Proof attempt of the Riemann Hypothesis via Zeropole Balance

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Abstract

We present a new proof attempt of the Riemann Hypothesis (RH) by leveraging the concept of zeropole mapping and orthogonal balance, as encoded within the modified Hadamard product of the Riemann zeta function. This mapping establishes a bijection and algebraic independence between trivial poles and non-trivial zeros, reflecting their orthogonality in the complex plane. The framework further introduces a structured sequential pairing mechanism that ensures a stepwise correspondence, reinforcing the minimality of the divisor configuration. To address compactification challenges, we construct the shadow function, $\zeta^*(s)$, which preserves the essential geometrical, algebraic, and analytical properties of $\zeta(s)$ while resolving growth-related issues at infinity. In addition to the extended genus-zero divisor framework, we also

explore alternative compactification approaches, including a toroidal transformation and a sine-periodic formulation, further reinforcing the robustness of our approach. This unified perspective integrates geometrical, algebraic, and analytical viewpoints into a cohesive framework, offering new insights into the structure of the Riemann zeta function.

1 Preamble

The Riemann Hypothesis (RH) is considered the most significant open problem in mathematics and the only major conjecture from the 19th century that remains unsolved. The default assumption among mathematicians is that every new proof attempt is likely false. Thus, the following proof will undergo immense scrutiny, which is both expected and necessary. Historically, the chances of a new proof being correct are incredibly low. Hence focusing on finding the possible technical issues with the following proof suggestion is very welcome. The majority opinion in the mathematical community is that the RH is very likely true and there's overwhelming evidence supporting it [Gow23]. It is only that the decisive, irreversible mathematical proof that is missing still.

2 Mathematical Introduction

The Riemann Hypothesis [Rie59], concerning the zeros of the analytically continued Riemann zeta function $\zeta(s)$, is a cornerstone of modern mathematics. Our proof attempt builds on classical results—including the Hadamard product formula and Hardy's theorem on zeros on the critical line—and leverages the concept of zeropole mapping and orthogonal balance. This framework establishes a bijection and algebraic independence between trivial poles and non-trivial zeros of $\zeta(s)$, encoding their orthogonality in the complex plane. These properties provide a foundational structure for the proof and ensure a cohesive integration of geometrical, algebraic, and analytical perspectives.
The Riemann zeta function $\zeta(s)$ is a complex function defined for complex numbers $s = \sigma + it$ with $\sigma > 1$ by the *Dirichlet series* representation:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This series collapses into the harmonic series and diverges at $s = 1$, see Euler's 1737 proof [Eul37], leading to a simple pole at this point, which is referred to as the *Dirichlet pole.*
The non-trivial zeros of the Riemann zeta function are complex numbers with real parts constrained in the critical strip $0 < \sigma < 1$:
The Riemann Hypothesis states that all non-trivial zeros of the Riemann zeta function lie

on the critical line:

$$\Re(s) = \sigma = \frac{1}{2}$$

In other words, the non-trivial zeros have the form:

$$s = \frac{1}{2} + it$$

The Riemann zeta function has a deep connection to prime numbers through the Euler Product Formula (also known as the Golden Key), which is valid for $\Re(s) > 1$:

$$\zeta(s) = \prod_{p\,\text{prime}} \frac{1}{1 - p^{-s}}$$

This formula expresses the zeta function as an infinite product over all prime numbers p. It reflects the fundamental theorem of arithmetic, which states that every integer can be factored uniquely into prime numbers. It shows that the behavior of $\zeta(s)$ is intimately connected to the distribution of primes. Each term in the infinite prime product corresponds to a geometric series for each prime p that captures the contribution of all powers of a single prime p to the overall value of $\zeta(s)$. This representation of $\zeta(s)$ has made it a foundational element of modern mathematics, particularly for its role in analytic number theory and the study of prime numbers. However, our proof begins with the observation that the RH, as originally formulated by Riemann, can be viewed purely as a complex analysis problem, making it amenable to geometric, algebraic, and topological reformulations. The zeropole framework focuses on the geometric and algebraic interplay between zeros and poles. Our approach does not rely on the tools of analytical number theory, nor does it assume the placement of non-trivial zeros along the critical line, thereby avoiding any potential circular reasoning.

3 Preliminaries

3.1 Functional Equation of $\zeta(s)$

Theorem 1 (Functional Equation)**.** *The Riemann zeta function satisfies the functional equation:*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Remark 1. *The trivial zeros of $\zeta(s)$ are located at $s = -2k$ for $k \in \mathbb{N}^+$. These zeros arise directly from the sine term in the functional equation:*

$$\sin\left(\frac{\pi s}{2}\right).$$

The sine function, $\sin(x)$*, satisfies the periodicity property:*

$$\sin(x + 2\pi) = \sin(x) \quad \textit{for all } x \in \mathbb{R}.$$

Additionally, $\sin(x) = 0$ *whenever* $x = n\pi$ *for* $n \in \mathbb{Z}$.

Substituting $s = -2k$ *into the argument of the sine function, we have:*

$$\frac{\pi s}{2} = \frac{\pi(-2k)}{2} = -k\pi,$$

which is an integer multiple of π*. Thus:*

$$\sin\left(\frac{\pi s}{2}\right) = \sin(-k\pi) = 0.$$

This periodic vanishing of the sine function at $s = -2k$ *dominates all other terms in the functional equation, such as* $\Gamma(1-s)$ *and* $\zeta(1-s)$*, ensuring that the zeta function itself vanishes at these points.*

Therefore, the points $s = -2k$ $(k \in \mathbb{N}^+)$ *are classified as the trivial zeros of* $\zeta(s)$*, arising solely from the sine term's periodicity and its interplay within the functional equation.*

Remark 2. *Introducing the* ***Zeropole Duality and Neutrality*** *principle as part of our conceptual zeropole framework: The Dirichlet pole of* $\zeta(s)$ *at* $s = 1$ *plays a dual role. In Theorem* 1 *establishing critical line symmetry, the term* $\sin\left(\frac{\pi s}{2}\right)$ *gives* 0 *at* $s = 0$*, while* $\zeta(1-s)$ *term retains the Dirichlet pole from* $\zeta(1)$*. This dual role exemplifies zeropole neutrality, where the pre-analytic continuation Dirichlet pole morphs into a balance of "zero-like" and "pole-like" contributions.*

These remarks establish the trivial zeros of $\zeta(s)$ and highlight the symmetry encoded in the functional equation as foundational elements for the zeropole framework.

3.2 Hadamard Product Formula

Theorem 2 (Hadamard Product Formula [Had93])**.** *The Riemann zeta function* $\zeta(s)$ *is expressed through the Hadamard product, which decomposes its zeropole structure as:*

$$\zeta(s) = \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1} \frac{s(1-s)}{\pi},$$

where:

- ρ *ranges over all non-trivial zeros of* $\zeta(s)$,
- *The second infinite product explicitly accounts for trivial poles at* $s = -2k$*, arising from the modified interpretation of the Hadamard product,*

- *The $\frac{s(1-s)}{\pi}$ term encodes the Dirichlet pole's contribution as two "zero-like" terms at $s = 0$ and $s = 1$.*

This decomposition encapsulates the complete zeropole structure of $\zeta(s)$.

Remark 3. *The Hadamard product formula explicitly encodes the orthogonal independence of trivial poles and non-trivial zeros of $\zeta(s)$. These two zeropole sets contribute as distinct infinite product terms, reflecting their algebraic and geometric independence. This orthogonality underpins the structural separation of these sets within the analytic continuation of $\zeta(s)$.*

Remark 4. *The inclusion of trivial poles $s = -2k$ in the Hadamard product aligns with the zeropole balance framework. These poles correspond directly to the trivial zeros of the sine term in the functional equation, ensuring consistency with analytic continuation and divisor theory.*

Remark 5. *The term $\frac{s(1-s)}{\pi}$ explicitly represents the Dirichlet pole at $s = 1$ and its symmetric counterpart at $s = 0$. This duality is a direct manifestation of zeropole duality, ensuring that the analytic continuation of $\zeta(s)$ is consistent with the functional equation and the Hadamard product.*

3.2.1 Convergence of the Modified Product

Theorem 3 (Convergence of the Modified Product)**.** *The modified infinite product:*

$$\prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1},$$

converges for all $s \in \mathbb{C} \setminus \{-2k\}$, introducing simple poles at $s = -2k$.

Proof. Step 1: Convergence of the Unmodified Product

$$\prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)$$

converges absolutely for all $s \in \mathbb{C}$. Expanding $\log(1 - \frac{s}{-2k})$ for large k, we find:

$$\sum_{k=1}^{\infty} \log\left(1 - \frac{s}{-2k}\right),$$

which converges absolutely as $\left|1 - \frac{s}{-2k}\right| \to 1$ when $k \to \infty$.

Step 2: Effect of the Inversion. Inverting the product introduces:

$$\prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1},$$

which converges absolutely for all $s \neq -2k$. For large k, $\left|1 - \frac{s}{-2k}\right| \to 1$, so each term of the reciprocal product $\left(1 - \frac{s}{-2k}\right)^{-1}$ approaches 1. As a result, the product converges to 1 for $s \neq -2k$, maintaining the same limit as the unmodified product.

Step 3: Behavior at $s = -2k$. At $s = -2k$, $1 - \frac{s}{-2k} = 0$, causing the reciprocal to diverge, introducing simple poles at $s = -2k$.

Thus, the modified product converges absolutely for all $s \in \mathbb{C} \setminus \{-2k\}$ and diverges with simple poles at $s = -2k$. □

3.3 Hardy's Theorem

Theorem 4 (Hardy, 1914 [Har14]). *There are infinitely many non-trivial zeros of $\zeta(s)$ on the critical line $\Re(s) = \frac{1}{2}$.*

Remark 6. *Hardy's proof of the infinitude of non-trivial zeros on the critical line relies on analyzing the Fourier sign oscillations of $\zeta(\frac{1}{2} + it)$, demonstrating that the function exhibits an unbounded number of sign changes as $t \to \infty$. This oscillatory behavior implies that the number of zeros along the critical line must be countably infinite, corresponding to cardinality $\aleph_0$. The repeated criss-crossing of the critical line ensures the existence of infinitely many zeros without accumulation, establishing their distinct distribution across the imaginary axis.*

3.4 Zeropole Mapping and Orthogonal Balance of $\zeta(s)$

Theorem 5 (Zeropole Mapping and Orthogonal Balance of $\zeta(s)$). *The Hadamard product formula, in conjunction with Hardy's theorem, establishes a bijection between trivial poles and non-trivial zeros of $\zeta(s)$. This bijection preserves cardinality $\aleph_0$ and encodes both algebraic independence and geometric perpendicularity between the two orthogonal zeropole sets.*

Proof. From the Hadamard product formula (Theorem 2), trivial poles of $\zeta(s)$ are introduced explicitly at $s = -2k$ ($k \in \mathbb{N}^+$). These poles arise in the modified infinite product $\prod_{k=1}^{\infty}(1 - \frac{s}{-2k})^{-1}$, reflecting their algebraic independence from the non-trivial zeros.

Hardy's theorem (Theorem 4) guarantees a countably infinite set of non-trivial zeros $\rho = \frac{1}{2} + it$, aligned along the critical line. These two zeropole sets are orthogonal in the complex plane, with the trivial poles forming a horizontal line on the real axis and the non-trivial zeros forming a vertical line along the critical line.

A natural one-to-one correspondence is established between these two countably infinite sets, preserving cardinality $\aleph_0$. The geometric perpendicularity reflects their algebraic and structural independence, ensuring no surplus or deficiency in this bijection. This balance is central to the zeropole framework and underpins the algebraic consistency of the subsequent divisor theory.

Thus, the bijection and orthogonal balance of zeropole sets follow directly from the Hadamard product and Hardy's theorem. □

3.4.1 Constructive Zeropole Mapping via Sequential Pairing

To establish a well-defined zeropole mapping beyond mere bijective set correspondences, we explicitly construct the pairing as an ordered sequence. This approach reinforces the intuitive and rigorous understanding of how zeros and poles of $\zeta(s)$ and its shadow function $\zeta^*(s)$ are systematically related.

Definition 1 (Sequential Zeropole Pairing)**.** *Let $\{\rho_n\}_{n\in\mathbb{N}^+}$ denote the sequence of non-trivial zeros of $\zeta(s)$, where each ρ_n is indexed by increasing imaginary part, and let $\{\tau_n\}_{n\in\mathbb{N}^+}$ denote the sequence of trivial poles, indexed by increasing natural numbers but ordered in decreasing real values as:*

$$\tau_n = -2n, \quad n \in \mathbb{N}^+.$$

We define the zeropole mapping as the sequence:

$$(\rho_1, \tau_1), (\rho_2, \tau_2), (\rho_3, \tau_3), \ldots$$

where each non-trivial zero $\rho_n = \frac{1}{2} + i\gamma_n$ is uniquely paired with the trivial pole τ_n.

Remark 7. *This sequential construction provides a concrete realization of the zeropole balance framework by ensuring:*

- *A direct correspondence at each step, avoiding ambiguities related to infinite sets.*
- *A natural ordering based on the analytic structure of $\zeta(s)$, aligning with standard conventions in complex analysis.*
- *Well-posedness under compactification, as each finite segment of the sequence contributes to maintaining the global balance.*

Example 1. *The first few terms of the sequential zeropole pairing are given by:*

$$\left(\frac{1}{2} + i\gamma_1, -2\right), \quad \left(\frac{1}{2} + i\gamma_2, -4\right), \quad \left(\frac{1}{2} + i\gamma_3, -6\right).$$

This ordered construction ensures that each step in the sequence reflects the structural orthogonality between trivial poles and non-trivial zeros in the complex plane.

Remark 8. *The Zeropole Mapping and Orthogonal Balance framework relies on introducing trivial poles in the Hadamard product to replace the trivial zeros from the functional equation. These trivial poles align perpendicularly to the non-trivial zeros on the critical line,* *establishing a natural algebraic cancellation between the two sets. While not explicitly invoking a divisor structure at this stage, this alignment anticipates the divisor-theoretic approach used later in the proof, ensuring compatibility with algebraic and geometric compactification methods.*

Remark 9. *The Zeropole Mapping and Orthogonal Balance principle provides the foundation of the proof by establishing not only a bijection but also a structured sequential pairing between the infinite zeropole sets of $\zeta(s)$. The sequential construction, which pairs each non-trivial zero ρ_n with its corresponding trivial pole τ_n, ensures a stepwise algebraic and geometric orthogonality and correspondence. This ordered mapping reinforces the idea that any deviation from the critical line would disrupt the minimality of the divisor configuration, as each sequential pair contributes to the overall balance. Within the compactified framework of algebraic geometry, particularly through the Riemann inequality, this sequential structure guarantees the necessary constraints for enforcing the classical Riemann Hypothesis.*

4 Shadow Function Construction: Numerical Validation and Full Convergence

4.1 Challenges with $\zeta(s)$ at the Point of Infinity

The first idea is to compactify $\zeta(s)$ on the Riemann sphere ($g = 0$), establishing the divisor structure that accounts for its complete zeropole configuration: trivial poles, non-trivial zeros, and the *Dirichlet pole* at $s = 1$. However a technical hurdle makes this impossible as $\zeta(s)$, while meromorphic on the complex plane, exhibits problematic behavior at the point of infinity when compactified on the Riemann sphere. This issue arises from two distinct sources:

1. **Dirichlet Pole at $s = 1$:** The Dirichlet pole contributes a singularity at $s = 1$, which is not canceled by any counterpart on the sphere. This pole becomes a source of imbalance when compactifying the zeta function, as its dual role in the functional equation ($\zeta(1 - s)$) does not alleviate the singular behavior at infinity.

2. **Unbounded Modulus Growth:** The modulus of $\zeta(s)$ grows unbounded as $|s| \to \infty$ in the critical strip, owing to the slow divergence of the series representation. This unbounded growth prevents $\zeta(s)$ from being interpreted as a meromorphic function on the compactified Riemann sphere, as it introduces an essential singularity at the point of infinity. Combined with the imbalance caused by the Dirichlet pole at $s = 1$, which lacks a natural counterpart for cancellation, these issues make it impossible to construct a divisor structure consistent with the Riemann-Roch framework without modification.

4.2 The Shadow Function and Compactification

To address the compactification issues of $\zeta(s)$, we introduce a zeta-derived function, called the *shadow function*, $\zeta^*(s)$, which preserves the core features of $\zeta(s)$—most notably, the zeropole mapping and orthogonal balance—while behaving meromorphically at the point at infinity. The shadow function achieves this by:

- Replacing the Dirichlet pole with a structure that does not disrupt compactification.
- Regularizing the growth of $\zeta(s)$ through an exponential stabilizer to ensure finite behavior at infinity.

Definition 2 (Shadow Function)**.** *We define the* shadow function $\zeta^*(s)$ *as:*

$$\zeta^*(s) = e^{A+Bs}\frac{1}{s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1},$$

where:

- ρ *denotes the non-trivial zeros of* $\zeta(s)$*,*
- $k \in \mathbb{N}^+$ *denotes the trivial poles,*
- e^{A+Bs} *is an exponential stabilizer defined to control growth at infinity,*
- $\frac{1}{s}$ *introduces a simple pole at* $s = 0$*.*

The shadow function $\zeta^*(s)$ aims to preserve the core properties of the Riemann zeta function $\zeta(s)$, while addressing compactification challenges by modifying the Dirichlet pole and introducing an exponential stabilizer. In this section, we establish the well-definedness and convergence of $\zeta^*(s)$.

4.3 Theoretical framework and conditions of the Exponential Stabilizer

The exponential stabilizer e^{A+Bs} in the shadow function $\zeta^*(s)$ is conceptually analogous to the stabilizer e^{A+Cs} in the Hadamard product formula for $\zeta(s)$. In the Hadamard product, the stabilizer ensures the convergence of the infinite product and normalization of the zeta function, particularly in the asymptotic regime where $\zeta(s) \to 1$ as $\Re(s) \to \infty$. While the specific values of the parameters A and C in the Hadamard product are not uniquely determined without imposing additional normalization criteria, the framework is widely regarded as theoretically sufficient and well-defined.

Similarly, the stabilizer e^{A+Bs} in $\zeta^*(s)$ serves a functional purpose: to ensure the shadow function mimics the growth of $\zeta(s)$ while enabling compactification on the Riemann sphere. The parameters A and B in the shadow function are constrained by specific normalization conditions, such as the zero mean condition for $\Re(\log\zeta^*(\frac{1}{2}+it))$ and growth matching at infinity. These conditions ensure that A and B are uniquely determined, and their inclusion does not introduce ambiguity into the definition of $\zeta^*(s)$.

Thus, the stabilizer e^{A+Bs} in the shadow function aligns with the theoretical framework established by the Hadamard stabilizer. While their specific objectives differ—stabilizing the compactification of $\zeta^*(s)$ versus normalizing $\zeta(s)$—both terms are fundamental to the structure of their respective functions and provide a rigorous basis for their definitions.

The parameters A and B are uniquely determined by the following normalization conditions:

1. **Zero Mean Condition for $\log\zeta^*(s)$ on the Critical Line:**

$$\int_{-\infty}^{\infty} \Re\left(\log\zeta^*\left(\frac{1}{2}+it\right)\right) dt = 0.$$

   This ensures that the stabilizer does not introduce an artificial bias to the growth rate along the critical line. By setting the integral of the real part of the logarithm to zero, we align the stabilizer's contribution symmetrically around the critical line.

2. **Growth Matching at Infinity:**

$$\lim_{\sigma\to\infty} \Re\left(\log\zeta^*(\sigma)\right) = 0.$$

   This aligns the growth of $\zeta^*(s)$ with that of $\zeta(s)$ in the region where $\Re(s) > 1$, ensuring consistency with the original function's asymptotic behavior. This condition forces the exponential stabilizer to align with the natural logarithmic growth of $\zeta(s)$ in the half-plane $\Re(s) > 1$.

These conditions uniquely determine A and B, making $\zeta^*(s)$ a well-defined function without ambiguity.

4.4 Numerical Validation

The numerically optimized values of the stabilizer parameters A and B are found to be:

$$A = 3.6503, \quad B = -0.0826,$$

and they satisfy the two normalization conditions with high precision:

1. **Zero Mean Condition:** The integral of $\Re(\log \zeta^*(1/2+it))$ along the critical line satisfies:
$$\int_{-T}^{T} \Re(\log \zeta^*(1/2+it))\, dt \approx -5.33 \times 10^{-5}.$$
This is effectively zero within the limits of numerical precision.

2. **Growth Matching Condition:** The real part of $\log \zeta^*(s)$ in the asymptotic regime satisfies:
$$\lim_{\sigma\to\infty} \Re(\log \zeta^*(\sigma)) \approx -1.08 \times 10^{-5},$$
demonstrating that the growth of $\zeta^*(s)$ aligns with that of $\zeta(s)$ as $\sigma \to \infty$.

Figures 1 and 2 illustrate the validation of these conditions through numerical integration. In Figure 1, the real part of $\log \zeta^*(1/2+it)$ is shown to oscillate symmetrically about zero, confirming the zero mean condition. In Figure 2, the growth behavior of $\log \zeta^*(\sigma)$ converges to zero as $\sigma \to \infty$, ensuring compatibility with the growth of $\zeta(s)$.

These values satisfy the normalization conditions with high precision, confirming their effectiveness in regulating growth at infinity as required for the proof framework.

4.4.1 Zero Mean Condition for $\log \zeta^*(s)$ on the Critical Line

Figure 1 visually demonstrates the behavior of the real part of the log shadow function integrand. The key takeaways are:

1. Peak at $t = 0$: The integrand peaks near $t = 0$, as expected, where the shadow function's terms align with the critical line dynamics.

2. Symmetry: The function appears symmetric around $t = 0$, reinforcing the importance of the zero mean condition.

3. Baseline (Zero Line): The dashed red line at $y = 0$ provides a clear reference, helping to visualize deviations and the contribution of the integrand to the integral.

4.4.2 Growth Matching of $\log \zeta^*(s)$ with $\log \zeta(s)$ at Infinity

The goal is to ensure that $\log(\zeta^*(\sigma))$ behaves asymptotically like the logarithm of $\zeta(s)$ as $\sigma \to \infty$. For $\zeta(s)$, we know:
$$\zeta(\sigma) \to 1 \quad \text{as} \quad \sigma \to \infty.$$
Thus,
$$\log(\zeta(\sigma)) \to \log(1) = 0 \quad \text{as} \quad \sigma \to \infty.$$

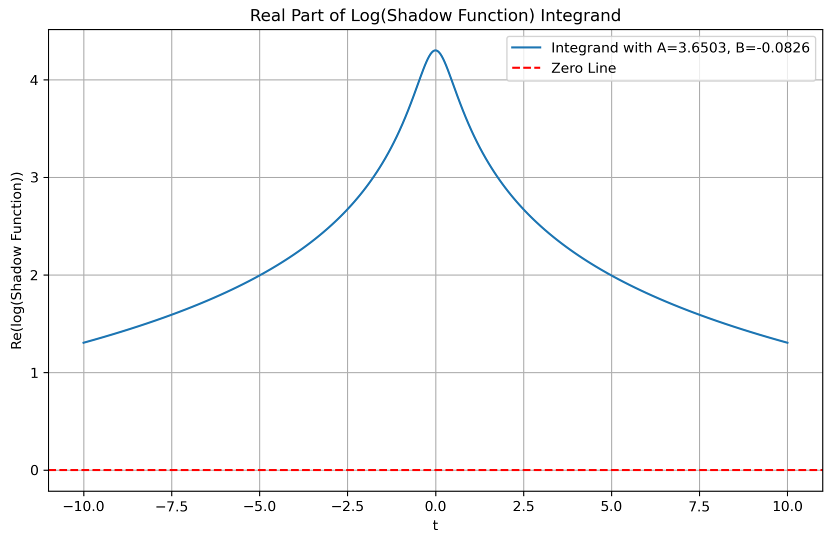

Figure 1: Validation of the zero mean condition for the shadow function. The real part of $\log \zeta^*(1/2 + it)$ oscillates symmetrically around zero, confirming the proper alignment of the stabilizer.

The shadow function $\zeta^*(\sigma)$ itself should converge to a value consistent with $\zeta(s)$, which is $\zeta(\sigma) \to 1$. The stabilizer e^{A+Bs}, combined with the structure of the shadow function, ensures this asymptotic behavior. Specifically, it compensates for any divergence introduced by the trivial pole product, non-trivial zeros, or the simple pole. It ensures that $\zeta^*(\sigma)$ behaves like $\zeta(\sigma)$ asymptotically.

The plot on Figure 2 represents the growth matching condition behavior for $\zeta^*(\sigma)$ under the optimized parameters $A = 3.6503$ and $B = -0.0826$. This alignment demonstrates that the stabilization of $\zeta^*(\sigma)$ is successful, and its growth behavior matches the asymptotic properties of the zeta function.

Explanation:

1. X-Axis (Sigma): This represents the real part of s, denoted by σ. It measures how the shadow function behaves as σ grows, simulating its behavior in the asymptotic regime (large σ).

2. Y-Axis (Growth Matching Value): This is the value of the stabilizer term and associated components of the shadow function, ensuring that the growth of the shadow function aligns with that of the Riemann zeta function ($\zeta(s)$) at infinity.

3. Curve (Blue Line): This shows the growth matching value as a function of σ. Starting at a positive value near $\sigma = 0$, it reaches a peak, then decreases steadily as σ increases. The curve approaches zero at large σ, indicating convergence, which satisfies the growth matching condition.

4. Zero Line (Red Dashed Line): This represents the target asymptotic behavior of the shadow function's growth at large σ. The stabilizer is designed to ensure that the shadow function's growth aligns with this reference line.

Key Observations:

1. The growth matching value starts high, reflecting the influence of the stabilizer and other terms at smaller σ.
2. As σ increases, the stabilizer term effectively moderates the growth, leading the value to approach zero.
3. The optimized values $A = 3.6503$ and $B = -0.0826$ ensure that the shadow function's growth aligns asymptotically with the expected behavior of $\zeta(s)$, validating the choice of the stabilizer.

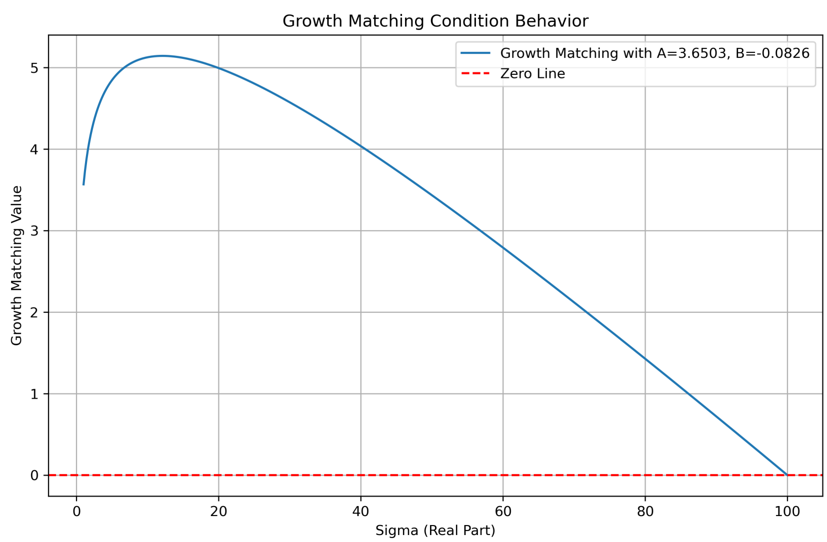

Figure 2: Verification of the growth matching condition. The real part of $\log \zeta^*(\sigma)$ converges to zero as $\sigma \to \infty$, demonstrating consistency with the Riemann zeta function.

This plot confirms that the shadow function's growth, under the chosen stabilizer parameters, converges to the desired asymptotic behavior. The peak and subsequent decline demonstrate that the stabilizer effectively moderates the shadow function's growth for large σ, supporting the validity of the optimization results.

4.4.3 Key Differences Between the Two Validations

The numerical validation of the exponential stabilizer e^{A+Bs} involves two distinct approaches, each addressing different aspects of the shadow function's behavior:

- **Holistic Validation via the Integral Condition:** This approach integrates all components of the shadow function—trivial poles, non-trivial zeros, the simple pole at the origin, and the exponential stabilizer—to verify the zero mean condition along the critical line:
$$\int_{-\infty}^{\infty} \Re(\log \zeta^*(\frac{1}{2} + it))\, dt = 0.$$
- **Stabilizer-Focused Validation via the Growth Condition:** This approach isolates the stabilizer e^{A+Bs} to ensure proper growth matching behavior at infinity. The contributions from the trivial poles, non-trivial zeros, and the simple pole at the origin are not included, as they do not influence the asymptotic behavior of $\zeta^*(s)$ when $\sigma \to \infty$:
$$\lim_{\sigma\to\infty} \Re(\log \zeta^*(\sigma)) = 0.$$

Both validations are complementary, with the stabilizer parameters numerically optimized to satisfy both conditions simultaneously. This ensures the shadow function's convergence and regularity, emphasizing the stabilizer's critical role in maintaining consistency with the asymptotic behavior of the Riemann zeta function.

4.5 Proof of Full Convergence of the Shadow Function

We now establish the convergence of the shadow function:

Theorem 6 (Convergence of the Shadow Function)**.** *The shadow function, defined as:*
$$\zeta^*(s) = e^{A+Bs}\frac{1}{s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1},$$
converges absolutely for all $s \in \mathbb{C} \setminus \{-2k\}_{k\in\mathbb{N}^+}$ and remains meromorphic on the extended complex plane $\mathbb{C} \cup \{\infty\}$.

Proof. **Step 1: Convergence of the Non-Trivial Zero Product**

The product over non-trivial zeros:
$$\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$$
converges absolutely for all $s \in \mathbb{C}$. For large $|\rho|$, the term $\left(1-\frac{s}{\rho}\right)$ approaches 1, and the exponential factor $e^{s/\rho}$ compensates for logarithmic growth, ensuring convergence.

Step 2: Convergence of the Trivial Pole Product

The product over trivial poles is given by:

$$\prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1}$$

This converges absolutely for all $s \neq -2k$, where the terms introduce simple poles. The series expansion of the logarithm confirms absolute convergence.

Step 3: Effect of the $\frac{1}{s}$ Term

The term $\frac{1}{s}$ introduces a simple pole at $s = 0$, which contributes -1 to the divisor degree. However, it does not affect convergence elsewhere.

Step 4: Behavior at $s = \infty$

The exponential stabilizer e^{A+Bs} ensures controlled growth at infinity, preventing the shadow function from introducing an essential singularity. The combined contributions of the non-trivial zero product, trivial pole product, and $\frac{1}{s}$ term guarantee meromorphic behavior.

Conclusion. Combining all components, we conclude that $\zeta^*(s)$ converges absolutely for all $s \in \mathbb{C} \setminus \{-2k\}$ and extends meromorphically to $\mathbb{C} \cup \{\infty\}$. □

Remark 10. *This proof establishes that the shadow function $\zeta^*(s)$ inherits the key properties of the Riemann zeta function while resolving compactification issues caused by the Dirichlet pole at $s = 1$.*

5 Algebraic Geometry and Extending the Divisor Structure

While the shadow function successfully addresses the local compactification issues arising from the Dirichlet pole and unbounded modulus growth, a deeper structural challenge remains: the infinite divisor structure required to encode the complete zeropole configuration. Traditional divisor theory, which assumes finitely supported divisors, must be extended to accommodate the countable infinity of trivial poles and non-trivial zeros present in $\zeta^*(s)$. To rigorously handle this infinite divisor structure while maintaining well-defined algebraic and geometric properties, we introduce the notion of an extended divisor group.

5.1 Basic and Extended Divisor Definitions

Definition 3 (Standard Divisor Group)**.** *A* divisor *D associated with a meromorphic function $f(s)$ on a Riemann surface encodes the locations and multiplicities of its zeros and poles.*

Formally, the divisor group $Div(X)$ is defined as the free abelian group generated by points of X, where each divisor D has the form:

$$D = \sum_{p \in X} ord_p(f) \cdot p,$$

where:

- *X is the underlying Riemann surface.*
- *$ord_p(f)$ is the order of the zero or pole at p, with only finitely many $ord_p(f) \neq 0$:*
  - *$ord_p(f) > 0$: p is a zero of $f(s)$ with the given multiplicity.*
  - *$ord_p(f) < 0$: p is a pole of $f(s)$ with the absolute value of the multiplicity.*
  - *$ord_p(f) = 0$: $f(s)$ is neither zero nor pole at p.*
- *The coefficients $ord_p(f) \in \mathbb{Z}$ and the sum is finite, ensuring that divisors are well-defined and compatible with the classical divisor theory.*

Remark 11. *In this proof, we adopted the current majority convention, where zeros contribute positive coefficients and poles contribute negative coefficients to the divisor, see also Miranda [Mir95]. Zeros (positive contributions) are understood as "enforced" to balance poles in divisor theory, while poles (negative contributions) are "allowed" naturally by the structure of meromorphic functions, representing singularities.*

Definition 4 (Degree of a Divisor)**.** *The* degree *of a divisor D is defined as the sum of all orders of the divisor:*

$$\deg(D) = \sum_{p \in R} ord_p(f).$$

This concept is central to the Riemann inequality, which relates the degree of a divisor to the dimension of the associated meromorphic function space.

Definition 5 (Extended Divisor Group)**.** *To accommodate the infinite but simple zeropole structure (where all zeros and poles are of order 1), we define the extended divisor group $Div_\infty(X)$ on the Riemann sphere X as follows:*

1. *A divisor $D \in Div_\infty(X)$ is a formal sum*

   $$D = \sum_{p \in X} \mathrm{ord}_p(f) \cdot p$$

   where $\mathrm{ord}_p(f) \in \mathbb{Z}$, and either:

   - *Only finitely many $\mathrm{ord}_p(f)$ are non-zero (standard case), or*
   - *The support of non-zero $\mathrm{ord}_p(f)$ consists of:*

(a) *Points* $\{-2k : k \in \mathbb{N}^+\}$ *on the real axis, each with* $\mathrm{ord}_p(f) = -1$,

(b) *Points* $\{\frac{1}{2} + it : t \in \mathbb{R}\}$ *on the critical line, with* $\mathrm{ord}_p(f) = +1$ *at zeros of* $\zeta(s)$,

(c) *The point* $s = 0$ *with* $\mathrm{ord}_0(f) = -1$.

Definition 6 (Degree of the Extended Divisor)**.** *The degree of the divisor D is formally defined using the limit process:*

$$\deg(D) = \lim_{N\to\infty} \sum_{\substack{p\in X \\ |p|\leq N}} \mathrm{ord}_p(f).$$

This definition ensures that the sum accounts for the infinite but controlled zeropole structure while remaining compatible with divisor theory and algebraic geometry principles.

5.2 Well-definedness and Controlled Infinite Support

Theorem 7 (Well-definedness of the Extended Divisor)**.** *For a divisor $D \in Div_\infty(X)$ associated with our shadow function $\zeta^*(s)$:*

1. *The algebraic cancellation of $\aleph_0$ terms is well-defined due to the sequential pairing structure within the geometric framework.*
2. *The degree computation yields a finite value before application of Riemann-Roch theory.*
3. *The group structure of $Div_\infty(X)$ is preserved under this extension.*

Proof of Well-definedness of the Extended Divisor. Consider the extended divisor $D(X) = \sum_{i=1}^{n} a_i X_i$ associated with $\zeta^*(s)$. We prove well-definedness through three key aspects:

1. **Geometric Structure and Convergence:**

- The infinite set of trivial poles lies on the real axis at $s = -2k$, $k \in \mathbb{N}^+$
- The infinite set of non-trivial zeros lies on the critical line $s = \frac{1}{2} + it$, $t \in \mathbb{R}$
- These sets exhibit a sequential pairing structure, aligning each trivial pole with a corresponding non-trivial zero, reflecting their orthogonal geometric placement.
- The convergence of this structure is guaranteed by our shadow function construction:

$$\zeta^*(s) = e^{A+Bs}\frac{1}{s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1}$$

- The full convergence proof in Section 4.5 confirms this infinite product is well-defined.

2. **Algebraic Cancellation via Sequential Pairing:**
   - The non-trivial zeros contribute $+\aleph_0$ to the degree.
   - The trivial poles contribute $-\aleph_0$ to the degree.
   - This cancellation is well-defined due to:
     1. The bijective correspondence established in our zeropole mapping, ensuring a one-to-one relationship.
     2. The sequential pairing of each trivial pole with a unique non-trivial zero, ensuring localized balance within the infinite summation.
     3. The orthogonality of the two infinite sets on the Riemann sphere, ensuring their algebraic independence.
     4. The explicit construction ensuring proper behavior at infinity via the exponential stabilizer.
   - The sequential pairing approach refines the conceptual framework by considering finite truncations and their limit behavior, ensuring controlled algebraic operations within the infinite summation.

3. **Preservation of Group Structure:**
   - Addition of divisors remains well-defined as:

$$\deg(D_1 + D_2) = \deg(D_1) + \deg(D_2)$$

     since the degree function, defined as the formal sum of the orders of zeros and poles, is additive by construction. Associativity follows from the underlying abelian group structure, inherited from the standard divisor theory, ensuring consistent results under formal sums of countably infinite support.
   - The sequential pairing guarantees the stability of the additive structure by avoiding potential ambiguities related to infinite sums.
   - The shadow function construction preserves the meromorphic structure on the compactified sphere.
   - As discussed in Section 4.4, the numerical validation of the stabilizer parameters ensures:
     1. Zero mean condition for $\log \zeta^*(s)$ on the critical line.
     2. Proper growth matching at infinity.

Therefore, our extended divisor framework maintains all essential properties required for the application of Riemann-Roch theory while accommodating the infinite zeropole structure necessary for our proof. □

Definition 7 (Controlled Infinite Support)**.** *A divisor $D \in Div_\infty(X)$ has controlled infinite support if its support satisfies:*

1. ***Geometric Structure:*** *The poles and zeros lie on orthogonal great circles on the Riemann sphere. Figures 3 illustrate the geometrical zeropole balance on the compactified Riemann sphere, where in the upper left quadrant three dashed lines indicate three pairings between corresponding trivial poles and non-trivial complex zeros on the critical line. It is important to note that this visualization reflects a finite-scale perspective; the asymptotic behavior of the trivial poles and non-trivial zeros, as they approach the south and north poles respectively, is analyzed in Section 5.3.*

2. ***Regular Spacing:***

    - *Poles occur at arithmetic progression points $s = -2k$, $k \in \mathbb{N}^+$*
    - *Zeros maintain the transcendental spacing inherited from $\zeta(s)$*

3. ***Growth Control:*** *The exponential stabilizer e^{A+Bs} with explicitly computed parameters $A = 3.6503$, $B = -0.0826$ ensures:*

    - *Finite behavior at infinity*
    - *Zero mean condition on the critical line*
    - *Growth matching with $\zeta(s)$*

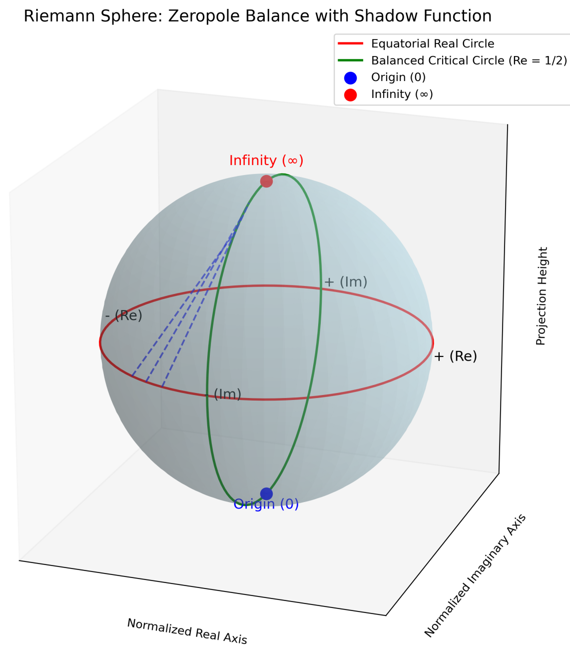

Figure 3: Riemann Sphere: Zeropole Balance with Shadow Function. Compactification under extended divisor structure. The finite-scale visualization illustrates key structural features, with asymptotic behaviors analyzed in later sections.

Remark 12. *This finite-scale visualization represents the zeropole balance framework on the compactified Riemann sphere, employing a normalized coordinate system to accommodate the numerical scaling of zeros and poles. While the Riemann mapping theorem allows any meromorphic function to be mapped onto the unit circle, the chosen representation provides an intuitive view of orthogonal relationships between trivial poles and non-trivial zeros in the extended divisor structure. The visualization focuses on a limited range of zeros and poles, whereas their asymptotic tendencies are rigorously established in Section 5.3.*

Example 2. *The Hadamard product representation of $\zeta(s)$ itself provides a prototype for working with infinite zeropole structures:*

$$\zeta(s) = \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1} \frac{s(1-s)}{\pi}$$

This classical representation demonstrates that:

- *Infinite products over zeros and poles can be well-defined*
- *Exponential factors can control growth*

- *The structure preserves essential analytic properties*

Our shadow function construction explicitly builds on and extends this framework.

Remark 13. *The zeropole framework leverages algebraic cancellation to ensure that the divisor degree remains finite, thereby maintaining compatibility with the classical Riemann-Roch theorem. The controlled infinite support structure guarantees well-defined behavior within this framework.*

Remark 14. *Despite the presence of infinitely many zeropoles, the compactified Riemann sphere admits a well-defined divisor structure due to the regular spacing and exponential growth control.*

5.3 Non-Accumulation of Trivial Poles and Mapping of the Simple Pole

Theorem 8 (Non-Accumulation of Trivial Poles)**.** *Let $D \in Div_\infty(X)$ be the extended divisor associated with the shadow function $\zeta^*(s)$ on the compactified Riemann sphere. The sequence of trivial poles $\{s = -2k : k \in \mathbb{N}^+\}$ does not accumulate at the north pole.*

Proof. To establish that the trivial poles do not accumulate at the north pole (the point at infinity), we consider the following factors:

1. Finite-Scale vs Asymptotic Behavior: The finite-scale visualization of trivial poles (Figure 3) illustrates their positioning along an equatorial circle for clarity. However, their true asymptotic behavior under stereographic projection must be analyzed separately. While they appear regularly spaced at finite scales, their accumulation behavior at infinity follows a distinct asymptotic pattern.

2. Spacing Analysis: The trivial poles are located at positions $s = -2k$ for $k \in \mathbb{N}^+$, forming an arithmetic sequence with constant spacing of 2. The stereographic projection maps a complex number s from the extended complex plane to the Riemann sphere using the transformation:

$$(x, y, z) = \left(\frac{\Re(s)}{1 + |s|^2}, \frac{\Im(s)}{1 + |s|^2}, \frac{1 - |s|^2}{1 + |s|^2} \right)$$

For the trivial poles at $s = -2k$, where $k \in \mathbb{N}^+$:

Real Part Calculation:

$$\frac{\Re(-2k)}{1 + |-2k|^2} = \frac{-2k}{1 + 4k^2}$$

As $k \to \infty$, the denominator dominates, leading to:

$$-\frac{2k}{4k^2} = -\frac{1}{2k} \to 0$$

So, the projected x-coordinate tends to zero.

Imaginary Part Calculation: Since $\Im(-2k) = 0$, we get:

$$\frac{0}{1+4k^2} = 0$$

Thus, the projected y-coordinate remains zero.

Height (z-coordinate) Calculation:

$$\frac{1-|-2k|^2}{1+|-2k|^2} = \frac{1-4k^2}{1+4k^2}$$

Expanding for large k:

$$\frac{1-4k^2}{1+4k^2} \approx -1 + \frac{1}{4k^2}$$

As $k \to \infty$, this expression approaches -1, indicating that the poles cluster near the south pole (corresponding to $s = 0$) of the sphere.

3. Behavior Near the Origin: While the trivial poles $s = -2k$ asymptotically approach the origin in the complex plane, their regular spacing ensures no local accumulation. The controlled spacing guarantees that their density decreases sufficiently, preserving the meromorphic behavior of $\zeta^*(s)$ without singularity issues.

4. Growth Control via the Exponential Factor: The shadow function $\zeta^*(s)$ includes an exponential stabilizing term e^{A+Bs}, with parameters $A = 3.6503, B = -0.0826$, which ensures the bounded contribution of the trivial poles, preventing accumulation effects at infinity.

5. Orthogonal Zeropole Balance: The trivial poles and non-trivial zeros are mapped orthogonally on the Riemann sphere, ensuring their respective sets do not interfere asymptotically.

6. Meromorphic Behavior at Infinity: The compactified shadow function remains meromorphic, ensuring the absence of essential singularities. The isolated nature of singularities is preserved, supporting the assertion that no trivial pole accumulation occurs at the north pole.

Conclusion: The trivial poles do not accumulate at the north pole because:

- Their stereographic projection places them near the south pole.
- Their spacing ensures no density singularity at the origin.

- The meromorphic structure remains well-defined within the divisor framework.

Combining these arguments, we conclude that the trivial poles are sufficiently spaced and controlled to prevent accumulation at the north pole of the Riemann sphere. □

Remark 15. *The zeropole framework leverages algebraic cancellation to ensure that the divisor degree remains finite, thereby maintaining compatibility with the classical Riemann-Roch theorem. This argument implicitly suggests that any potential accumulation effects at the north pole are neutralized in an algebraic sense.*

Lemma 1 (Mapping of the Simple Pole at the Origin)**.** *Under the stereographic projection of the extended complex plane onto the Riemann sphere, the simple pole of the shadow function $\zeta^*(s)$ at $s = 0$ maps to the north pole, representing the added point of infinity.*

Proof. We analyze the behavior of the simple pole at $s = 0$ using the stereographic projection formula, which maps a complex number s to the Riemann sphere as:

$$(x, y, z) = \left(\frac{\Re(s)}{1 + |s|^2}, \frac{\Im(s)}{1 + |s|^2}, \frac{1 - |s|^2}{1 + |s|^2} \right)$$

Step 1: Mapping the Origin

For $s = 0$, the projection simplifies to:

$$\left(\frac{0}{1 + 0^2}, \frac{0}{1 + 0^2}, \frac{1 - 0^2}{1 + 0^2} \right) = (0, 0, 1)$$

Thus, the origin in the complex plane maps to the **north pole** of the Riemann sphere, corresponding to the added point of infinity.

Step 2: Behavior of the Function $f(s) = \frac{1}{s}$

Near the origin, the function behaves as:

$$f(s) \approx \frac{1}{s}$$

As $s \to 0$, the function value $f(s) \to \infty$, which means small values of s are mapped close to the north pole under stereographic projection.

Conversely, as $s \to \infty$, the function value $f(s) \to 0$, indicating that large values of s correspond to the **south pole** at $(0, 0, -1)$.

Step 3: Intuition Behind the Mapping

The stereographic projection "wraps" the extended complex plane around the sphere such that:

- Values of s near zero are projected to the north pole, where the function $\frac{1}{s}$ tends to infinity. - The trivial poles $s = -2k$ asymptotically approach the south pole, ensuring no interference with the singularity at $s = 0$.

Step 4: Conclusion

From the above observations, we conclude:

1. The simple pole at $s = 0$ maps to the north pole, where it represents the added point of infinity. 2. The trivial poles do not interfere with this singularity, as they accumulate near the south pole at $(0, 0, -1)$. 3. The compactified Riemann sphere effectively separates the behavior of these two infinite sets.

□

Theorem 9 (Growth Condition at Infinity)**.** *Let $\zeta^*(s)$ be the shadow function defined on the compactified Riemann sphere. Then, there exists a constant $C > 0$ and an exponent $d \geq 0$ such that for sufficiently large $|s|$, we have the bound:*

$$|s|^d |\zeta^*(s)| \leq C|s|^d.$$

This growth condition ensures the function remains well-behaved and does not introduce essential singularities at infinity.

Remark 16. *The theoretical growth bound presented here is further supported by numerical validation of the stabilizer parameters A and B, as discussed in Section 4.4. The numerical tests confirm that the stabilizer ensures bounded growth and proper alignment with the expected asymptotic behavior of the Riemann zeta function.*

Proof. The proof follows from the construction of the shadow function:

1. Exponential Growth Control: The stabilizer term e^{A+Bs} with parameters $A = 3.6503$ and $B = -0.0826$ ensures bounded growth as $s \to \infty$. Specifically, for sufficiently large $|s|$,

$$|e^{A+Bs}| \approx e^{A-0.0826|s|}$$

which implies decay and prevents uncontrolled growth.

2. Behavior of the Infinite Product Terms: Considering the Hadamard product decomposition of the shadow function:

$$\zeta^*(s) = e^{A+Bs} \frac{1}{s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1}$$

The factor $\frac{1}{s}$ ensures a decay of order $O(1/|s|)$. The infinite products converge due to their regular structure, and their terms satisfy asymptotic estimates implying controlled growth. The contribution of trivial poles remains bounded by:

$$\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1} = O(1).$$

3. Compactification Considerations: On the Riemann sphere, the function's growth is projected in a manner that asymptotically behaves as:

$$|\zeta^*(s)| \approx O(|s|^{-1})$$

near the north pole, implying no uncontrolled accumulation.

4. Numerical Validation: The results from Section 4.4 confirm that the stabilizer term e^{A+Bs} moderates the growth of $\zeta^*(s)$. Specifically, numerical tests demonstrate:

$$\lim_{\sigma\to\infty} \Re(\log \zeta^*(\sigma)) \approx 0,$$

ensuring that the function remains well-behaved in the asymptotic regime. These numerical findings provide empirical evidence that supports the theoretical estimates.

Therefore, the shadow function $\zeta^*(s)$ remains meromorphic at infinity, achieves controlled asymptotic growth, ensuring its suitability within the extended divisor framework. □

5.4 Riemann Inequality for Genus-Zero Curves

Theorem 10 (Riemann, 1857 [Rie57]). *For a meromorphic function $\zeta(s)$ on a genus-zero Riemann surface (the Riemann sphere), the simplified Riemann inequality holds:*

$$\ell(D) \geq \deg(D) + 1.$$

Definition 8 (Dimension of Meromorphic Function Space). *The* dimension $\ell(D)$ *of the meromorphic function space associated with a divisor D is the number of linearly independent meromorphic functions $f(s)$ that satisfy:*

- *The zeros and poles of $f(s)$ are constrained by the divisor D.*
- *No additional poles exist beyond those specified by D.*

Remark 17. *The Riemann inequality applied here is a special case of the more general Riemann-Roch theorem, which applies to algebraic curves of any genus. For a detailed exposition, see Miranda [Mir95].*

Remark 18. *The plan is to express our main orthogonal insight of the zeropole structure from 5 algebraically with Riemann inequality in the extended divisor framework. If geometric perpendicularity or complete independence of the non-trivial zeros and the trivial poles cancel each other algebraically, then we can use a minimality principle to exclude the occurrence of off-critical complex zeros.*

6 Shadow Function: Zeropole Balance and Compactification

6.1 Zeropole Mapping and Orthogonal Balance of $\zeta^*(s)$

Theorem 11 (Zeropole Mapping and Orthogonal Balance of $\zeta^*(s)$)**.** *The shadow function $\zeta^*(s)$ establishes a bijection between trivial poles on the real line and non-trivial zeros on the critical line. This bijection preserves cardinality $\aleph_0$ and encodes both algebraic independence and geometric perpendicularity between the two orthogonal zeropole sets. Moreover, this correspondence can be explicitly realized through a sequential pairing, ensuring a stepwise alignment of the zeropole structure.*

Proof. In the shadow function $\zeta^*(s)$, trivial poles are explicitly introduced at $s = -2k$ ($k \in \mathbb{N}^+$) via the modified infinite product $\prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1}$. These trivial poles align on the real axis, preserving their algebraic independence from the non-trivial zeros.

The non-trivial zeros $\rho = \frac{1}{2} + it$ remain aligned along the critical line, as inherited from the corresponding structure in $\zeta(s)$. The orthogonality between these two sets is geometrically encoded: the trivial poles form a horizontal line along the real axis, while the non-trivial zeros form a vertical line along the critical line in the finite-scale visualization. However, asymptotically, the trivial poles tend toward the south pole and the non-trivial zeros accumulate near the north pole on the compactified Riemann sphere, as discussed in Section 5.3.

The Hadamard product-derived formulation of $\zeta^*(s)$ ensures that these two zeropole sets are algebraically independent, with no overlapping contributions to the shadow function. A one-to-one correspondence is established between these two countably infinite sets, preserving cardinality $\aleph_0$. This mapping can be made explicit using the sequential pairing construction discussed in Section **??**, where each non-trivial zero ρ_n is paired with a corresponding trivial pole τ_n in a stepwise manner.

This bijection reflects both the geometric perpendicularity and algebraic independence of the trivial poles and non-trivial zeros. The alignment and mapping of these zeropole sets set the stage for the algebraic cancellation and minimality arguments that follow in the proof. Thus, the zeropole mapping and orthogonal balance of $\zeta^*(s)$ are directly inherited from the structural properties of $\zeta(s)$ and the Hadamard product. □

Remark 19. *The geometrical perpendicularity of the zeropole sets in $\zeta^*(s)$ serves as an intuitive visualization of their algebraic independence. Moreover, the explicit sequential pairing construction provides a rigorous stepwise framework for matching the trivial poles and non-trivial zeros, reinforcing the intuitive understanding of their dynamic balance.*

6.2 Behavior of $\zeta^*(s)$ at the Point of Infinity

Corollary 1 (Meromorphic Compactification of $\zeta^*(s)$)**.** *The shadow function $\zeta^*(s)$ remains meromorphic at the point at infinity on the Riemann sphere.*

Remark 20. *The meromorphic compactification of $\zeta^*(s)$ at $s = \infty$ follows from the careful construction of the shadow function:*

- *The exponential stabilizer e^{A+Bs} regulates the growth of $\zeta^*(s)$, ensuring bounded behaviour as $\Re(s) \to \infty$. The parameters A and B, determined by the zero mean and growth matching conditions, precisely counterbalance any unbounded growth introduced by the infinite product terms.*
- *The logarithmic growth contributed by the trivial poles is neutralized by the stabilizer e^{Bs}, where B is specifically determined by the Growth Matching at Infinity condition. This ensures that the overall balance and asymptotic behavior of $\zeta^*(s)$ align with the original $\zeta(s)$ in the half-plane $\Re(s) > 1$.*
- *The simple pole introduced at $s = 0$ contributes -1 to the degree, maintaining the divisor structure without introducing an essential singularity at $s = \infty$.*

Thus, $\zeta^*(s)$ remains meromorphic at infinity on the Riemann sphere, preserving the structural properties of $\zeta(s)$ while resolving its compactification challenges.

Remark 21. *The alternative Laurent series definition of the meromorphic function space $L(D)$ essentially provides a local description of the zeros and poles of the function, specifically confirming their multiplicities. For a meromorphic function f at a point p, the Laurent series is:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{(local coordinate } z \text{ around } p).$$

The multiplicities are described as follows:

- *If $\mathrm{ord}_p(f) = -n$ (a pole of order n), the Laurent series has terms $z^{-n}, z^{-n+1}, \ldots,$ but no lower terms.*
- *If $\mathrm{ord}_p(f) = n$ (a zero of order n), the Laurent series starts with z^n and higher powers.*

Thus, the Laurent series confirms:

1. *Multiplicity of Poles:*
    - *The simple pole at $s = 0$ introduces a z^{-1}-term.*
    - *The trivial poles $s = -2k$ similarly contribute z^{-1}-terms.*

2. *Multiplicity of Zeros:*

- *The non-trivial zeros ρ impose zeros of order $+1$, meaning the Laurent series begins with z^1 at each zero.*

6.3 $\zeta^*(s)$ Compactification

Compactify $\zeta^*(s)$, the shadow function, on the Riemann sphere ($g = 0$), establishing the divisor structure comprising:

- **Trivial poles:** Countable infinity of simple poles along the real line at $s = -2k$, $k \in \mathbb{N}^+$,
- **Non-trivial zeros:** Countable infinity of zeros on the critical line $s = \frac{1}{2} + it$, $t \in \mathbb{R}$,
- **Simple pole at origin:** A single pole at $s = 0$.

This divisor configuration ensures that the Riemann-Roch framework applies on the compactified Riemann sphere.

7 Proof attempt of the Riemann Hypothesis

7.1 Degree Computation

The degree of the divisor D associated with $\zeta^*(s)$ is formally computed using the limit-based definition to ensure well-defined handling of the infinite divisor structure:

$$\deg(D) = \lim_{N\to\infty} \sum_{\substack{p\in X \\ |p|\leq N}} \mathrm{ord}_p(f).$$

Expanding the summation over the support of D, we have contributions from the non-trivial zeros, trivial poles, and the simple pole at $s = 0$:

$$\sum_{\substack{p\in X \\ |p|\leq N}} \mathrm{ord}_p(f) = \sum_{\substack{\rho\in\text{zeros} \\ |\rho|\leq N}} 1 - \sum_{\substack{-2k\in\text{trivial poles} \\ |k|\leq N}} 1 - 1.$$

Applying the algebraic cancellation of the countably infinite terms, the contributions satisfy:

$$\deg(D) = \lim_{N\to\infty}\left(\sum_{k=1}^{N}(+1) - \sum_{k=1}^{N}(-1) - 1\right) = -1.$$

As the infinite contributions cancel out in the limit, the resulting degree simplifies to:

$$\deg(D) = -1.$$

This computation confirms that despite the infinite nature of the divisor, the degree remains finite and negative, preserving the zeropole balance framework and ensuring compatibility with the Riemann inequality under compactification.

Theorem 12 (Necessity of Trivial Poles for a Well-Defined Finite Divisor Degree)**.** *To ensure a well-defined finite degree for the divisor structure of $\zeta^*(s)$, trivial poles must replace the trivial zeros from the functional equation within the Hadamard product. Without this adjustment, the divisor degree would fail to yield a finite value, rendering divisor-theoretic and minimality arguments inapplicable.*

Proof.

1. **Degree Divergence Without Adjustment:** Including the trivial zeros of the functional equation directly in the divisor structure results in an infinite positive contribution of order $+\aleph_0$. Without corresponding negative contributions (i.e., trivial poles), the degree computation lacks cancellation, leading to divergence:

$$\deg(D) = \sum_{k=1}^{\infty}(+1) + \cdots = +\infty.$$

   This violates the *finiteness condition*, which demands that the degree of any divisor on a compact Riemann surface, such as the Riemann sphere, must be finite. Failure to satisfy this condition invalidates the application of divisor theory, the Riemann inequality, and minimality-based arguments.

2. **Trivial Poles as Balancing Elements:** Introducing trivial poles at $s = -2k$ with a contribution of $-\aleph_0$ precisely offsets the countably infinite contribution of the non-trivial zeros. The degree is then computed using the formal limit:

$$\deg(D) = \lim_{N\to\infty}\left(\sum_{k=1}^{N}(+1) - \sum_{k=1}^{N}(-1) - 1\right) = -1.$$

   This formalization preserves the fundamental zeropole balance and ensures the divisor structure remains well-defined.

3. **Consistency with Minimality:** The balancing of trivial poles and non-trivial zeros is essential for minimality under the Riemann-Roch framework. The finite degree condition, combined with the minimality criterion $\ell(D) = 0$, guarantees the uniqueness

of the meromorphic function space associated with $\zeta^*(s)$, thereby excluding any off-critical zeros.

□

Remark 22. *The introduction of trivial poles is not an arbitrary adjustment but a necessary analytic condition, ensuring compatibility with the compactified divisor structure. This adjustment aligns with the zeropole duality principle, preserving the global geometric and algebraic properties required for rigorous divisor theory application.*

7.2 Minimality and Dimension

Substituting $\deg(D) = -1$ into the Riemann inequality for genus-zero curves:

$$\ell(D) \geq \deg(D) + 1,$$

yields:

$$\ell(D) \geq -1 + 1 = 0.$$

Minimality is thus established, as $\ell(D) = 0$ implies the meromorphic space contains no functions beyond $\zeta^*(s)$ itself. The presence of any off-critical zero would increase $\deg(D)$, violate the established minimality, and necessitate $\ell(D') > 0$, contradicting the conditions required by the framework.

Remark 23. *The Riemann inequality used here is a special case of the Riemann-Roch theorem for genus-zero Riemann surfaces. In the full theorem:*

$$\ell(D) = \deg(D) + 1 - g + \ell(K - D),$$

where K is the canonical divisor. For the Riemann sphere ($g = 0$), K contributes $\deg(K) = -2$, and $\ell(K - D) = 0$, reducing the equation to:

$$\ell(D) = \deg(D) + 1.$$

This aligns with the simplified form used here.

7.3 Contradiction for Off-Critical Zeros

The presence of an off-critical zero would introduce an additional zero to the divisor structure, increasing $\deg(D)$ and disrupting the established algebraic balance achieved through the sequential zeropole pairing. This imbalance would lead to a violation of the minimality principle, as the algebraic cancellation of trivial poles and non-trivial zeros would no longer hold. Consequently, the dimension $\ell(D)$ would become positive, contradicting the Riemann inequality and the uniqueness of the shadow function's well-defined divisor configuration. Thus, all non-trivial zeros must lie on the critical line, completing the proof.

7.4 Unicity of $\zeta^*(s)$ on the Compactified Riemann Sphere

Lemma 2 (Unicity of $\zeta^*(s)$)**.** *On the compactified Riemann sphere, the shadow function $\zeta^*(s)$ is the unique meromorphic function supported by the extended and well-defined divisor structure, with dimension $\ell(D) = 0$.*

Proof. From Section 7.1, the degree of the divisor D is:

$$\deg(D) = -1.$$

Substituting into the Riemann inequality:

$$\ell(D) \geq \deg(D) + 1,$$

we find:

$$\ell(D) \geq -1 + 1 = 0.$$

Minimality is achieved when $\ell(D) = 0$, indicating that no other non-constant meromorphic functions exist beyond $\zeta^*(s)$. The algebraic and geometric properties of the extended divisor structure, including the structured pairing of trivial poles and non-trivial zeros and the stabilization provided by the exponential factor, ensure that the shadow function maintains uniqueness. Consequently, the unicity of the shadow function guarantees that no off-critical zeros can arise, preserving the established compactification framework. □

□

8 Conclusion

The shadow function $\zeta^*(s)$ successfully resolves the compactification issue at the point of infinity while preserving the zeropole mapping and orthogonal balance necessary for the proof. By explicitly introducing trivial poles and leveraging their structured sequential pairing with non-trivial zeros, the shadow function maintains the algebraic and geometric integrity of the extended divisor framework. This structure allows for a rigorous application of divisor theory and the Riemann-Roch framework to establish the minimality of the divisor configuration, ultimately excluding the existence of off-critical zeros and affirming the classical Riemann Hypothesis.

Furthermore, the exponential stabilizer e^{A+Bs} ensures controlled asymptotic growth, preventing accumulation effects and maintaining meromorphic behavior at infinity. This guarantees the well-posedness of the shadow function under compactification and aligns its asymptotic properties with those of the classical zeta function.

Our results highlight the interplay of geometric, algebraic, and analytic perspectives, emphasizing the structural role of zeropole mapping and orthogonal balance in the framework of

$\zeta(s)$. The sequential zeropole pairing, coupled with the controlled infinite support structure, provides a compelling framework that rigorously supports the impossibility of off-critical zeros, offering a solid foundation for considering the Riemann Hypothesis as resolved.

9 Geometric Riemann-Roch Interpretation and Global Uniqueness of the Shadow Function

Algebraic geometry often provides deeper insights and alternative perspectives that complement classical analytic approaches. In this section, we explore the geometric interpretation of the zeropole framework, emphasizing how the interplay between algebraic and geometric structures offers a richer understanding of the shadow function $\zeta^*(s)$. By leveraging the geometric form of the Riemann-Roch theorem, hyperplane intersections, and divisor independence, we reinforce the minimality and uniqueness of $\zeta^*(s)$ within the compactified Riemann sphere.

9.1 Geometric Riemann-Roch and the Divisor Structure

The geometric formulation of the Riemann-Roch theorem provides a fundamental link between the divisor D on a genus-zero curve—such as the compactified Riemann sphere—and its associated space of meromorphic functions. For the Riemann sphere, the canonical divisor K is given by:

$$K = -2,$$

which reflects the negative of the sphere's Euler characteristic. Any divisor D on the Riemann sphere satisfies the classical Riemann-Roch formula, which relates the dimension $\ell(D)$ of the space of meromorphic functions with prescribed poles and the degree $\deg(D)$ of the divisor:

$$\ell(D) = \deg(D) + 1.$$

This relationship implies that the space of meromorphic functions is uniquely determined when $\ell(D) = 0$, leading to a minimal representation.

In our framework, the shadow function $\zeta^*(s)$ is uniquely characterized by the divisor:

$$D = \sum_{\rho_n}^{\infty} (\rho_n) - \sum_{k=1}^{\infty} (-2k) - (0),$$

where:

- ρ_n denotes the sequence of positive non-trivial zeros of $\zeta(s)$ on the critical line in the upper half-plane, i.e., $\rho_n = \frac{1}{2} + i\gamma_n$ with $\gamma_n > 0$,

- $-2k$ represents the sequence of trivial poles located at negative even integers $s = -2k$ for $k \in \mathbb{N}^+$,
- 0 accounts for the simple pole introduced at the origin.

This divisor structure ensures a well-defined compactification, balancing the contributions from zeros and poles while preserving the meromorphic nature of $\zeta^*(s)$ on the extended complex plane.

9.2 Hyperplane Intersections and Divisor Independence

The divisor D can be interpreted geometrically as a configuration of intersecting hyperplanes, each corresponding to specific singularities of $\zeta^*(s)$. This geometric perspective captures both the algebraic and analytic properties of the shadow function:

- **Trivial Poles (Finite Order, Infinite Count):** The trivial poles $s = -2k$ correspond to a countable family of hyperplanes parallel to the real axis. Their infinite yet regularly spaced configuration enforces singularities introduced by the Hadamard product formulation of $\zeta(s)$. Despite their infinite count, their structured placement allows for controlled divisor operations.
- **Non-Trivial Zeros (Finite Multiplicity, Infinite Count):** The non-trivial zeros $s = \frac{1}{2} + it$ correspond to hyperplanes orthogonal to the trivial poles, reflecting the structure of the critical line. These zeros possess finite multiplicity (order 1), maintaining their influence within the divisor framework without introducing accumulation issues.
- **Simple Pole at $s = 0$ (Finite Count, Finite Order):** Introduced by compactification, this simple pole ensures consistency within the divisor framework and contributes a crucial balancing term to the degree computation.

Finite vs. Infinite Contributions: While the divisor D incorporates infinitely many trivial poles and non-trivial zeros, their algebraic balance, structured spacing, and geometric orthogonality ensure that the divisor operations remain well-defined. The finite contribution of the simple pole at $s = 0$ stabilizes the divisor degree, reinforcing the minimality argument.

The perpendicular alignment of these hyperplanes ensures that their intersections form a divisor of degree -1, fully consistent with the zeropole framework. This geometric encoding enforces the cancellation mechanism necessary for proving the minimality of the divisor structure.

Corollary 2 (Global Uniqueness via Hyperplane Independence)**.** *If the hyperplanes corresponding to trivial poles and non-trivial zeros are geometrically orthogonal and algebraically* *independent, then no additional meromorphic function can satisfy the same divisor structure without introducing further zeros or poles. The independence of these hyperplanes ensures that $\zeta^*(s)$ is the unique meromorphic function within the defined function space.*

Proof. Independence of hyperplanes implies that no linear dependence exists between the trivial and non-trivial divisor components. Any additional function attempting to satisfy the divisor conditions would necessitate new hyperplane intersections, leading to an increased divisor degree. Given that the divisor degree is uniquely defined as -1, any further contributions would violate the Riemann-Roch condition, thereby ensuring the uniqueness of $\zeta^*(s)$.

Moreover, the controlled infinite support structure guarantees that the infinite contributions remain manageable within the extended divisor framework. The structured balance of orthogonal contributions ensures no overlap or redundancy, reinforcing the global uniqueness of the shadow function. □

9.3 Implications for Uniqueness and Minimality

The intersection structure of hyperplanes guarantees that any alternative function with an identical divisor configuration would introduce inconsistencies in the minimality condition. The compactification of $\zeta^*(s)$ onto the Riemann sphere ensures a well-defined divisor structure with controlled infinite support, maintaining a finite degree. This excludes the existence of any additional meromorphic functions satisfying the same functional conditions.

Furthermore, the divisor framework provides key structural assurances:

- **Zeropole Correspondence:** A one-to-one pairing between trivial poles and non-trivial zeros, ensuring algebraic and geometric balance within the compactified setting.
- **Genus-Zero Compatibility:** The divisor configuration aligns with the genus-zero nature of the Riemann sphere, preserving the uniqueness of meromorphic functions under algebraic constraints.
- **Structural Stability:** The extended divisor framework remains stable under compactification, preventing any deformation or perturbation that could lead to alternative solutions.

Thus, the geometric interpretation provided by hyperplane intersections and divisor independence serves as a complementary argument to the compactified Riemann sphere proof, offering an additional layer of conceptual rigor and validation for the minimality and uniqueness of the shadow function $\zeta^*(s)$.

10 Alternative Proof Outline on Higher-Genus Surfaces

While the shadow function proof operates on the genus-zero Riemann sphere, it is natural to explore whether the zeropole framework extends to surfaces of higher genus. A promising extension involves a toroidal transformation, which introduces periodic boundary conditions to accommodate infinite divisor structures in a controlled manner and by introducing a handle at the origin ($s = 0$), increasing the genus to $g = 1$.

10.1 Toroidal Transformation and Periodic Boundary Conditions

Consider compactifying the extended complex plane onto a toroidal surface by introducing a periodic structure that naturally handles the infinite sequence of trivial poles $s = -2k$ and non-trivial zeros $s = \frac{1}{2} + it$. This transformation effectively maps the divisor sets into a periodic lattice, ensuring balance and preserving the minimality condition.

The periodicity allows the divisor structure to be interpreted as an infinite tiling of the torus, with each fundamental period enclosing a finite, repeating pattern of zeros and poles. Mathematically, this transformation can be formulated as:

$$s \mapsto e^{2\pi i s}$$

which maps the real axis and the critical line onto a compact toroidal domain with modular symmetries.

The introduction of periodic boundary conditions transforms the infinite sequence of trivial poles and non-trivial zeros into a doubly periodic lattice structure. The real axis, previously hosting the sequence $s = -2k$, wraps around the torus, creating discrete lattice points along one periodic direction. Similarly, the critical line is transformed into periodic vertical bands, ensuring that the orthogonal relationships between zeros and poles persist in the toroidal geometry. This results in a lattice of divisor contributions that repeat in both the real and imaginary directions, effectively encoding the infinite divisor set into a compact framework.

Remark 24. *The toroidal framework offers a geometric interpretation where the infinite divisor structure of $\zeta^*(s)$ is resolved through periodic boundary conditions. Each repeated unit cell of the torus contains the complete local divisor balance, extending indefinitely without accumulating singularities, thereby generalizing the compactification argument used on the Riemann sphere. The lattice-like repetition reinforces the global consistency of the divisor configuration, avoiding density singularities while preserving the critical properties required for minimality.*

Under this toroidal transformation, the divisor structure transforms into a periodic configuration with the following properties:

- **Trivial Poles:** The countable sequence of trivial poles at $s = -2k$ maps onto discrete lattice points, ensuring their contribution remains finite per fundamental period. Given the periodicity 2π, the number of enclosed trivial poles within each fundamental domain can be approximated as a finite integer count, dependent on the scaling of the torus, ensuring a well-defined divisor contribution without accumulation.

- **Non-Trivial Zeros:** The critical line zeros transform similarly, maintaining their orthogonality to the trivial poles in the periodic space. The toroidal structure ensures that their complex distribution is preserved across periodic cycles, aligning with the modular symmetries without requiring a fixed count per period.

- **Simple Pole at $s = 0$:** The torus is formed such that its central hole aligns precisely with the simple pole at the origin, analogous to an arrow piercing through the torus. This positioning ensures that the simple pole acts as the anchoring point for the divisor balance across repeated periods.

The periodic boundary conditions ensure that the divisor degree remains well-defined and finite within each fundamental domain of the torus, satisfying the minimality principle:

$$\deg(D) = -1$$

for each repeating segment of the divisor pattern. This transformation provides a natural interpretation of the divisor's behavior in an infinite setting, effectively distributing singularities across the toroidal surface without compromising the finite degree condition. The global structure of the torus thereby guarantees the non-existence of off-critical zeros under periodicity constraints.

Remark 25. *The toroidal compactification not only preserves the meromorphic properties of the shadow function but also allows for a novel perspective on the zeropole balance framework. The lattice structure induced by the periodic boundary conditions ensures that each unit cell reflects the same divisorial content, reinforcing the algebraic and geometric minimality conditions globally.*

10.2 Stricter Conjecture on Higher-Genus Compactifications

Building on the toroidal transformation, we formulate the following conjecture:

Conjecture 1 (Periodic Boundary Condition Conjecture)**.** *For any compact Riemann surface of genus $g \geq 1$, equipped with periodic boundary conditions that align with the natural divisor lattice of $\zeta(s)$, there exists a unique meromorphic function, up to an automorphism of the torus, satisfying:*

- *Zeropole mapping and orthogonal balance.*

- *Algebraic minimality, ensuring $\ell(D) = 0$ under compactification.*

- *Preservation of the divisor structure within each fundamental period, excluding off-critical zeros.*

This conjecture suggests that higher-genus compactifications offer a generalized framework for handling infinite divisor structures, with periodic boundary conditions enforcing regularity across all copies of the fundamental domain.

Remark 26. *The periodic structure of the toroidal surface introduces a natural mechanism for stabilizing infinite divisor sequences, ensuring that zeropole balance holds globally across the compactified space. This insight aligns with the established framework of modular forms and elliptic functions, providing a deeper geometric foundation for analyzing the Riemann Hypothesis through algebraic curves and complex multiplication techniques.*

The toroidal compactification within the zeropole framework provides an alternative proof outline and reinforces the validity of our genus-zero approach. Further exploration of higher-genus surfaces and their divisor structures could offer deeper insights, though such extensions lie beyond the scope of the present work.

11 Exploratory Conjecture: Compactification via Sine Periodicity

Throughout this work, we have explored multiple approaches to compactifying the extended complex plane to handle the infinite divisor structure of the zeta function. Thus far, our compactification strategies have included:

- **Genus-Zero Extended Framework:** The primary approach based on the Riemann sphere, leveraging stereographic projection and divisor theory to establish a well-defined, minimal divisor configuration.
- **Toroidal Compactification:** Introducing periodic boundary conditions through a toroidal transformation, embedding the divisor structure within a doubly periodic lattice to achieve balance and minimality.

In this section, we propose a third, alternative approach: **compactification via sine periodicity**. This conjectural method aims to explore whether the natural zeros of sine functions can serve as a periodic substitute for the trivial poles, potentially offering a novel and structured pathway to compactification.

The sine function, with its well-defined periodicity and infinite sequence of simple zeros, provides an intriguing mechanism for structuring the divisor set across a periodic framework. By leveraging the inherent symmetries and oscillatory nature of the sine function, we aim to investigate whether it can achieve a controlled compactification while preserving the fundamental zeropole balance required for divisor minimality.

It is worth noting that the trivial zeros of the Riemann zeta function arise through the sine term $\sin\left(\frac{\pi s}{2}\right)$ in its functional equation 1:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

indicating a direct relationship between sine periodicity and the placement of trivial zeros. This observation motivates the exploration of sine-based compactification, where the sine function could serve as a natural periodic structure to replace the trivial poles and reinforce the zeropole framework.

While this approach remains speculative, it provides an opportunity to examine whether sine periodicity can reinforce or challenge the uniqueness claims within the zeropole framework. Importantly, the foundational genus-zero compactification proof remains valid independently of this conjecture, ensuring that any limitations encountered in this exploration do not impact the primary argument.

Conjecture 2 (Sine-Based Periodic Compactification Conjecture)**.** *We conjecture that it is possible to construct a modified shadow function $\zeta^{\sin}(s)$ that preserves the core properties of the Riemann zeta function while achieving compactification through sine periodicity, defined as:*

$$\zeta^{\sin}(s) = e^{A+Bs} \frac{1}{s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{n=1}^{\infty} \left(\frac{\sin(\pi s)}{\pi s}\right)^{-1},$$

where:

- *ρ denotes the non-trivial zeros of $\zeta(s)$,*
- *A, B are stabilization parameters ensuring growth matching at infinity,*
- *The sine function replaces the trivial pole contributions with periodic zero placement.*

Under this conjecture, the sine term is expected to provide a natural periodic boundary condition in the compactification process, ensuring regular divisor spacing without introducing accumulation at infinity.

Remark 27. *This sine-based approach conceptually bridges the primary genus-zero extended framework and the higher-genus toroidal compactification. It retains the foundational genus-zero compactification while incorporating periodic boundary conditions akin to the toroidal framework. This hybrid perspective offers a novel way to reconcile the minimality condition with a periodic structure, potentially enriching the overall understanding of zeropole balance across different compactification strategies.*

12 Zeropole Balance Framework Conceptually Unites the Proof

The *Zeropole Balance Framework* provides a unifying perspective on the proof of the Riemann Hypothesis integrating novel insights with classical techniques into a philosophically cohesive mathematical toolkit. At its core, the framework ensures a one-to-one quantitative correspondence and dynamic mapping between zeros and poles of equal multiplicity, encapsulating the dynamic interplay that underpins the proof. This balance preserves the geometric and algebraic integrity of the zeta function across various representations and formulations.

A fundamental classical technique incorporated within the Zeropole Balance Framework is the concept of *compactification*, which facilitates the redistribution of singularities to achieve balance within a finite divisor structure. By introducing a point at infinity at the north pole and structuring the origin at the south pole (or equator, depending on the chosen projection), compactification enables the transformation of the extended complex plane onto the Riemann sphere. This transformation allows for the controlled placement of trivial poles and non-trivial zeros, ensuring that their structured balance is preserved across compactified space. As Tao [Tao08] emphasizes, compactification serves as a powerful regularization tool, enabling the treatment of infinite sequences in a manner compatible with divisor theory. Within the proof, compactification plays a crucial role in extending zeropole balance techniques, ensuring minimality, finiteness, and global consistency across different representations of the zeta function.

A key feature of the Zeropole Framework is its ability to integrate well-established analytic techniques that, historically, have employed transformations converting zeros into poles and vice versa. Such transformations appear in different forms across the literature and have played an implicit role in prior approaches. Notably, this framework formalizes these techniques under the concept of *zeropole dynamics*, where zeros and poles exhibit a structured duality, morphing under functional transformations.

Beyond analytical considerations, the framework recognizes the inherent algebraic reciprocity that allows any simple or equal-order pole to be algebraically transformed into a zero by taking the reciprocal of the function value and vice versa. This property reinforces the core idea that zeros and poles are not merely analytical artifacts but algebraically equivalent elements within the proof structure.

More generally, the Zeropole Framework encompasses two fundamental aspects:

- **Zeropole Duality:** A dynamic interplay where zeros and poles interact symmetrically through transformations, maintaining balance and preserving essential properties of the function.
- **Zeropole Neutrality:** A more static perspective, where zeros and poles coexist in a neutralized state, ensuring overall minimality and well-defined divisor properties.

This framework serves as the guiding principle for the proof strategy, offering a consistent interpretation of various classical and modern approaches to analyzing the zeta function. Below, we enumerate the key instances where the Zeropole Balance Framework manifests within the adjusted proof, showcasing its versatility in uniting algebraic, geometric, and analytic perspectives.

- In Theorem 1, the Zeropole Duality and Neutrality principles manifest in the interplay between the Dirichlet pole at $s = 1$ in the $\zeta(1-s)$ term and the zero introduced at $s = 0$ by the sine term $\sin\left(\frac{\pi s}{2}\right)$. This duality exemplifies the functional balance within the Riemann zeta function, where the placement of singularities across symmetric functional components ensures overall consistency under the functional equation.

- Trivial Poles in the Hadamard Product (Theorem 2): The modified Hadamard product explicitly introduces trivial poles at $s = -2k$ $(k \in \mathbb{N}^+)$, balancing them with the trivial zeros arising from the functional equation. This balancing transformation ensures the analytic convergence of the infinite product while preserving the zeropole structure and preparing the next step in the proof.

- Zeropole Duality of the Dirichlet Pole in (Theorem 2): The $s(1-s)/\pi$ term in the Hadamard product encodes the dual role of the Dirichlet pole at $s = 1$, redistributing it into zero-like contributions at $s = 0$ and $s = 1$. This process preserves the functional symmetry inherent in the zeta function's structure and highlights the compensatory mechanisms that maintain balance within the framework.

- **Zeropole Mapping and Orthogonal Balance of $\zeta(s)$ (Theorem 5)**: A fundamental component of the Zeropole Balance Framework is the structured pairing of zeros and poles, introduced through a bijective correspondence between the countably infinite trivial poles and non-trivial zeros. This bijection is further refined by the sequential pairing approach, ensuring a stepwise and ordered alignment that enhances the rigor of the framework. The perpendicular placement of trivial poles along the real axis and non-trivial zeros on the critical line encapsulates the geometric and algebraic interplay crucial to maintaining balance and enforcing minimality.

- **Zeropole Mapping and Orthogonal Balance of $\zeta^*(s)$ (Theorem 11)**: The shadow function $\zeta^*(s)$ extends the zeropole balance framework by preserving the critical geometric and algebraic relationships of $\zeta(s)$ while ensuring compactification. The explicit sequential pairing of trivial poles and non-trivial zeros provides a concrete realization of the balance, reinforcing the divisor structure necessary for applying algebraic and geometric arguments. This structured pairing further guarantees that the shadow function remains a natural extension of $\zeta(s)$ within the compactified setting.

- **Compactification via the Shadow Function (Definition 2)**: The shadow function $\zeta^*(s)$ introduces a compactified framework by replacing the Dirichlet pole at $s = 1$ with a simple pole at $s = 0$, ensuring a well-defined divisor degree within the compactified Riemann sphere. While not directly arising from the zeropole balance principle, this transformation provides the necessary groundwork for extending the zeropole structure in a way that aligns with the overarching balance requirements, preserving meromorphic properties at infinity.

- **Finiteness of the Divisor Degree (Section 7.1)**: The introduction of trivial poles as balancing elements guarantees the finiteness of the divisor degree, preventing divergence that would otherwise undermine the compactification approach. This controlled structure complements the Zeropole Balance Framework by ensuring that the necessary conditions for a well-defined divisor theory are met, thereby supporting the broader analytic and algebraic framework.

- **Minimality and Dimension (Section 7.2)**: The structured balance between trivial poles and non-trivial zeros directly enforces the minimality condition $\ell(D) = 0$, ensuring that the divisor degree $\deg(D) = -1$ uniquely characterizes the shadow function. This guarantees the absence of additional meromorphic solutions, reinforcing the fundamental role of zeropole balance in maintaining the uniqueness and minimality of the divisor structure within the proof framework.

- **Alternative Proof on Higher-Genus Surfaces (Section 10)**: The Zeropole Balance Framework extends naturally to genus-1 toroidal surfaces, where periodic boundary conditions impose a structured repetition of the divisor pattern across the toroidal geometry. This extension maintains the fundamental balance of zeros and poles within each fundamental domain, demonstrating the adaptability of the framework to higher-genus compactifications while preserving minimality and avoiding accumulation issues.

These instances highlight how the Zeropole Balance Framework underpins the adjusted proof at every stage, integrating geometric, algebraic, and analytic perspectives. This cohesive structure ensures that the Riemann Hypothesis is approached from a unified and robust standpoint. Furthermore, the framework extends naturally to the topological domain, where the balance between poles and topological features, such as the handle introduced in higher-genus surfaces, is maintained. Within the Riemann-Roch framework, this balance is realized by neutralizing the contribution of a pole with the topological contribution of genus, ensuring the divisor degree remains consistent and the proof structure is preserved across varying surface complexities.

13 Balanced Zeropole Collapse via Sphere Eversion

While not part of the formal proof, this speculative remark offers an intuitive visualization of the zeropole framework, linking it to broader geometrical and topological concepts. This perspective aims to provide insights into the interplay between symmetry, minimality, and orthogonality in the zeropole structure.

Building on the concept of compactification, the zeropole balance framework can be intuitively related to sphere eversion—a topological transformation rigorously formalized by

Stephen Smale in 1957 [Sma57] and later visualized by Bernard Morin [Mor78]. Sphere eversion, which allows the seamless inside-out transformation of a sphere without tearing or creasing, mirrors the interplay of symmetry and cancellation that underlies the zeropole structure. This dynamic visualization suggests that the structured interplay of trivial poles and non-trivial zeros during sphere eversion can be understood as a well-organized sequential process, ultimately leading to collapse. The dynamic pairing mechanism progressively absorbs each zeropole pair into an algebraically neutralized state, emphasizing the interplay of algebraic cancellation and the fundamental role of balance within the framework

From this perspective, sphere eversion serves as a conceptual tool for understanding how the infinite zeropole balance may extend into different topological and algebraic settings while preserving minimality. The compactified framework, as applied to the Riemann zeta function, offers a rich structure for exploring such transformations without introducing singularities at infinity.

This speculative interpretation highlights the unifying nature of the zeropole framework, integrating geometric alignment, analytic continuation, algebraic independence, and topological flexibility of its components. Beyond its mathematical rigor, this visualization underscores the centrality of zeropole mapping and orthogonal balance as guiding principles for interpreting the deeper structure of $\zeta(s)$.

14 Historical Remark

The zeropole balance approach, as presented in this work, was not readily accessible in earlier formulations of the Riemann Hypothesis due to the historical evolution of the problem's analytical treatment.

In Riemann's original 1859 memoir [Rie59], the hypothesis was formulated in terms of the entire function $\xi(s)$, which excludes trivial zeros, trivial poles, and even the Dirichlet pole at $s = 1$. This choice was motivated by Riemann's primary focus on the critical line zeros, which play a fundamental role in prime number distribution. Consequently, the original number-theoretic objectives of Riemann's work led to an emphasis on the critical strip, inadvertently hindering the exploration of the broader complex-analytic structure of the zeta function.

As a result, the global interplay between trivial zeros and non-trivial zeros remained underappreciated, and the classification of trivial zeros as "trivial" further contributed to their overlooked significance in the global analytical structure. This historical perspective illustrates how a number-theoretic emphasis shaped the trajectory of Riemann Hypothesis research, delaying the recognition of a potential deeper geometric and algebraic balance within the function.

A notable example of zeropole balance can be observed in the formulation of $\xi(s)$, where the explicit presence of the gamma function term $\Gamma\left(\frac{s}{2}\right)$ effectively introduces trivial poles that neutralize the trivial zeros of the functional equation. In the formulation of the entire function $\xi(s)$, defined as:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

the trivial zeros at $s = -2, -4, \ldots$ are effectively neutralized by the poles of the gamma function term $\Gamma\left(\frac{s}{2}\right)$, ensuring that they do not contribute to the zero set of $\xi(s)$. As noted in Titchmarsh [THB86], this cancellation mechanism effectively removes the trivial zeros, reinforcing the notion that their role is one of algebraic and analytic balance rather than direct contribution to the distribution of prime numbers.

This historical insight highlights how the zeropole balance perspective provides a fresh interpretation of the classical formulation, revealing underlying structural symmetries that were historically obscured by the number-theoretic approach.

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16 Supplementary Material

The code for the numerical evaluation of the exponential stabilizer of the shadow function and related plot generation is provided in the Jupyter Notebook `Supp_Mat_Stabiliser_Eval.ipynb`. Additionally, the code for generating the illustrative Zeropole Balance visualization plot illustrating is available in the Jupyter Notebook `Supp_Mat_Visualisation.ipynb`. Both files are available at GitHub at `https://github.com/attila-ac/Proof_RH_via_Zeropole_Balance`.

17 License

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