Simplicial Sets

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Abstract

When one is first exposed to the definition of a Simplicial Set, it is not clear at all how this seemingly abstract object has a geometric interpretation. The goal of this papers is to motivate why the definition of a simplicial set is a good definition for the kind of object we want to study. For this, we will take a step back and show that the category of simplicial sets is the free cocompletion of the simplex category. This will be the main result of this paper. We will use this formalism to shed some light on singular homology, and show how it can be used to study categories themselves. The main reference for this paper is [1].

1 The Simplex Category

In order to illustrate the geometry behind the definition of simplicial sets, we will formalize the following slogan: a simplicial set is an objects built from n-dimensional triangles. In order to do this, we first need to formalize what we mean by an n-dimensional triangle. Moreover, since we want to built objects out of this triangles, we need to understand how this triangles interact with each other. In this section, our goal will be to understand the simplex category Δ, which is the category of n-dimensional triangles.

Definition 1. The simplex category Δ is the category of n-dimensional triangles. Its objects are totally ordered sets

$$[n] := 0 \to 1 \to 2 \to \cdots \to n \tag{1}$$

and its morphisms are order preserving functions. It is essential to note that the object $[n]$ can be interpreted as a category in itself. The objects of $[n]$ are the numbers $\{0, 1, \ldots, n\}$ and there exists a unique morphism $m \to k$ if $m \leq k$. Following this line of though, we can interpret Δ as the category of categories $[n]$, whose morphisms are all possible natural transformations.

While the first definition of Δ is in some sense more elementary, I believe that the second definition is better to understand the geometry of simplicial sets. Since with the second definition it becomes clear that $[2]$ is the two dimensional triangle, as depicted by the diagram bellow

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 \arrow[rr] & & 2 \end{tikzcd} \tag{2}$$

In the same way that understanding the objects of Δ is important, it is equally important to understand the morphisms between the objects. We will see that one can interpret a morphism $[n] \to [m]$ as a way in which one can paste $[n]$ onto $[m]$. For this, we now provide the definition of two very important classes of morphisms.

Definition 2. There are two very important classes of morphisms in Δ, the coface and codegeneracy morphisms:

$$\begin{aligned} d^i &: [n-1] \to [n] \quad 0 \leq i \leq n \quad \text{(cofaces)} \\ s^j &: [n+1] \to [n] \quad 0 \leq j \leq n \quad \text{(codegeneracies)} \end{aligned} \tag{3}$$

This morphisms can be described in terms of the underlying sets of the objects it is mapping

$$d^i(0 \to 1 \to \cdots \to n-1) = (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n) \tag{4}$$

(i.e. compose $i-1 \to i \to i+1$, giving a string of arrows of length $n-1$ in $[n]$), and

$$s^j(0 \to 1 \to \cdots \to n+1) = (0 \to 1 \to \cdots \to j \to j \to \cdots \to n) \tag{5}$$

(i.e. insert the identity Id_j in the j^{th} place, giving a string of length $n+1$ in $[n]$).

The reason why this classes of morphisms are important is because of the following proposition, which informally says that d^j and s^j are the set of generators of morphisms in Δ.

Proposition 1. *Every morphisms $[m] \to [n]$ in Δ can be presented as a composition of maps of the form d^j and s^j. In this sense we can think of the coface and codegeneracy maps as the generators of the morphisms in Δ. Moreover, it is immediate from the definition that they satisfy the following relations, called the cosimplicial identities*

$$\begin{cases} d^j d^i = d^i d^{j-1} & \textit{if } i < j \\ s^j d^i = d^i s^{j-1} & \textit{if } i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \textit{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \textit{if } i \leq j \end{cases} \tag{6}$$

Proof. This is a fairly simple result, so we will leave the proof as an exercise to the reader. □

At this point, the reader my ask themselves: why should we care about the simplex category Δ, that has objects and morphisms that are not familiar to us; as opposed to the category Δ_{Top}, whose objects and morphisms are more familiar to us. The category Δ_{Top} is the category of n-dimensional topological triangles. More precisely, the objects are

$$[n]_{\mathrm{Top}} := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0 + \cdots + x_n = 1, x_i \geq 0\} \tag{7}$$

and the morphisms $[n]_{\mathrm{Top}} \to [m]_{\mathrm{Top}}$ are continuous functions between them. The reason why we prefer Δ is that it is a much simpler category than Δ_{Top}. While the objects may be in a canonical correspondence, the morphisms are not; in fact, there are a lot more morphisms in Δ_{Top} than in Δ.

In order to illustrate the simplicity of the morphisms of Δ we will perform some computations.

Example 1. The set of morphisms $[1] \to [0]$ is:

$$\mathrm{Hom}_{\Delta}([1], [0]) = \{0 \to 0\} \tag{8}$$

A word must be said regarding the notation used above. The morphism $[1] \to [0]$ denoted by $0 \to 0$, refers to the morphisms $(0 \to 1) \Rightarrow (0 \to 0)$, in which $[1] \ni 0 \mapsto 0 \in [0]$ and $[1] \ni 1 \mapsto 0 \in [1]$. More precisely this morphisms corresponds to the map $s^0 : [1] \to [0]$. We will keep describing the maps with the notation above, since it provides allows for geometric intuition to be useful.

Example 2. The set of morphisms $[1] \to [2]$ is:

$$\mathrm{Hom}_{\Delta}([1], [2]) = \{0 \to 0, 1 \to 1, 2 \to 2, 0 \to 1, 1 \to 2, 0 \to 2\} \tag{9}$$

Note how simple it is to describe the morphisms between to objects on Δ. This is in fact what makes Δ so special as opposed to Δ_{Top}. Continuous functions can be very badly behave, for example it is a well known result that there exists a continuous morphism $[1]_{\mathrm{Top}} \to [2]_{\mathrm{Top}}$ which is surjective, while we see that no morphisms in $\mathrm{Hom}_{\Delta}([1], [2])$ is surjective (i.e. an epimorphism).

Due to the simplicity of the category Δ is that mathematicians have decided to use it as the building blocks for simplicial sets.

2 Simplicial Sets as Presheaves

In this section we will finally define what a simplicial set is: a simplicial set X is a functor $\Delta^{\mathrm{op}} \to Set$. As you can see, this definition does not seem to encode any geometric meaning. The goal of this chapter will be to motivate why this is a natural definition to have, and hopefully illustrate some of the geometry behind it. In order to do this we will take a step back and consider the category of all simplicial sets. We will show that the category of all simplicial sets enjoys a certain kind of universal property, which informally means that: it is the category whose objects are obtained by freely pasting n-dimensional triangles.

Definition 3. A *presheaf* on a (small) category $\mathcal{C}$ is a contravariant functor $\mathcal{C}^{op} \to Set$; morphisms of presheaves are just natural transformations of functors. The category of presheaves on $\mathcal{C}$ will be denoted $Pre(\mathcal{C})$.

Definition 4. A simplicial set X is a functor $\Delta^{\mathrm{op}} \to Set$, and the category of simplicial sets is denoted by $sSet$. This is equivalent to saying that a simplicial set is an object in the presheaf category $Pre(\Delta)$.

We aim to justify the slogan that says that the category of simplicial sets is the category obtained by freely pasting n-dimensional triangles. The first requirement for this is, of course that to each $[n]$ there exists a corresponding simplicial set. This is resolved by the following two famous results in category theory.

Theorem 1 (Yoneda Lemma). *For any functor $F : \mathcal{C} \to Set$, whose domain is locally small and any object $c \in \mathcal{C}$, there is a bijection*

$$\mathrm{Nat}(\mathcal{C}(-,c), F) \cong F(c) \tag{10}$$

That associated a natural transformation $\alpha : C(-,c) \Rightarrow F$ to an element $\alpha_c(1_c) \in F(c)$. Moreover this correspondence is natural in both c and F.

Corollary 1 (Yoneda Embedding). *There exists a fully faithful embedding $\mathcal{Y} : \mathcal{C} \hookrightarrow Pre(\mathcal{C})$. This is the Yoneda embedding. It maps $c \mapsto \hom_{\mathcal{C}}(-,c)$. The presheaves on the image of $\mathcal{Y}$ are called representable functors.*

The Yoneda Embedding in particular shows that to each $[n]$ we can assign it a simplicial set $\Delta^n = \mathrm{Hom}_{\Delta}(-,[n])$ that will interact in the same way with the rest of n-dimensional triangles. We call the objects in the image of $\mathcal{Y}$ representables. Moreover we will follow the notation that $\mathcal{Y}(X) = rX$.

Before proceeding with more general theory, we wish to discuss some concrete examples of simplicial sets. For this we need a good mechanism to write down simplicial sets. The classical way of doing this is by writing down sets Y_n, together with maps

$$\begin{aligned} &d_i : Y_n \to Y_{n-1} \quad 0 \geq i \geq n \quad \text{(faces)} \\ &s_j : Y_n \to Y_{n+1} \quad 0 \geq j \geq n \quad \text{(degeneracies)} \end{aligned} \tag{11}$$

satisfying the *simplicial identities*:

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j \end{cases} \tag{12}$$

The existence of this morphisms are just formal consequence of the definition of a simplicial set as a functor $\Delta^{\mathrm{op}} \to Set$, and the description of the morphisms in the simplex category.

Example 3. Consider the standard n-simplex denoted by $\Delta^n = (-, [n])$. Note that $\mathrm{Id}_{[n]} \in \Delta^n([n])$, this is just saying that the identity morphism $id_{[n]} : [n] \to [n]$ is a morphism in Δ. What is interesting is that one can generate every element of $\Delta^n([m])$, for any $[m]$ by just applying a combination of face and degeneracy maps to $\mathrm{Id}_{[n]} \in \Delta^n([n])$. To see this more concretely we will write down the values that Δ^2 takes when evaluated at $[2]$. We have

$$\Delta^2([2]) = \begin{cases} 0 \to 0 \to 1 & 0 \to 1 \to 1 \\ 0 \to 0 \to 2 & 0 \to 2 \to 2 \\ 1 \to 1 \to 2 & 1 \to 2 \to 2 \\ 0 \to 0 \to 0 & 1 \to 1 \to 1 \\ 2 \to 2 \to 2 & 0 \to 1 \to 2 \end{cases} \tag{13}$$

It is easy to see that every element of $\Delta^2([n])$ can be written down as the image of the map $\mathrm{Id}_{[2]}$ under face and degeneracy maps.

Example 4. Another central example is the boundary of the standard n-simplex, denoted by $\partial\Delta^n$. In this example, unlike in the previous one, I cannot yet justify why this simplicial set I am about to describe corresponds to the boundary of Δ^n. However, the reader should have an intuitive understanding of what this geometric object is, and I think it illustrates an important point of how simplicial sets behave. In general we have that $\partial\Delta^n([n]) = \Delta^n([n]) \setminus \{\mathrm{Id}_{[n]}\}$, and every element of $\Delta^n([m])$ be obtained by applying face and degeneracy maps to the elements of $\partial\Delta^n([n])$. To see this more concretely we will write down the values that $\partial\Delta^2$ takes when evaluated at $[2]$. We have

$$\partial\Delta^2([2]) = \begin{cases} 0 \to 0 \to 1 & 0 \to 1 \to 1 \\ 0 \to 0 \to 2 & 0 \to 2 \to 2 \\ 1 \to 1 \to 2 & 1 \to 2 \to 2 \\ 0 \to 0 \to 0 & 1 \to 1 \to 1 \\ 2 \to 2 \to 2 & \end{cases} \tag{14}$$

It is important to note that the only difference between Δ^2 and $\partial\Delta^2$ is the existence of the morphism $0 \to 1 \to 2$ on $\Delta^2([2])$ and not on $\partial\Delta^2([2])$. The non existence of this map in $\partial\Delta^2$ can be interpreted as the fact that there exists a hole.

The main goal of this chapter is to justify the slogan: the category of simplicial sets is the category of objects from n-dimensional triangles in the most free fashion. Of course, the first point to justify would be to show that to each n-dimensional triangle $[n]$ there exists a simplicial set corresponding to it. This is readily resolved by the following standard results in category theory.

Theorem 2 (Yoneda Lemma). *For any functor $F : \mathcal{C} \to Set$, whose domain is locally small and any object $c \in \mathcal{C}$, there is a bijection*

$$\mathrm{Nat}(\mathcal{C}(-, c), F) \cong F(c) \tag{15}$$

That associated a natural transformation $\alpha : C(-, c) \Rightarrow F$ to an element $\alpha_c(1_c) \in F(c)$. Moreover this correspondence is natural in both c and F.

Corollary 2 (Yoneda Embedding). *There exists a fully faithful embedding $\mathcal{Y} : \mathcal{C} \hookrightarrow Pre(\mathcal{C})$. This is the Yoneda embedding. It maps $c \mapsto \hom_{\mathcal{C}}(-, c)$. The presheaves on the image of $\mathcal{Y}$ are called representable functors.*

We would also need to show that every pasting of n-dimensional triangles exists in the category of simplicial sets. More formally this would mean that the colimit of any diagram that takes values on representable functors exist in the category of simplicial sets. The following result is stronger than the statement described above.

Proposition 2. *Let $\mathcal{C}$ be a (small) category. The category $Pre(\mathcal{C})$ cocomplete (i.e. closed under colimits).*

Proof. Let $D : J \to Pre(\mathcal{C})$ be a diagram in the category $Pre(\mathcal{C})$. We will first construct a presheaf F from this diagram, and then we will note that it is actually the colimit. For every $x \in \mathcal{C}$ define

$$F(x) = \operatorname{colim}_J D_j(x) \tag{16}$$

this is equivalent as saying that $F(x)$ is the coequalizer of the following diagram

$$\coprod_{g \in \operatorname{mor} J} D_{\operatorname{dom}(g)}(x) \underset{d}{\overset{c}{\rightrightarrows}} \coprod_{j \in J} D_j(x) \longrightarrow F(x) \tag{17}$$

where c is the identity map $c : D_{\operatorname{dom}(g)} \to D_j(x)$ for $j = \operatorname{dom}(g)$, and $d = D(g) : D_{\operatorname{dom}(g)} \to D_j$ for $j = \operatorname{cod}(g)$. With this description of the action of F on the objects of $\mathcal{C}$, we will show that there is a canonically induced action on the morphisms of $\mathcal{C}$. For any morphism $y \to x$ we have

$$\begin{tikzcd} \coprod_{g \in \operatorname{mor} J} D_{\operatorname{dom}(g)}(x) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_{j \in J} D_j(x) \arrow[r] \arrow[d] & F(x) \arrow[d, dashed] \\ \coprod_{g \in \operatorname{mor} J} D_{\operatorname{dom}(g)}(y) \arrow[r, shift left] \arrow[r, shift right] & \coprod_{j \in J} D_j(y) \arrow[r] & F(y) \end{tikzcd} \tag{18}$$

By the universal property of the coequalizer there is an induced map $F(x) \to F(y)$. It is easy to see that F is indeed the colimit of the diagram $J \to Pre(\mathcal{C})$. ☐

Next, we would like to justify that indeed every simplicial set can be obtained as by pasting n-dimensional triangles. More formally this means that every simplicial set is a colimit of representables. But we will proof something stronger, besides showing that every simplicial set is a colimit of representables, we will show that any morphisms of simplicial sets is also obtained as a functor between diagram categories.

Definition 5. To each presheaf $F \in Pre(\mathcal{C})$, we can functorialy assign a category $\mathcal{D}_F$ which is define to be the overcategory of representables in F. More concretely, and thanks to the Yoneda Lemma one can give a complete description of $\mathcal{D}_F$. It can be described as:

- as objects, pairs (c, x), where $c \in \mathcal{C}$ and $x \in F(c)$.
- a morphisms $(c, x) \to (c', x')$ is a morphism $f : c \to c'$ in $\mathcal{C}$ such that $Ff : x' \to x$.

Functoriality of this assignment follows directly from the Yoneda Lemma as well.

Proposition 3. *Every presheaf F is a colimit of a $\mathcal{D}_F$ shape diagram of representables. And $\mathcal{D}_- : Pre(\mathcal{C}) \to Cat$ is a faithful functor.*

Proof. Let $F \in Pre(\mathcal{C})$ be a presheaf on $\mathcal{C}$. For each $X \in \mathcal{C}$, we let rX be the representable presheaf defined, as above, by $rX(Y) = \operatorname{Hom}_{\mathcal{C}}(Y, X)$. Now we form the category $\mathcal{D}_F$ whose objects are morphisms of presheaves $rX \to F$, such that the morphisms between $rX \to F$ and $rY \to F$ are given by commutative diagrams

$$\begin{tikzcd} rX \arrow[rr] \arrow[dr] & & rY \arrow[dl] \\ & F & \end{tikzcd} \tag{19}$$

It is clear that this category we are defining is the over category of F of representables. And by the Yoneda lemma one can see that it coincides with the more concrete description we gave above of $\mathcal{D}_F$. There is a functor

$\varphi : \mathcal{D}_F \to Pre(\mathcal{C})$ sending $rX \to F$ to rX. The image of this functor consists of representable presheaves, and by construction there exists an induced map

$$\operatorname{colim}_{\mathcal{D}_F} \varphi(rX \to F) \to F \tag{20}$$

This is a map from a colimit of representable presheaves to F.

We claim that this induced map is an isomorphism. By the Yoneda lemma, if $X \in \mathcal{C}$ and $\alpha \in F(X)$, then there exists a map $rX \to F$ in $Pre(\mathcal{C})$ such that the identity in $rX(X)$ is sent to α in $F(X)$. It follows that we can hit every elements in any part of the presheaf by a representable presheaf. Thus the map $\operatorname{colim} \varphi(rX \to F) \to F$ is surjective.

Now let $X \in \mathcal{C}$ be a fixed object. We want to show that the map

$$\operatorname{colim}_{\mathcal{D}_F} \varphi(rY \to F)(X) \to F(X) \tag{21}$$

is injective. Since colimits in the presheaf category can be computed objectwise, this will show that the required morphism is indeed an isomorphism. Suppose two elements $\alpha_1 \in \varphi(rY_1 \to F)(X)$ and $\alpha_2 \in \varphi(rY_2 \to F)(X)$ are mapped to the same element of $\gamma \in F(X)$. We will unpack this statement. The morphisms $rY_1 \to F$ and $rY_2 \to F$ correspond to objects $\beta_1 \in F(Y_1)$ and $\beta_2 \in F(Y_2)$ respectively. Similarly α_1 and α_2 correspond to maps $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$. The fact that α_1 and α_2 are mapped to the same thing in $F(X)$ means that $f^*(\beta_1) = f^*(\beta_2)$, where the star means pulling back.

We are now going to show that α_1 and α_2 are identified in the colimit. To see this ww note that the work done above implies that the following diagrams commutes

$$\begin{tikzcd} rX \arrow[rr, "f_1"] \arrow[dr, "\gamma"'] & & rY_1 \arrow[dl, "\beta_1"] \\ & F & \end{tikzcd} \tag{22}$$

and

$$\begin{tikzcd} rX \arrow[rr, "f_1"] \arrow[dr, "\gamma"'] & & rY_2 \arrow[dl, "\beta_2"] \\ & F & \end{tikzcd} \tag{23}$$

The first diagram comes from the map $f_1 : X \to Y_1$ and the map $rX \to F$ given by γ. The second diagram is similar. The first diagram shows that the object $\alpha_1 \in rY_1(X)$ of the colimit is identified with the identity of $rX(X)$ by f_1 in the diagram. Similarly α_2 is identified with this in the colimit. So α_1 and α_2 are identified. It follows that the morphism $\operatorname{colim} \varphi(rX \to F) \to F$ is an isomorphism.

We have shown that every presheaf is a colimit of a canonical diagram of representables. Too see that $\mathcal{D}_{-}$ is a faithful functor, it suffices to see that the functor $\mathcal{D}_h : \mathcal{D}_F \to \mathcal{D}_G$ induced a morphism $\operatorname{colim}_{\mathcal{D}_F} U_F(c, x) \to \operatorname{colim}_{\mathcal{D}_G} U_G(c', x')$ which is precisely $h : F \to G$. $\square$

Finally, we wish to show that not only does the category of simplicial sets is in some sense the category of objects built from n-dimensional triangles, it also enjoys a certain universal property that tells you that it is done in the most free way possible.

Theorem 3 (Universal Property of $Pre(\mathcal{C})$). *Let $\mathcal{C}$ be a small category, $\mathcal{E}$ be any category, such that $\mathcal{E}$ is cocomplete, and let $\Gamma : \mathcal{C} \to \mathcal{E}$ be a functor. Then there exists a left adjoint functor $Re : Pre(\mathcal{C}) \to \mathcal{E}$ that makes the following diagram commute:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{Y}"] \arrow[dr, "\Gamma"'] & Pre(\mathcal{C}) \arrow[d, dashed, "Re"] \\ & \mathcal{E} \end{tikzcd} \tag{24}$$

In particular Re is a colimit preserving functor. Moreover, for any two such functors there is a unique natural isomorphism between them. We will denote the right adjoint functor of Re by $Sing : \mathcal{D} \to Pre(\mathcal{C})$.

Proof. For each object $F \in Pre(\mathcal{C})$ we have a induced diagram $\mathcal{D}_F \to \mathcal{C}$, such that $\mathcal{Y}$ and Γ induce $\mathcal{D}_F$ shaped diagrams in $Pre(\mathcal{C})$ and $\mathcal{E}$ respectively. Since we want Re to be colimit preserving, it follows that $Re(F)$ is determined up to isomorphism by the colimit of the $\mathcal{D}_F$ shaped diagram in $\mathcal{E}$. Note that the reason that Re is not unique, is because colimits are only unique up to a unique isomorphisms, and for any two colimit preserving functors $Re_1, Re_2 : Pre(\mathcal{C}) \to \mathcal{E}$ this induces a unique isomorphism between them.

Next, we will show that Re is a left adjoint to the functor $Sing : \mathcal{E} \to Pre(\mathcal{C})$ that maps $d \mapsto \mathrm{Hom}_{\mathcal{E}}(\Gamma-, d)$. We need to show that there is a bijection

$$\mathrm{Hom}_{Pre(\mathcal{C})}(F, Sing(d)) \cong \mathrm{Hom}_{\mathcal{E}}(Re(F), d) \tag{25}$$

For this consider the following identity

$$\mathrm{Hom}_{Pre(\mathcal{C})}(F, Sing(d)) \cong \lim_{\mathcal{D}_F} \mathrm{Hom}_{Pre(\mathcal{C})}(U_F(j), \mathrm{Hom}_{\mathcal{E}}(\Gamma-, d)) \tag{26}$$

Since $U_F(j)$ is a representable presheaf, by the Yoneda lemma we obtain

$$\cong \lim_{\mathcal{D}_F} \hom_{\mathcal{E}}(\Gamma U_F(j), d) \cong \mathrm{Hom}_{\mathcal{E}}(ReF, d) \tag{27}$$

It follows that Re is left adjoint, and therefore it is colimit preserving. □

Although the definition we have for Re is very nice conceptually, it is not very computable. We proceed to give an explicit formula to compute the action of Re on the objects of $Pre(\mathcal{C})$ as a coequalizer of a simple diagram. For this we will introduce the following definitions.

Definition 6. For any set S and object $d \in \mathcal{D}$, the *copower* or *tensor* of d by S, denoted $S \otimes d$ is simply the coproduct $\amalg_S d$ of copies of d indexed by S. In particular if F is a presheaf, we may form copowers

$$F(c) \otimes \Gamma(c') \tag{28}$$

for any c, c' in $\mathcal{C}$. A morphism $f : c' \to c$ of $\mathcal{C}$ induces a map

$$f_* : F(c) \otimes \Gamma(c') \longrightarrow F(c) \otimes \Gamma(c) \tag{29}$$

which applies Γf to the copy of $\Gamma(c')$ in the component corresponding to $x \in F(c)$ and includes it in the component corresponding to x in $F(c) \otimes \Gamma(c)$, and also a map

$$f^* : F(c) \otimes \Gamma(c') \longrightarrow F(c') \otimes \Gamma(c') \tag{30}$$

which maps the component corresponding to $x \in F(c)$ to the component corresponding to $f^* x \in F(c')$.

Proposition 4. *The object $Re(F)$ can be computed as the coequalizer of the following diagram.*

$$\coprod_{f \in \mathrm{mor}\,\mathcal{C}} F(c) \otimes \Gamma(c') \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{c \in \mathcal{C}} F(c) \otimes \Gamma(c) \dashrightarrow Re(F) \tag{31}$$

This is indeed a very nice description of the functor $Re : Pre(\mathcal{C}) \to \mathcal{D}$. And recall it has a right adjoint, the functor $Sing : \mathcal{D} \to Pre(\mathcal{C})$ that maps $d \to \hom_{\mathcal{D}}(\Gamma-, d)$.

Finally we want to conclude this section by formalizing the relation between simplicial sets and topological spaces.

Definition 7. Let Γ_{Top} be the functor $\Delta \to Top$ that sends $[n]$ to the standard topological n-simplex

$$\Delta_n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | x_0 + \dots + x_n = 1, x_i \geq 0\} \tag{32}$$

The morphisms $d^i : \Delta_{n-1} \to \Delta_n$ insert a zero in the ith coordinate, while the morphisms $s^i : \Delta_{n+1} \to \Delta_n$ add the x_i and x_{i+1} coordinates. Geometrically, d^i inserts Δ_{n-1} as the ith face of Δ_n and s^i projects the $n+1$ simplex Δ_{n+1} onto the n-simplex that is orthogonal to its ith face.

Since Top is a cocomplete category we obtain the following diagram

$$\begin{tikzcd} \Delta \arrow[r, "\mathcal{Y}"] \arrow[dr, "\Gamma_{\text{Top}}"'] & sSet \arrow[d, "Re"] \\ & Top \end{tikzcd} \tag{33}$$

Such that $Re : sSet \rightleftarrows Top : Sing$ is an adjunction, as shown by the Theorem about the universal property of sSe. By this adjunction we can assign a simplicial set to a topological space Y. Concretely we have that

$$SingY_n = \text{Hom}_{\text{Top}}(\Gamma_{\text{Top}}([n]), Y) \tag{34}$$

Which is the set of continuous maps from the standard topological n-simplex to Y . Elements of this set are called n-simplices of Y in algebraic topology, which coincides with our terminology. The morphisms are canonically induced by the functoriality of $Sing- = \text{Hom}_{\text{Top}}(\Gamma_{\text{Top}}-, Y)$.

The functor $Re : sSet \to Top$ is also interesting in its own right. This functor captures the intuitive notion that simplicial sets are in some sense the categorification of CW-complexes. At the end of the previous section we provided an explicit formula to compute the image of the functor Re. Recall that for a simplicial set X, its image under the functor Re can be computed by the equation

$$Re(X) = \text{coeq}\left[\coprod_{f:[n]\to[m]} X_m \otimes \Delta_n \overset{f_*}{\underset{f^*}{\rightrightarrows}} \coprod_{[n]} X_n \otimes \Delta_n\right] \tag{35}$$

With some thought we can see that $Re(X)$ is a CW-complex. It is standard to denote this functor by $|-| : sSet \to Top$, and it is called the geometric realization functor.

We will now show that homology is an invariant that is intrinsically defined for simplicial sets. From a simplicial set X, one may construct a simplicial abelian group $\mathbb{Z}X$ (i.e. a contravariant functor $\Delta^{op} \to Ab$), with $\mathbb{Z}X_n$ set equal to the free abelian group on X_n. $\mathbb{Z}X$ has associated to it a chain complex, called its Moore complex and also written $\mathbb{Z}X$, with

$$\cdots \xrightarrow{\partial} \mathbb{Z}X_2 \xrightarrow{\partial} \mathbb{Z}X_1 \xrightarrow{\partial} \mathbb{Z}X_0 \tag{36}$$

and

$$\partial = \sum_{i=0}^{n} (-1)^i d_i \tag{37}$$

in degree n. Recall that the integral singular homology groups $H_*(Y, \mathbb{Z})$ of the space Y are defined to be the homology groups of the chain complex $\mathbb{Z}Sing(Y)$.

3 Higher Category Theory

Let Cat denote the category of small categories. Consider the canonical inclusion $\Delta \hookrightarrow Cat$, since Cat is a cocomplete category we consider the following diagram

$$\begin{tikzcd} \Delta \arrow[r, "\mathcal{Y}"] \arrow[dr, hook] & sSet \arrow[d, "Re"] \\ & Cat \end{tikzcd} \tag{38}$$

Such that $Re : sSet \rightleftarrows Cat : Sing$ is an adjunction. The nerve of a category $\mathcal{C}$ is precisely $Sing(\mathcal{C})$. It is standard to denote the nerve of the category $\mathcal{C}$ by $N\mathcal{C}$. Concretely we have that

$$N\mathcal{C}_n = \hom_{Cat}([n], \mathcal{C}) \tag{39}$$

The face and degeneracy are induced by the functoriality of $\mathrm{Hom}_{Cat}(-, \mathcal{C})$. This shows that categories can be studied through simplicial sets, this is the main object of the book Higher Topos Theory by Jacob Lurie [2]

Definition 8. The simplicial set generated by S is then the smallest simplicial subset of X that contains S. Its k-simplices will be the union of those k-simplices of X that are in the image of S under the right action by some $f : [k] \to [n]$ in Δ.

Definition 9. We say that a simplicial set Y is a subset of a simplicial set X if there is a monomorphism $Y \to X$, i.e., if $Y_n \subset X_n$ for all $[n] \in \Delta$ and if

$$Xf|_{Y_n} = Yf \tag{40}$$

for all $f : [m] \to [n]$ in Δ. This second condition says that the subsets Y_n are closed under the right action by the face and degeneracy operations and furthermore that these operations agree with their definition for X.

Example 5. The ith face $\partial_i\Delta^n$ of Δ^n is the simplicial subset generated by the image of

$$\Delta_k^{n-1} \xrightarrow{d^i} \Delta_k^n \tag{41}$$

for all k.

Example 6. The simplicial n-sphere $\partial\Delta^n$ is the simplicial subset of Δ^n given by the union of the faces $\partial_0\Delta^n, \ldots, \partial_n\Delta^n$. The sphere $\partial\Delta^n$ has the property that $(\partial\Delta^n)_k = \Delta_k^n$ for all $k < n$; all higher simplices of $\partial\Delta_n$ are degenerate. In other words, $\partial\Delta^n$ is the $(n-1)$-skeleton of Δ^n.

Example 7. The simplicial horn Λ_k^n is the simplicial subset of Δ^n given by the union of all the faces except of Δ^n except for the k-th face. The horn Λ_k^n has the property that $(\Lambda_k^n)_j = \Delta_j^n$ for $j < n-1$ and $(\Lambda_k^n)_{n-1} = \Delta_{n-1} \backslash \partial_k\Delta^n$, with higher simplices again being degenerate.

Remark. For each of these simplicial sets, their geometric realization is the topological object suggested by their name; $|\partial_i\Delta^n|$ is the ith face of the standard topological n-simplex $\Delta_n = |\Delta^n|$, $|\partial\Delta^n|$ is its boundary, and $|\Lambda_k^n|$ is the union of all faces but the kth.

Definition 10. A Kan complex is a simplicial set X such that every horn has a filler (which is not assumed to be unique). This means that for each horn $\Lambda_k^n \to X$ in X there exists an extension along the inclusion $\Lambda_k^n \hookrightarrow \Delta^n$ as shown

$$\begin{tikzcd} \Lambda_k^n \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{42}$$

By the Yoneda lemma, the map $\Delta^n \to X$ identifies an n-simplex in X whose faces agree with those specified by the horn.

Definition 11. A $(\infty, 1)$ category is a simplicial set X such that every inner horn, i.e., horn Λ_k^n with $0 < k < n$, has a filler.

Example 8. For any category $\mathcal{C}$, its nerve $N\mathcal{C}$ is a $(\infty, 1)$ category. In fact, it is a $(\infty, 1)$ category with the special property that every inner horn has an unique filler. Conversely, any $(\infty, 1)$ category such that every inner horn has a unique filler is isomorphic to the nerve of a category.

We won't give formal proofs of these facts here, which can be found in [2] as Proposition 1.1.2.2, but we will at least provide some intuition for why the nerve of a category has a unique filler for horns $\Lambda^2_1 \to N\mathcal{C}$. This horn is often represented by the following picture:

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] & & x_2 \end{tikzcd} \qquad \begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & x_2 \end{tikzcd} \tag{43}$$

Here $f, g \in N\mathcal{C}_1$ are morphisms in $\mathcal{C}$ and $x_0, x_1, x_2 \in N\mathcal{C}_0$ are objects in $\mathcal{C}$. $f \circ d_1 = x_0$ and $f \circ d_0 = x_1$, colloquially, x_0 is the domain of f and x_1 is its codomain, and similarly for g. The essential point that this picture communicates is that if f and g are the generating 1-simplices of a horn $\Lambda^2_1 \to N\mathcal{C}$, then f and g are a composable pair of arrows in $\mathcal{C}$. The statement that this horn can be filled then simply expresses the fact that this pair necessarily has a composite $g \circ f$. Composition is unique in a category, so this horn can be filled uniquely. However in a $(\infty, 1)$ category, composition need not be unique. This lack of uniqueness, reflects some of the philosophy of homotopy theory, which is that morphisms are "equal" only up to homotopy.

References

[1] Paul G Goerss and John Jardine. *Simplicial homotopy theory.* Springer Science & Business Media, 2009.

[2] Jacob Lurie. *Higher Topos Theory.* Number 170. Princeton University Press, 2009.