Higher Algebraic Structures

Attilio Castano

Cartesian and coCartesian Fibrations

Consider a functor of ordinary categories $F : \widetilde{\mathcal{C}} \to \mathcal{C}$. An arrow $\widetilde{y} \to \widetilde{z}$ is called cartesian if for every diagram (of solid arrows) as below, there exists a unique map $\widetilde{x} \to \widetilde{y}$ that makes the diagram commute.

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[dd, "F"'] & \widetilde{x} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[dd, mapsto] & & \\
& & \widetilde{y} \arrow[r] \arrow[dd, mapsto] & \widetilde{z} \arrow[dd, mapsto] \\
\mathcal{C} & x \arrow[dr] \arrow[drr, bend left] & & \\
& & y \arrow[r] & z
\end{tikzcd} \tag{1}$$

Notice that the upper triangle is happening in the category $\widetilde{\mathcal{C}}$ and the lower triangle is happening in $\mathcal{C}$. And the arrows between the objects of different categories correspond to the map induced by the functor $F : \widetilde{\mathcal{C}} \to \mathcal{C}$.

The morphism $\widetilde{y} \to \widetilde{z}$ can be thought of as the base change of $y \to z$ along $\widetilde{z}$. The base change $\widetilde{y} \to \widetilde{z}$ is unique up to unique isomorphisms. As this will be important for us, I would like to highlight that this implies making a choice when talking about "the base change". On the other hand, the morphism $\widetilde{y} \to \widetilde{z}$ being cartesian is a property of the morphism and it does not involve any choice.

Since we will be interested in the higher categorical case, lets rewrite this definition in a way appropriate for the formalism of ∞-categories.

Definition 1. Let $F : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a functor of ∞-categories. We say that a morphism $\widetilde{y} \to \widetilde{z}$ in $\widetilde{\mathcal{C}}$ is Cartesian over $\mathcal{C}$ if for every $\widetilde{x} \in \widetilde{\mathcal{C}}$, the map

$$\mathrm{Map}_{\widetilde{\mathcal{C}}}(\widetilde{x}, \widetilde{y}) \to \mathrm{Map}_{\widetilde{\mathcal{C}}}(\widetilde{x}, \widetilde{z}) \times_{\mathrm{Map}_{\mathcal{C}}(x,z)} \mathrm{Map}_{\mathcal{C}}(x, y) \tag{2}$$

is an isomorphism in Spc. After staring at this definition for a bit you will realize that it coincides with the definition above. The main difference is that we now have to take into account the higher morphisms.

Definition 2. A functor $F : \widetilde{\mathcal{C}} \to \mathcal{C}$ of ∞-categories is said to be a Cartesian fibration if for every $y \to z$ in $\mathcal{C}$ and every $\widetilde{z}$ in $\widetilde{\mathcal{C}}$ there exists a Cartesian morphism $\widetilde{y} \to \widetilde{z}$. In other words, for every $y \to z$ in $\mathcal{C}$ and every $\widetilde{z}$ in $\widetilde{\mathcal{C}}$ there exists a base change.

Definition 3. The category $\mathrm{Cart}_{/\mathcal{C}}$ has as objects cartesian fibrations $\widetilde{\mathcal{C}} \to \mathcal{C}$ of ∞-categories, and as morphisms diagrams of the form

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{D}} \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd} \tag{3}$$

that preserve cartesian fibrations. In order to define the higher morphisms in $\mathrm{Cart}_{/\mathcal{C}}$ we consider it as the 1-full subcategory of $\mathrm{Cat}_{/\mathcal{C}}$, which means that we specify the objects and the one morphisms, and then everything that connects them.

So why do we care about the category $\mathrm{Cart}_{/\mathcal{C}}$? Lets motivate this by presenting some classical material. For the next paragraph we will be working with ordinary categories. Recall the following classical result

Theorem 4. *There exists an equivalence of ordinary categories*

$$\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \simeq \mathrm{Cart}^{\mathrm{Set}}_{/\mathcal{C}} \tag{4}$$

where $\mathrm{Cart}^{\mathrm{Set}}_{/\mathcal{C}} \subset \mathrm{Cart}_{/\mathcal{C}}$ *is the full subcategory spanned by objects* $\widetilde{\mathcal{C}} \to \mathcal{C}$ *where the fibers are sets.*

Explicitly, for a Cartesian fibration $\widetilde{\mathcal{C}} \to \mathcal{C}$, the value of the corresponding functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ on $c \in \mathcal{C}$ equals the fiber $\widetilde{\mathcal{C}}_c$ of $\widetilde{\mathcal{C}}$ over c. And the morphism $c_0 \to c_1$ induces a map of sets $\widetilde{\mathcal{C}}_{c_1} \to \widetilde{\mathcal{C}}_{c_0}$, where $\widetilde{c_1} \in \widetilde{\mathcal{C}}_{c_1}$ is mapped to its base change $\widetilde{c_0} \in \widetilde{\mathcal{C}}_{c_0}$. Notice that there is no need to make choices of pullback, since the Cartesian fibration $\widetilde{\mathcal{C}} \to \mathcal{C}$ is fibered on sets, the choice of a pullback is unique, and not only unique up to unique isomorphism. The other direction is similar, given a functor $\varphi : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ we construct a Cartesian fibration $\widetilde{\mathcal{C}} \to \mathcal{C}$ where the fiber over c corresponds to the set where $\varphi(c)$, and the maps $\varphi(c_1) \to \varphi(c_0)$ induce the corresponding Cartesian arrows.

However, things become more complicated for a Cartesian fibration $\widetilde{\mathcal{C}} \to \mathcal{C}$ fibered over groupoids, as pullbacks are now only defined up to unique isomorphisms. Lets consider an explicit example to understand what I am trying to say

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[d] & \widetilde{y_1} \simeq \widetilde{y_0} \arrow[r] \arrow[d, mapsto] & \widetilde{z} \arrow[d, mapsto] \\ \mathcal{C} & y \arrow[r] & z \end{tikzcd} \tag{5}$$

It is clear to which groupoids the induced functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Groupoid}$ will map its objects. The more difficult question is what is the induced map

$$\{\widetilde{z}\} \longrightarrow \{\widetilde{y_1} \simeq \widetilde{y_0}\} \tag{6}$$

Of course there are only two options, and both options are isomorphic. However, we notice that we are being forced to make a choice, at the very core of the definition of a functor we find ourselves being forces to specify an image of $\widetilde{z}$. This becomes more prominent in more complicated situations, as the pullbacks we choose might not be coherent with one another, therefore our induces functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Groupoid}$ will only commute up to some natural isomorphism.

When defining the higher algebraic structures, the advantages of this formalism will become more apparent. Lets now return to the world of higher categories. The analog in higher category theory of the previous theorem is the following

Theorem 5. *There is a canonical equivalence of* ∞*-categories*

$$\mathrm{Cart}_{/\mathcal{C}} \simeq \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}) \tag{7}$$

where Cat *is the* ∞*-category of* ∞*-categories.*

Consider a functor of ordinary categories $F : \widetilde{\mathcal{C}} \to \mathcal{C}$. An arrow $\widetilde{y} \to \widetilde{z}$ is called coCartesian if for every diagram (of solid arrows) as below, there exists a unique map $\widetilde{z} \to \widetilde{x}$ that makes the diagram commute.

$$\begin{tikzcd}
 & & & & \widetilde{x} \arrow[dd, mapsto] \\
\widetilde{\mathcal{C}} \arrow[dd, "F"'] & & \widetilde{y} \arrow[r] \arrow[dd, mapsto] \arrow[urr, bend left] & \widetilde{z} \arrow[ur, dashed] \arrow[dd, mapsto] & \\
 & & & & x \\
\mathcal{C} & & y \arrow[r] \arrow[urr, bend left] & z \arrow[ur] &
\end{tikzcd} \tag{8}$$

and Ill leave it to the reader to extrapolate the definition of a coCartesian fibration $\widetilde{C} \to \mathcal{C}$, and the definition of the ∞-category $\mathrm{coCart}_{/\mathcal{C}}$. However, let me record the following theorem

Theorem 6. *There is a canonical equivalence of ∞-categories*

$$\mathrm{coCart}_{/\mathcal{C}} \simeq \mathrm{Funct}(\mathcal{C}, \mathrm{Cat}) \tag{9}$$

where Cat *is the ∞-category of ∞-categories.*

Finally, we will use this formalism to provide a nice and clean definition of adjoint functors. Let $F : \mathcal{C}_0 \to \mathcal{C}_1$ be a functor of ∞-categories. We can view F as a functor $[1] \to \mathrm{Cat}$, and using the previous theorem, this corresponds to a coCartesian fibration

$$\widetilde{\mathcal{C}} \to [1] \tag{10}$$

We shal say that F admits a right adjoint if it above functor is a biCartesian fibration, i.e., if it happens to be a Cartesian and coCartesian fibration. Applying the previous theorem, it gives us a functor $[1]^{\mathrm{op}} \to \mathrm{Cat}$, which corresponds to the right adjoint $F^R : \mathcal{C}_1 \to \mathcal{C}_0$. Staring at the following diagram for a bit

$$\begin{tikzcd}
c_0 \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, mapsto] & & & & & c_1 \arrow[ddl, mapsto] \\
 & F^R(c_1) \arrow[r, dashed] \arrow[d, mapsto] & c_1 \arrow[d, mapsto] & c_0 \arrow[r, dashed] \arrow[d, mapsto] \arrow[urr, bend left] & F(c_0) \arrow[ur, dashed] \arrow[d, mapsto] & \\
 & 0 \arrow[r] & 1 & 0 \arrow[r] & 1 &
\end{tikzcd} \tag{11}$$

we obtain the usual equivalence

$$\mathrm{Map}_{\mathcal{C}_0}(c_0, F^R(c_1)) \simeq \mathrm{Map}_{\mathcal{C}_1}(F(c_0), c_1) \tag{12}$$

This perspective has been very useful for me, as it allows me to this about morphisms between objects on different categories, which I have found useful when thinking about adjunction.

Monoidal ∞-Categories

In this section we will introduce the notion of a monoidal ∞-category. The idea is simple: a monoidal category will be encoded by a functor $\Delta^{\mathrm{op}} \to \mathrm{Cat}$. You may recall that when defining the axioms of a monoidal category we usually have all this ugly diagrams which must commute, all of this is nicely encoded in this functor.

Definition 7. A monoidal ∞-category is a functor

$$\mathcal{A}^{\otimes} : \Delta^{\mathrm{op}} \to \mathrm{Cat} \tag{13}$$

subject to the following conditions

- $\mathcal{A}^{\otimes}([0]) = *$
- For any n, the functor, given by the n-tuple of maps in Δ

$$[1] \to [n] \qquad 0 \mapsto i, 1 \mapsto i+1 \tag{14}$$

  defines an equivalence

$$\mathcal{A}^{\otimes}([n]) \to \mathcal{A}^{\otimes}([1]) \times \cdots \times \mathcal{A}^{\otimes}([1]) \tag{15}$$

  in other words this maps correspond to the projections $\mathcal{A}^{\otimes}([1]) \times \cdots \times \mathcal{A}^{\otimes}([1]) \to \mathcal{A}^{\otimes}([1])$.

If $\mathcal{A}^{\otimes}$ is a monoidal ∞-category, we shall denote by $\mathcal{A}$ the underlying ∞-category, i.e., $\mathcal{A}^{\otimes}([1])$. The map

$$[1] \to [2] \qquad 0 \mapsto 0, 1 \mapsto 2 \tag{16}$$

defines a functor

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A} \tag{17}$$

This functor is the monoidal operation on $\mathcal{A}$, correspond to $\mathcal{A}^{\otimes}$. And the map $[1] \to [0]$ defined a functor $* \to \mathcal{A}$; the corresponding object is the unit of the monoidal structure $1_{\mathcal{A}} \in \mathcal{A}$.

We will now put the power of coCartesian fibrations to use, in order to define algebra objects in an monoidal ∞-category. We have a fully faithful functor

$$\mathrm{Cat}^{\mathrm{Mon}} \hookrightarrow \mathrm{coCart}_{/\Delta^{\mathrm{op}}} \tag{18}$$

and the essential image is singled out by the conditions of a monoidal category described above.

Remark 8. The definition we have provided above is that of associative monoidal category, and the category Δ^{op} is an example of an operad. In order to define a symmetric monoidal category we must replace Δ^{op} by Fin_* the category of pointed sets, and a slight modifications the conditions mentioned in the definition. For the sake of time, I will not define this category. But for the purposes of this talk, what I have mentioned is enough.

Example 9. Any ∞-category $\mathcal{C}$ that admits cartesian products has a canonically defined (symmetric) monoidal structure, where the monoidal operation is defined by

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad (c_0, c_1) \to c_0 \times c_1 \tag{19}$$

and dually if $\mathcal{C}$ admits coproducts there is a canonical (symmetric) monoidal structure. In this latter situation, there is an equivalence of categories between the category of commutative algebra objects on $\mathcal{C}$ and $\mathcal{C}$, which says that ever object $c \in \mathcal{C}$ has a uniquely defined structure of commutative algebra given by $c \sqcup c \to c$.

We now define what it means to have a commutative algebra object on a monoidal ∞-category $\mathcal{A}$.

Definition 10. Given a monoidal ∞-category $\mathcal{A}^{\otimes}$, an associative algebra is a right-lax monoidal functor

$$*^{\otimes} \to \mathcal{A}^{\otimes} \tag{20}$$

over Δ^{op}, where $*^{\otimes}$ can be described as the constant functor $\Delta^{\mathrm{op}} \to *$. But more importantly, what is a right-lax monoidal functor? The idea is that the morphism $*^{\otimes} \to \mathcal{A}^{\otimes}$ of coCartesian fibrations over Δ^{op} need not preserve all coCartesian arrows, as it was the case before, the only coCartesian arrows it must preserve are the ones corresponding to the projections $\mathcal{A}^{\otimes}([n]) \simeq \mathcal{A}^{\otimes}([1]) \times \cdots \times \mathcal{A}^{\otimes}([1]) \to \mathcal{A}^{\otimes}([1])$.

A natural question to ask is: why is it natural to require right-lax monoidal functors? Lets work out an example to understand this better. Let $\mathrm{Spc}^{\otimes}$ be the monoidal ∞-category of spaces, endowed with the Cartesian monoidal structure, i.e., the monoidal operation is given by $(X, Y) \to X \times Y$. Recall that, an associative algebra object X must has a monoidal operation $X \times X \to X$, lets see if we are able to write this down in our framework: the monoidal operations should correspond to a morphism $(X, X) \to X$ as described in the diagram below

$$\begin{tikzcd} & & X \arrow[dl] \\ (X,X) \arrow[r] \arrow[urr, bend left] \arrow[d, mapsto] & X \times X \arrow[ur, dashed] \arrow[d, mapsto] & \\ 2 \arrow[r] & 1 & \end{tikzcd} \tag{21}$$

However, this morphisms is not coCartesian, and indeed it factors through $X \times X \to X$. Being right lax-monoidal provides us this flexibility to have morphisms between the fibers of $\mathrm{Spc}^{\otimes} \to \Delta^{\mathrm{op}}$ which are not coCartesian.

Ill conclude this section with the following cool result, which roughly says that a commutative monoid in a stable ∞-category is also a commutative group.

Lemma 11. *Let $\mathcal{C}$ be a stable ∞-category, endowed with the Cartesian/coCartesian (they both coincide as it is a stable category) monoidal structure, the inclusion*

$$\mathrm{ComGrp}(\mathcal{C}) \hookrightarrow \mathrm{ComMonoid}(\mathcal{C}) \tag{22}$$

is an equivalence.

Lurie Tensor Product

Denote by StCat the ∞-category of stable cocomplete categories, and we only consider the colimit preserving functors between them. We will endow StCat with a symmetric monoidal structure, given by a tensor product of stable categories. In the process we will find the most basic stable category, that of spectra, which will be the unit object under this tensor product.

For a pairs of stable categories $\mathcal{C}_1$ and $\mathcal{C}_2$ and a third one $\mathcal{D}$, the space of exact continuous functors

$$\mathcal{C}_1 \otimes \mathcal{C}_2 \to \mathcal{D} \tag{23}$$

is the full subspace in

$$\mathrm{Map}_{\mathrm{Cat}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \tag{24}$$

that consist of functors that are exact and continuous in each variable. This operation endows StCat with a symmetric monoidal structure. The corresponding monoidal operation

$$(\mathcal{C}_i)_{i \in I} \mapsto \otimes_{i \in I} \mathcal{C}_i \tag{25}$$

is the Lurie tensor product.

Definition 12. The category Spctr of spectra can be defined as the unit object in the symmetric monoidal ∞-category StCat. Moreover, using this tensor product we can consider the commutative algebra objects in StCat, this endows the category of spectra with a symmetric monoidal structure. The sphere spectrum is the unit object in Spctr.

Definition 13. An $\mathbb{E}_\infty$ ring spectrum is an algebra object in the symmetric monoidal category Spctr.

Lets take a small detour and talk about colimits. Let $X \in \mathrm{Spc}$, we can also consider X as a category itself, and define the constant functor valued in $*$

$$X \to \mathrm{Spc} \tag{26}$$

then the colimit of this functor is exactly X. More generally, let $\mathcal{C}$ be an arbitrary ∞-category, and consider any functor $\mathcal{C} \to \mathrm{Spc}$, what is the colimit? Recall that we can associate to it a coCartesian fibration

$$\widetilde{\mathcal{C}} \to \mathcal{C} \tag{27}$$

and recall that in some sense $\widetilde{\mathcal{C}}$ totalizes the image of the functor. Then one can proof that the colimit of $\mathcal{C} \to \mathrm{Spc}$ is isomorphic to the geometric realization $|\widetilde{\mathcal{C}}|$. I would like to use this to emphasize that a colimit in the world of ∞-categories behaves like it remembers the whole diagram.

Let $\mathcal{C}$ be any ∞-category, then the category of colimit preserving functors $\mathrm{Spc} \to \mathcal{C}$ is isomorphic to $\mathcal{C}$, as the image is completely determined by the image of $*$. Therefore sending $* \to \mathbb{S}$ completely determines an colimit preserving functor

$$\Sigma^\infty : \mathrm{Spc} \to \mathrm{Spctr} \tag{28}$$

and this functor has the following universal property.

Lemma 14. *For any $\mathcal{C} \in \mathrm{StCat}$, there is an equivalence of categories between $\mathrm{Funct}_{\mathrm{StCat}}(\mathrm{Spctr}, \mathcal{C})$ and the full subcategory of $\mathrm{Funct}(\mathrm{Spc}, \mathcal{C})$ consisting of colimit preserving functors.*

The above lemma expresses the universal property of the category Spctr as the stabilization of Spc. In here, we are implicitly using the fact that any spectrum can be obtained as a colimit of a constant diagram $\mathbb{S}$.

Ill conclude the talk by stating two very cool theorems.

Theorem 15. *There exists an equivalence of categories*

$$\mathrm{Spctr}^{\leq 0} \simeq \mathrm{ComGrp(Spc)} \tag{29}$$

Theorem 16. *If R is a ring of characteristic zero, then there exists an equivalence of categories between commutative differential graded algebras over R, and $\mathbb{E}_\infty$-ring spectrum over HR.*