A_{inf} and Deformation Theory

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We begin by recalling the classical theory of Kahler differentials. Suppose $f : X \to Y$ is a map of schemes over $\operatorname{Spec} k$, then we have the following exact sequence associated to it

$$f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0 \tag{1}$$

The fist map on the right is generally not injective. However, if $f : X \to Y$ is smooth, then we get an honest short exact sequence

$$0 \longrightarrow f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0 \tag{2}$$

Recall that every time we see an right exact sequence, we should imagine that there is a way of completing the exact sequence into a long exact sequence. This is usually done via the theory of derived functors. This the idea underlying the contangent complex, however, it has the following extra complication: we can't take resolutions of schemes (or rings), as they don't form an abelian category.

Remark 1. In order to see that the category of rings is not abelian, simply notice that any map of rings $A \to B$ is either zero, or has a kernel which is not a ring. This is due to the presence of a multiplicative unit.

This is exactly where the theory of homotopical algebra, as developed by Quillen enters the game. It provides us with a framework to talk about resolutions in categories which are not abelian. Using this machinery, we are able to derived the module of kahler differentials and obtain the cotangent complex.

Definition 2. For any map of schemes $f : X \to Y$, the cotangent complex $L_{X/Y}$ is an object of the derived category $D(X)$ of X. More concretely, if $A \to B$ is a map of rings, then $L_{B/A}$ is a complex of B-modules.

We will not describe the construction here, but we do mention some properties that the cotangent complex enjoys.

1. If $A \to B$ is smooth, then $L_{B/A} \simeq \Omega_{B/A}$. And for any map $A \to B$ we have that $H^0(L_{B/A}) \simeq \Omega_{B/A}$.
2. Given a composite of maps $A \to B \to C$, there is a canonical exact triangle

$$L_{B/A} \otimes^{\mathbb{L}}_B C \longrightarrow L_{C/A} \longrightarrow L_{C/B} \tag{3}$$

   we recall that exact triangles are the correct analog for short exact sequences in derived categories.

3. If $A \to B$ is etale, then $L_{B/A} \simeq 0$

The reason why we are introducing the cotangent complex is because of it controls deformation theory in complete generality. By the properties described above one can conclude that the module of kahler differentials controls the deformation of smooth schemes.

Theorem 3 (Topological invariance of etale site). For any ring A, write $\mathcal{C}_A$ for the category of flat A-algebras B, such that $L_{B/A} \simeq 0$. Then for any surjective map $\widetilde{A} \to A$ with nilpotent kernell, base change induces an equivalence $\mathcal{C}_A \simeq \mathcal{C}_{\widetilde{A}}$.

Geometrically, this means that for a morphism of schemes $X \to Y$, which is flat and $L_{X/Y} \simeq 0$, for any nilpotent thickening $Y \hookrightarrow Y'$ there exists a unique morphism $X' \to Y'$ such that we have the pullback diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, hook] & Y' \end{tikzcd} \tag{4}$$

moreover the map $X' \to Y'$ is also flat, and $L_{X'/Y'} \simeq 0$. Notice the lack of finiteness conditions for this result, this is essential for our purposes as perfectoid algebras tend to be pretty big.

Remark 4. This theorem states that the etale site of X is invariant under nilpotent extensions. That is $Y_{\text{Et}} \simeq Y'_{\text{Et}}$, given by base changing. However, the definition of etale usually includes some finiteness conditions, so the condition of being flat and $L_{X/Y} \simeq 0$ is a more general notion of etalness, which recovers the usual notion of etalness when we require the schemes to be locally of finite presentation. Moreover, it also implies the lifting condition of formally etale, but recall that formally etale does not implies flat.

The following proposition will provides us with a essential computational tool, as it will imply that a large class of examples which are of interest for us are etale.

Proposition 5. Assume A has characteristic p. Let $f : A \to B$ be a flat map that is relatively perfect, i.e, the relative frobenious $F_{B/A}$ is an isomorphism. Then $L_{B/A} \simeq 0$. Recall that the relative frobenious is defined by

$$\begin{tikzcd} A \arrow[r, "x \mapsto x^p"] \arrow[d] & A \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right, "x \mapsto x^p"'] & B^{(1)} \arrow[dr, "F_{B/A}"] & \\ & & B \end{tikzcd} \tag{5}$$

Example 6. Perfect algebras in characteristic p have vanishing cotangent complex.

We will now proceed to define Witt vectors via deformation theory. Recall that Witt vectors provide us with a functor

$$W(-) : \{\text{Perfect algebras over } \mathbb{F}_p\} \longrightarrow \{\text{p-adically complete } \mathbb{Z}_p \text{ algebras}\} \tag{6}$$

And we have the following commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, hook] \arrow[d] & \operatorname{Spec} W_1(A) \arrow[r, hook] \arrow[d] & \operatorname{Spec} W_2(A) \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \operatorname{Spf} W(A) \arrow[d] \\ \operatorname{Spec} \mathbb{F}_p \arrow[r, hook] & \operatorname{Spec} \mathbb{Z}/p^2 \arrow[r, hook] & \operatorname{Spec} \mathbb{Z}/p^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spf} \mathbb{Z}_p \end{tikzcd} \tag{7}$$

In fact, we have functors

$$\operatorname{colim}(\mathcal{C}_{\mathbb{F}_p} \longrightarrow \mathcal{C}_{\mathbb{Z}/p^2} \longrightarrow \cdots) \simeq \mathcal{C}_{\mathbb{Z}_p} \tag{8}$$

Remark 7. The relevance of Spf, or equivalently the importance of the p-adic topology on $\mathbb{Z}_p$, comes from the fact that without it we would be "picking up things at infinity". In other words, the result above would not follow.

Remark 8. Notice that there is an strict inclusion

$$\mathcal{C}_{\mathbb{F}_p} \simeq \mathcal{C}_{\mathbb{Z}_p} \longhookrightarrow \{\text{p-adically complete } \mathbb{Z}_p \text{ algebras}\} \tag{9}$$

Hence, the witt vector functor defines an fully faitful functor.

We now proceed to investigate the relation with the tilt functor. Recall that tilting provides us with a functor

$$-^\flat : \{\text{p-adically complete } \mathbb{Z}_p \text{ algebras}\} \longrightarrow \{\text{Perfect algebras over } \mathbb{F}_p\} \tag{10}$$

given by

$$R \mapsto R^\flat := \lim_{x \mapsto x^p} R/pR \tag{11}$$

The tilt functor, and the will vectors functor are intimately related, as shown by the following result.

Proposition 9. The witt vector functor $W(-)$ is the left adjoint to the tilt functor $-^\flat$.

$$-^\flat : \{\text{p-adically complete } \mathbb{Z}_p \text{ algebras}\} \underset{\text{left}}{\overset{\text{right}}{\rightleftarrows}} \{\text{Perfect algebras over } \mathbb{F}_p\} : W(-) \tag{12}$$

Recall that by the (co)unit of adjunction, we get morphisms

$$W(R^\flat) \to R \qquad \text{in p-adically complete } \mathbb{Z}_p \text{ algebras} \tag{13}$$

$$S \to W(S)^\flat \qquad \text{in perfect algebras over } \mathbb{F}_p \tag{14}$$

the second morphism is an isomorphism, again by general nonsense as the Witt vector functor was fully faithful. On the other hand the morphism $W(R^\flat) \to R$ is highly nontrivial.

Definition 10. This morphism is in fact so important that it has its own name. We denote by $A_{\text{inf}}(R) := W(R^\flat)$, and the morphism by $\theta : A_{\text{inf}}(R) \to R$.

Corollary 11 (Etale nature of A_{inf}). By construction, the map $\mathbb{Z}_p \to A_{\text{inf}}(R)$ is etale, as it is a deformation of $\mathbb{F}_p \to R^\flat$, which is etale by the results above. Etale means that it is flat and the cotangent complex $L_{A_{\text{inf}}(R)/\mathbb{Z}_p}$ vanishes. An important subtle point is that since we are working with p-adically complete $\mathbb{Z}_p$-algebras, the correct cotangent complex is to consider is the p-adically complete one, again, because we don't want to pick up things at infinity.

We proceed by investigating more closely Fontaine's map $\theta : A_{\text{inf}}(R) \to R$. We will show that in some sense, it is the lift of the map

$$R^\flat := \lim_{x \mapsto x^p} R/p \xrightarrow{\overline{\theta}} R/p \tag{15}$$

In order to do this, we recall that we have an adjunction

$$-^\flat : \{\text{algebras over } \mathbb{F}_p\} \underset{\text{left}}{\overset{\text{right}}{\rightleftarrows}} \{\text{Perfect algebras over } \mathbb{F}_p\} \tag{16}$$

In particular, this means that for any morphism $S \to R$ of $\mathbb{F}_p$, where S is a perfect, the morphism factors as

$$S \longrightarrow R^\flat \longrightarrow R \tag{17}$$

Example 12. The previous result puts us on the situation we want, we have a map $R^\flat \to R/p$ of $\mathbb{F}_p$ algebras. This map then factors through the identity map $R^\flat \to R^\flat$ of perfect $\mathbb{F}_p$ algebras, and by the Witt-tilt adjunction we obtain a $W(R^\flat) \to R$ of p-adically complete $\mathbb{Z}_p$ algebras. By construction, and general non-sense, this map coincides with $\theta : A_{\text{inf}}(R) \to R$.

We can actually fit the maps θ and $\overline{\theta}$ into the following commutative diagram

$$\begin{array}{ccc} A_{\text{inf}}(R) & \xrightarrow{\theta} & R \\ \downarrow & & \downarrow \\ R^\flat & \xrightarrow{\overline{\theta}} & R/p \end{array} \tag{18}$$

We condlude this talk by showing how one can use witt vectors to understand the untilts of a perfectoid pair. We begin by recalling some of the basics of the perfectoid theory.

Example 13. We begin by recalling some basic examples of perfectoid Tate rings

1. The cyclotomic extension $\mathbb{Q}_p^{\text{cycl}}$, the completion of $\mathbb{Q}_p(\mu_{p^\infty})$.
2. The t-adic completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$, which we will write as $\mathbb{F}_p((t^{1/p^\infty}))$.

The moral of perfectoid Tate rings is that we are breaking the prime so much, that every field extension is almost unramified. This is the content of the almost purity theorem.

Remark 14. Recall that an essential property of perfectoid rings is that the frobenious map $R/p \to R/p$ is surjective. Moreover, by construction, the kernel is nilpotent. Therefore we can think of the map $\overline{\theta} : R^\flat \to R/p$ as a limit of infinitesimal thickenings of R/p. In fact, all maps of the previous diagram are limits of infinitesimal thickenings. And in a very precise sense, the map $A_{\text{inf}}(R) \to R/p$ is the universal such thickening of R/p.

In order to set some notation, we recall that by construction, tilting comes equipped with a multiplicative map

$$R^\flat \xrightarrow{\simeq} \lim_{x\mapsto x^p} R \xrightarrow{-^\sharp} R \qquad\qquad f \mapsto f^\sharp \tag{19}$$

Lemma 15. The set of rings of integral elements $R^+ \subset R^\circ$ is in bijection with the set of rings of integral elements $R^{\flat+} \subset R^{\flat\circ}$, via $R^{\flat+} = \lim_{x\mapsto x^p} R^+$. This bijection is inclusion preserving. Also, $R^{\flat+}/\widetilde{\omega}^\flat = R^+/\widetilde{\omega}$, where $\widetilde{\omega}$ is a pseudo-uniformizer that divides p.

Lemma 16. Let $(R^\sharp, R^{\sharp+})$ be an untilt of (R, R^+); i.e., a perfectoid tate ring $R^\sharp$ together with an isomorphism $R^{\sharp\flat} \to R$, such that $R^{\sharp+}$ and R^+ are identified under the previous lemma.

1. There is a canonical surjective ring homomorphism
$$\theta : W(R^+) \to R^{\sharp+} \qquad \sum_{n\geq 0}[r_n]p^n \mapsto \sum_{n\geq 0} r_n^\sharp p^n \tag{20}$$
2. The kernel of θ is generated by a non-zero divisor ξ of the form $\xi = p + [\widetilde{\omega}]\alpha$, where $\widetilde{\omega} \in R^+$ is a pseudo-uniformizer, and $\alpha \in W(R^+)$. In fact, one can show that ξ is a non-zero divisor.

Theorem 17. There is an equivalence of categories between:

1. Perfectoid Tate-Huber pairs (S, S^+)
2. Triples $(R, R^+, \mathcal{J})$, where (R, R^+) is a perfectoid Tate-Huber pair of characteristic p, and $\mathcal{J} \subset W(R^+)$ is an ideal generated by some $\xi = p + [\widetilde{\omega}]\alpha$ as before.

In one direction the map is $(S, S^+) \mapsto (S^\flat, S^{\flat+}, \ker\theta)$ and in the other direction, it is $(R, R^+, \mathcal{J}) \mapsto (W(R^+)[[\widetilde{\omega}]^{-1}]/\mathcal{J}, W(R^+)/\mathcal{J})$.

References

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