

A K-theoretic approach to the representation theory of finite groups

Attilio Castano

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One of the main players in this note is the spectrum KU . In order to make this note more accessible, we begin with some background material about spectra. We will not provide a rigorous definition of a spectrum (in the sense of homotopy theory), but I hope that the following discussion will be enough for the reader to understand the rest of the notes.

Definition 1. A spectrum behaves very much like a chain complex of abelian groups. Some properties that the category of spectra enjoys are the following

1. One can ask for the homotopy group of a spectrum E , which are denoted by $\pi_i(E)$. And they should be thought as analogs of the cohomology groups of a spectrum.
2. Given two spectrum, E_1 and E_2 one can form a direct sum $E_1 \oplus E_2$, and a tensor product $E_1 \otimes E_2$.
3. Given a map of spectra $E_1 \rightarrow E_2$ one can take the kernel and cokernel of this map.
4. Each spectrum, determines a collection of functors

$$E^i : \text{Spc}^{\text{op}} \rightarrow \text{Ab} \tag{1}$$

which one should think about as determining a cohomology theory in the category of spaces. In particular we have the identity

$$E^i(\text{pt}) = \pi_i(E) \tag{2}$$

5. One can also consider ring objects in the category of spectra. In the case that E admits a sort of ring structure, we can amalgamate all cohomology groups $E^i(X)$ into a single ring

$$E^* : \text{Spc}^{\text{op}} \rightarrow \text{Rings} \quad X \mapsto E^*(X) \tag{3}$$

Example 2. A familiar example is given by the spectrum $H\mathbf{Q}$, which following (4), determined a cohomology theory

$$H\mathbf{Q}^i : \text{Spc}^{\text{op}} \rightarrow \text{Ab} \quad X \mapsto H^i(X, \mathbf{Q}) \tag{4}$$

that can be identified with singular cohomology. Moreover, since \mathbf{Q} is not just an abelian group but a ring, we can conclude that $H\mathbf{Q}$ admits the structure of a ring spectrum, and then following (5), it determined a multiplicative cohomology theory

$$H\mathbf{Q}^* : \text{Spc}^{\text{op}} \rightarrow \text{Rings} \quad X \mapsto H^*(X, \mathbf{Q}) \tag{5}$$

A ring spectrum that will be of central importance to us is the ring spectrum KU , which goes by the name of complex K-theory. In order to get acquainted more with this ring spectrum a good first step is to ask what are its homotopy groups, indeed we have that

$$\pi_i(KU) = KU^i(\text{pt}) = \begin{cases} \mathbf{Z} & \text{if } i \text{ is even} \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

and since it has a ring structure, we can organize all this cohomology groups into a ring graded ring

$$\pi_*(KU) = KU^*(\text{pt}) = \mathbf{Z}[\beta^{\pm}] \quad |\beta| = 2 \quad (7)$$

In particular, we can see that this ring spectrum has a property called 2-periodic which will be of central importance for us. It is closely related to the following theorem, which goes by the name of Bott periodicity.

Proposition 3. For any space X , the Bott element β induces an isomorphism

$$\beta : KU^i(X) \xrightarrow{\cong} KU^{i+2}(X) \quad (8)$$

In order to relate KU to the representation theory of finite groups, it will be convenient to have the following description of $KU^0(X)$. We consider the category of complex vector bundles on a space X , and denote it by $\text{Vect}(X)$. The category $\text{Vect}(X)$ comes equipped with a direct sum operation, so we can organize the isomorphism classes of objects of $\text{Vect}(X)$ into a monoid. Elements of this monoids are denoted by $[V]$, where V is a complex vector bundle over X , and with sum given by

$$[V_1 \oplus V_2] = [V_1] + [V_2] \quad (9)$$

We temporarily denote this monoid by the letter $M(X)$. Notice, however, that $KU^0(X)$ is at least an abelian group, and not just a monoid, so in order to relate this two objects we will proceed by transforming M to an abelian group in a universal way. Recall that there exists a forgetful functor $\text{ComGrp} \rightarrow \text{Mon}$ from abelian groups to monoids, and this functor admits a left adjoint

$$-^{gp} : \text{Mon} \longrightarrow \text{ComGrp} \quad (10)$$

called group completion. One then has the following result

Proposition 4. There exists an isomorphism

$$M(X)^{gp} \cong KU^0(X) \quad (11)$$

Moreover, $M(X)^{gp}$ inherits a ring structure coming from the tensor product of vector bundles, which under the above isomorphism, coincides with the ring structure of $KU^0(X)$.

Example 5. Using the results above we can recover the calculation that $KU^0(\text{pt}) = \mathbf{Z}$. Simply notice that over a point there is only one vector bundle of each non-negative dimension. Then one has to observe that if V_i is the vector bundle of dimension i over a point, then $V_i \oplus V_j = V_{i+j}$ and that $V_i \otimes V_j = V_{ij}$.

Now we will show how to extract a representation of a group G , out of vector bundles on a certain space BG . For simplicity, consider the additive group \mathbf{Z} , then the associated space is $B\mathbf{Z} = S^1$. Fix a distinguished point $\text{pt} \rightarrow B\mathbf{Z}$, then out of a vector bundle $V \rightarrow B\mathbf{Z}$ we can extract a complex vector space V_{pt} , by looking at the fiber at the distinguished point of the map $V \rightarrow B\mathbf{Z}$. Then, moving our vector bundle $V \rightarrow B\mathbf{Z}$ around the loop, gives us an automorphism $V_{\text{pt}} \rightarrow V_{\text{pt}}$, which corresponds to an action of $1 \in \mathbf{Z}$ on V_{pt} . In other words, it is possible to extract a \mathbf{Z} representation out of a vector bundle $V \rightarrow B\mathbf{Z}$. Granted, \mathbf{Z} is not a finite group, but it seems good for pedagogical reasons.

Question 6. Can we obtain all complex representations of a finite group G , from complex vector bundles on BG ?

The answer to this question is negative, but the complex representations of G and complex vector bundles over BG are intimately related, by the following theorem.

Theorem 7. Let G be a finite group, and let $I_G \subset \text{Rep}(G)$ be the augmentation ideal, defined as the kernel of the ring homomorphism

$$\text{Rep}(G) \rightarrow \mathbf{Z} \quad [V] \mapsto \dim_{\mathbf{C}}(V) \quad (12)$$

The Atiyah-Segal comparison map

$$\zeta : \text{Rep}(G) \longrightarrow \text{KU}^0(BG) \quad (13)$$

exhibits $\text{KU}^0(BG)$ as the I_G completion of $\text{Rep}(G)$.

Definition 8. Let $\text{Rep}(G)$ be the ring of complex representations of the finite group G . As an abelian group it is the abelian group generated by the symbols $[V]$, where V is a finite dimensional complex representation of G , subject to the relations

$$[V] = [V_1] + [V_2] \quad (14)$$

for any isomorphism $V \cong V_1 \oplus V_2$. So in other words is the free abelian group of finite rank generated by $[W]$, where W is a finite dimensional irreducible representation. In order to consider $\text{Rep}(G)$ as a ring, we declare that multiplication is given by the rule

$$[V_1] \cdot [V_2] = [V_1 \otimes_{\mathbf{C}} V_2] \quad (15)$$

We would like to highlight that even though the representations we are considering are complex representations, i.e. V is a vector space over \mathbf{C} , the ring $\text{Rep}(G)$ is most naturally regarded as a \mathbf{Z} algebra.

In this notes, the goal is to provide a direct K-theoretic construction of the ring $\text{Rep}(G)$. This means that we will try to explain how to construct a "decompleted" version of KU , which we will denote by $\underline{\text{KU}}$, that satisfies

$$\underline{\text{KU}}^0(BG) \cong \text{Rep}(G) \quad (16)$$

equipped with a natural map

$$\underline{\text{KU}}^0(BG) \longrightarrow \text{KU}^0(BG) \quad (17)$$

which recovers the Atiyah-Segal comparison map. But before delving into the theory, let me explain some applications. Recall that in particular, the cohomology theory $\underline{\text{KU}}$ determines a functor

$$\underline{\text{KU}}^0 : \text{Spc}^{\text{op}} \rightarrow \text{Ab} \quad (18)$$

Remark 9. A reader familiar with the theory of tempered cohomology would realize that Spc is very much not the domain of the functor $\underline{\text{KU}}$, but rather one needs to renormalize the category of spaces, such that it is not generated by a single point pt, but rather by spaces of the form BM , where M is a finite abelian group. However, we choose to ignore this issue in this section, hoping it will not cause too much confusion. We will try to come back to this later.

Then, for any inclusion of finite groups $H \subset G$, the functoriality of $\underline{\mathbf{KU}}$ recovers the restriction map

$$\begin{array}{ccc} \underline{\mathbf{KU}}^0(BG) & \longrightarrow & \underline{\mathbf{KU}}^0(BH) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{Rep}(G) & \xrightarrow{\mathrm{Res}} & \mathrm{Rep}(H) \end{array} \quad (19)$$

However, $\underline{\mathbf{KU}}$ comes equipped with extra functoriality, it also comes equipped with transfer maps $\underline{\mathbf{KU}}(BH) \rightarrow \underline{\mathbf{KU}}(BG)$, which fit into the following commutative diagram

$$\begin{array}{ccc} \underline{\mathbf{KU}}^0(BH) & \longrightarrow & \underline{\mathbf{KU}}^0(BG) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{Rep}(H) & \xrightarrow{\mathrm{Ind}} & \mathrm{Rep}(G) \end{array} \quad (20)$$

More informally, we can say that the transfer maps $\underline{\mathbf{KU}}^0(BH) \rightarrow \underline{\mathbf{KU}}^0(BG)$ coincide with induction from the classical representation theory of finite groups. And finally, we can recover a variant of the Artin-Brauer induction theorem

Theorem 10. For any finite group G , the ring $\mathbf{C} \otimes_{\mathbf{Z}} \underline{\mathbf{KU}}^0(BG) = \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{Rep}(G)$ can be generated as a \mathbf{C} vector space by the image of the induction maps

$$\mathbf{C} \otimes_{\mathbf{Z}} \underline{\mathbf{KU}}^0(BT) \xrightarrow{\mathrm{Ind}} \mathbf{C} \otimes_{\mathbf{Z}} \underline{\mathbf{KU}}^0(BG) \quad (21)$$

for all maps $BT \rightarrow BG$, where T is a finite cyclic group.

This naturally leads to the following question

Question 11. Can one extend such induction results to a certain kind of infinite groups like $\mathrm{Gal}(\mathbf{Q})$? Or the infinite symmetric group S_{∞} ? As stated, the answer to this question seems to be negative, but maybe, by somehow incorporating the profinite topology we could make this work.

In order to achieve this goal, we will make use of Lurie's theory of tempered cohomology. A key ingredient of the theory of tempered cohomology is the close relationship between 2-periodic cohomology theories, like \mathbf{KU} , and the geometry of p -divisible groups. In particular, we hope to provide an algebro-geometric explanation for the failure of the Atiyah-Segal comparison map to be an isomorphism.

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We begin by studying a p -complete version of \mathbf{KU} , namely

$$\mathbf{KU}_{\widehat{p}} = \lim(\cdots \rightarrow \mathbf{KU}/p^n \rightarrow \mathbf{KU}/p^{n-1} \rightarrow \cdots \rightarrow \mathbf{KU}/p) \quad (22)$$

To make sense of this operations, recall that \mathbf{KU} is a spectrum, which is a homotopical analog of an abelian group. Therefore there exists a multiplication by p^n map $\mathbf{KU} \rightarrow \mathbf{KU}$, and the category of spectra behaves in some ways like an abelian category, in particular it has a notion of cokernel, so one can realize

$$\mathbf{KU}/p^n = \mathrm{coker}(\mathbf{KU} \xrightarrow{\times p^n} \mathbf{KU}) \quad (23)$$

Moreover, we still have that

$$\mathbf{KU}_{\widehat{p}}^*(\mathrm{pt}) = \mathbf{Z}_p[\beta^{\pm}] \quad |\beta| = 2 \quad (24)$$

In particular, $KU_{\widehat{p}}^*(pt)$ has infinitely many non-zero cohomology groups, and enjoys a form of bott periodicity. The bott periodicity property is essential to many of the favorable properties that $KU_{\widehat{p}}$ has.

Our goal now is to explain a relation between the deformation problem of a the p-divisible group $(\mu_{p^\infty}, \text{Spec } \mathbf{F}_p)$, and the spectrum $KU_{\widehat{p}}$. Naturally, we begin introducing the p-divisible group μ_{p^∞} , which will play a central role in this notes.

Later, we will provide a more detailed definition of p-divisible groups, but for now we will content with the following discussion. A p-divisible group, is in particular a group scheme, and as such it admits the following descriptions

1. The functor of points perspective, $\mu_{p^\infty} : \text{Aff}_{\text{Spec } \mathbf{F}_p}^{\text{op}} \rightarrow \text{Ab}$, defined by

$$\mu_{p^\infty}(\text{Spec } R) = \{x \in R, x^{p^m} = 1 \text{ for } m \gg 0\} \quad (25)$$

2. This functor is represented by an ind-affine scheme

$$\mu_{p^\infty} = \text{colim } \text{Spec } \mathbf{F}_p[t]/(t^{p^m} - 1) = \text{colim } \text{Spec } \mathbf{F}_p[t]/(t - 1)^{p^m} \quad (26)$$

Notice how in the second equality we are using in an essential way the fact that we are working over the field \mathbf{F}_p .

However μ_{p^∞} is not only a group scheme, but also what is called a p-divisible group, so let me tell a bit about what a p-divisible group is. A p-divisible group is a special kind of group scheme, which satisfies the following property: to any group scheme we can ask for its p^n torsion points

$$\mu_{p^\infty}[p^n] = \text{Ker}(\mu_{p^\infty} \xrightarrow{\times p^n} \mu_{p^\infty}) = \text{Spec } \mathbf{F}_p[t]/(t^{p^n} - 1) \quad (27)$$

then, we say that the group scheme is a p-divisible group if it is completely determined by its p-torsion points, namely it satisfies the identity

$$\mu_{p^\infty} = \text{colim} \left(\mu_{p^\infty}[p] \hookrightarrow \mu_{p^\infty}[p^2] \hookrightarrow \dots \right) \quad (28)$$

And such that all the p-torsion parts are finite flat group schemes. In our case this is fine as $\mu_{p^\infty}[p^n] = \mu_{p^n}$ is a finite flat group scheme.

Remark 12. It will be pretty important for us to consider the p-divisible group μ_{p^∞} over more general bases, an important one for us will be $\text{Spec } \mathbf{Z}_p$. A similar definition of μ_{p^∞} will work in this more general situations, but an important difference that could be useful to mention is that, as opposed to the situation over $\text{Spec } \mathbf{F}_p$, over $\text{Spec } \mathbf{Z}_p$ the p-divisible group will not be connected.

- Draw a picture of μ_{p^∞} over \mathbf{F}_p
- Draw picture of μ_{p^∞} over $\text{Spec } \mathbf{Z}_p$.

Maybe more seriously, is the fact that we will need to consider p-divisible groups over ring spectra like $KU_{\widehat{p}}$. There are some technical issues one has to deal with here, but since $\pi_0(KU_{\widehat{p}}) = \mathbf{Z}_p$, it could be convenient to visualize μ_{p^∞} over $KU_{\widehat{p}}$ as if it lived over $\text{Spec } \mathbf{Z}_p$.

Now, we will start talking about deformations of p-divisible groups, as we want to describe $KU_{\widehat{p}}$ as the oriented deformation space of the p-divisible group $(\mu_{p^\infty}, \text{Spec } \mathbf{F}_p)$.

Definition 13. Let \mathcal{M}_p be the moduli space of p -divisible groups over $\mathrm{Spec} \mathbf{Z}$, that is, it is given by a functor

$$\mathcal{M}_p : \mathrm{Aff}_{\mathrm{Spec} \mathbf{Z}}^{\mathrm{op}} \longrightarrow \mathrm{Spc} \quad (29)$$

which maps $\mathrm{Spec} R \mapsto \{\text{the set of } p\text{-divisible groups over } \mathrm{Spec} R\}$. We warn the reader that this object is not representable by a scheme, or even artin stack, it is a more subtle object. But, we know that giving a map $\mathrm{Spec} \mathbf{F}_p \rightarrow \mathcal{M}_p$ corresponds to specifying a p -divisible group over $\mathrm{Spec} \mathbf{F}_p$. In particular, the p -divisible group $(\mu_{p^\infty}, \mathrm{Spec} \mathbf{F}_p)$ determines a map $\mathrm{Spec} \mathbf{F}_p \rightarrow \mathcal{M}_p$.

Recall that we want to understand deformations of our p -divisible group $(\mu_{p^\infty}, \mathrm{Spec} \mathbf{F}_p)$, for us, this means that we want to understand the completion of \mathcal{M}_p along the map $\mathrm{Spec} \mathbf{F}_p \rightarrow \mathcal{M}_p$. This may seem weird, since as we said before \mathcal{M}_p is not representable in any way, however, there is still a way of making sense of this.

Proposition 14. Then there is a result of Lubin and Tate that tells us what the deformation space of $(\mu_{p^\infty}, \mathrm{Spec} \mathbf{F}_p)$, it is $\mathrm{Spec} \mathbf{Z}_p$.

So now by taking the completion we have a map $\mathrm{Spec} \mathbf{Z}_p \rightarrow \mathcal{M}_p$, which is given by the p -divisible group $(\mu_{p^\infty}, \mathrm{Spec} \mathbf{Z}_p)$. However, there is an important distinction between what happened in characteristic p , and over $\mathrm{Spec} \mathbf{Z}_p$. Our p -divisible group is no longer connected

- Draw picture of μ_{p^∞} over $\mathrm{Spec} \mathbf{Z}_p$.

In fact, we have the following result of Deligne

Proposition 15. All finite flat group schemes are etale over $\mathrm{Spec} \mathbf{Q}$. In particular, $\mu_{p^\infty} \times \mathrm{Spec} \mathbf{Q}_p$ is etale.

Remark 16. Notice that μ_{p^∞} is not finite, so it does not directly fit into the setting of the result above, however, μ_{p^∞} is built out of finite flat group schemes, later we will take a perspective on p -divisible groups that emphasizes the perspective that it is built out of finite flat group schemes.

At the beginning of this sections, we said we wanted to realize $\mathrm{KU}_{\widehat{p}}$ as an oriented deformation space of $(\mu_{p^\infty}, \mathrm{Spec} \mathbf{F}_p)$, but so far all we have gotten is $\mathrm{Spec} \mathbf{Z}_p$, granted we haven't said anything about orientations. But notice that \mathbf{Z}_p and $\mathrm{KU}_{\widehat{p}}$ are not so distinct, recall that there is an embedding

$$\mathrm{Rings} \hookrightarrow \mathrm{ComAlg}(\mathrm{Spctr}) \quad R \mapsto HR \quad (30)$$

where HR is the Eilenberg-MacLane spectrum associated to R , which represents singular cohomology with coefficients in the rings R , in particular we have that

$$H\mathbf{Z}_p^*(\mathrm{pt}) = \mathbf{Z}_p \quad \text{concentrated in degree zero} \quad (31)$$

which is pretty close to

$$\mathrm{KU}_{\widehat{p}}(\mathrm{pt}) = \mathbf{Z}_p[\beta^{\pm}] \quad |\beta| = 2 \quad (32)$$

So in some sense, one can think of $\mathrm{KU}_{\widehat{p}}$ as a 2-periodic refinement of \mathbf{Z}_p . And this is precisely how one obtains a universal deformation ring

1. One first deforms its p -divisible group in the classical sense. In our case we have $(\mu_{p^\infty}, \mathrm{Spec} \mathbf{F}_p)$, which we deformed to $(\mu_{p^\infty}, \mathrm{Spec} \mathbf{Z}_p)$.

2. One then forces the classical deformation ring to satisfy a form of Bott periodicity, which can be informally described by adding a certain invertible element β in degree two, we will call this element the Bott element. In our specific example, this amounts to passing from \mathbf{Z}_p to $\mathbf{Z}_p[\beta^\pm]$.

The existence of such a Bott element, is exactly what it amounts for a ring spectrum E to be oriented, it must come equipped with a certain symmetry

$$\beta : E \xrightarrow{\simeq} E[2] \quad (33)$$

which in particular implies that for any space X we have a form of Bott periodicity

$$\beta : E^i(X) \xrightarrow{\simeq} E^{i+2}(X) \quad (34)$$

The notion of orientation is rather opaque, at least for me, and it seems like somewhat of a miracle that it is so ubiquitous. However, even if one does not understand what an orientation is, one can still appreciate some properties that it enjoys. We will specialize these properties to our example of $\mathrm{KU}_{\widehat{p}}$ as the oriented deformation ring of $(\mu_{p^\infty}, \mathrm{Spec} \mathbf{F}_p)$.

1. It still contains the information of the classical deformation ring. It can be recovered as $\mathbf{Z}_p = \mathrm{KU}_{\widehat{p}}^0(\mathrm{pt})$.
2. Moreover, $\mathrm{KU}_{\widehat{p}}$ knows about the p -divisible group μ_{p^∞} over $\mathrm{Spec} \mathbf{Z}_p$, namely we have that

$$\mathrm{Spec} \left(\mathrm{KU}_{\widehat{p}}^0(B\mathbf{Z}/p^n) \right) = \mu_{p^\infty}[p^n] \quad (35)$$

One can even recover the multiplicative structure, however this is a bit more subtle as one really needs to work with spectra and not just its homotopy groups.

Remark 17. It is important to remember that in this section we have only discussed about a specific p -divisible group μ_{p^∞} , which happens to be connected over $\mathrm{Spec} \mathbf{F}_p$. This connectivity condition is essential, and will play a crucial role in the Atiyah-Segal completion theorem.

Finally, let's conclude this section discussing exactly how much of the representation theory of G can $\mathrm{KU}_{\widehat{p}}$ see. Recall that we have a morphism

$$\zeta : \mathrm{Rep}(G) \longrightarrow \mathrm{KU}_{\widehat{p}}^0(BG) \quad (36)$$

and we notice that $\mathrm{Rep}(G)$ is an algebra over \mathbf{Z} , while $\mathrm{KU}_{\widehat{p}}^0(BG)$ is a \mathbf{Z}_p algebra, therefore the Atiyah-Segal comparison map factors as

$$\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G) \xrightarrow{\zeta} \mathrm{KU}_{\widehat{p}}^0(BG) \quad (37)$$

One can then ask the following question, for which G is this an isomorphism? The Atiyah-Segal comparison map is an isomorphism in this case if and only if G is a p -group.

We would like to provide an algebro-geometric explanation for this, relying on the geometry of the p -divisible group μ_{p^∞} over $\mathrm{KU}_{\widehat{p}}$. Yes, we have not provided a definition of μ_{p^∞} over general ring spectra, but let's just assume that there exists a good notion of p -divisible groups over rings like $\mathrm{KU}_{\widehat{p}}$. To any p -divisible group \mathbb{G} , and any abelian group M , one can ask for the M torsion points, which we denote by $\mathbb{G}[M]$.

Example 18. Some relevant examples for us are the following

1. $\mu_{p^\infty}[\mathbf{Z}/p^n] = \mu_{p^\infty}[p^n]$
2. $\mu_{p^\infty}[\mathbf{Z}/l^n] = \emptyset$

And moreover, when one asks for $\mathrm{KU}_{\widehat{p}}(BM)$ for a finite abelian group M , one is really asking for the M torsion points of $\mu_{p^\infty}[M]$ over $\mathrm{KU}_{\widehat{p}}$. We hope this provides an geometric explanation for the failure of $\mathrm{KU}_{\widehat{p}}$ to see the whole representation of G .

3

In the last section, we saw that at the core of the failure of the Atiyah-Segal comparison map to be an isomorphism

$$\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G) \xrightarrow{\zeta} \mathrm{KU}_{\widehat{p}}^0(BG) \quad (38)$$

was the fact that there is a natural p -divisible group μ_{p^∞} associated to $\mathrm{KU}_{\widehat{p}}$, and that this p -divisible group, only has p -torsion points. Which in turn implied that $\mathrm{KU}_{\widehat{p}}$ could only detect the representation theory of the p -singular subgroup $G^{(p)} \subset G$, that is, the subgroup of elements $g \in G$ such that $g^{p^n} = 1$.

In this section, we will define a tempered version of $\mathrm{KU}_{\widehat{p}}$, which we will denote by $\underline{\mathrm{KU}}_{\widehat{p}}$, in which satisfies the following identity

$$\underline{\mathrm{KU}}_{\widehat{p}}(BM) = \bigoplus_{p \in P} \mu_{p^\infty}[M] = \mu_{\mathbf{P}^\infty}[M] \quad (39)$$

namely, instead of just recording the M torsion points of μ_{p^∞} , for a single prime p , it records the M torsion points for all primes p . This marks a departure from the kind of p -divisible groups that can be recovered from $\mathrm{KU}_{\widehat{p}}$, namely the p -divisible group $\mu_{\mathbf{P}^\infty}$ is not connected, as opposed to μ_{p^∞} .

The goal of this section is to provide some background in order to understand the definition of $\underline{\mathrm{KU}}_{\widehat{p}}$. We begin by introducing the notion of a \mathbf{P} -divisible group, which is a generalization of the classical notion of p -divisible group, which works for all p , simultaneously, an example of this kind of object will be $\mu_{\mathbf{P}^\infty}$.

Definition 19. A \mathbf{P} -divisible group over a ring A , is functor

$$X : \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \longrightarrow \mathrm{FiniteFlatGroupSchemes}_{/\mathrm{Spec} A} \quad (40)$$

which satisfies certain conditions

- The commutative finite flat group scheme $X(0) = A$
- The functor is symmetric monoidal, that is, the group structure of $X(M)$ is completely determined by the group structure of M . So in some sense, it is redundant to require that the functor lands in finite flat group schemes, as finite flat algebras will suffice
- The functor X is exact, that is, for a short exact sequence of finite abelian groups $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$, the induced diagram

$$\begin{array}{ccc} X(M_3) & \longrightarrow & X(M_2) \\ \downarrow & & \downarrow \\ X(0) & \longrightarrow & X(M_1) \end{array} \quad (41)$$

is a exact sequence of finite flat group schemes

- Moreover, one can define a notion of heigh. We say that a \mathbf{P} -divisible group has height h , if for every finite abelian group M , the finite flat group scheme $X(M)$ has degree $|M|^h$ over A . Where $|M|$ is the cardinality of M .

Remark 20. This definitions even make sense over a ring spectrum, like $\mathrm{KU}_{\widehat{\mathbf{p}}}$. This will be of central importance to us, since to construct $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}$, we will need to consider \mathbf{P} -divisible groups over $\mathrm{KU}_{\widehat{\mathbf{p}}}$. However, since this may seem to abstract, and the reader might enjoy having some pictures to visualize what is happening, since $\pi_0(\mathrm{KU}_{\widehat{\mathbf{p}}}) = \mathbf{Z}_p$, it is reasonable to imagine that a \mathbf{P} -divisible group over $\mathrm{KU}_{\widehat{\mathbf{p}}}$ looks like a \mathbf{P} -divisible group over \mathbf{Z}_p .

Example 21. A central example of a \mathbf{P} -divisible group, will be

$$\mu_{\mathbf{P}^\infty}[M] := \bigoplus_{p \in \mathbf{P}} \mu_{p^\infty}[M] \quad (42)$$

In order to get a bit more familiar with this notion, notice that any finite abelian group M can be factored into its "prime pieces"

$$M = \bigoplus_{p \in \mathbf{P}} \left(\bigoplus \mathbf{Z}/p^n \right) \quad (43)$$

and then one gets that

$$\mu_{\mathbf{P}^\infty}[M] = \bigoplus_{p \in \mathbf{P}} \left(\bigoplus \mu_{p^\infty}[\mathbf{Z}/p^n] \right) \quad (44)$$

We would now like to use this example to further explain why $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}$ is able to see all the representation theory of a group finite abelian group M . We will come back to the general case for non-abelian G later. Recall that we have been saying that to our cohomology theories, namely, $\mathrm{KU}_{\widehat{\mathbf{p}}}$ and $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}$, they come equipped with a \mathbf{P} -divisible group

1. $\mathrm{KU}_{\widehat{\mathbf{p}}}$ comes equipped with μ_{p^∞} . That means that for a finite abelian group M , we can recover the M -torsion points $\mu_{p^\infty}[M]$ simply by computing the cohomology $\mathrm{KU}_{\widehat{\mathbf{p}}}(BM)$.
2. $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}$ comes equipped with $\mu_{\mathbf{P}^\infty}$. That means that for a finite abelian group M , we can recover the M -torsion points $\mu_{\mathbf{P}^\infty}[M]$ simply by computing the cohomology $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}(BM)$.

If one is not comfortable with \mathbf{P} -divisible groups over ring spectra like $\mathrm{KU}_{\widehat{\mathbf{p}}}$, one can extract, in both situations, a \mathbf{P} -divisible group over \mathbf{Z}_p simply by taking the 0th cohomology $\mathrm{KU}_{\widehat{\mathbf{p}}}^0(BM)$ or $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}^0(BM)$.

Then one can clearly see why $\mathrm{KU}_{\widehat{\mathbf{p}}}^0(BM)$ is only able to see the p -singular subgroup $M^{(p)} \subset M$, its associated p -divisible group can only see p -power torsion elements, while $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}$ is able to see p -power torsion for all $p \in \mathbf{P}$, as it comes equipped with the \mathbf{P} -divisible group $\mu_{\mathbf{P}^\infty}$.

We would now want to explain what kind of object this tempered cohomology $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}$ is. As a warm up, we start with a discussion about $\mathrm{KU}_{\widehat{\mathbf{p}}}$. Recall that $\mathrm{KU}_{\widehat{\mathbf{p}}}$ is a ring spectrum, in particular this means that it determines a functor

$$\mathrm{KU}_{\widehat{\mathbf{p}}} : \mathrm{Spc}^{\mathrm{op}} \longrightarrow \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathbf{p}}}} \quad X \mapsto \mathrm{KU}_{\widehat{\mathbf{p}}}(X) \quad (45)$$

Moreover, this functor preserves limits. In particular, this implies that the cohomology of any space X , is completely determined by the cohomology of a point pt . An important property of the category of spaces Spc , in which X lives, is that

$$X = \mathrm{colim}_X \mathrm{pt} \quad (46)$$

In other words, we can construct X , as the colimit of the constant diagram of shape X . One then has the following property of a spectrum

$$\mathrm{KU}_{\widehat{\mathbf{p}}}(X) = \mathrm{KU}_{\widehat{\mathbf{p}}}(\mathrm{colim}_X \mathrm{pt}) = \lim_X \mathrm{KU}_{\widehat{\mathbf{p}}}(\mathrm{pt}) = \lim_X \mathrm{KU}_{\widehat{\mathbf{p}}} \quad (47)$$

that is, the cohomology of X is completely determined by the cohomology of a point. So in some sense, knowing the cohomology of, say, BM is in no way extra data to $\mathrm{KU}_{\widehat{\mathbf{p}}}$. And this perspective seems to be at the core of the definition of $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}$, we want that the cohomology of BM to be extra data coming from the \mathbf{P} -divisible group $\mu_{\mathbf{P}^\infty}$.

In order to do this, we will define $\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}$ to be a functor from the category \mathcal{T} , where the objects are spaces of the form BM , where M is a finite abelian group. In formulas can be written as

$$\underline{\mathrm{KU}}_{\widehat{\mathbf{p}}} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{ComAlg}_{\mathrm{KU}_p} \quad BM \mapsto \mu_{\mathbf{P}^\infty}[M] = \underline{\mathrm{KU}}_{\widehat{\mathbf{p}}}(BM) \quad (48)$$

In order to better understand this construction, it will be important to review some categorical constructions.

Construction 22. To any (small) category \mathcal{C} , there is a way in which one can construct a "free closure under colimits" of \mathcal{C} , which we denote by $\widehat{\mathcal{C}}$.