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Tutorial 1 (Matrix chain ordering problem)
Matrix multiplication is associative, so given three matrices A, B and C, $(AB)C = A(BC)$. However, when it comes to the computational cost of these two expressions, one may be more efficient than the other.

(a) Write down the element-wise notation of matrix multiplication and consider its computational cost.

(b) Given three matrices $A \in \mathbb{R}^{10\times 30}$, $B \in \mathbb{R}^{30\times 2}$ and $C \in \mathbb{R}^{2\times 7}$, compare the computational cost of $(AB)C$ and $A(BC)$.

(c) If we add one more matrix D to the product chain, how many orderings to compute $ABCD$ are there? (It may be useful to draw a tree diagram).

(d) Argue why using a brute force approach to select the least expensive ordering requires an exponential run-time in the number of matrices.

(e) One recursive algorithm to select the best ordering works as follows:

1. Take the chain of matrices and split it into two subsequences.
2. Find the minimum cost of multiplying each subsequence.
3. Compute the final cost of multiplying the whole matrix chain.
4. Repeat for each possible split of the chain and choose the one with minimum cost.

Is this any better than the brute force approach? Why / why not?

(f) Say we want to find the optimal ordering of the matrix product $ABCDE$ and we use the above algorithm. Identify any redundant operations performed and propose a solution.

(g) Analyze how the following algorithm fits with the answer to the previous question:

```
// Matrix A[i] has dimension dims[i-1] x dims[i] for i = 1..n
MatrixChainOrder(int dims[])
{
    // length[dims] = n + 1
    n = dims.length - 1;
    // m[i,j] = Minimum number of scalar multiplications (i.e., cost)
    // needed to compute the matrix A[i]A[i+1]...A[j] = A[i..j]
    // The cost is zero when multiplying one matrix
    for (i = 1; i <= n; i++)
        m[i, i] = 0;

    for (len = 2; len <= n; len++) { // subsequence lengths
        for (i = 1; i <= n - len + 1; i++) {
            j = i + len - 1;
            m[i, j] = MAXINT;
            for (k = i; k <= j - 1; k++) {
                cost = m[i, k] + m[k+1, j] + dims[i-1]*dims[k]*dims[j];
                if (cost < m[i, j]) {
                    m[i, j] = cost;
                    s[i, j] = k; // index of the subsequence split that achieved minimal cost
                }
            }
        }
    }
}
```

Exercise 1.1 (Complex number arithmetic)
This exercise should refresh your knowledge and proficiency with complex numbers. Given $a = 3 + 4i$ and $b = 2 - i$:

(a) Compute

- $a + b$
- ab (product of a and b)
- $1/a$
- a^* (complex conjugate of a)
- $|a|$ and $\arg(a)$ (argument), such that $a = |a|\,\mathrm{e}^{i\arg(a)}$
- the Euclidean length of the vector $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$, denoted $\|\psi\|$

(b) Draw a in the complex plane, and interpret a^*, $|a|$ and $\arg(a)$ geometrically.

(c) How can one construct $a + b$ and ab geometrically in the complex plane?

Exercise 1.2 (Linear algebra basics)

(a) Compute (with "pen and paper") the matrix-vector product

$$\begin{pmatrix} 2 & -i & 5 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ i \\ -3 \end{pmatrix},$$

and the matrix-matrix product

$$\begin{pmatrix} -2 & 7 \\ 3 & 1+2i \end{pmatrix} \cdot \begin{pmatrix} 5 & -4 \\ 6i & 0 \end{pmatrix}.$$

(b) Find a 2×2 matrix which is not normal.
Hint: you can restrict your search to real-valued matrices.

(c) Show that the following matrix is unitary (with $\theta \in \mathbb{R}$ a real parameter):

$$\begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

(d) Let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix. Show that

$$|\det(U)| = 1,$$

where $|\cdot|$ denotes the absolute value.
Hint: You can use without proof that $\det(A^T) = \det(A)$ and $\det(AB) = \det(A)\det(B)$ for any $A, B \in \mathbb{C}^{n\times n}$, and that the determinant of the identity matrix is 1. Derive $\det(A^*) = \det(A)^*$ based on the definition of the determinant given in the lecture.