Appendix: Geometric Riemann-Roch, Hyperplane Intersections and the Zeropole Framework

Supplementary Material for the Manuscript: ”Proof of the Riemann Hypothesis via Zeropole Balance”

Attila Csordas
Affiliation: AgeCurve Limited, Cambridge, UK

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1 Introduction

This appendix provides a geometric interpretation of the proof of the Riemann Hypothesis via the zeropole framework. Specifically, we reinterpret the divisor structure of $\zeta^*(s)$ using the geometric form of the Riemann-Roch theorem. The interpretation leverages the canonical divisor, hyperplane intersections, and the correspondence between trivial poles and non-trivial zeros to provide additional conceptual clarity and robustness to the proof.

2 Geometric Form of Riemann-Roch and the Canonical Divisor

The Riemann-Roch theorem, in its geometric form, relates the divisor D on a curve X to its associated linear system of hyperplanes. For a genus-zero curve (such as the Riemann sphere), the canonical divisor K is given by:

$$K = -2.$$

This implies that the degree of the canonical divisor is -2, and any divisor D on the Riemann sphere satisfies:

$$\ell(D) = \deg(D) + 1,$$

where $\ell(D)$ represents the dimension of the space of meromorphic functions associated with D.

The shadow function $\zeta^*(s)$ is constructed as a unique meromorphic function on the compactified Riemann sphere consistent with the divisor:

$$D = \sum_{\rho}(\rho) - \sum_{k=1}^{\infty}(-2k) - (0),$$

where ρ are the non-trivial zeros, $-2k$ are the trivial poles, and 0 denotes the simple pole at the origin.

2.1 Roles of Compactification in the Zeropole Framework

Compactification plays two crucial roles in the context of the Riemann sphere and the divisor structure associated with $\zeta^*(s)$:

- **Completeness**: Compactification ensures that the Riemann sphere is a complete surface, meaning that every meromorphic function (including $\zeta^*(s)$) is defined globally. This eliminates ambiguities in the behavior of the function, particularly at $|s| \to \infty$, and allows for a coherent divisor structure.
- **Finiteness**: The compactified Riemann sphere imposes a finite divisor framework, where the degree of the divisor D is well-defined and finite. This constraint is essential for applying the geometric Riemann-Roch theorem, as infinite degrees would violate the assumptions of the theorem and break the balance required by the zeropole framework.

Compactification introduces the point at infinity on the Riemann sphere, which integrates the global behavior of the function into the divisor structure. The simple pole at $s = 0$ in $\zeta^*(s)$ replaces the Dirichlet pole at $s = 1$, aligning the function's properties with the compactified framework. This adjustment ensures that all hyperplanes (trivial poles, non-trivial zeros, and the simple pole) interact coherently without introducing inconsistencies.

2.2 Linearity in the System of Hyperplanes

In this context, "linearity" refers to the property that the hyperplanes associated with a divisor D form a linear system. A linear system is a family of divisors that can be expressed as a linear combination of a fixed base divisor and a parameterized set of variations. Specifically, the associated hyperplanes are defined by linear equations in projective space, which encode the conditions imposed by the zeros and poles of the meromorphic function on the divisor.

This linearity ensures that the intersections of these hyperplanes correspond to the zeros and poles of the function, preserving the structure of the divisor and enabling a clear geometric interpretation of the Riemann-Roch theorem.

3 Hyperplane Intersections and Divisor Structure

The divisor D can be visualized geometrically as the intersection of hyperplanes corresponding to the trivial poles, non-trivial zeros, and the simple pole at $s = 0$. These hyperplanes encode linear conditions on the meromorphic functions associated with D:

- **Trivial Poles:** The trivial poles $s = -2k$ correspond to a set of hyperplanes enforcing singularities introduced by the gamma factor $\Gamma(1 - s)$.
- **Non-Trivial Zeros:** The non-trivial zeros $s = \frac{1}{2} + it$ correspond to hyperplanes aligned with the critical line.
- **Simple Pole at $s = 0$:** The simple pole at the origin corresponds to an additional hyperplane condition introduced by compactification.

These hyperplanes intersect on the canonical curve, forming a divisor of degree -1 that satisfies the zeropole framework. The cancellation mechanism is enforced geometrically by the perpendicularity of the hyperplanes associated with the trivial poles and non-trivial zeros.

4 Independence of Hyperplanes and Zeropole Structure

Definition 1 (Independence of Hyperplanes)**.** *Hyperplanes in projective space are said to be independent if their intersections are distinct and form a unique divisor consistent with the specified conditions. Independence ensures that no hyperplane can be expressed as a linear combination of others within the divisor system.*

In the zeropole framework:

- The trivial poles $s = -2k$ and non-trivial zeros $s = \frac{1}{2} + it$ correspond to distinct hyperplanes that are geometrically orthogonal, preserving their independence.
- The simple pole at $s = 0$ introduces an additional independent hyperplane, ensuring compatibility with the compactified genus-zero Riemann sphere.

Independence of hyperplanes is crucial for preserving the minimality and uniqueness of the divisor structure. It prevents degeneracy in the intersection process and aligns with the bijective correspondence between zeros and poles.

5 Topological Remarks

Although the geometric form of Riemann-Roch does not explicitly involve topology, the divisor framework aligns naturally with the topological properties of the Riemann sphere. The finite degree of D ensures compatibility with the compactified genus-zero surface. The intersection of hyperplanes provides an intuitive visualization of the zeropole framework, complementing the algebraic and analytical perspectives.

6 Conclusion

Recasting the zeropole framework in terms of hyperplane intersections and the canonical divisor enhances the geometric understanding of the proof. Independence of hyperplanes ensures the integrity of the divisor structure, reinforcing the minimality and uniqueness of $\zeta^*(s)$ while maintaining consistency with the zeropole balance framework.

While the geometric Riemann-Roch perspective offers a complementary framework, particularly useful for higher-dimensional generalizations or proofs, its visual intuition may not rival the elegance and simplicity of the default Riemann sphere depiction with orthogonal great circles. Each approach brings unique strengths: the orthogonal circles provide a direct and visually compelling representation of zeropole perpendicularity, while the hyperplane framework lends itself to deeper algebraic and geometric generalizations.

Further exploration of these interpretations may yield additional insights into the geometric structure of $\zeta(s)$ and its zeros, bridging analytical, geometric, and topological perspectives of the Riemann Hypothesis.

7 Speculative Remark: Degree -1 and the Empty Set

In the geometric form of Riemann-Roch, the empty set is assigned a degree of -1, a convention that ensures the consistency and completeness of divisor computations. This convention reflects the minimality inherent in the construction of divisors and their associated linear systems.

Similarly, the degree computation for the shadow function $\zeta^*(s)$ yields:

$$\deg(D) = +\aleph_0 \,(\text{non-trivial zeros}) - \aleph_0 \,(\text{trivial poles}) - 1 \,(\text{simple pole at } s = 0) = -1.$$

This parallel highlights the intrinsic minimality encoded in the zeropole framework. The introduction of the simple pole at $s = 0$ mirrors the role of the empty set's degree in geometric Riemann-Roch, reinforcing the conceptual alignment between the two perspectives.

By leveraging this correspondence, the zeropole framework connects the analytical properties of $\zeta(s)$ with geometric insights from divisor theory, providing a unified lens through which the Riemann Hypothesis can be explored.

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