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Tutorial 1 (Matrix chain ordering problem)
Matrix multiplication is associative, so given three matrices A, B and C, $(AB)C = A(BC)$. However, when it comes to the computational cost of these two expressions, one may be more efficient than the other.

- Write down the element-wise notation of matrix multiplication and consider its computational cost.
- Given three matrices $A \in \mathbb{R}^{10\times 30}$, $B \in \mathbb{R}^{30\times 2}$ and $C \in \mathbb{R}^{2\times 7}$, compare the computational cost of $(AB)C$ and $A(BC)$.
- If we add one more matrix D to the product chain, how many orderings to compute $ABCD$ are there? (It may be useful to draw a tree diagram).
- Argue why using a brute force approach to select the least expensive ordering requires an exponential run-time in the number of matrices.
- One recursive algorithm to select the best ordering works as follows:
  - Take the chain of matrices and split it into two subsequences.
  - Find the minimum cost of multiplying each subsequence.
  - Compute the final cost of multiplying the whole matrix chain.
  - Repeat for each possible split of the chain and choose the one with minimum cost.

  Is this any better than the brute force approach? Why / why not?
- Say we want to find the optimal ordering of the matrix product $ABCDE$ and we use the above algorithm. Identify any redundant operations performed and propose a solution.
- Analize how the following algorithm fits with the answer to the previous question:

```
// Matrix A[i] has dimension dims[i-1] x dims[i] for i = 1..n
MatrixChainOrder(int dims[])
{
    // length[dims] = n + 1
    n = dims.length - 1;
    // m[i,j] = Minimum number of scalar multiplications (i.e., cost)
    // needed to compute the matrix A[i]A[i+1]...A[j] = A[i..j]
    // The cost is zero when multiplying one matrix
    for (i = 1; i <= n; i++)
        m[i, i] = 0;

    for (len = 2; len <= n; len++) { // Subsequence lengths
        for (i = 1; i <= n - len + 1; i++) {
            j = i + len - 1;
            m[i, j] = MAXINT;
            for (k = i; k <= j - 1; k++) {
                cost = m[i, k] + m[k+1, j] + dims[i-1]*dims[k]*dims[j];
                if (cost < m[i, j]) {
                    m[i, j] = cost;
                    s[i, j] = k; // Index of the subsequence split that achieved minimal cost
                }
            }
        }
    }
}
```

Exercise 1.1 (Complex number arithmetic)
This exercise should refresh your knowledge and proficiency with complex numbers. Given $a = 3 + 4i$ and $b = 2 - i$:

(a) Compute

- $a + b$
- ab (product of a and b)
- $1/a$
- a^* (complex conjugate of a)
- $|a|$ and $\arg(a)$ (argument), such that $a = |a|\,\mathrm{e}^{i\arg(a)}$
- the Euclidean length of the vector $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$, denoted $\|\psi\|$

(b) Draw a in the complex plane, and interpret a^*, $|a|$ and $\arg(a)$ geometrically.

(c) How can one construct $a + b$ and ab geometrically in the complex plane?

Exercise 1.2 (Linear algebra basics)

(a) Compute (with "pen and paper") the matrix-vector product

$$\begin{pmatrix} 2 & -i & 5 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ i \\ -3 \end{pmatrix},$$

and the matrix-matrix product

$$\begin{pmatrix} -2 & 7 \\ 3 & 1+2i \end{pmatrix} \cdot \begin{pmatrix} 5 & -4 \\ 6i & 0 \end{pmatrix}.$$

(b) Find a 2×2 matrix which is not normal.
Hint: you can restrict your search to real-valued matrices.

(c) Show that the following matrix is unitary (with $\theta \in \mathbb{R}$ a real parameter):

$$\begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

(d) Let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix. Show that

$$|\det(U)| = 1,$$

where $|\cdot|$ denotes the absolute value.
Hint: You can use without proof that $\det(A^T) = \det(A)$ and $\det(AB) = \det(A)\det(B)$ for any $A, B \in \mathbb{C}^{n\times n}$, and that the determinant of the identity matrix is 1. Derive $\det(A^*) = \det(A)^*$ based on the definition of the determinant given in the lecture.