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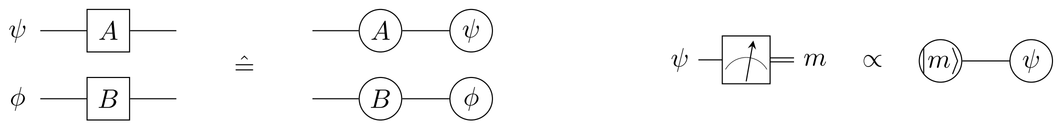
Tutorial 7 (Quantum teleportation as tensor network)
Mathematically, a qubit ψ is a vector in $\mathbb{C}^2$ with norm 1. Basis states in quantum (bra-ket) notation are written as $|0\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|1\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, such that a qubit can be represented as $\psi = \alpha\,|0\rangle + \beta\,|1\rangle$ with $\alpha, \beta \in \mathbb{C}$.

Quantum circuit diagrams are read from left to right, and the wires can be interpreted as tensor legs. As example, translating to the notation used in the lecture, with $\psi, \phi \in \mathbb{C}^2$, matrices $A, B \in \mathbb{C}^{2\times 2}$ and $m \in \{0, 1\}$:

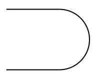


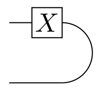
Here the "input" of the left two diagrams is the outer product $\psi \circ \phi$, or equivalently (when interpreted as vector) the Kronecker product $\psi \otimes \phi$. For basis states one writes $|ab\rangle = |a\rangle \otimes |b\rangle$. On the right, m is the "measurement outcome", and $\propto$ means "proportional to".

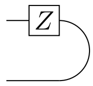
The *Pauli matrices* are both Hermitian and unitary, and given by

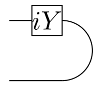
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the following, we also define the so-called *Bell states* β_{ab}, which can be regarded as vectorizations of the identity and Pauli matrices:

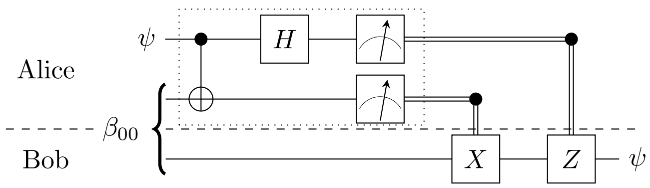
$$\beta_{00} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \hat{=} \tfrac{1}{\sqrt{2}}$$ 

$$\beta_{01} = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \hat{=} \tfrac{1}{\sqrt{2}}$$ 

$$\beta_{10} = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \hat{=} \tfrac{1}{\sqrt{2}}$$ 

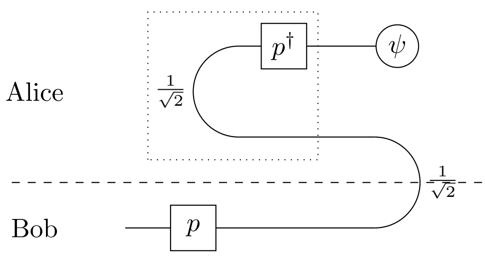
$$\beta_{11} = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \hat{=} \tfrac{1}{\sqrt{2}}$$ 

Quantum teleportation involves two parties called Alice and Bob, and allows Alice to transmit the data (the coefficients α and β) of a qubit ψ to Bob. Initially they share the Bell state β_{00}. Alice performs local operations on ψ and her half of β_{00}, then transmits two bits of classical information to Bob, who can then recover the original qubit ψ. The protocol is summarized by the following circuit:



(a) Describe the components and individual steps of the quantum teleportation circuit.

(b) According to Exercise 7.1, the dotted region in the circuit can be interpreted as projection onto one of the Bell states (depending on the measurement outcomes). With this information, map the circuit to the following tensor network diagram (read from right to left), where p is the identity matrix, X, iY or Z:



What is the final state at Bob according to this diagram?

(c) A tensor network is deterministic. However, the measurement outcome and the correction unitary is not. How can this be reconciled?

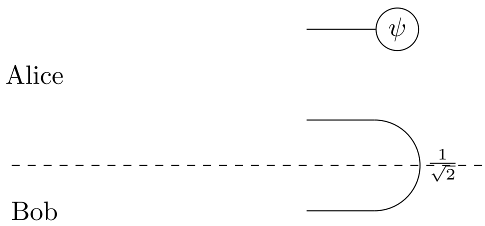
Solution

(a) The quantum teleportation protocol allows to transfer the quantum state information of a qubit from Alice to Bob, without requiring a physical transfer nor destroying its quantum superposition by a measurement. It starts from a single-qubit quantum state ψ and an entangled Bell state β_{00}, which is split between Alice and Bob.

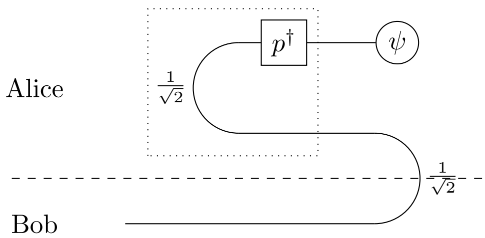
The CNOT and Hadamard gates map from computational basis states $|ab\rangle$ $(a, b \in \{0, 1\})$ to the Bell states (see Exercise 7.1). Thus Alice effectively projects the overall quantum state of her two qubits onto one of the Bell states. Which one is determined by the measurement outcome, i.e., two bits of classical information. Alice has to communicate these bits to Bob (via conventional classical means). As final step, Bob applies X and/or Z gates (depending on the measurement outcome) to recover the original ψ.

Note that, due to the required classical communication, no "faster-than-light" information transfer is possible via this protocol, although the "wavefunction collapse" at the measurement is instantaneous and also affects Bob's qubit.

(b) We begin with the three-qubit quantum state consisting of the state ψ and the entangled pair β_{00}:

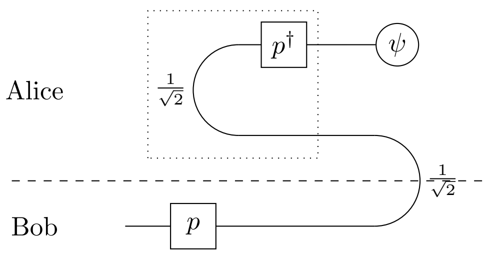


As just described, the measurement performed by Alice effectively projects her two qubits onto one of the Bell states β_{ab}. This "projection" is the inner product between β_{ab} and Alice's two qubis. Written as tensor diagram:



Here p is one of I, X, iY or Z, and the dotted box is β^*_{ab}. We have to take the complex conjugate of β_{ab} since it appears as first argument of the inner product.

By inspection, we can see that the correction step to be performed by Bob consists of applying p:



According to the diagram, the final state at Bob is $\frac{1}{2}\psi$, but actually one renormalizes quantum states after a measurement.

Note the following map between the two bits ab of the measurement outcome and p (parametrizing the Bell state):

$$p = \begin{cases} I, & ab = 00 \\ Z, & ab = 01 \\ X, & ab = 10 \\ iY = ZX, & ab = 11 \end{cases}$$

Thus Bob has to apply X precisely if $a = 1$, and then Z precisely if $b = 1$.

(c) Since the matrix p depends on the measurement outcome, it is known only after the measurement has occurred. The tensor network representation summarizes the protocol for all possible cases.