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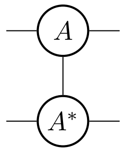
Exercise 10.1 (Properties of the AKLT state)
We can use the methods from Tutorial 10 to analyze the AKLT state, denoted ψ here. First recall that its MPS tensors $A \in \mathbb{C}^{3\times2\times2}$ (which are both left- and right-orthonormal) are given by

$$A_{\hat{1},:,:} = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \end{pmatrix}, \quad A_{\hat{0},:,:} = \begin{pmatrix} -\sqrt{\frac{1}{3}} & 0 \\ 0 & \sqrt{\frac{1}{3}} \end{pmatrix}, \quad A_{-\hat{1},:,:} = \begin{pmatrix} 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}.$$

(a) Calculate the corresponding 4×4 transfer matrix

$E =$  $= \sum_{\sigma\in\{1,0,-1\}} A_{\hat{\sigma},:,:} \otimes A^*_{\hat{\sigma},:,:},$

and compute its spectral decomposition. Why can we infer from the orthonormalization of the MPS tensors that the largest eigenvalue of E must be 1?

(b) It turns out that the AKLT state has a "hidden order", which is indicated by fact that the "string correlation function" $\langle \psi, S_j^z \left(\prod_{j<\ell<j+k} \mathrm{e}^{i\pi S_\ell^z}\right) S_{j+k}^z \psi\rangle$ does not tend to 0 with increasing k. Draw the tensor diagram for evaluating this correlation function.

(c) (Voluntary) Evaluate the string correlation function symbolically for arbitrary integer $k \geq 2$.

Hint: You can use a symbolic algebra system like Wolfram Alpha (wolframalpha.com) or Mathematica to solve this task. Your final result should be independent of k.

Solution

(a) We explicitly compute each term

$$A_{\hat{1},:,:} \otimes A^*_{\hat{1},:,:} = \begin{pmatrix} 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{\hat{0},:,:} \otimes A^*_{\hat{0},:,:} = \operatorname{diag}\left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$$

$$A_{-\hat{1},:,:} \otimes A^*_{-\hat{1},:,:} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 \end{pmatrix}$$

and then add them together

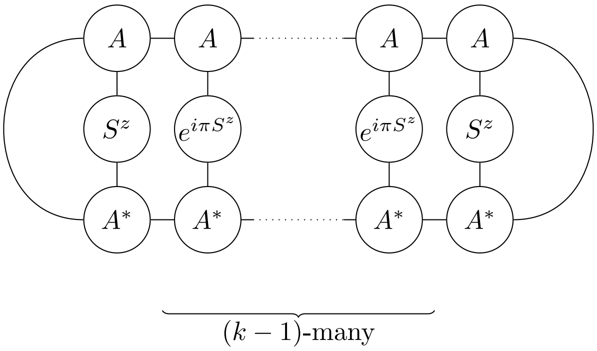
$$E = \sum_{\sigma\in\{1,0,-1\}} A_{\hat{\sigma},:,:} \otimes A^*_{\hat{\sigma},:,:} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Next we find the spectrum $\sigma(E)$ of E. From the structure of E one directly observes that $(0,1,0,0)^T$ and $(0,0,1,0)^T$ are eigenvectors with eigenvalue $-\frac{1}{3}$. Therefore the last two eigenvectors' first and last entry are the only non-zero entries. By explicit computation or once more by observation one finds that the last two eigenvectors are $(1,0,0,1)^T$ and $(-1,0,0,1)^T$ with eigenvalues 1 and $-\frac{1}{3}$ respectively. Thus we found

$$\sigma(E) = \left\{1, -\frac{1}{3}\right\}.$$

We can infer that 1 is the largest eigenvector, since by definition, the orthonormalisation of the MPS tensors means that the identity map (interpreted as the vector $(1,0,0,1)$ of length 4) is an eigenvector of the transfer operator with eigenvalue 1. This is confirmed by the explicit calculation shown above.

(b) We know that A is both left- and right-orthonormal. Therefore



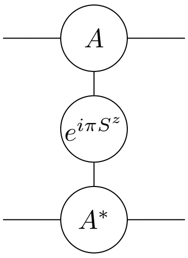
For an advanced discussion, see also section 5.4 *Symmetry Respecting Phases* in: J. C. Bridgeman, C.T. Chubb, *Hand-waving and Interpretive Dance: An Introductory Course on Tensor Networks*, J. Phys. A: Math. Theor. 50, 223001 (2017)

(c) (Voluntary) As a first step note that

$$e^{i\pi S^z} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We then sandwich the exponential between A and A^*

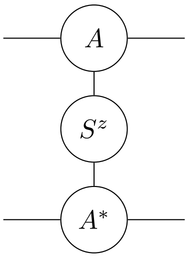
$$E_{\mathrm{exp}} =$$



$$= \sum_{\sigma,\tau} \left(e^{i\pi S^z}\right)_{\sigma,\tau} A_{\sigma,:,:} \otimes A^*_{\sigma,:,:} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & -\frac{2}{3} \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ -\frac{2}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

and do the same with S^z

$$E_{S^z} =$$



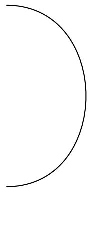
$$= \sum_{\sigma,\tau} \left(S^z\right)_{\sigma,\tau} A_{\sigma,:,:} \otimes A^*_{\sigma,:,:} = \begin{pmatrix} 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & 0 \end{pmatrix}.$$

We can now compute the power

$$E^{\ell}_{\mathrm{exp}} = \begin{pmatrix} \frac{1}{2}\left(1+\left(-\frac{1}{3}\right)^{\ell}\right) & 0 & 0 & \frac{1}{2}\left(-1+\left(-\frac{1}{3}\right)^{\ell}\right) \\ 0 & \left(-\frac{1}{3}\right)^{\ell} & 0 & 0 \\ 0 & 0 & \left(-\frac{1}{3}\right)^{\ell} & 0 \\ \frac{1}{2}\left(-1+\left(-\frac{1}{3}\right)^{\ell}\right) & 0 & 0 & \frac{1}{2}\left(1+\left(-\frac{1}{3}\right)^{\ell}\right) \end{pmatrix}$$

Remember that we can reinterpret the identity as a vector and normalise it such that

$$|\mathcal{I}\rangle =$$



$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore we can rewrite

$$\langle\psi| S^z_j \left(\prod_{j<\ell<j+k} \mathrm{e}^{i\pi S^z_\ell}\right) S^z_{j+k} |psi\rangle = \langle\mathcal{I}| E_{S^z} E^{k-1}_{\mathrm{exp}} E_{S^z} |\mathcal{I}\rangle = -\frac{4}{9},$$

where we evaluated the matrix multiplication for the last equality.