Serre's Intersection Formula and Derived Nonsense

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Serre's Intersection Formula

Let $\mathbb{C}[x,y] = R$. We consider the (possibly non smooth) curves $C = \operatorname{Spec} R/I$ and $C' = \operatorname{Spec} R/J$ on $\mathbb{A}^2_{\mathbb{C}}$. We would like to understand the multiplicity of the intersection $C \cap C'$ at a point $p \in \mathbb{A}^2_{\mathbb{C}}$. If C and C' smooth and transverse then the multiplicity at a point of intersection is one. However, if they are smooth but non-transverse then the multiplicity of the intersection is given by

$$\dim_{\mathbb{C}} R/I \otimes_R R/J \tag{1}$$

This is actually happening on the localized rings at $p \in \mathbb{A}^2_{\mathbb{C}}$. This formula measures the non-reduced structure on the ring. However, there are more general situations in which the above formula does not gives you the right multiplicity. Serre tells us that the right multiplicity at $p \in \mathbb{A}^2_{\mathbb{C}}$ is given by

$$\sum (-1)^i \operatorname{Tor}_i^R(R/I, R/J) \tag{2}$$

The underlying reason why the scheme theoretic multiplicity fails, seems to be because $- \otimes_R R/J$ is not an exact functor. More explicitly, for the curve C we have the short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0 \tag{3}$$

But when we apply the functor $- \otimes_R R/J$ we only get right exactness

$$I \otimes_R R/J \longrightarrow R \otimes_R R/J \longrightarrow R/I \otimes_R R/J \longrightarrow 0 \tag{4}$$

For the rest of the talk we will take for granted that the reason why we don't get the right multiplicity is because the tensor product is not well behaved (exact), and therefore does not have nice formal properties.

However, there are situations when the tensor product $- \otimes_R M$ is exact, i.e. when M is a free (projective) module. It would be nice if there was a way to approximate poorly behaved objects by good ones, in order to extend the nice properties to more general situations. This is exactly what the derived framework allows us to do. Once in this setting, the higher Tor groups appear naturally. Explaining this last paragraph is the real goal of this lecture.

What is Tor again?

Lets turn our attention to modules, as modules form an abelian category so the procedures we will use later to take the derived intersection are more familiar in this setting. Taking free resolutions is exactly the approximation procedure we need. Lets quickly recall how we get free resolutions of a module M

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \tag{5}$$

In here, P_0 is a free module on a set of generators of M, and P_1 is a free module on a set of generators of the kernel $P_0 \to M$.

Remark 1. There is a slogan that I would like to emphasize: taking a free resolution is a way of recording why some elements of M are equal, as it records all the relations of certain objects. But it also takes it to higher levels, it records relations of relations, as well as all higher relations.

Lets think categorically for a second, in which category does free resolutions live? At least for me, this was not clear at first. Resolutions happen in the category of chain complexes $\mathrm{Ch}_{\geq 0}(R)$. So to be more precise a free resolution is a morphism

$$
\begin{tikzcd}
P_\bullet \arrow[d] & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
M & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}
\tag{6}
$$

where M is a chain complex concentrated in degree zero. There are two essential properties of this construction I would like to highlight

- The map $P_\bullet \to M$ induces isomorphism at the level of homology

$$H_0(P_\bullet) \cong H_0(M) = M \tag{7}$$

$$H_n(P_\bullet) \cong H_n(M) = 0 \qquad \text{for} \quad n > 0 \tag{8}$$

- $P_\bullet \to M$ is an isomorphism in the homotopy category $\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R))$.

- An isomorphism on homology is equivalent to an homotopy equivalence.

Warning 2. There is a small lie here. When I say $\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R))$ I actually mean $\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R)_{\text{proj}})$, which is actually equivalent to the classical derived category. In particular, in $\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R)_{\text{proj}})$ the homotopy type is completely determined by its homology. However, this construction is better, as it gives us control over the category we are constructing. In general, naively inverting morphisms is a terrible operation.

Warning 3. In the usual $\mathrm{Ch}_{\geq 0}(R)$ homotopies do not form an equivalence relation. Only in $\mathrm{Ch}_{\geq 0}(R)_{\text{proj}}$ they do. This is another reason why we should think that projective objects are the well behaved ones.

From this, we see that we can identify Mod_R with a full subcategory of $\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R))$. In other words there exists a fully faithful embedding

$$\mathrm{Mod}_R \hookrightarrow \mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R)) \cong \mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R)_{\text{proj}}) \tag{9}$$

And there are two essential properties from this embedding I would like you to remember

- We don't lose any information about Mod_R by passing to $\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R))$

- For every $M \in \mathrm{Mod}_R$ we have a well behaved object (a chain complex of projective) in its isomorphism class.

All we have done so far is to describe a procedure which allow us to approximate poorly behaved objects by good ones.

Definition 4. The **derived tensor product** $- \otimes^{\mathbf{L}}_R M$ is defined to be the tensor product $- \otimes_R P_\bullet$. Where $P_\bullet \to M$ is a projective resolution. In fact, it is a functor

$$- \otimes^{\mathbf{L}}_R M : \mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R)) \longrightarrow \mathrm{Ho}(\mathrm{Ch}_{\geq 0}(R)) \tag{10}$$

So if we have a module $N \in \mathrm{Mod}_R$ when we apply the derived tensor product, we get a chain complex. Now we can understand Tor as the homology groups of this chain complex. We have

$$H_0(N \otimes^{\mathbf{L}}_R M) = N \otimes_R M = \mathrm{Tor}^0_R(N, M) \tag{11}$$

$$H_n(N \otimes^{\mathbf{L}}_R M) = \mathrm{Tor}^n_R(N, M) \qquad \text{for} \quad n > 0 \tag{12}$$

So what I would like you to take away from this is that the fundamental operation is actually taking the derived tensor product, because it allows us to extend nice properties of certain well behaved objects, by approximating poorly behaved objects by good ones. The Tor groups are just the homology groups, it can be thought as the disembodied pieces of a chain complex.

Going back to Serre's intersection formula, recall I mentioned that the reason why $\dim_{\mathbb{C}} R/I \otimes_R R/J$ does not gives the right multiplicity, is because it does not have good formal properties, i.e. it is not exact. So is the derived tensor product is exact? In a sense yes. It preserves triangles, which is the homotopical analog of short exact sequences.

Derived Intersections

So in order to obtain a better behaved theory, we should be taking the derived intersection, which corresponds to taking the derived tensor product

$$R/I \otimes^{\mathbf{L}}_R R/J \tag{13}$$

You may notice that there is a slight problem, we cannot make chain complexes of rings. Indeed assume that we have some morphisms of rings

$$A \xrightarrow{\ \partial\ } B \xrightarrow{\ \partial\ } C \tag{14}$$

then $\partial^2 = 0$, implies that one of the boundary maps is zero, as the unit needs to be mapped to zero at some point. Maybe worst, is the fact that the category of rings is not closed under kernels. More abstractly, the category of rings is not abelian, so chain complexes are not well behaved.

There is an alternate approach to homological algebra through simplicial objects. This approach allows us to do homological algebra on non-abelian settings.

A **simplicial module** is like a CW-complex with an R-module structure. Indeed, you have an addition map $+ : M \times M \to M$, and an action of R on M. Usually in CW-complexes you require this morphisms to be continuous at the point set level. However, a simplicial module is somehow a simpler object. On a simplicial module M there is a chosen CW-structure, and the operations happen level-wise. This means that you have two collection of operations

$$+ : \{n - \text{cells}\} \times \{n - \text{cells}\} \longrightarrow \{n - \text{cells}\} \tag{15}$$

$$R \times \{n - \text{cells}\} \longrightarrow \{n - \text{cells}\} \tag{16}$$

This operations at each level are required to satisfy some compatibility conditions. And continuity is then a consequence of this compatibility. We actually have the following amazing theorem

Theorem 5 (Dold-Kan Correspondence). *There exists an equivalence of categories*

$$\mathrm{Ch}_{\geq 0}(R) \simeq \mathrm{sMod}_R \tag{17}$$

which is homotopically meaningful.

This equivalence has two special properties:

- Suppose we have have two corresponding objects $M \leftrightarrow sM$, then we have that $H_n(M) = \pi_n(sM)$.
- The equivalence decays to an equivalence of their homotopy categories.

In fact, everything we have done so far has analogs in simplicial modules, including a notion of free resolutions and well behaved objects.

Somehow, the simplicial formalism allow us to do homological algebra in non-abelian settings. For example, we can now define a simplicial ring, in an analogous way to how we define a simplicial module. And in particular we have the following result

$$\pi_n(R/I \otimes^{\mathbf{L}}_R R/J) = \mathrm{Tor}^n_R(R/I, R/J) \tag{18}$$

I hope this explains the appearance of the Tor terms in Serre's intersection formula.

A natural question to ask is what does this higher homotopy groups represent geometrically? They are analogs of nilpotent elements, usually referred as "higher nilpotents". Let R be a simplicial commutative ring, then the derived scheme associated to is has the following property

$$|\operatorname{Spec} R| = |\operatorname{Spec} \pi_0 R| \tag{19}$$

So the underlying topological space coincides with the underlying topological space of the usual scheme $\operatorname{Spec} \pi_0 R$. The difference is on their structure sheaf. This should be reminiscing of the fact that in classical algebraic geometry we have

$$|\operatorname{Spec} R| = |\operatorname{Spec} R_{\mathrm{red}}| \tag{20}$$

Where the nilpotents don't change the underlying topological space.