First-Order Logic
Equality

Predicate logic with equality

Predicate logic

$+$

distinguished predicate symbol "=" of arity 2

Semantics: A structure $\mathcal{A}$ of predicate logic with equality always maps the predicate symbol = to the identity relation:

$$\mathcal{A}(=) \ = \ \{(d, d) \mid d \in U_{\mathcal{A}}\}$$

Expressivity

Fact

A structure is model of $\exists x \forall y \; x{=}y$ *iff its universe is a singleton.*

Theorem

Every satisfiable formula of predicate logic
has a countably infinite model.

Proof Let F be satisfiable.
We assume w.l.o.g. that $F = \forall x_1 \ldots \forall x_n F^*$ and the variables occurring in F^* are exactly $x_1, \ldots, x_n$.
(If necessary bring F into closed Skolem form).
We consider two cases:
$n = 0$. Exercise.
$n > 0$. Let $G = \forall x_1 \ldots \forall x_n F^*[f(x_1)/x_1]$, where f is a function symbol that does not occur in F^*. G is satisfiable (why?) and $T(G)$ is countably infinite. It follows from the fundamental theorem that G has a countably infinite model.

Modelling equality

Let F be a formula of predicate logic with equality.
Let Eq be a predicate symbol that does not occur in F.
Let E_F be the conjunction of the following formulas:

$$\forall x\ Eq(x, x)$$

$$\forall x \forall y\ (Eq(x, y) \rightarrow Eq(y, x))$$

$$\forall x \forall y \forall z\ ((Eq(x, y) \wedge Eq(y, z)) \rightarrow Eq(x, z))$$

For every function symbol f in F of arity n and every $1 \leq i \leq n$:

$$\forall x_1 \dots \forall x_n \forall y\ (Eq(x_i, y) \rightarrow Eq(f(x_1, \dots, x_i, \dots x_n), f(x_1, \dots, y, \dots, x_n)))$$

For every predicate symbol P in F of arity n and every $1 \leq i \leq n$:

$$\forall x_1 \dots \forall x_n \forall y (Eq(x_i, y) \rightarrow (P(x_1, \dots, x_i, \dots, x_n) \leftrightarrow P(x_1, \dots, y, \dots, x_n)))$$

E_F expresses that Eq is a *congruence relation* on the symbols in F.

Quotient structure

Definition

Let $\mathcal{A}$ be a structure and $\sim$ an equivalence relation on $U_{\mathcal{A}}$ that is a congruence relation for all the predicate and function symbols defined by $I_{\mathcal{A}}$. The quotient structure $\mathcal{A}/_{\sim}$ is defined as follows:

- $U_{\mathcal{A}/_{\sim}} = \{[u]_{\sim} \mid u \in U_{\mathcal{A}}\}$ where $[u]_{\sim} = \{v \in U_{\mathcal{A}} \mid u \sim v\}$
- For every function symbol f defined by $I_{\mathcal{A}}$:
  $f^{\mathcal{A}/\sim}([d_1]_{\sim}, \ldots, [d_n]_{\sim}) = [f^{\mathcal{A}}(d_1, \ldots, d_n)]_{\sim}$
- For every predicate symbol P defined by $I_{\mathcal{A}}$:
  $P^{\mathcal{A}/\sim}([d_1]_{\sim}, \ldots, [d_n]_{\sim}) = P^{\mathcal{A}}(d_1, \ldots, d_n)$
- For every variable x defined by $I_{\mathcal{A}}$: $x^{\mathcal{A}/\sim} = [x^{\mathcal{A}}]_{\sim}$

Lemma

$\mathcal{A}/_{\sim}(t) = [\mathcal{A}(t)]_{\sim}$

Lemma

$\mathcal{A}/_{\sim}(F) = \mathcal{A}(F)$

Theorem

The formulas F and $E_F \wedge F[Eq/{=}]$ are equisatisfiable.

Proof We show that if $E_F \wedge F[Eq/{=}]$ is sat., then F is satisfiable.
Assume $\mathcal{A} \models E_F \wedge F[Eq/{=}]$.
$\Rightarrow$ $Eq^{\mathcal{A}}$ is an congruence relation.

Let $\mathcal{B} = \mathcal{A}/_{Eq^{\mathcal{A}}}$ (extended with $=$ interpreted as identity).
$\Rightarrow \mathcal{B} \models F[Eq/{=}]$

By construction $Eq^{\mathcal{B}}$ is identity:
$Eq^{\mathcal{B}}([a], [a']) = Eq^{\mathcal{A}}(a, a') = ([a]_{Eq^{\mathcal{A}}} = [a']_{Eq^{\mathcal{A}}})$
$\Rightarrow \mathcal{B}(F[Eq/{=}]) = \mathcal{B}(F)$
$\Rightarrow \mathcal{B} \models F$

Conversely, it is easy to see that any model of F can be turned into a model of $E_F \wedge F[Eq/{=}]$ by interpreting Eq as equality.