MAT150 - Summer 2023

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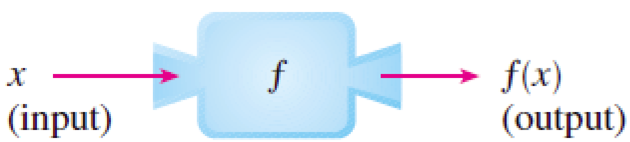
Content

Functions

A function is a **rule** that assigns to each element x in a set D exactly one element, called y, in a set E.
$f : x \in D \to y \in E$.



- Input: independent variable(s): x

- Output: dependent variable: y

- D : domain

- E : range (the set of all possible values of $f(x)$ as x varies throughout the domain)

- Usually $D \in \mathfrak{R}$ and $E \in \mathfrak{R}$

Algebraic representations

- Explicit form $y = f(x)$

- Parametric form $\left\lbrace
```
\begin{array}{ll}
  x(t) \\
```

```
    y(t)
\end{array}
\right\rbrace$
```

- Implicit form $F(x, y) = 0$

Example 1:

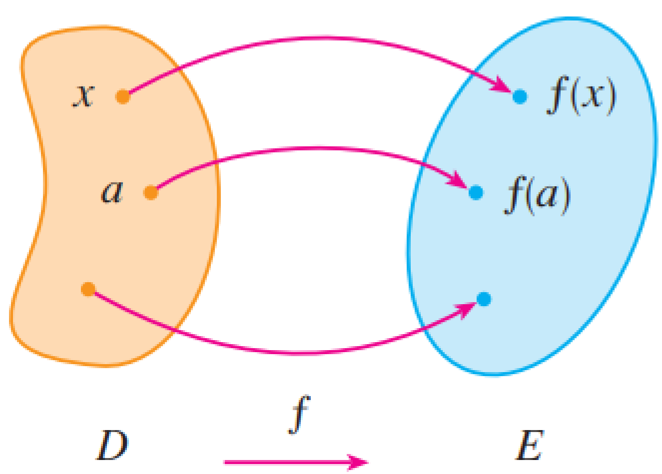
A straight line

- Explicit representation: $y = x + 2$
- Implicit representation: $y - x - 2 = 0$
- Parametric equation: $\left\lbrace

```
 \begin{array}{ll}
    x=t \\
    y=t+2
\end{array}
```
\right\rbrace$ (there are ∞ many ways)

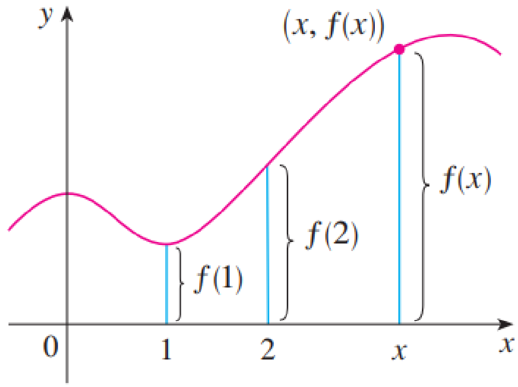
Graphic representations

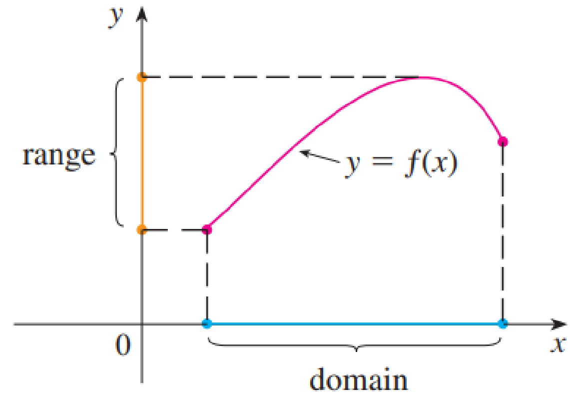
Arrow diagram



Graph

set of ordered pairs $\{(x, f(x)) | x \in D\}$

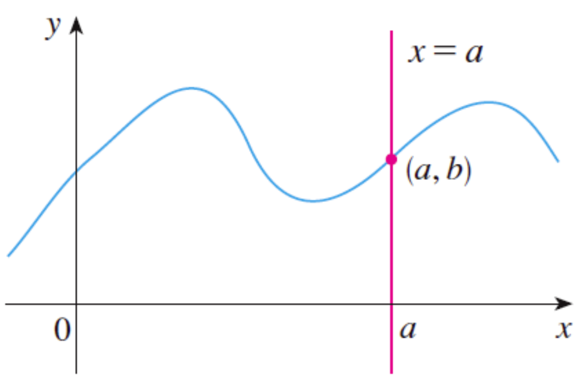


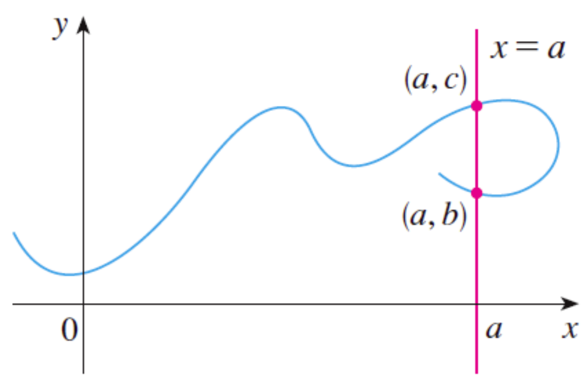


The relation between a graph and its algebraic expression must be completely **univocal**, that is, we need to get the same information from both without ambiguities.

THE VERTICAL LINE TEST

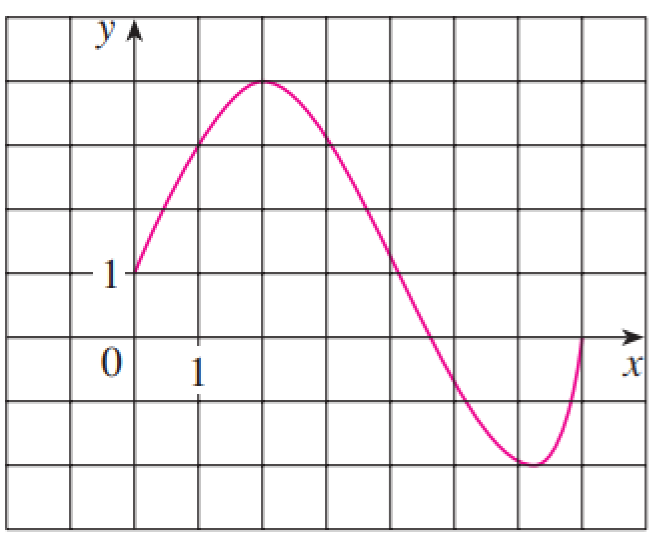
A curve in the xy-plane is the graph of a function if and only if no vertical line intersects the curve more than once.





Example 2

What are the domain and range of f?



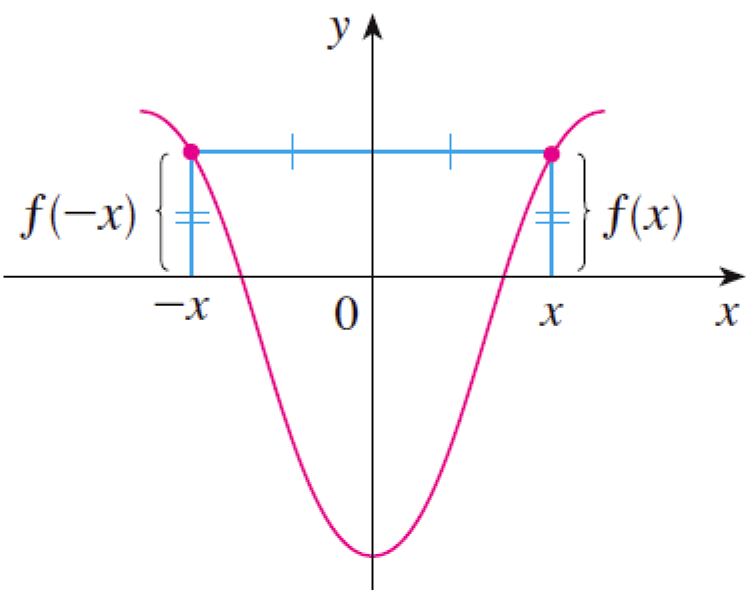
Solution:

Domain: $\{x|0 \leq x \leq 7\} = [0, 7]$

Range: $\{y| -2 \leq y \leq 4\} = [-2, 4]$

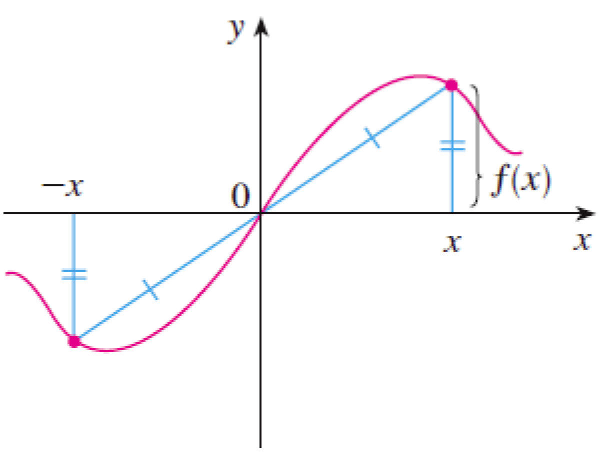
Simmetry

If a function satisfies $f(-x) = f(x)$ for every number in its domain, then is called an **even** function.



For instance $f(x) = x^2$

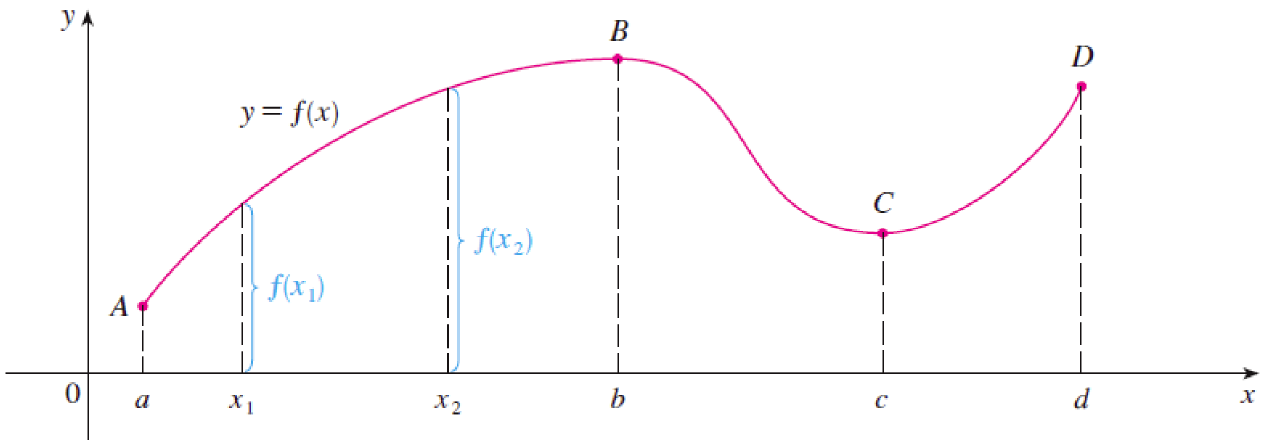
If f satisfies $f(-x) = -f(x)$ for every number in its domain, then is called an **odd** function.



For instance $f(x) = x^3$

Increasing and Decrasing functions

- A function is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
- It is called **decreasing** on if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.



Basic Functions

Basic explicit functions: $y = f(x)$

- Polinomials $\Leftrightarrow$ Irrational
- Exponentials $\Leftrightarrow$ Logaritmic
- Trigonometrical $\Leftrightarrow$ Transcendental

Implicit functions: Circles and parabolas (quadratic-conic sections)

Polynomials

- In general, polynomials are represented as: $P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3^3 + \cdots + a_n x^n$
- Where, $n > 0$, is the order of the polynomial and $a_0, a_1, \ldots, a_n$ are constants called the coefficients of the polynomial.
- The domain of any polynomial is $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.

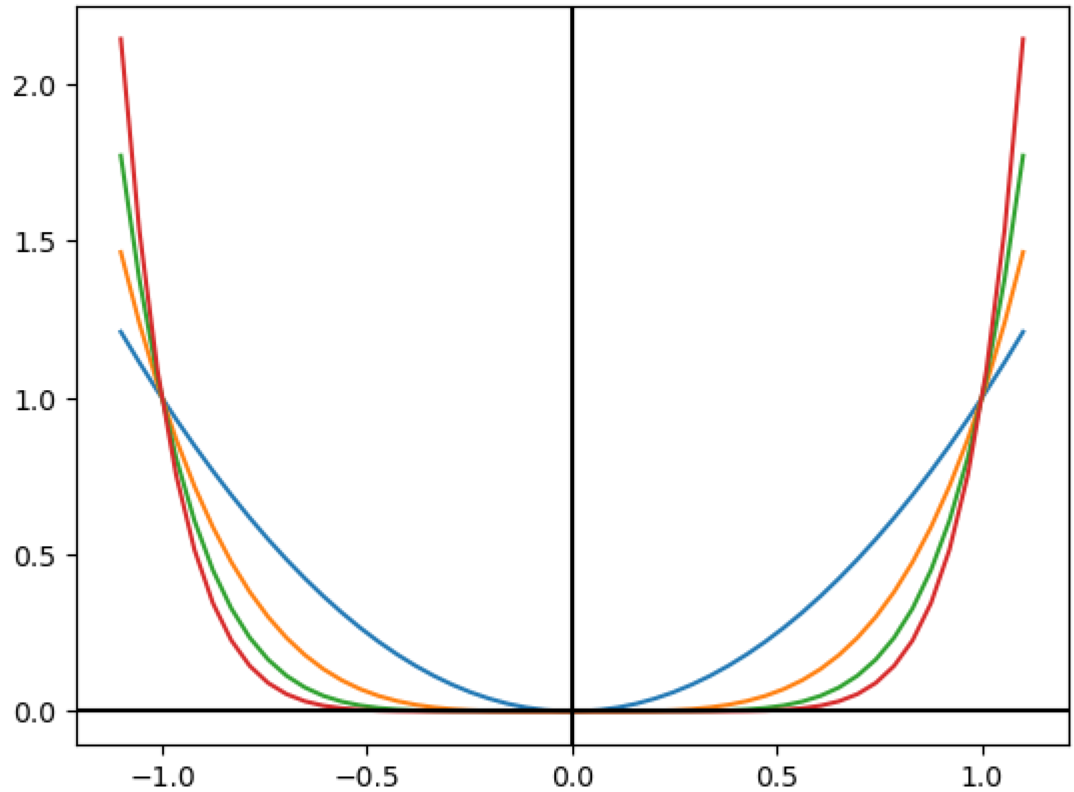
Let's plot some polynomials...

In [1]:
```
import matplotlib.pyplot as plt
import numpy as np

x=np.linspace(-1.1,1.1)

plt.plot(x,x**2)
plt.plot(x,x**4)
plt.plot(x,x**6)
plt.plot(x,x**8)
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
```

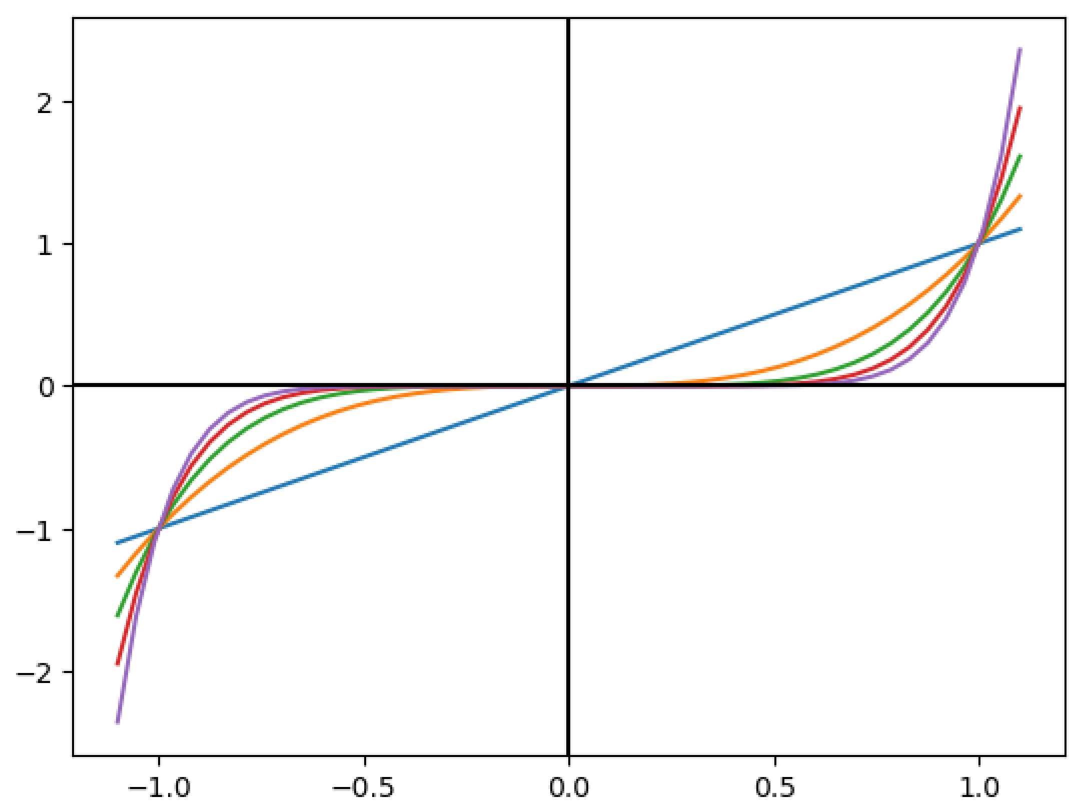
Out[1]:
```
<matplotlib.lines.Line2D at 0x16063c05cf0>
```



As n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq$1.

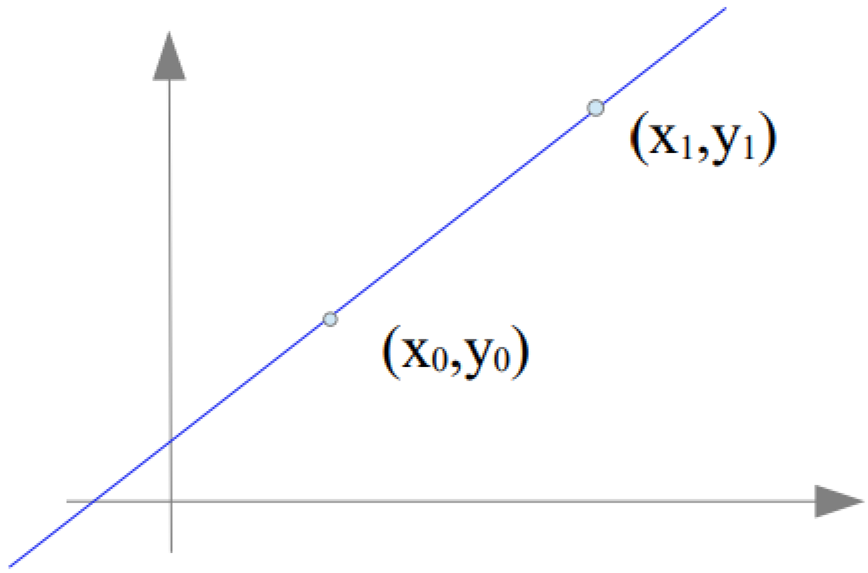
In [2]:
```
plt.plot(x,x)
plt.plot(x,x**3)
plt.plot(x,x**5)
plt.plot(x,x**7)
plt.plot(x,x**9)
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
```

Out[2]:
```
<matplotlib.lines.Line2D at 0x16063c6b4f0>
```



Straight lines: $n = 1$

- Explicit representation: $y = P_1(x) = a_1 x + a_0$
- They are defined through 2 points, in general the intersection with the axes.



- Alternatively, straight lines are given through a point (x_0, y_0) and its slope

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}, \tag{1}$$

where the symbol Δ means "variation, change"

- Alternative: $y - y_0 = m(x - x_0)$

Example 3

Express the line through the points $(-1, 3)$ and $(5, 2)$ following the expression $y - y_0 = m(x - x_0)$ and sketch it. Indicate the **intersections** with x and y axis.

Parabolas: $n = 2$

$$y = P_2(x) = a_0 + a_1 x + a_2 x^2 \tag{2}$$

Example 4

Find the intersections with the axis of the parabola $y = x^2 - 2x - 3$. With these data, can you easily sketch it?

Vertex

The **vertex** of a parabola is the extreme of the curve, and identifying it will help up to sketch the parabola intuitively.

Example 5

Sketch and find the vertex of:

1. $y = x^2$
2. $y = (x-1)^2$
3. $y = -x^2$

What is produced by these operations from geometrical point of view?

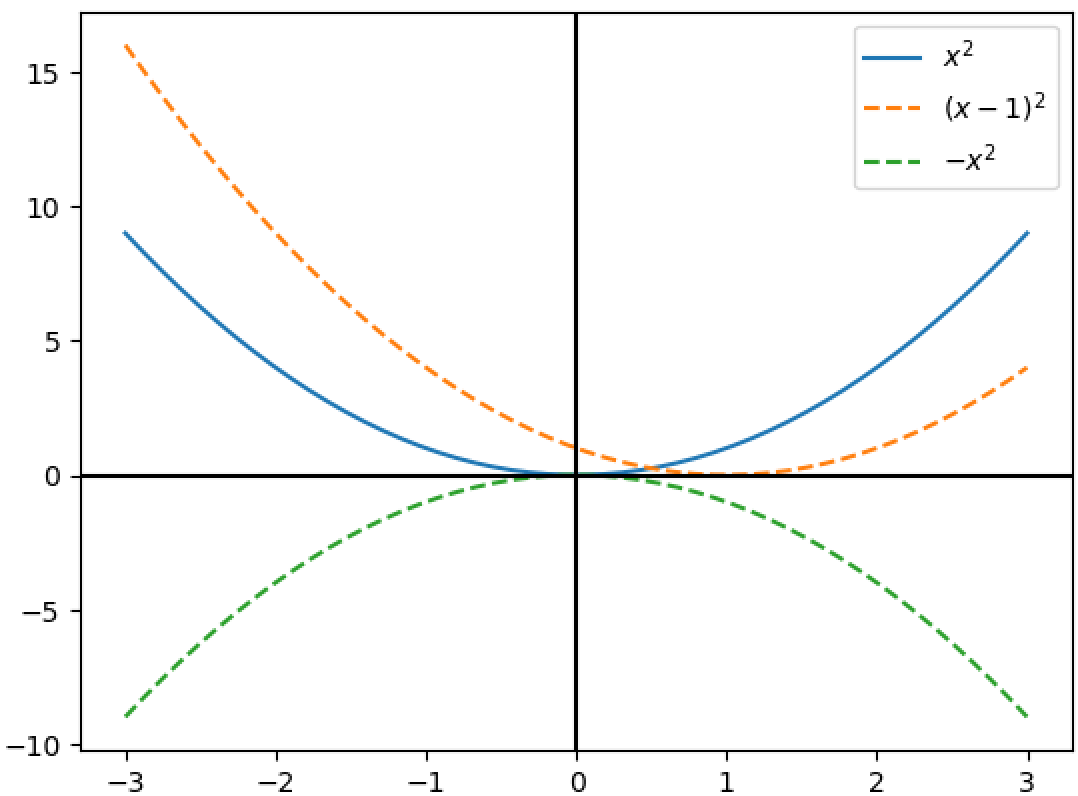
In [3]:
```
import matplotlib.pyplot as plt
import numpy as np

x=np.linspace(-3,3)

def y(x):
    return x**2

plt.plot(x,y(x))
plt.plot(x,y(x-1),'--')
plt.plot(x,-y(x),'--')
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
plt.legend(['$x^2$','$(x-1)^2$','$-x^2$'])
```

Out[3]: <matplotlib.legend.Legend at 0x16064461630>



Basic operations: New functions from old functions

Vertical and horizontal shifts

- $f(x-a)$: translation to the right (delay)
- $f(x+a)$: translation to the left
- $y+b = f(x)$: translation upward

- $y - b = f(x)$: translation downward

Vertical and horizontal stretching and reflecting

Suppose $c > 1$

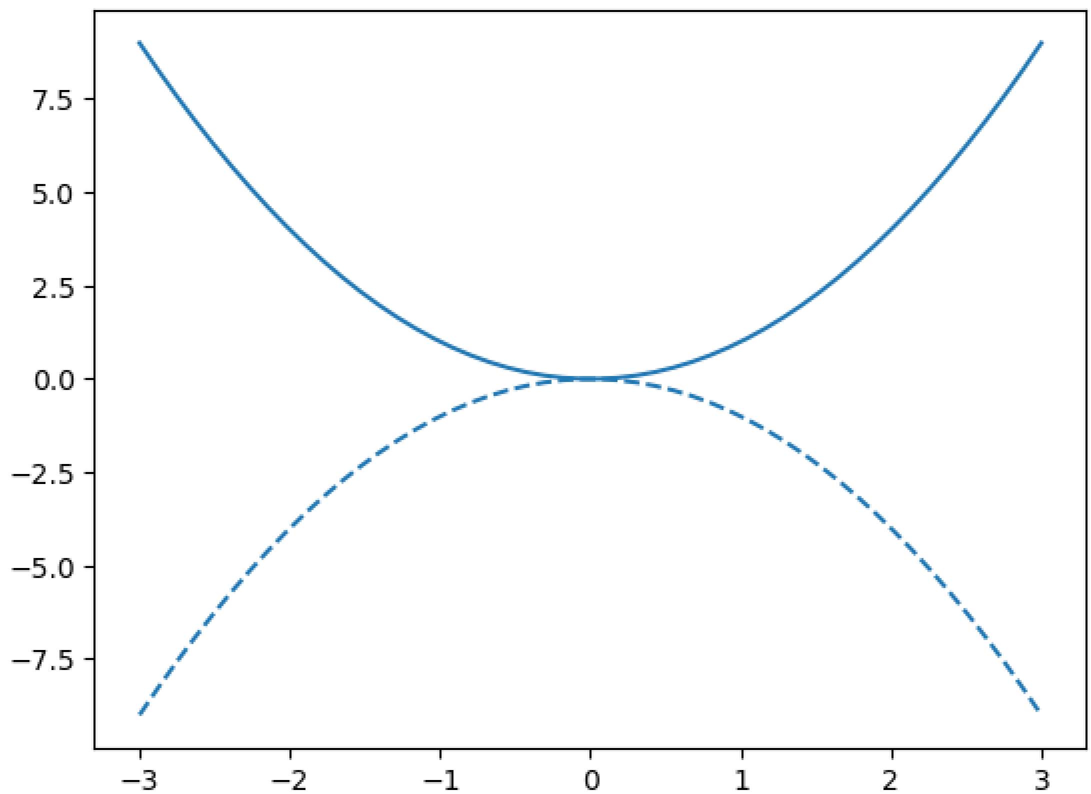
- $y = cf(x)$: stretch the graph vertically by a factor of c.
- $y = (1/c)f(x)$: compress the graph vertically by a factor of c.
- $y = f(cx)$: compress the graph horizontally by a factor of c.
- $y = f(x/c)$: stretch the graph horizontally by a factor of c.
- $f(-x)$: reflection about the y-axis.
- $- f(x)$: reflection about x-axis
- The inverse: $f^{-1}(x)$ reflection about $y = x$.

Let's plot some transformations...

In [4]:
```
x=np.linspace(-3,3)

plt.plot(x,y(x))
plt.plot(x,-y(x),'--',color='tab:blue')
```

Out[4]: [<matplotlib.lines.Line2D at 0x160646b50f0>]



Example 6

Find the vertex of the following parabolas **completing the square.**

1. $y = x^2 - 2x$
2. $y = x^2 + 3x + 1$
3. $y = 2x^2 + 6x - 1$
4. $y - 6x + x^2 = 0$

Solutions

1. $y = \left(x-1\right)^2 - 1 \Rightarrow V = (1,-1)$

2. $y = \left(x+3/2\right)^2 - 5/4 \Rightarrow V = (-3/2,-5/4)$

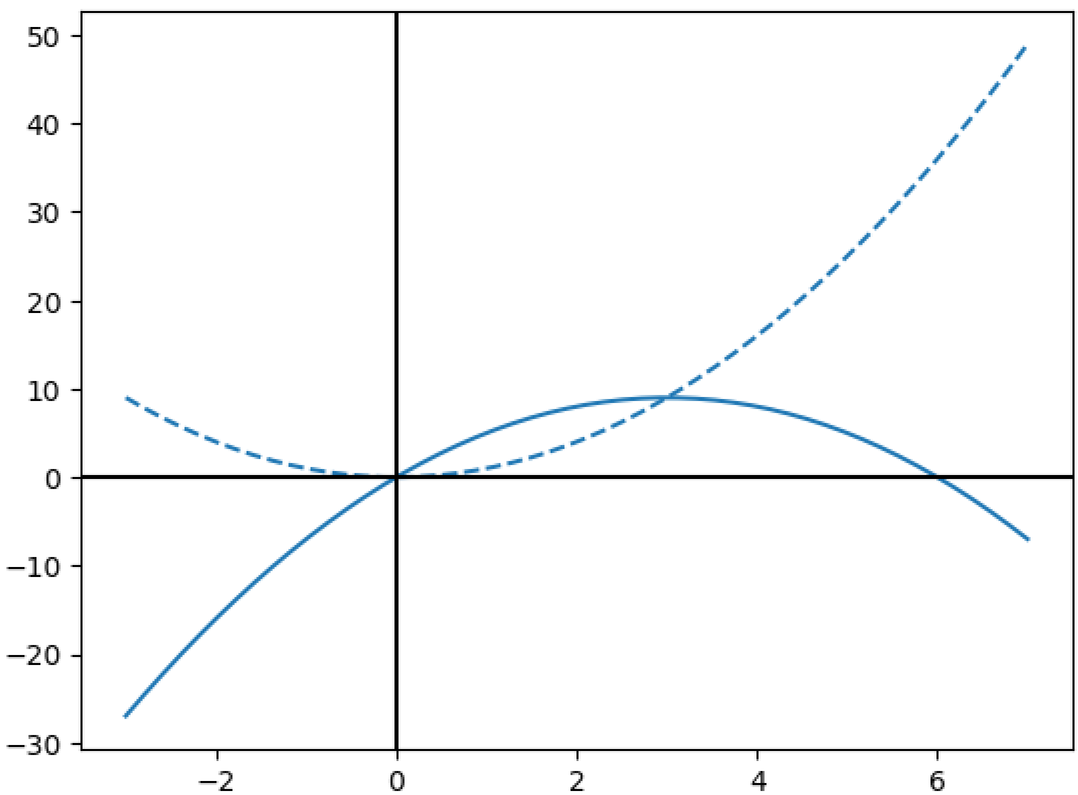
3. $y = 2\left[\left(x+3/2\right)^2 - 11/4\right] \Rightarrow V = (-3/2,-11/2)$

4. $y = -\left(x-3\right)^2 + 9 \Rightarrow V = (3,9)$

In [5]:
```
x=np.linspace(-3,7)

plt.plot(x,y(x),'--')
plt.plot(x,-y(x-3)+9,color='tab:blue')
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
```

Out[5]:
```
<matplotlib.lines.Line2D at 0x160643f2dd0>
```



The inverse of a function

The inverse of a function is the operation that does just the opposite of the original one. In other words, the inverse **undoes** what the function have done before.

Definition

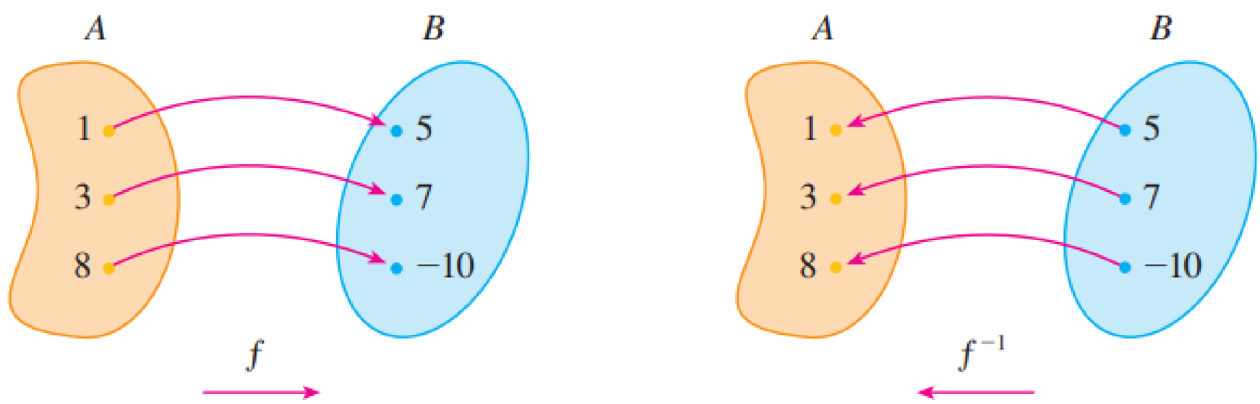
Let f be a one-to-one function with domain A and range B. Then its **inverse function** has domain B and range A and is defined by

$$f^{-1} = x \Leftrightarrow f(x) = y \tag{3}$$

for any y in B.

Then ...

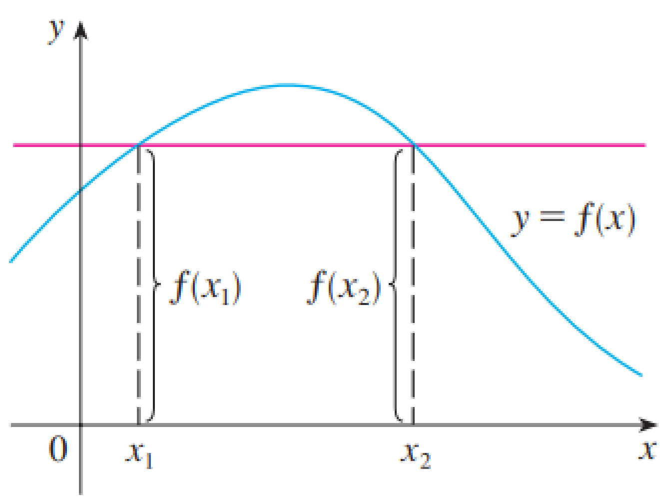
- domain of f^{-1} = range of f
- range of f^{-1} = domain of f



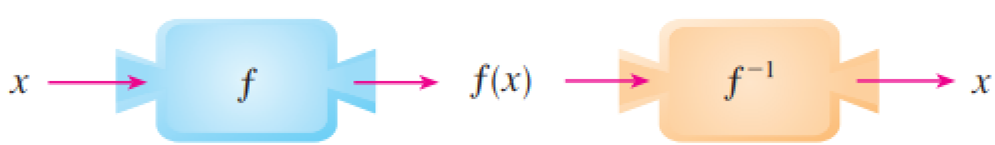
Definition

A function is called a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2), \ whenever \ x_1 \neq x_2 \tag{4}$$



Horizontal line test



- $f^{-1}\big(f(x)\big) = x$ for every $x \in A$ (domain of f)
- $f\big(f^{-1}(x)\big) = x$ for every $x \in B$ (range of f)

How to find the inverse of a function f

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y.
3. interchange x and y.

Example 7

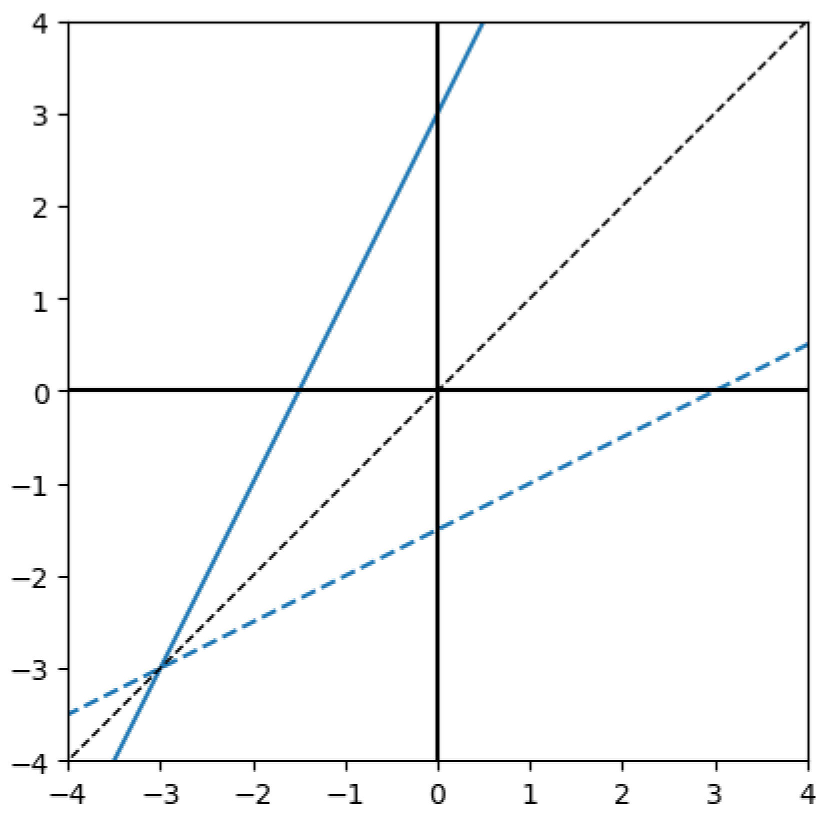
Find the inverse of $y = 2x + 3$. Sketch both.

Solution

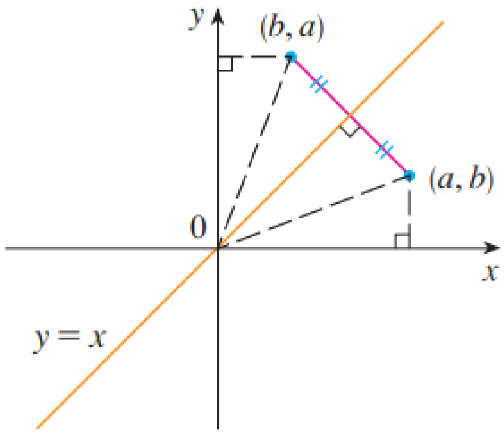
$$y = \frac{x-3}{2} \tag{5}$$

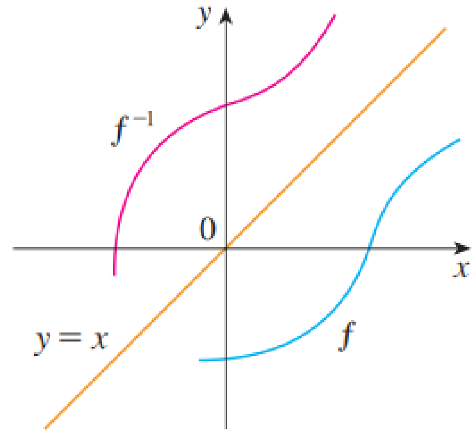
In [29]:
```
x=np.linspace(-5,5)

plt.plot(x,2*x+3)
plt.plot(x,(x-3)/2,'--',color='tab:blue')
plt.plot(x,x,'--',color='k', linewidth=1)
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
# plt.axhline(y = 3, color = 'r', linestyle = '--')
# plt.axvline(x = -3/2, color = 'r', linestyle = '--')
plt.xlim([-4,4])
plt.ylim([-4,4])
ax = plt.gca()
ax.set_aspect('equal')
```



The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.





Example 8

Find the **inverse** of the following functions analytically (first, we solve x in the function of y and after that, we exchange x and y).

a. $y = 3x - 1$

b. $y = 3x^5$

c. $y = \frac{3x-1}{2x+3}$

d. $y = x^2$

Solution

a. $y = (x + 1)/3$

b. $y = (x/3)^{1/5}$

c. $y = (3x + 1)/(3 - 2x)$

d. $y = \pm\sqrt{x}$

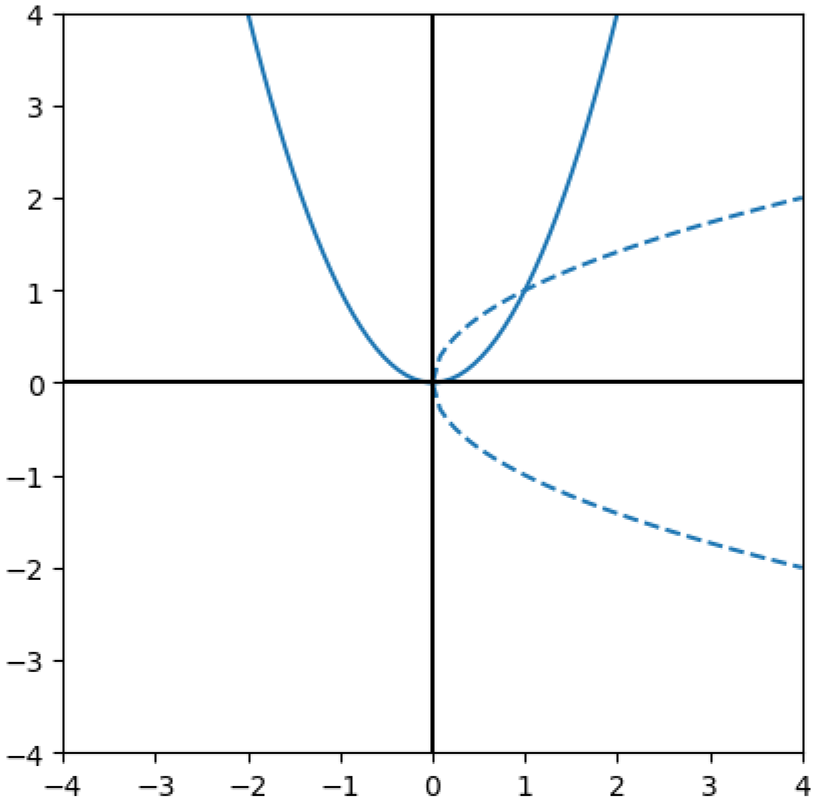
Notice that, the inverse of $y = x^2$ cannot be expressed in a singular explicit function. It has two definition (o parts), $y = \sqrt{x}$ and $y = -\sqrt{x}$.

If we would like to refer to a horizontal parabola with a unique expression we should use its **Implicit form**: $y^2 = x$. Note that x is still the input, and y the output.

In [7]:
```
x=np.linspace(-2,2)
x2=np.linspace(0,4)

plt.plot(x,x**2)
plt.plot(x2,-np.sqrt(x2),'--',color='tab:blue')
plt.plot(x2,np.sqrt(x2),'--',color='tab:blue')

plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
# plt.axhline(y = 3, color = 'r', linestyle = '--')
# plt.axvline(x = -3/2, color = 'r', linestyle = '--')
plt.xlim([-4,4])
plt.ylim([-4,4])
ax = plt.gca()
ax.set_aspect('equal')
```



Combination of functions

Given $f(x)$ and $g(x)$ with domains A and B, respectively:

- $(f+g)(x) = f(x) + g(x)$, domain: $A \cap B$
- $(f-g)(x) = f(x) - g(x)$, domain: $A \cap B$
- $(fg)(x) = f(x)g(x)$, domain: $A \cap B$
- $(f/g)(x) = f(x)/g(x)$, domain: $\{x \in A \cap B | g(x) \neq 0\}$

More Functions

Exponential Functions

$y(x) = a^x$, the variable x is the exponent, $a > 0$ and $x \in \Re$.

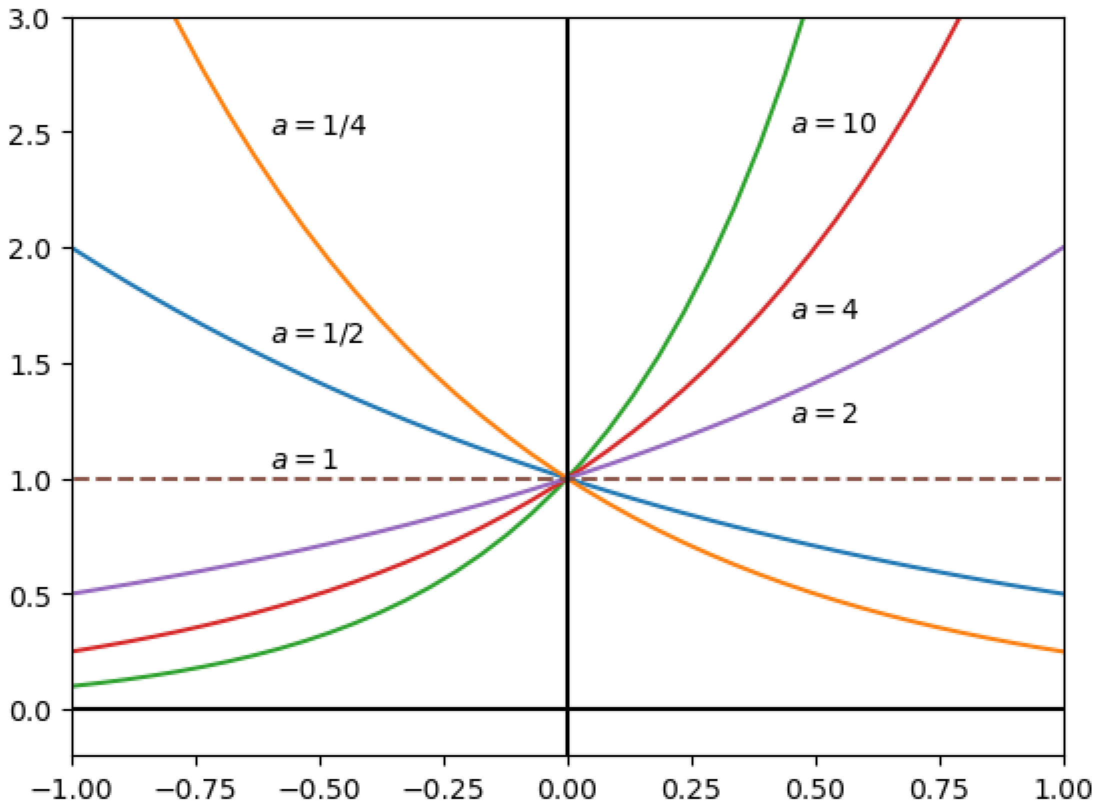
In [8]:
```
x=np.linspace(-1.5,1.0)

def exp_func(a,x):
    return a**x
def plot_exps():
    plt.plot(x,exp_func(0.5,x))
    plt.plot(x,exp_func(0.25,x))
    plt.plot(x,exp_func(10,x))
    plt.plot(x,exp_func(4,x))
    plt.plot(x,exp_func(2,x))
    # plt.plot(x,exp_func(1.5,x))
    plt.plot(x,exp_func(1,x),'--')
    plt.xlim([-1,1])
    plt.ylim([-0.2,3])
    plt.axhline(y = 0, color = 'k', linestyle = '-')
```

```
    plt.axvline(x = 0, color = 'k', linestyle = '-')

    plt.text(-0.6, 2.5, '$a=1/4$')
    plt.text(-0.6, 1.6, '$a=1/2$')
    plt.text(-0.6, 1.05, '$a=1$')
    plt.text(0.45, 2.5, '$a=10$')
    plt.text(0.45, 1.7, '$a=4$')
    plt.text(0.45, 1.25, '$a=2$')
```

In [9]:
```
plot_exps()
```



Properties:

- Domain: $(-\infty, \infty)$.
- In the previous graph, all of the curves pass through the same point (0,1) since $a^0 = 1$.
- As the base a gets larger, the exponential function grows more rapidly (for x>0).
- There are basically three kinds of exponential functions:
  - if $a = 1$, it is a constant.
  - if $0 < a < 1$, the exponential function decreases.
  - if $1 > a$, the exponential function increases.
  - if $a \neq 1$, the range is: $(0, \infty)$.

Laws of exponents

- $a^{x+y} = a^x a^y$
- $a^{x-y} = a^x / a^y$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$

Proof: Appendix G from Stewart - Calculus - Early Transcendentals (Thomson, 2008).

Example 9

Modeling growing populations:

- Population of bacteria in a homogeneous nutrient medium.
- The population doubles every hour.
- $P(0) = 1000$

$$P(1) = 2P(0) = 2 \times 1000 \tag{6}$$
$$P(2) = 2P(1) = 2^2 \times 1000 \tag{7}$$
$$P(3) = 2P(2) = 2^3 \times 1000 \tag{8}$$

In general...

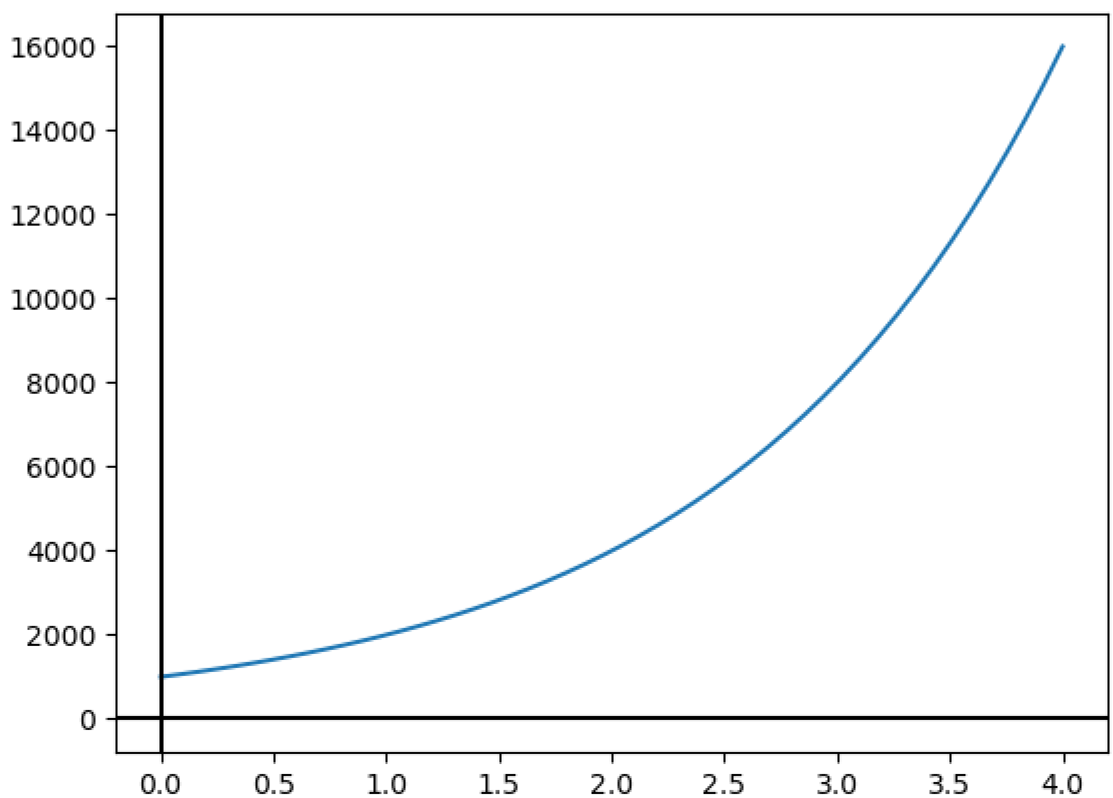
$$P(t) = 1000 \times 2^t \tag{9}$$

In [10]:
```
t=np.linspace(0,4)

def P(t):
    return 1000*2**t

plt.plot(t,P(t))
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
```

Out[10]: <matplotlib.lines.Line2D at 0x16064852590>



What about the human population?

In [11]:
```
import pandas as pd
year=range(1900,2000,10)
population=[1650,1750,1860,2070,2300,2560,3040,3710,4450,5280] #millions

human_population=pd.DataFrame({'year':year, 'Population':population})
human_population
```

Out[11]:

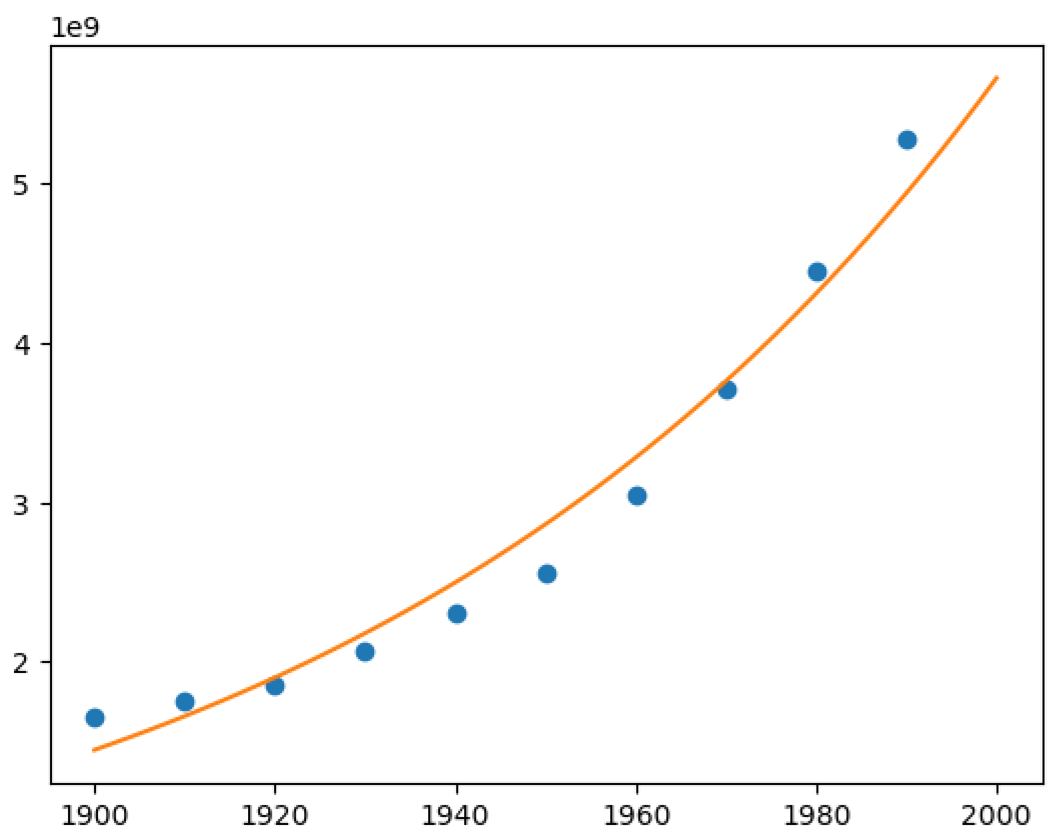
	year	Population
0	1900	1650
1	1910	1750
2	1920	1860
3	1930	2070
4	1940	2300
5	1950	2560
6	1960	3040
7	1970	3710
8	1980	4450
9	1990	5280

In [31]:

```
time=np.linspace(1900,2000)
plt.plot(human_population['year'],human_population['Population']*10**6,'o')
plt.plot(time,0.008079266*(1.013731)**time)
```

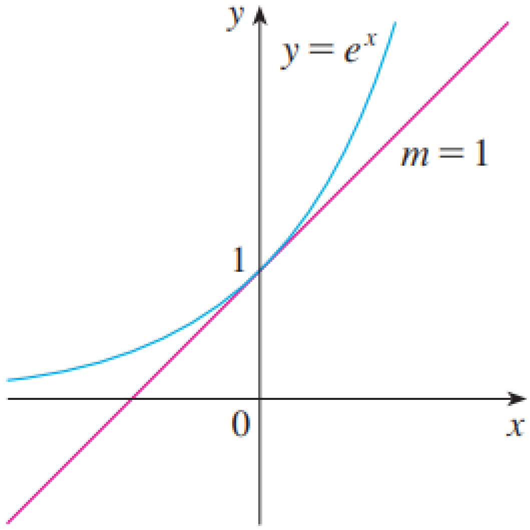
Out[31]:

```
[<matplotlib.lines.Line2D at 0x16067da9600>]
```



The number e

- The most well know exponential is the natural exponential: $y = e^x$ with $e = euler\ number$.
- The natural exponential function crosses the y-axis with a slope of 1.
- The inverse are natural logarithmic functions $\Rightarrow e^{ln(y)} = y$



Logarithmic Function

Definition

$$log_a x = y \Leftrightarrow a^y = x,\ if a > 0\ and\ a \neq 1 \tag{10}$$

- $log_a(a^x) = x$ for every $x \in \Re$
- $a^{log_a x} = x$ for every $x > 0$
- It is the inverse function to exponentiation.
- The logarithm of a number x to the base a, $\log_a(x)$, is the exponent to which b must be raised, to produce x.
- The logarithm of base e is the $natural\ logarithm$, $\ln(x)$.
- $x > 0,\ a > 0$

Properties:

- $\log_a\left(xy\right) = \log_b x + \log_b y$
- $\log_a\left(\frac{x}{y}\right) = \log_b x - \log_b y$
- $\log_a\left(x^p\right) = p\ \log_a x$
- $\log_a \sqrt[p]{x} = \frac{\log_a x}{p}$

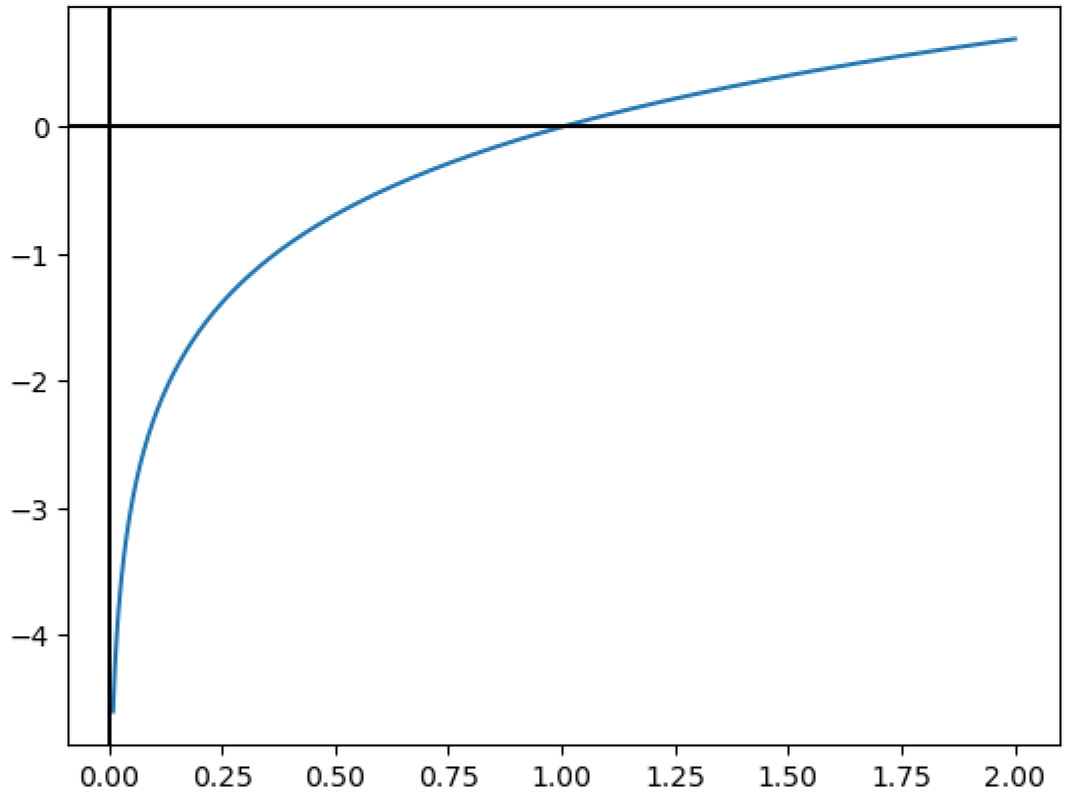
Change of base $b \to k$:

$$\log_k x = \frac{\log_b x}{\log_b k} \tag{11}$$

In [13]:
```
t=np.linspace(0.01,2,1000)

plt.plot(t,np.log(t))
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
```

Out[13]: `<matplotlib.lines.Line2D at 0x160647d7880>`



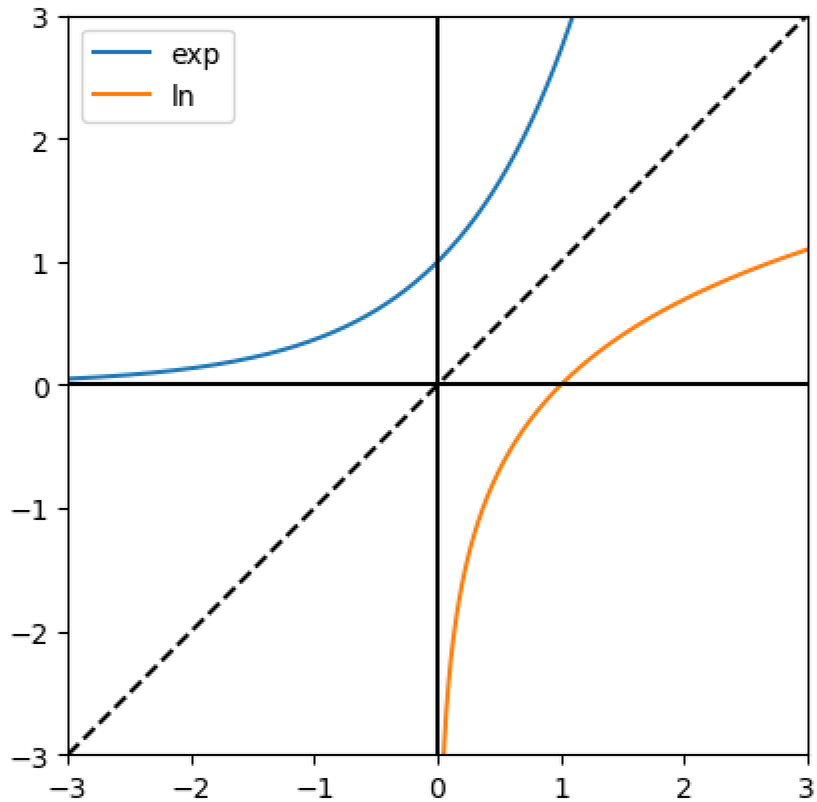
Domain: $\{x|0 < x\} = (0, +\infty)$

Range: $\{y|\ y \in \Re\}$

Exponential & Logarithmic

In [14]:
```
t1=np.linspace(-3,2,1000)
t2=np.linspace(0.01,3,1000)
t3=np.linspace(-3,3,1000)

plt.plot(t1,np.exp(t1))
plt.plot(t2,np.log(t2))
plt.plot(t3,t3,'--' ,color='k')
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
plt.legend(['exp','ln'])
plt.xlim([-3,3])
plt.ylim([-3,3])
ax = plt.gca()
ax.set_aspect('equal')
```



Example 10

Find the **inverse** of $y = 2 \cdot e^{3x}$

Solution $y = (1/3)\ln(x/2)$

Example 11

Sketch the following exponentials. Indicate their domain and range

a. $y = e^{x+1} - 5$

b. $y = e^{-x+1} + 2$

c. Repeat the same with their inverse

Solution

- Inverse of a: $y = \ln(x+5) - 1$
- Inverse of a: $y = 1 - \ln(x-2)$

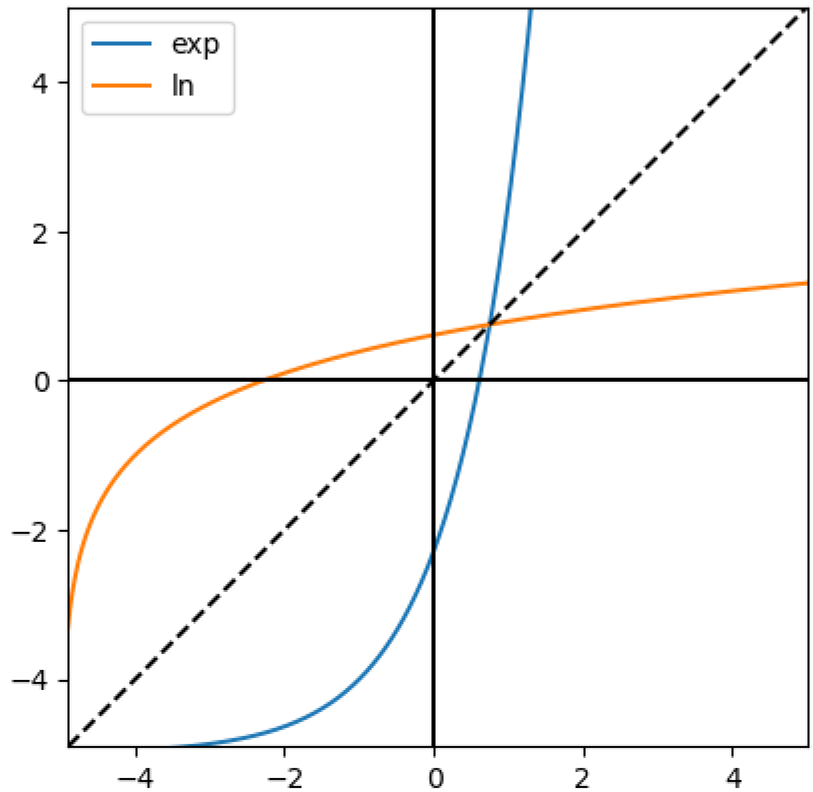
How would you check it?

In [15]:
```
x=np.linspace(-4.9,5,1000)

plt.plot(x,np.exp(x+1)-5)
plt.plot(x,np.log(x+5)-1)
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
plt.legend(['exp','ln'])

plt.plot(x,x,'--' ,color='k')
```

```
plt.xlim([-4.9,5])
plt.ylim([-4.9,5])
ax = plt.gca()
ax.set_aspect('equal')
```

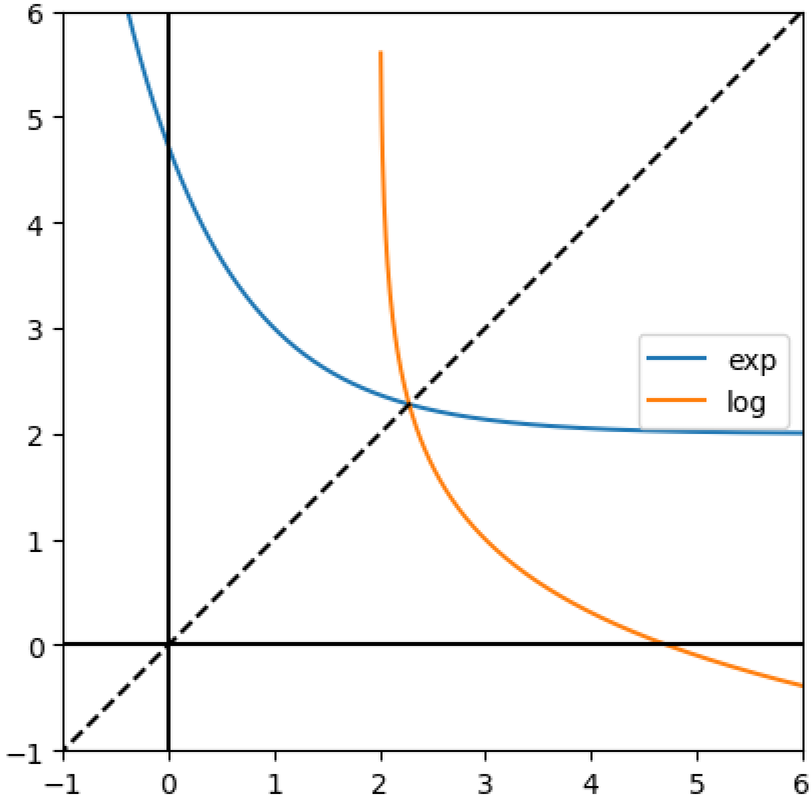


In [16]:
```
x=np.linspace(-3,10,1000)
x1=np.linspace(2.01,10,1000)

plt.plot(x,np.exp(-x+1)+2)
plt.plot(x1,-np.log(x1-2)+1)
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
plt.legend(['exp','log'])

plt.plot(x,x,'--' ,color='k')

plt.xlim([-1,6])
plt.ylim([-1,6])
ax = plt.gca()
ax.set_aspect('equal')
```



Power functions

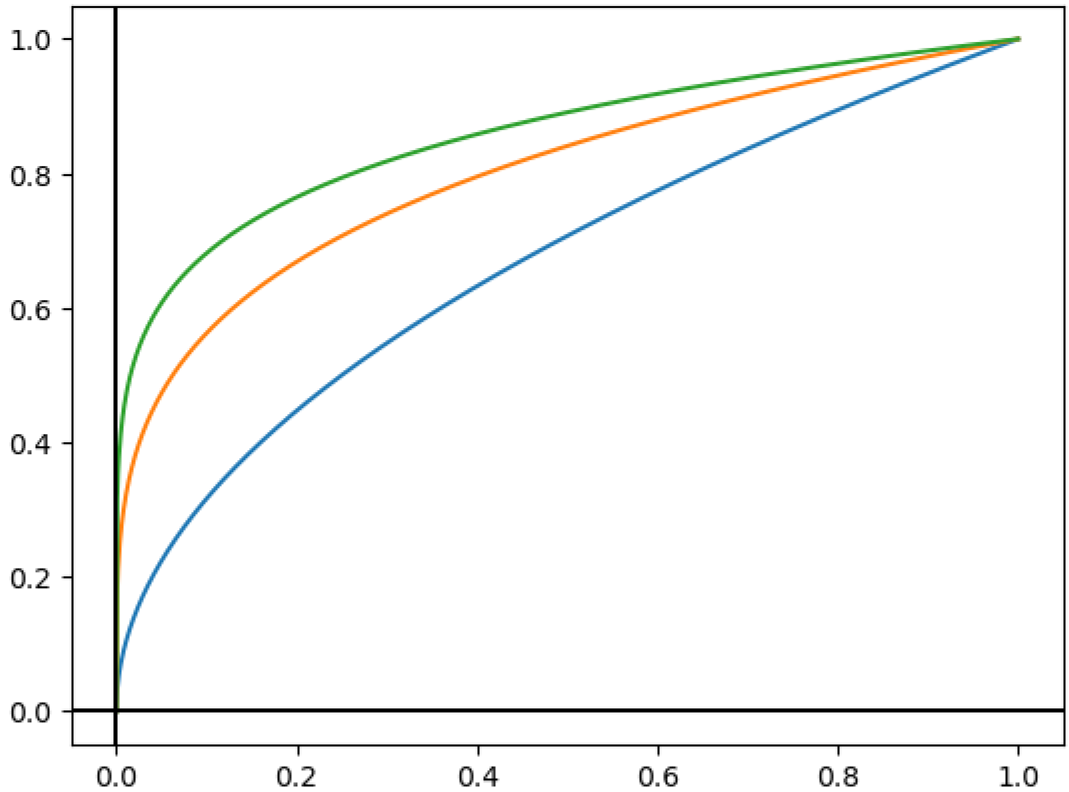
$$f(x) = X^a, \text{ where } a \text{ is a consant.} \tag{12}$$

- When $a = 1, 2, 3, \ldots n$, they are polynomials
- If $a = 1/n$ where n is a positive integer, the function is a **root function**.

In [17]:
```
x=np.linspace(0,1,1000)

plt.plot(x,x**(1/2))
plt.plot(x,x**(1/4))
plt.plot(x,x**(1/6))
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
# plt.legend(['exp','log'])
```

Out[17]:
```
<matplotlib.lines.Line2D at 0x16066867be0>
```

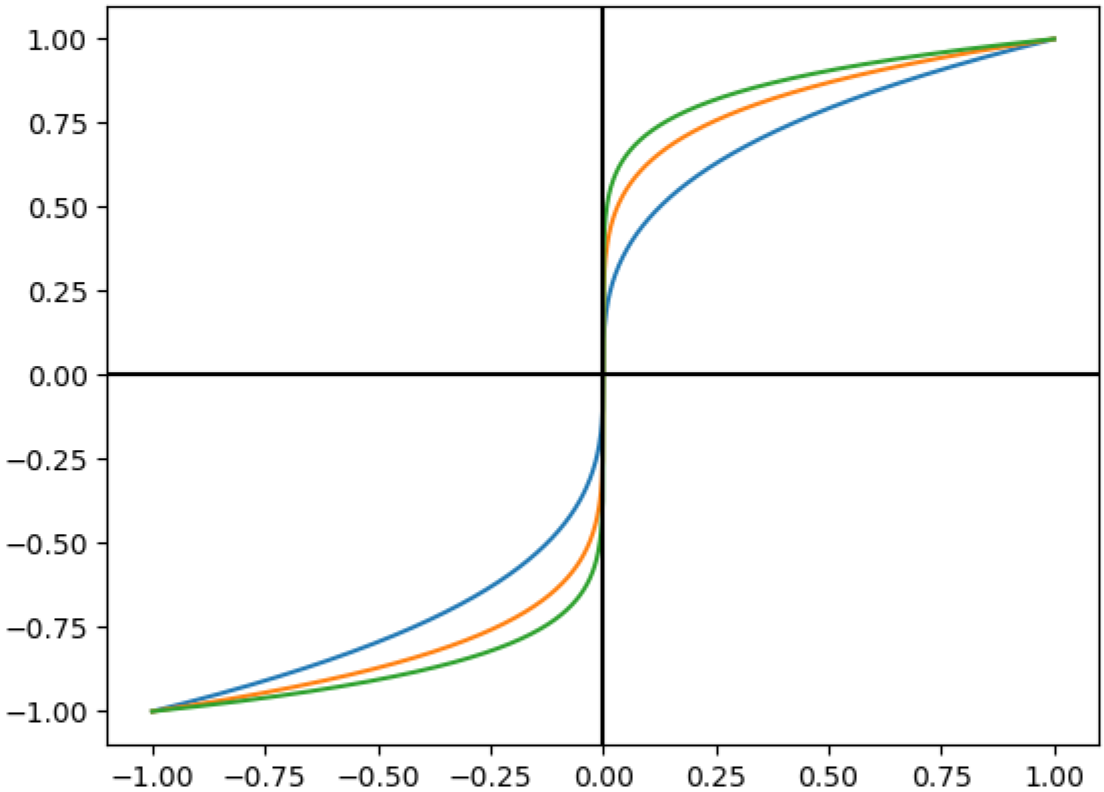


In [18]:

```
x=np.linspace(-1,1,1000)
def root(x,n):
    result = []
    for val in x:
        if val > 0:
            result.append(val ** (1./n))
        elif val < 0:
            result.append(-np.abs(val) ** (1./n))
    return result

plt.plot(x,root(x,3))
plt.plot(x,root(x,5))
plt.plot(x,root(x,7))
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
# plt.legend(['exp','log'])
```

Out[18]: <matplotlib.lines.Line2D at 0x160668fa0e0>

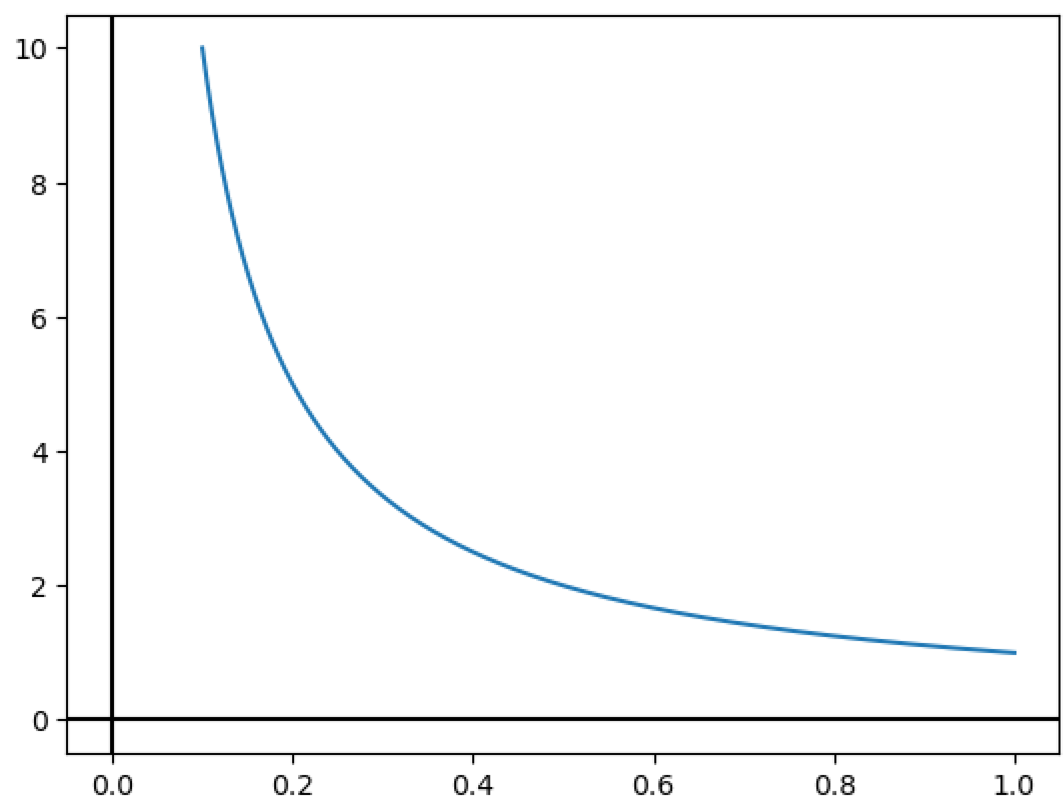


- If $a = -1$, $f(x)$ is a reciprocal function.

In [19]:
```
x=np.linspace(0.1,1,1000)

plt.plot(x,x**(-1))
plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
# plt.legend(['exp','log'])
```

Out[19]: `<matplotlib.lines.Line2D at 0x16066992110>`



Rational Functions

$$f(x) = \frac{P(x)}{Q(x)},\ P(x),\ Q(x)\ polynomials \tag{13}$$

- Domain: $\{x|x \in \Re\ and\ Q(x) \neq 0\}$
- $f(x) = 1/x$ is also a rational function.

For example, let's plot the function:

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4} \tag{14}$$

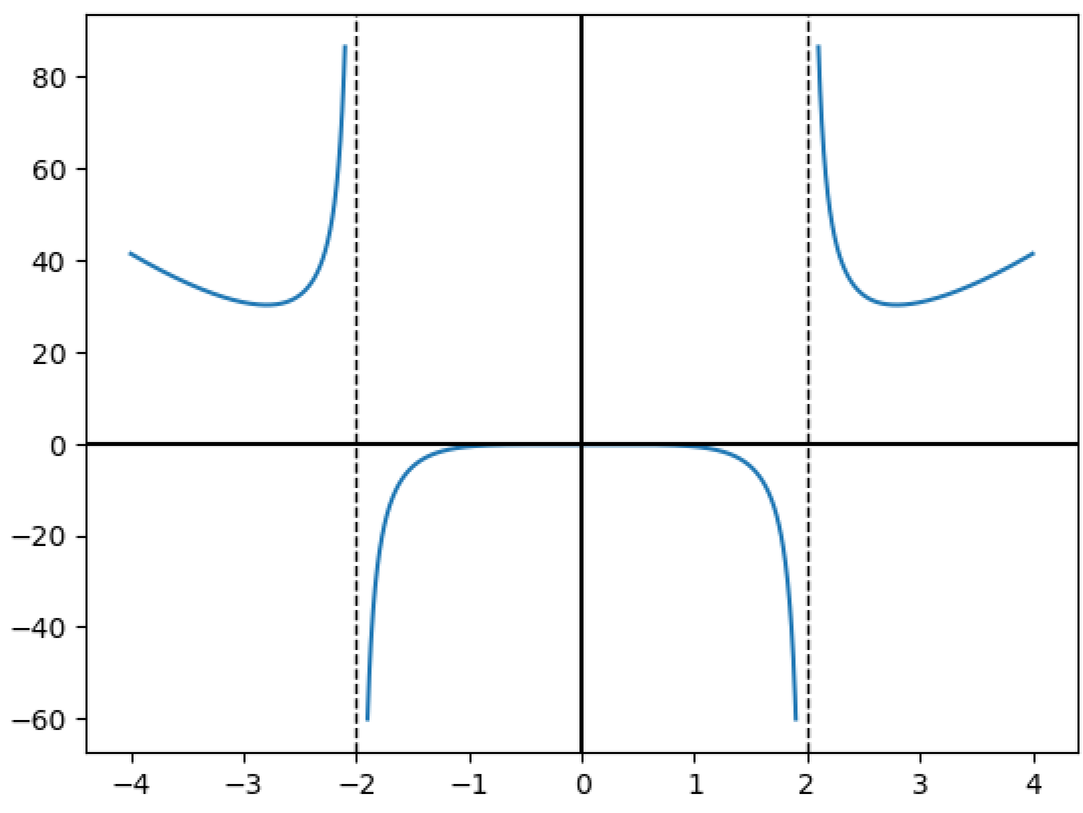
In [20]:
```
x=np.linspace(-1.9,1.9,1000)
x1=np.linspace(2.1,4,1000)
x2=np.linspace(-4,-2.1,1000)

plt.plot(x,(2*x**4-x**2+1)/(x**2-4))
plt.plot(x1,(2*x1**4-x1**2+1)/(x1**2-4),color='tab:blue')
plt.plot(x2,(2*x2**4-x2**2+1)/(x2**2-4),color='tab:blue')

plt.axvline(x = 2, color = 'k', linestyle = '--',linewidth=1)
plt.axvline(x = -2, color = 'k', linestyle = '--',linewidth=1)

plt.axhline(y = 0, color = 'k', linestyle = '-')
plt.axvline(x = 0, color = 'k', linestyle = '-')
```

Out[20]:
```
<matplotlib.lines.Line2D at 0x16067a70a30>
```



Algebraic Functions

A function is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function.

Examples:

$$f(x) = \sqrt{x^2 + 1} \tag{15}$$

$$f(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)^3 \sqrt[3]{x + 1} \tag{16}$$