# Calibrated Forecasting and Persuasion

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**Abstract**

How should an expert send forecasts to maximize her utility subject to passing a calibration test? We consider a dynamic game where an expert sends probabilistic forecasts to a decision maker. The decision maker uses a calibration test based on past outcomes to verify the expert's forecasts. We characterize the optimal forecasting strategy by reducing the dynamic game to a static persuasion problem. A distribution of forecasts is implementable by a calibrated strategy if and only if it is a mean-preserving contraction of the distribution of conditionals (honest forecasts). We characterize the value of information by comparing what an informed and uninformed expert can attain. Moreover, we consider a decision maker who uses regret minimization, instead of the calibration test, to take actions. We show that the expert can achieve the same payoff against a regret minimizer as under the calibration test, and in some instances, she can achieve strictly more.

*JEL classification*: C72, C73.

*Keywords*: strategic forecasting, calibration, regret, Bayesian persuasion, approachability.

# 1 Introduction

Probability forecasts are widely used by experts to provide information about uncertain events. The forecasts shape the beliefs of decision makers and persuade them to take specific actions. For example, investors rely on forecasts by financial analysts to determine which asset will achieve the best performance. However, the decision maker follows the expert's forecasts only if they are credible. One way to determine credibility is to perform statistical tests on the outcomes and verify the claims of the expert. We focus on an objective and reasonable test: *calibration*. It is based on the frequency interpretation of probability. The basic idea is to check if the forecast of a state are close to the actual proportion of times the state occurred when the forecast was announced. For example, the investor checks if an asset outperformed others 70% of the days on which an analyst claimed the chance of it being the best was 0.7. Calibration is central to forecasting and is used to assess the accuracy of prediction markets (Page and Clemen, 2012). Decision makers rely on accurate forecasters to take optimal actions. But in many settings, the expert herself has skin in the game and is impacted by the decision maker's action. This preference misalignment leads to strategic forecasting. For instance, if a financial analyst earns a substantial commission upon the purchase of a certain asset, her forecasts may be biased in favor of this asset. We study the extent of an expert's utility gain from strategic forecasting in an infinite horizon dynamic game. Our main focus is on an *informed expert* who knows the data-generating process and can pass any complex statistical test, including calibration. Given that she is tested by a calibration test (and failure leads to a large loss), how should an expert send forecasts to maximize her utility?

To do this, we develop a dynamic sender-receiver model. The state of nature evolves over time according to a stochastic process. At every period, the sender sends a probabilistic forecast about that day's state to the receiver. The receiver performs the calibration test to determine the sender's credibility. If the sender passes the calibration test, the receiver takes the forecasts at face value and takes an action as if the state is drawn according to the forecast. Otherwise, he takes a default action and the sender incurs a punishment cost. The sender seeks to persuade the receiver to choose actions that are aligned with her preferences.

One of the crucial assumptions made in dynamic sender-receiver models is that the receiver either knows the distribution of the process or has a prior belief over the states. Thus, given the sender's strategy, he can perfectly analyze the messages, deduce the posterior beliefs and take actions. Handling such beliefs in equilibrium, even in simple situations, is a daunting task. In contrast to the Bayesian setting, we focus on the case where the receiver has no prior information. He simply verifies the sender's claims by performing the calibration test on the data so far: the forecasts and the states.

We first analyze the case of an informed sender who knows the data-generating process. She can always pass the calibration test by reporting honestly. However, there are other strategies she can use to pass the test. We show passing the calibration test constrains the distribution of forecasts that can be implemented. For a stationary and ergodic process, a calibrated forecasting strategy implements a distribution of forecasts if and only if it is less informative than the distribution of conditionals (or honest forecasts). Overall, the forecasts need to be accurate but can be less precise than truth-telling.

Our main result shows that the sender's maximum payoff equals her equilibrium payoff in a static persuasion problem. The state, signal and prior of the persuasion game represent the conditional probability, the forecast and the distribution of conditionals of the dynamic game. In the persuasion problem, the sender persuades a receiver by committing to a signaling policy. Previous work including Kamenica and Gentzkow (2011) and Arieli et al. (2023) characterize the optimal signaling policy that maximizes the sender's ex-ante expected payoff. We show that the optimal forecasting strategy can be constructed using the optimal signaling policy of the persuasion problem. At each period, the forecasts are sent according to the optimal signaling policy and depend solely on that period's conditional. To summarize, we solve the dynamic forecasting game by reducing it to a static persuasion problem.

Next, we consider an uninformed sender who does not know the data-generating process. As shown by Foster and Vohra (1998), an uninformed sender can also pass the calibration test for any process. As in the informed case, we characterize the maximum payoff an uninformed sender can achieve in terms of the persuasion problem. However, for an uninformed sender, the prior is determined by the empirical distribution of the states. Initially, we model nature as an adversary attempting to prevent the sender from passing the test. In an adversarial environment, the maximal payoff that she can (approximately) guarantee corresponds to the sender's worst signaling policy in the persuasion problem. This payoff is lower than the payoff an informed sender gets by honest forecasting. Next, we show that if the environment is non-adversarial, the sender can guarantee much more. For a stationary and ergodic process, the sender can (approximately) achieve the payoff corresponding to the no information (or babbling) signaling policy in the persuasion problem. This implies that the uninformed sender can always learn the empirical distribution of the states. Thus, we compare the payoffs that an informed and uninformed sender can guarantee, and characterize the value of information for a stochastic process.

We apply our model to analyze a financial application that provides forecasts to its users regarding a state evolving according to a Markov chain. The app's payoff from a forecast depends on its precision and the user's time engagement. While precise forecasts enhance the app's reputation, they reduce the amount of time a user spends on it. We characterize the optimal forecasting strategy for the app using the concavification approach. The optimal strategy is to provide accurate forecasts which are less precise than honest forecasts.

Finally, we also model the receiver's behavior using regret, owing to its close connection with calibration (see Perchet (2014)). The receiver's regret measures the difference in payoff he could have gotten and what he actually got. Heuristics based on regret minimization ensure that, in hindsight, the receiver could not have done better by playing any fixed action repeatedly. We show that a receiver has no regret if he follows the recommendations of any calibrated forecasting strategy.

This justifies the use of calibration test as a heuristic in non-Bayesian environments. Conversely, we show that when facing a regret minimizing receiver, the sender can guarantee the equilibrium payoff in the persuasion problem, as in the case of calibration test. In fact, we provide an example, for a natural class of regret minimizing algorithms, where she can guarantee strictly more. Our work contributes to different strands of literature:

**Strategic Information transmission:** Our work contributes to the literature on communication between an informed sender and an uninformed receiver. In cheap talk (Crawford and Sobel, 1982), the sender's message is unverifiable, while in Bayesian persuasion (Kamenica and Gentzkow, 2011), the sender commits to how the message is generated. In our dynamic setup, the sender does not have commitment power and the messages are probabilistic forecasts. Our model bridges the cheap talk and persuasion models. In particular, our work contributes in investigating the role of communication in a dynamic environment (Best and Quigley, 2022, Renault et al., 2013, Kuvalekar et al., 2022). Our main result shows that the dynamic forecasting game can be reduced to a static persuasion problem (with a large state space)(Arieli et al., 2023). Thus, our work provides a micro-foundation for the commitment assumption in the Bayesian Persuasion setup. Our paper is also related to Guo and Shmaya (2021), which considers a static model of forecasting and introduces an exogenous cost of miscalibration.

**Calibration and Expert Testing:** The initial focus of this literature has been to design a statistical test that can distinguish between an informed expert, who knows the data-generating process, and an uninformed expert, who does not. The calibration test is an objective and popular criterion for evaluating experts. However, Foster and Vohra (1998) show that even an uninformed expert can pass the calibration test. Despite this, calibration is crucial for decision-making based on forecasts (see (Foster and Hart, 2021)). Our contribution is to introduce the calibration test as a heuristic for decision-making. We characterize the extent of strategic forecasting when the calibration test is used to determine the credibility of the expert. In line with the literature, we compare the payoffs that an informed and uninformed expert can achieve for a given process. Our work is related to Echenique and Shmaya (2007), Gradwohl and Salant (2011) and Olszewski and Pęski (2011), which also examines the expert's forecasts in the context of a decision problem. They show that a test exists which passes the informed expert, and if an uninformed expert also passes this test, her forecasts do not lead to unfavorable outcomes. Calibration also has important applications in machine learning (see (Gupta and Ramdas, 2021)). We refer curious readers to Foster and Vohra (2013) and Olszewski (2015) for comprehensive surveys on this topic.

**No-Regret and Online Learning:** There has been a growing interest in investigating regret minimization in dynamic strategic interactions. We focus on optimizing against regret minimizing agents. The closest paper is Deng et al. (2019). We consider a natural class of regret learning algorithms called *mean-based learning algorithms* introduced by Braverman et al. (2018). A common theme in both papers is that they show the optimizer can obtain a higher utility than the rational benchmark. Similarly, we show the sender can obtain a higher utility than the calibration benchmark against a mean-based leaner. Our contribution is to show a novel connection between regret minimization and the calibration test as heuristics for decision-making. Also, we use tools from online learning to solve the dynamic game (see Bernstein et al., 2014, Mannor et al., 2009).

The rest of the paper is organized as follows. In Section 2, we introduce the model and the calibration test. In Section 3, we present the persuasion problem, prove our main results for both informed and uninformed senders, and finally provide an application. In Section 4, we consider a receiver who minimizes regret instead of using the calibration test. In Section 5, we conclude and discuss future work. All omitted proofs are in Appendix A. In Appendix B, we consider an environment where the receiver's action affects how the states evolves. In Appendix C, we provide a reformulation of the persuasion problem in terms of Blackwell experiments.

# 2 Model

We consider a dynamic game between a sender (she) and a receiver (he), in which the state of nature changes over time. At each period, the sender sends a forecast about that period's state which is unknown. The receiver then chooses an action. The state and action are observed before proceeding to the next stage.

Let $\Omega$ denote the finite set of states, $F \subseteq \Delta\Omega$ [1] denote the set of feasible forecasts over the states, and $A$ denote the finite set of actions. Unless specified otherwise, all forecasts are feasible, i.e., $F = \Delta\Omega$. Denote a *play*, i.e., an infinite sequence of states, by $\omega^\infty = (\omega_1, \ldots.) \in \Omega^\infty$. The state $\{\omega_t\}_{t\in\mathbb{N}}$ evolves over time and is governed by a stochastic process with distribution $\mu \in \Delta\Omega^\infty$. Denote by $\omega_t$ and $\omega^t = (\omega_1, \ldots, \omega_{t-1})$ the state and the (public) history of the states at period $t$ respectively. We assume the sender is informed and knows the distribution of the process while the receiver is uninformed and does not. Given the history $\omega^t$, the sender can compute the conditional probability $p_t \in \Delta\Omega$ of the states in that period ,i.e., $p_t = \mu(\cdot \mid \omega^t) \in \Delta\Omega$. So, the sender knows the objective probability of the states at each period. We assume that the conditional probability $p_t$ takes values from a finite set $D \subset \Delta\Omega$ (for example, this assumptions holds for finite Markov chains).[2]

At each period $t \in \mathbb{N}$, the sender publicly announces a forecast $f_t \in \Delta\Omega$ based on the history of the states and forecasts. Formally, the sender's forecasting strategy is a map $\sigma : \cup_{t\geq 1}(F \times \Omega)^{t-1} \to \Delta F$. After observing the forecast, the receiver takes an action $a_t \in A$ and finally the state $\omega_t \in \Omega$ is

[1] $\Delta F$ denotes the set of all probability distributions over the set $F$.

[2] If $D$ is not finite, we can construct a finite $\varepsilon$-grid $L := \{p_l; l \in L\}$ such that for any $p \in \Delta\Omega$ there exists an $l \in L$ such that $\|p - p_l\| \leq \varepsilon$.

observed. The payoff $u_S(\omega_t, a_t)$ and $u_R(\omega_t, a_t)$ for the sender and the receiver in a given period are determined by the state $\omega_t$ and the receiver's action $a_t$.

**Calibration Test:** To model the receiver's behavior, we take a frequentist approach using the calibration test. The receiver has no prior information nor belief regarding the sender's strategy. He simply verifies the claims of the sender using the calibration test on the data he has. At each stage, he checks if the predicted forecasts are close to the realized frequency of the states when the forecast was made. Formally, in period $t$, fixing an error margin $\varepsilon_t$, the receiver performs the $\varepsilon_t$-calibration test. Based on the history of states and forecasts $h_t = (f_1, \omega_1, ...\omega_{t-1}, f_t)$, he checks if the forecasts are $\varepsilon_t$-close to the empirical distribution of the states.

**Definition 1** (Finite)**.** *A $T$-sequence of forecasts $(f_t)_{t=1}^T$ is $\varepsilon_T$-calibrated if*

$$\sum_{f \in F} \frac{|\mathbb{N}_T[f]|}{T} \|\overline{\omega}_T[f] - f\| \leq \varepsilon_T, \tag{1}$$

*where $\mathbb{N}_T[f]$ and $\overline{\omega}_T[f]$ refers to the set of periods and the empirical distribution of states when the forecast is $f$ up to period $T$ respectively and $\|\cdot\|$ is the Euclidean norm, i.e.,*

$$\mathbb{N}_T[f] := \{t \in \{1,...,T\} : f_t = f\}, \qquad \overline{\omega}_T[f] := \frac{\sum_{t \in \mathbb{N}_T[f]} \delta_{\omega_t}}{|\mathbb{N}_T[f]|}, \tag{2}$$

*where $\delta_\omega \in \Delta\Omega$ denotes the Dirac distribution on state $\omega$.*

A $T$-sequence of forecasts is $\varepsilon_T$-calibrated if the empirical distribution of states $\overline{\omega}_T[f]$ is close to $f$ for all forecast $f$ that were sent sufficiently often. [3] Let us look at a simple example of rain forecasting.

**Example 1.** *Let $\mu$ be a Markov chain over the state space $\Omega = \{0,1\}$. The sender predicts the chances of rain ($\omega = 1$). Suppose that the weather remains the same next day with probability $0.8$, i.e., transition matrix $T(1 \mid 1) = T(0 \mid 0) = 0.8$. Consider three sequence of forecasts: $F1$, $F2$ and $F3$ (Table 1). $F1$ are honest forecasts, where the sender announces $80\%$ if it rained yesterday and $20\%$ if it did not. $F2$ are coarse forecasts,which are less precise than the honest ones. While, $F3$ correspond to extreme forecasts, which predicts $100\%$ if it rained yesterday and $0\%$ if it did not.*

*For the error margin $\varepsilon_{10} = 0.05$, both $F1$ and $F2$ pass the $\varepsilon_{10}$-calibration test, while $F3$ does not. This is because even on days where $F3$ predicts $100\%$ chance of rain, it does not rain. In contrast, both the forecasts $F1$ and $F2$ exactly match the realized frequencies. On days that $F2$ predicts $60\%$ chance of rain, it indeed rains $3$ times out of $5$. Both forecasts $F1$ and $F2$ have the*

[3] This holds even if the set $F$ is infinite, as the sum in the definition is implicitly constrained to the forecasts actually sent.

**Table 1:** Sequence of forecasts and the calibration test.

| **Period** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **State** | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | |
| **F1** | 80% | 80% | 20% | 20% | 20% | 20% | 20% | 80% | 80% | 80% | ✓ |
| **F2** | 60% | 60% | 40% | 40% | 60% | 40% | 40% | 60% | 60% | 40% | ✓ |
| **F3** | 100% | 100% | 0% | 0% | 0% | 0% | 0% | 100% | 100% | 100% | ✗ |

*same mean of* 50%*, with the only difference that the forecasts $F2$ are less precise than the honest forecasts $F1$. Later, we show that any distribution of forecasts with correct mean and less precision than honest forecasts can pass the calibration test.*

**Pass**: A sender passes the calibration test at period $t$ if the sequence of forecasts $\{f_i\}_{i=1}^t$ is $\varepsilon_t$-calibrated. In this case, the receiver takes the forecast at face value and responds as if the state $\omega_t$ is drawn according to the forecast $f_t$. He takes the action $\hat{a}(f_t)$, where $\hat{a}(f_t)$ denotes the receiver's optimal action when his belief over states is $f_t$, i.e.,[4]

$$\hat{a}(f_t) := \underset{a \in A}{\operatorname{argmax}} \{ \sum_{\omega \in \Omega} f_t(\omega) u_R(\omega, a) \}. \tag{3}$$

**Fail**: If the sender fails the calibration test, she incurs a *punishment cost* $c < 0$ in that stage. Consequently, the receiver refrains from playing according to the sender's forecast until she passes the test at a later stage. One interpretation is that this cost results from the receiver's default action.[5] For example, in the case of financial forecasting, it could correspond to not buying any asset, which results in zero commission for the analyst. Another interpretation is that this cost arises from loss in credibility from inaccurate predictions. For example, consider an intermediary (like a platform) who only forwards the sender's forecast if she passes the calibration test. Our aim is to examine the sender's persuasive capability when her credibility hinges on passing the calibration test. So, we focus on high punishment costs that ensure that the forecasts need to pass the calibration test.

Note that, for some $\varepsilon_T > 0$, no forecasting strategy will be $\varepsilon_T$-calibrated for *all* possible $T$-sequences of states and forecasts. Even if a sender provides *honest forecasts, i.e.*, $f_t = p_t$ for all $t \in \mathbb{N}$, there is a non-negligible chance that the forecasts will not be $\varepsilon_T$-calibrated. For instance, in Example 1 for $\varepsilon_1 = 0.1$, the honest forecasts $F1$ fails the $\varepsilon_1$-calibration test, regardless of the realized state $\omega_1$. However, as one collects more data, one expects the forecasts to become closer to the empirical distribution of states. This motivates the definition of the (asymptotic) calibration test.

**Definition 2** (Asymptotic)**.** *A forecasting strategy $\sigma$ is $\varepsilon$-calibrated if*

[4]Given multiple optimal actions, we arbitrarily choose to break ties in favor of the sender.

[5]Formally, $u_S(\omega, a_d) = -c$ and $u_R(\omega, a_d) = 0$ for all $\omega \in \Omega$, where $a_d$ denotes the default action.

$$\limsup_{T\to\infty} \sum_{f\in F} \frac{|\mathbb{N}_T[f]|}{T} \|\overline{\omega}_T[f]-f\| \leq \varepsilon, \quad \mathbb{P}_{\sigma,\mu}-a.s. \tag{4}$$

*A forecasting strategy $\sigma$ is calibrated if it is $\varepsilon$-calibrated, for every $\varepsilon > 0$.*

A forecasting strategy is calibrated if the limit empirical distribution of states exactly matches with forecast $f$ for all possible $f$ that were used sufficiently often. We have already seen that the requirement to pass the $\varepsilon_T$-calibration test for all stages $T$ is too demanding. But is it possible for a forecasting strategy to pass the (asymptotic) calibration test? Yes, an honest sender passes the calibration test almost surely (Dawid, 1982). Infact, there exists a sequence of error margins for the finite stages such that the error margins converges to zero, and an honest sender only fails the $\varepsilon_T$-calibration test in finitely many stages (see Proposition 5 in Appendix A). Throughout the paper, we assume the sequence of error margins satisfies this property.

**Assumption 1.** *The sequence of error margins $\{\varepsilon_T\}_{T=1}^{\infty}$ is such that $\lim_{T\to\infty}\varepsilon_T = 0$ and the sequence of honest forecasts ($f_t = p_t$) only fails the $\varepsilon_T$ -calibration test finitely many times (almost surely).*

This assumption ensures that the calibration test does not reject an honest sender. Or else, even an honest sender might be punished in infinitely many stages. In summary, our results remain robust across various sequences of error margins, as long as Assumption 1 holds.[6]

The sender's goal is to find the forecasting strategy that maximizes her long-run average payoff:

$$\liminf_{T\to\infty} \frac{\sum_{t=1}^{T} u_S(\omega_t, a_t)}{T}. \tag{5}$$

We refer to it as the *optimal forecasting strategy* $\sigma^*$. If the punishment cost $c$ is small, the optimal strategy can belong outside the class of calibrated strategies. Our motivation is to use the calibration test as the credibility criterion for the sender. So, we focus on high punishment costs where the optimal forecasting strategy needs to pass the calibration test.

**Proposition 1.** *There exists a punishment cost $\bar{c} = -\max_{p\in\Delta\Omega, a\in A} \mathbb{E}_p[u_S(\omega,a) - u_S(\omega,\hat{a}(p))]$ such that for all $c \leq \bar{c}$ the optimal forecasting strategy needs to pass the (asymptotic) calibration test.*

If the forecasting strategy is not calibrated, it implies that she failed the $\varepsilon_T$-calibration test and incurred the cost $c$ in infinite periods along some possible play. The proposition tells us that if the punishment cost is sufficiently high, the sender can always do better by providing honest forecasts and getting punished only in finitely many periods. This implies that the (asymptotic) calibration

[6] This assumption will not be applicable in the case of a uninformed sender (in Section 3.3) where nature acts as an adversary and the stochastic process is not fixed.

test, which does not depend on the error margins, is a necessary criterion for the forecasting strategy to be optimal. As we shall see, the calibration test imposes limitations on the distribution of forecasts a sender can implement without being punished. Throughout the paper, we assume the punishment cost $c$ satisfies the following condition.

**Assumption 2.** *The punishment cost $c \leq \bar{c}$.*

Thus, the optimal forecasting strategy needs to pass the calibration test, i.e., equation (4) holds.

# 3 Main results

Our main result solves the dynamic forecasting game by reducing it to a static persuasion problem. Before presenting our main result, we introduce the persuasion problem.

## 3.1 Persuasion problem

We consider a static model of persuasion between a sender and a receiver. The parameters of the persuasion problem are linked to that of the dynamic forecasting game. In the persuasion problem, the set of states and signals are both given by $\Delta\Omega$. The players have an atomic common prior $P \in \Delta(\Delta\Omega)$. The state, signal and prior represent the conditional, the forecast and the distribution of conditionals of the dynamic game respectively.

The sender *commits* to a signaling policy $\pi : \Delta\Omega \to \Delta(\Delta\Omega)$. [7] Once the conditional $p \in \Delta\Omega$ is realized, the sender sends a forecast (or signal) $q \in \Delta\Omega$ according to the signaling policy $\pi(p)$. Each forecast $q \in \Delta\Omega$ results in a posterior belief about the conditionals and consequently a posterior mean. [8] Without loss of generality, we can consider signaling policies such that the posterior mean, given forecast $q$, equals $q$. Any signaling policy results in a distribution of posterior means (or forecasts) $Q \in \Delta(\Delta\Omega)$. A distribution $Q \in \Delta(\Delta\Omega)$ is implementable by a signaling policy if and only if $Q$ is a *mean-preserving contraction* of $P$ (Arieli et al., 2023, Kolotilin, 2018).

The players' utility only depends on the posterior mean (or forecast). [9] Denote by $\hat{u}_i(q)$, the indirect utility of player $i \in \{S, R\}$ when the forecast is $q \in \Delta\Omega$. This corresponds to players' expected stage payoff of the dynamic game when the state is drawn according to $q$ and the receiver takes the optimal action $\hat{a}(q)$.

A probability distribution is defined in terms of its mass and support. Let $P := ((\boldsymbol{\lambda}), \boldsymbol{p})$ be a probability distribution on a finite support $\boldsymbol{p} = (p_1, ...., p_n) \in \mathbb{R}^{|\Omega| \times n}$ with mass $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^{1 \times n}$ such that $\lambda_i > 0 \quad \forall i \in \{1, ..., n\}$ and $\sum_{i=1}^{n} \lambda_i = 1$. Let $Q := (\boldsymbol{\mu}, q)$ be a probability distribution with support on $m$ points $\boldsymbol{q} = (q_1, ..., q_m) \in \mathbb{R}^{|\Omega| \times m}$ with mass $\boldsymbol{\mu} = (\mu_1, ...., \mu_m) \in \mathbb{R}^{1 \times m}$. We adopt

[7] To clarify, the sender does not have commitment power in the dynamic forecasting game.

[8] Formally, signaling policy $\pi$ and forecast (or signal) $f$ results in posterior mean $q = \sum_{p \in Supp(P)} \lambda(p)\pi(f \mid p)p \in \Delta\Omega$, where $P := (\boldsymbol{\lambda}, \boldsymbol{p})$ is the prior with mass $\boldsymbol{\lambda}$ and support $\boldsymbol{p}$.

[9] This assumption is key to identify the optimal policy (Arieli et al., 2023, Dworczak and Martini, 2019, Kleiner et al., 2021).

the definition of mean-preserving contractions as provided by Elton and Hill (1992) and Whitmeyer and Whitmeyer (2021).

**Definition 3.** *A probability distribution $Q = (\boldsymbol{\mu}, \boldsymbol{q})$ is a simple mean-preserving contraction (smpc) of $P = (\boldsymbol{\lambda}, \boldsymbol{p})$, if there exists a row-stochastic matrix $G \in \mathbb{R}^{n\times m}$ such that:*

$$\boldsymbol{\lambda} G = \boldsymbol{\mu}, \tag{6}$$

$$(\boldsymbol{\lambda p})G = (\boldsymbol{\mu q}), \tag{7}$$

*where $\boldsymbol{\lambda p} = (\lambda_1 p_1, \ldots, \lambda_n p_n) \in \mathbb{R}^{|\Omega|\times n}$ and $\boldsymbol{\mu q} = (\mu_1 q_1, \ldots, \mu_m q_m)\mathbb{R}^{|\Omega|\times m}$.*

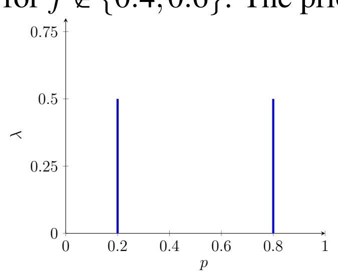
Intuitively, a simple mean-preserving contractions takes fraction $G_{ij}$ of mass $\lambda_i$ at $p_i$ for all $i \in \{1, \ldots, n\}$ and merges them together to get mass $\mu_j$ at $q_j$ for all $j \in \{1, \ldots, m\}$.

**Definition 4.** *$Q$ is a mean-preserving contraction (mpc) of $P$ if there exists a sequence of spmcs $\{Q_m\}_{m=1}^{\infty}$ that satisfies $Q_m \to_w Q$ (weak convergence). Denote by $\mathcal{M}(P)$ the set of all mpcs of $P$.*

The sender's goal is to find the *optimal distribution* $Q^* \in \mathcal{M}(P)$ that maximizes her expected indirect utility. Let $\pi^*$ denote the *optimal signaling policy* which implements the optimal distribution $Q^*$. Given prior distribution $P \in \Delta(\Delta\Omega)$ and sender's indirect utility $\hat{u}_S$, the solution to the persuasion problem $(P, \hat{u}_S)$ is given by:

$$Per(P, \hat{u}_S) = \max_{Q\in\mathcal{M}(P)} \mathbb{E}_Q[\hat{u}_S] = \max_{Q\in\mathcal{M}(P)} \sum_{q\in Supp(Q)} \mu(q)\hat{u}_S(q). \tag{8}$$

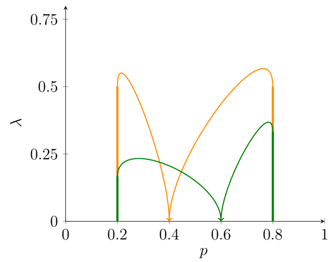
For example, let $\Omega = \{0, 1\}$ [10] and let the sender's utility be given by $\hat{u}_S(0.4) = \hat{u}_S(0.6) = 1$ and $\hat{u}_S(f) = 0$ for $f \notin \{0.4, 0.6\}$. The prior $P$ is given and shown below:




$$P = \begin{bmatrix} \lambda \\ p \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

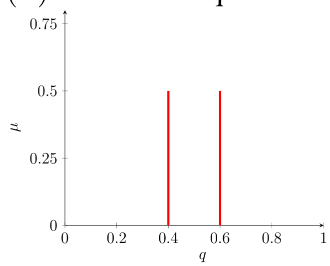
The optimal signaling policy only sends forecasts $q = 0.4$ and $q = 0.6$ and is given by $\pi^*(q = 0.4 \mid p = 0.2) = \pi^*(q = 0.6 \mid p = 0.8) = \frac{2}{3}$. This results in the distribution of forecasts $Q^*$ which is a mpc of the distribution of conditionals $P$. The distribution $Q$ is obtained by merging the distribution $P$ according to the matrix $G$. The orange and green lines represent the weight of the conditionals that are merged to get the forecasts 0.4 and 0.6 respectively.

[10]As the state space is binary, we use the probability of state 1 to denote the forecast of the states.




$$G = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

The only way for the sender to get expected utility of 1 is by implementing the optimal distribution $Q^* \in \mathcal{M}(P)$ which has equal mass on the forecasts $q = 0.4$ and $q = 0.6$.




$$Q^* = \begin{bmatrix} \mu \\ q \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

Now, we provide a sufficient condition when the persuasion problem can be solved using the concavification approach (Aumann et al., 1995, Kamenica and Gentzkow, 2011) applied to a restricted domain.

**Proposition 2.** *If $Supp(P)$ is affinely independent, then the solution of the persuasion problem $(P, \hat{u}_S)$ is given by*

$$Per(P, \hat{u}_S) = Cav\ \hat{u}_S\big|_C(\mathcal{B}(P)) \tag{9}$$

*where, $Cav\ \hat{u}_S\big|_C$ denotes the concave envelope of $\hat{u}_S$ restricted to domain $C = Co(Supp(P)))$ and $\mathcal{B}(P) = \sum_{i=1}^{n} \lambda_i p_i$ denotes the barycenter (or mean) of the distribution $P$.* [11]

The proof relies on the simple characterization of $\mathcal{M}(P)$ when $Supp(P)$ is affinely independent. The feasibility condition corresponds to Bayes plausibility in the restricted domain (see Proposition 6 in the appendix). Hence, the solution is given by the concave envelope restricted to $Co(Supp(P))$. A speical case is when the sender is perfectly informed and she can commit to a signaling policy as a function of the underlying state ($Supp(P) = \cup_{\omega \in \Omega} \delta_\omega$). This corresponds to the standard Bayesian persuasion model (Kamenica and Gentzkow, 2011).

**Corollary 1.** *If the sender is perfectly informed, the solution of the persuasion problem $(P, \hat{u}_S)$ is given by*

$$Per(P, \hat{u}_S) = Cav\, \hat{u}_S(\mathcal{B}(P)). \tag{10}$$

The persuasion problem attains its maximum at an extreme point of $\mathcal{M}(P)$. Whitmeyer and Whitmeyer (2021) show that if $|Supp(P)| = n$, then it suffices to restrict the search to distributions

[11] $Co(A)$ refers to the convex hull of set $A$.

$Q$ with $|Supp(Q)| \leq n$. Dworczak and Martini (2019) and Arieli et al. (2023) study the persuasion problem for a non-atomic prior $P$ and interval state space ($\Delta\Omega = [0,1]$). They show it is sufficient to search within the class of bi-pooling policies to find the optimal policy. In Appendix C, we provide a reformulation of the persuasion problem using Blackwell experiments (mean-preserving spreads).

## 3.2 Optimal forecasting strategy

In this subsection, we consider an informed sender in the dynamic game, who knows the data-generating process. We provide a necessary and sufficient condition for a forecasting strategy to be calibrated. Specifically, for a stationary ergodic process, the condition is that the distribution of forecasts is a mean-preserving contraction of the distribution of conditionals. This can be thought of as merging low-value and high-value forecasts with appropriate weights such that the overall forecast is calibrated. Finally, we characterize the optimal forecasting strategy.

Let $C_\mu \in \Delta(\Delta\Omega)$ denote the *distribution of conditionals* for stochastic process $\mu$

$$C_\mu(p) = \lim_{T\to\infty} \frac{\sum_{t=1}^{T} \mathbf{1}_{\{p_t = p\}}}{T} \qquad \text{(if limit exists)} \tag{11}$$

where $p_t = \mu(\cdot \mid \omega_1, \ldots, \omega_{t-1}) \in \Delta\Omega$. Note, both $p_t$ and $C_\mu$ are random variables and depend on the realization of $\omega^t$ and $\omega^\infty$ respectively. Given a forecasting strategy $\sigma$ and stochastic process $\mu$, let $F_{\mu,\sigma} \in \Delta(\Delta\Omega)$ denote the *distribution of forecasts*:

$$F_{\mu,\sigma}(f) = \lim_{T\to\infty} \frac{\sum_{t=1}^{T} \mathbf{1}_{\{f_t = f\}}}{T} \qquad \text{(if limit exists)} \tag{12}$$

where $f_t$ is chosen randomly according to $\sigma(f_1, \omega_1 \ldots, f_{t-1}, \omega_{t-1})$. If the sender reports honestly then the distribution of forecasts equals the distribution of conditionals.

For any stochastic process, an informed sender can always pass the calibration test by forecasting honestly. However, without imposing restrictions on the process, characterizing the set of feasible outcomes that pass the calibration test is difficult. Consequently, we focus on the class of stationary and ergodic processes, for which the distribution of conditionals is well defined. Recall Example 1 of a Markov chain over binary states to illustrates this.

**Example 1** (continued)**.** *There are two states $\Omega = \{0,1\}$ which evolve according to transition matrix $T(1 \mid 1) = T(0 \mid 0) = 0.8$. As the Markov chain is aperiodic and irreducible, the limit distribution over states converges to the (unique) invariant distribution $\pi^*(1) = \pi^*(0) = 0.5$ for any initial distribution $\pi_0$.*

$$\lim_{n\to\infty}\sum_{i\in\Omega}\pi_0(i)T^n(i,j)=\pi^*(j)\quad\forall\pi_0\in\Delta\Omega.$$

*In the long run, both rain ($\omega=1$) and no rain ($\omega=0$) occur with equal probability* $0.5$ *leading to conditional probabilities of* $80\%$ *and* $20\%$*, respectively. Hence, the distribution of conditionals $C_\mu$ exists and is constant $\mu$-almost surely, with equal mass of* $0.5$ *on the supports of* $20\%$ *and* $80\%$*. An informed sender who knows the process $\mu$ also knows the distribution of conditionals $C_\mu$. The distribution $C_\mu$ corresponds to the prior of the persuasion problem described in Section 3.1.*

Recall that a stochastic process $\{\omega_t\}_{t\in\mathbb{N}}$ is *stationary* if, for any $k\in\mathbb{N}$, the joint distribution of the $k-$tuple $(\omega_t,\omega_{t+1},\ldots,\omega_{t+k-1})$ does not depend on $t$. Let $\mu\in\Delta\Omega^\infty$ denote the distribution of the stationary process. Let $T:\Omega^\infty\to\Omega^\infty$ be the shift transformation given by $T(\omega)_t=\omega_{t+1}$ for all $t\in\mathbb{Z}$. Let $\mathcal{I}$ denote the $\sigma-$algebra of all invariant Borel sets for the transformation $T$. The stationary process $\{\omega_t\}_{t\in\mathbb{N}}$ is *ergodic* if $\mathcal{I}$ is trivial, that is, $\mathbb{P}(A)\in\{0,1\}$ for all $A\in\mathcal{I}$. All the statistical properties can be deduced from a single, long realization of the stationary ergodic process. This makes the search for the optimal forecasting strategy tractable.

Recall that the indirect utility $\hat{u}_i(f)$ is the expected payoff when the states are drawn according to $f\in\Delta\Omega$ and the receiver plays the optimal action $\hat{a}(f)$. Given a calibrated strategy, the empirical distribution of states conditional on forecast $f$ exactly match with the forecast $f$ and the receiver plays the action $\hat{a}(f)$ on (almost) all such periods. Thus, for a calibrated strategy, the long-run average payoff for player $i$ when the forecast was $f$ equals $\hat{u}_i(f)$. Now, we state our main result that characterizes the optimal forecasting strategy.

**Theorem 1.** *For a stationary ergodic process $\mu$, the informed sender can achieve the solution of the persuasion problem $Per(C_\mu,\hat{u}_S)$.*

*Proof.* First, we characterize the set of feasible outcomes for the class of stationary ergodic processes. To do so, we fist show that the distribution of conditionals $C_\mu$ converges and is constant for almost every play (see Lemma 4 in the appendix). So, an informed sender knows the distirbution of conditionals $C_\mu$. Next, we provide, in Lemma 1, a sufficient and necessary condition for a forecasting strategy to be calibrated. The feasible distributions of forecasts are precisely the set of mean-presrving contractions of the distribution of conditionals.

**Lemma 1.** *For a stationary ergodic process $\mu$, if a forecasting strategy $\sigma$ is calibrated then $F_{\mu,\sigma}\in\mathcal{M}(C_\mu)$. Conversely, for any $Q\in\mathcal{M}(C_\mu)$, there exists a calibrated strategy $\sigma$ such that $F_{\mu,\sigma}=Q$.*

Now, we construct the optimal forecasting strategy $\sigma^*$. As the sender knows the distribution of conditionals $C_\mu$ , she can compute the optimal signaling policy $\pi^*$ of the persuasion problem $(C_\mu,\hat{u}_S)$. Consider the forecasting strategy $\sigma^*_t(f_t=f\mid p_t=p)=\pi^*(f\mid p)$ for all $t\in\mathbb{N}$. At any period $t$, the strategy only depends on the conditional $p_t$ of that period. We show that it achieves the solution of the persusion problem $Per(C_\mu,\hat{u}_s)$.

$$= \liminf_{T\to\infty} \frac{\sum_{t=1}^{T} u_S(\omega_t, a_t)}{T} \tag{13}$$

$$= \liminf_{T\to\infty} \frac{\sum_{f\in Supp(Q^*)} \sum_{p\in D} \pi^*(f \mid p) \sum_{t=1}^{T} \mathbf{1}_{\{p_t=p\}} u_S(\omega_t, \hat{a}(f))}{T} \tag{14}$$

$$= \sum_{p\in D} C_\mu(p) \sum_{\omega} p(\omega) \pi^*(f \mid p) u_S(\omega, \hat{a}(f)) \tag{15}$$

$$= Per(C_\mu, \hat{u}_S) \tag{16}$$

As the set of feasible distribution of forecasts are the set of mpcs of the distribution of conditionals, the sender cannot achieve a higher long-run average payoff.

□

Finally, we also provide a sufficient and necessary condition for a forecasting strategy to pass the calibration test for *any* stochastic process $\mu$.

**Lemma 2.** *A forecasting strategy $\sigma$ passes the calibration test if and only if*

$$\limsup_{T\to\infty} \sum_{f\in F} \frac{|N_T[f]|}{T} \|f - \sum_{p\in D} p\mu_T(f,p)\| = 0 \quad \mu\text{-a.s. where, } \mu_T(f,p) = \frac{\sum_{t=1}^{T} 1_{\{p_t=p, f_t=f\}}}{\sum_{t=1}^{T} 1_{\{f_t=f\}}} \tag{17}$$

The term $\mu_T(f,p)$ corresponds to the relative weight on conditional $p$ given forecast $f$. In the long-run, combining all conditionals $p$ with their respective weights $\mu_T(f,p)$ equals forecast $f$. This resembles the definition of a mean-preserving contraction. But even though equation (17) holds for any stochastic process, the distributions $C_\mu$ and $F_{\mu,\sigma}$ might not be well defined. This makes it intractable to characterize the set of feasible outcomes for a calibrated forecasting strategy.

## 3.3 Uninformed expert

In this section, we consider an *uninformed* sender who does not know the data-generating process. Foster and Vohra (1998) show that an uninformed sender can come up with a calibrated forecasting strategy for *any* stochastic process. We ask: what is the maximum payoff that the sender can guarantee? How does this payoff compare with that of an informed sender? First, we tackle the worst case scenario where nature acts as an adversary. Then, we analyze the case where the stochastic process is stationary and ergodic but still unknown to the sender.

The sender's goal is to maximize her long-run average payoff while passing the calibration test. nature acts as an adversary trying to prevent the sender from doing so. The receiver's behavior can be summarized by the calibration test criterion. Given that the sender passes the calibration test, the receiver plays according to the forecast almost surely. So, our focus is on the interaction between the sender and nature.

At every period $t$, the sender and nature simultaneously choose $f_t \in F$ and $\omega_t \in \Omega$ respectively. A strategy $\tau$ for nature is a mapping from the set of all possible past histories to the set of mixed states, i.e, $\tau : \bigcup_{t\geq 0}(F\times\Omega)^{t-1} \to \Delta\Omega$. Let $\overline{\omega}_T \in \Delta\Omega$ denote the *empirical distribution* of states by period $T$:

$$\overline{\omega}_T = \frac{1}{T}\sum_{i=1}^{T}\delta_{\omega_i} \tag{18}$$

For this section, we focus on a less ambitious goal: $\varepsilon-$calibration test.[12] This allows us to consider a finite set of forecasts. Given $\varepsilon > 0$, the set of feasible forecasts $F_\varepsilon$ is given by the regular $\varepsilon$-grid:

$$F_\varepsilon = \{\sum_{\omega\in\Omega} n_\omega \delta_\omega \in \Delta\Omega \mid n_\omega \in \{0, \frac{1}{L}, ..., 1\} \text{ and } \sum_{\omega\in\Omega} n_\omega = 1\}. \tag{19}$$

where, $L = \lceil \frac{\sqrt{|\Omega|-1}}{2\varepsilon} \rceil \in \mathbb{N}$. [13] This ensures (generically) for any $p \in \Delta\Omega$ there exists a unique pure forecast $f \in F_\varepsilon$ such that $\|f - p\| \leq \varepsilon$. We denote this forecast in the $\varepsilon$-neighborhood of $p$ as $f^*(p)$.

If the sender knew beforehand that the empirical distribution $\overline{\omega}_T$ would be equal to $p \in \Delta\Omega$, she could repeatedly send the fixed forecast $f^*(p)$ and pass the calibration test. This would allow the sender to achieve $\hat{u}_S(f^*(p))$. We characterize what the uninformed sender can attain as a function of the (limit) empirical distribution of states.

**Definition 5.** *A function $h : \Delta\Omega \to \mathbb{R}$ is attainable by the sender if there exists a forecasting strategy $\sigma$ such that for any nature's strategy $\tau$, we have*

1. $\liminf_{T\to\infty}(\frac{\sum_{t=1}^{T} u_S(\omega_t, a_t)}{T} - h(\overline{\omega}_T)) \geq 0 \quad \mathbb{P}_{\sigma,\tau} - a.s,$
2. $\limsup_{T\to\infty}\sum_{f\in F}\frac{|\mathbb{N}_T[f]|}{T}\|\overline{\omega}_T[f] - f\| \leq \varepsilon \quad \mathbb{P}_{\sigma,\tau} - a.s.$

The first condition states that the long-run average payoff is higher than the function $h(p)$, where $p$ is the (limit) empirical distribution of states. While the second condition states that the forecasting strategy passes the $\varepsilon-$calibration test. Both conditions need to be satisfied almost surely with respect to the probability distributions induced by the strategy profile $(\sigma, \tau)$.

In the next theorem, we show the closed convex hull of the indirect utility function is attainable. Denote by $\overline{Co}(h)$ the closed convex hull of function $h$ .[14] The function $\overline{Co}\ \hat{u}_S(f^*(p))$ is always less

[12]This weaker criterion is common in literature, atleast as a first step. Using $\varepsilon-$calibrated strategies and the "doubling trick" one can obtain calibrated strategies (Perchet, 2014, Mannor and Stoltz, 2010, Cesa-Bianchi and Lugosi, 2006).
[13]where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

than or equal to $\hat{u}_S(f^*(p))$ for all $p \in \Delta\Omega$. Furthermore, it is continuous on $\Delta\Omega$. Unlike the concave envelope of the sender's utility function, which represents the best mpc maximizing the sender's expected utility, the closed convex hull represents the sender's worst mpc, minimizing the expected utility. Finally, we show it is the highest function that an uninformed sender can attain.

**Theorem 2.** *For an uninformed sender, $\overline{Co}\ \hat{u}_S(f^*(p))$ is the highest attainable function, where $p \in \Delta\Omega$ is the (limit) empirical distribution of states.*

*Sketch of Proof.* We use the technique of approachability to prove the theorem. We first show the function $\overline{Co}\ \hat{u}_S(f^*(p))$ is attainable and then show it is the highest attainable function. To do so, we reformulate the criteria of an attainable function in terms of a vector-valued payoff function. We show the sender can come up with a strategy such that this payoff converges to the target set, no matter what nature does.

At every period, the action of the sender and nature result not only in the sender's payoff but also a *calibration cost*. Given forecast $f \in F_\varepsilon$ and state $\omega \in \Omega$ the calibration cost is given by:

$$c(f,\omega) = (\underline{0},...,f-\delta_\omega,....,\underline{0}) \in \mathbb{R}^{|F_\varepsilon||\Omega|}. \tag{20}$$

It is a vector of $|F_\varepsilon|$ elements of size $\mathbb{R}^{|\Omega|}$ with one non-zero element (at the position for $f$) while the rest are equal to $0 \in \mathbb{R}^{|\Omega|}$. The calibration condition (2) can be rewritten as follows: the average of the sequence of vector-valued calibration costs $c(f_t,\omega_t)$ converges to the set $E_\varepsilon$ almost surely, where

$$E_\varepsilon = \{x \in \mathbb{R}^{|F_\varepsilon||\Omega|} : \sum_{f \in F} \|\underline{x}_f\| \leq \varepsilon\} \tag{21}$$

Thus, the goal of the sender is to maximize the average payoff, such that the average calibration cost converge to $E_\varepsilon$ pathwise.[15] The crux of the proof is to combine the sender's payoff and calibration cost to form a vector-valued payoff. Then, we use the dual condition of approachability to show the closed and convex function $\overline{Co}\ \hat{u}_S(f^*(p))$ is attainable. A set is *approachable* if the sender has a forecasting strategy such that the average vector-valued payoff converges to the target set no matter how nature plays. Convergence rate results follow from approachability theory and are used to determine the sequence of error margins $\{\varepsilon_T\}_{T=1}^{\infty}$ (see appendix for the complete proof).

To show it is also the highest attainable function, we construct a nature's strategy that prevents the sender from attaining any higher function without failing the calibration test. Let $\alpha_l$ and $p_l$ denote the weight and support of the closed convex hull of the sender's utility function, i.e.,

[14] Given a function $f : X \to \mathbb{R}$, over a convex domain $X$, its closed convex hull is the function whose epigraph is $\overline{Co}(\{(x,r)) : r \geq f(x)\})$ where, $\overline{Co}(X)$ is the closed convex hull of set $X$.

[15] Formally, this implies $\limsup_{t\to\infty} \text{dist}(\frac{\sum_{t=1}^{T} c(f_t,\omega_t)}{T}, E) \to 0$ a.s. where, $dist(.)$ is the Euclidean distance.

$\overline{Co}\ \hat{u}_S(f^*(p)) = \sum_{l=1}^{k} \alpha_l \hat{u}_S(f^*(p_l))$. Nature plays in a sequence of $k$ blocks, where the relative size of block $l$ equals $\alpha_l$. In block $l$, nature plays repeatedly the mixed action $p_l \in \Delta\Omega$. So, the states in block $l$ are drawn i.i.d according to $p_l$. The only calibrated strategy is to repeatedly send the forecast $f^*(p_l)$ almost surely. The crucial step is to show that the sender must pass the calibration test within each block to pass the overall calibration criterion. If she does not, then nature can come up with a punishment strategy that ensures that the sender fails the overall calibration test. □

So far, nature acted as an adversary preventing the sender from passing the calibration test. In particular, we assume nature can condition her actions based on the sender's previous forecasts. Can an uninformed sender do better when nature is non-adversarial and states are drawn according to a fixed process? We show that this indeed possible. If the stochastic process is stationary and ergodic, the sender is able to attain the indirect utility function $\hat{u}_S(f^*(p))$. This allows us to compare the attainable payoffs of an informed and uninformed sender.

**Lemma 3.** *For a stationary ergodic process, the function $\hat{u}_S(f^*(p))$ is attainable, where $p \in \Delta\Omega$ is the empirical distribution of states.*

*Sketch of Proof.* The proof uses the concept of *opportunistic approachability* (see Bernstein et al. (2014)). A set that is not approachable in general can be approached if nature plays favorably or in a non-adversarial manner. Nature can no longer use a punishment strategy that forces the sender to fail the calibration test on deviation. Given the process is stationary and ergodic, any play results in the empirical distribution being in the neighborhood around $p$. The sender comes up with a forecasting strategy that learns the (limit) empirical distribution and thus attains the function $\hat{u}_S(f^*(p))$. □

The sender is not required to know that the process is stationary and ergodic beforehand to achieve this benchmark. If nature's strategy proves favorable, she achieves the favorable payoff; otherwise, she still achieves the lower benchmark of the general setting. [16] In summary, an uninformed sender can attain $\overline{Co}\ \ \hat{u}_S(f^*(p))$ in general, and $\hat{u}_S(f^*(p))$ if the process is stationary and ergodic, where $p$ denotes the empirical distribution of states.

## 3.4 Application: Financial Application

In this section, we consider a financial app (sender) that sends forecasts about a binary state to a user (receiver). The app's utility from a forecast depends on its precision and the user's time engagement. Our focus is on the app and the maximal utility it can get when it needs to pass the calibration test. The app must balance precise forecasts with time users spend on it. We show it is optimal to provide accurate but less precise forecasts as compared to honest forecasting.

[16] Bernstein et al. (2014) show that this property holds true even in general settings.

Each day, the user has to decide whether to invest in the financial market or not. The market can be in two states: $\Omega = \{H, L\}$. The user only wants to invest if the state is $H$.[17] The state changes through time according to a Markov chain with the transition matrix: $T(H \mid H) = T(L \mid L) = 0.95$. After observing the app's forecast, the user can acquire more information by paying a fee, which can be thought of as the time spent on the app.

The timing of the forecasting and the information acquisition is as follows. Each day $t \in \mathbb{N}$, the app sends a forecasts the state of that day $f_t \in \Delta\Omega$. Let $f_t$ simply denote $\mathbb{P}(\omega_t = H) \in [0,1]$. The user checks for the credibility of the app using the calibration test. If the app fails the test, he does not use the app. If the app passes the test, he uses the app's forecast as his *prior belief* $p_t \in \Delta\Omega$ and acquires signals by paying a cost to the app. The cost of information acquisition is equal to the expected change in the prior and posterior log-likelihood ratio (Morris and Strack, 2019).

$$C(G) = \begin{cases} \kappa_1 \int_0^1 L(q)d\mu(q) - L(p) & \text{if } \mathcal{B}(G) = p \\ \infty & \text{else} \end{cases}$$

where, $\kappa_1 > 0$, $L(q) = q\log(\frac{1}{1-q}) + (1-q)\log(\frac{1-q}{q})$ denotes the log-likelihood ratio for belief $q$, and $p$ and $G$ denote the prior belief and the posterior distribution of the user respectively. The user acquires information till he is certain of the state upto a 95% threshold. Let $G_p^*$ denote the probability distribution with support 0.05 and 0.95, and mean $p$ (whenever feasible). Starting with prior belief $p$, the app's utility from signal acquisition (see Fig. 1) is given by :

$$\hat{u}_S^{sig}(p) = \begin{cases} C(G_p^*) & 0.05 \leq p \leq 0.95 \\ 0 & \text{otherwise} \end{cases}$$

The app also gains utility from announcing precise forecasts, which can be interpreted as reputation. The utility is higher when the forecast is more precise. The app's utility gain in reputation from announcing forecast $p$ (see Fig. 2) is given by:

$$\hat{u}_S^{rep}(p) = \kappa(p - 0.5)^2 \tag{22}$$

for some constant $\kappa > 0$. The app's total (indirect) utility $\hat{u}_S(p)$ from announcing forecast $p$ is the sum of these two components (Fig. 3):

$$\hat{u}_S(p) = \hat{u}_S^{sig}(p) + \hat{u}_S^{rep}(p) \tag{23}$$

The user always prefers accurate and precise forecasts to take the optimal action and minimize

[17] Formally, we have $u_R(H,b) = 1, u_R(L,b) = -1$ and $u_R(H,db) = u_R(L,db) = 0$ where $b$ and $db$ denote the actions buy and don't buy.

the information acquisition cost. But the app faces a trade-off. On the one hand, revealing precise information increases the app's reputation. While on the other, a precise forecast reduces signal acquisition utility. On average, the user spends lesser time on the app. The app's goal is to find the optimal forecasting strategy that maximizes its long-run average payoff subject to passing the calibration test.

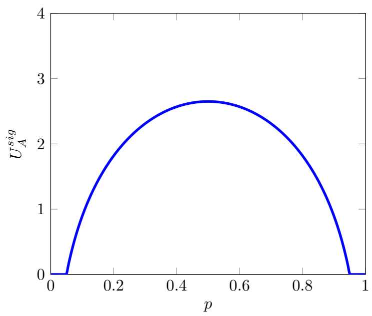



**Figure 1:** $\hat{u}_S^{sig}(p)$

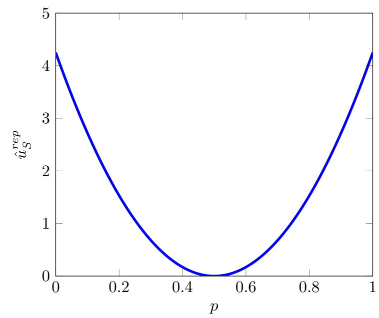



**Figure 2:** $\hat{u}_S^{rep}(p)$

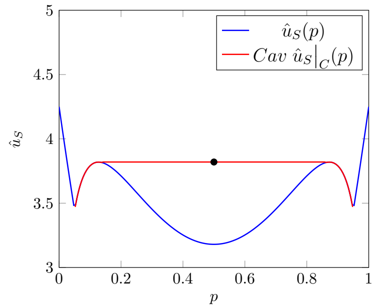



**Figure 3:** $\hat{u}_S(p)$

Using Theorem 1, we know the optimal calibrated strategy corresponds to the solution to the persuasion problem $(C_\mu, \hat{u}_S)$. Given $Supp(C_\mu) = \{0.05, 0.95\}$ is affinely independent, we can use Proposition 2 to find the optimal distribution and the optimal forecasting strategy (see Fig. 3, where $C = [0.05, 0.95]$ and $\mathcal{B}(C_\mu) = 0.5$). The solution is given by:

$$Per\ (C_\mu, \hat{u}_S) = Cav\ \hat{u}_S\big|_{[0.05,0.95]}(0.5) \tag{24}$$

The distribution of conditionals $C_\mu$ and the optimal distribution $Q^*$ are given by:

$$C_\mu = \begin{bmatrix} \lambda \\ p \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{5}{100} & \frac{95}{100} \end{bmatrix} \qquad Q^* = \begin{bmatrix} \mu \\ q \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{15}{100} & \frac{85}{100} \end{bmatrix}$$

The optimal forecasting strategy $\sigma_*$, for all $t \in \mathbb{N}$, is given by:

$$\sigma_*(f_t = 15\% \mid p_t = 5\%) = \frac{8}{9} \qquad \sigma_*(f_t = 85\% \mid p_t = 5\%) = \frac{1}{9} \tag{25}$$

$$\sigma_*(f_t = 15\% \mid p_t = 95\%) = \frac{1}{9} \qquad \sigma_*(f_t = 85\% \mid p_t = 95\%) = \frac{8}{9} \tag{26}$$

If the app was perfectly informed and knew the state $\omega \in \{H, L\}$, it would want to reveal it honestly at the start of each day. This corresponds to the standard Bayesian persuasion solution (Kamenica and Gentzkow (2011)), i.e., the concave envelope of the utility function (with no restriction on the domain). Even though the user spends no time on the app, the utility from reputation is unmatched. But given the distribution $C_\mu$, the app cannot announce the true states without failing the calibration test. Thus, for a partially informed app ($p_t = \{5\%, 95\%\}$), the optimal strategy is to announce accurate but coarse forecasts ($f_t = \{15\%, 85\%\}$). This corresponds to the concanve envelope of the utility function restricted to the support induced by the conditionals (red line in Fig 4).

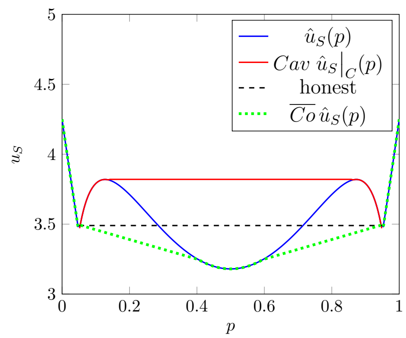



**Figure 4:** Financial App

What could the app achieve if it was uninformed? The app would be able to approximately attain the indirect utility function $\hat{u}_S(p)$ (blue line).[18] This corresponds to the no information signaling policy of the persuasion problem. The difference between the red and the blue line gives us the value of information for the Markov chain. It quantifies what an uninformed sender is willing to pay to become informed.

What could the app guarantee if the stochastic process was not stationary and ergodic? For a general process, an informed app could always pass the calibration test by forecasting honestly (black line). The app would be able to attain the mean of the utility function. On the other hand, an uninformed app would be able to approximately attain the closed convex hull of the utility function, i.e., $\overline{Co}\ \hat{u}_S(p)$ (green line). This corresponds to the worst mpc of a perfectly informed sender in the persuasion problem. The difference between these two functions gives us the value of information for any general process.

# 4 Forecasting against regret minimizers

In this section, we consider a receiver who uses regret minimization, instead of the calibration test, as the heuristic for decision-making. Apart from that, the setting remains the same as in Section 2. At each period, the sender sends a forecast and the receiver uses a regret minimizing strategy to take action. Regret is used as an exogenous criterion to evaluate a strategy in non-Bayesian environments (Cesa-Bianchi and Lugosi, 2006). It measures the difference in the average payoff a player got and what he could have got if he had chosen a fixed action repeatedly. Regret minimizing strategies ensure that a player has no regret in the long-run. Given the close connection between regret and calibration (see Perchet (2014)), we compare what the sender can guarantee in each case. First, we show that the sender can always achieve the solution of the persuasion problem

[18] As the indirect utility function is continuous, we can choose an appropriate $\varepsilon > 0$ so that $\hat{u}_S(f^*(p)) \approx \hat{u}_S(p)$.

against a regret minimizer, as in case of the calibration test. However, we show that, in some cases, the sender can guarantee much more when the receiver uses a mean-based learning algorithm, a natural class of regret minimizing strategies.

A key distinction when playing against a regret minimizer is that the forecast $f$ no longer has intrinsic meaning but simply acts as a message. Given our aim to compare the analysis with the calibration test, we assume the receiver minimizes his regret for each forecast. This is a special case of *contextual regret*, where the forecast acts as context or information that the receiver has in each period. The receiver has no regret with respect to forecast $f$ if on the period where the forecast was $f$ he cannot shift to a fixed action $a^* \in A$ repeatedly and obtain a higher payoff. We denote by $\overline{u}_{R,T}[f]$ the receiver's average payoff up to period $T$ when the forecast was $f$, i.e.,

$$\overline{u}_{R,T}[f] := \frac{\sum_{t \in \mathbb{N}_T[f]} u_R(\omega_t, a_t)}{|\mathbb{N}_T[f]|}. \tag{27}$$

**Definition 6.** *The receiver has no regret with respect to forecast $f$ if*

$$\limsup_{T \to \infty} \frac{|\mathbb{N}_T[f]|}{T} \Big( \max_{a^* \in A} u_R(\overline{\omega}_T[f], a^*) - \overline{u}_{R,T}[f] \Big) \leq 0. \tag{28}$$

The next proposition provides a justification for using the calibration test as a heuristic for decision making.

**Proposition 3.** *The receiver has no regret with respect to any forecast if he follows the recommendations of a calibrated forecasting strategy.*

*Proof.* Fix any calibrated strategy and assume the receiver plays according to the forecast $a_t = \hat{a}(f_t)$ almost surely. Then, the receiver's regret with respect to forecast $f$ is given by:

$$= \limsup_{T \to \infty} \frac{|\mathbb{N}_T[f]|}{T} \Big( \max_{a^* \in \Delta A} u_R(\overline{\omega}_T[f], a^*) - u_R(\overline{\omega}_T[f], \hat{a}(f)) \Big) \tag{29}$$

$$= \limsup_{T \to \infty} \frac{|\mathbb{N}_T[f]|}{T} \Big( \max_{a^* \in \Delta A} \sum_{\omega \in \Omega} f(\omega) [u_R(\omega, a^*) - u_R(\omega, \hat{a}(f))] \Big) \leq 0 \quad \text{(using calibration)} \tag{30}$$

□

For a stationary ergodic process, the highest payoff the sender can get under the calibration test is the solution of a persuasion problem. We show that she can also guarantee this payoff when facing a regret minimizer.

**Proposition 4.** *For a stationary ergodic process $\mu$, the sender can guarantee Per $(C_\mu, \hat{u}_S)$ against a regret minimizer.*

The sender can use the same optimal forecasting strategy as in the calibration test to guarantee this payoff. The sender sends forecast $f$ according to optimal signaling policy $\pi^*(f \mid p)$ of the persuasion problem $Per\ (C_\mu, \hat{u}_s)$ when the conditional is $p$. Given any forecast $f$, if the receiver uses an action $a \notin \hat{a}(f)$ on a non-negligible fraction of periods, then he has positive regret. So, in the long-run, to ensure no regret he has to play the fixed action $\hat{a}(f)$ almost surely.

We now provide an example where the sender can guarantee much more. We follow the approach used by Deng et al. (2019) and Braverman et al. (2018) and focus on the natural class of *mean-based learning algorithms*. This class of no-regret strategies includes Multiplicative Weights algorithm, the Follow-the-Perturbed-Leader algorithm, and the EXP3 algorithm. Intuitively, mean-based strategies play the action that historically performs the best. For the next result, we assume the receiver uses a mean-based learning algorithm for a $T-$period game, where $T >> 0$.

**Definition 7.** *Let $\sigma_{a,t} = \sum_{s=1}^{t} u_R(\omega_s, a)$ be the cumulative payoff for action $a$ for the first $t$ periods. An algorithm is mean-based if whenever $\sigma_{a,t} < \sigma_{b,t} - \gamma T$ for some $b \in A$, the probability to play action $a$ on period $t$ is at most $\gamma$. An algorithm is mean-based if it is $\gamma$-mean-based for some $\gamma = o(1)$.*

The next theorem provides an instance where the sender can attain a payoff higher than the calibration benchmark against a mean-based learner.

**Theorem 3.** *There exists a game in which the sender can guarantee $V > Per\ (C_\mu, \hat{u}_S)$ against a mean-based learner.*

*Proof.* Consider the following payoff matrix, which represents $u_S(\omega, a)$ and $u_R(\omega, a)$:

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $\omega_1$ | $(2,8)$ | $(0,7)$ | $(4,3)$ | $(2,0)$ |
| $\omega_2$ | $(2,0)$ | $(4,3)$ | $(0,7)$ | $(2,8)$ |

The receiver's optimal action depends on her belief over the states. Let $p = \mathbb{P}(\omega_2)$, it is optimal to play $a_1$ when $p \in [0, 0.25]$, to play $a_2$ when $p \in [0.25, 0.5]$, to play $a_3$ when $p \in [0.5, 0.75]$ and to play $a_4$ otherwise (Fig. 5). Using this, we can compute the indirect utilities $\hat{u}_R$ and $\hat{u}_S$ (Fig. 5 and 6).

The state evolves according to a Markov chain with transition matrix $T(\omega_1 \mid \omega_1) = T(\omega_2 \mid \omega_2) = 0.8$. So, the distribution of conditionals $C_\mu$ has equal mass on the conditionals 20% and 80%. The optimal signaling policy is to either forecast honestly or babble, which implies $Per\ (C_\mu, \hat{u}_S) = 2$ (Fig. 6). We now describe a strategy that will guarantee the sender an average payoff $V > 2$.

As the receiver is a mean-based learner, he plays (with a high probability) the best response to the distribution of states $\overline{\omega}_t[f]$. This is because for any period $t$, we have $\sigma_{a,t}^{f} = [u_R(\overline{\omega}_t[f], a)] \times t$.

The game is divided into two stages, where the sender uses only two forecasts (or messages) $l$ and $h$. For the first $\frac{T}{2}$ periods, the sender announces $l$ when $p_t = 20\%$ and announces $h$ when $p_t = 80\%$. Based on the mean-based algorithm, the receiver responds by playing action $a_1$ and $a_4$

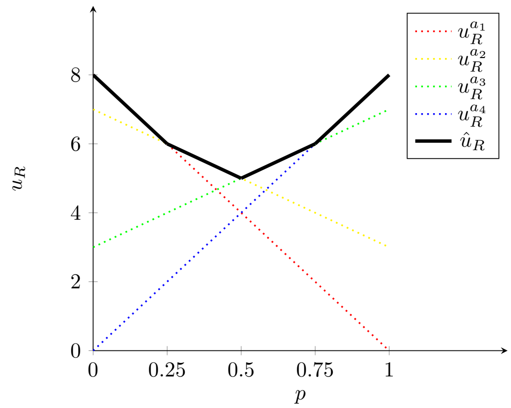



**Figure 5:** Receiver's indirect utility: $\hat{u}_R$

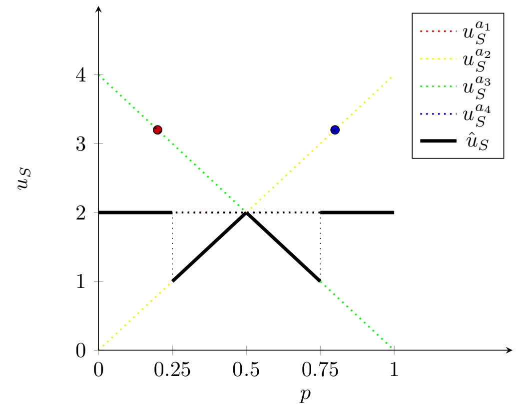



**Figure 6:** Sender's indirect utility: $\hat{u}_S$

(with a high probability) on seeing forecasts $l$ and $h$ respectively. As $T >> 0$, the sender's average payoff equals $\approx 2$ in the first $\frac{T}{2}$ periods.

For the remaining $\frac{T}{2}$, the sender swaps the forecasts, i.e., she announces $l$ when $p_t = 80\%$ and announces $h$ when $p_t = 20\%$. The empirical distribution $\overline{\omega}_t[l]$ begins at 20% and gradually increases until it reaches 50%. However, once it crosses 25%, action $a_2$ results in the highest cumulative payoff. Consequently, the receiver switches to playing action $a_2$ (with a high probability). Similarly given forecast $h$, when the empirical distribution decreases from 75%, the receiver switches to playing action $a_3$. The sender's average payoff equals $\approx \frac{173}{55}$ in the last $\frac{T}{2}$ periods. This is because, for most of the $\frac{T}{2}$ periods, the receiver plays action $a_2$ and $a_4$ when the true distribution of states is 80% and 20% respectively. The receiver's responses are sub-optimal and so the average payoff does not lie in the graph of indirect utilities $\hat{u}_S$ and $\hat{u}_R$ (Fig. 6). Overall, across the $T$-period, the sender is able to guarantee an average payoff $V \approx \frac{283}{110} > 2$. Thus, against a mean-based learner, the sender is able to achieve a payoff higher than the solution of the persuasion problem. □

This example holds even in the case when the receiver does not observe the state $\{\omega_t\}_{t\in\mathbb{N}}$ and/or only observes the payoff from the chosen action. Using standard tools from bandits problem, such as inverse propensity score estimator, it is possible to build unbiased estimator of the reward of unplayed actions (Bubeck and Cesa-Bianchi, 2012). The main idea consists in slightly perturbing the receiver's strategy, say by playing randomly with some small probability. This perturbation can actually be quite small, with probability of order $T^{-1/3}$ and even $T^{-1/2}$, so that the actual cost of estimation is negligible as $T$ increases. These techniques are now considered standard and are well documented (Lattimore and Szepesvári, 2020).

# 5 Conclusion

We studied a dynamic forecasting game, where the sender maximizes her payoff given she has to pass the calibration test. Within the class of stationary and ergodic processes, we identified the optimal calibrated forecasting strategy. This was achieved by transforming the dynamic forecasting game into a static persuasion problem. We showed that the dynamic interaction of a sender and a receiver performing the calibration test substitutes for ex-ante commitment in persuasion models. We compared what an informed and uninformed expert can attain. Additionally, we compared regret minimization and the calibration test as heuristics for decision-making.

Many problems remain open for the setting that we study. In particular, the optimal calibrated strategy for *any* stochastic process. Given this problem might be intractable, we could investigate, for what class of stochastic processes, we can (or cannot) find a calibrated strategy that does better than honest forecasting. Also, the complete characterization for attainable payoffs against no-regret learners remains open. Furthermore, there are natural extensions of the model: the receiver can use other statistical tests to verify the credibility of the sender. This raises the question of identifying tests where honest forecasting is optimal or which minimize the extent of strategic misreporting.

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# A Omitted Proofs

## A.1 Proof of Proposition 1

Fix $c \leq \bar{c}$, where $\bar{c} = -\max_{p \in \Delta\Omega, a \in A} \mathbb{E}_p[u_S(\omega, a) - u_S(\omega, \hat{a}(p))]$. Let us assume the optimal forecasting strategy is not calibrated. This implies that there exists an $\varepsilon > 0$ and a possible play such that for infinitely many periods $T \geq T_0$, we have

$$\sum_{f \in F} \frac{|\mathbb{N}_T[f]|}{T} \|\overline{\omega}_T[f] - f\| > \varepsilon \tag{31}$$

This implies that the sender incurs the punishment cost $c$ in infinitely many periods. Thus, her long-run average payoff equals $c$. Recall, $\hat{a}(p)$ is the receiver's optimal action given his belief over state is $p$. The punishment cost $c \leq \bar{c}$ ensures that for any conditional $p \in \Delta\Omega$, the sender prefers to send honest forecasts and pass the calibration test than incur the punishment cost.

We show in the following proposition that there exists a sequence of error margins $\{\varepsilon_T\}_{T=1}^{\infty}$ such that a honest sender only incurs the punishment cost $c$ in finitely many periods.

**Proposition 5.** *There exists a sequence of error margins $\{\varepsilon_T\}_{T=1}^{\infty}$ such that $\lim_{T\to\infty} \varepsilon_T = 0$ and an honest sender only fails the $\varepsilon_T$-calibration test finitely many time almost surely.*

*Proof.* Let

$$\overline{d}_T^f = 1_{\{f_t = f\}}(\delta_{\omega_t} - f), \qquad \overline{x}_T^f = \frac{1}{T}\sum_{t=1}^{T} \overline{d}_T^f. \tag{32}$$

If the calibration error $\sum_{f \in F} \|\overline{x}_T^f\|$ is greater than the error margin $\varepsilon_T$, then the sender fails the calibration test in period $T$. Let's consider an honest sender who predicts the conditional probability, i.e., $f_t = p_t$ for all $t \in \mathbb{N}$. Then, $\overline{d}_T^f$ is a martingale difference sequence adapted to the process $\mu$. We have $\mathbb{E}[\overline{d}_T^f] = \mathbf{0}$ and that $\|\overline{d}_T^f\| \leq 1$ a.s.. Using Lemma 2 of Foster et al. (2011), we have

$$\mathbb{P}(\|\overline{x}_T^f\| \geq \varepsilon_T) \leq 2e^{\frac{T\varepsilon_T^2}{8c}}$$

for some constant $c > 0$.

The Borel-Cantelli lemma states that if the sum of the probability of a sequence of events is finite then the probability that infinitely many of them occur is zero. Given the forecasts exactly match with the conditionals, where $|D| < \infty$, we can put a bound on the event $E_T = (\mathbb{P}(\max_{f \in D})\|\overline{x}_T^f\| \geq \varepsilon_T)$. This bound represents the probability that the honest sender fails the calibration test in period $T$. Taking the sum across all periods, we have $\sum_{T=1}^{\infty} \mathbb{P}(E_T) = \sum_{T=1}^{\infty} \mathbb{P}(\max_{f \in D} \|\overline{x}_T^f\| > \varepsilon_T) \leq \sum_{T=1}^{\infty} e^{\frac{-T\varepsilon_T^2}{8c}}$. Choosing $\varepsilon_T = o(T^{-\frac{1}{3}})$ suffices to complete the proof. □

We have shown that the honest sender only fails the calibration test in finitely many periods. Thus, assuming the error margins satisfy the property, the sender can always avoid the punishment cost $c$ and improve her payoff by honest forecasting.

## A.2 Proof of Proposition 2

The proof relies on the simple characterization of the set of mean-preserving contractions $\mathcal{M}(P)$, when $Supp(P)$ is affinely independent. We show in the following Proposition that a probability distribution is a smpc if and only if the mean remains fixed $\mathcal{B}(P) = \mathcal{B}(Q)$ and $q \in Co(Supp(P)) \forall q \in Supp(Q)$. This feasibility condition only holds when the support is affinely independent. The feasibility condition for distribution $Q$ simply boils down to Bayes plausibility in the restricted domain. So, using the results of Kamenica and Gentzkow (2011), this implies that that the solution of the persuasion problem is given by the concave envelope restricted to $Co(Supp(P))$.

**Proposition 6.** *Suppose $Supp(P)$ is affinely independent, then*

$$Q \in \mathcal{M}(P) \iff Supp(Q) \subset Co(Supp(P)) \text{ and } \mathcal{B}(P) = \mathcal{B}(Q) \tag{33}$$

*Proof.* ($\Leftarrow$) Choose a (finite) probability distribution $Q$ such that $Supp(Q) \subset Co(Supp(P))$ and $\mathcal{B}(P) = \mathcal{B}(Q)$. For all $q \in Supp(Q)$, we can write $q$ as a convex combination of $Supp(P)$.

$$f_j = \sum_{i=1}^{n} \alpha_{ij} p_i \text{ such that } \alpha_{ij} \geq 0, \sum_{i=1}^{n} \alpha_{ij} = 1 \quad \forall i \in \{1, ..., n\}, j \in \{1, ..., m\}, \tag{34}$$

$$\sum_{j=1}^{m} \mu_j f_j = \sum_{j=1}^{m} \sum_{i=1}^{n} \mu_j \alpha_{ij} p_i. \tag{35}$$

Under our assumption, we have $\mathcal{B}(P) = \mathcal{B}(Q)$, i.e.,

$$\sum_{i=1}^{n} \lambda_i p_i = \sum_{j=1}^{m} \mu_j f_j. \tag{36}$$

Combining this with the above equation we get

$$\sum_{i=1}^{n} \big(\lambda_i - \sum_{j=1}^{m} \mu_j \alpha_{ij}\big) p_i = 0. \tag{37}$$

As $Supp(P)$ is affinely independent, we have

$$\Rightarrow \lambda_i = \sum_{j=1}^{m} \mu_j \alpha_{ij} \quad \forall i \in \{1, .., n\}. \tag{38}$$

Let $G_{ij} = \frac{\mu_j \alpha_{ij}}{\lambda_i}$. The matrix $G$ is a row-stochastic. Using this matrix, we show that the distribution $Q$ is a simple mean-preserving contraction of the distribution $P$. Formally, we show it satisfies equation (6):

$$\sum_{i=1}^{n} \lambda_i G_{ij} = \sum_{i=1}^{n} \mu_j \alpha_{ij}, \tag{39}$$

$$= \mu_j. \tag{40}$$

Also, we show it satisfies equation (7):

$$\sum_{i=1}^{n} \lambda_i p_i G_{ij} = \sum_{i=1}^{n} \mu_j p_i \alpha_{ij}, \tag{41}$$

$$= \mu_j f_j. \tag{42}$$

($\Rightarrow$) Given each $f_j$ is constructed by merging weights of $Supp(P)$, it is necessary that $f_j \in Co(Supp(P))$. We only need to show that $\mathcal{B}(P) = \mathcal{B}(Q)$. Assume $Q \in \mathcal{M}(P)$, so we know that there exists a row-stochastic matrix $G$ such that:

$$\mu_j f_j = \sum_{i=1}^{n} \lambda_i p_i G_{ij} \quad \forall j \in \{1, ..., m\}, \tag{43}$$

$$\mathcal{B}(Q) = \sum_{j=1}^{m} \mu_j f_j = \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i p_i G_{ij}, \tag{44}$$

$$= \sum_{i=1}^{m} \lambda_i p_i = \mathcal{B}(P). \tag{45}$$

□

## A.3 Proof of Lemma 2

($\Rightarrow$) First, we show that if the forecasting strategy $\sigma$ is calibrated, then equation (17) holds. We have

$$\frac{|N_T[f]|}{T} \| f - \sum_{p \in D} p \mu_T(f, p) \| \tag{46}$$

$$\leq \frac{|N_T[f]|}{T} \| f - \frac{\sum_{t=1}^{T} 1_{\{f_t = f\}} \delta_{\omega_t}}{\sum_{t=1}^{T} 1_{\{f_t = f\}}} \| + \| \frac{\sum_{t=1}^{T} \sum_{p \in D} 1_{\{f_t = f, p_t = p\}} [\delta_{\omega_t} - p]}{T} \| \tag{47}$$

Both the terms converge to zero as $T \to \infty$. The first term vanishes from our assumption that the strategy is calibrated. For the second term, we apply Azuma-Hoeffding inequality. Let

$$\bar{x}_T = \frac{\sum_{p \in D} \sum_{t=1}^{T} 1_{\{f_t = t, p_t = p\}} [\delta_{\omega_t} - p]}{T}.$$

We have $\mathbb{E}[\bar{x}_T] = 0$ and that $\|\bar{x}_T - \bar{x}_1\| \leq 1$. Using the Azuma-Hoeffding inequality, we have $\mathbb{P}(\|\bar{x}_T\| > \eta) \leq \exp^{-2T\eta^2}$. Choosing $\eta = o(T^{-\frac{1}{3}})$ suffices in our case.

($\Leftarrow$) For the converse, we apply the triangle inequality to show that the if equation (17) holds, then the forecasting strategy is calibrated. The calibration error is given by

$$\frac{|N_T[f]|}{T} \|f - \frac{\sum_{t=1}^{T} 1_{\{f_t = f\}} \delta_{\omega_t}}{\sum_{t=1}^{T} 1_{\{f_t = f\}}}\| \tag{48}$$

$$\leq \frac{|N_T[f]|}{T} \|f - \sum_{p} p \mu_T(f,p)\| + \|\frac{\sum_{t=1}^{T} \sum_p 1_{\{f_t = t, p_t = p\}} [\delta_{\omega_t} - p]}{T}\| \tag{49}$$

As $T$ goes to infinity, the first term goes to zero from assumption and the second terms goes to zero due to the Azuma-Hoeffding inequality.

## A.4 Proof of Lemma 4

**Lemma 4.** *For a stationary ergodic process $\mu$, the distribution of conditionals $C_\mu$ exists and is constant $\mu$-a.s.*

*Proof.* Consider the two-sided extension of the stationary process i.e., $\mu \in \Delta(\Omega^{\mathbb{Z}})$. Let $\omega_a^b = (\omega_a, \ldots.., \omega_{b-1})$ where $a < b - 1$.

$$f_n = \mu(\omega_0 = \cdot \mid \omega_{-n}^0), \tag{50}$$

$$f_\infty = \mu(\omega_0 = \cdot \mid \omega_{-\infty}^0). \tag{51}$$

Using the martingale convergence theorem we have that $f_n \to f_\infty$ $\mu$-a.s.. Given $\mu$ is stationary, using the shift transformation $T$, we have

$$f_n \circ T^n = \mu(\omega_n = \cdot \mid \omega_0^n) = p_n. \tag{52}$$

Since $f_n$ and $p_n = f_n \circ T^n$ have the same distribution for all $n \in \mathbb{N}^+$, we can conclude that $p_n \to p_\infty = \mu(\omega_\infty = \cdot \mid \omega_0^\infty)$ $\mu$-a.s. and $\mathbb{E}[\mu(\omega_0 = \cdot \mid \omega_{-\infty}^0)] = \mathbb{E}[\mu(\omega_\infty = \cdot \mid \omega_0^\infty)]$. Now, as $\mu$ is stationary and ergodic, we apply the Maker's Ergodic theorem to get:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mu(\omega_n=\cdot\mid\omega_0^n)=\mathbb{E}[\mu(\omega_\infty=\cdot\mid\omega_0^\infty)]\quad \mu-a.s. \tag{53}$$

In particular, this implies for any $p\in D$, we have

$$C_\mu(p)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_{\{p_n=p\}}=\mathbb{E}[\mathbf{1}_{\{p_\infty=p\}}]\quad \mu-a.s. \tag{54}$$

For reference, the Maker's Ergodic Theorem Kallenberg (2002):

**Theorem 4.** *(Maker's Ergodic Theorem) Let $\mu\in\Delta\Gamma$ be a stationary distribution and let $f_0,f_1,\ldots:\Gamma\to\mathbb{R}$ be such that $sup_n|f_n|\in L_1(\mu)$ and $f_n\to f_\infty\quad\mu$-a.s. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}f_n\circ T^n\to\mathbb{E}[f_\infty\mid\mathcal{I}]\quad \mu-a.s. \tag{55}$$

□

## A.5 Proof of Lemma 1

First, we show that for any calibrated strategy $\sigma$, the resultant distribution of forecasts $F_{\mu,\sigma}$ is a mpc of the distribution of conditionals $C_\mu$, i.e., $F_{\mu,\sigma}\in\mathcal{M}(C_\mu)$. Given $C_\mu$, consider

$$\sum_{f\in F}\|\frac{\sum_{t=1}^T\mathbf{1}_{\{f_t=f\}}}{T}f-\sum_p pC_\mu(p)\frac{\sum_{t=1}^T\mathbf{1}_{\{f_t=f,p_t=p\}}}{\sum_{t=1}^T\mathbf{1}_{\{p_t=p\}}}\| \tag{56}$$

$$\leq\sum_{f\in F}\|\frac{\sum_{t=1}^T\mathbf{1}_{\{f_t=f\}}}{T}f-\sum_p p\frac{\sum_{t=1}^T\mathbf{1}_{\{f_t=f,p_t=p\}}}{T}\|+\sum_{f\in F}\sum_p p(\frac{\sum_{t=1}^T\mathbf{1}_{\{f_t=f,p_t=p\}}}{T})\|1-\frac{C_\mu(p)T}{\sum_{t=1}^T\mathbf{1}_{\{p_t=p\}}}\|. \tag{57}$$

From Lemma 2, we have that the first term goes to zero as $T\to\infty$. The second term also goes to zero as the distribution of conditionals $C_\mu$ exists. Thus, we have

$$\limsup_{T\to\infty}\sum_{f\in F}\|\frac{\sum_{t=1}^T\mathbf{1}_{\{f_t=f\}}}{T}f-\sum_p pC_\mu(p)\frac{\sum_{t=1}^T\mathbf{1}_{\{f_t=f,p_t=p\}}}{\sum_{t=1}^T\mathbf{1}_{\{p_t=p\}}}\|=0 \tag{58}$$

For any $n\in\mathbb{N}$, let the row stochastic matrix $G_n(f,p)$ be equal to $\frac{\sum_{t=1}^n\mathbf{1}_{\{f_t=f,p_t=p\}}}{\sum_{t=1}^T\mathbf{1}_{\{p_t=p\}}}$. This matrix will result in a simple mean-preserving contraction $F_n\in\mathcal{M}(P)$ for each $n$. As $\Delta(\Delta\Omega)$ is compact, we know there exists a subsequence $n_1,n_2,....$ such that this distribution $F_n$ converges to a limit point $F$, i.e., $F = \lim_{t\to\infty} F_{n_t}$. However, from equation (58), this implies that the distribution of forecasts also converges to $F$. Thus, from definition 4, as $F$ is a limit of the sequence of simple mean-preserving contractions, we have $F \in \mathcal{M}(C_\mu)$.

On the other hand, let $F \in \mathcal{M}(C_\mu)$. Given Lemma 4, the sender knows the distribution of conditionals $C_\mu$. Let $\pi : \Delta\Omega \to \Delta(\Delta\Omega)$ be the signaling policy that given prior $C_\mu$ results in the distribution $F$. Consider the forecasting strategy given by $\sigma_t(f_t = f \mid p_t = p) = \pi(f \mid p)$ for all $t$. We show the forecasting strategy $\sigma$ is calibrated. For any $f \in Supp(F)$, we have

$$\frac{|N_T[f]|}{T}\|\frac{\sum_{t=1}^T 1_{\{f_t=f\}}\delta_{\omega_t}}{\sum_{t=1}^T 1_{\{f_t=f\}}} - f\| \tag{59}$$

$$\leq \|\frac{\sum_p \sum_{t=1}^T 1_{\{f_t=f,p_t=p\}}[\delta_{\omega_t} - p]}{T}\| + \|\frac{\sum_{p\in P}\sum_{t=1}^T 1_{\{f_t=f,p_t=p\}}(p-f)}{T}\| \tag{60}$$

$$\leq \|\frac{\sum_p \sum_{t=1}^T 1_{\{f_t=f,p_t=p\}}[\delta_{\omega_t} - p]}{T}\| + \|\frac{\sum_p \sum_{t=1}^T 1_{\{p_t=p\}}\pi(f \mid p)[p-f]]}{T}\| \tag{61}$$

Again, using the Azuma-Hoeffding inequality, the first term vanishes to zero. For the second term, we know that $C_\mu$ is well defined and constant, so the second term also converges to zero from the definition of mean-preserving contraction. Hence, we have proved that for a stationary ergodic process, the sender can implement any distribution $F \in \mathcal{M}(C_\mu)$.

## A.6 Proof of Theorem 2

We first show that the function $\overline{Co}\ \hat{u}_S(f^*(p))$ is attainable. Then, we show that is the highest continuous attainable function.

(a) The main idea of the proof is to combine the payoff and calibration cost function such that the overall set is a closed and convex set (a similar proof can be found in Mannor et al. (2009)). Then, we use the dual condition of approachability to show that the set is approachable.

Given forecast $f \in F_\varepsilon$ and state $\omega \in \Omega$ the calibration cost $c$ is given by:

$$c(f,\omega) = (\underline{0},\ldots, f - \delta_\omega, \ldots., \underline{0}) \in \mathbb{R}^{|F_\varepsilon||\Omega|}. \tag{62}$$

It is a vector of $|F_\varepsilon|$ elements of size $\mathbb{R}^{|\Omega|}$ with one non-zero element (at the position for $f$) while the rest are equal to $0 \in \mathbb{R}^{|\Omega|}$. The $\varepsilon$-calibration condition (2) can be rewritten as follows: the average of the sequence of vector-valued calibration costs $c_t = c(f_t, \omega_t)$ converges to the set $E_\varepsilon$ almost surely, where

$$E_\varepsilon = \{x \in \mathbb{R}^{|F_\varepsilon||\Omega|} : \sum_{f\in F} \|\underline{x}_f\| \leq \varepsilon\}.$$

For each period $t$, the sender and nature simultaneously choose $f_t \in F_\varepsilon$ and $\omega_t \in \Omega$ respectively.

This results in a reward $r_t = \hat{u}_S(f_t)$ and penalty $c_t = c(f_t, \omega_t)$ for the sender in period $t$.

Unlike the (exact) calibration test, the forecast and the limit empirical distribution of states do not have to exactly match but can differ up to an error margin $\varepsilon$. Even an informed sender who sends forecast honestly can fail the exact calibration test if she was restricted to send forecasts from the finite set $F_\varepsilon$.

We now show the function $\overline{Co}\ \hat{u}_S(f^*(p))$ is attainable. We show this using the dual condition of approachability. For any period $t$, consider the vector-valued payoff $m(f_t, \omega_t)$ constructed using the sender's payoff and the calibration cost:

$$m(f_t, \omega_t) := (r_t, c_t, \delta_{\omega_t}) \in \mathbb{R} \times \mathbb{R}^{|F_\varepsilon||\Omega|} \times \Delta\Omega. \tag{63}$$

Now, consider the sets:

$$D_1 = \{(r, c, p) \in D : r \geq \overline{Co}\ \hat{u}_S(f^*(p))\} \quad D_2 = \{(r, c, p) \in D : c \in E_\varepsilon\}. \tag{64}$$

and let $D^* = D_1 \cap D_2$. The set $D^*$ is closed and convex. To show the average of the sequence of the vector $m(f_t, \omega_t)$ approaches $D^*$, we need to verify the dual condition of approachability (Blackwell, 1953) is satisfied, i.e., for any $p \in \Delta(\Omega) \exists \mu \in \Delta(F_\varepsilon)$ such that $m(\mu, p) \in D^*$. We have

$$m(\mu, p) = (\sum_{\omega, f} \mu(f) \hat{u}_S(f), \sum_{\omega, f} \mu(f) p(\omega) c(f, \omega), p) \in D^*. \tag{65}$$

By definition $f^*(p)$ satisfies Condition (65) and the set $D^*$ is approachable. The set $D_1$ ensures that the sender can attain $\overline{Co}\ \hat{u}_S(f^*(p))$ while the set $D_2$ ensures that the $\varepsilon$-calibration condition is met.

Moreover, approachability theory provides convergence rates of the vector payoff (see Mannor and Stoltz (2010)). For every strategy of nature and for every $\delta > 0$ , with probability at least $1 - \delta$, we have

$$\sum_{f \in F_\varepsilon} \frac{|\mathbb{N}_T[f]|}{T} \|\overline{\omega}_T[f] - f\| \leq \varepsilon + \gamma \sqrt{|\Omega|} \sqrt{\frac{\log(1/\delta)}{\varepsilon^{(|\Omega|-1)} T}} \tag{66}$$

for some constant $\gamma > 0$.

(b) Now, we show that the function $\overline{Co}\ \hat{u}_S(f^*(p))$ is the highest continuous attainable function. We construct nature's strategy $\tau$ that prevents the sender from attaining $\tilde{r}$. Let $\{p_j\}_{j=1}^k$ denote the support points corresponding to the closed convex hull, i.e., $\overline{Co}\ \hat{u}_S(f^*(p)) = \sum_{j=1}^k \alpha_j \hat{u}_S(f^*(p_j))$. From Caratheodorys Theorem we can take $k$ to be equal to $|\Omega| + 2$. Consider a game with $T$

periods where nature plays in a sequence of $k$ blocks, where the size of block $l$ is $\alpha_l T$. In block $l$, nature plays i.i.d. according to $p_l$.

First, we show that for any i.i.d process with distribution $p$ the only forecasting strategy that passes the $\varepsilon-$calibration test sends the pure forecast $f^*(p)$ almost surely. In other words, a sender cannot send a forecast $f \neq f^*(p)$ (with positive probability) and pass the $\varepsilon-$ calibration test. From the calibration condition and the law of large numbers, we have

$$\limsup_{T\to\infty} \frac{|\mathbb{N}_T[f]|}{T} \|f-p\| > \varepsilon \quad \forall f \neq f^*(p) \tag{67}$$

Now, consider sender's play in any block $l$. We claim that for the sender to pass the overall calibration test, she has to pass the test in block $l$ and thus can only send the forecast $f^*(p_l)$ almost surely. Assume that is not the case and let $l$ denote the first block where the sender's strategy is not calibrated. Then, we have

$$\limsup_{T\to\infty} \sum_{f\in F_\varepsilon} \frac{|\mathbb{N}_{n_l T}[f]|}{n_l T} \|\overline{\omega}_{n_l T}[f] - f\| > \varepsilon$$

where, $n_l = \sum_{j=1}^{l} \alpha_j$. If this happens, then nature can play according to $\delta_\omega$ for the rest of the game, where $\delta_\omega \neq p_j$ for $j = 1, \ldots, l$. Even if the sender repeatedly forecasts $\delta_\omega$, the calibration cost will be positive. Even if the sender knew the sequence $p_1, \ldots, p_k$ in advance, she could not guarantee a higher payoff without failing the calibration test. Using this block strategy, nature restricts the sender to announce forecast $f^*(p_l)$ in each block $l$. Thus, under nature's strategy $\tau$, the sender's average payoff cannot be higher than $\overline{Co}\ \hat{u}_S(f^*(p)) = \sum_{j=1}^{k} \alpha_j \hat{u}_S(f^*(p_l))$, where $p$ is the empirical distribution of the states.

## A.7 Proof of Lemma 3

We use the notion of *opportunistic approachability*, which was developed in Bernstein et al. (2014). They devise algorithms that in addition to approaching the convex hull of the target set, seek to approach strict subsets thereof when the opponents play turns out to be restricted in an appropriate sense (either statistically or empirically).

**Definition 8.** *The play of the opponent is empirically Q-restricted with respect to a partition $\{\pi_m\}$, if there exists a convex subset $Q \subset \Delta(B)$ such that, for the given sample path*

$$\lim_{M\to\infty} \frac{1}{n_M} \sum_{m=1}^{M} \pi_m dist(\overline{\omega}_m, Q) = 0 \tag{68}$$

*where, $\pi_m$ denote the length of block $m$, $n_M = \sum_{m=1}^{M} \pi_m$ denotes the period at the end of the block $M$ and $\overline{\omega}_m$ denotes the empirical distribution of states in block $m$.*

Bernstein et al. (2014) (see Theorem 5) show that if nature's play is empirically $Q$-restricted with respect to a partition with subexponentially increasing blocks, then

$$\lim_{n\to\infty} d(\hat{r}_n, R^+(Q)) = 0 \quad \text{where, } R^+(Q) = \cap_{\varepsilon>0} Co\{\hat{u}_S(p) : d(p,Q) \leq \varepsilon\} \tag{69}$$

Here, $R^+(Q)$ denotes the closed convex image of the indirect utility restricted to the set $Q$. If $Q = \Delta(\Omega)$, then we obtain the same bounds as in the case of an adversarial environment.

Given $\eta > 0$, let $N_\eta(p)$ denote the $\eta-$neighborhood around $p$ where $p$ is the empirical distribution of states. We show that for a stationary ergodic process the play is empirically $N_\eta(p)$-restricted with respect to a partition with finite blocks $\pi$. This results directly from the ergodic theorem: given any $\eta > 0$, there exists $\pi_\omega \in \mathbb{N}$ such that for all $\tilde{\pi} \geq \pi_\omega$ we have

$$|\frac{1}{\tilde{\pi}} \sum_{t=1}^{\tilde{\pi}} \mathbf{1}_{\{\omega_t=\omega\}} - p(\omega)| \leq \eta \tag{70}$$

The empirical distribution $p$ exists and is constant for a stationary ergodic process. Choosing $\pi^* = \max_{\omega\in\Omega} \pi_\omega$ as the block size ensures that nature's play in each block is empirically restricted to $N_\eta(p)$. This implies that the average reward function $\hat{r}_n$ converges to $R^+(N_\eta(p))$. Thus, choosing an appropriate $\eta > 0$ such that $f^*(q) = f^*(p)$ for all $q \in N_\eta(p)$, the sender can attain the indirect utility function evaluated at the empirical distribution $p$. [19] Additionally, Bernstein et al. (2014) show that the results hold without knowing if nature's play is empirically restricted nor knowing the restriction set $Q$.

## A.8 Proof of Proposition 4

Fix $Q \in \mathcal{M}(C_\mu)$ and let $\sigma$ denote the signaling policy that results in $Q$. From Theorem (1), we know the forecasting strategy $\sigma_t(f_t = f \mid p_t = p) = \pi(f \mid p) \forall t \in \mathbb{N}$ passes the calibration test and results in the distribution of forecasts $F_{\mu,\sigma} = Q$ . In particular, we have

$$\limsup_{T\to\infty} \frac{\sum_{t=1}^{T} \mathbf{1}_{\{f_t=f,\omega_t=\omega\}} u_R(\omega, a_t)}{T} = f(\omega) \limsup_{T\to\infty} \frac{\sum_{t=1}^{T} \mathbf{1}_{\{f_t=f\}} u_R(\omega, a_t)}{T} \tag{71}$$

This follows as if $\lim_n a_n = a$, then $\limsup_n a_n b_n = a \limsup_n b_n$. Given the receiver minimizes regret, we have

[19] Note, this is valid as long as $f^*(p_0)$ corresponds to a unique forecast. This condition is met generically.

$$\limsup_{T\to\infty}\max_{a^*\in A}\frac{\sum_{t=1}^{T}\mathbf{1}_{\{f_t=f\}}[u_R(\omega_t,a^*)-u_R(\omega_t,\hat{a}(f))]}{T}\leq 0 \tag{72}$$

$$\Rightarrow \limsup_{T\to\infty}\max_{a^*\in A}\frac{\sum_{t=1}^{T}\mathbf{1}_{\{f_t=f\}}\mathbb{E}_f[u_R(\omega,a^*)-u_R(\omega,a_t)]}{T}\leq 0 \tag{73}$$

If the receiver plays actions $a_t \notin \hat{a}(f)$ on periods with non-negligible weight, it results in a positive regret. Hence, if the receiver uses any no-regret learning algorithm, it will play the recommended action $\hat{a}(f)$ almost surely . Thus, for a stationary ergodic process, the sender can ensure that she guarantee the payoff corresponding to for any distribution of forecasts $Q \in \mathcal{M}(C_\mu)$. In particular, this implies she can guarantee the solution of the persuasion problem $Per(C_\mu, \hat{u}_S)$.

# B Extension: MDP

In this section, we consider an environment where the receiver's action affects how the states evolves. We consider the stochastic process $\mu$ is a Markov Decision Process with transition matrix $T(\omega_{t+1} \mid \omega_t, a_t)$. The behavioral assumption of the receiver remains the same: he uses the calibration test to verify the claims of the sender. He plays according to the sender's forecast $a_t = \hat{a}(f_t)$ if she passes the test and punishes her if she fails. [20] While characterizing the set of feasible policies that pass the calibration test, it suffices to consider policies with memory 1, i.e., policies that maps the past forecast and state into a possible random forecast $\sigma : F\times\Omega \to \Delta F$. Any such policy induces a Markov chain over the set $F\times\Omega$, whose transition matrix $f_\sigma$ is given by

$$f_\sigma(f,\omega \mid \tilde{f},\tilde{\omega}) = T(\omega \mid \tilde{\omega},\tilde{f})\,\sigma(f \mid \tilde{\omega},\tilde{f}) \tag{74}$$

Let us denote by $\mathcal{F}$ the set of feasible distributions $\mu \in \Delta F$, i.e., $\mu$ corresponds to the marginal over $F$ of the invariant distribution of the Markov chain induced by a calibrated forecasting strategy $\sigma : F\times\Omega \to \Delta F$.

**Proposition 7.** *The set of feasible distributions $\mathcal{F}$ is a convex polytope. It is the marginal of $\eta \in \Delta(F\times\Omega\times F)$ over $F$ where $\eta$ satisfies:*

$$\sum_{\tilde{\omega},\tilde{f}}\eta(\tilde{f},\tilde{\omega},f)T(\omega \mid \tilde{\omega},\tilde{f}) = \eta(f,\omega) \tag{75}$$

$$\eta(f,\omega) = \eta(f)f(\omega) \tag{76}$$

[20] For simplicity, we assume the receiver plays according to the sender's forecasts and only performs the calibration test at the end of the game. This is because the punishment action also affects the state transition and can be difficult to keep track of.

Equation (75) states that $\eta$ is the invariant distribution for some forecasting strategy. Equation (76) states that the invariant distribution $\eta$ is calibrated.

*Proof.* Given a forecasting strategy $\sigma : F \times \Omega \to \Delta F$, we know the (time-averaged) distribution of outcomes (forecasts and states) converges almost surely. For simplicity, we assume the MDP is unichain. This implies that distribution of outcomes is the unique invariant distribution $\eta$ of the Markov chain induced by $\sigma$. [21] As $\eta$ is the invariant distribution of the induced Markov chain, we have

$$\eta(\tilde{f}, \tilde{\omega}) = \lim_{T\to\infty} \frac{\sum_{t=1}^{T} 1_{\{\omega_t=\tilde{\omega}, f_t=\tilde{f}\}}}{T} \tag{77}$$

$$\eta(f, \omega) = \sum_{\tilde{\omega}, \tilde{f}} \eta(\tilde{f}, \tilde{\omega}) f_\sigma(f, \omega \mid \tilde{f}, \tilde{\omega}) \tag{78}$$

Also, as we assume the forecasts are calibrated, we have

$$\eta(\omega \mid f) = \lim_{T\to\infty} \frac{\sum_{t=1}^{T} 1_{\{f_t=f, \omega_t=\omega\}}}{\sum_{t=1}^{T} 1_{\{f_t=f\}}} = f(\omega) \tag{79}$$

Equivalently, a distribution $\eta \in \Delta(F \times \Omega \times F)$ is feasible and invariant for some forecasting strategy $\sigma$ if

$$\sum_{f''} \eta(f, \omega, f'') = \sum_{\tilde{\omega}, \tilde{f}} \eta(\tilde{f}, \tilde{\omega}, f) T(\omega \mid \tilde{\omega}, \tilde{f}) \tag{80}$$

$$\eta(f, \omega) = \eta(f) f(\omega) \tag{81}$$

where, the policy (with memory 1) to induce the distribution $\eta$ is given by

$$\sigma(f \mid \tilde{\omega}, \tilde{f}) = \begin{cases} \frac{\eta(\tilde{f}, \tilde{\omega}, f)}{\eta(\tilde{f}, \tilde{\omega})} & \text{if } \eta(\tilde{f}, \tilde{\omega}) > 0 \\ \eta(f) & \text{if } \eta(\tilde{f}, \tilde{\omega}) = 0 \end{cases}$$

$\square$

The sender's maximization problem is given by the following linear program:

$$\max_{\mu \in \mathcal{F}} \sum_f \mu(f) \hat{u}_S(f) \tag{82}$$

[21] An MDP is unichain if every pure policy gives rise to a Markov chain with at most one recurrent class.

Thus, we can extend our model to situations where the receiver's action affects the distribution of future states. Furthermore, we can characterize the set of outcomes that result from calibrated strategies and solve for the optimal forecasting strategy.

# C Persuasion problem: Blackwell experiments

In this section, we define an equivalent way of describing the persuasion problem in terms of Blackwell experiments. This is the standard way of modeling the persuasion problem in the case of a perfectly informed sender(see Kamenica and Gentzkow (2011)). We extend it to the case, where the sender is imperfectly informed and can only use experiments less informative than a prior experiment.

**Definition 9.** *An experiment $F : \Omega \to \Delta T$ is a garbling of the experiment $E : \Omega \to \Delta S$ if there exists a row-stochastic matrix (or mapping) $G : S \to \Delta T$ such that $EG = F$.*

This defines a partial ordering in the set of Blackwell experiments. We say $F \precsim E$ when experiment $F$ is a garbling of experiment $E$. A prior belief $p_0 \in \Delta\Omega$ and an experiment $E : \Omega \to \Delta S$ give rise to a probability distribution $\mathcal{P}(p_0, E) := (\lambda_s, p_s)_{s \in S}$. Conversely, given any probability distribution $Q = (\mu, q)$ you can define a prior belief (or mean) $\mathcal{B}(Q)$ and an experiment $\mathcal{E}(Q)$.

$$\lambda_s = \sum_{\omega \in \Omega} p_0(\omega) E(s \mid \omega) \qquad\qquad \mathcal{B}(Q) = \sum_{i=1}^{n} \mu_i f_i$$

$$p_s(\omega) = \frac{p_0(\omega) E(s \mid \omega)}{\sum_{\omega \in \Omega} p_0(\omega) E(s \mid \omega)} \qquad\qquad \mathcal{E}(Q)(s_i \mid \omega) = \frac{\mu_i f_i(\omega)}{\sum_{i=1}^{n} \mu_i f_i(\omega)}$$

Lemma 5 shows the equivalence between the simple mean-preserving contraction of a probability distribution (Elton and Hill (1998),Whitmeyer and Whitmeyer (2021)) and the garbling of an experiment (Blackwell (1953)) with a fixed prior belief.

**Lemma 5.** *A probability distribution $Q$ is a mean-preserving contraction of $P$ if and only if $\mathcal{B}(P) = \mathcal{B}(Q)$ and the experiment $\mathcal{E}(Q)$ is a garbling of $\mathcal{E}(P)$.*

$$\begin{tikzcd}
P \arrow[r, dash, dashed] \arrow[d, "G"'] & (p_0, E) \arrow[d, "G"] \\
Q \arrow[r, dash, dashed] & (p_0, EG)
\end{tikzcd} \tag{83}$$

*Proof.* ($\Rightarrow$) Assume probability distribution $Q$ is a mean-preserving contraction of $P$. First, we show that the mean of the two distributions is equal.

$$
\begin{aligned}
\mathcal{B}(Q) &= \sum_{j=1}^{m} \mu_j f_j && (84)\\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i p_i G_{ij} && (85)\\
&= \sum_{i=1}^{n} \lambda_i p_i = \mathcal{B}(P) && (86)
\end{aligned}
$$

Now, we show the resulting experiment $F$ is a garbling of $E$.

$$
\begin{aligned}
F(t_j \mid \omega_n) &= \frac{\mu_j f_j(\omega_n)}{\sum_{j=1}^{m} \mu_j f_j(\omega_n)} && (87)\\
&= \frac{\sum_{i=1}^{n} \lambda_i p_i(\omega_n) G_{ij}}{\sum_{i=1}^{n} \lambda_i p_i(\omega_n)} && (88)\\
&= \sum_{i=1}^{n} E(s_i \mid \omega_n) G_{ij} && (89)
\end{aligned}
$$

($\Leftarrow$) Assume the Blackwell experiment $F$ is a garbling of $E$. We show that the resulting distribution $Q$ is a simple mean-preserving contraction of $P$.

$$
\begin{aligned}
\mu_j &= \sum_{k=1}^{s} p_0(\omega_k) F(t_j \mid \omega_k) && (90)\\
&= \sum_{k=1}^{s} p_0(\omega_k) \sum_{i=1}^{n} E(s_i \mid \omega_k) G_{ij} && (91)\\
&= \sum_{i=1}^{n} \lambda_i G_{ij} && (92)\\
\mu_j f_j &= p_0(\omega_k) F(t_j \mid \omega_k) && (93)\\
&= p_0(\omega_k) \sum_{i=1}^{n} E(s_i \mid \omega_k) G_{ij} && (94)\\
&= \sum_{i=1}^{n} \lambda_i p_i G_{ij} && (95)
\end{aligned}
$$

□

Thus, solving the persuasion problem $(P, \hat{u}_S)$ is equivalent to finding the optimal garbling of the experiment $\mathcal{E}(P)$ with prior $\mathcal{B}(P)$.

$$\max_{F \precsim \mathcal{E}(P)} \mathbb{E}_Q[u] \text{ where } Q = \mathcal{P}(\mathcal{B}(P), F) \tag{96}$$

For a perfectly informed sender, $\mathcal{E}(\mathcal{P})$ corresponds to the fully informative experiment and $\mathcal{B}(P)$ is simply the common prior belief of the players. Any distribution $Q$ such that $\mathcal{B}(P) = \sum_j \mu_j f_j$ (Bayes plausibility condition) can be implemented.

Note that the optimization problem (96) is in terms of mean-preserving spreads (splittings) while the equivalent problem (8) is in terms of mean-preserving contractions (garblings). Thus, we provide an alternate formulation of the persuasion problem in terms of garblings of an experiment. This formulation offers useful insights when the sender is not perfectly informed.