MAT150 - Summer 2023

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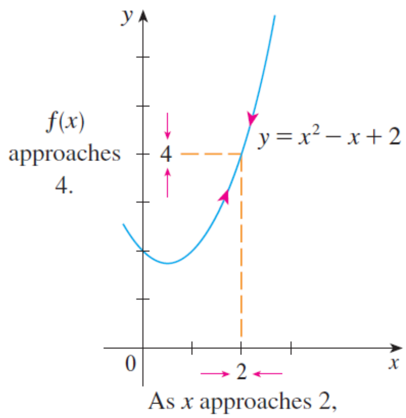
Digipen, Bilbao

The limit of a Function



x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

It appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying "the limit of the function as approaches 2 is equal to 4."

Notation:

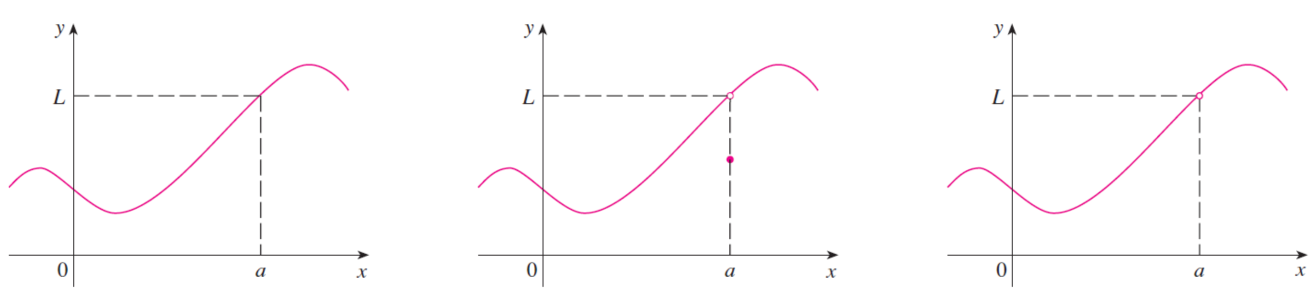
$$\lim_{x \to 2}(x^2 - x + 2) = 4 \tag{1}$$

Definition

$$\lim_{x \to a} f(x) = L \tag{2}$$

Means that we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

- In finding the limit of $f(x)$ as x approaches a, we never consider $x = a$.
- In fact, $f(x)$ need not even be defined when $x = a$.
- The only thing that matters is how it is defined near a.



(a) (b) (c)

In each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.

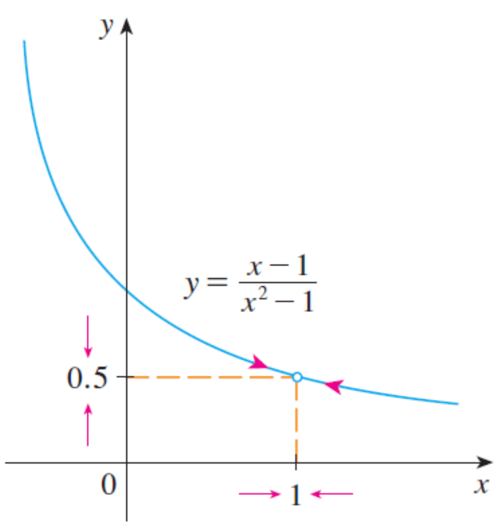
Example: Guess the value of

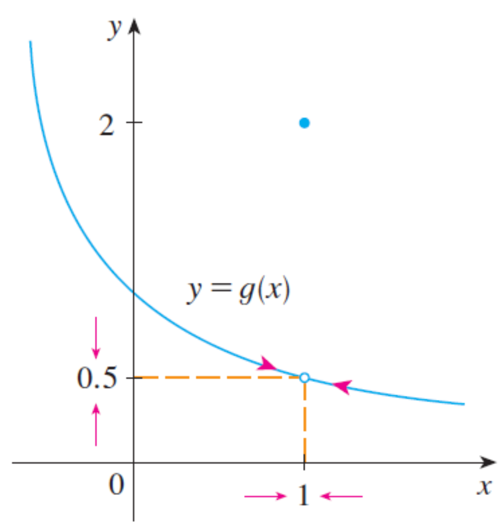
$$\lim_{x \to 1} \frac{x-1}{x^2-1} \tag{3}$$

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

What is the limit of $f(x) = \begin{cases} \frac{x-1}{x^2-1}, & if x \neq 1 \\ 2, & x = 1 \end{cases}$?





In both cases, $\lim_{x \to 1} f(x) = 0.5$

Example: Estimate the value of

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} \tag{4}$$

It seems to be $1/6$

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.16228
± 0.5	0.16553
± 0.1	0.16662
± 0.05	0.16666
± 0.01	0.16667

If we try with even smaller values ...

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 0.0005	0.16800
± 0.0001	0.20000
± 0.00005	0.00000
± 0.00001	0.00000

In [1]:

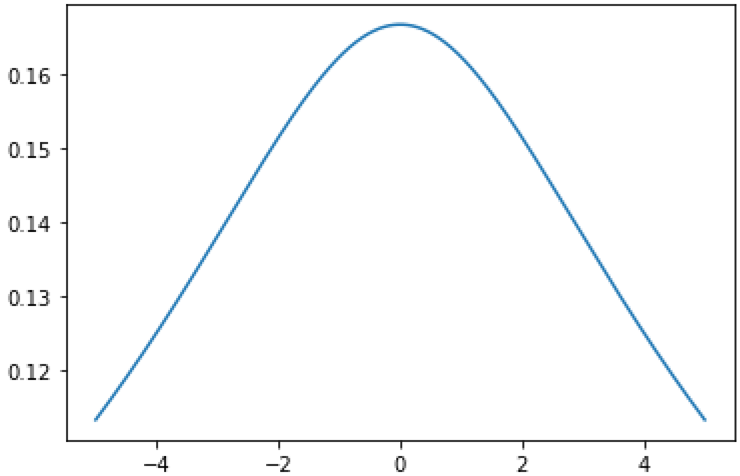
```
import matplotlib.pyplot as plt
import numpy as np

t=np.linspace(-5,5,1000)
# t=np.linspace(-0.1,0.1,1000)
# t=np.linspace(-10**(-6),10**(-6),1000)
# t=np.linspace(-10**(-7),10**(-7),1000)

plt.plot(t,(np.sqrt(t**2+9)-3)/t**2)
```

Out[1]:

```
[<matplotlib.lines.Line2D at 0x2a2e0d78b50>]
```



Definition of Limit

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x-1, & if x \neq 3 \\ 6, & x = 3 \end{cases} \tag{5}$$

Intuitively, $\lim_{x \to 3} f(x) = 5$

- Such phrases as "x is close to 3" and "$f(x)$ gets closer and closer to 5" are vague.
- We ask instead, "How close to 3 does x have to be so that $f(x)$ differs from 5 by less than $0.l$?"
- So, our problem is to find a number δ such that:

$$|f(x) - 5| < 0.1 \quad if \quad |x-3| < \delta \quad but\ x \neq 3 \tag{6}$$

$$|f(x) - 5| < 0.1 \Rightarrow |2x - 1 - 5| < 0.1 \tag{7}$$

$$\Rightarrow |2x - 6| < 0.1 \Rightarrow |2(x-3)| < 0.1 \tag{8}$$

$$\Rightarrow |x - 3| < 0.005 \tag{9}$$

That is,

$$|f(x) - 5| < 0.1 \quad if \quad 0 < |x-3| < 0.005 \tag{10}$$

In a simmilar way:

$$|f(x) - 5| < 0.01 \quad if \quad 0 < |x-3| < 0.0005 \tag{11}$$

$$|f(x) - 5| < 0.001 \quad if \quad 0 < |x-3| < 0.00005 \tag{12}$$

For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below any positive number.

By the same reasoning:

$$|f(x) - 5| < \epsilon \quad if \quad 0 < |x - 3| < \delta = \frac{\epsilon}{2} \tag{13}$$

Or...

$$if \quad 3 - \delta < x < 3 + \delta \quad (x \neq 3) \qquad then \qquad 5 - \epsilon < f(x) < 5 + \epsilon \tag{14}$$

Precise definition of a limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches** a is L, and we write

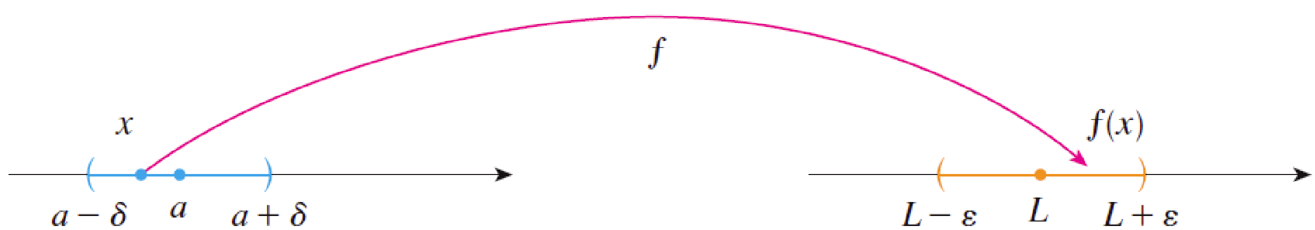
$$\lim_{x \to a} f(x) = L \tag{15}$$

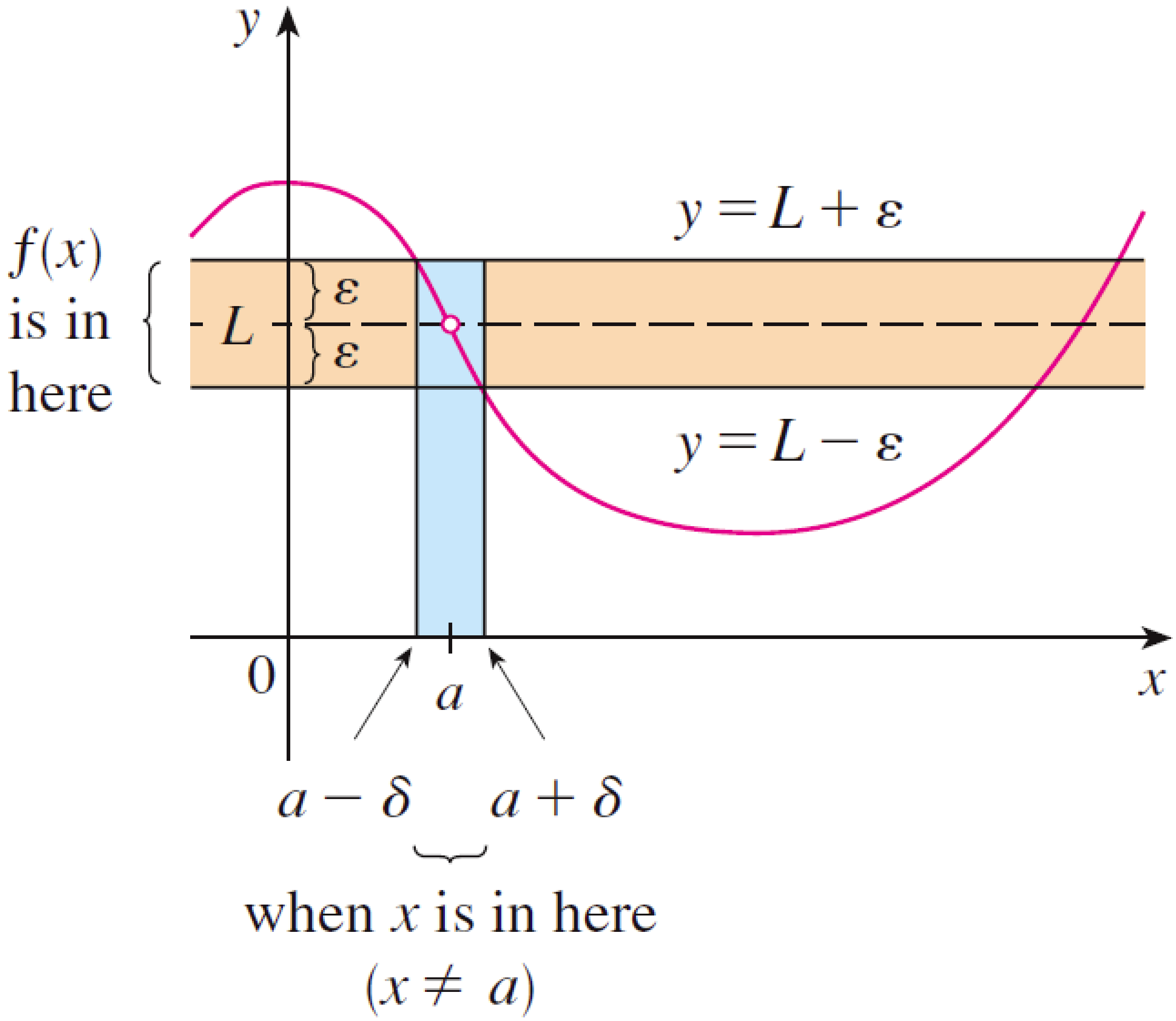
if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$if \quad 0 < |x - a| < \delta, \quad then \quad |f(x) - L| < \epsilon \tag{16}$$

Another way to say the same thing is:

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \quad \epsilon > 0 \quad \exists \delta > 0 / |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon \tag{17}$$



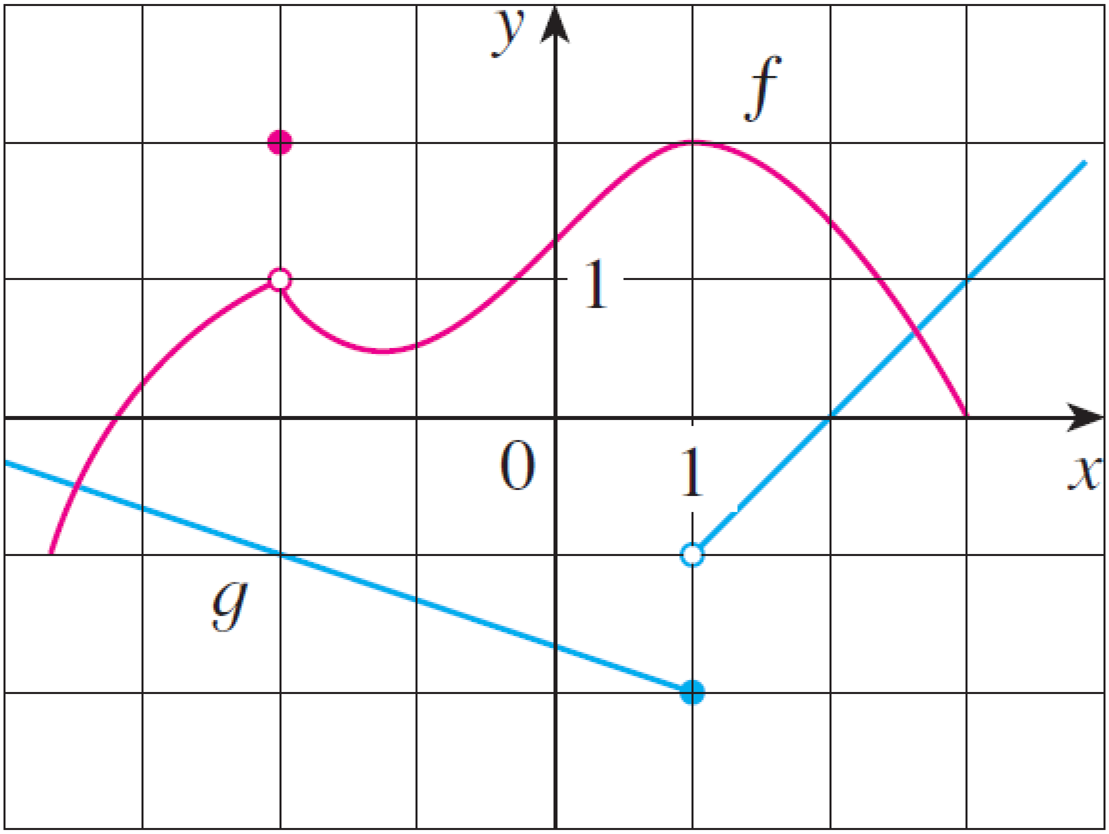


Left-hand limit

$$\lim_{x \to a^-} f(x) = L \Leftrightarrow \forall \ \ \epsilon > 0 \ \ \exists \ \delta > 0 \ / \ a - \delta < x < a \Rightarrow |f(x) - L| < \epsilon \tag{18}$$

Right-hand limit

$$\lim_{x \to a^+} f(x) = L \Leftrightarrow \forall \ \ \epsilon > 0 \ \ \exists \ \delta > 0 \ / \ a < x < a + \delta \Rightarrow |f(x) - L| < \epsilon \tag{19}$$



Theorem: Existence of a limit

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \quad \& \quad it\, is\, UNIQUE \tag{20}$$

Example:

Calculate

$$\lim_{x\to 1} \frac{2x}{x-3} = \frac{2}{1-3} = -1 \tag{21}$$

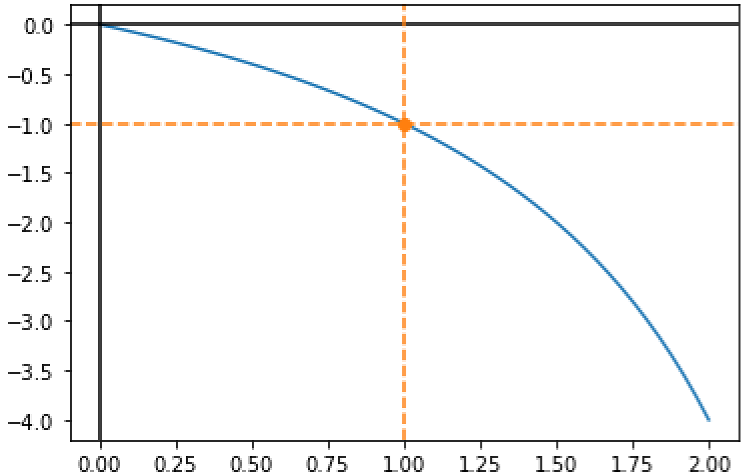
In [11]:
```
import matplotlib.pyplot as plt
import numpy as np


x=np.linspace(0,2)

plt.plot(x,2*x/(x-3))
plt.plot(1,-1,'o')

plt.axvline(x = 0, color = 'k')
plt.axhline(y = 0, color = 'k')
plt.axhline(y = -1,linestyle ='--', color = 'tab:orange')
plt.axvline(x = 1,linestyle ='--', color = 'tab:orange')
```

Out[11]:
```
<matplotlib.lines.Line2D at 0x2a2e11f0ee0>
```



Definition

Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = +\infty \tag{22}$$

means that the values of $f(x)$ can be made arbitrarily large positive by taking x sufficiently close to a, but not equal to a.

That is,

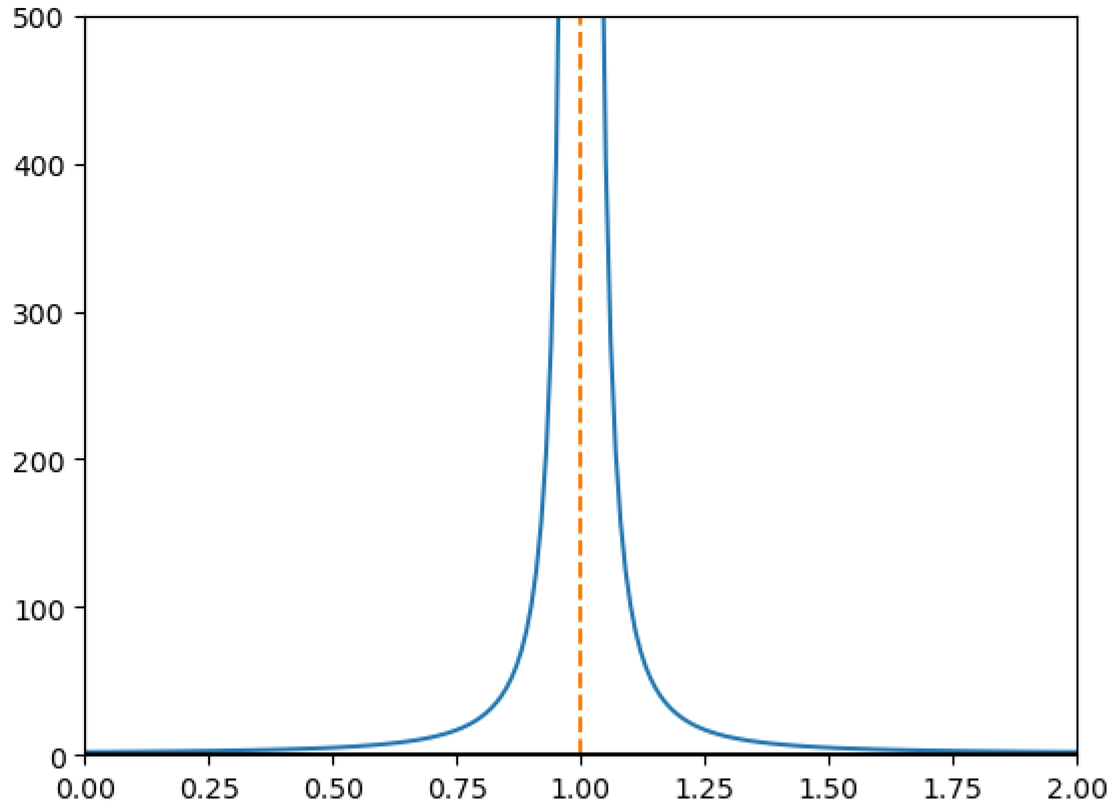
$$\lim_{x \to a} f(x) = +\infty \Leftrightarrow \forall \quad M > 0 \quad \exists \ \delta > 0 \ / \ 0 < |x - a| < \delta \Rightarrow f(x) > M \tag{23}$$

In [46]:

```
import matplotlib.pyplot as plt
import numpy as np

x = np.linspace (1.01,2,100)
x1 = np.linspace (0,0.99,100)
plt.plot(x, 1/(x-1)**2+1)
plt.plot(x1, 1/(x1-1)**2+1,color='tab:blue')
plt.axvline(x = 1, color = 'tab:orange', linestyle = '--')
plt.axvline(x = 0, color = 'k')
plt.axhline(y = 0, color = 'k')
plt.ylim(0,500)
plt.xlim(0,2)
```

Out[46]: (0.0, 2.0)



Definition

Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty \tag{24}$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

That is,

$$\lim_{x \to a} f(x) = -\infty \Leftrightarrow \forall \quad N < 0 \quad \exists \ \delta > 0 \ \big/ \ 0 < |x - a| < \delta \Rightarrow f(x) < N \tag{25}$$

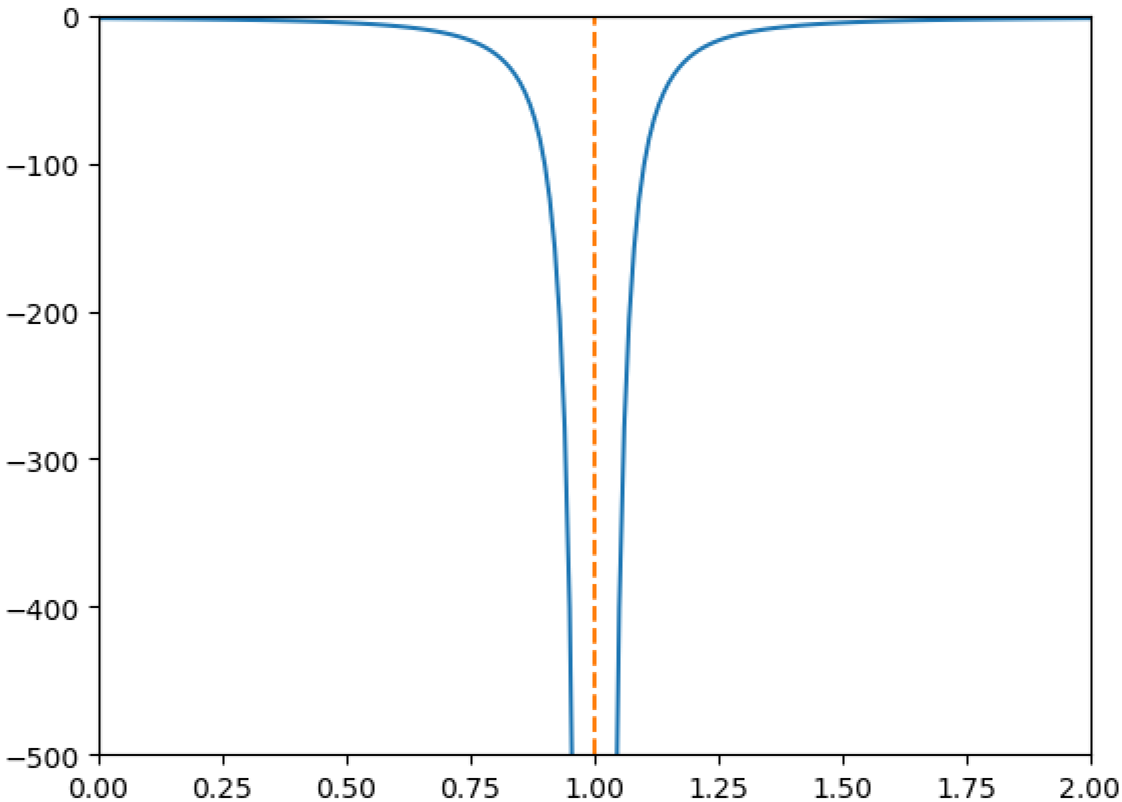
In [47]:

```
import matplotlib.pyplot as plt
import numpy as np

x = np.linspace (1.01,2,100)
x1 = np.linspace (0,0.99,100)
plt.plot(x, -(1/(x-1)**2+1))
plt.plot(x1, -(1/(x1-1)**2+1),color='tab:blue')
plt.axvline(x = 1, color = 'tab:orange', linestyle = '--')
plt.axvline(x = 0, color = 'k')
plt.axhline(y = 0, color = 'k')
plt.ylim(-500,0)
plt.xlim(0,2)
```

Out[47]:

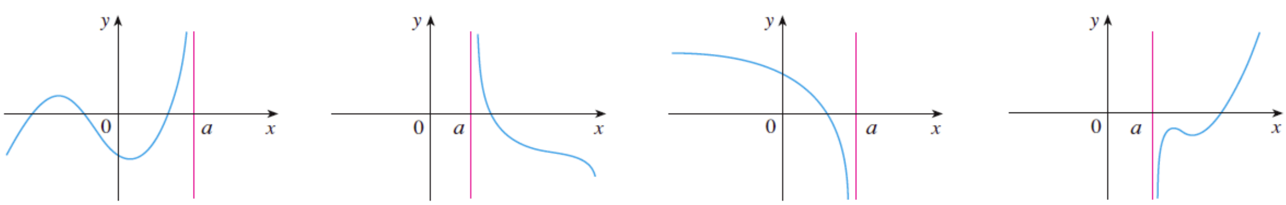
```
(0.0, 2.0)
```



Similar definitions can be given for the one-sided infinite limits.

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty \tag{26}$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty \tag{27}$$



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

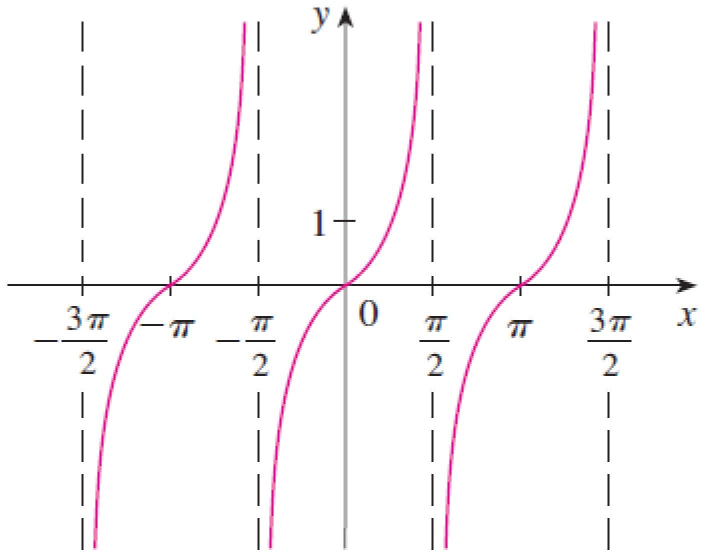
Definition

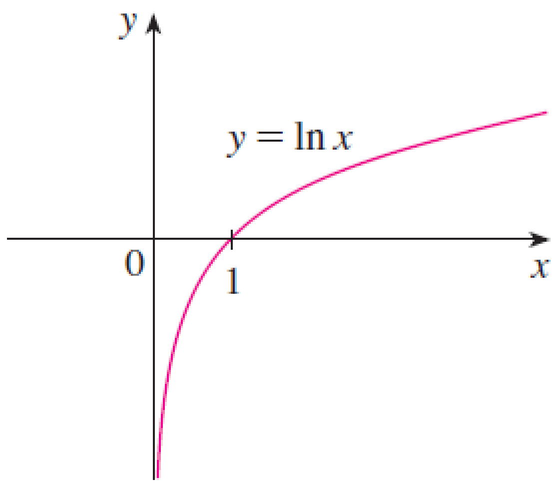
The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Examples





- $\lim_{x \to 0^+} \ln x = -\infty$
- $\lim_{x \to \pi/2^-} \tan x = \infty$
- $\lim_{x \to \pi/2^+} \tan x = -\infty$

Definition

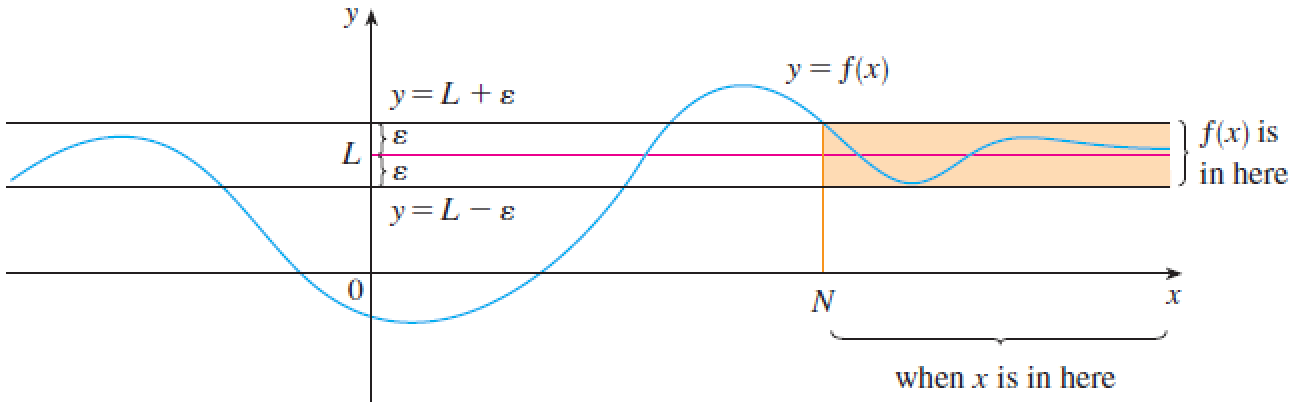
Let f be defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L \tag{28}$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large positive.

That is,

$$\lim_{x \to \infty} f(x) = L \Leftrightarrow \forall \quad \epsilon > 0 \quad \exists \quad M > 0 \; / x > M \Rightarrow |f(x) - L| < \epsilon \tag{29}$$

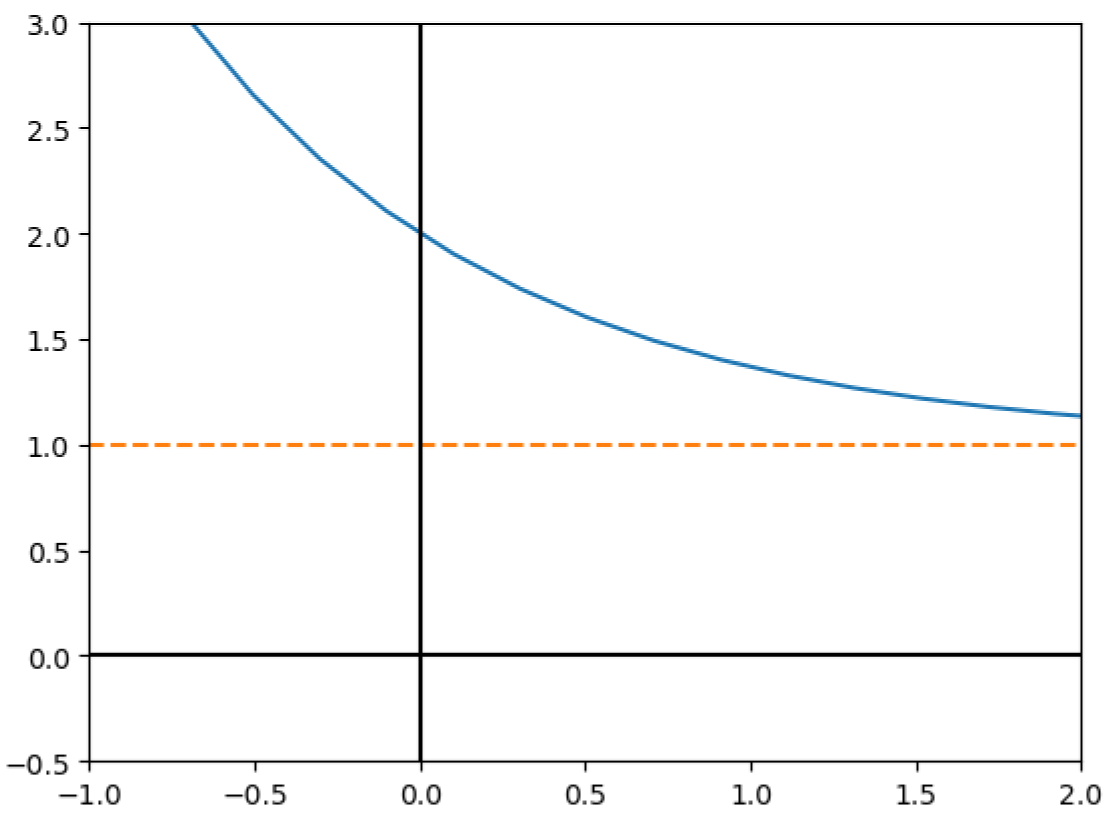


In [80]:
```
import matplotlib.pyplot as plt
import numpy as np

x = np.linspace (-10,10,100)
x1 = np.linspace (0,0.99,100)
plt.plot(x, np.exp(-x)+1)

plt.axhline(y = 1, color = 'tab:orange', linestyle = '--')
plt.axvline(x = 0, color = 'k')
plt.axhline(y = 0, color = 'k')
plt.ylim(-0.5,3)
plt.xlim(-1,2)
```

Out[80]:
```
(-1.0, 2.0)
```



Definition

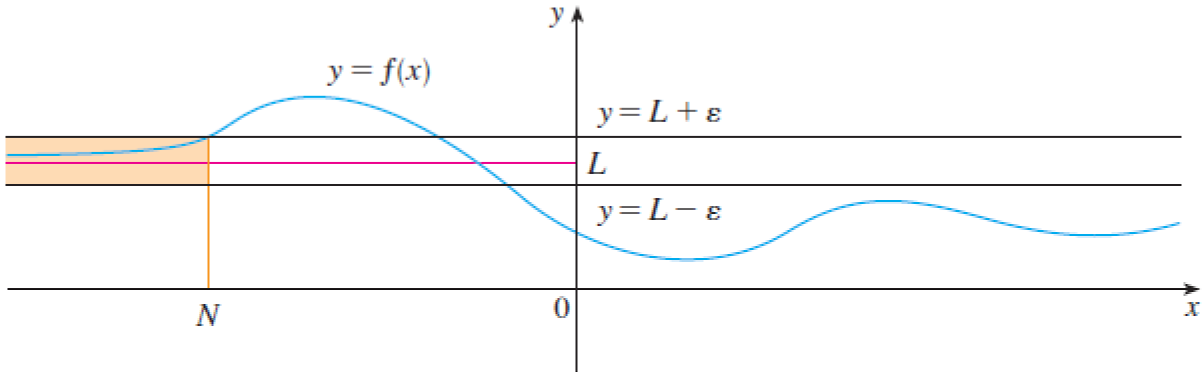
Let f be defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L \tag{30}$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

That is,

$$\lim_{x \to -\infty} f(x) = L \Leftrightarrow \forall \quad \epsilon > 0 \quad \exists \quad M < 0 \ / x < M \Rightarrow |f(x) - L| < \epsilon \tag{31}$$



Definition

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad or \quad \lim_{x \to -\infty} f(x) = L \tag{32}$$

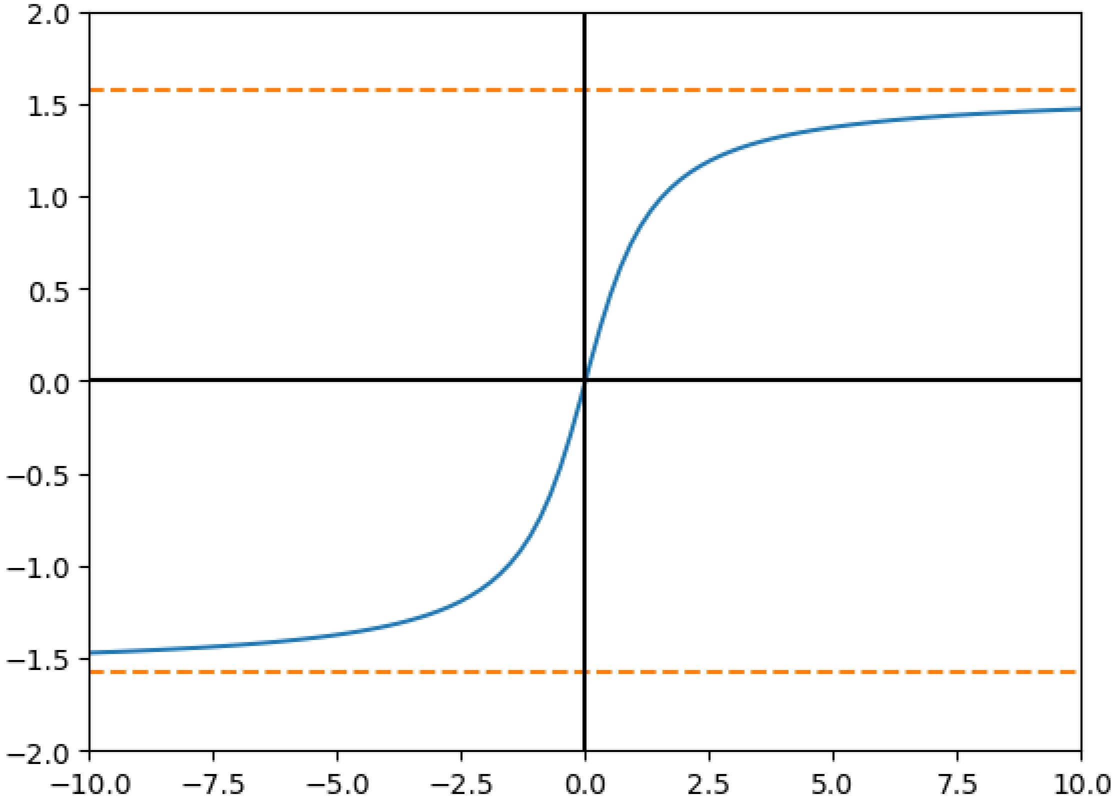
In [45]:
```
import matplotlib.pyplot as plt
import numpy as np

x = np.linspace (-10,10,100)

plt.plot(x, np.arctan(x))

plt.axhline(y = np.pi/2, color='tab:orange', linestyle = '--')
plt.axhline(y = -np.pi/2, color='tab:orange', linestyle = '--')
plt.axvline(x = 0, color = 'k')
plt.axhline(y = 0, color = 'k')
plt.ylim(-2,2)
plt.xlim(-10,10)
```

Out[45]:
```
(-10.0, 10.0)
```



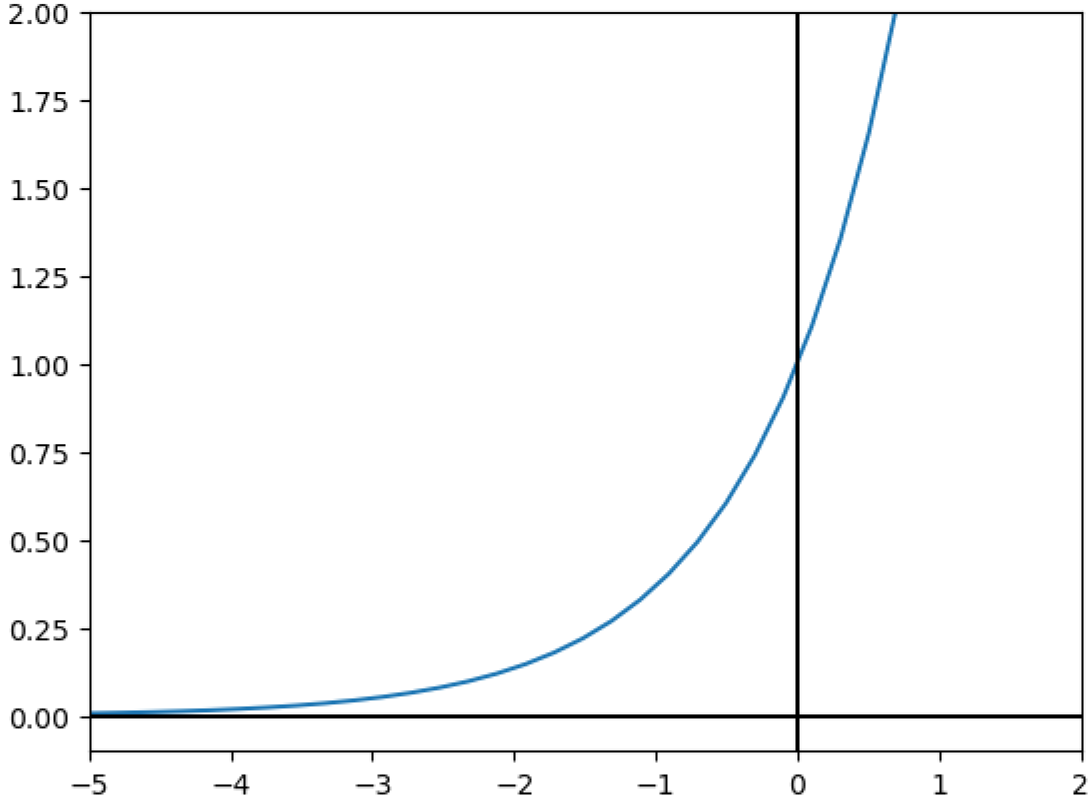
$$\lim_{x \to -\infty} e^x = 0 \tag{33}$$

In [56]:
```
import matplotlib.pyplot as plt
import numpy as np

x = np.linspace (-10,10,100)
```

In [60]:
```
plt.plot(x, np.exp(x))
plt.axvline(x = 0, color = 'k')
plt.axhline(y = 0, color = 'k')
plt.ylim(-0.1,2)
plt.xlim(-5,2)
```

Out[60]:
```
(-5.0, 2.0)
```



Exercise 1

Find the value of the limit if it does exist.

a. $\lim_{x\to 3} \frac{2x}{x-3}$

b. $\lim_{x\to 3} \frac{2x}{(x-3)^2}$

c. $\lim_{x\to 0} f(x)$, $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

d.$\lim_{x\to 0} f(x)$, $f(x) = |x|$

e. $\lim_{x\to 0} f(x)$, $f(x) = \begin{cases} x - x^2, & x < 0 \\ 1, & x = 0 \\ \sin x, & x > 0 \end{cases}$

Plot of (a):

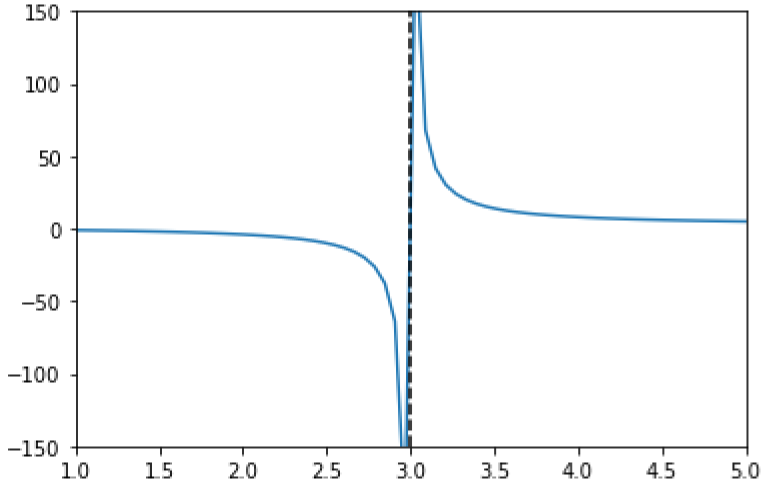
In [6]:
```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0,6,100)

plt.plot(x1,2*x1/(x1-3))
plt.axvline(x = 3, color = 'k', linestyle = '--')

plt.xlim(1,5)
plt.ylim(-150,150)
```

Out[6]:
```
(-150.0, 150.0)
```



Plot of (b):

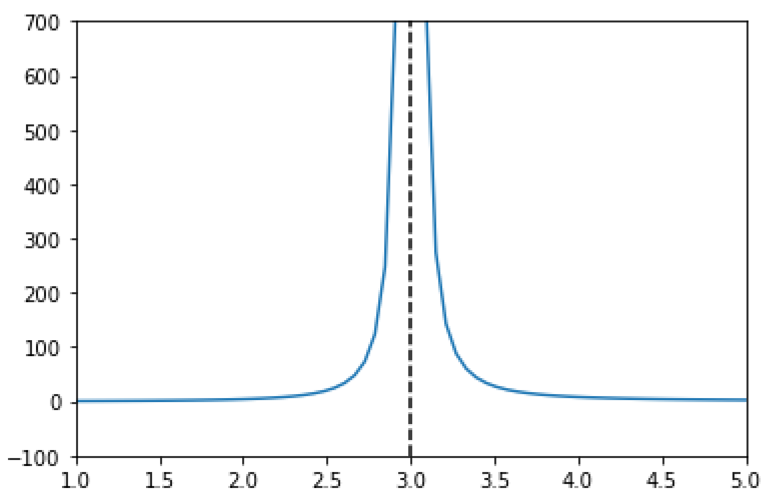
In [13]:
```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0,6,100)

plt.plot(x1,2*x1/(x1-3)**2)
plt.axvline(x = 3, color = 'k', linestyle = '--')

plt.xlim(1,5)
plt.ylim(-100,700)
```

Out[13]:
```
(-100.0, 700.0)
```



Limits Laws

Suppose that c is a constant and the limits below exist

$$\lim_{x\to a} f(x) \quad and \quad \lim_{x\to a} g(x) \tag{34}$$

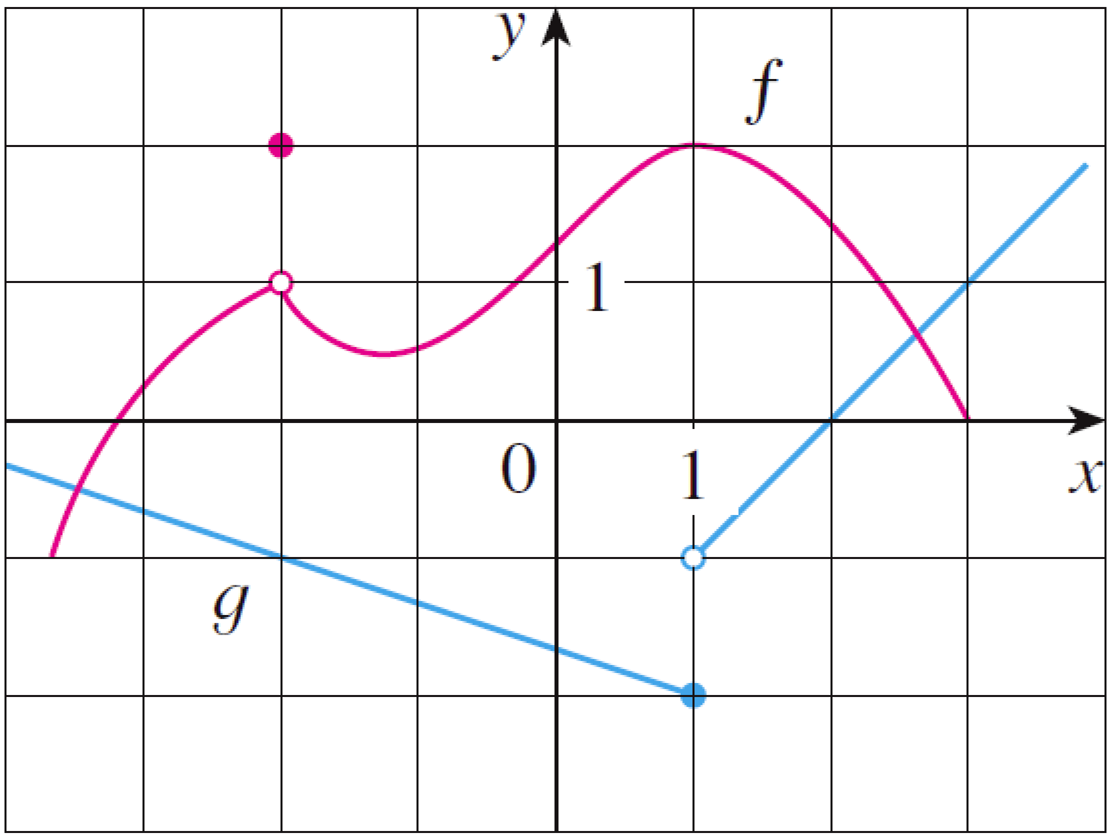
Then,

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
1. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
1. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

1. $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$

1. $\lim_{x\to a}[f(x)/g(x)] = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$ if $\lim_{x\to a} g(x)\neq 0$

Exercise 2



Use the Limit Laws and the graphs of f and g in the figure to evaluate the following limits, if they exist.

a. $\lim_{x\to -2}[f(x)+5g(x)]$

b. $\lim_{x\to 1}[f(x)\cdot g(x)]$

c. $\lim_{x\to 2}[f(x)/g(x)]$

More laws...

1. $\lim_{x\to a}[f(x)^n] = \left[\lim_{x\to a} f(x)\right]^n$, $n=1,2,3,\ldots$

1. $\lim_{x\to a} c = c$

1. $\lim_{x\to a} x = a$

1. $\lim_{x\to a}[x^n] = a^n$

1. $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$, if n is even $a\geq 0$

1. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, if n is even $\lim_{x\to a} f(x)\geq 0$

1. $\lim_{x\to a}\ln\left(f(x)\right) = \ln\left(\lim_{x\to a} f(x)\right)$, $\lim_{x\to a} f(x) > 0$

1. $\lim_{x\to a}\sin\left(f(x)\right) = \sin\left(\lim_{x\to a} f(x)\right)$

Direct substitution property

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a) \tag{35}$$

Example

Find

$$\lim_{x \to \infty} \frac{1}{x} \;\; and \;\; \lim_{x \to -\infty} \frac{1}{x} \tag{36}$$

When x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \quad \frac{1}{10,000} = 0.0001 \quad \frac{1}{1,000,000} = 0.000001 \tag{37}$$

In fact, by taking x large enough, we can make $f(x)$ as close to 0 as we please.

$$\frac{1}{x} < \epsilon \Rightarrow x > \frac{1}{\epsilon} = M \tag{38}$$

Therefore, we can make $1/x$ arbitrarily close to 0 by taking x suficiently large. We can conclude that,

$$\lim_{x \to \infty} \frac{1}{x} = 0 \tag{39}$$

Theorem

If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0 \tag{40}$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0 \tag{41}$$

Theorem

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \tag{42}$$

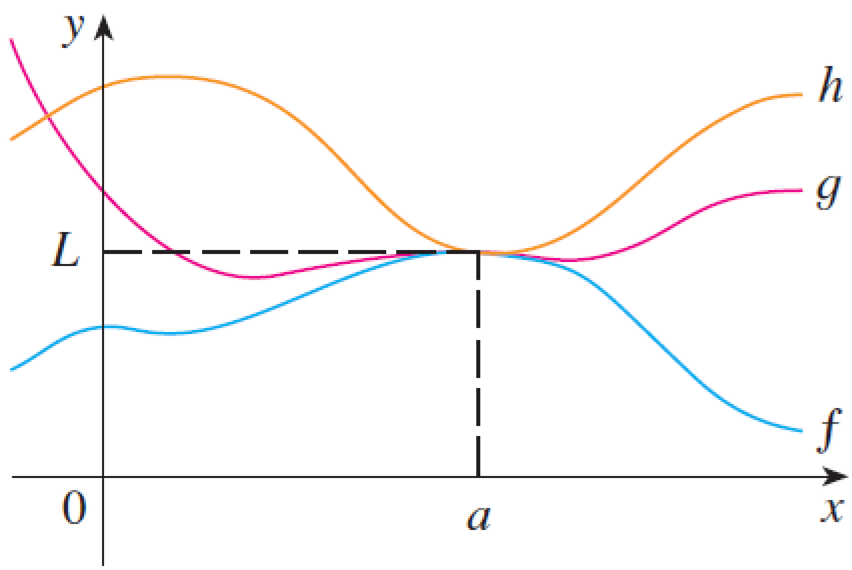
THE SQUEEZE THEOREM

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \tag{43}$$

then

$$\lim_{x \to a} g(x) = L \tag{44}$$



Indeterminations

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 1^{\infty}, \quad \infty^0, \quad 0^0 \tag{45}$$

1. $\frac{0}{0} \to$ simplify

1. $\frac{\infty}{\infty} \to$ comparison, $x^x > a^x > x^a > \log_a x$

1. $\infty - \infty \to$ convert into a $\frac{0}{0}$ or $\frac{\infty}{\infty}$

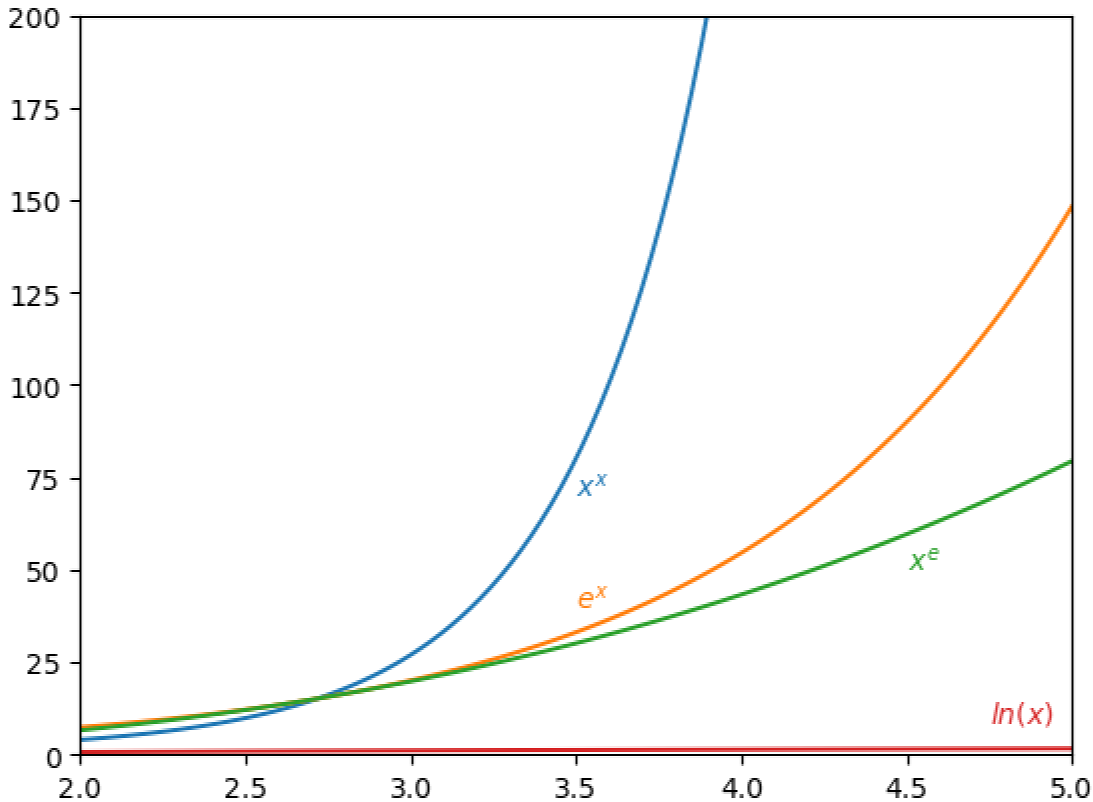
In [97]:
```
import matplotlib.pyplot as plt
import numpy as np

x=np.linspace(2,5,100)

plt.plot(x,x**x)
plt.plot(x,np.e**x)
plt.plot(x,x**np.e)
plt.plot(x,np.log(x))

plt.ylim(0,200)
plt.xlim(2,5)
plt.annotate('$x^x$',xy=(3.5,70), color='tab:blue')
plt.annotate('$e^x$',xy=(3.5,40), color='tab:orange')
plt.annotate('$x^e$',xy=(4.5,50), color='tab:green')
plt.annotate('$ln(x)$',xy=(4.75,9), color='tab:red')
```

Out[97]:
```
Text(4.75, 9, '$ln(x)$')
```



Exercise 3

Find

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \tag{46}$$

Exercise 4

Evaluate the following limits. $\frac{0}{0} \to$ simplify

a. $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$

b. $\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4}$

c. $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$

Exercise 5

Evaluate the following limits. $\frac{\infty}{\infty} \to$ Comparison

a. $\lim_{x \to \infty} \frac{x^3}{3^x}$

b. $\lim_{x \to \infty} \frac{2^x + \ln x}{x^x}$

c. $\lim_{x \to \infty} \frac{\sqrt{x^2+1}-3}{x}$

Exercise 6

Evaluate the following limits. $\infty - \infty \to$ Convert

a. $\lim_{x\to\infty} \sqrt{x+2} - \sqrt{x-2}$

b. $\lim_{x\to 2} \ln\left(x^2-4\right) - \ln\left(x-2\right)$

c. $\lim_{x\to 2} \left[\log_{10}\left(\sqrt{x}+1\right) - 2\log_{10}\sqrt{x} + \log_{10}\left(\sqrt{x}-1\right)\right]$

Review

5 DEFINITION e is the number such that $\ln e = 1$.

$$\log_a x = y = \frac{\ln x}{\ln a}$$

$$\log_a x = y \iff a^y = x$$

$$\log_e x = \ln x$$

3 LAWS OF LOGARITHMS If x and y are positive numbers and r is a rational number, then

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
3. $\ln(x^r) = r\ln x$

11 LAWS OF EXPONENTS If x and y are real numbers and r is rational, then

1. $e^{x+y} = e^x e^y$
2. $e^{x-y} = \frac{e^x}{e^y}$
3. $(e^x)^r = e^{rx}$

$$a^x = e^{x\ln a}$$

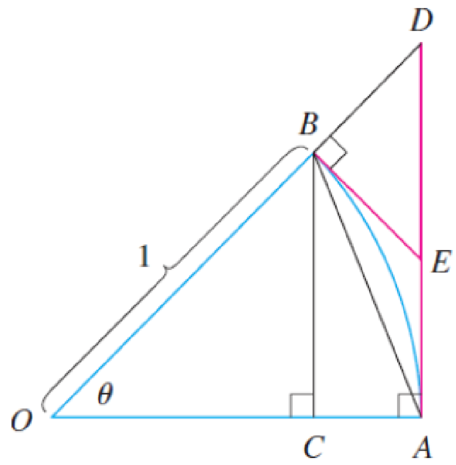
15 LAWS OF EXPONENTS If x and y are real numbers and $a, b > 0$, then

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = a^x/a^y$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

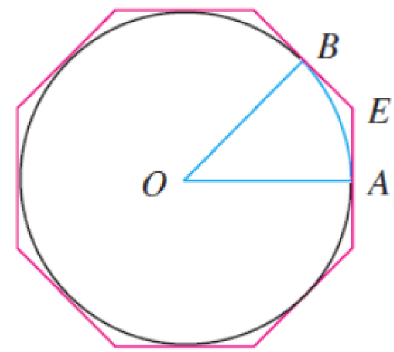
Special Limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{47}$$

Proof:



(a)



(b)

From (b)

$arclenght = radius \cdot \theta(rad) \Rightarrow \ arc\ AB = 1 \cdot \theta$

From (a)

$BC < AB < arc\ AB$ and $BC = \sin\theta$

$$\Rightarrow sin\theta < \theta \tag{48}$$

$$or \ \ \frac{sin\theta}{\theta} < 1 \tag{49}$$

On the other hand,

$$\tan\theta = \frac{AD}{AO} = \frac{AD}{1} > \theta \tag{50}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} > \theta \Rightarrow \cos\theta < \frac{\sin\theta}{\theta} \tag{51}$$

then ...

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \tag{52}$$

taking limits $\theta \to 0$

$$\lim_{x\to 0} \cos\theta < \lim_{x\to 0} \frac{\sin\theta}{\theta} < \lim_{x\to 0} 1 \tag{53}$$

$$\Rightarrow 1 < \lim_{x \to 0} \frac{\sin \theta}{\theta} < 1 \tag{54}$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin \theta}{\theta} = 1 \tag{55}$$

Exercise 7

a. $\lim_{x \to 0} \frac{sin(3x)}{x}$

b. $\lim_{x \to 0} \frac{sin(2x^2)}{-x}$

c. $\lim_{x \to 0} \frac{cos(x)-1}{x}$

d. $\lim_{x \to 0} \frac{cos(2x)-1}{x^2}$

Some useful approximations

- if $f(x) \to 0 \Rightarrow \sin\left(f(x)\right) \approx f(x) \Rightarrow \sin \approx$ line close to the origin
- if $f(x) \to 0 \Rightarrow \cos\left(f(x)\right) \approx 1 - \left[f(x)\right]^2/2 \Rightarrow \cos \approx$ parabola close to the origin

Indeterminations 1^∞, 0^0, ∞^0

This limits are solved applying logarithms to them in both sides. This limits are solved applying and approximating $\ln\left(f(x)\right) \approx f(x) - 1$

1.

$$\lim_{x \to \infty} \left(\frac{x-4}{x-1}\right)^{x-3} \tag{56}$$

1.

$$\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x \tag{57}$$

1.

$$\lim_{x \to 0^+} \left(1 + \sin(4x)\right)^{\cot x} \tag{58}$$

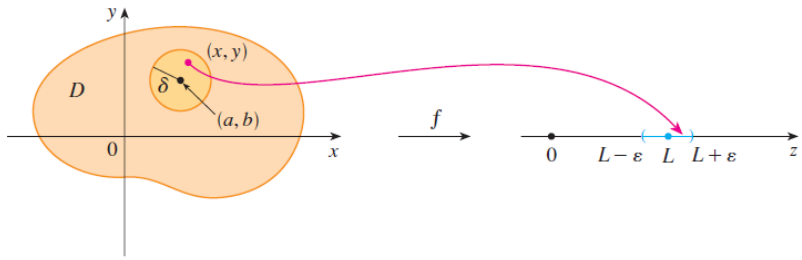
Limits in $\Re^3$

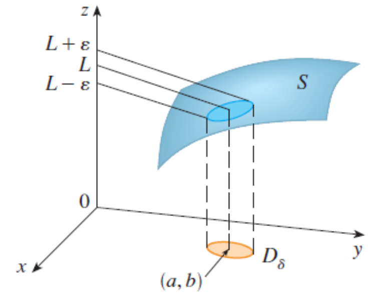
Definition

Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the **limit of** $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \tag{59}$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \epsilon$.





- In $\Re^3$ there are ∞ many directions to approach to the limit, not only left and right.
- If the limit depend on the direction, this is, it depends on θ, it will not exist.
- The limit exists if

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{\rho\to 0 \ \ \forall\, \theta} f(x_0 + \rho\cos\theta, y_0 + \rho\sin\theta) \tag{60}$$

Exercise 8

1.

$$\lim_{(x,y)\to(1,2)} (5x^3 - x^2y^2) \tag{61}$$

1.

$$\lim_{(x,y)\to(1,2)} \frac{x+1}{y-2} \tag{62}$$

1.

$$\lim_{(x,y)\to(0,1)} \left(\frac{x}{y}\cos\frac{x}{y}\right) \tag{63}$$

Exercise 9 Indeterminations ∞/∞

1.

$$\lim_{(x,y)\to(\infty,\infty)} (2x/y^2) \tag{64}$$

1.

$$\lim_{(x,y)\to(\infty,\infty)} (2^x/y^5) \tag{65}$$

1.

$$\lim_{(x,y)\to(\infty,\infty)} \frac{x^3 - 2xy}{xy^2 + x^2y} \tag{66}$$

1.

$$\lim_{(x,y)\to(\infty,\infty)} \frac{2^x + x^2}{3^y + xy} \tag{67}$$

1.

$$\lim_{(x,y)\to(\infty,\infty)} \frac{2^{xy}}{3^x} \tag{68}$$

1.

$$\lim_{(x,y)\to(\infty,\infty)} \frac{\sin x}{y} \tag{69}$$

Exercise 10 Indeterminations $0/0$

1.

$$\lim_{(x,y)\to(9,9)} \frac{x - y}{\sqrt{x} - \sqrt{y}} \tag{70}$$

1.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} \tag{71}$$

1.

$$\lim_{(x,y)\to(0,0)} \frac{\cos x - 1}{y^2} \tag{72}$$

1.

$$\lim_{(x,y)\to(2,0)} \frac{\sin y}{2x - 4} \tag{73}$$
