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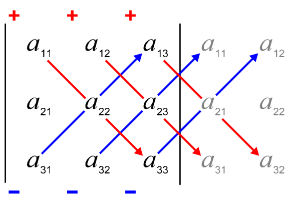
Exercise 2.1 (Spectral decomposition)

(a) Compute the characteristic polynomial and spectral decomposition of the normal matrix

$$A = \begin{pmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -\frac{3}{5} & 0 & 0 \\ -\frac{4}{5} & 0 & 0 \end{pmatrix}.$$

Hint: The following "rule of Sarrus" might be helpful for calculating the determinant of a 3×3 matrix:



Source: Wikipedia

(b) Let $A \in \mathbb{C}^{n \times n}$ be a normal matrix, and $\{\lambda_1, \dots, \lambda_n\}$ the eigenvalues of A. Show that

$$\mathrm{tr}[A] = \sum_{j=1}^{n} \lambda_j,$$

where $\mathrm{tr}[\cdot]$ denotes the matrix trace (sum of its diagonal entries).

Hint: Start from the spectral decomposition of A, and use that $\mathrm{tr}[AB] = \mathrm{tr}[BA]$ for any square matrices A and B.

Solution

(a) To determine the characteristic polynomial of A we use the rule of Sarrus to compute

$$\begin{aligned} \chi_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & \frac{3}{5} & \frac{4}{5} \\ -\frac{3}{5} & -\lambda & 0 \\ -\frac{4}{5} & 0 & -\lambda \end{pmatrix} \\ = -\lambda^3 - \frac{9}{25}\lambda - \frac{16}{25}\lambda \\ = -\lambda^3 - \lambda. \end{aligned}$$

When setting $\chi_A(\lambda) = 0$ we find the eigenvalues to be $\lambda_1 = 0$, $\lambda_2 = i$ and $\lambda_3 = -i$. Next we determine the eigenvectors. For example for λ_1 we need to find $a, b, c \in \mathbb{C}$ such that

$$A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{3}{5}b + \frac{4}{5}c \\ -\frac{3}{5}a \\ -\frac{4}{5}a \end{pmatrix} = 0.$$

This implies $a = 0$ and $b = -\frac{4}{3}c$. So one possible eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 0 \\ -\frac{4}{3} \\ 1 \end{pmatrix}$$

with norm $||\vec{v}_1|| = \frac{5}{3}$. Thus a normalized eigenvector is

$$\hat{v}_1 = \frac{1}{5} \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}.$$

In an analogous way we can determine normalized eigenvectors for the other two eigenvalues yielding

$$\hat{v}_2 = \frac{1}{5\sqrt{2}} \begin{pmatrix} 5i \\ 3 \\ 4 \end{pmatrix} \quad , \quad \hat{v}_3 = \frac{1}{5\sqrt{2}} \begin{pmatrix} -5i \\ 3 \\ 4 \end{pmatrix} .$$

If we choose

$$U = (\hat{v}_1 \vdots \hat{v}_2 \vdots \hat{v}_3) = \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -4 & \frac{3}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \\ 3 & \frac{2\sqrt{2}}{5} & \frac{2\sqrt{2}}{5} \end{pmatrix}$$

and

$$D = \mathrm{diag}(0, i, -i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$$

we get a spectral decomposition of A as

$$A = UDU^\dagger.$$

(b) As A is normal we can write a spectral decomposition of it as

$$A = UDU^\dagger.$$

with U unitary and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Thus

$$\mathrm{tr}[A] = \mathrm{tr}[UDU^\dagger] = \mathrm{tr}[U^\dagger UD] = \mathrm{tr}[D] = \sum_{i=1}^{n} \lambda_i,$$

where the hint was used in the second equality.