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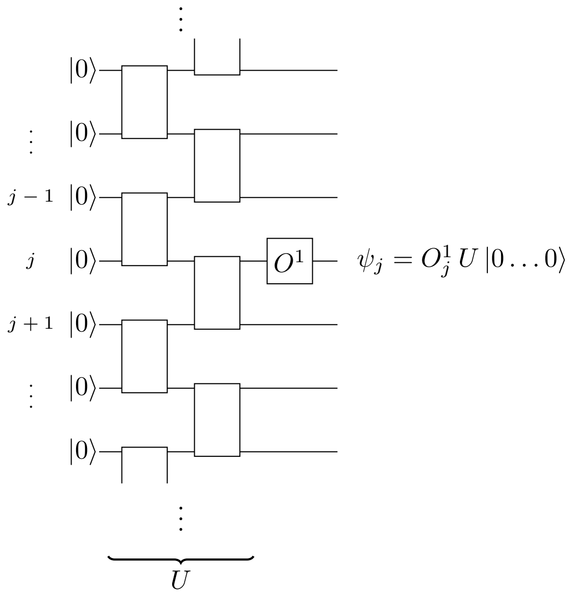
Exercise 12.1 (Light cone pattern of dynamical correlations in brick wall quantum circuits)

A common quantum circuit layout is a "brick wall" arrangement of two-qubit gates, as illustrated on the right. One can interpret such a layout as discretization of a quantum time evolution step, $U \approx \mathrm{e}^{-iH\Delta t}$ (cf. the TEBD algorithm).

Our goal here is to compute the so-called *dynamical correlation function* between the j-th and k-th qubit ($k \geq j$). This is achieved via the following construction: The initial state $|0\dots0\rangle$ time-evolves for a step Δt; we then apply a single-qubit gate O^1 to the j-qubit. O^1 is also assumed to be Hermitian, like for example one of the Pauli matrices, and plays the role of an observable. We denote the overall output state of this protocol by ψ_j, as illustrated on the right. (The total number of qubits is assumed to be large enough such that boundary effects are not relevant. The following considerations are valid even if all the two-qubit gates are different.)



Regarding the k-th qubit, we again start from the state $|0\dots0\rangle$, then first apply another gate O^2 to the k-qubit, and finally perform a time step. The overall output of this protocol is thus $\chi_k = U O_k^2 |0\dots0\rangle$.

The dynamical correlation function[1] is now defined as

$$c_{j,k} = \langle \psi_j, \chi_k \rangle.$$

(a) Draw the diagram for evaluating $c_{j,k}$ for the cases $k = j+2$ and $k = j+3$, and simplify it as much as possible by canceling two-qubit gates (i.e., a two-qubit gate V directly followed by $V^\dagger$).

(b) For $k = j+3$, compute the so-called connected correlation function

$$c_{j,k}^{\mathrm{conn}} = c_{j,k} - \langle 0\dots0|\, U^\dagger O_j^1 U\, |0\dots0\rangle \cdot \langle 0\dots0|\, O_k^2\, |0\dots0\rangle\,.$$

Does your result change when considering $k \geq j+3$ in general?

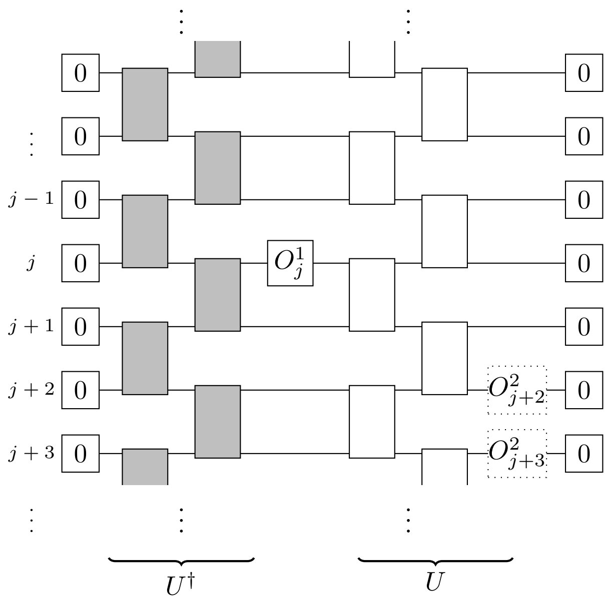
(c) (Voluntary) Repeat the calculations using two time steps, by applying U twice at each occurrence.

Remark: You should learn form this exercise that the brick wall pattern leads to a "light cone" spreading of correlations, i.e., qubits j and k can only be correlated (non-zero $c_{j,k}^{\mathrm{conn}}$) if there are sufficiently many time steps to reach k from j.

[1] In the physics literature, the correlation function is typically written as $c_{j,k} = \left\langle O_j^1(\Delta t)\, O_k^2 \right\rangle$, with $O_j^1(\Delta t) = U^\dagger O_j^1 U$ the time-evolved operator in the Heisenberg picture.

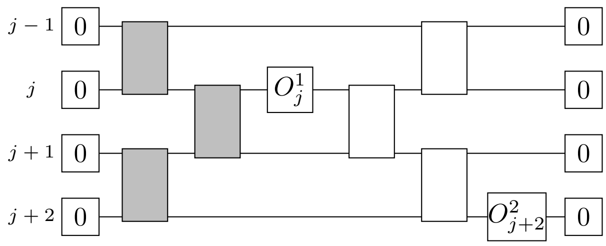
Solution

(a) We first draw the overall tensor diagram representing the contraction operation $c_{jk} = \langle \psi_j, \chi_k \rangle$, with the possible positions of O^2 in dotted lines for $k = j+2$ and $k = j+3$.

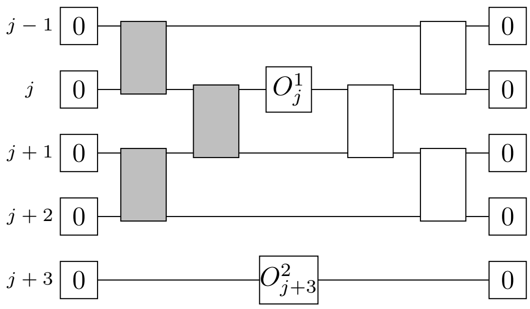


The grey gates denote $V^\dagger$. After removing each $V^\dagger V$ pair and noting that $\langle 0|0\rangle$ equivalent to multiplying by a factor of 1, the tensor diagram for the two given cases simplify respectively into:

Case 1: $k = j+2$



Case 2: $k = j+3$



(b) We define the time evolved operator as follows:

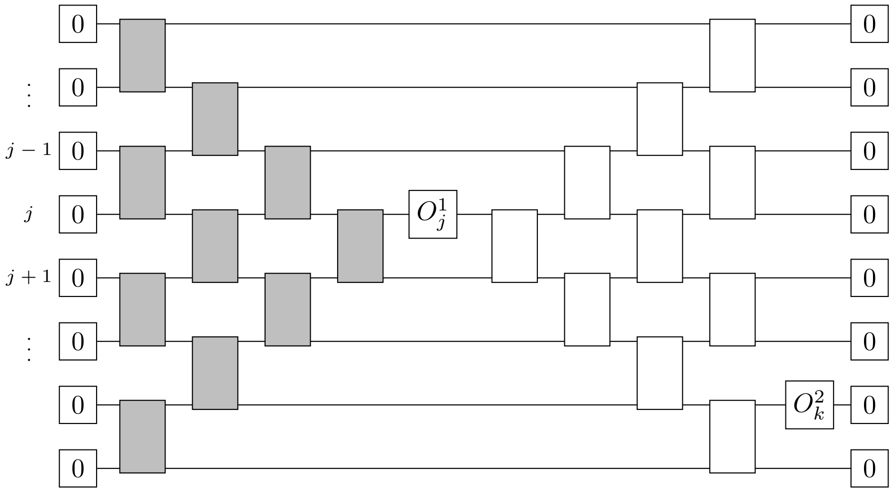
$$O_j^1(\Delta t) = U^\dagger O_j^1 U = I \otimes I ... I \otimes \widetilde{O}^1 \otimes I \otimes I ... I$$

where $\widetilde{O}^1$ is a multi-qubit operator on wires $j-1, j, j+1, j+2$. In the last equality we used the fact, seen in part (a), that pairs $V \dagger V$ cancel if they are applied outside of wires $j-1, j, j+1, j+2$. Thus, $c_{j,j+3}^{\text{conn}}$ simplifies to:

$$\begin{aligned} c_{j,j+3}^{\text{conn}} =& c_{j,j+3} - \langle 0 \ldots 0| U^\dagger O_j^1 U |0 \ldots 0\rangle \cdot \langle 0 \ldots 0| O_{j+3}^2 |0 \ldots 0\rangle . \\ =& c_{j,j+3} - \langle 0000| \widetilde{O}^1 |0000\rangle \cdot \langle 0| O_{j+3}^2 |0\rangle . \end{aligned}$$

We can then observe that the second term in the equation is then exactly equivalent to the contraction performed in the Case 2, meaning that $c_{j,j+3}^{\text{conn}} = 0$ and further note that this remains true as long as $k \geq j+3$ or $k \leq j-2$.

(c) In this example, each added layer extends the "light cone" by two additional qubits in the +/− directions as additional rows of unitaries in the "brick wall" no longer contract with their adjoints.



In this case, $c_{j,k}^{\text{conn}} = 0$ when $k \geq j+5$ or $k \leq j-4$.