Deforming Kunz Theorem to mixed characteristic

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Recollection of characteristic p case

We begin by recalling Kunz theorem

Theorem 1. *Let R be a noetherian $\mathbb{F}_p$-algebra. Then R is regular if and only if* $\mathrm{Fr} : R \to R$ *is flat.*

Remark 2. Since frobenius is the identity on the underlying topological space of $\operatorname{Spec} R$, requiring that $\mathrm{Fr} : \operatorname{Spec} R \to \operatorname{Spec} R$ is flat is equivalent to requiring it is faithfully flat.

The goal of this lecture is to formulate an analogous theorem in mixed characteristic, however, a problem quickly arises, there is no Frobenius in mixed characteristic. So our first goal is to formulate an equivalent statement which is more amenable to such generalization. This is obtained by the following:

Theorem 3. *Let R be a noetherian $\mathbb{F}_p$-algebra. Then R is regular if and only if the map*

$$R \longrightarrow R_{\mathrm{perf}} := \operatorname{colim}(R \xrightarrow{\mathrm{Fr}} R \xrightarrow{\mathrm{Fr}} R \xrightarrow{\mathrm{Fr}} \cdots) \tag{1}$$

is flat.

So now, all we need to do is to find the correct analog of perfect rings in the mixed characteristic setting, and then we will be able to formulate a mixed characteristic version of Kunz theorem. Perfectoid rings will provide the mixed characteristic analog for perfect rings. The goal of this lecture is to introduce the material necessary to understand the following statement

Theorem 4 (Bhatt-Iyengar-Ma). *Let R be a p-adically complete noetherian ring. Then R is regular if and only if there exists a faithfully flat map $R \to A$ with A perfectoid.*

But first, let's review some properties that perfect rings enjoy.

Example 5. Consider the $\mathbb{F}_p$ algebra $\mathbb{F}_p[T]$, then its perfection is $\mathbb{F}_p[T^{1/p^\infty}]$. So one should think about the perfection as adding pth power roots. We highlight the fact that $\mathbb{F}_p[T^{1/p^\infty}]$ is not noetherian.

After being presented with such an example, one may reasonable ask, why? Non noetherian rings are outside the comfort zone of most commutative algebraist, and algebraic geometry; what makes perfect rings so important that would make people want to leave their comfort zone?

A first answer could be provided by the following result.

Proposition 6. *Let R be a perfect $\mathbb{F}_p$-algebra, then it has vanishing cotangent complex $L_{R/\mathbb{F}_p} = 0$.*

Proof. This will not be a full proof. But let me just say that working naively that it follows from $d(y) = d(x^p) = px^{p-1}dx = 0$. □

This result is basically stating that perfect $\mathbb{F}_p$-algebras are very smooth.

If you have not seen the cotangent complex before, let me say a few words about it. You should think about it as controlling, or describing the tangent spaces of $\operatorname{Spec} R$. More precisely, it is a derived version of the usual Kahler differentials $\Omega_{R/\mathbb{F}_p}$.

Remark 7. In order to understand this talk, all one needs to remember is that if A is perfect then then map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{F}_p$ induces a bijection on tangent spaces. This is unusual, since for example $R = \mathbb{F}_p[T^{1/p^\infty}]$ has Krull dimension 1, but its tangent space at every point is zero dimensional. This may seem like a pathology, but it will be essential for us to deform to be able to deform our situation to characteristic zero.

Deforming to mixed characteristic

In this section, our goal is to formulate a way to deform perfect $\mathbb{F}_p$-algebras to p-adically complete $\mathbb{Z}_p$-algebras. In other words we are after a functor

$$W(-) : \{\text{Perfect algebras over } \mathbb{F}_p\} \longrightarrow \{\text{p-adically complete } \mathbb{Z}_p \text{ algebras}\} \tag{2}$$

The W stands for Witt-vectors, which will now realize using deformation theory.

Let A be a perfect $\mathbb{F}_p$-algebra. We will construct our $\mathbb{Z}_p$-algebra $W(A)$ in the following infinite sequence of steps

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, hook] \arrow[d] & \operatorname{Spec} W_1(A) \arrow[r, hook] \arrow[d] & \operatorname{Spec} W_2(A) \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \operatorname{Spf} W(A) \arrow[d] \\
\operatorname{Spec} \mathbb{F}_p \arrow[r, hook] & \operatorname{Spec} \mathbb{Z}/p^2 \arrow[r, hook] & \operatorname{Spec} \mathbb{Z}/p^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spf} \mathbb{Z}_p
\end{tikzcd} \tag{3}$$

Where each square is a cartesian square, in other words $W_n(A) \simeq W_{n+1}(A) \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n$.

But one may quickly ask, why is this functor well defined? As it is not clear that there is a unique $W(A)$ which satisfies $W(A) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$. But we can provide a heuristic explanation for this phenomenon: recall that the map $\mathbb{F}_p \to A$ induces a bijection on tangent spaces, or more formally, has vanishing cotangent complex $L_{A/\mathbb{F}_p}$. Then one realizes that the map $\operatorname{Spec} \mathbb{Z}/p \hookrightarrow \operatorname{Spec} \mathbb{Z}/p^2$ induces an isomorphism in the underlying topological spaces, and in fact, it only differs in tangent information. So there is a unique way in which we can deform A to $W_1(A)$ such that the map $\mathbb{Z}/p^2 \to W_1(A)$ induces a bijection on tangent information, i.e., such that the cotangent complex $L_{W_1(A)/\mathbb{Z}/p^2} \simeq 0$ vanishes. By repeating this procedure through the maps $\operatorname{Spec} \mathbb{Z}/p^n \hookrightarrow \operatorname{Spec} \mathbb{Z}/p^{n+1}$ we obtain $W(A)$.

- Draw a picture of the deformation of $\operatorname{Spec} \mathbb{Z}/p^n \hookrightarrow \operatorname{Spec} \mathbb{Z}/p^{n+1}$, and how it picks up a prime at infinity.

Remark 8. One can also provide a definition of Witt vectors via Teichmuller expansions. Recall that for any perfect ring A of characteristic p, each $f \in W(A)$ can be written uniquely as

$$f = \sum_{i=0}^{\infty} [a_i] p^i \tag{4}$$

where $a_i \in A$, and $[-] : A \to W(A)$ is the Teichmuller map, which is multiplicative but not additive, and it is a section of the quotient map $W(A) \to A$ defined by sending $p \mapsto 0$.

Question 9. What is $W(\mathbb{F}_p)$?

Example 10. We can provide a very concrete description of $W(\mathbb{F}_p[T^{1/p^\infty}])$, it can be identified with $\mathbb{Z}_p[T^{1/p^\infty}]^\wedge$. The reason why we need to p-adically complete, may be justifies by the fact that we are now allowing power series of p, which need to interact well with T.

- Draw a picture of how $\mathbb{F}_p[T^{1/p^\infty}]$ deforms to characteristic zero.

Perfectoids

So what do perfectoid rings have to do with this? Using the machinery we have developed, we can now produce some simple examples of perfectoid rings.

Example 11. Here are some examples of perfectoid rings. Remember to have the pictures with you.

- $\mathbb{Z}_p[T^{1/p^\infty}]^\wedge/(p-[T]) = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge$

Definition 12. A ring B is perfectoid, if there exists a perfect $\mathbb{F}_p$-algebra A, such that

$$W(A)/(\xi) = B \tag{5}$$

where $\xi = p + \sum_{i \neq 1}[a^i]p^i$, and such that A is ξ/p complete. Or equivalently, A is complete with respect to the image of ξ under the map $W(A) \to A$. In particular, we have that perfectoid rings are p-adically complete.

- Draw a picture of A_{inf}.

Remark 13. So in the example above, we can also realize $\mathbb{Z}_p[T^{1/p^\infty}]^\wedge/(p-[T])$ as the p-adic completion of $\mathbb{Z}_p[[T^{1/p^\infty}]]^\wedge/(p-[T])$.

I hope that this justifies the slogan that perfectoids are deformations to mixed characteristic of perfect rings.

Proposition 14. *Perfectoid rings also have vanishing cotangent complex.*

Theorem 15. *Let R be a p-adically complete noetherian ring. Then R is regular if and only if there exists a faithfully flat map $R \to A$ with A perfectoid.*

References

1. Bhatt, Iyengar, Ma - Regular rings and perfectoid algebras