Off-Critical Riemann Zeta Zeros Cannot Seed Symmetric Entire Functions: A Hyperlocal Proof of Constructive Impossibility

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Full proof available at GitHub (`https://github.com/attila-ac/hyperlocal`).

This proof establishes the Riemann Hypothesis by showing, via *reductio ad absurdum*, that assuming an off-critical zero ρ' for a transcendental entire function $H(s)$ with Riemann ξ-function symmetries—the Functional Equation (FE) and Reality Condition (RC) —leads to an unavoidable analytic contradiction for a zero of any order.

The strategy is a constructive hyperlocal test. It uses the combination of the Imaginary Derivative Condition (IDC) and the Line-to-Line Mapping Theorem as an analytical engine. This engine translates the global symmetries (FE and RC) into fatal local constraints on the derivative $H'(s)$ in the immediate neighborhood of the hypothetical off-critical zero ρ'. The main proof follows a hybrid strategy, using the argument best suited for each case.

1. **Part I: Refutation of Multiple Zeros ($k \geq 2$).** Holds for *any* entire function.
   - *Premise:* Assume a multiple off-critical zero $\rho' = \sigma + it$ of order $k \geq 2$. This means $H^{(j)}(\rho') = 0$ for $j < k$, but $H^{(k)}(\rho') \neq 0$.
   - *Mapping Constraint:* The reparametrized derivative $P(w) = H'(\rho' + w)$ must map the line $L_A = \{(1/2 - \sigma) + iu : u \in \mathbb{R}\}$ into the imaginary axis $i\mathbb{R}$. The Line-to-Line Mapping Theorem forces $P(w)$ to be affine or constant.
   - *Contradiction:* The Taylor series for $P(w)$ begins with a non-zero term of order w^{k-1} where $k - 1 \geq 1$. A direct algebraic analysis shows that for such a series to represent an affine function mapping $L_A \to i\mathbb{R}$, its leading coefficient must be zero. This is a contradiction. Thus, multiple off-critical zeros are impossible.
2. **Part II: Refutation for Simple Zeros ($k = 1$).** Specific to the class of transcendental functions.
   - *The Pincer Movement:* The proof establishes two mutually exclusive properties for the derivative $H'(s)$:
     (a) **It Must Be Affine:** The IDC, when applied to the entire function $H'(s)$, forces it to be an affine polynomial via the Line-to-Line Mapping Theorem.
     (b) **It Cannot Be Affine:** The existence of the zero ρ' allows the factorization $H(s) = R_{\rho'}(s)G(s)$, where $R_{\rho'}(s)$ is the minimal model polynomial. A rigorous analysis of this structure proves that if $H(s)$ is transcendental, its derivative $H'(s)$ cannot be an affine polynomial.
   - *Contradiction:* A function cannot be both affine and non-affine. This logical impossibility refutes the existence of simple off-critical zeros in any such transcendental function.
3. **Conclusion: Riemann Hypothesis Holds.** Since the assumption of an off-critical zero of any order leads to a contradiction for the class of functions to which the Riemann $\xi(s)$ function belongs, all of its non-trivial zeros must lie exclusively on the critical line.