Mathematical Induction

Introduction

Mathematical induction, or just induction, is a proof technique. Suppose that for every natural number n, $P(n)$ is a statement. We wish to show that all statements $P(n)$ are true.

In a proof by induction, we show that $P(1)$ is true, and that whenever $P(n)$ is true for some n, $P(n+1)$ must also be true. In other words, we show that the conditional $P(n) \rightarrow P(n+1)$ is true for all n.

This creates a logical "chain reaction" – the truth of $P(1)$ implies the truth of $P(2)$, which implies the truth of $P(3)$, which implies the truth of $P(4)$, and so on.

On the next slide, we will restate this so-called principle of induction, prove it, and introduce some terminology.

The Principle of Induction (1)

Theorem (Principle of Induction):
Suppose $P(n)$ is a statement for every natural number n. Further suppose that

1. The first statement, $P(1)$ is true.
2. For every n, the conditional $P(n) \rightarrow P(n+1)$ is true.

Then $P(n)$ is true for all n.

Proof: Suppose $P(n)$ is not true for all n. Then there must be at least one n for which it is false. Let n_0 be the smallest of all these n. Since we know that $P(1)$ is true, $n_0 \geq 2$. Since n_0 is the smallest n for which $P(n)$ is false, and $n_0 - 1$ is a natural number, $P(n_0 - 1)$ is true. By assumption, the conditional $P(n_0 - 1) \rightarrow P(n_0)$ is true as well, hence $P(n_0)$ is true. That is a contradiction because $P(n_0)$ is false.

This proves that $P(n)$ is true for all n.

In an inductive proof, verifying condition 1 is called the base case or basis step. Verifying condition 2 is called the induction step or inductive step. We verify condition 2 by assuming that $P(n)$ is true (called the inductive hypothesis) for **some arbitrary** n, and showing that then $P(n+1)$ is true as well.

The Principle of Induction (2)

To simplify the explanation, we stated the principle of induction for the domain of the natural numbers. There is a similar principle of induction for every domain D that is a subset of the integers with a lowest element, i.e. $D \subseteq \mathbb{Z}$ and D has a minimum d.

If we can prove $P(d)$, and that, whenever $P(n)$ is true for some $n \in D$, then $P(m)$ must also be true, where $m \in D$ is the next higher number in D after n, then $P(n)$ is true for all $n \in D$. Examples:

- $P(0)$ and $P(n) \rightarrow P(n+1)$ for all non-negative integers n proves $P(n)$ for all non-negative integers n.
- $P(2)$ and $P(n) \rightarrow P(n+2)$ for all positive, even integers n proves $P(n)$ for all positive, even integers n.

There is also bidirectional induction. For example, $P(1)$ and $P(n) \rightarrow P(n+2) \land P(n-2)$ for all odd integers n proves $P(n)$ for all odd integers n.

The Renaming Ritual

Many textbooks, including Rosen, highlight the fact that the inductive hypothesis is made about some arbitrary n value by giving this n a new name such as k. In the inductive hypothesis, rather than assuming $P(n)$ “for some arbitrary n”, they assume $P(n)$ “for some arbitrary $n = k$" and then demonstrate that $P(k + 1)$ must be true as well.

This renaming ritual is not necessary, and has the disadvantage that it may cause serious confusion when the statement about n already involves some variable named k. The slide titled “Common Mistakes (4)” illustrates this.

Example Ia

Let us prove by induction that for all natural numbers n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(This equation is the statement $P(n)$.)

1. Base case: we verify $P(1)$. $\sum_{k=1}^{1} k = \frac{1 \cdot 2}{2}$ is true.
2. Inductive step: let us assume $P(n)$ for **some arbitrary** n, i.e.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

We now add the next term of the summation, <u>which in this case happens to be</u> $n+1$, to both sides. On the left side, that "upgrades" the sum to the next higher upper limit. On the right, we hope that the right side of $P(n+1)$ emerges. **(Note that generally, when you write an inductive proof, the statement $P(n+1)$ will NOT be produced by adding $n+1$ to both sides of $P(n)$. That algebraic manipulation is SPECIFIC to this example.)**

$$\sum_{k=1}^{n+1} k = \frac{n(n+1)}{2} + n + 1 = \left(\frac{n}{2} + 1\right)(n+1) = \frac{(n+1)(n+2)}{2}$$

This statement,

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2},$$

is the statement $P(n+1)$. We just completed the inductive step: we showed that if $P(n)$ is true, then $P(n+1)$ is true as well. That completes the proof by induction.

$P(n+1)$ is **not** generally obtained by adding $n+1$ to both sides of $P(n)$.

It bears repeating: the algebraic manipulation we used to obtain $P(n+1)$ from $P(n)$ in the first example is **specific** to that example and only works because the quantity under examination grows by $n+1$ as we step from the case n to the case $n+1$.

Each inductive proof requires its own, individual algebraic steps to conclude that $P(n+1)$ must be true if $P(n)$ is true.

The next page demonstrates a slightly different algebraic perspective for proving summation formulas. Instead of adding to the summation formula we are assuming is already true, we can split the next summation into what we already know about, plus the next term.

Example Ia, done a little differently

Let us prove by induction that for all natural numbers n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

1. Base case: we verify the summation for n=1: $1 = \frac{1\cdot 2}{2}$. This is true.
2. Inductive step: let us assume that we already know $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for **some arbitrary** n. Then we split the next sum into the sum we already know, plus the next term:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1)$$

We now use the inductive hypothesis on the sum we already know:

$$\sum_{k=1}^{n+1} k = \frac{n(n+1)}{2} + n + 1 = \left(\frac{n}{2} + 1\right)(n+1) = \frac{(n+1)(n+2)}{2}.$$

This statement

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2},$$

is the case n+1 of our summation formula. That completes the proof by induction.

Common Mistakes (1)

Students often confuse the statement $P(n)$ with the algebraic expression about which a statement is being made. ***They abuse the notation $P(n)$*.**

In the previous example, we proved the statement

$$P(n) := \left(\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \right)$$

for all n. Notice the parentheses - $P(n)$ is not the sigma sum, and it's not the expression $\frac{n(n+1)}{2}$. It is the <u>statement</u> that these two quantities are equal.

Therefore, you must **not** write equations like $P(n) = \frac{n(n+1)}{2}$ in your inductive proofs, if P(n) has been agreed upon to be the statement you are proving. If you must have a notation for an expression that is frequently referred to, you are free to define your own notation for it, for example,

$$a_n = \frac{n(n+1)}{2}$$

but you must not use $P(n)$. You must also not write equations like this:

$$P(n) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Without the critical parentheses, $P(n)$ is no longer the statement that the middle and right quantity are equal. It is the common value of that quantity. So on top of using wrong notation, you are assuming the conclusion of the proof with this.

It’s best to avoid the abstraction of “P(n)” in actual inductive proofs

To describe the method of induction in theory, it was useful to have a notation – P(n) – to refer to the sequence of statements we are proving. In practice though, using this notation is often worse than useless. It’s just as easy, and better writing style, to refer verbally to the statement we are proving. This way, you avoid any risk of making the mistake described on the previous page, of abusing “P(n)” to refer to both a quantity involved in the statement we are proving and the statement itself.

Observe how the following example proofs all do without the notation “P(n)” and are better for it.

Example Ib

Let us prove by induction that for all natural numbers n,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

1. Base case: we verify the summation formula for n=1: $1 = \frac{1\cdot 2\cdot 3}{6}$. This is true.
2. Inductive step: let us assume that we have already proved the summation for **some arbitrary** n, i.e.

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

We now add the next term of the summation, which in this case is $(n+1)^2$, to both sides. On the left side, that "upgrades" the sum to the next higher upper limit. On the right, we expect that the expression $\frac{(n+1)(n+2)(2n+3)}{6}$ will emerge. **(Observe that the next term of the summation is NOT $\boldsymbol{n+1}$ here.)**

$$\sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \left(\frac{n(2n+1)}{6} + n + 1\right)(n+1) = \frac{n(2n+1)+6(n+1)}{6}(n+1)$$

$$= \frac{2n^2+7n+6}{6}(n+1) = \frac{(n+2)(2n+3)}{6}(n+1) = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

Observe that prudent, early factoring, rather than brute-force distributing, is the algebraic key to an efficient, readable solution here.

This statement,

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

,

is the summation formula for the case n+1. We just completed the inductive step: we showed that if the formula holds for one n, then it also holds for the next. That completes the proof by induction.

Common Mistakes (2)

A second, somewhat common mistake is to confuse *stating* $P(n+1)$ with *proving* it. Merely stating it does not prove it.

The following variation of the inductive step of Example 1 illustrates that mistake:

"Let us assume that $P(n)$ is true for some n, i.e.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Then $P(n+1)$ is the statement

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

This completes the proof by induction. "

The psychology of this mistake seems to be that merely writing $P(n+1)$ required some algebra work. You had to replace each n by n+1 and simplify, so it may feel like "you did something". You did, you just didn't prove $P(n+1)$.

Again – the notation P(n) is best avoided completely when you write inductive proofs. It is used here only because it is convenient to describe the mistake in question.

Common Mistakes (3)

When $P(n)$ is a summation formula, people may confuse n with the running variable in the summation. We proved that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

is true for all *n.* Notice that both sides are functions of *n* alone: *n* is the ONLY free variable in this formula. You may think that you are seeing another free variable, *k*, on the left, but this hallucination is easily dispelled by remembering the meaning of the sigma notation: it means simply $1 + 2 + 3 + \cdots + n$. Our formula is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

There is no "k" there. *If you replace k by $k + 1$, you simply break the sigma sum and turn it into something unrelated.*

People who are unclear about the distinction between k and n may think that they need to increment k when they form $\sum_{k=1}^{n+1} k$, either instead of incrementing n, or in addition to it:

$$\sum_{k=1}^{n} (k+1) \text{ or } \sum_{k=1}^{n+1} (k+1).$$

Neither is equal to $\sum_{k=1}^{n+1} k$.

From a programming perspective, you can think of the k inside the sigma notation as a local variable inside a function that is invisible from the outside:

```
function sum(int n)
  int sum = 0;
  for (int k=1;k<=n;k++)
     sum +=k;
```

Common Mistakes (3a)

A mistake that is closely related to the previous one is evaluating a sigma sum in the base case by substituting the first n value into both n and k, like this: if

$$S(n) = \sum_{k=1}^{n} k^2 ,$$

then

$$S(1) = \sum_{k=1}^{1} 1^2 .$$

The equation is not even incorrect, it's just that it was not logically derived from the definition of S. S is a function of n alone. Evaluating it at $n = 1$ means substituting $n = 1$ into the definition, and *only* $n = 1$:

$$S(1) = \sum_{k=1}^{1} k^2$$

If you also replace k by 1, then you are not evaluating the definition. You are *changing* the definition and evaluating a different sum at $n = 1$, namely

$$\sum_{k=1}^{n} 1 .$$

Common Mistakes (4)

If you practice the “renaming ritual” of assuming in the inductive step that $P(n)$ is true “for some $n = k$ “ rather than just “for some n”, and if your $P(n)$ involves a sigma sum that uses k as a running variable, you’re in danger of writing wholly nonsensical expressions like this:

$$\sum_{k=1}^{k} k$$

If you don’t see what’s wrong with that, consider the following equivalent code:

```
function sum(int k)
  int sum = 0;
  for (int k=1;k<=k;k++)
     sum +=k;
```

The best way to avoid this is to not rename the induction variable n and to assume that $P(n)$ is true for some n in the inductive step.

Common Mistakes (4a)

Practicing the "renaming ritual" can make you feel like you assumed something in the inductive step when you *assumed nothing*. Some students write the following incorrect inductive hypothesis: "Suppose n = k. "

This is meaningless, because it is assuming that an undefined variable n is equal to an undefined variable k; or that the undefined variable n now has the new name k. It's not even a proposition. It's omitting the entire point of the inductive hypothesis, which is the assumption that *what we are trying to prove has already been proved for some arbitrary n value*.

If you insist, you may give this arbitrary n value the special name k (unless k is already being used as a running variable in a sigma sum.) However, you must not think that "Suppose n = k" means the same as "Suppose we already know that P(n) is true for some arbitrary n = k".

Common Mistakes (5)

What's wrong with the following inductive step?

"Inductive Step. Let us assume $P(n)$ is true for all n.."

In the inductive step, we prove that the conditional $P(n) \rightarrow P(n+1)$ is true for all n, i.e. we prove

$$\forall n\big(P(n) \rightarrow P(n+1)\big).$$

We do this by assuming that $P(n)$ is true for some (arbitrary) n, which sets up universal generalization. Then we show that $P(n+1)$ must be true as well. By universal generalization, since our n was arbitrary, that shows

$$\forall n\big(P(n) \rightarrow P(n+1)\big).$$

In the proof fragment above, it is assumed that $P(n)$ is true for all n. **That is assuming the conclusion.** If we already know that $P(n)$ is true for all n, we don't need a proof.

Common Mistakes (6)

Some students who practice the renaming ritual confuse the act of giving the specific n for which we assume $P(n)$ is already proved in the inductive step *a new name* with the act of assuming that $P(n)$ has already been proved.

Instead of writing "Inductive Step. Let us assume $P(n)$ for some $n = k$",

they write "Inductive Step. Assume $n = k$".

This is meaningless since in that "assumption", n and k are just two undefined variables. Even if n was defined, giving n a new name is not the same as assuming that $P(n)$ is true.

The general theme of this misunderstanding is to not distinguish properly between numbers and the statements they refer to. Don't write "we proved n=1". Write "we proved the case n=1". Don't write "n=1 is true" when you mean "the first statement is true". Numbers don't have truth values.

Likewise, in the inductive step, don't write "assume n". n is an integer and therefore can't be assumed. "Assume n is an arbitrary integer" is still wrong as the inductive hypothesis, because it leaves out what we're actually assuming about that integer. Correct: "assume P(n) for some arbitrary integer n."

Common Mistakes (7)

Do not confuse the conclusion of the base case with the reason you reach this conclusion. For example, in example 1a, the summation formula P(1) reduces to the identity 1=1. That 1=1 is a known fact, a *premise*, and not in need of proof. P(1) is the *conclusion* that follows from that.

The following, wrong base case reverses this relationship:

Base case: for n=1, the summation formula reduces to $1 = \frac{1 \cdot 2}{2}$. This is true. Therefore, 1=1.

We already know that 1=1, and we are not proving that a number is equal to itself. If you are going to write a formal conclusion to the base case, it must be that the statement you are proving is true in that case.

Correct version:

Base case: for n=1, the summation formula reduces to $1 = \frac{1 \cdot 2}{2}$. This is true. Therefore, P(1) is true.

Common Mistakes (8)

What's wrong with the following inductive step?

Inductive Step. Since we have proved the case n = 1, we can now assume that the statement is true for some n.

This reasoning ties the inductive hypothesis to the base case. If your only justification for assuming that the statement has been shown for some n is the fact that it has been shown for n = 1, then you are really only assuming that the statement is true for n = 1.

Based on this premise, the inductive step then merely shows that the statement is true for n = 2. The logical chain reaction stops after one step.

The psychology of this mistake appears to be the attempt to re-explain the logic of the principle of induction inside an inductive proof. This is neither necessary nor appropriate. In an inductive proof, we *use* the principle of induction. We don't justify it.

Example IIa

Theorem: For any positive integer n, $4^n - 1$ is divisible by 3.

Proof by induction: Base Case: for $n = 1$, $4^n - 1 = 4 - 1 = 3$, which is divisible by 3.

Inductive Step: suppose we have already proved that $4^n - 1$ is divisible by 3 for some positive integer n. Our goal is to show that $4^{n+1} - 1 = 4 \cdot 4^n - 1$ must then be divisible by 3 as well. There are different ways to proceed from here.

Variation 1: We can split four times 4^n into three times that quantity, plus one times that quantity:

$$4 \cdot 4^n - 1 = 3 \cdot 4^n + (4^n - 1).$$

According to the inductive hypothesis, $4^n - 1$ is a multiple of 3. Since $3 \cdot 4^n$ is a multiple of 3 as well, so is the sum of the two terms, which is $4^{n+1} - 1$.

Variation 2: According to the inductive hypothesis, $4^n - 1$ is a multiple of 3, i.e. $4^n - 1 = 3k$ for some integer k, or $4^n = 3k + 1$. Substituting that into $4^{n+1} - 1$, we get

$$4^{n+1} - 1 = 4 \cdot (3k + 1) - 1 = 12k + 3 = 3(4k + 1).$$

Thus $4^{n+1} - 1$ is a multiple of 3.

Example IIb

Theorem: 21 divides $4^{n+1} + 5^{2n-1}$ for all natural numbers n.
Proof by induction: we first verify the statement for $n = 1$: 21 divides $4^2 + 5^1 = 21$.

Now suppose that 21 divides $4^{n+1} + 5^{2n-1}$ for some natural number n. This means $4^{n+1} + 5^{2n-1} = 21k$ for some integer k. Let us consider

$$4^{n+2} + 5^{2(n+1)-1} = 4^{n+2} + 5^{2n+1}$$

By using laws of exponentiation, we can rewrite this as $4 \cdot 4^{n+1} + 25 \cdot 5^{2n-1}$.

It would be nice here if we could just factor out a common factor in order to substitute the inductive hypothesis, but that is not possible. So we split the second term to make at least a partial factorization possible:

$$4 \cdot 4^{n+1} + 25 \cdot 5^{2n-1} = 4 \cdot 4^{n+1} + 4 \cdot 5^{2n-1} + 21 \cdot 5^{2n-1}$$

Therefore,

$$4^{n+2} + 5^{2(n+1)-1} = 4(4^{n+1} + 5^{2n-1}) + 21 \cdot 5^{2n-1} = 21(4k + 5^{2n-1}).$$

We have demonstrated that $4^{n+2} + 5^{2(n+1)-1}$ is divisible by 21. This completes the proof.

Example III

Theorem: if S is a finite set with $|S| = n$, then $|\mathcal{P}(S)| = 2^n$, for $n = 0,1,2,3, \ldots$

Proof by induction:

1. Base Case (n=0): if S is a set with $|S| = 0$, then $|\mathcal{P}(S)| = 1$. This statement is true, because $|S| = 0$ implies $S = \emptyset$, hence $\mathcal{P}(S) = \{\emptyset\}$, so $|\mathcal{P}(S)| = 1$.

2. Inductive Step: suppose we already know for some arbitrary n that if a set has n elements, its power set must have 2^n elements. We must then verify that if S is a set with $|S| = n + 1$, then $|\mathcal{P}(S)| = 2^{n+1}$. To verify this conditional, we will assume its premise and show that the conclusion must be true.

Let S be a set with $|S| = n + 1$. Let us single out an arbitrary element of S and call it x. Define $R = S - \{x\}$. Then $|R| = n$. Now let us count how many subsets S has. The number of subsets of S equals $A + B$, where

$A =$ the number of subsets of S that don't contain x,
$B =$ the number of subsets that do contain x.

A subset of S that doesn't contain x is a subset of R, hence $A = |\mathcal{P}(R)| = 2^n$ by the inductive hypothesis.

A subset of S that contains x is the union of a subset of R with $\{x\}$, hence $B = |\mathcal{P}(R)| = 2^n$. Therefore, $|\mathcal{P}(S)| = A + B = 2^n + 2^n = 2^{n+1}$. That completes our proof.

Example IV

Theorem: $n^2 < 2^n$ for all integers $n \geq 5$.

Proof by induction: we first verify the statement for $n = 5$: $5^2 < 2^5$ because $25 < 32$. This was the base case.

Inductive step: Now suppose that $n^2 < 2^n$ for some natural number $n \geq 5$. We will show $(n+1)^2 < 2^{n+1}$.

Since $(n+1)^2 = n^2 + 2n + 1$, we only need to show that $2n + 1 < 2^n$ for $n \geq 5$. Then, $n^2 + 2n + 1 < 2^n + 2^n = 2^{n+1}$ by the inductive hypothesis and thus $(n+1)^2 < 2^{n+1}$.

It remains to show that $2n + 1 < 2^n$ for all $n \geq 5$. We do this once again by induction.

Base case: $2n + 1 < 2^n$ is true for $n = 5$ because $11 < 32$.

Inductive step: suppose $2n + 1 < 2^n$ for some $n \geq 5$. Our goal is to show $2(n+1) + 1 < 2^{n+1}$. Let us add two to the inequality of the inductive hypothesis: $2n + 3 < 2^n + 2$. Since $2 < 2^n$ for $n \geq 2$, and certainly for $n \geq 5, 2n + 3 < 2^n + 2 < 2^n + 2^n = 2^{n+1}$. Therefore, $2(n+1) + 1 < 2^{n+1}$.

An Example of an Incorrect Proof by Induction

False Theorem: for all nonnegative integers n, $n = 2n$.

This statement is of course absurd. $n = 2n$ is true for only one number: $n = 0$. Now let us consider the following (incorrect) proof by induction. Define $P(n)$ for all nonnegative integers to be the statement $n = 2n$.

Base Case: for $n = 0$, the statement $n = 2n$ is correct.

Inductive Step: suppose $n = 2n$ for some arbitrary nonnegative integer n has already been shown. Multiply both sides of this equation by $\frac{n+1}{n}$ to get $n + 1 = 2(n + 1)$. That completes the proof.

It is clear that this proof is incorrect, because any alleged proof of a false statement must be incorrect. Let us try to pinpoint the mistake. The base case was correctly verified. $0 = 2 \cdot 0$ is a true statement.

The argument of the inductive step is correct for all n except the first. When $n = 0$, then the argument asks us to multiply by the undefined quantity $\frac{1}{0}$.

Indeed, any argument that claims to prove $P(n) \rightarrow P(n + 1)$ for all n must at least be invalid for $n = 0$, because $P(0)$ is a true statement and $P(1)$ is a false statement, so the conditional $P(0) \rightarrow P(1)$ is false. (The following conditionals, $P(1) \rightarrow P(2)$, etc, are actually true, and our proof above showed that.)

Example V

Theorem: Suppose $\{a_n\}$ is a sequence recursively defined by $a_1 = 5$ and $a_{n+1} = a_n + 3$ for all positive integers n. Then $a_n = 2 + 3n$ for all positive integers n.

Proof by Induction:

Base case: we verify the statement for $n = 1$: $a_1 = 2 + 3 \cdot 1 = 5$, which agrees with the definition of a_1.

Inductive step: suppose that $a_n = 2 + 3n$ has already been shown for some arbitrary positive integer n. We wish to show that then $a_{n+1} = 2 + 3(n + 1)$.

By definition of $\{a_n\}$, $a_{n+1} = a_n + 3$. Substituting our inductive hypothesis, we get

$$a_{n+1} = a_n + 3 = (2 + 3n) + 3 = 2 + 3(n + 1).$$

This is what we needed to show.

Example VI

Theorem: Suppose $\{a_n\}$ is a sequence recursively defined by $a_1 = 3$ and $a_{n+1} = 2a_n$ for all positive integers n. Then $a_n = 3 \cdot 2^{n-1}$ for all positive integers n.

Proof by Induction:

Base case: we verify the statement for $n = 1$: $a_1 = 3 \cdot 2^{1-1} = 3$, which agrees with the definition of a_1.

Inductive step: suppose that $a_n = 3 \cdot 2^{n-1}$ has already been shown for some arbitrary positive integer n. We wish to show that then $a_{n+1} = 3 \cdot 2^n$.

By definition of $\{a_n\}$, $a_{n+1} = 2a_n$. Substituting our inductive hypothesis, we get

$$a_{n+1} = 2a_n = 2 \cdot 3 \cdot 2^{n-1} = 3 \cdot 2^n.$$

This is what we needed to show.

Example VII – Part 1

Our final example has some important points to teach us.
Theorem: For all positive integers n,

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \leq 2\sqrt{n}-1$$

Proof by induction:

Base case: we verify the statement for $n=1$: $\frac{1}{\sqrt{1}} \leq 2\sqrt{1}-1$ is true. The two sides are equal.

Inductive step: suppose that $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \leq 2\sqrt{n}-1$ has already been proved for some arbitrary positive integer n.

We need to show $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}-1$.

We start with $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}$ and use the inductive hypothesis.

Example VII – Part 2

By the inductive hypothesis, the first n terms of our sum are at most $2\sqrt{n}-1$. Thus,

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}} \le 2\sqrt{n}-1+\frac{1}{\sqrt{n+1}}.$$

If we can show that $2\sqrt{n}-1+\frac{1}{\sqrt{n+1}} \le 2\sqrt{n+1}-1$, that would complete the inductive step. We show this in a separate calculation.

It is clear that $4n^2+4n \le 4n^2+4n+1$. We will see shortly that this is the inequality we need, in disguise. (If you are wondering how we came up with that, we did scratch work that is not shown here, in which we provisionally assumed the conclusion and worked backwards.)

The inequality is equivalent to $4n(n+1) \le (2n+1)^2$.

Example VII – Part 3

Knowing $4n(n+1) \leq (2n+1)^2$, we can take the square root and get

$$2\sqrt{n(n+1)} \leq 2n+1.$$

An important algebra point: we can apply the square root function to the inequality and expected the inequality to still hold because the square root function is **increasing**. If f is an increasing function, $a \leq b$ implies $f(a) \leq f(b)$. For a general function *f*, we cannot draw this conclusion.

Example VII – Part 4

The inequality $2\sqrt{n(n+1)} \leq 2n+1$ is equivalent to

$$2\sqrt{n}\sqrt{n+1}+1 \leq 2n+2 = 2(n+1).$$

Dividing by $\sqrt{n+1}$, we obtain

$$2\sqrt{n}+\frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}.$$

Finally, we subtract 1 and get the inequality we were seeking:

$$2\sqrt{n}-1+\frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}-1.$$

Example VII – Part 5

You may wonder why we did not start with what we wanted, $2\sqrt{n} - 1 + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1} - 1$, and simplified until we got the obviously correct inequality $0 \leq 1$. The short answer is, because drawing correct conclusions from an assumption does not prove the assumption. For example, $0 \leq -1$ is **false**, but by squaring it, we can draw the correct conclusion $0 \leq 1$ from it.

The only situation in which drawing a correct conclusion from an assumption proves the assumption is when every deduction we made was reversible, i.e. an *if and only if*.

So yes, we could have start with the inequality we want and worked our way to $0 \leq 1$, but then it would have been required in each step that we justify why our algebraic operation is reversible. It is easier, both for the proof writer and the reader who needs to understand the logic of the argument, to just go in the correct logical direction in the first place.

Example VII – Part 6

The inductive proof of our inequality demonstrates how induction often lets us prove statements without giving us any hint why they might be true. If the problem had been: find an upper bound in terms of n for $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$, induction would have been of no use.

To find inequalities for summations, we usually use **calculus**. Recall integration theory and Riemann sums. The function $f:(0,\infty)\to\mathbb{R}; f(x)=\frac{1}{\sqrt{x}}$ is strictly decreasing. Therefore, its left Riemann sum overestimates values of the definite integral. The sum we are investigating is a left sum with n subdivisions for the integral of the function from 1 to n+1 (make a sketch to clarify this.)

Example VII – Part 7

This means

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>\int_1^{n+1}\frac{1}{\sqrt{x}}\,dx=2\sqrt{n+1}-2.$$

Our first attempt based on the left sum produced a bound very similar to what we wanted, but it is a lower, not an upper bound. The *right* sum would have lead to an upper bound, because it underestimates integrals of strictly decreasing functions; unfortunately, our sum is not a right sum for the function f. If the first x value, $x=1$, was the right end point of an interval, and $\Delta x=1$, then the left interval end point would be $x=0$, but f is not defined there.

Example VII – Part 8

The resolution of this conundrum is simple. While $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$ is **not** a right sum for f, $\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$ is. It is the right sum with n-1 subdivisions for the integral of the function from 1 to n. Therefore,

$$\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}<\int_{1}^{n}\frac{1}{\sqrt{x}}\,dx=2\sqrt{n}-2$$

for $n \geq 2$. Now we just add 1 to the inequality and get

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1$$

for $n \geq 2$. Observing that the two sides are equal for $n=1$, we conclude

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\leq 2\sqrt{n}-1$$

for $n \geq 2$.

Example VII – Part 9

There is one more interesting thing this example can teach us. If $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$ for all positive integers, then

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n+1} - 1$$

is also true, because we made the upper bound just a little larger. A less precise version of the inequality should be easier to prove, but surprisingly, a straightforward proof attempt for the new inequality by induction fails. The base case is easy enough, 1 is less than $2\sqrt{2} - 1$. We run into a problem though in the inductive step.

Example VII – Part 10

Assuming $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n+1} - 1$ for some arbitrary n, we would then apply this inductive hypothesis to the sum up until n+1 and get:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1} - 1 + \frac{1}{\sqrt{n+1}}$$

and then hope to prove

$$2\sqrt{n+1} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+2}.$$

That inequality cannot be proved though, because it is **false**. Take the convenient value $n = 3$ to get a counter-example:

$2\sqrt{4} + \frac{1}{\sqrt{4}} = 4.5$ is not less than $2\sqrt{5} = 4.472\ldots$.

This seems like a paradox- how can the inequality be true, yet the inequality we need to prove in the inductive step be false?

Example VII – Part 11

To resolve the apparent paradox, consider that

$$2\sqrt{n+1} - 1 + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+2} - 1$$

is only a *sufficient* condition for

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+2} - 1,$$

not a *necessary* one.

$2\sqrt{n+1} - 1$ is not the sum of the first n terms, it is only an upper bound for it. Just because that upper bound plus the next term fails to be less than $2\sqrt{n+2} - 1$ does not mean that the sum up to n+1 itself is not less than $2\sqrt{n+2} - 1$. Indeed, we already know that it is less than $2\sqrt{n+2} - 1$.

The Principle of Induction (3)

Observe that when we discussed the principle of induction, we did not claim that if $P(n)$ is true for every natural number n, it must necessarily be practical to prove $P(n)$ for all n by proving $P(1)$, and $P(n) \rightarrow P(n+1)$ for all n. It may not be.

There are examples where a convenient algebraic argument for showing $P(n) \rightarrow P(n+1)$ is only valid for $n \geq N$, for some number $N > 1$, even though $P(n) \rightarrow P(n+1)$ is true for all $n \geq 1$.

In this situation, it is best to use induction to prove the statement for all $n \geq N$ (with $n = N$ being the base case), and to prove the "singular" cases $P(n)$ for $n < N$ individually, one at a time.

The following example gives one such situation.

Example VIII – Part 1

Let us prove $5^n > n^2$ for all non-negative integers n.

Before we write an inductive proof, let us consider what we will have to show in the inductive step: that if $5^n > n^2$ for some $n \geq 0$, then $5^{n+1} > (n+1)^2$.

We know $5^{n+1} = 5 \cdot 5^n > 5n^2$ by the inductive hypothesis. To complete the inductive step, we will need to show that $5n^2 \geq (n+1)^2$, or equivalently, $4n^2 \geq 2n+1$.

Unfortunately, the quadratic inequality $4n^2 \geq 2n+1$ is false for $n = 0$. It is true for $n \geq 1$.

Therefore, we prove the "singular" case $n = 0$ separately, and then prove $5^n > n^2$ for all positive integers n by induction. In practice, this means that we show the inequality for $n = 0$ and $n = 1$, and then show that if it is true for some $n \geq 1$, then it is also true for $n + 1$.

Example VIII – Part 2

We now prove $5^n > n^2$ for all non-negative integers n.

The inequality is true for $n = 0$ because $1 > 0$, and it is true for $n = 1$ because $5 > 1$.

Now assume that we have already proved the inequality for some $n \geq 1$.

We know $5^{n+1} = 5 \cdot 5^n > 5n^2$ by the inductive hypothesis.

Since $n \geq 1$,

$$\begin{aligned} 5n^2 &= n^2 + 2n^2 + 2n^2 \\ &\geq n^2 + 2n + 2 \\ &> n^2 + 2n + 1 \\ &= (n+1)^2. \end{aligned}$$

Therefore, $5^{n+1} > (n+1)^2$.