# III. Difference Equations.

Differential Eqn — continuous process — Rate of change one variable is varying w.r.t other continuosly.

Difference eqn — Discrete process. $y_n \quad y_{n+1} \quad y_{n+2}$

A Relation b/w differences of an unknown function at one or more discrete values of arguments.

$y_{n+2} - y_n = y_{n+1} + 2$ — difference Eqn.

$\Delta y = y_{n+1} - y_n$ (standard) $\quad$ $y$ — dependent variable

$\Delta y = y_{n+2} - y_n$ $\quad$ $n$ — Independent variable / Argument.

$$\Delta^2 y = \Delta(\Delta y) \qquad \Delta n = n+1-n.$$

$$= \Delta(y_{n+1} - y_n) = \Delta y_{n+1} - \Delta y_n$$

$$= y_{n+2} - y_{n+1} - y_{n+1} + y_n$$

$$\Delta^2 y = y_{n+2} - 2y_{n+1} + y_n.$$

## Order of Difference Equation :-

$$\text{Order} = \frac{\text{largest argument} - \text{Smallest argument}}{\text{Argument unit of increment}}$$

↓

find order eqn should be free of $\Delta$'s.

Ex: $\Delta y_{n+1} + \Delta^2 y_{n-1} = 1.$

$$y_{n+2} - y_{n+1} + \Delta(y_n - y_{n-1}) = 1$$

$$y_{n+2} - \cancel{y_{n+1}} + \cancel{y_{n+1}} - y_n - y_n + y_{n-1} = 1.$$

$$y_{n+2} - 2y_n + y_{n-1} = 1$$

$$\text{Order} = \frac{\cancel{n}+2-\cancel{n}+1}{1} = 3$$

* Linear difference equation: If $y_{n+1}, y_{n+2}, \ldots$ appear in first degree only & also not multiplied together.

Homogeneous equation: Each term has dependent variable. (No constant.)

$y_{n+1} - y_n + 2y_{n-1} = 0.$

$y_{n+1} - y_n + 2y_{n-1} = 2$ — Non Homogeneous Eq$^n$.

Solution: Satisfies the difference Equation.

General Solution: Number of arbitary constants = order of the difference equation

Particular solution: Particular value to constants.

Formation of Difference Equation:

$y = ax^2 + bx$

Removing constants a, b we form Difference Equation.

1) $y = ax + bx^2$.

dependent independent

$$y = ax + bx^2.$$

$$\Delta y = \Delta(ax + bx^2)$$

$$\Delta y = a\Delta x + b\Delta x^2.$$

$$= a(x+1-x) + b((x+1)^2 - x^2)$$

$$\Delta y = a + b(2x+1).$$

$$\Delta^2 y = \Delta(a + b(2x+1))$$

$$= 0 + 2b(x+1-x).$$

$$\underline{\Delta^2 y = 2b} \qquad a = \Delta y - \frac{(2x+1)\Delta^2 y}{2}$$

$$y = \frac{1}{2}\Delta^2 y\, x^2 + \left(\Delta y - \frac{1}{2}\Delta^2 y(2x+1)\right)x = \frac{1}{2}\Delta^2 y\, x^2 + \Delta y\, x - \frac{2\Delta^2 y^2}{2} - \frac{1}{2}\Delta^2 y\, x$$

$$y = -\frac{\Delta^2 y}{2}\left[x + x^2\right] + \Delta y\, x.$$

$$2y = 2x\Delta y - x^2\Delta^2 y - x\Delta^2 y.$$

$$(x^2+x)\Delta^2 y - 2x\Delta y + 2y = 0.$$

$$(x^2+x)\left[y_{x+2} - 2y_{x+1} + y_x\right] - 2x\left[y_{x+1} - y_x\right] + 2y = 0.$$

$$\text{Order} = \frac{x+2-x}{1} = 2.$$

2) $y_n = A2^n + B(-3)^n$.

$$y_{n+1} = A2^{n+1} + B(-3)^{n+1} = 2A2^n - 3B(-3)^n.$$

$$y_{n+2} = 4A2^n + 9B(-3)^n.$$

$$y_{n+1} + y_{n+2} = 6A2^n + 6B(-3)^n = 6y_n.$$

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* <u>Difference Equation</u>:-

General form of linear difference equation with constant coeff.

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_n y_n = F(n).$$

$a_1, a_2, \dots a_n$ are constants.

$y_n$ solution $= y_{nCF} + y_{nPI}$.

<u>Rules of finding complementary function</u>:-

Assume first order linear difference equation.

$y_{n+1} - \lambda y_n = 0$. $\quad \lambda =$ constant.

$y_{n+1} = Ey_n$

$y_{n+2} = E^2 y_n \quad \dots \quad y_{n+k} = E^k y_n$

$y_{n+1} - \lambda y_n = 0.$

$Ey_n - \lambda y_n = 0 \Rightarrow \quad (E-\lambda)y_n = 0.$

$y_{n+1} - \lambda y_n = 0.$

Divide $\lambda^{n+1}$.

$\Delta y_n = y_{n+1} - y_n$

$\frac{y_{n+1}}{\lambda^{n+1}} - \frac{\lambda y_n}{\lambda^{n+1}} = 0.$

$\Delta\left(\frac{y_n}{\lambda^n}\right) = \frac{y_{n+1}}{\lambda^n} - \frac{y_n}{\lambda^n}$

$\frac{y_{n+1}}{\lambda^{n+1}} - \frac{y_n}{\lambda^n} = 0.$

$\Delta\left(\frac{y_n}{\lambda^n}\right) = 0. \longrightarrow$ only when it is constant.

$\therefore y_n = c\lambda^n$

Represent in E — & find Roots.

<u>Second order LDE.</u>

$y_{n+2} + ay_{n+1} + by_n = 0.$

$E^2 y_n + aEy_n + by_n = 0.$

$(E^2 + aE + b)y_n = 0.$

$E^2 + aE + b.$

<u>Case I</u> :- $\lambda_1, \lambda_2$ are Real & distinct.

$$(E-\lambda_1)(E-\lambda_2)\, y_n = 0$$

$$\downarrow \qquad \downarrow$$

$$c_1\lambda_1^n \qquad c_2\lambda_2^n.$$

$y_{nCF} = c_1\lambda_1^n + c_2\lambda_2^n.$

<u>Case(II)</u> :- $\lambda_1, \lambda_2$ are Equal $\lambda_1 = \lambda_2 = \lambda.$

$y_{nCF} = (c_1 + c_2 n)\lambda^n.$

Case (iii): Pair of $\alpha \pm i\beta$.

$\lambda_1 = \alpha + i\beta$

$\lambda_2 = \alpha - i\beta$.

$(E - \lambda_1)(E - \lambda_2) y_n = 0.$

$x + iy \rightarrow r(\cos\theta + i\sin\theta)$.

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$y_{nCF} = C_1(\alpha + i\beta)^n + C_2(\alpha - i\beta)^n.$

$= C_1\left[(r(\cos\theta + i\sin\theta))\right]^n + C_2\left[r(\cos\theta - i\sin\theta)\right]^n$

$= C_1 r^n(\cos n\theta + i\sin n\theta) + C_2 r^n(\cos n\theta - i\sin n\theta).$

$= r^n\left[(C_1 + C_2)\cos n\theta + (C_1 - iC_2) i\sin n\theta\right]$

$y_{nCF} = r^n\left[A\cos n\theta + B\sin n\theta\right].$

$r = \sqrt{\alpha^2+\beta^2}$

$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right).$

$\alpha \pm i\beta,\ \alpha \pm i\beta$

$y_{nCF} = r^n\left[(A_1 + A_2 n)\cos n\theta + (B_1 + B_2 n)\sin n\theta\right].$

Complementary function.

| | |
|---|---|
| ① Real & Distinct $\lambda_1, \lambda_2, \ldots \lambda_n$. | $y_{nCF} = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_n\lambda_n^n.$ |
| ② Repeated Roots. $\lambda_1 = \lambda_2 = \ldots \lambda_n = \lambda$. | $y_{nCF} = (C_1 + C_2 n + C_3 n^2 + \ldots C_n n^n)\lambda^n.$ |
| ③ Imaginary Roots $\alpha_1 \pm i\beta_1,\ \alpha_1 \pm i\beta_2, \ldots$ | $y_{nCF} = r_1^n[A_1\cos n\theta_1 + B_1\sin n\theta_1] + r_2^n[A_2\cos n\theta_2 + B_2\sin n\theta_2] + \ldots$ <br> $r_1 = \sqrt{\alpha_1^2+\beta_1^2}$, $\theta_1 = \tan^{-1}\left(\frac{\beta_1}{\alpha_1}\right)$ <br> $r_2 = \sqrt{\alpha_2^2+\beta_2^2}$, $\theta_2 = \tan^{-1}\left(\frac{\beta_2}{\alpha_2}\right)$ |

Ex: 1) $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0.$

$E^3u_n - 2E^2u_n - 5Eu_n + 6u_n = 0 \Rightarrow (E^3 - 2E^2 - 5E + 6)u_n = 0.$

$$E^3 - 2E^2 - 5E + 6 = 0$$

$E = 1 \quad 1 - 2 - 5 + 6 = 0$

$$E - 1 \,\big|\, E^3 - 2E^2 - 5E + 6 \,\big|\, E^2 - E - 6$$

$$\begin{array}{r} E^3 - E^2 \\ - \quad + \\ \hline -E^2 - 5E + 6 \\ -E^2 + E \\ + \quad - \\ \hline -6E + 6 \\ -6E + 6 \\ \hline 0 \end{array}$$

$$E^2 - E - 6 = 0.$$

$$\frac{+1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2}$$

$$3, -2$$

Roots are $1, -2, 3$.

$$y_{nCF} = c_1(1)^n + c_2(-2)^n + c_3(3)^n.$$

2) $(E^2 - 2E + 1)y_n = 0$.

Roots are $1, 1$

$$y_{nCF} = (c_1 + nc_2)(1)^n.$$

## * Method of undetermined Coefficients :-

| $F(n)$ | | $y_{n\,trail}$ |
|---|---|---|
| 1) $a^n$ | $a =$ constant. | $c_1 a^n$. |
| 2) $n^k$ | $k =$ constant | $c_0 + c_1 n + c_2 n^2 + \ldots + c_k n^k$. |
| 3) $n^k a^n$. | | $(c_0 + c_1 n + \ldots + c_k n^k) a^n$. |
| 4) $\sin bn$ or $\cos bn$ | | $c_1 \sin bn + c_2 \cos bn$ |
| 5) $a^n \sin bn$ or $a^n \cos bn$ | | $(c_1 \sin bn + c_2 \cos bn) a^n$. |
| 6) $a^n n^k \sin bn$ or $a^n n^k \cos bn$. | | $(c_0 + c_1 n + \ldots + c_k n^k) a^n \sin bn + (d_0 + d_1 n + \ldots + d_k n^k) a^n \cos bn$. |

3) $(E-2)(E-3)y_n = 2^n$

$$y_{nCF} = c_1(2)^n + c_2(3)^n.$$

$y_{PI} = (c_1 2^n) \cdot n$

$n 3^n$

$(c_0 + c_1 n) 3^n$

4) $y_{n+2} + y_{n+1} - 12 y_n = n 2^n$.

5) $(E-3)(E+2) y_n = 5(3^n)$

4 sol) $E^2 y_n + E y_n - 12 y_n = n 2^n$

$(E^2 + E - 12) y_n = n 2^n$

$\downarrow$

$\frac{-1 \pm \sqrt{1+48}}{2} = \frac{-1 \pm 7}{2}, \ 3, -4$.

$y_{nCF} = c_1 (3)^n + c_2 (-4)^n$

$y_{nPI} = (c_0 + c_1 n) 2^n$.

$y_{n+2} + y_{n+1} - 12 y_n = n 2^n$.

$(c_0 + c_1 (n+2)) 2^{n+2} + (c_0 + c_1 (n+1)) 2^{n+1}$

$- 12 (c_0 + c_1 n) 2^n = n 2^n$.

5 sol)

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1) $y_{n+2} + y_{n+1} - 12 y_n = n 2^n$.

$2^n (4c_0 + 8c_1 + 2c_0 + 2c_1 - 12 c_0) = 0$.

$+ n 2^n (4c_2 + 2c_2 - 12 c_2) = n 2^n$.

$\cancel{c_2 = -1/6}$ $c_1 = -1/6$, $c_0 = -5/18$.

$y_{complete} = c_1 3^n + c_2 (-4)^n - \frac{5}{18} 2^n - \frac{1}{6} n 2^n$.

1) $(E-3)^2 y_n = 2^n + n 3^n$.

$y_{CF} = (c_1 + c_2 n) 3^n$ same

$y_{PI} = c_1 2^n + (c_2 + c_3 n) 3^n$

$= c_1 2^n + (c_2 + c_3 n) n 3^n$ — same X

$= c_1 2^n + (c_2 + c_3 n) n^2 3^n$.

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2.) $(E-3)(E+2)y_n = 5(3^n)$

$3, -2$

$y_{nCF} = c_1(3)^n + c_2(-2)^n$.

$y_{nR} = c_1(3^n)(n)$

$(E^2 - E - 6)y_n = 5(3^n)$

$y_{n+2} - y_{n+1} - 6y_n = 5(3^n)$

$c_1(n+2)3^{n+2} - c_1(n+1)3^{n+1} - 6c_1(n)(3^n) = 5 3^n$

$nc_1(9)3^n + 18c_1 3^n - 3c_1 n 3^n - 3c_1 3^n - 6c_1 n 3^n = 5 3^n$

$18c_1 - 3c_1 = 5 \Rightarrow 15c_1 = 5$

$9c_1 - 3c_1 - 6c_1 = 0 \qquad c_1 = 1/3$.

$y_{complete} = c_1(3)^n + c_2(-2)^n + \frac{1}{3} n 3^n$.

* Simultaneous Equations:-

$u_{x+1} + v_x - 3u_x = x \quad \rightarrow \quad Eu_x + v_x - 3u_x = x$

$3u_x + v_{x+1} - 5v_x = 4^x$. $\qquad (E-3)u_x + v_x = x$.

$3u_x + Ev_x - 5v_x = 4^x \qquad 3u_x + (E-5)v_x = 4^x$

$(E-5)(E-3)u_x + \cancel{(E-5)v_x} = (E-5)x$

$y_{x+1} = Ey_x \qquad 3u_x + \cancel{(E-5)v_x} = 4^x$

$- \qquad - \qquad -$

$Ex = x+1$. $\qquad (E-5)(E-3)u_x - 3u_x = Ex - 5x - 4^x$

$E x^2 = (x+1)^2 \qquad (E^2 - 5E - 3E + 15 - 3)u_x = x+1 - 5x - 4^x$.

$E^2 x = (x+2)$. $\qquad (E^2 - 8E + 12)u_x = 1 - 4x - 4^x$.

$6, 2 \qquad u_{nCF} = c_1(6)^n + c_2(2)^n$.

$u_{x\,PI} = c_1 + c_2 x + c_3 4^x$

$u_{x+2} - 8u_{x+1} + 12u_x = 1 - 4x - 4^x$

$(c_1 + c_2(x+2) + c_3(4)^{x}\cdot 16) - 8(c_1 + c_2(x+1) + c_3 4(4^x)) + 12(c_1 + c_2 x + c_3 4^x)$
$= 1 - 4x - 4^x$

$(c_1 + 2c_2 - 8c_1 - 8c_2 + 12c_1) + x(c_2 - 8c_2 + 12c_2) + (16c_3 - 32c_3 + 12c_3)4^x$
$= 1 - 4x - 4^x$

$5c_1 - 6c_2 = 1$ $\qquad$ $-4c_3 = -1$ $\qquad$ $5c_2 = -4$

$5c_1 = 1 + 6\left(-\frac{4}{5}\right)$ $\qquad$ $\underline{c_3 = 1/4}$ $\qquad$ $\underline{c_2 = -4/5}$

$5c_1 = 1 - \frac{24}{5}$

$\underline{c_1 = \frac{-19}{25}}$

$u_{x\,complete} = c_1 2^x + c_2 6^x - \frac{19}{25} - \frac{4}{5}x + 4^{x-1}$

$v_x = x - u_{x+1} + 3u_x$

$v_x = x - c_1 2^{x+1} - c_2 6^{x+1} + \frac{19}{25} + \frac{4}{5}(x+1) - 4^{x+1-1} + 3(c_1 2^x) + 3c_2 6^x$
$- 3\left(\frac{19}{25}\right) - 3\left(\frac{4}{5}\right)x + 34^{x-1}$

$v_{x\,complete} = c_1 2^x - 3c_2 6^x - \frac{38}{25} + \frac{4}{5} - \frac{3}{5}x - 4^x + 34^{x-1}$

$\underline{v_x = c_1 2^x - 3c_2 6^x - \frac{18}{25} - \frac{3}{5}x - 4^x + 3\cdot 4^{x-1}}$

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# Application & Difference Equation:-

### * Growth/ Decay Problems :-

$\frac{dx}{dt} \propto$ amount that is remaining.

$$\frac{dx}{dt} \propto x.$$

Ex:- Radioactive element decays at the rate of 1% every 25 years.

$y_n \rightarrow$ Amount present at the begining of 25 years.

$y_{n+1} \rightarrow$ " " " " end of 25 years.

$$y_{n+1} - y_n = -(1\%)y_n$$

since decay.

$$y_{n+1} - y_n = -0.01 y_n.$$

$$y_{n+1} - 0.99 y_n = 0.$$

$$(E - 0.99) y_n = 0 \Rightarrow \quad y_n = C_1 (0.99)^n.$$

$t = 0. \quad y = y_0.$

$$y_n = C_1 (0.99)^n.$$

$$y_0 = C_1. \qquad \boxed{y_n = y_0 (0.99)^n}$$

After 200 years ($n = 8$).

$$y_n = y_0 (0.99)^8 = 0.92 y_0.$$

92.2% Amount present after 200 years.

* Varying at regular intervals use difference equations.

* Newton's law of Cooling:-

$$\frac{dT}{dt} \propto (T-T_s)$$

change in temperature over fixed unit of time is $\propto (T_n - T_s)$

$$T_{n+1} - T_n \propto T_n - T_s.$$

$$\boxed{T_{n+1} - T_n = k(T_n - T_s)}$$

LIKHI

Ex:- Suppose a cup of tea, initially at a temperature 180°F, is placed in a room of 80°F. Suppose after 1 min tea has cooled to 175°F. what will be the temp of the tea after 20 min?

Sol)

$$T_{n+1} - T_n = k(T_n - T_s)$$

$T_0 = 180°F \quad T_1 = 175°F.$

$$T_{n+1} - T_n = k(T_n - 80)$$

$n=0. \quad T_1 - T_0 = k(T_0 - 80)$

$$175 - 180 = k(180 - 80)$$

$$-5 = k(100) \Rightarrow k = -1/20.$$

$$20(T_{n+1} - T_n) = -T_n + 80.$$

$$20T_{n+1} - 20T_n + T_n = 80.$$

$$(20E - 19)T_n = 80. \Rightarrow \left(E - \frac{19}{20}\right)T_n = 4.$$

$$T_{nCF} = c_1\left(\frac{19}{20}\right)^n$$

$$T_{ntrial} = c_2 \rightarrow c_2 - \frac{19}{20}c_2 = 4.$$

$$c_2 = 80.$$

$$T_{ncomplete} = c_1\left(\frac{19}{20}\right)^n + 80.$$

$T_0 = 180 \quad 180 = c_1 + 80 \Rightarrow c_1 = 100.$

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$$T_n = 100\left(\frac{19}{20}\right)^n + 80.$$

$$T_{20} = 100\left(\frac{19}{20}\right)^{20} + 80. = 115.8°F.$$

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# Unit 4

## Numerical Solutions of ODE's.

ODE's — First order 1st degree.
First order higher degree
Linear DE's.

y = elementary function.
y = exponential / polynomial / high.
closed form (or).
finite form of solution.

Numerical Solution:-

1.) Picard's Method.
2.) Taylor Series Method.
3) Runge Method.
4) Runge Kutta Method (RK Method)
5) Milli's Method.
6) Adon Bachfus Method.

Tabular.

Numerical Solutions.

Single step method.
y = power series of x
(i) Picard's Method
(ii) Taylor series Method

we need helpful in finding initial values

Step by step Method.

| x | y |
|---|---|
| | |

y = 1 at x = 0
x = 1

1) Euler's Method

*** Picard's Method:-**

Consider $\frac{dy}{dx} = f(x,y)$  $y(x_0) = y_0$.

$$dy = f(x,y)\,dx \qquad (x_0, x).$$

$$\int dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$y\Big|_{y_0}^{x} = \int_{x_0}^{x} f(x,y)\,dx$$

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx}$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx.$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx.$$

Ex: Using Picard's Method obtain solution of $\frac{dy}{dx} = y + x$ $\quad y(0) = 1.$ (upto 4 approximations.)

Using Picard's Method

$$\int dy = \int (y+x)\,dx.$$

$$y(x) = y_0 + \int_0^x (y+x)\,dx.$$

$$y(x) = y_0 + \int_0^x (1+x)\,dx \Rightarrow \quad y_1(x) = 1 + \int_0^x (1+x)\,dx.$$

$$y_1(x) = 1 + x + \frac{x^2}{2}$$

$$y_2 = 1 + \int_0^x 1 + x + \frac{x^2}{2} + x\,dx = 1 + x + \frac{x^3}{6} + x^2.$$

$$y_3 = 1 + \int_0^x 1 + x + x^2 + \frac{x^3}{6} + x\,dx. = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

$$y_4 = 1 + \int_0^x 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} + x\,dx = 1 + x + x^2 + \frac{x^4}{12} + \frac{x^5}{120} + \frac{x^3}{3}.$$

$$\frac{dy}{dx} - y = x \qquad e^{-\int 1 dx} = e^{-x}$$

$$y \cdot e^{-x} = \int x e^{-x} + c$$

$$y \cdot e^{-x} = x(-e^{-x}) + \int e^{-x} + c$$

$$y e^{-x} = -x e^{-x} - e^{-x} + c$$

$$y = -1 - x + c e^{x}$$

$y(0) = 1 \qquad 1 = -1 - 0 + c \Rightarrow c = 2$

$$\boxed{y = 2e^{x} - x - 1} \longrightarrow \text{Exact Solution.}$$

$$\boxed{y_4(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}} \longrightarrow \text{Approximate Solution}$$

$$y = 2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - x - 1$$

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12}$$

## * Taylor Series Method :-

$$y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \ldots + \frac{(x - x_0)^n}{n!} y^n(x_0)$$

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

$$y' = f(x, y)$$

Ex:- Find a solution of $y$ at $x = 0.1, 0.2$ if $\frac{dy}{dx} = x^2 y - 1$ $y(0) = 1$. upto 4 approximations.

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0)$$

$$\frac{dy}{dx} = x^2 y - 1 = y' \qquad y'(0) = -1$$

$y'' = 2xy + x^2 y'$ $\qquad$ $y''' = 2y + 2xy' + 2xy' + x^2 y''$

$y''(0) = 0$ $\qquad$ $y'''(0) = 2$

$y^{iv} = 2y' + 2y' + 2xy'' + 2y' + 2xy'' + 2xy'' + x^2 y''' + 2y'$

$\cancel{y^{iv}(0) = 2(-1) + 8(-1) = 2}$.

$y^{iv}(0) = 6y' = -6$.

$$y(x) = 1 - x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}.$$

$$y(0.1) = 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.9003.$$

$$y(0.2) = 1 - (0.2) + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} = 0.802266$$

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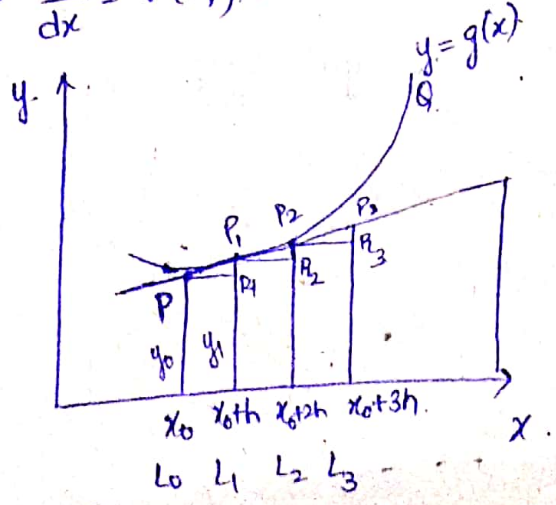
## * Euler's Method :-

Consider $\frac{dy}{dx} = f(x,y)$. $y(x_0) = y_0$.

Consider solution of the DE is represented by a curve $y = g(x)$

Such that $\frac{dy}{dx} = f(x,y)$.




$x = 0 \rightarrow y = y_0$.

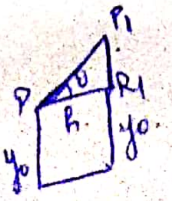
$x = 1 \rightarrow y = ?$

$y_1 = L_1 P_1$

$= L_1 R_1 + R_1 P_1$

$= y_0 + R_1 P_1$

$= y_0 + h\frac{dy}{dx}$




$\tan\theta = \frac{P_1 R_1}{h}$.

$P_1 R_1 = h\tan\theta$.

$$\boxed{y_1 = y_0 + hf(x_0, y_0)}$$

$y_2 = y_1 + hf(x_1, y_1)$

$x_1 = x_0 + h$.

$$\boxed{y_n = y_{n-1} + h f(x_0 + (n-1)h, y_{n-1})}$$

Ex:- $\frac{dy}{dx} = y + x \quad y(0) = 1.$

Find solution of $y$ at $x=1$ using Euler's Method.

$x = 0 \rightarrow x = 1.$

Step Size $h = 0.1$

| $x$ | $y$ | $\frac{dy}{dx} = f(x,y) = y + x$ | $y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$ |
|---|---|---|---|
| $x_0 = 0$ | $y_0 = 1$ | $f(x_0, y_0) = 1 + 0 = 1$ | $y_1 = y_0 + h f(x_0, y_0) = 1 + 0.1 = 1.1$ |
| $x_1 = 0 + h = 0.1$ | $y_1 = 1.1$ | $f(x_1, y_1) = 1.2$ | $y_2 = y_1 + h f(x_1, y_1) = 1.1 + (0.1)(1.2) = 1.22.$ |
| $x_2 = 0.2$ | $y_2 = 1.22$ | $f(x_2, y_2) = 1.42$ | $y_3 = 1.22 + (0.1)(1.42) = 1.362.$ |
| ⋮ | | | |
| $x = 1.$ | | | $y(1) = 3.18.$ |

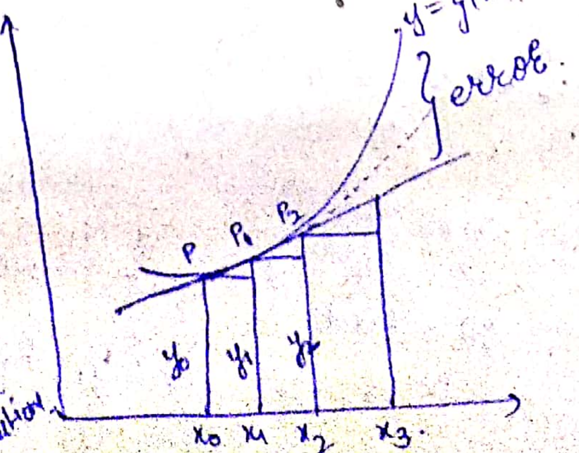
## * Modified Euler's Method:-

Correct $y_1$ using Mean Slopes.

$y_1 = y_0 + h f(x_0, y_0)$

After getting $y_1$ take avg slope at points $P$ & $P_1$.

* Always take avg with initial condition.



$\frac{dy}{dx} = f(x,y) \quad y(x) = y_0$

$y_1 = y_0 + hf(x_0, y_0) \longrightarrow$ predictor value.

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$\downarrow$

Continue this until you get the same value for two consecutive terms (atleast 4 decimals)

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$.

$y_2 = y_1 + hf(x_0+h, y_1) \longrightarrow$ corrected value.

$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_0+h, y_1) + f(x_2, y_2)\right]$

$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$.

Ex:- $\frac{dy}{dx} = y + x \quad y(0) = 1$. Find $y$ at $x = 0.3 \quad h = 0.1$
Till 4 decimal points.

| $x$ | $y$ | $f(x,y) = y+x$ | Meanslope | $y_{new} = y_{old} + h(\text{Meanslope})$ |
|---|---|---|---|---|
| $0$ | $1 = y_0$ | $1 = f(x_0, y_0)$ | - | $y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1 \times 1 = 1.1$. |
| $x_1 = 0.1$ | $y_1 = 1.1$ | $1.2 = f(x_1, y_1)$ | $\frac{1}{2}(1+1.2) = 1.1$ | $y_1' = y_0 + hf(x_1, y_1) = 1 + 0.1(1.1) = 1.11$ |
| | $y_1' = 1.11$ | $1.21$ | $\frac{1}{2}(1+1.21) = 1.105$ | $y_1^2 = 1 + (0.1)(1.105) = 1.1105$ |
| | $y_1^2 = 1.1105$ | $1.2105$ | $\frac{1}{2}(1+1.2105) = 1.1052$ | $y_1^3 = 1 + (0.1)(1.1052) = 1.1105$ |
| $x_2 = 0.2$ | $y_1 = 1.1105$ | $1.2105$ | - | $y_2 = 1.1105 + (0.1)(1.2105) = 1.2316$ |

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# Runge-Kutta (RK) Method

Doesnot ~~inv~~ require the determination of higher order derivatives.

* Good accuracy.

* R.K methods agree with Taylor's series solution upto the terms in $h^r$ where $r$ differs from method to method is called as order of that method.

$r$ — order of RK Method.

$y(x_0+h)$ — From Taylor Series.

$$y(x_1) = y(x_0+h) = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y^2(x_0) + \dots$$

R.K method of order 1 = Euler's Method.

R.K method of order 2 = Modified Euler's Method.

**Euler's Method:- (RK order 1)**

$\frac{dy}{dx} = f(x,y)$. $\quad y(x_0) = y_0$.

$$y_1 = y_0 + \underbrace{h f(x_0, y_0)}_{K_1}$$

$$y_1 = y_0 + K_1.$$

**Modified Euler's Method (RK order 2)**

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1' = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_1)\right]$$

Here We just stick to $y_1'$. Not no need to find mean slopes again.

$$K_1 = hf(x_0, y_0).$$

$$K_2 = hf(x_0+h, y_1).$$

$$y_1' = y_0 + \frac{1}{2}(K_1+K_2)$$

**3rd order RK Method :- (or) Runge Method**

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0+\tfrac{1}{2}h, y_0+\tfrac{1}{2}K_1)$$

$$K_3 = hf(x_0+h, y_0+2K_2-K_1) \quad \text{(or)}$$

$$K' = hf(x_0+h, y_0+K_1)$$

$$K_3 = hf(x_0+h, y_0+K')$$

$$y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3).$$

* Fourth order RK Method :- (or) RK method

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_1)$

$k_3 = hf(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_2)$

$k_4 = hf(x_0+h, y_0+k_3)$

$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

Ex: $\frac{dy}{dx} = y+x \quad y(0)=1$ at $x=0.2$

$h=0.2$ using $3^{rd}$ order RK Method.

4th R.K

$k_1 = (0.2)(0+1) = 0.2 \qquad k_2 = (0.2)\left((1+\frac{0.2}{2})+(\frac{0.2}{2})\right) = 0.24$

$k_3 = (0.2)\left[0.1+(1+\frac{0.24}{2})\right] = 0.244$

$k_4 = (0.2)(0.2+1+0.244) = 0.2888$

$y_1 = 1+\frac{1}{6}\left[0.2+2(0.24)+2(0.244)+0.2888\right] = 1.2428.$

3rd R.K

$k_1 = 0.2 \qquad k_2 = 0.24 \qquad k_3 = (0.2)[0.2+1+0.48-0.2] = 0.296$

$y_1 = 1+\frac{1}{6}(0.2+4\times0.24+0.296) = 1.2426.$

$\boxed{h=0.1}$

3rd R.K

$k_1 = (0.1)(1) = 0.1 \qquad k_2 = (0.1)\left(\frac{0.1}{2}+1+\frac{0.1}{2}\right) = 0.11$

$k_3 = (0.1)[\cancel{0.1}+1+2(0.11)-\cancel{0.1}] = 0.122$

$x=0.1, \quad y_1 = 1+\frac{1}{6}(0.1+4(0.11)+0.122) = 1.1103.$

$k_1 = (0.1)[0.1+1.1103] = 0.12103. \qquad k_2 = (0.1)\left[0.1+\frac{0.1}{2}+1.1103+\frac{0.12103}{2}\right]$

$= 0.13103.$

$k_3 = (0.1)(0.1+0.1+1.1103+2(0.13103)-0.12103)$

$= 0.145133.$

$$y_2 = 1.1103 + \frac{1}{6}\left(0.12103 + 4(0.13103) + 0.1451\right)$$

$$y_2 = 1.2420.$$

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Unit 5

Partial Differential Equations :-

2/more independent variables

ODE - 1 independent variable.

$F\left(x, y, \frac{dx}{dy}, \frac{d^2x}{dy^2}, ---\right)$

$z = f(x, y)$

dependent variables. Independent variables.

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, --, \frac{\partial z}{\partial y}, ---\right)$$

Order of PDE :- order of Highest derivative appearing in equation is order of PDE.

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x \partial y} + zy = 0. \quad \text{Order} = 2.$$

Degree of PDE :- Highest degree of the highest order derivative appearing in equation. [Free of fractions and Radicals.]

Formation of PDE :-

ODE :- For Eliminating arbitary constants ODE is formed.

PDE :- (i) arbitary constants. (ii) arbitary functions.

Ex :- $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$

a, b are constants.

Form PDE by eliminating Constants.

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow \frac{1}{a^2} = \frac{1}{x}\cdot\frac{\partial z}{\partial x} \qquad \text{llly} \qquad \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y}.$$

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$$2z = x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}.$$

Order = 1
No of constants Eliminated = 2.

* If No of constants = No of indendependent variables — Order = 1.
No of arbitary constants >. " " " — Order ≥ 2.

Ex:① $z = f(x^2+y^2)$    f — Arbitary function.

P. diff w.r.t x.

$$\frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot 2x \quad \text{— ①}$$

P.diff w.r.t y

$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\cdot 2y \quad \text{— ②}.$$

$$\frac{①}{②} \quad \frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{2x}{2y}\cdot\frac{f'(x^2+y^2)}{f'(x^2+y^2)} \Rightarrow \boxed{y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.}$$

② $z = f(x+ct) + g(x-ct)$.    c — arbitary consts
f, g — functions.

Form PDE by eliminating functions.

P.d w.r.t x.

$$\frac{\partial z}{\partial x} = f'(x+ct) + g'(x-ct).$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct)$$

P.diff w.r.t t

$$\frac{\partial z}{\partial t} = f'(x+ct)\cdot c + g'(x-ct)(-c)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2 g''(x-ct).$$

$$\boxed{\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}}$$

* When u can form more PDE's — we consider the PDE with least order.

# Classification.

**Linear PDE :-** Dependent variable $z$ & partial derivatives appear in degree 1 and also not multiplied together.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + 3 = 0 \text{ — LDE} \qquad z\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} + 2z = 0 \rightarrow \text{Not Linear PDE}$$

**Quasilinear :-** Degree of the highest partial derivatives involved in the equation is 1 & also not multiplied with lower order PDE.

$$z\left(\frac{\partial z}{\partial x}\right)^{8} + \frac{\partial^2 z}{\partial y^2} + 2z = 0 \text{ — Quasi linear}$$

$$z\left(\frac{\partial z}{\partial x}\right)^{3} + \frac{\partial^2 z}{\partial y^2} + 2z = 0 \text{ — Quasi linear}$$

degree = 3. order = 2, degree = 1

**Semi-linear :-** A Quasi linear equation is semi linear if the coefficients of the highest order partial derivatives are $x$ & $y$ alone

$$y\frac{\partial z}{\partial x} + x\left(\frac{\partial^2 z}{\partial y^2}\right)^{1} + z^3 = 0 \text{ — Semi linear.}$$

$$z\cdot\frac{\partial z}{\partial y} + z\cdot\frac{\partial^2 z}{\partial y^2} + z^3 = 0 \text{ — Quasilinear.}$$

**Non linear :-** $\frac{\partial z}{\partial x} + \left(\frac{\partial^2 z}{\partial y^2}\right)^3 + z^3 = 0$ — Non linear.