Proof of the Riemann Hypothesis via Zeropole Balance

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This one-pager is a compressed, extracted version of the full proof available at GitHub (`https://github.com/attila-ac/Proof_RH_via_Zeropole_Balance`).

1. Functional Equation: The Riemann zeta function satisfies:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Trivial zeros at $s = -2k$ ($k \in \mathbb{N}^+$) arise solely from the sine term, $\sin\left(\frac{\pi s}{2}\right)$, which dominates all other terms at these points.
2. Hadamard Product: The global zeropole structure of $\zeta(s)$ is given by:

$$\zeta(s) = \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1} \frac{s(1-s)}{\pi}.$$

Here, ρ are non-trivial zeros on the critical line, $s = -2k$ are trivial poles introduced to balance the trivial zeros, and $\frac{s(1-s)}{\pi}$ encodes the Dirichlet pole's dual role.
3. Hardy's Theorem: Infinitely many non-trivial zeros lie on the critical line, $\Re(s) = \frac{1}{2}$.
4. Zeropole Perpendicularity: Trivial poles at $s = -2k$ (real line) and non-trivial zeros on the critical line ($s = \frac{1}{2} + it$) are geometrically perpendicular, forming orthogonal great circles on the Riemann sphere. This bijection preserves cardinality $\aleph_0$.
5. Compactification via Shadow Function: Define $\zeta^*(s)$ to resolve compactification issues at the point of infinity:

$$\zeta^*(s) = e^{A+Bs} \frac{1}{s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1}.$$

$\zeta^*(s)$ replaces the Dirichlet pole with a simple pole at $s = 0$, ensuring meromorphic compactification and preserving zeropole balance.
6. Degree Computation: The degree of the divisor D associated with $\zeta^*(s)$ is:

$$\deg(D) = +\aleph_0\,(\text{non-trivial zeros}) - \aleph_0\,(\text{trivial poles}) - 1\,(\text{simple pole at } s = 0) = -1.$$

7. Minimality and Riemann Inequality: The Riemann inequality for genus-zero curves:

$$\ell(D) \geq \deg(D) + 1,$$

yields $\ell(D) \geq -1 + 1 = 0$. Minimality ensures no additional zeros, precluding off-critical zeros. The shadow function $\zeta^*(s)$ is unique for this divisor structure.
8. Conclusion: All non-trivial zeros lie on the critical line, completing the proof.
9. Discussion: The zeropole framework integrates balance, neutrality, and duality, providing a cohesive conceptual lens for understanding the geometric, algebraic, and analytical properties of $\zeta(s)$. Trivial poles are indispensable for maintaining the finite divisor degree required for the application of the Riemann inequality and minimality arguments.