Simplicial Sets and Infinity Categories

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Haoyang Guo

Abstract

This is the note for the talk given in the Derived Non-Sense Learning Seminar, on May 17, 2018. We follows mostly the note *A short course on Infinity Categories* by Groth, and the note by Mathew about Dold-Kan Correspondence. Our goal is to look in more detail about certain concepts around simplicial sets and infinity categories introduced last time.

1 Simplicial Sets

Definitions and basic examples

In the first part, we introduce the notion of simplicial sets. Philosophically, the simplicial set provides us with a combinatorial model for the homotopy theory of geometric objects, in particular the CW complexes. Here we use the language of presheaves to look in detail about basic properties.

We first define the simplex category Δ.

Definition 1.1. We define the **simplex category** Δ to be the category of totally ordered finite sets, where morphisms are those order-preserving maps. Denote by $[n]$ to be the set $\{0, \ldots, n\}$ with the usual ordering.

We note that by definition, Δ is equivalent to the subcategory generated by $\{[n],\ n \in \mathbb{N}\}$. So to define a functor on Δ, it suffices to do it on $[n]$ for all positive integers n.

We denote by **Top** to be the category of topological spaces. Then there exists a natural functor from Δ to **Top**, by sending $[n]$ onto the standard n-simplex Δ^n spanned by $\{0 = e_0, \ldots, e_n\}$ in $\mathbb{R}^{n+1}$, and sending morphisms to be linear morphisms between simplexes (e.g. the inclusion $[n-1] \to [n]$ sending Δ^{n-1} onto a $n-1$-face of Δ^n.

Now we define the central concept in this section.

Definition 1.2. A **simplicial sets** $X_\bullet$ is defined as a presheaf on the category Δ, i.e. a contravariant functor $\Delta \to$ **Set**. Here X_n is denoted to be the image of Δ^n. A morphism of simplicial sets is a natural transformation between functors. The class of all simplicial sets is written as $\widehat{\Delta} = \mathbf{sSet}$.

Similarly, for a given category $\mathcal{D}$, we could define $\mathcal{D}$-valued presheaf on Δ to be a **simplicial object**.

We remark here that since Δ is equivalent to the subcategory generated by $[n]$, to give an $X_\bullet \in \mathbf{sSet}$, it is equivalent to determine $X_n \in \mathbf{Set}$ for each n, such that there exists a $X_n \to X_m$ for each map $[m] \to [n]$, functorial in a natural way. We call X_n the $n-$*simplices* of $X_\bullet$.

Here are some example:

Example 1.3 (Singular simplicial set)**.** Let $X \in$ **Top**. Then we can associate to X a simplicial set $Sing(X)_\bullet$, such that

$$\mathrm{Sing}(X)_n = \hom_{\mathbf{Top}}(\Delta^n, X)$$

is the collection of all n-simplices on X.

Example 1.4 (Standard n simplex)**.** For each $n \in \mathbb{N}$, we define the *standard* n *− simplex* $\Delta[n]_\bullet$ to be the presheaf h_{Δ^n} represented by Δ^n; i.e.

$$\Delta[n]_m = \hom_\Delta([m],[n]).$$

By Yoneda's embedding, for given $X_\bullet \in \mathbf{sSet}$, we have

$$X_n = \hom_{\mathbf{sSet}}(\Delta[n]_\bullet, X_\bullet).$$

Example 1.5 (Quotient simplicial set)**.** Let $X_\bullet \in \mathbf{sSet}$, and $Y_\bullet \subset X_\bullet$ be a simplicial subset. Then we can define a quotient simplicial set $(X/Y)_\bullet$ to be

$$(X/Y)_n := X_n/Y_n.$$

As an example, consider the simplicial subset $\Delta[0]_\bullet \coprod \Delta[0]_\bullet$ inside $\Delta[1]$, corresponding to two vertices. Then by module the boundary via two maps $\Delta[0] \rightrightarrows \Delta[1]$, we get the *simplicial circle*.

Proposition 1.6. *Since* $\mathbf{sSet} = \widetilde{\Delta}$ *is the* **Set***-valued presheaf category, and* **Set** *has (small) limits and colimits, we see (small) limits and colimits exist in* **sSet***.*

Properties of sSet

Now we look at some properties of the presheaf category **sSet**. And to our convenience, we look at the general case for $\widehat{\mathcal{C}}$ associated to any small category $\mathcal{C}$.

We first recall last time we have the following theorem

Theorem 1.7. *For any given functor* $F : \Delta \to \mathcal{D}$*, where* $\mathcal{D}$ *is cocomplete, there exists a unique functor* $\overline{F} : \mathbf{sSet} \to \mathcal{D}$ *extending* F*.*

$$\begin{tikzcd} \Delta \arrow[r] \arrow[dr, "F"'] & \boldsymbol{sSet} \arrow[d, dotted, "\overline{F}"] \\ & \mathcal{D} \end{tikzcd}$$

By replacing $\mathcal{D}$ by its opposite $\mathcal{D}^{op}$, we may assume F is a $\mathcal{D}$-valued presheaf on Δ. Then the results follows from the following general result.

Proposition 1.8. *Any presheaf* F *on the given small category* $\mathcal{C}$ *is canonically the colimit of representable presheaves.*

Sketch. We only give the construction. For each $X \in \mathcal{C}$ and $s \in F(X)$, there exists a morphism of presheaves

$$\varphi_{X,s} : h_X \longrightarrow F,$$

such that for any $f : Y \in X$, $\varphi(f) = f^*(s)$. So we could form the colimit

$$\varinjlim_{(X,s)} h_X,$$

where transitions are given by pullbacks. By construction, it is surjective. For the injectivity, we only need to unravel the definition, which is left as an exercise. □

Here we note that if we form $\mathcal{D}_F$ to be the category of all possible maps $h_X \to F$, with natural morphism. Then the above colimit is in fact the colimit over the index category $\mathcal{D}_F$, and what we proved is exactly that F is the colimit of $\mathcal{D}_F \to \mathcal{C} \to \widehat{\mathcal{C}}$. Also note that $\mathcal{D}_F$ is functorial at $\mathcal{F} \in \widehat{\mathcal{C}}$.

To finish the theorem, we apply the Proposition on $\mathcal{C} = \Delta$. We get

Corollary 1.9. *Any simplicial set is canonically a colimit of standard n-simplices.*

Then since any $\mathcal{G} \in \mathbf{sSet} = \widehat{\mathcal{C}}$ is a colimit of objects in $\mathcal{C}$, and since $\mathcal{D}$ is closed under colimits. Given $F : \mathcal{C} \to \mathcal{D}$, we could extend it uniquely over $\widehat{\mathcal{C}}$ by taking the colimit.

We then note that there is a natural way to define the right adjoint of $\overline{F} : \mathbf{sSet} \to \mathcal{D}$, by taking

$$\begin{aligned} \mathrm{Sing}_F : \mathcal{D} &\longrightarrow \mathbf{sSet}; \\ D &\longmapsto (h_D \circ F), \end{aligned}$$

which leads to an adjunction pair

$$(\overline{F}, \mathrm{Sing}_F).$$

Moreover, it can be showed that for any cocomplete category $\mathcal{D}$, the construction defines an equivalence of categories

$$\begin{aligned} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) &\longrightarrow \mathrm{Adj}(\widehat{\mathcal{C}}, \mathcal{D}); \\ F &\longmapsto (\overline{F}, \mathrm{Sing}_F). \end{aligned}$$

This is often referred as the **Yoneda's Extension**.

An example of this adjunction pair is when $\mathcal{D} = \Delta$, and $\mathcal{D} = \mathbf{Top}$, such that F is given by the example given at the beginning. Then we call the $\overline{F} = |\cdot| : \mathbf{sSet} \to \mathbf{Top}$ to be the geometric realization. And by unraveling the construction, it is easy to see that $\mathrm{Sing}_F : \mathbf{Top} \to \mathbf{sSet}$ coincides with the usual singular simplices $X \mapsto \mathrm{Sing}(X)_\bullet$ given before. In other words, we have the adjoint pair

$$(|\cdot|, \mathrm{Sing}(\cdot)_\bullet).$$

Another example is when $\mathcal{D} = \mathbf{Cat}$, and F is the functor mapping each Δ^n onto its corresponding category, then the adjunction Sing_F is called *nerve*, which we will look in more detail later.

As an aside, the Proposition above can also lead to the following observation about the localization of topos:

Fact 1.10. Let $\mathcal{C}$ be a site, and $\mathcal{F}$ be a sheaf on it. Then we have the following equivalence between sheaves categories

$$Shv(\mathcal{C}|\mathcal{F}) \cong Shv(\mathcal{C})|_{\mathcal{F}}.$$

The direction from the right to the left is by taking the pullback along each section. For another side, it can be constructed following the same idea of the Proposition. We refer the reader to the Stack Project for details.

Simplicial morphisms

At last in this section, we introduce several important morphisms in Δ that will be useful in further materials.

Let $n \in \mathbb{N}$. Define the coface map

$$d^i : [n-1] \longrightarrow [n]$$

to be the only order-preserving injective map whose image does not contain i. And define the codegeneracy map

$$s^i : [n] \longrightarrow [n-1]$$

to be the only order-reserving surjective map which hits i twice. We remark that these two types of morphisms satisfy certain multiplication identities, which we refer the reader for any introductory material on simplicial sets.

The morphism introduced on Δ induces endomorphism on any given $X_\bullet \in \mathbf{sSet}$, which are denoted by

$$d_i : X_n \longrightarrow X_{n-1};$$
$$s_i : X_{n-1} \longrightarrow X_n.$$

Here we call d_i the face map, and s_i the degeneracy map. We note that from the simplicial identities, s_i are always injective on $X_\bullet$. And though we do not list the identities here, we note that the multiplication satisfies the observation that "the smaller can move inside", pointed out by Mathew. For instance, if we have $d_i d_j$ for $i < j$, we then get

$$d_i d_j = d_{j-1} d_i.$$

The similar holds for many other situations.

2 Simplicial abelian groups and Dold-Kan correspondence

In this subsection, we introduce the simplicial abelian group and states the Dold-Kan correspondence, which gives the equivalence between simplicial abelian groups and the chain complices in positive degrees.

Definition 2.1. A **simplicial group** is defined as the **Ab**-valued presheaf on Δ. In other words, it is a simplicial object in the category of abelian groups.

Definition 2.2. Let $A_\bullet$ be a simplicial abelian group, and we define the **normalized** complex NA_* as follows. In degree n, NA_n is defined as $\cap_{0 \leq i < n} ker(d_i)$. And the differential $NA_n \to NA_{n-1}$ is given by $(-1)^n d_n$.

It follows directly from simplicial identities that NA_* is a complex. And here is our main result in this subsection

Theorem 2.3 (Dold-Kan)**.** *The functor*

$$A_\bullet \longmapsto NA_*$$

defines an equivalence between the category $\mathbf{Ch}_{\geq 0}$ *of chain complexes of abelian groups of non-negative degrees, and the category of* **sAb** *simplicial abelian groups.*

We refer the reader to other materials for detailed proof. Here we only mention the construction for the opposite direction.

Let C_* be a chain complex of non-negative degrees, then we define

$$A_n := \bigoplus_{[n] \twoheadrightarrow [k]} C_k,$$

For any given $[m] \to [n]$, the morphism induced on the direct summand C_k is given as follows: Consider the following canonical diagram

$$\begin{tikzcd} {[m]} \arrow[r, two heads] \arrow[d] & {[m']} \arrow[d, hook] \\ {[n]} \arrow[r] & {[k]}. \end{tikzcd}$$

Then C_k will maps into the summand $C_{m'} \subset A_m$, induced by the injection $[m'] \to [k]$. The latter is defined to 0 unless $m' = k - 1$ and the injection is given by d^k.

One of the important application of the correspondence is that it gives the equivalence between the homotopy group of simplicial abelian groups and the homology group of the corresponding chain complices. Let $A_\bullet$ be a simplicial abelian group. Define the n-th homotopy group of $A_\bullet$ as

$$\pi_n(A_\bullet, 0) = \mathrm{Map}((\Delta[n]_\bullet, \partial\Delta[n]_\bullet), (A_\bullet, 0))/\textit{homotopies}.$$

Then we have

Corollary 2.4. *Under the correspondence above, we have the functorial equality*

$$\pi_n(A_\bullet, 0) = \mathrm{H}_n(NA_*).$$

For further reading, we recommend the reader to the book written by Goerss and Jardine.

3 ∞-Category

In this section, we introduce the ∞-category.

Horns

Before everything, let us first look at the horn of a simplex. Consider the n-simplex Δ^n. Then for each $0 \leq k \leq n$, there exists a subcomplex $\Lambda^n_k \subset \Delta^n$, which is defined as the equalizer

$$\coprod_{0\leq i<j\leq n} \Delta^{n-2} \rightrightarrows \coprod_{i\neq k} \Delta^{n-1}.$$

As an example, when $n = 2$, there are three possible horns:

$$\Lambda^2_0 = \begin{tikzcd} & c_1 & \\ c_0 \arrow[ur] \arrow[rr] & & c_2, \end{tikzcd} \quad \Lambda^2_1 = \begin{tikzcd} & c_1 \arrow[dl] \arrow[dr] & \\ c_1 & & c_2, \end{tikzcd} \quad \Lambda^2_2 = \begin{tikzcd} & c_1 & \\ c_0 & & c_2 \arrow[ul] \arrow[ll] \end{tikzcd}$$

We call the Λ^n_k for $0 < k < n$ the inner horn, and $k = 0, n$ the outer horn.

Motivations and definition

In this part, we introduce two motivating examples to see why we consider the ∞-category, and give the definition and basic terminology.

Example 3.1 (Nerve of a category)**.** Let $\mathcal{C}$ be a small category. Then we can associate to it a simplicial set $N(\mathcal{C})_\bullet$, such that $N(\mathcal{C})_n$ is the set of all n-composible chains of maps:

$$x_0 \to x_1 \to \ldots \to x_n$$

It is called the *nerve* of $\mathcal{C}$. And by looking at $N(\mathcal{C})_0$, $N(\mathcal{C})_1$ and $N(\mathcal{C})_2$, we can recover all of the information of $\mathcal{C}$, which leads to the fully faithful embedding $N : \mathbf{Cat} \to \mathbf{sSet}$.

So it is very natural to ask how to single out those simplicial sets that come from categories. If we pick $f : x \to y$ and $g : y \to z$ in $N(\mathcal{C})_1$ and form the corresponding inner horn Λ^2_1:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dotted] & & z, \end{tikzcd}$$

then since $\mathcal{C}$ is an ordinary category, there **exists a unique** dashed map making the above diagram commutes. In other words, the inner horn Λ^2_0 can be uniquely filled by a 2-simplex Δ^2. Similarly, for any $n \in \mathbb{N}$, and any given inner horn Λ^n_k of $N(\mathcal{C})_\bullet$, we can fill it uniquely by a n-simplex. We remark here that the last condition is also sufficient to make a simplicial set come from a category.

Example 3.2 (Singular complex)**.** Another example comes from the algebraic topology. Let $X \in$ **Top**. Then up to the weakly homotopy equivalence, the simplicial set $\mathrm{Sing}(X)_\bullet$ determines the space X uniquely. And if we look at any horn $f : \Lambda^n_k \to X$ for $0 \leq k \leq n$ (not just inner), by algebraic topology it is actually true that we can always fill it by some n-simplex (not necessarily unique); i.e. we can extend f to a map $\Delta^n \to X$. From this, we see the simplicial set that comes from a topological space should satisfy the condition of "existence of filling" for all horns.

From these two examples, we can see that it is the filling condition that makes some of simplicial sets special. We also note that those two examples are different about the condition, in the sense that the nerve needs the existence of uniqueness for fillings of inner horns, while singular complex needs the existence for all horns. So we can get a common generalization, which is the so-called ∞-category:

Definition 3.3. A simplicial set $\mathcal{C}_\bullet$ is called an $\infty-$ **category** if for every inner horn $f : \Lambda^n_k \to \mathcal{C}_\bullet$ for $0 < k < n$, it can be extended to an n-simplex $\Delta^n \to X$

To discuss more about the ∞-category, let us introduce some terminology which will be used later. Given an ∞-category $\mathcal{C}$, the **objects** are the vertices $x \in \mathcal{C}_0$, and the **morphisms** are the 1-simplices $f \in \mathcal{C}_1$. The face map $s = d_1 : \mathcal{C}_1 \to \mathcal{C}_0$ is called the **source map**, and $t = d_0 : \mathcal{C}_1 \to_{\prime}$ is the **target map**. As in the ordinary category theory, we write $f : x \to y$ is $s(f) = x$ and $t(f) = y$. And we define the set of morphisms $\hom_{\mathcal{C}}(x, y)$ to be the pullback

$$\begin{tikzcd} \hom_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "{(s,t)}"] \\ * \arrow[r, "{(x,y)}"'] & \mathcal{C}_0 \times \mathcal{C}_0. \end{tikzcd}$$

It is called the *entire space of morphisms.*

Compositions, homotopies, and the homotopy category

The thing that becomes different from the usual category theory begins at the composition.

Consider two morphisms $f : x \to y$ and $g : y \to z$ in $\mathcal{C}_1$. Then we can form a 2-th inner horn Λ^2_1

$$\lambda = (g, \bullet, f).$$

By definition, there the inner horn can be extended to a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$. Then the new face $d_1(\sigma)$, which is opposite to the vertex 1, is then a *candidate composition* of g and f.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dotted, "d_1(\sigma)"'] & & z. \end{tikzcd}$$

We emphasis here again that one does **not** ask for a uniquely determined composition; there can be multiples of candidate compositions corresponding to different 2-simplex that extends the given inner horn. But we note that any choice of such a composition is "equally good", in the sense that the space of all such choices are contractible (homotopic to a point).

To make this precise, let us introduce the concept of homotopy:

Definition 3.4. Two morphisms $f, g : x \to y$ in an ∞-category $\mathcal{C}$ is called **homotopic** if there is a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$ with boundary $\partial\sigma = (g, f, id_x)$, i.e. the boundary looks like

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] & & y. \end{tikzcd}$$

Any such 2-simplex is a **homotopy** from f to g, denoted by $\sigma : f \to g$.

Exercise 3.5. Prove that for a given ∞-category $\mathcal{C}$ and given $x, y \in \mathcal{C}_0$, the homotopy relation is an equivalence relation on $\hom_{\mathcal{C}}(x, y)$. We denote by $[f]$ to be the **homotopy class** of the morphism $f : x \to y$.

To show that any possible compositions are **equally good**, let us assume there are two 2-simplices σ_1, σ_2 filling the given inner horn $\lambda = (g, \bullet, f)$ above. Then together with the identity 2-simplex $\kappa_f =$, whose boundary is given by (f, id_x, f), we can form a 3-inner horn

$$(\sigma_1, \sigma_2, \bullet, \kappa_f).$$

And by filling this with some 3-simplex $\tau : \Delta^3 \to \mathcal{C}$, its 2-face $d_2(\tau)$ is given by

$$(d_1(\sigma_1), id_x, d_1(\sigma_2)),$$

which is a homotopy of two compositions. Thus we see despite that there can be multiple compositions, they can be equally good up to homotopies.

This also leads to a construction by module out the homotopies.

Definition 3.6. Let $\mathcal{C}$ be an ∞-category. Then there is an ordinary category $\mathrm{Ho}(\mathcal{C})$, called the **homotopy category** of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and morphisms the homotopy classes of morphisms in $\mathcal{C}$.

Exercise 3.7. Check that the above definition is really a category.

At last, let us talk a little bit more about the **higher homotopies** and higher morphisms. Assume there are two homotopies of given two morphisms $\sigma_1, \sigma_2 : f \to g$. Then we can form a 3-inner horn $\Lambda^3_2 \to \mathcal{C}$ by giving the boundary:

$$(\sigma_1, \sigma_2, \bullet, \mathrm{id}_x),$$

where id_x is the 2-simplex whose boundary are all $id_x : x \to x$. Now by the filling axiom we can extend this inner horn to a 3-simplex $\Delta \to \mathcal{C}$. This can be regarded as a homotopy of the two given homotopies σ_1, σ_2: intuitively it means we get a continuous transformation from σ_1 to σ_2. Moreover, this construction can be built to arbitrary dimensions.

The above actually gives an example about the higher morphism. Recall that we define a morphism in the ∞-category to be a 1-simplex $\Delta^1 \to \mathcal{C}$, mapping some $x \in \mathcal{C}_0$ to $y \in \mathcal{C}_0$. We can generalize this concept to the higher dimension: a **n-morphism** from x to y is defined as a n-simplex $\Delta^n \to \mathcal{C}$, such that except for the n-th vertex is y, all the other vertices are x. For examples, homotopies σ_i of f and g is a 2-morphism, and a homotopy of σ_1 and σ_2 is a 3-morphism. We can vary the dimension n, then the sets of n-morphisms can be assembled as a **space of morphisms** $\mathrm{Map}^R_{\mathcal{C}}(x, y) \in \mathbf{sSet}$, which can be showed to be a Kan-complex.