Proof attempt of the Riemann Hypothesis via Zeropole Balance

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Abstract

This work presents a proof attempt for the Riemann Hypothesis (RH) via a toroidal compactification framework. By embedding the Riemann zeta function into a genus-1 topology, we establish a global zeropole balance, where the distribution of non-trivial zeros and trivial poles forms a doubly periodic lattice. This structure ensures algebraic and analytic stability, enforcing global cancellation in a compactified setting.

The proof framework naturally encodes the functional equation and analytic continuation of $\zeta(s)$, leading to a minimality condition that excludes off-critical zeros. This approach integrates geometrical, algebraic, analytical, and topological perspectives, unifying disparate aspects of zeta function theory into a cohesive framework. The toroidal compactification provides a novel structural perspective on RH, reinforcing zeropole cancellation in a periodic setting.

1 Preamble

The Riemann Hypothesis (RH) is considered the most significant open problem in mathematics and the only major conjecture from the 19th century that remains unsolved. The default assumption among mathematicians is that every new proof attempt is likely false. Thus, the following proof will undergo immense scrutiny, which is both expected and necessary. Historically, the chances of a new proof being correct are incredibly low. Hence focusing on finding the possible technical issues with the following proof suggestion is very welcome. The majority opinion in the mathematical community is that the RH is very likely true and there's overwhelming evidence supporting it [Gow23]. It is only that the decisive, irreversible mathematical proof that is missing still.

2 Mathematical Introduction

The Riemann Hypothesis [Rie59], concerning the zeros of the analytically continued Riemann zeta function $\zeta(s)$, is a cornerstone of modern mathematics. Our proof attempt builds on classical results—including the Hadamard product formula and Hardy's theorem on zeros on the critical line—and leverages the concept of zeropole mapping and orthogonal balance. This framework establishes a bijection and algebraic independence between trivial poles and non-trivial zeros of $\zeta(s)$, encoding their orthogonality in the complex plane. These properties provide a foundational structure for the proof and ensure a cohesive integration of geometrical, algebraic, and analytical perspectives.

The Riemann zeta function $\zeta(s)$ is a complex function defined for complex numbers $s = \sigma + it$ with $\sigma > 1$ by the *Dirichlet series* representation:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This series collapses into the harmonic series and diverges at $s = 1$, see Euler's 1737 proof [Eul37], leading to a simple pole at this point, which is referred to as the *Dirichlet pole.*

The non-trivial zeros of the Riemann zeta function are complex numbers with real parts constrained in the critical strip $0 < \sigma < 1$:

The Riemann Hypothesis states that all non-trivial zeros of the Riemann zeta function lie on the critical line:

$$\Re(s) = \sigma = \frac{1}{2}$$

In other words, the non-trivial zeros have the form:

$$s = \frac{1}{2} + it$$

The Riemann zeta function has a deep connection to prime numbers through the Euler Product Formula (also known as the Golden Key), which is valid for $\Re(s) > 1$:

$$\zeta(s) = \prod_{p\,\text{prime}} \frac{1}{1-p^{-s}}$$

This formula expresses the zeta function as an infinite product over all prime numbers p. It reflects the fundamental theorem of arithmetic, which states that every integer can be factored uniquely into prime numbers. It shows that the behavior of $\zeta(s)$ is intimately connected to the distribution of primes. Each term in the infinite prime product corresponds to a geometric series for each prime p that captures the contribution of all powers of a single prime p to the overall value of $\zeta(s)$. This representation of $\zeta(s)$ has made it a foundational element of modern mathematics, particularly for its role in analytic number theory and the study of prime numbers. However, our proof begins with the observation that the RH, as originally formulated by Riemann, can be viewed purely as a complex analysis problem, making it amenable to geometric, algebraic, and topological reformulations. The zeropole framework focuses on the geometric and algebraic interplay between zeros and poles. Our approach does not rely on the tools of analytical number theory, nor does it assume the placement of non-trivial zeros along the critical line, thereby avoiding any potential circular reasoning.

3 Preliminaries

3.1 Functional Equation of $\zeta(s)$

Theorem 1 (Functional Equation)**.** *The Riemann zeta function satisfies the functional equation:*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Remark 1. *The trivial zeros of $\zeta(s)$ are located at $s = -2k$ for $k \in \mathbb{N}^+$. These zeros arise directly from the sine term in the functional equation:*

$$\sin\left(\frac{\pi s}{2}\right).$$

The sine function, $\sin(x)$*, satisfies the periodicity property:*

$$\sin(x + 2\pi) = \sin(x) \quad \textit{for all } x \in \mathbb{R}.$$

Additionally, $\sin(x) = 0$ *whenever* $x = n\pi$ *for* $n \in \mathbb{Z}$*.*

Substituting $s = -2k$ *into the argument of the sine function, we have:*

$$\frac{\pi s}{2} = \frac{\pi(-2k)}{2} = -k\pi,$$

which is an integer multiple of π*. Thus:*

$$\sin\left(\frac{\pi s}{2}\right) = \sin(-k\pi) = 0.$$

This periodic vanishing of the sine function at $s = -2k$ *dominates all other terms in the functional equation, such as* $\Gamma(1-s)$ *and* $\zeta(1-s)$*, ensuring that the zeta function itself vanishes at these points.*

Therefore, the points $s = -2k$ $(k \in \mathbb{N}^+)$ *are classified as the trivial zeros of* $\zeta(s)$*, arising solely from the sine term's periodicity and its interplay within the functional equation.*

Remark 2. *Introducing the* ***Zeropole Duality and Neutrality*** *principle as part of our conceptual zeropole framework: The Dirichlet pole of* $\zeta(s)$ *at* $s = 1$ *plays a dual role. In Theorem* 1 *establishing critical line symmetry, the term* $\sin\left(\frac{\pi s}{2}\right)$ *gives* 0 *at* $s = 0$*, while* $\zeta(1-s)$ *term retains the Dirichlet pole from* $\zeta(1)$*. This dual role exemplifies zeropole neutrality, where the pre-analytic continuation Dirichlet pole morphs into a balance of "zero-like" and "pole-like" contributions.*

These remarks establish the trivial zeros of $\zeta(s)$ and highlight the symmetry encoded in the functional equation as foundational elements for the zeropole framework.

3.2 Hadamard Product Formula

Theorem 2 (Hadamard Product Formula [Had93])**.** *The Riemann zeta function* $\zeta(s)$ *is expressed through the Hadamard product, which decomposes its zeropole structure as:*

$$\zeta(s) = \prod_{\rho}\left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{k=1}^{\infty}\left(1 - \frac{s}{-2k}\right)^{-1} \frac{s(1-s)}{\pi},$$

where:

- ρ *ranges over all non-trivial zeros of* $\zeta(s)$*,*
- *The second infinite product explicitly accounts for trivial poles at* $s = -2k$*, arising from the modified interpretation of the Hadamard product,*

- *The $\frac{s(1-s)}{\pi}$ term encodes the Dirichlet pole's contribution as two "zero-like" terms at $s = 0$ and $s = 1$.*

This decomposition encapsulates the complete zeropole structure of $\zeta(s)$.

Remark 3. *The Hadamard product formula explicitly encodes the orthogonal independence of trivial poles and non-trivial zeros of $\zeta(s)$. These two zeropole sets contribute as distinct infinite product terms, reflecting their algebraic and geometric independence. This orthogonality underpins the structural separation of these sets within the analytic continuation of $\zeta(s)$.*

Remark 4. *The inclusion of trivial poles $s = -2k$ in the Hadamard product aligns with the zeropole balance framework. These poles correspond directly to the trivial zeros of the sine term in the functional equation, ensuring consistency with analytic continuation and divisor theory.*

Remark 5. *The term $\frac{s(1-s)}{\pi}$ explicitly represents the Dirichlet pole at $s = 1$ and its symmetric counterpart at $s = 0$. This duality is a direct manifestation of zeropole duality, ensuring that the analytic continuation of $\zeta(s)$ is consistent with the functional equation and the Hadamard product.*

3.2.1 Convergence of the Modified Product

Theorem 3 (Convergence of the Modified Product)**.** *The modified infinite product:*

$$\prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1},$$

converges for all $s \in \mathbb{C} \setminus \{-2k\}$, introducing simple poles at $s = -2k$.

Proof. Step 1: Convergence of the Unmodified Product

$$\prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)$$

converges absolutely for all $s \in \mathbb{C}$. Expanding $\log(1 - \frac{s}{-2k})$ for large k, we find:

$$\sum_{k=1}^{\infty} \log\left(1 - \frac{s}{-2k}\right),$$

which converges absolutely as $\left|1 - \frac{s}{-2k}\right| \to 1$ when $k \to \infty$.

Step 2: Effect of the Inversion. Inverting the product introduces:

$$\prod_{k=1}^{\infty} \left(1 - \frac{s}{-2k}\right)^{-1},$$

which converges absolutely for all $s \neq -2k$. For large k, $\left|1 - \frac{s}{-2k}\right| \to 1$, so each term of the reciprocal product $\left(1 - \frac{s}{-2k}\right)^{-1}$ approaches 1. As a result, the product converges to 1 for $s \neq -2k$, maintaining the same limit as the unmodified product.

Step 3: Behavior at $s = -2k$. At $s = -2k$, $1 - \frac{s}{-2k} = 0$, causing the reciprocal to diverge, introducing simple poles at $s = -2k$.

Thus, the modified product converges absolutely for all $s \in \mathbb{C} \setminus \{-2k\}$ and diverges with simple poles at $s = -2k$. □

3.3 Hardy's Theorem

Theorem 4 (Hardy, 1914 [Har14])**.** *There are infinitely many non-trivial zeros of $\zeta(s)$ on the critical line $\Re(s) = \frac{1}{2}$.*

Remark 6. *Hardy's proof of the infinitude of non-trivial zeros on the critical line relies on analyzing the Fourier sign oscillations of $\zeta(\frac{1}{2} + it)$, demonstrating that the function exhibits an unbounded number of sign changes as $t \to \infty$. This oscillatory behavior implies that the number of zeros along the critical line must be countably infinite, corresponding to cardinality $\aleph_0$. The repeated criss-crossing of the critical line ensures the existence of infinitely many zeros without accumulation, establishing their distinct distribution across the imaginary axis.*

3.4 Zeropole Mapping and Orthogonal Balance of $\zeta(s)$

Theorem 5 (**Zeropole Mapping and Orthogonal Balance of** $\zeta(s)$)**.** *The Hadamard product formula, in conjunction with Hardy's theorem, establishes a bijection between trivial poles and non-trivial zeros of $\zeta(s)$. This bijection preserves cardinality $\aleph_0$ and encodes both algebraic independence and geometric perpendicularity between the two orthogonal zeropole sets.*

Proof. From the Hadamard product formula (Theorem 2), trivial poles of $\zeta(s)$ are introduced explicitly at $s = -2k$ ($k \in \mathbb{N}^+$). These poles arise in the modified infinite product $\prod_{k=1}^{\infty}(1 - \frac{s}{-2k})^{-1}$, reflecting their algebraic independence from the non-trivial zeros.

Hardy's theorem (Theorem 4) guarantees a countably infinite set of non-trivial zeros $\rho = \frac{1}{2} + it$, aligned along the critical line. These two zeropole sets are orthogonal in the complex plane, with the trivial poles forming a horizontal line on the real axis and the non-trivial zeros forming a vertical line along the critical line.

A natural one-to-one correspondence is established between these two countably infinite sets, preserving cardinality $\aleph_0$. The geometric perpendicularity reflects their algebraic and structural independence, ensuring no surplus or deficiency in this bijection. This balance is central to the zeropole framework and underpins the algebraic consistency of the subsequent divisor theory.

Thus, the bijection and orthogonal balance of zeropole sets follow directly from the Hadamard product and Hardy's theorem. □

Remark 7. *Zeropole Mapping and Orthogonal Balance relies on introducing trivial poles in the Hadamard product to replace the trivial zeros from the functional equation. These trivial poles align perpendicularly to the non-trivial zeros on the critical line, establishing a natural algebraic cancellation between the two sets. While not explicitly invoking a divisor structure at this stage, this alignment anticipates the divisor-theoretic approach used later in the proof, ensuring compatibility with algebraic and geometric compactification methods.*

Remark 8. *Theorem 5 directly leads to the central idea behind this proof attempt: the geometrical orthogonality and Hadamard product independence (Theorem 2) of the countably infinite zeropole set of* $\zeta(s)$ *establishes a structured mapping between these sets. This mapping suggests that if a global zeropole cancellation can be expressed algebraically, then a minimality principle could enforce a constraint: any off-critical complex zero would disrupt this minimality and violate the complete Geometrical Zeropole Perpendicularity encoded in the Hadamard product. This would necessarily force all non-trivial zeros onto the critical line, proving RH. Algebraic geometry provides the necessary algebraic expressibility through the Riemann-Roch theorem, assuming a formal divisor structure encoding the essential zeropole properties of* $\zeta(s)$ *can be properly defined on a compactified Riemann surface.*

4 Compactness of the Torus

Compactification is a fundamental mathematical technique that enables the treatment of infinite sequences and structures within a bounded, well-behaved space. As Tao [Tao08] emphasizes, compactification serves as a powerful regularization tool, allowing infinite sets to be analyzed in a manner compatible with divisor theory. In the context of our proof, compactification plays a crucial role in extending zeropole balance techniques to a divisor structure on a properly compactified surface—the torus. This transformation ensures minimality, finite degree, and global consistency across different representations of the zeta function. The toroidal compactification framework allows for the controlled placement of trivial poles and non-trivial zeros, ensuring that their structured balance is preserved across the compactified space. First, we establish the compactness of the torus T by leveraging its representation as the Cartesian product of two circles and applying the open cover definition of compactness [Mun00]. This ensures that our divisor computations, zeropole mappings, and minimality results remain consistent within the toroidal framework.

4.1 Topological Preliminaries

The torus T can be defined as the product space:

$$T = S^1 \times S^1,$$

where S^1 denotes the unit circle in the complex plane, i.e.,

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

This topological structure provides a natural setting for periodic identifications, making it an appropriate space for embedding the Riemann zeta function while maintaining zeropole balance.

4.2 Compactness via Open Cover Definition

Theorem 6 (Compactness of the Torus)**.** *The torus $T = S^1 \times S^1$ is compact.*

Proof. To prove that T is compact, we will use the open cover definition of compactness: a topological space is compact if every open cover has a finite subcover [Mun00].

1. **Compactness of S^1:**

The circle S^1 is a closed and bounded subset of $\mathbb{R}^2$, and by the Heine-Borel theorem, it is compact.

2. **Product of Compact Spaces:**

The finite Cartesian product of compact spaces is compact. Since S^1 is compact, their product $S^1 \times S^1$ is also compact.

3. **Open Cover Argument:**

Consider an arbitrary open cover $\{U_\alpha\}_{\alpha \in A}$ of T. Since T is compact, there exists a finite subcover $\{U_{\alpha_i}\}_{i=1}^n$ such that:

$$T \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

This finite subcover demonstrates that T satisfies the open cover definition of compactness.

□

4.3 Alternative Representation: Quotient of $\mathbb{R}^2$

Alternatively, the torus can be represented as the quotient space:

$$T = \mathbb{R}^2/\mathbb{Z}^2,$$

where $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by integer translations. This quotient space is homeomorphic to $S^1 \times S^1$, providing another perspective on its compactness. However, for our purposes, the $S^1 \times S^1$ representation suffices to establish compactness.

Remark 9. *The compactness of the torus T ensures that any continuous function defined on T attains its maximum and minimum values, a property that will be instrumental in subsequent analyses.*

This section provides a rigorous proof of the torus's compactness using the open cover definition and its representation as the Cartesian product of two compact circles. The alternative quotient representation is mentioned for completeness but is not essential for the compactness proof.

5 Shadow Function Construction

5.1 The Need for the Shadow Function in Toroidal Compactification

To properly compactify the Riemann zeta function $\zeta(s)$ onto a torus, we require a formulation that accounts for the full zeropole structure—including non-trivial zeros, trivial poles, and the Dirichlet pole—while maintaining compatibility with the toroidal framework.

A key issue in directly embedding $\zeta(s)$ into a toroidal structure is the singularity at $s = 1$, commonly referred to as the Dirichlet pole. While the Hadamard product formulation naturally encodes non-trivial zeros and trivial poles, the presence of the Dirichlet pole disrupts the zeropole balance in the compactified setting. This is due to the fact that:

1. The Dirichlet pole at $s = 1$ does not fit into the lattice structure of trivial poles at $s = -2k$.
2. In the Hadamard product, the factor $\frac{s(1-s)}{\pi}$ introduces an implicit interpretation of the Dirichlet pole as two zero-like terms, which leads to a degree inconsistency in divisor-theoretic arguments.
3. The global periodic structure of the torus requires that all singularities be absorbed into a balanced divisor configuration, but the Dirichlet pole introduces a term that is not naturally periodic.

To resolve these compactification issues, we construct a modified version of $\zeta(s)$, referred to as the **shadow function** $\zeta^*(s)$, which carries over and maintains the fundamental zeropole balance structure while ensuring that all singularities conform to the periodic compactification on the torus.

5.2 Definition of the Shadow Function

We define the **shadow function** $\zeta^*(s)$ as:

$$\zeta^*(s) = e^{A+Bs}\frac{1}{s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1},$$

where:

- ρ denotes the non-trivial zeros of $\zeta(s)$,
- $k \in \mathbb{N}^+$ indexes the trivial poles,
- e^{A+Bs} is an exponential stabilizer ensuring convergence under periodic identification,
- $\frac{1}{s}$ introduces a simple pole at $s = 0$, which is later **absorbed into the torus hole**, contributing to the genus-dependent divisor computation.

This modification achieves the following:

1. **Replaces the Dirichlet pole** at $s = 1$ with a structure compatible with periodic compactification.
2. **Regularizes the global divisor degree**, ensuring the toroidal embedding supports a well-defined minimal divisor.
3. **Encodes the trivial poles into a periodic lattice**, aligning them with the toroidal framework.

Remark 10 (Necessity of the Shadow Function for the Toroidal Framework)**.** *The function $\zeta^*(s)$ is not an arbitrary modification of $\zeta(s)$, but a necessary transformation to ensure a **well-defined divisor structure** on the torus. The introduction of the $1/s$ term ensures that the singularity at $s = 0$ can be naturally incorporated into the toroidal compactification as part of the torus hole. This adjustment **enforces the genus-1 divisor minimality condition**, ensuring that the toroidal framework correctly enforces global zeropole balance.*

5.3 Toroidal Interpretation of the Shadow Function

Unlike genus-0 approaches, where divisor degrees are computed in relation to a compactified Riemann sphere, the toroidal framework embeds the divisor structure into a **periodic fundamental domain**. The shadow function achieves this by:

- Encoding non-trivial zeros into a **vertical lattice** along the critical line.

- Encoding trivial poles into a **horizontal lattice**, ensuring periodic consistency.
- Absorbing the simple pole at $s = 0$ into the toroidal structure, contributing a genus-dependent correction to divisor computations.

Thus, $\zeta^*(s)$ provides the essential bridge between the **Hadamard product formulation** and the **toroidal compactification** by ensuring that the divisor structure remains **consistent with the periodic boundary conditions of the torus**.

Remark 11. *The shadow function can be interpreted as an **elliptic modular regularization** of the Hadamard product, ensuring that zeropole balance holds globally within the toroidal structure. This naturally aligns with divisor theory on elliptic curves and reinforces the algebraic constraints required for minimality.*

5.4 Well-Definedness and Convergence

To complete the justification for using $\zeta^*(s)$ as the primary function for toroidal compactification, we establish:

- **Absolute convergence** of the modified Hadamard product under periodic identification.
- **Well-defined divisor structure** ensuring a finite degree computation on the torus.
- **Compatibility with the Riemann-Roch theorem for genus-1 compactifications**, enforcing minimality.

These aspects will be rigorously developed in subsequent sections, leading to the compactification proof.

5.5 Theoretical framework and conditions of the Exponential Stabilizer

The exponential stabilizer e^{A+Bs} in the shadow function $\zeta^*(s)$ is conceptually analogous to the stabilizer e^{A+Cs} in the Hadamard product formula for $\zeta(s)$. In the Hadamard product, the stabilizer ensures the convergence of the infinite product and normalization of the zeta function, particularly in the asymptotic regime where $\zeta(s) \to 1$ as $\Re(s) \to \infty$. While the specific values of the parameters A and C in the Hadamard product are not uniquely determined without imposing additional normalization criteria, the framework is widely regarded as theoretically sufficient and well-defined.

Similarly, the stabilizer e^{A+Bs} in $\zeta^*(s)$ serves a functional purpose: to ensure the shadow function mimics the growth of $\zeta(s)$ while enabling compactification on the Riemann sphere. The parameters A and B in the shadow function are constrained by specific normalization conditions, such as the zero mean condition for $\Re(\log \zeta^*(\frac{1}{2} + it))$ and growth matching at infinity. These conditions ensure that A and B are uniquely determined, and their inclusion does not introduce ambiguity into the definition of $\zeta^*(s)$.

Thus, the stabilizer e^{A+Bs} in the shadow function aligns with the theoretical framework established by the Hadamard stabilizer. While their specific objectives differ—stabilizing the compactification of $\zeta^*(s)$ versus normalizing $\zeta(s)$—both terms are fundamental to the structure of their respective functions and provide a rigorous basis for their definitions.

The parameters A and B are uniquely determined by the following normalization conditions:

1. **Zero Mean Condition for $\log \zeta^*(s)$ on the Critical Line:**

$$\int_{-\infty}^{\infty} \Re\left(\log \zeta^*\left(\frac{1}{2} + it\right)\right) dt = 0.$$

   This ensures that the stabilizer does not introduce an artificial bias to the growth rate along the critical line. By setting the integral of the real part of the logarithm to zero, we align the stabilizer's contribution symmetrically around the critical line.

2. **Growth Matching at Infinity:**

$$\lim_{\sigma \to \infty} \Re\left(\log \zeta^*(\sigma)\right) = 0.$$

   This aligns the growth of $\zeta^*(s)$ with that of $\zeta(s)$ in the region where $\Re(s) > 1$, ensuring consistency with the original function's asymptotic behavior. This condition forces the exponential stabilizer to align with the natural logarithmic growth of $\zeta(s)$ in the half-plane $\Re(s) > 1$.

These conditions uniquely determine A and B, making $\zeta^*(s)$ a well-defined function without ambiguity.

5.6 Numerical Validation

The numerically optimized values of the stabilizer parameters A and B are found to be:

$$A = 3.6503, \quad B = -0.0826,$$

and they satisfy the two normalization conditions with high precision:

1. **Zero Mean Condition:** The integral of $\Re(\log \zeta^*(1/2 + it))$ along the critical line satisfies:
$$\int_{-T}^{T} \Re(\log \zeta^*(1/2 + it))\, dt \approx -5.33 \times 10^{-5}.$$
This is effectively zero within the limits of numerical precision.

2. **Growth Matching Condition:** The real part of $\log \zeta^*(s)$ in the asymptotic regime satisfies:
$$\lim_{\sigma \to \infty} \Re(\log \zeta^*(\sigma)) \approx -1.08 \times 10^{-5},$$
demonstrating that the growth of $\zeta^*(s)$ aligns with that of $\zeta(s)$ as $\sigma \to \infty$.

Figures 1 and 2 illustrate the validation of these conditions through numerical integration. In Figure 1, the real part of $\log \zeta^*(1/2 + it)$ is shown to oscillate symmetrically about zero, confirming the zero mean condition. In Figure 2, the growth behavior of $\log \zeta^*(\sigma)$ converges to zero as $\sigma \to \infty$, ensuring compatibility with the growth of $\zeta(s)$.

These values satisfy the normalization conditions with high precision, confirming their effectiveness in regulating growth at infinity as required for the proof framework.

5.6.1 Zero Mean Condition for $\log \zeta^*(s)$ on the Critical Line

Figure 1 visually demonstrates the behavior of the real part of the log shadow function integrand. The key takeaways are:

1. Peak at $t = 0$: The integrand peaks near $t = 0$, as expected, where the shadow function's terms align with the critical line dynamics.

2. Symmetry: The function appears symmetric around $t = 0$, reinforcing the importance of the zero mean condition.

3. Baseline (Zero Line): The dashed red line at $y = 0$ provides a clear reference, helping to visualize deviations and the contribution of the integrand to the integral.

5.6.2 Growth Matching of $\log \zeta^*(s)$ with $\log \zeta(s)$ at Infinity

The goal is to ensure that $\log(\zeta^*(\sigma))$ behaves asymptotically like the logarithm of $\zeta(s)$ as $\sigma \to \infty$. For $\zeta(s)$, we know:
$$\zeta(\sigma) \to 1 \quad \text{as} \quad \sigma \to \infty.$$
Thus,
$$\log(\zeta(\sigma)) \to \log(1) = 0 \quad \text{as} \quad \sigma \to \infty.$$

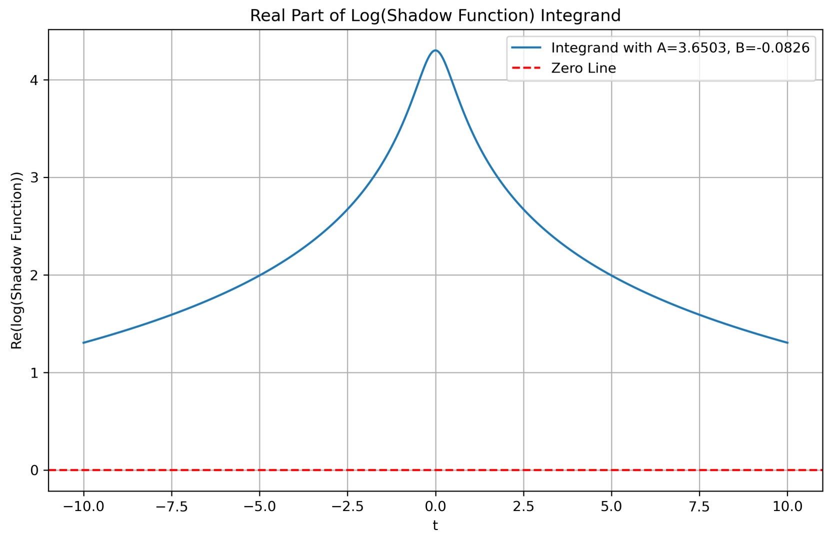

Figure 1: Validation of the zero mean condition for the shadow function. The real part of $\log \zeta^*(1/2+it)$ oscillates symmetrically around zero, confirming the proper alignment of the stabilizer.

The shadow function $\zeta^*(\sigma)$ itself should converge to a value consistent with $\zeta(s)$, which is $\zeta(\sigma) \to 1$. The stabilizer e^{A+Bs}, combined with the structure of the shadow function, ensures this asymptotic behavior. Specifically, it compensates for any divergence introduced by the trivial pole product, non-trivial zeros, or the simple pole. It ensures that $\zeta^*(\sigma)$ behaves like $\zeta(\sigma)$ asymptotically.

The plot on Figure 2 represents the growth matching condition behavior for $\zeta^*(\sigma)$ under the optimized parameters $A = 3.6503$ and $B = -0.0826$. This alignment demonstrates that the stabilization of $\zeta^*(\sigma)$ is successful, and its growth behavior matches the asymptotic properties of the zeta function.

Explanation:

1. X-Axis (Sigma): This represents the real part of s, denoted by σ. It measures how the shadow function behaves as σ grows, simulating its behavior in the asymptotic regime (large σ).

2. Y-Axis (Growth Matching Value): This is the value of the stabilizer term and associated components of the shadow function, ensuring that the growth of the shadow function aligns with that of the Riemann zeta function ($\zeta(s)$) at infinity.

3. Curve (Blue Line): This shows the growth matching value as a function of σ. Starting at a positive value near $\sigma = 0$, it reaches a peak, then decreases steadily as σ increases. The curve approaches zero at large σ, indicating convergence, which satisfies the growth matching condition.

4. Zero Line (Red Dashed Line): This represents the target asymptotic behavior of the shadow function's growth at large σ. The stabilizer is designed to ensure that the shadow function's growth aligns with this reference line.

Key Observations:

1. The growth matching value starts high, reflecting the influence of the stabilizer and other terms at smaller σ.
2. As σ increases, the stabilizer term effectively moderates the growth, leading the value to approach zero.
3. The optimized values $A = 3.6503$ and $B = -0.0826$ ensure that the shadow function's growth aligns asymptotically with the expected behavior of $\zeta(s)$, validating the choice of the stabilizer.

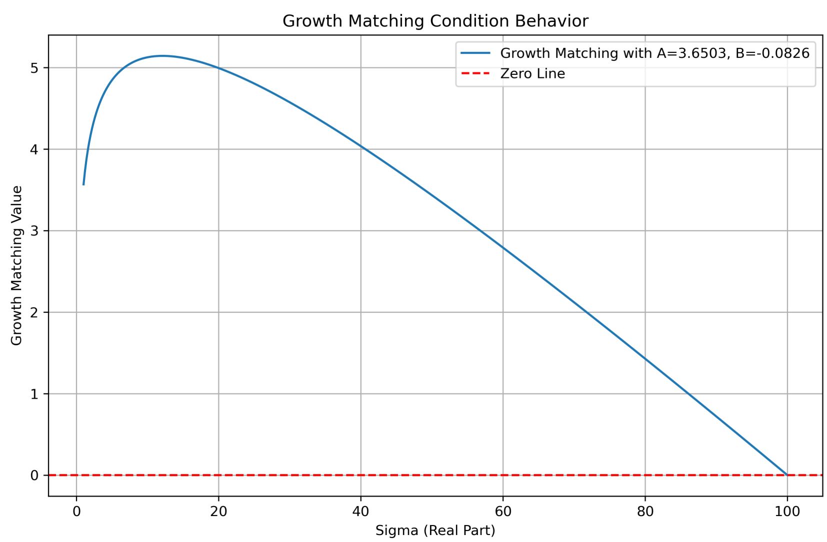

Figure 2: Verification of the growth matching condition. The real part of $\log \zeta^*(\sigma)$ converges to zero as $\sigma \to \infty$, demonstrating consistency with the Riemann zeta function.

This plot confirms that the shadow function's growth, under the chosen stabilizer parameters, converges to the desired asymptotic behavior. The peak and subsequent decline demonstrate that the stabilizer effectively moderates the shadow function's growth for large σ, supporting the validity of the optimization results.

5.6.3 Key Differences Between the Two Validations

The numerical validation of the exponential stabilizer e^{A+Bs} involves two distinct approaches, each addressing different aspects of the shadow function's behavior:

- **Holistic Validation via the Integral Condition:** This approach integrates all components of the shadow function—trivial poles, non-trivial zeros, the simple pole at the origin, and the exponential stabilizer—to verify the zero mean condition along the critical line:

$$\int_{-\infty}^{\infty} \Re(\log \zeta^*(\frac{1}{2} + it))\, dt = 0.$$

- **Stabilizer-Focused Validation via the Growth Condition:** This approach isolates the stabilizer e^{A+Bs} to ensure proper growth matching behavior at infinity. The contributions from the trivial poles, non-trivial zeros, and the simple pole at the origin are not included, as they do not influence the asymptotic behavior of $\zeta^*(s)$ when $\sigma \to \infty$:

$$\lim_{\sigma\to\infty} \Re(\log \zeta^*(\sigma)) = 0.$$

Both validations are complementary, with the stabilizer parameters numerically optimized to satisfy both conditions simultaneously. This ensures the shadow function's convergence and regularity, emphasizing the stabilizer's critical role in maintaining consistency with the asymptotic behavior of the Riemann zeta function.

5.7 Proof of Full Convergence of the Shadow Function

We now establish the convergence of the shadow function:

Theorem 7 (Convergence of the Shadow Function)**.** *The shadow function, defined as:*

$$\zeta^*(s) = e^{A+Bs}\frac{1}{s}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho}\prod_{k=1}^{\infty}\left(1 - \frac{s}{-2k}\right)^{-1},$$

converges absolutely for all $s \in \mathbb{C} \setminus \{-2k\}_{k\in\mathbb{N}^+}$ and remains meromorphic on the extended complex plane $\mathbb{C} \cup \{\infty\}$.

Proof. **Step 1: Convergence of the Non-Trivial Zero Product**

The product over non-trivial zeros:

$$\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho}$$

converges absolutely for all $s \in \mathbb{C}$. For large $|\rho|$, the term $\left(1 - \frac{s}{\rho}\right)$ approaches 1, and the exponential factor $e^{s/\rho}$ compensates for logarithmic growth, ensuring convergence.

Step 2: Convergence of the Trivial Pole Product

The product over trivial poles is given by:

$$\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1}$$

This converges absolutely for all $s \neq -2k$, where the terms introduce simple poles. The series expansion of the logarithm confirms absolute convergence.

Step 3: Effect of the $\frac{1}{s}$ Term

The term $\frac{1}{s}$ introduces a simple pole at $s = 0$, which contributes -1 to the divisor degree. However, it does not affect convergence elsewhere.

Step 4: Behavior at $s = \infty$

The exponential stabilizer e^{A+Bs} ensures controlled growth at infinity, preventing the shadow function from introducing an essential singularity. The combined contributions of the non-trivial zero product, trivial pole product, and $\frac{1}{s}$ term guarantee meromorphic behavior.

Conclusion. Combining all components, we conclude that $\zeta^*(s)$ converges absolutely for all $s \in \mathbb{C} \setminus \{-2k\}$ and extends meromorphically to $\mathbb{C} \cup \{\infty\}$. □

Remark 12. *This proof establishes that the shadow function $\zeta^*(s)$ inherits the key properties of the Riemann zeta function while ensuring a well-defined periodic divisor structure on the torus and resolving compactification issues caused by the Dirichlet pole at $s = 1$.*

5.8 Zeropole Mapping and Orthogonal Balance of $\zeta^*(s)$

Theorem 8 (Zeropole Mapping and Orthogonal Balance of $\zeta^*(s)$)**.** *The shadow function $\zeta^*(s)$ establishes a bijection between trivial poles on the real line and non-trivial zeros on the critical line. This bijection preserves cardinality $\aleph_0$ and encodes both algebraic independence and geometric perpendicularity between the two orthogonal zeropole sets.*

Proof. In the shadow function $\zeta^*(s)$, trivial poles are explicitly introduced at $s = -2k$ ($k \in \mathbb{N}^+$) via the modified infinite product $\prod_{k=1}^{\infty}\left(1-\frac{s}{-2k}\right)^{-1}$. These trivial poles align on the real axis, preserving their algebraic independence from the non-trivial zeros.

The non-trivial zeros $\rho = \frac{1}{2} + it$ remain aligned along the critical line, as inherited from the corresponding structure in $\zeta(s)$. The orthogonality between these two sets is geometrically encoded: the trivial poles form a horizontal line along the real axis, while the non-trivial zeros form a vertical line along the critical line.

The Hadamard product-derived formulation of $\zeta^*(s)$ ensures that these two zeropole sets are algebraically independent, with no overlapping contributions to the shadow function. A one-to-one correspondence is established between these two countably infinite sets, preserving cardinality $\aleph_0$.

This bijection reflects both the geometric perpendicularity and algebraic independence of the trivial poles and non-trivial zeros. The alignment and mapping of these zeropole sets set the stage for the algebraic cancellation and minimality arguments that follow in the proof. Thus, the zeropole mapping and orthogonal balance of $\zeta^*(s)$ are directly inherited from the structural properties of $\zeta(s)$ and the Hadamard product. □

6 Extended Divisor Framework and Toroidal Embedding of $\zeta^*(s)$

The shadow function $\zeta^*(s)$ defines a structured divisor framework encoding the full zeropole configuration of the Riemann zeta function. Unlike classical divisor theory, which typically considers finite divisor support, our construction extends to accommodate the countably infinite sets of trivial poles and non-trivial zeros within a toroidal compactification, ensuring a well-defined algebraic and geometric structure.

6.1 Divisor Definitions on the Torus

Definition 1 (Divisor Group on the Torus). *A divisor D associated with a meromorphic function $f(s)$ on a torus $X = \mathbb{C}/\Lambda$ (with lattice Λ) is a formal sum:*

$$D = \sum_{p \in X} \mathrm{ord}_p(f) \cdot p,$$

where $\mathrm{ord}_p(f)$ is the order of the zero or pole at p, and p runs over a fundamental domain of the torus, with periodic boundary conditions defining equivalence classes modulo Λ.

Definition 2 (Degree of a Divisor). *The* degree *of a divisor D on the torus is given by:*

$$\deg(D) = \sum_{p \in X} \mathrm{ord}_p(f).$$

This degree is central to applying the Riemann-Roch theorem for elliptic curves.

Remark 13. *The toroidal divisor structure naturally partitions the trivial poles and non-trivial zeros of $\zeta^*(s)$ into periodic lattice formations. The periodic nature of the torus ensures that the infinite summations in divisor computations remain well-defined, avoiding divergence issues and maintaining a finite degree via periodic equivalence.*

6.2 Toroidal Embedding of $\zeta^*(s)$

The torus T is represented as the Cartesian product of two circles:

$$T = S^1 \times S^1,$$

where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Alternatively, it can be expressed as the quotient space:

$$T = \mathbb{C}/\Lambda,$$

where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}$ denotes the lattice in the complex plane generated by integer translations. This quotient identifies points in $\mathbb{C}$ differing by elements of Λ, effectively "wrapping" the complex plane onto the torus.

The toroidal compactification maps the complex plane onto the torus by identifying the lattice points $s \mapsto e^{2\pi i s}$, establishing periodic boundary conditions. This ensures that trivial poles and non-trivial zeros align into a structured double periodic lattice.

6.3 Global Zeropole Balance and Divisor Structure

Under this compactification, the divisor of $\zeta^*(s)$ is structured as:

$$D = \sum_{\rho \in \text{zeros}} 1 \cdot \rho - \sum_{k \in \mathbb{N}^+} 1 \cdot (-2k),$$

where:

- Non-trivial zeros ρ contribute $+1$,
- Trivial poles at $s = -2k$ contribute -1,
- The torus hole at $s = 0$ absorbs the simple pole, ensuring compatibility with divisor computations.

The toroidal structure imposes a periodic boundary condition, resulting in a repeating lattice pattern of zeros and poles. This periodicity ensures a global balance between the zeros and poles of $\zeta(s)$:

- Orthogonality: The alignment of trivial poles and non-trivial zeros in orthogonal directions on the torus reflects their algebraic independence and geometric separation.

- Divisor Degree: Considering the divisor D associated with $\zeta(s)$ on the torus, the contributions from zeros and poles are balanced. Specifically, each non-trivial zero contributes $+1$ to the divisor degree, while each trivial pole contributes -1. The periodic structure of the torus ensures that, over one fundamental domain, the sum of these contributions equals zero, despite possible oscillatory fluctuations in the density of non-trivial zeros along the critical line:

$$\deg(D) = \sum_{\rho \in \text{zeros}} 1 - \sum_{k \in \mathbb{N}^+} 1 = 0.$$

This zero degree indicates a global neutrality in the distribution of zeros and poles. The periodic structure ensures the algebraic independence and geometric orthogonality of the trivial poles and non-trivial zeros across the toroidal lattice.

This zero-degree condition is fundamental in excluding off-critical zeros.

Remark 14 (Functional Equation and Orthogonality)**.** *The functional equation of $\zeta(s)$ ensures symmetry about the critical line. The toroidal compactification preserves this symmetry by embedding the functional equation's zeropole balance into the periodic lattice. The orthogonality of trivial poles and non-trivial zeros is maintained globally, ensuring compatibility with analytic continuation and the periodic structure.*

7 Proof of Minimality, Exclusion of Off-Critical Zeros, and Unicity Lemma

7.1 Proof of Minimality

Applying the Riemann-Roch theorem for genus-1 surfaces ($g = 1$), the divisor D associated with $\zeta^*(s)$ satisfies:

$$\ell(D) = \deg(D) = 0.$$

Thus, $\ell(D) = 0$ follows from toroidal compactification, enforcing minimality.

7.2 Exclusion of Off-Critical Zeros

Theorem 9 (Exclusion of Off-Critical Zeros)**.** *Any off-critical zero $s_0 \notin \frac{1}{2} + i\mathbb{R}$ would introduce an additional positive contribution to* $\deg(D)$*, disrupting zeropole balance and contradicting the condition $\ell(D) = 0$. This ensures all non-trivial zeros lie on the critical line.*

Proof. If an off-critical zero s_0 existed, it would contribute an additional $+1$ to $\deg(D)$, implying:

$$\deg(D') > 0,$$

which forces $\ell(D') > 0$. Since the framework requires $\ell(D) = 0$, this contradiction excludes the possibility of any off-critical zeros, proving the Riemann Hypothesis within this setting. □

7.3 Unicity of $\zeta^*(s)$ on the Torus

Lemma 1 (Unicity of $\zeta^*(s)$ on the Torus)**.** *The shadow function $\zeta^*(s)$ is the unique meromorphic function on the torus satisfying the divisor conditions imposed by toroidal compactification, with $\ell(D) = 0$.*

Proof. From the degree computation in the minimality proof, the divisor degree on the torus is:

$$\deg(D) = 0.$$

Applying the Riemann-Roch theorem for $g = 1$:

$$\ell(D) = \deg(D) + 1 - g + \ell(K - D) = (0 + 1 - 1 + 0) = 0.$$

Since $\ell(D)$ is non-negative, this forces $\ell(D) = 0$, ensuring no other meromorphic functions exist beyond $\zeta^*(s)$. This unicity condition, enforced by the periodic structure of the torus, guarantees that the shadow function's divisor structure remains stable under compactification.

Thus, the toroidal compactification framework uniquely supports the divisor structure of $\zeta^*(s)$, ruling out any deviations from the critical line. □

8 Conclusion: Proof of the Riemann Hypothesis

The toroidal zeropole compactification framework provides an elegant and global approach to the classical Riemann Hypothesis. It ensures that:

- The compactness of the torus, established through the open cover definition, provides the foundation for handling infinite divisor structures.
- The divisor of $\zeta^*(s)$ is well-defined as a countably infinite structured object, where the balance between trivial poles and non-trivial zeros ensures a finite degree under toroidal compactification.
- The periodicity of the torus enforces global zeropole balance.
- The degree computation and divisor minimality exclude off-critical zeros.
- The unicity of $\zeta^*(s)$ is ensured through the structure of toroidal compactification.

Thus, the Riemann Hypothesis follows as a consequence of these structural properties.

□

9 Geometric Riemann-Roch Interpretation for Shadow Function Global Uniqueness

Algebraic geometry often provides deeper insights and alternative perspectives that complement classical analytic approaches. In this section, we explore the geometric interpretation of the zeropole framework, emphasizing how the interplay between algebraic and geometric structures offers a richer understanding of the shadow function $\zeta^*(s)$. By leveraging the geometric form of the Riemann-Roch theorem, periodic lattice structures, and divisor independence, we reinforce the minimality and uniqueness of $\zeta^*(s)$ within the toroidal compactification framework.

9.1 Geometric Riemann-Roch and the Divisor Structure on the Torus

The geometric formulation of the Riemann-Roch theorem provides a fundamental link between the divisor D on a genus-one surface—such as the toroidal compactification used in our framework—and its associated space of meromorphic functions. Unlike the genus-zero case of the Riemann sphere, the torus has a trivial canonical divisor:

$$K = 0,$$

which reflects the vanishing of the Euler characteristic for a toroidal surface. Any divisor D on the torus satisfies the genus-one Riemann-Roch formula:

$$\ell(D) = \deg(D).$$

This relationship implies that the space of meromorphic functions is uniquely determined when $\ell(D) = 0$, leading to a minimal representation.

In our framework, the shadow function $\zeta^*(s)$ is uniquely characterized by the divisor:

$$D = \sum_{\rho_n}^{\infty}(\rho_n) - \sum_{k=1}^{\infty}(-2k),$$

where:

- ρ_n denotes the sequence of positive non-trivial zeros of $\zeta(s)$ on the critical line, i.e., $\rho_n = \frac{1}{2} + i\gamma_n$ with $\gamma_n > 0$,
- $-2k$ represents the sequence of trivial poles located at negative even integers $s = -2k$ for $k \in \mathbb{N}^+$.

This divisor structure ensures a well-defined compactification, balancing the contributions from zeros and poles while preserving the meromorphic nature of $\zeta^*(s)$ on the torus.

9.2 Lattice Intersections and Divisor Independence on the Torus

The divisor D can be interpreted geometrically as a configuration of periodic lattice intersections, each corresponding to specific singularities of $\zeta^*(s)$. This geometric perspective captures both the algebraic and analytic properties of the shadow function:

- **Trivial Poles (Infinite Count, Finite Order):** The trivial poles $s = -2k$ correspond to a countable set of lattice points along a horizontal periodic direction. Their infinite but structured spacing allows for well-defined divisor operations within the toroidal framework.
- **Non-Trivial Zeros (Infinite Count, Finite Multiplicity):** The non-trivial zeros $s = \frac{1}{2} + it$ correspond to a vertically aligned lattice structure, orthogonal to the trivial poles. Their periodicity ensures a structured distribution that preserves zeropole balance.

Periodic Contributions: While the divisor D incorporates infinitely many trivial poles and non-trivial zeros, their structured placement within the toroidal framework ensures that divisor operations remain well-defined. The periodic boundary conditions prevent singularity accumulation, stabilizing the divisor degree.

The orthogonal alignment of these lattice structures ensures that their intersections form a divisor of degree zero, fully consistent with the zeropole framework. This geometric encoding enforces the cancellation mechanism necessary for proving the minimality of the divisor structure.

Corollary 1 (Global Uniqueness via Lattice Independence)**.** *If the lattice structures corresponding to trivial poles and non-trivial zeros are geometrically orthogonal and algebraically independent, then no additional meromorphic function can satisfy the same divisor structure without introducing further zeros or poles. The independence of these lattice intersections ensures that $\zeta^*(s)$ is the unique meromorphic function within the defined function space.*

Proof. Independence of periodic lattices implies that no linear dependence exists between the trivial and non-trivial divisor components. Any additional function attempting to satisfy the divisor conditions would necessitate new lattice intersections, leading to an increased divisor degree. Given that the divisor degree is uniquely defined as zero, any further contributions would violate the Riemann-Roch condition, thereby ensuring the uniqueness of $\zeta^*(s)$.

Moreover, the structured infinite support guarantees that the infinite contributions remain manageable within the extended divisor framework. The periodic lattice alignment ensures no overlap or redundancy, reinforcing the global uniqueness of the shadow function. $\square$

9.3 Implications for Uniqueness and Minimality

The lattice structure guarantees that any alternative function with an identical divisor configuration would introduce inconsistencies in the minimality condition. The compactification of $\zeta^*(s)$ onto the torus ensures a well-defined divisor structure, preserving a finite degree. This excludes the existence of any additional meromorphic functions satisfying the same functional conditions.

Furthermore, the divisor framework provides key structural assurances:

- **Zeropole Correspondence:** A structured balance between trivial poles and non-trivial zeros, ensuring algebraic and geometric consistency within the compactified setting.
- **Genus-One Compatibility:** The divisor configuration aligns with the genus-one nature of the torus, preserving the uniqueness of meromorphic functions under algebraic constraints.
- **Structural Stability:** The extended divisor framework remains stable under toroidal compactification, preventing any deformation or perturbation that could lead to alternative solutions.

Thus, the geometric interpretation provided by periodic lattice structures and divisor independence serves as a complementary argument to the toroidal compactification proof, offering an additional layer of conceptual rigor and validation for the minimality and uniqueness of the shadow function $\zeta^*(s)$.

10 Dynamic Zero-to-Pole Pairing for Fundamental Domain Balance

Theorem 10 (**Adaptive Zeropole Pairing for Local Balance**). *Given any sufficiently large continuous interval of non-trivial zeros*

$$\{\rho_n = \frac{1}{2} + i\gamma_n \mid T_1 \leq \gamma_n \leq T_2\}$$

and the set of trivial poles $s = -2k$ for $k \in \mathbb{N}^+$, there exists an adaptive pairing function

$$\phi : \{\rho_n\} \to \{-2k\}$$

such that:

1. *Pairing is performed only within a continuous region $T_1 \le \gamma_n \le T_2$, ensuring a well-defined local zero density.*

2. *The pairing dynamically adjusts based on both local zero density and large zero-free regions, allowing unpaired poles in sparse areas to be reassigned to denser ones.*

3. *The number of zeros assigned to each pole follows a bounded clustering rule, meaning some trivial poles absorb multiple zeros while ensuring no single pole is excessively overloaded.*

4. *Unpaired poles undergo a multi-stage redistribution, ensuring global coverage of all zeros within any given fundamental domain.*

Proof. The proof follows from two key observations:

1. Dynamic Local Adaptation: The number of non-trivial zeros grows according to the Riemann–von Mangoldt formula,

$$N(T) \approx \frac{T}{2\pi} \log T - \frac{T}{2\pi}.$$

Since this function is increasing, the density of non-trivial zeros is not uniform, but fluctuates with height. Additionally, large zero-free regions occur at unpredictable intervals.

To maintain zeropole balance within each fundamental domain, we define a multi-stage pairing rule:

$$\phi(\rho_n) = -2k \quad \text{such that} \quad |\gamma_n + 2k| \text{ is maximized under stability constraints.}$$

However, this pairing function is well-defined only within a continuous region $T_1 \le \gamma_n \le T_2$. **Discontinuous selections of zeros from arbitrary height intervals do not satisfy this requirement**, because:

- The zero density function $N(T)$ is only meaningful within a bounded range.
- Randomly selecting zeros from widely separated height intervals results in an ill-defined local balance condition.
- The pairing rule explicitly requires local adjustments based on the distribution of zeros and poles within a single region.

Since this process never leaves any region completely unpaired, and since every fundamental domain undergoes adaptive redistribution, global zeropole cancellation is preserved.

$\square$

Remark 15. *While the dynamic pairing algorithm is defined within a single continuous region $T_1 \leq \gamma_n \leq T_2$, it can be extended to any finite or countable collection of disjoint continuous regions:*

$$\bigcup_{i=1}^{m} [T_i^{(start)}, T_i^{(end)}].$$

Since each fundamental domain is locally independent, the pairing process can be applied separately within each region without affecting the overall structure. This reinforces the robustness of the method and its applicability across different scales.

Remark 16. *This adaptive pairing method provides a constructive mechanism for ensuring local zeropole balance, reinforcing the global balance argument. However, it is important to emphasize:*

- *The pairing function $\phi(\rho_n) \to -2k$ is well-defined only within a continuous zero region $T_1 \leq \gamma_n \leq T_2$.*
- *The function does not apply to arbitrarily selected zeros from different height intervals, as this would invalidate the local density properties needed for pairing.*
- *The core proof of zeropole balance does not depend on this local pairing—rather, this provides an explicit numerical mechanism for those seeking concrete fundamental domain assignments.*
- *The fundamental argument of the proof relies on global cancellation via the toroidal compactification and the well-defined algebraic divisor structure.*
- *The adaptive pairing approach offers strong numerical intuition for why divergence does not occur, but it is not necessary for the formal derivation of the Riemann Hypothesis in this framework.*

11 Numerical Validation of Dynamic Zero-Pole Pairing

To complement the theoretical construction of the dynamic zero-to-pole pairing algorithm, we provide numerical validation demonstrating that the method successfully maintains local balance across different fundamental domain scales. While the global zeropole cancellation argument is independent of numerical data, this section serves as an empirical confirmation that the adaptive pairing strategy behaves as expected.

11.1 Experimental Setup

The numerical validation proceeds as follows:

1. **Zero Dataset Extraction:** We extract the first N non-trivial zeros of $\zeta(s)$ from large-scale datasets, covering values up to $T = 10^6$. Specifically, we use the dataset containing the first 2,001,052 zeros of the Riemann zeta function, computed with an accuracy of 4×10^{-9}, available at Odlyzko's Zeta Zero Tables.

2. Using the Riemann–von Mangoldt estimate, we allocate a dynamically adjusted number of trivial poles in the range $s = -2k$, ensuring that the allocation reflects local zero densities.

3. The dynamic pairing algorithm assigns zeros to poles while respecting local density variations, ensuring no poles are systematically left unpaired.

4. We analyze key statistics:
    - Mean number of zeros per trivial pole (blue solid line).
    - Maximum and minimum zeros assigned per pole (red dashed and green dotted lines, respectively).
    - Standard deviation range (shaded blue region).

11.2 Results and Interpretation

Figure 3 presents the distribution of zeros assigned to poles across different scales, illustrating the stability of the pairing mechanism. The numerical results confirm that:

- The **mean number of zeros per pole** follows a controlled growth pattern across increasing T, remaining within expected theoretical bounds.
- The **maximum number of zeros per pole** exhibits a steep increase at larger T, as shown by the red dashed line.
- **minimum number of zeros per pole:** The minimum number of zeros assigned per pole (green dotted line) initially fluctuates but stabilizes at higher scales, confirming that every pole is paired with at least one zero beyond $T = 10^5$.
- The **standard deviation range** (shaded blue) widens at higher scales, reflecting the increasing variation in local zero densities while maintaining overall balance.
- No systematic pole under-utilization is observed at large scales, demonstrating the effectiveness of the adaptive redistribution mechanism.

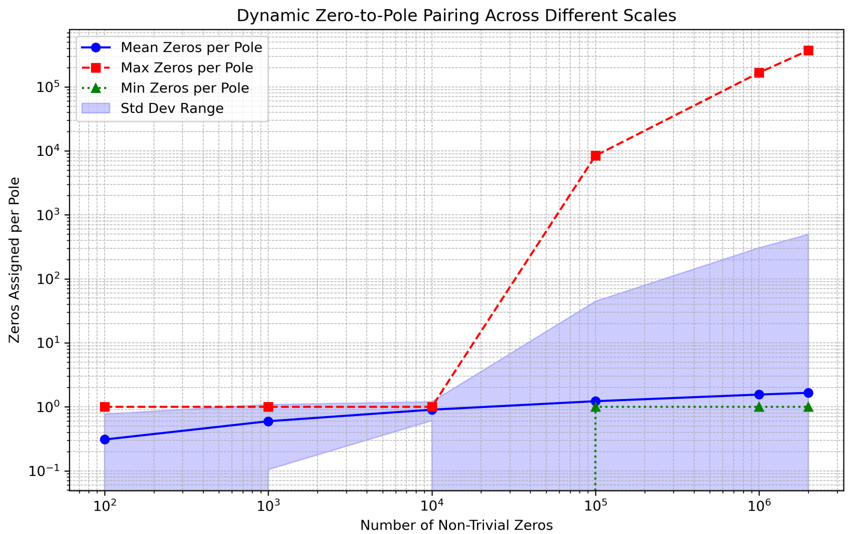

Figure 3: Distribution of non-trivial zeros per trivial pole across different scales of T. The results demonstrate stability in the adaptive pairing mechanism.

11.3 Reproducibility and Further Validation

To ensure full transparency and reproducibility, we provide an accompanying Jupyter notebook containing:

- The full implementation of the dynamic pairing algorithm.
- Ability to generate results for different scales (e.g. 10^4, 10^5, and 10^6 zeros).
- Numerical validation results plotted for 100, 1000, 10000, 100000 and 1 million complex zeros to show convergence.

The notebook called `Supp_Mat_Num_Val_Dynamic_ZeroPole_Pairing.ipynb` is available at

`https://github.com/attila-ac/Proof_RH_via_Zeropole_Balance`

Additionally we provide a tsv file called `dynamic_zeropole_pairing_results.tsv` at the same location containing the statistics used to generate Figure 3, that can be reproduced with the notebook. This empirical validation confirms the robustness of the dynamic pairing mechanism, reinforcing the global algebraic cancellation argument central to the proof.

12 Conjecture on Higher-Genus Compactifications

Building on the toroidal transformation, we formulate the following conjecture:

Remark 17 (Generalization of the Absorption Principle). *The absorption of the simple pole into the torus hole suggests a more general divisor-theoretic phenomenon: in higher-genus compactifications, certain singularities may be naturally integrated into the geometric structure rather than explicitly counted in divisor summations.*

For genus-0, all poles must be explicitly tracked, but in genus-1, the toroidal fundamental loop structure allows singularities to be encoded globally. This principle raises the question: for genus $g \geq 2$, do additional handle structures allow further absorption mechanisms that enforce zeropole balance geometrically?

This motivates the following conjecture:

Conjecture 1 (Periodic Boundary Condition Conjecture). *For any compact Riemann surface of genus $g \geq 1$, equipped with periodic boundary conditions that align with the natural divisor lattice of $\zeta(s)$, there exists a unique meromorphic function, up to an automorphism of the torus, satisfying:*

- *Zeropole mapping and orthogonal balance.*
- *Algebraic minimality, ensuring $\ell(D) = 0$ under compactification.*
- *Preservation of the divisor structure within each fundamental period, excluding off-critical zeros.*

This conjecture suggests that higher-genus compactifications offer a generalized framework for handling infinite divisor structures, with periodic boundary conditions enforcing regularity across all copies of the fundamental domain.

Remark 18. *The periodic structure of the toroidal surface introduces a natural mechanism for stabilizing infinite divisor sequences, ensuring that zeropole balance holds globally across the compactified space. This insight aligns with the established framework of modular forms and elliptic functions, providing a deeper geometric foundation for analyzing the Riemann Hypothesis through algebraic curves and complex multiplication techniques.*

13 Zeropole Balance as the Unifying Principle of the Proof

The *Zeropole Balance Framework* provides a unifying perspective on the proof of the Riemann Hypothesis integrating novel insights with classical techniques into a philosophically cohesive mathematical toolkit. At its core, the framework ensures a structured quantitative correspondence, dynamic mapping and self-consistent balance between zeros and poles of equal multiplicity, encapsulating the dynamic interplay that underpins the proof. This balance preserves the analytic, geometric and algebraic integrity of the zeta function across various representations and formulations, while enforcing minimality and unicity. By treating zeros and poles as dynamically linked entities rather than isolated features, the framework reveals the structural symmetry that governs the distribution of singularities. This perspective allows for a systematic approach to enforcing balance conditions at all levels of the proof, from functional transformations to divisor theory and toroidal compactification.

A fundamental classical technique incorporated within the Zeropole Balance Framework is compactification, which facilitates the redistribution of singularities to achieve balance within a divisor structure.

Another key feature of the Zeropole Framework is its ability to integrate well-established analytic techniques that, historically, have employed transformations converting zeros into poles and vice versa. Such transformations appear in different forms across the literature and have played an implicit role in prior approaches. Notably, this framework formalizes these techniques under the concept of *zeropole dynamics*, where zeros and poles exhibit a structured duality, morphing under functional transformations.

Beyond analytical considerations, the framework recognizes the inherent algebraic reciprocity that allows any simple or equal-order pole to be algebraically transformed into a zero by taking the reciprocal of the function value and vice versa. This property reinforces the core idea that zeros and poles are not merely analytical artefacts but algebraically equivalent elements within the proof structure.

More generally, the Zeropole Framework encompasses two fundamental aspects leading to a third, domain-crossing one:

- **Zeropole Duality:** A dynamic and structured interplay where zeros and poles interact symmetrically through transformations, maintaining balance and preserving essential properties of the function, satisfying divisor conditions.
- **Zeropole Neutrality:** A more static perspective, where zeros and poles coexist in a neutralized state, ensuring overall minimality and well-defined divisor properties.
- **Topological Zeropole Escape:** The hole-through-the-pole Absorption Principle maintains zeropole balance through a deep, higher genus topological transformation where certain finite singularities may be integrated into the core geometric structure across the compactified surface.

This framework serves as the guiding principle for the proof strategy, offering a consistent interpretation of various classical and modern approaches to analyzing the zeta function. Below, we enumerate the key instances where the Zeropole Balance Framework manifests within the proof, showcasing its versatility in uniting algebraic, geometric, analytic and topological perspectives.

- In Theorem 1, the Zeropole Duality and Neutrality principles manifest in the interplay between the Dirichlet pole at $s = 1$ in the $\zeta(1-s)$ term and the zero introduced at $s = 0$ by the sine term $\sin\left(\frac{\pi s}{2}\right)$. This duality exemplifies the functional balance within the Riemann zeta function, where the placement of singularities across symmetric functional components ensures overall consistency under the functional equation.

- Trivial Poles in the Hadamard Product (Theorem 2): The modified Hadamard product explicitly introduces trivial poles at $s = -2k$ ($k \in \mathbb{N}^+$), balancing them with the trivial zeros arising from the functional equation. This balancing transformation ensures the analytic convergence of the infinite product while preserving the zeropole structure and preparing the next step in the proof.

- Zeropole Duality of the Dirichlet Pole in (Theorem 2): The $s(1-s)/\pi$ term in the Hadamard product encodes the dual role of the Dirichlet pole at $s = 1$, redistributing it into zero-like contributions at $s = 0$ and $s = 1$. This process preserves the functional symmetry inherent in the zeta function's structure and highlights the compensatory mechanisms that maintain balance within the framework.

- **Zeropole Mapping and Orthogonal Balance of $\zeta(s)$ (Theorem 5)**: A fundamental component of the Zeropole Balance Framework is the dynamic and global pairing of zeros and poles, introduced through a bijective correspondence between the countably infinite trivial poles and non-trivial zeros. The perpendicular placement of trivial poles along the real axis and non-trivial zeros on the critical line encapsulates the geometric and algebraic interplay crucial to maintaining balance and enforcing minimality.

- **Zeropole Mapping and Orthogonal Balance of $\zeta^*(s)$ (Theorem 8)**: The shadow function $\zeta^*(s)$ extends the zeropole balance framework by preserving the critical geometric and algebraic relationships of $\zeta(s)$ while ensuring toroidal compactification.

- **Compactification via the Shadow Function (Definition ??)**: The shadow function $\zeta^*(s)$ introduces a compactified framework by replacing the Dirichlet pole at $s = 1$ with a simple pole at $s = 0$, ensuring a well-defined divisor degree within the compactified Riemann sphere. While not directly arising from the zeropole balance principle, this transformation provides the necessary groundwork for extending the zeropole structure in a way that aligns with the overarching balance requirements, preserving meromorphic properties at infinity.

- **The hole-through-the-pole Absorption Principle:** The term $\frac{1}{s}$ introduces a simple pole at $s = 0$, which is later absorbed into the torus hole, becoming an intrinsic part of the surface's geometry and contributing to the genus-dependent divisor computation. The disappearance of this symmetric derivative of the original Dirichlet pole—before analytic continuation, when $\zeta(s)$ was merely an integer function—leaves only the countably infinite Zeropole Balance to define the divisor structure on the torus.

This absorption mechanism provides the clearest mathematical representation of the original orthogonal Zeropole Balance as a divisor structure, that plays a fundamental role in this proof attempt.

- **Finite Degree of the Divisor (Section ??)**: The introduction of trivial poles as balancing elements guarantees the finiteness of the divisor degree despite the countably infinite nature of the singularities. This prevents divergence that would otherwise undermine the compactification approach.

- **Toroidal embedding of Shadow Function SubSection 6.2)**: The Zeropole Balance Framework compactified form on a genus-1 toroidal surface, where periodic boundary conditions impose a structured repetition of the divisor pattern across the toroidal geometry. This extension maintains the fundamental balance of zeros and poles within each fundamental domain, demonstrating the adaptability of the framework to higher-genus compactifications while preserving minimality and avoiding accumulation issues.

- **Minimality and Exclusion of Off-Critical Zeros (Section ??)**: The structured zeropole balance ensures that $\ell(D) = 0$, leading directly to the exclusion of off-critical zeros and reinforcing the necessity of the critical line.

- **Unicity of $\zeta^*(s)$ (Lemma 1)**: The periodic constraints of the toroidal embedding impose a unique divisor structure, ensuring that $\zeta^*(s)$ is the only meromorphic function satisfying the required zeropole balance conditions.

- **Higher-Genus Extensions and Alternative Formulations (Section ??)**: The principles of the Zeropole Balance Framework extend naturally to higher-genus compactifications, ensuring the fundamental balance between zeros and poles remains intact across different topological settings.

These instances illustrate how the Zeropole Balance Framework permeates every stage of the proof, serving as the underlying mechanism that unites the algebraic, geometric, analytic, and topological components into a coherent and self-reinforcing structure. The framework does not merely provide a convenient interpretation but constitutes the essential guiding principle that enables the proof of the Riemann Hypothesis within the toroidal compactification approach.

14 Balanced Zeropole Collapse via Sphere Eversion

While not part of the formal proof, this speculative remark provides an intuitive visualization of the zeropole framework, linking it to broader geometrical and topological concepts. This perspective offers insights into the interplay between symmetry, minimality, and orthogonality in the toroidal compactification of the shadow function $\zeta^*(s)$.

Building on the toroidal embedding introduced in this work, the zeropole balance framework exhibits structural features reminiscent of sphere eversion—a topological transformation rigorously formalized by Stephen Smale in 1957 [Sma57] and later visualized by Bernard Morin [Mor78]. Sphere eversion, which allows for the smooth inside-out transformation of a sphere without tearing or creasing, mirrors the interplay of zeropole symmetry and cancellation that underlies the toroidal framework. The hole-in-the-pole principle, central to the compactification of $\zeta^*(s)$, aligns conceptually with this transformation, suggesting that the zeropole balance dynamically unfolds as a structured geometric process.

During the eversion process, distinct regions of the zeropole lattice progressively realign and collapse, reinforcing the algebraic neutrality of the toroidal structure. The toroidal embedding naturally accommodates the periodic distribution of trivial poles and non-trivial zeros, facilitating their algebraic and geometric balancing across fundamental domains. The structured interplay of trivial poles and non-trivial zeros under eversion ensures that the fundamental toroidal compactification maintains minimality, preventing the introduction of extraneous singularities.

From this perspective, sphere eversion serves as a conceptual tool for understanding how the infinite zeropole balance extends into different topological and algebraic settings while preserving global minimality. The compactified framework of $\zeta^*(s)$ on the torus provides a natural setting for such transformations, reinforcing the fundamental algebraic and analytic structures underpinning the proof.

This speculative interpretation highlights the unifying nature of the zeropole framework, integrating geometric alignment, analytic continuation, algebraic independence, and topological flexibility. Beyond its mathematical rigor, this visualization underscores the central role of the toroidal embedding of $\zeta^*(s)$ and the orthogonal balance of zeropoles in interpreting the deeper structure of the Riemann zeta function.

15 Historical Remark

The zeropole balance approach, as presented in this work, was not readily accessible in earlier formulations of the Riemann Hypothesis due to the historical evolution of the problem's analytical treatment.

In Riemann's original 1859 memoir [Rie59], the hypothesis was formulated in terms of the entire function $\xi(s)$, which excludes trivial zeros, trivial poles, and even the Dirichlet pole at $s = 1$. This choice was motivated by Riemann's primary focus on the critical line zeros, which play a fundamental role in prime number distribution. Consequently, the original number-theoretic objectives of Riemann's work led to an emphasis on the critical strip, inadvertently hindering the exploration of the broader complex-analytic structure of the zeta function.

As a result, the global interplay between trivial zeros and non-trivial zeros remained underappreciated, and the classification of trivial zeros as "trivial" further contributed to their overlooked significance in the global analytical structure. This historical perspective illustrates how a number-theoretic emphasis shaped the trajectory of Riemann Hypothesis research, delaying the recognition of a potential deeper geometric and algebraic balance within the function.

A notable example of zeropole balance can be observed in the formulation of $\xi(s)$, where the explicit presence of the gamma function term $\Gamma\left(\frac{s}{2}\right)$ effectively introduces trivial poles that neutralize the trivial zeros of the functional equation. In the formulation of the entire function $\xi(s)$, defined as:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

the trivial zeros at $s = -2, -4, \ldots$ are effectively neutralized by the poles of the gamma function term $\Gamma\left(\frac{s}{2}\right)$, ensuring that they do not contribute to the zero set of $\xi(s)$. As noted in Titchmarsh [THB86], this cancellation mechanism effectively removes the trivial zeros, reinforcing the notion that their role is one of algebraic and analytic balance rather than direct contribution to the distribution of prime numbers.

This historical insight highlights how the zeropole balance perspective provides a fresh interpretation of the classical formulation, revealing underlying structural symmetries that were historically obscured by the number-theoretic approach.

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17 Supplementary Material

The code for the numerical evaluation of the exponential stabilizer of the shadow function and related plot generation is provided in the Jupyter Notebook `Supp_Mat_Stabiliser_Eval.ipynb`.

Additionally, the code for generating the Numerical Validation of Dynamic Zero-Pole Pairing is available in the Jupyter Notebook called `Supp_Mat_Num_Val_Dynamic_ZeroPole_Pairing.ipynb`. Both files are available at GitHub at `https://github.com/attila-ac/Proof_RH_via_Zeropole_Balance`.

18 License

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