Integral K-theory and Rep(G)

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Abstract

In this note we survey some aspects of the theory of tempered cohomology, as developed by Lurie. The notes where written as an exercise to solidify the author's understanding of the theory, trying to focus in the specific case of KU. As an application, we endow $\mathrm{KU}_{\widehat{\mathrm{p}}}$ with the structure of a genuine equivariant spectrum, whose genuine fix points recover $\mathrm{Rep}(G)$. Our approach is algebro-geometric and it works for all groups G simultaneously.

Introduction

One of the main players in this note is the spectrum KU. In order to make this note more accessible, we begin with some background material about spectra. We will not provide a rigorous definition of a spectrum (in the sense of homotopy theory), but I hope that the following discussion will be enough for the reader to understand the rest of the notes.

Definition 1. A spectrum behaves very much like a chain complex of abelian groups. Some properties that the category of spectra enjoys are the following

1. One can ask for the homotopy group of a spectrum E, which are denoted by $\pi_i(E)$. And they should be thought as analogs of the cohomology groups of a spectrum.

2. Given two spectrum, E_1 and E_2 one can form a direct sum $E_1 \oplus E_2$, and a tensor product $E_1 \otimes E_2$.

3. Given a map of spectra $E_1 \to E_2$ one can take the kernel and cokernel of this map.

4. Each spectrum, determines a collection of functors

$$E^i : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Ab} \tag{1}$$

which one should think about as determining a cohomology theory in the category of spaces. In particular we have the identity

$$E^i(\mathrm{pt}) = \pi_i(E) \tag{2}$$

5. One can also consider ring objects in the category of spectra. In the case that E admits a sort of ring structure, we can amalgamate all cohomology groups $E^i(X)$ into a single ring

$$E^* : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Rings} \qquad X \mapsto E^*(X) \tag{3}$$

Example 2. A familiar example is given by the spectrum $H\mathbf{Q}$, which following (4), determined a cohomology theory

$$H\mathbf{Q}^i : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Ab} \qquad X \mapsto H^i(X, \mathbf{Q}) \tag{4}$$

that can be identified with singular cohomology. Moreover, since Q is not just an abelian group but a ring, we can conclude that $H\mathbf{Q}$ admits the structure of a ring spectrum, and then following (5), it determined a multiplicative cohomology theory

$$H\mathbf{Q}^* : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Rings} \qquad X \mapsto H^*(X, \mathbf{Q}) \tag{5}$$

A ring spectrum that will be of central importance to us is the ring spectrum KU, which goes by the name of complex K-theory. In order to get acquainted more with this ring spectrum a good first step is to ask what are its homotopy groups, indeed we have that

$$\pi_i(\mathrm{KU}) = \mathrm{KU}^i(\mathrm{pt}) = \begin{cases} \mathbf{Z} & \text{if i is even} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and since it has a ring structure, we can organize all this cohomology groups into a ring graded ring

$$\pi_*(\mathrm{KU}) = \mathrm{KU}^*(\mathrm{pt}) = \mathbf{Z}[\beta^{\pm}] \qquad |\beta| = 2 \tag{7}$$

In particular, we can see that this ring spectrum has a property called 2-periodic which will be of central importance for us. It is closely related to the following theorem, which goes by the name of Bott periodicity.

Proposition 3. For any space X, the Bott element β induces an isomorphism

$$\beta : \mathrm{KU}^i(X) \xrightarrow{\simeq} \mathrm{KU}^{i+2}(X) \tag{8}$$

In order to relate KU to the representation theory of finite groups, it will be convenient to have the following description of $\mathrm{KU}^0(X)$. We consider the category of complex vector bundles on a space X, and denote it by $\mathrm{Vect}(X)$. The category $\mathrm{Vect}(X)$ comes equipped with a direct sum operation, so we can organize the isomorphism classes of objects of $\mathrm{Vect}(X)$ into a monoid. Elements of this monoids are denotes by $[V]$, where V is a complex vector bundle over X, and with sum given by

$$[V_1 \oplus V_2] = [V_1] + [V_2] \tag{9}$$

We temporarily denote this monoid by the letter $M(X)$. Notice, however, that $\mathrm{KU}^0(X)$ is at least an abelian group, and not just a monoid, so in order to relate this two objects we will proceed by transforming M to an abelian group in a universal way. Recall that there exists a forgetful functor $\mathrm{ComGrp} \to \mathrm{Mon}$ from abelian groups to monoids, and this functor admits a left adjoint

$$-^{gp} : \mathrm{Mon} \longrightarrow \mathrm{ComGrp} \tag{10}$$

called group completion. One then has the following result

Proposition 4. There exists an isomorphism

$$M(X)^{gp} \cong \mathrm{KU}^0(X) \tag{11}$$

Moreover, $M(X)^{gp}$ inherits a ring structure coming from the tensor product of vector bundles, which under the above isomorphism, coincides with the ring structure of $\mathrm{KU}^0(X)$.

Example 5. Using the results above we can recover the calculation that $\mathrm{KU}^0(\mathrm{pt}) = \mathbf{Z}$. Simply notice that over a point there is only one vector bundle of each non-negative dimension. Then one has to observe that if V_i is the vector bundle of dimension n over a point, then $V_i \oplus V_j = V_{i+j}$ and that $V_i \otimes V_j = V_{ij}$.

Now we will show how to extract a representation of a group G, out of vector bundles on a certain space BG. For simplicity, consider the additive group $\mathbf{Z}$, then the associated space is $B\mathbf{Z} = S^1$. Fix a distinguished point $\mathrm{pt} \to B\mathbf{Z}$, then out of a vector bundle $V \to B\mathbf{Z}$ we can extract a complex vector space V_{pt}, by looking at the fiber at the distinguished point of the map $V \to B\mathbf{Z}$. Then, moving our vector bundle $V \to B\mathbf{Z}$ around the loop, gives us an automorphism $V_{\mathrm{pt}} \to V_{\mathrm{pt}}$, which corresponds to an action of $1 \in \mathbf{Z}$ on V_{pt}. In other words, it is possible to extract a $\mathbf{Z}$ representation out of a vector bundle $V \to B\mathbf{Z}$. Granted, $\mathbf{Z}$ is not a finite group, but it seems good for pedagogical reasons.

Question 6. Can we obtain all complex representations of a finite group G, from complex vector bundles on BG?

The answer to this question is negative, but the complex representations of G and complex vector bundles over BG are intimately related, by the following theorem.

Theorem 7. Let G be a finite group, and let $I_G \subset \mathrm{Rep}(G)$ be the augmentation ideal, defined as the kernel of the ring homomorphism

$$\mathrm{Rep}(G) \to \mathbf{Z} \qquad [V] \mapsto \dim_{\mathbf{C}}(V) \tag{12}$$

The Atiyah-Segal comparison map

$$\zeta : \mathrm{Rep}(G) \longrightarrow \mathrm{KU}^0(BG) \tag{13}$$

exhibits $\mathrm{KU}^0(BG)$ as the I_G completion of $\mathrm{Rep}(G)$.

Definition 8. Let $\mathrm{Rep}(G)$ be the ring of complex representations of the finite group G. As an abelian group it is the abelian group generated by the symbols $[V]$, where V is a finite dimensional complex representation of G, subject to the relations

$$[V] = [V_1] + [V_2] \tag{14}$$

for any isomorphism $V \cong V_1 \oplus V_2$. So in other words is the free abelian group of finite rank generated by $[W]$, where W is a finite dimensional irreducible representation. In order to consider $\mathrm{Rep}(G)$ as a ring, we declare that multiplication is given by the rule

$$[V_1] \cdot [V_2] = [V_1 \otimes_{\mathbf{C}} V_2] \tag{15}$$

We would like to highlight that even though the representations we are considering are complex representations, i.e. V is a vector space over $\mathbf{C}$, the ring $\mathrm{Rep}(G)$ is most naturally regarded as a $\mathbf{Z}$ algebra.

In this notes, the goal is to provide a direct K-theoretic construction of the ring $\mathrm{Rep}(G)$. This means that we will try to explain how to construct a "decompleted" version of KU, which we will denote by $\underline{\mathrm{KU}}$, that satisfies

$$\underline{\mathrm{KU}}^0(BG) \cong \mathrm{Rep}(G) \tag{16}$$

equipped with a natural map

$$\underline{\mathrm{KU}}^0(BG) \longrightarrow \mathrm{KU}^0(BG) \tag{17}$$

which recovers the Atiyah-Segal comparison map. But before delving into the theory, let me explain some applications. Recall that in particular, the cohomology theory $\underline{\mathrm{KU}}$ determines a functor

$$\underline{\mathrm{KU}}^0 : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Ab} \tag{18}$$

Remark 9. A reader familiar with the theory of tempered cohomology would realize that Spc is very much not the domain of the functor $\underline{\mathrm{KU}}$, but rather one needs to renormalize the category of spaces, such that it is not generated by a single point pt, but rather by spaces of the form BM, where M is a finite abelian group. However, we choose to ignore this issue in this section, hoping it will not cause to much confusion. We will try to come back to this later.

Then, for any inclusion of finite groups $H \subset G$, the functoriality of $\underline{\mathrm{KU}}$ recovers the restriction map

$$\begin{array}{ccc} \underline{\mathrm{KU}}^0(BG) & \longrightarrow & \underline{\mathrm{KU}}^0(BH) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \mathrm{Rep}(G) & \xrightarrow{\mathrm{Res}} & \mathrm{Rep}(H) \end{array} \tag{19}$$

However, $\underline{\mathrm{KU}}$ comes equipped with extra functoriality, it also comes equipped with transfer maps $\underline{\mathrm{KU}}(BH) \to \underline{\mathrm{KU}}(BG)$, which fit into the following commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{KU}}^0(BH) & \longrightarrow & \underline{\mathrm{KU}}^0(BG) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \mathrm{Rep}(H) & \xrightarrow{\mathrm{Ind}} & \mathrm{Rep}(G) \end{array} \tag{20}$$

More informally, we can say that the transfer maps $\underline{\mathrm{KU}}^0(BH) \to \underline{\mathrm{KU}}^0(BG)$ coincide with induction from the classical representation theory of finite groups. And finally, we can recover a variant of the Artin-Brauer induction theorem

Theorem 10. For any finite group G, the ring $\mathbf{C} \otimes_{\mathbf{Z}} \underline{\mathrm{KU}}^0(BG) = \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{Rep}(G)$ can be generated as a $\mathbf{C}$ vector space by the image of the induction maps

$$\mathbf{C} \otimes_{\mathbf{Z}} \underline{\mathrm{KU}}^0(BT) \xrightarrow{\mathrm{Ind}} \mathbf{C} \otimes_{\mathbf{Z}} \underline{\mathrm{KU}}^0(BG) \tag{21}$$

for all maps $BT \to BG$, where T is a finite cyclic group.

This naturally leads to the following question

Question 11. Can one extend such induction results to a certain kind of infinite groups like $\mathrm{Gal}(\mathbf{Q})$? Or the infinite symmetric group S_∞? As stated, the answer to this question seems to be negative, but maybe, by somehow incorporating the profinite topology we could make this work.

In order to achieve this goal, we will make use of Lurie's theory of tempered cohomology. A key ingredient of the theory of tempered cohomology is the close relationship between 2-periodic cohomology theories, like KU, and the geometry of p-divisible groups. In particular, we hope to provide a algebro-geometric explanation for the failure of the Atiyah-Segal comparison map to be an isomorphism.

Local Systems

We begin this section by trying to provide a geometric interpretation for the spectrum $\mathrm{KU}(X)$, for some space X. This marks a point of departure from the previous section, in which we only considered the disembodied pieces of the spectrum, namely the abelian groups $\mathrm{KU}^i(X)$, or the ring $\mathrm{KU}^*(X)$. This difference is analogous to the difference between a chain complex and its cohomology groups. Lets recall some facts about spectra

- For any space X, we can construct a new ring spectrum $\mathrm{KU}(X)$. If you recall from the previous section, to any spectrum E we can consider its homotopy groups $\pi_i(E) = E^i(\mathrm{pt})$, which get identified with the ith E-cohomology of a point. In this case, we have the identification $\pi_i(\mathrm{KU}(X)) = \mathrm{KU}^i(X)$.

This is relevant for us, because in the same way that we have been considering complex vector bundles $V \to X$, we now would like to consider bundles over X, whose fibers are not $\mathbf{C}$-vector spaces, but rather KU-modules.

Remark 12. The reader not familiar with the category of spectra might not feel at ease with the suggestion of considering modules over a beast like KU, however this is a legitimate operation. Remember that we had the analogy that a spectrum should behave in some ways like an abelian group, a ring spectrum should be like a ring, and to any ring one can consider modules over it, so its reasonable to expect that one can consider modules over a ring spectrum.

Our geometric interpretation of $\mathrm{KU}(X)$ will rely on the notion of a local system over a space X. As a warm up we consider the following example

Example 13. Consider the space $B\mathbf{Z} = S^1$, and we are going to consider $\mathbf{Q}$-values local systems on S^1. This local systems can be arranged in a category which we will denote by $\mathrm{LocSys}_{\mathbf{Q}}(B\mathbf{Z})$. An important difference between a vector bundle over $B\mathbf{Z}$ and a local system, is that when we are considering local systems, we do not consider $B\mathbf{Z}$ as a topological space in the usual way, but rather, as some sort of "virtual topological space".

I don't want to provide a general definition of what this means, but let me try to illustrate what this means in this specific case. The space $B\mathbf{Z}$ will only have one point, and the rest of the "usual" S^1, will be some sort of virtual loop around the point.

Then, a $\mathbf{Q}$-valued local system on $B\mathbf{Z}$, simply consists on a $\mathbf{Q}$-vector space V, together with automorphisms $V \to V$, given by rotating around the "virtual loop". Notice however, that there are not vector spaces over the "virtual loop", it only serves to record the way in which the vector space V changes after rotation around the loop.

Proposition 14. The category $\mathrm{LocSys}_{\mathbf{Q}}(BG)$, for a finite group G, can be identified with the category of $\mathbf{Q}$-representation of G.

Now, we will try to make a connection between the category $\mathrm{LocSys}_{\mathbf{Q}}(X)$ and the singular cohomology $H\mathbf{Q}(X)$. Notice, however, that we are talking about $H\mathbf{Q}(X)$ as a spectrum, and we are not just considering its individual cohomology groups. This is analogous as considering a chain complex as opposed to its disembodies cohomology groups. However one can always recover the ith singular cohomology groups by taking $\pi_i(H\mathbf{Q}(X))$.

To any space X, which for concreteness we can assume is BG for a finite group G, or $B\mathbf{Z}$. We can consider the trivial local system

$$H\mathbf{Q}_X \in \mathrm{LocSys}_{\mathbf{Q}}(X) \tag{22}$$

Lets try to unpack to see what this means. For each point $\mathrm{pt} \in X$, we are going to have the vector space $\mathbf{Q}_{/\,\mathrm{pt}} = \mathbf{Q}$ over it, and for each path in X between to points, which we denote by $\mathrm{pt}_1 \to \mathrm{pt}_2$, there is a "parallel transport"

$$\mathbf{Q} = \mathbf{Q}_{/\,\mathrm{pt}_1} \xrightarrow{\mathrm{Id}} \mathbf{Q}_{/\,\mathrm{pt}_2} = \mathbf{Q} \tag{23}$$

which will correspond to the identity. In the case over BG, this will correspond to the trivial representation of G. As each path in BG, corresponds to a $g \in G$, and transporting along this path amounts to acting by $g \in G$.

Now that we have an explicit example of a local system $H\mathbf{Q}_X \in \mathrm{LocSys}_{\mathbf{Q}}(X)$, I would like to introduce the notion of global sections

$$\Gamma(H\mathbf{Q}_X) \tag{24}$$

For now, let me tell you about a certain set which will serve as a first approximation to the spectrum $\Gamma(H\mathbf{Q}_X)$. Consider all the tuples of rational numbers $(q_1, q_2, \dots)$, one for each point in X. A way to think about this is to think that we are picking an element $q_i \in \mathbf{Q}_{/\,\mathrm{pt}_i}$ for each point in X. But we only want to consider a subset of this tuples, we only want tuples $(q_1, q_2, \dots)$ such that if there is a path connecting $\mathrm{pt}_1 \to \mathrm{pt}_2$, then the induced map

$$\mathbf{Q}_{/\,\mathrm{pt}_1} \longrightarrow \mathbf{Q}_{/\,\mathrm{pt}_2} \qquad q_1 \mapsto q_2 \tag{25}$$

So in the special case when we are considering the trivial local system $H\mathbf{Q}_X$, if two points pt_1 and pt_2 are connected by a path, then $q_1 = q_2$. So the set that we have described can be canonically identified with

$$\bigoplus_{\pi_0(X)} \mathbf{Q} = \pi_0(H\mathbf{Q}(X)) = \pi_0\Gamma(H\mathbf{Q}_X) \tag{26}$$

For the first equality one can simply compute the 0th singular cohomology of X, and for the second equality it's ok to just take it as the definition. Unfortunately, I don't think I'll be able to provide a similar concrete description for the higher homotopy groups of $\Gamma(H\mathbf{Q}_X)$. So we will proceed by doing some general category theory

Definition 15. We can define the spectrum $\Gamma(H\mathbf{Q}_X)$, as a limit of the constant diagram of shape X

$$\Gamma(H\mathbf{Q}_X) := \lim_{\mathrm{pt}\to X} H\mathbf{Q} \tag{27}$$

Notice that there is an important difference between the more lax notion of homotopy limit, and the usual notion of limit. And this differences has concrete consequences, as if we were only taking the usual limit we will simply get $\pi_0\Gamma(H\mathbf{Q}_X)$ loosing all the higher homotopy groups.

Proposition 16. There is a canonical identification of spectra

$$H\mathbf{Q}(X) \simeq \Gamma(H\mathbf{Q}_X) \tag{28}$$

which induces an isomorphism at the level of homotopy groups

$$\pi_i(H\mathbf{Q}(X)) = \pi_i\Gamma(H\mathbf{Q}_X) \tag{29}$$

Example 17. Ok, now let's try to use what we know about singular cohomology to better understand $\Gamma(H\mathbf{Q}_X)$. A particular space of interest for us is the space $B\mathbf{Z}/n$, for some n. Then we know, by usual computations in singular cohomology that

$$\pi_i\Gamma(H\mathbf{Q}_{B\mathbf{Z}/n}) = \begin{cases} \mathbf{Q} & i = 0 \\ 0 & \text{else} \end{cases} \tag{30}$$

In particular, we see that we are not able to extract any torsion information of the space $B\mathbf{Z}/n$ from $H\mathbf{Q}$. This phenomenon is fairly well understood in homotopy theory, and it is usually explained by saying that the spectrum $H\mathbf{Q}$ has height 0.

Now, we want to start talking about KU-valued local systems on a space X. This discussion is completely analogous to the one we did just now, but with the difference that instead of considering $\mathbf{Q}$-vector spaces, we will be considering KU-modules. That is, an element of $\mathrm{LocSys}_{\mathrm{KU}}(X)$ is a collection of KU-modules $(V_1, V_2, \dots)$, one for each point $\mathrm{pt}_i \in X$. And where for each path $\mathrm{pt}_1 \to \mathrm{pt}_2$, there is an associated isomorphism of KU-modules.

$$V_1 \xrightarrow{\simeq} V_2 \tag{31}$$

Other, similarities that it enjoys are the following

1. There is a trivial local system, which we denote by KU_X
2. We can consider the global section spectrum $\Gamma(\mathrm{KU}_X)$, which comes with an identification $\Gamma(\mathrm{KU}_X) \simeq \mathrm{KU}(X)$.

But what I would like to try to convince you now is that the global sections $\Gamma(\mathrm{KU}_X)$ of the trivial local system, is a much richer invariant of the space than $\Gamma(H\mathbf{Q}_X)$. And this could be tracked back to the fact that there are much more global sections of KU_X, due to the extra symmetry enjoyed by KU due to Bott periodicity.

Remark 18. I don't think I could provide a better conceptual explanation for the fact that $\Gamma(\mathrm{KU}_X)$ is has more information than $\Gamma(H\mathbf{Q}_X)$, besides the fact that KU, being 2-periodic, has much more symmetries and thus picks up more information about a space than $H\mathbf{Q}$. This kind of phenomenon is usually explained by saying that KU is of height 1, and the higher the height the more torsion information about that space X it picks up.

However, I would still like to illustrate, maybe through some computations, how much more information about a space X, in particular we will be interested in spaces of the form BG, the cohomology theory KU picks up, as opposed to $H\mathbf{Q}$.

In order to perform some computations, we will have to introduce the p-completed complex K-theory spectrum, which we will denote by $\mathrm{KU}_{\widehat{\mathrm{p}}}$. Before providing a definition, let me just say that in the same way that going from $\mathbf{Z} \to \mathbf{Z}_p$ looses some integral information, in particular all primes $l \in \mathbf{Z}_p$ will be invertible; going from $\mathrm{KU} \to \mathrm{KU}_{\widehat{\mathrm{p}}}$ is also a loss of integral information. In particular, if I am able to convince you that $\mathrm{KU}_{\widehat{\mathrm{p}}}(X)$ has more information than $H\mathbf{Q}(X)$, then it should follow that $\mathrm{KU}(X)$ has more information than $H\mathbf{Q}(X)$.

Definition 19. The spectrum $\mathrm{KU}_{\widehat{\mathrm{p}}}$ can be defined as

$$\mathrm{KU}_{\widehat{\mathrm{p}}} = \lim(\cdots \to \mathrm{KU}/p^n \to \mathrm{KU}/p^{n-1} \to \cdots \to \mathrm{KU}/p) \tag{32}$$

To make sense of this operations, recall that KU is a spectrum, which is a homotopical analog of an abelian group. Therefore, there should exists a multiplication by p^n map $\mathrm{KU} \to \mathrm{KU}$, and the category of spectra behaves in some ways like an abelian category, in particular it has a notion of cokernel, so one can realize

$$\mathrm{KU}/p^n = \mathrm{coker}(\mathrm{KU} \xrightarrow{\times p^n} \mathrm{KU}) \tag{33}$$

Moreover, we still have that

$$\pi_* \mathrm{KU}_{\widehat{p}} = \mathrm{KU}^*_{\widehat{p}} = \mathbf{Z}_p[\beta^{\pm}] \qquad |\beta| = 2 \tag{34}$$

In particular, $\pi_* \mathrm{KU}_{\widehat{p}}$ has infinitely many non-zero homotopy groups, and still enjoys a form of Bott periodicity. The Bott periodicity property is essential to many of the favorable properties that $\mathrm{KU}_{\widehat{p}}$ has. And it will be particularly important for the moduli-theoretic construction of $\mathrm{KU}_{\widehat{p}}$ we hope to provide later.

Example 20. After completing at a prime p, we can actually show more clearly what the relation between $\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G)$ and $\pi_0 \mathrm{KU}_{\widehat{p}}(BG)$ is. Recall that we have a subgroup $G^{(p)} \subset G$ of p-singular elements, that is, elements $g \in G$, such that $g^{p^n} = 1$ for some $n \gg 0$. Then

$$\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G^{(p)}) = \pi_0 \mathrm{KU}_{\widehat{p}}(BG) \tag{35}$$

That is, $\mathrm{KU}_{\widehat{p}}$ is only able to see the p-power torsion elements of G. So our goal of providing a K-theoretic interpretation of $\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G)$ could be interpreted as finding an integral analog of $\mathrm{KU}_{\widehat{p}}$, which is able to detect torsion elements of G for any prime p.

Remark 21. Before finishing the section, I would like maybe say some words about why we started talking about local systems, when we had a good enough definition of $\mathrm{KU}(X)$. The reason is that our goal, is to explain the construction of the tempered version of complex K-theory, which we denote by $\underline{\mathrm{KU}}$. And we will be able to realize $\underline{\mathrm{KU}}(X)$ as the global sections of a trivial local system, however, in this case it will not be a local system on a space X, but rather, what we will call an orbispace. An orbispace X, is like a space, but now we allow our points to be self-folded (or to have monodromy). Then the Atiyah-Segal comparison map

$$\underline{\mathrm{KU}}(X) \longrightarrow \mathrm{KU}(|X|) = \underline{\mathrm{KU}}(|X|) \tag{36}$$

is simply obtained by the map $|X| \to X$. Where we call $|X|$ the underlying space of the orbispace X, and it basically amounts to the same space after we forget the monodromy at the points of the orbispace X.

But the reader might be confused, we have said that the Atiyah-Segal comparison map was $\mathrm{Rep}(G) \to \mathrm{KU}(BG)$, how is this related to the map above? One can see that by setting $X = BG$ (as an orbispace) the map above specializes to

$$\mathrm{Rep}(G) = \underline{\mathrm{KU}}(BG) \longrightarrow \mathrm{KU}(|BG|) \tag{37}$$

We hope to come back to a more detailed discussion later in the notes.

A renormalized category of spaces

In this section the goal is to explain a fundamental difference between KU and $\underline{\mathrm{KU}}$, which is related to the domain on which this define cohomology theories. The cohomology theory KU determines a functor

$$\mathrm{KU} : \mathrm{Spc}^{\mathrm{op}} \longrightarrow \mathrm{Spctr} \qquad X \mapsto \mathrm{KU}(X) \tag{38}$$

however, the tempered cohomology theory we are interested in understanding better is defined in a certain category of orbispaces, which we denote by OSpc

$$\underline{\mathrm{KU}} : \mathrm{OSpc}^{\mathrm{op}} \longrightarrow \mathrm{Spctr} \qquad X \mapsto \underline{\mathrm{KU}}(X) \tag{39}$$

So how are the categories Spc and OSpc related? The category of orbispaces OSpc is a sort of renormalization of the category of Spc. We will talk more about this, but let me just say that the category of spaces is freely generated by a point, while the category of orbispaces is freely generated by certain points with prescribed monodromy. In order to make sense of this we need to review some category theory

Definition 22. Given a (small) category $\mathcal{C}$, there is always a way in which we can take a free closure under colimits, we will denote this operation by $\widehat{\mathcal{C}}$. This category $\widehat{\mathcal{C}}$ can be characterized by the universal property that for any functor $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a category already closed under colimits, there is a unique factorization

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr] & \widehat{\mathcal{C}} \arrow[d, dashed] \\ & \mathcal{D} \end{tikzcd} \tag{40}$$

and where the resulting functor $\widehat{\mathcal{C}} \to \mathcal{D}$ is colimit preserving. This universal property completely characterizes the operation $\mathcal{C} \to \widehat{\mathcal{C}}$, however we can provide a more concrete description of it as

$$\widehat{\mathcal{C}} := \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc}) \tag{41}$$

and the functor $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ is the yoneda embedding.

Remark 23. An important property of this construction, that will be useful for us, is that the embedding $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ does not preserve colimits, in fact, it destroys almost all colimits, which already existed in $\mathcal{C}$. While this may seem unfortunate, it will be very useful for us.

Example 24. Let pt be the trivial category, it has one object and the identity morphism. Then almost by definition we have that $\widehat{\mathrm{pt}} = \mathrm{Spc}$. This can be more informally phrased by saying that the category of spaces is freely generated by one point. This characterization has some concrete consequences for us: let X be a space and consider the trivial functor

$$X \longrightarrow \mathrm{Spc} \tag{42}$$

that maps every object of X to the point $\mathrm{pt} \in \mathrm{Spc}$, and every morphisms to the identity morphism. Then we have that the colimit of this diagram, is $X \in \mathrm{Spc}$ itself, when considered as a space. Another concrete consequence it has is that any space X, is completely determined by its mapping space from a point. In other words, given a space X, we can completely determined what space it is, by simply knowing the mapping space from a point

$$\mathrm{Map}_{\mathrm{Spc}}(\mathrm{pt}, X) = X \tag{43}$$

Example 25. Using a similar construction, we can now construct the category of orbispaces OSpc. Let $\mathcal{T} \subset \mathrm{Spc}$ to be the full subcategory of objects of the form BM, where M is an abelian group. Then we can define

$$\mathrm{OSpc} := \widehat{\mathcal{T}} = \mathrm{Funct}(\mathcal{T}^{\mathrm{op}}, \mathrm{Spc}) \tag{44}$$

In other words, it can be considered as a renormalization of the category of spaces, which now has as generators spaces of the form BM, rather than just a point. This construction can be a bit subtle if you have not seen it before, so let's try to unpack it a bit. Recall that this construction comes equipped with a Yoneda embedding

$$\mathcal{T} \hookrightarrow \mathrm{OSpc} \qquad BM \mapsto BM^{(-)} = \mathrm{Map}_{\mathcal{T}}(-, BM) \tag{45}$$

It is now that it becomes seriously important that the Yoneda embedding does not preserve colimits. In the category $\mathcal{T}$, we can realize BM as the constant colimit of shape BM

$$BM = \mathrm{colim}_{BM}\, \mathrm{pt} \qquad \text{in } \mathcal{T} \tag{46}$$

However, after embedding $\mathcal{T} \hookrightarrow \mathrm{OSpc}$, we have that the colimit is

$$|BM| = \mathrm{colim}_{BM}\, \mathrm{pt} \qquad \text{in } \mathrm{OSpc} \tag{47}$$

where $|BM| \neq BM^{(-)}$. So there are two objects in OSpc that could reasonably called BM: there is the space $|BM|$ which we are going to call flat BM, and the space $BM^{(-)}$ which we are going to call monodromic BM. And this to spaces are generally not the same, however they come equipped with a canonical map

$$|BM| \longrightarrow BM^{(-)} \tag{48}$$

which will be essential in our discussion of the Atiyah-Segal comparison map. Another related point of difference between OSpc and Spc, is that an orbispace X is no longer determined by the mapping space from a point, in fact we have

$$\mathrm{Map}_{\mathrm{OSpc}}(\mathrm{pt}, X) = |X| \tag{49}$$

where we call $|X|$ the underlying space of X. In order to determine the orbispace X, from some mapping spaces, one really needs to know the mapping spaces for all BM, and not just the point. This is maybe why we are calling OSpc a renormalized category of spaces, since it is now freely generated by spaces of the form BM, rather than just a point.

Remark 26. Notice that all this different categories can be organized into the following commutative diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r] \arrow[d] & \mathcal{T} \arrow[d] \\ \mathrm{Spc} \arrow[r] & \mathrm{OSpc} \end{tikzcd} \tag{50}$$

where the functor $\mathrm{Spc} \to \mathrm{OSpc}$ maps a space X to a certain flat orbispace $|X|$.

Our next goal is to explain that the tempered cohomology $\underline{\mathrm{KU}}$ is a cohomology theory which is naturally defined in OSpc, and use it to rephrase the Atiyah-Segal comparison map in this setting. One of the reasons why we needed this renormalization is that, if you recall, the cohomology of a space $X \in \mathrm{Spc}$ is completely determined by the cohomology of a point

$$\Gamma(\mathrm{KU}_X) = \lim_{\mathrm{pt} \to X} \mathrm{KU} = \mathrm{KU}(X) \tag{51}$$

which basically boiled down to the fact that $\mathrm{KU}(X)$ can be realized as the global sections of the trivial KU-valued local system on a space X, and that any space X can be completely built out of points.

However, now we want to consider local systems in orbispaces X, and not just in spaces. A general good theory of local systems on orbispaces can be a bit subtle, so we content ourselves with knowing that

1. There exists a trivial $\underline{\mathrm{KU}}$-values local system on any orbispace X, which we denote by $\underline{\mathrm{KU}}_X$.

2. To the trivial local system $\underline{\mathrm{KU}}_X$, there is an associated spectrum of global sections $\Gamma(\underline{\mathrm{KU}}_X)$, which satisfy the identity
$$\Gamma(\underline{\mathrm{KU}}_X) = \lim_{BM^{(-)}\to X} \underline{\mathrm{KU}}(BM^{(-)}) = \underline{\mathrm{KU}}(X) \tag{52}$$

In particular, it is important to notice that we can no longer extract the cohomology of the orbispace $BM^{(-)}$, from the cohomology of a point. The cohomology of $BM^{(-)}$ is now an extra piece of data what one needs to input into $\underline{\mathrm{KU}}$, to get the theory off the ground. In this section we will not explain how to do this, we hope to come back to it later, but let me just say that in particular it satisfies

$$\underline{\mathrm{KU}}^0(BM^{(-)}) = \mathrm{Rep}(M) \tag{53}$$

for any abelian group M. However, at the beginning of this note we claimed that we would be able to recover the representation theory of all finite groups G, and not only abelian groups. We would now like to explain how to extract this kind of information. We can now ask ourselves, does there exists a natural orbispace X, such that $\underline{\mathrm{KU}}^0(X) = \mathrm{Rep}(G)$? A first guess could be the flat orbispace $|BG|$, but a flat orbispace is completely built out of points, and in particular it does not uses any of the new information stored in $\underline{\mathrm{KU}}$. Concretely we have that

$$\underline{\mathrm{KU}}(|BG|) = \lim_{\mathrm{pt}\to|BG|} \mathrm{KU} = \mathrm{KU}(BG) \tag{54}$$

which we already know does not recover the whole representation theory of G. In the case of abelian groups M, we had to input the monodromic orbispace $BM^{(-)}$, however, this is no longer available in this case, since G is not abelian. So we proceed as follows: in the category of spaces, consider the category of morphisms $BM \to BG$, for all spaces of the form BM for M abelian, and denote this category by $\mathcal{T}/BG$. Then we have the following
$$\mathrm{Rep}(G) = \pi_0\Big(\lim_{\mathcal{T}/BG} \underline{\mathrm{KU}}(BM)\Big) \tag{55}$$

In other words, we are able to recover the representation theory of all finite groups, only from the representation theory of finite abelian groups.

Finally, we would like to present a form of the Atiyah-Segal comparison map which fits well into the setting of orbispaces, and the discussion in this section. Let X be an orbispace, then we can explicitly construct the associated flat orbispace $|X|$ by taking the following colimit

$$|X| = \mathrm{colim}_{\mathrm{pt}\to X}\, \mathrm{pt} \longrightarrow X \tag{56}$$

In particular, by construction we notice that it comes equipped with a natural map $|X| \to X$. Then applying the functor $\underline{\mathrm{KU}}$ we get a map
$$\underline{\mathrm{KU}}(X) \longrightarrow \underline{\mathrm{KU}}(|X|) = \mathrm{KU}(|X|) \tag{57}$$

In particular this recovers the usual Atiyah-Segal comparison map by setting $X = BM$ for an abelian group M. And for a non-abelian G, simply set $X = \mathrm{colim}_{\mathcal{T}/BG}\, BM$.

P-divisible groups

In this section there will be a change of gears in our perspective on the subject. So far, we have talked about KU and $\underline{\mathrm{KU}}$, without defining then, but relying on several nice properties they enjoy. In particular, we have said the following

1. The spectrum KU determines a functor $\mathrm{Spc}^{\mathrm{op}} \to \mathrm{Spctr}$, and this functor sends colimits in Spc to limits in Spctr, in particular the functor is completely determined by its values on $\mathrm{pt} \in \mathrm{Spc}$
2. Similarly, the tempered spectrum $\underline{\mathrm{KU}}$ determines a functor $\mathrm{OSpc}^{\mathrm{op}} \to \mathrm{Spctr}$, and it sends colimits in orbispaces to limits in spectra. In particular, this functor is completely determined by its values on the monodromic $BM^{(-)} \in \mathrm{OSpc}$.

The goal for the rest of this notes is to explain a close relationship between the structure that this functors have when we restrict then to spaces of the form BM, with some objects in algebraic geometry called p-divisible groups. In fact, we will see that the spectrum $\mathrm{KU}_{\widehat{\mathrm{p}}}$ comes equipped with the p-divisible group μ_{p^∞}, and that $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$ comes equipped with the p-divisible group $\mu_{\mathbf{P}^\infty}$. We will say more about this later, but for now we will try to get comfortable with some relevant examples of this notions.

Example 27. The p-divisible group μ_{p^∞} will be of central importance for us. This geometric object can in fact be defined over any ring spectrum, but the most relevant cases for us will be when it is defined over $\mathbf{F}_p, \mathbf{Z}_p$ and $\mathrm{KU}_{\widehat{\mathrm{p}}}$.

1. Over any ring R, the p-divisible group μ_{p^∞}, admits a description as a functor $\mu_{p^\infty} : \mathrm{Aff}^{\mathrm{op}}_{\mathrm{Spec}\, R} \to \mathrm{Ab}$, defined by
$$\mu_{p^\infty}(\mathrm{Spec}\, S) = \{x \in S, x^{p^m} = 1 \text{ for } m \gg 0\} \tag{58}$$
for all R-algebras S.
2. This functor is in fact represented by and ind-affine scheme
$$\mu_{p^\infty} = \mathrm{colim}\, \mathrm{Spec}\, R[t]/(t^{p^m} - 1) \tag{59}$$
3. This functor admits a very special identity when we specialize ourselves to $\mathbf{F}_p$-algebras. Namely, $\mu_{p^\infty} / \mathrm{Spec}\, \mathbf{F}_p$ is represented by the ind-affine scheme
$$\mu_{p^\infty} = \mathrm{colim}\, \mathrm{Spec}\, \mathbf{F}_p[t]/(t^{p^m} - 1) = \mathrm{colim}\, \mathrm{Spec}\, \mathbf{F}_p[t]/(t-1)^{p^m} \tag{60}$$
In particular, it is connected.

However, I would like to highlight that all the descriptions we have provided above only rely in the fact that μ_{p^∞} is a group scheme, but a p-divisible group has more structure than a group scheme. As you may notice μ_{p^∞} is a bit infinite, that is, if we ask for the degree of the map $\mu_{p^\infty} \to \mathrm{Spec}\, R$, we get that it has infinite degree. But it is a very controlled kind of infinite, it is built from very nice finite pieces, namely $\mathrm{Spec}\, R[t]/(t^{p^m} - 1)$. This kind of extra structure, in which we record the kind of nice finite pieces μ_{p^∞} is built from, is what makes it a p-divisible group. In particular, we can extract the p^n-torsion points

$$\mu_{p^\infty}[p^n] = \mathrm{Ker}(\mu_{p^\infty} \xrightarrow{\times p^n} \mu_{p^\infty}) = \mathrm{Spec}\, R[t]/(t^{p^n} - 1) \tag{61}$$

and completely rebuilt μ_{p^∞} from this pieces

$$\mu_{p^\infty} = \mathrm{colim}\left(\mu_{p^\infty}[p] \hookrightarrow \mu_{p^\infty}[p^n] \hookrightarrow \cdots\right) \tag{62}$$

Remark 28. Recall that for our needs it will be important to consider $\mu_{p^\infty} / \mathrm{KU}_{\widehat{\mathrm{p}}}$, which might make the reader a bit uncomfortable, but this is in fact a legitimate operation. In order to help get some geometric intuition about this kind of object, simply recall that $\pi_0(\mathrm{KU}_{\widehat{\mathrm{p}}}) = \mathbf{Z}_p$, so as a first approximation, it is reasonable to visualize $\mu_{p^\infty} / \mathrm{KU}_{\widehat{\mathrm{p}}}$ as $\mu_{p^\infty} / \mathbf{Z}_p$.

The following calculation marks the beginning of the close relation between $\mathrm{KU}_{\widehat{\mathrm{p}}}$ and μ_{p^∞}

$$\mathrm{Spec}\Big(\mathrm{KU}_{\widehat{\mathrm{p}}}(B\mathbf{Z}/n)\Big) = \mu_{p^\infty}[p^n] \tag{63}$$

This is interesting, as it is saying that μ_{p^∞} is already encoded in $\mathrm{KU}_{\widehat{\mathrm{p}}}$. However, the reader might object that μ_{p^∞} is not only a scheme, but has other structure, like a multiplicative structure. Our next goal is to provide a way in which we can extract this extra information. But in order to do this, we will need a better understanding of p-divisible groups.

But for our needs, the notion of p-divisible group is not enough, we are going to need a similar notion, called $\mathbf{P}$-divisible groups, which work for all primes p-simultaneously.

Definition 29. A $\mathbf{P}$-divisible group over a ring A, is functor

$$X : \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \longrightarrow \mathrm{FiniteFlatGroupSchemes}_{/\,\mathrm{Spec}\,A} \tag{64}$$

which satisfies certain conditions

- The commutative finite flat group scheme $X(0) = A$
- We call the finite flat group scheme $X(M)$, the M-torsion points of the $\mathbf{P}$-divisible group X.
- The functor is symmetric monoidal, that is, the group structure of $X(M)$ is completely determined by the group structure of M. So in some sense, it is redundant to require that the functor lands in finite flat group schemes, as finite flat algebras will suffice
- The functor X is exact, that is, for a short exact sequence of finite abelian groups $0 \to M_1 \to M_2 \to M_3 \to 0$, the induced diagram.

$$\begin{tikzcd} X(M_3) \arrow[r] \arrow[d] & X(M_2) \arrow[d] \\ X(0) \arrow[r] & X(M_1) \end{tikzcd} \tag{65}$$

  is a exact sequence of finite flat group schemes

- Moreover, one can define a notion of heigh. We say that a $\mathbf{P}$-divisible group has height h, if for every finite abelian group M, the finite flat group scheme $X(M)$ has degree $|M|^h$ over A. Where $|M|$ is the cardinality of M.

Example 30. We would now like to explain how to encode μ_{p^∞} as a $\mathbf{P}$-divisible group. Recall that any finite abelian group M can be decomposed as

$$M = \bigoplus_{p\in\mathbf{P}} \Big(\oplus\, \mathbf{Z}/p^n\Big) \tag{66}$$

then we have that

$$\mu_{p^\infty}(M) = \mu_{p^\infty}(\oplus\mathbf{Z}/p^n) = \oplus\mu_{p^\infty}(\mathbf{Z}/p^n) = \oplus\mu_{p^\infty}[p^n] \tag{67}$$

in particular we have that

$$\mu_{p^\infty}(\mathbf{Z}/p^n) = \mu_{p^\infty}[p^n] \qquad \mu_{p^\infty}(\mathbf{Z}/l^n) = \emptyset \tag{68}$$

Example 31. This notion also allow us to produce other very interesting objects, like

$$\mu_{\mathbf{P}^\infty} = \bigoplus_{p \in \mathbf{P}} \mu_{p^\infty} \tag{69}$$

which satisfies

$$\mu_{\mathbf{P}^\infty}(M) = \bigoplus_{p \in \mathbf{P}} \Big(\oplus \, \mu_{p^\infty}[\mathbf{Z}/n] \Big) \tag{70}$$

provided that M decomposes as in the previous example.

But there is a small disconnect between the kind of objects we have been discussing. We claimed that $\mathrm{KU}_{\widehat{\mathrm{p}}}$ already has the information of the p-divisible group μ_{p^∞}. However, notice that the p-divisible group is defined as a functor

$$\mu_{p^\infty} : \mathrm{Ab}_{\mathrm{fin}} \longrightarrow \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \qquad M \mapsto \mu_{p^\infty}[M] \tag{71}$$

while we said that to extract the information of μ_{p^∞} from $\mathrm{KU}_{\widehat{\mathrm{p}}}$ we need to consider the functor

$$\mathrm{KU}_{\widehat{\mathrm{p}}} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \qquad BM \mapsto \mathrm{KU}_{\widehat{\mathrm{p}}}(BM) \tag{72}$$

How are this two related? Do they encode the same amount of information? We begin by constructing a functor relating both domains of this functors

$$\mathrm{Ab}_{\mathrm{fin}} \xrightarrow{M \mapsto \widehat{M}} \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \xrightarrow{\widehat{M} \mapsto B\widehat{M}} \mathcal{T}^{\mathrm{op}} \tag{73}$$

where $\widehat{M}$ is called the Pontryagin dual of M, and it is defined as

$$\widehat{M} = \mathrm{Map}(M, \mathbf{Q}/\mathbf{Z}) \tag{74}$$

So when we say that $\mathrm{KU}_{\widehat{\mathrm{p}}}$ already encodes the information of the $\mathbf{P}$-divisible group μ_{p^∞}, what we really mean is that the composite functor

$$\mathrm{Ab}_{\mathrm{fin}} \xrightarrow{M \mapsto \widehat{M}} \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \xrightarrow{\widehat{M} \mapsto B\widehat{M}} \mathcal{T}^{\mathrm{op}} \xrightarrow{B\widehat{M} \mapsto \mathrm{KU}_{\widehat{\mathrm{p}}}(B\widehat{M})} \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \tag{75}$$

determines the $\mathbf{P}$-divisible μ_{p^∞} group over $\mathrm{KU}_{\widehat{\mathrm{p}}}$.

Warning 32. We are now talking about $\mathrm{KU}_{\widehat{\mathrm{p}}}$ as a functor whose domain is $\mathcal{T}$, what we are doing here is simply to restrict the functor $\mathrm{KU}_{\widehat{\mathrm{p}}} : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}}$, via the natural inclusion $\mathcal{T} \hookrightarrow \mathrm{Spc}$. So in particular, we are considering BM as the flat $|BM|$. Nonetheless there is a natural way to extend this functor $\mathrm{KU}_{\widehat{\mathrm{p}}} : \mathcal{T}^{\mathrm{op}} \to \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}}$ to a functor via the process of right kan extension to a functor

$$\begin{tikzcd} \mathcal{T}^{\mathrm{op}} \arrow[r, "\mathrm{KU}_{\widehat{\mathrm{p}}}(-)"] \arrow[d, "BM \mapsto BM^{(-)}"'] & \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \\ \mathrm{OSpc}^{\mathrm{op}} \arrow[ur, "\mathrm{RKE}"'] & \end{tikzcd} \tag{76}$$

which would imply that we are considering BM as the monodromic one, as opposed to the flat one. However, this is not a contradiction, recall that there is a natural map $|BM| \to BM^{(-)}$ in OSpc, and then in this special situation we have that the Atiyah-Segal comparison map is an isomorphism

$$\mathrm{KU}_{\widehat{\mathrm{p}}}(BM^{(-)}) \xrightarrow{\simeq} \mathrm{KU}_{\widehat{\mathrm{p}}}(|BM|) \tag{77}$$

and this is due to the fact that $\mathbf{P}$-divisible group associated $\mathrm{KU}_{\widehat{\mathrm{p}}}$, namely μ_{p^∞}, is formally connected. More informally, we could say that there is a natural way to enhance $\mathrm{KU}_{\widehat{\mathrm{p}}}$ to a tempered spectrum, however, we don't gain anything by doing this.

Example 33. Since I am personally not very familiar with Pontryagin duality, it would like to work out an example to get a better understanding of what is happening under the functor

$$\mathrm{Ab}_{\mathrm{fin}} \to \mathcal{T}^{\mathrm{op}} \qquad M \mapsto B\widehat{M} \tag{78}$$

Notice that at the level of objects Pontryagin duality it is not doing anything very interesting, as all all finite abelian groups are self dual. Lets verify this fact, notice it suffices to check it for finite abelian groups of the form $\mathbf{Z}/p^n$ and then one can formally extend the results to all finite abelian groups. The self duality map of $\mathbf{Z}/p^n$ can be described via the map

$$\mathbf{Z}/p^n \to \mathrm{Map}(\mathbf{Z}/p^n, \mathbf{Q}/\mathbf{Z}) \qquad n \mapsto \big(1 \mapsto k/p^n\big) \tag{79}$$

However, it is doing something rather interesting at the level of morphisms. Consider the quotient map $\mathbf{Z}/p^n \to \mathbf{Z}/p^{n-1}$, then by applying Pontryagin duality we get a map

$$\mathrm{Map}(\mathbf{Z}/p^{n-1}, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Map}(\mathbf{Z}/p^n, \mathbf{Q}/\mathbf{Z}) \qquad (1 \mapsto k/p^{n-1}) \mapsto (1 \mapsto kp/p^n) \tag{80}$$

which after applying the self duality becomes

$$\mathbf{Z}/p^{n-1} \to \mathbf{Z}/p^n \qquad k \mapsto pk \tag{81}$$

Moreover, taking inspiration from the classical theory of Fourier analysis, we know that Pontryagin duality is most interesting when one tries to take limits, that is, when we try to make sense of $\widehat{\mathbf{Z}_p}$, which should be something resembling a circle. What we are going to say next, does not really make sense as $\mathbf{Z}_p$ is not a finite abelian group, but it is closed to being so, and maybe it could be put into solid footing if we considered the pro-finite completion (or maybe condensed objects) of the category $\mathrm{Ab}_{\mathrm{fin}}$. We want to compute the Pontryagin dual of

$$\mathbf{Z}_p = \lim\Big(\cdots \to \mathbf{Z}/p^n \to \mathbf{Z}/p^{n-1}\Big) \tag{82}$$

which will give us something like

$$\mathrm{colim}\left(\mathbf{Z}/p^{n-1} \to \mathbf{Z}/p^n \to \cdots\right) = \mathbf{Q}_p/\mathbf{Z}_p = C_{p^\infty} \tag{83}$$

which should resemble the circle $\mathbb{T}$ from classical Fourier analysis. Finally, we need to explain what happens when we do the operation $\widehat{M} \to B\widehat{M}$. We begin by noticing the following

$$\mathrm{Map}_{\mathrm{Spc}}(BM, BN) \simeq \mathrm{Map}_{\mathrm{Ab}}(M, N) \times BN \tag{84}$$

that is, the homotopy classes of maps $\mathrm{Map}_{\mathrm{Spc}}(BM, BN)$ are in bijection with the maps $\mathrm{Map}_{\mathrm{Ab}}(M, N)$, however, every map $BM \to BN$ comes equipped with some non-trivial self homotopies. We don't have much more to say about this, but would just like to highlight that the functor $\mathrm{Ab}_{\mathrm{fin}} \to \mathcal{T}^{\mathrm{op}}$ is doing something rather non-trivial.

From the identification

$$\mathrm{Map}_{\mathrm{Spc}}(BM, BN) \simeq \mathrm{Map}_{\mathrm{Ab}}(M, N) \times BN \tag{85}$$

we can see that providing a factorization of the $\mathbf{P}$-divisible group $\mu_{p^\infty} / \mathrm{KU}_{\widehat{p}}$

$$\mathrm{Ab}_{\mathrm{fin}} \to \mathcal{T}^{\mathrm{op}} \to \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{p}}} \tag{86}$$

requires us to provide extra information, since even though every object of $\mathcal{T}$ is in the essential image of the functor $\mathrm{Ab}_{\mathrm{fin}} \to \mathcal{T}^{\mathrm{op}}$, it does not induces an equivalence at the level of mapping spaces. Such extra information is called a pre-orientation of the $\mathbf{P}$-divisible group, and there are many choices of pre-orientations (we will come back to this in the next section). But in the case of the $\mathbf{P}$-divisible group $\mathrm{KU}_{\widehat{\mathrm{p}}}$ we lucked out, as we can simply define the map

$$\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \qquad BM \mapsto \mathrm{KU}_{\widehat{\mathrm{p}}}(BM) \tag{87}$$

and this will gives us the $\mathbf{P}$-divisible group μ_{p^∞}. This phenomenon is very special to the situation we have put ourselves on, we are using the fact that $\mathrm{KU}_{\widehat{\mathrm{p}}}$ is a $K(1)$-local spectrum, and that μ_{p^∞} identifies with the Quillen $\mathbf{P}$-divisible group of $\mathrm{KU}_{\widehat{\mathrm{p}}}$.

We can now use some of this language to explain the kind of information that is retained by $\mathrm{KU}_{\widehat{\mathrm{p}}}$ when we evaluate it at BM. We have that

$$\mathrm{KU}_{\widehat{\mathrm{p}}}(BM) = \mu_{p^\infty}[M] = \mu_{p^\infty}[M^{(p)}] \tag{88}$$

where $M^{(p)} \subset M$ are the p-singular elements of M, that is, the elements of $m \in M$ such that $m^{p^n} = 1$. In fact, recall that

$$\pi_0 \, \mathrm{KU}_{\widehat{\mathrm{p}}}(BG) = \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G^{(p)}) \tag{89}$$

I hope that the above discussion provides some explanation for this phenomenon. What is happening is that $\mathrm{KU}_{\widehat{\mathrm{p}}}$ is only able to detect the p-power torsion elements of G. In order to remedy this, that is, in order to construct $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$, we will need to replace by hand the $\mathbf{P}$-divisible group μ_{p^∞} by $\mu_{\mathbf{P}^\infty}$, such that $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$ is able to detect torsion elements of G for all primes.

Tempered KU

In this final section, we would like to construct the tempered spectrum $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$. We have chosen to complete at p we believe that it provides us with some technical simplifications, and our construction would not really work if we tried doing it integrally. If we try to tun the same story fully integrally, we will notice that there is no canonical $\mathbf{P}$-divisible group associated to KU, as the spectrum is no longer $K(1)$-local. Having $\mathrm{KU}_{\widehat{\mathrm{p}}}$ being $K(1)$-local really has concrete simplifications for us.

We now proceed with the construction of the tempered spectrum

$$\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}} : \mathrm{OSpc}^{\mathrm{op}} \longrightarrow \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \tag{90}$$

In order to construct such a functor, it suffices to specify its values at $\mathcal{T} \hookrightarrow \mathrm{OSpc}$. The values of this functor at BM cannot simply be $\mathrm{KU}_{\widehat{\mathrm{p}}}(BM)$, because by the discussion of the previous section, then then we would get that the $\mathbf{P}$-divisible group associated to it would be μ_{p^∞}, and we would like it to be $\mu_{\mathbf{P}^\infty}$. So we proceed differently, we begin with the data of the $\mathbf{P}$-divisible group

$$\mu_{\mathbf{P}^\infty} : \mathrm{Ab}_{\mathrm{fin}} \longrightarrow \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \tag{91}$$

and we need to specify a factorization of this functor through $\mathrm{Ab}_{\mathrm{fin}} \to \mathcal{T}^{\mathrm{op}}$. Recall the following terminology

Definition 34. A factorization of a $\mathbf{P}$-divisible group through the functor

$$\mathrm{Ab}_{\mathrm{fin}} \to \mathcal{T}^{\mathrm{op}} \qquad M \mapsto B\widehat{M} \tag{92}$$

is called a pre-orientation on the $\mathbf{P}$-divisible group. And recall that there are many possible pre-orientations.

Example 35. A first example of such a pre-orientation is given by the trivial pre-orientation, given by the composite

$$\mathcal{T}^{\mathrm{op}} \xrightarrow{\pi_1} \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{op}} \xrightarrow{\mu_{\mathbf{P}^\infty}} \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \tag{93}$$

However, this ends up being too naive to work. In order to see a concrete example where this fails, let's try to implement this on $\mu_{p^\infty} / \mathrm{KU}_{\widehat{\mathrm{p}}}$. In this situation, we still have an Atiyah-Segal comparison map

$$\mathrm{KU}_{\widehat{\mathrm{p}}}(BM^{(-)}) \to \mathrm{KU}_{\widehat{\mathrm{p}}}(|BM|) \tag{94}$$

but it will factor as

$$\mathrm{KU}_{\widehat{\mathrm{p}}}(BM^{(-)}) \to \mathrm{KU}_{\widehat{\mathrm{p}}} \to \mathrm{KU}_{\widehat{\mathrm{p}}}(|BM|) \tag{95}$$

in particular it would would make it impossible for it to present $\mathrm{KU}_{\widehat{\mathrm{p}}}(|BM|)$ as as a completion of $\mathrm{KU}_{\widehat{\mathrm{p}}}(BM^{(-)})$. This is showing that the canonical pre-orientation of $\mathrm{KU}_{\widehat{\mathrm{p}}}$ we have described above, is not the trivial one.

Definition 36. We say that a pre-orientation of the $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty} / \mathrm{KU}_{\widehat{\mathrm{p}}}$ is an orientation, if for every monodromic $BM^{(-)}$ the Atiyah-Segal comparison

$$\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}(BM^{(-)}) \longrightarrow \underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}(|BM|) \tag{96}$$

presents $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}(|BM|)$ as the completion of $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}(BM^{(-)})$ with respect to the ideal

$$\mathrm{Ker}\left(\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}(BM^{(-)}) \to \underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}(\mathrm{pt})\right) \tag{97}$$

Remark 37. So the fact that the canonical pre-orientation of $\mathrm{KU}_{\widehat{\mathrm{p}}}$ is an orientation is a rather special phenomenon. In fact, a $\mathbf{P}$-divisible group over a ring spectrum A, can only admit an orientation if A is 2-periodic.

So how are we going to construct a pre-orientation on $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$ which is an orientation? The trivial pre-orientation will not work as it will not satisfy the Atiyah-Segal condition, while the canonical one we used for $\mathrm{KU}_{\widehat{\mathrm{p}}}$ will also not work as we need to obtain a the $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$ as opposed to μ_{p^∞}. However, we have lucked out again, as the theory of (pre)-orientations enjoys some properties that will allow us to boot-strap an orientation for $\mu_{\mathbf{P}^\infty}$ from the one in μ_{p^∞}. We will need the following definition

Definition 38. To any $\mathbf{P}$-divisible group, we can extract its connected part, from the connected-etale sequence. In particular, we have that

$$\mu^\circ_{\mathbf{P}^\infty} = \mu_{p^\infty} \tag{98}$$

where $\mu^\circ_{\mathbf{P}^\infty}$ denotes the connected part of $\mu_{\mathbf{P}^\infty}$.

Now we can say what kind of favorable properties the theory of pre-orientation enjoys

1. Let $\mathrm{Pre}(\mu_{\mathbf{P}^\infty})$ be the space of pre-orientations on $\mu_{\mathbf{P}^\infty} / \mathrm{KU}_{\widehat{\mathrm{p}}}$. Then there is a canonical isomorphism

$$\mathrm{Pre}(\mu_{\mathbf{P}^\infty}) \simeq \mathrm{Pre}(\mu^\circ_{\mathbf{P}^\infty}) \simeq \mathrm{Pre}(\mu_{p^\infty}) \tag{99}$$

2. An orientation $e \in \mathrm{Pre}(\mu_{\mathbf{P}^\infty})$ is an orientation if and only if the corresponding $e^\circ \in \mathrm{Pre}(\mu_{p^\infty})$ is an orientation

Then using this results, we can endow $\mu_{\mathbf{P}^\infty}$ with an orientation, concluding the construction of the tempered spectrum

$$\begin{tikzcd} \mathcal{T}^{\mathrm{op}} \arrow[r, "\mu_{\mathbf{P}^\infty}"] \arrow[d, "BM \mapsto BM^{(-)}"'] & \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \\ \underline{\mathrm{KU}}_{\widehat{\mathrm{p}}} : \mathrm{OSpc}^{\mathrm{op}} \arrow[ur, "\mathrm{RKE}"'] & \end{tikzcd} \tag{100}$$

References

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