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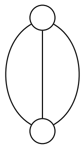
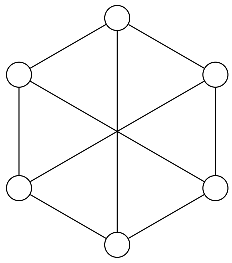
Tutorial 3 (Counting graph colorings)
Consider the following problem: given a 3-regular graph (i.e., a graph where each vertex has three connected edges), how many edge colorings using three colors exist, such that the edges at each vertex have distinct colors?

(a) Explicitly enumerate the allowed edge colorings for the following graph:



(b) Interpreting a 3-regular graph as tensor network, with each vertex a tensor of degree 3, how can we define these tensors such that contracting the tensor network yields the number of edge colorings?

(c) Compute how many ways one can color the following graph:



As a remark, it turns out that if the vertices are described by the *Levi-Civita symbol* ϵ defined as

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

the contraction will count the colorings for planar graphs, but yield 0 for non-planar ones.[1] You can test this statement for the graph above. (As voluntary homework puzzle, try to prove this statement in general.)

Solution

(a) The are 6 different colorings:

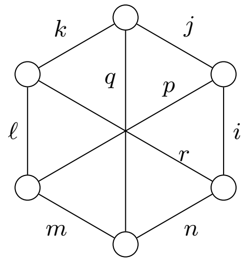
(b) We denote the edge colors by $\{1,2,3\}$, and define the vertex tensors as indicators for valid colorings, i.e., giving 1 precisely if the incident edges have pairwise different colors:

$$t_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ are pairwise different} \\ 0 & \text{otherwise} \end{cases}$$

with $i,j,k \in \{1,2,3\}$. Since contracting the network corresponds to enumerating all possible edge colorings, a network contraction will count the number of admissible colorings.

[1]See Roger Penrose: *Applications of negative dimensional tensors.* Combinatorial Mathematics and its Applications (1971)

(c) We label the edge indices as:



Using this convention, the number of colorings equals

$$
\begin{aligned}
\sharp\text{colorings} &= \sum_{i,j,k,\ell,m,n,p,q,r=1}^{3} t_{ijp}\, t_{jkq}\, t_{k\ell r}\, t_{\ell mp}\, t_{mnq}\, t_{nir} \\
&\overset{i\to 1}{=} 3 \cdot \sum_{j,k,\ell,m,n,p,q,r=1}^{3} t_{1jp}\, t_{jkq}\, t_{k\ell r}\, t_{\ell mp}\, t_{mnq}\, t_{n1r} \\
&\overset{j\to 2}{=} 3 \cdot 2 \cdot \sum_{k,\ell,m,n,p,q,r=1}^{3} t_{12p}\, t_{2kq}\, t_{k\ell r}\, t_{\ell mp}\, t_{mnq}\, t_{n1r} \\
&\overset{p\to 3}{=} 3 \cdot 2 \cdot \sum_{k,\ell,m,n,q,r=1}^{3} t_{123}\, t_{2kq}\, t_{k\ell r}\, t_{\ell m3}\, t_{mnq}\, t_{n1r} \\
&\overset{k\to 1}{=} 3 \cdot 2 \cdot 2 \cdot \sum_{\ell,m,n,q,r=1}^{3} t_{123}\, t_{21q}\, t_{1\ell r}\, t_{\ell m3}\, t_{mnq}\, t_{n1r} \\
&\overset{q\to 3}{=} 3 \cdot 2 \cdot 2 \cdot \sum_{\ell,m,n,r=1}^{3} t_{123}\, t_{213}\, t_{1\ell r}\, t_{\ell m3}\, t_{mn3}\, t_{n1r} \\
&\overset{\ell\to 2}{=} 3 \cdot 2 \cdot 2 \cdot \sum_{m,n,r=1}^{3} t_{123}\, t_{213}\, t_{12r}\, t_{2m3}\, t_{mn3}\, t_{n1r} \\
&\overset{m\to 1}{=} 3 \cdot 2 \cdot 2 \cdot \sum_{n,r=1}^{3} t_{123}\, t_{213}\, t_{12r}\, t_{213}\, t_{1n3}\, t_{n1r} \\
&\overset{n\to 2}{=} 3 \cdot 2 \cdot 2 \cdot \sum_{r=1}^{3} t_{123}\, t_{213}\, t_{12r}\, t_{213}\, t_{123}\, t_{21r} \\
&\overset{r\to 3}{=} 3 \cdot 2 \cdot 2 \cdot t_{123}\, t_{213}\, t_{123}\, t_{213}\, t_{123}\, t_{213} \\
&= 12.
\end{aligned}
$$

For the second equal sign, we have (arbitrarily) set $i = 1$; the factor 3 stems from the three possibilities $i \in \{1, 2, 3\}$. Regarding the third equal sign, j can either be 2 or 3 (but not 1, since (i, j, p) must be pairwise different); we have set $j = 2$ here. This scheme is repeated until all summation variables have disappeared.