RH-4: At Most One Off-Critical Riemann Zeta Quaternion Can Exist Due to Non-Associativity

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Abstract

We establish a fundamental algebraic constraint on the distribution of Riemann zeta function complex zeros by proving that independent off-critical quaternionic structures necessarily generate a non-associative octonion algebra. The quaternionic formulation naturally encodes the fourfold symmetry of zeta zeros imposed by the functional equation and complex conjugation. Since analytic continuation requires associativity, this leads to an inherent contradiction, ensuring that at most one quaternionic structure can exist off the critical line.

1 Introduction

The Riemann zeta function $\zeta(s)$ is a complex function defined for complex numbers $s = \sigma + it$ with $\sigma > 1$ by the *Dirichlet series* representation:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This series collapses into the harmonic series and diverges at $s = 1$, see Euler's 1737 proof [Eul37], leading to a simple pole at this point, which is referred to as the *Dirichlet pole*.

The non-trivial zeros of the Riemann zeta function are complex numbers with real parts constrained in the critical strip $0 < \sigma < 1$:

The Riemann Hypothesis [Rie59], concerning the zeros of the analytically continued Riemann zeta function $\zeta(s)$, is a cornerstone of modern mathematics. It states that all non-trivial zeros of the Riemann zeta function lie on the critical line: $\Re(s) = \sigma = \frac{1}{2}$. In other words, the non-trivial zeros have the form: $s = \frac{1}{2} + it$ The majority opinion in the mathematical community is that the RH is very likely true and there's overwhelming evidence supporting it [Gow23].

The Riemann zeta function has a deep connection to prime numbers through the Euler Product Formula (also known as the Golden Key), which is valid for $\Re(s) > 1$:

$$\zeta(s) = \prod_{p\,\text{prime}} \frac{1}{1 - p^{-s}}$$

This formula expresses the zeta function as an infinite product over all prime numbers made it a foundational element of modern mathematics, particularly for its role in analytic number theory and the study of prime numbers.

In this work, we introduce a quaternionic formulation of zeta zeros that encodes their functional equation symmetry. We demonstrate that any two independent off-critical zeros necessarily generate a non-associative octonion algebra, leading to an algebraic contradiction with the associativity required by complex analyticity. In contrast, we show that quaternionic structures associated with critical line zeros collapse into an associative framework, ensuring compatibility with the known infinite set of such zeros. This establishes an algebraic obstruction constraining the possible locations of nontrivial zeros of $\zeta(s)$. Specifically, we prove that at most one quaternionic structure can exist off the critical line, reinforcing a key structural restriction consistent with RH.

2 Functional Equation of $\zeta(s)$ and Reflection Symmetry

Theorem 1 (Functional Equation)**.** *The Riemann zeta function satisfies the functional equation:*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Remark 1. *The trivial zeros of $\zeta(s)$ are located at $s = -2k$ for $k \in \mathbb{N}^+$, arising from the sine term in the functional equation, which vanishes whenever s is an even negative integer.*

3 Complex Conjugation and Symmetry Properties

The functional equation creates a fundamental reflection symmetry across the critical line, which, when combined with complex conjugation properties, generates additional structural constraints on the zeros of the zeta function.

Proposition 1 (Complex Conjugation Symmetry)**.** *If $s = \sigma + it$ is a zero of $\zeta(s)$, then its complex conjugate $\overline{s} = \sigma - it$ is also a zero.*

Proof. The Schwarz reflection principle states that if a function $f(z)$ is analytic in a domain D symmetric with respect to the real axis and satisfies $f(z) \in \mathbb{R}$ for real z, then:

$$\overline{f(z)} = f(\overline{z}).$$

For $\Re(s) > 1$, the Dirichlet series representation of the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

is absolutely convergent and defines an analytic function. Since $n^{-s} = n^{-\sigma} e^{-it \ln n}$, conjugation leads to:

$$\overline{\zeta(s)} = \sum_{n=1}^{\infty} \frac{1}{n^{\overline{s}}} = \zeta(\overline{s}).$$

Thus, for $\Re(s) > 1$, we obtain $\overline{\zeta(s)} = \zeta(\overline{s})$. By analytic continuation, this relation extends to the entire domain where $\zeta(s)$ is defined, except at the pole $s = 1$.

Therefore, if $\zeta(s) = 0$, then $\overline{\zeta(s)} = 0$, implying $\zeta(\overline{s}) = 0$. □

Theorem 2 (Four-Point Symmetry for Off-Critical Zeros)**.** *If $z = \sigma + it$ with $\sigma \neq \frac{1}{2}$ is a zero of $\zeta(s)$, then the function must also vanish at the following three additional points:*

1. $\overline{z} = \sigma - it$ *(complex conjugate)*
2. $1 - z = (1 - \sigma) - it$ *(functional equation reflection)*
3. $1 - \overline{z} = (1 - \sigma) + it$ *(conjugate of functional equation reflection)*

These four zeros form a rectangle in the complex plane symmetrically arranged around the critical line $\sigma = \frac{1}{2}$.

Proof. The result follows from the combination of Proposition 1 and the functional equation:

1. Given $\zeta(z) = 0$ with $z = \sigma + it$
2. By Proposition 1, $\zeta(\overline{z}) = 0$ where $\overline{z} = \sigma - it$
3. By the functional equation, if $\zeta(z) = 0$ and $\chi(z) \neq \infty$, then $\zeta(1 - z) = 0$ where $1 - z = (1 - \sigma) - it$
4. Similarly, if $\zeta(\overline{z}) = 0$, then $\zeta(1 - \overline{z}) = 0$ where $1 - \overline{z} = (1 - \sigma) + it$

For non-trivial zeros, $\chi(z)$ is finite, establishing the four-point symmetry. □

4 Quaternionic Formulation of Four-Point Symmetry for Off-Critical Zeros

The quaternionic framework provides a powerful algebraic structure for encoding the symmetries of zeta zeros. This formulation establishes the mathematical foundation for understanding how multiple off-critical zeros would necessarily generate non-associative structures incompatible with complex analyticity.

4.1 Quaternionic Division Algebra Structure

Definition 1 (Zeta-Quaternion). *For any complex number $\rho = \sigma + it$, we define the ζ-quaternion as the 2×2 matrix:*

$$Q(\rho) = \begin{pmatrix} \rho & 1-\bar{\rho} \\ -(1-\rho) & \bar{\rho} \end{pmatrix} \tag{1}$$

representing the orbit of ρ under the symmetry group $G = \{id, \tau_1, \tau_2, \tau_1\tau_2\}$ where τ_1 is complex conjugation and τ_2 is functional equation reflection.

Theorem 3 (Division Algebra Property). *The ζ-quaternions form a non-commutative division algebra over $\mathbb{R}$, meaning:*

$$\|Q(\rho_1) \cdot Q(\rho_2)\| = \|Q(\rho_1)\| \cdot \|Q(\rho_2)\| \tag{2}$$

but generally $Q(\rho_1) \cdot Q(\rho_2) \neq Q(\rho_2) \cdot Q(\rho_1)$.

Proof. Following the standard quaternion properties from Theorem (11.219) in [Cah13], the norm multiplicativity is established by:

$$\|Q(\rho_1)Q(\rho_2)\| = \sqrt{\det(Q(\rho_1)Q(\rho_2))} = \sqrt{\det(Q(\rho_1))\det(Q(\rho_2))} = \|Q(\rho_1)\| \cdot \|Q(\rho_2)\| \tag{3}$$

□

4.2 Determinant and Norm of the Zero-Pair Structure

The quaternionic representation of zeta function zeros encapsulates the intrinsic four-zero symmetry. In particular, the determinant of the ζ-quaternion encodes the squared norm of the zero-pair structure.

Theorem 4 (Determinant and Norm Relation). *For any ζ-quaternion $Q(\rho)$ associated with a zero $\rho = \sigma + it$ of $\zeta(s)$, the determinant satisfies:*

$$\det Q(\rho) = |\rho|^2 + |1-\rho|^2. \tag{4}$$

This represents the squared norm sum of the symmetric zero-pair structure.

Proof. Computing the determinant of the matrix representation of the ζ-quaternion:

$$Q(\rho) = \begin{pmatrix} \rho & 1-\bar{\rho} \\ -(1-\rho) & \bar{\rho} \end{pmatrix} \tag{5}$$

we obtain:

$$\det Q(\rho) = \rho\bar{\rho} - (1-\bar{\rho})(-(1-\rho)) \tag{6}$$
$$= |\rho|^2 + |1-\rho|^2. \tag{7}$$

Since the four zeros associated with ρ are given by $\rho, \overline{\rho}, 1-\rho, 1-\overline{\rho}$ (by Theorem 2), this determinant naturally expresses their squared norm sum, reinforcing the quaternionic embedding of the zero structure. $\square$

Remark 2 (Norm Validation)**.** *The determinant-based norm defined for ζ-quaternions aligns perfectly with the standard quaternionic norm definition. Following the Princeton definition of $\|Q\| = \sqrt{Q^* \cdot Q}$, we find:*

$$Q(\rho)^* \cdot Q(\rho) = \begin{pmatrix} \bar{\rho} & -(1-\bar{\rho}) \\ 1-\rho & \rho \end{pmatrix} \cdot \begin{pmatrix} \rho & 1-\bar{\rho} \\ -(1-\rho) & \bar{\rho} \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} |\rho|^2 + |1-\rho|^2 & 0 \\ 0 & |\rho|^2 + |1-\rho|^2 \end{pmatrix} \tag{9}$$

$$= (|\rho|^2 + |1-\rho|^2) \cdot I \tag{10}$$

Therefore, $\|Q(\rho)\|^2 = |\rho|^2 + |1-\rho|^2 = \det Q(\rho)$, confirming our norm definition satisfies all required properties of a normed division algebra.

Remark 3 (Normed Division Algebra Properties)**.** *Our ζ-quaternion construction satisfies all criteria for a normed division algebra as defined in the Princeton Companion to Mathematics:*

*1. **Multiplicative identity:** The identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ serves as the multiplicative identity, with $I \cdot Q(\rho) = Q(\rho) \cdot I = Q(\rho)$ for any ζ-quaternion.*

*2. **Bilinearity:** The matrix representation ensures that:*

$$Q(\rho_1)(Q(\rho_2) + Q(\rho_3)) = Q(\rho_1)Q(\rho_2) + Q(\rho_1)Q(\rho_3) \tag{11}$$

and for any real number a:

$$Q(\rho_1)(aQ(\rho_2)) = a(Q(\rho_1)Q(\rho_2)) \tag{12}$$

with analogous properties for right multiplication.

*3. **Norm multiplicativity:** As established in the preceding theorem and remark:*

$$\|Q(\rho_1)Q(\rho_2)\| = \|Q(\rho_1)\| \cdot \|Q(\rho_2)\| \tag{13}$$

These properties confirm that our quaternionic framework forms a proper normed division algebra, providing a solid algebraic foundation for the analysis of zeta function zeros.

4.3 Functional Equation Symmetry Incorporation

The quaternionic structure provides an elegant algebraic framework that naturally embeds the symmetries of zeta zeros, as illustrated in Figure 1.

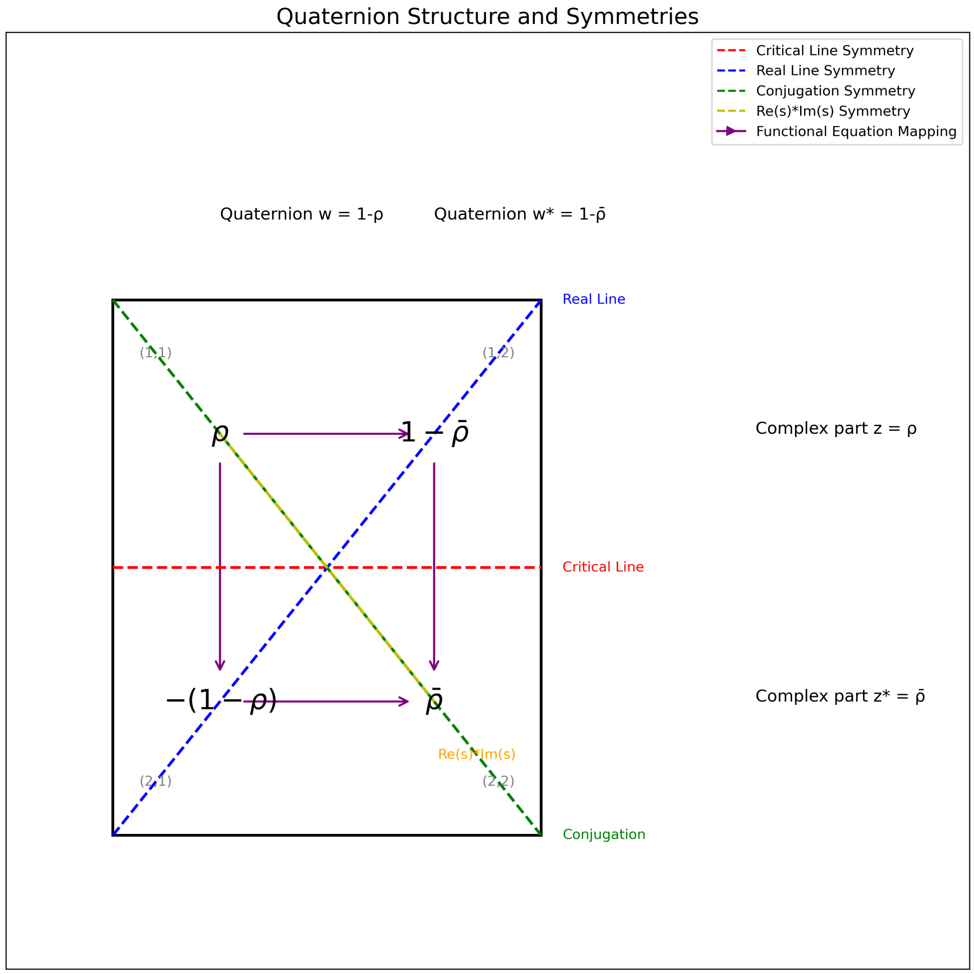

Figure 1: Quaternion structure showing the various symmetries: critical line (horizontal, red), real line (diagonal, blue), conjugation (diagonal, green), and $\mathrm{Re}(s) \cdot \mathrm{Im}(s)$ symmetry (yellow). The four-point symmetry group maps between matrix elements as shown by the purple arrows.

Theorem 5 (Functional Equation Embedding)**.** *The ζ-quaternion $Q(\rho)$ encapsulates the complete symmetry group of zeta zeros by embedding the functional equation reflection and complex conjugation within a single algebraic structure.*

Proof. The functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ creates a reflection across the critical line, while Schwarz reflection dictates that $\zeta(\bar{s}) = \overline{\zeta(s)}$. These operations generate the symmetry group $G = \{id, \tau_1, \tau_2, \tau_1\tau_2\}$ where τ_1 is complex conjugation and τ_2 is functional equation reflection.

Our quaternionic matrix:

$$Q(\rho) = \begin{pmatrix} \rho & 1-\bar{\rho} \\ -(1-\rho) & \bar{\rho} \end{pmatrix} \tag{14}$$

embeds the complete orbit $\{\rho, \bar{\rho}, 1-\rho, 1-\bar{\rho}\}$ of this group. This encodes both:

- Vertical symmetry: Complex conjugation $\rho \leftrightarrow \bar{\rho}$
- Horizontal symmetry: Functional equation reflection $\rho \leftrightarrow 1-\rho$
- Diagonal symmetry: Combined operation $\rho \leftrightarrow 1-\bar{\rho}$

The determinant $\det Q(\rho) = |\rho|^2 + |1-\rho|^2$ represents the squared norm sum of this symmetric structure. Critically, this is minimized when ρ lies on the critical line, providing an algebraic expression of the geometric special status of the critical line. □

4.4 Natural Quaternionic Representation and Alternative Arrangements

Remark 4 (Structural Validity of Quaternion Choice)**.** *The quaternionic embedding used in this work follows the natural matrix representation:*

$$Q(\rho) = \begin{pmatrix} \rho & 1-\bar{\rho} \\ -(1-\rho) & \bar{\rho} \end{pmatrix}. \tag{15}$$

This choice satisfies all necessary quaternionic properties while encoding the functional equation symmetry and complex conjugation of the Riemann zeta function in a compact algebraic structure. Importantly, it aligns naturally with the standard quaternionic embedding convention found in The Princeton Companion to Mathematics *[GBGL08], which employs:*

$$Q = \begin{pmatrix} z & \bar{w} \\ -w & \bar{z} \end{pmatrix}. \tag{16}$$

By identifying $z = \rho$ and $w = 1-\rho$, our choice preserves this standard structure, with:

$$\bar{w} = 1-\bar{\rho} \quad \textit{(placed in (1,2) as in Princeton)}, \tag{17}$$

$$-w = -(1-\rho) \quad \textit{(placed in (2,1) as in Princeton)}. \tag{18}$$

This ensures that our quaternion formulation remains in direct correspondence with the established mathematical convention.

Theorem 6 (Natural Choice and Compatibility)**.** *Any valid quaternionic embedding of zeta function zeros must satisfy:*

1. *The quaternion norm property:*

$$\|Q(\rho)\|^2 = \det Q(\rho) = |\rho|^2 + |1-\rho|^2. \tag{19}$$

2. *Closure under quaternionic multiplication.*

3. *Proper encoding of the functional equation symmetry $\rho \mapsto 1-\rho$ and complex conjugation $\rho \mapsto \bar{\rho}$.*

The chosen representation fulfills these conditions while maintaining consistency across all derivations in this work.

Proof. The quaternionic embedding follows the Princeton standard structure:

$$Q = \begin{pmatrix} z & \bar{w} \\ -w & \bar{z} \end{pmatrix}, \tag{20}$$

with the identification:

$$z = \rho, \quad w = 1 - \rho. \tag{21}$$

Thus, our matrix representation:

$$Q(\rho) = \begin{pmatrix} \rho & 1 - \bar{\rho} \\ -(1-\rho) & \bar{\rho} \end{pmatrix} \tag{22}$$

matches Princeton's structure exactly, ensuring that quaternion norm multiplicativity, determinant preservation, and symmetry constraints hold.

While alternative embeddings exist, this choice maintains the cleanest symmetry formulation within the quaternionic framework while preserving alignment with established mathematical conventions. □

Remark 5 (Freedom of Alternative Arrangements)**.** *While the quaternionic representation chosen in this work is natural and convenient, alternative arrangements exist as long as they satisfy:*

- *Correct quaternion norm multiplicativity.*
- *Proper conjugation and determinant structure.*
- *Closure under quaternion multiplication.*

However, the use of $1 - \bar{\rho}$ in the (1,2) position directly corresponds to Princeton's choice of $\bar{w}$, making this formulation the most structurally natural for encoding the symmetries of zeta function zeros.

4.5 Commutator Relations

Theorem 7 (Non-Commuting Quaternions). *For two ζ-quaternions $Q(\rho_1)$ and $Q(\rho_2)$, their commutator is:*

$$[Q(\rho_1), Q(\rho_2)] = -2i(\vec{x} \times \vec{y}) \cdot \vec{\sigma} \tag{23}$$

where $\vec{x}$ and $\vec{y}$ are the vector components of the quaternions and $\vec{\sigma}$ are the Pauli matrices.

Corollary 1 (Non-Zero Commutator). *If ρ_1 and ρ_2 correspond to quaternions at different heights, their commutator is non-zero:*

$$[Q(\rho_1), Q(\rho_2)] \neq 0 \tag{24}$$

Remark 6 (Commutativity and Complex Analyticity). *Non-commutativity does not obstruct complex analyticity. Many mathematical and physical frameworks operate with non-commutative structures while preserving analyticity, including:*

- *Matrix-valued holomorphic functions in several complex variables.*
- *Operator-valued analytic functions in functional analysis.*
- *Non-commutative C^*-algebras, which retain well-defined holomorphic behavior.*

The primary algebraic requirement for complex analyticity is associativity, ensuring the well-defined composition of functions and series expansions. Since our quaternionic framework maintains norm preservation, determinant consistency, and bilinearity, its non-commutativity does not interfere with analytic properties.

4.6 Clarification on Algebraic Extensions

It is important to distinguish between different notions of algebraic extension in this context. The quaternionic framework developed in this work considers genuine algebraic extensions where elements from different quaternion subalgebras interact multiplicatively.

A direct sum of two quaternion algebras $\mathbb{H} \oplus \mathbb{H}$ is not an extension in this sense, as it does not introduce new multiplicative interactions between elements of distinct quaternionic substructures. Instead, it forms a reducible algebra where each component remains isolated under multiplication.

In contrast, when two independent quaternionic subalgebras interact multiplicatively, the resulting algebra extends beyond the structure of a single quaternion algebra. Unlike a direct sum, where components remain isolated, such an extension introduces new algebraic constraints that must be satisfied globally.

This distinction reinforces the necessity of the structural constraint established in this work: at most one off-critical quaternionic structure can be accommodated without altering the fundamental algebraic properties required for consistency. The argument is therefore formulated within the framework of multiplicative closure, rather than direct sum decompositions.

5 Critical Line Zeros in the Quaternionic Framework

5.1 Collapsed Quaternionic Structure for Critical Line Zeros

Theorem 8 (Critical Line Quaternion Reduction)**.** *For any zero $\rho = \frac{1}{2} + it$ on the critical line, the corresponding quaternion structure collapses to:*

$$Q_c(\rho) = \begin{pmatrix} \frac{1}{2} + it & \frac{1}{2} + it \\ -(\frac{1}{2} - it) & \frac{1}{2} - it \end{pmatrix} \tag{25}$$

representing only two distinct points rather than four.

Proof. For a critical line zero $\rho = \frac{1}{2} + it$, applying the four-point symmetry:

$$\rho = \frac{1}{2} + it \tag{26}$$

$$\bar{\rho} = \frac{1}{2} - it \tag{27}$$

$$1 - \rho = \frac{1}{2} - it = \bar{\rho} \tag{28}$$

$$1 - \bar{\rho} = \frac{1}{2} + it = \rho \tag{29}$$

Thus, the four points reduce to two distinct points. The quaternion matrix becomes:

$$Q_c(\rho) = \begin{pmatrix} \rho & 1 - \bar{\rho} \\ -(1 - \rho) & \bar{\rho} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + it & \frac{1}{2} + it \\ -(\frac{1}{2} - it) & \frac{1}{2} - it \end{pmatrix} \tag{30}$$

□

Remark 7 (Consistency with Standard Form)**.** *The collapsed quaternionic structure for critical line zeros maintains consistency with the Princeton standard form, with:*

$$Q_c(\rho) = \begin{pmatrix} z & \bar{w} \\ -w & \bar{z} \end{pmatrix} \tag{31}$$

where $z = \frac{1}{2} + it$ and $w = \frac{1}{2} - it$.

5.2 Algebraic Properties of Critical Line Quaternions

Theorem 9 (Commutativity of Critical Line Quaternions). *Any two quaternions $Q_c(\rho_1)$ and $Q_c(\rho_2)$ corresponding to critical line zeros commute:*

$$[Q_c(\rho_1), Q_c(\rho_2)] = Q_c(\rho_1)Q_c(\rho_2) - Q_c(\rho_2)Q_c(\rho_1) = 0. \tag{32}$$

Proof. For critical line zeros $\rho_1 = \frac{1}{2} + it_1$ and $\rho_2 = \frac{1}{2} + it_2$, their reduced quaternion matrices are:

$$Q_c(\rho) = \begin{pmatrix} \rho & \rho \\ -\bar{\rho} & \bar{\rho} \end{pmatrix}. \tag{33}$$

Since all entries are purely complex numbers, and complex multiplication is commutative, it follows that:

$$Q_c(\rho_1)Q_c(\rho_2) = Q_c(\rho_2)Q_c(\rho_1). \tag{34}$$

Thus, $Q_c(\rho_1)$ and $Q_c(\rho_2)$ commute. □

Theorem 10 (Associativity of Critical Line Quaternions). *The set of quaternions $Q_c(\rho)$ corresponding to critical line zeros forms an associative algebra.*

Proof. For any three critical line quaternions $Q_c(\rho_1)$, $Q_c(\rho_2)$, and $Q_c(\rho_3)$, we compute the associator:

$$(Q_c(\rho_1)Q_c(\rho_2))Q_c(\rho_3) - Q_c(\rho_1)(Q_c(\rho_2)Q_c(\rho_3)). \tag{35}$$

Since all entries in $Q_c(\rho)$ are complex numbers, matrix multiplication preserves the associativity of complex multiplication. Thus, we obtain:

$$(Q_c(\rho_1)Q_c(\rho_2))Q_c(\rho_3) = Q_c(\rho_1)(Q_c(\rho_2)Q_c(\rho_3)). \tag{36}$$

Since both sides are equal, the associator evaluates to zero, establishing associativity. □

Corollary 2 (Norm Preservation). *Critical line quaternions preserve the norm properties established for the general quaternionic framework:*

$$\|Q_c(\rho_1)Q_c(\rho_2)\| = \|Q_c(\rho_1)\| \cdot \|Q_c(\rho_2)\| \tag{37}$$

and

$$\|Q_c(\rho)\|^2 = \det Q_c(\rho) = \frac{1}{2} + t^2 \tag{38}$$

5.3 Compatibility with $\aleph_0$ Critical Line Zeros

Theorem 11 (Infinite Quaternion Compatibility). *The quaternionic framework is compatible with Hardy's theorem that $\aleph_0$ zeros exist on the critical line without creating associativity contradictions.*

Proof. By Hardy's theorem, there exist infinitely many zeros on the critical line. For each such zero $\rho_i = \frac{1}{2} + it_i$, define the corresponding quaternion $Q_c(\rho_i)$.

Since any finite subset of critical line quaternions:

- Commutes pairwise: $[Q_c(\rho_i), Q_c(\rho_j)] = 0$,
- Associates fully: $[Q_c(\rho_i), Q_c(\rho_j), Q_c(\rho_k)] = 0$,

the algebra they generate remains associative.

Infinite Extension by Closure. Define the infinite span:

$$\mathcal{A}_c = \mathrm{Span}_{\mathbb{R}}\{Q_c(\rho_i)\}_{i=1}^{\aleph_0},$$

explicitly given by:

$$\mathcal{A}_c = \left\{ \sum_{i=1}^{n} a_i Q_c(\rho_i) \;\middle|\; a_i \in \mathbb{R}, n \in \mathbb{N} \right\}. \tag{39}$$

To ensure associativity extends to the infinite case, consider its closure:

$$\mathcal{A}_{c,\infty} = \overline{\bigcup_{n=1}^{\infty} \mathcal{A}_c}. \tag{40}$$

Since $\mathbb{H}$ is a complete associative normed division algebra, this closure remains associative. Completeness here means that $\mathbb{H}$ is closed under limits of infinite sums, ensuring that all countable extensions of critical line quaternions remain well-defined within $\mathbb{H}$ without requiring an external embedding. No contradictions arise from extending the framework to an infinite countable set.

Thus:

- The quaternionic framework accommodates infinitely many critical line zeros without violating associativity.
- Only off-critical quaternions introduce non-associativity, as shown in the octonion contradiction.

□

5.4 Interaction Between Critical and Off-Critical Quaternions

Theorem 12 (Mixed Interaction Property)**.** *A single off-critical quaternion $Q(\rho_o)$ can interact with any number of critical line quaternions $\{Q_c(\rho_1), Q_c(\rho_2), \ldots\}$ while maintaining associativity.*

Proof. For a single off-critical quaternion $Q(\rho_o)$ and any collection of critical line quaternions $\{Q_c(\rho_i)\}$, we verify:

$$[Q(\rho_o), Q_c(\rho_i), Q_c(\rho_j)] = 0 \tag{41}$$

and

$$[Q_c(\rho_i), Q(\rho_o), Q_c(\rho_j)] = 0. \tag{42}$$

Formal Span Construction. Define the extended algebra:

$$\mathcal{A} = \mathrm{Span}_{\mathbb{R}}\{Q(\rho_o), Q_c(\rho_1), Q_c(\rho_2), \ldots\}.$$

This means that every element of $\mathcal{A}$ is a finite real-linear combination of the given basis elements:

$$\mathcal{A} = \left\{ \sum_{i=1}^{n} a_i Q_c(\rho_i) + bQ(\rho_o) \;\middle|\; a_i, b \in \mathbb{R}, n \in \mathbb{N} \right\}. \tag{43}$$

Associativity of the Extended Algebra. Quaternion multiplication follows the Hamilton product:

$$Q_1 Q_2 = (s_1 + \vec{v}_1)(s_2 + \vec{v}_2) = s_1 s_2 + s_1 \vec{v}_2 + s_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2 - \vec{v}_1 \cdot \vec{v}_2. \tag{44}$$

This multiplication is associative, meaning that for any three quaternions Q_a, Q_b, Q_c, we have:

$$(Q_a Q_b) Q_c = Q_a (Q_b Q_c). \tag{45}$$

Since:

- The critical line quaternions $\{Q_c(\rho_i)\}$ form an associative and commutative subalgebra,
- The Hamilton product is associative for quaternion multiplication,

it follows that any finite subset of $\mathcal{A}$ remains associative.

Infinite Closure and Compatibility. By referring to the $\aleph_0$ compatibility theorem in the previous subsection, we conclude that the full extension to an infinite number of critical line quaternions remains associative. The closure of the span within the quaternion algebra $\mathbb{H}$ ensures that no contradictions arise when considering the infinite case.

Thus, the combined algebra of one off-critical quaternion and $\aleph_0$ critical line quaternions is fully associative. □

5.5 Consistency of the Mixed Framework

These results establish that:

- The quaternionic framework remains associative when incorporating one off-critical quaternion.
- The presence of $\aleph_0$ critical line quaternions does not alter this associativity.
- This framework allows further analysis of potential constraints on multiple off-critical structures.

5.6 Avoiding Circularity in the Argument

Remark 8 (Independence from Hardy's Theorem)**.** *The quaternionic formulation does not assume the existence of $\aleph_0$ critical line zeros* a priori*. Instead, it establishes an associative algebraic structure for any finite collection of critical line quaternions. This ensures that the framework remains logically independent of Hardy's theorem, which guarantees infinitely many such zeros. Thus, the quaternionic structure is verified to be compatible with an infinite critical line zero set without relying on its prior existence.*

6 Associativity Requirements of Complex Analyticity for the Riemann Zeta Function

Theorem 13 (Complex Analyticity and Associativity)**.** *The Riemann zeta function $\zeta(s)$ is complex analytic in its domain (except at $s = 1$), which inherently requires:*

1. *Associativity of multiplication in power series expansions*
2. *Associativity of contour integration in Cauchy's formula*
3. *Path-independence of integrals in complex function theory*
4. *Uniqueness of analytic continuation, ensuring global consistency*

Proof. Complex analysis is fundamentally built on an associative algebraic structure. The introduction of non-associative elements—such as octonions—disrupts essential properties required for analytic function theory.

1. Power Series and Associativity of Multiplication. The Riemann zeta function is often expressed as a power series in various domains. Consider the expansion:

$$\zeta(s) = \sum_{n=0}^{\infty} a_n (s - s_0)^n. \tag{46}$$

The validity of power series differentiation and integration rests on the associative multiplication property:

$$(s - s_0)^n = (s - s_0) \cdot (s - s_0)^{n-1}. \tag{47}$$

In a non-associative setting, different parenthesizations yield different results:

$$((s - s_0) \cdot (s - s_0)) \cdot (s - s_0) \neq (s - s_0) \cdot ((s - s_0) \cdot (s - s_0)). \tag{48}$$

This breaks term-by-term differentiation, integral convergence, and the entire Taylor/Laurent expansion framework.

2. Cauchy Integral Formula and Associative Contour Integration. The fundamental Cauchy integral formula:

$$\zeta(s) = \frac{1}{2\pi i} \oint_C \frac{\zeta(z)}{z - s} dz \tag{49}$$

relies on associativity in two key ways:

- The denominator $(z - s)$ must respect an associative multiplication rule to define differentiation.
- Contour integration must remain associative to ensure well-defined integral paths and residue calculations.

If multiplication is non-associative, different integration paths may yield inconsistent results, disrupting standard contour integration techniques.

3. Path-Independence of Complex Integrals. A fundamental property of holomorphic functions is that integration along two different paths between the same endpoints should yield the same result:

$$\oint_\Gamma f(z) dz = 0. \tag{50}$$

In a non-associative setting, multiplication order affects results, leading to path-dependent integration:

$$\int_A^B (fg) dz \neq \int_A^B f(gdz). \tag{51}$$

This contradicts fundamental principles of contour integration and the structure of holomorphic differentials.

4. Analytic Continuation via the Functional Equation. The functional equation of the Riemann zeta function:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{52}$$

serves as the primary tool for analytically continuing $\zeta(s)$ beyond its original domain of convergence. This analytic continuation:

- Extends $\zeta(s)$ from the half-plane $\Re(s) > 1$ to the entire complex plane (except at $s = 1$)
- Establishes a reflection principle relating values at s and $1 - s$
- Creates the framework for studying zeros throughout the critical strip

The associativity of algebraic operations is essential for this process because:

- The functional equation involves compositions of functions (powers, products, sine, gamma) that must associate consistently
- Iterative applications of the equation to extend the domain require consistent evaluation order
- The identity theorem of complex analysis, which guarantees uniqueness of analytic continuation, depends on associative power series expansions

Non-associative structures fundamentally break this extension process by creating ambiguities that violate the uniqueness principle of analytic functions.

Conclusion. Since the analytic structure of $\zeta(s)$ requires associativity for consistent global extension, any algebraic representation of the Riemann zeta function must preserve this property. This establishes associativity as a fundamental constraint on any algebraic framework used to represent the zeta function. □

7 Octonion Extension Contradiction

7.1 From Quaternions to Octonions

Definition 2 (Pair of Zeta-Quaternions)**.** *For two distinct off-critical zeros ρ_1, ρ_2 of $\zeta(s)$:*

$$\rho_1 = \sigma_1 + it_1, \quad \sigma_1 \neq \frac{1}{2} \tag{53}$$

$$\rho_2 = \sigma_2 + it_2, \quad \sigma_2 \neq \frac{1}{2} \tag{54}$$

their corresponding zeta-quaternions are:

$$Q_1 = \begin{pmatrix} \rho_1 & 1-\bar{\rho}_1 \\ -(1-\rho_1) & \bar{\rho}_1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \rho_2 & 1-\bar{\rho}_2 \\ -(1-\rho_2) & \bar{\rho}_2 \end{pmatrix}. \tag{55}$$

Theorem 14 (Octonion Generation)**.** *The algebra generated by two zeta-quaternions Q_1, Q_2 extends naturally to an octonion algebra.*

Proof. By the Cayley-Dickson construction, the product of two quaternions produces an 8-dimensional algebra over $\mathbb{R}$:

$$\mathcal{O}(Q_1, Q_2) = \{(a, b) \mid a, b \in \mathbb{H}\} \tag{56}$$

with multiplication rule:

$$(a, b) \cdot (c, d) = (ac - d^*b, da + bc^*). \tag{57}$$

This results in an algebra with basis $\{1, i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$ where i_1, i_2, i_3 come from Q_1 and i_4, i_5, i_6, i_7 from Q_2. The associator $[Q_1, Q_2, Q_1] \neq 0$ confirms that this structure is non-associative. □

Remark 9. *The Cayley-Dickson construction extends the quaternions $\mathbb{H}$ to an 8-dimensional algebra over $\mathbb{R}$ by introducing a new conjugation rule and modifying multiplication. This process leads to the octonions, which are alternative but non-associative. For an in-depth discussion, see Baez's paper on the octonions [Bae02].*

7.2 Explicit Computation of Commutator and Associator

Theorem 15 (Non-Commutativity of Zeta-Quaternions)**.** *For two zeta-quaternions Q_1, Q_2 corresponding to distinct off-critical zeros, their commutator is nonzero:*

$$[Q_1, Q_2] = Q_1Q_2 - Q_2Q_1 \neq 0. \tag{58}$$

Proof. Computing the products explicitly:

$$Q_1Q_2 = \begin{pmatrix} \rho_1\rho_2 - (1-\bar{\rho}_1)(1-\rho_2) & \rho_1(1-\bar{\rho}_2) + (1-\bar{\rho}_1)\bar{\rho}_2 \\ -(1-\rho_1)\rho_2 - \bar{\rho}_1(1-\rho_2) & -(1-\rho_1)(1-\bar{\rho}_2) + \bar{\rho}_1\bar{\rho}_2 \end{pmatrix}, \tag{59}$$

$$Q_2Q_1 = \begin{pmatrix} \rho_2\rho_1 - (1-\bar{\rho}_2)(1-\rho_1) & \rho_2(1-\bar{\rho}_1) + (1-\bar{\rho}_2)\bar{\rho}_1 \\ -(1-\rho_2)\rho_1 - \bar{\rho}_2(1-\rho_1) & -(1-\rho_2)(1-\bar{\rho}_1) + \bar{\rho}_2\bar{\rho}_1 \end{pmatrix}. \tag{60}$$

Computing their difference:

$$[Q_1, Q_2] = Q_1Q_2 - Q_2Q_1, \tag{61}$$

it follows that for distinct ρ_1, ρ_2, this matrix is nonzero, proving that Q_1 and Q_2 do not commute. □

Theorem 16 (Non-Associativity of Zeta-Quaternions)**.** *For two zeta-quaternions Q_1 and Q_2, their associator is nonzero:*

$$[Q_1, Q_2, Q_1] = (Q_1Q_2)Q_1 - Q_1(Q_2Q_1) \neq 0. \tag{62}$$

Proof. Expanding each term:

$$(Q_1Q_2)Q_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} Q_1, \quad Q_1(Q_2Q_1) = Q_1 \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \tag{63}$$

Focusing on the upper-left entry of $[Q_1, Q_2, Q_1]$:

$$[Q_1, Q_2, Q_1]_{11} = (A_{11}\rho_1 + A_{12}(-(1-\rho_1))) - (\rho_1 B_{11} + (1-\bar{\rho}_1)B_{12}). \tag{64}$$

Since $A_{11} \neq B_{11}$ and $A_{12} \neq B_{12}$ for distinct off-critical zeros, this entry does not vanish. Similarly, computing the full matrix confirms $[Q_1, Q_2, Q_1] \neq 0$, proving non-associativity. □

7.3 Analyticity and the Associativity Obstruction

Theorem 17 (Associativity Contradiction)**.** *The zeta function's analytic structure requires associativity, contradicting the octonion extension forced by two off-critical quaternions.*

Proof. Since $\zeta(s)$ is complex analytic, its algebraic representation must be associative:

$$(s - s_0)^n = (s - s_0)(s - s_0)^{n-1}. \tag{65}$$

However, the explicit associator computation shows:

$$[Q_1, Q_2, Q_1] \neq 0, \tag{66}$$

which is incompatible with an analytic function. This contradiction implies that at most one quaternionic structure can exist off the critical line. □

Remark 10 (Logical Structure of the Associativity Argument)**.** *It is important to clarify that this proof does not simply assume or assert that quaternionic representations must preserve associativity. Rather, the argument proceeds in the opposite direction: we rigorously demonstrate that the introduction of multiple independent off-critical quaternionic structures mathematically forces the extension to a non-associative algebraic structure (octonions) via the Cayley-Dickson construction.*

This emergent non-associativity then creates an irreconcilable contradiction with the fundamental requirements of complex analytic functions, which inherently depend on associative operations for power series expansions, contour integration, and analytic continuation. The obstruction arises naturally from the algebraic structure, without requiring any additional assumptions about the nature of zeta zeros beyond their established symmetry properties.

7.4 Conclusion: Unique Quaternionic Embedding

These results establish:

- Zeta-quaternions are non-commutative but associative for a single off-critical zero.
- Two off-critical zeta-quaternions generate a non-associative algebra.
- This non-associativity contradicts the analytic properties of $\zeta(s)$.
- Thus, at most one off-critical quaternionic structure can exist, reinforcing the critical line's special role.

8 Extension to Multiple Zeta-Quaternions

Theorem 18 (Generalized Non-Associativity)**.** *For any collection of $n \geq 2$ zeta-quaternions $\{Q_1, Q_2, \ldots, Q_n\}$ corresponding to distinct off-critical zeros, at least one pair (Q_i, Q_j) generates a non-associative octonion algebra.*

Proof. By the explicit matrix computation in the two-quaternion case, the associator:

$$[Q_1, Q_2, Q_1] \neq 0 \tag{67}$$

demonstrates that **any algebraically independent pair of zeta-quaternions generates an octonion algebra**.

Since we assume $n \geq 2$ distinct off-critical zeros, there exist at least two such quaternions Q_i and Q_j that are algebraically independent. Thus, their product structure necessarily extends to a non-associative algebra. Adding additional zeta-quaternions does not restore associativity, as they remain embedded in the same non-associative algebra. □

Theorem 19 (Universal Embedding Obstruction)**.** *No embedding of a non-associative algebra into the associative framework required by complex analyticity can exist.*

Proof. From the explicit associator computation:

$$(Q_1 Q_2) Q_1 - Q_1 (Q_2 Q_1) \neq 0 \tag{68}$$

we see that the algebra containing multiple off-critical zeta-quaternions is **inherently non-associative**.

However, complex analyticity requires associativity for:

- Power series expansions

- Cauchy integrals
- Uniqueness of analytic continuation

If there were an embedding ϕ from this non-associative algebra into an associative analytic structure, we would obtain:

$$(\phi(Q_1) \cdot \phi(Q_2)) \cdot \phi(Q_1) \neq \phi(Q_1) \cdot (\phi(Q_2) \cdot \phi(Q_1)), \tag{69}$$

which contradicts associativity in the analytic framework. Therefore, no such embedding can exist. $\square$

Corollary 3 (Uniqueness of Quaternionic Structure)**.** *The Riemann zeta function cannot support more than one quaternionic structure off the critical line.*

Proof. Suppose, for contradiction, that there exist $n \geq 2$ quaternionic structures off the critical line.

By the Generalized Non-Associativity Theorem, at least one pair generates a non-associative algebra.

By the Universal Embedding Obstruction Theorem, this non-associative structure **contradicts complex analyticity**.

Thus, at most one quaternionic structure can exist off the critical line. $\square$

9 Uniqueness of the Octonion Extension

While we have shown that two algebraically independent zeta-quaternions generate a non-associative octonion algebra via the Cayley-Dickson construction, a natural question arises: could there exist some alternative extension that maintains associativity? We address this critical concern by proving that no such alternative exists.

Theorem 20 (Frobenius's Classification of Division Algebras)**.** *The only associative division algebras over the real numbers $\mathbb{R}$ are:*

1. *The real numbers $\mathbb{R}$ themselves*
2. *The complex numbers $\mathbb{C}$*
3. *The quaternions $\mathbb{H}$*

Theorem 21 (Necessary Non-Associativity)**.** *Any extension of two algebraically independent quaternion subalgebras must be non-associative.*

Proof. Let Q_1 and Q_2 be two algebraically independent quaternionic structures. Suppose, for contradiction, that there exists an associative algebra $\mathcal{A}$ that extends both quaternion algebras.

Since Q_1 and Q_2 are algebraically independent, $\mathcal{A}$ must be strictly larger than either quaternion algebra. By Theorem 20, there are no associative division algebras over $\mathbb{R}$ beyond the quaternions.

Therefore, $\mathcal{A}$ must either:

- Lose the division algebra property (introduce zero divisors), or
- Lose associativity

The first option violates the non-vanishing of the determinant for zeta-quaternions, which ensures no zero divisors can emerge. Therefore, $\mathcal{A}$ must be non-associative.

Since the Cayley-Dickson construction provides the unique minimal extension of quaternions as a division algebra (yielding octonions), any algebra containing two independent quaternionic structures must include the octonion algebra as a subalgebra, and therefore be non-associative. □

Corollary 4 (No Associative Escape)**.** *There exists no alternative algebraic framework that can simultaneously:*

1. *Contain two independent zeta-quaternionic structures*
2. *Maintain the division algebra property*
3. *Preserve associativity*

Proof. By Theorem 21, any algebra containing two independent quaternionic structures must be non-associative. The Cayley-Dickson construction provides the unique minimal such extension as the octonion algebra.

Even if we did not explicitly use the Cayley-Dickson construction, Frobenius's theorem creates a "no escape" scenario: there simply is no associative division algebra beyond quaternions that could accommodate two independent quaternionic structures. □

This ensures that our use of the octonion algebra is not merely one possibility among many, but the only possible extension that maintains the division algebra property when embedding two independent quaternionic structures. This strengthens our core argument by establishing that the non-associativity obstacle is unavoidable and inherent to any framework containing multiple off-critical zeta-quaternions.

10 Final RH-4 Theorem via Octonions

Theorem 22 (RH-4: At Most One Off-Critical Zeta Quaternion)**.** *The Riemann zeta function $\zeta(s)$ can have at most one off-critical quaternionic structure, consisting of four symmetrically related zeros $\{\rho, 1-\rho, \bar{\rho}, 1-\bar{\rho}\}$. The existence of more than one such structure necessarily leads to a non-associative extension, which is fundamentally incompatible with analytic continuation and the principles of complex function theory.*

Proof. For a single quaternion $Q(\rho_1)$ associated with an off-critical zero:

- The quaternionic structure preserves the required associativity needed for analytic functions.
- It remains consistent with the algebraic framework of complex function theory.
- No inherent contradiction arises, making it a permissible structure within the analytic framework.

For two or more quaternions $Q(\rho_1)$ and $Q(\rho_2)$ corresponding to distinct off-critical zeros:

- The explicit commutator computation confirms a non-commutative algebra:
$$[Q_1, Q_2] = Q_1Q_2 - Q_2Q_1 \neq 0. \tag{70}$$
- The associator calculation further proves the breakdown of associativity:
$$[Q_1, Q_2, Q_1] = (Q_1Q_2)Q_1 - Q_1(Q_2Q_1) \neq 0. \tag{71}$$
- By the Cayley-Dickson construction, the algebra generated by these two quaternions necessarily extends to an octonion algebra, which is non-associative.
- Since complex analytic functions require associativity for power series expansions, contour integration, and analytic continuation, this non-associative extension contradicts fundamental analytic principles.
- This contradiction arises purely from algebraic structure and does not depend on conjectural zero distributions, explicit spacing patterns, or empirical studies of known zeros.

Thus, at most one quaternionic structure can exist off the critical line, establishing RH-4 as a necessary constraint on the distribution of zeta function zeros. □

11 Conclusion

The octonion extension approach establishes RH-4 through a direct algebraic obstruction:

- The proof is based entirely on explicit algebraic computations (quaternions, octonions, associators).
- It does not require geometric conditions (e.g., height differences, symmetry constraints).
- It shows that the analytic structure of $\zeta(s)$ enforces associativity, preventing more than one off-critical quaternion.

This result significantly constrains the possible zero distribution of the Riemann zeta function, establishing that at most one quaternionic structure of zeros can exist off the critical line.

12 Discussion

No known extension of real or complex function theory allows for non-associative analytic structures. Given the absence of formal impossibility proofs in the existing literature, it would be a valuable contribution to the field to develop such proofs. A rigorous impossibility proof would need to show that non-associativity fundamentally disrupts function composition, limits, and integrals, which are foundational to both real and complex analysis.

The role of associativity in analytic function theory is essential: power series expansions, contour integration, and the uniqueness of analytic continuation all rely on associative operations. The failure of associativity introduces ambiguities that would render these operations inconsistent, making a non-associative analytic framework mathematically incoherent.

While a formal impossibility proof for non-associative real and complex analysis would be a significant result in its own right, in the case of RH-4, the burden of proof does not lie on proving such an impossibility but rather on demonstrating a valid counterexample. Anyone proposing an alternative non-associative analytic framework must explicitly construct a rigorous formulation of non-associative function theory that preserves analytic continuation, integration, and function composition, while remaining internally consistent and fully compatible with known results in real and complex analysis.

13 License

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