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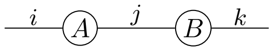
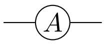
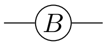
Exercise 3.1 (Tensor diagrams)

(a) Recall that the matrix product AB with entries $(AB)_{ik} = \sum_j a_{ij} b_{jk}$ translates to the following diagram:

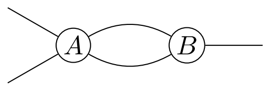


Connect the legs of  and  accordingly to represent AB^T, $B^T A^T$ and $\mathrm{tr}[AB]$ graphically.

(b) Given a matrix A, a tensor B of degree 3, and a vector C, express the following tensor contraction in graphical form:

$$m_{ik} = \sum_{j,\ell} a_{ij}\, b_{kj\ell}\, c_\ell.$$

(c) Let A and B be tensors of degree d_A and d_B, respectively, such that each individual dimension is equal to some $n \in \mathbb{N}$. (In other words, $A \in \mathbb{C}^{n\times\cdots\times n}$ where n appears d_A times, and likewise for B.) Now A and B are contracted along c of these dimensions, as illustrated for $d_A = 4$, $d_B = 3$ and $c = 2$ in the following diagram:

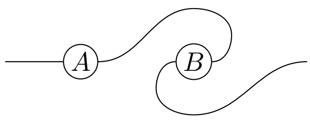


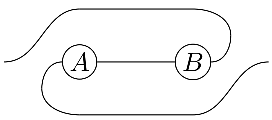
What is the asymptotic computational cost of this contraction (in the form $\mathcal{O}(n^\ell)$ with to-be determined exponent ℓ) based on a literal implementation of the summation formulation?

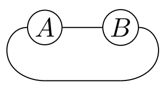
Hint: Determine the required number of nested for-loops from 1 to n to compute the entries of the resulting tensor.

Solution

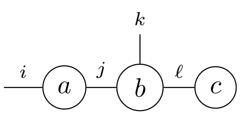
(a) Following the given diagram:

$AB^T \equiv$ 

$B^T A^T \equiv$ 

$\mathrm{tr}[AB] \equiv$ 

(b) In $m_{ik} = \sum_{j,\ell} a_{ij}\, b_{kj\ell}\, c_\ell$, there is a contraction over j and ℓ, while i and k are free indexes.

$m_{ik} \equiv$ 

(c) The asymptotic computational cost is $\mathcal{O}(n^{d_A+d_B-c})$, since there are n^c summation terms for each index assignment to the "open" legs. A has $d_A - c$ uncontracted dimensions (n^{d_A-c} indices) and B has $d_B - c$ uncontracted dimensions (n^{d_B-c} indices).