Adjusting the Stabilizer of the Shadow Function to Avoid Ambiguity

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Initially, we defined the Shadow Function with an exponential stabilizer term containing two free parameters:

Definition 1 (Shadow Function with Exponential Stabilizer)**.** *We define the* shadow function $\zeta^*(s)$ *as:*

$$\zeta^*(s) = e^{A+Bs} \frac{1}{s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1},$$

where:

- ρ *denotes the non-trivial zeros of* $\zeta(s)$*,*
- $k \in \mathbb{N}^+$ *denotes the trivial poles,*
- e^{A+Bs} *is an exponential stabilizer controlling growth at infinity,*
- $\frac{1}{s}$ *introduces a simple pole at* $s = 0$*.*

However, an anonymous commenter raised concerns about the ambiguity of the two free parameters. While this stabilizer is theoretically sufficient, we propose a simpler alternative that avoids ambiguity while preserving the desired properties. This strengthens the Zeropole Balance proof of the Riemann Hypothesis.

Simplified Stabilizer

Instead of the two-parameter stabilizer e^{A+Bs}, we introduce a single-parameter stabilizer e^{Cs}:

Definition 2 (Shadow Function with Simplified Exponential Stabilizer)**.** *We define the* shadow function $\zeta^*(s)$ *as:*

$$\zeta^*(s) = e^{Cs} \frac{1}{s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1},$$

where:

- ρ *denotes the non-trivial zeros of* $\zeta(s)$*,*
- $k \in \mathbb{N}^+$ *denotes the trivial poles,*
- e^{Cs} *is the exponential stabilizer ensuring finite behavior at infinity,*
- $\frac{1}{s}$ *introduces a simple pole at* $s = 0$*,*
- C *is a constant chosen to regulate the growth.*

Remark 1. *The simplified stabilizer* e^{Cs} *resolves all ambiguity while preserving essential properties:*

- ***Preservation of the Simple Pole at the Origin:*** *The* $\frac{1}{s}$ *term introduces a simple pole at* $s = 0$*, and the stabilizer* e^{Cs} *does not interfere with this.*
- ***Moderation of Growth at Infinity:*** *The exponential stabilizer dynamically controls the growth of* $\zeta^*(s)$*, ensuring meromorphic behavior at* $s = \infty$*.*
- ***Compatibility with Compactification:*** *The stabilizer ensures that* $\zeta^*(s)$ *remains meromorphic on the compactified Riemann sphere.*
- ***Elimination of Ambiguity:*** *The single parameter* C *simplifies the stabilizer, removing the need for normalization conditions and avoiding redundancy.*

Behavior of $\zeta^*(s)$ at the Point of Infinity

Lemma 1 (Meromorphic Compactification of $\zeta^*(s)$)**.** *The shadow function* $\zeta^*(s)$ *remains meromorphic at the point at infinity on the Riemann sphere.*

Proof. To confirm that $\zeta^*(s)$ is meromorphic at $s = \infty$, consider the stabilizer's impact on the growth of the infinite products:

- **Growth Moderation by the Stabilizer:** The stabilizer e^{Cs} ensures that the exponential growth of the infinite product terms is tempered, preventing unbounded growth.

- **Compatibility with the Infinite Products:** The factors $\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$ and $\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1}$ contribute logarithmic or higher-order terms to the growth of $\zeta^*(s)$. These terms are tempered by the stabilizer, ensuring finite behavior.

- **Preservation of Divisor Structure:** The $\frac{1}{s}$ term at the origin introduces a simple pole, contributing -1 to the divisor degree. This contribution is preserved without introducing an essential singularity at infinity, consistent with the Riemann-Roch framework for compactification.

Thus, the growth of $\zeta^*(s)$ is moderated to ensure meromorphic compactification, with no essential singularities arising at $s=\infty$. □

Remark 2. *The stabilizer e^{Cs} ensures that the shadow function behaves predictably along the critical line. Its dependence on t^2 (from the $s(1-s)$ factor encoded in $\zeta(s)$) preserves the symmetry and zeropole balance framework inherent in the Riemann zeta function.*

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