Professorship for Quantum Computing
Department of Informatics
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Eexam
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Tensor Networks

Exam:	IN2388 / Final Exam	**Date:**	Monday 2nd August, 2021
Examiner:	Christian Mendl	**Time:**	11:30 – 13:00

Working instructions

- This exam consists of **10 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 60 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: open book
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.

Problem 1 **(20 credits)**

0 1 2 3 4

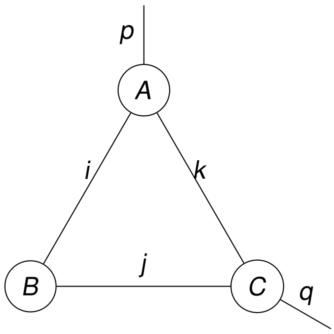
a) Represent the following contraction operation as graphical tensor diagram, labeling each tensor and tensor leg:

$$e_{pq} = \sum_{i,j,k,\ell} a_{\ell ik} b_{kjq} c_{j\ell p} d_i.$$

The legs should be drawn counter-clockwise with respect to the ordering of the tensor indices.

0 1 2 3

b)* We consider the following to-be contracted tensor network:



Find an optimal contraction order that minimizes contraction complexity, i.e., the overall computational cost, assuming that the dimensions of the tensor legs obey

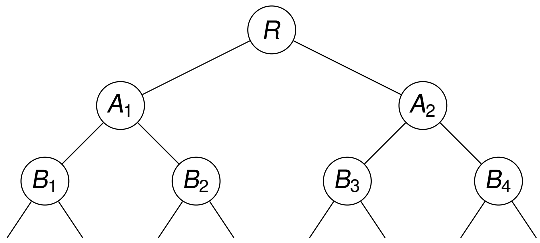
$$\dim(p) = \dim(q) =: \ell \ll \dim(j) = \dim(k) =: m \ll \dim(i) =: n,$$

and that at each contraction step, two tensors are contracted together. (An optimality proof is not required here.)

0 1 2 3 4

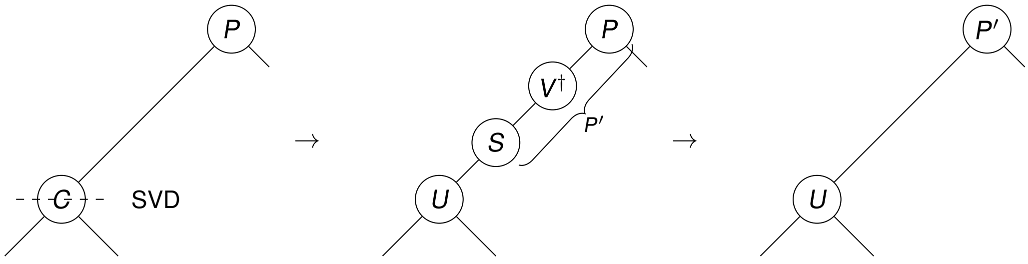
c) What is the asymptotic computational cost (in $\mathcal{O}$-notation) of the contractions you found in (b)? Assume a literal implementation of contractions based on the summation formulation.

d)* We define T as the following tree tensor network:

$T =$ 

Draw the tensor diagram for evaluating $\langle T, T \rangle$.

e) In the context of part (d), we consider a local SVD-splitting operation performed on a node C of the tree (parent node called P), which updates both the child and parent node:



This local operation is applied to each node in the tree (except the root) given a particular ordering. We distinguish between two realizations:

(i) The operation is first performed on the A tensors in the middle layer before proceeding to the B tensors.

(ii) The operation is first performed on the B tensors in the bottom layer before proceeding to the A tensors.

We denote the resulting tree tensor network by T'. For each of the two cases, simply the tensor diagram for evaluating $\langle T', T' \rangle$ as far as possible. Also provide a short explanation of your simplifications.

Problem 2 **(20 credits)**

We consider the following Hamiltonian on a one-dimensional lattice with L sites and open boundary conditions, where J is a real parameter:

$$H = -J \sum_{j=2}^{L-1} X_{j-1} Y_j Z_{j+1}.$$

a) What is the matrix dimension of H?

0 1 2

b)* For a given index $j \in \{2, \dots, L-1\}$, represent $X_{j-1} Y_j Z_{j+1}$ in terms of Kronecker products of identity matrices and the Pauli matrices X, Y, Z.

0 1 2 3

c)* Construct a finite state automaton and corresponding MPO tensors for representing H as matrix product operator. You should separately specify the MPO tensors A^j for $j = 2, \dots, L-1$ and the boundary tensors A^1, A^L. Also provide the dimensions of these tensors.

0 1 2 3 4 5 6 7

0 1 2 3 4

d)* Specify an operator of the form $P_k = A_{k-1} B_k C_{k+1}$ with each A, B, C a Pauli matrix and $k \in \{2, \dots, L-1\}$, such that P_k commutes with H. (A proof of the commuting property is not required.)

0 1 2 3 4

e)* Consider the partitioning of the Hamiltonian as $H = H_a + H_b$ with

$$H_a = -J \sum_{j=2,\mathrm{mod}(j,4)=2}^{L-2} \left(X_{j-1} Y_j Z_{j+1} + X_j Y_{j+1} Z_{j+2} \right) \quad \text{and} \quad H_b = -J \sum_{j=4,\mathrm{mod}(j,4)=0}^{L-2} \left(X_{j-1} Y_j Z_{j+1} + X_j Y_{j+1} Z_{j+2} \right).$$

Is it possible to represent the matrix exponentials $e^{-iH_a t}$ or $e^{-iH_b t}$ (with $t \in \mathbb{R}$) exactly in quantum circuit form, assuming that you can use arbitrary one-, two- and three-qubit gates? Briefly justify your answer.

Problem 3 **(20 credits)**

We define a linear map $\mathcal{E} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ ($n \in \mathbb{N}$) as

$$\mathcal{E}(\rho) = \sum_{j=1}^{s} K_j \rho K_j^{\dagger}$$

with given matrices $K_j \in \mathbb{C}^{n\times n}$ satisfying $\sum_{j=1}^{s} K_j^{\dagger} K_j = I$.

a) Show that $\mathcal{E}$ maps

(i) Hermitian matrices to Hermitian matrices, and

(ii) positive semi-definite matrices to positive semi-definite matrices.

0 1 2 3 4

b)* For the following, we define two new tensors $\mathcal{K}$ and $\mathcal{K}'$ of degree 3 by $\mathcal{K}_{j,:,:} = K_j$ and $\mathcal{K}'_{j,:,:} = K_j^{\dagger}$ for all $j = 1, \dots, s$. Draw the tensor network representing the application of $\mathcal{E}$ to ρ in terms of ρ, $\mathcal{K}$ and $\mathcal{K}'$, clearly indicating the dimension of each leg.

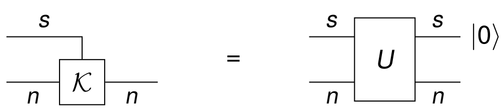
0 1 2 3 4

c) Given a unitary matrix $V = (v_{ij}) \in \mathbb{C}^{s\times s}$, we introduce the new linear map $\mathcal{F} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ by

$$\mathcal{F}(\rho) = \sum_{i=1}^{s} L_i \rho L_i^{\dagger} \quad \text{with } L_i = \sum_{j=1}^{s} v_{ij} K_j \text{ for } i = 1, \dots, s.$$

Show that $\mathcal{E}$ and $\mathcal{F}$ represent the same map.

d) $\mathcal{K}$ interpreted as $(sn) \times n$ matrix is an isometry by definition; we can thus extend it to a unitary matrix $U \in \mathbb{C}^{(sn)\times(sn)}$ such that (with labels at the legs denoting dimensions, and $|0\rangle$ the first unit vector in $\mathbb{C}^s$):



Draw a tensor diagram that represents $\mathcal{E}(\rho)$ in terms of ρ, U, $U^{\dagger}$ and $|0\rangle$.

e) Based on your solution of part (d), express $\mathcal{E}(\rho)$ using a partial trace operation.

0
1
2
3
4

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.