Homological Algebra from the perspective of ∞-Categories

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Basics of ∞-Categories

We will use the model of quasi-categories to model the idea of an ∞-category. In many cases we will be using terminology we haven't defined, like that of colimits in the setting of ∞-categories. If you don't know that they are, don't worry, we will just need certain calculations, which you should take for granted.

Definition 1. An ∞-category $\mathcal{C}$ is a simplicial set $\mathcal{C}$ such that every inner horn $\Lambda^n_i \to \mathcal{C}$ has a filling. In other words, for $0 < i < n$ there exists a lift

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{1}$$

The filling condition is the higher categorical analog of the existence of composition of morphisms in classical category theory. It has the following properties:

- In higher category theory we do not require the filling to be unique, so it may appear that composition is not well defined. However, the analog of unique in higher categories is: there exists a contractible space of choices. More concretely, one can proof that if there are two fillings to a inner horn, then there exists a homotopy between them. By extending this idea to higher dimensions, we see that there is only a contractible space of choices.

- If we want to be more precise, this is a model for $(\infty, 1)$-categories, meaning that all higher morphisms are invertible. Invertibility of higher morphisms is again a consequence of the filling axioms.

This definition is nice, but how do we find ∞-categories in the wild? A source of ∞-categories come from simplicial model categories. Simplicial model categories, are model categories which are enriched over simplicial sets. Let $\mathcal{M}$ be a simplicial model category, one could think that taking the nerve $N(\mathcal{M})$ would give you an ∞-category. While it is correct that it gives you an ∞-category, it does not take into account the simplicial enrichment, which is what makes the category "homotopical". To be more explicit, $N(\mathcal{M})$ is completely determined by its two-skeleton. However, there a construction called the coherent nerve, denoted by $N_\Delta(-)$, which takes into account the simplicial enrichment. We will not describe this construction. But we recall the following result, and some of its properties.

Theorem 2. *Let $\mathcal{M}$ be a simplicial model category, and let $\mathcal{M}_{cf}$ be the full subcategory of $\mathcal{M}$ spanned by the fibrant-cofibrant objects. Then $N_\Delta(\mathcal{M}_{cf})$ is an ∞-category.*

- In general, being weak equivalent is stronger than being homotopy equivalent. However, if we restrict ourselves to $\mathcal{M}_{cf}$ then both notions coincide. In fact, in the world of ∞-categories, there is no notion of weak equivalences, only homotopy equivalence.

- In ∞-categories, objects and morphisms are somehow in the same footing. They are all present simultaneously, while in simplicial model categories to access the higher morphisms one can only talk about the higher morphisms between two objects. This is problematic when one wants to take colimits, as we would need to do some sort of dark magic to access the higher morphisms. Having objects and morphisms in the same footing, resolves this problem in the world of ∞-categories.
- The magic of the coherent nerve is that it somehow extracts and amalgamates this two worlds, of objects and higher morphisms. The fundamental idea is to enrich Δ^n over simplicial sets, such that even if two morphisms $\Delta^n \rightrightarrows \mathcal{M}_{cf}$ coincide at on objects and one-morphisms, the induces map on the simplicial enrichment can somehow differentiate them.

Remark 3. There is also a notion of a differential graded nerve, which produces a ∞-category from a category enriched over chain complexes. We will come back to this point later.

Localization

An important advantage of the formalism of ∞-categories over the one of model categories, is that we are allowed to restrict ourselves to full subcategories without having to worry about the model structure. In particular, this provides a more natural notion of localization.

These ideas are due to Bousfield. Suppose we have a collection S of morphisms in a ∞-category C which we would like to invert. In other words, we seek a ∞-category $S^{-1}\mathcal{C}$ equipped with a functor $\pi : \mathcal{C} \to S^{-1}\mathcal{C}$ which is "universal" with respect to the fact that every morphism in S becomes an equivalence in $S^{-1}\mathcal{C}$. It is an observation of Bousfield that we can often find $S^{-1}\mathcal{C}$ inside of $\mathcal{C}$, as the set of "local" objects of $\mathcal{C}$.

Before proceeding, we recall certain notions that will be useful.

Definition 4. A morphism $f : X \to Y$ in $\mathcal{C}$ is an equivalence if it induces an isomorphism in $h\mathcal{C}$. In particular, being an equivalence is only a property of one morphisms, which makes sense as all the higher morphisms are invertible.

Proposition 5. *Let $X \to Y$ be a morphism on a ∞-category $\mathcal{C}$, then the following are equivalent*

- *$X \to Y$ is an equivalence.*
- *For any element Z, there is an equivalence* $\mathrm{Map}(Z, X) \to \mathrm{Map}(Z, Y)$
- *For any element Z, there is an equivalence* $\mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$

Definition 6. Let $\mathcal{C}$ be an ∞-category and S a collection of morphisms of $\mathcal{C}$. An object $Z \in \mathcal{C}$ is S-local if for any morphism $s : X \to Y$ in S, the induced morphism $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is an equivalence. A morphism $f : X \to Y$ in $\mathcal{C}$ is an S-equivalence if the natural map $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is an equivalence whenever Z is S-local.

It is then a tautology that the category of S-local objects $\mathcal{C}_S$ is a full subcategory of $\mathcal{C}$ where the maps S become an equivalence. However, we needed a bit more than that, we need a universal property. One can show that $\mathcal{C}_S \subset \mathcal{C}$ is a reflective subcategory of $\mathcal{C}$. The notion of localization becomes much clearer from the perspective of ∞-categories (as opposed to simplicial model categories), as passing to subcategories is now allowed (which is not allowed in model categories).

Example 7. Fix a spectrum E. We say that another spectrum X is E-acyclic if the smash product $X \otimes E$ is zero. We say that a spectrum X is E-local if every map $Y \to X$ is nullhomotopic whenever Y is E-acyclic.

Stable ∞-Categories

Stable ∞-categories are the higher categorical analog of abelian categories. In fact, its axioms should resemble the axioms of abelian categories.

Definition 8. A stable ∞-category, is an ∞-category with the following properties

- It is pointed. That is, its initial and final object are equivalent, this object is called the zero object and denoted by 0.
- A triangle is a pushout if and only if it is a pullback. Recall that a triangle is a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{2}$$

  If it is a pushout/pullback then it is called a fiber/cofiber sequence. In other words, a in the world of stable ∞-categories fiber and cofiber sequences coincide. The object Z is often called the mapping cone.

- Every morphism admits a fiber and cofiber.

Example 9. Let $f : X \to Y$ be a morphism of CW-complexes, then its cofiber is given by $C(f)$, where $C(f)$ is the mapping cone of f.

Example 10. Let $f : M \to N$ be a morphism in Mod_R of projective objects, then its cofiber $C(f)$ is the chain complex $0 \to M \to N \to 0$. An important property of this example is that $H_0C(f) = M/N$. This will appear later in the talk.

Notice that a fiber/cofiber sequence seem to behave like exact sequences. I would like to highlight that a fiber/cofiber sequence $X \to Y \to Z$, come equipped with the data of a nullhomotopy of $X \to Z$. Being more precise, fiber/cofiber sequences are the ∞-categorical enhancement of exact triangles. Lets set up some notation, and we will come back to this. We will see that given a stable ∞-category $\mathcal{C}$, then $h\mathcal{C}$ comes equipped with a canonical triangulated structure, coming from the fiber/cofiber sequences.

Remark 11. One attractive feature of the theory of stable ∞-categories is that stability is a property of ∞-categories, rather than additional data. As opposed to triangulated categories, where triangles are additional data in the category.

Before proceeding with the relation between stable ∞-categories and triangulated categories I include some examples of stable ∞-categories.

Example 12. Recall that a spectrum consists of an infinite sequence of pointed topological spaces $\{X_i\}_{i\geq 0}$, together with homeomorphisms $X_i \simeq \Omega X_{i+1}$, where Ω denotes the loop space functor. The collection of spectra can be organized into a stable ∞-category Sp. Moreover, Sp is in some sense the universal example of a stable ∞-category.

Example 13. Let $\mathcal{A}$ be an abelian category. Under mild hypotheses, we can construct a stable ∞-category $\mathcal{D}(\mathcal{A})$ whose homotopy category $h\mathcal{D}(\mathcal{A})$ can be identified with the derived category of $\mathcal{A}$, in the sense of classical homological algebra.

One important feature of the category of spectra is that the suspension Σ and loop functor Ω define equivalences, and are inverses of each other. What are suspensions and loops in an arbitrary ∞-category? Given objects X and Y in a pointed category $\mathcal{C}$, consider the following pushout (resp. pullback) squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} \Omega Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} \tag{3}$$

In fact, we can set this up to get well define functors

$$\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega \tag{4}$$

And from the following result, we can conclude that they define equivalences in the setting of stable ∞-categories

Proposition 14. *Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is stable if and only if*

- *The ∞-category $\mathcal{C}$ admits finite limits and colimits*
- *A square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd} \tag{5}$$

in $\mathcal{C}$ is a pushout if and only if it is a pullback.

Notation 15. If $\mathcal{C}$ is a stable ∞-category, and $n \geq 0$, we let

$$X \mapsto X[n] \tag{6}$$

denote the nth power of the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$. And by $X \to X[-n]$ the nth power of the loop functor $\Omega : \mathcal{C} \to \mathcal{C}$.

One of the main advantages of the language of higher categories, is that the operation of taking limits of ∞-categories is well behaved. In fact, there is a universal way of stabilizing a pointed category $\mathcal{C}$. The category $\mathrm{Sp}(\mathcal{C})$ defined as

$$\mathrm{Sp}(\mathcal{C}) := \lim \left(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right) \tag{7}$$

is a stable ∞-category, where the limit is takes in the ∞-category of ∞-categories. The category $\mathrm{Sp}(\mathcal{C})$ can be characterized as the universal stabilization of $\mathcal{C}$.

We now come back to the fact that the homotopy category of a stable ∞-category has a canonically induced triangulated structure.

Definition 16. Let $\mathcal{C}$ be a stable ∞-category. We say that a diagram

$$X \to Y \to Z \to X[1] \tag{8}$$

in $h\mathcal{C}$ is a distinguished triangle, if it is the image of the following diagram in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd} \tag{9}$$

where both squares are pullback/pushout squares.

Theorem 17. *Let $\mathcal{C}$ be a stable ∞-category, then $h\mathcal{C}$ comes equipped with a canonical triangulated structure, given by the distinguish triangles of $\mathcal{C}$.*

Definition 18. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. And suppose that F preserves the zero object, so it sends triangles to triangles. If in addition, F preserves fiber/cofiber sequences then we say it is exact.

Proposition 19. *Let $F : \mathcal{C} \to \mathcal{C}'$ is a functor between stable ∞-categories. Then the following are equivalent*

- *The functor F is left exact. That is, F commuted with finite limits.*
- *The functor F is right exact. That is, F commutes with finite colimits.*
- *The functor F is exact.*

Remark 20. One can show, without much trouble that every stable ∞-category is canonically enriched over spectra.

Derived Categories and Homological Algebra

Homological algebra provides a rich supplement of stable ∞-categories. Suppose that $\mathcal{A}$ is an abelian category with enough projective objects. In this section, we will explain how to associate to $\mathcal{A}$ an ∞-category $\mathcal{D}^-(\mathcal{A})$, which we call the derived ∞-category of $\mathcal{A}$, whose objects can be identified with (right-bounded) chain complexes with values in $\mathcal{A}$. The ∞-category $\mathcal{D}^-(\mathcal{A})$ is stable, and its homotopy category $h\mathcal{D}^-(\mathcal{A})$ can be identified (as a triangulated category) with the usual derived category of $\mathcal{A}$.

Jumping a bit ahead, the stable ∞-category $\mathcal{D}^-(\mathcal{A})$ is equipped with a t-structure, and there is a canonical equivalence of abelian categories $\mathcal{A} \to \mathcal{D}^-(\mathcal{A})^\heartsuit$. Moreover, we will show that if $\mathcal{C}$ is any stable ∞-category equipped with a left-complete t-structure, then any right exact functor $\mathcal{A} \to \mathcal{C}^\heartsuit$ extends (in an essentially unique way) to an exact functor $\mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$. By an entirely parallel discussion, if $\mathcal{A}$ is abelian category with enough injective objects, we can associate to $\mathcal{A}$ a left-bounded derived ∞-category $\mathcal{D}^+(\mathcal{A})$.

Definition 21. A differential graded category $\mathcal{C}$ over k, is a category enriched over chain complexes over k.

Remark 22. In the same way that we where able to use a coherent nerve to extract an infinity category from a simplicial model category, there is a construction named the differential graded nerve $N_{dg}(-)$ which extracts a ∞-category from a differential graded category. While this is a direct construction, you could convince yourself that such a thing exists, by the relation between chain complexes and the simplicial abelian groups. However, there is a subtle point, since chain complexes may have non-trivial negative cohomology. We ignore this fact, as it has an easy fix.

Definition 23. Let $\mathcal{A}$ be an abelian category with enough projective objects. We let $\mathcal{D}^-(\mathcal{A})$ denote the ∞-category $N_{dg}(\mathrm{Ch}^-(\mathcal{A}_{\text{proj}}))$. Dually, we can define $\mathcal{D}^+(\mathcal{A})$ if $\mathcal{A}$ has enough injective objects.

Proposition 24. *Let $\mathcal{C}$ be a differential graded category, then $N_{dg}(\mathcal{C})$ is an ∞-category. In particular, $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^+(\mathcal{A})$ are stable ∞-categories (if they exist).*

Remark 25. Notice, for example, that this last result tells us that for any abelian category $\mathcal{A}$ we have that $N_{dg}(\mathrm{Ch}(\mathcal{A}))$ is a stable ∞-category, without any boundedness conditions or projective objects. However, the reason why we restrict to projective objects, is such that the notions of quasi-isomorphism and equivalence coincide, since in homological algebra we usually only consider objects up to quasi-isomorphism.

In order to proceed, we would need the notion of a t-structure. However, I believe that it might be to heavy to include in this "soft" talk. So lets include some "organic" features of what a t-structure allows us to do, while providing analogs to more familiar ideas in homological algebra.

Let $\mathcal{C}$ be a stable ∞-category with a t-structure, then are able to do the following things:

- Part of the data given is that of full subcategories $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$. For example, in $\mathcal{D}^-(\mathcal{A})$ we have that $\mathcal{D}^-(\mathcal{A})_{\geq 0}$ correspond to chain complexes with non-zero homology concentrated in non-negative degree. Similarly we can define $\mathcal{D}^-(\mathcal{A})_{\leq 0}$, or replace the zero by any n.

- This full subcategories come equipped with truncation functors $\tau_{\geq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$. In fact, in this abstract setting, we use this truncation functors to define homotopy/homology groups as

$$\tau_{\geq 0} \circ \tau_{\leq 0} \simeq \tau_{\leq 0} \circ \tau_{\geq 0} = \pi_0 : \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit} = \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0} \tag{10}$$

  We can define π_n be precomposing π_0 with $X \mapsto X[-n]$. Of course, in $\mathcal{D}^-(\mathcal{A})$ this would coincide with the usual homology groups.

- The category $\mathcal{C}^{\heartsuit}$ is an abelian category. In order to show this, one notices that for $X, Y \in \mathcal{C}^{\heartsuit}$ the homotopy groups $\pi_n \operatorname{Map}(X, Y)$ vanish for $n > 0$. In fact, we have that $\mathcal{D}^-(\mathcal{A})^{\heartsuit} \simeq \mathcal{A}$. However, it might be important to note that in general $\mathcal{C} \not\simeq \mathcal{D}^-(\mathcal{C}^{\heartsuit})$.

- It may be worth pointing out that the inclusion $\mathcal{A} \subset \mathcal{D}^-(\mathcal{A})$ amounts to taking projective resolutions.

We can finally talk about the theory of derived functors in this setting.

Theorem 26. *Any right exact functor from $\mathcal{A}$ to $\mathcal{C}^{\heartsuit}$ can be extended (in an essentially unique way) to a functor $\mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$. In particular, if $\mathcal{C}^{\heartsuit}$ has enough projective objects, then we obtain an induced map $\mathcal{D}^-(\mathcal{C}^{\heartsuit}) \to \mathcal{C}$.*

Lets work out an example in order to get a better sense of what is happening here. Let $\mathcal{A} = \mathrm{Mod}_R$. Then we will take our favorite right exact functor

$$- \otimes_R M : \mathcal{A} \longrightarrow \mathcal{A} \tag{11}$$

Then the theorem is saying that there exists a essentially unique exact functor $- \otimes_R^{\mathbb{L}} M$ that makes the following diagram commute

$$\begin{tikzcd} \mathcal{A} \arrow[r, "- \otimes_R M"] \arrow[d] & \mathcal{A} \arrow[d] \\ D^-(\mathcal{A}) \arrow[r, "- \otimes_R^{\mathbb{L}} M"] & D^-(\mathcal{A}) \end{tikzcd} \tag{12}$$

It may be worth mentioning, that this requires the functor $- \otimes_R^{\mathbb{L}} M$ to map projective objects of the heart to the heart.

When doing homological algebra, from a more classical perspective, we usually start with a short exact sequence and a right exact functor. We then take the derived functor, and obtain an exact triangle. How does this translate to this setting? Does the inclusion $\mathcal{A} \subset \mathcal{D}^-(\mathcal{A})$ maps exact sequences to cofiber sequences? How? Let me first point out that it can be easily seen that the cofiber of a map of projectives $M \to N$ is the mapping cone, and not the quotient.

Proposition 27. *The inclusion $\mathcal{A} \subset \mathcal{D}^-(\mathcal{A})$ does not map exact sequences to cofiber sequences.*

We are now faced with the challenge of producing a cofiber sequence from an exact sequence. However, from what we know now, there is no clear candidate. Lets spell this out, given an exact sequence in $\mathcal{A}$

$$0 \to A \to B \to C \to 0 \tag{13}$$

we will lose exactness when mapped to $\mathcal{D}^-(\mathcal{A})$, how do we extend our create our cofiber sequence? There are two clear candidate

$$A \to B \to C(A \to B) \to A[1] \to B[1] \to \cdots \tag{14}$$

$$B \to A \to C(B \to C) \to B[1] \to C[1] \to \cdots \tag{15}$$

which one is the correct one? In order to see this, we need a more precise statement of the theorem regarding the unique extension of the right exact functor.

Theorem 28. *Let $\mathcal{A}$ be an abelian category with enough projectives, let $\mathcal{C}$ be an stable ∞-category equipped with a t-structure, and let $\mathcal{E} \subset \mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$ spanned by those exact functors which map projective objects of $\mathcal{A} \subset \mathcal{D}^-(\mathcal{A})$ to the heart of $\mathcal{C}$. The construction, $F \mapsto \tau_{\leq 0} \circ (F|\mathcal{A})$ determines an equivalence from $\mathcal{E}$ to the ordinary category of right exact functors $\mathcal{A} \to \mathcal{C}^{\heartsuit}$.*

It then becomes clear that an exact sequence is extended to a cofiber sequence as

$$A \to B \to C(A \to B) \to A[1] \to B[1] \to \cdots \tag{16}$$