Geometric Criterion for Automorphy

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This are extended notes for the talk given by the author at a learning seminar following [Dr]. The paper concerns the unramified Langands correspondence for GL_2, from the Galois to the Automorphic side. An interesting feature of [Dr] is that the constructions are done purely geometrically.

We first set up some notation. Let $X/\mathbf{F}_q$ be a smooth projective and geometrically connected curve. Write v for a closed point of X (or equivalently a place of K) with corresponding complete local ring O_v and fraction field K_v. Let $\mathbf{O}$ be the integral adeles, and $\mathbf{A}$ the ring of adeles of K.

The main theorem of the paper is the following:

Theorem 1. *Let $\rho : \pi_1(X) \to GL_2(\overline{\mathbf{Q}}_l)$ be an absolutely irreducible representation continuous on the l-adic topology. For every closed point $v \in X$, put $t_v = \operatorname{Tr} \rho(\mathrm{Fr}_v)$, $u_v = q_v^{-1} \det \rho(\mathrm{Fr}_v)$. Then there exists a non-zero unramified cusp form*

$$GL_2(\mathbf{O})\backslash GL_2(\mathbf{A})/GL_2(K) \xrightarrow{\ f\ } \overline{\mathbf{Q}}_l \tag{1}$$

such that for every closed points $v \in X$ the equalities $T_v f = t_v f$ and $U_v f = u_v f$ hold.

The process of geometrization begins with Weil uniformation theorem, which states that there is a canonical bijection

$$GL_2(\mathbf{O})\backslash GL_2(\mathbf{A})/GL_2(K) \simeq \mathrm{Bun}_2 \tag{2}$$

where Bun_2 denotes the set of isomorphism classes of rank two vector bundles on X. Using this bijection, the above theorem of Drinfeld can be rephrased as constructing some function

$$\mathrm{Bun}_2 \longrightarrow \overline{\mathbf{Q}}_l \tag{3}$$

with some additional properties. So far, in the seminar, given an appropriate representations

$$\rho : \pi_1(X) \longrightarrow GL_2(\overline{\mathbf{Q}}_l) \tag{4}$$

we have been able to construct a function

$$GL_2(\mathbf{O})\backslash GL_2(\mathbf{A})/B \simeq \mathrm{Flag}_2 \longrightarrow \overline{\mathbf{Q}}_l \tag{5}$$

where $B \subset GL_2(K)$ is the set of upper triangular matrices over k. Moreover this function is a Hecke eigenfunction, with the appropriate eigenvalues. Recall that Flag_2 parametrizes flags $(\mathcal{A}, \mathcal{L})$ where $\mathcal{L}$ is a rank 2 vector bundle on X and $\mathcal{A}$ is a line-subbundle of $\mathcal{L}$ such that $\mathcal{L}/\mathcal{A}$ is a invertible (i.e. $\mathcal{A}$ is a maximal line sub-bundle of $\mathcal{L}$).

Ultimately, the goal is to decent the function $\mathrm{Flag}_2 \to \overline{\mathbf{Q}}_l$ we have constructed to Bun_2 through the canonical map

$$\mathrm{Flag}_2 \longrightarrow \mathrm{Bun}_2 \qquad (\mathcal{L}, \mathcal{A}) \mapsto \mathcal{L} \tag{6}$$

In this talk we will provide a criterion under which a function $\mathrm{Flag}_2 \to \overline{\mathbf{Q}}_l$ can be descended to Bun_2. However, we will not be showing that the function we have constructed satisfied this criterion.

Modifications

Before proceeding to the actual content of this talk, we will briefly discuss an operation on the vector bundles on X, which will be essential for our needs. This operation is called modifications, and we will spend some time trying to understand them.

Let $\mathcal{A}$ be a line bundle on X. We can construct the line bundle $\mathcal{A}(-v) \subset \mathcal{A}$ whose global sections precisely correspond to the global sections of $\mathcal{A}$ which vanish at $v \in X$. We will provide an explicit construction of this line bundle, using a procedure we will need for the more general situation we are interested in. To construct a line bundle, it suffices to constructed on a open cover, with specified isomorphisms on the interceptions, and such that the isomorphisms satisfy the cocycle condition. Today we will not be using a cover of open sets, but rather, a more exotic cover, whose gluing procedure goes by the name of descent. Consider the following cover of X, where D_v is the completion at $v \in X$.

$$\begin{tikzcd} \operatorname{Spec} k((\pi_v)) \arrow[r, "\simeq"] & D_v^\circ \arrow[r] \arrow[d, hook] & X \setminus v \arrow[d, hook] \\ \operatorname{Spec} k[[\pi_v]] \arrow[r, "\simeq"] & D_v \arrow[r] & X \end{tikzcd} \tag{7}$$

Notice this is an fpqc cover. And $\mathcal{A}(-v)$ can be constructed by modifying the isomorphisms D_v°, as follows

$$\mathcal{A}_{D_v}|_{D_v^\circ} \xrightarrow{\times \pi_v^{-1}} \mathcal{A}_{D_v}|_{D_v^\circ} \xrightarrow{\simeq} \mathcal{A}_{X\setminus v}|_{D_v^\circ} \tag{8}$$

If you stare at this for a bit, you will realize that the line bundle we have constructed is exactly $\mathcal{A}(-v) \simeq A \otimes \mathcal{O}_X(-v)$.

Remark 2. One can show that we can obtain every line bundle on X by twisting $\mathcal{O}_X$ at finitely many D_v. In fact, this results holds more generally. This is exactly the principle under which one shows the bijection

$$G(\mathbf{O})\backslash G(\mathbf{A})/G(K) \cong \operatorname{Bun}_G \tag{9}$$

The reader can obtain a more detail account of this discussion at [Ca].

This procedure generalized immediately, let $\mathcal{L}$ be a rank two vector bundle on X, we want to construct $\mathcal{L}(-v)$. We have the datum of $\mathcal{L}$ on the same cover of X as above, and we will the isomorphism at D_v° as follows

$$\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & \pi_v^{-1} \end{pmatrix} : \mathcal{L}_{D_v}|_{D_v^\circ} \xrightarrow{\simeq} \mathcal{L}_{D_v}|_{D_v^\circ} \xrightarrow{\simeq} \mathcal{L}_{X\setminus v}|_{D_v^\circ} \tag{10}$$

Using descent we obtain $\mathcal{L}(-v) \subset \mathcal{L}$. As you can see the procedure is very similar as in line bundles. One can also show that $\mathcal{L}(-v) \simeq \mathcal{L} \otimes \mathcal{O}_X(-v)$, however, our perspective using descent will be useful in getting a good understanding of modifications.

Definition 3. A lower modification of $\mathcal{L}$, is a rank two vector bundle $\mathcal{L}'$ such that

$$\mathcal{L}(-v) \subset \mathcal{L}' \subset \mathcal{L} \tag{11}$$

Similarly an upper modification is a rank two vector bundle $\mathcal{L}'$ such that

$$\mathcal{L} \subset \mathcal{L}' \subset \mathcal{L}(v) \tag{12}$$

We highlight that the inclusions are proper.

Example 4. We construct a lower modification explicitly, it can be obtain by modifying the descent datum as follows

$$\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} : \mathcal{L}_{D_v}|_{D_v^\circ} \xrightarrow{\simeq} \mathcal{L}_{D_v}|_{D_v^\circ} \xrightarrow{\simeq} \mathcal{L}_{X\setminus v}|_{D_v^\circ} \tag{13}$$

Its important to note here that to this lower modifications we can assign a canonical line in $K_v^{\oplus 2}$ which corresponds to the eigenspace of 1. In fact, one can show that the space of lower modifications is in bijection with lines on $K_v^{\oplus 2}$, by the correspondence described above. While this is a priori just locally free in the fpqc topology, by the smoothness of GL_2 we conclude that they are locally free in the etale topology.

Remark 5. In order to understand the proofs of the results in the next section, one needs to understand how $\mathcal{A} \cap \mathcal{L}'$ (where $\mathcal{A}$ is a line subbundle) behaves as $\mathcal{L}'$ ranges over the lower modifications of $\mathcal{L}$ at v. Recall that a lower modification corresponds to a line on $K_v^{\oplus 2}$, as described above. Similarly, we have that line subbundles $\mathcal{A} \subset \mathcal{L}$ also correspond to lines in $K^{\oplus 2} \subset K_v^{\oplus 2}$. We can conclude that there is exactly one lower modification $\mathcal{L}'$ for which $\mathcal{A} \subset \mathcal{L}'$, this lower modification corresponds to the same line that $\mathcal{A}$ corresponds. Otherwise we have that $\mathcal{A} \cap \mathcal{L}' = \mathcal{A}(-v)$. This is interesting, notice that we are not requiring the eigenspace with eigenvalue π_v^{-1} to correspond to $\mathcal{A}$, but it still behave as it was. "In a lower modification we choose a line in $K_v^{\oplus 2}$ where we force the zeroes".

On the other hand, there exists exactly one upper modification in which $\mathcal{A}$ is not maximal, namely the one where $\mathcal{A}$ corresponds to the eigenspace of $K^{\oplus 2} \subset K_v^{\oplus 2}$ with eigenvalue π_v. "In an upper modification we choose a line in $K_v^{\oplus 2}$ where we allow poles".

Geometric Criterion for Automorphy

Recall the Hecke operators act on functions $f : \mathrm{Bun}_2 \to \overline{\mathbf{Q}}_l$ in the following way

$$(U_v f)(\mathcal{L}) = f(\mathcal{L}(-v)) \tag{14}$$

$$(T_v f)(\mathcal{L}) = \sum_{\mathcal{L}' \in S} f(\mathcal{L}') \tag{15}$$

Where S is the set of lower modifications of $\mathcal{L}$ at v. Similarly, the Hecke operators act on functions $f : \mathrm{Flag}_2 \to \overline{\mathbf{Q}}_l$

$$(U_v f)(\mathcal{L}, \mathcal{A}) = f(\mathcal{L}(-v), \mathcal{A}(-v)) \tag{16}$$

$$(T_v f)(\mathcal{L}, \mathcal{A}) = \sum_{\mathcal{L}' \in S} f(\mathcal{L}', \mathcal{A} \cap \mathcal{L}') \tag{17}$$

Where S is again the set of lower modifications of $\mathcal{L}$ at v.

Definition 6. For every $\mathcal{L} \in \mathrm{Bun}_2$, we denote by $h(\mathcal{L})$ the least degree of invertible quotient sheaves of $\mathcal{L} \otimes \overline{\mathbf{F}}_q$ is base change of $\mathcal{L}$ to $X \otimes \overline{\mathbf{F}}_q$.

We claim that that $h(\mathcal{L}) > -\infty$. Let $\mathcal{B}$ be an invertible quotient sheave of $\mathcal{L}$, then it fits into the following short exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{L} \longrightarrow \mathcal{B} \longrightarrow 0 \tag{18}$$

It suffices to show that $\dim H^1(X, \mathcal{B})$ is bounded below. Indeed, this follows from Riemann-Roch. And boundedness can be see by the induced surjection $H^1(X, \mathcal{L}) \twoheadrightarrow H^1(X, \mathcal{B})$.

The main goal of this talk is to proof the following proposition.

Proposition 7. *Let $f : \mathrm{Flag}_2 \to \overline{\mathbf{Q}}_l$ be a eigenfunction of the Hecke operators. Suppose for that for some $N \in \mathbf{Z}$ the following condition is satisfied: $f(\mathcal{L}, \mathcal{A}) = f(\mathcal{L}, \mathcal{A}')$ for all $\mathcal{L}, \mathcal{A}, \mathcal{A}'$ such that the degrees of $\mathcal{A}$ and $\mathcal{A}'$ are less than $h(\mathcal{L}) - N$. Then there exists a function $g : \mathrm{Bun}_2 \to \overline{\mathbf{Q}}_l$ such that $f(\mathcal{L}, \mathcal{A}) = g(\mathcal{L})$.*

In other words, if $f(\mathcal{L}, \mathcal{A})$ is independent of $\mathcal{A}$, when the degree of $\mathcal{A}$ is sufficiently negative, then $f(\mathcal{L}, \mathcal{A})$ is completely independent of $\mathcal{A}$. Lets sketch the idea of the proof: for a fixed line bundle $\mathcal{A}$ on X consider the following statement: "For every rank two vector bundle $\mathcal{L}$ on X containing $\mathcal{A}$ as a maximal invertible subsheaf and such that $h(\mathcal{L}) \geq m$, $\deg \mathcal{L} \geq n$ the equality $f(\mathcal{L}, \mathcal{A}) = g(\mathcal{L})$ holds". From the hypothesis of the proposition, $P(m, n)$ is true for $m > \deg \mathcal{A} + N$.

In order for this statement to not be vacuous, we show that there is an $\mathcal{A} \subset \mathcal{L}$ of arbitrary negative degree.

Lemma 8. *Let $\mathcal{L}$ be a rank two vector bundle on X. Then there exists maximal invertible subsheaves $\mathcal{A} \subset \mathcal{L}$ of arbitrary negative degree.*

Proof. The set of degrees of invertible subsheaves of $\mathcal{L}$ is bounded from above because $\dim H^0(X, \mathcal{L}) < \infty$, one can then use Riemann Roch to provide the bound. The set of invertible subsheaves of $\mathcal{L}$ having a fixed degree is finite since $\mathrm{Pic}^0 X$ is finite (the rational points of a finite type variety over a finite field), and the set $\mathrm{Hom}(\mathcal{A}, \mathcal{L})$ is finite (as the sheaf hom is a coherent sheaf, and we are working over a finite field, i.e. the global sections form a finite vector space over a finite field).

However the set of maximal line subbundles is infinite, as they are in bijection with lines in the generic fiber of $\mathcal{L}$ (which is isomorphic to $K^{\oplus 2}$). Indeed any such line can be extended to a line subbundle over an open set of X, and then completed using the descent datum of $\mathcal{L}$. $\square$

Lemma 9. *Let $(\mathcal{L}, \mathcal{A}) \in \mathrm{Flag}_2$ and let $\mathcal{L}'$ be an upper modification of $\mathcal{L}$ at v such that $\mathcal{A}$ is maximal as a subsheaf of $\mathcal{L}'$. Suppose that for every rank two vector bundle $\overline{\mathcal{L}} \supseteq \mathcal{L}'$ such that $\mathcal{A}$ is maximal as a subsheaf of $\overline{\mathcal{L}}$, the equality $f(\overline{\mathcal{L}}, \mathcal{A}) = g(\overline{\mathcal{L}})$ holds. Then $f(\mathcal{L}, \mathcal{A}) = g(\mathcal{L})$.*

This lemma implies

$$P(m, n+1) \Longrightarrow P(m, n) \tag{19}$$

as taking an upper modification increases the degree of $\mathcal{L}$ by one. One can show this by decomposing our vector bundle into line bundles

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{L} \longrightarrow \mathcal{B} \longrightarrow 0 \tag{20}$$

which correspond to the eigenspaces of the local twist at v.

Proof. Denote by S the set of lower modifications of $\mathcal{L}'$ at v different from $\mathcal{L}$. We have

$$f(\mathcal{L}, \mathcal{A}) + \sum_{\mathcal{L}'' \in S} f(\mathcal{L}'', \mathcal{A} \cap \mathcal{L}'') = t_v f(\mathcal{L}', \mathcal{A}) = t_v g(\mathcal{L}') \tag{21}$$

$$g(\mathcal{L}) + \sum_{\mathcal{L}'' \in S} g(\mathcal{L}'') = t_v g(\mathcal{L}') \tag{22}$$

Recall from the discussion above that $\mathcal{L}'' \cap \mathcal{A} = \mathcal{A}(-v)$. It suffices to show that $f(\mathcal{L}'', \mathcal{A}(-v)) = g(\mathcal{L}'')$. Set $\overline{\mathcal{L}} = \mathcal{L}''(v)$, therefore

$$f(\mathcal{L}'', \mathcal{A} \cap \mathcal{L}'') = u_v f(\mathcal{L}''(v), \mathcal{A}) = u_v g(\mathcal{L}''(v)) = g(\mathcal{L}'') \tag{23}$$

for every $\mathcal{L}'' \in S$. It follows that $f(\mathcal{L}, \mathcal{A}) = g(\mathcal{L})$. For the benefit of the reader we record a graph encoding the hierarchy of the modifications used in here

$$\begin{tikzcd}
& & \mathcal{L}''(v) \arrow[dl, dash] \arrow[dd, dash] \\
& \mathcal{L}' \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{L} & & \mathcal{L}''
\end{tikzcd} \tag{24}$$

The result follows. $\square$

Lemma 10. *Let $(\mathcal{L}, \mathcal{A}) \in \mathrm{Flag}_2$, $\deg \mathcal{L} > 2h(\mathcal{L})$. Then there exists an upper modification $\mathcal{L}'$ of $\mathcal{L}$ at v such that $h(\mathcal{L}') > h(\mathcal{L})$ and $\mathcal{A}$ is maximal as a subsheaf of $\mathcal{L}'$.*

This lemma together with the previous one, implies

$$P(m+1, n) \Longrightarrow P(m, n) \tag{25}$$

Indeed, this lemma says we can find an upper modification $\mathcal{L}$ preserving the maximality of $\mathcal{A}$ but changing $m \mapsto m+1$. We also notice that any other upper modification can only increase h, so by the previous lemma the claim holds. This completes the proof of the main proposition.

Proof. Let $\mathcal{M} \subset \mathcal{L} \otimes \overline{\mathbf{F}}_q$ be an invertible subsheaf of the highest degree. Its existence follows by Riemann-Roch and the injection $H^0(X, \mathcal{M}) \hookrightarrow H^0(X, \mathcal{L})$. Then by the additive property of degree on coherent sheaves on a curve we get

$$\deg \mathcal{M} = \deg \mathcal{L} - h(\mathcal{L}) > h(\mathcal{L}) \tag{26}$$

By the work done above there exists an upper modification $\mathcal{L}' \supset \mathcal{L}$ such that $\mathcal{A}$ and $\mathcal{M}$ are maximal subsheaves of $\mathcal{L}'$ and $\mathcal{L}' \otimes \overline{\mathbf{F}}_q$ respectively. Indeed, this upper modification is obtained by modifying

$$\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} : \mathcal{L}_{D_v}|_{D_v^\circ} \xrightarrow{\simeq} \mathcal{L}_{D_v}|_{D_v^\circ} \xrightarrow{\simeq} \mathcal{L}_{X \setminus v}|_{D_v^\circ} \tag{27}$$

where the eigenspace of π_v does not coincide with the lines of $\mathcal{M}$ or $\mathcal{L}$ in $K^{\oplus 2} \subset K_v^{\oplus 2}$. We claim that $h(\mathcal{L}') > h(\mathcal{L})$. Assume that there exists an invertible quotient $\mathcal{N}$ of $\mathcal{L}' \otimes \overline{\mathbf{F}}_q$ such that $\deg \mathcal{N} \leq h(\mathcal{L})$. Then the composition $\mathcal{M} \hookrightarrow \mathcal{L}' \otimes \overline{\mathbf{F}}_q \twoheadrightarrow \mathcal{N}$ must be equal to zero, indeed this follows from the tensor-hom adjunction and the fact that line bundles of negative degree have no non-zero global sections. By the maximality of $\mathcal{M}$ we have that $(\mathcal{L}' \otimes \overline{\mathbf{F}}_q)/\mathcal{M} = \mathcal{N}$, hence $\deg \mathcal{L}' = \deg \mathcal{M} + \deg \mathcal{M} \leq \deg \mathcal{L}$, which is as contradiction as upper modifications raise the degree. $\square$

References

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