Functor of Points Geometry

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Redefining Geometry

Geometry could be thought as the study of objects, which are obtained by pasting together local models. In other words, when we are doing geometry, we have a category of local models, which can be embedded into the category of global objects, that locally look like our local models. For example

(1) The category of affine schemes $\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{Sch}$ is the main example of local models we will be using, and the category of schemes Sch is the category of global objects.

(2) The category of n-manifolds, is another example of a category of global objects, whose local models are open subsets of of $\mathbb{R}^n$.

(3) The category of CW-complexes is yet another category of global objects, whose local models are the n-dimensional unit disk, for every n.

You may be thinking that a CW-complex X does not look locally like an n-dimensional disk. However, once we reinterpret categorically what we mean by local model, we will see that they fit right into our framework.

We will begin by making some very general construction: for any category $\mathcal{C}$ we will construct a category $\mathrm{PreStk}(\mathcal{C})$, which will be the free cocompletion of $\mathcal{C}$, i.e., the category which is obtained by freely adding colimits to $\mathcal{C}$. In this setting the category $\mathcal{C}$ will be the category of local models, and $\mathrm{PreStk}(\mathcal{C})$ will be the category of global objects.

Example 1. To the category of affine schemes $\mathrm{Sch}^{\mathrm{aff}}$ we can assign the free cocompletion $\mathrm{PreStk}(\mathrm{Sch}^{\mathrm{aff}})$. You may ask yourself, is this category the category of schemes? No. But it contains the category of schemes $\mathrm{Sch} \subset \mathrm{PreStk}(\mathrm{Sch}^{\mathrm{aff}})$ as a full subcategory. But it also includes other very interesting categories, like the category of algebraic spaces as a full subcategory. One of the advantages of working with $\mathrm{PreStk}(\mathrm{Sch}^{\mathrm{aff}})$ is that it is a well behaved category, for example, it is closed under colimits, while Sch is not. In order to see this consider the following example: Let $\mathbb{A}^1_k$ be the affine line over a field k, and denote by $\mathrm{Spec}\,\eta \hookrightarrow \mathbb{A}^1_k$ its generic point. Then the following diagram does not have a colimit in Sch

$$\begin{tikzcd} \mathrm{Spec}\,\eta \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^1_k \\ \mathbb{A}^1_k & \end{tikzcd} \tag{1}$$

the idea is that this space is too non-separated to be a scheme. However, this colimit does exists in $\mathrm{PreStk}(\mathrm{Sch}^{\mathrm{aff}})$.

We begin by defining what free cocompletion means, and study this construction more closely.

Definition 2. Let $\mathcal{C}$ be a category, a morphisms $\mathcal{Y} : \mathcal{C} \to \mathrm{PreStk}(\mathcal{C})$ presents the category $\mathrm{PreStk}(\mathcal{C})$ as its free completion, if it satisfies the following universal property: for any map $\mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is closed under colimits, we have a diagram of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "\mathcal{Y}"] \arrow[dr] & \mathrm{PreStk}(\mathcal{C}) \arrow[d, dashed] \\ & \mathcal{D} \end{tikzcd} \tag{2}$$

and the map $\mathrm{PreStk}(\mathcal{C}) \to \mathcal{D}$ is colimit preserving and unique.

Remark 3. Its important to highlight that it is not the category $\mathrm{PreStk}(\mathcal{C})$ which we say enjoys a certain universal property, it is the datum of the morphisms $\mathcal{C} \hookrightarrow \mathrm{PreStk}(\mathcal{C})$ which enjoys the universal property.

Theorem 4. The yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ presents the category $\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ as the free cocompletion of $\mathcal{C}$. Recall that the yoneda embedding is defined by the formula $c \mapsto \mathrm{Map}_{\mathcal{C}}(-, c)$.

Notice that under this theorem I could have defined the free cocompletion as $\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$. However, through this week, we will try to avoid making constructions through explicit formulas. As we have already done, we will make our constructions through a universal property, and then we will perform computations in order to understand the construction better.

Proving the theorem above requires some work, we will proceed to discuss some of the ingredients that enter the proof.

Lemma 5. Limits and colimits in $\mathrm{PreStk}(\mathcal{C})$ can be computed object wise. Therefore, the category $\mathrm{PreStk}(\mathcal{C})$ inherits the property that it is closed under limits and colimits from Sets.

Lets unpack what it means for limits and colimits to be computed objectwise. Let $X = \operatorname{colim} X_i$ be a colimit diagram taken on $\mathrm{PreStk}(\mathcal{C})$, colimits being computable objectwise means that we have the following identity

$$X(A) := (\operatorname{colim} X_i)(A) = \operatorname{colim}(X_i(A)) \tag{3}$$

the first equality is direct from the definition, and the second one is exactly the fact that colimits can be computed objectwise. A completely analogous statement is also true for limits.

The content of the proof of the above theorem consists in proving that all the information on $\mathrm{PreStk}(\mathcal{C})$ is actually recorded on $\mathcal{C}$. The following result is a clear example of what I mean by this.

Lemma 6. Every object on $\mathrm{PreStk}(\mathcal{C})$ can be functorially realized as a colimit in $\mathcal{C}$.

In other words, for each $X \in \mathrm{PreStk}(\mathcal{C})$, we can functorially construction a category $\mathrm{ob}(X)$ together with a functor $\mathrm{ob}(X) \to \mathcal{C} \hookrightarrow \mathrm{PreStk}(\mathcal{C})$, such that the colimit of the diagram is exactly X. The functoriality of the construction is saying that for every morphisms $X \to Y$, we can construct a diagram

$$\begin{tikzcd} \mathrm{ob}(X) \arrow[rr] \arrow[dr] & & \mathrm{ob}(Y) \arrow[dl] \\ & \mathrm{PreStk}(\mathcal{C}) & \end{tikzcd} \tag{4}$$

which only commutes up to some non invertible two morphisms, such that the colimit of this diagrams produces $X \to Y$. Before defining the category $\mathrm{ob}(X)$, it may be worth remembering the yoneda lemma: it states that there exists a canonical bijection

$$X(A) \cong \mathrm{Map}_{\mathrm{PreStk}(\mathcal{C})}(\mathrm{Map}_{\mathcal{C}}(-, A), X) \tag{5}$$

In other words, this is saying that we can think of the set $X(A)$ as ways in which one can paste A into X.

With the yoneda lemma in mind, we define $\mathrm{ob}(X) = \mathcal{C}_{/X}$, in other words, its objects are morphisms $\mathrm{Map}_{\mathcal{C}}(-, A) \to X$, and the morphisms are diagrams of the form

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(-, A) \arrow[rr] \arrow[dr] & & \mathrm{Map}_{\mathcal{C}}(-, B) \arrow[dl] \\ & X & \end{tikzcd} \tag{6}$$

then the functor $\mathrm{ob}(X) \to \mathrm{PreStk}(\mathcal{C})$ is given by forgetting the map to X. This construction should solidify the intuition that $X(A) \cong \mathrm{Map}_{\mathrm{PreStk}(\mathcal{C})}(\mathrm{Map}_{\mathcal{C}}(-, A), X)$ are the ways in which one can map A into X.

Example 7. When doing algebraic geometry, we have our usual category of local models, $\mathrm{Sch}^{\mathrm{aff}} \simeq \mathrm{CRing}^{\mathrm{op}}$, and we can construct $\mathrm{PreStk}(\mathrm{Sch}^{\mathrm{aff}})$, which contains the category of schemes as a full subcategory. Therefore for any scheme X, we can consider its associated functor $X : \mathrm{Sch}^{\mathrm{aff,op}} \to \mathrm{Sets}$. The values of the functor $X(\mathrm{Spec}\, A)$ are usually referred to as $\mathrm{Spec}\, A$ points of X.

Another common way of considering this functor of points perspective is to consider the embedding $\mathrm{Sch} \hookrightarrow \mathrm{PreStk}(\mathrm{Sch})$. Then to each scheme X we associate the functor $\mathrm{Map}_{\mathrm{Sch}}(-, X) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$. So which functor of points is the correct one? One can show that they encode exactly the same information. Let $Y \in \mathrm{Sch}$, then we can realize $Y = \mathrm{colim}\, \mathrm{Spec}\, B$ as a colimit of affines. And considering the following identities

$$\mathrm{Map}_{\mathrm{Sch}}(Y, X) = \mathrm{Map}_{\mathrm{Sch}}(\mathrm{colim}\, \mathrm{Spec}\, B, X) = \lim \mathrm{Map}_{\mathrm{Sch}}(\mathrm{Spec}\, B, X) = \lim X(\mathrm{Spec}\, B) \tag{7}$$

we have shown that both notions of the functor associated to X contain the same amount of information.

In order to get some familiarity with the notions we have introduces so far, I will introduce the notion of a simplicial set. Let Δ be the simplex category, it as objects $[n]$ for each positive integer n. The object $[n]$ is supposed to be a categorification of the n-simplex, and it is itself a category generated by the following objects and morphisms

$$[n] := 0 \to 1 \to 2 \to \ldots \to n \tag{8}$$

It is important to notice that since $[n]$ is a category itself, every object has an identity morphism, and morphisms are closed under composition, so the diagram above has incomplete information. In order to understand why this is supposed to categorify the n-simplex lets write down a diagram with more complete information

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \tag{9}$$

I hope this picture makes it clear why this is supposed to model the 2-simplex. The morphisms in the category Δ are the functors between the categories $[n] \to [m]$, i.e., they must be order preserving.

Definition 8. The category of simplicial sets is defined to be $\mathrm{sSet} := \mathrm{PreStk}(\Delta)$.

The category of sSet has a very special full subcategory which we denote by Spc, short for spaces. We can informally say that, an object $X \in \mathrm{sSet}$ is a space if all its arrows are invertible. For example, the object

$[n] \in \mathrm{sSet}$ is not a space. More formally, we say that a simplicial set X is a space if for every diagram

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{10}$$

we can find a lift. The simplicial set $\Delta^n := \mathrm{Map}_\Delta(-, [n])$ is called the n-simplex, and Λ_i^n is called the n-dimensional ith horn. It is obtained by removing the ith face of Δ^n.

In order to justify why we call the category Spc spaces, we recall that it is equivalent to the usual category of spaces, in some appropriate sense. This equivalence is obtained by the universal property of $\mathrm{PreStk}(-)$ as the free cocompletion, and the following diagram

$$\begin{tikzcd} \Delta \arrow[r, hook, "\mathcal{Y}"] \arrow[dr] & \mathrm{sSet} \arrow[d, dashed, "|-|"] \\ & \mathrm{Top} \end{tikzcd} \tag{11}$$

The functor $|-| : \mathrm{sSet} \to \mathrm{Top}$ is called geometric realization, and it defined an equivalence of categories (in an appropriate sense) between $\mathrm{Spc} \simeq \mathrm{Top}$.

Prestacks

Given our treatment, incorporating higher categories would not add much difficulty. Therefore, we will be working in the world of higher categories, and derived algebraic geometry. We do this in order to show the reader that if one does things categorically, most of the same arguments one would do classically will also apply to higher categories.

One of the main difficulties that working with higher categories introduces, is the fact that one can no longer define things by explicit formulas. For example, it would be impossible to define a functor $\mathcal{C} \to \mathcal{D}$ between higher categories by specifying where objects go, and where n-morphisms go, its just to much data. Hence, we are forced to always make constructions by some sort of canonical procedure. However, explicit descriptions are also very useful, so once the object has been constructed, one proceeds to compute it. This will become very apparent in our construction, and computation of the category of stacks.

Definition 9. An ∞-category is a category which besides objects and morphisms, it also has n-morphisms for every positive n.

A very nice feature of higher categories is that they behave very similarly as classical categories, one just needs to know how to translate statements from the classical world to the higher world. Probably the most important translation tools we have is that the role of sets is taken by the category of spaces. For example, we no longer have a mapping set between object, but rather a mapping space.

However, what I would like to emphasize now is that everything is always constructed by some canonical procedure. Even the algebraic objects we will be working on. In order to describe an algebraic object, one no longer defines a set with some operations, we instead, first consider a category $\mathcal{C}$ with a symmetric monoidal structure, and then we consider commutative algebra objects on it. In particular, we will realize the category of commutative differential graded algebras by this procedure.

Definition 10. A symmetric monoidal category $\mathcal{C}$ is a category together with a unit object $1 \in \mathcal{C}$ and a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, such that it is commutative and associative in some appropriate sense.

Example 11. The category of spaces Spc have a symmetric monoidal structure, where the multiplication functor $\mathrm{Spc} \times \mathrm{Spc} \to \mathrm{Spc}$ is given by the rule $(X, Y) \mapsto X \times Y$. This particular monoidal structure is called the cartesian monoidal structure, and it can be given to any category $\mathcal{C}$ with finite limits.

Definition 12. A commutative algebra object on a symmetric monoidal category $\mathcal{C}$ is an object $c \in \mathcal{C}$ together with a multiplication

$$\mathcal{C} \times \mathcal{C} \ni (c, c) \to c \in \mathcal{C} \tag{12}$$

and a map $1 \to c$ in $\mathcal{C}$. Given $\mathcal{C}$ with a symmetric monoidal structure we can consider the category of commutative algebra objects, which we will denote by $\mathrm{CAlg}(\mathcal{C})$.

Example 13. Let k be a field of characteristic zero. We can then consider the category of chain complexes of vector spaces over k, we will denote this category by Vect. The category Vect has a symmetric monoidal structure (not the cartesian monoidal structure), and one can identify the category $\mathrm{CAlg}(\mathrm{Vect})$ with the category of commutative differential graded algebras.

This category will be essential to our treatment as it will take the place that CRing takes in classical algebraic geometry. The reason why I defined it like this, is because I want to highlight that it was obtained by a canonical procedure, and not by explicit formulas. Moreover, the category Vect and its symmetric monoidal structure are also obtained by a canonical procedure.

Remark 14. Notice, for example, that if $\mathcal{C}$ had the cartesian monoidal structure, then the multiplication map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ would map $(c, c) \mapsto c \times c$, but as you can see by the definition above, a commutative algebra object must map $(c, c) \to c$.

Example 15. Let $X \in \mathrm{Spc}$, and endow Spc with the cartesian monoidal structure. Then the loop space ΩX is an associative algebra object. Recall that multiplication ΩX is given by concatenation of paths. This does not have an associative algebra structure in the classical sense since

$$(\gamma_1\gamma_2)\gamma_3 \neq \gamma_1(\gamma_2\gamma_3) \tag{13}$$

but rather

$$(\gamma_1\gamma_2)\gamma_3 \simeq \gamma_1(\gamma_2\gamma_3) \tag{14}$$

and it is here where the higher categorical structure enters.

We will now provide a more concrete description of commutative differential graded algebras. We will be using cohomological grading convention, this means that a chain complex is graded as

$$\cdots \to A^{-1} \to A^0 \to A^1 \to A^2 \to \cdots \tag{15}$$

Definition 16. An element of $\mathrm{CAlg}(\mathrm{Vect})$, also called a commutative differential graded algebra, is a chain complex $A^\bullet$ of k-vector spaces, together with a k-bilinear map

$$A^n \times A^m \to A^{n+m} \qquad (a, b) \mapsto ab \tag{16}$$

such that

$$d^{n+m}(ab) = d^n(a)b + ad^m(b) \tag{17}$$

and such that $\oplus A^n$ has the structure of a unital k-algebra. And commutativity is given by $ab = (-1)^{nm}ba$.

Remark 17. You should not worry to much about the definition of an element of $\mathrm{CAlg}(\mathrm{Vect})$, but you should remember some of the properties it has: it has cohomology groups, defined as you would expect. And when its only non-zero cohomology group is $H^0(A^\bullet)$ we say it is discrete. And the category of discrete elements of $\mathrm{CAlg}(\mathrm{Vect})$ is equivalent to the category of rings.

Definition 18. We define the category of (derived) affine schemes over k to be

$$\mathrm{Sch}^{\mathrm{aff}} := \mathrm{CAlg}(\mathrm{Vect}^{\leq 0}) \tag{18}$$

In other words, the cohomology of our $A \in \mathrm{CAlg}(\mathrm{Vect}^{\leq 0})$ is concentrated in non-positive degrees. We will drop the adjective derived from now on.

Remark 19. Recall that classically we say that $A, B \in \mathrm{CAlg}(\mathrm{Vect})$ are homotopic equivalent if there exists a morphism $A \to B$ which induces isomorphism in cohomology. Since we are actually working on the ∞-category $\mathrm{CAlg}(\mathrm{Vect})$, we say that they are isomorphic, and from our perspective, they behave as isomorphic objects do in classical category theory.

You may be asking yourself, how does $\operatorname{Spec} A$ for $A \in \mathrm{CAlg}(\mathrm{Vect}^{\leq 0})$ looks like geometrically. Indeed, we have that

$$|\operatorname{Spec} A| \simeq |\operatorname{Spec} H^0(A)| \tag{19}$$

In other words, the underlying topological space of $\operatorname{Spec} A$ and $\operatorname{Spec} H^0(A)$ are the same, but their structure sheaves differ. This should remind you of what happens in Grothendieck style algebraic geometry, in which for a classical ring A, we have

$$|\operatorname{Spec} A| \simeq |\operatorname{Spec} A^{\mathrm{red}}| \tag{20}$$

which is why we may think of the higher cohomology groups as higher nilpotents. This explanation of the topological space associated to $\operatorname{Spec} A$ is mostly for phycological comfort. Since in our treatment we will never see a topological space.

Definition 20. By a (derived) prestack we shall mean a functor $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$. We denote by

$$\mathrm{PreStk} := \mathrm{Funct}((\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathrm{Spc}) \tag{21}$$

and recall that the yoneda embedding defined a fully faithful functor

$$\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk} \qquad \operatorname{Spec} A \mapsto \mathrm{Map}_{\mathrm{Sch}^{\mathrm{aff}}}(-, \operatorname{Spec} A) \tag{22}$$

such that for $S \in \mathrm{Sch}^{\mathrm{aff}}$ and $\mathcal{Y} \in \mathrm{PreStk}$ we have

$$\mathrm{Map}_{\mathrm{PreStk}}(S, \mathcal{Y}) \simeq \mathcal{Y}(S) \tag{23}$$

by the Yoneda lemma. From now on we will drop the adjective derived.

We will now explain a difference between working derived and working classically, and it has to do with the tensor product. For ant $n \geq 0$, we consider the full subcategory

$$\mathrm{Vect}^{\geq -n, \leq 0} \subset \mathrm{Vect}^{\leq 0} \tag{24}$$

which consists of chain complexes whose nonzero cohomologies are concentrated on $[-n, 0]$. This is a fully faithful embedding which admits a left adjoint

$$\tau^{\geq -n} : \mathrm{Vect}^{\leq 0} \to \mathrm{Vect}^{\geq -n, \leq 0} \tag{25}$$

in particular, this means that any map $A \to \tau^{\leq n}(B)$ must factor as

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \tau^{\leq n}(A) \arrow[d] \\ & \tau^{\leq n}(B) \end{tikzcd} \tag{26}$$

We say that $\mathrm{Vect}^{\geq -n, \leq 0}$ is a localization of $\mathrm{Vect}^{\leq 0}$.

By passing to the opposite category, in other words passing to the geometric world, we have a full subcategory

$$^{\leq n}\,\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{Sch}^{\mathrm{aff}} \tag{27}$$

where we have, by definition $^{\leq n}\,\mathrm{Sch}^{\mathrm{aff}} := (\mathrm{CAlg}(\mathrm{Vect}^{\leq n}))^{\mathrm{op}}$. Moreover, we have that the inclusion admits a right adjoint

$$\tau^{\leq n} : \mathrm{Sch}^{\mathrm{aff}} \to^{\leq n} \mathrm{Sch}^{\mathrm{aff}} \tag{28}$$

In particular, this means that any map $\tau^{\leq n}(\mathrm{Spec}\, B) \to \mathrm{Spec}\, A$ must factor as

$$\begin{tikzcd} \tau^{\leq n}(\mathrm{Spec}\, B) \arrow[d] \arrow[dr] & \\ \tau^{\leq n}(\mathrm{Spec}\, A) \arrow[r] & \mathrm{Spec}\, A \end{tikzcd} \tag{29}$$

We say that $^{\leq n}\,\mathrm{Sch}^{\mathrm{aff}}$ is a colocalization of $\mathrm{Sch}^{\mathrm{aff}}$.

For $n = 0$ one recovers

$$^{\mathrm{cl}}\,\mathrm{Sch}^{\mathrm{aff}} :=^{\leq 0} \mathrm{Sch}^{\mathrm{aff}} \tag{30}$$

the category of classical affine schemes.

This colocalization discussion should remind you of the classical fact that a morphism of schemes $X^{\mathrm{red}} \to Y$ must always factor as

$$\begin{tikzcd} X^{\mathrm{red}} \arrow[d] \arrow[dr] & \\ Y^{\mathrm{red}} \arrow[r] & Y \end{tikzcd} \tag{31}$$

where X^{red} is the canonically associated reduced scheme to X.

This colocalization relation can be extended to prestacks. Indeed, can define a functor

$$\mathrm{Funct}(^{\leq n}\mathrm{Sch}^{\mathrm{aff,op}}, \mathrm{Spc}) = {}^{\leq n}\,\mathrm{PreStk} \hookrightarrow \mathrm{PreStk} = \mathrm{Funct}(\mathrm{Sch}^{\mathrm{aff,op}}, \mathrm{Spc}) \tag{32}$$

This functor is obtained by the process of Kan extension, more explicitely functor $^{\leq n}\,\mathrm{Sch}^{\mathrm{aff,op}} \to \mathrm{Spc}$ can be extended to $\mathrm{Sch}^{\mathrm{aff,op}} \to \mathrm{Spc}$ by using left kan extensions. This information can be recorded in the following diagram

$$\begin{tikzcd} ^{\leq n}\,\mathrm{Sch}^{\mathrm{aff,op}} \arrow[r] \arrow[d] & \mathrm{Spc} \\ \mathrm{Sch}^{\mathrm{aff,op}} \arrow[ur, "\mathrm{LKE}"'] & \end{tikzcd} \tag{33}$$

The reason why we want left kan extensions is that then, by definition, we have that the induced map

$$^{\leq n}\,\mathrm{PreStk} \hookrightarrow \mathrm{PreStk} \tag{34}$$

is a left adjoint, i.e., we have that it presents $^{\leq n}\,\mathrm{PreStk}$ as a colocalization of PreStk.

Example 21. A subtle point which I have not been able to mention so far is that the inclusion

$$\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk} \tag{35}$$

does not preserve colimits, in fact, it destroys most colimits in $\mathrm{Sch}^{\mathrm{aff}}$. Lets do a simple example. Recall that the category of affine schemes is closed under coproduct, this can be deduced from the fact that coproduct is a specific example of a colimit, and by passing to the opposite category we can compute it in $\mathrm{CAlg}(\mathrm{Vect}^{\leq 0})$. Indeed we have

$$\coprod_{\mathrm{Sch}^{\mathrm{aff}}} \mathrm{Spec}(A_i) \simeq \mathrm{Spec}(\times A_i) \tag{36}$$

In particular we have that it is a quasicompact topological space. However, if we take the coproduct in PreStk we get an actual disjoint union

$$\coprod_{\mathrm{PreStk}} \mathrm{Spec}(A_i) \not\simeq \mathrm{Spec}(\times A_i) \tag{37}$$

the fact that this are different can be deduced from one space being quasicompact, while the other one is not. We have shown that the embedding $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$ does not preserve colimits. A slogan is that taking the free completion destroys all known colimits, and add free ones. A simple example could be that we $2+3$ is no longer equal to 5, but equal to the formal sum $2+3$.

Example 22. Let $\mathcal{Y} \in \mathrm{PreStk}$, and consider the category ${}^{\leq n}\mathrm{PreStk}_{/\mathcal{Y}}$. This category consists of objects $\tau^{\leq n}X \to Y$, and morphisms commutative diagrams of the form

$$\begin{tikzcd} \tau^{\leq n}\mathcal{X}_1 \arrow[rr] \arrow[dr] & & \tau^{\leq n}\mathcal{X}_2 \arrow[dl] \\ & \mathcal{Y} & \end{tikzcd} \tag{38}$$

then there is a canonical map ${}^{\leq n}\mathrm{PreStk}_{/\mathcal{Y}} \to \mathrm{PreStk}$ which maps $(\tau^{\leq n}\mathcal{X} \to Y) \mapsto \tau^{\leq n}\mathcal{X}$. One could ask what is the colimit of this diagram, one first notices that the diagram ${}^{\leq n}\mathrm{PreStk}_{/\mathcal{Y}}$ has a final object, namely $\tau^{\leq n}\mathcal{Y} \to \mathcal{Y}$, where the map is given the the counit of adjunction. Therefore the colimit is exactly $\tau^{\leq n}\mathcal{Y}$.

Example 23. Let (A, I) be an I-adically complete ring, with the I-adic topology. Then one can forget the topological structure and just construct the geometric space $\mathrm{Spec}\, A$. However, sometimes we want to consider something different, the formal scheme $\mathrm{Spf}\, A$. We want to explain how to construct $\mathrm{Spf}\, A$ using the functor of points, and without any reference to the topology. In particular, we will see that the difference between $\mathrm{Spec}\, A$ and $\mathrm{Spf}\, A$ rely on the fact that the embedding $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$ does not preserves colimits.

If A is an I-adically complete ring, then we have

$$A \simeq \lim(\cdots \to A/I^3 \to A/I^2 \to A/I) \tag{39}$$

where the limit is taken in CRing. Equivalently we have that

$$\mathrm{Spec}\, A \simeq \mathrm{colim}(\mathrm{Spec}\, A/I \hookrightarrow \mathrm{Spec}\, A/I^2 \hookrightarrow \cdots) \tag{40}$$

where the colimit is taken if $\mathrm{Sch}^{\mathrm{aff}}$. However, if we take the colimit on PreStk we obtain

$$\mathrm{Spf}\, A \simeq \mathrm{colim}(\mathrm{Spec}\, A/I \hookrightarrow \mathrm{Spec}\, A/I^2 \hookrightarrow \cdots) \tag{41}$$

which corresponds to the formal scheme. Notice we have been able to define the a formal scheme without mentioning the I-adic topology in A. This illustrates that the I-adic topology on A is mostly used to obtain good behavior "at infinity".

Example 24. One important difference between $\mathrm{Sch}^{\mathrm{aff}}$ and ${}^{\mathrm{cl}}\,\mathrm{Sch}^{\mathrm{aff}}$ is how they differ when base changing. In particular, we will show that the classical base change is a truncated version of the derived one. Indeed, consider the following diagram

$$\begin{tikzcd}
\tau^{\leq 0}(Z \times_X Y) \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Z \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& Z \arrow[r] & X
\end{tikzcd} \tag{42}$$

where X, Y, Z are classical affine schemes. One can conclude, from the fact that $\tau^{\leq 0} : \mathrm{Sch}^{\mathrm{aff}} \to^{\mathrm{cl}} \mathrm{Sch}^{\mathrm{aff}}$ is a right adjoint, that the limit when taken in $\mathrm{Sch}^{\mathrm{aff}}$ is $Z \times_X Y$, while when we take it in ${}^{\mathrm{cl}}\,\mathrm{Sch}^{\mathrm{aff}}$ is the truncated version. In fact, the higher cohomology groups of $Z \times_X Y$ are extremely important when doing intersection theory, as this correspond to the higher Tor groups which appear in Serre's intersection formula.

Stacks

An essential point about the category PreStk is that the embedding $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$ destroys most colimits. As we saw in the examples above, and this was very useful, as it allowed us to define $\mathrm{Spf}\, A$. However, this also comes with undesirable consequences: there are certain colimits in $\mathrm{Sch}^{\mathrm{aff}}$ which we would like to preserve in the embedding $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$, because we intuitively feel that certain colimits are an essential part of the geometry.

Example 25. Consider the category of n-dimensional manifolds Man, and let M_1, M_2 be elements in this category. We can embed, via the Yoneda embedding $\mathrm{Man} \hookrightarrow \mathrm{PreStk}(\mathrm{Man})$, by sending $M_i \mapsto \mathrm{Map}_{\mathrm{Man}}(-, M_i)$. The fact that the embedding destroys almost all colimits is readily present. Indeed, one can see that

$$\mathrm{Map}_{\mathrm{Man}}(-, M_1 \sqcup M_2) \not\simeq \mathrm{Map}_{\mathrm{Man}}(-, M_1) \sqcup \mathrm{Map}_{\mathrm{Man}}(-, M_2) \tag{43}$$

The equation on the left we first take the coproduct in Man and then we embed, while one the right hand side we first embed and then take the coproduct. This is saying the the embedding does not preserve colimits.

Remark 26. However, it may be worth mentioning that the embedding $\mathcal{C} \hookrightarrow \mathrm{PreStk}(\mathcal{C})$ preserves all limits.

A solution to this problem was first proposed by Grothendieck by using Grothendieck topologies. A Grothendieck topology can be thought as instructions for which colimits we wish to preserve when taking the free closure under colimits.

Definition 27. In the same way that $\mathrm{PreStk}(\mathcal{C})$ was the free closure under colimits, if we endow $\mathcal{C}$ with a Grothendieck topology, we denote by $\mathrm{Stk}(\mathcal{C})$ as the free closure under colimits that preserves certain colimits.

The goal of this section is to investigate the category $\mathrm{Stk}(\mathrm{Sch}^{\mathrm{aff}})$, which we denote by Stk; how it interacts with $\mathrm{PreStk}(\mathrm{Sch}^{\mathrm{aff}})$, which we denote by PreStk; and how they are related the more classical notion of schemes.

Definition 28. A Grothendieck topology in a category $\mathcal{C}$, with fiber products, is a collection of covers $\{U_\alpha \to X\}$ which satisfy the following properties

(1) If $Y \to X$ is an isomorphism, then $\{Y \to X\}$ is a cover.

(2) If $\{U_\alpha \to X\}$ is a cover, and $\{U_{\alpha,\beta} \to U_\alpha\}$ is cover for every U_α, then $\{U_{\alpha,\beta} \to X\}$ is a cover.

(3) If $\{U_\alpha \to X\}$ is a cover, then for any map $T \to X$, the collection of maps $\{T \times_X U_\alpha \to T\}$ is a cover.

A Grothendieck site is a category $\mathcal{C}$ together with a Grothendieck topology.

Example 29. The category $\mathrm{Sch}^{\mathrm{aff}}$ have many interesting grothendieck topologies. We have the Zarisky topology, in which covers are collection of maps $\{U_\alpha \to X\}$ such that they are jointly surjective, and each $U_\alpha \to X$ is a Zarisky embedding.

As we have said before, our goal is to develop algebraic geometry without ever using the classical notion of a topology in our foundations. And in the definition above we have used the notion of surjectivity, which is a priory only defined for certain type of space. So we proceed to develop the notion of surjectivity without reference classical topology.

We begin by noticing how to abstract the notion of surjectivity from the classical setting. Recall we say that a map of schemes $X \to Y$ is surjective, if the underlying map of topological spaces $|X| \to |Y|$ is surjective. How do we talk about surjectivity from the perspective of functor of points? Notice that if the map $X \to Y$ is surjective, then we can make a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \kappa_x \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \kappa_y \arrow[r] & Y \end{tikzcd} \tag{44}$$

for every residue field $\operatorname{Spec} k_y \hookrightarrow Y$. But in fact, we have something more general, for any $\operatorname{Spec} A_y \to Y$ we can find a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B_x \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A_y \arrow[r] & Y \end{tikzcd} \tag{45}$$

where $\operatorname{Spec} B_x \subset \operatorname{Spec} A_y \times_Y X$.

Definition 30. We define the notion of surjective morphisms for a general prestack. Let $X \to Y$ be a morphisms of prestacks. Consider the following diagram

$$\begin{tikzcd} X(\operatorname{Spec} A) \arrow[r] \arrow[d] & X(\operatorname{Spec} B) \arrow[d] \\ Y(\operatorname{Spec} A) \arrow[r] & Y(\operatorname{Spec} B) \end{tikzcd} \tag{46}$$

We say that $X \to Y$ is surjective if for every element $\varphi \in Y(\operatorname{Spec} A)$, there exists a map $\operatorname{Spec} B \to \operatorname{Spec} A$ such that the image of φ under the map $Y(\operatorname{Spec} A) \to Y(\operatorname{Spec} B)$ is in the image of $X(\operatorname{Spec} B) \to$

$Y(\operatorname{Spec} B)$. In order words, we do not require objectwise surjectivity, but something more flexible. We could also say the following, a morphism $X \to Y$ is surjective, if for any morphism $\operatorname{Spec} A \to Y$ there exists a $\operatorname{Spec} B$ such that the following diagram commutes

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd} \tag{47}$$

this is equivalent to the definition provided above.

Remark 31. The attentive reader may notice that we for us a prestack is a functor $X : \mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}$, and may ask, what does it mean for a map of spaces $X(\operatorname{Spec} A) \to Y(\operatorname{Spec} A)$ to have an element on its image? Well, one could certainly require that it is an actual surjective map of topological spaces, but this is not a homotopy invariant notion, indeed notice that $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$ is surjective, but the map $\mathrm{pt} \to \mathbb{R}$ is not. However, we can also say that a map of spaces $X(\operatorname{Spec} A) \to Y(\operatorname{Spec} A)$ is surjective, if the corresponding map of sets $\pi_0 X(\operatorname{Spec} A) \to \pi_0 Y(\operatorname{Spec} A)$ is surjective.

Remark 32. The reason why we need Spc is because the category $\mathrm{Sch}^{\mathrm{aff}}$ forms an ∞-category. However, if one only cares about classical schemes ${}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}} =^{\leq 0} \mathrm{Sch}^{\mathrm{aff}}$, then any functor ${}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}$ factors as

$$ {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Sets} \to \mathrm{Spc} \tag{48}$$

putting us back in a more familiar situation. This is because ${}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$ forms classical category.

Moreover, we also notice that it is possible to define what a Zarisky embedding $\operatorname{Spec} A \to \operatorname{Spec} B$ is, without referencing a topological space. Indeed, we say that $\operatorname{Spec} A \to \operatorname{Spec} B$ is a Zarisky embedding, if the corresponding map $B \to A$ presents $A = B[f^{-1}]$ for some non nilpotent $f \in B$.

Remark 33. The fact that we only localize at non-nilpotent elements is important, since you may ask yourself, what is the analog of localization for $\mathrm{CAlg}(\mathrm{Vect}^{\leq 0})$, also known as commutative differential graded algebras? If we follow the idea that the higher terms of an element of $\mathrm{CAlg}(\mathrm{Vect}^{\leq 0})$ are higher nilpotents, the we can conclude that localization will only happen at the zeroth level. The operations of localization in this derived rings is a bit more subtle than this, but due to time constrains, Ill leave it there.

Our goal now is to introduce stacks in a natural way. We will first try to understand what we want, and then provide a definition. We will endow $\mathrm{Sch}^{\mathrm{aff}}$ with any of the following topologies: fppf, smooth, etale, or zarisky. This topologies have in common that they are subcanonical, that is, every affine scheme will be a stack with respect to this topologies. In particular, we will have a fully faithful embedding

$$\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Stk} \tag{49}$$

What do want out of a category Stk? Recall that the issue with PreStk is that the embedding $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$ destroys all colimits, including the ones we think are essential to the geometry we are trying to understand. Lets begin by introducing what kind of colimits we would want to preserve.

Given a morphism $S' \to S$ we can construct a diagram

$$\cdots \qquad S' \times_S S' \times_S S' \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; S' \times_S S' \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; S' \tag{50}$$

where the maps going to the right are projection maps. For example the two maps $S' \times_S \to S'$ correspond to the projections, which can be obtained from the diagram

$$\begin{tikzcd} S' \times_S S' \arrow[r] \arrow[d] & S' \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \tag{51}$$

and the map $S' \to S' \times_S S'$ corresponds to the diagonal map. Can you construct such a map?

Definition 34. Let $S' \to S$ be a morphisms in $\mathrm{Sch}^{\mathrm{aff}}$. We can construct a diagram as above which we call the Cech nerve, and is denoted by $S'^{\bullet}/S$. This diagram can be interpreted as a functor $\Delta^{\mathrm{op}} \to \mathrm{Sch}^{\mathrm{aff}}$. And we denote by $|S'^{\bullet}/S|_{\mathrm{PreStk}}$ its colimit taken in the category PreStk. The subscript is important to us, as we will also consider this colimit in Stk and we will like to be able to differentiate them.

Definition 35. Consider $\mathrm{Sch}^{\mathrm{aff}}$, endowed with one of the above topologies. We define the category Stk to be the universal closure under colimits of $\mathrm{Sch}^{\mathrm{aff}}$, such that for any cover $S' \to S$, we have $|S'^{\bullet}/S|_{\mathrm{Stk}} \simeq S$. In other words, for any functor $F : \mathrm{Sch}^{\mathrm{aff}} \to \mathcal{D}$, where $\mathcal{D}$ is closed under colimits, and $|F(S')^{\bullet}/F(S)|_{\mathcal{D}} \simeq F(S)$, we have a diagram of the form

$$\begin{tikzcd} \mathrm{Sch}^{\mathrm{aff}} \arrow[r, hook] \arrow[dr, "F"'] & \mathrm{Stk} \arrow[d] \\ & \mathcal{D} \end{tikzcd} \tag{52}$$

such that F factors uniquely through a colimit preserving map $\mathrm{Stk} \to \mathcal{D}$.

How would one go about finding such a category Stk? Well, lets first notice that the category PreStk has almost all the properties, except from requiring $|S^{'\bullet}/S|_{\mathrm{PreStk}} \to S$ being an isomorphism. Therefore, we can reinterpret the universal property of Stk, as the existence of a universal colimit preserving functor

$$L : \mathrm{PreStk} \longrightarrow \mathrm{Stk} \tag{53}$$

that maps $L(|S^{'\bullet}/S|_{\mathrm{PreStk}}) \to L(S)$ into an isomorphism. It is then a general principle, originally due to Bousfield, that one can usually find Stk as a full subcategory of PreStk. Where the functor $L : \mathrm{PreStk} \to \mathrm{Stk}$ will be a left adjoint to the inclusion $\mathrm{Stk} \hookrightarrow \mathrm{PreStk}$.

Theorem 36. The category Stk can be identified with the full subcategory of PreStk, spanned by the objects X such that the induced morphism

$$\mathrm{Map}_{\mathrm{PreStk}}(S, X) \to \mathrm{Map}_{\mathrm{PreStk}}(|S^{'\bullet}/S|_{\mathrm{PreStk}}, X) \tag{54}$$

is an isomorphism. Moreover, the inclusion $\mathrm{Stk} \hookrightarrow \mathrm{PreStk}$ has a left adjoint $L : \mathrm{PreStk} \to \mathrm{Stk}$. In other words, Stk is the minimal fix of PreStk, in order to have the map $|S^{'\bullet}/S|_{\mathrm{Stk}} \to S$ being an isomorphism.

Remark 37. The Yoneda philosophy provides a good explanation for the existence of such a theorem. Recall that the Yoneda philosophy says that an objects can be completely determined by how other objects perceive it. In other words, an objects can be completely determined by the mapping space to (or from) other objects. And this is exactly what we are doing, we are restricting to the objects of PreStk which perceive the morphism $|S^{'\bullet}/S|_{\mathrm{PreStk}} \to S$ as an isomorphism.

While this definition is satisfying, it does not coincide with the usual definition of a stack. One usually defines a stack as an object $X \in \mathrm{PreStk}$ such that for a cover $S' \to S$, the induced map

$$X(S) \to \mathrm{Tot}\, X(S'^{\bullet}/S) \tag{55}$$

is an isomorphism. Where $\mathrm{Tot}\, X(S'^{\bullet}/S)$ is by definition the limit of the diagram $X(S'^{\bullet}/S)$. However, it is not hard to proof that they are equivalent. Indeed we have that

$$X(S) \simeq \mathrm{Map}_{\mathrm{PreStk}}(S, X) \to \mathrm{Map}_{\mathrm{PreStk}}(|S^{'\bullet}/S|_{\mathrm{PreStk}}, X) \simeq \mathrm{Tot}\,\mathrm{Map}_{\mathrm{PreStk}}(S^{'\bullet}/S, X) \simeq \mathrm{Tot}\, X(S'^{\bullet}/S) \tag{56}$$

it follows that both conditions are equivalent. We have chosen this definition as we think that it is the most natural one, or in other words, the easier one to come up with.

So far we have constructed the category Stk from PreStk, by specifying a Grothendieck topology on $\mathrm{Sch}^{\mathrm{aff}}$ and forcing certain relations. One may ask, what other relations are forced by the procedure? We begin by providing the following definition

Definition 38. A morphisms of prestacks $Y_2 \to Y_1$ is a topological surjection (with respect to the topology of $\mathrm{Sch}^{\mathrm{aff}}$, if for every morphism $\mathrm{Spec}\, A \to Y_1$ there exits a cover $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Spec}\, B \arrow[r] \arrow[d] & Y_2 \arrow[d] \\ \mathrm{Spec}\, A \arrow[r] & Y_1 \end{tikzcd} \tag{57}$$

Proposition 39. Let $Y_2 \to Y_1$ be a topological surjection in Stk, then the following morphism

$$|Y_2^{\bullet}/Y_1|_{\mathrm{Stk}} \to Y_1 \tag{58}$$

is an isomorphism.

Schemes

In this section we will introduce some more classical notions as schemes and algebraic spaces. We will also introduce certain type of morphisms, whose usual definitions tends to rely heavily on the existence of a topological space, and phrase them in our language. For example, one usually defines an etale morphisms of schemes $X_2 \to X_1$ to be etale if it is etale on stalks. However, we dont have a notion of stalks, therefore we proceed differently.

Definition 40. We say that a map of prestack $Y_1 \to Y_2$ is affine schematic (classically called affine), if for any morphism $S \to Y_2$ from an affine scheme, the pullback of the following diagram

$$\begin{tikzcd} Y_1 \times_{Y_2} S \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ S \arrow[r] & Y_2 \end{tikzcd} \tag{59}$$

is representable by an affine scheme.

Definition 41. A morphism of prestacks $Y_1 \to Y_2$ is a closed embedding if it is affine schematic, and for every morphisms from an affine scheme $S \to Y_2$, the resulting base change morphism

$$\begin{tikzcd} Y_1 \times_{Y_2} S \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ S \arrow[r] & Y_2 \end{tikzcd} \tag{60}$$

is a close embedding of affines. Recall that the notion of being a closed embedding between affines has an intrinsic definition in terms of commutative algebra.

Lemma 42. If Z is a separated scheme, then any morphism from an affine scheme $S_i \to X$ is affine schematic. In fact, one only needs the diagonal morphism $Z \to Z \times Z$ to be affine schematic in order for this to hold.

Definition 43. A morphisms of prestacks $Y_1 \to Y_2$ is an open embedding, if it is affine schematic, and for every morphisms from an affine scheme $S \to Y_2$ the resulting base change morphism

$$\begin{tikzcd} Y_1 \times_{Y_2} S \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ S \arrow[r] & Y_2 \end{tikzcd} \tag{61}$$

is an open embedding. Being an open embedding of affine schemes has an intrinsic definition in terms of commutative algebra.

Remark 44. However, recall that we are working with some sort of derived rings, in our case with commutative differential graded algebras, then what does it mean for a morphisms $A \to B$ of this derived rings to be an open/closed embedding? Since this are intuitively topological properties, and the higher cohomology groups are just higher nilpotents, we say that $A \to B$ is an open/closed embedding if $H^0(A) \to H^0(B)$ are open/closed embeddings.

Definition 45. We say that a morphism $Y_1 \to Y_2$ is etale (resp. smooth, resp. unramified), if for any affine scheme $\operatorname{Spec} A$, and any closed subscheme $\operatorname{Spec} A_0$, defined by an ideal I such that $I^2 = 0$, the canonical map

$$\operatorname{Map}_{Y_2}(\operatorname{Spec} A, Y_1) \to \operatorname{Map}_{Y_2}(\operatorname{Spec} A_0, Y_1) \tag{62}$$

is bijective (resp. surjective, resp. injective). This statement is essentially about the ways in which we can lift the following diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y_2 \end{tikzcd} \tag{63}$$

Since we are working with mapping spaces rather than mapping sets, this are statements on the connected components of the mapping spaces.

Remark 46. The idea behind this definition is that etale morphisms must induce a bijection on tangent spaces. And recall that for a scheme Y, the tangent space at $\operatorname{Spec} k \to Y$ corresponds to the number of lifts

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & Y \\ \operatorname{Spec} k[\varepsilon] \arrow[ur, dashed] & \end{tikzcd} \tag{64}$$

where $\varepsilon^2 = 0$.

We proceed to define schemes and algebraic spaces. We notice that they are basically the same object from our perspective.

Definition 47. We say that $Z \in \mathrm{Stk}_{\mathrm{Zar}}$ is a scheme if there exists a collection of affine schemes S_i and open embeddings $f_i : S_i \to Z$ (called Zarisky atlas), such that they form a cover.

Definition 48. We say that $Z \in \mathrm{Stk}_{\mathrm{Et}}$ is a algebraic space if there exists a collection of affine schemes S_i, and a collection of affine schematic etale morphisms $f_i : S_i \to Z$ that form a cover.

Remark 49. One usually requires the diagonal morphism to be affine schematic, but we don't want to put such restrictions right on the definition. In fact, if one requires the diagonal to be affine schematic, then one recovers the usual definition of an algebraic space.

Definition 50. For an scheme Z (resp. algebraic space), we say that Z is quasicompact if it admits a Zarisky (resp. etale) atlas made up of finite affine schemes.

In order to get some practice, we will proof (in our language) that isomorphisms can be checked at stalks. We will use the fact from the previous section that if $Y_2 \to Y_1$ is a topological surjection in Stk, then the morphism

$$|Y_2^\bullet / Y_1|_{\mathrm{Stk}} \to Y_1 \tag{65}$$

is an isomorphism. In particular, we have the following result.

Proposition 51. Let X be a scheme (resp. algebraic space), and we live in the category $\mathrm{Stk}_{\mathrm{Zar}}$ (resp. $\mathrm{Stk}_{\mathrm{Et}}$), and consider the Zarisky (resp. etale) cover by affines $\sqcup S_i \to X$, then this is a topological surjection, and we have that

$$|\sqcup S_i^\bullet / X|_{\mathrm{Stk}} \to X \tag{66}$$

is an isomorphism.

This result is very interesting in its own right, as it is telling us that schemes (resp. algebraic space) are objects created by a very specific set of pastings. In particular we have that the diagram

$$\cdots \qquad (\sqcup S_i) \times_X (\sqcup S_i) \times_X (\sqcup S_i) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} (\sqcup S_i) \times_X (\sqcup S_i) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \sqcup S_i \tag{67}$$

all the maps going down are zarisky embeddings (resp. etale). As we said before, this shows that schemes or algebraic spaces, are a subcategory of Stk, of objects that can be constructed in a very specific way.

Proposition 52. A morphism $Y_2 \to Y_1$ of schemes (resp. algebraic spaces) is an isomorphism, if and only if for any affine schematic cover $\sqcup S_i \to Y_1$ the induced map $(\sqcup S_i) \times_{Y_2} Y_1 \to \sqcup S_i$ is an isomorphism.

Indeed if $Y_2 \to Y_1$ is an isomorphism, then it is clear that $(\sqcup S_i) \times_{Y_2} Y_1 \to \sqcup S_i$ is an isomorphism. However, the other direction is a bit more subtle. Indeed, if each $(\sqcup S_i) \times_{Y_2} Y_1 \to \sqcup S_i$ are isomorphisms, then we would have an isomorphism of diagrams

$$(\sqcup S_i \times_{Y_1} Y_2)^\bullet / Y_2 \to (\sqcup S_i)^\bullet / Y_1 \tag{68}$$

By taking the colimit, it follows that $Y_2 \to Y_1$ is an isomorphism.

References

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