Teoria do Risco
Aula 12

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https://atuaria.github.io/portalhalley

Modelo de Risco individual	Modelo de Risco coletivo
X_i Independentes	X_i Independentes e identicamente distribuídas
$S_{ind} = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} B_i I_i$	$s_{col} = \sum_{i=1}^{N} X_i$
$E(S_{ind}) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$	$E(S_{col}) = E\left(\sum_{i=1}^{N} X_i\right)$
S_{ind}, X_i, B_i, I_i	s_{col}, X_i, N
$M_{S_{ind}}(t) = \prod_{i=1}^{n} M_{X_i}(t)$	$M_{S_{col}}(t) = M_N(ln(M_X(t)))$
$E(S_{ind}) = \sum_{i=1}^{n} E(B_i)q_i$	$E(S_{col}) = E(N)E(X)$
$var(S_{ind}) = \sum_{i=1}^{n} var(B_i)q_i + \sum_{i=1}^{n} E(B_i)^2 q_i(1-q_i)$	$var(S_{col}) = E(X)^2\, var(N) + E(N)var(X)$

Modelos de risco Coletivo- A distribuição de S_{col}, os sinistros coletivos.

- O método da convolução a partir da distribuição de X e N

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s)\, p(N = k)$$

$$f_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s)\, p(N = k)$$

$$P^{*k}(s) = P(X_1 + X_2 + \ldots + X_k \leq s)$$

$$p^{*k}(s) = p(\boldsymbol{X_1} + \boldsymbol{X_2} + \ldots + \boldsymbol{X_k} = \boldsymbol{s})$$

Modelos de risco Coletivo

- O processo de convolução no modelo de risco coletivo leva em consideração a convolução entre os sinistros ocorridos dado que quantidade ocorrida também é uma variável aleatória.

Modelo de risco individual	Modelo de risco coletivo
$\mathrm{F}^{(\mathrm{k})} = \mathrm{F}_{\mathrm{k}} * \mathrm{F}^{(\mathrm{k}-1)}$	$\mathrm{P}^{(\mathrm{k})} = \mathrm{P}_{\mathrm{k}} * \mathrm{P}^{(\mathrm{k}-1)}$
$\mathrm{F}_{\mathrm{S}_{\mathrm{ind}}}^{(2)}(\mathrm{s}) = \sum_{j=0}^{s} F_X(s-y_j)p_Y(y_j)$	$\mathrm{F}_{\mathrm{S}_{\mathrm{col}}}^{(2)}(\mathrm{s}) = \sum_{\mathrm{k}=0}^{2} \mathrm{P}^{*\mathrm{k}}(\mathrm{s})\mathrm{p}_{\mathrm{N}}(\mathrm{k})$

$$X\,(discreto) \rightarrow S_{col}\,(discreto)$$
$$X(contínuo) \rightarrow S_{col}(contínuo)$$

Modelos de risco Coletivo

Pelo método de convolução a partir da distribuição de X e N.

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s) p_N(k)$$

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) p_N(k)$$

$$P^{*k}(s) = P(X_1 + X_2 + .. + X_k \leq s)$$

$$p^{*k}(s) = p(\boldsymbol{X_1} + \boldsymbol{X_2} + .. + \boldsymbol{X_k} = \boldsymbol{s})$$

Quando X são contínuos.

$$P^{*k}(s) = \int_0^s P^{*k-1}(s-h)p(h)dh$$

$$P^{*k-1*}P$$

$$p^{*k}(s) = \int_0^s p^{*k-1}(s-h)p(h)dh$$

$$p^{*k-1*}p$$

- Exemplo

Calcular $F_{S_{col}}(s)$, quando X possui distribuição Exponencial (α) e N possui distribuição de Poisson(λ).

Lembrando que:

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s)\, p(N = k)$$

$$P^{*k}(s) = \int_0^s P^{*k-1}(s-h)p(h)dh$$

Assim:

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s) \frac{e^{-\lambda}\lambda^k}{k!}$$

Cálculo de $P^{*k}(x)$

Sabendo que se X tem distribuição Exponencial (α), então:

$$p(x) = \alpha e^{-\alpha x} \qquad P(x) = 1 - e^{-\alpha x}\ ; x > 0$$

$$P^{*k}(s) = \int_h P^{*k-1}(s-h)\, p(h) dh$$

$$P^{*(2)}(s) = \int_0^s P^{*2-1}(s-h)p(h)dh = \int_0^s P\,(s-h)p(h)dh$$

$$P^{*(2)}(s) = \int_0^s \left(1 - e^{-\alpha(s-h)}\right)\alpha e^{-\alpha h} dh$$

...

$$P^{*(2)}(s) = 1 - e^{-\alpha s}(1 + \alpha s)$$

$$P^{*3}(s) = \int_0^s P^{*3-1}(s-h)p(h)dh = \int_0^s P^{*2}(s-h)p(h)dh$$

$$P^{*3}(s) = \int_0^s \left[1 - e^{-\alpha(s-h)}(1 + \alpha(s-h))\right]\alpha e^{-\alpha h}dh$$

...

$$P^{*3}(s) = 1 - e^{-\alpha s}\left(1 + \alpha s + \frac{(\alpha s)^2}{2!}\right)$$

- Desta forma, então, chega-se à seguinte formula de $\mathrm{P}^{*\mathrm{n}}$

$$\boldsymbol{P(s) = 1 - e^{-\alpha s}}$$

$$\boldsymbol{P^{*(2)}(s) = 1 - e^{-\alpha s}(1 + \alpha s)}$$

$$\boldsymbol{P^{*3}(s) = 1 - e^{-\alpha s}\left(1 + \alpha s + \frac{(\alpha s)^2}{2!}\right)}$$

$$\cdots$$

$$P^{*k}(s) = 1 - e^{-\alpha s}\sum_{i=0}^{k-1}\frac{(\alpha s)^i}{i!}$$

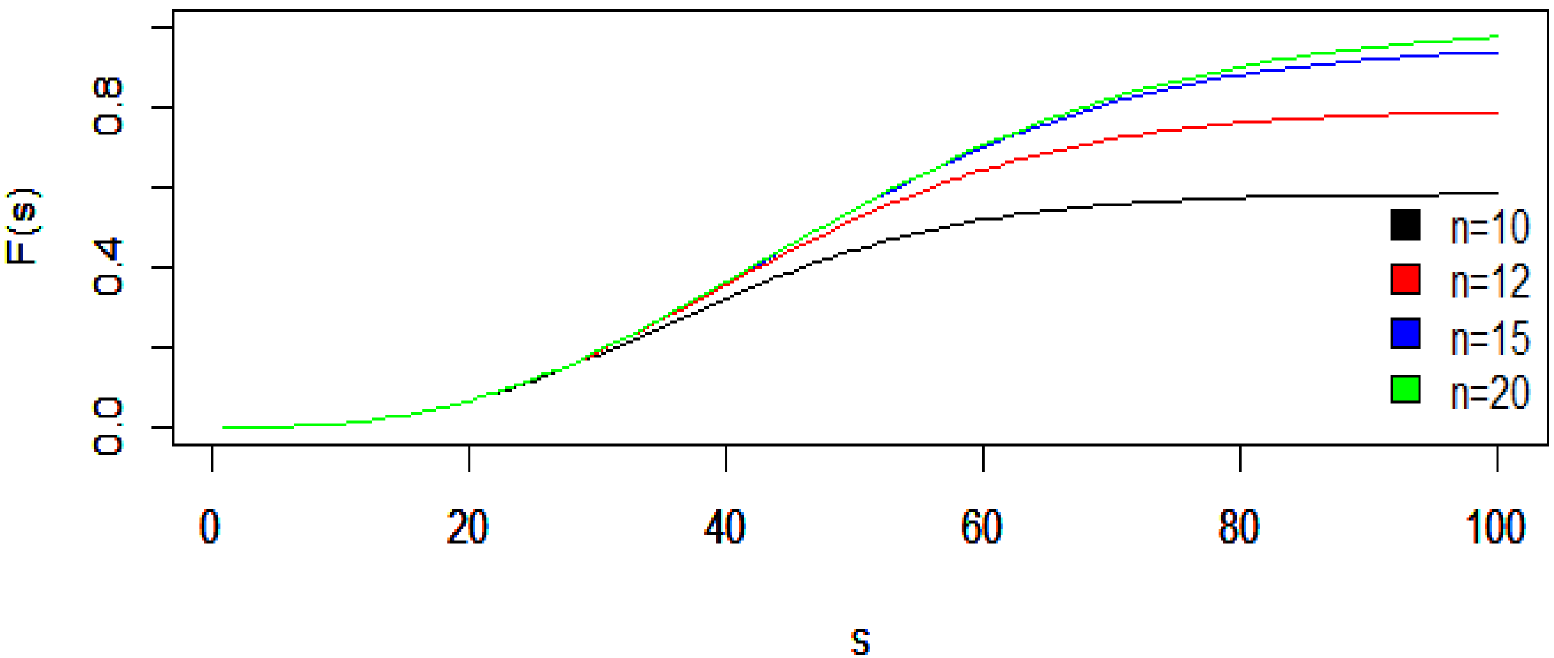
Como:

$$F_{S_{col}}(s) = \sum_{k=1}^{n} P^{*k}(s)\,\frac{e^{-\lambda}\lambda^{k}}{k!}$$

Tem-se que:

$$F_{S_{col}}(s) = \sum_{k=1}^{n}\left[1 - e^{-\alpha s}\sum_{i=0}^{k-1}\frac{(\alpha s)^{i}}{i!}\right]\frac{e^{-\lambda}\lambda^{k}}{k!}$$

$$F_{S_{col}}(s) = \sum_{k=1}^{n}\left[1 - e^{-\alpha s}\sum_{i=0}^{k-1}\frac{(\alpha s)^i}{i!}\right]\frac{e^{-\lambda}\lambda^k}{k!}$$



Comportamento de $F_{s_{col}}(S)$ com $\alpha = 0,2, \lambda = 10$ para diferentes quantidade de apólices n.

- Exemplo

Adicionalmente pode-se calcular p^{*k} e $f_{S_{col}}(s)$, quando X possui distribuição *Exponencial* (α) e *N* possui distribuição de *Poisson*(λ).

Lembrando que:

$$f_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s)\, p(N = k)$$

$$p^{*k}(s) = \int_0^s p^{*k-1}(s-h)p(h)dh$$

Assim:

$$f_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) \frac{e^{-\lambda}\lambda^k}{k!}$$

Cálculo de $\boldsymbol{P^{*k}(x)}$

Sabendo que se X tem distribuição Exponencial (α), então:

$$p(x) = \alpha e^{-\alpha x} \;; x > 0$$

$$p^{*k}(s) = \int_h p^{*k-1}(s-h)\, p(h) dh$$

$$p^{*(2)}(s) = \int_0^s p^{*2-1}(s-h) p(h) dh = \int_0^s p(s-h) p(h) dh$$

$$p^{*(2)}(s) = \int_0^s \left(\alpha e^{-\alpha(s-h)}\right) \alpha e^{-\alpha h} dh = \alpha^2 s\, e^{-\alpha s},$$

Cálculo de $P^{*k}(x)$

Sabendo que se X tem distribuição Exponencial (α), então:

$$p(x) = \alpha e^{-\alpha x} \ ; x > 0$$

$$p^{*k}(s) = \int_h p^{*k-1}(s-h)\, p(h) dh$$

$$p^{*(2)}(s) = \alpha^2 s\, e^{-\alpha s}$$

$$p^{*3}(s) = \int_0^s p^{*3-1}(s-h) p(h) dh = \int_0^x p^{*2}(s-h) p(h) dh$$

$$p^{*3}(s) = \int_0^s \alpha^2 (s-h)\, e^{-\alpha(s-h)} \alpha e^{-\alpha h} dh = \frac{\alpha^3 s^2\, e^{-\alpha s}}{2}$$

Cálculo de $\boldsymbol{P^{*k}(x)}$

Sabendo que se X tem distribuição Exponencial (α), então:

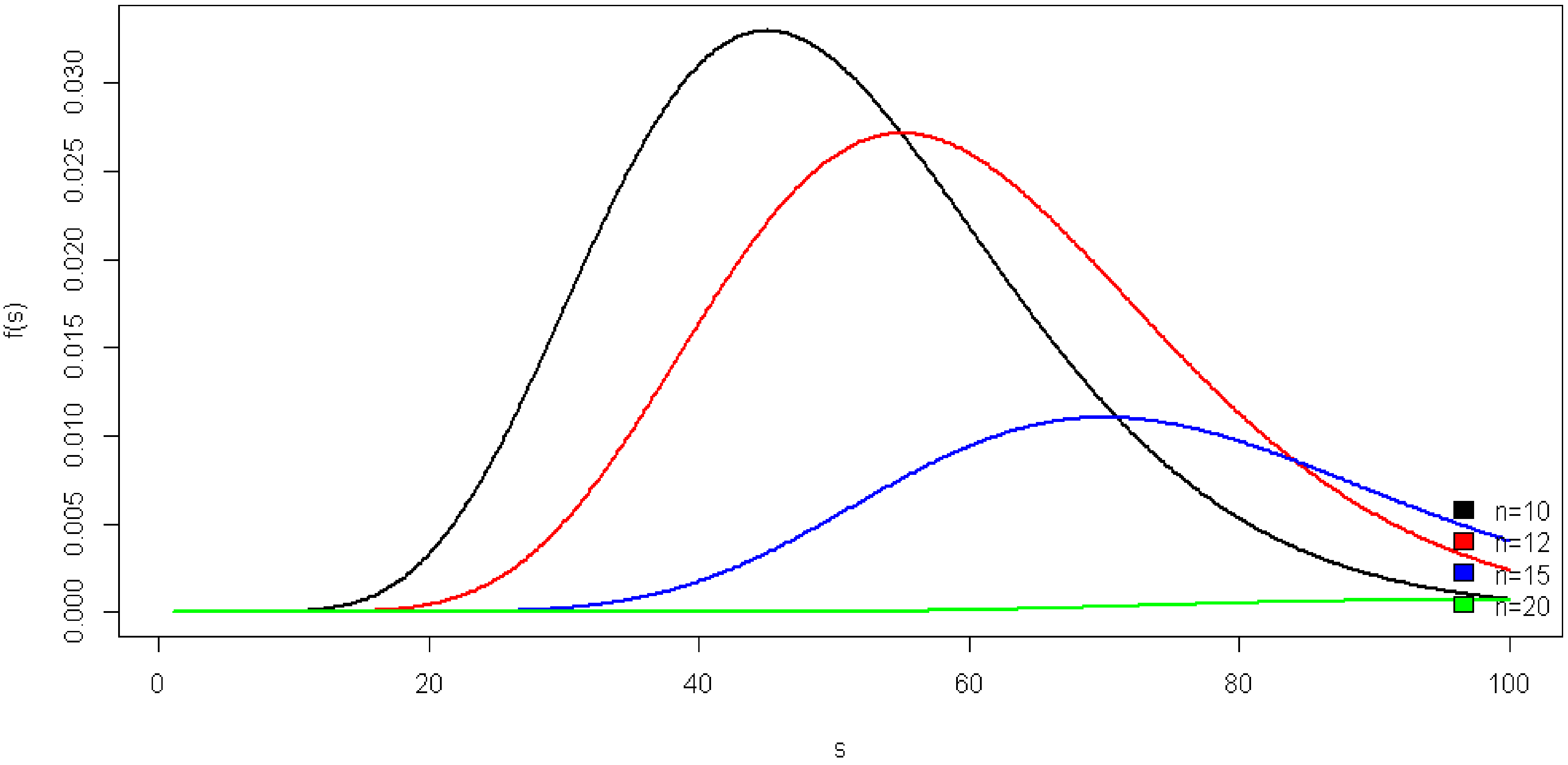
$$p(x) = \alpha e^{-\alpha x} \;; x > 0$$

$$p^{*(2)}(s) = \alpha^2 s\, e^{-\alpha s},$$

$$p^{*3}(s) = \frac{\alpha^3 s^2\, e^{-\alpha s}}{2}.$$

$$p^{*n}(s) = \frac{\alpha^n s^{n-1} e^{-\alpha s}}{(n-1)!}$$

$$f_{S_{col}}(s) = \sum_{k=1}^{n} \left[\frac{\alpha^k s^{k-1} e^{-\alpha s}}{(k-1)!}\right] \frac{e^{-\lambda}\lambda^k}{k!}$$



Comportamento de $f_{s_{col}}(S)$ com $\alpha = 0,2, \lambda = 10$ para diferentes quantidade de apólices n.

Modelos de risco Coletivo

Pelo método de convolução a partir da distribuição de X e N.

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s) p_N(k)$$

$$P^{*k}(s) = P(X_1 + X_2 + .. + X_k \leq s)$$

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) p_N(k)$$

$$p^{*k}(s) = p(\boldsymbol{X_1} + \boldsymbol{X_2} + .. + \boldsymbol{X_k} = \boldsymbol{s})$$

Modelos de risco Coletivo

Pelo método de convolução a partir da distribuição de $\mathbf{X}$ e $\mathbf{N}$.

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) p_N(k)$$

$$p^{*k}(s) = p(X_1 + X_2 + .. + X_k = s)$$

Quando $\mathbf{X}$ é discreto tem-se

$$p^{*0}(s) = \begin{cases} 0 \; se \; s \neq 0 \\ 1 \; se \; s = 0 \end{cases}$$

$$p^{*k}(s) = \sum_{h \leq s} p^{*k-1}(s-h) p(h)$$

Considere h como um dos valores possíveis para X.

- Exemplo

Uma carteira de seguros produz 0,1 ou 2 sinistros com as respetivas probabilidades: 20%,50% e 30%. Um sinistro dessa carteira assume os valores $R\$100$, $R\$200$ ou $R\$300$, com as respectivas probabilidades: $20\%, 70\%$ e 10%.

Construa a distribuição convoluta dos sinistros agregados S_{col}.

X_i	$R\$100$	$R\$200$	$R\$300$
$P_{X_i}(x_i)$	0,2	0,7	0,1

N	$P(N)$	S_{col}	
0	0,2	$S_{col} = 0$	
1	0,5	$S_{col} = X_1$	$\{R\$100, R\$200, R\$300\}$
2	0,3	$S_{col} = X_1 + X_2$	$\{R\$200, R\$300, R\$400, R\$500, R\$600\}$

Em primeiro lugar, computemos todas as combinações possíveis de frequência e severidades e assim explicitemos os valores possíveis de sinistros agregados e associados as probabilidades de ocorrência

Por definição tem-se que $p^{*0}(s) = \begin{cases} 0 \; se \; s \neq 0 \\ 1 \; se \;\; s = 0 \end{cases}$

- Logo para $\boldsymbol{k = 0}$:

$p^{*0}(0) = 1$;
$p^{*0}(100) = 0$;
$p^{*0}(200) = 0$;
$p^{*0}(300) = 0$;
$p^{*0}(400) = 0$;
$p^{*0}(500) = 0$
$p^{*0}(600) = 0$

Para $\boldsymbol{k} = \mathbf{1}$:

Usando $p^{*\mathrm{k}}(s) = \sum_{\mathrm{h} \leq s} p^{*\mathrm{k}-1}(\mathrm{s}-\mathrm{h})p(\mathrm{h})$ sendo k os possíveis valores assumidos por N.

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{0}) = \sum_{h=0}^{0} p^{*1-1}(0-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{100}) = \sum_{h=0}^{100} p^{*1-1}(100-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{200}) = \sum_{h=0}^{200} p^{*1-1}(200-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{300}) = \sum_{h=0}^{300} p^{*1-1}(300-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{400}) = \sum_{h=0}^{400} p^{*1-1}(400-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{500}) = \sum_{h=0}^{500} p^{*1-1}(500-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{600}) = \sum_{h=0}^{600} p^{*1-1}(600-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{0}) = p^{*0}(\,0)p(0) = 0$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{100}) = p^{*0}(100)p(0) + \boldsymbol{p}^{*\mathbf{0}}(\mathbf{0})\boldsymbol{p}(\mathbf{100}) = \mathbf{0,2}$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{200}) = p^{*0}\ (200)p(0) + p^{*0}(100)p(100) + \boldsymbol{p}^{*\mathbf{0}}(\mathbf{0})\boldsymbol{p}(\mathbf{200}) = 0{,}7$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{300}) = p^{*0}(300)p(0) + p^{*0}(200)p(100) + p^{*0}(100)p(200) + \boldsymbol{p}^{*\mathbf{0}}(\mathbf{0})\boldsymbol{p}(\mathbf{300}) = \mathbf{0,1}$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{400}) = p^{*0}(400)p(0) + p^{*0}(300)p(100) + p^{*0}(200)p(200) + p^{*0}(100)p(300) + p^{*0}(0)p(400) = \mathbf{0}$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{500}) = p^{*0}(500)p(0) + p^{*0}(400)p(100) + p^{*0}(300)p(200) + p^{*0}(200)p(300) + p^{*0}(100)p(400) + p^{*0}(0)p(500) = \mathbf{0}$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{600}) = p^{*0}(600)p(0) + p^{*0}(500)p(100) + p^{*0}(400)p(200) + p^{*0}(300)p(300) + p^{*0}(200)p(400) + p^{*0}(100)p(500) + p^{*0}(0)p(600) = \mathbf{0}$$

S_{col}	$N = 0$	$N = 1$
0	$\mathbf{p}^{*\mathbf{0}}(\mathbf{0}) = 1$	$\mathbf{p}^{*\mathbf{1}}(\mathbf{0}) = 0$
100	$\mathbf{p}^{*\mathbf{0}}(\mathbf{100}) = \mathbf{0}$	$\mathbf{p}^{*\mathbf{1}}(\mathbf{100}) = \mathbf{0},2$
200	$\boldsymbol{p}^{*\mathbf{0}}(\mathbf{200}) = 0$	$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{200}) = 0,7$
300	$\mathbf{p}^{*\mathbf{0}}(\mathbf{300}) = \mathbf{0}$	$\mathbf{p}^{*\mathbf{1}}(\mathbf{300}) = \mathbf{0},\mathbf{1}$
400	$\mathbf{p}^{*\mathbf{0}}(\mathbf{400}) = \mathbf{0}$	$\mathbf{p}^{*\mathbf{1}}(\mathbf{400}) = \mathbf{0}$
500	$\mathbf{p}^{*\mathbf{0}}(\mathbf{500}) = \mathbf{0}$	$\mathbf{p}^{*\mathbf{1}}(\mathbf{500}) = \mathbf{0}$
600	$\mathbf{p}^{*\mathbf{0}}(\mathbf{600}) = \mathbf{0}$	$\mathbf{p}^{*\mathbf{1}}(\mathbf{600}) = \mathbf{0}$

Para $\boldsymbol{k} = \mathbf{2}$:

$$\mathbf{p}^{*\mathbf{2}}(\mathbf{0}) = \sum_{\mathrm{h}=0}^{0} \mathrm{p}^{*2-1}(0 - \mathrm{h})\mathrm{p}(\mathrm{h})$$

$$\mathbf{p}^{*\mathbf{2}}(\mathbf{100}) = \sum_{\mathrm{h}=0}^{100} \mathrm{p}^{*2-1}(100 - \mathrm{h})\mathrm{p}(\mathrm{h})$$

$$\mathbf{p}^{*\mathbf{2}}(\mathbf{200}) = \sum_{h=0}^{200} p^{*2-1}(200 - h)p(h)$$

$$\mathbf{p}^{*\mathbf{2}}(\mathbf{300}) = \sum_{h=0}^{300} p^{*2-1}(300 - h)p(h)$$

$$\mathbf{p}^{*\mathbf{2}}(\mathbf{400}) = \sum_{h=0}^{400} p^{*2-1}(400 - h)p(h)$$

$$\mathbf{p}^{*\mathbf{2}}(\mathbf{500}) = \sum_{h=0}^{500} p^{*2-1}(500 - h)p(h)$$

$$\mathbf{p}^{*\mathbf{2}}(\mathbf{600}) = \sum_{h=0}^{600} p^{*2-1}(600 - h)p(h)$$

Para $\boldsymbol{k} = \mathbf{2}$:

$\mathbf{p}^{*2}(\mathbf{0}) = \mathrm{p}^{*1}(\,0)\mathrm{p}(0) = 0$

$\mathbf{p}^{*2}(\mathbf{100}) = \mathrm{p}^{*1}(100)\mathrm{p}(0) + \mathrm{p}^{*1}(0)\mathrm{p}(100) = 0$

$\boldsymbol{p}^{*2}(\mathbf{200}) = \; p^{*1}\,(200)p(0) + \boldsymbol{p}^{*\mathbf{1}}(\mathbf{100})\boldsymbol{p}(\mathbf{100}) + p^{*1}(0)p(200) = 0{,}04$

$\mathbf{p}^{*2}(\mathbf{300}) = \mathrm{p}^{*1}(300)\mathrm{p}(0) + \mathbf{p}^{*\mathbf{1}}(\mathbf{200})\mathbf{p}(\mathbf{100}) + \mathbf{p}^{*\mathbf{1}}(\mathbf{100})\mathbf{p}(\mathbf{200}) + \mathbf{p}^{*\mathbf{1}}(\mathbf{0})\mathbf{p}(\mathbf{300}) = \mathbf{0,28}$

$\mathbf{p}^{*2}(\mathbf{400}) = \mathrm{p}^{*1}(400)\mathrm{p}(0) + \mathbf{p}^{*\mathbf{1}}(\mathbf{300})\mathbf{p}(\mathbf{100}) + \mathbf{p}^{*\mathbf{1}}(\mathbf{200})\mathbf{p}(\mathbf{200}) + \mathbf{p}^{*\mathbf{1}}(\mathbf{100})\mathbf{p}(\mathbf{300}) + \mathrm{p}^{*1}(0)\mathrm{p}(400) = 0{,}53$

$\mathbf{p}^{*2}(\mathbf{500}) = \mathrm{p}^{*1}(500)\mathrm{p}(0) + \mathrm{p}^{*1}(400)\mathrm{p}(100) + \mathbf{p}^{*\mathbf{1}}(\mathbf{300})\mathbf{p}(\mathbf{200}) + \mathbf{p}^{*\mathbf{1}}(\mathbf{200})\mathbf{p}(\mathbf{300}) + \mathrm{p}^{*1}(100)\mathrm{p}(400)+\mathrm{p}^{*1}(0)\mathrm{p}(500) = 0{,}14$

$\mathbf{p}^{*2}(\mathbf{600}) = \mathrm{p}^{*1}(600)\mathrm{p}(0) + \mathrm{p}^{*1}(500)\mathrm{p}(100) + \mathrm{p}^{*1}(400)\mathrm{p}(200) + \mathbf{p}^{*\mathbf{1}}(\mathbf{300})\mathbf{p}(\mathbf{300}) + \mathrm{p}^{*1}(200)\mathrm{p}(400)+\mathrm{p}^{*1}(100)\mathrm{p}(500) + \mathrm{p}^{*1}(0)\mathrm{p}(600) = 0{,}01$

S_{col}	$P(N=0)=0,2$	$P(N=1)=0,5$	$P(N=2)=0,3$
	$N=0$	$N=1$	$N=2$
0	$\mathbf{p}^{*0}(\mathbf{0})=1$	$\mathbf{p}^{*1}(\mathbf{0})=0$	$\mathbf{p}^{*2}(\mathbf{0})=0$
100	$\mathbf{p}^{*0}(\mathbf{100})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{100})=\mathbf{0},2$	$\mathbf{p}^{*2}(\mathbf{100})=\mathbf{0}$
200	$p^{*0}(\mathbf{200})=0$	$p^{*1}(\mathbf{200})=0{,}7$	$p^{*2}(\mathbf{200})=0{,}04$
300	$\mathbf{p}^{*0}(\mathbf{300})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{300})=\mathbf{0,1}$	$\mathbf{p}^{*2}(\mathbf{300})=\mathbf{0,28}$
400	$\mathbf{p}^{*0}(\mathbf{400})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{400})=\mathbf{0}$	$\mathbf{p}^{*2}(\mathbf{400})=\mathbf{0,53}$
500	$\mathbf{p}^{*0}(\mathbf{500})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{500})=\mathbf{0}$	$\mathbf{p}^{*2}(\mathbf{500})=\mathbf{0,14}$
600	$\mathbf{p}^{*0}(\mathbf{600})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{600})=\mathbf{0}$	$\mathbf{p}^{*2}(\mathbf{600})=\mathbf{0,01}$
	1	**1**	**1**

Agora se faz necessário sumarizar todas as combinações que resultam no mesmo valor de sinistros.

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s)p_N(k)$$

Logo

$$\mathbf{P_{S_{col}}(0)} = p^{*0}(0)p_N(0) + p^{*1}(0)p_N(1) + p^{*2}(0)p_N(2) = 0{,}2$$

$$\mathbf{P_{S_{col}}(100)} = p^{*0}(100)p_N(0) + p^{*1}(100)p_N(1) + p^{*2}(100)p_N(2) = 0{,}1$$

$$\mathbf{P_{S_{col}}(200)} = p^{*0}(200)p_N(0) + p^{*1}(200)p_N(1) + p^{*2}(200)p_N(2) = 0{,}362$$

$$\mathbf{P_{S_{col}}(300)} = p^{*0}(300)p_N(0) + p^{*1}(300)p_N(1) + p^{*2}(300)p_N(2) = 0{,}134$$

$$\mathbf{P_{S_{col}}(400)} = p^{*0}(400)p_N(0) + p^{*1}(400)p_N(1) + p^{*2}(400)p_N(2) = 0{,}159$$

$$\mathbf{P_{S_{col}}(500)} = p^{*0}(500)p_N(0) + p^{*1}(500)p_N(1) + p^{*2}(500)p_N(2) = 0{,}042$$

$$\mathbf{P_{S_{col}}(600)} = p^{*0}(600)p_N(0) + p^{*1}(600)p_N(1) + p^{*2}(600)p_N(2) = 0{,}003$$

$$p_{S_{col}}(s)=\begin{bmatrix}1 & 0 & 0\\ 0 & 0,2 & 0\\ 0 & 0,7 & 0,04\\ 0 & 0,1 & 0,28\\ 0 & 0 & 0,53\\ 0 & 0 & 0,14\\ 0 & 0 & 0,01\end{bmatrix}\begin{bmatrix}0,2\\ 0,5\\ 0,3\end{bmatrix}\begin{array}{l}\rightarrow \boldsymbol{P_N(0)}\\ \rightarrow \boldsymbol{P_N(1)}\\ \rightarrow \boldsymbol{P_N(2)}\end{array}$$

$\mathrm{p}^{*\mathbf{2}}(\mathrm{s})$

$\mathrm{p}^{*\mathbf{1}}(\mathrm{s})$

$\mathrm{p}^{*\mathbf{0}}(\mathrm{s})$

$$\mathbf{P}_{\mathbf{S_{col}}}(\mathbf{0}) = 1(0,2) + 0(0,5) + 0(0,3) = 0,2$$

$$\mathbf{P}_{\mathbf{S_{col}}}(\mathbf{100}) = 0(0,2) + 0,2(0,5) + 0(0,3) = 0,1$$

...

$$\mathbf{P}_{\mathbf{S_{col}}}(\mathbf{600}) = 0(0,2) + 0(0,5) + 0,01(0,3) = 0,003$$

$$
p_{Scol}(s) = \begin{cases} 0{,}2 & s = 0 \\ 0{,}1 & s = 100 \\ 0{,}362 & s = 200 \\ 0{,}134 & s = 300 \\ 0{,}159 & s = 400 \\ 0{,}042 & s = 500 \\ 0{,}003 & s = 600 \end{cases}
$$

$$
F_{Scol}(s) = \begin{cases} 0 & s < 0 \\ 0{,}2 & 0 \le s < 100 \\ 0{,}2 + 0{,}1 = 0{,}3 & 100 \le s < 200 \\ 0{,}3 + 0{,}362 = 0{,}662 & 200 \le s < 300 \\ 0{,}662 + 0{,}134 = 0{,}796 & 300 \le s < 400 \\ 0{,}796 + 0{,}159 = 0{,}955 & 400 \le s < 500 \\ 0{,}955 + 0{,}042 = 0{,}997 & 500 \le s < 600 \\ 1 & s \ge 600 \end{cases}
$$

Modelos de risco Coletivo-Convolução

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s) p_N(k)$$

$$P^{*k}(s) = P(X_1 + X_2 + .. + X_k \leq s)$$

Quando X é discreto tem-se

$$P^{*0}(s) = \begin{cases} 0 \; se \; s \leq 0 \\ 1 \; se \; s > 0 \end{cases}$$

$$P^{*k}(s) = \sum_{h \leq s} P^{*k-1}(s-h) p(h)$$

Considere h como um dos valores possíveis para X.

- Exemplo

Uma carteira de seguros produz 0,1 ou 2 sinistros com as respetivas probabilidades: 20%,50% e 30%. Um sinistro dessa carteira assume os valores $R\$100$, $R\$200$ ou $R\$300$, com as respectivas probabilidades: $20\%, 70\%$ e 10%.

Construa a distribuição convoluta dos sinistros agregados S_{col}.

X_i	$R\$100$	$R\$200$	$R\$300$
$P_{X_i}(x_i)$	0,2	0,7	0,1

N	$P(N)$	S_{col}	
0	0,2	$S_{col} = 0$	
1	0,5	$S_{col} = X_1$	$\{R\$100, R\$200, R\$300\}$
2	0,3	$S_{col} = X_1 + X_2$	$\{R\$200, R\$300, R\$400, R\$500, R\$600\}$

Por definição tem-se que $P^{*0}(s) = \begin{cases} 0 \ se \ s \leq 0 \\ 1 \ se \ \ s > 0 \end{cases}$

Logo para $\boldsymbol{k = 0}$:

$P^{*0}(0) = 0$;

$P^{*0}(100) = 1$;

$P^{*0}(200) = 1$;

$P^{*0}(300) = 1$;

$P^{*0}(400) = 1$;

$P^{*0}(500) = 1$

$P^{*0}(600) = 1$

Para $\boldsymbol{k} = \mathbf{1}$:

Usando $P^{*\mathrm{k}}(\mathrm{s}) = \sum_{\mathrm{h} \leq \mathrm{s}} P^{*\mathrm{k}-1}(\mathrm{s} - \mathrm{h})p(\mathrm{h})$ sendo k os possíveis valores assumidos por N.

$$P^{*1}(0) = \sum_{h=0}^{0} P^{*1-1}(0 - h)p(h)$$

$$P^{*1}(100) = \sum_{h=0}^{100} P^{*1-1}(100 - h)p(h)$$

$$P^{*1}(200) = \sum_{h=0}^{200} P^{*1-1}(200 - h)p(h)$$

$$P^{*1}(300) = \sum_{h=0}^{300} P^{*1-1}(300 - h)p(h)$$

$$P^{*1}(400) = \sum_{h=0}^{400} P^{*1-1}(400 - h)p(h)$$

$$P^{*1}(500) = \sum_{h=0}^{500} P^{*1-1}(500 - h)p(h)$$

$$P^{*1}(600) = \sum_{h=0}^{600} P^{*1-1}(600 - h)p(h)$$

$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{0}) = P^{*0}(\ 0)p(0) = 0$

$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{100}) = P^{*0}(100)p(0) + P^{*0}(0)p(100) = 0$

$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{200}) = P^{*0}\ (200)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{100})\boldsymbol{p}(\mathbf{100}) + P^{*0}(0)p(200) = 0{,}2$

$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{300}) = P^{*0}(300)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{200})\boldsymbol{p}(\mathbf{100}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{100})\boldsymbol{p}(\mathbf{200}) + P^{*0}(0)p(300) = 0{,}9$

$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{400}) = P^{*0}(400)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{300})\boldsymbol{p}(\mathbf{100}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{200})\boldsymbol{p}(\mathbf{200}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{100})\boldsymbol{p}(\mathbf{300}) +$
$P^{*0}(0)p(400) = \mathbf{1}$

$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{500}) = P^{*0}(500)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{400})\boldsymbol{p}(\mathbf{100}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{300})\boldsymbol{p}(\mathbf{200}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{200})\boldsymbol{p}(\mathbf{300}) +$
$P^{*0}(100)p(400) + P^{*0}(0)p(500) = \mathbf{1}$

$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{600}) = P^{*0}(600)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{500})\boldsymbol{p}(\mathbf{100}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{400})\boldsymbol{p}(\mathbf{200}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{300})\boldsymbol{p}(\mathbf{300}) +$
$P^{*0}(200)p(400) + P^{*0}(100)p(500) + P^{*0}(0)p(600) = \mathbf{1}$

S_{col}	$N = 0$	$N = 1$
0	$P^{*0}(0) = 0$	$P^{*1}(0) = 0$
100	$P^{*0}(100) = 1$	$P^{*1}(100) = 0$
200	$P^{*0}(200) = 1$	$P^{*1}(200) = 0{,}2$
300	$P^{*0}(300) = 1$	$P^{*1}(300) = 0{,}9$
400	$P^{*0}(400) = 1$	$P^{*1}(400) = 1$
500	$P^{*0}(500) = 1$	$P^{*1}(500) = 1$
600	$P^{*0}(600) = 1$	$P^{*1}(600) = 1$

Para $k = 2$:

$$P^{*2}(0) = \sum_{h=0}^{0} P^{*2-1}(0-h)p(h)$$

$$P^{*2}(100) = \sum_{h=0}^{100} P^{*2-1}(100-h)p(h)$$

$$P^{*2}(200) = \sum_{h=0}^{200} P^{*2-1}\,(200-h)p(h)$$

$$P^{*2}(300) = \sum_{h=0}^{300} P^{*2-1}\,(300-h)p(h)$$

$$P^{*2}(400) = \sum_{h=0}^{400} P^{*2-1}\,(400-h)p(h)$$

$$P^{*2}(500) = \sum_{h=0}^{500} P^{*2-1}\,(500-h)p(h)$$

$$P^{*2}(600) = \sum_{h=0}^{600} P^{*2-1}\,(600-h)p(h)$$

Para $\boldsymbol{k} = \mathbf{2}$:

$P^{*2}(0) = P^{*1}(0)p(0) = 0$

$P^{*2}(100) = P^{*1}(100)p(0) + P^{*1}(0)p(100) = 0$

$P^{*2}(200) = P^{*1}(200)p(0) + P^{*1}(100)p(100) + P^{*1}(0)p(200) = 0$

$P^{*2}(300) = P^{*1}(300)p(0) + \boldsymbol{P^{*1}(200)p(100)} + P^{*1}(100)p(200) + P^{*1}(0)p(300) = 0{,}04$

$P^{*2}(400) = P^{*1}(400)p(0) + \boldsymbol{P^{*1}(300)p(100)} + \boldsymbol{P^{*1}(200)p(200)} + P^{*1}(100)p(300) + P^{*1}(0)p(400) = 0{,}32$

$P^{*2}(500) = P^{*1}(500)p(0) + \boldsymbol{P^{*1}(400)p(100)} + \boldsymbol{P^{*1}(300)p(200)} + \boldsymbol{P^{*1}(200)p(300)} + P^{*1}(100)p(400) + P^{*1}(0)p(500) = 0{,}85$

$P^{*2}(600) = P^{*1}(600)p(0) + \boldsymbol{P^{*1}(500)p(100)} + \boldsymbol{P^{*1}(400)p(200)} + \boldsymbol{P^{*1}(300)p(300)} + P^{*1}(200)p(400) + P^{*1}(100)p(500) + P^{*1}(0)p(600) = 0{,}99$

$\boldsymbol{S_{col}}$	$\boldsymbol{P(N=0)=0,2}$	$\boldsymbol{P(N=1)=0,5}$	$\boldsymbol{P(N=2)=0,3}$
	$\boldsymbol{N=0}$	$\boldsymbol{N=1}$	$\boldsymbol{N=2}$
0	$P^{*0}(0)=0$	$P^{*1}(0)=0$	$P^{*2}(0)=0$
100	$P^{*0}(100)=1$	$P^{*1}(100)=0$	$P^{*2}(100)=0$
200	$P^{*0}(200)=1$	$P^{*1}(200)=0{,}2$	$P^{*2}(200)=0$
300	$P^{*0}(300)=1$	$P^{*1}(300)=0{,}9$	$P^{*2}(300)=0{,}04$
400	$P^{*0}(400)=1$	$P^{*1}(400)=1$	$P^{*2}(400)=0{,}32$
500	$P^{*0}(500)=1$	$P^{*1}(500)=1$	$P^{*2}(500)=0{,}85$
600	$P^{*0}(600)=1$	$P^{*1}(600)=1$	$P^{*2}(600)=0{,}99$

$$P_{S_{col}}(s) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0{,}2 & 0 \\ 1 & 0{,}9 & 0{,}04 \\ 1 & 1 & 0{,}32 \\ 1 & 1 & 0{,}85 \\ 1 & 1 & 0{,}99 \end{bmatrix} \begin{bmatrix} 0{,}2 \\ 0{,}5 \\ 0{,}3 \end{bmatrix} \begin{array}{l} \Rightarrow \boldsymbol{P_N(0)} \\ \Rightarrow \boldsymbol{P_N(1)} \\ \Rightarrow \boldsymbol{P_N(2)} \end{array}$$

$\mathbf{P}^{*\mathbf{2}}(s)$

$\mathbf{P}^{*\mathbf{1}}(s)$

$\mathbf{P}^{*\mathbf{0}}(s)$

$$F_{S_{col}}(s) = \sum_{k=0}^{n} {\color{red} P^{*k}(s)}\, p_N(k)$$

$$F_{Scol}(s) = \begin{cases} 0 & s < 0 \\ 0{,}2 & 0 \leq s < 100 \\ 0{,}3 & 100 \leq s < 200 \\ 0{,}662 & 200 \leq s < 300 \\ 0{,}796 & 300 \leq s < 400 \\ 0{,}955 & 400 \leq s < 500 \\ 0{,}997 & 500 \leq s < 600 \\ 1 & s \geq 600 \end{cases}$$

$$F_{Scol}(s) = \begin{cases} 0 & s < 0 \\ 0{,}2 & 0 \leq s < 100 \\ 0{,}2 + 0{,}1 = 0{,}3 & 100 \leq s < 200 \\ 0{,}3 + 0{,}362 = 0{,}662 & 200 \leq s < 300 \\ 0{,}662 + 0{,}134 = 0{,}796 & 300 \leq s < 400 \\ 0{,}796 + 0{,}159 = 0{,}955 & 400 \leq s < 500 \\ 0{,}955 + 0{,}042 = 0{,}997 & 500 \leq s < 600 \\ 1 & s \geq 600 \end{cases}$$

➢ Exemplo

Suponha uma carteira composta por 2 apólices identicamente distribuídas e independentes.

X_i	$R\$0{,}00$	$R\$1000{,}00$	$R\$2000{,}00$	$R\$3000{,}00$
$P(X_i)$	0,6	0,02	0,06	0,32

Modelando essa carteira de acordo com modelo de risco individual. Obtenha a função de probabilidade de S_{ind}.

Solução:

Uma vez definido X_i, é feito o processo de convoluçao entre $X_1 + X_2$, por meio de

$$p_S(s) = p_{X_1} * p_{X_2}(s) = \sum_{\forall\, x_1 \leq s} p_{X_2}(s - x_1)\, p_{X_1}(x_1)$$

X_i	$R\$0,00$	$R\$1000,00$	$R\$2000,00$	$R\$3000,00$
$P(X_i)$	0,6	0,02	0,06	0,32

$$p_S(s) = p_{X_1} * p_{X_2}(s) = \sum_{\forall\, x_1 \leq s} p_{X_2}(s - x_1) p_{X_1}(x_1)$$

S	$S(X_1, X_2)$	P_S
0	(0,0)	0,36
1000	(1000,0) (0,1000)	0,024
2000	(2000,0)(1000,1000)(0,2000)	0,0724
3000	(3000,0)(2000,1000)(1000,2000)(0,3000)	0,3864
4000	(3000,1000)(2000,2000)(1000,3000)	0,00164
5000	(3000,2000)(2000,3000)	0,0384
6000	(3000,3000)	0,1024

➢ Exemplo

Suponha uma carteira composta por 2 apólices identicamente distribuídas e independentes.

X_i	$R\$0,00$	$R\$1000,00$	$R\$2000,00$	$R\$3000,00$
$P(X_i)$	0,6	0,02	0,06	0,32

Modelando essa carteira de acordo com modelo de risco coletivo. Obtenha a função de probabilidade de S_{col}.

- Exemplo

X_i	$R\$0{,}00$	$R\$1000{,}00$	$R\$2000{,}00$	$R\$3000{,}00$
$P(X_i)$	0,6	0,02	0,06	0,32

Solução:

X_i	$P(X_i = x_i)$	I_i	$P(I_i = i_i)$	$B_i = (X_i \mid I_i = 1)$	$P(B_i = b_i)$
R$0,00	0,6	0	0,6		
R$1000,00	0,02	1	0,4	R$1000,00	$\frac{0{,}02}{0{,}4} = 0{,}05$
R$2000,00	0,06			R$2000,00	$\frac{0{,}06}{0{,}4} = 0{,}15$
$R\$3000{,}00$	0,32			R$3000,00	$\frac{0{,}32}{0{,}4} = 0{,}8$

N	$P(N) = \binom{2}{n} 0{,}4^n 0{,}6^{2-n}$	S_{col}	Possíveis valores para S_{col}.
0	0,36	$S_{col} = 0$	
1	0,48	$S_{col} = X_i \; \forall \text{ i} = 1{,}2$	$\{R\$1000, R\$2000, R\$3000\}$
2	0,16	$S_{col} = X_1 + X_2$	$\{R\$2000, R\$3000, R\$4000, R\$5000, R\$6000\}$

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) \cdot p_N(k)$$

$$p^{*k}(s) = p[X_1 + X_2 + .. + X_k = s]$$

Quando X é discreto tem-se

$$p^{*0}(s) = \begin{cases} 0 \; se \; s \neq 0 \\ 1 \; se \; s = 0 \end{cases}$$

$p^{*k}(s) = \sum_{h \leq s} p^{*k-1}(s-h)p(h)$

Considere h *como um dos valores possíveis para* X.

S_{col}	$P(N=0)=0,36$	$P(N=1)=0,48$	$P(N=2)=0,16$
	$N=0$	$N=1$	$N=2$
0	$\mathbf{p}^{*0}(\mathbf{0})=1$	$\mathbf{p}^{*1}(\mathbf{0})=0$	$\mathbf{p}^{*2}(\mathbf{0})=0$
1000	$\mathbf{p}^{*0}(\mathbf{1000})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{1000})=\mathbf{0},05$	$\mathbf{p}^{*2}(\mathbf{1000})=\mathbf{0}$
2000	$p^{*0}(\mathbf{2000})=0$	$p^{*1}(\mathbf{0200})=0{,}15$	$p^{*2}(\mathbf{2000})=0{,}0025$
3000	$\mathbf{p}^{*0}(\mathbf{3000})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{3000})=\mathbf{0,8}$	$\mathbf{p}^{*2}(\mathbf{3000})=\mathbf{0,015}$
4000	$\mathbf{p}^{*0}(\mathbf{4000})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{4000})=\mathbf{0}$	$\mathbf{p}^{*2}(\mathbf{4000})=\mathbf{0,1025}$
5000	$\mathbf{p}^{*0}(\mathbf{5000})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{5000})=\mathbf{0}$	$\mathbf{p}^{*2}(\mathbf{5000})=\mathbf{0,24}$
6000	$\mathbf{p}^{*0}(\mathbf{6000})=\mathbf{0}$	$\mathbf{p}^{*1}(\mathbf{6000})=\mathbf{0}$	$\mathbf{p}^{*2}(\mathbf{6000})=\mathbf{0,64}$
	1	**1**	**1**

$$P_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) \cdot p_N(k)$$

$$P_{S_{col}}(s) = \begin{array}{ccc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0{,}2 & 0 \\ 0 & 0{,}7 & 0{,}04 \\ 0 & 0{,}1 & 0{,}28 \\ 0 & 0 & 0{,}53 \\ 0 & 0 & 0{,}14 \\ 0 & 0 & 0{,}01 \end{bmatrix} & \begin{bmatrix} 0{,}36 \\ 0{,}48 \\ 0{,}16 \end{bmatrix} & \begin{array}{l} \boldsymbol{P_N(0)} \\ \boldsymbol{P_N(1)} \\ \boldsymbol{P_N(2)} \end{array} \end{array}$$

$$p^{*2}(s) \qquad p^{*1}(s) \qquad p^{*0}(s)$$

$$p_{Scol}(s) = \begin{cases} 0{,}36 & s = 0 \\ 0{,}0240 & s = 1000 \\ 0{,}0724 & s = 2000 \\ 0{,}3864 & s = 3000 \\ 0{,}0164 & s = 4000 \\ 0{,}0384 & s = 5000 \\ 0{,}1024 & s = 6000 \end{cases}.$$