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Exercise 11.1 (Conservation laws for the Ising model)
For this exercise we use an alternative convention for the Ising Hamiltonian (on a one-dimensional lattice with open boundary conditions), namely

$$H_{\text{ising}} = -J \sum_{j=1}^{L-1} X_j X_{j+1} - g \sum_{j=1}^{L} Z_j.$$

(It results from the hitherto version by a conjugation with the unitary matrix $H \otimes \cdots \otimes H$, with H the Hadamard gate. This form simplifies the discussion of quantum numbers.) The corresponding symmetry operator is now

$$P = \prod_{j=1}^{L} Z_j = Z \otimes \cdots \otimes Z.$$

(a) Show that the eigenstates of P are precisely the computational basis states $|a_1\rangle \otimes \cdots \otimes |a_L\rangle \equiv |a_1 \dots a_L\rangle$ with $a_j \in \{0, 1\}$ for all j, with corresponding eigenvalues ± 1. How is the eigenvalue related to the bit string $a_1 \dots a_L$?

In the following, we associate the overall quantum number 0 with eigenvalue 1 of P, and the overall quantum number 1 with eigenvalue -1, such that $\lambda = (-1)^q$, where λ is the eigenvalue and q the quantum number. Addition of quantum numbers is understood modulo 2.

(b) At each lattice site, we likewise attach the (local) quantum number 0 to $|0\rangle$ and 1 to $|1\rangle$. Based on the sparsity pattern of I_2, X and Z, decide which of these matrices leave the quantum number invariant, and which change it (equivalent to incrementing it by 1 modulo 2).

Recall that the Ising Hamiltonian admits a MPO representation, with tensors (for the current convention)

$$A^1 = \begin{pmatrix} -gZ & -JX & I_2 \end{pmatrix}, \quad A^j = \begin{pmatrix} I_2 & 0 & 0 \\ X & 0 & 0 \\ -gZ & -JX & I_2 \end{pmatrix} \ \forall j = 2, \dots, L-1, \quad A^L = \begin{pmatrix} I_2 \\ X \\ -gZ \end{pmatrix}.$$

(c) Assign quantum numbers 0 and 1 to the virtual bonds (the finite automaton "states" a, b, c from the lecture, or equivalently the input and output channels of A^j) such that they are compatible with the behavior of the entries I_2, X and Z of A^j found in (b).

Remark: Since the initial finite automaton state in the MPO form is always a, and the final state always c, one can conclude also from the MPO representation (together with the result of (c)) that the Hamiltonian does not change the overall quantum number, as expected (since it leaves the eigenspaces of P invariant).

Solution

(a) $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, in particular, the computational basis states $|0\rangle = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $|1\rangle = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ are eigenvectors of Z with eigenvalues 1 and -1, respectively. Using the properties of Kronecker products, we obtain:

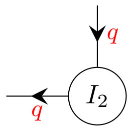
$$\begin{aligned} P\,|a_1 \dots a_L\rangle &= (Z \otimes \cdots \otimes Z)(|a_1\rangle \otimes \cdots \otimes |a_L\rangle) \\ &= (Z\,|a_1\rangle) \otimes \cdots \otimes (Z\,|a_L\rangle) \\ &= \big((-1)^{a_1}\,|a_1\rangle\big) \otimes \cdots \otimes \big((-1)^{a_L}\,|a_L\rangle\big) \\ &= (-1)^{a_1 + \cdots + a_L}\,|a_1 \dots a_L\rangle\,. \end{aligned}$$

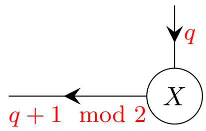
Thus each computational basis state $|a_1 \dots a_L\rangle$ is an eigenstate of P with eigenvalue $(-1)^{a_1+\cdots+a_L}$, and since these states form a basis, we have found all eigenstates of P. Regarding the eigenvalue, $(-1)^{a_1+\cdots+a_L} = (-1)^{a_1+\cdots+a_L \bmod 2}$, that is, the eigenvalue equals -1 for the case of an odd number of 1-bits in the bit string $a_1 \dots a_L$, otherwise the eigenvalue equals 1.

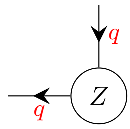
(b) I_2 and Z are diagonal matrices, and in particular their non-zero entries leave the quantum numbers 0 and 1 invariant. On the other hand, X interchanges $|0\rangle \leftrightarrow |1\rangle$, and likewise the quantum numbers $0 \leftrightarrow 1$:

$$I_2 = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad X = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \qquad Z = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

Summarizing these properties of I_2, X and Z:







(c) Assigning the quantum number 0 to a and c, and the quantum number 1 to b, is compatible with the properties found in part (b). Namely, the I_2 and Z blocks in A^j leave quantum numbers invariant, and the X blocks change them. Explicitly:

$$\begin{array}{ccc} & & \begin{array}{ccc} 0 & 1 & 0 \\ a & b & c \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} & \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} I_2 & 0 & 0 \\ X & 0 & 0 \\ -gZ & -JX & I_2 \end{pmatrix} \end{array}$$

The scalar prefactors J and g do not affect the sparsity pattern and are irrelevant for the quantum number considerations.