Model Categories :

These are notes for a talk on Model Categories as a part of Attilio's seminar.

Definition : A model Category is a category C with thr classes of maps – weak equivalences, fibrations and cofibr subject to the following axioms. An acyclic (co)fibr is a (co)fibration which is a weak equivalence. ~~There is~~

M1. The category C is closed under limits and colimit

M2. The three distinguished classes of maps are closed under retracts.

[A morphism $f: A \to B$ is a retract of a morphism $g: X \to Y$ if there is a commutative diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "="] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "="'] & Y \arrow[r] & B \end{tikzcd}$$

]

M3. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ so that any two of f, g or gf is a wk equivalence, then so is the third.

M4. Every lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where j is a cofibration and q is a fibration has a solution so that both diagrams commute if one of j or q is a wk-equivalence.

M5. Any $f: X \longrightarrow Y$ can be factored in two ways.

(i) $X \xrightarrow{i} Z \xrightarrow{q} Y$ where i is a cofibration and q is a wk-equivalence and a fibration.

(ii) $X \xrightarrow{j} Z \xrightarrow{p} Y$, where j is an acyclic cofibrati[on] and p is a fibration.

Rmk: ■ If C is a model category, then the cofibrations are exactly those morphisms with LLP with respect to acyclic fibrations; that is; $j: A \to B$ is a cofibration $\Leftrightarrow$ for every acyclic fibration $q: X \to Y$ and every lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a solution.

■ Let X be an object in the model category C. Then X is cofibrant if the unique morphism from the initial object to X is a cofibration. A cofibrant replacement for X is a wk equivalence $Z \longrightarrow X$ wi[th] Z cofibrant.

Examples:

1. The category $Ch_* R$ of chain complexes of modules over the ring R has the structure of a model category with a morphism $f_\bullet: M_\bullet \longrightarrow N_\bullet$

① A wk equivalence if $H_* f$ is an isomorphism.

② a fibration if $M_n \to N_n$ is surjective for $n \geq 1$; a[nd]

③ a cofibration if and only if for $n \geq 0$, the map $M_n \longrightarrow N_n$ is an injection with projective cokernel.

2. Recall the notion of simplicial sets from Haoyang's talk. Let sSet denote the category of simplicial sets. sSet has the structure of a model category with a morphism $f: X \to Y$

① a wk equivalence if $|f| : |X| \to |Y|$ is a wk-equivalence of topological spaces; where $|\cdot|$ den[otes] the geometric realization.

② a cofibration if $f_n : X_n \longrightarrow Y_n$ is a monomorphis[m] for all n; and,

③ a fibration if f has the left lifting property w.r.t. the inclusions of the horns $\Lambda^n_k \to \Delta^n$ $n \geq 1$, $0 \leq k \leq n$.

3. Let $sMod_R$ be the category of simplicial R-modules and $X \in sMod_R$. The Dold-Kan (P... equivalence says that the "normalized chain complex functor" $N : sMod_R \to Ch_* R$ is an equivalence of categories. Use this to provide $sMod_R$ with a model category structure from (i

4.

Homotopy Category of a model category:

Def. Let C be a model category and $A \in C$. A cylinder object for A is a factoring

$$\begin{tikzcd} A \sqcup A \arrow[r, "i"] \arrow[dr, "\nabla"'] & C(A) \arrow[d, "q", "\sim"'] \\ & A \end{tikzcd}$$

where i is a cofibration, q is a wk equivalence, and ∇ is the fold map. We'll refer to $C(A)$ as the cylinder object.

Def. Let $f, g : A \to X$ be two morphisms in a model category C. A left homotopy from f to g is a diagram in C.

$$\begin{tikzcd} A \sqcup A \arrow[r, "i"] \arrow[dr, "f \sqcup g"'] & C(A) \arrow[d, "H"] \\ & X \end{tikzcd}$$

where $C(A)$ is a cylinder object for A. We will denote the homotopy by H.

Rmk: Similarly we can dually form notions like path object and right homotopy.

Examples: We can define a natural cylinder object in $Ch_* R$ and then homotopies are just usual chain homotopies.

Def. (The homotopy category). Let C be a model category. Then the homotopy category $\underline{Ho(C)}$ is the category obtained from C by inverting the wk equivale

$$Ho(C)\,(X,Y) \simeq C\,(X_c, Y_f)/(\sim \text{homotopy}).$$

where $X_c \to X$ and $Y \to Y_f$ are cofibrant and fibrant replacements respectively. (There are thing to check but we omit them.)

Quillen Functors and Derived Functors:

Def. Let C and D be two model categories. Then a $\underline{\text{Quillen functor}}$ from C to D is an adjoint pair of functors

$$C \underset{G}{\overset{F}{\rightleftarrows}} D$$

with F the left adjoint so that

(1) The functor F preserves cofibrations and wk equivalences between cofibrant objects, and

(2) the functor G preserves fibrations and wk equivalences between fibrant objects.

A Quillen functor is a Quillen equivallence if for all cofibrant objects X in C and all fibrant objects Y in D, a morphism $X \longrightarrow GY$ is a wk equivalence in $C \iff$ the adjoint morphism $FX \longrightarrow Y$ is a wk equivalence in D.

Example: 1. $f: R \to S$ a homomorphism of commutative rings. $S \otimes_R - : Ch_* R \rightleftarrows : res_f$ is an example of a Quillen functor.

2. The geometric realization functor and the singular set functor give a Quillen equivalence.

$$|-|: sSets \rightleftarrows CGH : S(-).$$

(CGH is the cat. of compactly generated Hausdroff spaces)

$$S_n(X) := Top(\sigma_n, X).$$

Derived functors in the language of model categories.

Def. Let $F: C \rightleftarrows D: G$ be a Quillen functor. Then F has a total left derived functor LF defined as follows. If $X \in C$, let $X_c \to X$ be a cofibrant replacement. Set $LF(X) = F(X_c)$. This is well-defined in Homotopy Category.

Proposition Let $F: C \rightleftarrows D: G$ be a Quillen functor between two model categories. Then the total derived functors induce an adjoint pair

$$LF: Ho(C) \rightleftarrows Ho(D): RG.$$

this adjoint pair induces an equivalence of categories $\Leftrightarrow$ the Quillen functor is a Quillen equivalence.

Example In the previous situation,
$$L(S\otimes_R -)(N_\bullet) = S \otimes^L_R N_\bullet \ ,$$ where the RHS is the usual derived tensor product.

Applications: **1.** Let C be a category and C_{ab} be the category of its abelian objects. Assume that there are model category structures on both of them so that the inclusion $i : C_{ab} \to C$ is the right adjoint of a Quillen functor; that is
$$Ab : C \rightleftarrows C_{ab} : i$$

Def. Quillen homology of X is the object
$$L\,Ab(X) \in C_{ab}.$$

Example: 1. If $X \in sSets$, then X is cofibrant and $L\,Ab(X) = \mathbb{Z}X \in sMod_{\mathbb{Z}}$. If Y is a topological space then $\pi_n(L\,Ab(S(Y))) = \pi_n \mathbb{Z}S(Y) \simeq H_n(Y;\mathbb{Z})$, which recovers singular homology.

2. If G is a group, regard G as a constant simplicial group. There is a (degree-shifting) isomorphism between $\pi_* L\,Ab(G)$ and $\tilde{H}_*(BG)$, where BG is the classifying space of G.

2. One can prove that there is a model category structure on simplicial R-algebras such that the adjoint pair

$$S_R : sMod_R \rightleftarrows sAlg_R : \underline{Forget}$$

is a Quillen functor. (where S_R is the symmetric algebra functor). We will be deriving certain functors using this model structure on $sMod_R$:

<u>Set up</u>. We have an R-algebra A.

We have a functor from $Mod_A \longrightarrow Alg_R/A$.

by sending M to $A \ltimes M$, where

$A \ltimes M = A \oplus M$ with multiplication $(a,x)(b,y) = (ab, ay+bx)$. This extends to a functor $sMod_A \longrightarrow sAlg_R/A$.

This has a left adjoint –

$$\Omega_{(-)/R} : sAlg_R/A \longrightarrow sMod_A$$

which gives a Quillen functor:

$$\Omega_{(-)/R} : sAlg_R/A \rightleftarrows sMod_A : A \ltimes (-).$$

The <u>cotangent complex</u> $\boxed{L_{A/R} := L\Omega_{(-)/R}(A)}$

The André-Quillen homology of A is given by:

$$D_q(A/R) = \pi_q L_{A/R} = H_q N L_{A/R}.$$

(Dold-Kan)

<u>Rmk</u>: 1. Cotangent complex classifies all deformations. See the papers by Illusie.

2. We need to choose a resolution to really compute $L_{A/R}$. A natural resolution is given by:

$$\Big(\cdots R[R[R[A]]] \substack{\rightarrow\\ \rightarrow\\ \rightarrow} R[R[A]] \rightrightarrows R[A] \Big) \to A$$

3.

Let sCat denote the category of simplicially enriched categories 1

- There is a simplicial thickening $C[\Delta^n] \in$ sCat of $[n]$. For any $\mathcal{C} \in$ sCat, the coherent nerve $N_\Delta \mathcal{C}$ of $\mathcal{C}$ is the simplicial set

$$N_\Delta(\mathcal{C})_\bullet = \mathrm{Fun}_{sCat}(C[\Delta^\bullet], \mathcal{C}) \in sSet.$$

The above functor $N_\Delta : s\mathcal{C}at \longrightarrow sSet$ has a left adjoint $C[-]$ such that

$$C[-] : sSet \rightleftarrows sCat : N_\Delta$$ is a

Quillen equivalence w.r.t. the Joyal model structure on sSet and Bergner model structure on sCat. (Lurie)

For a simplicial model category (see Corner Axiom, Def 4.11 in Goerss, Schemmerhorn) $\mathcal{M}$, let $\mathcal{M}_{cf}$ be the full simplicial subcategory spanned by the fibrant and cofibrant (bifibrant) objects. then $N_\Delta(\mathcal{M}_{cf})$ is an ∞-category, called the ~~model~~ ∞-category associated to the simplicial model category $\mathcal{M}$.

[Infact, $N_\Delta(\mathcal{C})$ of any locally fibrant simplicial ~~model~~ category is an ∞-category. Here locally fibrant means all simplicial mapping spaces are Kan complexes.]

Examples:

1. Take $N_\Delta(sSet_{cf}) = N_\Delta(\mathcal{K}an)$. This is the ∞-category $\mathcal{S}$ of spaces.

2. Use the Dold-Kan equivalence to obtain the category $DK_*(Ch(A))$ enriched in simplicial abelian groups. Since simplicial abelian groups are Kan complexes, this is locally fibrant and so $N_\Delta(DK_*(Ch(A)))$ is an ∞-category defined to be the ∞-category $Ch(A)$ of chain complexes in A.