Higher Algebraic Structures

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The goal of this talk is to provide an exposition of Lurie's categorification of algebra into the setting of higher categories. There are two major definitions we will be using here

- The notion of a symmetric monoidal category $(\mathcal{C}, \otimes)$. This is simply a category with an appropriate notion of multiplication operation $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
- Given a monoidal category as above, we will define the notion of a commutative algebra object $\mathrm{ComAlg}(\mathcal{C}, \otimes)$, with respect to a monoidal category. Similarly, one can define commutative group objects.

In order to illustrate this notion, we show how one can recover abelian groups in this context. Let $(\mathrm{Set}, \times)$, be the category of sets, together with a multiplication map $\mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$, defined by the rule $(S_1, S_2) \mapsto S_1 \times S_2$. One can then consider the commutative algebra objects of this category, and in fact we obtain the identification

$$\mathrm{ComGrp}(\mathrm{Set}, \times) \simeq \{\text{Ab Groups}\} \tag{1}$$

In this talk we will develop this notions in the setting of higher categories, and we will use it to explain in which sense spectra is the higher analog of abelian groups. In fact, this is not so difficult to state. Let Spc be the category of spaces, and we endow it with a monoidal structure given by $(X, Y) \mapsto X \times Y$. Then the theory of infinite loop spaces, first developed by May, tells us that

$$\mathrm{ComGrp}(\mathrm{Spc}, \times) \simeq \mathrm{Spctr}^{\leq 0} \tag{2}$$

Where $\mathrm{Spctr}^{\leq 0}$ means connective spectra, as we are using cohomological grading.

Cartesian and coCartesian Fibrations

Consider a functor of ordinary categories $F : \widetilde{\mathcal{C}} \to \mathcal{C}$. An arrow $\widetilde{y} \to \widetilde{z}$ is called cartesian if for every diagram (of solid arrows) as below, there exists a unique map $\widetilde{x} \to \widetilde{y}$ that makes the diagram commute.

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[dd, "F"'] & \widetilde{x} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[dd, mapsto] & & \\
& & \widetilde{y} \arrow[r] \arrow[dd, mapsto] & \widetilde{z} \arrow[dd, mapsto] \\
\mathcal{C} & x \arrow[dr] \arrow[drr, bend left] & & \\
& & y \arrow[r] & z
\end{tikzcd} \tag{3}$$

Notice that the upper triangle is happening in the category $\widetilde{\mathcal{C}}$ and the lower triangle is happening in $\mathcal{C}$. And the arrows between the objects of different categories correspond to the map induced by the functor $F : \widetilde{\mathcal{C}} \to \mathcal{C}$.

The morphism $\widetilde{y} \to \widetilde{z}$ can be thought of as the base change of $y \to z$ along $\widetilde{z}$. The base change $\widetilde{y} \to \widetilde{z}$ is unique up to unique isomorphisms. As this will be important for us, I would like to highlight that this implies making a choice when talking about "the base change". On the other hand, the morphism $\widetilde{y} \to \widetilde{z}$ being cartesian is a property of the morphism and it does not involve any choice.

Since we will be interested in the higher categorical case, lets rewrite this definition in a way appropriate for the formalism of ∞-categories.

Definition 1. Let $F : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a functor of ∞-categories. We say that a morphism $\widetilde{y} \to \widetilde{z}$ in $\widetilde{\mathcal{C}}$ is Cartesian over $\mathcal{C}$ if for every $\widetilde{x} \in \widetilde{\mathcal{C}}$, the map

$$\mathrm{Map}_{\widetilde{\mathcal{C}}}(\widetilde{x}, \widetilde{y}) \to \mathrm{Map}_{\widetilde{\mathcal{C}}}(\widetilde{x}, \widetilde{z}) \times_{\mathrm{Map}_{\mathcal{C}}(x,z)} \mathrm{Map}_{\mathcal{C}}(x, y) \tag{4}$$

is an isomorphism in Spc. After staring at this definition for a bit you will realize that it coincides with the definition above. The main difference is that we now have to take into account the higher morphisms.

Definition 2. A functor $F : \widetilde{\mathcal{C}} \to \mathcal{C}$ of ∞-categories is said to be a Cartesian fibration if for every $y \to z$ in $\mathcal{C}$ and every $\widetilde{z}$ in $\widetilde{\mathcal{C}}$ there exists a Cartesian morphism $\widetilde{y} \to \widetilde{z}$. In other words, for every $y \to z$ in $\mathcal{C}$ and every $\widetilde{z}$ in $\widetilde{\mathcal{C}}$ there exists a base change.

Definition 3. The category $\mathrm{Cart}_{/\mathcal{C}}$ has as objects cartesian fibrations $\widetilde{\mathcal{C}} \to \mathcal{C}$ of ∞-categories, and as morphisms diagrams of the form

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{D}} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd} \tag{5}$$

that preserve cartesian fibrations. In order to define the higher morphisms in $\mathrm{Cart}_{/\mathcal{C}}$ we consider it as the 1-full subcategory of $\mathrm{Cat}_{/\mathcal{C}}$, which means that we specify the objects and the one morphisms, and then everything that connects them.

So why do we care about the category $\mathrm{Cart}_{/\mathcal{C}}$? Lets motivate this by presenting some classical material. For the next paragraph we will be working with ordinary categories. Recall the following classical result

Theorem 4. *There exists an equivalence of ordinary categories*

$$\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \simeq \mathrm{Cart}^{\mathrm{Set}}_{/\mathcal{C}} \tag{6}$$

where $\mathrm{Cart}^{\mathrm{Set}}_{/\mathcal{C}} \subset \mathrm{Cart}_{/\mathcal{C}}$ *is the full subcategory spanned by objects* $\widetilde{\mathcal{C}} \to \mathcal{C}$ *where the fibers are sets.*

Explicitly, for a Cartesian fibration $\widetilde{\mathcal{C}} \to \mathcal{C}$, the value of the corresponding functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ on $c \in \mathcal{C}$ equals the fiber $\widetilde{\mathcal{C}}_c$ of $\widetilde{\mathcal{C}}$ over c. And the morphism $c_0 \to c_1$ induces a map of sets $\widetilde{\mathcal{C}}_{c_1} \to \widetilde{\mathcal{C}}_{c_0}$, where $\widetilde{c_1} \in \widetilde{\mathcal{C}}_{c_1}$ is mapped to its base change $\widetilde{c_0} \in \widetilde{\mathcal{C}}_{c_0}$. Notice that there is no need to make choices of pullback, since the Cartesian fibration $\widetilde{\mathcal{C}} \to \mathcal{C}$ is fibered on sets, the choice of a pullback is unique, and not only unique up to unique isomorphism. The other direction is similar, given a functor $\varphi : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ we construct a Cartesian fibration $\widetilde{\mathcal{C}} \to \mathcal{C}$ where the fiber over c corresponds to the set where $\varphi(c)$, and the maps $\varphi(c_1) \to \varphi(c_0)$ induce the corresponding Cartesian arrows.

However, things become more complicated for a Cartesian fibration $\widetilde{\mathcal{C}} \to \mathcal{C}$ fibered over groupoids, as pullbacks are now only defined up to unique isomorphisms. Lets consider an explicit example to understand what I am trying to say

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[d] & \widetilde{y}_1 \simeq \widetilde{y}_0 \arrow[r] \arrow[d, mapsto] & \widetilde{z} \arrow[d, mapsto] \\ \mathcal{C} & y \arrow[r] & z \end{tikzcd} \tag{7}$$

It is clear to which groupoids the induced functor $\mathcal{C}^{\text{op}} \to \text{Groupoid}$ will map its objects. The more difficult question is what is the induced map

$$\{\widetilde{z}\} \longrightarrow \{\widetilde{y_1} \simeq \widetilde{y_0}\} \tag{8}$$

Of course there are only two options, and both options are isomorphic. However, we notice that we are being forced to make a choice, at the very core of the definition of a functor we find ourselves being forces to specify an image of $\widetilde{z}$. This becomes more prominent in more complicated situations, as the pullbacks we choose might not be coherent with one another, therefore our induces functor $\mathcal{C}^{\text{op}} \to \text{Groupoid}$ will only commute up to some natural isomorphism.

When defining the higher algebraic structures, the advantages of this formalism will become more apparent. The analog in higher category theory of the previous theorem is the following

Theorem 5. *There is a canonical equivalence of ∞-categories*

$$\text{Cart}_{/\mathcal{C}} \simeq \text{Funct}(\mathcal{C}^{\text{op}}, \text{Cat}) \tag{9}$$

where Cat *is the* ∞*-category of* ∞*-categories.*

Consider a functor of ordinary categories $F : \widetilde{\mathcal{C}} \to \mathcal{C}$. An arrow $\widetilde{y} \to \widetilde{z}$ is called coCartesian if for every diagram (of solid arrows) as below, there exists a unique map $\widetilde{z} \to \widetilde{x}$ that makes the diagram commute.

$$\begin{tikzcd}
 & & & \widetilde{x} \arrow[dd, mapsto] \\
\widetilde{\mathcal{C}} \arrow[dd, "F"'] & \widetilde{y} \arrow[r] \arrow[urr, bend left] \arrow[dd, mapsto] & \widetilde{z} \arrow[ur, dashed] \arrow[dd, mapsto] & \\
 & & & x \\
\mathcal{C} & y \arrow[r] \arrow[urr, bend left] & z \arrow[ur] &
\end{tikzcd} \tag{10}$$

and Ill leave it to the reader to extrapolate the definition of a coCartesian fibration $\widetilde{C} \to \mathcal{C}$, and the definition of the ∞-category $\text{coCart}_{/\mathcal{C}}$. However, let me record the following theorem

Theorem 6. *There is a canonical equivalence of ∞-categories*

$$\text{coCart}_{/\mathcal{C}} \simeq \text{Funct}(\mathcal{C}, \text{Cat}) \tag{11}$$

where Cat *is the* ∞*-category of* ∞*-categories.*

Finally, we will use this formalism to provide a nice and clean definition of adjoint functors. Let $F : \mathcal{C}_0 \to \mathcal{C}_1$ be a functor of ∞-categories. We can view F as a functor $[1] \to \text{Cat}$, and using the previous theorem, this corresponds to a coCartesian fibration

$$\widetilde{\mathcal{C}} \to [1] \tag{12}$$

We shal say that F admits a right adjoint if it above functor is a biCartesian fibration, i.e., if it happens to be a Cartesian and coCartesian fibration. Applying the previous theorem, it gives us a functor $[1]^{\text{op}} \to \text{Cat}$, which corresponds to the right adjoint $F^R : \mathcal{C}_1 \to \mathcal{C}_0$. Staring at the following diagram for a bit

$$\begin{tikzcd}
c_0 \arrow[drr, bend left] \arrow[dr, dashed] \arrow[ddr, mapsto] & & \\
 & F^R(c_1) \arrow[r, dashed] \arrow[d, mapsto] & c_1 \arrow[d, mapsto] \\
 & 0 \arrow[r] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & c_1 \arrow[ddl, mapsto] \\
c_0 \arrow[urr, bend left] \arrow[r, dashed] \arrow[d, mapsto] & F(c_0) \arrow[ur, dashed] \arrow[d, mapsto] & \\
0 \arrow[r] & 1 &
\end{tikzcd} \tag{13}$$

we obtain the usual equivalence

$$\mathrm{Map}_{\mathcal{C}_0}(c_0, F^R(c_1)) \simeq \mathrm{Map}_{\mathcal{C}_1}(F(c_0), c_1) \tag{14}$$

This perspective has been very useful for me, as it allows me to this about morphisms between objects on different categories, which I have found useful when thinking about adjunction.

Monoidal ∞-Categories

In this section we will introduce the notion of a monoidal ∞-category. The idea is simple: a monoidal category will be encoded by a functor $\Delta^{\mathrm{op}} \to \mathrm{Cat}$. You may recall that when defining the axioms of a monoidal category we usually have all this ugly diagrams which must commute, all of this is nicely encoded in this functor.

Definition 7. A monoidal ∞-category is a functor

$$\mathcal{A}^{\otimes} : \Delta^{\mathrm{op}} \to \mathrm{Cat} \tag{15}$$

subject to the following conditions

- $\mathcal{A}^{\otimes}([0]) = *$
- For any n, the functor, given by the n-tuple of maps in Δ

  $$[1] \to [n] \qquad 0 \mapsto i, 1 \mapsto i+1 \tag{16}$$

  defines an equivalence

  $$\mathcal{A}^{\otimes}([n]) \to \mathcal{A}^{\otimes}([1]) \times \cdots \times \mathcal{A}^{\otimes}([1]) \tag{17}$$

  in other words this maps correspond to the projections $\mathcal{A}^{\otimes}([1]) \times \cdots \times \mathcal{A}^{\otimes}([1]) \to \mathcal{A}^{\otimes}([1])$.

If $\mathcal{A}^{\otimes}$ is a monoidal ∞-category, we shall denote by $\mathcal{A}$ the underlying ∞-category, i.e., $\mathcal{A}^{\otimes}([1])$. The map

$$[1] \to [2] \qquad 0 \mapsto 0, 1 \mapsto 2 \tag{18}$$

defines a functor

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A} \tag{19}$$

This functor is the monoidal operation on $\mathcal{A}$, correspond to $\mathcal{A}^{\otimes}$. And the map $[1] \to [0]$ defined a functor $* \to \mathcal{A}$; the corresponding object is the unit of the monoidal structure $1_{\mathcal{A}} \in \mathcal{A}$.

Definition 8. A symmetric monoidal ∞-category is a functor $\mathrm{Fin}_* \to \mathrm{Cat}$. Where Fin_* is the category of finite pointed sets. We replace the condition above by the following

- $\mathcal{A}^{\otimes}(*) = *$
- For any finite pointed set $(* \in I)$ and any $i \in I - \{*\}$, we have the map $(* \in I) \to (* \in \{* \cup i\})$ given by $i \mapsto i$ and $j \mapsto *$ for $j \neq i$. We require the induced map

  $$\mathcal{A}^{\otimes}(* \in I) \to \prod_{i \in I \setminus \{*\}} \mathcal{A}^{\otimes}(* \in \{* \cup i\}) \tag{20}$$

  be an equivalence.

Lets again reinterpret the above definition in terms that resemble our usual notion of a monoidal category. we shall denote by $\mathcal{A}$ the underlying ∞-category, i.e., $\mathcal{A}^{\otimes}(\{* \cup i\})$. The reason for the existence of the point $* \in I$ is because we want to have a notion of projection $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, and the factors that we forget correspond to those mapped to $* \in I$. On the other hand, given a map of pointed sets $I \to J$, if $x, y \in I$ get mapped to the same object of $J \setminus \{*\}$, then the corresponding factors of $\prod \mathcal{A}$ get multiplied.

We will now put the power of coCartesian fibrations to use, in order to define algebra objects in an monoidal ∞-category. We have a fully faithful functor

$$\mathrm{Cat}^{\mathrm{Mon}} \hookrightarrow \mathrm{coCart}_{/\Delta^{\mathrm{op}}} \tag{21}$$

and the essential image is singled out by the conditions of a monoidal category described above.

Example 9. Any ∞-category $\mathcal{C}$ that admits cartesian products has a canonically defined (symmetric) monoidal structure, where the monoidal operation is defined by

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad (c_0, c_1) \to c_0 \times c_1 \tag{22}$$

and dually if $\mathcal{C}$ admits coproducts there is a canonical (symmetric) monoidal structure. In this latter situation, there is an equivalence of categories

$$\mathrm{oblv} : \mathrm{ComAlg}(\mathcal{C}) \to \mathcal{C} \tag{23}$$

informally, this says that ever object $c \in \mathcal{C}$ has a uniquely defined structure of commutative algebra given by $c \sqcup c \to c$.

We now define what it means to have a commutative algebra object on a monoidal ∞-category $\mathcal{A}$.

Definition 10. Given a monoidal ∞-category $\mathcal{A}^{\otimes}$, an associative algebra is a right-lax monoidal functor

$$*^{\otimes} \to \mathcal{A}^{\otimes} \tag{24}$$

over Δ^{op}, where $*^{\otimes}$ can be described as the constant functor $\Delta^{\mathrm{op}} \to *$. But more importantly, what is a right-lax monoidal functor? The idea is that the morphism $*^{\otimes} \to \mathcal{A}^{\otimes}$ of coCartesian fibrations over Δ^{op} need not preserve all coCartesian arrows, as it was the case before, the only coCartesian arrows it must preserve are the ones corresponding to the projections $\mathcal{A}^{\otimes}([n]) \simeq \mathcal{A}^{\otimes}([1]) \times \cdots \times \mathcal{A}^{\otimes}([1]) \to \mathcal{A}^{\otimes}([1])$.

Of course, we can analogously define commutative algebra objects replacing Δ^{op} by Fin_*.

A natural question to ask is: why is it natural to require right-lax monoidal functors? Lets work out an example to understand this better. Let $\mathrm{Spc}^{\otimes}$ be the monoidal ∞-category of spaces, endowed with the Cartesian monoidal structure, i.e., the monoidal operation is given by $(X, Y) \to X \times Y$. Recall that, an associative algebra object X must has a monoidal operation $X \times X \to X$, lets see if we are able to write this down in our framework: the monoidal operations should correspond to a morphism $(X, X) \to X$ as described in the diagram below

$$\begin{tikzcd} & & X \arrow[dl] \\ (X,X) \arrow[r] \arrow[rru, bend left] \arrow[d, mapsto] & X \times X \arrow[ru, dashed] \arrow[d, mapsto] & \\ 2 \arrow[r] & 1 & \end{tikzcd} \tag{25}$$

However, this morphisms is not coCartesian, and indeed it factors through $X \times X \to X$. Being right lax-monoidal provides us this flexibility to have morphisms between the fibers of $\mathrm{Spc}^{\otimes} \to \Delta^{\mathrm{op}}$ which are not coCartesian.

Example 11. An important example of a algebra object which requires this higher compatibility conditions appears naturally in the category Spc. For a space X, we can consider its loop space ΩX. This space has naturally the structure of an associative algebra object (when we endow Spc with the cartesian monoidal structure), but associativity only holds "up to homotopy", however, in our framework, this is associative up to "coherent homotopy". Similarly, we will later see that all the commutative algebra objects in Spc correspond to infinite loop spaces $\Omega^\infty X$, where multiplications is commutative up to coherent homotopy. This really cool result was first proved by May, using the machine of iterated loop spaces.

Spectra

Lets talk about colimits for a second. Let $X \in \mathrm{Spc}$, we can also consider X as a category itself, and define the constant functor valued in $*$

$$X \to \mathrm{Spc} \tag{26}$$

then the colimit of this functor is exactly X. More generally, let $\mathcal{C}$ be an arbitrary ∞-category, and consider any functor $\mathcal{C} \to \mathrm{Spc}$, what is the colimit? Recall that we can associate to it a coCartesian fibration

$$\widetilde{\mathcal{C}} \to \mathcal{C} \tag{27}$$

and recall that in some sense $\widetilde{\mathcal{C}}$ totalizes the image of the functor. Then one can proof that the colimit of $\mathcal{C} \to \mathrm{Spc}$ is isomorphic to the geometric realization $|\widetilde{\mathcal{C}}|$. I would like to use this to emphasize that a colimit in the world of ∞-categories behaves like it remembers the whole diagram.

Proposition 12. *The ∞-category Spc is generated under colimits by $*$.*

We recall the classical definition of a spectrum

Definition 13. The category of spectra can be obtained as

$$\mathrm{Spctr} \simeq \lim(\cdots \xrightarrow{\Omega} \mathrm{Spc} \xrightarrow{\Omega} \mathrm{Spc}) \simeq \mathrm{colim}(\mathrm{Spc} \xrightarrow{\Sigma} \mathrm{Spc} \xrightarrow{\Sigma} \cdots) \tag{28}$$

However, it may be worth mentioning that the colimit on the right is taken on a subcategory of Cat where we restrict ourselves to colimit preserving functors, plus the set theoretic condition of presentability. The operation done above goes by the name of passing to adjoints. Classically, a spectrum is a sequence of pointed spaces $\{X_n\}_{n\in\mathbb{Z}}$ equipped with homotopy equivalences $X_n \to \Omega X_{n+1}$. We say that X_{n+1} is the delooping of X_n.

This operation forces the loop and suspension functor to be inverses of each other. And recall that a stable ∞-category have the property that the suspension and loop functors define equivalences. Therefore we may think of the category Spctr as the universal stabilization of Spc. By the work done above we automatically have a spectrification functor

$$\Sigma^\infty : \mathrm{Spc} \to \mathrm{Spctr} \tag{29}$$

However, a more concrete description can be obtained. By the construction above we obtain that Σ^∞ is a left adjoint, hence it preserves colimits. Since the category of spaces is geneated by $*$ under colimits, it suffices to understand the image of $*$ under Σ^∞. By unwinding the definition we can conclude that $* \mapsto \mathbb{S}$ the sphere spectrum.

Definition 14. The category StCat is the category of cocomplete stable ∞-categories, and we only consider colimit preserving functors between them.

Lemma 15. *For any $\mathcal{C} \in \mathrm{StCat}$, there is an equivalence of categories between $\mathrm{Funct}_{\mathrm{StCat}}(\mathrm{Spctr}, \mathcal{C})$ and the full subcategory of $\mathrm{Funct}(\mathrm{Spc}, \mathcal{C})$ consisting of colimit preserving functors.*

Corollary 16. *For $\mathcal{C} \in \mathrm{StCat}$ we have the following equivalence*

$$\mathrm{Funct}_{\mathrm{StCat}}(\mathrm{Spctr}, \mathcal{C}) \simeq \mathcal{C} \tag{30}$$

Having in mind that the category Spctr is generated under colimits by $\mathbb{S}$, it is not hard to see why this result holds. As any colimit preserving functor $\mathrm{Spctr} \to \mathcal{C}$ is completely determined by the image of the sphere spectrum $\mathbb{S}$.

However, from the classical definition of spectra, constructing the spectrification functor is a more involved task. However, the classical definition provides us with a easy description of a right adjoint functor to the spectrification functor.

$$\Omega^{\infty} : \mathrm{Spctr} \longrightarrow \mathrm{Spc} \qquad \{X_n\} \mapsto X_0 \tag{31}$$

notice however, that $X_0 \simeq \Omega X_1 \simeq \Omega^n X_n$, i.e., it is an infinite loop space. In fact we have the following theorem.

Theorem 17. *There exists an equivalence of categories*

$$\mathrm{Spctr}^{\leq 0} \simeq \mathrm{ComGrp(Spc)} \tag{32}$$

where $\mathrm{ComGrp(Spc)}$ correspond to infinite loop spaces.

This theorem justifies why we think of spectra as the analog of abelian groups in spaces. Recall that in higher category theory we replace sets by spaces, and commutative group objects of sets are abelian groups, and now we see that commutative group objects in spaces correspond to connective spectra.

In order to compare this two categories, one shows that Ω^{∞} factors canonically as

$$\mathrm{Spctr} \longrightarrow \mathrm{ComGrp(Spc)} \longrightarrow \mathrm{Spc} \tag{33}$$

indeed this will follow by general nonsense. Recall that we can endow Spctr with the coCartesian monoidal structure, i.e., multiplication is given by the fold map $c \sqcup c \to c$, and that there is a equivalence of categories

$$\mathrm{ComAlg(Spctr)} \simeq \mathrm{Spctr} \tag{34}$$

under the cocartesian monoidal structure. Moreover, since Spctr is stable, the cocartesian monoidal structure coincides with the cartesian monoidal structure. Moreover, we have the following equivalence

Lemma 18. *For a stable ∞-category $\mathcal{C}$, endowed with a monoidal structure, the inclusion*

$$\mathrm{ComGrp}(\mathcal{C}) \hookrightarrow \mathrm{ComAlg}(\mathcal{C}) \tag{35}$$

is an equivalence

Therefore we actually have a map

$$\mathrm{Spctr} \simeq \mathrm{ComGrp(Spctr)} \xrightarrow{\Omega^{\infty}} \mathrm{Spc} \tag{36}$$

and since commutative group objects can only map to commutative group objects we conclude that the functor Ω^∞ actually factors as

$$\mathrm{Spctr} \longrightarrow \mathrm{ComGrp(Spc)} \longrightarrow \mathrm{Spc} \tag{37}$$

This is the desired functor. It may be worth to highlight the fact that through this section we have only been working with (co)cartesian monoidal structure, and we have shown that the category of spectra is completely made up of abelian group objects.

Lurie Tensor Product

Denote by StCat the ∞-category of stable cocomplete categories, and we only consider the colimit preserving functors between them. We will endow StCat with a symmetric monoidal structure, given by a tensor product of stable categories. In the process we will find the most basic stable category, that of spectra, which will be the unit object under this tensor product.

For a pairs of stable categories $\mathcal{C}_1$ and $\mathcal{C}_2$ and a third one $\mathcal{D}$, the space of exact continuous functors

$$\mathcal{C}_1 \otimes \mathcal{C}_2 \to \mathcal{D} \tag{38}$$

is the full subspace in

$$\mathrm{Map}_{\mathrm{Cat}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \tag{39}$$

that consist of functors that are exact and continuous in each variable. This operation endows StCat with a symmetric monoidal structure. The corresponding monoidal operation

$$(\mathcal{C}_i)_{i\in I} \mapsto \otimes_{i\in I}\mathcal{C}_i \tag{40}$$

is the Lurie tensor product.

Definition 19. The category Spctr of spectra can be defined as the unit object in the symmetric monoidal ∞-category StCat. Moreover, using this tensor product we can consider the commutative algebra objects in StCat, this endows the category of spectra with a symmetric monoidal structure. The sphere spectrum is the unit object in Spctr.

In fact, every commutative algebra object in StCat is itself a symmetric monoidal category.

Definition 20. An $\mathbb{E}_\infty$ ring spectrum is an algebra object in the symmetric monoidal category Spctr.

Definition 21. Denote by Vect the ∞-category of chain complexes over the field k, which is of characteristic zero. In fact, we have that Vect is stable and cocomplete. We are considering unbounded chain complexes, but the category is complete with respect to its t-structure.

Proposition 22. *The category of commutative differential algebras over k is equivalent to the category of* $\mathrm{ComAlg(Vect)}$.

Since Spctr is the unit object in the symmetric monoidal ∞-category StCat, we have a canonical colimit preserving map

$$\mathrm{Spctr} \to \mathrm{Vect} \tag{41}$$

since it is colimit preserving, it admits a right adjoint (which also preserves colimits), denoted by

$$\mathrm{Vect} \xrightarrow{\text{Dold-Kan}^{\mathrm{Spctr}}} \mathrm{Spctr} \tag{42}$$

In fact, this functor is t-exact so we obtain

$$\mathrm{Vect}^{\leq 0} \longrightarrow \mathrm{Spctr}^{\leq 0} \simeq \mathrm{ComGrp}(\mathrm{Spc}) \longrightarrow \mathrm{Spc} \tag{43}$$

which in coincides with the usual Dold-Kan functor. In fact this functor has a right adjoint

$$\mathrm{Spc} \to \mathrm{Spctr} \to \mathrm{Vect} \tag{44}$$

which maps $S \mapsto C_\bullet(S, k)$, which is the corresponding chain complex with coefficients in k. To conclude, we state the following theorem which provides us with a concrete model for working with $\mathbb{E}_\infty$ ring spectra over a field of characteristic zero.

Theorem 23. *If R is a ring of characteristic zero, then there exists an equivalence of categories between commutative differential graded algebras over R, and $\mathbb{E}_\infty$-ring spectrum over HR.*