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1.2

(b)

— $M \in \overset{\text{Hint}}{\mathbb{R}}{}^{2\times 2}$ normal $\Leftrightarrow M^\dagger M = MM^\dagger$

— want M s.t. $M^\dagger M \neq MM^\dagger$

— Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then (by Def. of adjoint)

$$M^\dagger = (M^*)^T = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^*\right)^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$\forall x \in \mathbb{R}: x^* = x$ since $\operatorname{Im}(a) = 0\ \forall a \in \mathbb{R}$

— $$M^\dagger M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$$

$$MM^\dagger = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+b^2 \end{bmatrix}$$

— When <u>is</u> $M^\dagger M = MM^\dagger$?

$$M^\dagger M = MM^\dagger \Leftrightarrow M^\dagger M - MM^\dagger = 0 \Leftrightarrow \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} - \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+b^2 \end{bmatrix} = 0 \Leftrightarrow$$

$$\underbrace{\begin{bmatrix} c^2-b^2 & ab+cd-ac-bd \\ ab+cd-ac-bd & -(c^2+d^2) \end{bmatrix}}_{=:M'} = 0$$

— For M to be normal, all entries $m'_{ij} \overset{!}{=} 0$ of M'

— Since we want M to be <u>not</u> normal, we choose $m'_{11} = c^2 - b^2 \neq 0$ by picking $c^2 \neq b^2$, for instance $c=0, b=1$. Assign $a=d=0$ randomly.

— We get $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which is not normal: $M^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$\Rightarrow MM^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq M^\dagger M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) — Let $M = \begin{bmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{bmatrix}$ for $\theta \in \mathbb{R}$.

— Let $$M^\dagger = \left(\begin{bmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{bmatrix}^*\right)^T = \begin{bmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{bmatrix}$$

— M unitary $\Leftrightarrow M^\dagger M = I \Leftrightarrow$

$$\begin{bmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta) - i^2\sin^2(\theta) & \overbrace{\cos(\theta)\, i\sin(\theta) - i\sin(\theta)\cos(\theta)}^{=0} \\ \underbrace{-i\sin(\theta)\cos(\theta) + \cos(\theta)\, i\sin(\theta)}_{=0} & -i^2\sin^2(\theta) + \cos^2(\theta) \end{bmatrix}$$

$$\underset{\uparrow\, i^2=-1}{=} \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & 0 \\ 0 & \sin^2(\theta) + \cos^2(\theta) \end{bmatrix} \underset{\uparrow\, \sin^2(a)+\cos^2(a)=1 \ \forall a \in \mathbb{R}}{=} I \quad \square$$

(d)
- Let $U \in \mathbb{C}^{n\times n}$ be unitary
- 1 | Hint: $\det(I) = 1$
  $= \det(I)$ | Def. unitary
  $= \det(U^\dagger U)$ | Hint: $\det(A\cdot B) = \det(A)\cdot\det(B)$
  $= \det(U^\dagger)\det(U)$ | Def. adjoint
  $= \det((U^*)^T)\cdot\det(U)$ | Hint: $\det(A^T) = \det(A)$
  $= \det(U^*)\det(U)$ | By application of (T1) using $A \leftarrow U$
  $= \det(U)^* \det(U)$
- Since $\det(U)$ exists for all $U \in \mathbb{C}^{n\times n}$, there exist $a, b \in \mathbb{R}$ s.t. $\det(U) = ae^{ib}$
- From above we have: $1 = \det(U)^* \cdot \det(U)$, by substituting $\det(U)$ we obtain:

$$1 = (ae^{ib})^*(ae^{ib}) \Leftrightarrow 1 = ae^{-ib}\cdot ae^{ib} = a^2 e^{ib-ib} = a^2 e^0 = a^2$$

by application of (T2)

- Thus follows: $\det(U) = \pm e^{ib}\ \forall b \in \mathbb{R}$
- Since $\forall b \in \mathbb{R}$. $|\det(U)| = |\pm e^{ib}| = |\pm 1| = 1$ $\square$ ✓

(T1) $\det(A^*) = \det(A)^* \quad \forall A \in \mathbb{C}^{n\times n}$

Proof: $\det(A^*)$ | Def. determinant from lecture

$= \sum_{\sigma \in S(n)} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a^*_{j,\sigma(j)}$ | By application of (T3) using $a_j \leftarrow a_{j,\sigma(j)}$

$= \sum_{\sigma \in S(n)} \operatorname{sgn}(\sigma) \left(\prod_{j=1}^{n} a_{j,\sigma(j)}\right)^*$ | By application of (T7) using $\alpha \leftarrow \operatorname{sgn}(\sigma)$, $x \leftarrow \left(\prod_{j=1}^{n} a_{j,\sigma(j)}\right)^*$

$= \sum_{\sigma \in S(n)} \left(\operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j,\sigma(j)}\right)^*$ | By application of (T4) using $S \leftarrow S(n)$, $f(\sigma) \leftarrow (\operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j,\sigma(j)})^*$

$= \left(\sum_{\sigma \in S(n)} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j,\sigma(j)}\right)^*$ | Def. determinant

$= \det(A)^*$ $\square$ ✓

You could have assumed the properties of complex no. with regards to *, but everything is fine.

(T2) complex conjugate: Let $x = ae^{ib} \in \mathbb{C}$, then $x^* = ae^{-ib}$

Proof: $x^* = (ae^{ib})^* \underset{\uparrow \text{ Euler's formula}}{=} (a(\cos(b) + i\sin(b)))^* \underset{\uparrow \text{ application of (T1) w/ } \alpha = a,\ x = \cos(b)+i\sin(b)}{=} a \cdot (\cos(b) + i\sin(b))^* \overset{\text{Def. complex conjugate}}{=} a(\cos(b) - i\sin(b))$

$= a \cdot (\cos(-b) + i\sin(-b)) = a(e^{i(-b)}) = ae^{-ib}$ □

$\forall a \in \mathbb{R}$: $\cos(a) = \cos(-a)$
$\sin(-a) = -\sin(a)$

(T3) "$\prod a^* = (\prod a)^*$" or:

Let $n \in \mathbb{N}$, $a_j \in \mathbb{C}$ for $j \in \{1, ..., n\}$, then $\prod_{j=1}^{n} a_j^* = \left(\prod_{j=1}^{n} a_j\right)^*$

Proof: by induction over n:

(IB) Let $n = 1$: $\prod_{j=1}^{n} a_j^* = \prod_{j=1}^{1} a_j^* = a_j^* = (a_j)^*$
$= \left(\prod_{j=1}^{1} a_j\right)^* = \left(\prod_{j=1}^{n} a_j\right)^*$

(IS) with (IH) assuming (T3) holds for n: $\prod_{j=1}^{n} a_j^* = \left(\prod_{j=1}^{n} a_j\right)^*$

Let $n \in \mathbb{N}$, show (T3) holds for $n+1$:

$$\prod_{j=1}^{n+1} a_j^* = a_{n+1}^* \prod_{j=1}^{n} a_j^* \overset{(IH)}{=} a_{n+1}^* \left(\prod_{j=1}^{n} a_j\right)^* = \left(a_{n+1} \prod_{j=1}^{n} a_j\right)^* = \left(\prod_{j=1}^{n+1} a_j\right)^* \quad \square$$

by application of (T5) using $x = a_{n+1}$, $y = \prod_{j=1}^{n} a_j$

(T4) "$\sum a^* = (\sum a)^*$"

Let $n \in \mathbb{N}$, S be a set with $|S| = n$, f a function $f: S \to \mathbb{C}$, then

$$\sum_{\theta \in S} f(\theta)^* = \left(\sum_{\theta \in S} f(\theta)\right)^*$$

Proof: By induction over n:

(IB) Let $n = 1 \Rightarrow$ Let WLOG $S = \{\alpha\}$ with $|S| = 1$ then

$$\sum_{\theta \in S} f(\theta)^* = \sum_{\theta \in \{\alpha\}} f(\theta)^* = f(\alpha)^* = (f(\alpha))^* = \left(\sum_{\theta \in \{\alpha\}} f(\theta)\right)^* = \left(\sum_{\theta \in S} f(\theta)\right)^*$$

(IS) with (IH) assuming (T4) holds for n: Let S with $|S| = n$:

$$\sum_{\theta \in S} f(\theta)^* = \left(\sum_{\theta \in S} f(\theta)\right)^*$$

Let WLOG $S' = S \cup \{\beta\}$ w/ $|S'| = |S| + 1 = n+1$

To show: $\sum_{\theta\in S'} f(\theta)^* = \left(\sum_{\theta\in S'} f(\theta)\right)^*$.

$$\sum_{\theta\in S'} f(\theta)^* = \sum_{\theta\in S\cup\{\beta\}} f(\theta)^* = f(\beta)^* + \sum_{\theta\in S} f(\theta)^*$$

$$\overset{\text{(IH)}}{=} f(\beta)^* + \left(\sum_{\theta\in S} f(\theta)\right)^* = \left(f(\beta) + \sum_{\theta\in S} f(\theta)\right)^*$$

by application of (T6) with
$x \leftarrow f(\beta)$
$y \leftarrow \sum_{\theta\in S} f(\theta)$

$$= \left(\sum_{\theta\in S\cup\{\beta\}} f(\theta)\right)^* = \left(\sum_{\theta\in S'} f(\theta)\right)^* \quad \square$$

(T5) $x^*y^* = (xy)^*$ for $x,y\in\mathbb{C}$

Proof:
- Let $x=a+bi$, $y=c+di$ with $x^*=a-bi$, $y^*=c-di$
- $x^*y^* = (a-bi)(c-di) = ac-bd-(ad+bc)i$
  $= (ac-bd+(ad+bc)i)^* = ((a+bi)(c+di))^* = (xy)^*$

(T6) $x^*+y^* = (x+y)^*$ for $x,y\in\mathbb{C}$

Proof:
- Let $x=a+bi$, $y=c+di$ with $x^*=a-bi$, $y^*=c-di$
- $x^*+y^* = a-bi+c-di = a+c-(b+d)i = (a+c+(b+d)i)^*$
  $= (a+bi+c+di)^* = (x+y)^*$

(T7) $\alpha x^* = (\alpha x)^*$ for $\alpha\in\mathbb{R}$, $x\in\mathbb{C}$

Proof:
- Let $x=a+bi$ with $x^*=a-bi$
- $\alpha x^* = \alpha(a-bi) = \alpha a - \alpha bi = (\alpha a+\alpha bi)^* = (\alpha(a+bi))^* = (\alpha x)^*$

(a)
- $\begin{bmatrix} 2 & -i & 5 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ i \\ -3 \end{bmatrix} = \begin{bmatrix} 8-i^2-15 \\ 12-3 \end{bmatrix} = \begin{bmatrix} 8+1-15 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$ ✓
- $\begin{bmatrix} -2 & 7 \\ 3 & 1+2i \end{bmatrix} \cdot \begin{bmatrix} 5 & -4 \\ 6i & 0 \end{bmatrix} = \begin{bmatrix} -10+42i & 8 \\ \underbrace{15+6i+12i^2}_{3+6i} & -12 \end{bmatrix} = \begin{bmatrix} -10+42i & 8 \\ 3+6i & -12 \end{bmatrix}$ ✓

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7.1 $a = 3+4i$, $b = 2-i$

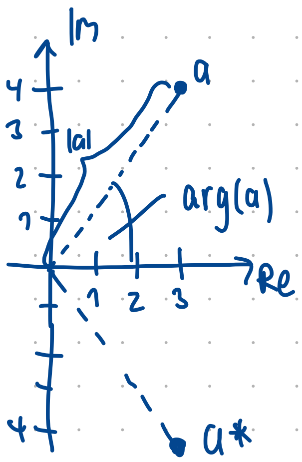
(a)

- $a+b = 3+4i+2-i = (3+2)+(4-1)i = 5+3i$
- $ab = (3+4i)(2-i) = 6-3i+8i-4i^2 = 10+5i$
- $1/a = \frac{1}{3+4i} = \frac{(3-4i)}{(3+4i)(3-4i)} = \frac{3-4i}{\underbrace{9-16i^2}_{9+16=25}} = \frac{3-4i}{25} = 3/25 - 4/25\, i$
- $a^* = 3-4i$
- $|a| = \sqrt{3^2+4^2} = \sqrt{9+16} = 5$ $\quad \arg(a) = \arctan(4/3) \approx 53°$
- $\psi = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3+4i \\ 2-i \end{bmatrix}$, $\|\psi\| = \sqrt{\langle \psi, \psi^* \rangle} = \sqrt{\langle [3+4i \;\; 2-i]^T, [3+4i \;\; 2-i]^{T*} \rangle}$

$= \sqrt{\langle [3+4i \;\; 2-i]^T, [3-4i \;\; 2+i]^T \rangle} = \sqrt{(3+4i)(3-4i)+(2-i)(2+i)}$

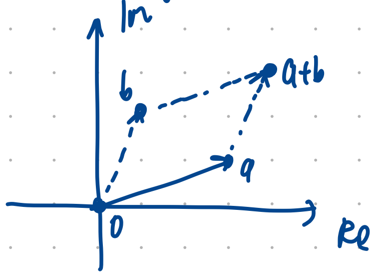
$= \sqrt{9-16i^2+4-i^2} = \sqrt{9+16+4+1} = \sqrt{30}$

(b)



(c)

- for complex numbers $a = x+yi$, $b = u+vi$
$a+b = x+yi+u+vi = x+u+(y+v)i$
→ Parallelogram law of two vectors:



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- for complex numbers $a = xe^{iy}$, $b = ue^{iv}$
$ab = xe^{iy}ue^{iv} = xue^{iy+iv} = xue^{i(y+v)}$