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Tutorial 2 (Spectral and singular value decomposition)
We consider the matrix

$$A = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

(a) Show that A is normal, and compute its characteristic polynomial and spectral decomposition.

(b) Compute the singular values of A. What is the rank of A?

(c) Let B be a unitary matrix. Provide a singular value decomposition of B.
Note: The matrices U and V are not unique.

Solution

(a) A is normal since it obeys

$$AA^\dagger = \begin{pmatrix} 2 & -2i & 0 \\ 2i & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A^\dagger A.$$

We use the block structure of A to simplify the calculation of the characteristic polynomial:

$$\begin{aligned} \chi_A(\lambda) = \det(\lambda I - A) &= \det \begin{pmatrix} \lambda - 1 & i & 0 \\ -i & \lambda - 1 & 0 \\ 0 & 0 & \lambda - i \end{pmatrix} \\ &= \big((\lambda-1)^2 - i(-i)\big)(\lambda - i) = (\lambda^2 - 2\lambda)(\lambda - i) = \lambda(\lambda-2)(\lambda-i). \end{aligned}$$

Thus $\chi_A(\lambda) = 0$ precisely if $\lambda \in \{0, i, 2\}$, which are the eigenvalues of A.

We can then use the eigenvalues to determine the eigenvectors of the matrix.

- $\lambda = 0$:

$$(0 \cdot I - A)v \stackrel{!}{=} 0 \quad \Leftrightarrow \quad Av = 0$$

$$Av = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 - iv_2 \\ iv_1 + v_2 \\ iv_3 \end{pmatrix} \stackrel{!}{=} 0, \quad v = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

- $\lambda = 2$:

$$(2 \cdot I - A)v \stackrel{!}{=} 0, \quad \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + iv_2 \\ -iv_1 + v_2 \\ (2-i)v_3 \end{pmatrix} \stackrel{!}{=} 0, \quad v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

- $\lambda = i$:

$$(i \cdot I - A)v \stackrel{!}{=} 0, \quad \begin{pmatrix} i-1 & i & 0 \\ -i & i-1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \stackrel{!}{=} 0, \quad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In summary, collecting the normalized eigenvectors as columns of a (unitary) matrix U results in the spectral decomposition $A = U\Lambda U^\dagger$ with

$$U = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & & \\ & 2 & \\ & & i \end{pmatrix}.$$

(b) The SVD decomposes a matrix as $A = USV^\dagger$, with U and V unitary matrices and S storing the singular values on its diagonal.

The squares of the singular values are the eigenvalues of $A^\dagger A$:

$$\begin{aligned}
\det(\lambda I - A^\dagger A) &= \det\begin{pmatrix} \lambda - 2 & 2i & 0 \\ -2i & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 1 \end{pmatrix} \\
&= \det\begin{pmatrix} \lambda - 2 & 2i \\ -2i & \lambda - 2 \end{pmatrix} \det(\lambda - 1) \\
&= \big((\lambda - 2)^2 - 2i(-2i)\big)(\lambda - 1) \\
&= (\lambda^2 - 4\lambda)(\lambda - 1) \\
&= \lambda(\lambda - 4)(\lambda - 1) \stackrel{!}{=} 0.
\end{aligned}$$

The eigenvalues of $A^\dagger A$ are thus 0, 4, 1. The singular values are the square roots of these eigenvalues (in descending order):

$$\sigma_1 = 2, \quad \sigma_2 = 1, \quad \sigma_3 = 0.$$

The rank of the matrix refers to the number of nonzero singular values, here $\operatorname{rank}(A) = 2$. Note that the columns of U are the eigenvectors of $AA^\dagger$ and the columns of V the eigenvectors of $A^\dagger A$.

Remark: We have shown how to compute the singular values of an arbitrary (square) matrix. For normal matrices (as is the case here), it turns out that the singular values are the absolute values of the eigenvalues (since one can start from the spectral decomposition and then absorb eigenvalue phase factors into U).

(c) Since B is unitary, we can just choose $U = B$ and $S = V = I$, since

$$B = B \cdot I \cdot I.$$

As mentioned in the note, this choice is not unique. We might alternatively set $U = I$ and $V^\dagger = B$, since likewise

$$B = I \cdot I \cdot B.$$

More generally, let C be another unitary matrix, then $U = C$, $S = I$ and $V^\dagger = C^\dagger B$ works as well, since

$$B = I \cdot B = CC^\dagger B = C \cdot I \cdot C^\dagger B.$$