July 26

Tannaka Duality (B. Bhatt)

From the paper Algebraization and Tannaka Duality.

thm A $X$ (always qcqs) scheme

(a) $A$ ring, $I$ ideal, $A$ is $I$-adically complete.

$$X(A) \longrightarrow \varprojlim X(A/I^n)$$

this is an equivalence.

Exc proof when $A$ local ~~and~~ or $X$ affine.

thm B $\{A_i\}$ set of rings, then

$$X(\prod A_i) \xrightarrow{\simeq} \prod X(A_i)$$

thm C $\pi: Y \to X$ map of schemes,
$i: Z \hookrightarrow X$ closed immersion (f.p.)

Assume:
$$\begin{array}{ccc} Z \times_X Y & \longrightarrow & Y \\ \simeq \downarrow & \lrcorner & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

then $X = \mathrm{colim}\left(\begin{array}{ccc} \pi^{-1}(X-Z) & \hookrightarrow & Y \\ \downarrow & & \\ X-Z & & \end{array}\right)$

Rem (1) Typical example of (c)

$X = \operatorname{Spec} A$ $\quad Y = \operatorname{Spec}(\hat{A})$
where $\hat{A}$ is the derived completion.

$Z = V(I)$ (Bousfield loc, then its always exact)

2) $X$ scheme

$J_n X =$ $n^{th}$ jet space of $X$

$(J_n X)(R) = \operatorname{Hom}(\operatorname{Spec} R[[t]]/t^n, X)$

have maps $J_{n+1}(X) \longrightarrow J_n(X)$

$\longrightarrow J_\infty(X) = \varprojlim J_n(X)$ by Reason A. (?)

~~this thing are~~ n this inverse limits are not "finite".

II) the classical setting

Fix $X$, $A = \varprojlim A/I^n$

Goal $X(A) \xrightarrow{\simeq} \varprojlim X(A/I^n)$ when $X$ is proj over a field $k$.

Affine Just by cont of $\operatorname{Hom}(-, X)$

Key part: $X$ has "enough functions."

Step 2 $\mathbb{P}^n(A) = \left\{ (L, s_0, \dots, s_n) \mid \begin{array}{l} L \in \operatorname{Pic}(A) \text{ which gen} \\ L \text{ as an } A\text{-module} \end{array} \right\}$

Fact $\operatorname{Pic}(A) \xrightarrow{\simeq} \varprojlim \operatorname{Pic}(A/I^n)$

and similarly for vector bundles.

Surjectivity on $\operatorname{Pic}(A) \iff$ surjectivity on $\operatorname{Pic}(A/I^n)$.

$E \to F$ map in $\operatorname{Vect}(A)$, then $E \to F$ surjective

$\iff E/I^nE \longrightarrow F/I^nF$ is surjective

for all / some $n > 0$.

Exc (1) + (2) $\Rightarrow \mathbb{P}^n(A) \simeq \varprojlim \mathbb{P}^n(A/I^n)$.

**Step 3** X projective (exercise).

**Key point**: (1) $\mathrm{Vect}(A) \simeq \varprojlim \mathrm{Vect}(A/I^n)$.

projective space has enough vector bundles to detect maps to it.

$$\mathrm{Hom}(S, \mathbb{P}^n) = \mathrm{Hom}^{ex}_{\otimes}(\mathrm{Vect}(\mathbb{P}^n), \mathrm{Vect}(S))$$

## III | Tannaka Duality

Let $X$ be a qcqs scheme.

$\rightsquigarrow$ $\mathrm{QCoh}(X)$ or $D(X)$

$$\mathrm{QCoh}(X) = \lim_{\mathrm{Spec} A \to X} \mathrm{QCoh}(A)$$

$\mathrm{QCoh}(X)$ is a symmetric monoidal $\infty$-category stable and presentable.

We will be working with

$\mathrm{QCoh}^{perf}(X)$ on an affine cover $K$
is finite complex of finite proj modules.
presented $\longleftrightarrow$ type $\rightsquigarrow$ So this projective complex.

**Ex** if $X$ smooth variety (separate)

$D(X) = D(\mathrm{QCoh}(X))$ (separated)
$D_{perf}(X) = D^b(\mathrm{Coh}(X))$ (smooth)

**thm B** $X$ is qcqs

$$\mathrm{Hom}(S, X) \simeq \mathrm{Fun}^{ex}_{\otimes}(D_{perf}(X), D_{perf}(S))$$
$$\simeq \mathrm{Fun}^{cart}_{\otimes}(D(X), D(S)) \text{ by ind completing.}$$

**thm B $\Rightarrow$ Thm A**

$X$ qcqs, $A$ ring, $A = \lim A/I^n$

**dos** $D_{perf}(A) \xrightarrow{\simeq} \varprojlim D_{perf}(A/I^n)$ only true for perfect

$$X(A) = \mathrm{Funct}^{ex}_{\otimes}(D_{perf}(X), D_{perf}(A))$$
$$= \mathrm{Funct}^{ex}_{\otimes}(D_{perf}(X), \varprojlim D_{perf}(A/I^n))$$
$$= \varprojlim \mathrm{Funct}^{ex}_{\otimes}(D_{perf}(X), D_{perf}(A/I^n))$$
$$= \varprojlim X(A/I^n).$$

Sketch of proof of thm B

$X$ cony qcqs scheme (classically).

$$\left\{ \begin{matrix} \mathcal{U} \subset X \text{ open} \\ j \text{ affine} \end{matrix} \right\} \longrightarrow \{ \mathrm{CAlg}(\mathrm{QCoh}(X)) \}$$

$$j : \mathcal{U} \to X \longmapsto j_* \mathcal{O}_{\mathcal{U}}.$$

in DAG

$$\left\{ \begin{matrix} \mathcal{U} \hookrightarrow X \\ \text{qc open} \end{matrix} \right\} \xrightarrow{F} \{ \mathrm{CAlg}(\mathrm{QCoh}(X)) \}$$

$$\mathcal{U} \hookrightarrow X \longmapsto Rj_* \mathcal{O}_{\mathcal{U}}.$$

(1) $F$ is fully faithful (think $\mathbb{A}^2 - \{0\} \hookrightarrow \mathbb{A}^2$)

(2) $\mathrm{Im}(F) = \left\{ A \;\middle|\; \begin{matrix} A \text{ compact object} \\ A \otimes^L_{\mathcal{O}_X} A \xrightarrow{\simeq} A \end{matrix} \right\}$