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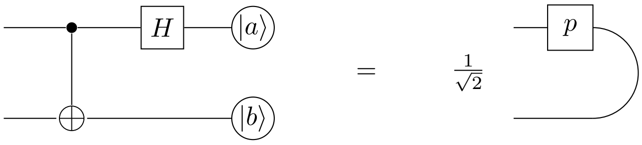
Exercise 7.1 (Bell circuit, part 2)
Generalize Exercise 4.1 by proofing the following equivalence for all $a, b \in \{0, 1\}$, where p is the identity matrix, X, iY or Z (depending on a and b). The circuit thus generates one of the Bell states. Hint: See also Exercise 4.2(d).



Solution We denote the matrix representing the overall quantum operation applied to $|ab\rangle$ as G. Since the Hadamard gate acts only on the top qubit, we take the Kronecker product with the identity matrix for the bottom qubit. Together with the matrix representation of the CNOT gate, this results in

$$
\begin{aligned}
G = U_{\mathrm{CNOT}} \cdot (H \otimes I_2) &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes I_2 \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.
\end{aligned}
$$

Recall that $|ab\rangle$ with $a, b \in \{0, 1\}$ is a shorthand notation for $|a\rangle \otimes |b\rangle$, and thus

$$
|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
$$

Applying G to $|ab\rangle$ thus selects the columns of G:

$$
G\,|00\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \qquad G\,|01\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X,
$$

$$
G\,|10\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \Leftrightarrow \quad p = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z, \qquad G\,|11\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = iY.
$$

We observe that the output vectors represent the four Bell states and, by matricizing them, we can see that p is I, X, iY or Z, depending on the intial input $|ab\rangle$.

Remark: Alternative solution based on explicit evaluation as tensor network diagram feasible as well.