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Exercise 1.2 (Linear algebra basics)

(a) Compute (with "pen and paper") the matrix-vector product

$$\begin{pmatrix} 2 & -i & 5 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ i \\ -3 \end{pmatrix},$$

and the matrix-matrix product

$$\begin{pmatrix} -2 & 7 \\ 3 & 1+2i \end{pmatrix} \cdot \begin{pmatrix} 5 & -4 \\ 6i & 0 \end{pmatrix}.$$

(b) Find a 2×2 matrix which is not normal.
Hint: you can restrict your search to real-valued matrices.

(c) Show that the following matrix is unitary (with $\theta \in \mathbb{R}$ a real parameter):

$$\begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

(d) Let $U \in \mathbb{C}^{n \times n}$ be a unitary matrix. Show that

$$|\det(U)| = 1,$$

where $|\cdot|$ denotes the absolute value.
Hint: You can use without proof that $\det(A^T) = \det(A)$ and $\det(AB) = \det(A)\det(B)$ for any $A, B \in \mathbb{C}^{n \times n}$, and that the determinant of the identity matrix is 1. Derive $\det(A^*) = \det(A)^*$ based on the definition of the determinant given in the lecture.

Solution

(a) Computing the matrix-vector product yields

$$\begin{pmatrix} 2 & -i & 5 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ i \\ -3 \end{pmatrix} = \begin{pmatrix} 8+1-15 \\ 12-3 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}.$$

Computing the matrix-matrix product results in:

$$\begin{pmatrix} -2 & 7 \\ 3 & 1+2i \end{pmatrix} \cdot \begin{pmatrix} 5 & -4 \\ 6i & 0 \end{pmatrix} = \begin{pmatrix} -10+42i & 8 \\ 3+6i & -12 \end{pmatrix}.$$

(b) By definition, a matrix A is normal if $A^\dagger \cdot A = A \cdot A^\dagger$. In case A is real, this is equivalent to $A^T \cdot A = A \cdot A^T$. There are many matrices that are not normal. A simple example is

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

since

$$A^T \cdot A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$A \cdot A^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(c) A square matrix U is called unitary if $U^\dagger \cdot U = \mathbb{1}$. So for

$$U = \begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix}$$

we get

$$U^\dagger = \begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Thus

$$U^\dagger \cdot U = \begin{pmatrix} \cos(\theta)^2 + \sin(\theta)^2 & 0 \\ 0 & \cos(\theta)^2 + \sin(\theta)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore the matrix is unitary.

(d) Observe that

$$1 = \det(\mathbb{1}) = \det\left(U^\dagger \cdot U\right) = \det\left(U^\dagger\right)\det(U) = \det(U)^* \det(U) = |\det(U)|^2.$$

This implies the statement.

Here we have used that for all $a, b \in \mathbb{C}$:

$$a^* b^* = (ab)^*$$
$$a^* + b^* = (a+b)^*.$$

Thus, for any $A \in \mathbb{C}^{n \times n}$:

$$\det(A^*) = \sum_\sigma \operatorname{sgn}(\sigma) \prod_{j=1}^n a^*_{j,\sigma(j)} = \left(\sum_\sigma \operatorname{sgn}(\sigma) \prod_{j=1}^n a_{j,\sigma(j)} \right)^* = \det(A)^*.$$