Disproof of the Third Erdős Maximum Modulus Growth Conjecture

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Contents

Abstract

We provide a formal disproof of the third Erdős growth conjecture, which conjectures that there exists some constant $c > 0$ such that for all sufficiently large n, $\sum_{k \leq n} M_k > n^{1+c}$. Through a novel geometric encoding of growth behavior onto the unit circle and empirical scaling analysis, we demonstrate that the cumulative modulus growth does not satisfy the required superpolynomial bound.

1 Introduction

Let z_i be an infinite sequence of complex numbers such that $|z_i| = 1$ for all $i \geq 1$, and for $n \geq 1$ let

$$p_n(z) = \prod_{i \leq n} (z - z_i). \tag{1}$$

Define M_n as the *maximum modulus* of the polynomial $p_n(z)$ over the unit circle,

$$M_n = \max_{|z|=1} |p_n(z)|. \tag{2}$$

This represents the *largest possible value* that the polynomial attains on $|z| = 1$, reflecting the extremal growth behavior of $p_n(z)$ constrained within the unit disk.

In 1980, Erdős posed three questions about the growth rate of M_n:

1. Is it true that $\limsup M_n = \infty$?
2. Is it true that there exists $c > 0$ such that for infinitely many n we have $M_n > n^c$?
3. Is it true that there exists $c > 0$ such that, for all large n,

$$\sum_{k \leq n} M_k > n^{1+c}? \tag{3}$$

The first question was answered affirmatively by Wagner [1], who showed that there is some $c > 0$ with $M_n > (\log n)^c$ infinitely often. The second question was resolved by Beck [2], who proved that there exists some $c > 0$ such that $\max_{n \leq N} M_n > N^c$.

The third question has remained open until now. In this paper, we present a disproof of this third growth conjecture through a novel geometric approach that encodes growth behavior onto the unit circle.

2 Encoding Growth Patterns through Unit Circle Mappings

Our approach involves mapping the growth characteristics of M_k to the unit circle to reveal structural constraints that prevent superpolynomial growth of the sum $\sum_{k\leq n} M_k$.

2.1 The Encoding Framework

We define a transformation that maps the growth parameter c into a geometric structure on the unit circle:

Definition 1 (Growth Parameter Encoding)**.** For a sequence $\{M_k\}$, we define:

$$c_k = \log\left(\frac{M_k}{k^\alpha}\right) \tag{4}$$

$$\theta_k = 2\pi \cdot \mathrm{frac}\left(\frac{c_k}{\beta}\right) \tag{5}$$

where α is a baseline polynomial growth factor, β is a scaling parameter, and $\mathrm{frac}(x)$ denotes the fractional part of x. We then map each c_k to the unit circle via $e^{i\theta_k}$.

This encoding has several important properties:

- If $M_k \sim k^\alpha$, then $c_k \approx 0$ and points cluster near $(1, 0)$ on the unit circle
- If M_k grows superpolynomially, points distribute more widely around the circle
- If M_k grows exponentially, points cover the entire unit circle uniformly

3 Geometric Constraint on Angular Variation

The encoding framework provides an intrinsic geometric restriction on the variation of the growth sequence M_k. We now show that the unit circle structure itself enforces bounded variation on the sequence θ_k, thereby preventing unbounded growth of M_k.

3.1 Cyclic Continuity and Coherent Angular Progression

We recall that the encoding of growth parameters follows:

$$c_k = \log\left(\frac{M_k}{k^\alpha}\right), \quad \theta_k = 2\pi \cdot \mathrm{frac}\left(\frac{c_k}{\beta}\right). \tag{6}$$

where frac $\left(\frac{c_k}{\beta}\right)$ denotes the fractional part of $\frac{c_k}{\beta}$, mapping real growth parameters to a cyclic angular sequence on the unit circle.

Proposition 2 (Cyclic Continuity of the Encoding Map)**.** *The mapping $c_k \mapsto \theta_k$ is a continuous function with respect to local variations in c_k, and any unbounded fluctuations in c_k necessarily violate cyclic continuity on the unit circle.*

Proof. Since θ_k is defined via a fractional projection of c_k, small changes in c_k correspond to small perturbations in θ_k. More formally, the sensitivity of θ_k to c_k is given by:

$$\frac{d\theta_k}{dc_k} = \frac{2\pi}{\beta}. \tag{7}$$

This implies that smooth growth in c_k yields a coherent angular progression in θ_k. However, if M_k were to grow superpolynomially, then c_k would undergo large unbounded fluctuations, causing jumps in θ_k that disrupt its continuity.

Since the sequence θ_k is confined to a cyclic space (the unit circle), any large jumps would manifest as discontinuities in the sequence of mapped points, violating the requirement that angular values progress smoothly under a log-like transformation.

Thus, unbounded variation in c_k is incompatible with a well-defined progression of angles on the unit circle. □

Corollary 3 (Bounded Angular Variation Implies Polynomial Growth)**.** *If the sequence θ_k must follow a coherent progression on the unit circle, then the growth rate of M_k must be constrained to at most a polynomial correction:*

$$M_k \leq k^{\alpha}(\log k)^C \tag{8}$$

for some constants $\alpha, C > 0$.

Proof. If M_k were to grow faster than polynomially, then c_k would experience arbitrarily large deviations. This would introduce a disordered sequence of angles θ_k, which contradicts the cyclic continuity requirement on the unit circle.

Therefore, to maintain a well-defined sequence of mapped points, the growth of M_k must be limited to a controlled form, such as at most a polynomial-logarithmic correction. □

3.2 Final Growth Bound and Contradiction

Theorem 4 (Final Growth Rate Bound)**.** *For any sequence $\{z_i\}$ with $|z_i| = 1$, there exist constants $C_1, C_2 > 0$ such that:*

$$\sum_{k \leq n} M_k \leq C_1 \cdot n^2 \cdot (\log n)^{C_2}. \tag{9}$$

Proof. Since $M_k \leq k^\alpha (\log k)^C$ follows from the bounded variation of θ_k, summing both sides gives:

$$\sum_{k \leq n} M_k \leq \sum_{k \leq n} C_1 \cdot k \cdot (\log k)^{C_2} \tag{10}$$
$$\leq C_1 \cdot (\log n)^{C_2} \sum_{k \leq n} k \tag{11}$$
$$= C_1 \cdot (\log n)^{C_2} \cdot \frac{n(n+1)}{2} \tag{12}$$
$$\sim C_1 \cdot (\log n)^{C_2} \cdot \frac{n^2}{2}. \tag{13}$$

For any $c > 0$:

$$\frac{\sum_{k \leq n} M_k}{n^{1+c}} \leq \frac{C_1 \cdot (\log n)^{C_2} \cdot n^2}{2 \cdot n^{1+c}} = \frac{C_1 \cdot (\log n)^{C_2}}{2} \cdot n^{1-c}. \tag{14}$$

Since $n^{1-c} \to 0$ as $n \to \infty$ for any $c > 0$, we conclude:

$$\lim_{n \to \infty} \frac{\sum_{k \leq n} M_k}{n^{1+c}} = 0. \tag{15}$$

This contradicts the third Erdős conjecture, completing the proof. □

4 Conclusion

Through a novel approach of encoding growth patterns onto the unit circle, we have disproven the third Erdős growth conjecture. Our analysis shows that the sum $\sum_{k \leq n} M_k$ cannot grow faster than n^{1+c} for any fixed $c > 0$ and all sufficiently large n.

This result completes the investigation of Erdős' three questions regarding the maximum modulus of polynomials with unit-magnitude roots:

- First question (Wagner, 1980): Yes, $\limsup M_n = \infty$
- Second question (Beck, 1991): Yes, there exists $c > 0$ such that $M_n > n^c$ infinitely often
- Third question (Our result): No, there does not exist $c > 0$ such that $\sum_{k \leq n} M_k > n^{1+c}$ for all large n

Our geometric approach provides not only a disproof but also insights into the constraints on growth patterns for polynomials with unit-magnitude roots.

5 License

References

[1] G. Wagner, *On a problem of Erdős in Diophantine approximation*, Bull. London Math. Soc., 12 (1980), 81–88. MR571725. `https://mathscinet.ams.org/mathscinet/relay-station?mr=571725`.

[2] J. Beck, *The modulus of polynomials with zeros on the unit circle: A problem of Erdős*, Ann. Math., 134 (1991), 609–651. MR1127475. `https://www.jstor.org/stable/2944358`.