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Tensor Networks

Exam:	IN2388 / Final Exam	**Date:**	Monday 2nd August, 2021
Examiner:	Christian Mendl	**Time:**	11:30 – 13:00

Working instructions

- This exam consists of **10 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: open book
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.

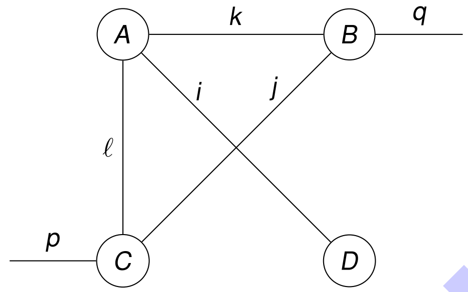
Problem 1 **(20 credits)**

0 1 2 3 4

a) Represent the following contraction operation as graphical tensor diagram, labeling each tensor and tensor leg:

$$e_{pq} = \sum_{i,j,k,\ell} a_{\ell ik} b_{kjq} c_{j\ell p} d_i.$$

The legs should be drawn counter-clockwise with respect to the ordering of the tensor indices.



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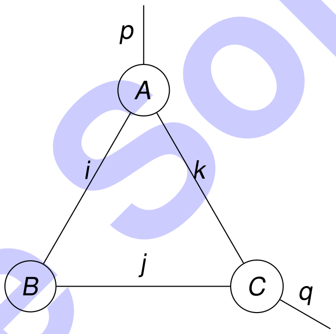
b)* We consider the following to-be contracted tensor network:



Find an optimal contraction order that minimizes contraction complexity, i.e., the overall computational cost, assuming that the dimensions of the tensor legs obey

$$\dim(p) = \dim(q) =: \ell \ll \dim(j) = \dim(k) =: m \ll \dim(i) =: n,$$

and that at each contraction step, two tensors are contracted together. (An optimality proof is not required here.)

Since i has the largest dimension, one should contract this bond first. The optimal order is thus i (i.e., tensor A with B), followed by (j, k) (i.e., C with the output of the first contraction).

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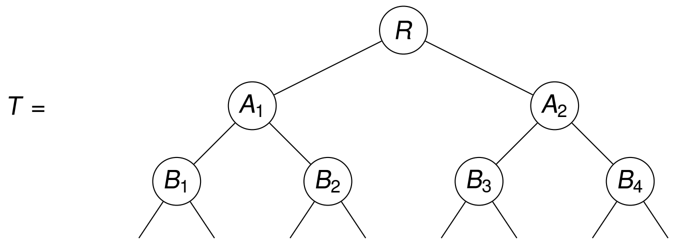
c) What is the asymptotic computational cost (in $\mathcal{O}$-notation) of the contractions you found in (b)? Assume a literal implementation of contractions based on the summation formulation.

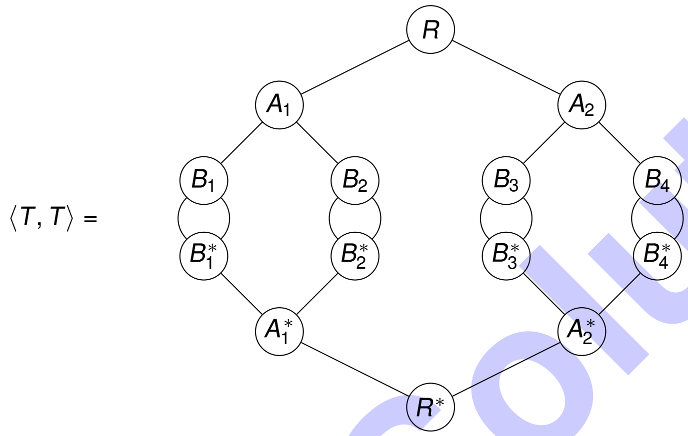
The contraction complexity of the A-B contraction is $\mathcal{O}(\ell m^2 n)$, and of the second contraction $\mathcal{O}(\ell^2 m^2)$.

d)* We define T as the following tree tensor network:

0 1 2 3



Draw the tensor diagram for evaluating $\langle T, T \rangle$.



e) In the context of part (d), we consider a local SVD-splitting operation performed on a node C of the tree (parent node called P), which updates both the child and parent node:

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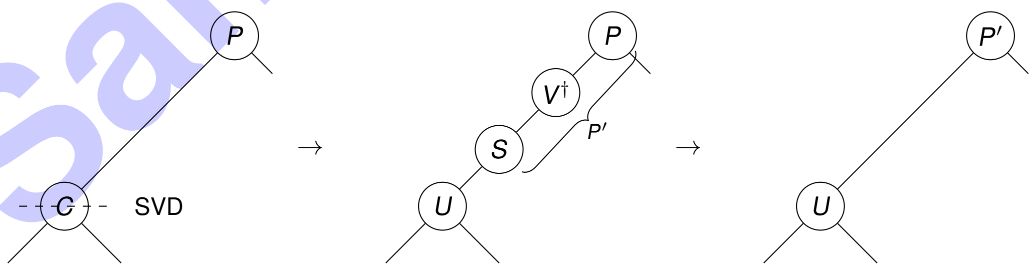


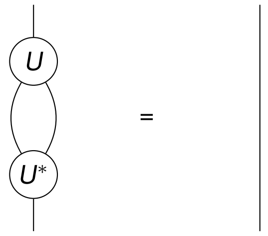
This local operation is applied to each node in the tree (except the root) given a particular ordering. We distinguish between two realizations:

(i) The operation is first performed on the A tensors in the middle layer before proceeding to the B tensors.

(ii) The operation is first performed on the B tensors in the bottom layer before proceeding to the A tensors.

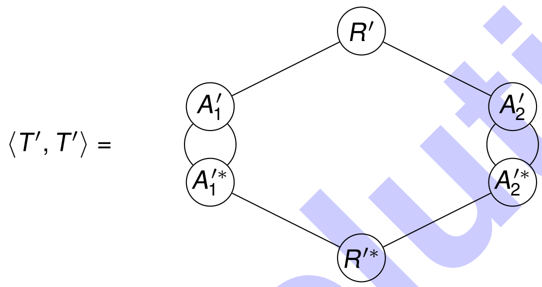
We denote the resulting tree tensor network by T'. For each of the two cases, simply the tensor diagram for evaluating $\langle T', T' \rangle$ as far as possible. Also provide a short explanation of your simplifications.

After each local operation, the updated child node U is an isometry, such that (interpreted as matrix) $U^\dagger U = I$. Graphically:

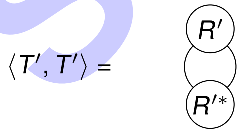


However, the parent node is (in general) no longer an isometry after performing the operation.

(i) In the first case, performing the local operations on the B tensors causes the parent A tensors to lose their isometric property. Hence, the inner product simplifies to the following contraction:



(ii) In the second case, the parent nodes are always updated after their respective children. This ensures that all nodes are isometries except for the root node. Hence, the inner product $\langle T', T' \rangle$ is equal to:



Problem 2 (20 credits)

We consider the following Hamiltonian on a one-dimensional lattice with L sites and open boundary conditions, where J is a real parameter:

$$H = -J \sum_{j=2}^{L-1} X_{j-1} Y_j Z_{j+1}.$$

a) What is the matrix dimension of H?

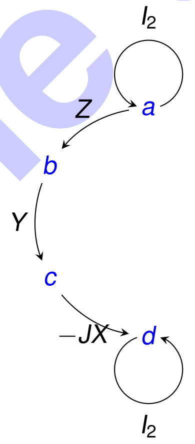
H is a $2^L \times 2^L$ matrix.

b)* For a given index $j \in \{2, \dots, L-1\}$, represent $X_{j-1} Y_j Z_{j+1}$ in terms of Kronecker products of identity matrices and the Pauli matrices X, Y, Z.

$$X_{j-1} Y_j Z_{j+1} = \underbrace{I_2 \otimes \cdots \otimes I_2}_{j-2 \text{ terms}} \otimes X \otimes Y \otimes Z \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{L-j-1 \text{ terms}}$$

c)* Construct a finite state automaton and corresponding MPO tensors for representing H as matrix product operator. You should separately specify the MPO tensors A^j for $j = 2, \dots, L-1$ and the boundary tensors A^1, A^L. Also provide the dimensions of these tensors.

An automaton with four states, denoted a, b, c, d, and the following transitions generates the Hamiltonian:



(The prefactor $-J$ can equivalently be attached to Y or Z as well.)
The corresponding MPO tensors are then

$$A^1 = \begin{pmatrix} 0 & 0 & -JX & I_2 \end{pmatrix}, \qquad A^j = \begin{pmatrix} I_2 & 0 & 0 & 0 \\ Z & 0 & 0 & 0 \\ 0 & Y & 0 & 0 \\ 0 & 0 & -JX & I_2 \end{pmatrix} \text{ for } j = 2, \dots, L-1, \qquad A^L = \begin{pmatrix} I_2 \\ Z \\ 0 \\ 0 \end{pmatrix}.$$

A^1 is equal to the last row of A^j (corresponding to d as final state), and A^L equal to the first column of A^j (corresponding to a as initial state).
With the convention that the leading two dimensions are the physical dimensions, $A^1 \in \mathbb{C}^{2\times2\times1\times4}$, $A^j \in \mathbb{C}^{2\times2\times4\times4}$ for $j = 2, \dots, L-1$, and $A^L \in \mathbb{C}^{2\times2\times4\times1}$.

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d)* Specify an operator of the form $P_k = A_{k-1} B_k C_{k+1}$ with each A, B, C a Pauli matrix and $k \in \{2, \dots, L-1\}$, such that P_k commutes with H. (A proof of the commuting property is not required.)

$P_k = Z_{k-1} Y_k X_{k+1}$ has the desired property, since it commutes with each summand $X_{j-1} Y_j Z_{j+1}$ of H. This is a consequence of the anti-commutation relation of the Pauli matrices:

$$XY = -YX, \quad YZ = -ZY, \quad ZX = -XZ.$$

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e)* Consider the partitioning of the Hamiltonian as $H = H_a + H_b$ with

$$H_a = -J \sum_{j=2, \mathrm{mod}(j,4)=2}^{L-2} \left(X_{j-1} Y_j Z_{j+1} + X_j Y_{j+1} Z_{j+2} \right) \quad \text{and} \quad H_b = -J \sum_{j=4, \mathrm{mod}(j,4)=0}^{L-2} \left(X_{j-1} Y_j Z_{j+1} + X_j Y_{j+1} Z_{j+2} \right).$$

Is it possible to represent the matrix exponentials $e^{-iH_a t}$ or $e^{-iH_b t}$ (with $t \in \mathbb{R}$) exactly in quantum circuit form, assuming that you can use arbitrary one-, two- and three-qubit gates? Briefly justify your answer.

Yes, this is possible since the individual terms in H_a pairwise commute, such that $e^{-iH_a t}$ becomes a product of individual matrix exponentials $e^{J X_{j-1} Y_j Z_{j+1} t}$, which can be realized as three-qubit gates. The same argument applies to $e^{-iH_b t}$.

Problem 3 **(20 credits)**

We define a linear map $\mathcal{E} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ ($n \in \mathbb{N}$) as

$$\mathcal{E}(\rho) = \sum_{j=1}^{s} K_j \rho K_j^\dagger$$

with given matrices $K_j \in \mathbb{C}^{n\times n}$ satisfying $\sum_{j=1}^{s} K_j^\dagger K_j = I$.

a) Show that $\mathcal{E}$ maps

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(i) Hermitian matrices to Hermitian matrices, and

(ii) positive semi-definite matrices to positive semi-definite matrices.

(i) Let $\rho \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, $\rho^\dagger = \rho$, then

$$\mathcal{E}(\rho)^\dagger = \sum_{j=1}^{s} \left(K_j \rho K_j^\dagger\right)^\dagger = \sum_{j=1}^{s} (K_j^\dagger)^\dagger \rho^\dagger K_j^\dagger = \sum_{j=1}^{s} K_j \rho K_j^\dagger = \mathcal{E}(\rho),$$

that is, $\mathcal{E}(\rho)$ is Hermitian as well. Here we have used that $(AB)^\dagger = B^\dagger A^\dagger$ for any two matrices A, B of compatible dimensions, and that $(A^\dagger)^\dagger = A$.

(ii) Let $\rho \in \mathbb{C}^{n\times n}$ be a positive semi-definite matrix, i.e., $\langle v, \rho v\rangle \geq 0 \ \forall v \in \mathbb{C}^n$. We have to show that $\langle v, \mathcal{E}(\rho) v\rangle \geq 0 \ \forall v \in \mathbb{C}^n$:

$$\langle v, \mathcal{E}(\rho) v\rangle = \sum_{j=1}^{s} \left\langle v, K_j \rho K_j^\dagger v\right\rangle = \sum_{j=1}^{s} \left\langle K_j^\dagger v, \rho K_j^\dagger v\right\rangle = \sum_{j=1}^{s} \langle \varphi_j, \rho \varphi_j\rangle \geq 0,$$

where we have defined $\varphi_j = K_j^\dagger v$ for $j = 1, \dots, s$.

b)* For the following, we define two new tensors $\mathcal{K}$ and $\mathcal{K}'$ of degree 3 by $\mathcal{K}_{j,:,:} = K_j$ and $\mathcal{K}'_{j,:,:} = K_j^\dagger$ for all $j = 1, \dots, s$. Draw the tensor network representing the application of $\mathcal{E}$ to ρ in terms of ρ, $\mathcal{K}$ and $\mathcal{K}'$, clearly indicating the dimension of each leg.

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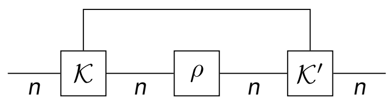
$\mathcal{E}(\rho) =$ 

The dimension of each horizontal leg is n, and the bond connecting $\mathcal{K}$ with $\mathcal{K}'$ has dimension s.

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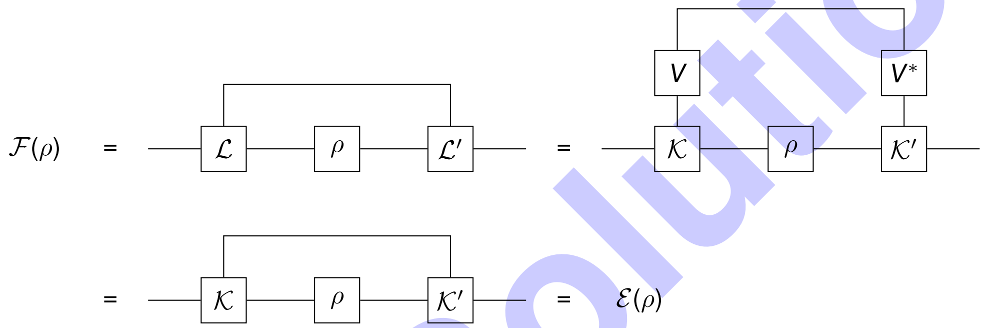
c) Given a unitary matrix $V = (v_{ij}) \in \mathbb{C}^{s\times s}$, we introduce the new linear map $\mathcal{F} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ by

$$\mathcal{F}(\rho) = \sum_{i=1}^{s} L_i \rho L_i^\dagger \quad \text{with } L_i = \sum_{j=1}^{s} v_{ij} K_j \text{ for } i = 1, \dots, s.$$

Show that $\mathcal{E}$ and $\mathcal{F}$ represent the same map.

Similar to $\mathcal{E}$, we can represent $\mathcal{F}$ as tensor network using analogous degree-3 tensors $\mathcal{L}_{i,:,:} = L_i$ and $\mathcal{L}'_{i,:,:} = L_i^\dagger$. By definition, $\mathcal{L}$ is related to $\mathcal{K}$ and $\mathcal{L}'$ to $\mathcal{K}'$ via

Thus, for any $\rho \in \mathbb{C}^{n\times n}$,



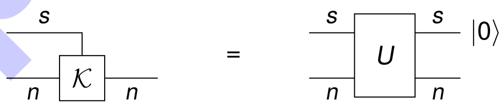
For the third equal sign we have used that V is unitary.

Alternative solution:

$$\mathcal{F}(\rho) = \sum_{i=1}^{s} L_i \rho L_i^\dagger = \sum_{i,j,j'=1}^{s} v_{ij} v_{ij'}^* K_j \rho K_{j'}^\dagger = \sum_{j,j'=1}^{s} \underbrace{\left(\sum_{i=1}^{s} v_{ij} v_{ij'}^*\right)}_{\delta_{j,j'}} K_j \rho K_{j'}^\dagger = \sum_{j=1}^{s} K_j \rho K_j^\dagger = \mathcal{E}(\rho).$$

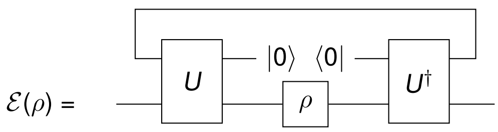
0 1 2 3 4

d) $\mathcal{K}$ interpreted as $(sn) \times n$ matrix is an isometry by definition; we can thus extend it to a unitary matrix $U \in \mathbb{C}^{(sn)\times(sn)}$ such that (with labels at the legs denoting dimensions, and $|0\rangle$ the first unit vector in $\mathbb{C}^s$):



Draw a tensor diagram that represents $\mathcal{E}(\rho)$ in terms of ρ, U, $U^\dagger$ and $|0\rangle$.

The tensor $\mathcal{K}'$, or equivalently $K_j^\dagger$ for $j = 1, \dots, s$, can be expressed by taking the adjoint of U. One arrives at:



e) Based on your solution of part (d), express $\mathcal{E}(\rho)$ using a partial trace operation.

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The diagram in part (d) can be interpreted as "tracing out" the top register with dimension s. As formula:

$$\mathcal{E}(\rho) = \mathrm{tr}_1\left[U\big((|0\rangle\langle 0|) \otimes \rho\big)U^\dagger\right]$$

Remark: This is a derivation of Stinespring's dilation theorem in finite dimensions.

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.