The Cotangent Complex and the Derived Philosophy

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Algebraic deRahm Cohomology

Let $X/\mathbb{C}$ be a smooth algebraic variety. Then we can consider the complex of differential forms, namely

$$\Omega_X^\bullet := \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^2 \longrightarrow \cdots \tag{1}$$

Note that this is a purely algebraic construction. In algebraic geometry there is this notion of a cohomology theory computing the right cohomology. The following theorem shows that this cohomology theory computes the right cohomology

Theorem 1 (Grothendieck). *We have an isomorphisms*

$$H^*(X, \Omega_X^\bullet) \cong H^*_{Sing}(X^{an}, \mathbb{C}) \tag{2}$$

This is a pretty satisfactory result, since we now have a complete algebraic description of deRahm cohomology, which is a priori an analytic object. Of course this definition works over every base field.

A natural question then arises: what happens when X is singular? Then this cohomology theory still makes sense, but it computes the wrong thing. The underlying reason is that Ω_X is no longer locally free, and therefore it does not have good properties.

There are several answers for deRahm cohomology for singular varieties, a classical answer is due to Hartshorne, which goes by the name of crystalline cohomology, and which relied on the following picture:

Remark 2. Draw picture of a nodal curve, and draw a tubular neighborhood around it.

More formally, the idea is the following: assume we can embedd $X \hookrightarrow Y$ where Y is smooth, and then we can take the formal completion of Y along X. This is the algebraic analog of a tubular neighborhood. We denote this completion as $\hat{Y}_X$.

Theorem 3 (Hartshorne). *The cohomology of this formal completion computes the right thing*

$$H^*(\hat{Y}_X, \Omega^\bullet_{\hat{Y}_X}) \cong H^*_{Sing}(X) \tag{3}$$

where the coefficients of the singular cohomology are on the base field of our scheme.

Once again we have a purely algebraic description of the right cohomology theory for algebraic varieties. A dissatisfying feature of this is that it requires a choice of embedding. However, Crystalline cohomology provides the right formalisms for this, in which we somehow quantify along all such thickenings.

There is another solution to this problem is provided by the derived picture. Recall that a major problem for the naive deRahm cohomology of singular varieties is that Ω_X is not locally free. The derived pictures provides an alternative solution to this problem: we can replace Ω_X by a much richer object, and this object is precisely the cotangent complex $\mathbb{L}_X$. This construction is due to Illusie, this is a complex of sheaves, and it lives in the derived category of coherent sheaves, but feel free to ignore that.

Theorem 4. *If X is smooth, then $\mathbb{L}_X \simeq \Omega_X$. So it computes the right thing.*

However when X is singular it has higher cohomology groups, and this higher cohomology groups somehow have some information about the singularities. With this we can form the derived deRahm complex

Definition 5. The derived deRahm complex is the following

$$\hat{\mathrm{dR}}_X := \mathcal{O}_X \longrightarrow \mathbb{L}_X \longrightarrow \bigwedge^2 \mathbb{L}_X \longrightarrow \cdots \tag{4}$$

Note that this is a priori a double complex, but there are ways of making sense of this as a complex. We ignore this technical difficulty and pretend this works as we expect. And this is a theorem of Bhargav Bhatt that it computes the right cohomology. It may be worth noting that this was proved by Illusie if X is a locally complete intersection.

Theorem 6 (Bhatt). *This computes the right cohomology always*

$$H^\bullet(X, d\hat{R}_X) \cong H^*_{Sing}(X) \tag{5}$$

Now that we have provided some motivation for this, we will spend the rest of the talk trying to construct this derived object, while trying to provide some intuition behind the ideas of this derived non sense.

Homotopical Algebra

Lets begin with a quick review of Tor and Ext, in order to get acquainted with the language of homotopical algebra, through examples.

We can define a functor

$$L : \mathrm{Mod}_R \longrightarrow \mathrm{Ch}_{\geq 0}(R) \tag{6}$$

which maps M to its free resolution. Recall that there exists a map $L(M) \to M$ such that we have a quasi isomorphisms. This is what I mean by approximating, we are obtaining a better behaved object $L(M)$, which is close enough to M.

While it is true that taking free resolutions involves a choice, it is possible to make this functorial. In fact, we can identify $\mathrm{Tor}^n_R(M, N)$ as the following composition of functors

$$\mathrm{Mod}_R \xrightarrow{L(-)} \mathrm{Ch}_{\geq 0}(R) \xrightarrow{-\otimes_R N} \mathrm{Ch}_{\geq 0}(R) \xrightarrow{H^n(-)} \mathrm{Mod}_R \tag{7}$$

Or better, we can talk about the left derived functor $- \otimes^{\mathbf{L}}_R N$, which is defined as the composition

$$- \otimes^{\mathbf{L}}_R N : \mathrm{Ch}_{\geq 0} \xrightarrow{L(-)} \mathrm{Ch}_{\geq 0}(R) \xrightarrow{-\otimes_R N} \mathrm{Ch}_{\geq 0}(R) \tag{8}$$

Why is this useful? For example, we know that if N is free, then $- \otimes_R N$ exact, and not only right exact. Since we are approximating our modules by free resolutions, we obtain that $- \otimes^{\mathbf{L}}_R N$ is always exact. In other words, by approximating M by better behaved objects we obtain a better behaved tensor product.

Remark 7. There is a cliche sentences that gets thrown around by homotopy theorist quiet a bit, and it is that to get finer results we must remember why things are equal. This is exactly what taking a free resolution does, we remember why our objects are equal.

Remark 8. Is there a way in which we can make precise the notion of approximating? Yes. Recall that we can consider the homotopy category $\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R))$ in which we are inverting the quasi isomorphism. In particular, notice that in $\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R))$, in the isomorphism class of M there is a well behaved object $L(M)$. In fact, we have the following fully faithful embedding

$$M \hookrightarrow \mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R)) \tag{9}$$

Similarly, we can define the right derived functor of $\mathbf{R}\,\mathrm{Hom}(-,N)$

$$\mathbf{R}\,\mathrm{Hom}(-,N) : \mathrm{Ch}_{\geq 0} \xrightarrow{L(-)} \mathrm{Ch}_{\geq 0}(R) \xrightarrow{\mathrm{Hom}(-N)} \mathrm{Ch}_{\geq 0}(R) \tag{10}$$

And where the homology groups are precisely Ext^n_R. You might be confused of why we have right derived functor and we are taking free resolutions, one would think we need to take injective resolutions. The reason is purely formal: the functor $\mathbf{R}\,\mathrm{Hom}(-,N)$ is contravariant.

Lets try to give a more geometric interpretation of a free resolution. Say we have a module M. We want to construct a space from this data, such that that it has the same information as M. For each abelian group M_i we consider a set of n-simplex, where each is identified with some element of M_i. We now want to paste this together, in the same way we construct CW complexes. The corresponding space will be denoted by sM, where the $s-$ stands for simplicial. We construct this space inductively, for the 0-skeleton we just consider the points M_0. Then we can construct the n-skeleton from the $n-1$-skeleton by pasting the boundary of $x \in M_n$ along $\partial x \in M_{n-1}$. So we have constructed a space sM, and there are some things to note about this space:

(1) As a topological space it is a CW complex, but it also some sort of topological abelian group. Where the addition of objects is happening at the level of cells, rather than at the level of points.

(2) More importantly, we get $\pi_n(sM) = H_i(M)$.

(3) In fact we can also translate the derived functor nonsense. However this is more difficult to make sense of, so we will leave it at that.

Definition 9. We will take this as our definition of simplicial abelian group.

The following theorem provides a deep connection between both perspectives.

Theorem 10 (Dold-Kan Correspondence). *There exists an equivalence of categories*

$$\mathrm{Ch}_{\geq 0}(R) \simeq \mathrm{sAb} \tag{11}$$

Moreover, this is homotopically meaningful, it decays to a equivalence of homotopy categories.

We hope that this provides information that homotopy theory subsumes homological algebra. The formalisms I am having in the back of my mind to make this precise is that of Model Categories as developed by Quillen. This allows you to do homological algebra in non abelian settings. In fact when one learns model categories, you quickly realize that the axioms are trying to model homological algebra.

Remark 11. One advantage of this formalism is that we don't require $\partial \circ \partial = 0$. However, we do have some transition data between cells of different dimensions, which will correspond to different face maps of the simplices. This is an important advantage, specially when we want to apply this to simplicial commutative rings. The category of rings is highly non abelian, so requiring $\partial \circ \partial = 0$ in a chain complex of rings, you would have to have one of them equal to zero.

Definition 12. The right analog of commutative rings are called simplicial commutative rings. In the same way as sM, this are space like objects, where we get a ring structure of the cells dimension wise. We can interpret our usual rings are zero dimensional objects in this category.

Proposition 13. *We can take free resolutions in the category of A-algebras. For example consider the following $A \to B$. Then we can present B as*

$$\cdots \longrightarrow A[x_0, x_1, \ldots, x_m] \longrightarrow A[x_0, x_1, \ldots, x_n] \longrightarrow B \tag{12}$$

where $x_i \mapsto f_i$, which cuts out B.

One may ask, what is the kind of geometric object associated to a simplicial commutative ring A. The underlying topological space is isomorphic to $\operatorname{Spec} \pi_0 A$. Recall that $\pi_0 A$ are the connected components of A, which reinforces our intuition that simplices of higher dimension just record in how many ways things are equal. However, the structure sheaf is different. In the same way that Grothendieck style algebraic geometry allows us to take nilpotents into account, derived algebraic geometry remembers higher nilpotents which can be interpreted as the higher homotopy groups of the simplicial ring.

Definition 14. A affine derived scheme is just a topological space $\operatorname{Spec} \pi_0 A$ endowed with a fancy sheaf of simplicial commutative rings. A derived scheme is just a scheme that looks locally like an affine scheme. Fancier pasting conditions are required here in order to maintain the homotopy type meaningful.

Example 15. In this example we will try to illustrate how limits/colimits are not well behaved with respect to homotopy type. Consider the following diagram

$$* \simeq D^1 \longleftarrow S^1 \longrightarrow D^1 \simeq * \tag{13}$$

This example illustrates that colimits are do not respect homotopy type. There are fancier analogs of this that do, we will not discuss this further.

The Cotangent Complex

We will now introduce the cotangent complex as some sort of left derived functor of the module of kahler differentials. Ultimately, we want to think about the cotangent complex as some sort of abelianization, or more precisely, the derived functor of the abelianization. We consider the category $\mathrm{CRing}_{A/}$ (the under category of commutative rings over A), or equivalently, the category of A-Algebras. However, as soon as we try to make this precice we find a problem

Definition 16. An abelian monoid object in a category $\mathcal{C}$ is an object X, together with a multiplication morphism $X \times X \to X$ and a unit $e : * \to X$ (where $*$ is the terminal object). Satisfying some conditions.

So ideally way we want is some sort of adjunction

$$\mathrm{Ab}(\mathcal{C}) \leftrightarrows \mathcal{C} \tag{14}$$

where one map is the canonical inclusion, and $\mathcal{C} \to \mathrm{Ab}(\mathcal{C})$ is a left adjoint, which is the abelianization functor. In other words, we want some sort of free forgetful adjunction. But we quickly notice that there are no interesting abelian group objects in $\mathrm{CRing}_{A/}$. The reason is that the final object in $\mathrm{CRing}_{A/}$ is zero, and only one object have a map from zero, namely zero itself. So we need to fix this.

Proposition 17. *There is an equivalence of categories between* $\mathrm{Ab}(\mathrm{CRing}_{A//B}) \simeq \mathrm{Mod}_B$.

Lets unpack this for a little bit. The objects of $\mathrm{CRing}_{A//B}$ are objects of the form $A \to X \to B$, and where morphisms are commutative diagrams

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X_1 \arrow[rr] \arrow[dr] & & X_2 \arrow[dl] \\ & B & \end{tikzcd} \tag{15}$$

And there is a morphisms $\mathrm{Mod}_B \hookrightarrow \mathrm{CRing}_{A//B}$ defined by $M \mapsto B \oplus M$, where the structure maps $A \to B \oplus M \to B$ are the canonical ones. This inclusion map defines an equivalence of categories $\mathrm{Ab}(\mathrm{CRing}_{A//B}) \simeq \mathrm{Mod}_B$. Then what is its left adjoint

Theorem 18. *The left adjoint to the inclusion functor* $\mathrm{Mod}_B \hookrightarrow \mathrm{CRing}_{A//B}$ *is defined by* $X \mapsto \Omega_{X/A} \otimes_X B$. *In particular, the image of* $A \to B \to B$ *under this functor is* $B \mapsto \Omega_{B/A}$.

Theorem 19. *This adjoint lifts to a homotopicaly meaningful adjunction* $\mathrm{Ch}(B)_{\geq B} \rightleftarrows \mathrm{sCRing}_{A//B}$. *Meaning that it is an adjunction of categories, and it induces an adjunction at the level of homotopy categories.*

Where the functor $\mathrm{sCRing}_{A//B}$ is just the kahler differential applied level wise. That is, given a simplicial commutative ring $P_\bullet$ (remember we are just thinking about this as spaces which have a ring structure at every dimension), then we have that

$$P_\bullet \longmapsto \Omega_{P_\bullet/A} \otimes_{P_\bullet} B \tag{16}$$

This is a left adjoint, so it preserves projectives, therefore we can talk about the left derived functor.

Definition 20. The Cotangent Complex $\mathbb{L}_{B/A}$ is the image of $A \to B \to B$ under the left derived functor of the module of kahler differentials. In other words, let $P_\bullet \to B$ be a free resolution, then

$$\mathbb{L}_{B/A} = \Omega_{P_\bullet/A} \otimes_{P_\bullet} B \tag{17}$$

Some remarks that maybe homotopy theorists care about. The cotangent complex, as defined here, lives in the derived category of Mod_B. Moreover, if we are just choosing a particular resolution of B, then $\Omega_{P_\bullet/A}$ is a cofibrant object in the derived category of $\mathrm{Mod}_{P_\bullet}$, so there is no distinction between the derived tensor product and the regular tensor product.

Remark 21. A natural question may arise, is $\mathbb{L}_{B/A}$ just a free resolution of $\Omega_{B/A}$? This is not true, and its completely formal. The cotangent complex is a left Quillen functor, so it preserves trivial cofibrations. But taking a cofibrant replacement $P_\bullet \to B$ is an trivial fibration.

Lets now state some facts about the cotangent complex

(1) If B is a free polynomial A algebra, then $\mathbb{L}_{B/A} \simeq \Omega_{B/A}$. This follows from the definition, since we dont need to take a free resolution.

(2) Given a composite $A \to B \to C$ of maps, we have a canonical exact triangle

$$\mathbb{L}_{B/A} \otimes^{\mathbf{L}}_{B} C \longrightarrow \mathbb{L}_{C/A} \longrightarrow \mathbb{L}_{C/B} \tag{18}$$

(3) For any map $A \to B$ we have $H^0(\mathbb{L}_{B/A}) \cong \Omega_{B/A}$

(4) If $A \to B$ is smooth then $\mathbb{L}_{B/A} \simeq \Omega_{B/A}$. Notice that this is not immediate from the definition, since we still need to take free resolutions. There is some work to be done here.