Efficiency in Games with Incomplete Information[*]

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Abstract

We study games with incomplete information and characterize when a feasible outcome is Pareto efficient. Outcomes with excessive randomization are inefficient: generically, the total number of action profiles across states must be strictly less than the sum of the number of players and the number of states. We consider three applications. A cheap talk outcome is efficient only if pure; with state-independent sender payoffs, it is efficient if and only if the sender's most preferred action is induced with certainty. In natural settings, Bayesian persuasion outcomes are inefficient across many priors. Finally, ranking-based allocation mechanisms are inefficient under mild conditions.

1 Introduction

A central question in economics is how individual incentives interact with social welfare. A natural benchmark for studying this relationship is Pareto efficiency—outcomes in which no individual can be made strictly better off without making another worse off. However, in many real-world settings, individuals make decisions under incomplete information, making it unclear when efficiency can be achieved. They face uncertainty about the underlying state of the world and about what others know. Each holds private information that could, in principle, improve everyone's welfare, yet individual incentives often stand in the way of efficiency. This tension raises two fundamental questions: *When is a feasible outcome Pareto efficient? When can strategic behavior lead to efficiency?*

A complementary motivation comes from information design (see Bergemann and Morris, 2019). In environments with incomplete information, a designer decides how to disclose information about the unknown state of the world. This information, together with the players' incentives, determines the resulting outcome—a mapping from states to distributions over action profiles—that fully determines the players' payoffs. Understanding when such outcomes are Pareto efficient is key to understanding the potential and the limits of information design for improving social welfare.

The contribution of this paper is to provide a necessary condition for efficiency in finite games with incomplete information. We derive a simple condition that depends only on the number of action profiles taken across states. We then apply this result to equilibrium outcomes in cheap talk, Bayesian persuasion, and an allocation problem without transfers, showing that incentive constraints often prevent efficiency.

Our main result shows that excessive randomization over action profiles leads to inefficiency. Generically, an outcome is ex-ante efficient only if the total number of action profiles played across all states is strictly less than the sum of the number of players and the number of states (Theorem 1). Notably, this condition does not depend on the prior, the action profiles used, or the weight of randomization. For instance, consider the case with two players. An outcome can be pure, quasi-pure—deterministic in all but one state with binary randomization in that state—or mixed, meaning neither pure nor quasi-pure. Generically, only pure or quasi-pure outcomes can be efficient.

This result builds on a link between ex-post and ex-ante efficiency that we establish using convex geometry (Proposition 1). An outcome is ex-post efficient in a given state if the state-contingent outcome maximizes some positive weighted sum of the players' payoffs in that state. By contrast, we show that an outcome is ex-ante efficient if and only if all state-contingent outcomes maximize a *common* positive weighted sum of the players' payoffs across *all* states. Intuitively, ex-ante efficiency therefore requires a single set of welfare weights to support all states simultaneously. Generically, when too many action profiles are played across states, no such common positive weight exists.

We next illustrate how our results apply in three canonical settings.

First, we consider the cheap talk model, where the sender cannot commit to a signaling policy. This lack of commitment imposes strict incentive constraints on the sender. We show that, generically, a cheap talk outcome is efficient only if it is pure (Proposition 2). Any stochastic equilibrium outcome requires the sender to be indifferent between multiple actions in a given state. However, the receiver generically prefers one of these actions, making the outcome inefficient. When the sender's payoff is state-independent, a cheap talk outcome is efficient if and only if the sender's most preferred action is chosen with certainty (Proposition 3). In this case, any equilibrium in which communication affects the receiver's

action is inefficient.

Second, we examine the Bayesian persuasion model in a natural environment with one safe action and several risky actions. There are as many states as actions, and in each state a distinct action is optimal for the receiver. Meanwhile, the sender prefers any risky action over the safe action. To increase the likelihood that some risky action is taken, the sender seeks to shift the receiver's beliefs toward states where a risky action is optimal. However, to satisfy the receiver's obedience constraint, the sender typically needs to randomize his recommendations. Building on this observation, we show that for a wide range of priors and preferences, the Bayesian persuasion outcome is necessarily mixed and therefore generically results in inefficiency (Proposition 4 and 5).

Finally, our insights extend beyond two-player games. We illustrate this by applying our results to the allocation problem studied by Niemeyer and Preusser (2024). The principal allocates a good among many agents with peer information. Under mild assumptions, the ranking-based mechanism assigns the good with positive probability to at least two eligible agents in each state. This unavoidable randomization violates our bound on the number of actions and generically implies inefficiency (Proposition 6).

Given its importance, a broad literature studies the conditions under which outcomes are Pareto efficient. Early work examined the efficiency of Nash equilibria (Case, 1974; Dubey, 1986; Cohen, 1998) and conditions for implementing efficient outcomes under incomplete information (Holmström and Myerson, 1983; Myerson, 1991, Ch. 10). Another line of research studies efficiency from a learning perspective, developing adaptive procedures that lead to efficient outcomes (Arieli and Babichenko, 2012; Pradelski and Young, 2012; Marden et al., 2014; Jindani, 2022), while Arieli et al. (2017) examine commitment procedures that can induce efficiency in extensive-form games. In a closely related paper, Rudov et al. (2025) analyze when a Nash equilibrium can be improved upon by a correlated equilibrium, using the convexity property of the set of correlated equilibria. Like us, they derive geometric conditions that restrict the extent of randomization. However, their analysis focuses on efficiency within the set of equilibria, whereas we study efficiency relative to the set of all feasible outcomes. Closer to our setting with incomplete information, they show that a Bayesian Nash equilibrium is generically an extreme point of the set of Bayes correlated equilibria if and only if it is pure, whereas in our framework even a pure Bayesian Nash equilibrium may fail to be efficient (Corollary 1).

Our work also contributes to the literature on strategic communication. In cheap talk (Crawford and Sobel, 1982), the sender's message is unverifiable, while in Bayesian persuasion (Kamenica and Gentzkow, 2011), the sender commits to how the message is generated.[1] The

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[1]There has been some work on selecting equilibria in cheap talk games that are Pareto dominant; see Crawford and Sobel (1982) and Antić and Persico (2023). We study efficiency relative to all feasible payoffs, not just equilibria.

closest work within this literature is Ichihashi (2019), who analyzes how restrictions on the sender's information in Bayesian persuasion affect the equilibrium outcome. In particular, he shows that the persuasion outcome is always efficient with binary actions, whereas our results demonstrate that efficiency may fail when there are more than two actions. Rayo and Segal (2010) study disclosure rules that maximize a weighted sum of sender and receiver welfare. Doval and Smolin (2024) analyze a welfare function over a heterogeneous population and characterize the Pareto frontier of outcomes achievable through information policies.

The remainder of the paper is organized as follows. Section 2 introduces the model and our notion of efficiency. Section 3 presents our main result. Section 4 applies the result to three economic environments: cheap talk, Bayesian persuasion, and an allocation problem without transfers. Finally, Section 5 offers concluding remarks. All omitted results and proofs are presented in the Appendix.

2   Model and Notion of Efficiency

We consider a finite game with incomplete information with $k \geq 2$ players. The state of the world is drawn from a finite set Ω according to a common prior $p \in \text{int}(\Delta\Omega)$. Each player $i \in \{1, \ldots, k\}$ has a finite set of actions A_i, and we write $A = \prod_{i=1}^{k} A_i$ for the set of pure action profiles. Each player i has private information represented by a type $t_i \in T_i$, where T_i is finite. Let $T = T_1 \times \ldots \times T_k$ denote the set of type profiles with distribution $\pi : \Omega \to \Delta T$. Each player i has a payoff function $u_i : \Omega \times A \to \mathbb{R}$, and we assume the collection of payoffs $(u_1, \ldots, u_k)$ is *generic*, meaning that the properties we establish hold for all bounded payoffs except on a subset of Lebesgue measure zero.

An *outcome* is a mapping $\mu : \Omega \to \Delta A$, which assigns to each state $\omega \in \Omega$ a probability distribution $\mu(\cdot \mid \omega)$ over action profiles. Equivalently, one can think of a mediator who observes the realized state and recommends an action (possibly at random) to the players, which they follow. Crucially, the recommendation need not be incentive compatible; we evaluate efficiency relative to the set of all feasible outcomes.

The *payoff vector* induced by outcome μ under prior p is

$$u(\mu) := \sum_{\omega \in \Omega} p(\omega) \sum_{a \in A} \mu(a \mid \omega) \left(u_1(\omega, a), \ldots, u_k(\omega, a) \right). \tag{1}$$

The set of feasible payoff vectors given prior p is defined as

$$F_p := \{ \overline{u} \in \mathbb{R}^k : \overline{u} = u(\mu), \text{ for some outcome } \mu : \Omega \to \Delta A \}. \tag{2}$$

The set F_p is a convex polytope whose extreme points correspond to pure or deterministic

recommendations.

Definition 1. *Given a compact convex set of feasible payoffs $F \subseteq \mathbb{R}^k$, a vector $\overline{u} \in F$ is efficient if there does not exist another feasible payoff vector $\overline{v} \in F$ such that $\overline{v} \geq \overline{u}$, with a strict inequality for at least one component.*

In our setting, a feasible payoff vector is efficient if and only if it maximizes a strictly positive weighted sum of the players' payoffs.[2] Moreover, we can identify the set of feasible payoffs with the set of outcomes. Thus, we can analyze efficiency in terms of outcomes instead of payoff vectors.

Definition 2. *An outcome $\mu : \Omega \to \Delta A$ is **efficient** if $u(\mu)$ is efficient with respect to F_p.*

Given an outcome μ, we refer to $\mu(\omega) \in \Delta A$ and $u(\mu \mid \omega) \in \mathbb{R}^k$ as the state-contingent outcome and state-contingent payoff vector, respectively. Let

$$F_\omega := \text{Co}\big\{(u_1(\omega, a), \ldots, u_k(\omega, a)) : a \in A\big\} \tag{3}$$

denote the feasible payoff vectors in state ω.[3] Any outcome can be decomposed in terms of its state-contingent outcomes. This follows as the set of the feasible payoff vectors given a prior can be written as a unique Minkowski sum of the set of the feasible payoff vectors for each state:[4]

$$F_p = \sum_{\omega \in \Omega} p(\omega) F_\omega. \tag{4}$$

Our notion of efficiency is based on the ex-ante perspective, that is, before the state is realized. However, efficiency can also be evaluated ex-post, once the state is realized. An outcome μ is ex-post efficient in state ω if its induced outcome $\mu(\omega)$ is efficient with respect to the set of feasible payoff vectors in that state.[5]

Definition 3. *An outcome $\mu : \Omega \to \Delta A$ is **ex-post efficient** in state ω if $u(\mu \mid \omega)$ is efficient with respect to F_ω.*

Efficiency implies ex-post efficiency but the converse does not hold. We establish a geometric relationship between ex-ante and ex-post efficiency. An outcome is ex-post efficient

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[2]This equivalence fails for general set of feasible payoff vectors but can be approximated by "near" weighted sum of the players' payoffs, as shown in Che et al. (2024). In our case, the equivalence holds because the set of feasible payoffs is a convex polytope.

[3]where $\text{Co}(A)$ stands for the convex hull of set A.

[4]The Minkowski sum of two sets A and B is given by $A + B = \{a + b \mid a \in A, b \in B\}$.

[5]There is not a unique way to define efficiency in environments with incomplete information. In particular, Holmström and Myerson (1983) propose several notions of efficiency based on timing and feasibility. Our notion corresponds to what they term *ex-ante classically efficient*, while ex-post efficiency corresponds to their *ex-post classically efficient*.

in a given state if it maximizes some strictly positive weighted sum of the players' payoffs in that state. The weights, however, may differ across states. In contrast, ex-ante efficiency requires that the outcome maximize a *common* strictly positive weighted sum of the players' payoffs across *all* states.

Proposition 1. *An outcome μ is efficient if and only if there exists a positive weight $n \in \mathbb{R}_{++}^k$ such that, for all $\omega \in \Omega$, the payoff $u(\mu \mid \omega)$ maximizes the common positive linear functional $n^T x$ over all $x \in F_\omega$.*

We illustrate the notions of ex-ante and ex-post efficiency in the following example.

Example 1. *Consider a game with two players: the sender and the receiver. The state space is $\Omega = \{\omega_0, \omega_1\}$ and the receiver's action space is $A = \{a_0, a_1, a_2, a_3, a_4\}$. The sender's and the receiver's payoffs are given by the following matrix:*

	a_0	a_1	a_2	a_3	a_4
ω_0	$(2,9)$	$(10,8)$	$(0,6.4)$	$(3,4)$	$(1,0)$
ω_1	$(2,0)$	$(10,4)$	$(0,6.4)$	$(3,8)$	$(1,9)$

We analyze the efficiency of three pairs of outcomes and priors $p = \mathbb{P}(\omega_1)$:

(a) *$p = 0.10$: an outcome where actions a_0 and a_1 are taken in ω_0 and action a_1 is taken in ω_1;*

(b) *$p = 0.30$: an outcome where action a_1 is taken with certainty in both states;*

(c) *$p = 0.70$: an outcome where action a_1 is taken in ω_0, and actions a_1 and a_4 are taken in ω_1.*

The feasible payoff vectors for states ω_0, ω_1, and the prior p are represented by the red, blue, and orange regions in Figure 1, respectively. We find that for (a), the outcome is ex-post efficient in both states but is not supported by a common positive normal. The unique normal n_{ω_0} does not belong to the normal cone spanned by $\overline{n}_{\omega_1}$ and $\underline{n}_{\omega_1}$, i.e., $n_{\omega_0} \notin \text{cone}\{\overline{n}_{\omega_1}, \underline{n}_{\omega_1}\}$. For (b), the outcome is ex-post efficient in both states and supported by a common positive normal, while for (c), the outcome is not ex-post efficient in state ω_1. Overall, the outcome is Pareto efficient in (b) but not in (a) or (c) (see the respective polytopes F_p in Figure 1). As we will later see, these correspond to the equilibrium outcomes of Bayesian persuasion for the respective priors.
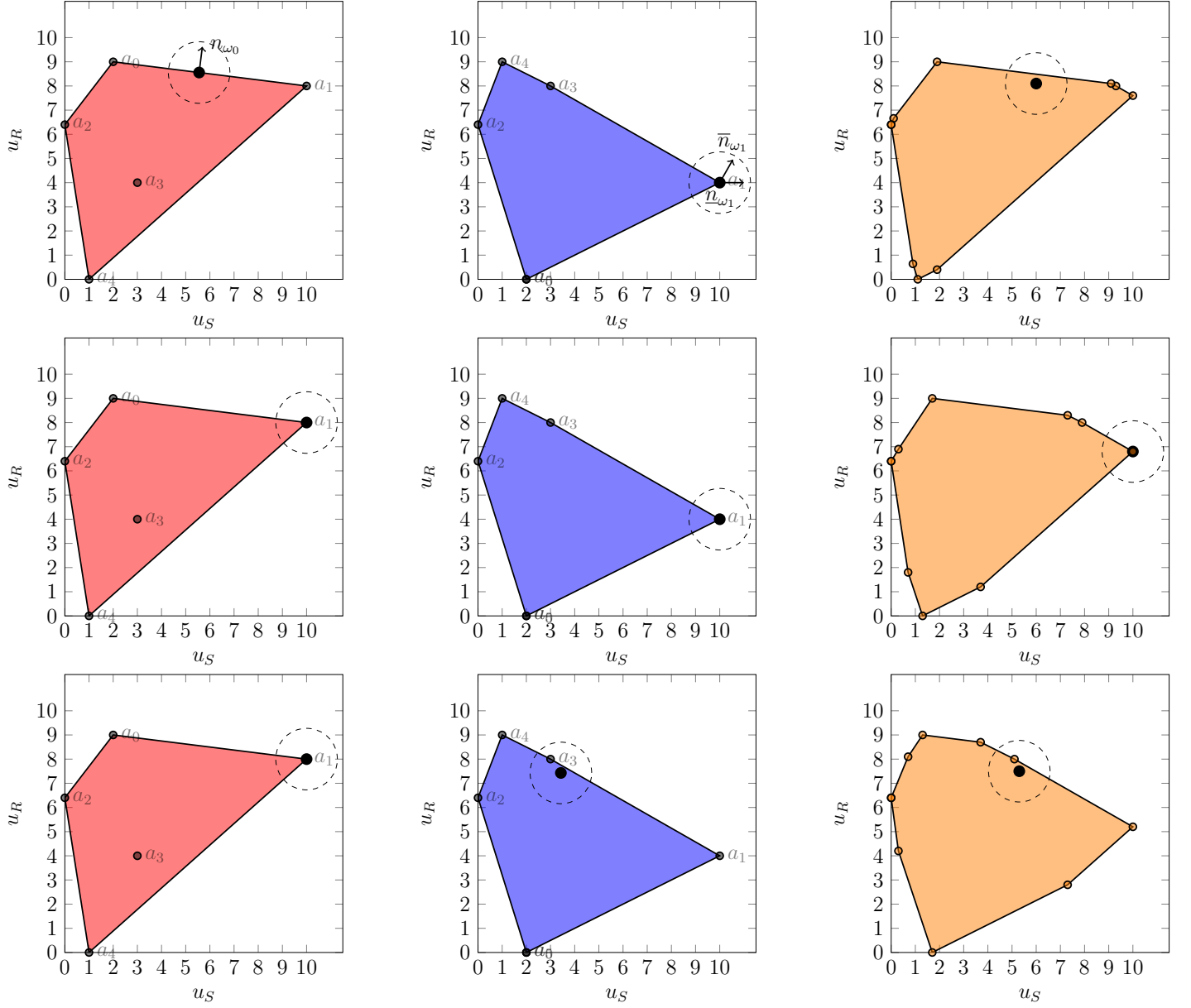
Figure 1: Each horizontal panel represents a different prior: (a) $p = 0.10$, (b) $p = 0.30$, and (c) $p = 0.70$. Within each panel, the red region represents F_{ω_0}, the blue region represents F_{ω_1}, and the orange region represents F_p. The black circled node denotes the outcome.

3 Necessary Condition for Efficiency

Our main result provides a necessary condition for efficiency based on the number of action profiles taken across states. Given an outcome μ and a state ω let $|\mu(\omega)|$ denote the size of the support of $\mu(\omega) \subseteq A$, namely the number of action profiles that are taken with positive probability in that state.

Generically, efficiency requires that the total number of action profiles taken across all states be strictly less than the sum of the number of players and states; excessive randomization therefore leads to inefficiency. Efficiency implies that the state-contingent outcomes must maximize a *common* positive weighted sum of the players' payoffs across all states. When too many action profiles are taken across states, no such weight can exist generically.

Theorem 1. *Generically, an outcome $\mu : \Omega \to \Delta A$ is efficient only if*

$$\sum_{\omega \in \Omega} |\mu(\omega)| < k + |\Omega|. \tag{5}$$

Proof. An outcome μ is efficient if and only if it is ex-post efficient in every state and supported by the same vector of strictly positive weights on players' payoffs across all states. Let $a_1, \ldots, a_{|\mu(\omega)|}$ denote the pure action profiles taken under outcome μ in state ω.

For μ to be ex-post efficient in state ω, there must exist a strictly positive weight vector $n \in \mathbb{R}^k_{++}$ such that the induced payoff vector $u(\mu \mid \omega)$ maximizes the weighted sum $n \cdot u(\omega, a)$ among all feasible action profiles in that state. When the outcome mixes several action profiles $a_1, \ldots, a_{|\mu(\omega)|}$, ex-post efficiency requires that all of them yield the same weighted sum under n:

$$n \cdot \big(u(\omega, a_i) - u(\omega, a_1)\big) = 0 \quad \text{for } i = 2, \ldots, |\mu(\omega)|. \tag{6}$$

Generically, each additional action profile used introduces one additional independent linear constraint on the set of admissible weights. Let $N_\omega(\mu)$ denote the outer normal cone of the outcome μ in state ω, that is, the set of strictly positive vectors $n \in \mathbb{R}^k_{++}$ for which $u(\mu \mid \omega)$ maximizes $n^T u(\omega, a)$ over F_ω. Thus, generically, this set has dimension $k - (|\mu(\omega)| - 1)$.

Efficiency requires that there exists a vector n that belongs to the outer normal cones of all states. Generically, the dimension of the intersection of these cones is

$$\dim\Big(\bigcap_{\omega \in \Omega} N_\omega(\mu) \Big) = k - \sum_{\omega \in \Omega} (|\mu(\omega)| - 1) = k + |\Omega| - \sum_{\omega \in \Omega} |\mu(\omega)|. \tag{7}$$

If $\sum_{\omega \in \Omega} |\mu(\omega)| \geq k + |\Omega|$, then this dimension is non-positive, implying that no common strictly positive vector n can support all state-contingent outcomes. $\qquad \square$

Remark 1. *The aggregate bound implies a state-wise bound: $|\mu(\omega)| \leq k$ for all $\omega \in \Omega$. Generically, an outcome is ex-post efficient in a state only if the number of action profiles taken in that state is weakly less than the number of players.*

Remark 2. *For two-player games, outcomes can be pure (a deterministic action profile in every state), quasi-pure (a deterministic action profile in all but one state, where two action profiles are taken), or mixed (any other case). Generically, an outcome is efficient only if it is pure or quasi-pure.*

The necessary condition in Theorem 1 is far from sufficient. For instance, in Example 1, the outcome is quasi-pure (satisfying the bound) in cases (a) and (c), yet it is still inefficient. Even if all state-contingent outcomes are pure and ex-post efficient, the overall outcome can still be inefficient. Efficiency is guaranteed when a player's most preferred outcome that is also ex-post efficient in every state is induced. In Example 1, this occurs when the sender fully reveals the state and the receiver takes his optimal action in each state, or when the sender's preferred action is chosen in every state, as in case (b).

A complete characterization of efficiency is provided in Proposition 7 in the Appendix. It offers a simple way to determine efficiency directly from the payoff functions. Although we take an ex-ante view, efficiency can be assessed by examining payoff changes from deviations in each state. An outcome is efficient if and only if no convex combination of these deviations across states makes all players weakly better off and at least one strictly better off. In short, no Pareto improvement is possible. Like Theorem 1, this condition is independent of both the interior prior and the weights of randomization.

Type-contingent decision rules Following Bergemann and Morris (2019), outcomes can be viewed as the result of type-contingent decision rules $\sigma : T \times \Omega \rightarrow \Delta A$, where action profiles depend on both the type profile and the state. Since payoffs depend only on states and the joint actions, the type profile and its distribution do not change the set of feasible payoff vectors. Type profiles become relevant only when incentive compatibility constraints are imposed.

A *Bayes correlated equilibrium (BCE)* is a type-contingent decision rule that satisfies the players' obedience constraints. A *Bayesian Nash equilibrium (BNE)* is a particular BCE in which each player randomizes over actions as a function of his type ($\sigma_i : T_i \rightarrow \Delta A_i$). If the distribution of type profiles $\pi : \Omega \rightarrow \Delta T$ has full support (assigns positive probability to every type at every state), then even pure decision rules can induce mixed action profiles across states.[6]

Corollary 1. *Assume $\pi : \Omega \rightarrow \Delta T$ has full support. If either*

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[6]In a generic game, both the prior p and the type distribution π have full support.

(i) $\sigma : T \times \Omega \rightarrow A$ and there are at least k states with types $t, t' \in T$ such that $\sigma(t, \omega) \neq \sigma(t', \omega)$, or

(ii) $\sigma_i : T_i \rightarrow A_i$ for all $i = 1, \ldots, k$ and at least q players use two or more actions across their types where $|\Omega| (2^q - 1) \geq k$,

then generically the induced outcome μ is inefficient.

The corollary shows that BCE or BNE outcomes may even fail to be efficient when players follow pure decision rules. Differences in types impose distinct incentive constraints, which can generate excessive randomization and lead to inefficiency. Rudov et al. (2025) show that within the set of BCEs, a BNE is extreme if and only if it is pure. In contrast, we show that efficiency may even fail for such pure rules when outcomes are evaluated relative to all feasible payoffs rather than only the set of equilibrium payoffs.

4 Applications

We now illustrate how the general framework and result apply to three canonical economic environments: cheap talk, Bayesian persuasion, and an allocation problem without transfers.

4.1 Cheap talk

We now consider the cheap talk model introduced in Crawford and Sobel (1982), which studies strategic communication between an informed sender and an uninformed receiver.

The timing is as follows. First, the state $\omega \in \Omega$ is drawn according to the common prior p. The sender then observes the realized state and chooses a message $m \in M$ to send. After observing the message, the receiver chooses an action $a \in A$. This results in payoffs $u_S(\omega, a)$ and $u_R(\omega, a)$ for the sender and receiver, respectively.

The sender's strategy is given by $\sigma : \Omega \rightarrow \Delta M$, where M is a finite set of messages. We assume there are at least as many messages as actions or states, i.e., $|M| \geq \max\{|A|, |\Omega|\}$. The receiver's strategy is given by $\tau : M \rightarrow \Delta A$. A strategy profile (σ, τ) induces an outcome $\mu : \Omega \rightarrow \Delta A$, specifying a probability distribution over actions for each state, where

$$\mu(a \mid \omega) = \sum_{m \in M} \sigma(m \mid \omega)\tau(a \mid m) \text{ for all } \omega \in \Omega, a \in A. \tag{8}$$

A strategy profile (σ, τ) is a *Perfect Bayesian Equilibrium* (PBE) if the sender chooses messages that maximize his expected payoff in every state given the receiver's strategy, the receiver chooses actions that maximize his expected payoff given his posterior belief after each message, and beliefs are updated according to Bayes' rule wherever possible.

Unlike Bayesian persuasion, cheap talk exhibits a multiplicity of equilibria. In particular, a *babbling equilibrium* always exists, in which no communication takes place and the receiver plays his best response to the prior.[7]

First, we show that, generically, a cheap talk outcome can be efficient only if it is pure. This is a stronger result than the one stated in Theorem 1, as any stochastic outcome, including quasi-pure, is inefficient.

Proposition 2. *Generically, a cheap talk outcome $\mu : \Omega \to \Delta A$ is efficient only if it is pure.*

For any stochastic cheap talk outcome, the sender's equilibrium condition requires him to be indifferent between the actions that are played in a given state. For the outcome to be ex-post efficient in that state, the receiver must also be indifferent between these actions. Hence, efficiency of any stochastic cheap talk outcome necessarily relies on knife-edge indifferences for both players. These break under generic payoffs, implying that only pure outcomes can be efficient.

Recently, Kamenica and Lin (2025) show that, generically, if the sender's preferred cheap talk outcome is necessarily stochastic, then he values commitment. Hence, this cheap talk outcome must be pure not only for efficiency but also for commitment to have no value.

Next, we consider the case where the sender's payoff is state-independent. Lipnowski and Ravid (2020) characterize the sender's preferred equilibrium using a belief-based approach. Define the receiver's best responses given his belief $p \in \Delta\Omega$ as $A^*(p) := \arg\max_{a \in A} \mathbb{E}_p[u_R(\omega, a)]$. The sender's value function

$$V(p) := \max_{a \in A^*(p)} \mathbb{E}_p[u_S(\omega, a)], \tag{9}$$

represents the sender's expected payoff when the receiver, with belief p, selects the sender's preferred best response. They show that the sender's preferred equilibrium corresponds to the quasiconcave envelope of the value function, evaluated at the prior, which we denote by Quasicav V. We graphically illustrate this equilibrium for Example 1 in Figure 2.

Let $A^* := \bigcup_{p \in \Delta\Omega} A^*(p)$ denote the set of actions that are a best response for the receiver under some belief, and let $a^* := \arg\max_{a \in A^*} u_S(a)$ denote the sender's most preferred action in A^*. To rule out non-generic cases, we assume that the sender's payoffs are distinct across actions, i.e., $u_S(a) \neq u_S(b)$ whenever $a \neq b$.

We now show that when the sender's payoff is state-independent, a cheap talk outcome is efficient if and only if the sender's most preferred action is induced with certainty.

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[7]Note that in the case of cheap talk, we cannot restrict attention to direct signaling policies, as some equilibria may require randomizing between actions for a given message.
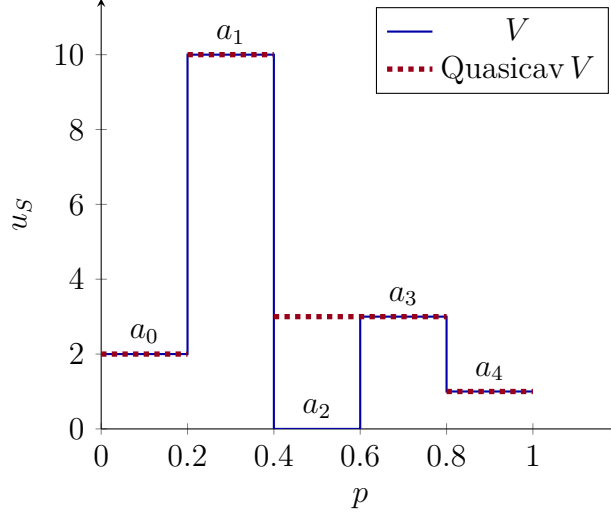
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Figure 2: Cheap talk: The value function (blue solid) and its quasiconcave envelope (red dotted). Vertical lines show the jumps at cutoff beliefs.

Proposition 3. *For a sender with state-independent payoff, a cheap talk outcome is efficient if and only if the sender's most preferred action a^* is induced with certainty.*

Any informative equilibrium requires the sender to be indifferent between the messages he sends. For this to hold, the receiver must mix between at least two actions in response to one of the messages. But then there exists a state in which both players strictly prefer the sender's most preferred action. Hence, any informative equilibrium fails to be efficient, as a deviation to this pure action would make both strictly better off in that state. A babbling equilibrium is efficient if and only if the sender's most preferred action is induced. If a^* is not induced, we can again find a state in which both players strictly prefer this action, implying inefficiency.

Consider the informative equilibrium in Example 1 given prior $p = 0.5$. The equilibrium induces two posterior beliefs, $q = 0.4$ and $q = 0.6$ (see Figure 2). At belief $q = 0.4$, the receiver mixes between actions a_1 and a_2 to make the sender indifferent. However, both players prefer $a^* = a_1$ to a_2 in state ω_0, so the cheap talk outcome is not ex-post efficient in that state. Similarly, consider the babbling equilibrium given prior $p = 0.7$. Again, both players prefer $a^* = a_1$ to the induced action a_3 in state ω_0, implying inefficiency. In contrast, the babbling equilibrium given prior $p = 0.3$ is efficient, as sender's most preferred action $a^* = a_1$ is induced.

Therefore, with state-independent sender payoffs, only the babbling equilibrium in which the receiver plays the sender's preferred action a^* is efficient. The same action would be taken even without communication, and the resulting outcome would be efficient. Thus, any non-trivial equilibrium in which communication influences the outcome necessarily leads to

inefficiency.

4.2   Bayesian persuasion

We now turn to Bayesian persuasion (Kamenica and Gentzkow, 2011), another model of strategic communication between a sender and a receiver. Unlike in cheap talk, the sender can commit to how messages are generated before the state is realized.

The timing is as follows. First, the sender chooses a message space M and a signaling policy $\sigma : \Omega \to \Delta M$ before the state is realized. Then, the state $\omega \in \Omega$ is drawn according to the common prior p, and the message $m \in M$ is sent according to the sender's policy. Upon observing the message, the receiver chooses an action $a \in A$. This results in payoffs $u_S(\omega, a)$ and $u_R(\omega, a)$ for the sender and receiver, respectively.

The sender's objective is to choose a signaling policy that maximizes his ex-ante expected payoff by influencing the receiver's action. As is standard, ties are broken in favor of the sender. Without loss, we restrict attention to *direct signaling policies* $\mu : \Omega \to \Delta A$, where messages correspond to action recommendations. Given prior p, let μ_p^* denote the equilibrium outcome of Bayesian persuasion.[8]

Definition 4. *The **Bayesian persuasion (BP) outcome** $\mu_p^* : \Omega \to \Delta A$ solves*

$$\max_{\mu:\Omega\to\Delta A} \sum_{\omega\in\Omega} p(\omega) \sum_{a\in A} \mu(a \mid \omega) u_S(\omega, a) \tag{10}$$

subject to

$$\sum_{\omega\in\Omega} p(\omega)\mu(a \mid \omega)\big(u_R(\omega, a) - u_R(\omega, b)\big) \geq 0 \quad \forall a, b \in A. \tag{11}$$

Equation (11) corresponds to the receiver's *obedience* condition: given a recommended action a, the receiver prefers following it to deviating to any other action b.

In Kamenica and Gentzkow (2011), the BP outcome is characterized using the concavification approach (Aumann and Maschler, 1995). Let Cav $V : \Delta\Omega \to \mathbb{R}$ denote the concave envelope of the value function V. The sender's expected payoff in the BP outcome is given by the evaluation of the concave envelope at the prior: $\mathbb{E}_{\mu_p^*}[u_S(\omega, a)] = \text{Cav } V(p)$.

The value function and its concave envelope for Example 1 are depicted in Figure 3. The outcomes in Example 1 correspond precisely to the BP outcomes for the respective priors: (a) $p = 0.10$, (b) $p = 0.30$, and (c) $p = 0.70$. As seen in Figure 1, the BP outcome is efficient for case (b), not for cases (a) or (c). Moreover, within each convex region of the concave envelope, the BP outcome μ_p^* has the same support—and therefore the same

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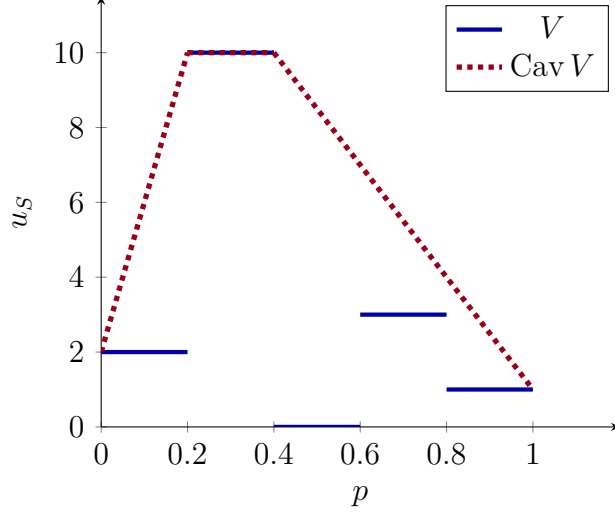[8]Generically, the equilibrium outcome in Bayesian persuasion is unique.

Figure 3: Bayesian persuasion: The value function (blue solid) and its concave envelope (red dotted).

number of actions—across all priors in that region. Hence, the BP outcome μ_p^* is efficient for $p \in [0.2, 0.4]$ and inefficient for $p \in (0, 0.2) \cup (0.4, 1)$.

Threshold environment with safe and risky actions   We study a natural environment in which the Bayesian persuasion outcome fails our necessary condition for efficiency across a wide range of priors and preferences.

The setting features one safe action and several risky actions, with as many states as actions. Each action corresponds to a particular state, and the receiver wants to take the matching action when he believes that state is sufficiently likely. A mismatched risky action pays less than the safe action, so the receiver defaults to the safe option when he is unsure about the state. By contrast, the sender prefers any risky action to the safe one, which is his least-preferred outcome. A natural example is a seller–buyer interaction in which the buyer (receiver) purchases a product only when sufficiently convinced it is the best choice, and otherwise prefers not to buy any product. Meanwhile, the seller (sender) prefers that some product be bought rather than none.

The action set is $A = \{a_0, a_1, \ldots, a_n\}$, where a_0 is the safe action and a_i for $i = 1, \ldots, n$ are risky actions. The state space is $\Omega = \{\omega_0, \omega_1, \ldots, \omega_n\}$, where ω_i is the state in which action a_i is optimal for the receiver for all i. The sender's payoff is state-independent: he gets $u_S(a_i) > 0$ for any risky action a_i with $i = 1, \ldots, n$, and $u_S(a_0) = 0$ for the safe action. We assume the prior $p \in \text{int}(\Delta\Omega)$ lies in the region where the sender's least preferred action a_0 is optimal. The receiver prefers the action that matches the state and is worse off when taking a mismatched risky action. We capture this by describing, for each action, the set of beliefs under which it is optimal.

Let $C_i \subseteq \Delta\Omega$ denote the convex subset of the beliefs where the receiver's optimal action is a_i.

$$C_i := \{p \in \Delta\Omega : \mathbb{E}_p[u_R(\omega, a_i)] \geq \mathbb{E}_p[u_R(\omega, a_j)] \quad \forall a_j \in A\}. \tag{12}$$

The sets $\mathcal{P} = \{C_0, C_1, \ldots, C_n\}$ form a partition of the belief space $\Delta\Omega$. We assume:

1. For each $i = 0, \ldots, n$, the state $\omega_i \in C_i$ and there exists an open neighborhood $N_{\omega_i} \subset C_i$ containing ω_i.

2. For any distinct indices $i \neq j$ with $i, j \neq 0$, we have $C_i \cap C_j = \emptyset$.

Condition 1 ensures that it is optimal to take action a_i when the receiver is sufficiently confident that the state is ω_i. Condition 2 ensures that when the receiver is unsure which risky action is optimal, he prefers the safe action.

To characterize the BP outcome μ_p^*, we follow Lipnowski and Mathevet (2017) and restrict the feasible posteriors to the finite set of *outer points* $\mathrm{Out}(\mathcal{P})$, defined as

$$\mathrm{Out}(\mathcal{P}) := \{p \in \Delta\Omega : p \in \mathrm{ext}(C_i) \text{ whenever } p \in C_i \in \mathcal{P}\}, \tag{13}$$

$$= \left(\bigcup_{i=0}^{n} \omega_i\right) \cup \left(\bigcup_{i=1}^{n} \bigcup_{j \neq i} o_{ij}\right), \tag{14}$$

where o_{ij} is the unique extreme point of the convex set C_i that lies on the line segment joining vertices ω_i and ω_j.[9] Lipnowski and Mathevet (2017) show that the BP outcome can always be supported using the set of posterior beliefs $Out(\mathcal{P})$. Furthermore, they show that only an affinely independent subset of posteriors is needed, so one can restrict attention to signaling policies that induce at most $n + 1$ beliefs.
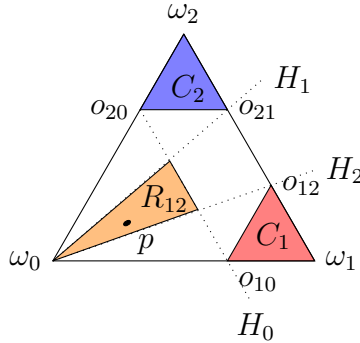

Figure 4: The set R_{12} (orange region) for $n = 2$.

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[9]As we break ties in favor of the sender, if the belief o_{ij} is induced, the receiver takes action a_i, since $u_S(a_i) > u_S(a_0)$.

Suppose $n = 2$ and the prior p is as shown in Figure 4. Any feasible signaling policy whose induced posteriors lie on the outer points—while satisfying Bayes plausibility—induces all three actions. In the BP outcome, the posterior ω_0 is induced while ω_1 and ω_2 are not, since inducing these posteriors would lead the receiver to choose the safe action a_0 with high probability. To increase the likelihood of a risky action (a_1 or a_2), the sender therefore assigns positive weight to non-degenerate beliefs such as o_{10}, o_{12}, o_{20}, or o_{21}. Hence, the BP outcome is necessarily mixed—either all three actions occur in ω_0, or both a_1 and a_2 are played in ω_1 and ω_2—and thus generically inefficient by Theorem 1.

This intuition extends naturally to settings with $n > 2$ risky actions. For any two distinct risky actions, there exists a convex region of priors where the BP outcome mixes between taking some risky action and the safe action. The union of these regions, denoted R_*, forms a full-dimensional set of priors for which the outcome is mixed and therefore generically inefficient.

Proposition 4. *For $n \geq 2$, there exists a set $R_* \subseteq C_0$, with $\dim(R_*) = \dim(\Delta\Omega)$, such that, generically, the BP outcome is inefficient for all $p \in \text{int}(R_*)$.*

The previous proposition established inefficiency for a range of priors under fixed preferences. We now fix the prior and vary the receiver's preferences. Specifically, we consider partitions induced by threshold-based decision rules and show that, for sufficiently high thresholds, the BP outcome must be mixed.

Consider a receiver who takes risky action a_i only when he is sufficiently confident about the state, that is, when his belief over the state ω_i is greater than some threshold $T > 0.5$.[10] Now, the convex subset C_i, where the receiver's optimal action is a_i, is given by:

$$C_i = \{p \in \Delta\Omega \mid p(\omega_i) \geq T\} \quad \text{for all } i = 1, \ldots, n, \tag{15}$$

$$C_0 = \Delta\Omega \setminus \bigcup_{i=1}^{n} C_i. \tag{16}$$

Let $\mathcal{P}_T = \{C_0, \ldots, C_n\}$ denote the partition induced by threshold T.

To build intuition, consider $n = 2$ and a prior $p \in \text{int}(\Delta\Omega)$. As T increases, the regions C_1 and C_2 shrink, and the prior eventually lies in a region where any feasible signaling policy must induce all three actions a_0, a_1 and a_2. By the previous arguments, for any such prior, the Bayesian persuasion outcome must be mixed.

The same logic extends to any $n > 2$: for each prior, there exists a bound $T_p < 1$ such that whenever $T > T_p$, the BP outcome becomes mixed and therefore inefficient.

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[10] A similar class of preferences is considered in Aybas and Turkel (2024).

Proposition 5. *For $n \geq 2$ and any prior $p \in \text{int}(\Delta\Omega)$, there exists a threshold $T_p < 1$ such that, generically, the BP outcome is inefficient with respect to the partition $\mathcal{P}_T$ for all $T > T_p$.*

Together, these results show that Bayesian persuasion outcomes are inefficient across a wide range of priors and preferences.

4.3   Allocation problem without transfers

We now turn to a mechanism design problem without transfers, studied by Niemeyer and Preusser (2024). A principal needs to allocate a good among agents with correlated types. This setting fits naturally within our framework of finite games with incomplete information.

A single indivisible good must be allocated among k players: one principal and $k-1$ agents, where $k \geq 3$. The state is $\omega = (\omega_1, \dots, \omega_{k-1}) \in \Omega = \prod_{i=1}^{k-1} \Omega_i$, and agent i's private type is $\omega_i \in \Omega_i$ with $|\Omega_i| \geq 2$. If agent i receives the good, his payoff is 1, and 0 otherwise. The principal's payoff from allocating to agent i in state ω is $u_i(\omega) \in [-1, 1]$, while keeping the good yields 0. Let $p \in \text{int}(\Delta\Omega)$ denote the common prior over states, and $A = \{a_0, a_1, \dots, a_{k-1}\}$ the set of actions, where a_0 denotes the principal keeping the good and a_i denotes allocating it to agent i.

A (direct) mechanism is a mapping $\mu : \Omega \to \Delta A$, where $\mu(a \mid \omega)$ is the probability that action a is taken when the state is ω. *Dominant-strategy incentive compatibility (DIC)* requires that truthful reporting maximizes each agent's allocation probability:

$$\mu(a_i \mid \omega_i, \omega_{-i}) \geq \mu(a_i \mid \omega_i', \omega_{-i}), \quad \forall \omega_i, \omega_i' \in \Omega_i, \ \forall \omega_{-i} \in \Omega_{-i}, \forall i \in \{1, \dots, k-1\}. \tag{17}$$

The principal's goal is to find the *optimal mechanism*: the DIC mechanism that maximizes his expected payoff. Because the objective is linear and the set of DIC mechanisms is convex, any optimal mechanism is a convex combination of extreme DIC mechanisms. When the state space is sufficiently rich, Niemeyer and Preusser (2024) show that almost all extreme mechanisms are stochastic, making randomization a natural property of optimal mechanisms in this environment. Although not the principal's objective, examining the welfare properties of the outcome helps determine when optimality is at odds with efficiency.

Ranking-based mechanism   Because optimal mechanisms lack closed-form descriptions, Niemeyer and Preusser (2024) introduce *ranking-based mechanisms*, which are simple yet approximately optimal.

These mechanisms work in two steps. First, fix a threshold $t \in (0, 1]$ and let $r_i(\omega)$ denote agent i's rank at state ω according to *peer value* $u_i(\omega_{-i}) := \mathbb{E}[u_i \mid \omega_{-i}]$. The principal randomly selects one of these eligible agents ranked among the top $t(k-1)$ agents. Second,

the principal allocates the good to the selected agent if and only if his *robust rank* $r_i^*(\omega_{-i}) = \max_{\omega_i \in \Omega_i} r_i(\omega_i, \omega_{-i})$ also lies within the top $t(k-1)$ and his peer value is non-negative. If the selected agent fails this test, the good is kept by the principal. Thus, each eligible agent is assigned the good with a probability of at most $1/(t(k-1))$, and any leftover probability mass is kept by the principal.

A crucial factor for the performance of ranking-based mechanisms is the impact that any individual agent's report has on his own rank, which they refer to as the *informational size*. For each state ω, the informational size is defined as

$$\delta(\omega) := \max_{i \in \{1,\dots,k-1\}} \max_{\omega_i' \in \Omega_i} \left| r_i(\omega_i, \omega_{-i}) - r_i(\omega_i', \omega_{-i}) \right|. \tag{18}$$

When informational size is large, many agents can manipulate their position above the threshold, and the mechanism may erroneously withhold the good. When it is small, no single agent can substantially affect rankings. This assumption is natural in large environments with many agents, where each individual's type has only limited influence.

We impose a mild regularity condition similar to Niemeyer and Preusser (2024). The threshold t is chosen large enough so that, in every state, there are at least two eligible agents and the principal allocates the good to at least one agent if selected. Ranking-based mechanisms are approximately optimal when informational size is small and the number of agents is large. Under similar assumptions, we show that these mechanisms are generically inefficient. Ranking-based mechanisms are approximately optimal when informational size is small and the number of agents is large. Under similar assumptions, we show that these mechanisms are generically inefficient.

Proposition 6. *Suppose that for every state ω, the threshold t satisfies $t \geq \max\left\{ \frac{1}{k-1} + \delta(\omega), \frac{2}{k-1} \right\}$, and that there exists at least one agent i with non-negative peer value $u_i(\omega_{-i}) \geq 0$. Then, generically, the ranking-based mechanism is inefficient.*

The principal randomizes over at least two agents. Given the eligibility assumption, either two agents or one agent and the principal are allotted the good with positive probability in each state. Summing over all states then violates the bound in Theorem 1, implying that the outcome induced by the ranking-based mechanism is inefficient.

5 Conclusion

This paper analyzed Pareto efficiency in games with incomplete information. We identified necessary conditions based on the number of action profiles taken across states—independent of the prior, the action profiles used, and the weight of randomization. We found that any

stochastic cheap talk outcome is generically inefficient, and that when the sender's preferences are state-independent, it is efficient if and only if the sender's preferred action is chosen with certainty. In Bayesian persuasion, equilibrium outcomes are also generically inefficient for a broad set of priors and preferences. These findings highlight that preference misalignment prevents efficiency in direct communication between the sender and the receiver. Beyond two-player games, our results extend to an allocation problem without transfers, where ranking-based mechanisms rely on unavoidable randomization that generically leads to inefficiency.

Several directions for future research remain open. A key challenge is to identify sufficient conditions under which persuasion outcomes can be efficient. Moreover, alternative economic models—such as mediation or delegation—warrant further study. A central question is whether any communication protocol can guarantee efficiency under incomplete information.

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A Appendix

A.1 Proof of Proposition 1

For a convex polytope $F \subseteq \mathbb{R}^k$ and for any vector $n \in \mathbb{R}^k$, let

$$S(F; n) := \{x \in F \mid n^T x = \max_{y \in F} n^T y\} \tag{19}$$

denote the set of maximizers x of the inner product $n^T x$ over F.

The characterization relies on the following relation for the Minkowski sum of convex compact sets, as shown by Fukuda (2004):

$$S(F_1 + ... + F_k; n) = S(F_1; n) + ... + S(F_k; n) \text{ for any } n \in \mathbb{R}^k \tag{20}$$

where, F_i is a compact convex set for all $i = 1, ..., k$ and $F_1 + F_2$ denotes the Minkowski sum of the sets F_1 and F_2.

Each point in the Pareto frontier of the set F_p maximizes some strictly positive linear functional $n^T x$ over F_p. Given $F_p = \sum_\omega p(\omega) F_\omega$, and using Equation (20), the payoff vector $u(\mu)$ lies on the Pareto frontier of F_p if and only if, for all $\omega \in \Omega$, the payoff vector $u(\mu \mid \omega)$ lies on the Pareto frontier of F_ω and there exists a common strictly positive vector n that is maximized. Ex-post efficiency ensures that the outcome maximizes a positive linear functional over the feasible payoff set for each state but does not ensure the existence of a common strictly positive functional.

A.2 Efficiency via payoff deviations

In the following proposition, we provide a necessary and sufficient condition for an outcome to be efficient. This condition is determined solely by the payoff functions of the players. To check for efficiency, one must look at the possible change in pair of payoffs when deviating from the recommended action for all possible states. Define

$$d_\mu(\omega, a) := \big(u_1(\omega, a) - u_1(\omega, \mu(\omega)), \cdots, u_k(\omega, a) - u_k(\omega, \mu(\omega))\big) \tag{21}$$

as the deviation in state ω when action profile a is taken instead of the recommended profile $\mu(\omega) \in \Delta A$. Let

$$\mathcal{D}_\mu = \{\tilde{d} \in \mathbb{R}^k : \tilde{d} = d_\mu(\omega, a) \text{ for some } \omega \in \Omega, a \in A\} \tag{22}$$

denote the set of all deviations given outcome μ. And, let

$$\text{cone}(\mathcal{D}_\mu) := \Big\{ \sum_{d \in \mathcal{D}_\mu} \lambda_d \, d : \lambda_d \geq 0 \text{ for all } d \in \mathcal{D}_\mu \Big\}. \tag{23}$$

denote the cone generated by the set of deviations $\mathcal{D}_\mu$. An outcome is efficient if and only if no convex combination of deviations across states leads to a Pareto improvement.

Proposition 7. *An outcome $\mu : \Omega \to \Delta A$ is efficient if and only if*

$$\text{cone}(\mathcal{D}_\mu) \cap \mathbb{R}^k_+ = \{\mathbf{0}\}. \tag{24}$$

Proof. ($\Rightarrow$) We prove by contradiction. If μ is efficient, then there exists $n \in \mathbb{R}^k_{++}$ such that $n \cdot d \leq 0$ for all $d \in \mathcal{D}_\mu$. Suppose instead that (24) does not hold. Then there exist non-negative coefficients $(\lambda_d)_{d \in \mathcal{D}_\mu}$ with $d_\lambda = \sum_{d \in \mathcal{D}_\mu} \lambda_d d \in \mathbb{R}^k_+ \setminus \{\mathbf{0}\}$. In particular, as $n \in \mathbb{R}^k_{++}$, this implies that $n \cdot d_\lambda > 0$. But $n \cdot d_\lambda = \sum_{d \in \mathcal{D}_\mu} \lambda_d(n \cdot d)$, so there exists some $d \in \mathcal{D}_\mu$ with $\lambda_d > 0$ for which $n \cdot d > 0$, leading to a contradiction.

($\Leftarrow$) Assume (24) holds, i.e., no strictly positive vector lies in the cone generated by the deviations. Let A denote the $k \times |\mathcal{D}_\mu|$ matrix whose columns are the deviation vectors $d_\mu(\omega, a)$. By Mangasarian's Theorem, as stated in Perng (2017), for any real matrix A exactly one of the following holds:

(i) There exists $x \geq \mathbf{0}$, $x \neq \mathbf{0}$ such that $Ax \geq \mathbf{0}$ (with at least one strictly positive component).

(ii) There exists $n > \mathbf{0}$ (strictly positive component wise) such that $n^\top A \leq \mathbf{0}$.

Since case (i) is ruled out by Equation (24), case (ii) must hold. Thus, there exists $n \in \mathbb{R}_{++}^k$ such that $n \cdot d \leq 0$ for all $d \in \mathcal{D}_\mu$.

Fix any state ω and action $a \in A$. Then $n \cdot d_\mu(\omega, a) = n \cdot (u(\omega, a) - u(\mu \mid \omega)) \leq 0$. This implies $u(\mu \mid \omega) \in S(F_\omega; n)$. Hence μ is ex-post efficient in every state, and the same strictly positive n suffices for all states. Thus, the outcome μ is efficient. $\qquad \square$

Remark 3. *For two players, the condition reduces to: there is no deviation with $d_\mu(\omega, a) \geq \mathbf{0}$ (strict in at least one component), and for any pair of deviations $(d, \tilde{d})$ where d benefits player 1 and $\tilde{d}$ benefits player 2, the internal angle between d and $\tilde{d}$ is at most $180°$. Equivalently, $\tilde{d} \cdot d^\dagger \geq 0$, where $d^\dagger$ denotes the vector obtained by rotating d clockwise by $90°$. In particular, if $d = (x, -y)$ with $x, y \geq 0$, then $d^\dagger = (-y, -x)$.*

The efficiency condition depends solely on the support of the outcome and is independent of the weight of randomization and the prior. This has two implications. First, if an outcome is efficient (or inefficient) for a prior $p \in \text{int}(\Delta\Omega)$, it remains so for all interior priors. Second, the support of the Bayesian persuasion outcome, for each state, remains fixed within a convex region, with only the weight of randomization varying. Thus, if the Bayesian persuasion outcome is efficient (or inefficient) for a given prior, it holds across all interior priors in that convex region.

A.3 Proof of Proposition 2

We prove by contradiction. Suppose there exists an efficient cheap talk outcome that is not pure.

First, consider the case where the signaling policy is necessarily stochastic. So, there exists $\omega^* \in \Omega$ and messages $m_1, m_2 \in M$ such that $\sigma(m_1 \mid \omega^*) \cdot \sigma(m_2 \mid \omega^*) > 0$. Note that $\tau(m_1) \neq \tau(m_2)$, since otherwise the same outcome could be induced using a single message. By Theorem 1, if more than two actions are played in a given state then the outcome is inefficient for a generic set of payoffs. Therefore, assume exactly two actions, a_1 and a_2, are played with positive probability. The sender's equilibrium condition implies that

$$u_S(\omega^*, a_1) = u_S(\omega^*, a_2). \tag{25}$$

For the outcome to be ω^*-efficient, the receiver must also be indifferent:

$$u_R(\omega^*, a_1) = u_R(\omega^*, a_2). \tag{26}$$

If this indifference did not hold, then deviating to a pure action would strictly improve the receiver's payoff in the state ω^*. Consider any perturbation of payoffs that preserves

the sender's indifference, so the outcome remains an equilibrium. For any such generic perturbation, the receiver's indifference conditions hold only on a subset of payoff vectors with Lebesgue measure zero, and thus represent a non-generic condition. For example, if $A = \{a_1, a_2\}$, both indifference conditions imply that the state ω^* is payoff-irrelevant.

Second, consider the case where the signaling policy is pure. As the outcome is stochastic, this implies that the receiver's response to some message m is mixed. This implies that the receiver has multiple best responses at the posterior belief q_m. Again, to ensure efficiency, assume exactly two actions a_1 and a_2 are played with positive probability and the sender is indifferent between them. Otherwise, deviating to a pure action would strictly improve the sender's payoff without reducing the receiver's. This implies that

$$\mathbb{E}_{q_m}[u_S(\omega, a_1)] = \mathbb{E}_{q_m}[u_S(\omega, a_2)], \qquad \mathbb{E}_{q_m}[u_R(\omega, a_1)] = \mathbb{E}_{q_m}[u_R(\omega, a_2)]. \tag{27}$$

As before, this corresponds to a non-generic condition. So, the cheap talk outcome is inefficient with respect to the prior q_m under all generic perturbations. This implies that there exists a deviation $d_\mu(\omega, a)$ in some state $\omega \in \text{supp}(q_m)$ and action a that violates the efficiency condition of Proposition 7. First, since $\text{supp}(q_m) \subseteq \text{supp}(p)$, this deviation is also feasible under the prior p. Second, condition (ii) in Proposition 7 imposes stronger restrictions under prior p as the set of possible deviations is larger. Hence, generically, the outcome is inefficient with respect to F_p.

A.4   Proof of Proposition 3

($\Rightarrow$) Following Sobel (2013), we say an equilibrium is *influential* if the receiver does not always take the same action. First, we show that any influential equilibrium is inefficient. Then, we show that any non-influential equilibrium that does not induce the sender's preferred action with certainty is inefficient.

Assume that the equilibrium is influential, that is, $\tau(m) \in \Delta A$ is not constant on the equilibrium path. For this to happen, at least two messages are sent with positive probability, resulting in different actions. First, observe that at least one message must induce a non-degenerate posterior belief $q \in \Delta\Omega$.[11] This is due to the sender's equilibrium condition as he must be indifferent between sending messages that result in the same expected payoff. And as we assume the sender's payoff is state-independent and each action leads to a different payoff, some randomization is necessary for the indifference condition to hold. Such a randomization can only occur at a non-degenerate belief where the receiver has multiple best response

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[11]A non-degenerate belief refers to a belief where the probability distribution assigns positive probability to more than one state.

actions, that is, $|A^*(q)| > 1$.[12] So, there must be at least two distinct actions a_1 and a_2 that are played with positive probability when the posterior belief q is induced. For example, consider the influential equilibrium for prior $p = 0.5$ in Figure 2, where the posterior beliefs $q = 0.4$ and $q = 0.6$ are induced. Given the non-degenerate posterior belief $q = 0.4$, the sender must randomize between a_1 and a_2 to satisfy the sender's indifference condition. However, the sender strictly prefers one action over the other, for instance, assume $u_S(a_1) > u_S(a_2)$. This implies that the sender's preferred action a_1 is induced with less than probability one at belief $q = 0.4$. Since both actions are receiver's best responses under some non-degenerate belief, there exists a hyperplane passing through the belief q that separates the simplex into two convex regions—one where the receiver prefers a_1 and one where he prefers a_2. In our example, for the pair of actions a_1 and a_2, these convex regions are given by the intervals $[0, 0.4]$ and $[0.4, 1]$ respectively. Using this partition, one can always identify a state where the receiver also prefers the sender's preferred action. Formally, given actions $a_1, a_2 \in A^*(q)$, there exists a state ω^* such that $q(\omega^*) > 0$ and $u_R(\omega^*, a_1) \geq u_R(\omega^*, a_2)$. In our example (see Figure 2), both the sender and the receiver prefer action a_1 over a_2 in state $\omega^* = \omega_0$. Hence, the equilibrium is not ex-post efficient in state ω^*. The deviation to play the sender's preferred action a_1 is profitable and does not satisfy condition (i) of Proposition 7.

Now, consider a non-influential (or babbling) equilibrium where the receiver always plays the action $a \neq a^*$.[13] As we assume the prior lies in the interior of the belief simplex, as before, we can identify a state ω^* where the receiver prefers the sender's preferred action a^* over a. For example, given a prior $p \in [0.6, 0.8]$ in Figure 2, the equilibrium is babbling, resulting in action a_3 with certainty. However, both players prefer the action $a^* = a_1$ over a_3 in state $\omega^* = \omega_0$. So, the equilibrium is not ex-post efficient in state ω^*. To summarize, any cheap talk equilibrium that does not induce the sender's preferred action a^* with certainty is inefficient.

($\Leftarrow$) The babbling equilibrium where the sender's preferred action a^* is induced with certainty is an efficient outcome. In this case, the sender gets the highest payoff within his feasible set, ensuring that the outcome lies on the Pareto frontier.

A.5   Proof of Proposition 4

Let H_i for $i \neq 0$ be the hyperplane defined by the set of points $\{\omega_0\} \bigcup_{j \neq i, 0} \{o_{ji}\}$ (see Figure 4). The hyperplane H_i separates the convex set C_i from C_j for all $j \neq i, 0$. Denote by R_i the region in the simplex given by the half-space of H_i that includes C_i. For any prior $p \in R_i$, it is necessary that any feasible outcome induces the risky action a_i. Similarly, let H_0 be the

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[12]We omit non-generic cases where the receiver has multiple best responses at any degenerate belief.

[13]If the receiver chooses a mixed action, select a pure action from the support that differs from the sender's preferred action, that is, pick action $a \in \bigcup_m \text{supp}(\tau(m))$ such that $a \neq a^*$.

hyperplane defined by $\{o_{10}, \ldots, o_{n0}\}$ and R_0 denote the convex region in the simplex that includes the node ω_0 and is separated by the half-space of H_0. For any prior $p \in R_0$, it is necessary to play the safe action a_0 under any feasible outcome.

For any $i, j \neq 0$ let $R_{ij} = R_i \cap R_j \cap R_0$ (see Figure 4). The convex set R_{ij} is non-empty as $\omega_0 \in R_{ij}$. In fact, we show that $\dim(R_{ij}) = \dim(\Delta\Omega) = n$ for all $i, j \neq 0$. Given a system of inequalities $Ax \leq b$, an inequality $a_i^T x \leq b_i$ in $Ax \leq b$ is an *implicit equality* if $a_i^T \bar{x} = b_i \quad \forall \bar{x} \in \{x : Ax \leq b\}$. A polyhedron $R \subseteq \mathbb{R}^l$ has full dimension ($\dim(R) = l$) if and only if it has no implicit equality, as shown by Conforti et al. (2014). The polyhedron R_{ij} is defined by the system of inequalities of the hyperspaces R_0, R_i and R_j. If the polyhedron R_{ij} has an implicit equality, then all points $p \in R_{ij}$ lie on the hyperplane H_0, H_i or H_j. But this happens only if $o_{ij} = o_{ji}$ or $o_{k0} = \omega_0$ where $k = i, j$. But as we assume (1) there is an open neighbourhood $N_{\omega_0} \subset C_0$ and (2) $C_i \cap C_j \neq \emptyset$, we can conclude there is no implicit equality for the polyhedron R_{ij}. Thus, we have $\dim(R_{ij}) = \dim(\Delta\Omega) = n$.

Given prior $p \in R_{ij}$, any feasible outcome induces the actions a_i, a_j and a_0. We claim that posteriors of the BP outcome cannot include the vertices ω_i or ω_j. We prove by contradiction, assume the feasible outcome μ_1 is optimal and its induced posteriors include the vertex ω_i. As $p \in R_0$, it needs to induce action a_0 and its support includes the node ω_0. Now, as $o_{i0} \in (\omega_0, \omega_i)$ and is separated from p by the hyperplane H_0, there exists a feasible outcome μ_2, where the belief o_{io} is induced instead of ω_i. This outcome leads to a higher probability of action a_i and conversely a lower probability of action a_0. Let λ_i denote the weight of outcome μ_i on its posteriors. We have $\lambda_2(o_{io}) = \frac{\lambda_1(\omega_i)}{o_{io}(\omega_i)} > \lambda_1(\omega_i)$ and $\lambda_2(\omega_0) = \lambda_1(\omega_0) - \frac{\lambda_1(\omega_i)o_{i0}(\omega_0)}{o_{i0}(\omega_i)} < \lambda_1(\omega_0)$. The weight on all other actions $a_j \neq i$ remains the same. Thus, μ_2 is a profitable deviation and the posteriors of the BP outcome cannot include ω_i or ω_j.

Using the above result, for any prior $p \in R_{ij}$, the BP outcome μ_p^* has either (a) three actions a_i, a_j and a_0 played in state ω_0 (e.g., $\bigcup_{m \in M} q_m = \{\omega_0, o_{10}, o_{20}\}$ in Figure 4) or (b) mixed outcomes in states ω_i and ω_j (e.g., $\bigcup_{m \in M} q_m = \{\omega_0, o_{12}, o_{21}\}$ in Figure 4). This violates the necessary condition for efficiency in Theorem 1. The set $R_* = \bigcup_{j \neq 0} \bigcup_{i \neq j, 0} R_i \cap R_j \cap R_0$ combines the regions R_{ij} for all pairs of distinct risky actions a_i and a_j.

A.6 Proof of Proposition 5

First, we show for any $p \in \text{int}(\Delta\Omega)$ and action a_i, there exists a threshold T_p^i such that whenever $T > T_p^i$, any feasible outcome induces that action. Using this characterization, we then show that there exist receiver preferences under which the BP outcome μ_p^* is inefficient for the partition $\mathcal{P}_T$.

Let q denote the point of intersection between the line joining the points ω_0 and p and the face $F_0 = \{p \in \Delta\Omega : p(\omega_0) = 0\}$. Recall, the hyperplane H_i defined by the points $\{\omega_0\} \bigcup_{j \neq i, 0}\{o_{ji}\}$. It separates C_i and the convex sets $\{C_j\}_{j \neq i, 0}$. The point p lies in the
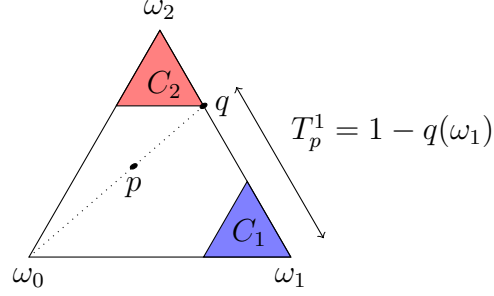
Figure 5: Threshold beliefs for $n = 2$.

region R_i if $T \geq T_p^i = 1 - q(\omega_i)$ (see Figure 5). This follows as $p \in R_i$ if and only if it's projection $q \in R_i$. And q belongs to R_i (for $i \neq 0$) if and only if $q(\omega_i) \geq 1 - T$. Similarly, the hyperplane H_0 separates the vertex ω_0 and the convex sets $\{C_j\}_{j \neq 0}$ and we have $p \in R_0$ if $T \geq T_p^0 = 1 - p(\omega_0)$.

Let $i^* = \operatorname*{argmax}_{i \neq 0} q(\omega_i)$ and let $j^* = \operatorname*{argmax}_{i \neq 0, i^*} q(\omega_i)$. The vertices ω_{i^*} and ω_{j^*} represent the states that are closest to the projection $q \in \Delta\Omega$. We have $p \in \operatorname{int}(R_{i^*} \cap R_{j^*} \cap R_0)$ if

$$T > T_p = \max\{1 - p(\omega_0), 1 - q(\omega_{i^*}), 1 - q(\omega_{j^*})\}. \tag{28}$$

Since the common prior $p \in \operatorname{int}(\Delta\Omega)$, we have $p(\omega_0) > 0$ and $q(\omega_i) < 1$ for $i \neq 0$. Hence each term in Equation (28) is strictly less than 1, and therefore $T_p < 1$.

Therefore, whenever $T > T_p$, the Bayesian persuasion outcome μ_p^* lies in R_* and is inefficient with respect to the partition $\mathcal{P}_T$.

A.7 Proof of Proposition 6

Under our assumptions, in the first step the principal randomly selects among at least two agents, i.e., $t(k-1) \geq 2$. At least one of these agents, if chosen, is allocated the good. Hence, with positive probability, either two distinct agents receive the good or one agent and the principal do. Thus, for all $\omega \in \Omega$, we have $|\mu(\omega)| \geq 2$. Moreover, since each agent has at least two types, $|\Omega| = \prod_{i=1}^{k-1} |\Omega_i| \geq 2^{k-1}$. Therefore, $\sum_{\omega \in \Omega} |\mu(\omega)| \geq 2|\Omega| > k + |\Omega|$, where the last inequality uses $2^{k-1} > k$ for $k \geq 3$. This contradicts the bound in Theorem 1, so the outcome induced by the ranking-based mechanism is inefficient.