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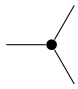
Exercise 4.1 (Bell circuit)
The delta tensor of degree d is defined via the Kronecker delta, with entries

$$\delta_{i_1,\dots,i_d} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_d \\ 0, & \text{otherwise} \end{cases}$$

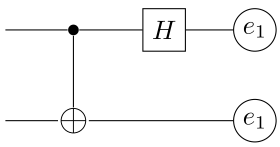
It can be regarded as generalization of the identity matrix, and is visualized by a filled dot, e.g., for $d = 3$:



The XOR tensor $\oplus \in \mathbb{C}^{2\times 2\times 2}$ has degree 3 and is given by (using zero-based indices $i, j, k \in \{0, 1\}$):

$$\oplus_{i,j,k} \begin{cases} 1, & i + j + k = 0 \mod 2 \text{ (even number of 1s)} \\ 0, & \text{otherwise} \end{cases}$$

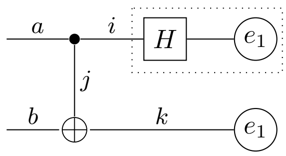
Evaluate and simplify the following diagram as far as possible, where $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ is the Hadamard matrix and $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$:



Solution We first apply the Hadamard matrix H to e_1:

$$He_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We now label the remaining legs as follows:



The contraction is then a summation over the indices i, j, k, resulting in a tensor $T = (t_{ab})$ with entries given by $(a, b \in \{0, 1\})$:

$$\begin{aligned} t_{ab} &= \sum_{i,j,k\in\{0,1\}} \delta_{a,i,j} \oplus_{b,j,k} (He_1)_i (e_1)_k \\ &= \sum_{k\in\{0,1\}} \oplus_{b,a,k} (He_1)_a (e_1)_k \\ &= \oplus_{b,a,0} (He_1)_a \\ &= \frac{1}{\sqrt{2}} \oplus_{b,a,0} \\ &= \frac{1}{\sqrt{2}} \delta_{a,b}. \end{aligned}$$

For the second equal sign we have used the properties of the Kronecker delta to conclude that $i = j = a$, for the third equal sign that $(e_1)_k$ is 1 precisely for $k = 0$, otherwise 0, for the fourth equal sign inserted He_1 from above, and for the last equal sign that $\oplus_{b,a,0}$ is 1 precisely if $a = b$. Thus

$$T = \frac{1}{\sqrt{2}} I_2,$$

where I_2 is the 2×2 identity matrix.

Remark: From a quantum computing viewpoint, the basis state e_1 is denoted $|0\rangle$, and the tensor diagram is a quantum circuit for generating one of the *Bell states*. The combination of the delta and XOR tensor is usually referred to as CNOT gate, see also Exercise 4.2(d). We will learn more about these topics later in the course.

Alternative solution We can also solve the exercise using matrix representations. A moment of thought confirms that U_{CNOT} from Exercise 4.2(d) is indeed the matrix representation of the CNOT gate when combining the two input tensor legs into one dimension, and likewise for the two output legs. For the present tensor diagram, the input to the CNOT gate (as vector) is

$$v = (He_1) \otimes e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Applying the CNOT gate to v results in

$$U_{\text{CNOT}} \cdot v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

This vector is the sought final output, and equal to the vectorization of the above matrix T, $\text{vec}(T)$, as expected.