Integral K-theory and Rep(G)

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Introduction

One of the main players in this note is the spectrum KU. In order to make this note more accessible, we begin with some background material about spectra. We will not provide a rigorous definition of a spectrum (in the sense of homotopy theory), but I hope that the following discussion will be enough for the reader to understand the rest of the notes.

Definition 1. A spectrum behaves very much like a chain complex of abelian groups. Some properties that the category of spectra enjoys are the following

1. One can ask for the homotopy group of a spectrum E, which are denoted by $\pi_i(E)$. And they should be thought as analogs of the cohomology groups of a spectrum.
2. Given two spectrum, E_1 and E_2 one can form a direct sum $E_1 \oplus E_2$, and a tensor product $E_1 \otimes E_2$.
3. Given a map of spectra $E_1 \to E_2$ one can take the kernel and cokernel of this map.
4. Each spectrum, determines a collection of functors
$$E^i : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Ab} \tag{1}$$
which one should think about as determining a cohomology theory in the category of spaces. In particular we have the identity
$$E^i(\mathrm{pt}) = \pi_i(E) \tag{2}$$
5. One can also consider ring objects in the category of spectra. In the case that E admits a sort of ring structure, we can amalgamate all cohomology groups $E^i(X)$ into a single ring
$$E^* : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Rings} \qquad X \mapsto E^*(X) \tag{3}$$

Example 2. A familiar example is given by the spectrum $H\mathbf{Q}$, which following (4), determined a cohomology theory
$$H\mathbf{Q}^i : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Ab} \qquad X \mapsto H^i(X, \mathbf{Q}) \tag{4}$$
that can be identified with singular cohomology. Moreover, since Q is not just an abelian group but a ring, we can conclude that $H\mathbf{Q}$ admits the structure of a ring spectrum, and then following (5), it determined a multiplicative cohomology theory
$$H\mathbf{Q}^* : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Rings} \qquad X \mapsto H^*(X, \mathbf{Q}) \tag{5}$$

A ring spectrum that will be of central importance to us is the ring spectrum KU, which goes by the name of complex K-theory. In order to get acquainted more with this ring spectrum a good first step is to ask what are its homotopy groups, indeed we have that

$$\pi_i(\mathrm{KU}) = \mathrm{KU}^i(\mathrm{pt}) = \begin{cases} \mathbf{Z} & \text{if i is even} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and since it has a ring structure, we can organize all this cohomology groups into a ring graded ring

$$\pi_*(\mathrm{KU}) = \mathrm{KU}^*(\mathrm{pt}) = \mathbf{Z}[\beta^{\pm}] \qquad |\beta| = 2 \tag{7}$$

In particular, we can see that this ring spectrum has a property called 2-periodic which will be of central importance for us. It is closely related to the following theorem, which goes by the name of Bott periodicity.

Proposition 3. For any space X, the Bott element β induces an isomorphism

$$\beta : \mathrm{KU}^i(X) \xrightarrow{\simeq} \mathrm{KU}^{i+2}(X) \tag{8}$$

In order to relate KU to the representation theory of finite groups, it will be convenient to have the following description of $\mathrm{KU}^0(X)$. We consider the category of complex vector bundles on a space X, and denote it by $\mathrm{Vect}(X)$. The category $\mathrm{Vect}(X)$ comes equipped with a direct sum operation, so we can organize the isomorphism classes of objects of $\mathrm{Vect}(X)$ into a monoid. Elements of this monoids are denotes by $[V]$, where V is a complex vector bundle over X, and with sum given by

$$[V_1 \oplus V_2] = [V_1] + [V_2] \tag{9}$$

We temporarily denote this monoid by the letter $M(X)$. Notice, however, that $\mathrm{KU}^0(X)$ is at least an abelian group, and not just a monoid, so in order to relate this two objects we will proceed by transforming M to an abelian group in a universal way. Recall that there exists a forgetful functor $\mathrm{ComGrp} \to \mathrm{Mon}$ from abelian groups to monoids, and this functor admits a left adjoint

$$-^{gp} : \mathrm{Mon} \longrightarrow \mathrm{ComGrp} \tag{10}$$

called group completion. One then has the following result

Proposition 4. There exists an isomorphism

$$M(X)^{gp} \cong \mathrm{KU}^0(X) \tag{11}$$

Moreover, $M(X)^{gp}$ inherits a ring structure coming from the tensor product of vector bundles, which under the above isomorphism, coincides with the ring structure of $\mathrm{KU}^0(X)$.

Example 5. Using the results above we can recover the calculation that $\mathrm{KU}^0(\mathrm{pt}) = \mathbf{Z}$. Simply notice that over a point there is only one vector bundle of each non-negative dimension. Then one has to observe that if V_i is the vector bundle of dimension n over a point, then $V_i \oplus V_j = V_{i+j}$ and that $V_i \otimes V_j = V_{ij}$.

Now we will show how to extract a representation of a group G, out of vector bundles on a certain space BG. For simplicity, consider the additive group $\mathbf{Z}$, then the associated space is $B\mathbf{Z} = S^1$. Fix a distinguished point $\mathrm{pt} \to B\mathbf{Z}$, then out of a vector bundle $V \to B\mathbf{Z}$ we can extract a complex vector space V_{pt}, by looking at the fiber at the distinguished point of the map $V \to B\mathbf{Z}$. Then, moving our vector bundle $V \to B\mathbf{Z}$ around the loop, gives us an automorphism $V_{\mathrm{pt}} \to V_{\mathrm{pt}}$, which corresponds to an action of $1 \in \mathbf{Z}$ on V_{pt}. In other words, it is possible to extract a $\mathbf{Z}$ representation out of a vector bundle $V \to B\mathbf{Z}$. Granted, $\mathbf{Z}$ is not a finite group, but it seems good for pedagogical reasons.

Question 6. Can we obtain all complex representations of a finite group G, from complex vector bundles on BG?

The answer to this question is negative, but the complex representations of G and complex vector bundles over BG are intimately related, by the following theorem.

Theorem 7. Let G be a finite group, and let $I_G \subset \mathrm{Rep}(G)$ be the augmentation ideal, defined as the kernel of the ring homomorphism

$$\mathrm{Rep}(G) \to \mathbf{Z} \qquad [V] \mapsto \dim_{\mathbf{C}}(V) \tag{12}$$

The Atiyah-Segal comparison map

$$\zeta : \mathrm{Rep}(G) \longrightarrow \mathrm{KU}^0(BG) \tag{13}$$

exhibits $\mathrm{KU}^0(BG)$ as the I_G completion of $\mathrm{Rep}(G)$.

Definition 8. Let $\mathrm{Rep}(G)$ be the ring of complex representations of the finite group G. As an abelian group it is the abelian group generated by the symbols $[V]$, where V is a finite dimensional complex representation of G, subject to the relations

$$[V] = [V_1] + [V_2] \tag{14}$$

for any isomorphism $V \cong V_1 \oplus V_2$. So in other words is the free abelian group of finite rank generated by $[W]$, where W is a finite dimensional irreducible representation. In order to consider $\mathrm{Rep}(G)$ as a ring, we declare that multiplication is given by the rule

$$[V_1] \cdot [V_2] = [V_1 \otimes_{\mathbf{C}} V_2] \tag{15}$$

We would like to highlight that even though the representations we are considering are complex representations, i.e. V is a vector space over $\mathbf{C}$, the ring $\mathrm{Rep}(G)$ is most naturally regarded as a $\mathbf{Z}$ algebra.

In this notes, the goal is to provide a direct K-theoretic construction of the ring $\mathrm{Rep}(G)$. This means that we will try to explain how to construct a "decompleted" version of KU, which we will denote by $\underline{\mathrm{KU}}$, that satisfies

$$\underline{\mathrm{KU}}^0(BG) \cong \mathrm{Rep}(G) \tag{16}$$

equipped with a natural map

$$\underline{\mathrm{KU}}^0(BG) \longrightarrow \mathrm{KU}^0(BG) \tag{17}$$

which recovers the Atiyah-Segal comparison map. But before delving into the theory, let me explain some applications. Recall that in particular, the cohomology theory $\underline{\mathrm{KU}}$ determines a functor

$$\underline{\mathrm{KU}}^0 : \mathrm{Spc}^{\mathrm{op}} \to \mathrm{Ab} \tag{18}$$

Remark 9. A reader familiar with the theory of tempered cohomology would realize that Spc is very much not the domain of the functor $\underline{\mathrm{KU}}$, but rather one needs to renormalize the category of spaces, such that it is not generated by a single point pt, but rather by spaces of the form BM, where M is a finite abelian group. However, we choose to ignore this issue in this section, hoping it will not cause to much confusion. We will try to come back to this later.

Then, for any inclusion of finite groups $H \subset G$, the functoriality of $\underline{\mathrm{KU}}$ recovers the restriction map

$$\begin{tikzcd} \underline{\mathrm{KU}}^0(BG) \arrow[r] \arrow[d, "\simeq"'] & \underline{\mathrm{KU}}^0(BH) \arrow[d, "\simeq"] \\ \mathrm{Rep}(G) \arrow[r, "\mathrm{Res}"] & \mathrm{Rep}(H) \end{tikzcd} \tag{19}$$

However, $\underline{\mathrm{KU}}$ comes equipped with extra functoriality, it also comes equipped with transfer maps $\underline{\mathrm{KU}}(BH) \to \underline{\mathrm{KU}}(BG)$, which fit into the following commutative diagram

$$\begin{tikzcd} \underline{\mathrm{KU}}^0(BH) \arrow[r] \arrow[d, "\simeq"'] & \underline{\mathrm{KU}}^0(BG) \arrow[d, "\simeq"] \\ \mathrm{Rep}(H) \arrow[r, "\mathrm{Ind}"] & \mathrm{Rep}(G) \end{tikzcd} \tag{20}$$

More informally, we can say that the transfer maps $\underline{\mathrm{KU}}^0(BH) \to \underline{\mathrm{KU}}^0(BG)$ coincide with induction from the classical representation theory of finite groups. And finally, we can recover a variant of the Artin-Brauer induction theorem

Theorem 10. For any finite group G, the ring $\mathbf{C} \otimes_{\mathbf{Z}} \underline{\mathrm{KU}}^0(BG) = \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{Rep}(G)$ can be generated as a $\mathbf{C}$ vector space by the image of the induction maps

$$\mathbf{C} \otimes_{\mathbf{Z}} \underline{\mathrm{KU}}^0(BT) \xrightarrow{\mathrm{Ind}} \mathbf{C} \otimes_{\mathbf{Z}} \underline{\mathrm{KU}}^0(BG) \tag{21}$$

for all maps $BT \to BG$, where T is a finite cyclic group.

This naturally leads to the following question

Question 11. Can one extend such induction results to a certain kind of infinite groups like $\mathrm{Gal}(\mathbf{Q})$? Or the infinite symmetric group S_∞? As stated, the answer to this question seems to be negative, but maybe, by somehow incorporating the profinite topology we could make this work.

In order to achieve this goal, we will make use of Lurie's theory of tempered cohomology. A key ingredient of the theory of tempered cohomology is the close relationship between 2-periodic cohomology theories, like KU, and the geometry of p-divisible groups. In particular, we hope to provide a algebro-geometric explanation for the failure of the Atiyah-Segal comparison map to be an isomorphism.

Local Systems

We begin this section by trying to provide a geometric interpretation for the spectrum $\mathrm{KU}(X)$, for some space X. This marks a point of departure from the previous section, in which we only considered the disembodied pieces of the spectrum, namely the abelian groups $\mathrm{KU}^i(X)$, or the ring $\mathrm{KU}^*(X)$. This difference is analogous to the difference between a chain complex and its cohomology groups. Lets recall some facts about spectra

- For any space X, we can construct a new ring spectrum $\mathrm{KU}(X)$. If you recall from the previous section, to any spectrum E we can consider its homotopy groups $\pi_i(E) = E^i(\mathrm{pt})$, which get identified with the ith E-cohomology of a point. In this case, we have the identification $\pi_i(\mathrm{KU}(X)) = \mathrm{KU}^i(X)$.

This is relevant for us, because in the same way that we have been considering complex vector bundles $V \to X$, we now would like to consider bundles over X, whose fibers are not $\mathbf{C}$-vector spaces, but rather KU-modules.

Remark 12. The reader not familiar with the category of spectra might not feel at ease with the suggestion of considering modules over a beast like KU, however this is a legitimate operation. Remember that we had the analogy that a spectrum should behave in some ways like an abelian group, a ring spectrum should be like a ring, and to any ring one can consider modules over it, so its reasonable to expect that one can consider modules over a ring spectrum.

Our geometric interpretation of $\mathrm{KU}(X)$ will rely on the notion of a local system over a space X. As a warm up we consider the following example

Example 13. Consider the space $B\mathbf{Z} = S^1$, and we are going to consider $\mathbf{Q}$-values local systems on S^1. This local systems can be arranged in a category which we will denote by $\mathrm{LocSys}_{\mathbf{Q}}(B\mathbf{Z})$. An important difference between a vector bundle over $B\mathbf{Z}$ and a local system, is that when we are considering local systems, we do not consider $B\mathbf{Z}$ as a topological space in the usual way, but rather, as some sort of "virtual topological space".

I don't want to provide a general definition of what this means, but let me try to illustrate what this means in this specific case. The space $B\mathbf{Z}$ will only have one point, and the rest of the "usual" S^1, will be some sort of virtual loop around the point.

Then, a $\mathbf{Q}$-valued local system on $B\mathbf{Z}$, simply consists on a $\mathbf{Q}$-vector space V, together with automorphisms $V \to V$, given by rotating around the "virtual loop". Notice however, that there are not vector spaces over the "virtual loop", it only serves to record the way in which the vector space V changes after rotation around the loop.

Proposition 14. The category $\mathrm{LocSys}_{\mathbf{Q}}(BG)$, for a finite group G, can be identified with the category of $\mathbf{Q}$-representation of G.

Now, we will try to make a connection between the category $\mathrm{LocSys}_{\mathbf{Q}}(X)$ and the singular cohomology $H\mathbf{Q}(X)$. Notice, however, that we are talking about $H\mathbf{Q}(X)$ as a spectrum, and we are not just considering its individual cohomology groups. This is analogous as considering a chain complex as opposed to its disembodies cohomology groups. However one can always recover the ith singular cohomology groups by taking $\pi_i(H\mathbf{Q}(X))$.

To any space X, which for concreteness we can assume is BG for a finite group G, or $B\mathbf{Z}$. We can consider the trivial local system

$$H\mathbf{Q}_X \in \mathrm{LocSys}_{\mathbf{Q}}(X) \tag{22}$$

Lets try to unpack to see what this means. For each point $\mathrm{pt} \in X$, we are going to have the vector space $\mathbf{Q}_{/\,\mathrm{pt}} = \mathbf{Q}$ over it, and for each path in X between to points, which we denote by $\mathrm{pt}_1 \to \mathrm{pt}_2$, there is a "parallel transport"

$$\mathbf{Q} = \mathbf{Q}_{/\,\mathrm{pt}_1} \xrightarrow{\mathrm{Id}} \mathbf{Q}_{/\,\mathrm{pt}_2} = \mathbf{Q} \tag{23}$$

which will correspond to the identity. In the case over BG, this will correspond to the trivial representation of G. As each path in BG, corresponds to a $g \in G$, and transporting along this path amounts to acting by $g \in G$.

Now that we have an explicit example of a local system $H\mathbf{Q}_X \in \mathrm{LocSys}_{\mathbf{Q}}(X)$, I would like to introduce the notion of global sections

$$\Gamma(H\mathbf{Q}_X) \tag{24}$$

For now, let me tell you about a certain set which will serve as a first approximation to the spectrum $\Gamma(H\mathbf{Q}_X)$. Consider all the tuples of rational numbers $(q_1, q_2, \dots)$, one for each point in X. A way to think about this is to think that we are picking an element $q_i \in \mathbf{Q}_{/\,\mathrm{pt}_i}$ for each point in X. But we only want to consider a subset of this tuples, we only want tuples $(q_1, q_2, \dots)$ such that if there is a path connecting $\mathrm{pt}_1 \to \mathrm{pt}_2$, then the induced map

$$\mathbf{Q}_{/\,\mathrm{pt}_1} \longrightarrow \mathbf{Q}_{/\,\mathrm{pt}_2} \qquad q_1 \mapsto q_2 \tag{25}$$

So in the special case when we are considering the trivial local system $H\mathbf{Q}_X$, if two points pt_1 and pt_2 are connected by a path, then $q_1 = q_2$. So the set that we have described can be canonically identified with

$$\bigoplus_{\pi_0(X)} \mathbf{Q} = \pi_0(H\mathbf{Q}(X)) = \pi_0\Gamma(H\mathbf{Q}_X) \tag{26}$$

For the first equality one can simply compute the 0th singular cohomology of X, and for the second equality it's ok to just take it as the definition. Unfortunately, I don't think I'll be able to provide a similar concrete description for the higher homotopy groups of $\Gamma(H\mathbf{Q}_X)$. So we will proceed by doing some general category theory

Definition 15. We can define the spectrum $\Gamma(H\mathbf{Q}_X)$, as a limit of the constant diagram of shape X

$$\Gamma(H\mathbf{Q}_X) := \lim_{\mathrm{pt}\to X} H\mathbf{Q} \tag{27}$$

Notice that there is an important difference between the more lax notion of homotopy limit, and the usual notion of limit. And this differences has concrete consequences, as if we were only taking the usual limit we will simply get $\pi_0\Gamma(H\mathbf{Q}_X)$ loosing all the higher homotopy groups.

Proposition 16. There is a canonical identification of spectra

$$H\mathbf{Q}(X) \simeq \Gamma(H\mathbf{Q}_X) \tag{28}$$

which induces an isomorphism at the level of homotopy groups

$$\pi_i(H\mathbf{Q}(X)) = \pi_i\Gamma(H\mathbf{Q}_X) \tag{29}$$

Example 17. Ok, now let's try to use what we know about singular cohomology to better understand $\Gamma(H\mathbf{Q}_X)$. A particular space of interest for us is the space $B\mathbf{Z}/n$, for some n. Then we know, by usual computations in singular cohomology that

$$\pi_i\Gamma(H\mathbf{Q}_{B\mathbf{Z}/n}) = \begin{cases} \mathbf{Q} & i = 0 \\ 0 & \text{else} \end{cases} \tag{30}$$

In particular, we see that we are not able to extract any torsion information of the space $B\mathbf{Z}/n$ from $H\mathbf{Q}$. This phenomenon is fairly well understood in homotopy theory, and it is usually explained by saying that the spectrum $H\mathbf{Q}$ has height 0.

Now, we want to start talking about KU-valued local systems on a space X. This discussion is completely analogous to the one we did just now, but with the difference that instead of considering $\mathbf{Q}$-vector spaces, we will be considering KU-modules. That is, an element of $\mathrm{LocSys}_{\mathrm{KU}}(X)$ is a collection of KU-modules $(V_1, V_2, \dots)$, one for each point $\mathrm{pt}_i \in X$. And where for each path $\mathrm{pt}_1 \to \mathrm{pt}_2$, there is an associated isomorphism of KU-modules.

$$V_1 \xrightarrow{\simeq} V_2 \tag{31}$$

Other, similarities that it enjoys are the following

1. There is a trivial local system, which we denote by KU_X
2. We can consider the global section spectrum $\Gamma(\mathrm{KU}_X)$, which comes with an identification $\Gamma(\mathrm{KU}_X) \simeq \mathrm{KU}(X)$.

But what I would like to try to convince you now is that the global sections $\Gamma(\mathrm{KU}_X)$ of the trivial local system, is a much richer invariant of the space than $\Gamma(H\mathbf{Q}_X)$. And this could be tracked back to the fact that there are much more global sections of KU_X, due to the extra symmetry enjoyed by KU due to Bott periodicity.

Remark 18. I don't think I could provide a better conceptual explanation for the fact that $\Gamma(\mathrm{KU}_X)$ is has more information than $\Gamma(H\mathbf{Q}_X)$, besides the fact that KU, being 2-periodic, has much more symmetries and thus picks up more information about a space than $H\mathbf{Q}$. This kind of phenomenon is usually explained by saying that KU is of height 1, and the higher the height the more torsion information about that space X it picks up.

However, I would still like to illustrate, maybe through some computations, how much more information about a space X, in particular we will be interested in spaces of the form BG, the cohomology theory KU picks up, as opposed to $H\mathbf{Q}$.

In order to perform some computations, we will have to introduce the p-completed complex K-theory spectrum, which we will denote by $\mathrm{KU}_{\widehat{\mathrm{p}}}$. Before providing a definition, let me just say that in the same way that going from $\mathbf{Z} \to \mathbf{Z}_p$ looses some integral information, in particular all primes $l \in \mathbf{Z}_p$ will be invertible; going from $\mathrm{KU} \to \mathrm{KU}_{\widehat{\mathrm{p}}}$ is also a loss of integral information. In particular, if I am able to convince you that $\mathrm{KU}_{\widehat{\mathrm{p}}}(X)$ has more information than $H\mathbf{Q}(X)$, then it should follow that $\mathrm{KU}(X)$ has more information than $H\mathbf{Q}(X)$.

Definition 19. The spectrum $\mathrm{KU}_{\widehat{\mathrm{p}}}$ can be defined as

$$\mathrm{KU}_{\widehat{\mathrm{p}}} = \lim(\cdots \to \mathrm{KU}/p^n \to \mathrm{KU}/p^{n-1} \to \cdots \to \mathrm{KU}/p) \tag{32}$$

To make sense of this operations, recall that KU is a spectrum, which is a homotopical analog of an abelian group. Therefore, there should exists a multiplication by p^n map $\mathrm{KU} \to \mathrm{KU}$, and the category of spectra behaves in some ways like an abelian category, in particular it has a notion of cokernel, so one can realize

$$\mathrm{KU}/p^n = \mathrm{coker}(\mathrm{KU} \xrightarrow{\times p^n} \mathrm{KU}) \tag{33}$$

Moreover, we still have that

$$\pi_* \mathrm{KU}_{\widehat{\mathrm{p}}} = \mathrm{KU}^*_{\widehat{p}} = \mathbf{Z}_p[\beta^{\pm}] \qquad |\beta| = 2 \tag{34}$$

In particular, $\pi_* \mathrm{KU}_{\widehat{\mathrm{p}}}$ has infinitely many non-zero homotopy groups, and still enjoys a form of Bott periodicity. The Bott periodicity property is essential to many of the favorable properties that $\mathrm{KU}_{\widehat{\mathrm{p}}}$ has. And it will be particularly important for the moduli-theoretic construction of $\mathrm{KU}_{\widehat{\mathrm{p}}}$ we hope to provide later.

Example 20. After completing at a prime p, we can actually show more clearly what the relation between $\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G)$ and $\pi_0 \mathrm{KU}_{\widehat{\mathrm{p}}}(BG)$ is. Recall that we have a subgroup $G^{(p)} \subset G$ of p-singular elements, that is, elements $g \in G$, such that $g^{p^n} = 1$ for some $n \gg 0$. Then

$$\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G^{(p)}) = \pi_0 \mathrm{KU}_{\widehat{\mathrm{p}}}(BG) \tag{35}$$

That is, $\mathrm{KU}_{\widehat{\mathrm{p}}}$ is only able to see the p-power torsion elements of G. So our goal of providing a K-theoretic interpretation of $\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G)$ could be interpreted as finding an integral analog of $\mathrm{KU}_{\widehat{\mathrm{p}}}$, which is able to detect torsion elements of G for any prime p.

Remark 21. Before finishing the section, I would like maybe say some words about why we started talking about local systems, when we had a good enough definition of $\mathrm{KU}(X)$. The reason is that our goal, is to explain the construction of the tempered version of complex K-theory, which we denote by $\underline{\mathrm{KU}}$. And we will be able to realize $\underline{\mathrm{KU}}(X)$ as the global sections of a trivial local system, however, in this case it will not be a local system on a space X, but rather, what we will call an orbispace. An orbispace X, is like a space, but now we allow our points to be self-folded (or to have monodromy). Then the Atiyah-Segal comparison map

$$\underline{\mathrm{KU}}(X) \longrightarrow \mathrm{KU}(|X|) = \underline{\mathrm{KU}}(|X|) \tag{36}$$

is simply obtained by the map $|X| \to X$. Where we call $|X|$ the underlying space of the orbispace X, and it basically amounts to the same space after we forget the monodromy at the points of the orbispace X.

But the reader might be confused, we have said that the Atiyah-Segal comparison map was $\mathrm{Rep}(G) \to \mathrm{KU}(BG)$, how is this related to the map above? One can see that by setting $X = BG$ (as an orbispace) the map above specializes to

$$\mathrm{Rep}(G) = \underline{\mathrm{KU}}(BG) \longrightarrow \mathrm{KU}(|BG|) \tag{37}$$

We hope to come back to a more detailed discussion later in the notes.

A renormalized category of spaces

In this section the goal is to explain a fundamental difference between KU and $\underline{\mathrm{KU}}$, which is related to the domain on which this define cohomology theories. The cohomology theory KU determines a functor

$$\mathrm{KU} : \mathrm{Spc}^{\mathrm{op}} \longrightarrow \mathrm{Spctr} \qquad X \mapsto \mathrm{KU}(X) \tag{38}$$

however, the tempered cohomology theory we are interested in understanding better is defined in a certain category of orbispaces, which we denote by OSpc

$$\underline{\mathrm{KU}} : \mathrm{OSpc}^{\mathrm{op}} \longrightarrow \mathrm{Spctr} \qquad X \mapsto \underline{\mathrm{KU}}(X) \tag{39}$$

So how are the categories Spc and OSpc related? The category of orbispaces OSpc is a sort of renormalization of the category of Spc. We will talk more about this, but let me just say that the category of spaces is freely generated by a point, while the category of orbispaces is freely generated by certain points with prescribed monodromy. In order to make sense of this we need to review some category theory

Definition 22. Given a (small) category $\mathcal{C}$, there is always a way in which we can take a free closure under colimits, we will denote this operation by $\widehat{\mathcal{C}}$. This category $\widehat{\mathcal{C}}$ can be characterized by the universal property that for any functor $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a category already closed under colimits, there is a unique factorization

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr] & \widehat{\mathcal{C}} \arrow[d, dashed] \\ & \mathcal{D} \end{tikzcd} \tag{40}$$

and where the resulting functor $\widehat{\mathcal{C}} \to \mathcal{D}$ is colimit preserving. This universal property completely characterizes the operation $\mathcal{C} \to \widehat{\mathcal{C}}$, however we can provide a more concrete description of it as

$$\widehat{\mathcal{C}} := \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc}) \tag{41}$$

and the functor $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ is the yoneda embedding.

Remark 23. An important property of this construction, that will be useful for us, is that the embedding $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ does not preserve colimits, in fact, it destroys almost all colimits, which already existed in $\mathcal{C}$. While this may seem unfortunate, it will be very useful for us.

Example 24. Let pt be the trivial category, it has one object and the identity morphism. Then almost by definition we have that $\widehat{\mathrm{pt}} = \mathrm{Spc}$. This can be more informally phrased by saying that the category of spaces is freely generated by one point. This characterization has some concrete consequences for us: let X be a space and consider the trivial functor

$$X \longrightarrow \mathrm{Spc} \tag{42}$$

that maps every object of X to the point $\mathrm{pt} \in \mathrm{Spc}$, and every morphisms to the identity morphism. Then we have that the colimit of this diagram, is $X \in \mathrm{Spc}$ itself, when considered as a space. Another concrete consequence it has is that any space X, is completely determined by its mapping space from a point. In other words, given a space X, we can completely determined what space it is, by simply knowing the mapping space from a point

$$\mathrm{Map}_{\mathrm{Spc}}(\mathrm{pt}, X) = X \tag{43}$$

Example 25. Using a similar construction, we can now construct the category of orbispaces OSpc. Let $\mathcal{T} \subset \mathrm{Spc}$ to be the full subcategory of objects of the form BM, where M is an abelian group. Then we can define

$$\mathrm{OSpc} := \widehat{\mathcal{T}} = \mathrm{Funct}(\mathcal{T}^{\mathrm{op}}, \mathrm{Spc}) \tag{44}$$

In other words, it can be considered as a renormalization of the category of spaces, which now has as generators spaces of the form BM, rather than just a point. This construction can be a bit subtle if you have not seen it before, so let's try to unpack it a bit. Recall that this construction comes equipped with a Yoneda embedding

$$\mathcal{T} \hookrightarrow \mathrm{OSpc} \qquad BM \mapsto BM^{(-)} = \mathrm{Map}_{\mathcal{T}}(-, BM) \tag{45}$$

It is now that it becomes seriously important that the Yoneda embedding does not preserve colimits. In the category $\mathcal{T}$, we can realize BM as the constant colimit of shape BM

$$BM = \mathrm{colim}_{BM}\, \mathrm{pt} \qquad \text{in } \mathcal{T} \tag{46}$$

However, after embedding $\mathcal{T} \hookrightarrow \mathrm{OSpc}$, we have that the colimit is

$$|BM| = \mathrm{colim}_{BM}\, \mathrm{pt} \qquad \text{in } \mathrm{OSpc} \tag{47}$$

where $|BM| \neq BM^{(-)}$. So there are two objects in OSpc that could reasonably called BM: there is the space $|BM|$ which we are going to call flat BM, and the space $BM^{(-)}$ which we are going to call monodromic BM. And this to spaces are generally not the same, however they come equipped with a canonical map

$$|BM| \longrightarrow BM^{(-)} \tag{48}$$

which will be essential in our discussion of the Atiyah-Segal comparison map. Another related point of difference between OSpc and Spc, is that an orbispace X is no longer determined by the mapping space from a point, in fact we have

$$\mathrm{Map}_{\mathrm{OSpc}}(\mathrm{pt}, X) = |X| \tag{49}$$

where we call $|X|$ the underlying space of X. In order to determine the orbispace X, from some mapping spaces, one really needs to know the mapping spaces for all BM, and not just the point. This is maybe why we are calling OSpc a renormalized category of spaces, since it is now freely generated by spaces of the form BM, rather than just a point.

Remark 26. Notice that all this different categories can be organized into the following commutative diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r] \arrow[d] & \mathcal{T} \arrow[d] \\ \mathrm{Spc} \arrow[r] & \mathrm{OSpc} \end{tikzcd} \tag{50}$$

where the functor $\mathrm{Spc} \to \mathrm{OSpc}$ maps a space X to a certain flat orbispace $|X|$.

Our next goal is to explain that the tempered cohomology $\underline{\mathrm{KU}}$ is a cohomology theory which is naturally defined in OSpc, and use it to rephrase the Atiyah-Segal comparison map in this setting. One of the reasons why we needed this renormalization is that, if you recall, the cohomology of a space $X \in \mathrm{Spc}$ is completely determined by the cohomology of a point

$$\Gamma(\mathrm{KU}_X) = \lim_{\mathrm{pt} \to X} \mathrm{KU} = \mathrm{KU}(X) \tag{51}$$

which basically boiled down to the fact that $\mathrm{KU}(X)$ can be realized as the global sections of the trivial KU-valued local system on a space X, and that any space X can be completely built out of points.

However, now we want to consider local systems in orbispaces X, and not just in spaces. A general good theory of local systems on orbispaces can be a bit subtle, so we content ourselves with knowing that

1. There exists a trivial $\underline{\mathrm{KU}}$-values local system on any orbispace X, which we denote by $\underline{\mathrm{KU}}_X$.

2. To the trivial local system $\underline{\mathrm{KU}}_X$, there is an associated spectrum of global sections $\Gamma(\underline{\mathrm{KU}}_X)$, which satisfy the identity

$$\Gamma(\underline{\mathrm{KU}}_X) = \lim_{BM^{(-)}\to X} \underline{\mathrm{KU}}(BM^{(-)}) = \underline{\mathrm{KU}}(X) \tag{52}$$

In particular, it is important to notice that we can no longer extract the cohomology of the orbispace $BM^{(-)}$, from the cohomology of a point. The cohomology of $BM^{(-)}$ is now an extra piece of data what one needs to input into $\underline{\mathrm{KU}}$, to get the theory off the ground. In this section we will not explain how to do this, we hope to come back to it later, but let me just say that in particular it satisfies

$$\underline{\mathrm{KU}}^0(BM^{(-)}) = \mathrm{Rep}(M) \tag{53}$$

for any abelian group M. However, at the beginning of this note we claimed that we would be able to recover the representation theory of all finite groups G, and not only abelian groups. We would now like to explain how to extract this kind of information. We can now ask ourselves, does there exists a natural orbispace X, such that $\underline{\mathrm{KU}}^0(X) = \mathrm{Rep}(G)$? A first guess could be the flat orbispace $|BG|$, but a flat orbispace is completely built out of points, and in particular it does not uses any of the new information stored in $\underline{\mathrm{KU}}$. Concretely we have that

$$\underline{\mathrm{KU}}(|BG|) = \lim_{\mathrm{pt}\to|BG|} \mathrm{KU} = \mathrm{KU}(BG) \tag{54}$$

which we already know does not recover the whole representation theory of G. In the case of abelian groups M, we had to input the monodromic orbispace $BM^{(-)}$, however, this is no longer available in this case, since G is not abelian. So we proceed as follows: in the category of spaces, consider the category of morphisms $BM \to BG$, for all spaces of the form BM for M abelian, and denote this category by $\mathcal{T}/BG$. Then we have the following

$$\mathrm{Rep}(G) = \pi_0\Big(\lim_{\mathcal{T}/BG} \underline{\mathrm{KU}}(BM)\Big) \tag{55}$$

In other words, we are able to recover the representation theory of all finite groups, only from the representation theory of finite abelian groups.

Finally, we would like to present a form of the Atiyah-Segal comparison map which fits well into the setting of orbispaces, and the discussion in this section. Let X be an orbispace, then we can explicitly construct the associated flat orbispace $|X|$ by taking the following colimit

$$|X| = \mathrm{colim}_{\mathrm{pt}\to X}\, \mathrm{pt} \longrightarrow X \tag{56}$$

In particular, by construction we notice that it comes equipped with a natural map $|X| \to X$. Then applying the functor $\underline{\mathrm{KU}}$ we get a map

$$\underline{\mathrm{KU}}(X) \longrightarrow \underline{\mathrm{KU}}(|X|) = \mathrm{KU}(|X|) \tag{57}$$

In particular this recovers the usual Atiyah-Segal comparison map by setting $X = BM$ for an abelian group M. And for a non-abelian G, simply set $X = \mathrm{colim}_{\mathcal{T}/BG}\, BM$.

4

Our goal now is to explain a relation between the deformation problem of a the p-divisible group $(\mu_{p^\infty}, \operatorname{Spec} \mathbf{F}_p)$, and the spectrum $\mathrm{KU}_{\widehat{\mathrm{p}}}$. Naturally, we begin introducing the p-divisible group μ_{p^∞}, which will play a central role in this notes.

Later, we will provide a more detailed definition of p-divisible groups, but for now we will content with the following discussion. A p-divisible group, is in particular a group scheme, and as such it admits the following descriptions

1. The functor of points perspective, $\mu_{p^\infty} : \mathrm{Aff}^{\mathrm{op}}_{\operatorname{Spec} \mathbf{F}_p} \to \mathrm{Ab}$, defined by

$$\mu_{p^\infty}(\operatorname{Spec} R) = \{x \in R, x^{p^m} = 1 \text{ for } m \gg 0\} \tag{58}$$

2. This functor is represented by an ind-affine scheme

$$\mu_{p^\infty} = \operatorname{colim} \operatorname{Spec} \mathbf{F}_p[t]/(t^{p^m} - 1) = \operatorname{colim} \operatorname{Spec} \mathbf{F}_p[t]/(t-1)^{p^m} \tag{59}$$

   Notice how in the second equality we are using in an essential way the fact that we are working over the field $\mathbf{F}_p$.

However μ_{p^∞} is not only a group scheme, but also what is called a p-divisible group, so let me tell a bit about what a p-divisible group is. A p-divisible group is a special kind of group scheme, which satisfies the following property: to any group scheme we can ask for its p^n torsion points

$$\mu_{p^\infty}[p^n] = \operatorname{Ker}(\mu_{p^\infty} \xrightarrow{\times p^n} \mu_{p^\infty}) = \operatorname{Spec} \mathbf{F}_p[t]/(t^{p^n} - 1) \tag{60}$$

then, we say that the group scheme is a p-divisible group if it is completely determined by its p-torsion points, namely it satisfies the identity

$$\mu_{p^\infty} = \operatorname{colim} \Big(\mu_{p^\infty}[p] \hookrightarrow \mu_{p^\infty}[p^n] \hookrightarrow \cdots \Big) \tag{61}$$

And such that all the p-torsion parts are finite flat group schemes. In our case this is fine as $\mu_{p^\infty}[p^n] = \mu_{p^n}$ is a finite flat group scheme.

Remark 27. It will be pretty important for us to consider the p-divisible group μ_{p^∞} over more general bases, an important one for us will be $\operatorname{Spec} \mathbf{Z}_p$. A similar definition of μ_{p^∞} will work in this more general situations, but an important difference that could be useful to mention is that, as opposed to the situation over $\operatorname{Spec} \mathbf{F}_p$, over $\operatorname{Spec} \mathbf{Z}_p$ the p-divisible group will not be connected.

- Draw a picture of μ_{p^∞} over $\mathbf{F}_p$
- Draw picture of μ_{p^∞} over $\operatorname{Spec} \mathbf{Z}_p$.

Maybe more seriously, is the fact that we will need to consider p-divisible groups over ring spectra like $\mathrm{KU}_{\widehat{\mathrm{p}}}$. There are some technical issues one has to deal with here, but since $\pi_0(\mathrm{KU}_{\widehat{\mathrm{p}}}) = \mathbf{Z}_p$, it could be convenient to visualize μ_{p^∞} over $\mathrm{KU}_{\widehat{\mathrm{p}}}$ as if it lived over $\operatorname{Spec} \mathbf{Z}_p$.

Now, we will start talking about deformations of p-divisible groups, as we want to describe $\mathrm{KU}_{\widehat{\mathrm{p}}}$ as the oriented deformation space of the p-divisible group $(\mu_{p^\infty}, \operatorname{Spec} \mathbf{F}_p)$.

Definition 28. Let $\mathcal{M}_p$ be the moduli space of p-divisible groups over $\operatorname{Spec}\mathbf{Z}$, that is, it is given by a functor

$$\mathcal{M}_p : \mathrm{Aff}^{\mathrm{op}}_{\operatorname{Spec}\mathbf{Z}} \longrightarrow \mathrm{Spc} \tag{62}$$

which maps $\operatorname{Spec} R \mapsto \{\text{the set of p-divisible groups over } \operatorname{Spec} R\}$. We warn the reader that this object is not representable by a scheme, or even artin stack, it a more subtle object. But, we know that giving a map $\operatorname{Spec}\mathbf{F}_p \to \mathcal{M}_p$ corresponds to specifying a p-divisible groups over $\operatorname{Spec}\mathbf{F}_p$. In particular, the p-divisible group $(\mu_{p^\infty}, \operatorname{Spec}\mathbf{F}_p)$ determines a map $\operatorname{Spec}\mathbf{F}_p \to \mathcal{M}_p$.

Recall that we want to understand deformations of our p-divisible group $(\mu_{p^\infty}, \operatorname{Spec}\mathbf{F}_p)$, for us, this means that we want to understand the completion of $\mathcal{M}_p$ along the map $\operatorname{Spec}\mathbf{F}_p \to \mathcal{M}_p$. This may seem weird, since as we said before $\mathcal{M}_p$ is not representable in any way, however, there is still a way of making sense of this.

Proposition 29. Then there is a result of Lubin and Tate that tells us what the deformation space of $(\mu_{p^\infty}, \operatorname{Spec}\mathbf{F}_p)$, it is $\operatorname{Spec}\mathbf{Z}_p$.

So now by taking the completion we have a map $\operatorname{Spec}\mathbf{Z}_p \to \mathcal{M}_p$, which is given by the p-divisible group $(\mu_{p^\infty}, \operatorname{Spec}\mathbf{Z}_p)$. However, there is an important distinction between between what happened in characteristic p, and over $\operatorname{Spec}\mathbf{Z}_p$. We our, p-divisible group is no longer connected

- Draw picture of μ_{p^∞} over $\operatorname{Spec}\mathbf{Z}_p$.

In fact, we have the following result of Deligne

Proposition 30. All finite flat group schemes are etale over $\operatorname{Spec}\mathbf{Q}$. In particular, $\mu_{p^\infty} \times \operatorname{Spec}\mathbf{Q}_p$ is etale.

Remark 31. Notice that μ_{p^∞} is not finite, so it does not directly fits into the setting of the result above, however, μ_{p^∞} is built out of finite flat group schemes, later we will take a perspective on p-divisible groups that emphasizes the perspective that it is built out of finite flat group schemes.

At the beginning of this sections, we said we wanted to realize $\mathrm{KU}_{\widehat{\mathrm{p}}}$ as an oriented deformation space of $(\mu_{p^\infty}, \operatorname{Spec}\mathbf{F}_p)$, but so far all we have gotten is $\operatorname{Spec}\mathbf{Z}_p$, granted we haven't said anything about orientations. But notice that $\mathbf{Z}_p$ and $\mathrm{KU}_{\widehat{\mathrm{p}}}$ are not so distinct, recall that there is an embedding

$$\mathrm{Rings} \hookrightarrow \mathrm{ComAlg}(\mathrm{Spctr}) \qquad\qquad R \mapsto HR \tag{63}$$

where HR is the Eilenberg-Maclane spectrum associated to R, which represents singular cohomology with coefficients in the rings R, in particular we have that

$$H\mathbf{Z}_p^*(\mathrm{pt}) = \mathbf{Z}_p \qquad \text{concentrated in degree zero} \tag{64}$$

which is pretty close to

$$\mathrm{KU}_{\widehat{\mathrm{p}}}(\mathrm{pt}) = \mathbf{Z}_p[\beta^{\pm}] \qquad |\beta| = 2 \tag{65}$$

So in some sense, one can think of $\mathrm{KU}_{\widehat{\mathrm{p}}}$ as a 2-periodic refinement of $\mathbf{Z}_p$. And this is precisely how one obtains a universal deformation ring

1. One first deforms its p-divisible group in the classical sense. In our case we have $(\mu_{p^\infty}, \operatorname{Spec}\mathbf{F}_p)$, which we deformed to $(\mu_{p^\infty}, \operatorname{Spec}\mathbf{Z}_p)$.

2. One then forces the classical deformation ring to satisfy a form of Bott periodicity, which can be informally described by adding a certain invertible element β in degree two, we will call this element the Bott element. In our specific example, this amount to passing from $\mathbf{Z}_p$ to $\mathbf{Z}_p[\beta^{\pm}]$.

The existence of such a Bott element, is exactly what it amounts for a ring spectrum E to be oriented, it must come equipped with a certain symmetry

$$\beta : E \xrightarrow{\simeq} E[2] \tag{66}$$

which in particular implies that for any space X we have a form of Bott periodicity

$$\beta : E^i(X) \xrightarrow{\simeq} E^{i+2}(X) \tag{67}$$

The notion of orientation is rather opaque, at least for me, and it seems like somewhat of a miracle that it is so ubiquitous. However, even if one does not understand what an orientation is, one can still appreciate some properties that it enjoys. We will specialize this properties to our example of $\mathrm{KU}_{\widehat{\mathrm{p}}}$ as the oriented deformation ring of $(\mu_{p^\infty}, \operatorname{Spec} \mathbf{F}_p)$.

1. It still contains the information of the classical deformation ring. It can be recovered as $\mathbf{Z}_p = \mathrm{KU}^0_{\hat{p}}(\mathrm{pt})$.

2. Moreover, $\mathrm{KU}_{\widehat{\mathrm{p}}}$ knows about the the p-divisible group μ_{p^∞} over $\operatorname{Spec} \mathbf{Z}_p$, namely we have that

$$\operatorname{Spec}\left(\mathrm{KU}^0_{\hat{p}}(B\mathbf{Z}/p^n)\right) = \mu_{p^\infty}[p^n] \tag{68}$$

One can even recover the multiplicative structure, however this is a bit more subtle as one really needs to work with spectra and not just its homotopy groups.

Remark 32. It is important to remember that in this section we have only discussed about a specific p-divisible group μ_{p^∞}, which happens to be connected over $\operatorname{Spec} \mathbf{F}_p$. This connectivity condition is essential, and will play a crucial role in the Atiyah-Segal completion theorem.

Finally, lets conclude this section discussing exactly how much of the representation theory of G can $\mathrm{KU}_{\widehat{\mathrm{p}}}$ see. Recall that we have a morphism

$$\zeta : \operatorname{Rep}(G) \longrightarrow \mathrm{KU}^0_{\hat{p}}(BG) \tag{69}$$

and we notice that $\operatorname{Rep}(G)$ is an algebra over $\mathbf{Z}$, while $\mathrm{KU}^0_{\hat{p}}(BG)$ is a $\mathbf{Z}_p$ algebra, therefore the Atiyah-Segal comparison map factors as

$$\mathbf{Z}_p \otimes_{\mathbf{Z}} \operatorname{Rep}(G) \xrightarrow{\zeta} \mathrm{KU}^0_{\hat{p}}(BG) \tag{70}$$

One can then ask the following question, for which G is this an isomorphism? The Atiyah-Segal comparison map is an isomorphism in this case is an isomorphism if and only if G is a p-group.

We would like to provide an algebro-geometric explanation for this, relying on the geometry of the p-divisible group μ_{p^∞} over $\mathrm{KU}_{\widehat{\mathrm{p}}}$. Yes, we have not provided a definition of μ_{p^∞} over general ring spectra, but lets just assume that there exists a good notion of p-divisible groups over rings like $\mathrm{KU}_{\widehat{\mathrm{p}}}$. To any p-divisible group $\mathbb{G}$, and any abelian group M, one can ask for the M torsion points, which we denote by $\mathbb{G}[M]$.

Example 33. Some relevant examples for us are the following

1. $\mu_{p^\infty}[\mathbf{Z}/p^n] = \mu_{p^\infty}[p^n]$

2. $\mu_{p^\infty}[\mathbf{Z}/l^n] = \emptyset$

And moreover, when one asks for $\mathrm{KU}_{\widehat{\mathrm{p}}}(BM)$ for a finite abelian group M, one is really asking for the M torsion points of $\mu_{p^\infty}[M]$ over $\mathrm{KU}_{\widehat{\mathrm{p}}}$. We hope this provides an geometric explanation for the failure of $\mathrm{KU}_{\widehat{\mathrm{p}}}$ to see the whole representation of G.

5

In the last section, we saw that at the core of the failure of the Atiyah-Segal comparison map to be an isomorphism

$$\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathrm{Rep}(G) \xrightarrow{\zeta} \mathrm{KU}^0_{\hat{p}}(BG) \tag{71}$$

was the fact that there is a natural p-divisible group μ_{p^∞} associated to $\mathrm{KU}_{\widehat{\mathrm{p}}}$, and that this p-divisible group, only has p-torsion points. Which in turn implied that $\mathrm{KU}_{\widehat{\mathrm{p}}}$ could only detect the representation theory of the p-singular subgroup $G^{(p)} \subset G$, that is, the subgroup of elements $g \in G$ such that $g^{p^n} = 1$.

In this section, we will define a tempered version of $\mathrm{KU}_{\widehat{\mathrm{p}}}$, which we will denote by $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$, in which satisfies the following identity

$$\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}(BM) = \bigoplus_{p \in P} \mu_{p^\infty}[M] = \mu_{\mathbf{P}^\infty}[M] \tag{72}$$

namely, instead of just recording the M torsion points of μ_{p^∞}, for a single prime p, it records the M torsion points for all primes p. This marks a departure from the kind of p-divisible groups that can be recovered from $\mathrm{KU}_{\widehat{\mathrm{p}}}$, namely the p-divisible group $\mu_{\mathbf{P}^\infty}$ is not connected, as opposed to μ_{p^∞}.

The goal of this section is to provide some background in order to to understand the definition of $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$. We begin by introducing the notion of a $\mathbf{P}$-divisible group, which is a generalization of the classical notion of p-divisible group, which works for all p, simultaneously, an example of this kind of object will be $\mu_{\mathbf{P}^\infty}$.

Definition 34. A $\mathbf{P}$-divisible group over a ring A, is functor

$$X : \mathrm{Ab}^{\mathrm{op}}_{\mathrm{fin}} \longrightarrow \mathrm{FiniteFlatGroupSchemes}_{/\,\mathrm{Spec}\,A} \tag{73}$$

which satisfies certain conditions

- The commutative finite flat group scheme $X(0) = A$
- The functor is symmetric monoidal, that is, the group structure of $X(M)$ is completely determined by the group structure of M. So in some sense, it is redundant to require that the functor lands in finite flat group schemes, as finite flat algebras will suffice
- The functor X is exact, that is, for a short exact sequence of finite abelian groups $0 \to M_1 \to M_2 \to M_3 \to 0$, the induced diagram

$$\begin{array}{ccc} X(M_3) & \longrightarrow & X(M_2) \\ \downarrow & & \downarrow \\ X(0) & \longrightarrow & X(M_1) \end{array} \tag{74}$$

  is a exact sequence of finite flat group schemes

- Moreover, one can define a notion of heigh. We say that a $\mathbf{P}$-divisible group has height h, if for every finite abelian group M, the finite flat group scheme $X(M)$ has degree $|M|^h$ over A. Where $|M|$ is the cardinality of M.

Remark 35. This definitions even make sense over a ring spectrum, like $\mathrm{KU}_{\widehat{\mathrm{p}}}$. This will be of central importance to us, since to construct $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$, we will need to consider $\mathbf{P}$-divisible groups over $\mathrm{KU}_{\widehat{\mathrm{p}}}$. However, since this may seem to abstract, and the reader might enjoy having some pictures to visualize what is happening, since $\pi_0(\mathrm{KU}_{\widehat{\mathrm{p}}}) = \mathbf{Z}_p$, it is reasonable to imagine that a $\mathbf{P}$-divisible group over $\mathrm{KU}_{\widehat{\mathrm{p}}}$ looks like a $\mathbf{P}$-divisible group over $\mathbf{Z}_p$.

Example 36. A central example of a $\mathbf{P}$-divisible group, will be

$$\mu_{\mathbf{P}^\infty}[M] := \bigoplus_{p\in\mathbf{P}} \mu_{p^\infty}[M] \tag{75}$$

In order to get a bit more familiar with this notion, notice that any finite abelian group M can be factored into its "prime pieces"

$$M = \bigoplus_{p\in\mathbf{P}} \Big(\oplus \mathbf{Z}/p^n \Big) \tag{76}$$

and then one gets that

$$\mu_{\mathbf{P}^\infty}[M] = \bigoplus_{p\in\mathbf{P}} \Big(\oplus \mu_{p^\infty}[\mathbf{Z}/p^n] \Big) \tag{77}$$

We would now like to use this example to further explain why $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$ is able to see all the representation theory of a group finite abelian group M. We will come back to the general case for non-abelian G later. Recall that we have been saying that to our cohomology theories, namely, $\mathrm{KU}_{\widehat{\mathrm{p}}}$ and $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$, they come equipped with a $\mathbf{P}$-divisible group

1. $\mathrm{KU}_{\widehat{\mathrm{p}}}$ comes equipped with μ_{p^∞}. That means that for a finite abelian group M, we can recover the M-torsion points $\mu_{p^\infty}[M]$ simply by computing the cohomology $\mathrm{KU}_{\widehat{\mathrm{p}}}(BM)$.

2. $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$ comes equipped with $\mu_{\mathbf{P}^\infty}$. That means that for a finite abelian group M, we can recover the M-torsion points $\mu_{\mathbf{P}^\infty}[M]$ simply by computing the cohomology $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}(BM)$.

If one is not comfortable with $\mathbf{P}$-divisible groups over ring spectra like $\mathrm{KU}_{\widehat{\mathrm{p}}}$, one can extract, in both situations, a $\mathbf{P}$-divisible group over $\mathbf{Z}_p$ simply by taking the 0th cohomology $\mathrm{KU}^0_{\widehat{p}}(BM)$ or $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}{}^0(BM)$.

Then one can clearly see why $\mathrm{KU}^0_{\widehat{p}}(BM)$ is only able to see the p-singular subgroup $M^{(p)} \subset M$, its associated p-divisible group can only see p-power torsion elements, while $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$ is able to see p-power torsion for all $p \in \mathbf{P}$, as it comes equipped with the $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$.

We would now want to explain what kind of object this tempered cohomology $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$ is. As a warm up, we start with a discussion about $\mathrm{KU}_{\widehat{\mathrm{p}}}$. Recall that $\mathrm{KU}_{\widehat{\mathrm{p}}}$ is a ring spectrum, in particular this means that it determines a functor

$$\mathrm{KU}_{\widehat{\mathrm{p}}} : \mathrm{Spc}^{\mathrm{op}} \longrightarrow \mathrm{ComAlg}_{\mathrm{KU}_{\widehat{\mathrm{p}}}} \qquad X \mapsto \mathrm{KU}_{\widehat{\mathrm{p}}}(X) \tag{78}$$

Moreover, this functor is preserves limits. In particular, this implies that the cohomology of any space X, is completely determined by the cohomology of a point pt. An important property of the category of spaces Spc, in which X lives, is that

$$X = \mathrm{colim}_X\, \mathrm{pt} \tag{79}$$

In other words, we can construct X, as the colimit of the constant diagram of shape X. One then has the following property of a spectrum

$$\mathrm{KU}_{\widehat{\mathrm{p}}}(X) = \mathrm{KU}_{\widehat{\mathrm{p}}}(\mathrm{colim}_X\, \mathrm{pt}) = \lim_X \mathrm{KU}_{\widehat{\mathrm{p}}}(\mathrm{pt}) = \lim_X \mathrm{KU}_{\widehat{\mathrm{p}}} \tag{80}$$

that is, the cohomology of X is completely determined by the cohomology of a point. So in some sense, knowing the cohomology of, say, BM is in no way extra data to $\mathrm{KU}_{\widehat{\mathrm{p}}}$. And this perspective seems to be at the core of the definition of $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$, we want that the cohomology of BM to be extra data coming from the $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$.

In order to do this, we will define $\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}$ to be a functor from the category $\mathcal{T}$, where the objects are spaces of the form BM, where M is a finite abelian group. In formulas can be written as

$$\underline{\mathrm{KU}}_{\widehat{\mathrm{p}}} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{ComAlg}_{\mathrm{KU}_p} \qquad BM \mapsto \mu_{\mathbf{P}^\infty}[M] = \underline{\mathrm{KU}}_{\widehat{\mathrm{p}}}(BM) \tag{81}$$

In order to better understand this construction, it will be important to review some categorical constructions.

Construction 37. To any (small) category $\mathcal{C}$, there is a way in which one can construct a "free closure under colimits" of $\mathcal{C}$, which we denote by $\widehat{\mathcal{C}}$.