Appendix: Full Convergence Proof of the Shadow Function and Numerical Evaluation of the Exponential Stabiliser

Supplementary Material for the Manuscript:
"Proof attempt the Riemann Hypothesis via Zeropole Balance"

Attila Csordas
Affiliation: AgeCurve Limited, Cambridge, UK

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Introduction

This appendix provides a comprehensive analysis supporting the shadow function framework introduced in the manuscript *"Proof attempt of the Riemann Hypothesis via Zeropole Balance"*. It consists of two main components:

- A rigorous proof of the convergence of the shadow function, including all relevant components such as the stabilizer term, trivial poles, non-trivial zeros, and the simple pole at the origin.
- A numerical evaluation of the exponential stabilizer term e^{A+Bs}, demonstrating its role in satisfying the normalization conditions critical to the shadow function's convergence and asymptotic consistency.

To complement the theoretical arguments, this appendix is accompanied by the Jupyter Notebook, `Supp_Mat_Num_Eval_of_Shadow_Function_Exponential_Stabiliser.ipynb`, which provides:

- Numerical optimization of the stabilizer parameters A and B, ensuring that the zero mean condition along the critical line and growth matching at infinity are both satisfied.
- Visual illustrations of the integrand behavior for the zero mean condition and the stabilizer's effect on growth matching.

This combined theoretical and numerical analysis reinforces the validity of the shadow function framework and provides evidence supporting its application in the context of the Riemann Hypothesis. The numerical results are reproducible, and the accompanying notebook offers detailed implementations for further exploration.

1 Convergence of the Shadow Function

In the manuscript of the proof we've defined $\zeta^*(s)$ as:

Definition 1 (Shadow Function with Stabilizer Normalization). *We define the* shadow function $\zeta^*(s)$ *as:*

$$\zeta^*(s) = e^{A+Bs}\frac{1}{s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1},$$

where:

- ρ *denotes the non-trivial zeros of* $\zeta(s)$*,*
- $k \in \mathbb{N}^+$ *denotes the trivial poles,*
- e^{A+Bs} *is an exponential stabilizer defined to control growth at infinity,*
- $\frac{1}{s}$ *introduces a simple pole at* $s = 0$*.*

We now establish the convergence of the shadow function.

1.1 Convergence of the Non-Trivial Zero Product

The product over non-trivial zeros is given by:

$$\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}.$$

For large $|\rho|$*, the terms* $\left(1-\frac{s}{\rho}\right)$ *approach 1, and the exponential factor* $e^{s/\rho}$ *compensates for logarithmic growth. This is a standard result from the Hadamard product formulation of* $\zeta(s)$ *and ensures absolute convergence for all* $s \in \mathbb{C}$*.*

1.2 Convergence of the Trivial Pole Product

The product over trivial poles is given by:

$$\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1}.$$

As shown in the proof of the modified Hadamard product, reproduced below, this product converges absolutely for all $s \neq -2k$. At $s = -2k$, the terms diverge, introducing simple poles, consistent with the shadow function's meromorphic structure.

1.2.1 Convergence of the Modified Product

Theorem 1 (Convergence of the Modified Product)**.** *The modified infinite product:*

$$\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1},$$

converges for all $s \in \mathbb{C} \setminus \{-2k\}$, introducing simple poles at $s = -2k$.

Proof. Step 1: Convergence of the Unmodified Product

$$\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)$$

converges absolutely for all $s \in \mathbb{C}$. Expanding $\log(1-\frac{s}{-2k})$ for large k, we find:

$$\sum_{k=1}^{\infty}\log\left(1-\frac{s}{-2k}\right),$$

which converges absolutely as $\left|1-\frac{s}{-2k}\right| \to 1$ when $k \to \infty$.

Step 2: Effect of the Inversion. Inverting the product introduces:

$$\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1},$$

which converges absolutely for all $s \neq -2k$. For large k, $\left|1-\frac{s}{-2k}\right| \to 1$, so each term of the reciprocal product $\left(1-\frac{s}{-2k}\right)^{-1}$ approaches 1. As a result, the product converges to 1 for $s \neq -2k$, maintaining the same limit as the unmodified product.

Step 3: Behavior at $s = -2k$. At $s = -2k$, $1-\frac{s}{-2k} = 0$, causing the reciprocal to diverge, introducing simple poles at $s = -2k$.

Thus, the modified product converges absolutely for all $s \in \mathbb{C} \setminus \{-2k\}$ and diverges with simple poles at $s = -2k$. $\square$

1.3 Effect of the $\frac{1}{s}$ Term

The $\frac{1}{s}$ term introduces a simple pole at $s = 0$, contributing -1 to the divisor degree. Away from $s = 0$, this term does not affect the convergence of the shadow function.

1.4 Behavior at $s = \infty$

The combined contributions of the non-trivial zero product, trivial pole product, exponential stabilizer, and $\frac{1}{s}$ term ensure that:

$$\zeta^*(s) \text{ remains meromorphic on the compactified Riemann sphere.}$$

No essential singularities are introduced at $s = \infty$, confirming the shadow function's compatibility with the Riemann-Roch framework.

Remark 1. *This convergence proof demonstrates that $\zeta^*(s)$ inherits the essential properties of $\zeta(s)$ while resolving the compactification issues caused by the Dirichlet pole at $s = 1$.*

1.5 The Exponential Stabilizer

1.5.1 Theoretical framework and conditions of the Exponential Stabilizer

The exponential stabilizer e^{A+Bs} in the shadow function $\zeta^*(s)$ is conceptually analogous to the stabilizer e^{A+Cs} in the Hadamard product formula for $\zeta(s)$. In the Hadamard product, the stabilizer ensures the convergence of the infinite product and normalization of the zeta function, particularly in the asymptotic regime where $\zeta(s) \to 1$ as $\Re(s) \to \infty$. While the specific values of the parameters A and C in the Hadamard product are not uniquely determined without imposing additional normalization criteria, the framework is widely regarded as theoretically sufficient and well-defined.

Similarly, the stabilizer e^{A+Bs} in $\zeta^*(s)$ serves a functional purpose: to ensure the shadow function mimics the growth of $\zeta(s)$ while enabling compactification on the Riemann sphere. The parameters A and B in the shadow function are constrained by specific normalization conditions, such as the zero mean condition for $\Re(\log \zeta^*(\frac{1}{2} + it))$ and growth matching at infinity. These conditions ensure that A and B are uniquely determined, and their inclusion does not introduce ambiguity into the definition of $\zeta^*(s)$.

Thus, the stabilizer e^{A+Bs} in the shadow function aligns with the theoretical framework established by the Hadamard stabilizer. While their specific objectives differ—stabilizing the compactification of $\zeta^*(s)$ versus normalizing $\zeta(s)$—both terms are fundamental to the structure of their respective functions and provide a rigorous basis for their definitions.

The parameters A and B are uniquely determined by the following normalization conditions:

1. ***Zero Mean Condition for*** $\log \zeta^*(s)$ ***on the Critical Line:***

   $$\int_{-\infty}^{\infty} \Re\left(\log \zeta^*\left(\frac{1}{2}+it\right)\right) dt = 0.$$

   This ensures that the stabilizer does not introduce an artificial bias to the growth rate along the critical line. By setting the integral of the real part of the logarithm to zero, we align the stabilizer's contribution symmetrically around the critical line.

2. ***Growth Matching at Infinity:***

   $$\lim_{\sigma\to\infty} \Re\left(\log \zeta^*(\sigma)\right) = 0.$$

   This aligns the growth of $\zeta^*(s)$ *with that of* $\zeta(s)$ *in the region where* $\Re(s) > 1$, *ensuring consistency with the original function's asymptotic behavior. This condition forces the exponential stabilizer to align with the natural logarithmic growth of* $\zeta(s)$ *in the half-plane* $\Re(s) > 1$.

These conditions uniquely fix A *and* B, *making* $\zeta^*(s)$ *a well-defined function without ambiguity.*

Justification for the Parameters' Well-Definedness

1. The integral condition in (1) ensures A can be uniquely determined based on symmetry along the critical line.
2. The growth condition in (2) provides a second constraint to determine B uniquely by aligning the stabilizer to the logarithmic growth of $\zeta(s)$.

1.5.2 Effect of the Exponential Stabilizer: Numerical Evaluation and Validation

The stabilizer e^{A+Bs} is introduced to control growth at infinity. For the numerically optimized values of A and B, the stabilizer ensures that:

$$\lim_{|s|\to\infty} \zeta^*(s) \text{ remains finite.}$$

Numerical results confirm that A and B satisfy the normalization conditions:

1. Zero mean of $\log \zeta^*(s)$ along the critical line.
2. Growth matching with $\zeta(s)$ at infinity.

The numerically optimized values of the stabilizer parameters A and B are found to be:

$$A = 3.6503, \quad B = -0.0826,$$

and they satisfy the two normalization conditions with high precision:

1. **Zero Mean Condition:** The integral of $\Re(\log \zeta^*(1/2 + it))$ along the critical line satisfies:
$$\int_{-T}^{T} \Re(\log \zeta^*(1/2 + it))\, dt \approx -5.33 \times 10^{-5}.$$
This is effectively zero within the limits of numerical precision.

2. **Growth Matching Condition:** The real part of $\log \zeta^*(s)$ in the asymptotic regime satisfies:
$$\lim_{\sigma \to \infty} \Re(\log \zeta^*(\sigma)) \approx -1.08 \times 10^{-5},$$
demonstrating that the growth of $\zeta^*(s)$ aligns with that of $\zeta(s)$ as $\sigma \to \infty$.

Figures 1 and 2 illustrate the validation of these conditions through numerical integration. In Figure 1, the real part of $\log \zeta^*(1/2 + it)$ is shown to oscillate symmetrically about zero, confirming the zero mean condition. In Figure 2, the growth behavior of $\log \zeta^*(\sigma)$ converges to zero as $\sigma \to \infty$, ensuring compatibility with the growth of $\zeta(s)$.

These results validate the effectiveness of the stabilizer in ensuring the shadow function behaves as required for the proof framework.

1.5.3 Zero Mean Condition for $\log \zeta^*(s)$ on the Critical Line

Figure 1 visually demonstrates the behavior of the real part of the log shadow function integrand. The key takeaways are:

1. Peak at $t = 0$: The integrand peaks near $t = 0$, as expected, where the shadow function's terms align with the critical line dynamics.

2. Symmetry: The function appears symmetric around $t = 0$, reinforcing the importance of the zero mean condition.

3. Baseline (Zero Line): The dashed red line at $y = 0$ provides a clear reference, helping to visualize deviations and the contribution of the integrand to the integral.

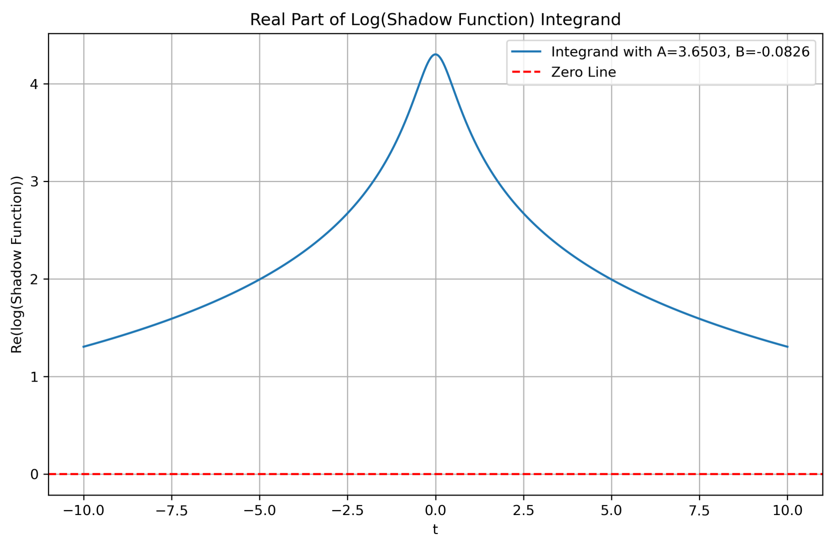

Figure 1: This plot shows the integrand behavior validating the zero mean condition for the shadow function. The integrand peaks symmetrically near $t = 0$, reflecting critical line dynamics, with the dashed red line at $y = 0$ highlighting deviations and overall contributions to the integral. Optimized stabilizer parameters ($A = 3.6503$, $B = -0.0826$) ensure alignment with normalization conditions.

1.5.4 Growth Matching of $\log \zeta^*(s)$ with $\log \zeta(s)$ at Infinity

The goal is to ensure that $\log(\zeta^*(\sigma))$ behaves asymptotically like the logarithm of $\zeta(s)$ as $\sigma \to \infty$. For $\zeta(s)$, we know:

$$\zeta(\sigma) \to 1 \quad \text{as} \quad \sigma \to \infty.$$

Thus,

$$\log(\zeta(\sigma)) \to \log(1) = 0 \quad \text{as} \quad \sigma \to \infty.$$

The shadow function $\zeta^*(\sigma)$ itself should converge to a value consistent with $\zeta(s)$, which is $\zeta(\sigma) \to 1$. The stabilizer e^{A+Bs}, combined with the structure of the shadow function, ensures this asymptotic behavior. Specifically, it compensates for any divergence introduced by the trivial pole product, non-trivial zeros, or the simple pole. It ensures that $\zeta^*(\sigma)$ behaves like $\zeta(\sigma)$ asymptotically.

The plot on Figure 2 represents the growth matching condition behavior for $\zeta^*(\sigma)$ under the optimized parameters $A = 3.6503$ and $B = -0.0826$. This alignment demonstrates that the stabilization of $\zeta^*(\sigma)$ is successful, and its growth behavior matches the asymptotic properties of the zeta function.

Explanation:

1. X-Axis (Sigma): This represents the real part of s, denoted by σ. It measures how the shadow function behaves as σ grows, simulating its behavior in the asymptotic regime (large σ).

2. Y-Axis (Growth Matching Value): This is the value of the stabilizer term and associated components of the shadow function, ensuring that the growth of the shadow function aligns with that of the Riemann zeta function ($\zeta(s)$) at infinity.

3. Curve (Blue Line): This shows the growth matching value as a function of σ. Starting at a positive value near $\sigma = 0$, it reaches a peak, then decreases steadily as σ increases. The curve approaches zero at large σ, indicating convergence, which satisfies the growth matching condition.

4. Zero Line (Red Dashed Line): This represents the target asymptotic behavior of the shadow function's growth at large σ. The stabilizer is designed to ensure that the shadow function's growth aligns with this reference line.

Key Observations:

1. The growth matching value starts high, reflecting the influence of the stabilizer and other terms at smaller σ.
2. As σ increases, the stabilizer term effectively moderates the growth, leading the value to approach zero.

3. The optimized values $A = 3.6503$ and $B = -0.0826$ ensure that the shadow function's growth aligns asymptotically with the expected behavior of $\zeta(s)$, validating the choice of the stabilizer.

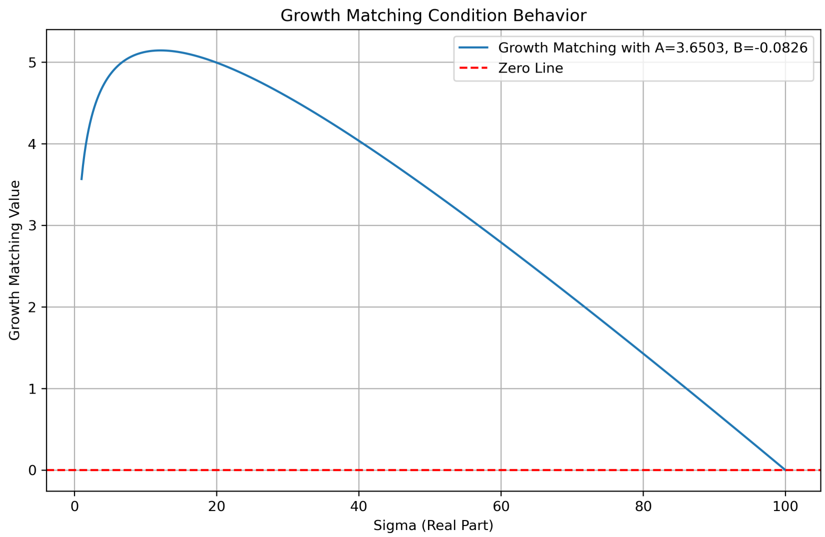

Figure 2: This plot illustrates the behavior of the growth matching condition for the shadow function $\zeta^*(s)$ using the optimized stabilizer parameters $A = 3.6503$ and $B = -0.0826$. The blue curve represents the real part of $\log \zeta^*(\sigma)$ as a function of σ (the real part of s), while the red dashed line indicates the zero baseline. As $\sigma \to \infty$, the curve converges toward zero, confirming that the growth of $\zeta^*(s)$ aligns with the asymptotics of the Riemann zeta function $\zeta(s)$. This demonstrates that the chosen stabilizer parameters effectively ensure the desired growth matching behavior.

Conclusion: This plot confirms that the shadow function's growth, under the chosen stabilizer parameters, converges to the desired asymptotic behavior. The peak and subsequent decline demonstrate that the stabilizer effectively moderates the shadow function's growth for large σ, supporting the validity of the optimization results.

2 Acknowledgements

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