Group 2:
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(2.1) $A = \begin{bmatrix} 0 & 3/5 & 4/5 \\ -3/5 & 0 & 0 \\ -4/5 & 0 & 0 \end{bmatrix}$

1a)

- $\chi_A(\lambda) = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -3/5 & -4/5 \\ 3/5 & \lambda & 0 \\ 4/5 & 0 & \lambda \end{bmatrix}\right)$

$= \lambda^3 + 0 + 0 - \frac{4}{5}\cdot\lambda\cdot\left(-\frac{4}{5}\right) - 0 - \lambda\cdot\left(\frac{3}{5}\right)\left(-\frac{3}{5}\right)$

$= \lambda^3 + \frac{16}{25}\lambda + \frac{9}{25}\lambda = \lambda^3 + \lambda = \lambda(1+\lambda^2) = \lambda(\lambda - i)(\lambda + i)$

- eigenvalues of A are $\lambda = 0,\ \lambda = \pm i$ (roots of $\chi_A(\lambda)$)
- proof for Spectral decomp. $\Rightarrow \exists U$ unitary: $A \cdot U = \mathrm{diag}(\lambda_1 \dots \lambda_n) A$

where $U = (u_1 | \dots | u_n)$ where u_j are eigenvectors of A

- Find $u_j \in \mathbb{C}^3$: $Au_j = \lambda_j u_j \quad \forall j \in \{1,2,3\}$
  - $\lambda_2 = i$: $Au_2 = iu_2 \Leftrightarrow Au_2 - iIu_2 = 0 \Leftrightarrow (A - iI)u_2 = 0$

$\Leftrightarrow \left(\begin{bmatrix} 0 & 3/5 & 4/5 \\ -3/5 & 0 & 0 \\ -4/5 & 0 & 0 \end{bmatrix} - iI\right)u_2 = 0 \Leftrightarrow \begin{bmatrix} -i & 3/5 & 4/5 \\ -3/5 & -i & 0 \\ -4/5 & 0 & -i \end{bmatrix} u_2 = 0$

using Gauß-Jacobi: $\begin{bmatrix} -i & 3/5 & 4/5 \\ -3/5 & -i & 0 \\ -4/5 & 0 & -i \end{bmatrix} \downarrow \cdot i\frac{3}{5} \cdot i\frac{4}{5} \sim \begin{bmatrix} -i & 3/5 & 4/5 \\ 0 & -16/25\,i & 12/25\,i \\ 0 & 12/25\,i & -9/25\,i \end{bmatrix} \cdot 25/4 \cdot 25/3$

using: $-\frac{3}{5} + \underbrace{(-i)\cdot i}_{+1}\cdot\frac{3}{5} = 0,\quad -\frac{4}{5} + (-i)\cdot i\cdot\frac{4}{5} = 0,$

$\sim \begin{bmatrix} -i & 3/5 & 4/5 \\ 0 & -4i & 3i \\ 0 & 4i & -3i \end{bmatrix} \cdot 5 \downarrow \sim \begin{bmatrix} -5i & 3 & 4 \\ 0 & -4i & 3i \\ 0 & 0 & 0 \end{bmatrix} \downarrow \cdot i \cdot \left(\frac{3}{4i}\right) \rightarrow \begin{bmatrix} -5i & 0 & 25/4 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \frac{4}{5} \sim \begin{bmatrix} -4i & 0 & 5 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\Leftrightarrow -4ix_1 + 5x_3 = 0 \wedge -4x_2 + 3x_3 = 0 \Leftrightarrow 4ix_1 = 5x_3 \wedge 4x_2 = 3x_3$

$\Leftrightarrow x_1 = -\frac{5}{4}i\,x_3 \wedge x_2 = \frac{3}{4}x_3$

$\Leftrightarrow \mathbb{L} = \left\{\left(-\frac{5}{4}i\,x_3 \quad \frac{3}{4}x_3 \quad x_3\right)^T \mid x_3 \in \mathbb{C}\right\}$

$\Rightarrow$ choose $v_2 = \left[-\frac{5}{4}i \quad \frac{3}{4} \quad i\right]^T$

  - $\lambda_3 = -i$: $Au_3 = \lambda_3 u_3 \Leftrightarrow Av_3 = -iv_3 \Leftrightarrow Av_3 + iv_3 = 0 \Leftrightarrow (A + iI)v_3 = 0$

let $A + iI = \begin{bmatrix} i & 3/5 & 4/5 \\ -3/5 & i & 0 \\ -4/5 & 0 & i \end{bmatrix}$ ← solve LGS:

$$\begin{bmatrix} i & 3/5 & 4/5 \\ -3/5 & i & 0 \\ -4/5 & 0 & i \end{bmatrix} \rightarrow (-\tfrac{3}{5}i) \rightarrow \sim \begin{bmatrix} i & 3/5 & 4/5 \\ 0 & 16/25\,i & -12/25\,i \\ 0 & -12/25\,i & 9/25\,i \end{bmatrix} \cdot \tfrac{25}{4} \cdot \tfrac{25}{3} \sim \begin{bmatrix} 5i & 3 & 4 \\ 0 & 4i & -3i \\ 0 & -4i & 3i \end{bmatrix} \rightarrow (-\tfrac{3}{4}i) \rightarrow \begin{bmatrix} 5i & 0 & 25/4 \\ 0 & 4i & -3i \\ 0 & 0 & 0 \end{bmatrix} \cdot \tfrac{4}{5}$$

$$\sim \begin{bmatrix} 4i & 0 & 5 \\ 0 & 4i & -3i \\ 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{matrix} 4ix_1 + 5x_3 = 0 \\ \wedge\ 4ix_2 - 3ix_3 = 0 \end{matrix} \Leftrightarrow \begin{matrix} 4ix_1 = -5x_3 \\ \wedge\ 4ix_2 = 3ix_3 \end{matrix} \Leftrightarrow \begin{matrix} x_1 = -\frac{5}{4i}x_3 \\ \wedge\ x_2 = \frac{3}{4}x_3 \end{matrix}$$

$$\Leftrightarrow x_1 = \tfrac{5}{4}ix_3 \wedge x_2 = \tfrac{3}{4}x_3 \Leftrightarrow \mathbb{L} = \left\{ \left(\tfrac{5}{4}ix_3 \;\; \tfrac{3}{4}x_3 \;\; x_3 \right) \mid x_3 \in \mathbb{C} \right\}$$

$$\Rightarrow \text{(choose } u_3 = (5/4\,i \;\; 3/4 \;\; 1)^T$$

- $\lambda_1 = 0$: $Au_1 = \lambda_1 u_1 \Leftrightarrow Au_1 = 0$

$$\begin{pmatrix} 0 & 3/5 & 4/5 \\ -3/5 & 0 & 0 \\ -4/5 & 0 & 0 \end{pmatrix} \rightarrow \left(\tfrac{5}{3} \cdot \tfrac{4}{5}\right) \begin{bmatrix} 0 & 3/5 & 4/5 \\ -3/5 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix} \cdot \tfrac{5}{3} \cdot (-\tfrac{3}{5}) \sim \begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow 3x_2 + 4x_3 = 0 \wedge x_1 = 0 \Leftrightarrow x_2 = -\tfrac{4}{3}x_3$$

$$\Leftrightarrow \mathbb{L} = \left\{ (0 \;\; -\tfrac{4}{3}x_3 \;\; x_3)^T \mid x_3 \in \mathbb{C} \right\}$$

- (choose unitary matrix $U = (u_1 \mid u_2 \mid u_3)^T$ for Spectral Decomp.:

$$U = \begin{pmatrix} 0 & -5/4\,i & 5/4\,i \\ -4/3 & 3/4 & 3/4 \\ 1 & 1 & 1 \end{pmatrix}$$

$\lambda = 0$ $\quad \lambda = i$ $\quad \lambda = -i$

(b) • A normal $\Rightarrow$ A is unitarily diagonalizable (Spectral decomp. Thm.):

• $\Rightarrow \exists U \in \mathbb{C}^{n\times n}$: $A \cdot U = U \cdot \mathrm{diag}(\lambda_1 \cdots \lambda_n)$ | $\cdot U^\dagger$

$\Leftrightarrow A \cdot UU^\dagger = U\,\mathrm{diag}(\lambda_1 \cdots \lambda_n)\,U^\dagger$ | Def. unitary: $UU^\dagger = I$, $\forall A \in \mathbb{C}^{n\times n}: A \cdot I = A$ w/o Pf.

$\Leftrightarrow A = U\,\mathrm{diag}(\lambda_1 \cdots \lambda_n)\,U^\dagger$

• $\mathrm{tr}[A] \overset{\Leftarrow}{=} \mathrm{tr}[U\,\mathrm{diag}(\lambda_1 \cdots \lambda_n)\,U^\dagger]$ | Hint

$= \mathrm{tr}[UU^\dagger\,\mathrm{diag}(\lambda_1 \cdots \lambda_n)]$ | Def. unitary $UU^\dagger = I$

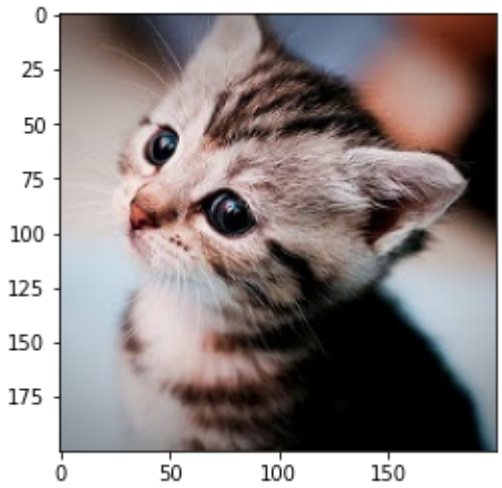
$= \mathrm{tr}[I \cdot \mathrm{diag}(\lambda_1 \cdots \lambda_n)]$ | $\forall A \in \mathbb{C}^{n\times n}: IA = A$ w/o Pf.

$= \mathrm{tr}[\mathrm{diag}(\lambda_1 \cdots \lambda_n)]$ | By def. matrix trace

$= \sum_{j=1}^{n} \lambda_j$ □

In [2]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [3]:
```
# read image from disk
imgc = plt.imread("kitten.jpg")
plt.imshow(imgc);
```



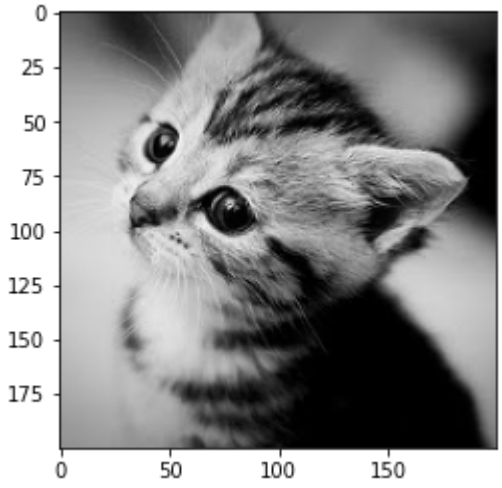
In [4]:
```
imgc.shape
```

Out[4]:
```
(200, 200, 3)
```

In [5]:
```
# convert to grayscale and [0, 1] value range
img = np.mean(imgc, axis=2) / 255.0
img.shape
```

Out[5]:
```
(200, 200)
```

In [6]:
```
plt.imshow(img, cmap="gray");
```

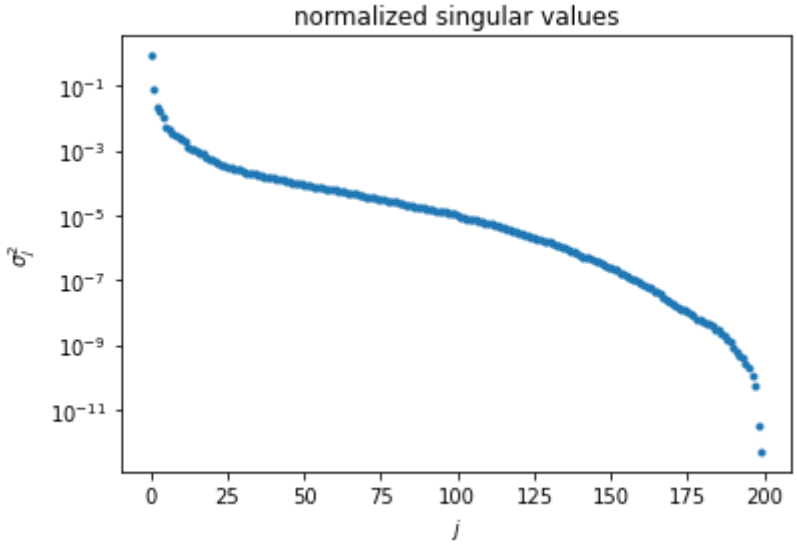


In [7]:
```
# singular value decomposition
u, s, vh = np.linalg.svd(img)
```

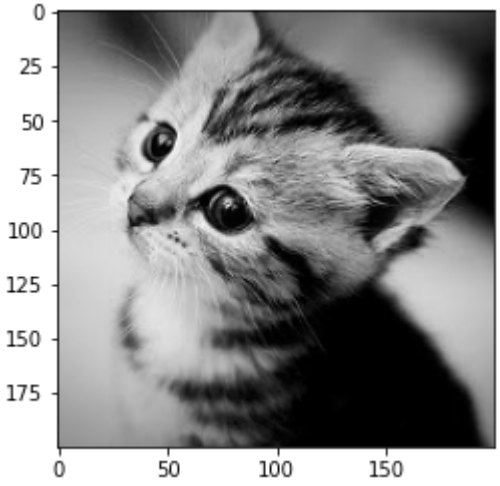
In [8]:
```
# check
np.allclose(img, (u * s) @ vh)
```

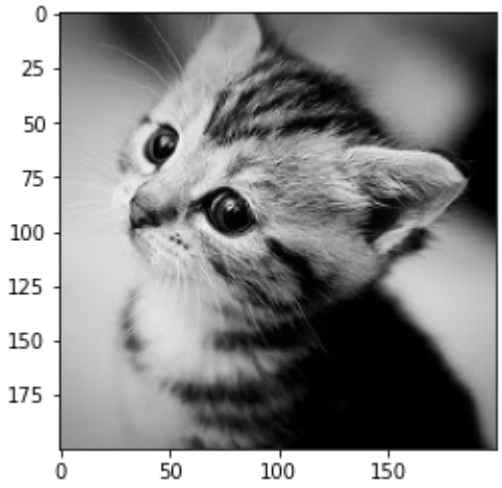
Out[8]:
```
True
```

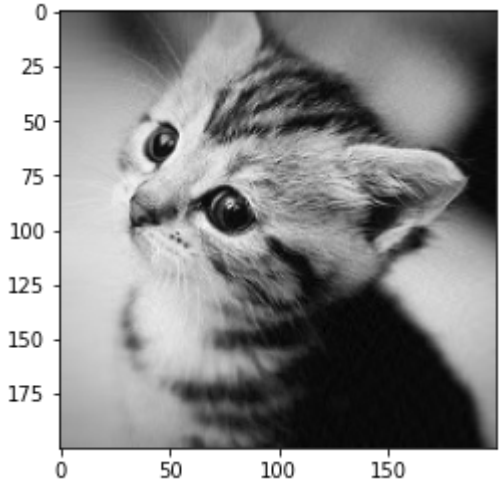
In [9]:
```
plt.semilogy(s**2 / np.sum(s**2), '.')
plt.ylabel("$\\sigma_j^2$")
plt.xlabel("$j$")
plt.title("normalized singular values");
```

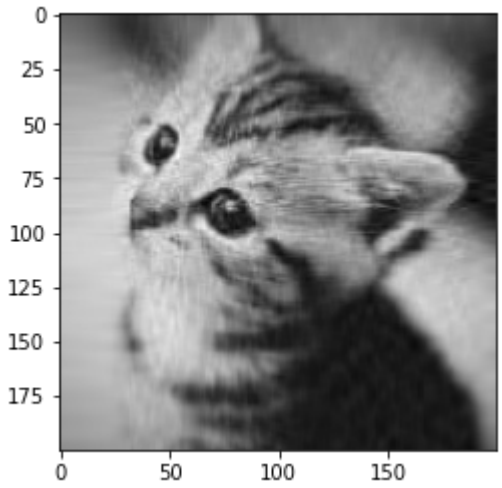


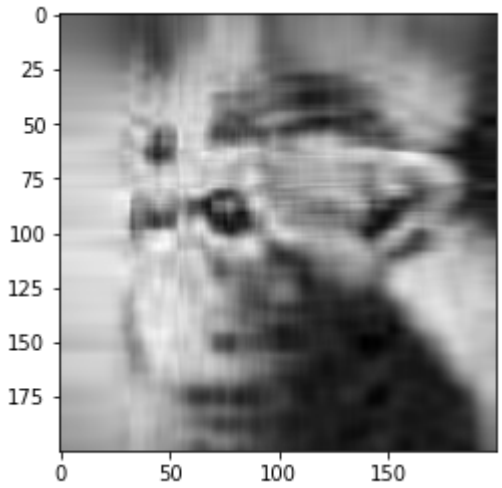
In [10]:
```
χlist = [len(s), 150, 75, 25, 10]

# TODO: Compute and plot the SVD-compressed approximation of `img` for all
# Hint: `u[:, :χ]` selects the leading `χ` columns of the matrix `u`.
for x in χlist:
    u_tilde = u[:, :x]
    s_tilde = s[:x]

    # manually create vh_tilde from v
    v = np.conj(vh.T)
    v_tilde = v[:, :x]
    vh_tilde = np.conj(v_tilde).T

    # reconstruct image
    img_tilde = u_tilde * s_tilde @ vh_tilde

    # show image
    fig, ax = plt.subplots()
    ax.imshow(img_tilde, cmap="gray")
```




0
25
50
75
100
125
150
175
0
50
100
150

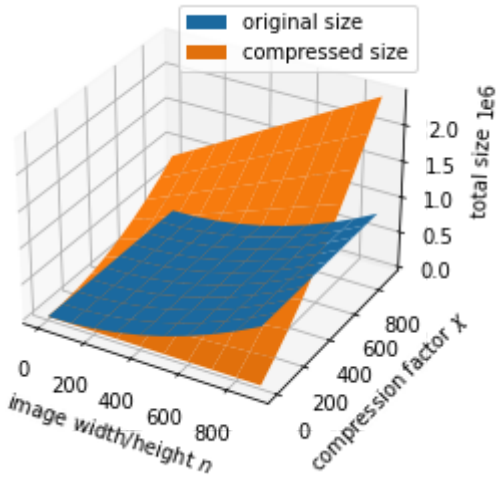

0
25
50
75
100
125
150
175
0
50
100
150


0
25
50
75
100
125
150
175
0
50
100
150


0
25
50
75
100
125
150
175
0
50
100
150

In [12]:
```
# bonus: show how compression level affects stored size
n = np.arange(0, 1000, 100)
chi = np.arange(0, 1000, 100)
n, chi = np.meshgrid(n, chi)
original = n*n
compressed = 2*n*chi + chi*chi
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
orig_surf = ax.plot_surface(n, chi, original, label="original size")
comp_surf = ax.plot_surface(n, chi, compressed, label="compressed size")

# create legend
orig_surf._edgecolors2d = orig_surf._edgecolor3d
orig_surf._facecolors2d = orig_surf._facecolor3d
comp_surf._edgecolors2d = comp_surf._edgecolor3d
comp_surf._facecolors2d = comp_surf._facecolor3d
ax.legend()

ax.set_xlabel("image width/height $n$")
ax.set_ylabel("compression factor $\\chi$")
ax.set_zlabel("total size")
```

Out[12]:
```
Text(0.5, 0, 'total size')
```



In [14]:
```
# exercise 2.2a
def spectral_decomp(A):
    # eigenvalues w and eigenvectors v
    w, v = np.linalg.eig(A)
    U = np.array(v)

    # calculate spectral decomposition: A*U = U*diag(lambda1 ... lambda3)
    print("Spectral Decomposition:")
    lhs = A@U
    print(f"A*U={lhs}")
    rhs = U@np.diag(w)
    print(f"U*d={rhs}")
    assert np.allclose(lhs, rhs)

def singular_val_decomp(A):
    print("Singular Value Decomposition:")
    u, s, vh = np.linalg.svd(A)
    print(f"U={u}")
    print(f"S=diag({s})")
    print(f"V^t={vh}")
    A_prime = u*s@vh
    print(f"U*S*V^t={A_prime}")
    assert np.allclose(A, A_prime)
```

```
# matrices from T2 and H2.1
A1 = np.array([[1, -1j, 0], [1j, 1, 0], [0, 0, 1j]])
A2 = np.array([[0, 3/5, 4/5], [-3/5, 0, 0], [-4/5, 0, 0]])
print(f"A={A1}")
spectral_decomp(A1)
singular_val_decomp(A1)
print(f"\nA={A2}")
spectral_decomp(A2)
singular_val_decomp(A2)
```

```
A=[[ 1.+0.j -0.-1.j  0.+0.j]
 [ 0.+1.j  1.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  0.+1.j]]
Spectral Decomposition:
A*U=[[0.00000000e+00-1.41421356e+00j 2.22044605e-16+0.00000000e+00j
  0.00000000e+00+0.00000000e+00j]
 [1.41421356e+00+0.00000000e+00j 0.00000000e+00+2.22044605e-16j
  0.00000000e+00+0.00000000e+00j]
 [0.00000000e+00+0.00000000e+00j 0.00000000e+00+0.00000000e+00j
  0.00000000e+00+1.00000000e+00j]]
U*d=[[0.         -1.41421356j 0.         +0.j         0.         +0.j        ]
 [1.41421356+0.j         0.         +0.j         0.         +0.j        ]
 [0.         +0.j         0.         +0.j         0.         +1.j        ]]
Singular Value Decomposition:
U=[[-0.70710678+0.j          0.         +0.j          0.         +0.70710678
j]
 [ 0.         -0.70710678j  0.         +0.j          0.70710678+0.j        ]
 [ 0.         +0.j          0.         -1.j          0.         +0.j        ]]
S=diag([2. 1. 0.])
V^t=[[-0.70710678+0.j          0.         +0.70710678j -0.         +0.j
]
 [-0.         +0.j          0.         +0.j         -1.         +0.j        ]
 [-0.70710678+0.j          0.         -0.70710678j  0.         +0.j        ]]
U*S*V^t=[[1.+0.j 0.-1.j 0.+0.j]
 [0.+1.j 1.+0.j 0.+0.j]
 [0.+0.j 0.+0.j 0.+1.j]]

A=[[ 0.   0.6  0.8]
 [-0.6  0.   0. ]
 [-0.8  0.   0. ]]
Spectral Decomposition:
A*U=[[ 0.00000000e+00+0.70710678j  0.00000000e+00-0.70710678j
   1.15463195e-16+0.j        ]
 [-4.24264069e-01+0.j         -4.24264069e-01+0.j
   0.00000000e+00+0.j        ]
 [-5.65685425e-01+0.j         -5.65685425e-01+0.j
   0.00000000e+00+0.j        ]]
U*d=[[ 0.         +0.70710678j  0.         -0.70710678j  0.         +0.j
]
 [-0.42426407+0.j         -0.42426407+0.j          0.         +0.j        ]
 [-0.56568542+0.j         -0.56568542+0.j          0.         +0.j        ]]
Singular Value Decomposition:
U=[[-1.00000000e+00  0.00000000e+00 -6.66133815e-18]
 [ 0.00000000e+00  6.00000000e-01 -8.00000000e-01]
 [ 1.11022302e-16  8.00000000e-01  6.00000000e-01]]
S=diag([1. 1. 0.])
V^t=[[-0.  -0.6 -0.8]
 [-1.  -0.  -0. ]
 [ 0.  -0.8  0.6]]
U*S*V^t=[[ 0.00000000e+00  6.00000000e-01  8.00000000e-01]
 [-6.00000000e-01  0.00000000e+00  0.00000000e+00]
 [-8.00000000e-01 -6.66133815e-17 -8.88178420e-17]]
```

In [ ]: