# Teoria do Risco
# Aula 13

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| Modelo de Risco individual | Modelo de Risco coletivo |
|---|---|
| $X_i$ Independentes | $X_i$ Independentes e identicamente distribuídas |
| $S_{ind} = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} B_i I_i$ | $s_{col} = \sum_{i=1}^{N} X_i$ |
| $E(S_{ind}) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$ | $E(S_{col}) = E\left(\sum_{i=1}^{N} X_i\right)$ |
| $S_{ind}, X_i, B_i, I_i$ | $s_{col}, X_i, N$ |
| $M_{S_{ind}}(t) = \prod_{i=1}^{n} M_{X_i}(t)$ | $M_{S_{col}}(t) = M_N(ln\, M_X(t)\,)$ |
| $E(S_{ind}) = \sum_{i=1}^{n} E(B_i) q_i$ | $E(S_{col}) = E(N)E(X)$ |
| $var(S_{ind}) = \sum_{i=1}^{n} var(B_i) q_i + \sum_{i=1}^{n} E(B_i)^2 q_i (1 - q_i)$ | $var(S_{col}) = E(X)^2\, var(N) + E(N) var(X)$ |

# Modelos de risco Coletivo- A distribuição de $S_{col}$, os sinistros coletivos.

- O método da convolução a partir da distribuição de X e N.

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s)\, p(N = k)$$

$$f_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s)\, p(N = k)$$

$$P^{*k}(s) = P(X_1 + X_2 + \ldots + X_k \leq s)$$

$$p^{*k}(s) = p(\boldsymbol{X_1} + \boldsymbol{X_2} + \ldots + \boldsymbol{X_k} = \boldsymbol{s})$$

# Modelos de risco Coletivo

- O processo de convolução no modelo de risco coletivo leva em consideração a convolução entre os sinistros ocorridos dado que quantidade ocorrida também é uma variável aleatória.

| Modelo de risco individual | Modelo de risco coletivo |
|---|---|
| $\mathrm{F}^{(\mathrm{k})} = \mathrm{F}_{\mathrm{k}} * \mathrm{F}^{(\mathrm{k}-1)}$ | $P^{(k)} = P_k * P^{(k-1)}$ |
| $\mathrm{F}_{\mathrm{S}_{\mathrm{ind}}}^{(2)}(\mathrm{s}) = \sum_{j=0}^{s} F_X(s - y_j) p_Y(y_j)$ | $F_{S_{col}}^{(2)}(s) = \sum_{k=0}^{2} P^{*k}(s) p_N(k)$ |

$$X\ (discreto) \rightarrow S_{col}\ (discreto)$$
$$X(contínuo) \rightarrow S_{col}(contínuo)$$

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# Modelos de risco Coletivo

## Pelo método de convolução a partir da distribuição de X e N.

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s) p_N(k)$$

$$P^{*k}(s) = P(X_1 + X_2 + .. + X_k \leq s)$$

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) p_N(k)$$

$$p^{*k}(s) = p(\boldsymbol{X_1} + \boldsymbol{X_2} + .. + \boldsymbol{X_k} = \boldsymbol{s})$$

**Quando $X$ são contínuos.**

$$P^{*k}(s) = \int_0^s P^{*k-1}(s-h) p(h) dh$$

$$P^{*k-1*}P$$

$$p^{*k}(s) = \int_0^s p^{*k-1}(s-h) p(h) dh$$

$$p^{*k-1*}p$$

# Exemplo 1

Calcular $F_{S_{col}}(s)$, quando $X$ possui distribuição Exponencial $(\alpha)$ e $N$ possui distribuição de Poisson$(\lambda)$.

Lembrando que:

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s)\, p(N = k)$$

$$P^{*k}(s) = \int_0^s P^{*k-1}(s-h)p(h)dh$$

*Assim:*

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s) \frac{e^{-\lambda}\lambda^k}{k!}$$

## Cálculo de $P^{*k}(x)$

Sabendo que se $X$ tem distribuição Exponencial $(\alpha)$, então:

$$p(x) = \alpha e^{-\alpha x} \qquad P(x) = 1 - e^{-\alpha x}\ ; x > 0$$

$$P^{*k}(s) = \int_h P^{*k-1}(s-h)\, p(h)dh$$

$$P^{*(2)}(s) = \int_0^s P^{*2-1}(s-h)p(h)dh = \int_0^s P\,(s-h)p(h)dh$$

$$P^{*(2)}(s) = \int_0^s \left[1 - e^{-\alpha(s-h)}\right]\alpha e^{-\alpha h}dh$$

...

$$P^{*(2)}(s) = 1 - e^{-\alpha s}(1 + \alpha s)$$

$$P^{*3}(s) = \int_0^s P^{*3-1}(s-h)p(h)dh = \int_0^s P^{*2}(s-h)p(h)dh$$

$$P^{*3}(s) = \int_0^s \{1 - e^{-\alpha(s-h)}[1 + \alpha(s-h)]\}\alpha e^{-\alpha h}dh$$

...

$$P^{*3}(s) = 1 - e^{-\alpha s}\left[1 + \alpha s + \frac{(\alpha s)^2}{2!}\right]$$

- Desta forma, então, chega-se à seguinte formula de $P^{*n}$

$$\boldsymbol{P(s) = 1 - e^{-\alpha s}}$$

$$\boldsymbol{P^{*(2)}(s) = 1 - e^{-\alpha s}(1 + \alpha s)}$$

$$\boldsymbol{P^{*3}(s) = 1 - e^{-\alpha s}\left[1 + \alpha s + \frac{(\alpha s)^2}{2!}\right]}$$

...

$$P^{*k}(s) = 1 - e^{-\alpha s}\sum_{i=0}^{k-1}\frac{(\alpha s)^i}{i!}$$

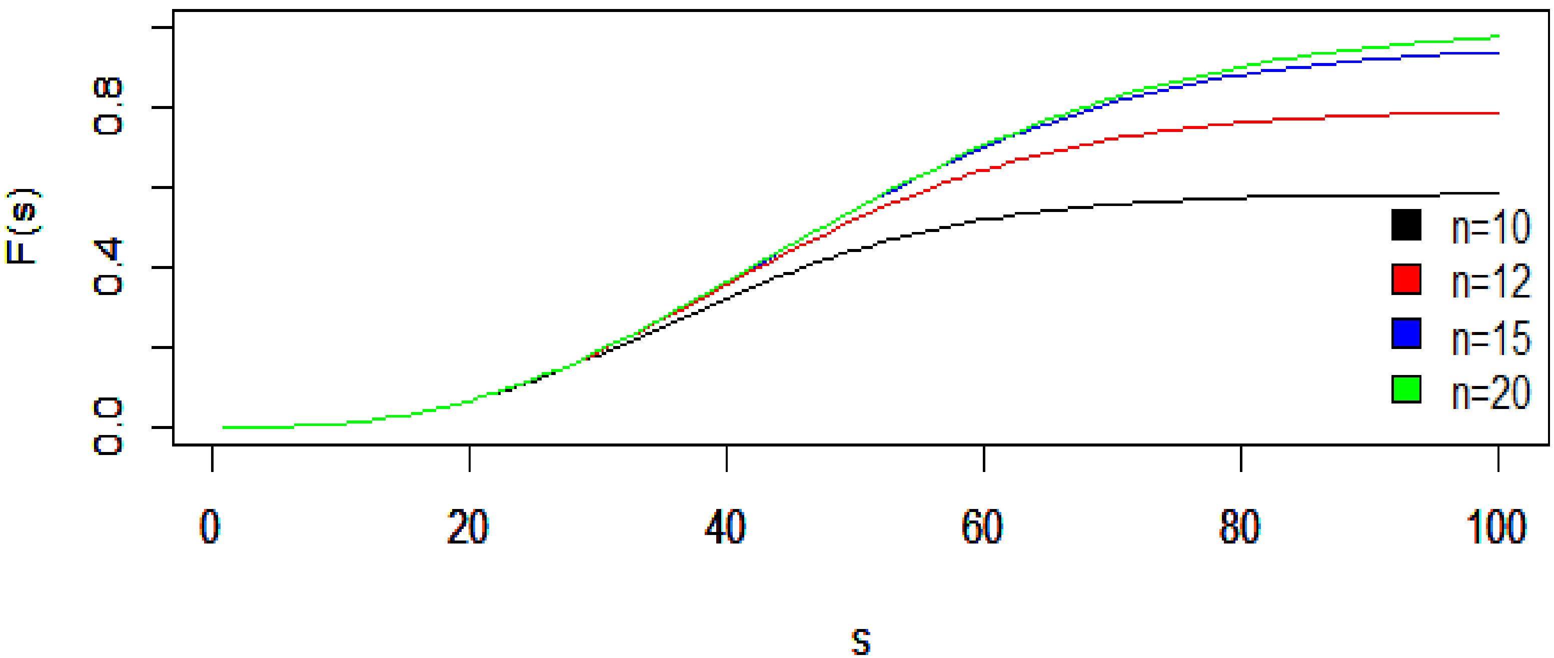
Como:

$$F_{S_{col}}(s) = \sum_{k=1}^{n} P^{*k}(s) \frac{e^{-\lambda}\lambda^{k}}{k!}$$

Tem-se que:

$$F_{S_{col}}(s) = \sum_{k=1}^{n} \left[1 - e^{-\alpha s} \sum_{i=0}^{k-1} \frac{(\alpha s)^{i}}{i!}\right] \frac{e^{-\lambda}\lambda^{k}}{k!}$$

$$F_{S_{col}}(s) = \sum_{k=1}^{n} \left[ 1 - e^{-\alpha s} \sum_{i=0}^{k-1} \frac{(\alpha s)^i}{i!} \right] \frac{e^{-\lambda} \lambda^k}{k!}$$




**Comportamento de $F_{s_{col}}(S)$ com $\alpha = 0,2, \lambda = 10$ para diferentes quantidade de apólices $n$.**

# Exemplo 2

Adicionalmente pode-se calcular $p^{*k}$ e $f_{S_{col}}(s)$, quando $X$ possui distribuição *Exponencial* $(\alpha)$ e *N* possui distribuição de *Poisson*$(\lambda)$.

Lembrando que:

$$f_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s)\, p(N = k)$$

$$p^{*k}(s) = \int_0^s p^{*k-1}(s-h)p(h)dh$$

*Assim:*

$$f_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) \frac{e^{-\lambda}\lambda^k}{k!}$$

## Cálculo de $P^{*k}(x)$

Sabendo que se $X$ tem distribuição Exponencial $(\alpha)$, então:

$$p(x) = \alpha e^{-\alpha x} \; ; x > 0$$

$$p^{*k}(s) = \int_h p^{*k-1}(s-h)\, p(h) dh$$

$$p^{*(2)}(s) = \int_0^s p^{*2-1}(s-h) p(h) dh = \int_0^s p(s-h) p(h) dh$$

$$p^{*(2)}(s) = \int_0^s \left[\alpha e^{-\alpha(s-h)}\right] \alpha e^{-\alpha h} dh = \alpha^2 s\, e^{-\alpha s},$$

## Cálculo de $P^{*k}(x)$

Sabendo que se $X$ tem distribuição Exponencial $(\alpha)$, então:

$$p(x) = \alpha e^{-\alpha x}\,; x > 0$$

$$p^{*k}(s) = \int_h p^{*k-1}(s-h)\,p(h)dh$$

$$p^{*(2)}(s) = \alpha^2 s\, e^{-\alpha s}$$

$$p^{*3}(s) = \int_0^s p^{*3-1}(s-h)p(h)dh = \int_0^x p^{*2}(s-h)p(h)dh$$

$$p^{*3}(s) = \int_0^s \alpha^2 (s-h)\, e^{-\alpha(s-h)} \alpha e^{-\alpha h} dh = \frac{\alpha^3 s^2\, e^{-\alpha s}}{2}$$

## Cálculo de $\boldsymbol{P^{*k}(x)}$

Sabendo que se $X$ tem distribuição Exponencial $(\alpha)$, então:

$$p(x) = \alpha e^{-\alpha x} \; ; x > 0$$

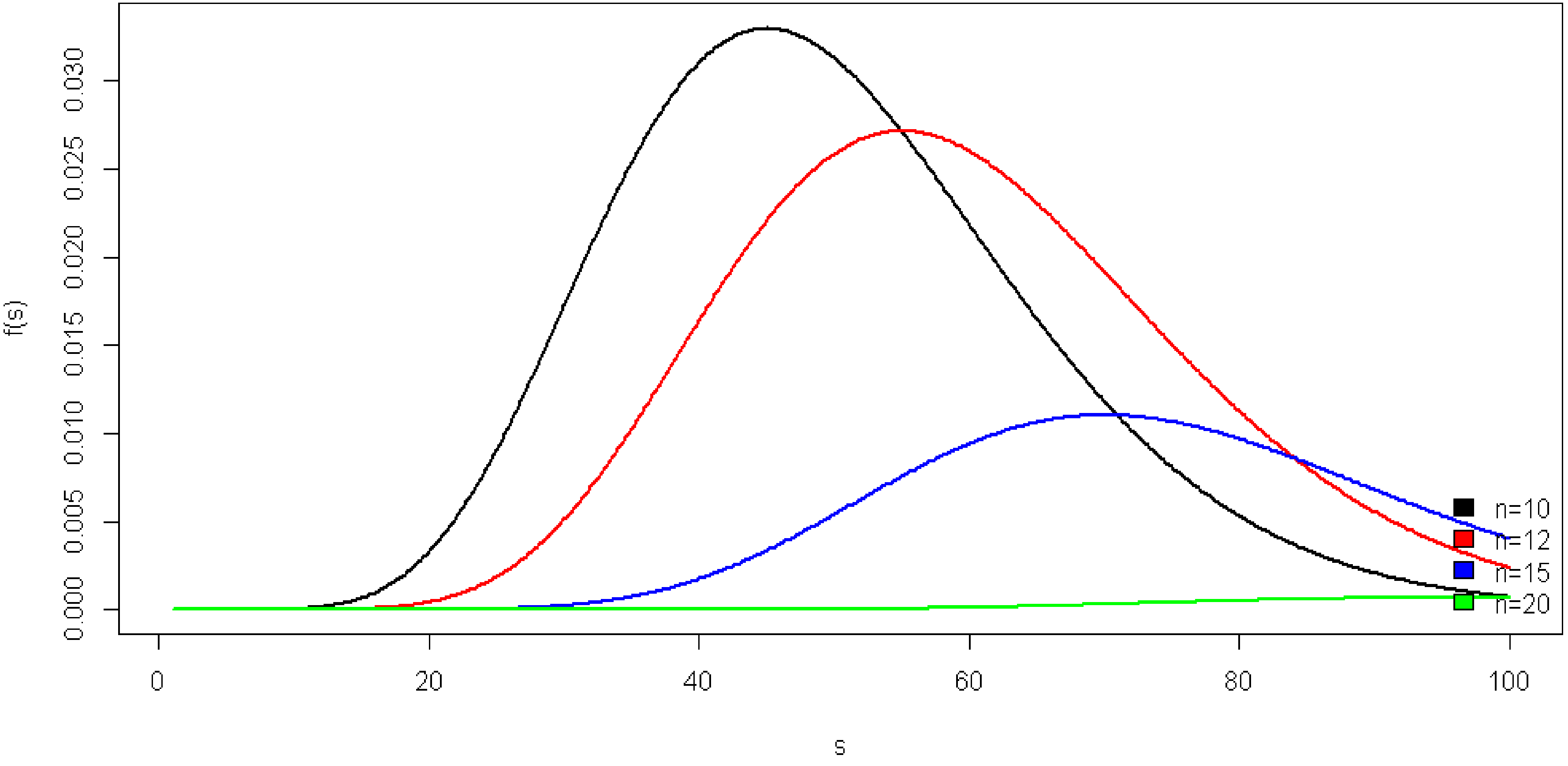
$$p^{*(2)}(s) = \alpha^2 s \, e^{-\alpha s},$$

$$p^{*3}(s) = \frac{\alpha^3 s^2 \, e^{-\alpha s}}{2}$$

$$\ldots$$

$$\boldsymbol{p^{*n}(s) = \frac{\alpha^n s^{n-1} e^{-\alpha s}}{(n-1)!}}$$

$$f_{S_{col}}(s) = \sum_{k=1}^{n} \left[ \frac{\alpha^k s^{k-1} e^{-\alpha s}}{(k-1)!} \right] \frac{e^{-\lambda} \lambda^k}{k!}$$




**Comportamento de $f_{s_{col}}(S)$ com $\alpha = 0,2, \lambda = 10$ para diferentes quantidade de apólices $n$.**

# Modelos de risco Coletivo

## Pelo método de convolução a partir da distribuição de X e N.

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s) p_N(k)$$

$$P^{*k}(s) = P(X_1 + X_2 + \ldots + X_k \leq s)$$

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) p_N(k)$$

$$p^{*k}(s) = p(\boldsymbol{X_1} + \boldsymbol{X_2} + \ldots + \boldsymbol{X_k} = \boldsymbol{s})$$

# Modelos de risco Coletivo

## Pelo método de convolução a partir da distribuição de X e N.

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s) p_N(k)$$

$$p^{*k}(s) = p(X_1 + X_2 + \ldots + X_k = s)$$

**Quando X é discreto tem-se**

$$p^{*0}(s) = \begin{cases} 0 \; se \; s \neq 0 \\ 1 \; se \; s = 0 \end{cases}$$

$$p^{*k}(s) = \sum_{h \leq s} p^{*k-1}(s-h) p(h)$$

*Considere $h$ como um dos valores possíveis para $X$.*

# Exemplo 4

Uma carteira de seguros produz 0,1 ou 2 sinistros com as respetivas probabilidades: 20%,50% e 30%. Um sinistro dessa carteira assume os valores $R\$100$, $R\$200$ ou $R\$300$, com as respectivas probabilidades: $20\%, 70\%$ e 10%.

Construa a distribuição convoluta dos sinistros agregados $S_{col}$.

| $X_i$ | $R\$100$ | $R\$200$ | $R\$300$ |
|---|---|---|---|
| $P_{X_i}(x_i)$ | 0,2 | 0,7 | 0,1 |

| $N$ | $P(N)$ | $S_{col}$ | |
|---|---|---|---|
| 0 | 0,2 | $S_{col} = 0$ | |
| 1 | 0,5 | $S_{col} = X_1$ | $\{R\$100, R\$200, R\$300\}$ |
| 2 | 0,3 | $S_{col} = X_1 + X_2$ | $\{R\$200, R\$300, R\$400, R\$500, R\$600\}$ |

Em primeiro lugar, computemos todas as combinações possíveis de frequência e severidades e assim explicitemos os valores possíveis de sinistros agregados e associados as probabilidades de ocorrência

Por definição tem-se que $\boldsymbol{p^{*0}(s)} = \begin{cases} \mathbf{0}\ \boldsymbol{se}\ \boldsymbol{s} \neq \mathbf{0} \\ \mathbf{1}\ \boldsymbol{se}\ \ \boldsymbol{s} = \mathbf{0} \end{cases}$

- Logo para $\boldsymbol{k} = \mathbf{0}$:

$p^{*0}(0) = 1$

$p^{*0}(100) = 0$

$p^{*0}(200) = 0$

$p^{*0}(300) = 0$

$p^{*0}(400) = 0$

$p^{*0}(500) = 0$

$p^{*0}(600) = 0$

Para $\boldsymbol{k = 1}$:

Usando $p^{*\mathrm{k}}(\mathrm{s}) = \sum_{\mathrm{h}\leq \mathrm{s}} p^{*\mathrm{k}-1}(\mathrm{s}-\mathrm{h})p(\mathrm{h})$ sendo $k$ os possíveis valores assumidos por $N$.

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{0}) = \sum_{h=0}^{0} p^{*1-1}(0-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{100}) = \sum_{h=0}^{100} p^{*1-1}(100-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{200}) = \sum_{h=0}^{200} p^{*1-1}(200-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{300}) = \sum_{h=0}^{300} p^{*1-1}(300-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{400}) = \sum_{h=0}^{400} p^{*1-1}(400-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{500}) = \sum_{h=0}^{500} p^{*1-1}(500-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{600}) = \sum_{h=0}^{600} p^{*1-1}(600-h)p(h)$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{0}) = p^{*0}(\,0)p(0) = 0$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{100}) = p^{*0}(100)p(0) + \boldsymbol{p}^{*\mathbf{0}}(\mathbf{0})\boldsymbol{p}(\mathbf{100}) = \mathbf{0,2}$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{200}) = p^{*0}\,(200)p(0) + p^{*0}(100)p(100) + \boldsymbol{p}^{*\mathbf{0}}(\mathbf{0})\boldsymbol{p}(\mathbf{200}) = 0{,}7$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{300}) = p^{*0}(300)p(0) + p^{*0}(200)p(100) + p^{*0}(100)p(200) + \boldsymbol{p}^{*\mathbf{0}}(\mathbf{0})\boldsymbol{p}(\mathbf{300}) = \mathbf{0,1}$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{400}) = p^{*0}(400)p(0) + p^{*0}(300)p(100) + p^{*0}(200)p(200) + p^{*0}(100)p(300) + p^{*0}(0)p(400) = \mathbf{0}$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{500}) = p^{*0}(500)p(0) + p^{*0}(400)p(100) + p^{*0}(300)p(200) + p^{*0}(200)p(300) + p^{*0}(100)p(400) + p^{*0}(0)p(500) = \mathbf{0}$$

$$\boldsymbol{p}^{*\mathbf{1}}(\mathbf{600}) = p^{*0}(600)p(0) + p^{*0}(500)p(100) + p^{*0}(400)p(200) + p^{*0}(300)p(300) + p^{*0}(200)p(400) + p^{*0}(100)p(500) + p^{*0}(0)p(600) = \mathbf{0}$$

| $S_{col}$ | $N = 0$ | $N = 1$ |
|---|---|---|
| $0$ | $p^{*0}(0) = 1$ | $p^{*1}(0) = 0$ |
| $100$ | $p^{*0}(100) = 0$ | $p^{*1}(100) = 0,2$ |
| $200$ | $p^{*0}(200) = 0$ | $p^{*1}(200) = 0,7$ |
| $300$ | $p^{*0}(300) = 0$ | $p^{*1}(300) = 0,1$ |
| $400$ | $p^{*0}(400) = 0$ | $p^{*1}(400) = 0$ |
| $500$ | $p^{*0}(500) = 0$ | $p^{*1}(500) = 0$ |
| $600$ | $p^{*0}(600) = 0$ | $p^{*1}(600) = 0$ |

Para $\boldsymbol{k = 2}$:

$$\boldsymbol{p^{*2}(0)} = \sum_{h=0}^{0} p^{*2-1}(0-h)p(h)$$

$$\boldsymbol{p^{*2}(100)} = \sum_{h=0}^{100} p^{*2-1}(100-h)p(h)$$

$$\boldsymbol{p^{*2}(200)} = \sum_{h=0}^{200} p^{*2-1}(200-h)p(h)$$

$$\boldsymbol{p^{*2}(300)} = \sum_{h=0}^{300} p^{*2-1}(300-h)p(h)$$

$$\boldsymbol{p^{*2}(400)} = \sum_{h=0}^{400} p^{*2-1}(400-h)p(h)$$

$$\boldsymbol{p^{*2}(500)} = \sum_{h=0}^{500} p^{*2-1}(500-h)p(h)$$

$$\boldsymbol{p^{*2}(600)} = \sum_{h=0}^{600} p^{*2-1}(600-h)p(h)$$

Para $\boldsymbol{k=2}$:

$\boldsymbol{p^{*2}(0)} = p^{*1}(0)p(0) = 0$

$\boldsymbol{p^{*2}(100)} = p^{*1}(100)p(0) + p^{*1}(0)p(100) = 0$

$\boldsymbol{p^{*2}(200)} = p^{*1}(200)p(0) + \boldsymbol{p^{*1}(100)p(100)} + p^{*1}(0)p(200) = 0{,}04$

$\boldsymbol{p^{*2}(300)} = p^{*1}(300)p(0) + \boldsymbol{p^{*1}(200)p(100)} + \boldsymbol{p^{*1}(100)p(200)} + \boldsymbol{p^{*1}(0)p(300) = 0,28}$

$\boldsymbol{p^{*2}(400)} = p^{*1}(400)p(0) + \boldsymbol{p^{*1}(300)p(100)} + \boldsymbol{p^{*1}(200)p(200)} + \boldsymbol{p^{*1}(100)p(300)} + p^{*1}(0)p(400) = 0{,}53$

$\boldsymbol{p^{*2}(500)} = p^{*1}(500)p(0) + p^{*1}(400)p(100) + \boldsymbol{p^{*1}(300)p(200)} + \boldsymbol{p^{*1}(200)p(300)} + p^{*1}(100)p(400) + p^{*1}(0)p(500) = 0{,}14$

$\boldsymbol{p^{*2}(600)} = p^{*1}(600)p(0) + p^{*1}(500)p(100) + p^{*1}(400)p(200) + \boldsymbol{p^{*1}(300)p(300)} + p^{*1}(200)p(400) + p^{*1}(100)p(500) + p^{*1}(0)p(600) = 0{,}01$

| $\boldsymbol{S_{col}}$ | $\boldsymbol{P(N=0)=0,2}$ | $\boldsymbol{P(N=1)=0,5}$ | $\boldsymbol{P(N=2)=0,3}$ |
|---|---|---|---|
| | $\boldsymbol{N=0}$ | $\boldsymbol{N=1}$ | $\boldsymbol{N=2}$ |
| **0** | $p^{*0}(0)=1$ | $\boldsymbol{p^{*1}(0)=0}$ | $\boldsymbol{p^{*2}(0)=0}$ |
| **100** | $p^{*0}(100)=0$ | $\boldsymbol{p^{*1}(100)=0,2}$ | $\boldsymbol{p^{*2}(100)=0}$ |
| **200** | $p^{*0}(200)=0$ | $\boldsymbol{p^{*1}(200)=0,7}$ | $\boldsymbol{p^{*2}(200)=0,04}$ |
| 300 | $p^{*0}(300)=0$ | $p^{*1}(300)=0{,}1$ | $p^{*2}(300)=0{,}28$ |
| 400 | $p^{*0}(400)=0$ | $p^{*1}(400)=0$ | $p^{*2}(400)=0{,}53$ |
| 500 | $p^{*0}(500)=0$ | $p^{*1}(500)=0$ | $p^{*2}(500)=0{,}14$ |
| 600 | $p^{*0}(600)=0$ | $p^{*1}(600)=0$ | $p^{*2}(600)=0{,}01$ |
| | **1** | **1** | **1** |

Agora se faz necessário sumarizar todas as combinações que resultam no mesmo valor de sinistros.

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s)p_N(k)$$

Logo

$$\boldsymbol{P_{S_{col}}(0)} = p^{*0}(0)p_N(0) + p^{*1}(0)p_N(1) + p^{*2}(0)p_N(2) = 0{,}2$$

$$\boldsymbol{P_{S_{col}}(100)} = p^{*0}(100)p_N(0) + p^{*1}(100)p_N(1) + p^{*2}(100)p_N(2) = 0{,}1$$

$$\boldsymbol{P_{S_{col}}(200)} = p^{*0}(200)p_N(0) + p^{*1}(200)p_N(1) + p^{*2}(200)p_N(2) = 0{,}362$$

$$\boldsymbol{P_{S_{col}}(300)} = p^{*0}(300)p_N(0) + p^{*1}(300)p_N(1) + p^{*2}(300)p_N(2) = 0{,}134$$

$$\boldsymbol{P_{S_{col}}(400)} = p^{*0}(400)p_N(0) + p^{*1}(400)p_N(1) + p^{*2}(400)p_N(2) = 0{,}159$$

$$\boldsymbol{P_{S_{col}}(500)} = p^{*0}(500)p_N(0) + p^{*1}(500)p_N(1) + p^{*2}(500)p_N(2) = 0{,}042$$

$$\boldsymbol{P_{S_{col}}(600)} = p^{*0}(600)p_N(0) + p^{*1}(600)p_N(1) + p^{*2}(600)p_N(2) = 0{,}003$$

$$p_{S_{col}}(s) = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0{,}2 & 0 \\ 0 & 0{,}7 & 0{,}04 \\ 0 & 0{,}1 & 0{,}28 \\ 0 & 0 & 0{,}53 \\ 0 & 0 & 0{,}14 \\ 0 & 0 & 0{,}01 \end{bmatrix} \\ \begin{array}{ccc} \mathrm{p}^{*\mathbf{0}}(\mathrm{s}) & \mathrm{p}^{*\mathbf{1}}(\mathrm{s}) & \mathrm{p}^{*\mathbf{2}}(\mathrm{s}) \end{array} \end{array} \begin{array}{cc} \begin{bmatrix} 0{,}2 \\ 0{,}5 \\ 0{,}3 \end{bmatrix} & \begin{array}{l} \boldsymbol{P_N(0)} \\ \boldsymbol{P_N(1)} \\ \boldsymbol{P_N(2)} \end{array} \end{array}$$

$$\boldsymbol{P_{S_{col}}(0)} = 1 \times 0{,}2 + 0 \times 0{,}5 + 0 \times 0{,}3 = 0{,}2$$

$$\boldsymbol{P_{S_{col}}(100)} = 0 \times 0{,}2 + 0{,}2 \times 0{,}5 + 0 \times 0{,}3 = 0{,}1$$

...

$$\boldsymbol{P_{S_{col}}(600)} = 0 \times 0{,}2 + 0 \times 0{,}5 + 0{,}01 \times 0{,}3 = 0{,}003$$

$$p_{Scol}(s) = \begin{cases} 0{,}2 & s = 0 \\ 0{,}1 & s = 100 \\ 0{,}362 & s = 200 \\ 0{,}134 & s = 300 \\ 0{,}159 & s = 400 \\ 0{,}042 & s = 500 \\ 0{,}003 & s = 600 \end{cases}$$

$$F_{Scol}(s) = \begin{cases} 0 & s < 0 \\ 0{,}2 & 0 \leq s < 100 \\ 0{,}2 + 0{,}1 = 0{,}3 & 100 \leq s < 200 \\ 0{,}3 + 0{,}362 = 0{,}662 & 200 \leq s < 300 \\ 0{,}662 + 0{,}134 = 0{,}796 & 300 \leq s < 400 \\ 0{,}796 + 0{,}159 = 0{,}955 & 400 \leq s < 500 \\ 0{,}955 + 0{,}042 = 0{,}997 & 500 \leq s < 600 \\ 1 & s \geq 600 \end{cases}$$

# Modelos de risco Coletivo-Convolução

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s) p_N(k)$$

$$P^{*k}(s) = P(X_1 + X_2 + \ldots + X_k \leq s)$$

**Quando X é discreto tem-se**

$$\boldsymbol{P^{*0}(s) = \begin{cases} 0 \; se \; s \leq 0 \\ 1 \; se \; s > 0 \end{cases}}$$

$$P^{*k}(s) = \sum_{h \leq s} P^{*k-1}(s-h) p(h)$$

*Considere $h$ como um dos valores possíveis para $X$.*

# Exemplo 5

Uma carteira de seguros produz 0,1 ou 2 sinistros com as respetivas probabilidades: 20%,50% e 30%. Um sinistro dessa carteira assume os valores $R\$100$, $R\$200$ ou $R\$300$, com as respectivas probabilidades: $20\%$, $70\%$ e 10%.

Construa a distribuição convoluta dos sinistros agregados $S_{col}$.

| $X_i$ | $R\$100$ | $R\$200$ | $R\$300$ |
|---|---|---|---|
| $P_{X_i}(x_i)$ | 0,2 | 0,7 | 0,1 |

| $N$ | $P(N)$ | $S_{col}$ | |
|---|---|---|---|
| 0 | 0,2 | $S_{col} = 0$ | |
| 1 | 0,5 | $S_{col} = X_1$ | $\{R\$100, R\$200, R\$300\}$ |
| 2 | 0,3 | $S_{col} = X_1 + X_2$ | $\{R\$200, R\$300, R\$400, R\$500, R\$600\}$ |

Por definição tem-se que $P^{*0}(s) = \begin{cases} 0 \; se \; s \leq 0 \\ 1 \; se \; \; s > 0 \end{cases}$

Logo para $\boldsymbol{k = 0}$:

$P^{*0}(0) = 0$

$P^{*0}(100) = 1$

$P^{*0}(200) = 1$

$P^{*0}(300) = 1$

$P^{*0}(400) = 1$

$P^{*0}(500) = 1$

$P^{*0}(600) = 1$

Para $\boldsymbol{k = 1}$:

Usando $P^{*\mathrm{k}}(\mathrm{s}) = \sum_{\mathrm{h}\leq \mathrm{s}} P^{*\mathrm{k}-1}(\mathrm{s}-\mathrm{h})p(\mathrm{h})$ sendo $k$ os possíveis valores assumidos por $N$.

$$P^{*1}(0) = \sum_{h=0}^{0} P^{*1-1}(0-h)p(h)$$

$$P^{*1}(100) = \sum_{h=0}^{100} P^{*1-1}(100-h)p(h)$$

$$P^{*1}(200) = \sum_{h=0}^{200} P^{*1-1}(200-h)p(h)$$

$$P^{*1}(300) = \sum_{h=0}^{300} P^{*1-1}(300-h)p(h)$$

$$P^{*1}(400) = \sum_{h=0}^{400} P^{*1-1}(400-h)p(h)$$

$$P^{*1}(500) = \sum_{h=0}^{500} P^{*1-1}(500-h)p(h)$$

$$P^{*1}(600) = \sum_{h=0}^{600} P^{*1-1}(600-h)p(h)$$

$$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{0}) = P^{*0}(\,0)p(0) = 0$$

$$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{100}) = P^{*0}(100)p(0) + P^{*0}(0)p(100) = 0$$

$$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{200}) = P^{*0}\ (200)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{100})\boldsymbol{p}(\mathbf{100}) + P^{*0}(0)p(200) = 0{,}2$$

$$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{300}) = P^{*0}(300)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{200})\boldsymbol{p}(\mathbf{100}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{100})\boldsymbol{p}(\mathbf{200}) + P^{*0}(0)p(300) = 0{,}9$$

$$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{400}) = P^{*0}(400)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{300})\boldsymbol{p}(\mathbf{100}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{200})\boldsymbol{p}(\mathbf{200}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{100})\boldsymbol{p}(\mathbf{300}) + P^{*0}(0)p(400) = \mathbf{1}$$

$$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{500}) = P^{*0}(500)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{400})\boldsymbol{p}(\mathbf{100}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{300})\boldsymbol{p}(\mathbf{200}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{200})\boldsymbol{p}(\mathbf{300}) + P^{*0}(100)p(400) + P^{*0}(0)p(500) = \mathbf{1}$$

$$\boldsymbol{P}^{*\mathbf{1}}(\mathbf{600}) = P^{*0}(600)p(0) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{500})\boldsymbol{p}(\mathbf{100}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{400})\boldsymbol{p}(\mathbf{200}) + \boldsymbol{P}^{*\mathbf{0}}(\mathbf{300})\boldsymbol{p}(\mathbf{300}) + P^{*0}(200)p(400) + P^{*0}(100)p(500) + P^{*0}(0)p(600) = \mathbf{1}$$

| $S_{col}$ | $N = 0$ | $N = 1$ |
|---|---|---|
| 0 | $P^{*0}(0) = 0$ | $P^{*1}(0) = 0$ |
| 100 | $P^{*0}(100) = 1$ | $P^{*1}(100) = 0$ |
| 200 | $P^{*0}(200) = 1$ | $P^{*1}(200) = 0{,}2$ |
| 300 | $P^{*0}(300) = 1$ | $P^{*1}(300) = 0{,}9$ |
| 400 | $P^{*0}(400) = 1$ | $P^{*1}(400) = 1$ |
| 500 | $P^{*0}(500) = 1$ | $P^{*1}(500) = 1$ |
| 600 | $P^{*0}(600) = 1$ | $P^{*1}(600) = 1$ |

Para $\mathbf{k = 2}$:

$$P^{*2}(0) = \sum_{h=0}^{0} P^{*2-1}(0-h)p(h)$$

$$P^{*2}(100) = \sum_{h=0}^{100} P^{*2-1}(100-h)p(h)$$

$$P^{*2}(200) = \sum_{h=0}^{200} P^{*2-1}\,(200-h)p(h)$$

$$P^{*2}(300) = \sum_{h=0}^{300} P^{*2-1}\,(300-h)p(h)$$

$$P^{*2}(400) = \sum_{h=0}^{400} P^{*2-1}\,(400-h)p(h)$$

$$P^{*2}(500) = \sum_{h=0}^{500} P^{*2-1}\,(500-h)p(h)$$

$$P^{*2}(600) = \sum_{h=0}^{600} P^{*2-1}\,(600-h)p(h)$$

Para $\boldsymbol{k = 2}$:

$$P^{*2}(0) = P^{*1}(0)p(0) = 0$$

$$P^{*2}(100) = P^{*1}(100)p(0) + P^{*1}(0)p(100) = 0$$

$$P^{*2}(200) = P^{*1}(200)p(0) + P^{*1}(100)p(100) + P^{*1}(0)p(200) = 0$$

$$P^{*2}(300) = P^{*1}(300)p(0) + \boldsymbol{P^{*1}(200)p(100)} + P^{*1}(100)p(200) + P^{*1}(0)p(300) = 0{,}04$$

$$P^{*2}(400) = P^{*1}(400)p(0) + \boldsymbol{P^{*1}(300)p(100)} + \boldsymbol{P^{*1}(200)p(200)} + P^{*1}(100)p(300) + P^{*1}(0)p(400) = 0{,}32$$

$$P^{*2}(500) = P^{*1}(500)p(0) + \boldsymbol{P^{*1}(400)p(100)} + \boldsymbol{P^{*1}(300)p(200)} + \boldsymbol{P^{*1}(200)p(300)} + P^{*1}(100)p(400) + P^{*1}(0)p(500) = 0{,}85$$

$$P^{*2}(600) = P^{*1}(600)p(0) + \boldsymbol{P^{*1}(500)p(100)} + \boldsymbol{P^{*1}(400)p(200)} + \boldsymbol{P^{*1}(300)p(300)} + P^{*1}(200)p(400) + P^{*1}(100)p(500) + P^{*1}(0)p(600) = 0{,}99$$

| $\boldsymbol{S_{col}}$ | $\boldsymbol{P(N=0)=0,2}$ | $\boldsymbol{P(N=1)=0,5}$ | $\boldsymbol{P(N=2)=0,3}$ |
|---|---|---|---|
| | $\boldsymbol{N=0}$ | $\boldsymbol{N=1}$ | $\boldsymbol{N=2}$ |
| 0 | $P^{*0}(0)=0$ | $P^{*1}(0)=0$ | $P^{*2}(0)=0$ |
| 100 | $P^{*0}(100)=1$ | $P^{*1}(100)=0$ | $P^{*2}(100)=0$ |
| 200 | $P^{*0}(200)=1$ | $P^{*1}(200)=0{,}2$ | $P^{*2}(200)=0$ |
| 300 | $P^{*0}(300)=1$ | $P^{*1}(300)=0{,}9$ | $P^{*2}(300)=0{,}04$ |
| 400 | $P^{*0}(400)=1$ | $P^{*1}(400)=1$ | $P^{*2}(400)=0{,}32$ |
| 500 | $P^{*0}(500)=1$ | $P^{*1}(500)=1$ | $P^{*2}(500)=0{,}85$ |
| 600 | $P^{*0}(600)=1$ | $P^{*1}(600)=1$ | $P^{*2}(600)=0{,}99$ |

$$\mathrm{P}_{S_{col}}(s) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0{,}2 & 0 \\ 1 & 0{,}9 & 0{,}04 \\ 1 & 1 & 0{,}32 \\ 1 & 1 & 0{,}85 \\ 1 & 1 & 0{,}99 \end{bmatrix} \begin{bmatrix} 0{,}2 \\ 0{,}5 \\ 0{,}3 \end{bmatrix} \begin{array}{l} \Rightarrow \boldsymbol{P_N(0)} \\ \Rightarrow \boldsymbol{P_N(1)} \\ \Rightarrow \boldsymbol{P_N(2)} \end{array}$$

$\mathbf{P}^{*\mathbf{2}}(s)$

$\mathbf{P}^{*\mathbf{1}}(s)$

$\mathbf{P}^{*\mathbf{0}}(s)$

$$F_{S_{col}}(s) = \sum_{k=0}^{n} P^{*k}(s)\, p_N(k)$$

$$F_{Scol}(s) = \begin{cases} 0 & s < 0 \\ 0{,}2 & 0 \leq s < 100 \\ 0{,}3 & 100 \leq s < 200 \\ 0{,}662 & 200 \leq s < 300 \\ 0{,}796 & 300 \leq s < 400 \\ 0{,}955 & 400 \leq s < 500 \\ 0{,}997 & 500 \leq s < 600 \\ 1 & s \geq 600 \end{cases}$$

$$F_{Scol}(s) = \begin{cases} 0 & s < 0 \\ 0{,}2 & 0 \leq s < 100 \\ 0{,}2 + 0{,}1 = 0{,}3 & 100 \leq s < 200 \\ 0{,}3 + 0{,}362 = 0{,}662 & 200 \leq s < 300 \\ 0{,}662 + 0{,}134 = 0{,}796 & 300 \leq s < 400 \\ 0{,}796 + 0{,}159 = 0{,}955 & 400 \leq s < 500 \\ 0{,}955 + 0{,}042 = 0{,}997 & 500 \leq s < 600 \\ 1 & s \geq 600 \end{cases}$$

- Exemplo 6

Suponha uma carteira composta por 2 apólices identicamente distribuídas e independentes.

| $X_i$ | $R\$0,00$ | $R\$1000,00$ | $R\$2000,00$ | $R\$3000,00$ |
|---|---|---|---|---|
| $P(X_i)$ | 0,6 | 0,02 | 0,06 | 0,32 |

Modelando essa carteira de acordo com modelo de **risco individual**. Obtenha a função de probabilidade de $S_{ind}$.

| $X_i$ | $R\$0,00$ | $R\$1000,00$ | $R\$2000,00$ | $R\$3000,00$ |
|---|---|---|---|---|
| $P(X_i)$ | 0,6 | 0,02 | 0,06 | 0,32 |

$$p_S(s) = p_{X_1} * p_{X_2}(s) = \sum_{\forall\, x_1 \leq s} p_{X_2}(s - x_1) p_{X_1}(x_1)$$

| $S$ | $S(X_1, X_2)$ | $P_S$ |
|---|---|---|
| 0 | (0,0) | 0,36 |
| 1000 | (1000,0) (0,1000) | 0,024 |
| 2000 | (2000,0)(1000,1000)(0,2000) | 0,0724 |
| 3000 | (3000,0)(2000,1000)(1000,2000)(0,3000) | 0,3864 |
| 4000 | (3000,1000)(2000,2000)(1000,3000) | 0,0164 |
| 5000 | (3000,2000)(2000,3000) | 0,0384 |
| 6000 | (3000,3000) | 0,1024 |

- Exemplo 6

Suponha uma carteira composta por 2 apólices identicamente distribuídas e independentes.

| $X_i$ | $R\$0{,}00$ | $R\$1000{,}00$ | $R\$2000{,}00$ | $R\$3000{,}00$ |
|---|---|---|---|---|
| $P(X_i)$ | 0,6 | 0,02 | 0,06 | 0,32 |

Modelando essa carteira de acordo com modelo de **risco coletivo.** Obtenha a função de probabilidade de $S_{col}$.

## Solução:

| $X_i$ | $P(X_i = x_i)$ | $I_i$ | $P(I_i = i_i)$ | $B_i = (X_i \mid I_i = 1)$ | $P(B_i = b_i)$ |
|---|---|---|---|---|---|
| R\$0,00 | 0,6 | 0 | 0,6 | | |
| R\$1000,00 | 0,02 | 1 | 0,4 | R\$1000,00 | $\frac{0,02}{0,4} = 0,05$ |
| R\$2000,00 | 0,06 | | | R\$2000,00 | $\frac{0,06}{0,4} = 0,15$ |
| R\$3000,00 | 0,32 | | | R\$3000,00 | $\frac{0,32}{0,4} = 0,8$ |

| $N$ | $P(N) = \binom{2}{n} 0,4^n 0,6^{2-n}$ | $S_{col}$ | Possíveis valores para $S_{col}$. |
|---|---|---|---|
| 0 | 0,36 | $S_{col} = 0$ | |
| 1 | 0,48 | $S_{col} = X_i \ \forall \ i = 1,2$ | $\{R\$1000, R\$2000, R\$3000\}$ |
| 2 | 0,16 | $S_{col} = X_1 + X_2$ | $\{R\$2000, R\$3000, R\$4000, R\$5000, R\$6000\}$ |

$$p_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s)\, p_N(k)$$

$$p^{*k}(s) = p(X_1 + X_2 + \ldots + X_k = s)$$

**Quando X é discreto tem-se**

$$p^{*0}(s) = \begin{cases} 0 \; se \; s \neq 0 \\ 1 \; se \; s = 0 \end{cases}$$

$$p^{*k}(s) = \sum_{h \leq s} p^{*k-1}(s-h)p(h)$$

*Considere* $h$ *como um dos valores possíveis para* $X$.

| $S_{col}$ | $P(N=0)=0,36$ | $P(N=1)=0,48$ | $P(N=2)=0,16$ |
|---|---|---|---|
| | $N=0$ | $N=1$ | $N=2$ |
| 0 | $p^{*0}(0)=1$ | $p^{*1}(0)=0$ | $p^{*2}(0)=0$ |
| 1000 | $p^{*0}(1000)=0$ | $p^{*1}(1000)=0{,}05$ | $p^{*2}(1000)=0$ |
| 2000 | $p^{*0}(2000)=0$ | $p^{*1}(0200)=0{,}15$ | $p^{*2}(2000)=0{,}0025$ |
| 3000 | $p^{*0}(3000)=0$ | $p^{*1}(3000)=0{,}8$ | $p^{*2}(3000)=0{,}015$ |
| 4000 | $p^{*0}(4000)=0$ | $p^{*1}(4000)=0$ | $p^{*2}(4000)=0{,}1025$ |
| 5000 | $p^{*0}(5000)=0$ | $p^{*1}(5000)=0$ | $p^{*2}(5000)=0{,}24$ |
| 6000 | $p^{*0}(6000)=0$ | $p^{*1}(6000)=0$ | $p^{*2}(6000)=0{,}64$ |
| | **1** | **1** | **1** |

$$P_{S_{col}}(s) = \sum_{k=0}^{n} p^{*k}(s)\, p_N(k)$$

$$P_{S_{col}}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0{,}2 & 0 \\ 0 & 0{,}7 & 0{,}04 \\ 0 & 0{,}1 & 0{,}28 \\ 0 & 0 & 0{,}53 \\ 0 & 0 & 0{,}14 \\ 0 & 0 & 0{,}01 \end{bmatrix} \begin{bmatrix} 0{,}36 \\ 0{,}48 \\ 0{,}16 \end{bmatrix} \begin{matrix} \Rightarrow \boldsymbol{P_N(0)} \\ \Rightarrow \boldsymbol{P_N(1)} \\ \Rightarrow \boldsymbol{P_N(2)} \end{matrix}$$

$p^{*0}(s)$ $p^{*1}(s)$ $p^{*2}(s)$

$$p_{Scol}(s) = \begin{cases} 0{,}36 & s = 0 \\ 0{,}0240 & s = 1000 \\ 0{,}0724 & s = 2000 \\ 0{,}3864 & s = 3000 \\ 0{,}0164 & s = 4000 \\ 0{,}0384 & s = 5000 \\ 0{,}1024 & s = 6000 \end{cases}$$

| $S$ | $S(X_1,X_2)$ | $P_S$ |
|---|---|---|
| 0 | (0,0) | 0,36 |
| 1000 | (1000,0) (0,1000) | 0,024 |
| 2000 | (2000,0)(1000,1000)(0,2000) | 0,0724 |
| 3000 | (3000,0)(2000,1000)(1000,2000)(0,3000) | 0,3864 |
| 4000 | (3000,1000)(2000,2000)(1000,3000) | 0,0164 |
| 5000 | (3000,2000)(2000,3000) | 0,0384 |
| 6000 | (3000,3000) | 0,1024 |

$$p_{Scol}(s) = \begin{cases} 0{,}36 & s = 0 \\ 0{,}0240 & s = 1000 \\ 0{,}0724 & s = 2000 \\ 0{,}3864 & s = 3000 \\ 0{,}0164 & s = 4000 \\ 0{,}0384 & s = 5000 \\ 0{,}1024 & s = 6000 \end{cases}$$

$$\boldsymbol{E(S_{ind}) = \sum_{i=1}^{2} E(B_i)q_i}$$

$$\boldsymbol{E(S_{ind}) = 2200}$$

$$\boldsymbol{var(S_{ind}) = \sum_{i=1}^{2}\left[var(B_i)q_i + E(B_i)^2 var(I_i)\right]}$$

$$\boldsymbol{var(S_{ind}) = 3860000}$$

$$\boldsymbol{E(S_{col}) = E(N)E(X)}$$

$$\boldsymbol{E(S_{col}) = 2200}$$

$$\boldsymbol{var(S_{col}) = E(X)^2 var(N) + E(N)var(X)}$$

$$\boldsymbol{var(S_{col}) = 3860000}$$