# On a theorem of Beauville-Laszlo

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**Abstract**

We discuss a theorem of Beauville-Laszlo which roughly states that given a scheme $X/S$ and an effective Cartier divisor $Z \hookrightarrow X$, we can "factor" $X$ as $X_{\hat{Z}}$ and $X \setminus Z$. As an application we provide an explicit construction of the line bundle associated with the effective Cartier divisor $Z$.

## Line bundles on a curve

Let $X/\operatorname{Spec} k$ be a smooth curve over a field $k$. A line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ is a quasicoherent sheaf such that there exists a Zariski open cover $\{i : U_i \to X\}$ where

$$i^*\mathcal{L} \cong \mathcal{O}_{U_i} \tag{1}$$

i.e. $\mathcal{L}$ is locally trivial in the Zariski topology. On the other hand, we can construct a line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ by gluing local data. Explicitly, this means that given a Zariski open cover $\{U_i \to X\}$, and line bundles $\mathcal{L}_i \in \operatorname{Pic}(U_i)$ together with transition maps

$$\varphi_{ij} : \mathcal{L}_i|_{U_i \cap U_j} \to \mathcal{L}_j|_{U_i \cap U_j} \tag{2}$$

which satisfy the cocycle condition, that is, we have the following equality

$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij} \tag{3}$$

Then there is a unique line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ such that restriction along $U_i \to X$ gives us $\mathcal{L}|_{U_i} \cong \mathcal{L}_i$, together with the extra compatibility. Succinctly this means that

**Proposition 1.** Pic forms a stack on the Zariski topology of $X$

However, this approach as am important downside, open sets $U \subset X$ are huge, they cover everything but a finite set of points. This can become cumbersome if we are trying to construct an explicit line bundle, like

$$\mathcal{O}_X(-v) := \{\text{rational functions of } X \text{ with at least a zero of order one at } v \in X\} \tag{4}$$

$$\mathcal{O}_X(v) := \{\text{rational functions of } X \text{ with at most a pole of order one at } v \in X\} \tag{5}$$

and it is not exactly clear how one can give decent datum in a Zariski cover of $X$ to construct $\mathcal{O}_X(-v)$. The theorem of Beauville-Laszlo will provide us with a more efficient procedure to construct this line bundles.

For future reference we record the following result

**Proposition 2.** Every line bundle of our curve $X$ can be realized as

$$\mathcal{O}_X\Big(\sum \eta_v v\Big) \tag{6}$$

for some finite sum $\sum \eta_v v$.

## Beauville-Laszlo Theorem

Before providing a precise statement of the theorem of Beauville-Laszlo let me first give an example how one can use it to construct $\mathcal{O}_X(-v)$ using the theorem.

Let $D_v := X_{\hat{v}}$ be the completion of $X$ at $v$. Non-canonically we have the identification $D_v \cong \operatorname{Spec} k[[\pi]]$. Then we have the following cartesian square

$$\begin{tikzcd} \operatorname{Spec} k((\pi)) \arrow[r, "\cong"] & D_v^\circ \arrow[r] \arrow[d] & X \setminus v \arrow[d] \\ \operatorname{Spec} k[[\pi]] \arrow[r, "\cong"] & D_v \arrow[r] & X \end{tikzcd} \tag{7}$$

the statement of Beauville Laszlo is saying that to construct a line bundle on $X$, it suffices to give line bundles $\mathcal{L}_v \in \operatorname{Pic}(D_v)$ and $\mathcal{L}_{X\setminus v} \in \operatorname{Pic}(X \setminus v)$ together with an isomorphism

$$\mathcal{L}_v|_{D_v^\circ} \cong \mathcal{L}_{X\setminus v}|_{D_v^\circ} \tag{8}$$

that is, the maps $D_v \to X$ and $X \setminus v \to X$ behave like a cover of $X$, for the purpose of constructing line bundles on $X$.

**Example 3**. We will now use Beauville-Laszlo to construct our line bundle $\mathcal{O}_X(-v)$. For this we begin with the data of the trivial line bundle $\mathcal{O}_X$, then by passing to the cover $D_v \to X$ and $X \setminus v \to X$, the trivial line bundles gives us the data of the trivial line bundle $\mathcal{O}_{D_v}$ on $D_v$ and the trivial line bundle $\mathcal{O}_{X\setminus v}$ on $X \setminus v$, together with the identity morphism

$$\mathrm{Id} : \mathcal{O}_{D_v}|_{D_v^\circ} \to \mathcal{O}_{X\setminus v}|_{D_v^\circ} \tag{9}$$

We can now construct $\mathcal{O}_X(-v)$ from $\mathcal{O}_X$ simply by twisting the isomorphism on $D_v^\circ$ of the trivial line bundle as follows

$$\begin{tikzcd} \mathcal{O}_{D_v}|_{D_v^\circ} \arrow[r] \arrow[d, "\cong"] & \mathcal{O}_{D_v}|_{D_v^\circ} \arrow[r, "\mathrm{Id}"] \arrow[d, "\cong"] & \mathcal{O}_{X\setminus v}|_{D_v^\circ} \\ k((\pi)) \arrow[r, "\times\pi"] & k((\pi)) & \end{tikzcd} \tag{10}$$

We are making the lattice of $k[[\pi]] \subset k((\pi))$ smaller.

Now we are finally ready to state the theorem of Beauville-Laszlo.

**Theorem 4**. Let $X/S$ be a qcqs scheme, and let $Z \hookrightarrow X$ be an effective cartier divisor, that is, a closed subscheme locally cut out by a nonzero divisor. Then

(1) The following cartesian square

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r] \arrow[d] & X_{\hat{Z}} \arrow[d] \\ U \arrow[r] & X \end{tikzcd} \tag{11}$$

is also a pushout square in the category of schemes. Where $U = X \setminus Z$.

(2) We can then apply the functor

$$\mathrm{QCoh}^* : \mathrm{Sch}_{/S}^{\mathrm{op}} \longrightarrow \text{Categories} \tag{12}$$

which maps $f : X \to Y$ to $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$. Then the induced diagram

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r] \arrow[d] & \mathrm{QCoh}(X_{\hat{Z}}) \arrow[d] \\ \mathrm{QCoh}(U) \arrow[r] & \mathrm{QCoh}(\pi^{-1}(U)) \end{tikzcd} \tag{13}$$

is a pullback square.

**Remark 5**. The factorization of $\mathrm{QCoh}(X)$ into $\mathrm{QCoh}(U)$ and $\mathrm{QCoh}(X_{\hat{Z}})$ could be interpreted as a motivic property of $\mathrm{QCoh}^*$.

## Effective Cartier divisors

Let $X/S$ be a qcqs scheme, to any effective Cartier divisor $Z \subset X$ we can associate a line bundle, which we denote by $\mathcal{O}_X(-Z)$, which has a global section whose vanishing locus is exactly $Z$. As an application of the theorem of Beauville-Laszlo we will provide an explicit construction of $\mathcal{O}_X(-Z)$ together with its corresponding global section.

For simplicity, assume that $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/f$ where $f$ is a non-zero divisor on $A$. Consider the following cartesian square

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r] \arrow[d] & X_{\hat{Z}} \arrow[d] \\ U \arrow[r] & X \end{tikzcd} \tag{14}$$

We can then obtain $\mathcal{O}_X(-Z)$ from the trivial line bundle by twisting the isomorphism of the trivial line bundle on $\pi^{-1}(U)$ as follows

$$\begin{tikzcd} \mathcal{O}_{X_{\hat{Z}}}|_{\pi^{-1}(U)} \arrow[r] \arrow[d, "\cong"] & \mathcal{O}_{X_{\hat{Z}}}|_{\pi^{-1}(U)} \arrow[r, "\mathrm{Id}"] \arrow[d, "\cong"] & \mathcal{O}_U|_{\pi^{-1}(U)} \\ A_{\hat{f}}[f^{-1}] \arrow[r, "\times f^{-1}"] & A_{\hat{f}}[f^{-1}] & \end{tikzcd} \tag{15}$$

This construction yields a line bundle on $X$, which we denote by $\mathcal{O}_X(-Z)$. We now need to show that it contains a global section with the desired vanishing locus. Consider $1 \in \mathcal{O}_U$, by the construction above, we notice that it extends to a global section of $\mathcal{O}_X(-Z)$, with the desired vanishing locus.

The general case can be bootstrapped from the simplified case described in the previous paragraph, by first passing to a fine enough Zariski cover of $X$, in which the effective Cartier divisor $Z$ is cut by a global section.

## References


1. Beauville, Laszlo - Un lemme de descente

2. Bhatt - Algebraization and Tannaka duality

3. Katz, Mazur - Arithmetic Moduli of Elliptic Curves