BER'S EMBEDDING

Aim of the talk

R surface of genus $g \geq 2$., H^* – lower half plane

→ Fuchsian model $\Gamma \subseteq PSL_2(\mathbb{R})$. fixing $0, 1, \infty$.

* ~~Embed Teichmuller~~

* Put a complex structure on Teich(R)
  - → Give a description of ~~Teich(R)~~ T(R) as q.c. maps on $\hat{\mathbb{C}}$, which are conformal on H^*
  - → Use Schwarzian derivatives to embed ~~Teich(R~~ T(R) in a complex vector space of dimension $3g-3$.
  - → Show that the map is ~~top~~ continuous, injective & argue using invariance of domain that it is a homeomorphism onto image
* Prove that the image is a bounded domain
* ~~Show~~ If R_1 is a surface of genus g, $T(R_1) \xrightarrow[\text{bihol}]{\cong} T(R)$

Beltrami Coefficients

$$B(H, \Gamma) = \left\{ \mu \in B(H) \;\middle|\; \mu = (\mu \circ \gamma) \frac{\overline{\gamma'}}{\gamma'}, \forall \gamma \in \Gamma \right\}$$

Want $f \circ \gamma \equiv f$
$\Downarrow$
$\mu_{f \circ \gamma} = \mu_f$

Given $\mu \in B(H, \Gamma)_1$, we can get

- $w^\mu : H \to H$ μ-q.c. map fixing $0, 1, \infty$
- $\tilde{\mu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $\tilde{\mu}(z) = \begin{cases} \mu(z) & z \in H \\ 0 & z \in \mathbb{C} \setminus H \end{cases}$

$\hookrightarrow \omega_\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

ω_μ is $\tilde{\mu}$-q.c. & fixes $0, 1, \infty$.

For $\gamma \in \Gamma$,

$\chi_\mu(\gamma) := \omega_\mu \circ \gamma \circ \omega_\mu^{-1}$.

$\hookrightarrow$ Compare Beltrami coefficients

$\chi_\mu \in \mathrm{Aut}(\hat{\mathbb{C}})$

$\Gamma_\mu = \{\chi_\mu(\gamma) \mid \gamma \in \Gamma\} \subseteq PSL_2(\mathbb{C})$

$\hookrightarrow$ quasi-Fuchsian group (fix a s.c.c.)

~~$\omega_\mu : H/\Gamma \longrightarrow H/\Gamma$~~ $\quad H_\mu = \omega_\mu(H)$, $H^*_\mu = \omega_\mu(H^*)$

$$\left.\begin{aligned} \omega_\mu &: H/\Gamma \xrightarrow[\text{q.c.}]{} H_\mu/\Gamma_\mu \\ &: H^*/\Gamma \xrightarrow[\text{biholo}]{\simeq} H^*_\mu/\Gamma_\mu \end{aligned}\right\} \text{Ber's simultaneous uniformization.}$$

$$\boxed{\begin{array}{l} \text{If } R_1, R_2 \text{ two Riemann surfaces of genus } g, \\ \text{let } f : R_1^* \to R_2^*, \quad \mu = \mu_f, \quad \Gamma \text{ fuchsian model for } R_1^* \\ \omega_\mu : H/\Gamma \longrightarrow H_\mu/\Gamma_\mu \qquad R_2 \xrightarrow[\text{biho}]{\simeq} H_\mu/\Gamma_\mu \\ \omega_\mu : H^*/\Gamma \underset{\cong}{\longrightarrow} H^*_\mu/\Gamma_\mu \qquad R_1 \xrightarrow[\text{biholo}]{\simeq} H^*_\mu/\Gamma_\mu \end{array}}$$

Description of the Teichmuller space

$$T(\Gamma) = \left\{ [S, f] \;\middle|\; f : H/\Gamma \to S \text{ q.c.} \right\} / \sim$$

$$\Updownarrow$$

$$\{ f : H \to H \text{ q.c.} \} / \sim \qquad f_1 \circ \Gamma \circ f_1^{-1} = f_2 \circ \Gamma \circ f_2^{-1}$$

$$\Updownarrow$$

$$\{ \mu \in B(H, \Gamma)_1 \} / \sim \qquad [\mu] = [\nu] \text{ if } \omega^\mu = \omega^\nu \text{ on } \mathbb{R}$$

TFAE

(i) $\omega^\mu = \omega^\nu$ on $\mathbb{R}$

(ii) $\omega_\mu = \omega_\nu$ on H^*

(i) $\Rightarrow$ (ii) $\quad f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$$f(z) = \begin{cases} (\omega^\mu)^{-1} \circ (\omega^\nu)(z) & z \in H \\ z & z \in \hat{\mathbb{C}} \setminus H \end{cases}$$

f is ACL & f is q.c.

$g = w_\mu \circ f \circ (w_\nu)^{-1}$ is 1-q.c. mapping of $\mathbb{C}$

g fixes $0, 1, \infty \Rightarrow g = id$

(ii) $\Rightarrow$ (i)

$w_\mu = w_\nu$ on H^* $\Rightarrow$ $w_\mu = w_\nu$ on $H^* \cup \hat{\mathbb{R}}$.

$h = w^\mu \circ (w_\mu)^{-1} \circ w_\nu \circ (w^\nu)^{-1} : H \longrightarrow H$

h is 1-q.c. $\Rightarrow h = id$

$\Rightarrow$ $w^\mu = w^\nu$ on $\mathbb{R}$

$\cancel{T_\beta(\Gamma) = \{}$

$T(\Gamma) = \{ w^\mu : \mu \in B(H,\Gamma)_1 \} / \sim$ $\quad w^\mu \sim w^\nu$ if $w^\mu|_{\mathbb{R}} = w^\nu|_{\mathbb{R}}$

$\updownarrow$

$T_\beta(\Gamma) = \{ w_\mu : \mu \in B(H,\Gamma)_1 \} / \sim$ $\quad w_\mu \sim w_\nu$ if $w_\mu|_{H^*} = w_\nu|_{H^*}$

Topologize $T_\beta(\Gamma)$

$\beta : B(H,\Gamma)_1 \longrightarrow T_\beta(\Gamma)$

[Can use results about conformal maps on H^*]

Schwarzian derivative

Suppose $[w_\mu]$ is a Möbius transformation, then w_μ leaves $0, 1, \infty$ ~~ag~~ fixed $\Rightarrow [w_\mu] = [id]$

"Measure how much w_μ differs from a Möbius transformation"

Find a differential equation satisfied by Möbius transformations

$$\gamma(z) = \frac{az+b}{cz+d}$$

$$\gamma'(z) = \frac{1}{(cz+d)^2}$$

$$\gamma''(z) = \frac{-2c}{(cz+d)^3}$$

Eliminate 'c'

$$\frac{\gamma'(z)}{\gamma''(z)} = \frac{-z}{2} - \frac{d}{2c}$$

$$\left(\frac{\gamma'(z)}{\gamma''(z)}\right)' = \frac{-1}{2}$$

$$\frac{\gamma'''(z)}{\gamma'(z)} - \frac{3}{2}\left(\frac{\gamma''(z)}{\gamma'(z)}\right)^2 = 0$$

For a conformal map f, define

$$\{f, z\} \cancel{\{f(z)\}} := \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

$\cancel{\{g \circ f, z\}}$

$\{f, \cancel{z\} = 0 \text{ iff } f}$

$$\{f, z\} = \left(\log f'(z)\right)'' - \frac{1}{2}\left(\left(\log f'(z)\right)'\right)^2$$

$\{f, z\} = 0$ iff f is a Möbius transformation

We also have

$$\{g \circ f, z\} = \{g, f(z)\} \cdot f'(z)^2 + \{f, z\}$$

<u>Bers Embedding</u>

Given $\mu \in B(H, \Gamma)_1$, define

$$\varphi_\mu(z) = \{w_\mu, z\} \qquad z \in H^*$$

$\cancel{\varphi_\mu(\gamma(z)) =}$

$$\varphi_\mu(\gamma(z))\, \gamma'(z)^2 = \varphi_\mu(z)$$

$\to$ φ_μ is a holomorphic Automorphic form of weight -4 on H^* for Γ.

$$\Phi : B(H, \Gamma)_1 \longrightarrow A_2(H^*, \Gamma) \qquad \text{[Bers' Projection]}$$
$$\mu \longmapsto \varphi_\mu$$

$$[w_\mu] = [w_\nu] \Rightarrow \varphi_\mu = \varphi_\nu$$

$$\mathcal{B} : T_B(\Gamma) \longrightarrow A_2(H^*, \Gamma) \qquad \text{[Bers' Embedding]}$$

$\boxed{\omega_\mu}$ $\quad \varphi_\mu = \varphi_\nu$

$F : \omega_\mu(H^*) \longrightarrow \omega_\nu(H^*)$

$F = \cancel{\omega_\varphi}\ \omega_\nu \circ (\omega_\mu)^{-1}$

$\Rightarrow \omega_\nu = F \circ \omega_\mu$

$$\{\omega_\nu, z\} = \{F \circ \omega_\mu, z\} = \{F, \omega_\mu(z)\} \cdot (\omega_\mu')^2(z) + \{\omega_\mu, z\}$$

$\{F, \omega_\mu(z)\} = 0 \Rightarrow F$ is a Möbius transformation & fixes $0, 1, \infty$

$\Rightarrow F = \mathrm{id}.$

Riemann-Roch $\Rightarrow A_2(H^*, \Gamma)$ $(3g-3)$ complex dimensional

Metric on $A_2(H^*, \Gamma)$

$$\operatorname{Im}(\gamma(z))^2 \, |\varphi(\gamma(z))| = \operatorname{Im}(z)^2 \, |\varphi(z)| \qquad \forall z \in H^*, \ \gamma \in \Gamma$$

$$\|\varphi\| := \sup_{z \in H^*} (\operatorname{Im}(z))^2 \, |\varphi(z)|$$

Bers' Proj & Bers' Embedding are continuous

$\mu_n \to \mu \Rightarrow \omega_{\mu_n} \longrightarrow \omega_\mu \Rightarrow \varphi_{\mu_n} \longrightarrow \varphi_\mu$ (uniformly on compacts).

Invariance of domain

$\mathcal{B}$ is a homeo onto its image.

$$T_\beta(\Gamma) \xrightarrow{\simeq} T_B(\Gamma) \subseteq A_2(H^*, \Gamma)$$

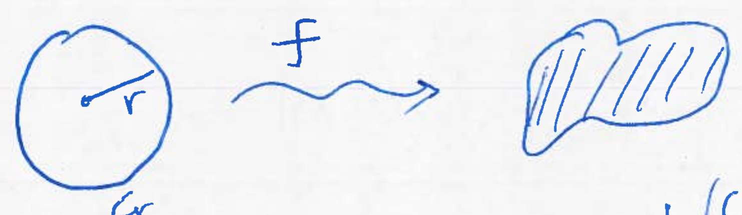
↪ complex manifold structure

$T_B(\Gamma)$ is bounded

Nehari – Krauss inequality

For a injective analytic function on H^* satisfies

$$\|\{f, z\}\|_\infty = \sup_{z \in H^*} (\operatorname{Im} z)^2 |\{f, z\}| \leq \frac{3}{2}$$



$$A_r = \frac{1}{2i}\left(\int_{C_r} \overline{f(w)} \cdot df(w)\right)$$

Invariance of complex structure

$$T(R) \xrightarrow{\simeq} T(R_1) \quad \text{homeo}$$

$$T(\Gamma) \xrightarrow{\simeq} T(\Gamma_1) \quad \text{homeo}$$

Want to show $T_B(\Gamma) \xrightarrow[\text{biholo}]{\sim} T_B(\Gamma_1)$

$$\varphi \in A_2(H^*, \Gamma)$$

$$\psi(z) = \overline{\varphi(\bar{z})} \rightsquigarrow \psi \in A_2(H, \Gamma)$$

Consider Beltrami coefficient

$$\mu_\varphi(z) = -2(\operatorname{Im} z)^2 \varphi(\bar{z})$$

$$\mu_\varphi \in B(H, \Gamma)$$

$$V = \{\varphi \in A_2(H^*, \Gamma) \mid \|\varphi\|_\infty < \tfrac{1}{2}\}$$

$$\varphi \in V \Rightarrow \mu_\varphi \in B(H, \Gamma)_1$$

$$\Psi(\varphi) = [w_{\mu_\varphi}]$$

$$\begin{tikzcd} T_\beta(\Gamma) \arrow[d, shift right, "\mathcal{B}"'] \\ A_2(H^*, \Gamma) \arrow[u, dashed, bend right, "\Psi"'] \end{tikzcd}$$

$\mathcal{B}$

Claim : $\mathcal{B}[\omega_{\mu_\varphi}] = \varphi$

Find two solutions of the diff eqn.

$$\eta'' + \frac{1}{2}\eta\varphi = 0 \qquad \eta_1, \eta_2$$

$$f(z) = \frac{\eta_1(z)}{\eta_2(z)} \quad \text{on } z \in H^*$$

$$F(z) = \frac{\eta_1(\bar z) + (z-\bar z)\,\eta_1'(\bar z)}{\eta_2(\bar z) + (z-\bar z)\,\eta_2'(\bar z)} \qquad z \in H.$$

$$\hat f(z) = \begin{cases} f(z), & z \in H^* \\ F(z), & z \in H \end{cases}$$

$\hat f$ is ω_{μ_φ} upto a Möbius transformation.

& $\{\hat f, z\} = \varphi$

- Construct a local inverse of the Bers embedding
- Study the derivative of the Bers projection
  ↳ Nice form (but not nice enough to write on the board).

<u>Reduce to the case of base point</u>

$$\mathcal{B}_1 \circ \langle w \rangle_* \circ \mathcal{B}^{-1} : T_B(\Gamma) \longrightarrow T_B(\Gamma_1)$$

biholomorphic in a nbhd of $\mathcal{B}([\omega_\mu]) \in T_B(\Gamma_0)$

$$\Gamma^\mu := \omega^\mu \, \Gamma \, (\omega^\mu)^{-1}$$

$$F_1 = \mathcal{B} \circ [(\omega^\mu)^{-1}]_* \circ \mathcal{B}_\mu^{-1} \quad , \quad F_2 :$$

$$\begin{tikzcd}
 & T_B(\Gamma) \arrow[dr] & & T_B(\Gamma_1) \\
\mathcal{B}[\omega_\mu] \arrow[r, mapsto] & \mathcal{B}[\mathrm{id}] & T_B(\Gamma^\mu) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
T_B(\Gamma') \arrow[r, dashed] \arrow[dr, dashed] & T_\beta(H) \arrow[r, leftrightarrow] & T_\beta(\Gamma_1') \arrow[r] & T_B(\Gamma_1') \\
 & B(H,\Gamma') \arrow[r, dash] & B(H,\Gamma_1') & \\
 & T_\beta(\Gamma') & T_\beta(\Gamma_1') &
\end{tikzcd}$$

$$\{ \varphi + t\psi : t \in \mathbb{D}_\varepsilon \}$$

$$\varphi(t) = \varphi + t\psi \qquad \mu_t = \mu_{\varphi(t)}$$

$$\lambda(t) = \left(\frac{\omega_z}{\overline{\omega_z}} \cdot \frac{\mu(t) - \mu_\omega}{1 - \overline{\mu_\omega} \cdot \mu(t)} \right) \circ \omega^{-1} \qquad \begin{array}{l} \omega : H \to H \\ H/\Gamma' \to H/\Gamma_1' \end{array}$$

$$\begin{tikzcd}
T_B(\Gamma') \arrow[rrr, bend left, "F"] \arrow[dr, dashed] & \overset{[\omega^{\mu_t}]}{T(\Gamma')} \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\cong"] & \overset{[\omega^{\lambda_t}]}{T(\Gamma_1')} \arrow[d, phantom, "\cong"] & T_B(\Gamma_1') \\
 & \underset{[\omega_{\mu_t}]}{T_\beta(\Gamma')} & \underset{[\omega_{\lambda_t}]}{T_\beta(\Gamma_1)} \arrow[ur] &
\end{tikzcd}$$

~~[illegible]~~

$\cancel{F(\psi)}$ $F(\varphi(t)) = \{ \omega_{\lambda_t}, z \}$