Digital Assignment - 1

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[Module 1]

Easy Question

Q3) Find constants a; b; c if $f(z) = x + ay + i(bx + cy)$ is analytic.

According to Cauchy - Riemann equation:-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$f(z) = u(x,y) + iv(x,y)$ &

$\Rightarrow u(x,y) = x + ay$

$v(x,y) = bx + cy$

$u_x = 1 \qquad v_x = b$

$u_y = a \qquad v_y = c$

$\because f(z)$ is analytic

$\Rightarrow u_x = v_y$

& $u_y = -v_x$

$\Rightarrow c = 1$ & $a = -b$ (Ans)

Q154 A stream function is given by x^2-y^2, find the potential function of the

$\psi(x,y) = x^2 - y^2$

$\because$,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial y}$$

First, partial derivatives:-

$$\frac{\partial \psi}{\partial x} = 2x$$

$$\frac{\partial \psi}{\partial y} = -2y$$

Using ϕ & ψ relation:-

$$\phi_x = \psi_y = -2y$$

$$\phi_y = \psi_x = -2x$$

On integrating,

$$\int \phi_x \, dx = -2yx + f(y)$$

diff ϕ with y

$$\phi_y = -2x + f'(y)$$

$$\Rightarrow -2x + f'(y) = -2x$$

$\Rightarrow f'(y) = 0 \Rightarrow f(y)$ is a constant (Assuming c)

$\Rightarrow \phi(x,y) = -2yx + c$

$\Rightarrow$ potential function $\phi(x,y)$ corresponding to given stream

is $\phi(x,y) = -2xy + C$

Moderate Questions

4.4 $f(z) = (r^2\cos(2\theta) + r\cos\theta) + i(r^2\sin(2\theta) + r\sin\theta)$

$f'(z) = ?$

$\Rightarrow f(z) = r^2\cos(2\theta) + i(r^2\sin(2\theta)) + r\cos\theta + i r\sin\theta$

$= r^2 e^{i2\theta} + re^{i\theta}$

Let $re^{i\theta} = z$

$\Rightarrow f(z) = z^2 + z$

Diff. $f(z)$

$\Rightarrow f'(z) = 2z + 1$

Q. 1.4 $f(z) = e^{\sqrt{2.01}} x + 100 i y$. Show that its not differentiable.

we can express it as:-

$f(z) = u(x,y) + iv(x,y)$

$\Rightarrow u = e^{\sqrt{2.01}} x \qquad v = 100y$

Diff of partial derivatives:-

$u_x = e^{\sqrt{2.01}} \qquad u_y = 0$

$v_x = 0 \qquad v_y = 100$

For Cauchy-Riemann,

$u_x = v_y$

$u_y = -v_x$

$\Rightarrow e^{\sqrt{2.01}} \neq 100$

$\Rightarrow$ It doesn't validate cauchy-Riemann,

therefore its not differentiable.

Q1.2.4 Show that $f(z) = x^2 + iy^3$ is not analytic anywhere. Reconcile this with fact that Cauchy-Riemann eqn are ~~satis~~ satisfied at $x=0, y=0$.

$f(z) = x^2 + iy^3$

$f(z) = u(x,y) + iv(x,y)$

$u(x,y) = x^2 \qquad v(x,y) = y^3$

$u_x = 2x \qquad u_y = 0$

$v_x = 0 \qquad v_y = 3y^2$

For Cauchy - Riemann to be true

$u_x = v_y \;\&\; v_x = -u_y$

$\& \; u_x = 2x$

$\Rightarrow$

$\& \; v_y = 3y^2$

$\Rightarrow$ Function can't be ~~true~~ analytic at any point except $x=0 \;\& \; y=0$

Hard Questions

Q.5.7 Find $f(z)$ when $2u + v = e^x(\cos y - \sin y)$

$f(z) = u + iv$

$\Rightarrow \; i f(z) = iu - v$

$\Rightarrow (2-i) f(z) = (2u+v) + i(2v-u)$

$\Rightarrow f(z) = \dfrac{(2u+v)}{(2-i)} + \dfrac{i(2v-u)}{2-i}$

$\Rightarrow F(z) = \underbrace{(2u+v)}_{U} + i\underbrace{(2v-u)}_{V}$

$U_x = e^x(\cos y - \sin y)$

$U_y = e^x(-\sin(y) - \cos(y))$

$F(z) = \int \cancel{U_x\,dx} + \int \cancel{U_x\,dy}$

$= \int \cancel{U_x\,dx} - \int U_y\,dy$

$= e^x(\cos y - \cancel{\sin y}) -$

By using milne Thomson,

$(x,y) \to (z,0)$

$U_x = e^z(\cos(0) - \sin(0))$

$= e^z(1-0) = e^z$

$U_y = e^z(0-1)$

$= -e^z$

$f(z) = \int e^z\,dz - i\int e^z\,dz$

$= e^z - ie^z$

3107 P.T Cauchy- Riemann can be written as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

In cartesian, CR eqn is

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$\Rightarrow u(x,y) = u(r,\theta)$ $\;$ & $\;$ $v(x,y) = v(r,\theta)$

$$\Rightarrow u_x = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\times\frac{\partial \theta}{\partial x}$$

$$u_y = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\times\frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Substituting into CR :-

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\times\cos\theta - \frac{\partial u}{\partial r}\frac{\sin\theta}{r}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\times\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}$$

$\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ becomes:-

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}$$

& $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ becomes

$$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} = -\left(\frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}\right)$$

$\Rightarrow$ On simplifying,

$$\cos\theta\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right) = \sin\theta\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)$$

$$\sin\theta\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right) = -\cos\theta\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)$$

For it to be true ~~of~~ for all θ, co-effecients of sin & cos must sep[arately]

be 0

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Q 114 ST. $f(z) = \begin{cases} \dfrac{x^3y^5 + ix^2y^6}{x^4+y^{10}} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$ satisfies CR at origin

but $f'(0)$ does not exist

$f(z) = u + iv$

$u = \dfrac{x^3y^5}{x^4+y^{10}}$ $\quad v = \dfrac{x^2y^6}{x^4+y^{10}}$

$$u_x = \frac{(3x^2y^5)(x^4+y^{10}) - (x^3y^5)(4x^3)}{(x^4+y^{10})^2}$$

$$u_y = \frac{(5x^3y^4)(x^4+y^{10}) - (x^3y^5)(10y^9)}{(x^4+y^{10})^2}$$

$$v_x = \frac{(2xy^6)(x^4+y^{10}) - (x^2y^6)(4x^3)}{(x^4+y^{10})^2}$$

$$v_y = \frac{(6x^2y^5)(x^4+y^{10}) - (x^2y^6)(10y^9)}{(x^4+y^{10})^2}$$

At $x=0$ & $y=0$

$u_x = 0 \;;\; u_y = 0 \;;\; v_x = 0 \;;\; v_y = 0$

$\Rightarrow$ It follows Cauchy-Riemann at origin.

Checking for $f'(0)$

~~for $y=0$~~

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

~~$f(0) =$~~

$$\Rightarrow \lim_{z\to 0} \frac{f(z)}{x+iy}$$

- for $y = 0$

$$f'(0) = \lim_{x\to 0} \frac{0}{x} = 0$$

- for $y = x$

$$f(x,x) = \frac{x^3x^3 + ix^2x^6}{x^4+x^{10}} = \frac{(1+i)}{(1+x^6)} \times x^4$$

$$= \lim_{x\to 0} \frac{f(x,x)}{x(1+i)} = \lim_{x\to 0} \frac{(1+i)x^4}{(1+x^6)x(1+i)}$$

$$= \lim_{x\to 0} \frac{x^3}{1+x^6} = 0$$

Even though specific paths give value of $f'(0) = 0$, the limit directionally dependent and therefore does not exist.

MODULE - 2]

Easy

Q. BT with $a/c = i$ & fixed points 1, 2

$$f(z) = \frac{az+b}{cz+d}$$

$f(z_0) = w$

$\Rightarrow f(z_0) = z_0$

$\Rightarrow az+b = cz^2 + dz$

$\Rightarrow cz^2 + (d-a)z - b = 0$

for $z = 1$,

$c + (d-a) - b = 0$

for $z = 2$,

$4c + 2d - 2a - b = 0$

$\because a = ic$

$\Rightarrow c + d - ic - b = 0$

& $4c + 2d - 2ic - b = 0$

$\Rightarrow 3c + d - ic = 0$

$\Rightarrow d = -3c + ic$

$\Rightarrow -2c - b = 0$

$\Rightarrow b = -2c$

$\Rightarrow a = ic \quad ; \quad b = -2c \quad ; d = -3c + ic$

On sub,

$$f(z) = \cancel{c(iz-2)} \quad \frac{ic \cdot z - 2c}{cz - 3c + ic}$$

$$= \frac{\cancel{c}\,(iz-2)}{\cancel{c}\,(z-3+i)}$$

$$\Rightarrow f(z) = \frac{iz-2}{z-3+i}$$

Q.34 Image of $x^2 + y^2 - 4y + 2 = 0$ under $\omega = \dfrac{z-i}{iz-1}$

$$\Rightarrow x^2 + (y-2)^2 - 4 + 2 = 0$$

$$\Rightarrow x^2 + (y-2)^2 = 2$$

$\Rightarrow$ circle of radius $\sqrt{2}$ at $(0,2)$

$\omega = \dfrac{z-i}{iz-1}$ & $z = x + iy$

$$\Rightarrow \omega = u + iv = \frac{(x+iy) - i}{i(x+iy) - 1}$$

$$= \frac{x + i(y-1)}{ix - y - 1}$$

$$\Rightarrow \omega = \frac{(x + i(y-1))(ix - y - 1)}{(ix - y - 1)(-ix - y - 1)}$$

$$w = \frac{-x^2 - (y-1)^2 + i(2x - y - 1)}{x^2 + (y+1)^2}$$

$$\Rightarrow u = \frac{-x^2 - (y-1)^2}{x^2 + (y+1)^2} \quad \& \quad v = \frac{2x - y - 1}{x^2 + (y+1)^2}$$

Moderate

Q24 Find image of square $z = 1+i,\ 3+i,\ 1+3i,\ 3+3i$ under $w = 1/z$

$\Rightarrow z_1 = 1+i$

$z_2 = 3+i$

$z_3 = 1+3i$

$z_4 = 3+3i$

for z_1

$$w_1 = \frac{1}{1+i} = \frac{1-i}{1-(-1)} = \frac{1}{2} - \frac{i}{2}$$

for z_2

$$w_2 = \frac{1}{3+i} = \frac{3-i}{4} = \frac{3}{4} - \frac{i}{4}$$

for z_3

$$w_3 = \frac{1}{1+3i} = \frac{1-3i}{10} = \frac{1}{10} - \frac{3i}{10}$$

for z_4

$$w_4 = \frac{1}{3+3i} = \frac{3-3i}{18} = \frac{3}{18} - \frac{3i}{6} = \frac{1}{6} - \frac{i}{6}$$

$\Rightarrow$ Image is $\frac{1-i}{2},\ \frac{3-i}{4},\ \frac{1-3i}{10},\ \frac{1-i}{6}$.

Q 3.4 Find image of $y=b$ & $x=a$ under $w=z^2$

$w=z^2$ & $z = x+iy$

$\Rightarrow w = (x+iy)^2 = x^2 - y^2 + 2ixy$

$\Rightarrow u = x^2 - y^2$ — (i)

$v = 2xy$ — (ii)

• $y=b$

On subst. in (i) & (ii)

$u = x^2 - b^2$

$v = 2xb$

$\Rightarrow v = 2xb \Rightarrow x = \frac{v}{2b}$

$\Rightarrow u = \left(\frac{v}{2b}\right)^2 - b^2$

$\Rightarrow u = \frac{v^2}{4b^2} - b^2$

$\Rightarrow$ ~~parabola~~ image of $y=b$ under $w=z^2$ is a parabola that opens in u-direction

• $x=a$

$u = a^2 - y^2$

$v = 2ay \Rightarrow y = \frac{v}{2a} \Rightarrow u = a^2 - \frac{v^2}{4a^2}$

$\Rightarrow$ image of $x=a$ under $w=z^2$ is a parabola that opens in the complete opposite direction of image of $y=b$.

Find B.T that maps $z = \infty, i, 0$ onto $w = 0, i, \infty$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\Rightarrow \frac{(z-z_1)(i-0)}{(z-0)(i-z_1)} = \frac{(w-0)(i-w_3)}{(w-w_3)(i-0)}$$

$$\Rightarrow \frac{\cancel{z_1}\left(\frac{z}{z_1}-1\right)(i)}{z \times \cancel{z_1}\left(\frac{i}{z_1}-1\right)} = \frac{w \times \cancel{w_3}\left(\frac{i}{w_3}-1\right)}{\cancel{w_3}\left(\frac{w}{w_3}-1\right) i}$$

$$\Rightarrow \frac{-i}{-z} = \frac{-w}{-i}$$

$$\Rightarrow \frac{i}{z} = \frac{w}{i} \Rightarrow w = -\frac{1}{z}$$

Hard

Q1.4 Find B.T which maps $z = 1, i, -1$ onto $w = i, 0, -i$

Find image of $|z| < 1$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\Rightarrow \frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{(w-i)(0+i)}{(w+i)(0-i)}$$

$$\Rightarrow \frac{(z-1)}{(z+1)}\frac{(i+1)}{(i-1)} = -\frac{(w-i)}{(w+i)}$$

$$\Rightarrow \frac{(z-1)}{(z+1)}\frac{(i+1)^2}{2} = \frac{(w-i)}{(w+i)}$$

$$\Rightarrow \frac{(z-1)}{z+1} \times \frac{2i}{\cancel{2}} = \frac{w-i}{w+i}$$

$\Rightarrow (zi - i)(w+i) = (w-i)(z+1)$

$\cancel{\Rightarrow wzi - wi - z + 1 = wz + w - iz - i}$

$\cancel{\Rightarrow w(zi-1) - z + 1 = w(z+1) - iz - i}$

$\cancel{\Rightarrow w(zi - 1 - z - 1) = z - 1 - iz - i}$

$$\cancel{\Rightarrow w = \frac{(z-1) - i(z+1)}{-(z+2) + iz}}$$

$$z(i\omega - 1 - (\omega - i)] = i(\omega+1) + (\omega - i)$$

$$\Rightarrow z[(i-1)\omega + (i-1)] = (i+1)\omega - (i+1)$$

$$\Rightarrow z = \frac{(i+1)(\omega-1)}{(i-1)(\omega+1)}$$

$$\Rightarrow z = \frac{2i}{-2}\cdot\frac{(\omega-1)}{(\omega+1)}$$

$$\Rightarrow z = i\left(\frac{1-\omega}{1+\omega}\right)$$

Image of $|z|<1 \Rightarrow |z|^2 < 1$

$$\Rightarrow z\bar{z} < 1$$

$$\Rightarrow i\left(\frac{1-\omega}{1+\omega}\right) \times -i\left(\frac{1-\bar{\omega}}{1+\bar{\omega}}\right) < 1$$

$$\Rightarrow (1-\omega)(1-\bar{\omega}) < (1+\bar{\omega})(1+\omega)$$

$$\Rightarrow \cancel{1} - (\omega+\bar{\omega}) + \cancel{\omega\bar{\omega}} < \cancel{1} + (\omega+\bar{\omega}) + \cancel{\omega\bar{\omega}}$$

$$\Rightarrow 0 < 2(\omega+\bar{\omega})$$

$$\Rightarrow 2(\omega+\bar{\omega}) > 0$$

$$\Rightarrow \text{2of } \because \omega = u + iv$$

$$\bar{\omega} = u - iv$$

$$\Rightarrow \omega + \bar{\omega} = 2u$$

$$\Rightarrow 2(2u) > 0 \Rightarrow 4u > 0$$

Q.15.4 BT. which maps $z = 0, 1, \infty$ to $w = i, 1, -i$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\Rightarrow \cancel{w} \; \frac{(z-0)(1-z_3)}{(z-z_3)(1-0)} = \frac{(w-i)(1+i)}{(w+i)(1-i)}$$

$$\Rightarrow \frac{z \times \cancel{z_3}\left(\frac{1}{z_3}-1\right)}{\cancel{z_3}\left(\frac{z}{z_3}-1\right)(1)} = \frac{(w-i)}{(w+i)} \times \frac{(1+i)}{(1-i)}$$

$$\Rightarrow \frac{-z}{1} = \frac{(w-i)}{(w+i)} \times \frac{2i}{2}$$

$$\Rightarrow -z = \frac{(w-i)i}{w+i}$$

$$\Rightarrow -z(w+i) = wi + 1$$

$\cancel{\Rightarrow w(z-i) = 1-i}$

$\Rightarrow -wz - iz = wi + 1$

$\Rightarrow w(-z-i) = 1 + iz$

$\cancel{\Rightarrow w = \frac{1-i}{z-iz}}$ $\quad \cancel{\Rightarrow w = \frac{1-i}{z(1-i)}}$ $\quad \Rightarrow w = \frac{1-iz}{z-i}$

$\cancel{\Rightarrow w = \frac{(1-i)(z+i)}{z^2+1}}$

$\Rightarrow w = \frac{1+iz}{z+i}$

$\cancel{\Rightarrow w = z+1+i-2i/z^2+1}$

$\Rightarrow w = \frac{1+iz}{z+i} \times \frac{(z-i)}{(z-i)}$

$\cancel{\Rightarrow w = \frac{z(1-i)+i+1}{z^2+1}}$

$$\Rightarrow w = \frac{z+i}{1+iz}$$

Find image of rectangle with lines $x=1, x=3, y=1, y=2$ under transformation $w = z^2$

Let $z = x + iy$ & $w = u + iv$

$w = z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$

$\Rightarrow u = x^2 - y^2$ & $v = 2xy$

for $x = 3$;

$u = 1^2 - y^2 = 1 - y^2$ $\quad v = 2(1)y = 2y$

$v = 2y \Rightarrow y = v/2$

$\Rightarrow u = 1 - \frac{v^2}{4} \Rightarrow v^2 = -4(u-1)$

for $x = 3$;

$u = 9 - y^2$ & $v = 6y$

$\Rightarrow u = 9 - \frac{v^2}{36} \Rightarrow v^2 = -36(u-9)$

for $y = 1$;

$u = x^2 - 1^2 = x^2 - 1$ & $v = 2x(1) = 2x$

$v = 2x \Rightarrow x = \frac{v}{2}$

$\Rightarrow u = \frac{v^2}{4} - 1 \Rightarrow v^2 = 4(u+1)$

for $y = 2$;

$u = x^2 - 4 \qquad v = 2x \times 2 = 4x$

$u = \frac{v^2}{16} - 4 \implies v^2 = 16(u+4)$

$\implies x = 1$ maps to parabola $v^2 = -4(u-1)$

$\implies x = 3$ maps to parabola $v^2 = -36(u-9)$

$\implies y = 1$ maps to parabola $v^2 = 4(u+1)$

$\implies y = 2$ maps to parabol. $v^2 = 16(u+4)$