Infinite versions of (p, q)-theorems

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The hypergraph $\mathcal{K}_d$

- Let $\mathcal{K}_d$ be the hypergraph whose vertices are the compact convex sets in $\mathbb{R}^d$.
- Edges represent intersecting families of convex sets.
- This edge set is **downwards closed**.

Helly's theorem

Theorem (Helly, 1923)

Let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^d$. If every $d+1$ of them intersect, then $\cap\mathcal{F} \neq \emptyset$.

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If $S \subset V(\mathcal{K}_d)$, and $\mathcal{K}_d^{(d+1)}[S]$ is a clique, then $S \in \mathcal{K}_d$.

- $V(\mathcal{H})$: vertex set of hypergraph $\mathcal{H}$.
- $\mathcal{H}^{(q)}$: q-uniform part — edges with exactly q vertices.
- $\mathcal{H}[S]$: subhypergraph induced by $S \subset V(\mathcal{H})$.

The Alon–Kleitman (p, q)-theorem

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Theorem (Alon and Kleitman, 1992)

For every $p \geq d+1$, there exists $C < \infty$ such that:
If $S \subset V(\mathcal{K}_d)$ and $\mathcal{K}_d^{(d+1)}[S]$ has no independent set of size p, then S can be covered with C edges of $\mathcal{K}_d$.

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Fractional Helly theorem

Theorem (Katchalski and Liu, 1979)

If $S \subset V(\mathcal{K}_d)$ is finite and

$$e(\mathcal{K}_d^{(d+1)}[S]) \geq \alpha \binom{|S|}{d+1}$$

for some $\alpha > 0$, then there exists an edge of $\mathcal{K}_d[S]$ of size $\beta|S|$, where $\beta = \beta(\alpha, d) > 0$.

- $e(\mathcal{H})$: number of edges.

Fractional Helly property (general form)

Definition

A q-uniform (possibly infinite) hypergraph $\mathcal{H}$ satisfies the *fractional Helly property* if: For all $\alpha > 0$ there exists $\beta > 0$ such that for every finite $S \subset V(\mathcal{H})$ with

$$e(\mathcal{H}[S]) \geq \alpha \binom{|S|}{q},$$

$\mathcal{H}[S]$ contains a q-uniform clique of size $\beta|S|$.

- Katchalski, Liu '79: $\mathcal{K}_d^{(d+1)}$ satisfies the fractional Helly property.

The hypergraph $\mathcal{B}_{d,k}$

- Vertices: compact balls in $\mathbb{R}^d$.
- Edges: families of balls that can be pierced by a single k-flat.

Theorem (Keller and Perles, 2022)

If $S \subset V(\mathcal{B}_{d,k})$ and $\mathcal{B}_{d,k}^{(k+2)}[S]$ has no infinite independent set, then S can be covered with finitely many edges of $\mathcal{B}_{d,k}$.

Our main result

- **Alon–Kleitman type hypergraph:** $\exists q \forall p \geq q \exists C < \infty$ such that if $\mathcal{H}^{(q)}[S]$ has no independent set of size p, then S can be covered with at most C edges of $\mathcal{H}$.
- **Keller–Perles type hypergraph:** If $\mathcal{H}^{(q)}[S]$ has no infinite independent set, then S can be covered with finitely many edges of $\mathcal{H}$.

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Theorem

If a q-uniform hypergraph satisfies the fractional Helly property and has arbitrarily large finite independent sets, then it has an infinite independent set.

Corollaries

Theorem

Let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^d$. If among every $\aleph_0$ members of $\mathcal{F}$ some $d+1$ are intersecting, then all the members of $\mathcal{F}$ can be pierced by finitely many points.

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Theorem (Chackraborty, Ghosh, Nandi '24)

Let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^d$. If among every $\aleph_0$ members of $\mathcal{F}$ some $d+1$ can be pierced by a hyperplane, then all the members of $\mathcal{F}$ can be pierced by finitely many hyperplanes.

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Theorem

Let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^d$. If among every $\aleph_0$ members of $\mathcal{F}$ some $d+1$ contain a point in their intersection with integer coordinates, then all the members of $\mathcal{F}$ can be pierced by finitely many points with integer coordinates.

Proof sketch

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1. **Fractional Helly property:** If a uniform hypergraph has edge density at least $\alpha > 0$ on a large vertex set, then it contains a clique on a β-fraction of its vertices (for some $\beta = \beta(\alpha) > 0$).

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3. Take an increasing sequence of finite independent sets $S_1 \subset S_2 \subset \dots$ with $|S_n| \to \infty$.
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4. Apply Ramsey-type arguments to find a large *homogeneous* subhypergraph: $S'_1 \subset S_{i_1}, \; S'_2 \subset S_{i_2}, \; \dots$
5. Use (2) to conclude that $\bigcup_i S'_i$ spans an independent set.

Key lemma

Let $\mathcal{H}$ be a k-uniform hypergraph with disjoin vertex sets $V_1, V_2, \ldots, V_n, \ldots \subset V(\mathcal{H})$ with $|V_i| \to \infty$.

Lemma

We can find subsets $V_i' \subset V_i$ with $w_n = \max\{|V_i| : i \leq n\} \to \infty$ and the following property.
If $i_1 < i_2 < \ldots < i_k$ and $v_j \in V_{i_j}$,
then $\{v_1, \ldots, v_k\} \in \mathcal{H}$ depends only on v_1.

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Thank you!