The Cotangent Complex and Infinitesimal Deformations

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Contents

0 Introduction **2**

1 The Cotangent Complex **2**

1.1 Simplicial Commutative Rings 2

1.2 Definition and First Propeties 6

2 Infintesimal Deformations **7**

2.1 Extensions of Algebras 7

2.2 Proof of the Fundamental Theorem 8

2.3 Deformation of Schemes and Morphisms 11

3 Application to Abelian Varieties **16**

0 Introduction

An important goal of algebraic geometry is to be able to understand moduli spaces of different geometric objects. One of the most important examples if the moduli space of elliptic curves $\mathcal{M}_{1,1}$. This moduli spaces tend to be quiet hard to get your hands on, and deformation theory provides a tool to understand them locally.

For example, say that you would like to understand the tangent space of $\mathcal{M}_{1,1}$ at a given point $\operatorname{Spec} k \hookrightarrow \mathcal{M}_{1,1}$, we know that the tangent space can be identified as lifts of the following diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d, hook] & \mathcal{M}_{1,1} \\ \operatorname{Spec} k[\varepsilon] \arrow[ur, dashed] & \end{tikzcd} \tag{1}$$

We can rephrase this in the lingo of deformation theory. Let E be the elliptic curve corresponding to $\operatorname{Spec} k \hookrightarrow \mathcal{M}_{1,1}$. Finding lifts amounts to finding elliptic curves $E' \to \operatorname{Spec} k[\varepsilon]$ such that the following diagram is cartesian

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & E' \arrow[d, hook] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} k[\varepsilon] \end{tikzcd} \tag{2}$$

This is an example of a first order deformation of E/k. In this paper we will present the machinery as developed by Illusie in [Ill06], to understand infinitesimal deformations. At the very end, we will use this heavy machinery to provide a simple proof that the tangent space of $\mathcal{M}_{1,1}$ at its rational points is one dimensional.

1 The Cotangent Complex

Through this section we will provide a quick introduction to some ideas we will need through this paper. First, we will introduce the notion of simplicial rings, and we will introduce its model structure. This model structure, will allow us to work with derived functors in a non abelian setting as the category of commutative rings. We will then use this formalism to define the cotangent complex as a left derived functor of the functor of Kahler differentials. In this section, we will not provide proofs for most of our statements, as it is mean just as a quick overview. References for more detailed accounts will be provided.

1.1 Simplicial Commutative Rings

In this section we provide a quick overview of the theory of simplicial commutative rings. In order to motivate this theory, lets first recall some fact about homological algebra. Recall that in the category of R-modules, $\operatorname{Mod} R$, the functor $- \otimes M$ is only right exact. By this we mean that given an exact sequence of R-modules

$$0 \longrightarrow N \longrightarrow N' \longrightarrow N'' \longrightarrow 0 \tag{3}$$

when we apply the functor $- \otimes M$ we get

$$N \otimes M \longrightarrow N' \otimes M \longrightarrow N'' \otimes M \longrightarrow 0 \tag{4}$$

loosing exactness on the left. However, there are good situations in which our sequence will be exact on the left; for example, if all three modules N, N', N'' are projective (i.e. locally free), then the sequence will be exact on the left. However, homological algebra provides us a way to solve this problem: by enlarging our category $\operatorname{Mod} R \hookrightarrow \operatorname{Ch}(R)$, we can define a projective resolution functor $Q : \operatorname{Ch}(R) \to \operatorname{Ch}(R)$, such that there exists a map $Q(N) \to N$ which is a quasi-isomorphism. By precomposing $- \otimes M$ with Q, we obtain an exact functor

$$0 \longrightarrow Q(N) \otimes M \longrightarrow Q(N') \otimes M \longrightarrow Q(N'') \otimes M \longrightarrow 0 \tag{5}$$

This functor is an example of a left derived functor, which we denote by $- \otimes^{\mathbb{L}} M$. This encodes the usefulness of homological algebra: by enlarging our category, and defining an appropriate notion of approximation by well behaved objects (in this case projective modules), we obtain a better behaved theory.

Our goal of this chapter is to define an analogous theory for commutative rings, we would like to enlarge our category of commutative rings to allow resolutions by well behaved rings (i.e. smooth rings), such that we have a better behaved theory. Unfortunately, resolutions (in the sense of homological algebra) do not exist in the category of rings, as the kernel of a map $A \to B$ is not a ring, unless the map is zero. However, Daniel Quillen was able to develop a formalism that allow us to use the techniques from homological algebra in highly non abelian settings, this is the theory of Model Categories, which we will use freely through this paper. A good reference is for this material is [Hov07]. For a more detailed discussion we refer the reader to [Mat12] of Akhil Mathew, from who the author learned the theory.

Definition 1.1.1. A simplicial commutative ring $A_\bullet$ is a simplicial object in the category of commutative rings. In other words, it is an object of the category $\mathrm{sCRing} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{CRing})$.

One could be wondering, why do we use simplicial commutative rings instead of some sort of chain complex? First, a characteristic property of chain complexes is that $\partial \circ \partial = 0$, and this will not be possible to accomplish in a natural way with commutative rings, since the relation $\partial \circ \partial = 0$ would only be possible if one of the maps where zero. Secondly, and more important, in the case where $\mathcal{A}$ is an abelian category, we have an equivalence of categories

$$\mathrm{Ch}_{\geq 0}(\mathcal{A}) \cong \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \tag{6}$$

which is moreover, a Quillen equivalence. This equivalence goes by the name, the Dold-Kan correspondence. Showing us that we can still obtain a good analogy of homological algebra using simplicial objects.

Example 1.1.2. In order to justify that sCRing is an enlargement of CRing we need a fully faithful embedding

$$\mathrm{CRing} \lhook\joinrel\longrightarrow \mathrm{sCRing} \tag{7}$$

In which a commutative ring R, is mapped to the constant simplicial commutative ring, where all the face and degeneracy maps are the identity.

We will now begin discussing the homotopy theory of sCRing from the perspective of Quillen's Model Categories. Every simplicial commutative ring, is in particular, a simplicial abelian group, hence also a Kan complex. This allows us to talk about homotopy groups of $R_\bullet \in \mathrm{sCRing}$ relative to a basepoint $* \in R_0$, which we take to be the zero object. Interestingly, from the ring structure we are able to say even more from the homotopy groups.

Proposition 1.1.3. *The homotopy groups $\pi_* R_\bullet$ form a graded commutative ring. In other words, we have a functor*

$$\pi_* : \mathrm{sCRing} \longrightarrow \mathrm{GrCommAlg}_{\geq 0} \tag{8}$$

where $\mathrm{GrCommAlg}_{\geq 0}$ *is the category of nonnegatively graded commutative rings.*

We want to think of the relationship between simplicial and ordinary commutative rings, as being something like the relationship between ordinary rings and reduced rings. In the same way that schemes allows us to include nilpotent into the picture, derived algebraic geometry allows us to have a fancier version of nilpotent: higher homotopy groups. The analog of the canonical map $R \to R_{\mathrm{red}}$ valid for any ring R is the map

$$R_\bullet \longrightarrow \pi_0 R_\bullet \tag{9}$$

where $\pi_0 R_\bullet$ can be identified with $R_0/(d_1 - d_0)R_1$. Moreover, we obtain an adjunction

$$\mathrm{CRing} \rightleftarrows \mathrm{sCRing} : \pi_0 \tag{10}$$

where π_0 is the right adjoint to the inclusion. Finally, we reinforce the analogy of higher homotopy groups with nilpotents, recall that the underlying topological space of $\operatorname{Spec} R$ and $\mathrm{Spec}_{\mathrm{red}}$ are canonically isomorphic. In the same way, the to a simplicial commutative ring we can associated a derived scheme, in the sense of Lurie, and we have that the underlying topological space of $\operatorname{Spec} R_\bullet$ is canonically isomorphic to $\operatorname{Spec} \pi_0 R_\bullet$.

Definition 1.1.4. A simplicial module $M_\bullet$ over a simplicial ring $R_\bullet$, is a simplicial abelian group, equipped with a simplicial map

$$R_\bullet \times M_\bullet \longrightarrow M_\bullet \tag{11}$$

satisfying all the usual axioms degreewise. By the same reasoning as above, we find that there is a natural map

$$\pi_* R_\bullet \times \pi_* M_\bullet \longrightarrow \pi_* M_\bullet \tag{12}$$

which makes π_* a graded module over the graded ring $\pi_* M_\bullet$. In particular, $\pi_0 M_\bullet$ is a $\pi_0 R_\bullet$ module.

In order to do homotopy theory in sCRing we will need a model structure. Indeed we have the following theorem.

Theorem 1.1.5. *(Model Structure on* sCRing*.) There exists a cofibrantly generated simplicial model structure on the category* sCRing *of simplicial commutative rings with*

(W) Weak equivalences are just weak equivalences of underlying simplicial sets (that is, an equivalence on π_*)

(F) Fibrations are just fibrations of the underlying simplicial sets

(C) The cofibrations are determined

One obtains a Quillen adjunction between sSet *and* sCRing *from the free-forgetful adjunction. In fact,* sCRing *is a monoidal model category under the tensor product, and it is proper.*

Remark 1.1.6. Underlying the proof of this theorem is the fact that we choose the generating cofibrations and trivial cofibrations to be the image of the generating cofibrations and trivial cofibrations in sSet which are well understood. This allows us to construct a cofibrant replacement in sCRing_A, where the simplicial ring is a made of free A-algebras level wise.

The existence of the model structure is not obvious. It can be stablished by a process called "transfer" which was already implicit in Quillen's work. The setup is as follows. One has an adjunction:

$$F : \mathrm{sSet} \rightleftarrows \mathrm{sCRing} : U \tag{13}$$

where F is the free functor and U is the forgetful functor. We already have a nice model structure on sSet; the strategy is to transfer this to sCRing in such a way that the above adjunction is actually a Quillen adjunction. Since U has to preserve fibrations and weak equivalence, this gives us a natural definition of those in sCRing. This process applies more generally to algebraic structure in a model category

Remark 1.1.7. In fact, it is a basic result in model category theory that an under or over category inherits a natural model structure from the original category. This will be particularly useful since we will be mostly working with the category $\mathrm{sCRing}_{A//R}$.

Definition 1.1.8. Let S be an R-algebra. Then a simplicial resolution for S is a cofibrant replacement for S in the category of simplicial R-algebras. In other words, it is the data of a trivial fibration of simplicial algebras

$$X_\bullet \longrightarrow S \tag{14}$$

such that $X_\bullet$ is a cofibrant simplicial R-algebra.

The existence of a model structure enables us to see very efficiently that $X_\bullet$ is unique up to homotopy. Understanding cofibrant replacements in the category sCRing will be the most important outcome of this section. Let $A_\bullet \to A'_\bullet$ be a cofibrations of simplicial sets (that is, a monomorphism). Then $\mathbb{Z}[A_\bullet] \to \mathbb{Z}[A'_\bullet]$ is a cofibrations of simplicial commutative rings, since the free functor is a left Quillen functor. On the other hand if we are working with $\mathrm{sCRing}_{R/}$ we would take $R[A_\bullet] \to R[A'_\bullet]$ instead. Moreover, as $A_\bullet \to A'_\bullet$ ranges over a set of generating cofibrations of simplicial sets (for instance, $\partial\Delta[n]_\bullet \to \Delta[n]_\bullet$), we get a set of generating cofibrations for $\mathrm{sCRing}_{R/}$. It follows that if we have a map $R \to X_\bullet$ of simplicial commutative rings, then to say that it is a cofibrations implies that each X_n is a retract of a polynomial algebra on R – in particular, each X_n is a (formally) smooth R-algebra. This smoothness condition suggests that taking a left derived functor of the Kahler differential might be a better behave theory (by making the analogy between "smoothness" and "projective"). We will discuss this further in the following section. We record out result

Proposition 1.1.9. *Let $X_\bullet \in \mathrm{sCRing}_{/R}$, then $X_\bullet$ is cofibrant if and only if each X_n is a (formally) smooth R-algebra.*

Proposition 1.1.10. *The category $\mathrm{Mod}_{R_\bullet}$ comes equipped with a simplicial model category, which can be described by*

- *(W) The weak equivalences are the weak equivalences of underlying simplicial sets (equivalently, the quasi-isomorphisms of chain complexes)*
- *(F) The fibrations are the underlying fibrations of simplicial sets (in particular, every simplicial module is fibrant)*
- *(C) The cofibrations are determined.*

We can obtain this model structure by transferring along the free-forgetfull adjunction from simplicial sets.

1.2 Definition and First Propeties

The goal of this section is to provide a well motivated definition for the cotangent complex, in fact we will see that it the cotangent complex is just the left derived functor of the Kahler differentials. Our interest from the cotangent complex arises from the fact that it controls infinitesimal deformations, as shown by several results by Illusie [Ill06], which will be discussed and proved in the next section.

Recall that given morphisms of commutative rings

$$R \longrightarrow S \longrightarrow T \tag{15}$$

there exists an associated exact sequence

$$\Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R} \longrightarrow \Omega_{T/S} \longrightarrow 0 \tag{16}$$

This exact sequence is the origin of the cotangent complex, as it hints that there should exists a left derived functor of the Kahler differentials that extends the sequence into a long exact sequence. We will need the theory of model categories to be able to make sense of the notion of a left derived functor in the non abelian setting of commutative rings.

Recall that if $S \to T$ is (formally) smooth, then the sequence is also exact on the left, namely, we have

$$0 \longrightarrow \Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R} \longrightarrow \Omega_{T/S} \longrightarrow 0 \tag{17}$$

From this we can infer that we should be taking smooth resolutions of objects, in order to obtain a better behaved notion of Kahler differentials. Giving further evidence that considering the left derived functor from the category of sCRing, with the model structure presented in the previous section, will provides us with a better behaved theory, as the cofibrant objects are precisely the (formally) smooth ones.

In order to be able to talk about the left derived functor of $\Omega_{-/R}$ we will need to first interpret it as a functor. Note that it is not so clear how to express the Kahler differentials as a functor, since $\Omega_{A/R}$ is an A-module; showing that the target category depends on the input. In order to solve this problem we will work on the category $\mathrm{CRing}_{A//B}$. Indeed we have the following result

Theorem 1.2.1. *There exists an adjunction*

$$\Omega_{-/A} \otimes_{-} B : \mathrm{CRing}_{A//B} \rightleftarrows \mathrm{Mod}_B \tag{18}$$

where the left adjoint maps $X \mapsto \Omega_{X/A} \otimes_X B$. And the right adjoint $\mathrm{Mod}_B \to \mathrm{CRing}_{A//B}$ is defined by $M \mapsto B \oplus M$; where $A \to B \oplus M$ comes from $A \to B$, $B \oplus M \to B$ is just the projection, and multiplication is defined by the rule

$$(b_1, m_1)(b_2, m_2) \longmapsto (b_1 b_2, b_1 m_2 + b_2 m_1) \tag{19}$$

In particular note that M satisfies $M^2 = 0$.

There is in fact a more this say about this theorem. For instance, the functor $\mathrm{Mod}_B \to \mathrm{CRing}_{A//B}$ is a fully faithful embedding, defining an equivalence of categories with the full subcategory of abelian group objects of $\mathrm{CRing}_{A//B}$. Therefore it is reasonable to say that module of Kahler differentials the abenialization of the category $\mathrm{CRing}_{A//B}$. We will now upgrade our adjunction to the simplicial setting.

Theorem 1.2.2. *There exists a Quillen adjunction*

$$\Omega_{-/A} \otimes_{-} B : \mathrm{sCRing}_{A//B} \rightleftarrows \mathrm{sMod}_B \tag{20}$$

where $\Omega_{P_\bullet/A}$, of a simplicial ring $P_\bullet$, is defined objectwise.

With all this machinery at hand, we can provide a nice definition of the cotangent complex.

Definition 1.2.3. The cotangent complex $L_{B/A}$ is defined to be

$$L_{B/A} := (\mathbb{L}\Omega_{-/A} \otimes_{-} B)(B) \tag{21}$$

In other words, we can pick a cofibrant replacement $P \to B$ in the category $\mathrm{sCRing}_{A//B}$, and define $L_{B/A} := \Omega_{P/A} \otimes_P B$. Since $L_{B/A}$ is simply an object in the derived category $D(B)$ this is well defined.

It may be worth pointing out that since $\Omega_{P/A}$ is a cofibrant object in Mod_P, there is not ambiguity in the distinction between the derived tensor product and the ordinary tensor product. In fact, the tensor product defines a Quillen equivalence

$$- \otimes_P B : \mathrm{Mod}_P \rightleftarrows \mathrm{sMod}_B \tag{22}$$

This is a direct consequence of the fact that $X \otimes_P B \to X$ is a weak equivalence in Mod_P.

We will now state some basic properties about the cotangent complex that we will use through the paper.

(1) Smooth algebras: If $A \to B$ is smooth, then $L_{B/A} \simeq \Omega_{B/A}[0]$. This is a direct consequence of the fact that B is already a cofibrant object.

(2) Transitivity Triangle: Given a composite $A \to B \to C$ of maps, we have a canonical exact triangle

$$L_{B/A} \otimes_B C \longrightarrow L_{C/A} \longrightarrow L_{C/B} \tag{23}$$

in $D(C)$.

(3) Relation to Kahler differentials: for any map $A \to B$, we have $H_0(L_{B/A}) \simeq \Omega_{B/A}$.

For a more through account of this results we refer the reader to [Ill06]. Analogous statements hold for schemes.

2 Infintesimal Deformations

In this chapter, we will investigate the relation between the cotangent complex, and infinitesimal deformation of schemes, as states in Illusie [Ill06]. However, we will only provide proofs for the affine case, as working with derived categories of sheaves turned out to be more subtle than expected by the author. However, the real meat of the theory can be found in the affine case, as globalizing it would just require an appropriate language to talk about the derived category of sheaves. Unfortunately for the author, the language of model categories turned out to be quiet cumbersome for this purpose, and therefore has decided to postpone its exposition until later in his mathematical career.

2.1 Extensions of Algebras

In this section we will develop some language that we will need thought the paper. The language is concerned with a certain category of extension of algebras, which will be essential through this paper.

Let $f : B \to C$ be a map of rings, and let I be a C-module. By a B-extension of C we mean a factorization of $B \to C$ trough C'. The following diagram encodes the information quiet well

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & C' \arrow[r, two heads, "i"] & C \arrow[r] & 0 \\ & & B \arrow[u] \arrow[ur] & & \end{tikzcd} \tag{24}$$

Definition 2.1.1. Denote by $\underline{\mathrm{Exal}}_B(C, I)$ the category of B-extensions of C by I.

$\underline{\mathrm{Exal}}$ is a historical notation, it stands for "Extension of Algebras" of C by I.

Proposition 2.1.2. *The category* $\underline{\mathrm{Exal}}_B(C, I)$ *is a Picard groupoid. Where the addition map is defined by the following diagram*

$$\begin{tikzcd} \underline{\mathrm{Exal}}_B(C,I) \times \underline{\mathrm{Exal}}_B(C,I) \arrow[r, "(pr_{1*}, pr_{2*})\cong"] \arrow[dr, "+"'] & \underline{\mathrm{Exal}}_B(C, I \oplus I) \arrow[d, "(I\oplus I \to I)_*"] \\ & \underline{\mathrm{Exal}}_B(C, I) \end{tikzcd} \tag{25}$$

Definition 2.1.3. Denote by $\mathrm{Exal}_B(C, I)$ the isomorphism classes of $\underline{\mathrm{Exal}}_B(C, I)$. Therefore $\mathrm{Exal}_B(C, I)$ is an abelian group.

We will now provide the definition of a deformation using the language of extension of algebras.

Definition 2.1.4. Consider some morphisms of rings

$$A \longrightarrow B_0 \longrightarrow C_0 \tag{26}$$

and a B_0 module I. We say that some $\underline{X} \in \mathrm{Exal}_A(C, I)$ is a deformation of C over $\underline{Y} \in \mathrm{Exal}_A(B, I)$, if it fits into a diagram of the form

$$\begin{tikzcd} \underline{X}: & \operatorname{Spec} C_0 \arrow[r, hook] \arrow[d] & \operatorname{Spec} C \arrow[d] \\ \underline{Y}: & \operatorname{Spec} B_0 \arrow[r, hook] \arrow[d] & \operatorname{Spec} B \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd} \tag{27}$$

and the upper square is cartesian.

Proposition 2.1.5. *In the diagram above,* $\operatorname{Spec} C \to \operatorname{Spec} B$ *is flat if and only if* $\operatorname{Spec} C_0 \to \operatorname{Spec} B_0$ *is flat.*

This construction can be globalized. Given a morphism of schemes $X \to Y$ we can define $\underline{\mathrm{Exal}}_Y(X, \mathcal{I})$ as the Picard category

$$\underline{\mathrm{Exal}}_Y(X, \mathcal{I}) = \underline{\mathrm{Exal}}_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{I}) \tag{28}$$

2.2 Proof of the Fundamental Theorem

The goal of this section is to proof Illusies "Fundamental Theorem", which provides a natural isomorphism $\mathrm{Exal}_A(C, I) \cong \mathrm{Ext}^1_A(L_{C/A}, I)$. This isomorphism provides a connection between the deformations of a A-algebra C, and its geometry encoded in the cotangent complex. More precisely, we will proof the following theorem in the case of affine schemes.

Theorem 2.2.1. *(Illusie's Fundamental Theorem) There exists a natural isomorphism between* $\mathrm{Exal}_Y(X, M)$, *equipped with the standard group structure, and the group* $\mathrm{Ext}^1_X(L_{X/Y}, M)$.

Recall that by an extension A-algebras $\underline{X}$, we mean a commutative diagram

$$\underline{X} = \begin{tikzcd} E \arrow[r, two heads, "p"] & B \\ A \arrow[u] \arrow[ur] & \end{tikzcd} \tag{29}$$

such that p is surjective, and has as kernel an square zero ideal I. There is a canonical B-module structure induced on I by p. We should also recall that for every extension of A-algebras $\underline{\mathrm{X}}$ there exists a functorially associated exact sequence

$$\mathrm{diff}(\underline{\mathrm{X}}) = I \longrightarrow \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0 \tag{30}$$

This associated exact sequence $\mathrm{diff}(\underline{\mathrm{X}})$, and the fact that $\mathrm{Ext}^1_C(\Omega_{C/A}, I)$ classifies isomorphism classes of short exact sequences of A-modules of the form

$$0 \longrightarrow I \longrightarrow K \longrightarrow \Omega_{C/A} \longrightarrow 0 \tag{31}$$

provides our connection between the groups we are trying to identify. It may be worth noting that the result about Ext^1_C classifying short exact sequences, is completely formal, and works in the generality of abelian categories with enough injectives and projectives.

Unfortunately, the exact sequence $\mathrm{diff}(\underline{\mathrm{X}})$ is not short exact, however, in good situations, the exact sequence $\mathrm{diff}(\underline{\mathrm{X}})$ is short exact.

Definition 2.2.2. We say that an A-algebra C satisfy condition (L) if for every A-extension $\underline{\mathrm{X}}$, the exact sequence $\mathrm{diff}(\underline{\mathrm{X}})$ is short exact.

We will first proof Illusie's fundamental Theorem for the case when C satisfies condition (L), and then we will use the power of homotopy theory to reduce the general problem to this well behaved case. As an example of A-algebras that satisfy (L), we recall the following result

Proposition 2.2.3. *If C is a smooth A-algebra, then it satisfies condition (L).*

This result is a direct consequence of the properties of the cotangent complex listed above. From the work done above, we have that of C satisfies (L), then we have a morphism of groups

$$\mathrm{diff} : \mathrm{Exal}_A(C, I) \longrightarrow \mathrm{Ext}^1_C(\Omega_{C/A}, I) \tag{32}$$

Moreover, this morphism is natural in I.

Next, we will define the inverse of diff. Let C be an A-algebra, and let

$$\underline{\mathrm{Y}} = 0 \longrightarrow I \xrightarrow{\ i\ } J \xrightarrow{\ f\ } \Omega_{C/A} \longrightarrow 0 \tag{33}$$

be a short exact sequence of C-modules. And define

$$\underline{\mathrm{Y}}' = 0 \longrightarrow I \xrightarrow{0 \oplus i} C \oplus J \xrightarrow{\mathrm{Id} \oplus f} C \oplus \Omega_{C/A} \longrightarrow 0 \tag{34}$$

and define $\mathrm{alg}(\underline{\mathrm{Y}})$ to be

$$\begin{tikzcd}
\mathrm{alg}(\underline{\mathrm{Y}}) = 0 \arrow[r] & I \arrow[r] \arrow[d, "\mathrm{Id}"'] & C \times_{(C \oplus \Omega_{C/A})} (C \oplus J) \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\mathrm{Id} \oplus d_{C/A}"] & 0 \\
\underline{\mathrm{Y}}' = 0 \arrow[r] & I \arrow[r] & C \oplus J \arrow[r] & C \oplus \Omega_{C/A} \arrow[r] & 0
\end{tikzcd} \tag{35}$$

It may be worth pointing out that both squares in the diagram above are cartesian, and that $\mathrm{alg}(\underline{\mathrm{Y}})$ is exact. This is a simple exercise in abstract nonsense. We claim that this defines a morphism of groups

$$\mathrm{alg} : \mathrm{Ext}^1_C(\Omega_{C/A}, I) \longrightarrow \mathrm{Exal}_A(C, I) \tag{36}$$

which is natural in I. For this, it suffices to show that $C \times_{(C\oplus\Omega_{C/A})} (C \oplus J)$ is an A-algebra. Note that $C \oplus J$ and $C \oplus \Omega_{C/A}$ are A-algebras, with the A-algebra structure induced by the map structure map $A \to C$. This construction assignment resembles the map

$$\mathrm{Mod}_C \hookrightarrow \mathrm{CRing}_{A//C} \tag{37}$$

defined above.

Proposition 2.2.4. *Let C be an A-algebra satisfying condition (L). The morphisms*

$$\mathrm{diff} : \mathrm{Exal}_A(C, I) \longrightarrow \mathrm{Ext}^1_C(\Omega_{C/A}, I) \qquad \textit{and} \qquad \mathrm{alg} : \mathrm{Ext}^1_C(\Omega_{C/A}, I) \longrightarrow \mathrm{Exal}_A(C, I) \tag{38}$$

defined above are inverses of each other.

Let $A \to B$ be any morphism of rings, and let M be a B-module. We want to stablish a natural isomorphism

$$\mathrm{Exal}_A(B, M) \cong \mathrm{Ext}^1_B(L_{B/A}, M) \tag{39}$$

Recall that $L_{B/A}$ is an element of the derived category $D(B)$, however, in order to proof this theorem, we will make an auxiliary choice of a representative element of $L_{B/A}$ in the category of simplicial B-modules. We will abuse notation and denote this object by $L_{B/A}$ as well. Consider a commutative diagram

$$\begin{tikzcd} P \arrow[r, "p"] & B \\ & A \arrow[ul] \arrow[u] \end{tikzcd} \tag{40}$$

where $p : P \to B$ is a cofibrant replacement of B. Recall that

$$L_{B/A} \cong \Omega_{P/A} \otimes_P B \tag{41}$$

in the category $D(B)$. It may be worth pointing out that since $\Omega_{P/A}$ is a cofibrant object in Mod_P, there is not ambiguity in the distinction between the derived tensor product and the ordinary tensor product. In fact, the tensor product defines a Quillen equivalence

$$- \otimes_P B : \mathrm{Mod}_P \rightleftarrows \mathrm{sMod}_B \tag{42}$$

This is a direct consequence of the fact that $X \otimes_P B \to X$ is a weak equivalence. This is useful because now we have a canonical isomorphism

$$\mathrm{Ext}^1_P(\Omega_{P/A}, M) \xrightarrow{\cong} \mathrm{Ext}_B(L_{B/A}, M) \tag{43}$$

which is natural on all its entries.

Therefore, in order to proof Illusies Fundamental Theorem, it suffices to define a natural isomorphism

$$\alpha : \mathrm{Exal}_A(B, M) \longrightarrow \mathrm{Ext}^1_P(\Omega_{P/A}, M) \tag{44}$$

Let $\alpha(\underline{X}) := \mathrm{diff}(\underline{p}^* X)$, where p^*X is defined by

$$\begin{tikzcd} p^*X = 0 \arrow[r] & M \arrow[r] \arrow[d] & P \times_B K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "p"] & 0 \\ X = 0 \arrow[r] & M \arrow[r] & K \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{45}$$

Moreover, α is natural on M.

Theorem 2.2.5. *(Illusie's Fundamental Theorem) The natural morphism*

$$\alpha : \mathrm{Exal}_A(B, M) \longrightarrow \mathrm{Ext}^1_P(\Omega_{P/A}, M) \cong \mathrm{Ext}^1_B(L_{B/A}, M) \tag{46}$$

is an isomorphism.

Proof. We see that by definition α factors as

$$\begin{tikzcd} \mathrm{Exal}_A(B, M) \arrow[r, "\alpha"] \arrow[d, "p^*"'] & \mathrm{Ext}^1_P(\Omega_{P/A}, M) \\ \mathrm{Exal}_A(P, M) \arrow[ur, "\cong"'] & \end{tikzcd} \tag{47}$$

And the isomorphism follows because P satisfies condition (L), as $A \to P$ is a smooth morphism. Therefore it suffices to show that $p^* : \mathrm{Exal}_A(B, M) \to \mathrm{Exal}_A(P, M)$ is an isomorphism.

We will show that p^* has an inverse which we will denote by $H_0 : \mathrm{Exal}_A(P, M) \to \mathrm{Exal}_A(B, M)$. Given an $\underline{\mathrm{X}} \in \mathrm{Exal}_A(P, M)$, we can take its homology groups and it would generate a short exact sequence

$$H_0(\underline{\mathrm{X}}) = 0 \longrightarrow M \longrightarrow K \longrightarrow B \longrightarrow 0 \tag{48}$$

of A-modules. Short exactness follows from the fact that all higher homology groups of P vanish, as it is weak equivalent to B. Moreover, we should point out that H_0 of an A-algebra inherits its A-algebra structure. Therefore $H_0(\underline{\mathrm{X}}) \in \mathrm{Exal}_A(B, M)$.

First, we will show that $H_0 \circ p^* = \mathrm{Id}$. For this consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\mathrm{Id}"'] & P \times_B K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "p"] & 0 \\ 0 \arrow[r] & M \arrow[r] & K \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{49}$$

As trivial fibrations are closed under base change, it follows that $P \times_B K \to K$ is a trivial fibration. It follows that $H_0 \circ p^* = \mathrm{Id}$. On the other hand, we can consider the following commutative diagram

$$\begin{tikzcd} \underline{\mathrm{X}} \arrow[d] & 0 \arrow[r] & M \arrow[r] \arrow[d, "\mathrm{Id}"'] & L \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ p^* H_0(\underline{\mathrm{X}}) \arrow[d] & 0 \arrow[r] & M \arrow[r] \arrow[d, "\mathrm{Id}"'] & P \times_B K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "p"] & 0 \\ H_0(\underline{\mathrm{X}}) & 0 \arrow[r] & M \arrow[r] & K \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{50}$$

A couple of words about this diagram are in order. A priori, one only the morphisms $\underline{\mathrm{X}} \to H_0(\underline{\mathrm{X}})$ and $p^* H_0(\underline{\mathrm{X}}) \to H_0(\underline{\mathrm{X}})$. However, the morphism $\underline{\mathrm{X}} \to H_0(\underline{\mathrm{X}})$ factors through $p^* H_0(\underline{\mathrm{X}})$ by the universal property of $P \times_B K$. And finally, by the five-lemma we can conclude that $\underline{\mathrm{X}} \cong p^* H_0(\underline{\mathrm{X}})$. We have shown that $p^* \circ H_0 = \mathrm{Id}$.

□

2.3 Deformation of Schemes and Morphisms

In this section we will use the machinery develop in the previous section to answer questions about the deformation of schemes. More precisely, given the following map of schemes

$$X_0 \xrightarrow{f} Y_0 \longrightarrow Z \tag{51}$$

we will provide a cohomological criterion, which is necessary and sufficient, for the existence of the deformation of X_0, over an square zero extension $Y_0 \hookrightarrow Y$. In other words, we want to find a morphism $X_0 \hookrightarrow X$ such that we have the following commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d] \\ Y_0 \arrow[r, hook] \arrow[d] & Y \arrow[dl] \\ Z & \end{tikzcd} \tag{52}$$

where the upper square is cartesian. Moreover, we will provide a complete description of the category of such deformations. More precicelly, we will proof the following theorem in the case of affine schemes

Theorem 2.3.1. *Given some morphisms of schemes*

$$X_0 \xrightarrow{f} Y_0 \longrightarrow Z \tag{53}$$

and $\underline{Y} \in \mathrm{Exal}_Z(Y_0, f_*\mathcal{I})$. *We can say the following:*

(1) There exists a deformation of $X_0 \hookrightarrow X$, *defined by the square zero ideal* $\mathcal{I}$, *over* $\underline{Y} : Y_0 \hookrightarrow Y$ *if and only if*

$$\partial(\underline{Y}) \in \mathrm{Ext}^2_{X_0}(L_{X_0/Y_0}, \mathcal{I}) \tag{54}$$

vanishes.

(2) If $\partial(\underline{Y})$ *vanishes, then the isomorphism classes of deformations* $X_0 \hookrightarrow X$, *defined by the square zero ideal* $\mathcal{I}$, *form a torsor under* $\mathrm{Exal}_{Y_0}(X_0, \mathcal{I}) \cong \mathrm{Ext}^1_{X_0}(L_{X_0/Y_0}, \mathcal{I})$.

(3) The group of automorphisms of a deformation $X_0 \hookrightarrow X$, *defined by the square zero ideal* $\mathcal{I}$, *is canonically identified with* $\mathrm{Ext}^0_{X_0}(L_{X_0/Y_0}, \mathcal{I})$

Since the category of such deformations form a groupoid, we have provided a complete description if the category of deformations of X_0 *over* $Y_0 \hookrightarrow Y$.

For this, we will use in an essential way Illusie's Fundamental Theorem, together with the theory of the Cotangent Complex. We will only provide a proof for the affine case. The biggest challenge will be to reinterpret the problem in more tractable terms; otherwise, most of this section is purely formal. Consider the maps of rings

$$A \longrightarrow B_0 \xrightarrow{f} C_0 \tag{55}$$

Let $\underline{\mathrm{Y}} \in \mathrm{Exal}_A(B_0, I)$, we claim that finding an $\underline{\mathrm{X}} \in \mathrm{Exal}_A(C_0, I)$ such that $f^*\underline{\mathrm{X}} = \underline{\mathrm{Y}}$ is equivalent to finding a deformation of $\operatorname{Spec} C_0$ over $\operatorname{Spec} B_0 \hookrightarrow \operatorname{Spec} B$. To see this, consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\mathrm{Id}"] & C \times_{C_0} B_0 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I \arrow[r] & C \arrow[r] & C_0 \arrow[r] & 0 \end{tikzcd} \tag{56}$$

We can see that the square

$$\begin{tikzcd} C \times_{C_0} B_0 \arrow[r] \arrow[d] & B_0 \arrow[d, "f"] \\ C \arrow[r] & C_0 \end{tikzcd} \tag{57}$$

is a pushout if and only if it is a pullback, without knowing a priori that $C \times_{C_0} B_0$ is a cartesian product. Indeed, if it is a pullback, it follows that it is a pushout from $C \to C_0$ being surjective. And if it is a pushout, it follows that it is a pullback from $I \cap \ker(C \times_{C_0} B_0 \to C) = \emptyset$. In more geometric terms, we have shown that the commutative diagram of square zero extensions

$$\begin{tikzcd} \operatorname{Spec} C_0 \arrow[r, hook] \arrow[d] & \operatorname{Spec} C \arrow[d] \\ \operatorname{Spec} B_0 \arrow[r, hook] & \operatorname{Spec} B \end{tikzcd} \tag{58}$$

is a pullback if and only it is a pushout. We have shown that finding a deformation of $\operatorname{Spec} C_0$ over $\operatorname{Spec} B_0 \hookrightarrow \operatorname{Spec} B$ amounts to finding an $\underline{\mathrm{X}}$ such that $f^*\underline{\mathrm{X}} = \underline{\mathrm{Y}}$. In other words, it amounts to understanding whether $\underline{\mathrm{X}}$ is in the image of

$$f^* : \operatorname{Exal}_A(C_0, I) \longrightarrow \operatorname{Exal}_A(B_0, I) \tag{59}$$

or not. Finally, we see that $\underline{\mathrm{X}}$ is canonically an element of $\operatorname{Exal}_B(B_0, I)$, and that $\underline{\mathrm{X}}$ is in the image of

$$f^* : \operatorname{Exal}_B(C_0, I) \longrightarrow \operatorname{Exal}_B(B_0, I) \tag{60}$$

if and only if it is in the image of $f^* : \operatorname{Exal}_A(C_0, I) \to \operatorname{Exal}_A(B_0, I)$. This final reduction step is not necessary to formulate the cohomological criterion for the existence of the deformation; however, it is necessary in order to provide a full description of the category of deformations.

We will now use the machinery we have develop so far to provide a simple solution to this problem. Recall that to $B \to B_0 \to C_0$, there is an associated exact triangle

$$f^* L_{B_0/B} \longrightarrow L_{C_0/B} \longrightarrow L_{C_0/B_0} \tag{61}$$

Where we can apply the right derived functor $R\operatorname{Hom}_{C_0}(-, I)$, to obtain the long exact sequence of Ext groups, namely

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ext}^0_{C_0}(L_{C_0/B_0}, I) \arrow[r] & \operatorname{Ext}^0_{C_0}(L_{C_0/B}, I) \arrow[r] & \operatorname{Ext}^0_{C_0}(f^* L_{B_0/B}, I) \arrow[dll] \\ & \operatorname{Ext}^1_{C_0}(L_{C_0/B_0}, I) \arrow[r] & \operatorname{Ext}^1_{C_0}(L_{C_0/B}, I) \arrow[r] & \operatorname{Ext}^1_{C_0}(f^* L_{B_0/B}, I) \arrow[dll] \\ & \operatorname{Ext}^2_{C_0}(L_{C_0/B_0}, I) \arrow[r] & \operatorname{Ext}^2_{C_0}(L_{C_0/B}, I) \arrow[r] & \operatorname{Ext}^2_{C_0}(f^* L_{B_0/B}, I) \end{tikzcd} \tag{62}$$

It may be worth pointing out that while I is a C_0-module, it can be considered a B_0-module through the map $f : B_0 \to C_0$. By Illusie's Fundamental Theorem we have the identifications

$$\operatorname{Exal}_B(C_0, I) \cong \operatorname{Ext}^1_{C_0}(L_{C_0/B}, I) \qquad \operatorname{Exal}_B(B_0, I) \cong \operatorname{Ext}^1_{B_0}(L_{B_0/B}, I) \tag{63}$$

And by the tensor-hom Quillen adjunction

$$\operatorname{sMod}_{C_0} \leftrightarrows \operatorname{sMod}_{B_0} : - \otimes_{B_0} C_0 \tag{64}$$

we conclude that $\operatorname{Ext}^1_{C_0}(f^* L_{B_0/B}, I) \cong \operatorname{Ext}^1_{B_0}(L_{B_0/B}, I)$. To summarize we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Exal}_B(C_0, I) \arrow[r, "f^*"] \arrow[dd, "\cong"'] & \mathrm{Exal}_B(B_0, I) \arrow[d, "\cong"] \\
& \mathrm{Ext}^1_{B_0}(L_{B_0/B}, I) \arrow[d, "\cong"] \\
\mathrm{Ext}^1_{C_0}(L_{C_0/B}, I) \arrow[r] & \mathrm{Ext}^1_{C_0}(f^* L_{B_0/B}, I)
\end{tikzcd} \tag{65}$$

where commutativity can by checked by unwrapping the definition of the morphisms; note that all we have done here is very functorial.

We have the following long exact sequence

$$\mathrm{Ext}^1_{C_0}(L_{C_0/B_0}, I) \cong \mathrm{Exal}_{B_0}(C_0, I) \longrightarrow \mathrm{Exal}_B(C_0, I) \xrightarrow{f^*} \mathrm{Exal}_B(B_0, I) \xrightarrow{\partial} \mathrm{Ext}^2_{C_0}(L_{C_0/B_0}, I) \tag{66}$$

It is now easy to see that given an $\underline{\mathrm{Y}} \in \mathrm{Exal}_B(B_0, I)$, there exists a $\underline{\mathrm{X}} \in \mathrm{Exal}_B(C_0, I)$ such that $f^*\underline{\mathrm{X}} = \underline{\mathrm{Y}}$ if and only if $\partial(\underline{\mathrm{Y}}) = 0$ in $\mathrm{Ext}^2_{C_0}(L_{C_0/B}, I)$. We record our result as part of the following theorem.

Theorem 2.3.2. *Given some morphisms of rings*

$$A \longrightarrow B_0 \longrightarrow C_0 \tag{67}$$

and $\underline{Y} \in \mathrm{Exal}_A(B_0, I)$*, where* I *is a* C*-module considered as a* B_0*-module. We can say the following:*

(1) There exists a deformation $\operatorname{Spec} C_0 \hookrightarrow \operatorname{Spec} C$*, defined by the square zero ideal* I*, over* $\underline{Y} : \operatorname{Spec} B_0 \hookrightarrow \operatorname{Spec} B$ *if and only if*

$$\partial(\underline{Y}) \in \mathrm{Ext}^2_{C_0}(L_{C_0/B_0}, I) \tag{68}$$

vanishes.

(2) If $\partial(\underline{Y})$ *vanishes, then the isomorphism classes of deformations* $\operatorname{Spec} C_0 \hookrightarrow \operatorname{Spec} C$*, defined by the square zero ideal* I*, form a torsor under* $\mathrm{Exal}_B(C_0, I) \cong \mathrm{Ext}^1_{C_0}(L_{C_0/B_0}, I)$.

(3) The group of automorphisms of a deformation $\operatorname{Spec} C_0 \hookrightarrow \operatorname{Spec} C$*, defined by the square zero ideal* I*, is canonically identified with* $\mathrm{Der}_{B_0}(C_0, I) \cong \mathrm{Ext}^0_{C_0}(L_{C_0/B_0}, I)$

Since the category of such deformations form a groupoid, we have provided a complete description if the category of deformations of $\operatorname{Spec} C_0$ *over* $\operatorname{Spec} B_0 \hookrightarrow \operatorname{Spec} B$.

Proof. Part (1) of the theorem has already been proven, so we proceed with (2). Recall we have the following exact sequence

$$\mathrm{Ext}^0_{C_0}(f^* L_{B_0/B}, I) \longrightarrow \mathrm{Exal}_{B_0}(C_0, I) \longrightarrow \mathrm{Exal}_B(C_0, I) \xrightarrow{f^*} \mathrm{Exal}_B(B_0, I) \xrightarrow{\partial} \mathrm{Ext}^2_{C_0}(L_{C_0/B_0}, I) \tag{69}$$

In order to proof (2) it suffices to show that $\mathrm{Ext}^0_{C_0}(f^* L_{B_0/B}, I) = 0$. Recall that $\mathrm{Ext}^0_{C_0}(f^* L_{B_0/B}, I) \cong \mathrm{Ext}^0_{B_0}(L_{B_0/B}, I)$ by the tensor-hom adjunction, and $H_0(L_{B/A}) = \Omega_{B_0/B} = 0$. We have the following sequence of isomorphisms

$$\mathrm{Ext}^0_{B_0}(L_{B_0/B}, I) \cong [L_{B_0/B}, I] \cong [H_0(L_{B_0/B}), I] = 0 \tag{70}$$

where the last isomorphism follows from the fact that I is concentrated in degree one. This proves (2).

In order to proof (3), we first consider a similar sequence of isomorphisms

$$\mathrm{Ext}^0_{C_0}(L_{C_0/B_0}, I) \cong [L_{C_0/B_0}, I] \cong [H_0(L_{C_0/B_0}), I] \cong Hom_{C_0}(\Omega_{C_0/B_0}, I) \cong \mathrm{Der}_{B_0}(C_0, I) \tag{71}$$

and since $\mathrm{Ext}^0_{C_0}(f^* L_{B_0/B}, I) = 0$ we get an isomorphism

$$\mathrm{Der}_{B_0}(C_0, I) \cong \mathrm{Ext}^0_{C_0}(L_{C_0/B}, I) \cong \mathrm{Hom}_{C_0}(\Omega_{C_0/B}, I) \tag{72}$$

We have reduced the problem to showing that $\mathrm{Hom}_{C_0}(\Omega_{C_0/B}, I)$ determines the automorphisms of an extension. The key insight here is that the category of deformations of $\mathrm{Spec}\, C_0$ over $\mathrm{Spec}\, B_0 \hookrightarrow \mathrm{Spec}\, B$ is a full subcategory of $\underline{\mathrm{Exal}}_B(C_0, I)$, which in a Picard groupoid. The additive structure on $\underline{\mathrm{Exal}}_B(C_0, I)$ allows us to determine that every connected component of $\underline{\mathrm{Exal}}_B(C_0, I)$ is equivalent. Therefore it suffices to show that there is a canonical isomorphism between $\mathrm{Hom}_{C_0}(\Omega_{C_0/B}, I)$ and the automorphisms of the identity element of $\underline{\mathrm{Exal}}_B(C_0, I)$. Recall that the identity element is $C_0 \oplus I$, and its automorphisms are exactly

$$\mathrm{Hom}_{\mathrm{CRing}_{B//C_0}}(C_0 \oplus I, C_0 \oplus I) \cong \mathrm{Hom}_{C_0}(\Omega_{C_0/B}, I) \tag{73}$$

where the last isomorphism follows by the adjunction in of the Kahler differentials. The result follows.

□

We will now investigate the question of deformation of morphisms of schemes. Given the following morphisms of S-schemes

$$X_0 \xrightarrow{f} Y_0 \xrightarrow{g} Z_0 \tag{74}$$

a $\mathcal{O}_{X_0}$-module $\mathcal{I}$, a S-extension $\underline{Z} : Z_0 \hookrightarrow Z$ by $\mathcal{I}$, and deformations $\underline{X} : X_0 \hookrightarrow X$ and $\underline{Y} : Y_0 \hookrightarrow Y$ over $Z_0 \hookrightarrow Z$, both by the square zero ideal $\mathcal{I}$. We would like to know weather we can present $\underline{X} : X_0 \hookrightarrow X$ as a deformation over $\underline{Y} : Y_0 \hookrightarrow Y$. In other words, we are looking for a morphism $X \to Y$, such that the following diagram is commutative

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d, "f"] \arrow[dd, bend right] & X \arrow[d, dashed, "f'"'] \arrow[dd, bend left] \\
Y_0 \arrow[r, hook] \arrow[d, "g"] & Y \arrow[d] \\
Z_0 \arrow[r, hook] \arrow[d] & Z \arrow[dl] \\
S &
\end{tikzcd} \tag{75}$$

and such that the upper square is cartesian. In this case, we say that f' is a deformation of f over $Y_0 \hookrightarrow Y$. We will provide a cohomological criterion for the existence of a deformation of f over $Y_0 \hookrightarrow Y$. More precisely we will proof the following theorem

Theorem 2.3.3. *In the situation described above, there exists a deformation of f over $Y_0 \hookrightarrow Y$ if and only if*

$$0 = f^*(\underline{X}) - \underline{Y} \in \mathrm{Exal}_Y(Y_0, f_*\mathcal{I}) \cong \mathrm{Ext}^1_{X_0}(f^* L_{Y_0/Y}, \mathcal{I}) \tag{76}$$

where $f^* : \mathrm{Exal}_Y(X_0, I) \to \mathrm{Exal}_Y(Y_0, I)$.

The proof of this result, in the affine case, is an easy application of the theory developed so far. So we will leave it as an excercise to the reader.

3 Application to Abelian Varieties

In this section we will use all the heavy machinery we have used to proof some facts about the moduli of elliptic curves. In particular, we compute the tangent space of the moduli space of elliptic curves $\mathcal{M}_{1,1}$, at its rational points. Recall that under the functor of points perspective we have that $\mathcal{M}_{1,1}(S)$ is the set of isomorphism classes of maps $E \to S$ such that it is flat, proper, and each fiber is an elliptic curve. Moreover, recall that a map $S' \to S$ induces a map $\mathcal{M}_{1,1}(S) \to \mathcal{M}_{1,1}(S')$ which is induced by the following cartesian diagram

$$\begin{tikzcd} E \times_{S'} S \arrow[r] \arrow[d] & E \arrow[d] \\ S \arrow[r] & S' \end{tikzcd} \tag{77}$$

it is standard to check that $E \times_{S'} S$ satisfies the required hypothesis. Recall that the tangent space of $\mathcal{M}_{1,1}$ at a point $\operatorname{Spec} k \hookrightarrow \mathcal{M}_{1,1}$ correspond to lifts of the map

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d, hook] & \mathcal{M}_{1,1} \\ \operatorname{Spec} k[\varepsilon] \arrow[ur, dashed] & \end{tikzcd} \tag{78}$$

Therefore to compute its tangent space it amounts to computing the first order deformations of E/k, that is, deformations over $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]$. Our goal is to compute the tangent space at every field valued point $\mathcal{M}_{1,1}(k)$.

Theorem 3.0.1. *The tangent space of $\mathcal{M}_{1,1}$ at a point $\mathcal{M}_{1,1}(k)$ is one dimensional.*

Proof. Let E/k be an elliptic curve. Our goal is to classify the first order deformations of E/k, in other words, we want to classify the deformations of E over $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]$. First, we see that elliptic curves can always be deformed since $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]$ has a section, this follows from $\operatorname{Ext}^1_k(k, [\varepsilon]) = 0$. In other words we have the following diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E \times_k k[\varepsilon] \arrow[r] \arrow[d] & E \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\varepsilon] \arrow[r] & \operatorname{Spec} k \end{tikzcd} \tag{79}$$

A priori we have that the outer and right most squares are cartesian, it follows that the left most square is cartesian. It follows that we can always deform E/k over $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]$.

We will now try to classify this first order deformations, in order to understand the tangent space of $\mathcal{M}_{1,1}$ better. First, we note that given the flatness of E/k the map $E \hookrightarrow E \times_k k[\varepsilon]$ is an extension by $\mathcal{O}_E$. So we can now use our machinery to classify this deformations. We have that the isomorphism classes of first order deformations form a torsor under $\operatorname{Ext}^1_E(L_{E/k}, \mathcal{O}_E)$. In order to compute this Ext group, we will use the sheaf $\underline{\operatorname{Ext}}_E(L_{E/k}, \mathcal{O}_E)$, and use the Leray spectral sequence that says

$$E_2^{p,q} = H^p(X, \underline{\operatorname{Ext}}^q_E(L_{E/k}, \mathcal{O}_E)) \Longrightarrow \operatorname{Ext}^{p+q}_E(L_{E/k}, \mathcal{O}_E) \tag{80}$$

we use the procedure because it is easier to compute $\underline{\operatorname{Ext}}$, since we understand the local behavior of this problem quiet well. Since E is smooth, $L_{E/k} \cong \Omega_{E/k}$ is a cofibrant object, hence $\underline{\operatorname{Ext}}^q_E(\Omega_{E/k}, \mathcal{O}_E) = 0$ for $q > 0$. As a result we get that

$$H^p(X, \underline{\operatorname{Ext}}^0_E(\Omega_{E/k}, \mathcal{O}_E)) \cong \operatorname{Ext}^p_E(\Omega_{E/k}, \mathcal{O}_E) \tag{81}$$

Moreover, we know that $\Omega_{E/k} \cong \mathcal{O}_E$, and therefore, $H^p(X, \mathcal{O}_E) \cong \operatorname{Ext}^p(\Omega_{E/k}, \mathcal{O}_E)$. In particular,

$$k \cong H^1(X, \mathcal{O}_E) \cong \operatorname{Ext}^1(\Omega_{E/k}, \mathcal{O}_E) \tag{82}$$

where the first isomorphism is a well known fact about elliptic curves. We have shown that the tangent space at a point $\mathcal{M}_{1,1}(k)$ is one dimensional. $\square$

A lot more could be said about deformations of abelian varieties. In fact, any abelian variety A/S, where $S \in \mathrm{Art}_k$ is unobstructed, in the sense that it can always be deformed over $\operatorname{Spec} S \hookrightarrow \operatorname{Spec} S'$, where S' is an square zero extension of S. However, showing that the obstruction classes vanish requires some work.

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