Formal Languages and Automata Theory

Lecture 2: Non Deterministic Finite Automata

Lecture Slide Reference: UC San Diego

Roadmap for Lecture

In this lecture we:

- Present the three regular operations.
- Present Non-Deterministic Finite Automata.
- Prove that NFA-s and DFA-s are ***equivalent.***
- Use NFA-s to show that when each of the regular operation is applied on regular languages it yields yet another regular language.

The Regular Operations

Let A and B be 2 regular languages above the same alphabet, Σ. We define the 3 ***Regular Operations***:

- **Union**: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- **Concatenation**: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$.
- **Star**: $A^* = \{x_1, x_2, ..., x_k \mid k \geq 0 \text{ and } x_k \in A\}$.

Elaboration

- **Union** is straight forward.
- **Concatenation** is the operation in which each word in A is concatenated with every word in B.
- **Star** is a **unary operation** in which each word in A is concatenated with every other word in A and this happens any **finite number** of times.

The Regular Operations - Examples

$$A = \{good, bad\} \quad B = \{\text{girl}, \text{boy}\}$$

- $A \cup B = \{good, bad, girl, boy\}$
- $A \circ B = \{goodgirl, goodboy, badgirl, badboy\}$
- $A^* = \left\{ \begin{array}{l} \varepsilon, good, bad, goodgood, goodbad, \\ goodgoodgoodbad, badbadgoodbad, \ldots \end{array} \right\}$

Motivation for Nondeterminism

The **Regular Operations** give us a way to construct all regular languages systematically.

In the previous lecture, we showed that the union operation preserves regularity:

Given two regular languages L_1 and L_2 and their recognizing automata, M_1 and M_2 ,we constructed an automaton that recognizes $L_1 \cup L_2$.

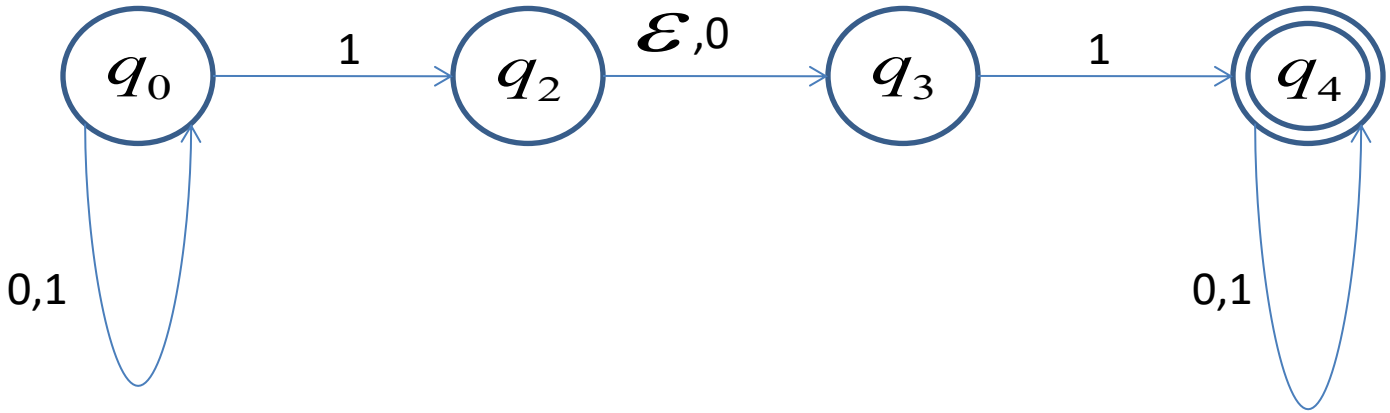
Motivation for Nondeterminism

This approach fails when trying to prove that **concatenation** and **star** preserve regularity.

To overcome this problem we introduce nondeterminism.

Example of an NFA

NFA – Nondeterministic Finite Automaton



1. A state may have 0 to many transitions labeled with the same symbol.
2. ε transitions are possible.

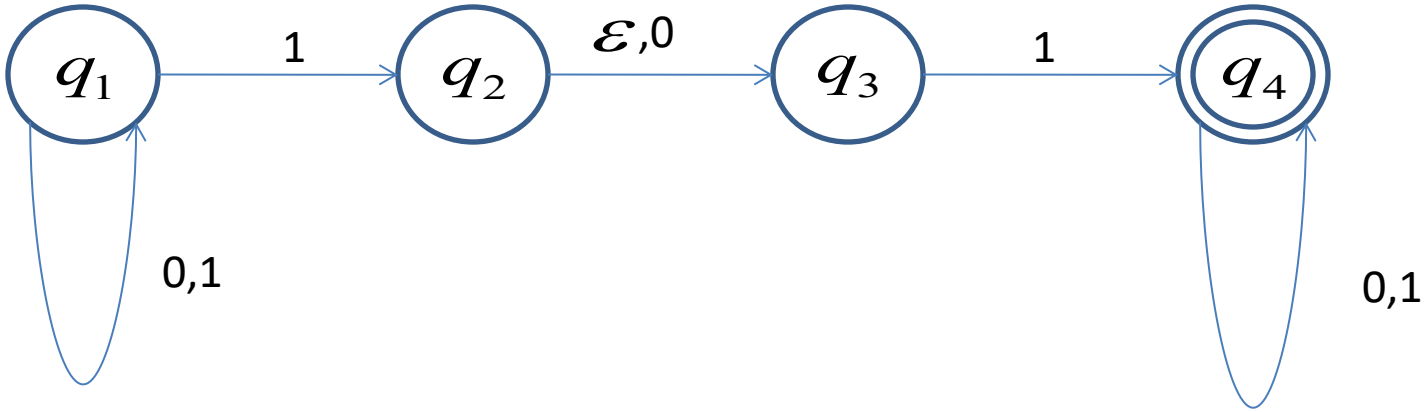
Computation of an NFA

- When several transitions with the same label exist, an input word may induce **several** paths.
- When 0 transition is possible a computation may get “stuck”.

Q:Which words are accepted and which are not?

A: If word w induces (at least) a single accepting computation, the automaton “chooses” this **accepting path** and w is accepted.

Computation of an NFA - Example



On the input word w=010110 there exist an accepting path and w is accepted.

Can we characterize (find) the language recognized by this automaton?

Words containing 101 or 11 as substrings

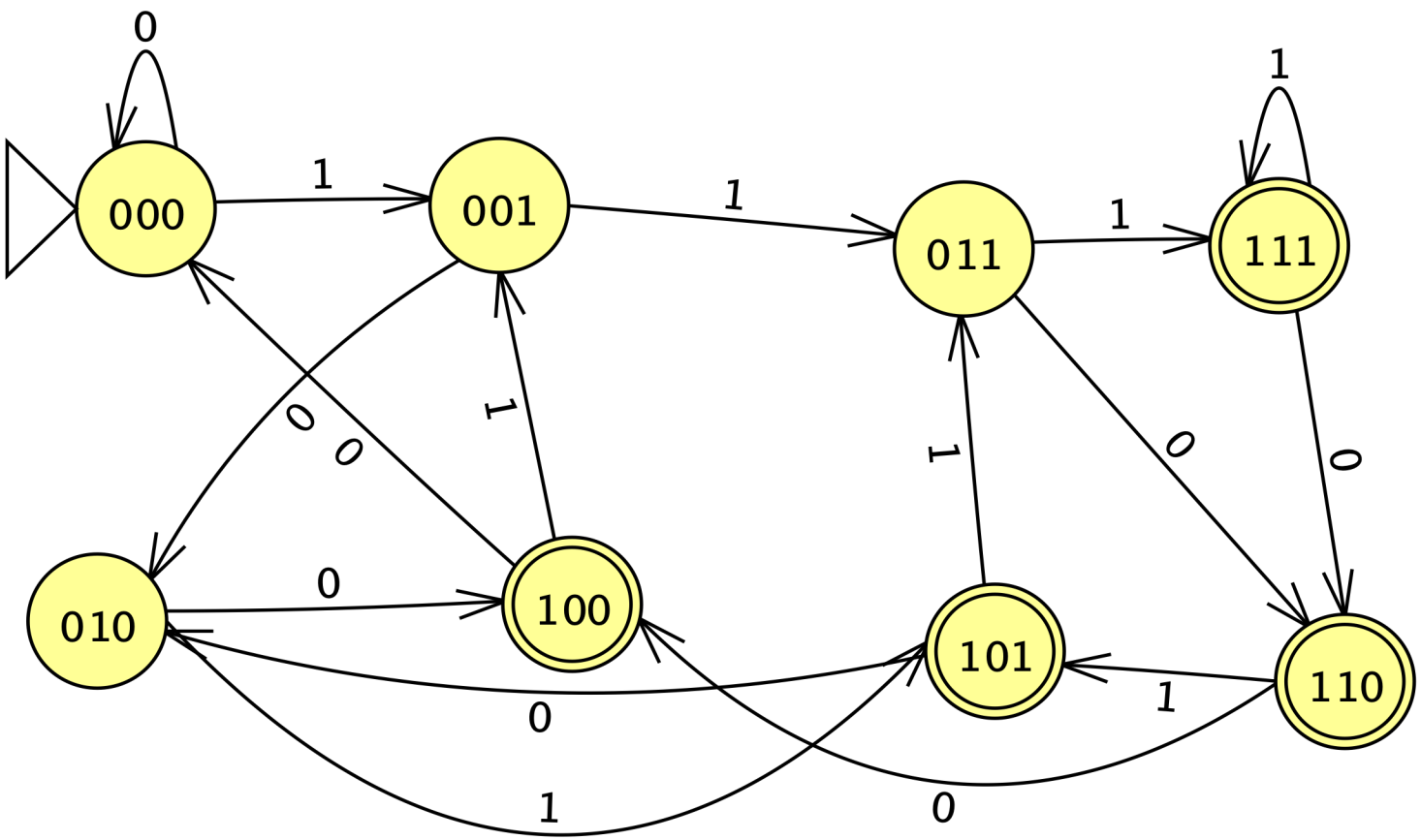
Why do we Care About NFAs?

- NFA-s and DFA-s are **equivalent** (Meaning: They recognize the same set of languages). In other words: Each NFA recognizing language L has an equivalent DFA recognizing L.

  (**Note:** This statement **must be proved**.)
- But usually, the NFA is much simpler.
- Enables the proof of theorems. (e.g. about regular operations)

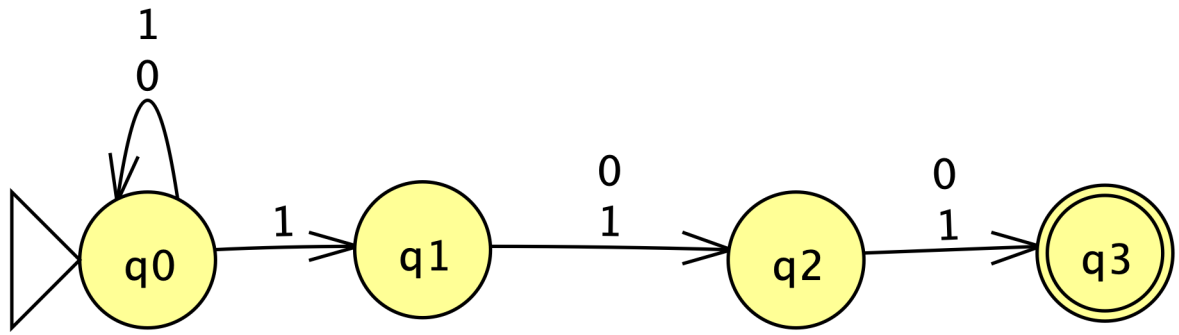
Example

- An automaton over unary alphabet accepting words whose length is divided either by 2 or by 3.



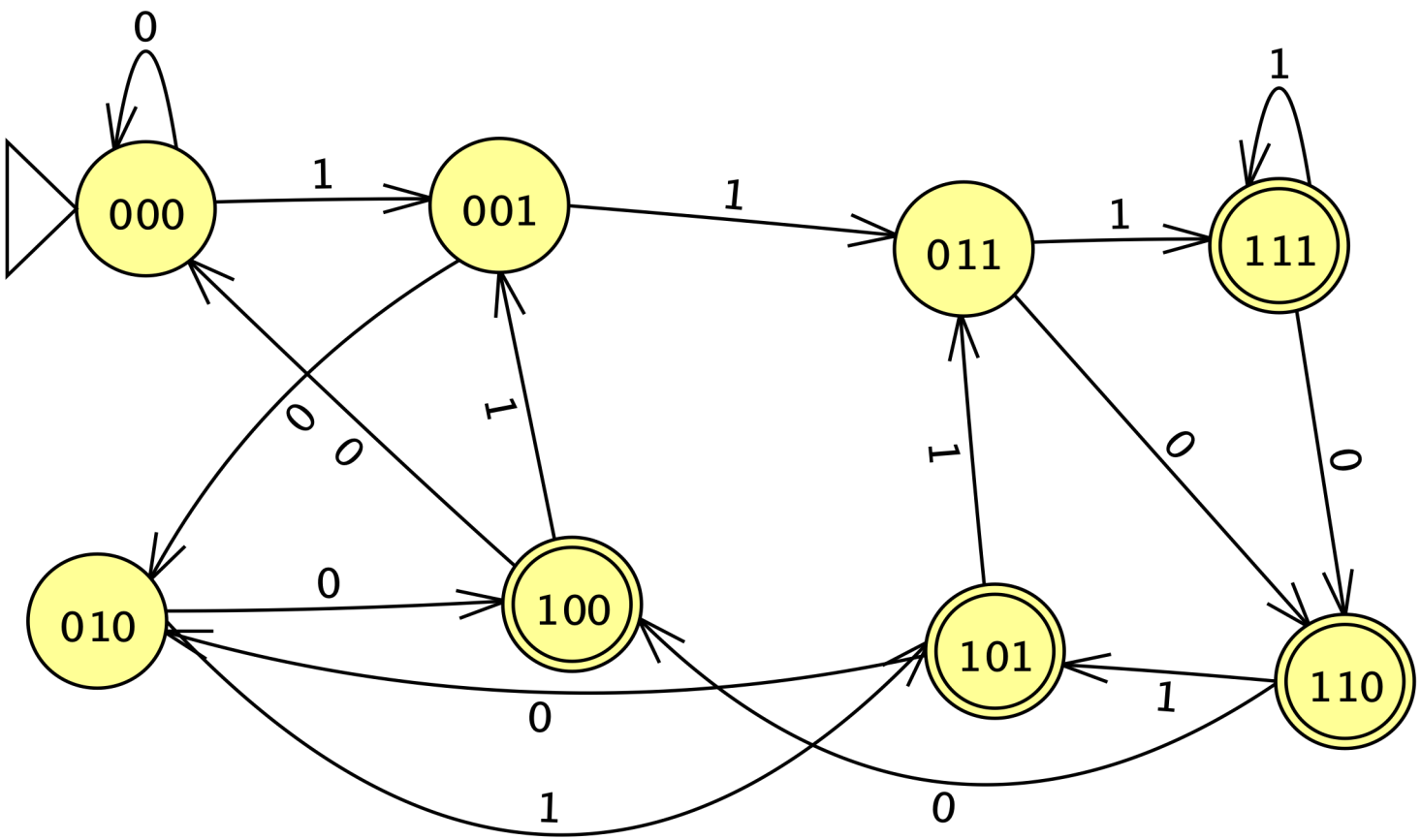
Example: Complicated DFA

What is the language of this DFA?



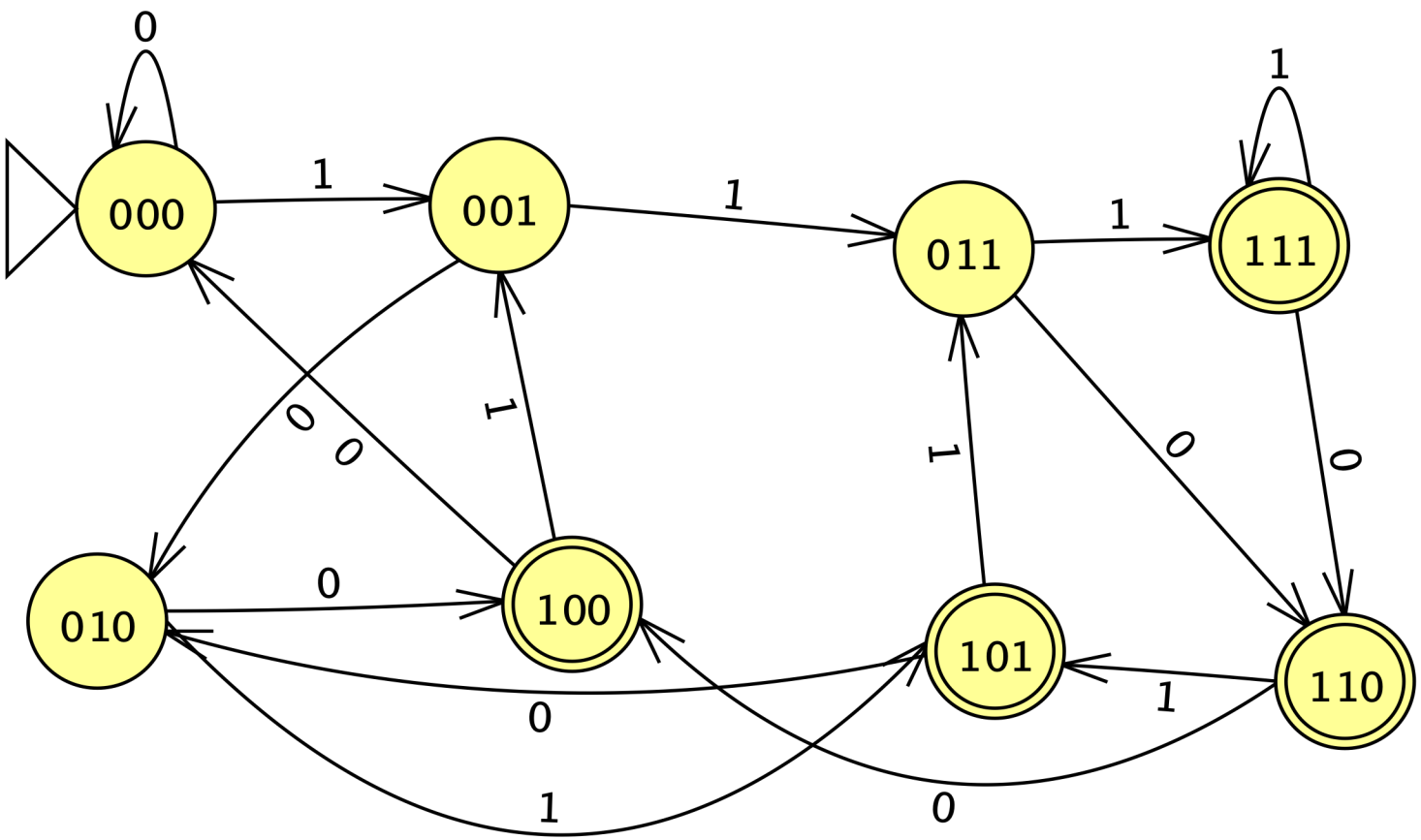
Much simpler NFA

L = { w | bits strings that have a 1 in the third position from the end }



Can you verify it now?

L = { w | bits strings that have a 1 in the third position from the end }



Verification Demonstration using JFLAP

L = { w | bits strings that have a 1 in the third position from the end }

DFA – A Formal Definition(Rerun)

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where:

1. Q is a finite set called the ***states***.
2. Σ is a finite set called the ***alphabet***.
3. $\delta : Q \times \Sigma \to Q$ is the ***transition function***.
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the set of ***accepting states***.

NFA – A Formal Definition

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1. Q is a finite set called the ***states***.
2. Σ is a finite set called the ***alphabet***.
3. $\delta : Q \times \Sigma_\varepsilon^* \to P(Q)$ is the ***transition function***.
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the set of ***accept states***.

* $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

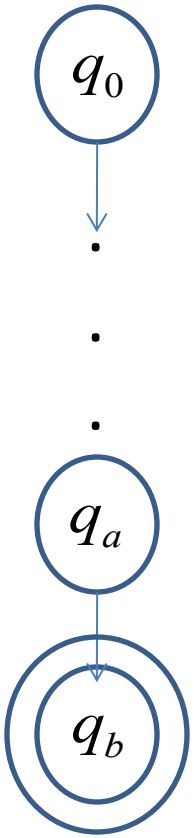
Differences between NFA-s and DFA-s

There are **two** differences:

1. The range of the **transition function** δ is now $P(Q)$. (The set of **subsets** of the state set Q)
2. The transition function allows ε transitions.

Possible Computations

DFA

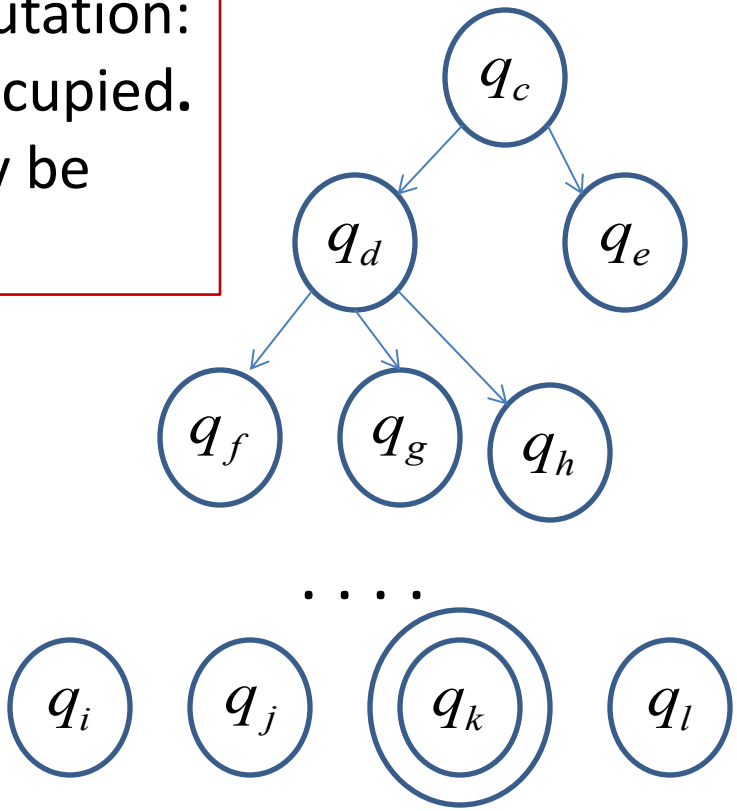


At each step of the computation:

DFA - A **single state** is occupied**.**

NFA - **Several states** may be occupied.

NFA



Computations of NFA-s

In general a computation of an NFA, N, on input w, induces a **computation tree**.

Each path of the computation tree represents a possible computation of N.

The NFA N accepts w, if its computation tree includes **at least** one path ending with an accepting state.

Computations of NFA-s

There are two ways to look at computations of an NFA:

- The first is to say that the NFA N "**chooses**" the right path on its tree of possible computations.
- The second is to say that the NFA N traverses its computation tree **"in parallel"**.

Equivalence Between DFAs and NFAs

Now we prove that the class of NFAs is

Equivalent to the class of DFA:

Theorem: For every NFA N, there exists a DFA $M = M(N)$, such that $L(N) = L(M(N))$.

Proof Idea: The proof is ***Constructive***: We assume that we know N, and construct a simulating DFA, M.

Proof

Let $N = (Q, \Sigma, \delta, q_0, F)$ recognizing some language A. the state set of the simulating DFA M, should reflect the fact that at each step of the computation, N may occupy **several sates**.

Thus we define the state set of M as $P(Q)$ the **power-set** of the state set of N.

Proof (cont.)

Let $N = (Q, \Sigma, \delta, q_0, F)$ recognizing some language A. First we assume that N has no ε - transitions.

Define $M = (Q', \Sigma, \delta', q_0', F)$ where $Q' = P(Q)$.

Proof (cont.)

Our next task is to define M's transition function, δ' :

For any $R \in Q'$ and $a \in \Sigma$ define

$$\delta'(R,a) = \{q \in Q \mid q \in \delta(r,a) \text{ for some } r \in R\}$$

If R is a state of M, then it is a set of states of N.

When M in state R processes an input symbol a, M goes to the set of states to which N will go in any of the branches of its computation.

Proof (cont.)

An alternative way to write the definition of M's transition function, δ' is:

For any $R \in Q'$ and $a \in \Sigma$ define

$$\delta'(R,a) = \bigcup_{r \in R} \delta(r,a)$$

And the explanation is just the same.

Note: if $\bigcup_{r \in R} \delta(r,a) = \phi$ than $\delta'(R,a) = \phi$

Which is OK since $\phi \in P(Q)$.

Proof (cont.)

The initial state of M is:

$q_0' = \{q_0\}$.

Finally, the final state of M is:

F’ = { R ∈ Q’ | R contains a final state of N }

Since M accepts if N reaches **at least one** accepting state.

The reader can verify for her/him self that M indeed **simulates** N .

Proof (cont.)

It remains to consider ε - transitions.

For any state R of M define $E(R)$ to be the collection of states of R unified with the states reachable from R by ε- transitions.

The old definition of $\delta'(R,a)$ is:

$$\delta'(R,a) = \{q \in Q \mid q \in \delta(r,a) \text{ for some } r \in R\}$$

And the new definition is:

$$\delta'(R,a) = \{q \in Q \mid q \in E(\delta(r,a)) \text{ for some } r \in R\}$$

Proof (end)

In addition, we have to change the definition of q_0', the initial state of M . The previous definition, $q_0' = \{q_0\}$, is replaced with $q_0' = E(\{q_0\})$.

Once again the reader can verify that the new definition of M satisfies all requirements.

Corollary

A language L is ***regular*** if and only if there exists an NFA recognizing L .

The Regular Operations (Rerun)

Let A and B be 2 regular languages above the same alphabet, Σ. We define the 3 ***Regular Operations***:

- **Union**: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
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The Regular Operations - Examples

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Theorem (rerun)

The class of Regular languages is **closed** under the all three **regular operations**.

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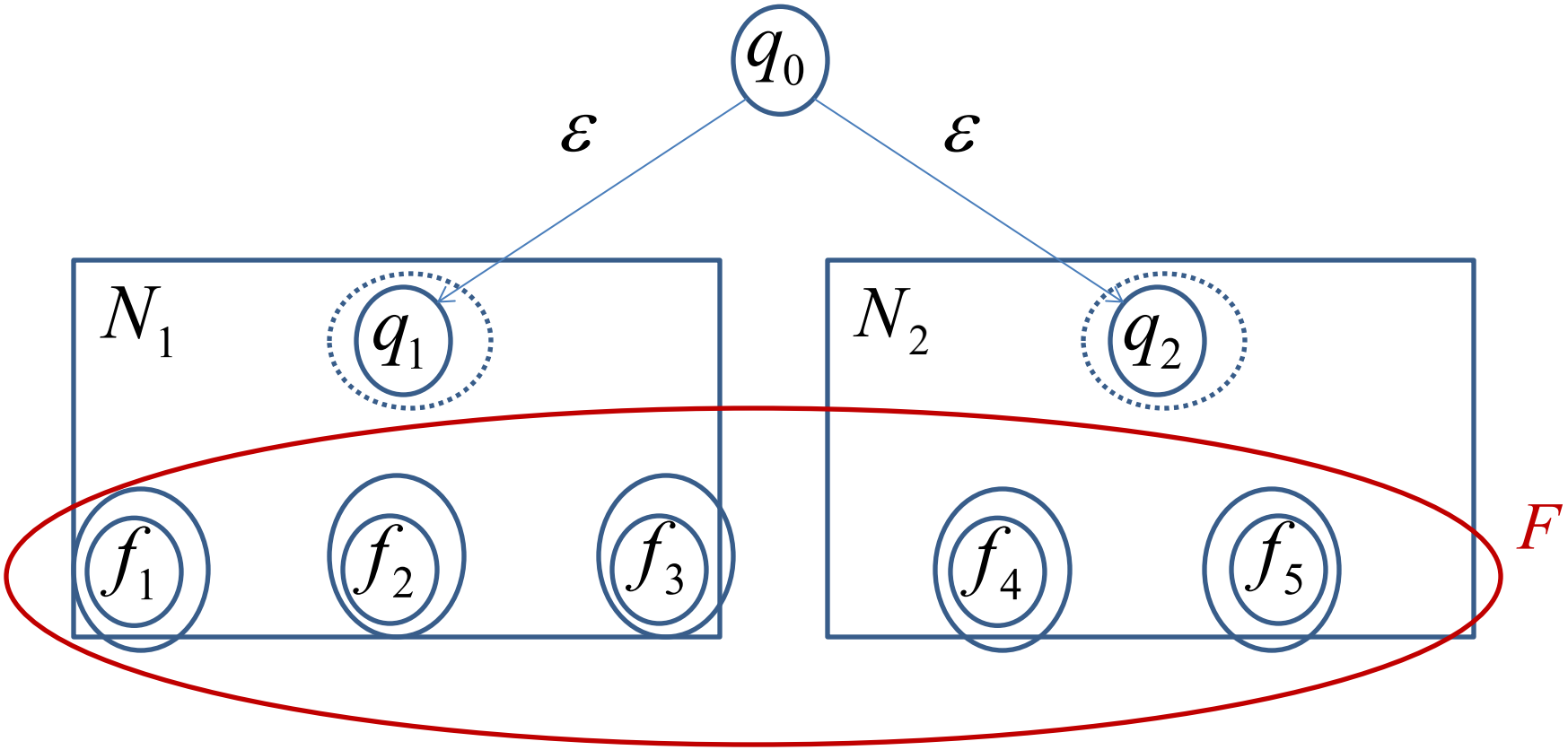
Proof for *union* Using NFA-s

If A_1 and A_2 are regular, each has its own recognizing automaton N_1 and N_2, respectively.

In order to prove that the language $A_1 \cup A_2$ is regular we have to construct an FA that accepts exactly the words in $A_1 \cup A_2$.

A Pictorial proof



Proof for *union* Using NFA-s

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognizing A_1, and

$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizing A_2.

Construct $N = (Q, \Sigma, \delta, q, F)$ to recognize $A_1 \cup A_2$,

Where $Q = \{q_0\} \cup Q_1 \cup Q_2$, $F = F_1 \cup F_2$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \\ \delta_2(q,a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon. \end{cases}$$

Theorem

The class of Regular languages is **closed** under the ***concatenation*** operation.

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Proof idea

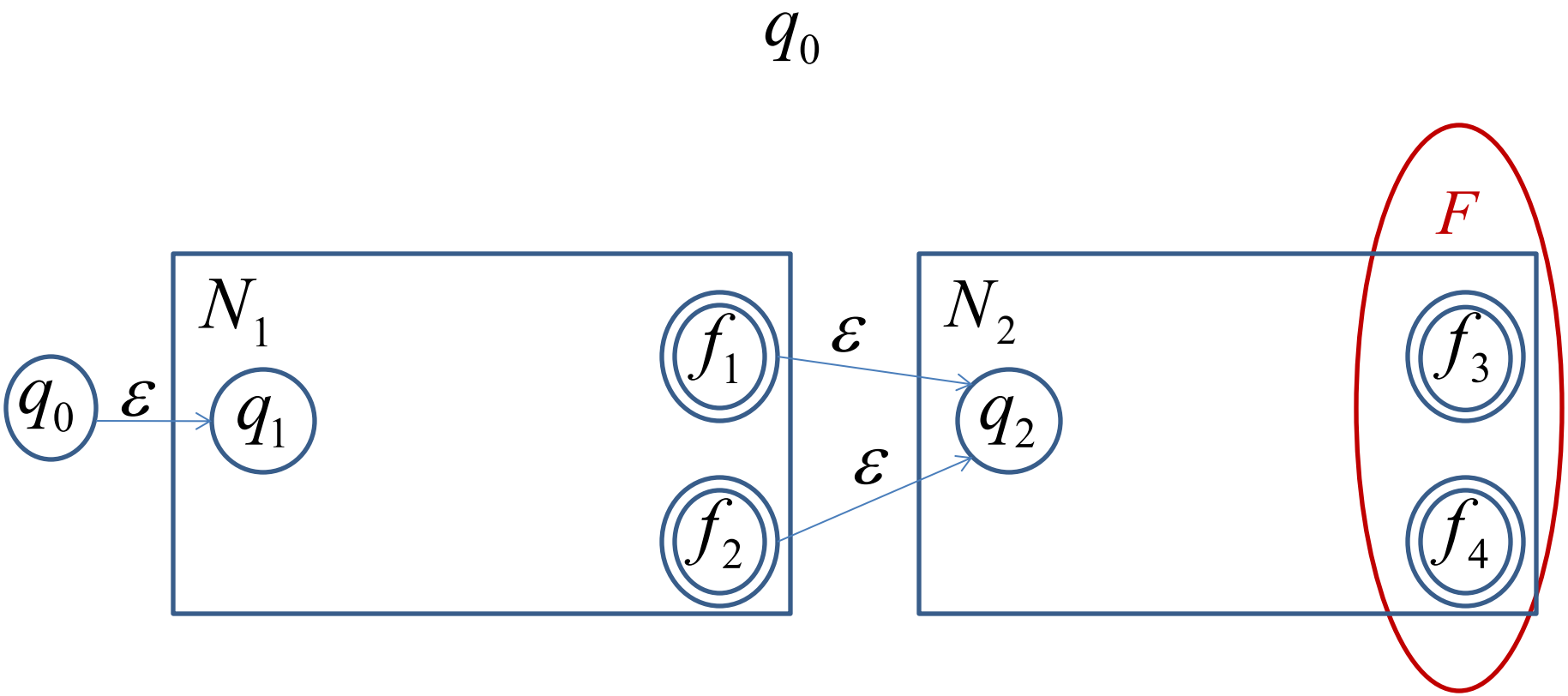
Given an input word to be checked whether it belongs to $A_1 \circ A_2$, we may want to run N_1 until it reaches an accepting state and then to move to N_2.

Proof idea

The problem: Whenever an accepting state is reached, we cannot be sure whether the word of A_1 is finished yet.

The idea: Use non-determinism to **choose** the right point in which the word of A_1 is finished and the word of A_2 starts.

A Pictorial proof



Proof using NFAs

Let $N_1 = (Q_1, \Sigma, \delta, q_1, F_1)$ recognizing A_1, and
$N_2 = (Q_2, \Sigma, \delta, q_2, F_2)$ recognizing A_2.
Construct $N = (Q, \Sigma, \delta, q_1, F)$ to recognize $A_1 \circ A_2$,
Where $Q = Q_1 \cup Q_2$, $F = F_2$. Define **δ in this way:**

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) & q \in Q_2. \end{cases}$$

Theorem

The class of Regular languages is **closed** under the ***star*** operation.

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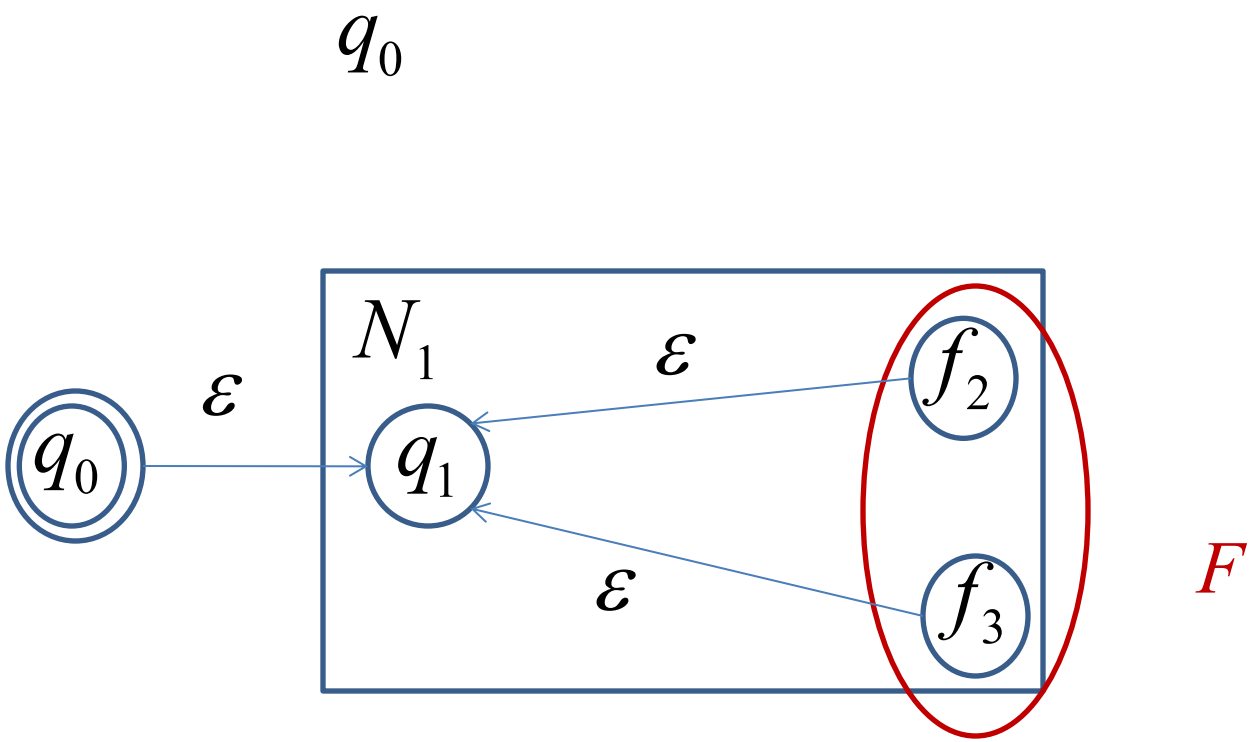
A Pictorial proof



Proof using NFAs

Let $N_1 = (Q_1, \Sigma, \delta, q_1, F_1)$ recognizing A_1.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*

Where $Q = \{q_0\} \cup Q_1$, $F = \{q_0\} \cup F_1$, and

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ q_1 & q = q_0 \text{ and } a = \varepsilon \\ \phi & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

Wrap Up

In this lecture we:

- Motivated and defined the three **Regular Operations.**
- Introduced **NonDeterministic Finite Automatons**.
- Proved equivalence between DFA-s and NFA-s.
- Proved that the regular operations **preserve regularity**.