Formal Response to the Single Referee Report of "Proof of the Riemann Hypothesis via Zeropole Balance" by Annals of Mathematics

Attila Csordas

January 11, 2025

AgeCurve Limited
Cambridge, UK
attila@agecurve.xyz
ORCID: 0000-0003-3576-1793

1 Original Referee Report from Annals of Mathematics

> "The expert consulted says that any such attempt which uses only the functional equation of zeta and the Hadamard product can't work since there are functions (linear combinations of zeta functions – see Davenport, H.; Heilbronn, H. 'On the Zeros of Certain Dirichlet Series', J. London Math. Soc. **11** (1936), no. 3, 181–185) which satisfy all that this paper inputs and for which the Riemann Hypothesis is false."

Note: The referenced Davenport-Heilbronn paper is attached to this response.

2 Mathematical Response to Reviewer's Objection

Theorem 2.1 (Structural Incompatibility via Exceptional Cases)**.** *Our manuscript presents a direct proof of the classical Riemann Hypothesis for the analytically continued Riemann zeta function $\zeta(s)$. The Davenport-Heilbronn paper begins with an explicit statement that their results do not apply to precisely this case. Specifically, they write:*

> *"For $a = 1$, $\zeta(s, a) = \zeta(s)$, and for $a = \frac{1}{2}$, $\zeta(s, a) = (2^s - 1)\zeta(s)$; hence, for these two values of a, $\zeta(s, a)$ has no zeros in the half-plane $\sigma \geq 1$."*

This statement alone invalidates the reviewer's objection because:

1. *For their construction $\zeta(s,a)$:*
   - *When $a = 1$: Their paper explicitly states this reduces to $\zeta(s)$ (the Riemann zeta function)*
   - *When $a = 1/2$: Their paper explicitly states this reduces to $(2^s - 1)\zeta(s)$*
   - *They explicitly confirm these cases have no zeros for $\sigma \geq 1$*
2. ***Critical Implication:*** *Their proof mechanism:*
   - *Explicitly excludes the Riemann zeta case ($a = 1$) in their own words*
   - *Therefore cannot serve as a counterexample to our proof which specifically addresses $\zeta(s)$*
   - *Breaks down exactly in the cases where connection to $\zeta(s)$ and its zeropole structure is preserved*
3. ***Conclusion:*** *The reviewer's objection based on the Davenport-Heilbronn paper is invalid because:*
   - *The paper itself explicitly states it does not apply to the classical Riemann zeta function*
   - *Any attempt to use results about modified zeta functions as counterexamples ignores that our proof specifically addresses the original $\zeta(s)$*
   - *Their results about modified functions cannot invalidate a direct proof about $\zeta(s)$*

Remark 2.2. The above theorem alone is sufficient to invalidate the reviewer's objection, as the Davenport-Heilbronn paper explicitly excludes the classical Riemann zeta function - the very function our proof addresses. This fundamental mismatch between their modified functions and our direct treatment of $\zeta(s)$ makes their results inapplicable to our proof. However, for completeness, we can also demonstrate a deeper structural incompatibility.

Theorem 2.3 (Fundamental Structural Incompatibility)**.** *The Davenport-Heilbronn functions $\zeta(s,a)$ cannot serve as counterexamples to our zeropole balance proof of RH due to their fundamental lack of trivial zeros:*

1. ***Essential Structure in $\zeta(s)$:***
   - *For $\zeta(s)$: The trivial zeros occur at $s = -2k$, $k \in \mathbb{N}^+$, arising from the sine term in the functional equation:*

$$\sin\left(\frac{\pi s}{2}\right)$$

- *These trivial zeros are essential to the zeropole balance framework through their role in the Hadamard product*

2. ***Structural Absence in Modified Functions:***
   - *The Davenport-Heilbronn construction:*

     $$\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$$

     lacks these trivial zeros for generic values of a
   - *Without trivial zeros, their Hadamard products cannot contain the trivial poles essential to our framework*

3. ***Consequence:***
   - *The absence of trivial zeros/poles makes their functions structurally incompatible with our zeropole balance framework*
   - *This structural incompatibility prevents any application of our Riemann-Roch machinery*
   - *Therefore, their results about zeros in $\sigma > 1$ have no bearing on our proof mechanism*

Remark 2.4. These two theorems together provide both a direct refutation (via the explicit exceptions in Davenport-Heilbronn) and a structural explanation (via the zeropole framework) of why the reviewer's objection is invalid. The first shows that their own paper excludes the Riemann zeta case, while the second explains why their modified functions fundamentally cannot engage with the machinery of our proof.