What are we trying to do?

Attilio Castano

Relation to Langlands

Ultimately, the goal of this seminar is to gain a better understanding of the geometry of

$$\mathcal{M}_0(N) \to \operatorname{Spec} \mathbb{Z} \tag{1}$$

called the moduli stack parametrizing generalized elliptic curves with level structures.

Why do we care? The space $\mathcal{M}_0(N) \to \operatorname{Spec} \mathbb{Z}$ has a lot of interesting symmetries, which have important consequences. Here is an example of the kind of symmetries it has.

For each prime number p we have a correspondence, i.e., a diagram of the form

$$\begin{tikzcd} & \mathcal{M}_0(pN) \arrow[dl] \arrow[dr] & \\ \mathcal{M}_0(N) & & \mathcal{M}_0(N) \end{tikzcd} \tag{2}$$

which induces a morphism at the level of derived categories, given by pull-push

$$T_p : \operatorname{Coh}(\mathcal{M}_0(N)) \longrightarrow \operatorname{Coh}(\mathcal{M}_0(N)) \tag{3}$$

called a Hecke operator.

Hecke operators are important because they provide a bridge between the galois and automorphic side of the langlands program. In what follows one should keep in mind the following dictionary

- Vector space - $\operatorname{Coh}(\mathcal{M}_0(N))$
- Linear transformation - T_p.
- Eigenvector - Hecke Eigensheaf

Let us say that, $\mathcal{F} \in \operatorname{Coh}(\mathcal{M}_0(N))$ is a Hecke eigensheaf if there exists a local system L_ρ associated to a galois representation, such that

$$T_p(\mathcal{F}) \simeq L_\rho \boxtimes \mathcal{F} \tag{4}$$

In the langlands program, when one tries to go from the galois to automorphic side, given a galois representation ρ one tries to construct a Hecke eigensheaf with eigenvalues given by the galois representation.

Ok, so now we know why we care about $\mathcal{M}_0(N) \to \operatorname{Spec} \mathbb{Z}$, it has interesting symmetries, and moreover, we can study this symmetries by studying the cohomology of $\mathcal{M}_0(N)$.

Geometry of $\mathcal{M}_0(N)$

You may ask, what makes the geometry of $\mathcal{M}_0(N)$ so interesting? Where are the symmetries coming from?

Recall that we said that $\mathcal{M}_0(N)$ is the moduli stack parametrizing generalized elliptic curves with level structure. We begin by providing rough definitions of what kind of object this is.

Definition 1. An elliptic curve $X \to \operatorname{Spec} \mathbb{Z}$ is a smooth proper map, of relative dimension one, together with a group structure on X.

- Draw a picture of $X \to \operatorname{Spec} \mathbb{Z}$, and explain what does smooth proper conditions imply.

So with this definition, one can quickly define a moduli problem, which will parametrize elliptic curves over $\operatorname{Spec} \mathbb{Z}$. But there is a problem

Theorem 2. There are no elliptic curves over the integers.

This is exactly why we need to add the adjective "generalized" elliptic curves, in order to have elliptic curves over $\operatorname{Spec} \mathbb{Z}$ to which we can add level structures. The problem above is that for every candidate elliptic curve $E \to \operatorname{Spec} \mathbb{Z}$, there will be finitely many points $\operatorname{Spec} \mathbb{F}_p \hookrightarrow \operatorname{Spec} \mathbb{Z}$ where $E_{\mathbb{F}_p} \to \operatorname{Spec} \mathbb{F}_p$ will not be smooth. The adjective generalized comes from the fact that we are allowing mild singular curves at finitely many $\operatorname{Spec} \mathbb{F}_p \hookrightarrow \operatorname{Spec} \mathbb{Z}$.

Definition 3. We now provide a definition of our main object of study:

$$\mathcal{M}_0(N) : \mathrm{Sch}^{\mathrm{op}}_{/\operatorname{Spec} \mathbb{Z}} \longrightarrow \mathrm{Sets} \qquad S \mapsto \{ \text{ generalized elliptic curves with } \Gamma_0(N) \text{ level structure } /S\} \tag{5}$$

and a $\Gamma_0(N)$ level structure is a diagram of the form

$$\begin{tikzcd} G \arrow[rr, hook] \arrow[dr] & & E \arrow[dl] \\ & S & \end{tikzcd} \tag{6}$$

where G is a finite flat group scheme, of degree n, which is also a cyclic subgroup of E.

Example 4. We specialize to the case when $S = \operatorname{Spec} \mathbb{Z}$. Then there are two main examples of finite flat group schemes

- Explain what does finite flat means from a geometric point of view.
- The constant group scheme $\mathbb{Z}/N\mathbb{Z}$. This group scheme is etale over $\operatorname{Spec} \mathbb{Z}$, i.e. it has no ramifications. Draw picture.
- The group scheme μ_N. This group ramifies exactly at the points where $p|N$. Draw picture.

So one could think that $\mathcal{M}_0(N)$ is parametrizing the quotients $E/G \to S$, but it quickly becomes confusing when $G = \mu_N$, what does it mean to take a quotient by something like μ_N? This is a difficult question, and it will require us to talk about cohomological deformations of schemes, we postpone this discussion for now.

Let's try to understand where the symmetries of $\mathcal{M}_0(N)$ come from. We can consider the functor, associated to a generalized elliptic curve $\mathcal{E} \to \operatorname{Spec} \mathbb{Z}$, defined by

$$\mathcal{E}_0(N) : \mathrm{Sch}^{\mathrm{op}}_{/\operatorname{Spec} \mathbb{Z}} \longrightarrow \mathrm{Sets} \qquad S \mapsto \{ \Gamma_0(N) \text{ level structure on } \mathcal{E} \times_{\mathbb{Z}} S \to S\} \tag{7}$$

There is clearly an embedding $\mathcal{E}_0(N) \hookrightarrow \mathcal{M}_0(N)$, so if we would like to understand the symmetries in the geometry of $\mathcal{M}_0(N)$, its reasonable to first try to understand the symmetries in the geometry of $\mathcal{E}_0(N)$.

One of the main theorems of Katz-Mazur says that

Theorem 5. The functor $\mathcal{E}_0(N)$ is represented by a finite flat scheme $\mathcal{E}_0(N) \to \operatorname{Spec}\mathbb{Z}$, and its etale over $\operatorname{Spec}\mathbb{Z}[1/N]$.

The etaleness of $\mathcal{E}_0(N)$ over $\operatorname{Spec}\mathbb{Z}[1/N]$ is related to the following phenomenon

Proposition 6. Any finite flat group scheme $G \to \operatorname{Spec}\mathbb{Z}$, of order N, is etale over $\operatorname{Spec}\mathbb{Z}[1/N]$.

Putting this together, this is saying that over $\operatorname{Spec}\mathbb{Z}[1/N]$ there are only finitely many embeddings

$$\begin{tikzcd} G \arrow[rr, hook] \arrow[dr] & & \mathcal{E} \arrow[dl] \\ & \mathbb{Z}[1/N] & \end{tikzcd} \tag{8}$$

and it is also saying that this embeddings are "discrete", i.e., they do not deform from one to the other. However, something subtle happens at the primes where $p|N$. This is because over this primes we can have non-etale finite flat group schemes. For example we can consider

- $\mathbb{Z}/p\mathbb{Z} \times \mu_p$
- μ_{p^2}

and the scheme $\mathcal{E}_0(N)$ not being etale over $\mathbb{F}_p$ where $p|N$ is saying that the different embeddings $G \hookrightarrow \mathcal{E}$ can deform from one to the other, or in other words that we can deform from E/G_1 to E/G_2. It is in this points, where $\mathcal{E}_0(N) \to \operatorname{Spec}\mathbb{Z}$ is not etale is where the symmetries from the Hecke operators T_p come from.

Cohomological Deformations

Recall that we would like to understand the cohomology of $\mathcal{M}_0(N)$, or more precisely, we would like to understand the derived category $\operatorname{Coh}(\mathcal{M}_0(N))$. In particular, there should exists an object $\omega \in \operatorname{Coh}(\mathcal{M}_0(N))$ such that the fiber at each point in $\mathcal{M}_0(N)$, which corresponds to an elliptic curve with level structure (E, G), should give us the de Rham cohomology of (E, G).

What is the de Rahm cohomology of (E, G)? de Rahm cohomology is a cohomological invariant assign to an object of algebraic geometry, and (E, G) is so far not an object of algebraic geometry. A naive guess could be that we are looking for the de Rahm cohomology of E/G, but this gives us the wrong answer. The problem relies in that one characterizes E/G by a certain universal property of the map $G \hookrightarrow E$, but this universal property depends on the ambient category we are working on, in this case Sch. So one could ask, is there a god given category where E/G characterized by the universal property would give us the right object?

Theorem 7. There exists a fully faithfull embedding, of symmetric monoidal categories

$$\operatorname{Coh}^! : \mathrm{Sch}_{/\operatorname{Spec}\mathbb{Z}} \longrightarrow \mathbb{Z}\text{ -linear Categories} \tag{9}$$

which maps $f : X \to Y$ to $f^! : \operatorname{Coh}(Y) \to \operatorname{Coh}(X)$.

Recall that we have that $G \curvearrowright E$, i.e., we have an action of G on E via the embedding $G \hookrightarrow E$. Now, we do not want to take the quotient by this action in this category, rather we want apply the functor $\mathrm{Coh}^!$ and then apply the universal property. After applying the functor $\mathrm{Coh}^!$ we get the following action

$$\mathrm{Coh}(G) \curvearrowright \mathrm{Coh}(E) \tag{10}$$

So now we have a group acting on a category. Now we would like to apply the universal property, and since the functor $\mathrm{Coh}^!$ is contravariant, instead of getting the orbits of the action, we need to take the fixed points so the desired category now is

$$\mathrm{Coh}(E)^G \tag{11}$$

the category derived category of coherent sheaves over E, which are equivariant with the action of G. One can then extract the de Rahm cohomology coming from this category, giving us the desired de Rahm cohomology of (E, G).

References

1. Katz, Mazur - Arithmetic Moduli of Elliptic Curves
2. Snowden - Course on Mazur's Theorem
3. Lurie - Elliptic Cohomology
4. Gaitsgory, Rozenblyum - A study in derived algebraic geometry