SCUM - Simplicial Sets

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April 12, 2017

Simplicial Sets are spaces constructed from n-dimensional triangles. Examples of what n dimensional triangles are. This triangles interact with each other in very simple ways, you only have face and degeneracy maps. Give examples of those. This is where the beauty resides, since this are much more simple than topological spaces.

While this is a noble idea, as everything in math, it needs to be formalized to meet the industry standards. Unfortunately, at first sight, the definition of a simplicial set looks nothing like the one of a space. The goal of this lecture is to show why it is a natural definition to have, and hopefully provide some geometric intuition.

The general idea to describe simplicial sets is the following: we will describe a category of building objects, the category of n dimensional triangles. Under abstract non sense it is possible to give instructions to paste objects together. This is closely related to the notion of a colimit. Unfortunately, the "pasted object" need not exist in the category of n-dimensional triangles. Give an example. So our goal is to describe a general framework in which one can take the "closure" of a category. And this I hope motivates the definition of a simplicial set.

We will start by presenting some background that we will need for the rest of the lecture. The background needed is mostly on category theory. We will also provide examples relevant to the talk to better understand this definitions.

Definition 1. A *category* $\mathcal{C}$ is a collection of objects, and morphisms between the objects. The morphism are closed under composition in the category. This means that is if $f : A \to B$ and $g : B \to C$ are in $\mathcal{C}$ then $g \circ f : A \to C$ is in $\mathcal{C}$.

Example 1. Category of Groups, and morphisms are homomorphisms. And the category of topological spaces, with morphisms being continuous functions. Morphisms are structure preserving.

Example 2. Our first example will be the simplex category Δ. This is the category of n dimensional triangles. The objects are totally ordered sets

$$[n] := 0 \to 1 \to 2 \to \cdots \to n \tag{1}$$

and the morphisms are order preserving functions. Also provide alternative definition where $[n]$ is a category, and the morphisms are functors.

We should think of $[n]$ as an n-dimensional triangle, as the following picture shows how we can think about [2]

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 \arrow[rr] & & 2 \end{tikzcd} \tag{2}$$

and the morphisms should be thought of as ways in which one can paste triangles with each other. For example the morphisms $[1] \to [2]$ defined by

$$0 \to 1 \longrightarrow 1 \to 2 \tag{3}$$

says that we can paste the 1-dimensional triangle onto the lower side of the 2-dimensional triangle. This is just a formalism we use to conveniently talk about n-dimensional triangles, and how they interact with each other. Note that this category is very very simple, and this is why simplicial sets have been so popular for describing this kinds of spaces as opposed to topological spaces.

Continuous functions in topological spaces are really weird. For example we know that there exists a surjective continuous function from $I \to I^2$. This is the Peano filling curve. We want something simpler that provides us with a clean description of the spaces, without having to deal with pathologies. Math is already hard, I dont need that kind of things in my life.

We will now define functors and natural transformations. If you had seen this before then it will make not difference, and if you havnt then this should be enough to understand.

Definition 2. A functor is a morphisms in the category *Cat* of categories. A natural transformation is a morphism of functors.

Example 3. A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathit{Set}$.

I told you it would look nothing like a space. The goal of this lecture is to illustrate the geometry behind this definition, and why it is a natural definition to have. Roughly speaking, we will show that a simplicial set is an object built from the objects of Δ. But to do this we need to consider the category of simplicial sets.

Definition 3. Let $\mathcal{C}$ be a (small) category. Then the category of presheaves $\mathit{Pre}(\mathcal{C})$ is the category, whose objects are functors $\mathcal{C}^{\mathrm{op}} \to \mathit{Set}$ and its morphisms are natural transformations.

Definition 4. The category of simplicial sets, denoted by *sSet* is the category $\mathit{Pre}(\Delta)$.

We will interpret the category $\mathcal{C}$ as the category of building objects, and $\mathit{Pre}(\mathcal{C})$ as the category of built objects. That is $\mathit{Pre}(\mathcal{C})$ is the category of objects built from $\mathcal{C}$.

A first requirement for this interpretation to make sense is that one can embeding $\mathcal{C} \hookrightarrow \mathit{Pre}(\mathcal{C})$. This is called the Yoneda Embedding.

Lemma 1 (Yoneda Embedding)**.** *There exists a fully faithful embedding $\mathcal{Y} : \mathcal{C} \hookrightarrow \mathit{Pre}(\mathcal{C})$. This is the Yoneda embedding. It maps $c \mapsto \hom_{\mathcal{C}}(-, c)$. The presheaves on the image of $\mathcal{Y}$ are called representable functors.*

Note that from this first examples we have that the values that Δ^n takes are the ways in which one can paste other objects onto it. This interpretation will hold for every simplicial set.

Example 4. To each $[n]$ in Δ there exists a simplicial set canonically assign to it. This is the functor $\mathrm{Hom}(-, [n])$. It takes an element $[m]$ and it gives of the set of maps $[m] \to [n]$. We will later see why this assignment is canonical. Say that we denote this simplicial set by Δ^n.

There has been a lot of abstract non sense so far, and this talk is mostly about abstract non sense. But I am going to provide some examples to try to get a hang of the geometry of simplicial sets. But before that we need another theorem.

Lemma 2 (Yoneda Lemma)**.** *For any functor $F : \mathcal{C} \to Set$, whose domain is locally small and any object $c \in \mathcal{C}$, there is a bijection*

$$\mathrm{Nat}(\mathcal{C}(-, c), F) \cong F(c) \tag{4}$$

That associated a natural transformation $\alpha : C(-, c) \Rightarrow F$ to an element $\alpha_c(1_c) \in F(c)$. Moreover this correspondence is natural in both c and F.

There exists a way of giving instructions for things to paste in the categorical setting. It is call a colimit, but all you need to know is that it means that we want things to paste.

Definition 5. A colimit is the most efficient pasting of some diagram. In any *Pre*$(\mathcal{C})$ the colimits are computed object wise.

Finally some examples.

Example 5. Compute $\partial\Delta^1(1), \Delta^1(1)$ and $\partial\Delta^2(2), \Delta^2(2)$ while making the connection with the Yoneda lemma.

Now back to abstract non sense. We want to keep justifying the fact that we interpret *sSet* as the objects built from Δ. Another requirement is that every object of *sSet* can be built from Δ. In more technical terms this means that every simplicial set is a colimit of representables. We have the following result.

Proposition 1. *Every presheaf is a colimit of representables. This says in some sense that the information of all the objects is intrinsic from $\mathcal{C}$, since it can be encoded on a diagram on $\mathcal{C}$.*

There is an analogous statement about morphisms in Pre$(\mathcal{C})$. Any morphism can be defined as a morphism of diagrams in $\mathcal{C}$. In some sense all the information of Pre$(\mathcal{C})$ is intrinsic from $\mathcal{C}$.

Note that the fact that morphisms also come in this form makes *Pre*$(\mathcal{C})$ a better candidate than any topological category.

And the final requirement, at least in my opinion is to show that any given pasting instruction that we give will be pastes on the category of simplicial sets. In other words this means that the category of simplicial sets is closed under colimits. We have the following result.

Proposition 2. *The category Pre$(\mathcal{C})$ is cocomplete. This means that it is closed under colimits. Cocompleteness is inherited from the category of Sets, and colimits can be computed objectwise.*

Ok. I think now we have a fairly good justification to interpret *sSet* as the category of objects built from Δ.

But I would still be a bit unsatisfied with this answers. There could be many categories with this properties that are not equivalent, why this one? For me, a satisfying answer would be some sort of universal property of *Pre*$(\mathcal{C})$.

The following result will show that *Pre*$(\mathcal{C})$ is not only a cocompletion of $\mathcal{C}$, but it is in some sense the simplest completion that one can do.

Theorem 1. *Let $\mathcal{C}$ be a small category, $\mathcal{D}$ be a cocomplete category, and let $\Gamma : \mathcal{C} \to \mathcal{D}$ be a functor. Then there exists a left adjoint functor Re : Pre$(\mathcal{C}) \to \mathcal{D}$ that makes the following diagram commute:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{Y}"] \arrow[dr, "\Gamma"'] & Pre(\mathcal{C}) \arrow[d, "Re"] \\ & \mathcal{D} \end{tikzcd} \tag{5}$$

Remark. $\mathit{Re} \dashv \mathit{Sing}$, this means that Re is the left adjoint, and Sing is the right adjoint. This correspond to the usual Geometric realization functor and singular functor when $\mathcal{D}$ is the category of topological spaces. Left adjoint functors preserve colimits. Maybe develop what it means for a functor to be colimit preserving.