Homological Algebra from the perspective of ∞-Categories

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Basics of ∞-Categories

In this section I will provide a quick overview of some of the ideas surrounding higher category theory. We will use the model of quasi-categories to model the idea of an ∞-category. Implicitly, I am saying here that you should not worry to much about the definition of what and ∞-category is, but rather understand what properties we would want such a theory to have. As a corollary, through the talk I will be using terminology that I haven't defined, as colimits, we will just assume that they behave as we expect. Sometimes we will need some computations involving colimits, but we will just take them for granted.

Definition 1. A simplicial set is an object built from directed n-dimensional triangles. Consider the category Δ, whose objects are themselves categories

$$[n] := 0 \to 1 \to 2 \to \cdots \to n \tag{1}$$

Notice that if $[n]$ are categories, then they must have composition, and identity morphisms. Drawing a diagram with all the morphisms provides a more convincing picture for why we say that $[n]$ are n-dimensional triangles. We include such a picture for $[2]$

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \tag{2}$$

The morphisms of Δ are simply functors between this categories, or in other words, they are order preserving morphisms. Our model independent characterization of the category of simplicial sets sSet is the free completion (closure under colimits) of the category Δ. Indeed, the category of functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$ enjoys this universal property.

Theorem 2 (Universal Property). *For any category $\mathcal{D}$ closed under colimits, and any functor $\Delta \to \mathcal{D}$ we have the following diagram*

$$\begin{tikzcd} \Delta \arrow[r, "y"] \arrow[dr] & \mathrm{sSet} \arrow[d, dashed] \\ & \mathcal{D} \end{tikzcd} \tag{3}$$

Such that the induced functor $\mathrm{sSet} \to \mathcal{D}$ *is colimits preserving and essentially unique.*

Example 3. If we consider the functor $\Delta \to \mathrm{Top}$ defined by $[n] \mapsto \Delta^n$, the induced functor $\mathrm{sSet} \to \mathrm{Top}$ is called geometric realization, and its image are precisely the CW-complexes.

We have characterized the category of simplicial sets as the free completion of Δ. Don't worry to much about the realization of the category of simplicial sets as functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$, this will be covered in detail next time. It will be more useful for this lecture to just have an intuitive understanding of what this category is supposed to be.

Example 4. An essential example for us will be the horn Λ^2_1 which can be realized as a certain colimit.

$$[1] \leftarrow [0] \rightarrow [1] \tag{4}$$

Draw a picture during the lecture. Which gives us the following figure

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd} \tag{5}$$

Ill now ask you to convince yourself that a category can be modeled by a simplicial set. A category has arrows and morphisms, and those have a direct translation to simplicial sets. But a category also has the notion of composition of morphisms, we proceed to discuss how we can encode this into simplicial sets. Given compossible morphisms f and g we say that h is the composition of f and g if there is a 2-simplex filling up the horn

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] & & \cdot \end{tikzcd} \qquad \begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"] & & \cdot \end{tikzcd} \tag{6}$$

Therefore, a category $\mathcal{C}$ has the following lifting property: for any morphism $\Delta^2_1 \to \mathcal{C}$ there exists a unique lift $\Delta^2 \to \mathcal{C}$. This can be better described by the following diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd} \tag{7}$$

In here one can notice an important point about the philosophy of higher categories, we do not require h to be equal to the composition of $g \circ f$, we just require it to be homotopic. However, if only one such homotopy exists, there is not much difference with requiring it to be equal.

Definition 5. An ∞-category $\mathcal{C}$ is a simplicial set $\mathcal{C}$ such that every inner horn $\Lambda^n_i \to \mathcal{C}$ has a filling. In other words, for $0 < i < n$ there exists a lift

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{8}$$

This construction has the following important properties:

- In higher category theory we do not require the filling to be unique, so it may appear that composition is not well defined. However, the analog of unique in higher categories is: there exists a contractible space of choices. More concretely, one can proof that if there are two fillings to a inner horn, then there exists a homotopy between them. By extending this idea to higher dimensions, we see that there is only a contractible space of choices.

- If we want to be more precise, this is a model for $(\infty, 1)$-categories, meaning that all higher morphisms are invertible. Invertibility of higher morphisms is again a consequence of the filling axioms.

- There exists a functor $\mathrm{Map}(x, y)$ for any two objects of an ∞-category $\mathcal{C}$. This map is now a simplicial set, where all the morphisms are invertible. This is called an ∞-groupoid and its homotopy theory is equivalent to that of spaces.

Example 6. Maybe the most important example of an ∞-category for us is the category of chain complexes on an abelian category $\mathcal{A}$. In order to see how this makes an ∞-category, recall that the category of chain complexes can be enriched over chain complexes. There is a procedure for constructing an ∞-category from a category enriched over chain complexes, we will not discuss this further.

Remark 7. In later talks, we will see how we can go between the world of chain complexes and spaces, therefore we now know that the category of chain complexes is a category enriched over spaces. You may be thinking, why is this not a good enough model for higher categories? Indeed, in a precise sense the model we of ∞-categories we have described here, and that of categories enriched over spaces are equivalent. However, the language of ∞-categories happen to be more convenient to work with. One of the reasons is that enriched categories are just ordinary categories, equipped with a functor $\mathrm{Map}(-, -)$ which gives you the enrichment. In other words, the higher morphisms, and the regular category somehow life separately. However, in the language of ∞-categories, objects, morphisms and higher morphisms are all put under the same footing. This happens to be very convenient.

Stable ∞-Categories

Stable ∞-categories are the higher categorical analog of abelian categories. In fact, its axioms should resemble the axioms of abelian categories.

Definition 8. A stable ∞-category, is an ∞-category with the following properties

- It is pointed. That is, its initial and final object are equivalent, this object is called the zero object and denoted by 0.
- A triangle is a pushout if and only if it is a pullback. Recall that a triangle is a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{9}$$

  If it is a pushout/pullback then it is called a fiber/cofiber sequence. In other words, a in the world of stable ∞-categories fiber and cofiber sequences coincide. The object Z is often called the mapping cone.

- Every morphism admits a fiber and cofiber.

Example 9. Let $f : X \to Y$ be a morphism of CW-complexes, then its cofiber is given by $C(f)$, where $C(f)$ is the mapping cone of f.

Example 10. Let $f : M \to N$ be a morphism in Mod_R of projective objects, then its cofiber $C(f)$ is the chain complex $0 \to M \to N \to 0$. An important property of this example is that $H_0 C(f) = N/M$. This shows another important point in the philosophy of higher categories, we remember why things are equal, ie, the chain complex $0 \to M \to N \to 0$ remembers why the elements of N/M are equal.

Notice that a fiber/cofiber sequence seem to behave like exact sequences. I would like to highlight that a fiber/cofiber sequence $X \to Y \to Z$, come equipped with the data of a nullhomotopy of $X \to Z$. Being more precise, fiber/cofiber sequences are the ∞-categorical enhancement of exact triangles. Lets set up some notation, and we will come back to this. We will see that given a stable ∞-category $\mathcal{C}$, then $h\mathcal{C}$ comes equipped with a canonical triangulated structure, coming from the fiber/cofiber sequences.

Remark 11. One attractive feature of the theory of stable ∞-categories is that stability is a property of ∞-categories, rather than additional data. As opposed to triangulated categories, where triangles are additional data in the category.

Before proceeding with the relation between stable ∞-categories and triangulated categories I include some examples of stable ∞-categories.

Example 12. Recall that a spectrum consists of an infinite sequence of pointed topological spaces $\{X_i\}_{i\geq 0}$, together with homeomorphisms $X_i \simeq \Omega X_{i+1}$, where Ω denotes the loop space functor. The collection of spectra can be organized into a stable ∞-category Sp. Moreover, Sp is in some sense the universal example of a stable ∞-category.

Example 13. Let $\mathcal{A}$ be an abelian category. Under mild hypotheses, we can construct a stable ∞-category $\mathcal{D}(\mathcal{A})$ whose homotopy category $h\mathcal{D}(\mathcal{A})$ can be identified with the derived category of $\mathcal{A}$, in the sense of classical homological algebra.

One important feature of the category of spectra is that the suspension Σ and loop functor Ω define equivalences, and are inverses of each other. What are suspensions and loops in an arbitrary ∞-category? Given objects X and Y in a pointed category $\mathcal{C}$, consider the following pushout (resp. pullback) squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} \Omega Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} \tag{10}$$

In fact, we can set this up to get well define functors

$$\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega \tag{11}$$

Notation 14. If $\mathcal{C}$ is a stable ∞-category, and $n \geq 0$, we let

$$X \mapsto X[n] \tag{12}$$

denote the nth power of the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$. And by $X \to X[-n]$ the nth power of the loop functor $\Omega : \mathcal{C} \to \mathcal{C}$. This coincides with the homological grading in classical homological algebra.

One of the main advantages of the language of higher categories, is that the operation of taking limits of ∞-categories is well behaved. In fact, there is a universal way of stabilizing a pointed category $\mathcal{C}$. The category $\mathrm{Sp}(\mathcal{C})$ defined as

$$\mathrm{Sp}(\mathcal{C}) := \lim \left(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right) \tag{13}$$

is a stable ∞-category, where the limit is taken in the ∞-category of ∞-categories. The category $\mathrm{Sp}(\mathcal{C})$ can be characterized as the universal stabilization of $\mathcal{C}$. In particular, $\mathrm{Sp}(\mathrm{Top}) =$ Spectra.

We now come back to the fact that the homotopy category of a stable ∞-category has a canonically induced triangulated structure.

Definition 15. Let $\mathcal{C}$ be a stable ∞-category. We say that a diagram

$$X \to Y \to Z \to X[1] \tag{14}$$

in $h\mathcal{C}$ is a distinguished triangle, if it is the image of the following diagram in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd} \tag{15}$$

where both squares are pullback/pushout squares.

Theorem 16. *Let $\mathcal{C}$ be a stable ∞-category, then $h\mathcal{C}$ comes equipped with a canonical triangulated structure, given by the distinguish triangles of $\mathcal{C}$.*

Definition 17. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. And suppose that F preserves the zero object, so it sends triangles to triangles. If in addition, F preserves fiber/cofiber sequences then we say it is exact.

Proposition 18. *Let $F : \mathcal{C} \to \mathcal{C}'$ is a functor between stable ∞-categories. Then the following are equivalent*

- *The functor F is left exact. That is, F commuted with finite limits.*
- *The functor F is right exact. That is, F commutes with finite colimits.*
- *The functor F is exact.*

Remark 19. One can show, without much trouble that every stable ∞-category is canonically enriched over spectra.

Derived Categories and Homological Algebra

Homological algebra provides a rich supplement of stable ∞-categories. Suppose that $\mathcal{A}$ is an abelian category with enough projective objects. In this section, we will explain how to associate to $\mathcal{A}$ an ∞-category $\mathcal{D}^-(\mathcal{A})$, which we call the derived ∞-category of $\mathcal{A}$, whose objects can be identified with (right-bounded) chain complexes with values in $\mathcal{A}$. The ∞-category $\mathcal{D}^-(\mathcal{A})$ is stable, and its homotopy category $h\mathcal{D}^-(\mathcal{A})$ can be identified (as a triangulated category) with the usual derived category of $\mathcal{A}$.

Jumping a bit ahead, the stable ∞-category $\mathcal{D}^-(\mathcal{A})$ is equipped with a t-structure, and there is a canonical equivalence of abelian categories $\mathcal{A} \to \mathcal{D}^-(\mathcal{A})^\heartsuit$. Moreover, we will show that if $\mathcal{C}$ is any stable ∞-category equipped with a left-complete t-structure, then any right exact functor $\mathcal{A} \to \mathcal{C}^\heartsuit$ extends (in an essentially unique way) to an exact functor $\mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$. By an entirely parallel discussion, if $\mathcal{A}$ is abelian category with enough injective objects, we can associate to $\mathcal{A}$ a left-bounded derived ∞-category $\mathcal{D}^+(\mathcal{A})$.

Definition 20. A differential graded category $\mathcal{C}$ over k, is a category enriched over chain complexes over k. As I mentioned before, there is a way of going between spaces and chain complexes, so you should convince yourself that a differential graded category makes a ∞-category.

Definition 21. Let $\mathcal{A}$ be an abelian category with enough projective objects. We let $\mathcal{D}^-(\mathcal{A})$ denote the ∞-category $\mathrm{Ch}^-(\mathcal{A}_{\text{proj}})$. Dually, we can define $\mathcal{D}^+(\mathcal{A})$ if $\mathcal{A}$ has enough injective objects. Notice that $\mathrm{Ch}^-(\mathcal{A}_{\text{proj}})$ is a priori just a category enriched over chain complexes, but there is a procedure to construct an ∞-category from this.

Remark 22. The reason why we restrict ourselves to projective objects is because by doing so, the notion of quasi isomorphism and homotopy equivalence coincide.

Proposition 23. *The ∞-categories $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^+(\mathcal{A})$ are stable ∞-categories (if they exist).*

In order to proceed, we would need the notion of a t-structure. However, I believe that it might be to heavy to include in this "soft" talk. So lets include some "organic" features of what a t-structure allows us to do, while providing analogs to more familiar ideas in homological algebra.

Let $\mathcal{C}$ be a stable ∞-category with a t-structure, then are able to do the following things:

- Part of the data given is that of full subcategories $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$. For example, in $\mathcal{D}^-(\mathcal{A})$ we have that $\mathcal{D}^-(\mathcal{A})_{\geq 0}$ correspond to chain complexes with non-zero homology concentrated in non-negative degree. Similarly we can define $\mathcal{D}^-(\mathcal{A})_{\leq 0}$, or replace the zero by any n.

- This full subcategories come equipped with truncation functors $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$. In fact, in this abstract setting, we use this truncation functors to define homotopy/homology groups as

$$\tau_{\geq 0} \circ \tau_{\leq 0} \simeq \tau_{\leq 0} \circ \tau_{\geq 0} = \pi_0 : \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit} = \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0} \tag{16}$$

  We can define π_n be precomposing π_0 with $X \mapsto X[-n]$. Of course, in $\mathcal{D}^-(\mathcal{A})$ this would coincide with the usual homology groups.

- The category $\mathcal{C}^{\heartsuit}$ is an abelian category. In order to show this, one notices that for $X, Y \in \mathcal{C}^{\heartsuit}$ the homotopy groups $\pi_n \operatorname{Map}(X, Y)$ vanish for $n > 0$. In fact, we have that $\mathcal{D}^-(\mathcal{A})^{\heartsuit} \simeq \mathcal{A}$. However, it might be important to note that in general $\mathcal{C} \not\simeq \mathcal{D}^-(\mathcal{C}^{\heartsuit})$.

- It may be worth pointing out that the inclusion $\mathcal{A} \subset \mathcal{D}^-(\mathcal{A})$ amounts to taking projective resolutions. This means that the choice of a projective resolution if unique up to a contractible space of choices.

We can finally talk about the theory of derived functors in this setting.

Theorem 24. *Any right exact functor $f : \mathcal{A} \to \mathcal{C}^{\heartsuit}$ can be extended (in an essentially unique way) to an exact functor $F : \mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$, such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, hook] & \mathcal{A} \\ \mathcal{D}^-(\mathcal{A}) \arrow[r, "F"] & \mathcal{D}^-(\mathcal{A}) \arrow[u, "\tau_{\leq 0}"'] \end{tikzcd} \tag{17}$$

In particular, if $\mathcal{C}^{\heartsuit}$ has enough projective objects, then we obtain an induced map $\mathcal{D}^-(\mathcal{C}^{\heartsuit}) \to \mathcal{C}$.

Warning 25. Notice that $\tau_{\leq 0}$ does not map $\mathcal{D}^-(\mathcal{A}) \to \mathcal{A}$ in general, however, it does if we restrict to the image of $F|_{\mathcal{A}}$. More precisely, we have that $f \leftrightarrow \tau_{\leq 0} F|_{\mathcal{A}}$. I am including the diagram because it is precise enough, and somewhat instructive.

Proposition 26. *The inclusion map $\mathcal{A} \hookrightarrow \mathcal{D}^-(\mathcal{A})$ is exact. This means that, if $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathcal{A}$ if and only if $X \to Y \to Z \to X[1]$ is a exact triangle in $\mathcal{D}^-(\mathcal{A})$.*