Christian B. Mendl, Richard M. Milbradt

Exercise 9.1 (Commuting matrices and conservation laws)
The *commutator* of two square matrices A and B is defined as

$$[A, B] = AB - BA.$$

We say that the matrices commute if $[A, B] = 0$. An important property of two commuting, normal matrices A and B is that they can be "simultaneously diagonalized", i.e., there exists a common basis of eigenvectors; in other words, we can find a unitary matrix U such that both $U^\dagger AU$ and $U^\dagger BU$ are diagonal. As sketch of the proof, note that the matrices leave eigenspaces invariant: namely, if (λ, v) is an eigenpair of A such that $Av = \lambda v$, then Bv remains in the λ-eigenspace of A, since $A(Bv) = (AB)v = BAv = B(\lambda v) = \lambda(Bv)$. Thus we can first find the eigenspaces of A, and then diagonalize B within each eigenspace (or other way around), which will result in a common eigenbasis.

As application to quantum physics with a given Hamiltonian H, one often searches for another Hermitian matrix P which commutes with H. P is then denoted "symmetry operator". This allows to partition the eigenvectors of H into symmetry subspaces, i.e., eigenspaces of P.

(a) Different Pauli matrices do not commute, but "anti-commute" instead: one can verify by an explicit calculation that

$$XY = -YX, \quad YZ = -ZY, \quad ZX = -XZ.$$

Nevertheless, it turns out that $X \otimes X$, $Y \otimes Y$ and $Z \otimes Z$ pairwise commute. Prove this statement.

Hint: You can work directly with matrix representations, or combine the general identity $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ with the anti-commuting property.

(b) A symmetry operator for the Ising model (see below) is

$$P = \prod_{j=1}^{L} X_j = X \otimes \cdots \otimes X,$$

where L denotes the number of lattice sites. Show that $[H, P] = 0$.

Solution

(a) Let us consider $X \otimes X$ and $Y \otimes Y$ first:

$$(X \otimes X)(Y \otimes Y) = (XY) \otimes (XY) = (-YX) \otimes (-YX) = (-1)^2 (YX) \otimes (YX) = (Y \otimes Y)(X \otimes X).$$

The commuting properties of the remaining pairwise combinations can be be derived analogously.

(b) We will show that P commutes with each individual term of the Hamiltonian

$$H = -J \sum_{\langle j,k \rangle} Z_j Z_k - g \sum_j X_j,$$

and thus in particular with the overall Hamiltonian H. First recall that, for fixed $j < k$,

$$Z_j Z_k = \underbrace{I_2 \otimes \cdots \otimes I_2}_{j-1 \text{ terms}} \otimes Z \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-j-1 \text{ terms}} \otimes Z \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{L-k \text{ terms}}.$$

We can represent the symmetry operator as

$$P = \left(\prod_{\ell \neq j,k} X_\ell \right) X_j X_k.$$

By an analogous calculation as in part (a), $Z_j Z_k$ commutes with $X_j X_k$, and thus

$$P\,(Z_j Z_k) = \left(\prod_{\ell \neq j,k} X_\ell \right) (Z_j Z_k)(X_j X_k) = (Z_j Z_k) \left(\prod_{\ell \neq j,k} X_\ell \right) (X_j X_k) = (Z_j Z_k)\,P.$$

For the second equal sign we have used that $Z_j Z_k$ and the terms in the product over ℓ act non-trivially on different lattice sites, and hence commute as well.

Finally, one observes that each X_j term in the Hamiltonian commutes with P since X commutes with itself.