Linear Algebra Cheat Sheet

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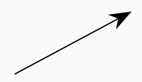
Basic definitions and examples

We use complex numbers as underlying field here due to the relevance for quantum mechanics.

Vector spaces

"Standard" example: $V = \mathbb{C}^n$

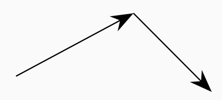
$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n$$



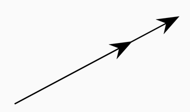
Another example: $V = C([0,1], \mathbb{C})$: space of continuous functions $f : [0,1] \to \mathbb{C}$

Basic operations:

vector addition: $+ : V \times V \to V$: $v + w$



scalar multiplication: $\cdot : \mathbb{C} \times V \to V$: $\alpha \cdot v$



Matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{C}^{m\times n}$$

Matrix vector product:

$$Av = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{pmatrix} \in \mathbb{C}^m$$

A can be interpreted as linear operator from $\mathbb{C}^n \to \mathbb{C}^m$

The **trace** of a square matrix $A \in \mathbb{C}^{n\times n}$ is the sum of its diagonal entries:

$$\operatorname{tr}(A) = \sum_{i=1}^n a_{ii}$$

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $A, B \in \mathbb{C}^{n\times n}$, thus $\operatorname{tr}(ABC) = \operatorname{tr}(CAB) = \operatorname{tr}(BCA)$

Inner product and norm

Inner product on a vector space V:

$$\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$$

with properties: linearity in second argument, conjugate symmetry: $\langle w|v\rangle = \langle v|w\rangle^*$, positive definiteness: $\langle v|v\rangle > 0$ for $v \neq 0$

Usual definition for $V = \mathbb{C}^n$:

$$\langle v|w\rangle = \sum_{i=1}^n v_i^* w_i$$

with x^* the complex conjugate of $x \in \mathbb{C}$.

Example on $V = C([0,1], \mathbb{C})$: $\langle f|g\rangle = \int_0^1 f(x)^* g(x)\,\mathrm{d}x$

Note: $\langle\cdot|\cdot\rangle$ is linear in second argument, but anti-linear in first: $\langle \alpha v|w\rangle = \alpha^*\langle v|w\rangle$

(Alternative convention: linearity in first and anti-linearity in second argument also used in the literature)

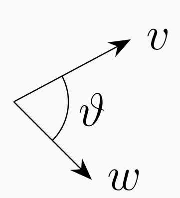
$\langle\cdot|\cdot\rangle$ induces a **norm** on V via

$$\|v\| = \sqrt{\langle v|v\rangle}$$

Cauchy-Schwarz inequality: for all $v, w \in V$:

$$|\langle v|w\rangle| \leq \|v\|\|w\|$$

Geometric interpretation (for real-valued vectors):



$$\cos(\vartheta) = \frac{\langle v|w\rangle}{\|v\|\|w\|}$$

Special matrices

The **adjoint** (or **conjugate transpose**) of a matrix is defined as:

$$A^\dagger = (A^*)^T$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{C}^{m\times n} \quad \rightsquigarrow \quad A^\dagger = \begin{pmatrix} a_{11}^* & \cdots & a_{m1}^* \\ \vdots & & \vdots \\ a_{1n}^* & \cdots & a_{mn}^* \end{pmatrix} \in \mathbb{C}^{n\times m}$$

Note: $\langle v|Aw\rangle = \langle A^\dagger v|w\rangle$ for all $v \in \mathbb{C}^m, w \in \mathbb{C}^n$ (follows directly from definitions)

A matrix A is called **Hermitian** (or **self-adjoint**) if

$$A^\dagger = A$$

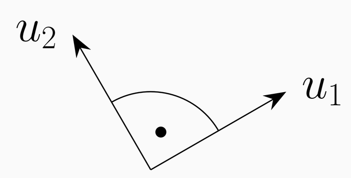
Notes: Hermitian property implies that A is a square matrix; by definition $(A^\dagger)^\dagger = A$

$U \in \mathbb{C}^{n\times n}$ is called **unitary** if

$$U^\dagger U = I \text{ (identity matrix)},$$

i.e., $U^\dagger$ is the inverse of U

Intuition: $U = (u_1|\cdots|u_n)$ consists of orthonormal column vectors: $\langle u_i|u_i\rangle = \|u_i\|^2 = 1$ and $\langle u_i|u_j\rangle = 0$ for $i \neq j$



U describes a change of basis which preserves the length and angles between vectors.

Note: $U^\dagger U = I \Leftrightarrow UU^\dagger = I$ due to uniqueness of the inverse matrix, thus U is unitary precisely if $U^\dagger$ is unitary.

A square matrix $A \in \mathbb{C}^{n\times n}$ is called **normal** if it commutes with its adjoint:

$$A^\dagger A = AA^\dagger$$

In particular, every unitary matrix and every Hermitian matrix is normal.

$P \in \mathbb{C}^{n\times n}$ is called an **orthogonal projection matrix** if it is Hermitian and $P^2 = P$.

Determinant

The **determinant** of a square matrix $A \in \mathbb{C}^{n\times n}$ is defined as

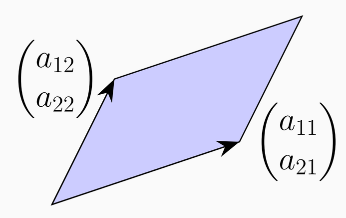
$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$$

with S_n the group of all permutations of $\{1,\dots,n\}$.

Example for $n = 2$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad S_2 = \{\mathrm{id}, \underbrace{(1,2)}_{1\leftrightarrow 2}\} \quad \rightsquigarrow \quad \det(A) = \underbrace{a_{11}a_{22}}_{\sigma=\mathrm{id}} - \underbrace{a_{12}a_{21}}_{\sigma=(1,2)}$$

Geometric interpretation:



area $= |\det(A)|$

Properties:

- A is invertible if and only if $\det(A) \neq 0$
- $\det(A^*) = \det(A)^*$ (follows directly from definition)
- $\det(A^T) = \det(A)$
- for all $A, B \in \mathbb{C}^{n\times n}$: $\det(AB) = \det(A)\det(B)$

Eigenvalues and -vectors

Let $A \in \mathbb{C}^{n\times n}$. A non-zero vector $v \in \mathbb{C}^n$ is called an **eigenvector** of A with corresponding **eigenvalue** $\lambda \in \mathbb{C}$ if

$$Av = \lambda v.$$

Rewriting this relation: $Av = \lambda v \Leftrightarrow (\lambda I - A)v = 0$; therefore λ is an eigenvalue of A precisely if $\lambda I - A$ is *not* invertible, i.e., its determinant is $0 \rightsquigarrow$ motivates definition of **characteristic polynomial** (polynomial of degree n in λ):

$$\chi_A(\lambda) = \det(\lambda I - A)$$

Thus: eigenvalues of A are the zeros of χ_A

Fundamental theorem of algebra garantees that there exist n complex roots of the characteristic polynomial.

Example:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \det(\lambda I - A) = \det\begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 + 1 \overset{!}{=} 0 \Leftrightarrow \lambda = \pm i$$

Corresponding eigenspaces are the kernels (null spaces) of $\pm iI - A$

Eigenvalues of Hermitian matrices are real: namely, taking the inner product of v and $Av = \lambda v$ leads to $\langle v|Av\rangle = \lambda\langle v|v\rangle$, thus $\lambda = \langle v|Av\rangle / \langle v|v\rangle$. By definition $\langle v|v\rangle > 0$. Furthermore $\langle v|Av\rangle$ is real as well since

$$\langle v|Av\rangle = \langle A^\dagger v|v\rangle \overset{A \text{ Hermitian}}{=} \langle Av|v\rangle \overset{\text{conjugate symmetry}}{=} \langle v|Av\rangle^*.$$

Spectral theorem and singular value decomposition

Theorem 1 (Spectral decomposition)**.** *Any normal matrix $A \in \mathbb{C}^{n\times n}$ is unitarily diagonalizable, i.e., there exists a unitary matrix $U = (u_1|\cdots|u_n) \in \mathbb{C}^{n\times n}$ of eigenvectors as columns and corresponding eigenvalues $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ such that*

$$A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^\dagger, \quad \textit{equivalently} \quad Au_i = \lambda_i u_i \textit{ for all } i = 1,\dots,n.$$

Conversely, every matrix representable in this form is normal.

Remarks:

- Eigenvalues $\lambda_1,\dots,\lambda_n$ need not be distinct
- Rewriting the spectral decomposition leads to (for any $v \in \mathbb{C}^n$)

$$Av = \sum_{i=1}^n \lambda_i u_i \langle u_i|v\rangle$$

- Generalized decomposition of arbitrary matrices: Jordan normal form

The **spectral radius** of a square matrix $A \in \mathbb{C}^{n\times n}$ is the largest absolute value of the eigenvalues of A:

$$\rho(A) = \max\{|\lambda_1|,\dots,|\lambda_n|\}$$

Theorem 2 (Singular value decomposition (SVD))**.** *Let $A \in \mathbb{C}^{n\times n}$ be a square matrix. Then there exist unitary matrices $U, V \in \mathbb{C}^{n\times n}$ and real numbers $\sigma_1,\dots,\sigma_n$ with $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ called* **singular values***, such that*

$$A = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} V^\dagger.$$