Fractional Helly Theorems

Attila Jung

ELTE and Rényi Institute

January 2026

Drafting Workshop in Discrete Mathematics, Number Theory, and Probability

Fractional Helly

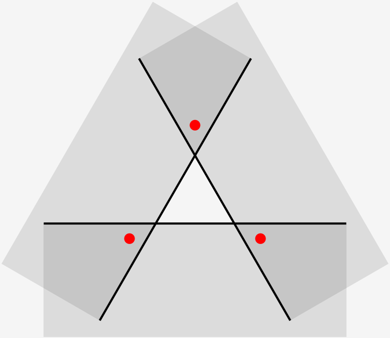
Helly's theorem (1923):

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
any $(d+1)$-tuple of sets have nonempty intersection
$\implies$ the whole family has a nonempty intersection

Fractional Helly

Helly's theorem (1923):

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
any $(d+1)$-tuple of sets have nonempty intersection
$\implies$ the whole family has a nonempty intersection



Fractional Helly

Helly's theorem (1923):

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
any $(d+1)$-tuple of sets have nonempty intersection
$\implies$ the whole family has a nonempty intersection

Fractional Helly theorem (Katchalski, Liu '79):

a positive fraction of $(d+1)$-tuples of sets have nonempty intersection
$\implies$ a positive fraction of the whole family has nonempty intersection.

Fractional Helly

Fractional Helly theorem (Katchalski, Liu '79):

a positive fraction of $(d+1)$-tuples of sets have nonempty intersection
$\implies$ a positive fraction of the whole family has nonempty intersection.

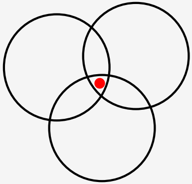
FH more precisely

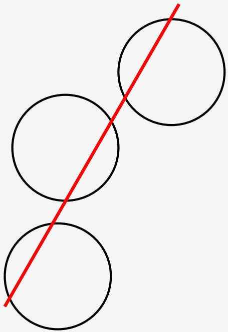
$\forall d \in \mathbb{N}, \alpha \in (0,1] \exists \beta \in (0,1]$ such that
if $\mathcal{F}$ is an n-element family of convex sets in $\mathbb{R}^d$
and $\alpha\binom{n}{d+1}$ of the $(d+1)$-tuples has nonempty intersection
then there exists βn sets with nonempty intersection.

Vincensini's question

Vincensini '35:

Is it possible to generalize Helly's theorem to k-dimensional affine subspaces (k-flats) hitting convex sets?





Vincensini's question

Vincensini '35:

Is it possible to generalize Helly's theorem to k-dimensional affine subspaces (k-flats) hitting convex sets?

Santaló '40:

Impossible even for lines hitting unit disks.

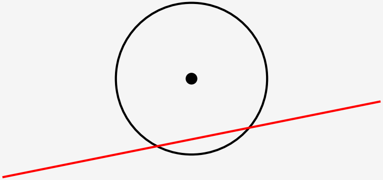
Vincensini's question

Vincensini '35:

Is it possible to generalize Helly's theorem to k-dimensional affine subspaces (k-flats) hitting convex sets?

Santaló '40:

Impossible even for lines hitting unit disks.



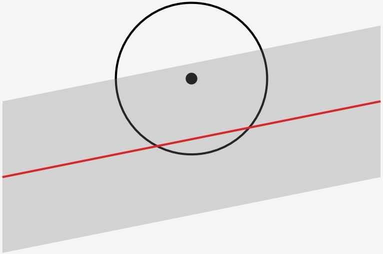
Vincensini's question

Vincensini '35:

Is it possible to generalize Helly's theorem to k-dimensional affine subspaces (k-flats) hitting convex sets?

Santaló '40:

Impossible even for lines hitting unit disks.



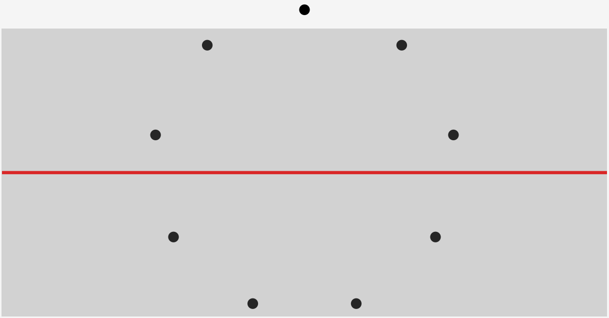
Vincensini's question

Vincensini '35:

Is it possible to generalize Helly's theorem to k-dimensional affine subspaces (k-flats) hitting convex sets?

Santaló '40:

Impossible even for lines hitting unit disks.



Fractional Helly for k-flats

- Alon and Kalai '95: FH for hyperplanes hitting convex sets.

Fractional Helly for k-flats

- Alon and Kalai '95: FH for hyperplanes hitting convex sets.
- Alon, Kalai, Matoušek, Meshulam '02: No FH for lines hitting convex sets in $\mathbb{R}^3$.

Fractional Helly for k-flats

- Alon and Kalai '95: FH for hyperplanes hitting convex sets.
- Alon, Kalai, Matoušek, Meshulam '02: No FH for lines hitting convex sets in $\mathbb{R}^3$.

J, Pálvölgyi '24+:

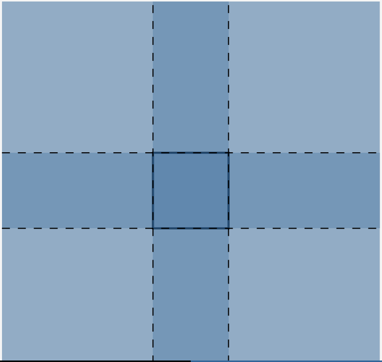
Let $0 \leq k < d$, and $\mathcal{F}$ a finite family of ρ-fat convex sets in $\mathbb{R}^d$, if a positive fraction of the $(k+2)$-tuples can be hit with a k-flat, then there exists a k-flat hitting a positive fraction of the family.

- ρ-fat: ratio of radii of min enclosing and max inscribed balls is at most ρ.

Quantitative Helly

Bárány, Katchalski, Pach '82

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
any $2d$-tuple has an intersection of volume at least 1
$\implies$ the whole family has an intersection of volume at least $v(d) > 0$.

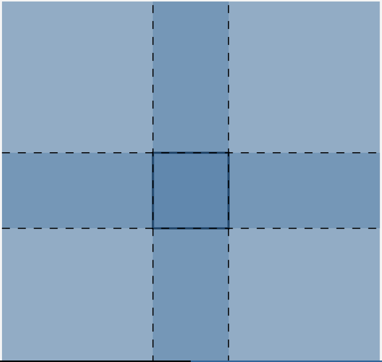


Quantitative Helly

Bárány, Katchalski, Pach '82

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
any $2d$-tuple has an intersection of volume at least 1
$\implies$ the whole family has an intersection of volume at least $v(d) > 0$.

- Naszódi '16: $v(d) \approx d^{-cd}$



Quantitative Fractional Helly

Sarkar, Xue, Soberón '21

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
a positive fraction of the $h = \frac{d(d+3)}{2}$-tuples have intersection of volume at least 1.
$\implies$ a positive fraction of the family has intersection of volume at least $v(d) \geq d^{-d}$.

Quantitative Fractional Helly

Sarkar, Xue, Soberón '21

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
a positive fraction of the $h = \frac{d(d+3)}{2}$-tuples have intersection of volume at least 1.
$\implies$ a positive fraction of the family has intersection of volume at least $v(d) \geq d^{-d}$.

- Conjecture: $h = 2d$ and $v(d) \approx d^{-cd}$.

Quantitative Fractional Helly

Sarkar, Xue, Soberón '21

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
a positive fraction of the $h = \frac{d(d+3)}{2}$-tuples have intersection of volume at least 1.
$\implies$ a positive fraction of the family has intersection of volume at least $v(d) \geq d^{-d}$.

- Conjecture: $h = 2d$ and $v(d) \approx d^{-cd}$.
- J, Naszódi '22: $h = 3d + 1$ and $v(d) \geq d^{-cd^2}$

Quantitative Fractional Helly

Sarkar, Xue, Soberón '21

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
a positive fraction of the $h = \frac{d(d+3)}{2}$-tuples have intersection of volume at least 1.
$\implies$ a positive fraction of the family has intersection of volume at least $v(d) \geq d^{-d}$.

- Conjecture: $h = 2d$ and $v(d) \approx d^{-cd}$.
- J, Naszódi '22: $h = 3d + 1$ and $v(d) \geq d^{-cd^2}$
- J '23: $h = 3d$ and $v(d) \geq d^{-cd^3}$

Quantitative Fractional Helly

Sarkar, Xue, Soberón '21

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
a positive fraction of the $h = \frac{d(d+3)}{2}$-tuples have intersection of volume at least 1.
$\implies$ a positive fraction of the family has intersection of volume at least $v(d) \geq d^{-d}$.

- Conjecture: $h = 2d$ and $v(d) \approx d^{-cd}$.
- J, Naszódi '22: $h = 3d + 1$ and $v(d) \geq d^{-cd^2}$
- J '23: $h = 3d$ and $v(d) \geq d^{-cd^3}$

Frankl, J, Tomon '25

$h = 2d$ and $v(d) \geq d^{-cd}$

Quantitative Fractional Helly

Sarkar, Xue, Soberón '21

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
a positive fraction of the $h = \frac{d(d+3)}{2}$-tuples have intersection of volume at least 1.
$\implies$ a positive fraction of the family has intersection of volume at least $v(d) \geq d^{-d}$.

- Conjecture: $h = 2d$ and $v(d) \approx d^{-cd}$.
- J, Naszódi '22: $h = 3d + 1$ and $v(d) \geq d^{-cd^2}$
- J '23: $h = 3d$ and $v(d) \geq d^{-cd^3}$

Frankl, J, Tomon '25

$h = 2d$ and $v(d) \geq d^{-cd}$

Frankl, J, Tomon '25

$h = d + 1$ and $v(d) > 0$

Axis parallel boxes

Helly for boxes

$\mathcal{B}$: finite family of axis parallel boxes in $\mathbb{R}^d$.
If any two boxes intersect,
then $\cap\mathcal{B} \neq \emptyset$.

Axis parallel boxes

Helly for boxes

$\mathcal{B}$: finite family of axis parallel boxes in $\mathbb{R}^d$.
If any two boxes intersect,
then $\cap\mathcal{B} \neq \emptyset$.

- Proof: project to each coordinate axis, use Helly for intervals.

Axis parallel boxes

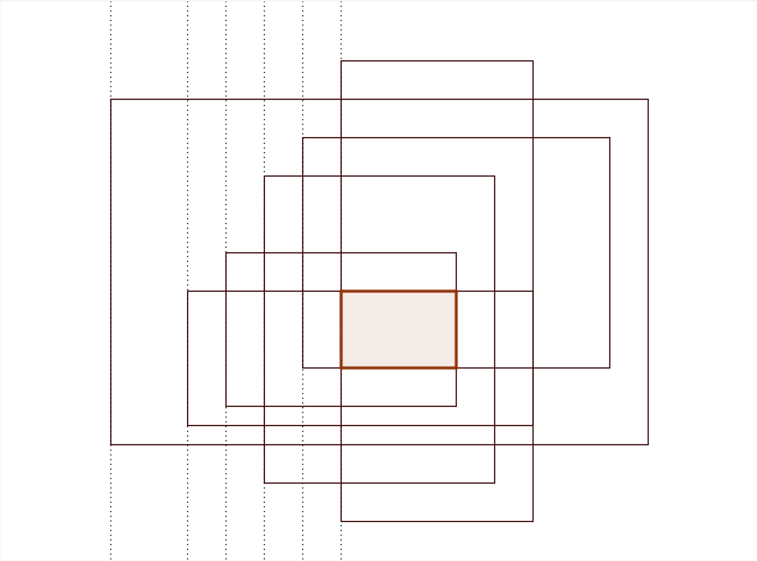
Helly for boxes

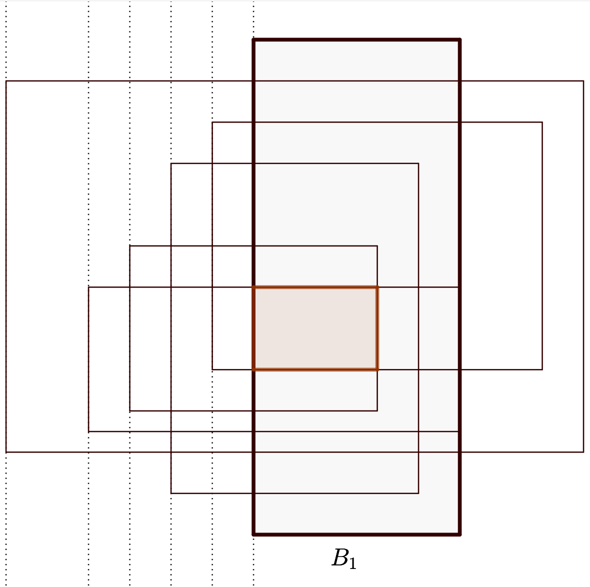
$\mathcal{B}$: finite family of axis parallel boxes in $\mathbb{R}^d$.
If any two boxes intersect,
then $\cap\mathcal{B} \neq \emptyset$.

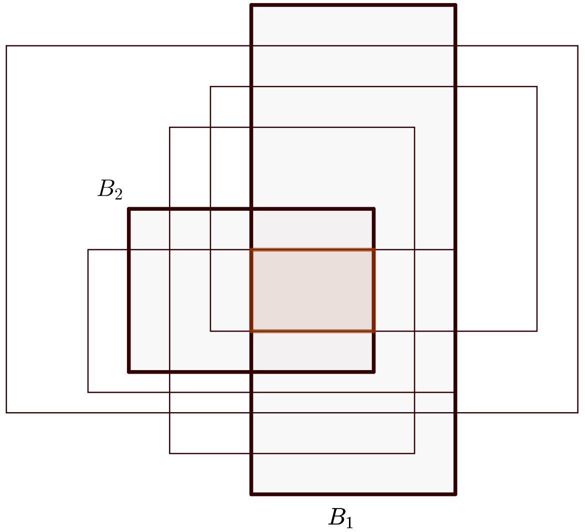
- Proof: project to each coordinate axis, use Helly for intervals.

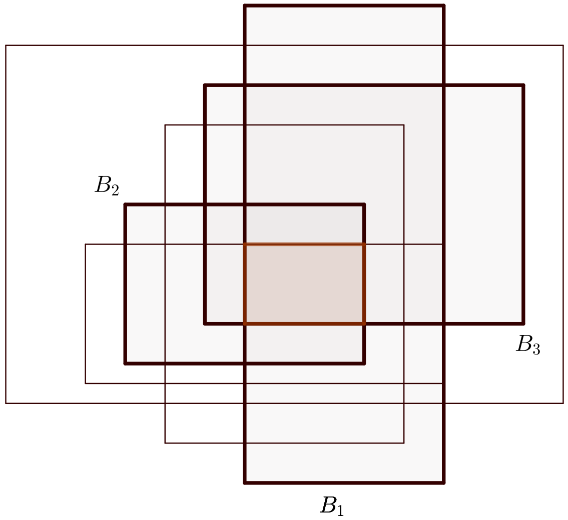
Quantitative Helly for boxes

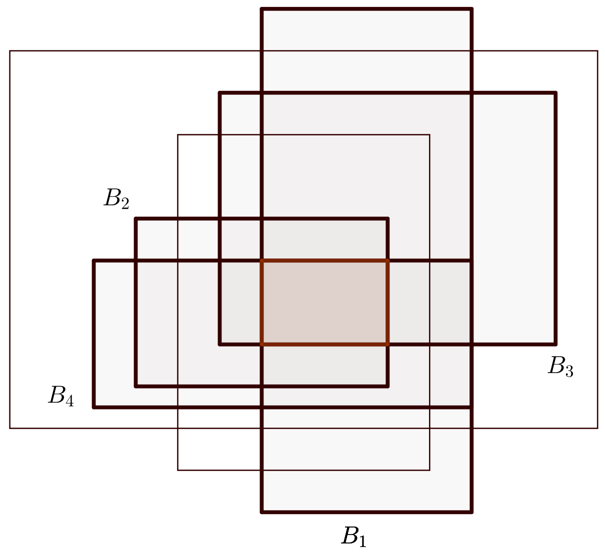
$\mathcal{B}$: finite family of axis parallel boxes in $\mathbb{R}^d$.
If any $2d$ boxes have intersection of volume at least 1,
then $\cap\mathcal{B}$ is of volume at least 1.




B_1


B_2
B_1


B_2
B_3
B_1


B_2
B_3
B_4
B_1

Helly for Monotone Properties of Boxes

Lemma

Any finite family $\mathcal{B}$ of boxes in $\mathbb{R}^d$ contains a subfamily $\mathcal{B}' \subseteq \mathcal{B}$ of at most $2d$ boxes, such that $\cap\mathcal{B} = \cap\mathcal{B}'$.

Helly for Monotone Properties of Boxes

Lemma

Any finite family $\mathcal{B}$ of boxes in $\mathbb{R}^d$ contains a subfamily $\mathcal{B}' \subseteq \mathcal{B}$ of at most $2d$ boxes, such that $\cap\mathcal{B} = \cap\mathcal{B}'$.

- Property P is monotone if $B \in P, B \subset B' \implies B' \in P$.

Helly for Monotone Properties of Boxes

Lemma

Any finite family $\mathcal{B}$ of boxes in $\mathbb{R}^d$ contains a subfamily $\mathcal{B}' \subseteq \mathcal{B}$ of at most $2d$ boxes, such that $\cap\mathcal{B} = \cap\mathcal{B}'$.

- Property P is monotone if $B \in P, B \subset B' \implies B' \in P$.
- $\mathcal{B}$ is P-intersecting, if $\cap\mathcal{B} \in P$.

Helly for Monotone Properties of Boxes

Lemma

Any finite family $\mathcal{B}$ of boxes in $\mathbb{R}^d$ contains a subfamily $\mathcal{B}' \subseteq \mathcal{B}$ of at most $2d$ boxes, such that $\cap\mathcal{B} = \cap\mathcal{B}'$.

- Property P is monotone if $B \in P, B \subset B' \implies B' \in P$.
- $\mathcal{B}$ is P-intersecting, if $\cap\mathcal{B} \in P$.

Corollary (Frankl, J '25+)

Let P be any property of boxes. If every subfamily of $\mathcal{B}$ of size $2d$ is P-intersecting, then $\mathcal{B}$ is P-intersecting.

Helly for Monotone Properties of Boxes

Lemma

Any finite family $\mathcal{B}$ of boxes in $\mathbb{R}^d$ contains a subfamily $\mathcal{B}' \subseteq \mathcal{B}$ of at most $2d$ boxes, such that $\cap\mathcal{B} = \cap\mathcal{B}'$.

- Property P is monotone if $B \in P, B \subset B' \implies B' \in P$.
- $\mathcal{B}$ is P-intersecting, if $\cap\mathcal{B} \in P$.

Corollary (Frankl, J '25+)

Let P be any property of boxes. If every subfamily of $\mathcal{B}$ of size $2d$ is P-intersecting, then $\mathcal{B}$ is P-intersecting.

- Halman '08 and Edwards-Soberón '25: when $P =$ containing n points from a given finite set

Fractional Helly for Monotone Properties of Boxes

Frankl, J '25+

$\mathcal{B}$: a finite family of boxes in $\mathbb{R}^d$.
P : a monotone property.
If a positive fraction of $(d+1)$-tuples is P-intersecting,
$\implies$ $\mathcal{B}$ has a P-intersecting subfamily of linear size.

Fractional Helly for Monotone Properties of Boxes

Frankl, J '25+

$\mathcal{B}$: a finite family of boxes in $\mathbb{R}^d$.
P : a monotone property.
If a positive fraction of $(d+1)$-tuples is P-intersecting,
$\implies$ $\mathcal{B}$ has a P-intersecting subfamily of linear size.

- $P =$ volume at least 1

Fractional Helly for Monotone Properties of Boxes

Frankl, J '25+

$\mathcal{B}$: a finite family of boxes in $\mathbb{R}^d$.
P : a monotone property.
If a positive fraction of $(d+1)$-tuples is P-intersecting,
$\implies$ $\mathcal{B}$ has a P-intersecting subfamily of linear size.

- $P =$ volume at least 1
- $P =$ contains $\geq n$ points from $S \subseteq \mathbb{R}^d$ (Edwards, Soberón '25)

Fractional Helly for Monotone Properties of Boxes

Frankl, J '25+

$\mathcal{B}$: a finite family of boxes in $\mathbb{R}^d$.
P : a monotone property.
If a positive fraction of $(d+1)$-tuples is P-intersecting,
$\Longrightarrow$ $\mathcal{B}$ has a P-intersecting subfamily of linear size.

- $P =$ volume at least 1
- $P =$ contains $\geq n$ points from $S \subseteq \mathbb{R}^d$ (Edwards, Soberón '25)
- $P =$ perimeter, diameter, area of projection, or height is at least t, etc.

Fractional Helly for Monotone Properties of Boxes

Frankl, J '25+

$\mathcal{B}$: a finite family of boxes in $\mathbb{R}^d$.
P : a monotone property.
If a positive fraction of $(d+1)$-tuples is P-intersecting,
$\implies$ $\mathcal{B}$ has a P-intersecting subfamily of linear size.

- $P =$ volume at least 1
- $P =$ contains $\geq n$ points from $S \subseteq \mathbb{R}^d$ (Edwards, Soberón '25)
- $P =$ perimeter, diameter, area of projection, or height is at least t, etc.

Fractional Helly for Monotone Properties of Boxes

Frankl, J '25+

$\mathcal{B}$: a finite family of boxes in $\mathbb{R}^d$.
P : a monotone property.
If a positive fraction of $(d+1)$-tuples is P-intersecting,
$\implies$ $\mathcal{B}$ has a P-intersecting subfamily of linear size.

- $P =$ volume at least 1
- $P =$ contains $\geq n$ points from $S \subseteq \mathbb{R}^d$ (Edwards, Soberón '25)
- $P =$ perimeter, diameter, area of projection, or height is at least t, etc.

Thank you for your attention!

Manuscripts

- Jung, Pálvölgyi (2024+). *k-dimensional transversals for fat convex sets.* SoCG 2025
- Jung, Pálvölgyi (2024+). *A note on infinite versions of (p, q)-theorems.* Eurocomb 2025
- Frankl, Jung (2025+). *Helly-type theorems for monotone properties of boxes.* Eurocomb 2025

Papers

1. Jung, Naszódi (2022). *Quantitative Fractional Helly and (p, q)-theorems*. European J. of Comb.
2. Jung (2023). *Shadow ratio of hypergraphs with bounded degree*. Graphs and Comb.
3. Ambrus, Balko, Frankl, Jung, Naszódi (2024). *On Helly numbers of exponential lattices*. European J. of Comb.
4. Barát, Grzesik, Jung, Nagy, Pálvölgyi (2024). *The double Hall property and cycle covers in bipartite graphs.* Disc. Math.
5. Jung (2024). *Quantitative Helly-type Theorems via Hypergraph Chains.* Electronic J. of Comb.
6. Frankl, Jung, Tomon (2025). *The Quantitative Fractional Helly theorem*. Israel J. of Math.