κ many elements between x and y in T.

Proposition 5. The κ-close relation is a convex equivalence relation.

Proof. It is clearly reflexive and symmetric. Let

$$I_{x,y} = [\min\{x, y\}, \max\{x, y\}]_T.$$

We have that $x \approx_\kappa y$ if and only if $|I_{x,y}| < \kappa$. Towards showing the transitivity of $\approx_\kappa$, suppose $x \approx_\kappa y$ and $z \approx_\kappa y$. It is easy to check by casework that

$$I_{x,z} \subseteq I_{x,y} \cup I_{y,z}.$$

Therefore, we have

$$|I_{x,z}| \leq |I_{x,y}| + |I_{y,z}| < \kappa + \kappa = \kappa.$$

Thus, $x \approx_\kappa z$.

Finally, let $x \leq y \leq z$ and $x \approx_\kappa z$. In this case, $I_{x,y} \subseteq I_{x,z}$. Hence, $|I_{x,z}| < \textit{kappa}$ impies $|I_{x,y}| < \kappa$. Thus, $\approx_\kappa$ is convex. $\square$

Lemma 6. For any infinite κ, if T is κ-balanced, then each equivalence class of $\approx_\kappa$ has size smaller than κ.

Proof. Fix an equivalence class C, and pick a point $a \in C$. Define $L = (-\infty, a]_C$ and $R = [a, \infty)_C$. Clearly $C = L \cup R$. Note that for any $b \in L$, we have that $[b, \infty)_L = [b, a]_C$. Since $\approx_\kappa$ is convex, $[b, a]_C = [b, a]_T$. Then since a, b are in the same $\approx_\kappa$ equivalence class, we have $|[b, a]_T| < \kappa$. Tracing this back, we have shown that the size of every right ray of L is smaller than κ. By the κ-balancedness of T, it follows that $|L| < \kappa$. We may use an analogous argument to show $|R| < \kappa$. Hence, $|C| = |L| + |R| < \kappa$. □

Corollary 6.1. If κ is regular, and T is a κ-balanced linear order then $|T/{\approx_\kappa}| \geq \kappa$.

Proof. Each equivalence class of $\approx_\kappa$ has size less than κ, and their union is T — a set of size at least κ. By the regularity of κ, there must be at least κ many equivalence classes. □

Lemma 7. If κ is regular and T is κ-balanced then T contains a κ-dense subset.

Proof. Choose some set $Q \subseteq T$ of representatives of $\approx_\kappa$. By corollary 6.1, $|Q| \geq \kappa$. Consider $a, b \in Q$ with $a < b$. We wish to show that $|[a, b]_Q| \geq \kappa$. Firstly, since $a \neq b$, we know that $|(a, b)_T| \geq \kappa$. Now, take any $x \in T$ with $a < x < b$. Since Q is a choice of representatives for $\approx_\kappa$, there is some $\tilde{x} \in Q$ with $x \approx_\kappa x$. We claim that from the convexity of $\approx_\kappa$, we must have $a \leq \tilde{x} \leq b$. Otherwise we would have either $\tilde{x} < a < x$ or $x < b < \tilde{x}$. In the former case, the convexity of $\approx_\kappa$ gives $a \approx_\kappa \tilde{x}$ and in the latter case $b \approx_\kappa \tilde{x}$. But since $a, b, \tilde{x}$ are representatives of equivalence classes, it would follow that $a = \tilde{x}$ or $b = \tilde{x}$ respectively.

We have just shown that if $a < x < b$, then for the $\tilde{x} \in Q$ with $x \approx_\kappa \tilde{x}$, we must have $a \leq \tilde{x} \leq b$. Thus, we have

$$(a, b)_T \subseteq \bigcup_{\tilde{x} \in [a,b]_Q} [\tilde{x}]_{\approx_\kappa}.$$

By lemma 6, each class $[\tilde{x}]_{\approx_\kappa}$ has size less than κ. Since $|(a, b)_T| \geq \kappa$ and κ is regular, the index set $[a, b]_Q$ must have size at least κ. □

Corollary 7.1. For any regular κ, every total order of size at least κ either:

- embeds κ
- embeds κ^*
- contains a κ-dense subset

Proof. If the order is κ-balanced then it contains a κ-dense subset by lemma 7. Otherwise, it embeds either κ or κ^* by corollary 2.1. □

3 Combinatorics

Lemma 8. For any infinite cardinal κ, we have

$$(2^\kappa)^+ \to (\kappa^+)^2_\kappa.$$

Proof. Let $\mu = (2^\kappa)^+$. Let $F : [\mu]^2 \to \kappa$ be any function. Our goal is to find a subset $H \subseteq \mu$ of size κ^+ such that F is constant on $[H]^2$. For each $a \in \kappa$, let $F_a : \kappa \setminus \{a\} \to \kappa$ take x to $F(\{a, x\})$. First we will prove the following claim:

(1) There exists a subset $A \subseteq \mu$ of size 2^κ such that for any $C \subseteq A$ with $|C| \leq \kappa$, and any $u \in \kappa \setminus C$, there exists a $v \in A \setminus C$ such that F_v agrees with F_u on C.

We will construct an ascending inclusion sequence $\{A_\alpha\}_{\alpha<\kappa^+}$ of subsets of μ of size 2^κ. First let A_0 be any subset of μ of size 2^κ. Suppose A_α is defined for some $\alpha < \kappa^+$. For any $C \subseteq A_\alpha$ of size at most κ, the number of functions from C to ω is

$$(|C| \cdot |C|)^\kappa \leq \kappa^\kappa \leq (2^\kappa)^\kappa = 2^{\kappa\times\kappa} = 2^\kappa.$$

The number of subsets $C \subseteq A_\alpha$ of size at most κ is $(2^\kappa)^\kappa = 2^\kappa$. Thus, there are only 2^κ many functions from any $C \subseteq A_\alpha$ of size at most κ to κ. In particular,

$$\{F_u \restriction C \mid C \subseteq A_\alpha, |C| \leq \kappa, u \in \mu \setminus C\}$$

has size at most 2^κ. Choosing some representative v for each function in the above set, we obtain a new set $A_{\alpha+1} \supseteq A_\alpha$ of size 2^κ. For any limit $\alpha < \kappa^+$, let $A_\alpha = \cup_{\beta<\alpha} A_\beta$. We have $|A_\alpha| = |\alpha| \cdot 2^\kappa \leq \kappa \cdot 2^\kappa = 2^\kappa$.

Let $A = \cup_{\alpha<\kappa^+} A_\alpha$. Let $C \subseteq A$ have size at most κ, and let $u \in \mu \setminus C$. Each $c \in C$ is in some A_{α_c}. Let $\alpha = \sup\{\alpha_c \mid c \in C\}$. Since $|C| \leq \kappa$, and each $\alpha_c < \kappa^+$, we have $\alpha < \kappa^+$. Then $C \subseteq A_\alpha$. By construction, there is some $v \in A_{\alpha+1} \setminus C$ such that F_u agrees with F_v on C. Furthermore, that v is also in A, therefore the claim is proven.

Fix some $b \in \mu \setminus A$. We will construct a sequence $\{x_\alpha\}_{\alpha<\kappa^+}$ in A. First, let x_0 be arbitrary. Given that x_β is defined for all $\beta < \alpha$ for some $\alpha < \kappa^+$, let $C = \{x_\beta \mid \beta < \alpha\}$. Choose some $x_\alpha \in A$ using (1) such that F_{x_α} agrees with F_b on C. Let $X = \{x_\alpha \mid \alpha < \kappa^+\}$. Define $G : X \to \kappa$ by $G(x) = F_b(x)$. For any $\alpha < \beta < \kappa$, we have $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha)$. By construction, F_{x_β} agrees with F_b on $\{x_\gamma \mid \gamma < \beta\}$, so $F_{x_\beta}(x_\alpha) = F_b(x_\alpha) = G(x_\alpha)$.

Since the codomain of G is κ and its domain has size κ^+, there must be some $y \in \kappa$ whose preimage under G has κ^+ many elements. In other words, there is an $H \subseteq X$ of size κ^+ that G is constant on. For all $x_\alpha, x_\beta \in H$ with $\alpha < \beta$, we have $F(\{x_\alpha, x_\beta\}) = G(x_\beta) = y$. Thus F is constant on $[H]^2$ (with value y). $\square$

Proposition 9. For any regular λ, we have

$$(2^{<\lambda})^+ \to (\lambda)^2_2$$

Proof. We use as a special case of 1.1 in [1] that if κ is some strongly compact cardinal, $\lambda > \kappa$ is regular, and $\zeta, \theta < \kappa$ then

$$(2^{<\lambda})^+ \to (\lambda + \zeta)^2_\theta.$$

As noted in the paper, the result is valid in the case $\kappa = \omega$. For our purposes, we only need the case $\zeta = 0$ and $\theta = 2$. Thus, we have the simplified statement that if $\lambda > \omega$ is regular then $(2^{<\lambda})^+ \to (\lambda)^2_2$. We note that even in the case $\lambda = \omega$, the statement becomes $\omega \to (\omega)^2_2$, which is also true. $\square$

4 Order Types

We introduce some notation which ease the proof of the next lemma. Suppose Ψ is some class of linear orders. Then we say Φ **holds convex hereditarily of** T if Ψ holds for every nontrivial convex subset of T. We notate the class of such T by $\mathcal{CH}[\Psi]$.

Definition 10. For a cardinal κ, let $\mathrm{Emb}(\kappa)$ be the class of orders that embed κ. Equivalently, the class of orders that contain a well order of size κ.

Lemma 11. If κ is a singular cardinal and $T \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cup \mathrm{Emb}(\lambda^*)]$ for all $\lambda < \kappa$, then $T \in \mathrm{Emb}(\kappa) \cup \mathrm{Emb}(\kappa^*)$.

Proof. First we will try to find a nontrivial convex $C \subseteq T$ so that either C is in $\mathcal{CH}[\mathrm{Emb}(\lambda)]$ for all $\lambda < \kappa$, or C^* is. Firstly, suppose $T \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cap \mathrm{Emb}(\lambda^*)]$ for all $\lambda < \kappa$. Then clearly $C = T$ is satisfactory. Otherwise, let λ_0 be the minimal cardinal less than κ so that $T \notin \mathcal{CH}[\mathrm{Emb}(\lambda_0) \cap \mathrm{Emb}(\lambda_0^*)]$. That means there is some nontrivial, convex $C \subseteq T$ so that $C \notin \mathrm{Emb}(\lambda_0)$ or $C \notin \mathrm{Emb}(\lambda_0^*)$. This is the same as saying either C^* or C is not in $\mathrm{Emb}(\lambda_0^*)$.

We may assume that $C \notin \mathrm{Emb}(\lambda_0^*)$, as we may use an analogous argument to the following in the case that $C^* \notin \mathrm{Emb}(\lambda_0)$. We will show $C \in \mathcal{CH}(\lambda)$ for all $\lambda < \kappa$. Fix a nontrivial convex $D \subseteq C$, and some cardinal $\lambda < \kappa$. If $\lambda < \lambda_0$ then by the minimality of λ_0 we have $T \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cap \mathrm{Emb}(\lambda^*)]$. In particular, D is a nontrivial convex subset of T, so $D \in \mathrm{Emb}(\lambda)$. On the other hand, suppose $\lambda \geq \lambda_0$. Since C cannot embed λ_0^* and $\lambda \geq \lambda_0$, clearly a subset $D \subseteq C$ cannot embed λ^*. That is, $D \notin \mathrm{Emb}(\lambda^*)$. However, by assumption $T \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cup \mathrm{Emb}(\lambda^*)]$. Then since D is a nontrivial convex subset of D, we must have $D \in \mathrm{Emb}(\lambda)$. Thus, the claim is shown.

Finally, we will use the fact that $C \in \mathcal{CH}[\mathrm{Emb}(\lambda)$ for all $\lambda < \kappa$ to build an embedding of κ into C. If on the other hand $C^* \in \mathcal{CH}[\mathrm{Emb}(\lambda)]$ for all $\lambda < \kappa$, then we would end up building an embedding of κ into C^*, and hence an embedding of κ^* into C. Either case achieves an embedding of κ or κ^* into T, which is what we are trying to show.

As κ is singular, we may take a cofinal sequence of cardinals $(\lambda_\alpha)_{\alpha<\mu}$ where $\mu = \mathrm{cf}(\kappa) < \kappa$. Since $\mu < \kappa$, we have $C \in \mathrm{Emb}(\mu)$. Hence, we may choose an embedding $f : \mu \to C$. Next, for each $\alpha < \mu$, we have that $[f(\alpha\cdot 2), f(\alpha\cdot 2+1)]_C$ is a nontrivial convex subset of C. Hence, $[f(\alpha\cdot 2), f(\alpha\cdot 2+1)]_C \in \mathrm{Emb}(\lambda_\alpha)$. Then we may choose some well ordered subset $W_\alpha \subseteq [f(\alpha \cdot 2), f(\alpha \cdot 2 + 1)]_C$ such that the order type of C_α is λ_α. Now, consider the set

$$W = \bigcup_{\alpha<\mu} W_\alpha.$$

Firstly, it is well ordered. Take any nonempty $X \subseteq W$. There is a least $\alpha < \mu$ so that $X \cap W_\alpha \neq \emptyset$. Since W_α is well ordered, there is a least element x of $X \cap W_\alpha$. Now since f is order preserving and $W_\alpha \subseteq [f(\alpha \cdot 2), f(\alpha \cdot 2 + 1)]_C$, we have that x is the least element of all of X. Secondly, we have

$$|W| = \sum_{\alpha<\mu} |W_\alpha| = \sum_{\alpha<\mu} \lambda_\alpha = \kappa.$$

Hence, C contains a well order of size κ. Thus, $C \in \mathrm{Emb}(\kappa)$. □

Lemma 12. If $\kappa \to (\lambda)^2_2$, then every order of size at least κ embeds either λ or λ^*.

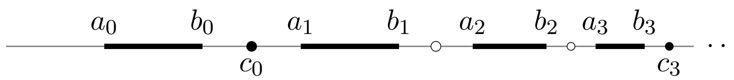
Proof. Assume without loss of generallity that the underlying set of T is κ, ordered by some relation $\prec$. Let $F : [\kappa]^2 \to \{0, 1\}$ send $\{a, b\}$ to 0 if either $a < b$ and $a \prec b$ or $a > b$ and $a \succ b$ and to 1 otherwise. In other words, $F(\{a, b\})$ is 0 if $<$ and $\prec$ agree and is 1 if they disagree. Using the above, there is an $H \subseteq \kappa$ of size λ such that F is constant on $[H]^2$. Since H is a subset of κ, and $|H| \geq \lambda$ it has $<$-order type some ordinal at least λ. Choose some subset $X \subseteq H$ with $<$-order type exactly λ. If $F[H] = \{0\}$ then $\prec$ agrees with $<$ everywhere on X, and so X also has $\prec$-order type λ. In other words, X is an ascending λ sequence in $\prec$. If $F[H] = \{1\}$ then $\prec$ disagrees with $<$ everywhere on X. Then the $\prec$-order type of X will be the dual of the $<$-order type of X. Thus, X is a descending λ sequence in $\prec$. $\square$

Lemma 13. If λ embeds into a dense order T then T contains at least 2^λ suborder types.

Proof. Choose some embedding $\iota : \lambda \to T$. Using the density of T, choose points $a_\alpha, b_\alpha, c_\alpha$ for $\alpha < \lambda$ such that $\iota(\alpha) = a_\alpha < b_\alpha < c_\alpha < \iota(\alpha + 1)$. Given a subset $X \subseteq \lambda$, define the set $S(X) \subseteq T$ by

$$S(X) = \bigcup_{\alpha<\lambda} [a_\alpha, b_\alpha]_T \cup \{c_\alpha \mid \alpha \in X\}.$$

The following is a depiction of $S(X)$ for some X that contains 0 and 3, but not 1 or 2:



We aim to show that if $X \neq Y$ then $S(X)$ and $S(Y)$ are not isomorphic. It follows that there are at least $|\mathcal{P}(\lambda)| = 2^\lambda$ subordertypes of T.

Suppose there is an order isomorphism $f : S(X) \to S(Y)$. By inspection, we see that the set of b points is a definable subset of $S(X)$ given by the first order formula

$$\varphi_b(x) = (\exists y < x \text{ s.t. the interval } [y, x] \text{ is dense}) \wedge (x \text{ has a successor}).$$

Since f is an order isomorphism, it must preserve definable subsets. In particular, x is a b point iff $f(x)$ is a b point. Since the set of b points is well ordered, it follows by transfinite induction that $f(b_\alpha) = b_\alpha$ for all $\alpha < \lambda$. Similarly, we see that the set of a points is defined by the formula

$$\varphi_a(x) = \exists y > x([\text{the interval } [x, y] \text{ is dense}] \wedge [\forall x' < x \text{ the interval } [x', y] \text{ is not dense}]),$$

and so $f(a_\alpha) = a_\alpha$ for all $\alpha < \lambda$.

Let $\alpha \in X$. There is exactly one point in $S(X)$ between b_α and $a_{\alpha+1}$, namely c_α. Therefore, there must be exactly one point between $f(b_\alpha) = b_\alpha$ and $f(a_{\alpha+1}) = a_{\alpha+1}$. This is only possible if $\alpha \in Y$. By the symmetric argument, if $\alpha \in Y$ then $\alpha \in X$. Thus, we have shown $X = Y$. $\square$

Theorem 14. Any order T has as at least as many suborder types as elements

Proof. Suppose not. Let κ be the least cardinality so that there is some order T of size κ that has fewer than κ many suborder types. Fix such an order T. From our minimality assumptions, for each $\lambda < \kappa$ there are at least λ many suborder types of T. It follows that there are at least $\sup\{\lambda \mid \lambda < \kappa\}$ suborder types of T. If κ were a limit cardinal then $\sup\{\lambda \mid \lambda < \kappa\} = \kappa$, so κ must be a successor cardinal. In particular, κ is regular.

Next, if T embeds either κ or κ^* then T embeds each ordinal less than κ, or the dual of each ordinal less than κ, giving at least κ many suborder types. Thus, neither can be the case. By corollary 7.1, T must contain a κ-dense subset S. It suffices to show that S contains κ many suborder types.

Let μ be the least cardinal such that $2^\mu \geq \kappa$. By lemma 13, it suffices to show that S has embeds either μ or μ^*. We now argue in two cases:

Case 1: μ is regular. By proposition 9, we have

$$(2^{<\mu})^+ \to (\mu)^2_2.$$

Since μ is minimal, for each $\lambda < \mu$ we have $2^\lambda < \kappa$. Then, since κ is a successor cardinal, we have $2^{<\mu} < \kappa$. It follows that $(2^{<\mu})^+ \leq \kappa$, so $\kappa \to (\mu)^2_2$. By lemma 12, either μ or μ^* embeds into S.

Case 2: μ is singular. By the minimality of μ, for any $\lambda < \mu$, we have $2^\lambda < \kappa$. It follows that $(2^\lambda)^+ \leq \kappa$ for each $\lambda < \mu$, so by lemma 8, we have $(2^\lambda)^+ \to (\lambda)^2_2$. By construction, S is κ-dense. In particular, if C is a nontrivial convex subset of S, then $|C|$ has size at least κ. Then lemma 12 gives us that $|C|$ embeds either λ or λ^* for each $\lambda < \mu$. Hence, $S \in \mathcal{CH}[\mathrm{Emb}(\lambda) \cup \mathrm{Emb}(\lambda^*)]$ for every $\lambda < \mu$. By lemma 11, we are done. $\square$

References

[1] Saharon Shelah. A partition relation using strongly compact cardinals. *Proc. Amer. Math. Soc.*, 131(8):2585–2592, 2003. arXiv: math/0103155.