Vowpal Wabbit (Langford et al. [2007]) update rules overview

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Overviews

- VW supports linear predictors with convex loss functions.
  - Linear regression
  - Logistic regression
  - Quantile regression
  - SVM regression

Optimization

VW solves optimization of the form

$$l(w) = \sum_i l(w^T x_i; y_i) + \lambda R(w)$$

Here, $l()$ is convex, $R(w) = \lambda_1 |w| + \lambda_2 ||w||^2$.

VW support a variety of loss function

Linear regression	$(y - w^T x)^2$
Logistic regression	$\log(1 + exp(-yw^T x))$
SVM regressin	$\max(0, 1 - yw^T x)$
Quantile regression	$\tau(w^T x - y) * I(y < w^T x) + (1 - \tau)(y - w^T x) I(y > w^T x)$

Generalized linear models

A generalized linear predictor specifies

- A linear predictor of the form $f(x) = w^T x$
- A mean estimate μ
- A link function $g(\mu)$ such that $g(\mu) = f(x)$ that relates the mean estimate to the linear predictor.

This framework supports a variety of regression problems

Linear regression	$\mu = w^T x$
Logistic regression	$\log(\frac{\mu}{1-\mu}) = w^T x$
Poisson regression	$\log(\mu) = w^T x$

Gradient descent

Given an estimator $\widehat{y} = f_w(x)$ and loss $l(y, \widehat{y})$, find the function that minimizes the expected loss function

$$E(f) = \int l(f(x), y) dP(x)$$

The expected risk measures the generalization performance. When the estimator is parametric, it is sufficient to minimize empirical risk

$$E_n(f) = \frac{1}{n} \sum_i^n l(f(x_i), y_i)$$

Gradient descent If the loss function is convex and parametized by w, we can minimize risk by gradient descent.

$$w_{t+1} = w_t - \eta \frac{1}{n} \sum_i^n \nabla_w l(f_w(x_i), y_i)$$

This converges in linear time ($-\log(residual) \sim t$).

We can speed up convergence by using second order information

$$w_{t+1} = w_t - H_w^{-1} \frac{1}{n} \sum_i^n \nabla_w l(f_w(x_i), y_i)$$

where $H_w = \nabla^2 l(f_w(x_i), y_i)$ This converges in linear time ($-\log\log(\text{residual}) \sim t$).

Stochastic gradient descent

We replace the real gradient $\frac{1}{n}\sum_i^n \nabla_w l(f_w(x_i), y_i)$ with a instantaneous estimate

$$w_{t+1} = w_t - \eta_t \nabla_w l(f_w(x_i), y_i)$$

Note that the scaling factor has also been replaced with a time variant version. At each step t, the example is **randomly** picked. Good convergence is obtained using $\eta_t \sim \frac{1}{t}$ or $\eta_t \sim \frac{1}{\sqrt{t}}$

The rate of convergence is much slower than batch version of gradient descent

	GD	2nd order GD	SGD
Iterations to accuracy (ρ)	$\log(\frac{1}{\rho})$	$\log\log(\frac{1}{\rho})$	$\frac{1}{\rho}$

Flavors of SGD Variants of SGD differ in several dimensions

- Learning rate schedule : Determines how η_t is updated.
  - Adaptive McMahan et al. [2013]
  - Normalized Ross et al. [2013]
  - Importance aware Karampatziakis and Langford [2010]
- Weight update: Determines how w_t is computed based on $w_1 \ldots w_t, \eta_1 \ldots \eta_t$
  - Ordinary SGD. Optimizes w_t
  - Averaged SGD. Averages $w_{1 \ldots t}$
- Loss functions: Determines how gradient is computed based on $l(x_1, y_1) \ldots l(x_t, y_t)$
  - Ordinary SGD. Optimizes using last w_t
  - RDA, FTRL. Optimizes based on all previous updates $w_{1 \ldots t}$.

Learning rate update schedule (`--sgd`)

–sgd

$$\eta_t = \lambda d^k \frac{t_0}{(t_0 + t)^p}$$

λ	`-l`
d	`--decay_learning_rate`
t_0	`--initial_t`
p	`--power_t`

Learning rate update schedule (`--adaptive`)

- Scales the update based on all the previous gradient values.
- Useful for different dynamic ranges

Data: λ, T
Initialization w=0, G=0 (diagonal matrix) ;
for *i = 1,2,. . . m* **do**
  Set $g = \nabla_w l(w^T x_i; y_i)$;
  Set $w = w - G^{-\frac{1}{2}} s(w, x, y)$;
  Set $G_{jj} = G_{jj} + g^2, \forall j \in 1 \ldots d$;
end

Learning rate update schedule (`--invariant`)

- Importance weights are useful in many applications: subsampling, boosting
- Algorithm invariant way of implementing importance weight is to replicate the example.
- Many implementations choose to scale the gradient instead. This may cause updates to overshoot and is equivalent to having a large learning rate.
- Importance **aware** updates ensure that the updates with importance weights **h** are equivalent to the updates applied when the instance is presented **h** times.

Learning rate update schedule (`--invariant`)

Common approach

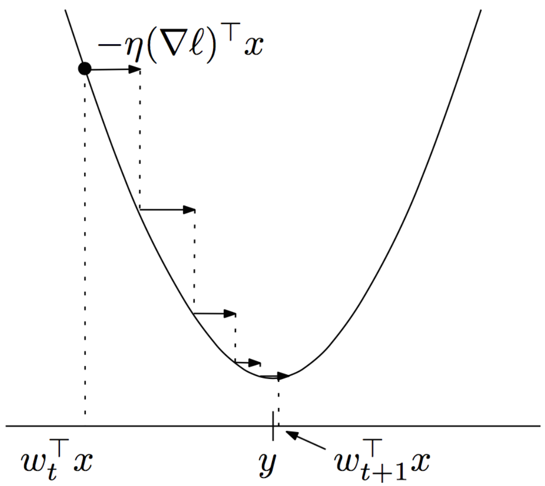
$$w_{t+1} = w_t - h\eta \nabla_w l(w_t^T x_t, y_t)$$

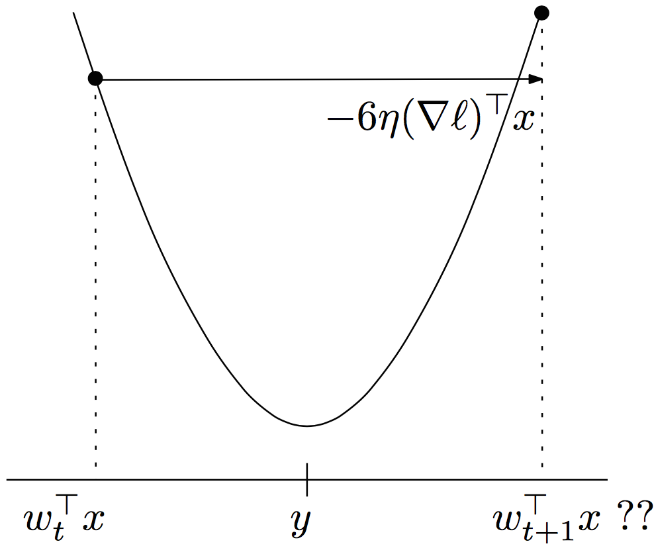
This is not the same as training on the example twice

$$v = w_{t+1} = w_t - \eta \nabla_w l(w_t^T x_t, y_t)$$

$$w_{t+2} = v - \eta \nabla_w l(v^T x_t, y_t)$$

Learning rate update schedule (`--invariant`)





Learning rate update schedule (`--invariant`)

For linear models $\nabla_w l = \frac{\partial l}{\partial p} x$.

$$w_{t+1} = w_t - s(h)x$$

. The scaling factor $s(h)$ has the recursive form

$$s(h+1) = s(h) + \eta \frac{\partial l}{\partial p}, p = (w_t - s(h)x)^T x$$

For squared loss, it turns out that

$$s(h) = \frac{w_t^T x - y}{x^T x}(1 - (1 - \eta x^T x)^h)$$

Learning rate update schedule (`--invariant`) The importance aware update can be determined for many loss functions Karampatziakis and Langford [2010]

Table 1: Importance Invariant and Imp[2] (cf. section 5) Updates for Various Loss Fu

Loss	$\ell(p,y)$	Invariant Update $s(h)$
Squared	$\frac{1}{2}(y-p)^2$	$\frac{p-y}{x^\top x}\left(1-e^{-h\eta x^\top x}\right)$
Logistic	$\log(1+e^{-yp})$	$\frac{W(e^{h\eta x^\top x+yp+e^{yp}})-h\eta x^\top x-e^{yp}}{yx^\top x}$ for $y\in\{-1,1\}$
Exponential	e^{-yp}	$\frac{py-\log(h\eta x^\top x+e^{py})}{x^\top xy}$ for $y\in\{-1,1\}$
Logarithmic	$y\log\frac{y}{p}+(1-y)\log\frac{1-y}{1-p}$	if $y=0$ $\quad\frac{p-1+\sqrt{(p-1)^2+2h\eta x^\top x}}{x^\top x}$ if $y=1$ $\quad\frac{p-\sqrt{p^2+2h\eta x^\top x}}{x^\top x}$
Hellinger	$2(1-\sqrt{py}-\sqrt{(1-p)(1-y)})$	if $y=0$ $\quad\frac{p-1+\frac{1}{4}(12h\eta x^\top x+8(1-p)^{3/2})^{2/3}}{x^\top x}$ if $y=1$ $\quad\frac{p-\frac{1}{4}(12h\eta x^\top x+8p^{3/2})^{2/3}}{x^\top x}$
Hinge	$\max(0,1-yp)$	$-y\min\left(h\eta,\frac{1-yp}{x^\top x}\right)$ for $y\in\{-1,1\}$
τ-Quantile	if $y>p$ $\quad\tau(y-p)$ if $y\le p$ $\quad(1-\tau)(p-y)$	if $y>p$ $\quad-\tau\min(h\eta,\frac{y-p}{\tau x^\top x})$ if $y\le p$ $\quad(1-\tau)\min(h\eta,\frac{p-y}{(1-\tau)x^\top x})$

Learning rate update schedule (`--normalized`)

- Features can have different dynamic ranges (scales) – usually got rid of by pre-scaling

- Offline mean-variance normalization may be expensive. No online version of normalization.

- Regret bounds for regular SGD algorithms depend on the norm of input.

Normalized updates

Intuition Ross et al. [2013]

- Keep track of the max value for each dimension
- If current value exceeds current max, scale down the weight as if new max was known all along
- Accumulate scaled value as pseudo-count to modulate learning rate.

Algorithm 1 NG(learning_rate η_t)

1. Initially $w_i = 0$, $s_i = 0$, $N = 0$
2. For each timestep t observe example (x, y)
   (a) For each i, if $|x_i| > s_i$
      i. $w_i \leftarrow \frac{w_i s_i^2}{|x_i|^2}$
      ii. $s_i \leftarrow |x_i|$
   (b) $\hat{y} = \sum_i w_i x_i$
   (c) $N \leftarrow N + \sum_i \frac{x_i^2}{s_i^2}$
   (d) For each i,
      i. $w_i \leftarrow w_i - \eta_t \frac{t}{N} \frac{1}{s_i^2} \frac{\partial L(\hat{y}, y)}{\partial w_i}$

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FTPRL (`--ftrl`**)**

Reformulation of gradient descent: Standard gradient descent update with learning rate η can be rewritten as the solution to

$$x_{t+1} = \operatorname*{argmin}_x \left(g_t \; x + \frac{1}{2\eta} ||x - x_t||_2^2 \right)$$

Solving the argmin yield the familiar update rule

$$x_{t+1} = x_t - \eta g_t$$

For adaptive updates, η is replaced by η_t.

Sparse updates

FOBOS (Duchi and Singer [2009]) explicitly adds L1 penalty to the optimization

$$x_{t+1} = \underset{x}{\mathrm{argmin}} \left(g_t x + \lambda ||x||_1 + \frac{1}{2\eta} ||x - x_t||_2^2 \right)$$

FTRL (Follow the regularized leader) RDA (Xiao [2010]) optimizes over all the previous gradient steps.

$$x_{t+1} = \operatorname*{argmin}_x \left(g_{1:t}x + \lambda ||x||_1 + \frac{1}{2\eta} ||x||_2^2 \right)$$

Note: In RDA, L2 regularization is proximal to the origin.

FTPRL (Follow the *proximal* regularized leader)

FTPRL provides regularization around the previous updates instead of the origin (like RDA).

$$x_{t+1} = \operatorname*{argmin}_x \left(g_{1:t}x + \lambda||x||_1 + \frac{1}{2}\sum_{s=1}^{t} \sigma_s ||x - x_s||_2^2 \right)$$

Adding L2 regularization, we have

$$x_{t+1} = \operatorname*{argmin}_x \left(g_{1:t}x + \lambda_1||x||_1 + \lambda_2||x||_2^2 + \frac{1}{2}\sum_{s=1}^{t} \sigma_s ||x - x_s||_2^2 \right)$$

Here $\sigma_{1:t} = \eta_t$

Optimization

$$x_{t+1} = \operatorname*{argmin}_x \left(g_{1:t} x + \lambda_1 ||x||_1 + \lambda_2 ||x||_2^2 + \frac{1}{2} \sum_{s=1}^{t} \sigma_s ||x - x_s||_2^2 \right)$$

$$x_{t+1} = \operatorname*{argmin}_x \left(\left(g_{1:t} - \sum_{s=1}^{t} \sigma_s x_s) \right) x + \left(\lambda_2 + \frac{\sigma_{1:t}}{2} \right) x^2 + \lambda_1 ||x||_1 \right)$$

Algorithm 1 Per-Coordinate FTRL-Proximal with L_1 and L_2 Regularization for Logistic Regression

 # With per-coordinate learning rates of Eq. (2).
Input: parameters α, β, λ_1, λ_2
$(\forall i \in \{1, \dots, d\})$, initialize $z_i = 0$ and $n_i = 0$
for $t = 1$ **to** T **do**
  Receive feature vector $\mathbf{x}_t$ and let $I = \{i \mid x_i \neq 0\}$
  For $i \in I$ compute

$$w_{t,i} = \begin{cases} 0 & \text{if } |z_i| \leq \lambda_1 \\ -\left(\frac{\beta+\sqrt{n_i}}{\alpha} + \lambda_2\right)^{-1}(z_i - \operatorname{sgn}(z_i)\lambda_1) & \text{otherwise.} \end{cases}$$

  Predict $p_t = \sigma(\mathbf{x}_t \cdot \mathbf{w})$ using the $w_{t,i}$ computed above
  Observe label $y_t \in \{0, 1\}$
  for all $i \in I$ **do**
    $g_i = (p_t - y_t)x_i$ *#gradient of loss w.r.t.* w_i
    $\sigma_i = \frac{1}{\alpha}\left(\sqrt{n_i + g_i^2} - \sqrt{n_i}\right)$ *#equals* $\frac{1}{\eta_{t,i}} - \frac{1}{\eta_{t-1,i}}$
    $z_i \leftarrow z_i + g_i - \sigma_i w_{t,i}$
    $n_i \leftarrow n_i + g_i^2$
  end for
end for

Summary

- Default behavior `--normalized --invariant --adaptive`
- If you have variable dynamic ranges, rely on `--adaptive`
- If using importance weights, rely on `--invariant`
- If you can't afford multiple passes through the data, rely on `--ftrl`

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